Stability Analysis and Controller Design for Switched Time-Delay Systems

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ABSTRACT

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In this thesis, the stability analysis and control synthesis for uncertain switched time-delay systems are investigated. It is known that a wide variety of real-world systems are subject to uncertainty and also time-delay in their dynamics. These characteristics, if not taken into consideration in analysis and synthesis, can lead to important problems such as performance degradation or instability in a control system. On the other hand, the switching phenomenon often appears in numerous applications, where abrupt change is inevitable in the system model. Switching behavior in this type of systems can be triggered either by time, or by the state of the system. A theoretical framework to study various features of switched systems in the presence of uncertainty and time-delay (both neutral and retarded) would be of particular interest in important applications such as network control systems, power systems and communication networks.

To address the problem of robust stability for the class of uncertain switched systems with unknown time-varying delay discussed above, sufficient conditions in the form of linear matrix inequalities (LMI) are derived. An adaptive switching control algorithm is then proposed for the stabilization of uncertain discrete time-delay systems subject to disturbance. It is assumed that the discrete time-delay system is highly uncertain, such that a single fixed controller cannot stabilize it effectively. Sufficient conditions are provided subsequently for the stability of switched time-delay systems with polytopic-type uncertainties. Moreover, an adaptive control scheme is provided to stabilize the uncertain neutral time-delay systems when the upper bounds on the system uncertainties are not available *a priori*. Simulations are provided throughout the thesis to support the theoretical results.

To my parents

for their love and sacrifice

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Last, but not least, I thank my beloved parents for their love and warm support, which brings belief and hope into my life. And to my brothers, whose love and patience have backed me up to keep moving on.

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List of Abbreviations

PWA	Piecewise Affine
LMI	Linear Matrix Inequality
SOS	Sum-of-Squares
ODE	Ordinary Differential Equation
FDE	Functional Differential Equation
NFDE	Neutral Functional Differential Equation
RFDE	Retarded Differential Equation
PDE	Partial Differential Equation
NCS	Networked Control System
PID	Proportional-Integral-Derivative
CPU	Central Processing Unit

Chapter 1

Introduction

1.1 Motivation

Stability analysis for time-delay systems has attracted many researchers in recent years due to its importance in several practical control systems (see e.g. [68], [21] and references therein). In numerous control applications such as multi-vehicle coordination, manufacturing systems, spacecraft exploration missions and network control systems, time-delay has a visible impact on the output response. Neglecting delay in control design procedure can result in poor closed-loop performance or even instability. In addition, all practical systems are subject to disturbances, and sometimes abrupt change in their parameters (due, for example, to changes in the operating point). These are important and inevitable factors in many physical control systems which are often neglected in performance analysis and control synthesis, leading to significant discrepancies between the real behavior of the system and what one might expect from simulations. Several methods are proposed in the literature to reduce (or compensate for) the effect of uncertainties and disturbances in control problems [53], [46], [32], [104], [63], [95]. It is to be noted that the magnitude of delay in a real-world system may also be subject to change and/or uncertainties, and no information about the range of delay may be available [34], [11], [47].

To study the class of time-delay systems which are subject to abrupt change in their parameters, the problem is often formulated in the framework of switched systems. This introduces a relatively new line of research, namely *switched time-delay systems*. Problems of this type occur, for example, in rate assignment in wireless communication networks [103], networked control systems (NCS) [61], power systems [51] and biological systems [82], [14]. From a more practical view-point, each model in the family of models considered in switched systems is also subject to parameter perturbation and uncertainties. While there is a rich literature on the stability analysis and controller design for time-delay systems [68], [21], switched systems [43], [44], [85], [12], uncertain time-delay systems [63], [10] and switched time-delay systems [81], [80], the proposed techniques cannot effectively handle the class of uncertain switched time-delay systems. This special class of control systems has important industrial applications, and is the main focus of the present thesis. It is desired to develop tractable stability conditions for this type of systems, and design efficient controllers for them accordingly.

While in most of the existing work on the stability of time-delay systems it is assumed that upper bounds on the magnitude of the delay and the parameter uncertainties are available and one single controller would be able to meet the design specifications, in many physical applications there is no *a priori* knowledge on these bounds. In fact, stability may not be achievable by one single controller when the magnitude of uncertainties (or parameter variation) is "large". On the other hand, supervisory switching control schemes (e.g. the ones in [26], [81], [108]) which are proposed to stabilize highly uncertain systems (or systems with parameter jump) are often ineffective in presence of time-delay. This signifies the importance of developing a novel technique to handle the class of uncertain switched time-delay systems noted in the previous paragraph as the main objective of this thesis.

Consider a highly uncertain time-delay system subject to disturbances. It is desired to design a stabilizing control scheme and develop stability conditions with the following properties:

- The effect of disturbances is rejected as much as possible.
- The scheme is robust to the parameter uncertainties as well as time-varying delays.
- The control strategy should be stable in the presence of sudden jumps in the magnitude of delay and also in system parameters (robustness to abrupt changes in system dynamics).

Briefly, the main objective of this research is to *analyze the stability and stabilization of highly uncertain time-delay systems.*

1.2 Background

Many Physical systems cannot be modeled with sufficient accuracy by an ordinary differential equation (ODE) [21]. In particular, in many systems the future evolution of the state variables not only depends on their current value, but also on their past values. Such systems are called *time-delay systems.* Time-delay systems are often described by functional differential equations (FDE). Such systems are sometimes referred to as systems with aftereffect, or systems with time lags.

Practical applications of FDEs in the 20th century motivate a lot of interest in the mathematical studies of FDEs. Some fundamental results on the zeros of quasiploynomials were developed by Pontryagin [68]. Chebotarev studied the Routh-Hurwitz type problems for quasipolynomials. On the other hand, Krasovskii extended the Lyapunov's theory to time-delay systems in 1956, and Razumikhin proposed a method to avoid functionals in the Lyapunov stability analysis [22]. While most of these results are for the case of retarded time-delay systems (which will be defined later), some extensions to the neutral functional differential equations (NFDE) are provided by Kolmanovskii and Nosov [40]. For a thorough study of the history of time-delay systems control, see the survey paper [68] and references therein. In particular, there are numerous books covering various aspects of retarded systems [7], [40], [50], [23].

1.3 Examples of Time-Delay Systems

A time-delay model can be used either to reflect the true time lag, or to approximate high-order systems. In this section, some practical examples of time-delay systems with emphasis on engineering problems are borrowed from [21] and [22] to show the applicability of this type of system.

Regenerative chatter in metal cutting. [21], [22] The metal cutting process on a lathe can be mathematically modeled as

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = -F_t(f + y(t) - y(t-\tau))$$

where m, c and k are the inertia, damping, and stiffness of the machine, the delay time $\tau = 2 * \pi/\omega$ corresponds to the time workpiece to complete one revolution, and $F_t(\cdot)$ is the thrust force depending on the instantaneous chip thickness $f + y(t) - y(t-\tau)$ (for a comprehensive study of this model see [83]).

Internal combustion engine. [21], [22] The Mean Torque Production Model is often used to describe an internal combustion engine. In such a system,, the crankshaft rotation is modeled by the following equation of motion

$$J\dot{\omega}(t) = T_i(t - \tau_i) - T_f(t) - T_{load}(t)$$

where T_i is the torque generated by the engine, τ_i is the delay due to engine cycle delays such as fuel-air mixing, ignition delay, and cylinder pressure force propagation. For more examples, see [36].

Communication Delay in Remote Control and Networks. Time delay also occurs in the real-time control systems which use a common serial network. Examples of such applications include networked control systems [55] and master-slave teleoperation in robotics [3].

Feedback Loop Delay. Sometimes delay appears in the output measurement and transmission in feedback control systems. This occurs, for instance, in finite-time observer design [49], and tuning of active vibration absorbers [64].

1.4 Mathematical Representation of Continuous Time-Delay Systems

A simple dynamic model in the form of FDE that is often used to describe retarded time-delay systems is as follows

$$\dot{x}(t) = f(t, x(t), x(t - r_1), x(t - r_2), ..., x(t - r_K))$$

where $x \in \mathbf{R}^n$ is the state variable, and $r_1, ..., r_K \in \mathbf{R}$ are bounded time-delays such that $0 < r_1 < ... < r_K$. The largest delay will be denoted by r, which in this case is r_K. In the above equation, it is clear that the system behavior depends on the current value of the state $x(t)$ and its past values $x(t - r_i)$, $i = 1, ..., K$. A more general form for describing time-delay systems with FDEs is as follows

$$\begin{aligned} \dot{x}(t) &= f(t, x_t) \\ x_t(\theta) &= x(t + \theta), \quad -r \leq \theta \leq 0 \\ x(\theta) &= \phi(\theta), \quad t_0 - r \leq \theta \leq t_0 \end{aligned} \tag{1.1}$$

where $\phi(\theta)$ is the initial condition of time-delay system (1.1) on $[t_0 - h, t_0]$. For a function $\phi \in C([a, b], \mathbf{R}^n)$, define the continuous norm $||\cdot||_c$ as

$$||\phi||_c = \max_{a \le \theta \le b} ||\phi(\theta)||$$

(Here $||\cdot||$ represents the 2-norm $||\cdot||_2$). The more general form of the time-delay systems, namely neutral time-delay systems, can be described by:

$$\dot{x}(t) = f(t, x_t, \dot{x}_t)$$

In a neutral time-delay system, $\dot{x}(t)$ and $\dot{x}(t+\theta)$ $(-r \le \theta \le 0)$ both appear in the differential equation.

1.5 Stability Analysis of Continuous Time-Delay Systems

Assume without loss of generality that the FDE presented in (1.1) admits the trivial solution $x(t) = 0$. The concept of a stable, asymptotically stable and uniformly asymptotically stable time-delay system is given in the following definition [21].

Definition 1.1. *For the system (1.1), the solution $x(t) = 0$ is said to be stable if for any $t_0 \in \mathbf{R}$ and any $\epsilon > 0$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that $||x_{t_0}||_c < \delta$ implies that $||x(t)|| < \epsilon$ for $t \ge t_0$. It is said to be asymptotically stable if it is stable, and for any $t_0 \in \mathbf{R}$, and any $\epsilon > 0$, there exists a $\delta_a = \delta_a(t_0, \epsilon) > 0$ such that $||x_{t_0}||_c < \delta_a$ implies $\lim_{t\to\infty} x(t) = 0$. It is said to be uniformly stable (US) if it is stable and $\delta(t_0, \epsilon)$ can be chosen independently of t_0. It is uniformly asymptotically stable (UAS) if it is US and there is a $\delta_a > 0$ such that for any $\eta > 0$, there exists a $T = T(\delta_a, \eta)$, such that $||x_{t_0}||_c < \delta$ implies $||x(t)|| < \eta$ for $t \ge t_0 + T$ and $t_0 \in \mathbf{R}$. It is globally (uniformly) asymptotically stable if it is (uniformly) asymptotically stable and δ_a can be any arbitrarily large (but finite) number.*

Stability analysis of continuous time-delay systems can be divided to two different categories: frequency domain analysis and time-domain approach. The main focus of this thesis is directed towards time-domain approaches, and hence some relevant techniques and tools for stability analysis of time-delay systems in time domain is reviewed briefly in the sequel.

1.5.1 Lyapunov-Krasovskii Functional Method

In the study of systems without delay, one of the most popular methods is utilizing Lyapunov function to determine the stability of a dynamical equation. For a system without delay this requires to find a Lyapunov function $V(t, x(t))$, which in some sense is a measure of deviating the solution $x(t)$ from the origin. Since for a time-delay system the value of state $x(t)$ in the interval $[t-r, t]$ (i.e. x_t) is needed to specify the system's evolution, the corresponding Lyapunov function turns out to be a functional $V(t, x_t)$ depending on x_t, which measures the deviation of x_t from the origin.

From the above discussion, one can extend the Lyapunov stability analysis to time-delay systems [41]. According to the Lyapunov-Krasovskii stability theorem, a time-delay system is asymptotically stable if there exists a Lyapunov-Krasovskii functional $V(t, x_t)$ such that

$$u(||x(t)||) \leq V(t, x_t) \leq v(||x_t||_c) \tag{1.2}$$

$$\dot{V}(t, x_t) \leq -w(||x(t)||)$$

where u, v, and w are strictly increasing functions satisfying $u(0) = v(0) = w(0) = 0$, $\lim_{s\to\infty} u(s) = \infty$, and $||\cdot||_c$ is defined as

$$||\phi|| = \max_{-r\leq\theta\leq 0} ||\phi(\theta)||$$

In fact, the Inverse Lyapunov Theorem is also valid in this case [41].

Various Lyapunov-Krasovskii functionals have been proposed in the literature.

For example, to find the stability conditions for the following linear time-delay system

$$\dot{x}(t) = Ax(t) + Bx(t-r) \tag{1.3}$$

the Lyapunov-Krasovskii functional given below can be used

$$V(x_t) = x^T(t)Px(t) + \int_{-r}^{0} x^T(t+\theta)Sx(t+\theta)d\theta$$

where P and S are positive definite matrices. This leads to the stability criterion

$$\begin{bmatrix} PA + A^TP + S & PB \\ B^TP & -S \end{bmatrix} < 0 \tag{1.4}$$

The stability condition given by the above inequality is delay independent. One of the most popular methods for finding a delay-dependent criterion is to rewrite (1.3) in the following form

$$\dot{x}(t) = (A+B)x(t) + B\int_{-r}^{0} [Ax(t+\theta) + Bx(t+\theta-r)]\theta \tag{1.5}$$

and use the following Lyapunov-Krasovskii functional

$$V(x_t) = x^T(t)Px(t) + \int_{-r}^{0}\int_{t+\theta}^{t} x^T(\theta)R_1x(\theta)d\theta ds + \int_{-r}^{0}\int_{t+\theta}^{t} x^T(\theta-r)R_2x(\theta-r)d\theta ds$$

where P, R_1 and R_2 are positive definite matrices with appropriate dimensions. This leads to the following stability criterion

$$\begin{bmatrix} P(A+B) + (A+B)^TP + rR_1 + rR_2 & rPAB & rPB^2 \\ * & -rR_1 & 0 \\ * & * & -rR_2 \end{bmatrix} < 0$$

where "*" denotes the symmetric entries. The dependency of this criterion to the value of the delay is evident. Note that different stability criteria with different degrees of conservatism are introduced in the literature using various Lyapunov-Krasovskii functionals.

1.5.2 Razumikhin Approach

Since the Lyapunov-Krasovskii functional uses the state variable $x(t)$ in the interval $[t-r, t]$, it requires to manipulate certain functionals, which makes the application of the Lyapunov-Krasovskii Theorem rather difficult. As an alternative, the Razumikhin Theorem uses only functions rather than functionals. This method is described next.

Razumikhin [22] showed that for a time-delay system to be asymptotically stable, it is sufficient to require

$$\dot{V}(x(t)) \leq -w(||x(t)||)$$

whenever

$$V(t+\theta, x(t+\theta)) \leq p(V(t, x(t))), \qquad \text{for all} \quad \theta \in [-r, 0]$$

where $p(\cdot)$ is a continuous nondecreasing function satisfying $p(s) > s$ for all $s > 0$. Choose the Lyapunov function $V(x(t)) = x^T(t)Px(t)$, and set $w(s) = \epsilon s^2$, $p(s) = (1+\epsilon)s$, with $\epsilon > 0$. Then

$$\dot{V}(x(t)) + \alpha[(1+\epsilon)V(x(t)) - V(x(t-r))] < -\epsilon||x(t)||^2$$

for some $\alpha \geq 0$. Since ϵ is an arbitrary value (which can be chosen sufficiently small), the following stability criterion is obtained

$$\begin{bmatrix} PA + A^TP + \alpha P & PB \\ * & -\alpha P \end{bmatrix} < 0 \tag{1.6}$$

A comparison between (1.6) and (1.4) reveals that the results obtained from the Razumikhin Theorem are more conservative than the ones obtained from the Lyapunov-Krasovskii Theorem. However, the Razumikhin Theorem is also applicable to systems with a large class of time-varying delays, which is its main advantage over the Lyapunov-Krasovskii functional-based method. Delay-dependent stability criteria using the Razumikhin Theorem can similarly be derived based on model transformation as stated in the previous subsection.

1.6 Discrete Time-Delay Systems

In this section, the problem of stability of discrete time-delay systems is investigated briefly. A function $\gamma : \mathbf{R}^+ \to \mathbf{R}^+$ is said to be of class $\mathcal{K}(\gamma \in \mathcal{K})$ if it is continuous, zero at zero, and strictly increasing. For a given real number $r > 0$, let $C([-r, 0], \mathbf{R}^n) = \{\Psi : [-r, 0] \to \mathbf{R}^n, \Psi \text{ is continuous}\}$ be given. Given a positive integer m, define $\mathbf{N}_{-m} = \{-m, ..., -1, 0\}$, and let $||\phi||_m = \max_{\theta \in \mathbf{N}_{-m}}\{||\phi(\theta)||\}$.

Consider the discrete time-delay system given below

$$\begin{aligned} x(n+1) &= f(n, x_n), \quad n \geq n_0 \\ x_{n_0} &= \phi \end{aligned} \tag{1.7}$$

where $x \in \mathbf{R}^n$, $n_0 \in \mathbf{N}$, and $f \in C(\mathbf{N} \times C([-m, 0], \mathbf{R}^n), \mathbf{R}^n)$. Furthermore, $\phi \in C([-m, 0], \mathbf{R}^n)$, where $m \in \mathbf{N}$ represents the delay in system (1.7), and $x_n \in C([-m, 0], \mathbf{R}^n)$ is defined by $x_n(s) = x(n + s)$ for any $s \in [-m, 0]$.

It is assumed that $f(n, 0) = 0$ so that (1.7) admits the trivial solution. It is also assumed that system (1.7) has a unique solution, denoted by $x(n) = x(n, n_0, \phi)$, for any given initial data: $n_0 \in \mathbf{N}$ and $\phi \in C([-m, 0], \mathbf{R}^n)$. The following definition is presented for the stability type of discrete time-delay systems [45].

Definition 1.2. *The trivial solution of system (1.7) is said to be uniformly stable if for any given initial data:* $n_0 \in \mathbf{N}, x_{n_0} = \phi$, *and for any* $\epsilon > 0$, *there exists a* $\delta = \delta(\epsilon) > 0$ *independent of* n_0 *such that when* $||\phi||_m \leq \delta$, *the following inequality holds:*

$$||x(n, n_0, \phi)|| \leq \epsilon \;\; \text{for any} \;\; n \geq n_0, \quad n_0 \in \mathbf{N} \tag{1.8}$$

The trivial solution is called uniformly attractive if for each given initial data: $n_0 \in \mathbf{N}$, $x_{n_0} = \phi$, *and any* $\eta > 0$, *there exist a positive real number* $\sigma = \sigma(\eta) > 0$

and a positive integer $K = K(\eta) > 0$*, where both* σ *and* K *are independent of* n_0*, such that when* $||\phi||_m < \sigma$ *and* $n \geq n_0 + K$*, the following inequality holds:*

$$||x(n, n_0, \phi)|| \leq \eta \quad i.e. \quad \lim_{n \to \infty} ||x(n, n_0, \phi)|| = 0 \tag{1.9}$$

The trivial solution of (1.7) is said to be uniformly asymptotically stable if for any initial data: $n_0 \in \mathbf{N}$, $x_{n_0} = \phi$*, the trivial solution of the system (1.7) is uniformly stable and uniformly attractive.*

In [45], several Razumikhin Theorems are introduced for different stability types of system (1.7). One of those theorems is given below, and the interested reader is referred to [45] and references therein for a comprehensive study.

Theorem 1.1. *Assume that there exist functions* $c_1, c_2 \in \mathcal{K}$ *and a positive definite function* $V(n, x)$ *such that the following conditions hold:*

- $c_1(||x||) \leq V(n, x) \leq c_2(||x||)$*;*
- *for any* $\phi \in C([-m, 0], \mathbf{R}^n)$ *and some* $s \in \mathbf{N}_{-m}$*,* $V(n, \phi(0)) \geq V(n + s, \phi(s))$ *implies* $V(n + 1, f(n, \phi)) \leq V(n, \phi(0))$*;*
- *for any* $\phi \in C([-m, 0], \mathbf{R}^n)$ *and some* $s \in \mathbf{N}_{-m} - \{0\}$*,* $V(n, \phi(0)) \leq V(n + s, \phi(s))$ *implies* $V(n + 1, f(n, \phi)) \leq \max_{\theta \in \mathbf{N}_{-m}}\{V(n + \theta, \phi(\theta))\}$.

Then system (1.7) is uniformly stable.

The discrete-time counterpart of the Lyapunov-Krasovskii Theorem is presented below [77].

Theorem 1.2. *Assume there exist continuous functional* $V : C \to \mathbf{R}$ *and continuous nondecreasing functions* v *and* $w : \mathbf{R}^+ \to \mathbf{R}^+$ *with the properties* $v(0) = w(0) = 0$*,* $v(s) > 0$ *and* $w(s) > 0$*,* $\forall s > 0$*, such that*

$$0 < V(x_n) \leq v(||x_n||_C), \quad V(0) = 0$$

$$\Delta V(x_n) := V(x_{n+1}) - V(x_n) \leq -w(||x_n||_C),$$

$\forall x_n \in C$ satisfying $x(n+1) = f'(x_n)$. Then the solution $x = 0$ of $x(n+1) = f'(x_n)$ is asymptotically stable.

1.7 Robust Stability Problem

In this section, some important problems concerning robust stability of time-delay systems are presented using a simple example [28]. Let the nominal model for a linear continuous system with time-varying delay be given by

$$\dot{x}(t) = Ax(t) + A_d x(t - d(t)), \quad t > 0$$
$$x(t) = \phi(t), \quad t \in [-h, 0]$$

where $x(t) \in \mathbf{R}^n$ is the state vector, A and A_d are constant matrices of appropriate dimensions. The delay $d(t)$ is assumed to be a time-varying differentiable function that satisfies

$$0 \leq d(t) \leq h, \qquad |\dot{d}(t)| \leq \mu \tag{1.10}$$

where h and μ are positive constants. The initial condition $\phi(t)$ is a continuous and differentiable vector-valued function of t over the interval $[-h, 0]$.

In what follows two classes of uncertainties, namely time-varying norm-bounded and polytopic-type uncertainties, are investigated. The first class is concerned with the uncertainties of the following form in the system model

$$[\Delta A(t) \;\; \Delta A_d(t)] = DF(t)[E_a \;\; E_b]$$

where D, E_a and E_d are appropriately-dimensioned constant matrices, and $F(t)$ is an unknown real matrix with Lebesgue measurable elements satisfying the inequality

$$F^T(t)F(t) \leq I, \qquad \forall t$$

The second class of uncertainties in the system matrices is formulated as the real convex polytopic constraint

$$[A \quad A_d] \in \Omega, \quad \Omega := \{[A(\xi) \quad A_d(\xi)] = \sum_{j=1}^{p} \xi_j [A_j \quad A_{dj}], \quad \sum_{j=1}^{p} \xi_j = 1, \quad \xi_j \geq 0\}$$

where $A_j, A_{dj}, \ j = 1, ..., p$, are constant matrices with appropriate dimensions, and $\xi_j, \ j = 1, ..., p$, are time-invariant uncertainties.

1.7.1 Switched Systems

The continuous-time dynamic model of many practical systems is governed by some discrete switching signals. This class of systems is referred to as *switched systems*, and its examples include biological systems, air traffic control, NCS and power systems, to name only a few (see, e.g. [43] and references therein). It is known that highly uncertain systems and the system which are subject to abrupt changes can be effectively modeled in the framework of switched systems [43], [30], [31], [2], [1], [75], [57].

A switched system is formulated as a family of plant models along with a switching event. The switching event can be divided into two categories: state-dependent and time-dependent. In state-dependent events, the state space is partitioned into finite or infinite number of operating regions. When the system state hits a region's boundary, the corresponding dynamics is activated. Piecewise-affine (PWA) systems are systems with state-dependent switching signals. In time-dependent events, the switching signal is a piecewise continuous signal which activates a particular subsystem in a set of subsystems (family of models). Time-dependent switched systems, on the other hand, are systems governed by prescribed switching signals which are explicit functions of time.

Several methods are developed recently to analyze the stability of switched systems. Common Lyapunov functions were proposed in [43] to provide sufficient conditions for the stability of the switched systems under arbitrary switching signals. The concept of stability under slow switching was first introduced by Hespanha in [30], [31], where the notion of the *average dwell time* was introduced, which provides a sufficient condition to maintain the stability of the switched system. The multiple Lyapunov functions approach proposed in the literature also provide sufficient conditions for the stability of switched systems [35], [80]. This approach, however, provides less conservative stability conditions compared to the common Lyapunov function technique.

Piecewise-Affine Systems

PWA systems are a class of switched systems with state-dependent switching events. This type of system model is often used to analyze the stability of certain real-world control problems with nondifferentiable nonlinearities such as saturation, dead-zone, backlash, etc. [75]. These systems consist of a family of subsystems and a state space partitioning. When the system state hits a certain boundary, the corresponding subsystem is activated. In [35], [5], quadratic Lyapunov functions are employed and linear matrix inequalities (LMI) are subsequently derived as stability conditions for the PWA systems. Sum-of-squares (SOS) polynomials were proposed recently as candidate Lyapunov functions (see e.g. [75] and references therein).

1.7.2 Switched Time-Delay Systems

A class of time-delay systems whose dynamics are subject to abrupt change has attracted considerable attention recently, due to its application to several engineering problems. This type of system can be modeled efficiently as a switched time-delay system; e.g., see [54], [33] for an example of rate-assignment in a wireless communication network, [38], [55], for examples of networked control systems, and [51] for power systems. Some of the results proposed in the literature for the analysis of switched time-delay systems are as follows. In [99], [105] common Lyapunov functionals are employed to obtain sufficient conditions for the stability of switched time-delay systems. The problem of finding proper switching signals to stabilize switched time-delay systems is addressed in [39], where it is also shown that a switching strategy which stabilizes a delay-free switched system, can also stabilize a switched time-delay system with a sufficiently small delay. The main shortcoming of the stability analysis proposed in the above works is their conservatism, as they employ common Lyapunov functionals.

A multiple Lyapunov functionals approach is used in [80] to obtain less conservative conditions for the exponential stability and L_2-gain analysis of switched time-delay systems with arbitrary switching signals. The above work also provides the minimum time required between the consecutive switching instants (dwell time) to guarantee the exponential stability. Furthermore, the multiple Lyapunov functionals approach is used in [56], [42] to find stability conditions for PWA systems with time-delay.

While there are several results available in the literature for the stability analysis of switched time-delay systems, most of the existing work is mainly focused on the systems with fixed models. However, it is known that in a real-world switched system, each subsystem in the family of plants is subject to uncertainties and perturbation. As one of the contributions of this thesis, the problem of switched time-delay systems with norm-bounded and also polytopic uncertainties is considered, and the corresponding stability criteria are provided accordingly.

1.8 Contributions of Thesis

The main contributions of this thesis are as follows:

- Chapter 2 addresses the problem of robust stability of PWA uncertain systems with unknown time-varying delay in the state. It is assumed that the uncertainties are norm-bounded, and that the upper bounds on the state delay and its rate of change are available. A set of LMIs is derived which provide sufficient conditions for the stability of the system. These conditions depend on the upper bound of the delay. First, new delay-dependent LMI conditions are derived for the stability of PWA time-delay systems. Second, the stability conditions are extended to the case of uncertain PWA time-delay systems. Numerical examples are presented to show the effectiveness of the approach.

- In Chapter 3 of the thesis, an adaptive switching control algorithm is proposed for the stabilization of uncertain discrete-time systems with time-varying delay in the presence of disturbance. It is assumed that the time-delay is unknown and time-varying, but is bounded with a known bound. It is also assumed that the system is highly uncertain, and that a set of controllers are designed (off-line) to stabilize the system in the whole uncertain parameter space. A switching scheme is subsequently developed to stabilize the uncertain time-delay system. A thorough stability analysis for the uncertain time-delay system under the mentioned control scheme is provided. Furthermore, upper bounds on the allowable rate of change of the system parameters and delay are obtained. Simulation results are presented to show the efficacy of the proposed switching scheme.

- Chapter 4 presents sufficient conditions for the stability of switched time-delay systems with polytopic-type uncertainties. It is assumed that the delay in the system dynamics is time-varying and bounded. Parameter-dependent Lyapunov functionals are employed to obtain exponential stability criteria in the form of LMIs for switched time-delay systems. Free weighting matrices are then provided to express the relationship between the system variables and the terms in the Leibniz-Newton formula. Numerical examples are presented to illustrate the efficacy of the results.

- The problem of robust state regulation for the class of neutral time-delay systems with uncertainties is investigated in Chapter 5. It is assumed that the uncertainties in the system matrices are bounded with unknown bounds. Using the estimates of the unknown parameters, an adaptive robust feedback controller is developed which guarantees the stability of the uncertain time-delay system. Simulation results demonstrate the efficacy of the proposed approach.

1.9 Publications

The results of different chapters of this Ph.D. thesis are published (or submitted for publication) in different journals and conferences, as listed below.

- Chapter 2
  1. Kaveh Moezzi, Luis Rodrigues and Amir G. Aghdam, Stability of Uncertain Piecewise Affine Systems with Time-Delay: Delay-Dependent Lyapunov Approach, *International Journal of Control*, vol. 82, no. 8, pp. 1423-1434, 2009.
  2. Kaveh Moezzi, Luis Rodrigues and Amir G. Aghdam, Stability of Uncertain Piecewise Affine Systems with Time-Delay, in Proceedings of *American Control Conference*, pp. 2373-2378, 2009.
- Chapter 3
  1. Kaveh Moezzi, Ahmadreza Momeni and Amir G. Aghdam, An Adaptive Switching Scheme for Uncertain Discrete Time-Delay Systems, *International Journal of Adaptive Control and Signal Processing*, vol. 24, no. 1,

pp. 1-11, 2010.

2. Kaveh Moezzi, Ahmadreza Momeni and Amir G. Aghdam, A Switching Supervisory Control Design for Uncertain Discrete Time-Delay Systems, in Proceedings of *the 47th IEEE Conference on Decision and Control*, pp. 2886-2893, 2008.

3. Kaveh Moezzi, Ahmadreza Momeni and Amir G. Aghdam, An Adaptive Switching Scheme for Network Control Systems Subject to Uncertainty, in Proceedings of *the 46th IEEE Conference on Decision and Control*, pp. 653-658, 2007.

- Chapter 4

1. Kaveh Moezzi and Amir G. Aghdam, Stability Analysis of Switched Time-Delay Systems with Polytopic Uncertainties, to appear in *49th IEEE Conference on Decision and Control*, 2010.

2. Kaveh Moezzi and Amir G. Aghdam, Sufficient Conditions for the Stability of Switched Time-Delay Systems with Polytopic Uncertainties, under review for journal publication.

- Chapter 5

1. Kaveh Moezzi and Amir G. Aghdam, Adaptive Robust Control of Uncertain Neutral Time-Delay Systems, in Proceedings of *American Control Conference*, pp. 5162-5167, 2008.

2. Kaveh Moezzi and Amir G. Aghdam, Adaptive Controller Designs for Uncertain Neutral Time-Delay Systems, under review for conference publication.

Chapter 2

Stability of Uncertain Piecewise Affine Systems with Time-Delay: Delay-Dependent Lyapunov Approach

2.1 Introduction

Continuous-time piecewise affine (PWA) systems have attracted considerable interest in the control literature in recent years [35], [42], [69], [26], [71], [73], [70], [72]. The theory of PWA systems has found important applications in CPU processing control [5], boost DC-DC converters [6] and aerospace [91], to name only a few. In brief, a PWA system consists of a set of affine subsystems (representing the different operating conditions of a system, or an approximation of a complex nonlinear system) and a switching law that enables switching between different subsystems. It is to be noted that switching is also used in control to stabilize and regulate highly uncertain systems [59], [1], [2], [84].

Many practical systems are subject to input and/or state delay. Examples of time-delay systems include power systems [8] and communication networks [106]. It is known that time-delay can cause poor performance or even instability if its effect is neglected in control design. The existing results for robust stability of time-delay systems can be categorized as delay independent and delay dependent results. Different delay independent robust stability criteria have been developed in [86] and [89]. Delay independent stability results are conservative, in general because they do not take into account any available information on the delay. Delay dependent approaches for the systems subject to parameter uncertainty, on the other hand, are investigated in [18], [10], [78], [24], [67]. Stability analysis for switched systems with time-delay is provided in [81], [105], [80]. In [81], a common Lyapunov functional is used for robust stability analysis of switched uncertain time-delay systems with arbitrary switching. However, any stability analysis using a common quadratic Lyapunov function is typically known to be conservative. In [80], sufficient conditions for exponential stability of linear time-delay systems with a class of switching signals is developed . To the best of the knowledge of the author, however, the stability problem for PWA time-delay systems has only been addressed in [42], where a piecewise quadratic Lyapunov function is used to derive LMIs for stability analysis following the approach of [35]. Nevertheless, the important and practically relevant case of robust stability of PWA time-delay systems in presence of parametric uncertainty has not been considered in [42]. Furthermore, the affine term of the dynamics did not have a delay in that paper.

Based on the considerations of the previous paragraph, PWA uncertain systems with unknown time-delay are investigated in this chapter, and LMI-based conditions for asymptotic stability are derived following the approach of [72]. It is assumed that the parameter uncertainties are norm bounded and that upper bounds on the time-varying delay and its rate of change are given. In order to reduce the conservativeness of the results, piecewise quadratic Lyapunov functions are employed for stability analysis. The main contributions of this work are as follows. First, new delay dependent LMI conditions are derived for the stability of PWA time-delay systems. Second, the stability conditions are extended to the case of uncertain PWA time-delay systems.

This chapter is organized as follows. The problem statement and formulation are given in Section 2.2. The main result of the chapter is provided in Section 2.3, followed by robustness analysis in Section 2.4. Simulation results are presented in Section 2.5.

2.2 Problem Formulation

Consider an uncertain piecewise affine system with time-delay described as

$$\begin{aligned} \dot{x}(t) &= (A_i + \Delta A_i)x(t) + (A_{di} + \Delta A_{di})x(t - \tau(t)) \\ &\quad + (a_i + \Delta a_i) + (b_i + \Delta b_i)\, 1(t - \tau(t)), \quad x(t) \in X_i \end{aligned} \tag{2.1}$$

where $A_i, A_{di} \in \mathbf{R}^{n\times n}$, $a_i,\ b_i \in \mathbf{R}^n$, and $\{X_i\} \subseteq \mathbf{R}^n$ form a partition of the state space into a number of open (possibly unbounded) polyhedral cells with pairwise empty intersection. The index set of the cells is denoted by $I = \{1, ..., M\}$. The set of cells that include the origin is denoted by $I_0 \subseteq I$, and its complement is represented by $I_1 = I/I_0$. It is assumed that $a_i = 0,\ \Delta a_i = 0,\ b_i = 0,\ \Delta b_i = 0$ for $i \in I_0$. In addition, $\Delta A_i,\ \Delta A_{di},\ \Delta a_i$ and Δb_i are norm-bounded uncertainties which will be defined later. Furthermore, $1(t)$ is the step function. In (2.1), $\tau(t)$ is a positive time-varying delay such that

$$0 \le \tau(t) \le h, \quad |\dot{\tau}(t)| \le d < 1 \tag{2.2}$$

where h and d are positive constants.

Assume the initial condition

$$x(\theta) = \phi(\theta), \qquad \theta \in [-h,\ 0]$$

for the system (2.1) such that $\phi(\theta)$ is a differentiable vector-valued initial function on $[-h, 0]$, $h > 0$. Assume also $x(t)$ is a continuous piecewise C^1 function of time. Following [35], [72], the state space is partitioned based on $x(t)$ such that $x(t) \in \bigcup X_i$ as follows. Let $\bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}$, ($e_i = 0$, $\forall i \in I_0$), such that

$$\bar{E}_i \begin{bmatrix} x(t) \\ 1 \end{bmatrix} \geq 0 \qquad \forall x(t) \in X_i, i \in I \tag{2.3}$$

Let $\mathcal{N}_i$ denote the set of neighbouring cells that share a common facet with the cell X_i. The facet boundary between the cells X_i and X_k is contained in the set $\{x \in \mathbf{R}^n \mid c_{ik}^T x(t) - d_{ik} = 0\}$, where $c_{ik} \in \mathbf{R}^n$, $d_{ik} \in \mathbf{R}$, for all $i \in I$, $k \in \mathcal{N}_i$. Moreover, assume the description of the boundaries as follows

$$\bar{X}_i \cap \bar{X}_k \subseteq \{l_{ik} + F_{ik}s | s \in \mathbf{R}^{n-1}\} \tag{2.4}$$

for all $i \in I$, $k \in \mathcal{N}_i$, where $F_{ik} \in \mathbf{R}^{n \times (n-1)}$ is a full rank matrix whose columns span the null space of c_{ik}^T and $l_{ik} \in \mathbf{R}^n$ is given by $l_{ik} = c_{ik}(c_{ik}^T c_{ik})^{-1} d_{ik}$.

The main objective of this chapter is to determine a set of computationally tractable conditions under which (2.1) is asymptotically stable. In the next section, a Lyapunov functional will be introduced to determine the stability of PWA systems.

2.3 Nominal Analysis

In this section, sufficient LMI conditions will be established for the stability of (2.1) without uncertainties. These conditions will be extended to the systems with uncertainties in Section 2.4. To proceed further, we define the following matrices and sets

$$\bar{A}_i := \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{di} := \begin{bmatrix} A_{di} & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{b}_i := \begin{bmatrix} b_i \\ 0 \end{bmatrix}$$

$$\bar{\mathcal{A}} = \left\{ \begin{bmatrix} A_j & a_j \\ 0 & 0 \end{bmatrix}, \ \forall j \in I \right\}, \qquad \mathcal{A} = \{A_j, \ \forall j \in I\}, \qquad \mathcal{E} = \{a_j, \ \forall j \in I_1\}$$

$$\bar{\mathcal{B}} = \left\{ \begin{bmatrix} b_j \\ 0 \end{bmatrix}, \ \forall j \in I_1 \right\}, \qquad \mathcal{B} = \{b_j, \ \forall j \in I_1\}$$

$$\bar{\mathcal{A}}_d = \left\{ \begin{bmatrix} A_{dj} & 0 \\ 0 & 0 \end{bmatrix}, \ \forall j \in I \right\}, \qquad \mathcal{A}_d = \{A_{dj}, \ \forall j \in I\}$$

Note that system (2.1) without uncertainties can be rewritten as follows

$$\dot{\bar{x}} = \bar{A}_i \bar{x}(t) + \bar{A}_{di} \bar{x}(t - \tau(t)) + \bar{b}_i \ 1(t - \tau(t)) \tag{2.5}$$

where $\bar{x}(t) = [x^T(t), \ 1]^T$ and $\bar{x}(t - \tau(t)) = [x^T(t - \tau(t)), \ 1]^T$, with $x(t) \in X_i$. We use the expression

$$\bar{x}(t - \tau(t)) = \bar{x}(t) - \int_{t-\tau(t)}^{t} \dot{\bar{x}}(s) ds \tag{2.6}$$

Hence, considering (2.5), the equation (2.6) can be rewritten as

$$\begin{aligned} \dot{\bar{x}}(t) =& (\bar{A}_i + \bar{A}_{di})\bar{x}(t) + \bar{b}_i \ 1(t - \tau(t)) - \bar{A}_{di} \int_{t-\tau(t)}^{t} \bar{A}_{j(s)} \bar{x}(s) \, ds \\ & - \bar{A}_{di} \int_{t-\tau(t)}^{t} \bar{A}_{dj(s)} \bar{x}(s - \tau(s)) \, ds - \bar{A}_{di} \int_{t-\tau(t)}^{t} \bar{b}_{j(s)} \ 1(s - \tau(s)) \, ds \end{aligned} \tag{2.7}$$

Note that $j(s)$ in (2.7) is a piecewise constant function which represents the index of the matrices $\bar{A}_{j(s)} \in \bar{\mathcal{A}}$, $\bar{b}_{j(s)} \in \bar{\mathcal{B}}$, $\bar{A}_{dj(s)} \in \bar{\mathcal{A}}_d$ at time s. In order to proceed further, the following well-known lemma is borrowed from [90].

Lemma 2.1. *For any vectors or matrices z and y with appropriate dimensions and any symmetric matrix $P > 0$, the following inequalities are satisfied:*

$$\begin{aligned} -z^T y - y^T z &\leq z^T P z + y^T P^{-1} y \\ z^T y + y^T z &\leq z^T P z + y^T P^{-1} y \end{aligned} \tag{2.8}$$

Proof: See [90].

The following Theorem presents sufficient conditions for the stability of the PWA system (2.5).

Theorem 2.1. *Consider symmetric matrices* $\bar{U}_i, U_i$ *and* $\bar{W}_i, W_i$*, where* $\bar{U}_i, U_i$ *and* $\bar{W}_i, W_i$ *are composed of non-negative entries, and*

$$\begin{bmatrix} H_i' & hP_iA_{di} & S_{31} + \begin{bmatrix} 0_{n\times n} & R_3 + S_3 \end{bmatrix} \\ * & -hM_{1i} & 0 \\ * & * & \bar{S}_2 - (1-d)\bar{R} + \begin{bmatrix} 0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & S_2 + R_2 \end{bmatrix} + \Lambda \end{bmatrix} < 0 \tag{2.9}$$

$$\Lambda = \begin{bmatrix} 0_{n\times(n+1)} \\ S_{32} \end{bmatrix} + \begin{bmatrix} 0_{(n+1)\times n} & S_{32}^T \end{bmatrix}$$

$$\begin{bmatrix} hQ_i & \begin{bmatrix} hP_iA_{di}A_j & hP_iA_{di}a_j \end{bmatrix} & \begin{bmatrix} hP_iA_{di}A_{dj} & 0 \end{bmatrix} \\ * & \bar{S}_1 - \begin{bmatrix} 0 & 0 \\ 0 & hb_j^TM_{1i}b_j \end{bmatrix} & \bar{S}_3 \\ * & \bar{S}_3^T & \bar{S}_2 \end{bmatrix} \geq 0 \tag{2.10}$$

$$P_i - E_i^TU_iE_i > 0, \qquad M_{1i} > 0 \tag{2.11}$$

where

$$H_i' = P_i(A_i + A_{di}) + (A_{di} + A_i)^TP_i + S_1 + R_1 + hQ_i + E_i^TU_iE_i \tag{2.12}$$

for any fixed $i \in I_0$ *and for all* $A_j \in \mathcal{A}$, $b_j \in \mathcal{B}$, $a_j \in \mathcal{E}$, $A_{dj} \in \mathcal{A}_d$*, such that*

$$\bar{S}_1 = \begin{bmatrix} S_1 & S_3 \\ S_3^T & S_2 \end{bmatrix}, \quad \bar{S}_3 = \begin{bmatrix} S_{31} \\ S_{32} \end{bmatrix}$$

$$S_1 \in \mathbf{R}^{n\times n},\ S_2 \in \mathbf{R},\ S_3 \in \mathbf{R}^{n\times 1},\ S_{31} \in \mathbf{R}^{n\times(n+1)},\ S_{32} \in \mathbf{R}^{1\times(n+1)}$$

$$\bar{R} = \begin{bmatrix} R_1 & R_3 \\ R_3^T & R_2 \end{bmatrix}, \; R_1 \in \mathbf{R}^{n\times n}, \; R_2 \in \mathbf{R}, \; R_3 \in \mathbf{R}^{n\times 1}$$

satisfying

$$\begin{bmatrix} \bar{S}_1 & \bar{S}_3 \\ \bar{S}_3^T & \bar{S}_2 \end{bmatrix} > 0, \qquad \bar{R} > 0 \tag{2.13}$$

for $\bar{S}_1$, $\bar{S}_2$ *and* $\bar{S}_3$. *Furthermore, let the following inequalities hold*

$$\begin{bmatrix} \bar{H}_i' & \bar{P}_i & h\bar{P}_i\bar{A}_{di} & \bar{S}_3 \\ * & -\bar{M}_{1i} & 0 & 0 \\ * & * & -h\bar{M}_{2i} & 0 \\ * & * & * & \bar{S}_2 - (1-d)\bar{R} \end{bmatrix} < 0 \tag{2.14}$$

$$\begin{bmatrix} h\bar{Q}_i & h\bar{P}_i\bar{A}_{di}\bar{A}_j & h\bar{P}_i\bar{A}_{di}\bar{A}_{dj} \\ * & \bar{S}_1 - \begin{bmatrix} 0 & 0 \\ 0 & h\bar{b}_j^T\bar{M}_{2i}\bar{b}_j \end{bmatrix} & \bar{S}_3 \\ * & * & \bar{S}_2 \end{bmatrix} \geq 0 \tag{2.15}$$

$$\bar{P}_i - \bar{E}_i^T\bar{W}_i\bar{E}_i > 0, \qquad \bar{M}_{ki} > 0, \; k = 1, 2 \tag{2.16}$$

for any fixed $i \in I_1$ *and for all* $\bar{A}_j \in \bar{\mathcal{A}}$, $\bar{b}_j \in \bar{\mathcal{B}}$, $\bar{A}_{dj} \in \bar{\mathcal{A}}_d$, *where*

$$\bar{H}_i' := \bar{P}_i(\bar{A}_i + \bar{A}_{di}) + (\bar{A}_{di} + \bar{A}_i)^T\bar{P}_i + \bar{S}_1 + \bar{R} + \begin{bmatrix} 0 & 0 \\ 0 & \bar{b}_i^T\bar{M}_{1i}\bar{b}_i \end{bmatrix} + h\bar{Q}_i + \bar{E}_i^T\bar{U}_i\bar{E}_i \tag{2.17}$$

Assume also that for all $i \in I$ *and* $k \in \mathcal{N}_i$,

$$F_{ik}^T(P_i - P_k)F_{ik} = 0 \tag{2.18a}$$

$$F_{ik}^T(P_i - P_k)l_{ik} + F_{ik}^T(q_i - q_k) = 0 \tag{2.18b}$$

$$l_{ik}^T(P_i - P_k)l_{ik} + 2(q_i - q_k)^Tl_{ik} + (r_i - r_k) = 0 \tag{2.18c}$$

where $\bar{P}_i := \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix}$, *for all* $i \in I$.

Under conditions (2.2), (2.3) and (2.9)-(2.18), every piecewise C^1 *trajectory* $x(t)$, *governed by (2.5) for* $t \geq 0$, *tends to zero asymptotically in the absence of sliding modes.*

Proof: Define the candidate Lyapunov-Krasovskii functional

$$\bar{V}_i = \bar{V}_{1i} + \bar{V}_{2i} + \bar{V}_{3i} \tag{2.19}$$

where, for $x(t) \in X_i$, $i \in I_1$

$$\bar{V}_{1i} = \bar{x}^T(t)\bar{P}_i\bar{x}(t) \tag{2.20a}$$

$$\bar{V}_{2i} = \int_{t-\tau(t)}^{t} \bar{x}^T(s)\bar{R}\bar{x}(s)ds \tag{2.20b}$$

$$\bar{V}_{3i} = h^{-1}\int_{-h}^{0}\int_{t+s}^{t} \begin{bmatrix} \bar{x}(\theta) \\ \bar{x}(\theta-\tau(\theta)) \end{bmatrix}^T \begin{bmatrix} \bar{S}_1 & \bar{S}_3 \\ \bar{S}_3^T & \bar{S}_2 \end{bmatrix} \begin{bmatrix} \bar{x}(\theta) \\ \bar{x}(\theta-\tau(\theta)) \end{bmatrix} d\theta ds \tag{2.20c}$$

The conditions that guarantee the continuity of the Lyapunov function at the boundaries are give in (2.18a-c), and can be obtained directly using the same approach as the one in [72]. Note that the candidate Lyapunov functional is positive definite because of (2.16) and (2.13). Applying Leibniz integral rule and using (2.2), the derivative of this Lyapunov functional will be obtained as

$$\begin{aligned} \dot{\bar{V}}_i \leq & 2\bar{x}(t)^T\bar{P}_i\dot{\bar{x}}(t) + \bar{x}^T(t)\bar{R}\bar{x}(t) - \bar{x}^T(t-\tau(t))(1-d)\bar{R}\bar{x}(t-\tau(t)) + \bar{x}^T(t)\bar{S}_1\bar{x}(t) \\ & + \bar{x}^T(t-\tau(t))\bar{S}_2\bar{x}(t-\tau(t)) + 2\bar{x}^T(t)\bar{S}_3\bar{x}(t-\tau(t)) \\ & - h^{-1}\int_{t-\tau(t)}^{t} \begin{bmatrix} \bar{x}(s) \\ \bar{x}(s-\tau(s)) \end{bmatrix}^T \begin{bmatrix} \bar{S}_1 & \bar{S}_3 \\ \bar{S}_3^T & \bar{S}_2 \end{bmatrix} \begin{bmatrix} \bar{x}(s) \\ \bar{x}(s-\tau(s)) \end{bmatrix} ds \end{aligned} \tag{2.21}$$

Substituting (2.7) in (2.21) leads to

$$\begin{aligned}
\dot{V}_i \leq\ & 2\bar{x}^T(t)\bar{P}_i(\bar{A}_i + \bar{A}_{di})\bar{x}(t) + 2\bar{x}^T(t)\bar{P}_i\bar{b}_i\ 1(t-\tau(t)) \\
& + \bar{x}^T(t)(\bar{S}_1 + \bar{R})\bar{x}(t) + 2\bar{x}^T(t)\bar{S}_3\bar{x}(t-\tau(t)) \\
& + \bar{x}^T(t-\tau(t))(\bar{S}_2 - (1-d)\bar{R})\bar{x}(t-\tau(t)) \\
& - 2\bar{x}^T(t)\bar{P}_i\bar{A}_{di}\int_{t-\tau(t)}^{t}\bar{A}_{j(s)}\bar{x}(s)\,ds - 2\bar{x}^T(t)\bar{P}_i\bar{A}_{di}\int_{t-\tau(t)}^{t}\bar{A}_{dj(s)}\bar{x}(s-\tau(s))\,ds \\
& - 2\bar{x}^T(t)\bar{P}_i\bar{A}_{di}\int_{t-\tau(t)}^{t}\bar{b}_{j(s)}\ 1(s-\tau(s))\,ds \\
& - h^{-1}\int_{t-\tau(t)}^{t}\begin{bmatrix}\bar{x}(s)\\ \bar{x}(s-\tau(s))\end{bmatrix}^T\begin{bmatrix}\bar{S}_1 & \bar{S}_3\\ \bar{S}_3^T & \bar{S}_2\end{bmatrix}\begin{bmatrix}\bar{x}(s)\\ \bar{x}(s-\tau(s))\end{bmatrix}ds
\end{aligned} \tag{2.22}$$

Now, considering positive-definite matrices $\bar{M}_{ki}$, $k = 1, 2$, $i \in I_1$ using Lemma 2.1 and the inequalities (2.2), (2.22) yields

$$\begin{aligned}
\dot{V}_i \leq\ & 2\bar{x}^T(t)\bar{P}_i(\bar{A}_i + \bar{A}_{di})\bar{x}(t) + \bar{x}^T(t)\bar{P}_i\bar{M}_{1i}^{-1}\bar{P}_i\bar{x}(t) + \bar{b}_i^T\bar{M}_{1i}\bar{b}_i\ 1(t-\tau(t)) \\
& + \bar{x}^T(t)(\bar{S}_1 + \bar{R})\bar{x}(t) + \bar{x}^T(t-\tau(t))(\bar{S}_2 - (1-d)\bar{R})\bar{x}(t-\tau(t)) \\
& + 2\bar{x}^T(t)\bar{S}_3\bar{x}(t-\tau(t)) \\
& - 2\bar{x}^T(t)\bar{P}_i\bar{A}_{di}\int_{t-\tau(t)}^{t}\bar{A}_{j(s)}\bar{x}(s)\,ds - 2\bar{x}^T(t)\bar{P}_i\bar{A}_{di}\int_{t-\tau(t)}^{t}\bar{A}_{dj(s)}\bar{x}(s-\tau(s))\,ds \\
& + h\bar{x}^T(t)\bar{P}_i\bar{A}_{di}\bar{M}_{2i}^{-1}\bar{A}_{di}^T\bar{P}_i\bar{x}(t) + \int_{t-\tau(t)}^{t}\bar{b}_{j(s)}^T\bar{M}_{2i}\bar{b}_{j(s)}\ 1(s-\tau(s))\,ds \\
& - h^{-1}\int_{t-\tau(t)}^{t}\begin{bmatrix}\bar{x}(s)\\ \bar{x}(s-\tau(s))\end{bmatrix}^T\begin{bmatrix}\bar{S}_1 & \bar{S}_3\\ \bar{S}_3^T & \bar{S}_2\end{bmatrix}\begin{bmatrix}\bar{x}(s)\\ \bar{x}(s-\tau(s))\end{bmatrix}ds
\end{aligned} \tag{2.23}$$

Note that from (2.2), there always exists a symmetric positive semi-definite matrix $\bar{Q}_i$ such that

$$h\bar{x}^T(t)\bar{Q}_i\bar{x}(t) - \int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{Q}_i\bar{x}(t)ds \geq 0 \tag{2.24}$$

Define now

$$\bar{H}_i := \bar{P}_i(\bar{A}_i + \bar{A}_{di}) + (\bar{A}_{di} + \bar{A}_i)^T\bar{P}_i + \bar{S}_1 + \bar{R} + \begin{bmatrix} 0 & 0 \\ 0 & \bar{b}_i^T\bar{M}_{1i}\bar{b}_i \end{bmatrix} + h\bar{Q}_i \quad (2.25a)$$

$$\bar{Z}_i := \begin{bmatrix} \bar{H}_i + \bar{P}_i\bar{M}_{1i}^{-1}\bar{P}_i + h\bar{P}_i\bar{A}_{di}\bar{M}_{2i}^{-1}\bar{A}_{di}^T\bar{P}_i & \bar{S}_3 \\ \bar{S}_3^T & \bar{S}_2 - (1-d)\bar{R} \end{bmatrix} \quad (2.25b)$$

$$\bar{Y}_{j(s)} := -\begin{bmatrix} h\bar{Q}_i & h\bar{P}_i\bar{A}_{di}\bar{A}_{j(s)} & h\bar{P}_i\bar{A}_{di}\bar{A}_{dj(s)} \\ * & \bar{S}_1 - \begin{bmatrix} 0 & 0 \\ 0 & h\bar{b}_{j(s)}^T\bar{M}_{2i}\bar{b}_{j(s)} \end{bmatrix} & \bar{S}_3 \\ * & * & \bar{S}_2 \end{bmatrix} \quad (2.25c)$$

Then by adding inequality (2.24) to the right hand side of (2.23) and considering (2.25a-c), one can write the following for $x(t) \in X_i$, $i \in I_1$

$$\begin{aligned} \dot{\bar{V}}_i \leq & \bar{\xi}^T(t,\tau(t))\bar{Z}_i\bar{\xi}(t,\tau(t)) \\ & + h^{-1}\int_{t-\tau(t)}^{t} \bar{\eta}^T(t,s,\tau(s))\bar{Y}_{j(s)}\bar{\eta}(t,s,\tau(s))ds \end{aligned} \quad (2.26)$$

where

$$\bar{\xi}(t,\tau(t)) = \left[\bar{x}^T(t),\ \bar{x}^T(t-\tau(t))\right]^T, \quad \bar{\eta}(t,s,\tau(s)) = \left[\bar{x}^T(t),\ \bar{x}^T(s),\ \bar{x}^T(s-\tau(s))\right]^T \quad (2.27)$$

Note that (2.14) and (2.16) imply

$$\bar{\xi}^T(\cdot)(\bar{Z}_i + \tilde{\bar{E}}_i^T\tilde{\bar{U}}_i\tilde{\bar{E}}_i)\bar{\xi}(\cdot) < 0 \quad (2.28)$$

using the Schur complement, where $\tilde{\bar{U}}_i = \text{diag}[\bar{U}_i, 0]$, $\tilde{\bar{E}}_i = [\bar{E}_i, 0]$, and $\bar{U}_i$ has only non-negative entries. Note also that from (2.3), the inequality $\bar{E}_i\bar{x}(t) \geq 0$ holds for $x(t) \in X_i$. This leads to

$$\tilde{\bar{E}}_i\bar{\xi} \geq 0,\ \forall x(t) \in X_i,\ i \in I_1$$

and consequently it follows that

$$\bar{\xi}^T(\cdot)\tilde{\bar{E}}_i^T\tilde{\bar{U}}_i\tilde{\bar{E}}_i\bar{\xi}(\cdot) \geq 0, \qquad x(t) \in X_i, \qquad i \in I_1 \quad (2.29)$$

Therefore, the relations (2.3), (2.16) and (2.14) imply $\bar{\xi}^T(\cdot)\bar{Z}_i\bar{\xi}(\cdot) < 0$, for all $x(t) \in X_i$, $i \in I_1$. Furthermore, (2.15) implies $\bar{Y}_{j(s)} \leq 0$ and from (2.26), $\dot{\bar{V}}_i < 0$, $x(t) \in X_i$, $i \in I_1$. A similar procedure can be repeated for the case when the switching index belongs to I_0 leading to (2.9)-(2.11) and $\dot{V}_i < 0$, $x(t) \in X_i$, $i \in I_0$. Thus the system is asymptotically stable. ■

Remark 2.1. *Theorem 2.1 assumes the absence of sliding modes. To avoid sliding modes at the boundaries the following conditions can be added. Let the set* $\{x \in \mathbf{R}^n | \sigma_{ik} = c_{ik}^T x - d_{ik} = 0\}$ *denote the sliding surface between the cells* X_i *and* X_k. *According to [72],* $\dot{\sigma}_{ik}$ *must be continuous across the boundary described in (2.4), which yields*

$$\begin{aligned} &c_{ik}^T \left[A_i(F_{ik}s + l_{ik}) + A_{di}x(t-\tau(t)) + a_i + b_i\ 1(t-\tau(t))\right] \\ &\quad = c_{ik}^T \left[A_k(F_{ik}s + l_{ik}) + A_{dk}x(t-\tau(t)) + a_k + b_k\ 1(t-\tau(t))\right] \end{aligned}$$

for all $s \in \mathbf{R}^{n-1}$, $k \in \mathcal{N}_i$. *The above equation can be rewritten as follows*

$$c_{ik}^T(A_i - A_k)F_{ik} = 0 \qquad (2.30a)$$

$$c_{ik}^T(A_{di} - A_{dk}) = 0 \qquad (2.30b)$$

$$c_{ik}^T\left[(A_i - A_k)l_{ik} + (a_i - a_k)\right] = 0 \qquad (2.30c)$$

$$c_{ik}^T(b_i - b_k) = 0 \qquad (2.30d)$$

Remark 2.2. *Using a procedure similar to the one presented here, one can apply the results of [60] and define the following Lyapunov-Krasovskii functional*

$$\begin{aligned} \bar{V}_i' =& \bar{x}^T(t)\bar{P}_i\bar{x}(t) + \int_{t-\tau(t)}^{t} \bar{x}^T(s)e^{\beta(s-t)}\bar{R}\bar{x}(s)ds \\ &+ h^{-1}\int_{-h}^{0}\int_{t+s}^{t} \begin{bmatrix} \bar{x}(\theta) \\ \bar{x}(\theta-\tau(\theta)) \end{bmatrix}^T e^{\beta(\theta-t)} \begin{bmatrix} \bar{S}_1 & \bar{S}_3 \\ \bar{S}_3^T & \bar{S}_2 \end{bmatrix} \begin{bmatrix} \bar{x}(\theta) \\ \bar{x}(\theta-\tau(\theta)) \end{bmatrix} d\theta ds \end{aligned}$$

to obtain the LMIs that determine the exponential stability of the system (2.5). It is to be noted that exponential stability typically is stronger than asymptotic stability, at the cost of more conservative LMIs.

2.4 Robustness Analysis

Consider now the system (2.1) and define the matrices $\tilde{A}_i = A_i + \Delta A_i$, $\tilde{A}_{di} = A_{di} + \Delta A_{di}$, $\tilde{a}_i = a_i + \Delta a_i$, $\tilde{b}_i = b_i + \Delta b_i$ $(i \in I)$ and

$$\bar{\tilde{A}}_i = \bar{A}_i + \Delta\bar{A}_i, \quad \Delta\bar{A}_i = \begin{bmatrix} \Delta A_i & \Delta a_i \\ 0 & 0 \end{bmatrix}$$

$$\bar{\tilde{A}}_{di} = \bar{A}_{di} + \Delta\bar{A}_{di}, \quad \Delta\bar{A}_{di} = \begin{bmatrix} \Delta A_{di} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{\tilde{b}}_i = \bar{b}_i + \Delta\bar{b}_i, \qquad \Delta\bar{b}_i = \begin{bmatrix} \Delta b_i \\ 0 \end{bmatrix}$$

$$\Delta\bar{\mathcal{A}} = \left\{ \begin{bmatrix} \Delta A_j & \Delta a_j \\ 0 & 0 \end{bmatrix}, \ \forall j \in I \right\}, \quad \Delta\mathcal{A} = \{\Delta A_j, \ \forall j \in I\}, \quad \Delta\mathcal{E} = \{\Delta a_j, \ \forall j \in I_1\}$$

$$\Delta\bar{\mathcal{B}} = \left\{ \begin{bmatrix} \Delta b_j \\ 0 \end{bmatrix}, \ \forall j \in I_1 \right\}, \qquad \Delta\mathcal{B} = \{\Delta b_j, \ \forall j \in I_1\}$$

$$\Delta\bar{\mathcal{A}}_d = \left\{ \begin{bmatrix} \Delta A_{dj} & 0 \\ 0 & 0 \end{bmatrix}, \ \forall j \in I \right\}, \qquad \Delta\mathcal{A}_d = \{\Delta A_{dj}, \ \forall j \in I\}$$

Let $||\cdot||$ denote the 2-norm. The following bounds are assumed to be given for the norm of relevant matrices

$$||\Delta A_i|| \leq \alpha_i. \qquad ||\Delta A_{di}|| \leq \beta_i$$

$$||\Delta\bar{A}_i|| \leq \bar{\alpha}_i, \qquad ||\Delta\bar{A}_{di}|| \leq \bar{\beta}_i, \qquad ||\Delta\bar{b}_i|| \leq \bar{\delta}_i$$

$$\max_{X\in\Delta\bar{\mathcal{A}}}||X|| \leq \bar{\alpha}^*, \quad \max_{X\in\Delta\bar{\mathcal{A}}_d}||X|| = \max_{X\in\Delta\mathcal{A}_d}||X|| \leq \beta^*$$

$$\max_{X\in\Delta\mathcal{A}}||X|| \leq \alpha^*, \quad \max_{X\in\Delta\mathcal{E}}||X|| \leq \gamma^*$$

$$\max_{X\in\Delta\bar{\mathcal{B}}}||X|| = \max_{X\in\Delta\mathcal{B}}||X|| \leq \delta^*$$

The following theorem presents sufficient conditions for the stability of uncertain PWA systems described by (2.1).

Theorem 2.2. *Consider symmetric matrices $\bar{U}_i, U_i$ and $\bar{W}_i, W_i$, where $\bar{U}_i, U_i$ and $\bar{W}_i, W_i$ are composed of non-negative entries. Then, the uncertain PWA time-delay system (2.1) is asymptotically stable in the absence of sliding modes, if (2.11), (2.16), (2.13) and (2.18a-c) hold, and there exist positive definite matrices L_{ki}, $k = 1, ..., 10$, $\bar{L}_{ki}$, ($k = 1, ..., 9$), M_{1i}, $i \in I_0$ and $\bar{M}_{pi}$, $p = 1, 2$, $i \in I_1$ such that*

$$\left[\begin{array}{cccccccc}
hQ_i & hP_i\beta_i\alpha^* & hP_i\beta_i & hP_i\alpha^* A_{di} & hP_i\beta_i\beta^* & hP_i\beta_i^2 & hP_i\beta^* A_{di} & hP_i\beta_i\delta^* \\
* & h\rho_{L_{1i}}I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & h\rho_{X_{1i}}I & 0 & 0 & 0 & 0 & 0 \\
* & * & * & h\rho_{L_{3i}}I & 0 & 0 & 0 & 0 \\
* & * & * & * & h\rho_{L_{4i}}I & 0 & 0 & 0 \\
* & * & * & * & * & h\rho_{X_{2i}}I & 0 & 0 \\
* & * & * & * & * & * & h\rho_{L_{6i}}I & 0 \\
* & * & * & * & * & * & * & hl_{2i}I \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & *
\end{array}\right.$$

$$\begin{bmatrix} hP_iA_{di}\delta^* & hP_i\beta_i^2 & hP_i\beta_i & hP_i\beta_i\gamma^* & hP_iA_{di}\gamma^* & hP_iA_{di}[A_j\ a_j] & hP_iA_{di}[A_{dj}\ 0] \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ hl_{3i}I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & h\rho_{L_{9i}}I & 0 & 0 & 0 & 0 & 0 \\ * & * & h\rho_{L_{10i}}I & 0 & 0 & 0 & 0 \\ * & * & * & hl_{1i}I & 0 & 0 & 0 \\ * & * & * & * & hl_{4i}I & 0 & 0 \\ * & * & * & * & * & S_1 - \Pi_j^1 & S_3 \\ * & * & * & * & * & * & S_2 - \Pi_i^2 \end{bmatrix} \geq 0 \tag{2.31}$$

where

$$\Pi_j^1 = h(L_{1i} + L_{2i} + L_{3i} + \begin{bmatrix} 0 & 0 \\ 0 & b_j^T(M_{1i} + L_{9i})b_j \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_j^T L_{10i} a_j \end{bmatrix})$$

$$\Pi_i^2 = h(L_{4i} + L_{5i} + L_{6i})$$

$$\begin{bmatrix} H_i' & hP_iA_{di} & P_i\alpha_i & P_i\beta_i & S_{31}+\begin{bmatrix}0_{n\times n} & R_3+S_3\end{bmatrix} \\ * & -hM_{1i} & 0 & 0 & 0 \\ * & * & -\rho_{L_{7i}}I & 0 & 0 \\ * & * & * & -\rho_{L_{8i}}I & 0 \\ * & * & * & * & \bar{S}_2-(1-d)\bar{R}+\begin{bmatrix}0_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & S_2+R_2\end{bmatrix}+\Lambda \end{bmatrix}<0 \tag{2.32}$$

$$\Lambda = \begin{bmatrix}0_{n\times(n+1)} \\ S_{32}\end{bmatrix} + \begin{bmatrix}0_{(n+1)\times n} & S_{32}^T\end{bmatrix}$$

$$\begin{bmatrix} X_{1i} & X_{1i}A_j \\ * & L_{2i} \end{bmatrix} > 0, \quad \begin{bmatrix} X_{2i} & X_{2i}A_{dj} \\ * & L_{5i} \end{bmatrix} > 0 \tag{2.33}$$

for any fixed $i \in I_0$ *and for all* $A_j \in \mathcal{A}$, $b_j \in \mathcal{B}$, $a_j \in \mathcal{E}$, $A_{dj} \in \mathcal{A}_d$. *In addition, let the following LMIs hold*

$$\left[\begin{array}{ccccccc} h\bar{Q}_i & h\bar{P}_i\bar{\beta}_i\bar{\alpha}^* & h\bar{P}_i\bar{\beta}_i & h\bar{P}_i\bar{\alpha}^*\bar{A}_{di} & h\bar{P}_i\bar{\beta}_i\bar{\beta}^* & h\bar{P}_i\bar{\beta}_i^2 & h\bar{P}_i\bar{\beta}^*\bar{A}_{di} \\ * & h\bar{\rho}_{\bar{L}_{1i}}I & 0 & 0 & 0 & 0 & 0 \\ * & * & h\bar{\rho}_{\bar{X}_{1i}}I & 0 & 0 & 0 & 0 \\ * & * & * & h\bar{\rho}_{\bar{L}_{3i}}I & 0 & 0 & 0 \\ * & * & * & * & h\bar{\rho}_{\bar{L}_{4i}}I & 0 & 0 \\ * & * & * & * & * & h\bar{\rho}_{\bar{X}_{2i}}I & 0 \\ * & * & * & * & * & * & h\bar{\rho}_{\bar{L}_{6i}}I \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{array}\right.$$

$$\begin{matrix} h\bar{P}_i\bar{\beta}_i\delta^* & h\bar{P}_i\bar{A}_{di}\delta^* & h\bar{P}_i\bar{\beta}_i & h\bar{P}_i\bar{A}_{di}\bar{A}_j & h\bar{P}_i\bar{A}_{di}\bar{A}_{dj} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ h\bar{l}_{2i}I & 0 & 0 & 0 & 0 \\ * & h\bar{l}_{3i}I & 0 & 0 & 0 \\ * & * & h\bar{\rho}_{\bar{L}_{9i}}I & 0 & 0 \\ * & * & * & \bar{S}_1 - \bar{\Pi}_j^1 & \bar{S}_3 \\ * & * & * & * & \bar{S}_2 - \bar{\Pi}_i^2 \end{matrix}\left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\\a\end{matrix}}\right] \geq 0 \qquad (2.34)$$

$$\bar{\Pi}_j^1 = h(\bar{L}_{1i} + \bar{L}_{2i} + \bar{L}_{3i} + \begin{bmatrix} 0 & 0 \\ 0 & \bar{b}_j^T(\bar{M}_{2i} + \bar{L}_{9i})\bar{b}_j \end{bmatrix}), \qquad \bar{\Pi}_i^2 = h(\bar{L}_{4i} + \bar{L}_{5i} + \bar{L}_{6i})$$

$$\begin{bmatrix} \hat{\bar{H}}_i & \bar{P}_i & h\bar{P}_i\bar{A}_{di} & \bar{P}_i\bar{\alpha}_i & \bar{P}_i\bar{\beta}_i & \bar{P}_i\bar{\delta}_i & \bar{S}_3 \\ * & -\bar{M}_{1i} & 0 & 0 & 0 & 0 & 0 \\ * & * & -h\bar{M}_{2i} & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{\rho}_{\bar{L}_{7i}}I & 0 & 0 & 0 \\ * & * & * & * & -\bar{\rho}_{\bar{L}_{8i}}I & 0 & 0 \\ * & * & * & * & * & -\bar{l}_{1i}I & 0 \\ * & * & * & * & * & * & \bar{S}_2 - (1-d)\bar{R} \end{bmatrix} < 0 \qquad (2.35)$$

$$\begin{bmatrix} \bar{X}_{1i} & \bar{X}_{1i}\bar{A}_j \\ * & \bar{L}_{2i} \end{bmatrix} > 0, \quad \begin{bmatrix} \bar{X}_{2i} & \bar{X}_{2i}\bar{A}_{dj} \\ * & \bar{L}_{5i} \end{bmatrix} > 0 \qquad (2.36)$$

for any fixed $i \in I_1$ *and for all* $\bar{A}_j \in \bar{\mathcal{A}}$, $\bar{b}_j \in \bar{\mathcal{B}}$, $\bar{A}_{dj} \in \bar{\mathcal{A}}_d$, *where the following inequalities are satisfied*

$$\rho_{L_{ki}} > 0, \quad L_{ki} - \rho_{L_{ki}} I > 0, \quad k = 1,\ 3,\ 4,\ 6,\ 7, \ldots,\ 10 \quad \forall i \in I_0 \tag{2.37}$$

$$\bar{\rho}_{\bar{L}_{ki}} > 0, \quad \bar{L}_{ki} - \bar{\rho}_{\bar{L}_{ki}} I > 0, \quad k = 1,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9 \quad \forall i \in I_1 \tag{2.38}$$

$$\rho_{X_{ki}} > 0, \quad \rho_{X_{ki}} I - X_{ki} < 0, \quad k = 1 \text{ and } 2 \quad \forall i \in I_0 \tag{2.39}$$

$$\bar{\rho}_{\bar{X}_{ki}} > 0, \quad \bar{\rho}_{\bar{X}_{ki}} I - \bar{X}_{ki} < 0, \quad k = 1 \text{ and } 2 \quad \forall i \in I_1 \tag{2.40}$$

where

$$\hat{H}_i = H_i' + L_{7i} + L_{8i} + \begin{bmatrix} 0 & 0 \\ 0 & hl_{1i} + hl_{4i} \end{bmatrix}, \qquad \hat{\bar{H}}_i = \bar{H}_i' + \bar{L}_{7i} + \bar{L}_{8i} + \begin{bmatrix} 0 & 0 \\ 0 & \bar{l}_{1i} + h\bar{l}_{2i} + h\bar{l}_{3i} \end{bmatrix}$$

(note that H_i' *and* $\bar{H}_i'$ *are defined in (2.12) and (2.17), respectively).*

Proof: The proof follows the steps of the proof of Theorem 2.1, after replacing $\bar{A}_i$ with, $\bar{\bar{A}}_i$, $\bar{A}_{di}$ with $\bar{\bar{A}}_{di}$ and $\bar{b}_i$ with $\bar{\bar{b}}_i$.

Doing this, expression (2.26) becomes for $x(t) \in X_i$, $i \in I_1$

$$
\begin{aligned}
\dot{V} \leq & \bar{\xi}^T(t,\tau(t))\bar{Z}_i\bar{\xi}(t,\tau(t)) \\
& + h^{-1}\int_{t-\tau(t)}^{t} \bar{\eta}^T(t,s,\tau(s))\bar{Y}_{j(s)}\bar{\eta}(t,s,\tau(s))ds \\
& + 2\bar{x}^T\bar{P}_i\Bigg[(\Delta\bar{A}_i + \Delta\bar{A}_{di})\bar{x}(t) + \Delta\bar{b}_i\ 1(t-\tau(t)) - \Delta\bar{A}_{di}\int_{t-\tau(t)}^{t} \Delta\bar{A}_{j(s)}\bar{x}(s)\,ds \\
& - \Delta\bar{A}_{di}\int_{t-\tau(t)}^{t} \bar{A}_{j(s)}\bar{x}(s)\,ds - \bar{A}_{di}\int_{t-\tau(t)}^{t} \Delta\bar{A}_{j(s)}\bar{x}(s)\,ds \\
& - \Delta\bar{A}_{di}\int_{t-\tau(t)}^{t} \Delta\bar{A}_{dj(s)}\bar{x}(s-\tau(s))ds - \bar{A}_{di}\int_{t-\tau(t)}^{t} \Delta\bar{A}_{dj(s)}\bar{x}(s-\tau(s))ds \\
& - \Delta\bar{A}_{di}\int_{t-\tau(t)}^{t} \bar{b}_{j(s)}1(s-\tau(s))ds - \Delta\bar{A}_{di}\int_{t-\tau(t)}^{t} \Delta\bar{b}_{j(s)}1(s-\tau(s))ds \\
& - \bar{A}_{di}\int_{t-\tau(t)}^{t} \Delta\bar{b}_{j(s)}1(s-\tau(s))ds - \Delta\bar{A}_{di}\int_{t-\tau(t)}^{t} \bar{A}_{dj(s)}\bar{x}(s-\tau(s))ds\Bigg]
\end{aligned}
\tag{2.41}
$$

The objective now is to find upper bounds to all terms of (2.41). Defining positive definite matrices $\bar{L}_{7i}$, $\bar{L}_{8i}$, a positive constant $\bar{l}_{1i}$ and using Lemma 2.1, yields

$$
\begin{aligned}
2\bar{x}^T(t)\bar{P}_i\left[(\Delta\bar{A}_i + \Delta\bar{A}_{di})\bar{x}(t) + \Delta\bar{b}_i\ 1(t-\tau(t))\right] \leq & \bar{x}^T(t)\bar{P}_i\Delta\bar{A}_i\bar{L}_{7i}^{-1}\Delta\bar{A}_i^T\bar{P}_i\bar{x}(t) \\
& + \bar{x}^T(t)\bar{P}_i\Delta\bar{A}_{di}\bar{L}_{8i}^{-1}\Delta\bar{A}_{di}^T\bar{P}_i\bar{x}(t) + \bar{x}^T(t)\bar{P}_i\Delta\bar{b}_i\bar{l}_{1i}^{-1}\Delta\bar{b}_i^T\bar{P}_i\bar{x}(t) \\
& + \bar{x}^T(t)\bar{L}_{7i}\bar{x}(t) + \bar{x}^T(t)\bar{L}_{8i}\bar{x}(t) + \bar{l}_{1i}\ 1(t-\tau(t))
\end{aligned}
\tag{2.42}
$$

Considering the fact that, $\lambda_{min}^{-1}(\bar{L}_{ki}) = \lambda_{max}(\bar{L}_{ki}^{-1})$, $k = 1, \ldots, 9$, and $||\bar{x}^T(t)\bar{P}_i||^2 = \bar{x}^T(t)\bar{P}_i\bar{P}_i\bar{x}(t)$, expression (2.42) can be rewritten as

$$
\begin{aligned}
& 2\bar{x}^T(t)\bar{P}_i\left[(\Delta\bar{A}_i + \Delta\bar{A}_{di})\bar{x}(t) + \Delta\bar{b}_i\ 1(t-\tau(t))\right] \leq \\
& \qquad \bar{x}^T(t)\bar{P}_i(\bar{\alpha}_i^2\lambda_{min}^{-1}(\bar{L}_{7i}) + \bar{\beta}_i^2\lambda_{min}^{-1}(\bar{L}_{8i}) + \bar{\delta}_i^2\bar{l}_{1i}^{-1})\bar{P}_i\bar{x}(t) \\
& \qquad + \bar{x}^T(t)(\bar{L}_{7i} + \bar{L}_{8i})\bar{x}(t) + \bar{l}_{1i}
\end{aligned}
\tag{2.43}
$$

Defining positive definite matrices $\bar{L}_{ki}$, $k = 1, 3, 4, 6, 9$, positive constants $\bar{l}_{mi}$,

$m = 2, 3$ and following the same procedure as stated above, it can be shown that

$$
\begin{aligned}
-2\bar{x}^T(t)\bar{P}_i\Delta\bar{A}_{di}\int_{t-\tau(t)}^{t}\Delta\bar{A}_{j(s)}\bar{x}(s)\,ds \le \int_{t-\tau(t)}^{t}\bar{x}^T(s)\bar{L}_{1i}\bar{x}(s)ds \\
+\int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i\bar{\beta}_i^2(\bar{\alpha}^*)^2\lambda_{min}^{-1}(\bar{L}_{1i})\bar{P}_i\bar{x}(t)ds
\end{aligned} \tag{2.44}
$$

$$
\begin{aligned}
-2\bar{x}^T(t)\bar{P}_i\bar{A}_{di}\int_{t-\tau(t)}^{t}\Delta\bar{A}_{j(s)}\bar{x}(s)\,ds \le \int_{t-\tau(t)}^{t}\bar{x}^T(s)\bar{L}_{3i}\bar{x}(s)ds \\
+\int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i\bar{A}_{di}(\bar{\alpha}^*)^2\lambda_{min}^{-1}(\bar{L}_{3i})\bar{A}_{di}^T\bar{P}_i\bar{x}(t)ds
\end{aligned} \tag{2.45}
$$

$$
\begin{aligned}
-2\bar{x}^T(t)\bar{P}_i\Delta\bar{A}_{di}\int_{t-\tau(t)}^{t}\Delta\bar{A}_{dj(s)}\bar{x}(s-\tau(s))\,ds \le \int_{t-\tau(t)}^{t}\bar{x}^T(s-\tau(s))\bar{L}_{4i}\bar{x}(s-\tau(s))ds \\
+\int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i\bar{\beta}_i^2(\bar{\beta}^*)^2\lambda_{min}^{-1}(\bar{L}_{4i})\bar{P}_i\bar{x}(t)ds
\end{aligned} \tag{2.46}
$$

$$
\begin{aligned}
-2\bar{x}^T(t)\bar{P}_i\bar{A}_{di}\int_{t-\tau(t)}^{t}\Delta\bar{A}_{dj(s)}\bar{x}(s-\tau(s))\,ds \le \int_{t-\tau(t)}^{t}\bar{x}^T(s-\tau(s))\bar{L}_{6i}\bar{x}(s-\tau(s))ds \\
+\int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i\bar{A}_{di}(\bar{\beta}^*)^2\lambda_{min}^{-1}(\bar{L}_{6i})\bar{A}_{di}^T\bar{P}_i\bar{x}(t)ds
\end{aligned} \tag{2.47}
$$

$$
\begin{aligned}
-2\bar{x}^T(t)\bar{P}_i\Delta\bar{A}_{di}\int_{t-\tau(t)}^{t}\bar{b}_{j(s)}1(s-\tau(s))\,ds \le \int_{t-\tau(t)}^{t}\bar{b}_{j(s)}^T\bar{L}_{9i}\bar{b}_{j(s)}\bar{1}(s-\tau(s))ds \\
+\int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i\bar{\beta}_i^2\lambda_{min}^{-1}(\bar{L}_{9i})\bar{P}_i\bar{x}(t)ds
\end{aligned} \tag{2.48}
$$

$$
\begin{aligned}
-2\bar{x}^T(t)\bar{P}_i\Delta\bar{A}_{di}\int_{t-\tau(t)}^{t}\Delta\bar{b}_{j(s)}1(s-\tau(s))\,ds \le \int_{t-\tau(t)}^{t}\bar{l}_{2i}1(s-\tau(s))ds \\
+\int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i(\delta^*)^2\bar{\beta}_i^2\bar{l}_{2i}^{-1}\bar{P}_i\bar{x}(t)ds
\end{aligned} \tag{2.49}
$$

$$
\begin{aligned}
-2\bar{x}^T(t)\bar{P}_i\bar{A}_{di}\int_{t-\tau(t)}^{t}\Delta\bar{b}_{j(s)}1(s-\tau(s))\,ds \le \int_{t-\tau(t)}^{t}\bar{l}_{3i}1(s-\tau(s))ds \\
+\int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i\bar{A}_{di}(\delta^*)^2\bar{l}_{3i}^{-1}\bar{A}_{di}^T\bar{P}_i\bar{x}(t)ds
\end{aligned} \tag{2.50}
$$

$$-2\bar{x}^T(t)\bar{P}_i\Delta\bar{A}_{di}\int_{t-\tau(t)}^{t}\bar{A}_{j(s)}\bar{x}(s)\,ds \le \int_{t-\tau(t)}^{t}\bar{x}^T(s)\bar{L}_{2i}\bar{x}(s)ds + \int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i\Delta\bar{A}_{di}\left[\bar{A}_{j(s)}\bar{L}_{2i}^{-1}\bar{A}_{j(s)}^T\right]\Delta\bar{A}_{di}^T\bar{P}_i\bar{x}(t)ds \tag{2.51}$$

If (2.36) is verified then there exist positive definite matrices $\bar{X}_{1i}$, $\bar{L}_{2i}$ and $\bar{X}_{2i}$, $\bar{L}_{5i}$ such that [15]

$$\bar{A}_{j(s)}\bar{L}_{2i}^{-1}\bar{A}_{j(s)}^T < \bar{X}_{1i}^{-1} \quad \text{and} \quad \bar{A}_{dj(s)}\bar{L}_{5i}^{-1}\bar{A}_{dj(s)}^T < \bar{X}_{2i}^{-1} \tag{2.52}$$

Using (2.52), and the fact that $\lambda_{max}(\bar{X}_{ki}^{-1}) = \lambda_{min}^{-1}(\bar{X}_{ki})$, $k = 1, 2$ in inequality (2.51) yields

$$-2\bar{x}^T(t)\bar{P}_i\Delta\bar{A}_{di}\int_{t-\tau(t)}^{t}\bar{A}_{j(s)}\bar{x}(s)\,ds \le \int_{t-\tau(t)}^{t}\bar{x}^T(s)\bar{L}_{2i}\bar{x}(s)ds + \int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i\bar{\beta}_i^2\lambda_{min}^{-1}(\bar{X}_{1i})\bar{P}_i\bar{x}(t)ds \tag{2.53}$$

Finally, applying the above argument to the last term of (2.41) and using (2.52) leads to

$$-2\bar{x}^T(t)\bar{P}_i\Delta\bar{A}_{di}\int_{t-\tau(t)}^{t}\bar{A}_{dj(s)}\bar{x}(s-\tau(s))\,ds \le \int_{t-\tau(t)}^{t}\bar{x}^T(s-\tau(s))\bar{L}_{5i}\bar{x}(s-\tau(s))ds + \int_{t-\tau(t)}^{t}\bar{x}^T(t)\bar{P}_i\bar{\beta}_i^2\lambda_{min}^{-1}(\bar{X}_{2i})\bar{P}_i\bar{x}(t)ds \tag{2.54}$$

Hence, substituting (2.43)-(2.50), (2.53) and (2.54) in (2.41) yields

$$\begin{aligned}
\dot{V} \leq & \bar{\xi}^T(t,\tau(t))\bar{Z}_i\bar{\xi}(t,\tau(t)) \\
& + \bar{x}^T(t)\bar{P}_i(\lambda_{min}^{-1}(\bar{L}_{7i})\bar{\alpha}_i^2 + \lambda_{min}^{-1}(\bar{L}_{8i})\bar{\beta}_i^2 + \bar{\delta}_{1i}^{-1})\bar{P}_i\bar{x}(t) \\
& + \bar{x}^T(t)(\bar{L}_{7i} + \bar{L}_{8i})\bar{x}(t) + \bar{l}_{1i} + h\bar{l}_{2i} + h\bar{l}_{3i} \\
& + h^{-1}\int_{t-\tau(t)}^{t} \eta^T(t,s,\tau(s))\bar{Y}_{j(s)}\eta(t,s,\tau(s))ds \\
& + h^{-1}\int_{t-\tau(t)}^{t} (\bar{x}^T(t)h\bar{P}_i\left[\lambda_{min}^{-1}(\bar{L}_{1i})\bar{\beta}_i^2(\bar{\alpha}^*)^2 + \lambda_{min}^{-1}(\bar{X}_{1i})\bar{\beta}_i^2\right. \\
& \qquad + \lambda_{min}^{-1}(\bar{L}_{3i})(\bar{\alpha}^*)^2\bar{A}_{di}\bar{A}_{di}^T + \lambda_{min}^{-1}(\bar{L}_{4i})\bar{\beta}_i^2(\bar{\beta}^*)^2 \\
& \qquad + \lambda_{min}^{-1}(\bar{X}_{2i})\bar{\beta}_i^2 + \lambda_{min}^{-1}(\bar{L}_{6i})(\bar{\beta}^*)^2\bar{A}_{di}\bar{A}_{di}^T \\
& \qquad \left. + (\delta^*)^2\beta_i^2\bar{l}_{2i}^{-1} + A_{di}A_{di}^T(\delta^*)^2\bar{l}_{3i}^{-1} + \beta_i^2\lambda_{min}^{-1}(\bar{L}_{9i})\right]\bar{P}_i\bar{x}(t))ds \\
& + h^{-1}\int_{t-\tau(t)}^{t} h\bar{x}^T(s)(\bar{L}_{1i} + \bar{L}_{2i} + \bar{L}_{3i} + \begin{bmatrix} 0 & 0 \\ 0 & b_{j(s)}^T\bar{L}_{9i}b_{j(s)} \end{bmatrix})\bar{x}(s)ds \\
& + h^{-1}\int_{t-\tau(t)}^{t} h\bar{x}^T(s-\tau(s))(\bar{L}_{4i} + \bar{L}_{5i} + \bar{L}_{6i})\bar{x}(s-\tau(s))ds
\end{aligned} \tag{2.55}$$

On the other hand, it is known that

$$\lambda_{min}(\bar{L}_{ki})\ I \leq \bar{L}_{ki}, \quad \lambda_{min}(\bar{X}_{pi})\ I \leq \bar{X}_{pi}$$

Inequalities (2.38) and (2.40) imply that one can find positive constants $\bar{\rho}_{L_{\bar{k}i}}$ and $\bar{\rho}_{\bar{X}_{pi}}$ such that [60]

$$\bar{\rho}_{\bar{L}_{ki}}I - \bar{L}_{ki} < 0, \quad \bar{\rho}_{\bar{X}_{pi}}I - \bar{X}_{pi} < 0 \tag{2.56}$$

where $k = 1,\ 3,\ 4,\ 6,\ 7,\ 8,\ 9$ and $p = 1, 2$. This implies that $\bar{\rho}_{\bar{L}_{ki}} < \lambda_{min}(\bar{L}_{ki})$ and $\bar{\rho}_{\bar{X}_{pi}} < \lambda_{min}(\bar{X}_{pi})$. Let us denote

$$\bar{\Omega} = \begin{bmatrix} \bar{\Omega}_{11} & 0 & 0 \\ 0 & \bar{\Omega}_{22} & 0 \\ 0 & 0 & \bar{\Omega}_{33} \end{bmatrix}, \qquad \bar{\Gamma} = \begin{bmatrix} \bar{\Gamma}_{11} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.57}$$

where

$$\begin{aligned}\bar{\Omega}_{11} =& h\bar{P}_i[\bar{\rho}_{\bar{L}_{1i}}^{-1}\bar{\beta}_i^2(\bar{\alpha}^*)^2 + \bar{\rho}_{\bar{X}_{1i}}^{-1}\bar{\beta}_i^2 + \bar{\rho}_{\bar{L}_{3i}}^{-1}(\bar{\alpha}^*)^2\bar{A}_{di}\bar{A}_{di}^T \\ &+ \bar{\rho}_{\bar{L}_{4i}}^{-1}\bar{\beta}_i^2(\bar{\beta}^*)^2 + \bar{\rho}_{\bar{X}_{2i}}^{-1}\bar{\beta}_i^2 + \bar{\rho}_{\bar{L}_{6i}}^{-1}(\bar{\beta}^*)^2\bar{A}_{di}\bar{A}_{di}^T \\ &+ (\delta^*)^2\beta_i^2\bar{l}_{2i}^{-1} + \bar{A}_{di}\bar{A}_{di}^T(\delta^*)^2\bar{l}_{3i}^{-1} + \bar{\beta}_i^2\bar{\rho}_{\bar{L}_{9i}}^{-1}]\bar{P}_i\end{aligned} \tag{2.58a}$$

$$\bar{\Omega}_{22} = h(\bar{L}_{1i} + \bar{L}_{2i} + \bar{L}_{3i} + \begin{bmatrix} 0 & 0 \\ 0 & b_{j(s)}^T\bar{L}_{9i}b_{j(s)}\end{bmatrix}) \tag{2.58b}$$

$$\bar{\Omega}_{33} = h(\bar{L}_{4i} + \bar{L}_{5i} + \bar{L}_{6i}) \tag{2.58c}$$

$$\bar{\Gamma}_{11} = \bar{P}_i(\bar{\rho}_{\bar{L}_{7i}}^{-1}\bar{\alpha}_i^2 + \bar{\rho}_{\bar{L}_{8i}}^{-1}\bar{\beta}_i^2 + \bar{\delta}_i^2\bar{l}_{1i}^{-1})\bar{P}_i + \bar{L}_{7i} + \bar{L}_{8i} + \begin{bmatrix} 0 & 0 \\ 0 & \bar{l}_{1i} + h\bar{l}_{2i} + h\bar{l}_{3i}\end{bmatrix} \tag{2.58d}$$

Therefore, from (2.55), (2.56) and (2.58a)-(2.58d) one can write

$$\begin{aligned}\dot{\bar{V}} \leq& \bar{\xi}^T(t,\tau(t))(\bar{Z}_i + \bar{\Gamma})\bar{\xi}(t,\tau(t)) \\ &+ h^{-1}\int_{t-\tau(t)}^{t} \eta^T(t,s,\tau(s))(\bar{Y}_{j(s)} + \bar{\Omega})\eta(t,s,\tau(s))ds\end{aligned} \tag{2.59}$$

Similar to the proof of Theorem 2.1, note that inequality (2.35) implies

$$\bar{\xi}^T(\cdot)(\bar{Z}_i + \bar{\Gamma} + \tilde{\bar{E}}_i^T\tilde{\bar{U}}_i\tilde{\bar{E}}_i)\xi(\cdot) < 0 \tag{2.60}$$

using the Schur complement, where $\tilde{\bar{U}}_i$ and $\tilde{\bar{E}}_i$ are defined in Theorem 2.1. Note also that (2.29) and (2.35) imply $\bar{\xi}^T(\cdot)(\bar{Z}_i + \bar{\Gamma})\bar{\xi}(\cdot) < 0$, $x(t) \in X_i$, $i \in I_1$. Furthermore, (2.34) implies $\bar{Y}_{j(s)} + \bar{\Omega} \leq 0$ and from (2.59), $\dot{\bar{V}}_i < 0$, $x(t) \in X_i$, $i \in I_1$. A similar procedure can be repeated for the case when the switching index belongs to I_0 leading to (2.31)-(2.33) and $\dot{V}_i < 0$, $x(t) \in X_i$, $i \in I_0$. Thus the system is asymptotic stable following the argument of Theorem 2.1. ■

Remark 2.3. *Extension of the results of Theorem 2.2 to PWA time-delay systems with the following dynamics*

$$\begin{aligned}\dot{x}(t) =& (A_i + \Delta A_i)x(t) + \sum_{l=1}^{L}(A_{di_l} + \Delta A_{di_l})x(t - \tau_l(t)) \\ &+ (a_i + \Delta a_i) + \sum_{l=1}^{L}(b_{il} + \Delta b_{il})\ 1(t - \tau_l(t))\end{aligned}$$

where $0 < \tau_l \leq h_l$, $\dot{\tau}_l(t) \leq d_l < 1$, $A_{di_l} \in \mathbf{R}^{n\times n}$, $\tau_l(t) \in \mathbf{R}^+$, *and* $l \in \mathbf{N}$, *is straightforward and is not developed in this chapter due to space constraints.*

2.5 Numerical Examples

In this section, three examples are provided to show the effectiveness of the proposed approach.

Example 2.1. *In this example the stability of a time-delay system is investigated and it is shown that while the LMIs proposed in [42] are infeasible, the ones introduced in this chapter are quite effective. Consider the piecewise linear time-delay system* $\dot{x}(t) = A_i x(t) + A_{di} x(t-\tau)$ *with the system matrices given by*

$$A_1 = A_3 = -\begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad A_2 = A_4 = -\begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}$$

$$A_{d1} = A_{d3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A_{d2} = A_{d4} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$$

and let the cell partition be given by

$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \quad E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.61}$$

One can verify that using the LMIs proposed in [42], stability of the system is guaranteed only for time-delays less than 0.005, which is a very small margin. However, the LMIs derived in Theorem 2.1 ensure the stability for the time-delays as large as $h = 10^5$.

Example 2.2. *Consider the piecewise linear time-delay system* $\dot{x}(t) = A_i x(t) + A_{di} x(t-\tau)$, *with the same cell partition as in (2.61), and the system matrices given by*

$$A_1 = A_3 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = A_4 = -\begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}$$

$$A_{d1} = A_{d3} = \begin{bmatrix} 0.1 & 5.0 \\ -5.0 & 0.1 \end{bmatrix}, \quad A_{d2} = A_{d4} = \begin{bmatrix} 1.0 & 5.0 \\ -5.0 & -1.0 \end{bmatrix}$$

The LMIs derived in Theorem 2.1 are feasible for time-delays less than or equal to $h = 0.0264$ *in this example. Using simulation the system is unstable for* $\tau_{max} = 0.031$. *This seems to indicate that the result obtained in this example using the approach proposed for systems with no uncertainty is not too conservative.*

Assume now that the matrices A_i *and* A_{di} $(i = 1, \ldots, 4)$ *in the above example are subject to uncertainty. It can be verified that for* $||\Delta A_i|| \leq 0.1$ *and* $||\Delta A_{di}|| \leq 0.1$ $(i = 1, \ldots, 4)$ *the LMIs given in Theorem 2.2 are feasible for the time-delays less than or equal to* $h = 0.024$.

Example 2.3. *Consider the equation of motion of a simple pendulum [16] as follows*

$$T(t - \tau(t)) - mgl\sin(\theta(t)) = ml^2\ddot{\theta}(t) \tag{2.62}$$

where l *is the length of the pendulum,* g *is the gravitational acceleration,* m *is the pendulum mass and* T *is the input torque. It is assumed that the communication link between the sensor and the controller has the delay of* $\tau(t)$. *It is desired to keep the pendulum at* $\theta = 0.3491$ *rad* $= 20$ *deg. To that end, the nonlinear model (2.62) is linearized around* $\theta_0 = 0.7854$ *rad and* $\theta_0 = 0$ *rad as follows*

$$\begin{bmatrix} \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2}T(t - \tau(t)) \end{bmatrix} + \Delta A_1 x(t), \quad 0 \leq \theta(t) < 0.3927 \tag{2.63a}$$

$$\begin{bmatrix} \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.7071\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2}T(t-\tau(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ -1.2625\frac{g}{l} \end{bmatrix} + \Delta A_2 x(t)$$

$$0.3927 \leq \theta(t) \leq 0.7854 \tag{2.63b}$$

where $||\Delta A_1|| \leq 1.4$ *and* $||\Delta A_2|| \leq 1.2$ *represent the approximation error due to the linearization, and can be treated as uncertainties. The controller input to be used is as follows*

$$T(t) = [-16 \quad -8] \begin{bmatrix} \theta(t-\tau(t)) \\ \dot{\theta}(t-\tau(t)) \end{bmatrix} + 0.3150\ 1(t), \quad 0 \leq \theta(t) < 0.3927 \tag{2.64}$$

$$T(t) = [13.1296 \quad -8] \begin{bmatrix} \theta(t-\tau(t)) \\ \dot{\theta}(t-\tau(t)) \end{bmatrix} + 6.0725\ 1(t), \quad 0.3927 \leq \theta(t) \leq 0.7854 \tag{2.65}$$

Now it is desired to find the upper bound on delay $\tau(t)$*, such that the system be stable. Using the LMIs in Theorem 2.2 the maximum value of delay* $\tau(t)$ *becomes* $h = 0.026\ s$.

Chapter 3

An Adaptive Switching Scheme for Uncertain Discrete Time-Delay Systems

3.1 Introduction

Robust stabilization of uncertain discrete-time systems with time-varying delay is well-documented [9], [19], [46], [100]. In most of the existing works in this area, delay-dependent approaches are presented to find a single controller which stabilizes the uncertain system with time-varying delay. However, these works are often unable to effectively handle large uncertainties in both system parameters and delay.

Furthermore, in conventional adaptive control techniques (even for the case of finite-dimensional LTI systems), a number of standard assumptions in the form of *a priori* knowledge (e.g., on the relative degree, non-minimum phase property, and the sign of the high-frequency gain) are required to be made ([42], [62]). Furthermore, such techniques are usually inefficient in presence of highly uncertain or rapidly changing parameters. In order to relax the above mentioned limitations of classical adaptive control methods, the supervisory switching control schemes are presented [26], [81], [108], [43], [78], [74], [4]. The main idea of such schemes is to switch among a family of pre-designed and fixed controllers in such a way that adaptive tracking of reference signals is achieved. One of the recent works in this discipline of research is localized-based switching adaptive control proposed in [108], [109] which result in a fast model falsification and an acceptable transient response.

On the other hand, classical adaptive methods and recently developed switched based controllers can stabilize the systems with only large uncertainties on the system parameters. For uncertain time-delay systems there are only a few references that can handle both large uncertainties in system parameters in addition to time-delay in the system dynamic [58]. In [58] a pre-routed switching approach is developed to stabilize a class of uncertain continuous-time system with time-delay while the delay is supposed to be known.

In this chapter it is assumed that the system is exposed to large parameter uncertainties and time-varying delay in the state with known upper and lower bound on the delay. Furthermore the system is subject to bounded disturbances. It is also assumed that the system is not stabilizable by one single state feedback controller (due to the large magnitude of the uncertainties). It is aimed to design a set of state feedback gains along with a supervisory algorithm such that the discrete-time system becomes stable. To that end, a decomposition of uncertain parameter space is considered such that a stabilizing state feedback controller exists for each region. In the following, similar to [108], a switching algorithm is proposed with fast falsification property. Based on the properties of the system and the designed controllers, an upper bound on the rate of the changes on the system parameters and delay is obtained.

The plan of the rest of this chapter is as follows. The problem formulation and some useful definitions are presented in Section 3.2. Then, in Section 3.3 the stability analysis is carried out in two steps. First, the uncertain system parameters and delay are all considered to be fixed. Then, the results are extended to the case when the uncertain parameters and delay are time-varying. A numerical example is presented in Section 3.4 which demonstrates the effectiveness of the proposed adaptive switching controller.

3.2 Problem Formulation

Consider the following uncertain time-varying discrete-time system

$$x(k+1) = A(k)x(k) + A_d(k)x(k-l(k)) + B(k)u(k) + \mu(k) \tag{3.1}$$

where $x(k) \in \mathbf{R}^n$ is the measurable state vector, $u(k) \in \mathbf{R}^m$ is the control input, $\mu(k) \in \mathbf{R}^n$ is the disturbance vector, $A(k) \in \mathbf{R}^{n\times n}$, $A_d(k) \in \mathbf{R}^{n\times n}$ and $B(k) \in \mathbf{R}^{n\times m}$ are uncertain time-varying norm-bounded system matrices and $l(k)$ is the time-varying delay in state dynamics. It is assumed that the disturbance vector $\mu(k) \in \mathbf{R}^n$ is norm-bounded; i.e. $||\mu(k)|| \leq \bar{\mu}$, where, $\bar{\mu}$ is a positive constant. Moreover, the following assumption is made on the size of delay:

$$\underline{l} \leq l(k) \leq \bar{l} \tag{3.2}$$

where $\underline{l}$ and $\bar{l}$ are known non-negative integers. Let the initial condition for (3.1) be given by

$$x(k) = \phi(k), \qquad k = k_0 - \bar{l}, k_0 - \bar{l} + 1 ...., k_0 \tag{3.3}$$

where $\phi(k)$ is a real valued function on $[k_0 - \bar{l}, k_0]$, and k_0 is the initial time. The following definitions for the type of stability of discrete-time systems will prove convenient in the development of the main results. Throughout this chapter, for a vector $x \in \mathbf{R}^n$, $||x||$ denotes p-norm of x where p can be any arbitrary integer number. Furthermore, for a matrix $A \in \mathbf{R}^{n\times m}$, $||A||$ represents the corresponding induced norm defined by $||A|| = \max_{||x||=1} ||Ax||$.

Definition 3.1. *The time-delay system (3.1) with* $u(k)$ *and* $\mu(k)$ *both set to zero,* $\forall k \geq k_0$*, is said to be exponentially stable if there exist constant scalars* $\rho \in (0,1)$ *and* $M_1 > 0$ *such that* $||x(k)|| \leq M_1 \rho^{(k-k_0)} \sup_{k_0-\bar{l}\leq v\leq k_0} ||x(v)||$*,* $\forall k \geq k_0$*.*

Definition 3.2. *The system (3.1) with* $u(k) = 0$ *is said to be globally* $\bar{\mu}$*-exponentially stable, if there exist constant scalars* $\rho \in (0,1)$ *and* $\tilde{M}_1 > 0$*, as well as a function* $\tilde{M}_2(\cdot) : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ *with* $\tilde{M}_2(0) = 0$ *such that* $||x(k)|| \leq \tilde{M}_1 \rho^{(k-k_0)} \sup_{k_0-\bar{l}\leq v\leq k_0} ||x(v)|| + \tilde{M}_2(\bar{\mu})$ *,* $\forall k \geq k_0$*, where* $\mathbf{R}^+$ *denotes the set of strictly positive real numbers.*

It is desired now to design a switching discrete-time controller under which the system (3.1) is exponentially $\bar{\mu}$-exponentially stable in the presence of uncertainties and time-varying delay.

3.3 Main Results

3.3.1 A Time-Invariant System with Fixed Delay

In this subsection, it is assumed that the system matrices are uncertain nevertheless time invariant, and that the delay is an unknown, bounded constant. Moreover, no single controller is assumed to exist with the property that it stabilizes the system within the whole uncertain parameter space pertaining to A, A_d, B, and l. The following assumption is essential for developing the main results.

Assumption 3.1. A*,* A_d *and* l*, denoted by* Ω*, is compact and can be decomposed into a finite cover* $\{\Omega_i\}_1^L$*, for which the following conditions hold*

i. $\Omega_i \subset \Omega, \quad \Omega_i \neq \{\}, \quad i = 1, \ldots, L$

ii. $\bigcup_{i=1}^{L} \Omega_i = \Omega$

iii. *For any* $i \in \{1, ..., L\}$, *there exist* $(A_i, A_{di}, B_i) \in \Omega_i$ *(center) and* K_i *(control gain) such that for any* $l \in [l_{m_i}, l_{M_i}]$, *the controller* $u(k) = K_i x(k)$ *exponentially stabilizes the system (3.1) for all* $(A, A_d, B) \in \Omega_i$ *satisfying*

$$||A - A_i|| \leq \alpha_i, \quad ||A_d - A_{di}|| \leq \beta_i, \quad ||B - B_i|| \leq \delta_i \tag{3.4}$$

Remark 3.1. *In Conditions (i) and (ii) given above, it is supposed that the uncertain set* Ω *can be constructed from the union of the sets* Ω_i, $i = 1, ..., L$. *In addition, it is to be noted that* $\underline{l} \leq l_{m_i}$ *and* $l_{M_i} \leq \bar{l}$ *in condition (iii) above represent the lower and upper bounds of* l *in* Ω_i, *respectively.*

In the next step, a condition is presented based on which a supervisory control scheme is obtained to $\bar{\mu}$-exponentially stabilize the system (3.1). Let the control input be $u(k) = K_{i(k)}x(k)$ for some $i(k)$. Suppose that the uncertain plant (3.1) lies in Ω_p, $p \in \{1, ..., L\}$. Then, there exists a delay $\hat{l} \in [l_{m_p}, l_{M_p}]$ that is equal to the plant delay (i.e., $\hat{l} = l$) which can be interpreted as the nominal delay corresponding to Ω_p. Suppose that $i(k) = p$, then the Condition (iii) implies that

$$\begin{aligned} ||x(k) - A_p x(k-1) - A_{dp} x(k - \hat{l} - 1) - B_p u(k-1)|| \leq \alpha_p ||x(k-1)|| \\ + \beta_p ||x(k - \hat{l} - 1)|| + \delta_p ||u(k-1)|| + \bar{\mu} \end{aligned} \tag{3.5}$$

The above inequality provides the core falsifying criterion for the switching rule proposed in this work. If this inequality is violated for $\forall \hat{l} \in [l_{m_p}, l_{M_p}]$, it implies that the switching index $i(k)$ is wrong, i.e. $i(k) \neq p$.

The following algorithm is proposed to design supervisory control based on (3.5).

Algorithm 3.1.

1. Let $k = k_0$, $k_0 > 0$

   $P = \{1, \; ..., \; L\}$

$Q_p = \{m_p, \ ..., \ M_p\}, \forall p \in P$

$H(k_0) = \{(p, \ l_q) \mid p \in P \text{ and } q \in Q_p\}$

Choose $i(k_0) \in P$

2. $k = k + 1$

3. $\hat{H}(k) = \left\{(p, \ l_q) \mid (3.5) \text{ holds for } p \in P, \ \hat{l} = l_q, q \in Q_p\right\}$

4. $H(k) = H(k-1) \cap \hat{H}(k)$

5. **if** $\exists \ q \in Q_{i(k-1)}$ such that $(i(k-1), l_q) \in H(k)$,

   then $i(k) = i(k-1)$. Go to step 2

   else

   $i(k)$= any entry of P such that $(p, \ l_q) \in H(k)$. Go to step 2

Remark 3.2. *Note that the controller gain K_i will not be replaced in step 5 unless all potential values of delay corresponded to decomposition Ω_i, i.e. $\forall l_i \in [l_{m_i}, l_{M_i}]$, are falsified.*

Remark 3.3. *Algorithm 3.1 guarantees that the whole compact set Ω (uncertain parameter space) with any plant delay corresponding to $l \in [\underline{l}, \bar{l}]$ is examined. It also guarantees the existence of a control law that stabilizes the plant (3.1), leading to the convergence of the switching sequence $i(k_0), \ i(k_0+1), \ \dots$.*

In the following two definitions are introduced, which are required to proceed further.

Definition 3.3. *Consider the following linear time-varying discrete-time system with an integer state delay $l(k) \in [\underline{l}, \bar{l}]$*

$$x(k+1) = A_0(k)x(k) + A_1(k)x\left(k - l(k)\right) + \mu(k), \tag{3.6}$$

where $x(k) \in \mathbf{R}^n$ and $\mu(k) \in \mathbf{R}^n$ are the state and input of the system, respectively. Assume that $l(k) \in [\underline{l}, \bar{l}]$. Let $\Lambda(k) \in \mathbf{R}^{n(\bar{l}+1)\times n(\bar{l}+1)}$ be defined as

$$\Lambda(k) = \begin{bmatrix} \Lambda_{1,1} & 0_n & \cdots & 0_n & \Lambda_{1,l(k)+1} & 0_n & \cdots & 0_n & 0_n & 0_n \\ I_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & 0_n & 0_n \\ \vdots & & & & \vdots & & & \vdots & & \\ 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & I_n & 0_n & 0_n \\ 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & I_n & 0_n \end{bmatrix} \tag{3.7}$$

where I_n and 0_n denote the $n \times n$ identity and zero matrices, respectively. If $l(k)$ is non-zero,

$$\Lambda_{1,1} = A_0(k), \qquad \Lambda_{1,l(k)+1} = A_1(k) \tag{3.8}$$

otherwise,

$$\Lambda_{1,1} = \Lambda_{1,l(k)+1} = A_0(k) + A_1(k) \tag{3.9}$$

Definition 3.4. *For the system (3.6), let $\Phi(k_1, k_2)$ be defined as*

$$\Phi(k_1, k_2) = \begin{cases} \Lambda(k_1 - 1)\Lambda(k_1 - 2)\ldots\Lambda(k_2), & k_1 > k_2 \\ I_{n(\bar{l}+1)\times n(\bar{l}+1)}, & k_1 = k_2 \end{cases} \tag{3.10}$$

where $k_1 \geq k_2$.

Five lemmas are presented in the sequel, which are required to develop the main results of the chapter.

Lemma 3.1. *Consider the system (3.6). The state $x(k)$ can be expressed by*

$$x(k) = \Psi(k, k_0)z(k_0) + \sum_{p=k_0+1}^{k} \Psi(k, p)E\mu(p-1) \tag{3.11}$$

for $k > k_0$, where

$$\Psi(k_1, k_2) = \Xi\Phi(k_1, k_2) \tag{3.12}$$

$$\Xi = \begin{bmatrix} I_n & 0_n & \cdots & 0_n \end{bmatrix}, \qquad E = \Xi^T$$

and

$$z(k_0) = \left[\begin{array}{cccc} \phi^T(k_0) & \phi^T(k_0-1) & \cdots & \phi^T(k_0-\bar{l}) \end{array} \right]^T \tag{3.13}$$

Note that $\Xi \in \mathbf{R}^{n\times n(\bar{l}+1)}$ *and* $\phi(\cdot)$ *is defined in (3.3).*

Proof: From

$$z(k) = \left[\begin{array}{cccc} x^T(k) & x^T(k-1) & \cdots & x^T(k-\bar{l}) \end{array} \right]^T \tag{3.14}$$

the equation (3.6) can be written as

$$\begin{aligned} z(k+1) &= \Lambda(k)z(k) + E\mu(k) \\ x(k+1) &= \Xi z(k+1) \end{aligned} \tag{3.15}$$

where E and Ξ are given by (3.12), and $\Lambda(k)$ is defined in (3.7). It is to be noted that the recursion (3.15) is initialized by $z(k_0)$ defined in (3.13). By solving the system (3.15) recursively, one can verify that $x(k)$ satisfies (3.11). ■

Lemma 3.2. *In system (3.6), let* $\mu(k) = 0$. *Thus, if this system is exponentially stable for* $l(k) \in [\underline{l}, \bar{l}]$, *for* $\Psi(k, k_0)$ *defined in Lemma 3.1, there exists a constant* $\tilde{M} > 0$ *such that*

$$\|\Psi(k, k_0)\| \leq \tilde{M}\rho^{k-k_0}, \qquad k \geq k_0 \tag{3.16}$$

Proof: Let the i-th column of $\Psi(k, k_0)$ be represented by $\psi_i(k, k_0)$. Denote the i-th column of the identity matrix $I_{n(\bar{l}+1)\times n(\bar{l}+1)}$ with e_i. In (3.11), assume that $\mu(k) = 0$ and $z(k_0) = e_i$. Therefore,

$$x(k) = \Psi(k, k_0)e_i = \psi_i(k, k_0) \tag{3.17}$$

Since the system (3.6) is exponentially stable, according to Definition 3.1, there exists a constant $M > 0$ such that

$$\|\psi_i(k, k_0)\| = \|x(k)\| \leq M\rho^{k-k_0}, \qquad k > k_0 \tag{3.18}$$

On the other hand, since all the vector norms are equivalent in a finite-dimensional space [20], it is concluded that there exist a constant $c > 0$ such that

$$\|\Psi(k, k_0)\| \le c \sum_{i=1}^{n(\bar{l}+1)} \|\psi_i(k, k_0)\| \tag{3.19}$$

By defining $\hat{M} = c(\bar{l} + 1)nM$, it is followed that

$$\|\Psi(k, k_0)\| \le \hat{M}\rho^{k-k_0}, \qquad k > k_0 \tag{3.20}$$

Let $\tilde{M} = \max\{1, \hat{M}\}$, then (3.16) is resulted. ■

Lemma 3.3. *Suppose that the system*

$$x(k+1) = A(k)x(k) + A_d(k)x(k - l(k)) + B(k)u(k) \tag{3.21}$$

is exponentially stable under the feedback control $u(k) = Kx(k)$ *for all* $l(k) \in [\underline{l}, \bar{l}]$. *Then the system*

$$x(k+1) = A(k)x(k) + A_d(k)x(k - l(k)) + B(k)u(k) + \mu(k) \tag{3.22}$$

where $\|\mu(k)\| \le \bar{\mu}$, *is* $\bar{\mu}$*-exponentially stable under the control law* $u(k) = Kx(k)$, *for all* $l(k) \in [\underline{l}, \bar{l}]$.

Proof: Substituting $u(k) = Kx(k)$ in (3.21), the following equation is obtained

$$x(k+1) = (A(k) + B(k)K)\, x(k) + A_d(k)x\,(k - l(k)) + \mu(k) \tag{3.23}$$

According to Lemma 3.1, $x(k)$ can be written as (3.11) for $k > k_0$, with $A_0(k) = A(k) + B(k)K$, and $A_1(k) = A_d(k)$. Since the system obtained in (3.23) is exponentially stable for all $l(k) \in [\underline{l}, \bar{l}]$,

$$\|\Psi(k, k_0)z(k_0)\| \le M\rho^{k-k_0} \sup_{k_0 - \bar{l} \le v \le k_0} \|x(v)\| \tag{3.24}$$

From (3.11) and (3.24), it follows that

$$\|x(k)\| \le M\rho^{k-k_0} \sup_{k_0 - \bar{l} \le v \le k_0} \|x(v)\| + \sum_{p=k_0+1}^{k} \|\Psi(k, p)\|\|E\|\bar{\mu} \tag{3.25}$$

Using Lemma 3.2, it can be concluded that

$$\|x(k)\| \leq M\rho^{k-k_0} \sup_{k_0-\bar{l}\leq v\leq k_0} \|x(v)\| + \sum_{p=k_0+1}^{k} \tilde{M}\rho^{k-p}\bar{\mu} \tag{3.26}$$

Thus,

$$\|x(k)\| \leq M\rho^{k-k_0} \sup_{k_0-\bar{l}\leq v\leq k_0} \|x(v)\| + \tilde{M}\bar{\mu}\frac{1-\rho^{k-k_0}}{1-\rho} \tag{3.27}$$

Define

$$\tilde{M}_2(\bar{\mu}) := \tilde{M}\bar{\mu}\frac{1}{1-\rho} \tag{3.28}$$

It is clear that

$$\tilde{M}_2(\bar{\mu}) : R^+ \to R^+, \qquad \tilde{M}_2(0) = 0 \tag{3.29}$$

Therefore, for $k > k_0$,

$$\|x(k)\| \leq M\rho^{k-k_0} \sup_{k_0-\bar{l}\leq v\leq k_0} \|x(v)\| + \tilde{M}_2(\bar{\mu}) \tag{3.30}$$

Let $\tilde{M}_1$ denote $\max\{1, M\}$; it can now be inferred from Definition 3.2 that (3.21) is $\bar{\mu}$-exponentially stable under the feedback $u(k) = Kx(k)$. ■

Lemma 3.4. *Consider the following system*

$$x(k+1) = A(k)x(k) + A_d(k)x(k-l(k)) + B(k)u(k) + \mu(k) \tag{3.31}$$

where $l(k) \leq l_M$, and $\mu(k)$ is a bounded disturbance ($\|\mu(k)\| \leq \bar{\mu}$). If (3.31) is exponentially stable under the feedback law $u(k) = Kx(k)$, $\forall l(k) \leq l_M$, then there exist constants $\tilde{M} > 0$, $\rho \in (0,1)$, and a function $\hat{M}(\cdot) :\ \mathbf{R}^+ \to \mathbf{R}^+$ with $\hat{M}(0) = 0$ such that for every $\bar{l} \geq l_M$, and $k \geq k_0$

$$\sup_{k-\bar{l}\leq v\leq k} \|x(v)\| \leq \tilde{M}\rho^{k-k_0} \sup_{k_0-\bar{l}\leq v\leq k_0} \|x(v)\| + \hat{M}(\bar{\mu}) \tag{3.32}$$

Proof: Since (3.31) is exponentially stable under the feedback $u = Kx$, it is $\bar{\mu}$-exponentially stable as well. Thus, for any $k \geq k_0$, there exist a positive constant M_1 and a function $\hat{M}_1$ such that

$$\|x(k)\| \leq M_1\rho^{k-k_0} \sup_{k_0-\bar{l}\leq v\leq k_0} \|x(v)\| + \hat{M}_1(\bar{\mu}) \tag{3.33}$$

If $k > k_0 + \bar{l}$, then for $0 \leq i \leq \bar{l}$

$$\|x(k-i)\| \leq M_1 \rho^{-i} \rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_1(\bar{\mu}) \tag{3.34}$$

Let $\tilde{M}_1$ be equal to $\frac{M_1}{\rho^{\bar{l}}}$. Then, it can be concluded that

$$\sup_{k-\bar{l} \leq v \leq k} \|x(v)\| \leq \tilde{M}_1 \rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_1(\bar{\mu}). \tag{3.35}$$

Consider now the case when $k \leq k_0 + \bar{l}$. Since $A(k)$, $A_d(k)$ and $B(k)$ are norm bounded and K is a constant matrix, one can find a positive constant M_2 and a function $\hat{M}_2(\cdot)$ such that

$$\sup_{k-\bar{l} \leq v \leq k} \|x(v)\| \leq M_2 \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_2(\bar{\mu}), \tag{3.36}$$

Let $\tilde{M}_2$ be equal to $\frac{M_2}{\rho^{\bar{l}}}$. Therefore, it is resulted that

$$\sup_{k-\bar{l} \leq v \leq k} \|x(v)\| \leq \tilde{M}_2 \rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_2(\bar{\mu}). \tag{3.37}$$

Define $\tilde{M}$ and $\hat{M}$ as

$$\begin{aligned} \tilde{M} &= \max_{i=1,2} \tilde{M}_i \\ \hat{M}(\bar{\mu}) &= \max_{i=1,2} \hat{M}_i(\bar{\mu}), \qquad \forall \bar{\mu} \geq 0 \end{aligned} \tag{3.38}$$

It can be deduced that for a constant $\tilde{M}$ and a function $\hat{M}(\cdot) : R^+ \to R^+$, $\hat{M}(0) = 0$ defined above, the inequality (3.32) holds. ■

Lemma 3.5. *Let P denote the finite set $\{1, ..., L\}$, and consider the following time-delay system*

$$x(k+1) = A(k)x(k) + A_d(k)x(k - l(k)) + B(k)u(k) + \mu(k) \tag{3.39}$$

where $l(k) \leq \bar{l}$, $A(k)$, $A_d(k)$ and $B(k)$ are norm-bounded system matrices and $||\mu(k)|| \leq \bar{\mu}$, $\forall k$. Suppose that $u(k) = K_i x(k)$, $i \in P$. Then

$$\sup_{k-\bar{l} \leq v \leq k} ||x(v)|| \leq M_0 \sup_{k-\bar{l}-1 \leq v \leq k-1} ||x(v)|| + \bar{\mu} \tag{3.40}$$

where M_0 is a positive constant.

Proof: Assume that $||A(k)|| \leq \epsilon_A$, $||B(k)|| \leq \epsilon_B$, $||A_d(k)|| \leq \epsilon_{A_d}$, $\forall k$ and $\max_{i \in P} ||K_i|| := \epsilon_K$. From (3.39), it can be concluded that

$$||x(k)|| \leq (\epsilon_A + \epsilon_B \epsilon_K) \; ||x(k-1)|| + \epsilon_{A_d} \; ||x(k - l(k-1) - 1)|| + \bar{\mu} \tag{3.41}$$

It is straightforward to show that

$$\sup_{k-\bar{l} \leq v \leq k} ||x(v)|| \leq \sup_{k-\bar{l}-1 \leq v \leq k-1} ||x(v)|| + ||x(k)|| \tag{3.42}$$

Therefore, by substituting (3.41) in (3.42) and considering the fact that $||x(k-1)||$ and $||x(k - l(k-1) - 1)||$ are less than or equal to $\sup_{k-\bar{l}-1 \leq v \leq k-1} ||x(v)||$, it yields that

$$\sup_{k-\bar{l} \leq v \leq k} ||x(v)|| \leq (\epsilon_A + \epsilon_{A_d} + \epsilon_B \epsilon_K + 1) \sup_{k-\bar{l}-1 \leq v \leq k-1} ||x(v)|| + \bar{\mu} \tag{3.43}$$

By defining $M_0 := \epsilon_A + \epsilon_{A_d} + \epsilon_B \epsilon_K + 1$, and using (3.43), one can obtain (3.40). ■

Theorem 3.1. *Consider the system (3.1) and suppose that the conditions of Assumption 3.1 hold. Then, using the proposed switching algorithm the resultant closed-loop system is $\bar{\mu}$-exponentially stable.*

Proof: Consider the finite set $\{k_1, \ldots, k_f\}$ as the sequence of switching instants. Consider also two consecutive instants k_s and k_{s+1}. From (3.1), the dynamics of the closed-loop system (with time-invariant parameters and fixed delay) for $k \in [k_s, k_{s+1})$ can be presented by

$$\begin{aligned} x(k+1) =& (A + BK_{i(k_s)})x(k) + A_d x(k-l) + \mu(k) \\ =& (A_{i(k_s)} + B_{i(k_s)} K_{i(k_s)})x(k) + A_{di(k_s)} x(k - \hat{l}(k)) + (A + BK_{i(k_s)})x(k) \\ &+ A_d x(k-l) - (A_{i(k_s)} + B_{i(k_s)} K_{i(k_s)})x(k) \\ &- A_{di(k_s)} x(k - \hat{l}(k)) + \mu(k) \end{aligned} \tag{3.44}$$

where $\hat{l}(k)$ is the nominal delay that satisfies (3.5) for $k \in [k_s,\ k_{s+1})$. Define $\psi(k)$ as

$$\begin{aligned}\psi(k) =&(A + BK_{i(k_s)})x(k) + A_d x(k-l) - (A_{i(k_s)} + B_{i(k_s)}K_{i(k_s)})x(k) \\ &- A_{di(k_s)}x(k-\hat{l}(k)) + \mu(k)\end{aligned} \tag{3.45}$$

Thus, on substituting (3.45) in (3.44) one will obtain

$$x(k+1) = (A_{i(k_s)} + B_{i(k_s)}K_{i(k_s)})x(k) + A_{di(k_s)}x(k-\hat{l}(k)) + \psi(k) \tag{3.46}$$

where $||\psi(k)|| \leq \alpha_{i(k_s)}||x(k)|| + \beta_{i(k_s)}||x(k-\hat{l}(k))|| + \delta_{i(k_s)}||u(k)|| + \bar{\mu}$, as (3.5) is not violated in the switching interval $[k_s,\ k_{s+1})$. Introducing fictitious matrices and parameter ΔA, ΔA_d, ΔB and $\hat{\mu}(t)$, it can be concluded from the structure of (3.46) that

$$\begin{aligned}x(k+1) =&(A_{i(k_s)} + \Delta A + B_{i(k_s)}K_{i(k_s)} + \Delta BK_{i(k_s)})x(k) \\ &+ (A_{di(k_s)} + \Delta A_d)x(k-\hat{l}(k)) + \hat{\mu}(k)\end{aligned} \tag{3.47}$$

where $||\hat{\mu}(k)|| \leq \bar{\mu}$, $||\Delta A|| \leq \alpha_{i(k_s)}$, $||\Delta A_d|| \leq \beta_{i(k_s)}$ and $||\Delta B|| \leq \delta_{i(k_s)}$.

Since $K_{i(k)} \in \{K_i\}_{i=1}^{L}$, and $i(k) \in P$ (P is a finite set), it follows from Lemma 3.5 that there exist finite positive constants M_0 and $\bar{\mu}$ satisfying

$$\sup_{k_s-\bar{l}\leq v\leq k_s} ||x(v)|| \leq M_0 \sup_{k_s-\bar{l}-1\leq v\leq k_s-1} ||x(v)|| + \bar{\mu} \tag{3.48}$$

and since (3.47) behaves like an exponentially stable system, for any $k \in [k_s,\ k_{s+1})$, it follows from Lemmas 3.3 and 3.4 that there exist constants $M_{i(k_s)}$, $\rho_{i(k_s)}$, and a function $\hat{M}_{i(k_s)}(\cdot)$ with $\hat{M}_{i(k_s)}(0) = 0$ such that

$$\sup_{k-\bar{l}\leq v\leq k} ||x(v)|| \leq M_{i(k_s)}\rho_{i(k_s)}^{k-k_s} \sup_{k_s-\bar{l}\leq v\leq k_s} ||x(v)|| + \hat{M}_{i(k_s)}(\bar{\mu}) \tag{3.49}$$

By substituting (3.48) in (3.49) one will obtain

$$\sup_{k-\bar{l}\leq v\leq k} ||x(v)|| \leq M_0 M_{i(k_s)}\rho_{i(k_s)}^{k-k_s} \sup_{k_s-\bar{l}-1\leq v\leq k_s-1} ||x(v)|| + \hat{M}_{i(k_s)}(\bar{\mu}) + M_{i(k_s)}\rho_{i(k_s)}^{k-k_s}\bar{\mu} \tag{3.50}$$

Define

$$\rho := \max_{i(k)} \rho_{i(k)}, \quad i(k) \in \{1, \ ..., \ L\} \tag{3.51a}$$

$$M := \max_{i(k)} M_{i(k)}, \quad i(k) \in \{1, \ ..., \ L\} \tag{3.51b}$$

and using the same argument as above, and for the time interval $k \in [k_{s-1}, \ k_s)$, it can be concluded that

$$\begin{aligned} \sup_{k_s-\bar{l}-1 \leq v \leq k_s-1} ||x(v)|| \leq M_0 M \rho^{k_s-k_{s-1}-1} & \sup_{k_{s-1}-\bar{l}-1 \leq v \leq k_{s-1}-1} ||x(v)|| \\ & + \hat{M}_{i(k_{s-1})}(\bar{\mu}) + M\rho^{k_s-k_{s-1}-1}\bar{\mu} \end{aligned} \tag{3.52}$$

Therefore, substituting inequality (3.52) in (3.50), and considering (3.51) the following can be obtained for $k \in [k_s, k_{s+1})$

$$\sup_{k-\bar{l} \leq v \leq k} ||x(v)|| \leq M^2 M_0^2 \rho^{k-k_{s-1}-1} \sup_{k_{s-1}-\bar{l}-1 \leq v \leq k_{s-1}-1} ||x(v)|| + \bar{M}_{i(k_s)}(\bar{\mu}) \tag{3.53}$$

where

$$\bar{M}_{i(k_s)}(\bar{\mu}) = \hat{M}_{i(k_s)}(\bar{\mu}) + M\rho^{k-k_s}[M_0\hat{M}_{i(k_{s-1})}(\bar{\mu}) + \bar{\mu}] + M^2 M_0 \rho^{k-k_{s-1}-1}\bar{\mu} \tag{3.54}$$

Using the above inequality iteratively yields

$$||x(k)|| \leq M^f M_0^f \rho^{k-k_0-f} \sup_{k_0-\bar{l} \leq v \leq k_0} ||x(v)|| + \bar{M}_{i(k_f)}(\bar{\mu}) \tag{3.55}$$

for $k \geq k_f$, where

$$\bar{M}_{i(k_f)}(\bar{\mu}) = \hat{M}_{i(k_f)}(\bar{\mu}) + M\rho^{k-k_f}[M_0\hat{M}_{i(k_{f-1})}(\bar{\mu}) + \bar{\mu}] + M^f M_0^{f-1}\rho^{k-k_0-f}\bar{\mu} \tag{3.56}$$

Define $\tilde{M}_1 = (\frac{MM_0}{\rho})^f$ and $\tilde{M}_2(\bar{\mu}) = \hat{M}_{i(k_f)}(\bar{\mu}) + M\rho^{k_0-k_f}[M_0\hat{M}_{i(k_{f-1})}(\bar{\mu}) + \bar{\mu}] + M^f M_0^{f-1}\rho^{-f}\bar{\mu}$. Note that $\tilde{M}_1$ is a bounded positive constant and $\tilde{M}_2(\bar{\mu})$ is a function of $\bar{\mu}$ such that $\tilde{M}_2(\cdot) : \mathbf{R}^+ \to \mathbf{R}^+$ with $\tilde{M}_2(0) = 0$. Therefore (3.55) can be written as

$$||x(k)|| \leq \tilde{M}_1 \rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} ||x(v)|| + \tilde{M}_2(\bar{\mu}) \tag{3.57}$$

Hence, it follows from Definition 3.2, that the system (3.1) is $\bar{\mu}$-exponentially stable. ■

3.3.2 A Time-Varying System with Time-Varying Delay

It is now assumed that the uncertain system (3.1) can have infrequent parameter jumps. The following assumption is made on the maximum allowable speed of parameter jumps.

Assumption 3.2. *The number of jumps in system parameters in (3.1) (i.e. $A(k)$, $A_d(k)$, $B(k)$ and $l(k)$) for any time interval $[k,\ k + \sigma N + \bar{l}]$ cannot exceed σ, where $\bar{l}$ is the maximum bound on the delay, and σ, N are strictly positive constants.*

Algorithm 3.1 can still be used in the case of a system with slowly vary parameters satisfying Assumption 3.2, after modifying step 4 as follows [108]

$$H(k) = \begin{cases} H(k-1) \cap \hat{H}(k), & \text{if } H(k-1) \cap \hat{H}(k) \neq \{\} \\ \hat{H}(k), & \text{otherwise} \end{cases} \tag{3.58}$$

This modification allows the algorithm to recheck the falsified items. In the following theorem, conditions for $\bar{\mu}$-exponential stability of the time-varying systems are presented.

Theorem 3.2. *Consider the system (3.1) and let the condition of Assumptions 3.1 and 3.2 holds for some σ and N (strictly positive). Then the closed system is globally $\bar{\mu}$-exponentially stable if $\tilde{M}_1 \rho^N < 1$, where $\tilde{M}_1$ and ρ are constant scalars defined in Definition 3.2.*

Proof: Consider the behavior of system (3.1) in the interval $[k,\ k+\sigma N+\bar{l}]$, where the number of parameter jumps is assumed to be less than σ. Let h denote the number of switchings carried out by the controller in the above interval. Consider the constant ρ and the function $\bar{M}_f(\cdot)$ such that $\rho \in (0,1)$, $\bar{M}_f(0)=0$, and $\bar{M}_f(\cdot): \mathbf{R}^+ \rightarrow \mathbf{R}^+$. For any interval $[k,\ k+\sigma N+\bar{l}]$ one can use (3.55) to obtain

$$||x(k+\sigma N+\bar{l})|| \leq (\frac{MM_0}{\rho})^h \rho^{\sigma N+\bar{l}} \sup_{k-\bar{l}\leq v\leq k} ||x(v)|| + \bar{M}_f(\bar{\mu}) \tag{3.59}$$

where f is the maximum number of switchings that can be made by the controller applied to the plant with unknown time-invariant parameters and fixed unknown time-delay. Then,

$$h \leq \sigma f + 1$$

which implies that

$$||x(k+\sigma N+\bar{l})|| \leq (\frac{MM_0}{\rho})((\frac{MM_0}{\rho})^f \rho^N)^\sigma \rho^{\bar{l}} \sup_{k-\bar{l}\leq v\leq k} ||x(v)|| + \bar{M}_f(\bar{\mu}) \tag{3.60}$$

Using the same argument as in [108] and for sufficiently large N, it can be shown that $(\frac{MM_0}{\rho})((\frac{MM_0}{\rho})^f \rho^N)^\sigma \leq 1$, then the above inequality leads to

$$\sup_{k+\sigma N\leq v\leq k+\sigma N+\gamma} ||x(v)|| \leq a \sup_{k-\bar{l}\leq v\leq k} ||x(v)|| + \bar{M}_f(\bar{\mu}) \tag{3.61}$$

where a and γ are scalar constants such that $0<a<1$ and $\gamma \in [0,\ \bar{l}]$, respectively.

Note that it can be concluded from (3.61) that

$$\sup_{k_0+k(\sigma N)+(k-1)\bar{l}\leq v\leq k_0+k(\sigma N+\bar{l})} ||x(v)|| \leq a^k \sup_{k_0-\bar{l}\leq v\leq k_0} ||x(v)|| + \sum_{i=1}^{i=k} a^{i-1}\bar{M}_f(\bar{\mu}) \tag{3.62}$$

for all $k \in \mathbf{N} \cup \{0\}$. Since $0<a<1$ and $\sum_{i=1}^{k} a^{i-1} < \infty$, it is easy to notice from Definition 3.2, that $||x(k_0+k(\sigma N+\bar{l}))||$ is $\bar{\mu}$-exponentially stable for all $k \in \mathbf{N}\cup\{0\}$.

Now, it is straightforward to observe that $\forall j \in [1, \sigma N + \bar{l} - 1]$ and $\forall k \in \mathbf{N} \cup \{0\}$, there exist positive constants G_1 and G_2 such that

$$||x(k_0 + k(\sigma N + \bar{l}) + j)|| \leq G_1 \sup_{k_0 + k\cdot(\sigma N) + (k-1)\cdot \bar{l} \leq v \leq k_0 + k\cdot(\sigma N + \bar{l})} ||x(v)|| + G_2 \quad (3.63)$$

This implies $\bar{\mu}$-exponential stability of the system (3.1) with time-varying parameters and time-varying delay satisfying Assumption 3.2. ■

3.4 Numerical Example

Example 3.1. *Consider the following discrete-time system with time-delay:*

$$x(k+1) = A(k)x(k) + A_d(k)x(k - l(k)) + B(k)u(k) + \mu(k) \quad (3.64)$$

where $l(k) \in [0, 24]$, $\mu(k)$ *is the uniformly bounded disturbance such that* $||\mu(k)|| < 0.1$, $\forall k \in \mathbf{N} \cup \{0\}$. *Furthermore, the uncertain matrices* $A(k)$, $A_d(k)$ *and* $B(k)$ *can switch between the following 7 sets of matrices*

1) $A_1 = \begin{bmatrix} 0.8 & 0 \\ 0.1 & 0.9 \end{bmatrix} + \Delta A_1, B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + \Delta B_1, A_{d1} = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} + \Delta A_{d1}$

(3.65a)

2) $A_2 = \begin{bmatrix} 0.3 & 1 \\ 0.1 & 0.6 \end{bmatrix} + \Delta A_2. B_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + \Delta B_2, A_{d2} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} + \Delta A_{d2}$

(3.65b)

3) $A_3 = \begin{bmatrix} 1 & 1 \\ 0.2 & 0.7 \end{bmatrix} + \Delta A_3, B_3 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + \Delta B_3, A_{d3} = \begin{bmatrix} 0.4 & 0 \\ 0.8 & 0.4 \end{bmatrix} + \Delta A_{d3}$

(3.65c)

4) $A_4 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \Delta A_4, B_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta B_4, A_{d4} = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 0.3 \end{bmatrix} + \Delta A_{d4}$

(3.65d)

5) $A_5 = \begin{bmatrix} 1 & 1 \\ -1 & -0.1 \end{bmatrix} + \Delta A_5, B_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta B_5, A_{d5} = \begin{bmatrix} -0.5 & 0 \\ 0.1 & 0.03 \end{bmatrix} + \Delta A_{d5}$

(3.65e)

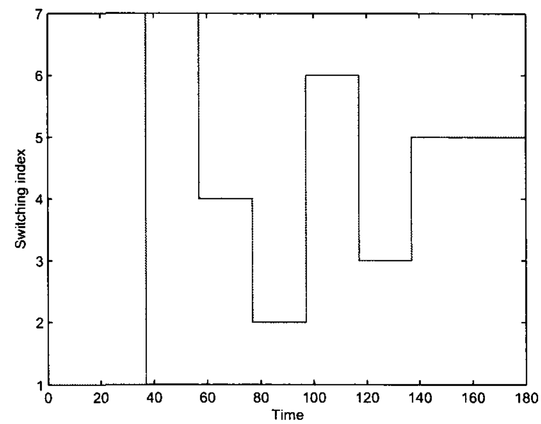
6) $A_6 = \begin{bmatrix} -0.2 & 0.1 \\ -0.5 & 1 \end{bmatrix} + \Delta A_6, B_6 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta B_6, A_{d6} = \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.5 \end{bmatrix} + \Delta A_{d6}$

(3.65f)

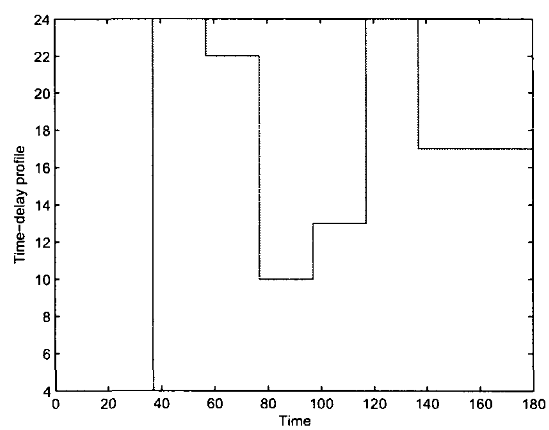
7) $A_7 = \begin{bmatrix} -0.1 & 1 \\ 0.1 & -1 \end{bmatrix} + \Delta A_7, B_7 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta B_7, A_{d7} = \begin{bmatrix} -0.4 & 0.4 \\ -0.1 & 0.1 \end{bmatrix} + \Delta A_{d7}$

(3.65g)

where $||\Delta A_i|| \leq 0.05$, $||\Delta A_{di}|| \leq 0.05$ *and* $||\Delta B_i|| \leq 0.05$ *for* $i = 1, ..., 7$ *such that* $||\cdot||$ *represents 2-norm. Using the Lyapunov approach described in [19], 50 controllers are designed to cover the whole parameter space corresponding to the uncertain time-delay and state-space matrices given above. Switching sequence between different* *subsystems in (3.65a)-(3.65g) along with time-delay profile are depicted in Fig. 3.1. By employing the controllers mentioned above and following Algorithm 3.1, the state trajectories sketched in Fig. 3.2 are obtained. These trajectories clearly show that the system is stable (or more precisely, $\bar{\mu}$-exponentially stable) in the presence of the uncertain time-delay and state-space matrices. The switching instants between different controllers are given in Fig. 3.3.*



(a)



(b)

Figure 3.1: (a) Switching sequence between different subsystems in (3.65a)-(3.65g); (b) time-delay profile.

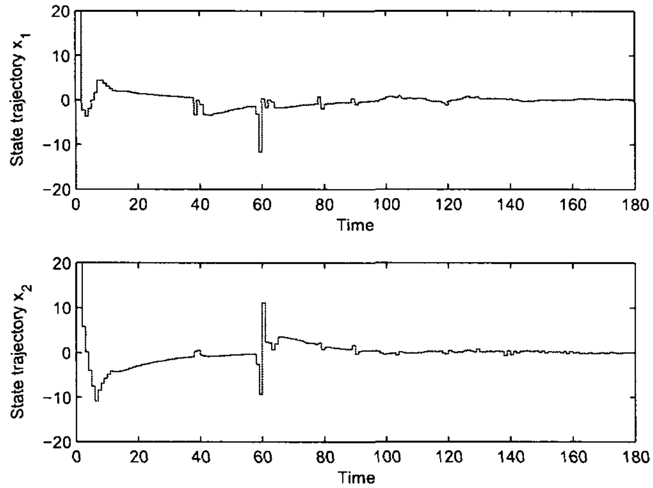


Figure 3.2: State trajectories

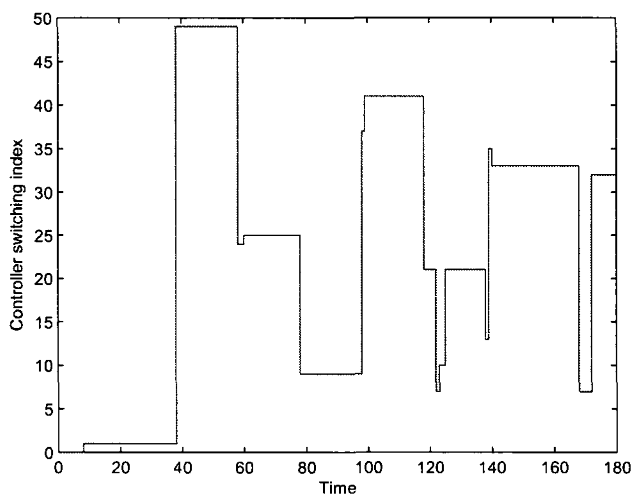


Figure 3.3: Controller index at switching instants

Chapter 4

Stability Analysis of Switched Time-Delay Systems with Polytopic Uncertainties

4.1 Introduction

Considerable attention has been paid in the past few years to the stability analysis of switched systems [43], [30], [31], [1], [59]. A broad range of engineering problems such as biological systems, air traffic control, networked control systems (NCS) and power systems can be modeled as switched systems (e.g., see [43] and references therein). It is also known that highly uncertain systems or systems subject to abrupt changes can be effectively described in the framework of switched systems [1]. Given the uncertain nature of real-world systems, a fixed model at any time interval may not describe the system accurately enough, and one may need to incorporate uncertainty in each individual model in the switched system formulation [88], [102], [55].

The common Lyapunov functional approach is proposed for systems with arbitrary switching signals [43], [92]. The main drawback of this method is the conservatism of the resultant stability conditions, in general. The piecewise Lyapunov functional method is introduced in [30] to remedy this shortcoming. In this method, a Lyapunov functional is assigned to any given subsystem; the overall system is stable if each Lyapunov functional decreases in a sufficiently short time interval. The notion of *dwell time* was introduced in [30] to show that a switched system (consisting of stable subsystems) is stable if the switching is sufficiently slow.

On the other hand, time-delay is an inevitable phenomenon in many physical systems [8], [106], [97]. It is known that time-delay, if not taken into account in the controller design, can result in poor performance of the closed-loop system or even instability. Time-delay systems are classified as retarded and neutral. The differential equations describing retarded time-delay systems involve delay in the state only, whereas the ones describing neutral time-delay systems involve delay in the derivatives of the state as well (see e.g. [23], [40]). In [76], [96], [29], [28], [98], [17], [18], retarded time-delay systems with polytopic-type uncertainties are considered, and delay-dependent stability criteria are developed to deal with this type of system. In particular, [96], [29], [28] introduce free-weighting matrices to utilize the Leibniz-Newton formula in the derivative of the Lyapunov functional in order to reduce the conservatism of the stability conditions and to avoid the multiplication of the system matrices by the Lyapunov functional matrices.

Stability of switched time-delay systems has been studied in [81], [80], [105], [39], [56]. In [56], sufficient conditions are derived for the stability of uncertain piecewise affine time-delay systems, using the Lyapunov functionals. Stability and stabilization of linear switched time-delay systems using a delay-dependent common Lyapunov functional is investigated in [39], [81], [105]. To the best of the knowledge of the author, no results are reported in the literature on the stability and stabilization of switched time-delay systems with polytopic uncertainties, due primarily to the complexity of the problem.

This chapter investigates the stability of switched time-delay systems with polytopic-type uncertainties. The idea of free-weighting matrices in the derivative of Lyapunov functional is employed to find stability criteria for this type of system. In contrast with [96], [29], [28], the *exponential* stability of switched time-delay systems is addressed here.

The remainder of this chapter is organized as follows. The problem of switched time-delay systems subject to polytopic uncertainties is formulated in Section 4.2. Then, in Section 4.3 stability analysis is carried out for both cases of switched retarded and neutral time-delay systems, and new linear matrix inequalities (LMI) are derived which provide sufficient conditions for the stability of switched time-delay systems with polytopic-type uncertainties. Numerical examples are presented in Section 4.4 which demonstrate the effectiveness of the proposed stability criteria.

4.2 Problem Formulation

Consider a switched time-delay system described by

$$\dot{x}(t) - D\dot{x}(t-\bar{\tau}) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t-\tau(t)) \tag{4.1}$$

with the initial condition

$$x(t) = \phi(t), \quad t \in [-h, 0] \tag{4.2}$$

where $x(t) \in \mathbf{R}^n$ is the state, and $\sigma(t)$ is the switching signal. Furthermore, $D \in \mathbf{R}^{n\times n}$ is a fixed matrix, A_i, $B_i \in \mathbf{R}^{n\times n}$ are the subsystem matrices, where the subscript $i \in \bar{\mathbf{m}} := \{1, ..., m\}$ represents the i-th mode of the switched system, $\bar{\tau}$ is fixed delay, $\tau(t)$ is time-varying delay, $h = \max\{\bar{\tau}, \tau(t)\}$, and $\phi(t)$ is a known continuous function defined on the interval $[-h, 0]$. Let $\{t_k\}$, $k \in \mathbf{Z}$ be the switching instants for system (4.4). Notice that in any time interval $[t_k, t_{k+1})$, the function $\sigma(t) = i$ ($i \in \bar{\mathbf{m}}$) is right continuous piecewise constant.

It is assumed in (4.4) that the system matrices A_i, B_i, $i \in \bar{\mathbf{m}}$, are subject to polytopic uncertainties which can be modeled as

$$[A_i\ B_i] \in \Omega,\quad \Omega := \left\{ [A_i(\xi)\ B_i(\xi)] = \sum_{v_i=1}^{p_i} \xi_i^{(v_i)} [A_i^{(v_i)}\ B_i^{(v_i)}] \right. \\ \left. \sum_{v_i=1}^{p_i} \xi_i^{(v_i)} = 1, \quad \xi_i^{(v_i)} \geq 0 \right\},\ \forall i \in \bar{\mathbf{m}} \tag{4.3}$$

where $A_i^{(v_i)}$, $B_i^{(v_i)}$ are known constant matrices, and $\xi_i^{(v_i)}$'s represent time-invariant uncertainties, $v_i \in \bar{\mathbf{p}}_i := \{1, ..., p_i\}$.

Remark 4.1. *The model given for each subsystem in (4.1) is the most general formulation for linear time-invariant (LTI) time-delay systems. Each subsystem modeled by (4.1) is often referred to as a neutral time-delay system. In many real-world applications, however, delay appears only in the state and not in the derivative of the state, i.e.*

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} x(t - \tau(t)) \tag{4.4}$$

This type of system is referred to as a retarded time-delay system. In this chapter both type of time-delay systems will be studied. In the case of retarded time-delay systems the delay can vary with time, while for the neutral time-delay systems it is assumed to be fixed.

The following assumption and two definitions will be used in developing the results for retarded time-delay systems.

Assumption 4.1. *It is assumed that the system delay $\tau(t)$ in (4.4) satisfies the* *following constraints*

$$0 \leq \tau(t) \leq h \tag{4.5a}$$

$$|\dot{\tau}(t)| \leq d \tag{4.5b}$$

for all t, where h and d are known positive constants.

Definition 4.1. *The switched time-delay system (4.4) under the switching signal $\sigma(t)$ is called exponentially stable with the rate $\alpha > 0$, if the solution $x(t)$ of the state equation (4.4) satisfies the inequality*

$$||x(t)|| \leq \kappa ||x_{t_0}||_{c_1} e^{-\alpha(t-t_0)} \qquad \forall t_o \geq 0, \quad t \geq t_0$$

for any $\kappa \geq 1$, where $||\cdot||$ represents the Euclidean norm, and $||x_t||_{c_1} = \sup_{-h \leq \theta \leq 0} ||x(t+\theta)||$.

Definition 4.2. *[30] The switching signal $\sigma(t)$ has an average dwell time T_d if there exists a nonnegative number N_0 such that the number of discontinuities of $\sigma(t)$ on an arbitrary interval (t_0, t), denoted by $N_{\sigma(t)}(t; t_0)$, satisfies*

$$N_{\sigma(t)}(t; t_0) \leq N_0 + \frac{t - t_0}{T_d}$$

Without loss of generality, it is assumed in this work that $N_0 = 0$.

In the next section, the stability conditions are derived for switched retarded and neutral time-delay systems with polytopic uncertainties.

4.3 Main Results

4.3.1 Stability of Uncertain Switched Retarded Time-Delay Systems

Assume for now that the system matrices A_i, B_i in (4.4) are constant and fixed. Using the Leibniz-Newton formula, one can write

$$x(t-\tau(t)) = x(t) - \int_{t-\tau(t)}^{t} \dot{x}(s)ds \tag{4.6}$$

Note that generally $\dot{x}(s)$ in (4.6) may be governed by different equations in the time interval $[t-\tau(t),\ t)$ (due to possible switchings between different subsystems in this interval). Denote the switching time instants with $t_1,\ t_2,\ \ldots,\ t_\mu$ in the interval $[t-\tau(t),\ t)$. In other words, in each time interval $[t-\tau(t),\ t_1)$, $[t_1,\ t_2)$, ..., $[t_\mu,\ t)$, the system dynamics is governed by a fixed state equation. Hence, using (4.6) the equation (4.4) can be rewritten as follows

$$\dot{x}(t) = (A_i + B_i)x(t) - B_i \int_{t-\tau(t)}^{t} \dot{x}(s)ds \tag{4.7a}$$

$$\begin{aligned}\int_{t-\tau(t)}^{t} \dot{x}(s)ds = &\int_{t-\tau(t)}^{t_1} [A_{(t-\tau(t))}x(s) + B_{(t-\tau(t))}x(s-\tau(s))]ds \\ &- \sum_{l=2}^{\mu} \int_{t_{l-1}}^{t_l} [A_{(t_{l-1})}x(s) + B_{(t_{l-1})}x(s-\tau(s))]ds \\ &- \int_{t_\mu}^{t} [A_{t_\mu}x(s) + B_{t_\mu}x(s-\tau(s))]ds\end{aligned} \tag{4.7b}$$

To simplify the notation, (4.7b) can be rewritten in the following form

$$\begin{aligned}\dot{x}(t) =&(A_i + B_i)x(t) - B_i \int_{t-\tau(t)}^{t} A_{q(s)}x(s)\, ds \\ &- B_i \int_{t-\tau(t)}^{t} B_{q(s)}x(s-\tau(s))\, ds\end{aligned} \tag{4.8}$$

Note that $q(s)$ in (4.8) is a piecewise constant function, representing the index of the system matrices in different time intervals. In order to proceed further, let the following positive-definite Lyapunov-Krasovskii functional for the i-th subsystem be defined

$$\begin{aligned} V_i =& x^T(t)P_ix(t) + \int_{t-\tau(t)}^{t} x^T(s)Q_ie^{\alpha(s-t)}x(s)ds \\ &+ \int_{-h}^{0}\int_{t+\theta}^{t} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix}^T Z_ie^{\alpha(s-t)} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix} dsd\theta \end{aligned} \tag{4.9}$$

where P_i, Z_i ($i \in \bar{\mathbf{m}}$) are symmetric positive-definite matrices, and Q_i is a symmetric positive-semidefinite matrix.

The following lemma presents the conditions for exponential decaying of Lyapunov-Krasovskii functional (4.9) in the time interval $[t_k, t_{k+1})$ for system (4.4) without uncertainties.

Lemma 4.1. *Consider the switched time-delay system (4.4) with fixed matrices A_i, B_i. Let $\sigma(t) = i$, $t \in [t_k, t_{k+1})$, $i \in \bar{\mathbf{m}}$. The Lyapunov-Krasovskii functional (4.9) is exponentially decaying in each time interval $[t_k, t_{k+1})$ (i.e. $V_i(x_t) \leq e^{\alpha(t_k-t)}V_i(x_{t_k^+})$), $\forall t \in [t_k, t_{k+1})$ if there exist symmetric positive-definite matrices $P_i \in \mathbf{R}^{n\times n}$, $\Gamma_i \in \mathbf{R}^{3n\times 3n}$, $Z_i \in \mathbf{R}^{2n\times 2n}$, a symmetric positive-semidefinite matrix $Q_i \in \mathbf{R}^{n\times n}$, and appropriately dimensioned matrices T_i, N_i, given by*

$$T_i = \begin{bmatrix} T_{1i} \\ \vdots \\ T_{3i} \end{bmatrix}, \quad N_i = \begin{bmatrix} N_{1i} \\ \vdots \\ N_{3i} \end{bmatrix}$$

for all $i \in \bar{\mathbf{m}}$, such that the LMIs

$$E_i + h\Gamma_i < 0 \tag{4.10a}$$

$$\begin{bmatrix} \Gamma_i & T_i \begin{bmatrix} A_q & B_q \end{bmatrix} \\ * & Z_i e^{-\alpha h} \end{bmatrix} \geq 0 \tag{4.10b}$$

hold for all $i, q \in \bar{\mathbf{m}}$, where

$$\Gamma_i = \begin{bmatrix} \Gamma_{11i} & \Gamma_{12i} & \Gamma_{13i} \\ * & \Gamma_{22i} & \Gamma_{23i} \\ * & * & \Gamma_{33i} \end{bmatrix} \geq 0, \; Z_i = \begin{bmatrix} Z_{11i} & Z_{12i} \\ * & Z_{22i} \end{bmatrix} \geq 0 \tag{4.11}$$

$$E_i = \begin{bmatrix} E_{11i} & E_{12i} & E_{13i} \\ * & E_{22i} & E_{23i} \\ * & * & E_{33i} \end{bmatrix} \tag{4.12}$$

$$\begin{aligned}
E_{11i} =& Q_i + \alpha P_i - N_{1i} A_i - A_i^T N_{1i}^T + h Z_{11i} + T_{1i} + T_{1i}^T \\
E_{12i} =& - N_{1i} B_i - A_i^T N_{2i}^T + h Z_{12i} - T_{1i} + T_{2i}^T \\
E_{13i} =& P_i + N_{1i} - A_i^T N_{3i}^T + T_{3i}^T \\
E_{22i} =& - (1-d) Q_i e^{-\alpha h} - N_{2i} B_i - B_i^T N_{2i}^T + h Z_{22i} - T_{2i} - T_{2i}^T \\
E_{23i} =& N_{2i} - B_i^T N_{3i}^T - T_{3i}^T \\
E_{33i} =& N_{3i} + N_{3i}^T
\end{aligned}$$

Proof: Take the derivative of the Lyapunov-Krasovskii functional (4.9) along the trajectories of (4.4) with fixed matrices A_i, B_i, to obtain

$$
\begin{aligned}
\dot{V}_i =& 2x^T(t)P_i\dot{x}(t) + x^T(t)Q_ix(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))Q_ie^{-\alpha h}x(t-\tau(t)) \\
& - \alpha\int_{t-\tau(t)}^{t} x^T(s)Q_ie^{\alpha(s-t)}x(s)ds + h\begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix}^T Z_i \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} \\
& - \int_{t-h}^{t} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix}^T Z_ie^{-\alpha h} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix} ds \\
& - \alpha\int_{-h}^{0}\int_{t+\theta}^{t} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix}^T Z_ie^{\alpha(s-t)} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix} dsd\theta \qquad (4.13) \\
\leq & -\alpha V_i + 2x^T(t)P_i\dot{x}(t) + x^T(t)Q_ix(t) + \alpha x^T(t)P_ix(t) \\
& - (1-d)x^T(t-\tau(t))Q_ie^{-\alpha h}x(t-\tau(t)) \\
& + h\begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix}^T Z_i \begin{bmatrix} x(t) \\ x(t-\tau(t)) \end{bmatrix} \\
& - \int_{t-h}^{t} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix}^T Z_ie^{-\alpha h} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix} ds
\end{aligned}
$$

for any $t \in [t_k, t_{k+1})$. From the Leibniz-Newton formula and for appropriately dimensioned matrices N_{li}, $l = 1, 2, 3$, one can write

$$
\begin{aligned}
2\left[x^T(t)N_{1i} + x^T(t-\tau(t))N_{2i} + \dot{x}^T(t)N_{3i}\right] \times& \\
[\dot{x}(t) - A_ix(t) - B_ix(t-\tau(t))] = 0&
\end{aligned} \qquad (4.14)
$$

Moreover, from (4.6) and (4.8), and for appropriately dimensioned matrices T_{li}, $l = 1, 2, 3$, the following is obtained

$$
\begin{aligned}
& 2\left[x^T(t)T_{1i} + x^T(t-\tau(t))T_{2i} + \dot{x}^T(t)T_{3i}\right] \times \\
& \quad \left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} A_{q(s)}x(s)\,ds - \int_{t-\tau(t)}^{t} B_{q(s)}x(s-\tau(s))\,ds\right] = 0
\end{aligned} \qquad (4.15)
$$

where $q(s)$ is an appropriate piecewise constant function, as mentioned earlier.

Define

$$\eta(t) = \left[x^T(t) \;\; x^T(t-\tau(t)) \;\; \dot{x}^T(t)\right]^T$$

and add (4.14), (4.15) to the right-hand side of the inequality (4.13) to arrive at

$$\begin{aligned} \dot{V}_i + \alpha V_i \leq\; & \eta^T(t) E_i \eta(t) \\ & - \int_{t-\tau(t)}^{t} \left(2\eta^T(t) T_i [A_{q(s)} B_{q(s)}] \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix}^T Z_i e^{-\alpha h} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix} \right) ds \end{aligned} \tag{4.16}$$

where E_i is defined in (4.12). Note that using the Leibniz-Newton formula, it yields that for any $\Gamma_i = \Gamma_i^T > 0$

$$h\eta^T(t)\Gamma_i\eta(t) - \int_{t-h}^{t} \eta^T(t)\Gamma_i\eta(t)ds \geq 0 \tag{4.17}$$

Now, adding (4.17) to the right-hand side of the inequality (4.16) yields

$$\dot{V}_i + \alpha V_i \leq \eta^T(t)E_i\eta(t) + h\eta^T(t)\Gamma_i\eta(t) - \int_{t-\tau(t)}^{t} \zeta^T(t,s)\Psi_i(q(s))\zeta(t,s)ds \tag{4.18}$$

where

$$\zeta(t,s) = [\eta^T(t) \;\; x^T(s) \;\; x^T(s-\tau(s))]^T$$

$$\Psi_i(q(s)) = \begin{bmatrix} \Gamma_i & T_i[A_{q(s)} B_{q(s)}] \\ * & Z_i e^{-\alpha h} \end{bmatrix}$$

If $E_i + h\Gamma_i < 0$ and $\Psi_i(q(s)) \geq 0$, then $\dot{V}_i + \alpha V_i < -\epsilon ||x(t)||^2$ for a sufficiently small ϵ, which ensures that $V_i(x_t) \leq e^{\alpha(t_k - t)} V_i(x_{t_k^+}))$, $t \in [t_k, t_{k+1})$. These conditions are, in fact, the ones provided in (4.10). ■

The following lemma presents conditions to guarantee the exponential stability of the switched time-delay system with fixed matrices A_i, B_i, $i \in \bar{\mathbf{m}}$.

Lemma 4.2. *Assume the conditions of Lemma 4.1 hold for all subsystems. Then, the switched system (4.4) is exponentially stable for any switching signal with the dwell time T_d satisfying*

$$T_d \geq \frac{\ln \mu}{\alpha} \tag{4.19}$$

where $\mu \geq 1$ meets the following relations

$$P_i \leq \mu P_j, \quad Q_i \leq \mu Q_j, \quad Z_i \leq \mu Z_j, \ \forall i, j \in \bar{\mathbf{m}} \tag{4.20}$$

Proof: Consider the Lyapunov-Krasovskii functional (4.9). Then, using Lemma 4.1, the switching signal $\sigma(t)$ meets the following condition

$$V_{\sigma(t)}(x_t) \leq e^{-\alpha(t-t_k)} V_{\sigma(t_k)}(x_{t_k^+}) \tag{4.21}$$

for any $t \in [t_k, t_{k+1})$. Note that from Lyapunov-Krasovskii functional (4.9) and the relations (4.20) at any switching instant $\hat{t}$ one can obtain

$$V_{\sigma(\hat{t})} \leq \mu V_{\sigma(\hat{t}^-)}(x_{\hat{t}^-})$$

The above inequality yields that

$$V(x_t) \leq \mu e^{-\alpha(t-t_k)} V_{\sigma(\hat{t}^-)}(x_{\hat{t}^-}) \leq ... \leq \mu^k e^{-\alpha(t-t_0)} V_{\sigma(t_0)}(x_{t_0})$$

at any time t. Now, if $k \leq \frac{t-t_0}{T_d} \leq \frac{(t-t_0)\alpha}{\ln \mu}$, where k is the number of switchings in the interval $[t_0, \ t)$, one arrives at

$$V(x_t) \leq e^{-(\alpha - \frac{\ln \mu}{T_d})(t-t_0)} V_{\sigma(t_0)}(x_{t_0}) \tag{4.22}$$

Considering (4.9), define positive numbers $k_1,\ k_2$ as given below

$$\begin{aligned} k_1 &= \min_{i \in \bar{\mathbf{m}}} \lambda_{\min}(P_i) \\ k_2 &= \max_{i \in \bar{\mathbf{m}}} \lambda_{\max}(P_i) + h \max_{i \in \bar{\mathbf{m}}} \lambda_{\max}(Q_i) + 2h^2 \max_{i \in \bar{\mathbf{m}}} \lambda_{\max}(Z_i) \end{aligned} \tag{4.23}$$

Then, from (4.9) and (4.23) it is concluded that

$$k_1 ||x(t)||^2 \leq V(x_t), \quad V_{\sigma(t_0)} \leq k_2 ||x_{t_0}||^2_{c_1} \tag{4.24}$$

The relations (4.22) and (4.24) lead to

$$||x(t)||^2 \leq \frac{1}{k_1} V(x_t) \leq \frac{k_2}{k_1} e^{-(\alpha - \frac{\ln \mu}{T_d})(t-t_0)} ||x_{t_0}||^2_{c_1} \tag{4.25}$$

which ensures the exponential stability of the switched system (4.4) with fixed matrices A_i and B_i. ∎

Remark 4.2. *It can be shown from (4.25) that if the conditions of Lemma 4.2 are satisfied, then the state trajectories of the system (4.4) meet, in particular, the following relation*

$$||x(t)|| \leq \sqrt{\frac{k_2}{k_1}} e^{-\rho(t-t_0)} ||x_{t_0}||_{c_1} \tag{4.26}$$

where k_1 and k_2 are positive constants defined in (4.23) and $\rho = \frac{1}{2}\left(\alpha - \frac{\ln \mu}{T_d}\right)$.

To proceed further, define the following positive-definite Lyapunov-Krasovskii functional

$$\begin{aligned} V_i &= \sum_{v_i=1}^{p_i} x^T(t) \xi_i^{(v_i)} \tilde{P}_i^{(v_i)} x(t) + \sum_{v_i=1}^{p_i} \int_{t-\tau(t)}^{t} x^T(s) \xi_i^{(v_i)} \tilde{Q}_i^{(v_i)} e^{\alpha(s-t)} x(s) ds \\ &+ \int_{-h}^{0} \int_{t+\theta}^{t} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix}^T \tilde{Z}_i e^{\alpha(s-t)} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix} ds d\theta \end{aligned} \tag{4.27}$$

for the i-th subsystem, where $\tilde{P}_i^{(v_i)}$, $\tilde{Z}_i$ are symmetric positive-definite matrices, and $\tilde{Q}_i^{(v_i)}$ is a symmetric positive-semidefinite matrix, for any $i \in \bar{\mathbf{m}}$ and $v_i \in \bar{\mathbf{p}}_i$.

The following lemma is the key to the subsequent analysis concerning the sufficient conditions for the exponential decaying of Lyapunov-Krasovskii functional (4.27) in time interval $[t_k, t_{k+1})$ for the uncertain switched time-delay system with polytopic uncertainties.

Lemma 4.3. *Consider the switched time-delay system (4.4) with polytopic uncertainties (4.3). Let $\sigma(t)$ be equal to i for $t \in [t_k, t_{k+1})$, $i \in \bar{\mathbf{m}}$. Then, the Lyapunov-Krasovskii functional (4.27) is exponentially decaying in each time interval $[t_k, t_{k+1})$ (i.e. $V_i(x_t) \le e^{\alpha(t_k - t)} V_i(x_{t_k^+})$, $\forall t \in [t_k, t_{k+1})$) if the following matrix inequalities are satisfied*

$$\tilde{E}_i^{(v_i)} + h\tilde{\Gamma}_i < 0 \tag{4.28a}$$

$$\begin{bmatrix} \tilde{\Gamma}_i & \tilde{T}_i \begin{bmatrix} A_j^{(v_j)} & B_j^{(v_j)} \end{bmatrix} \\ * & \tilde{Z}_i e^{-\alpha h} \end{bmatrix} \ge 0 \tag{4.28b}$$

for all $v_i \in \bar{\mathbf{p}}_i$, $j \in \bar{\mathbf{m}}$, and $v_j \in \bar{\mathbf{p}}_j$, where

$$\tilde{T}_i = \begin{bmatrix} \tilde{T}_{1i} \\ \vdots \\ \tilde{T}_{3i} \end{bmatrix}, \quad \tilde{N}_i = \begin{bmatrix} \tilde{N}_{1i} \\ \vdots \\ \tilde{N}_{3i} \end{bmatrix}$$

$$\tilde{\Gamma}_i = \begin{bmatrix} \tilde{\Gamma}_{11i} & \tilde{\Gamma}_{12i} & \tilde{\Gamma}_{13i} \\ * & \tilde{\Gamma}_{22i} & \tilde{\Gamma}_{23i} \\ * & * & \tilde{\Gamma}_{33i} \end{bmatrix} \ge 0, \quad \tilde{Z}_i = \begin{bmatrix} \tilde{Z}_{11i} & \tilde{Z}_{12i} \\ * & \tilde{Z}_{22i} \end{bmatrix} \ge 0 \tag{4.29}$$

$$\tilde{E}_i^{(v_i)} = \begin{bmatrix} \tilde{E}_{11i}^{(v_i)} & \tilde{E}_{12i}^{(v_i)} & \tilde{E}_{13i}^{(v_i)} \\ * & \tilde{E}_{22i}^{(v_i)} & \tilde{E}_{23i}^{(v_i)} \\ * & * & \tilde{E}_{33i}^{(v_i)} \end{bmatrix} \tag{4.30}$$

$$
\begin{aligned}
\tilde{E}_{11i}^{(v_i)} =& \tilde{Q}_i^{(v_i)} + \alpha \tilde{P}_i^{(v_i)} - \tilde{N}_{1i} A_i^{(v_i)} - \left(A_i^{(v_i)}\right)^T N_{1i}^T + h\tilde{Z}_{11i} + \tilde{T}_{1i} + \tilde{T}_{1i}^T \\
\tilde{E}_{12i}^{(v_i)} =& - \tilde{N}_{1i} B_i^{(v_i)} - \left(A_i^{(v_i)}\right)^T \tilde{N}_{2i}^T + h\tilde{Z}_{12i} - \tilde{T}_{1i} + \tilde{T}_{2i}^T \\
\tilde{E}_{13i}^{(v_i)} =& \tilde{P}_i^{(v_i)} + \tilde{N}_{1i} - \left(A_i^{(v_i)}\right)^T \tilde{N}_{3i}^T + \tilde{T}_{3i}^T \\
\tilde{E}_{22i}^{(v_i)} =& - (1-d)\tilde{Q}_i^{(v_i)} e^{-\alpha h} - \tilde{N}_{2i} B_i^{(v_i)} - \left(B_i^{(v_i)}\right)^T \tilde{N}_{2i}^T + h\tilde{Z}_{22i} - \tilde{T}_{2i} - \tilde{T}_{2i}^T \\
\tilde{E}_{23i}^{(v_i)} =& \tilde{N}_{2i} - \left(B_i^{(v_i)}\right)^T \tilde{N}_{3i}^T - \tilde{T}_{3i}^T \\
\tilde{E}_{33i}^{(v_i)} =& \tilde{N}_{3i} + \tilde{N}_{3i}^T
\end{aligned}
$$

Proof: Similar to Lemma 4.1, the derivative of the Lyapunov-Krasovskii functional (4.27) along the trajectories of (4.4) in the presence of uncertainties (4.3) is obtained as follows

$$
\begin{aligned}
\dot{V}_i + \alpha V_i \leq & 2 \sum_{v_i=1}^{p_i} x^T(t) \xi_i^{(v_i)} \tilde{P}_i^{(v_i)} \dot{x}(t) + \alpha \sum_{v_i=1}^{p_i} x^T(t) \xi_i^{(v_i)} \tilde{P}_i^{(v_i)} x(t) \\
& + \sum_{v_i=1}^{p_i} x^T(t) \xi_i^{(v_i)} \tilde{Q}_i^{(v_i)} x(t) \\
& - \sum_{v_i=1}^{p_i} (1 - \dot{\tau}(t)) x^T(t-\tau(t)) \xi_i^{(v_i)} \tilde{Q}_i^{(v_i)} e^{-\alpha h} x(t-\tau(t)) \\
& + h \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix}^T \tilde{Z}_i \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix} \\
& - \int_{t-h}^{t} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix}^T \tilde{Z}_i e^{-\alpha h} \begin{bmatrix} x(s) \\ x(s-\tau(s)) \end{bmatrix} ds
\end{aligned}
\tag{4.31}
$$

for any $t \in [t_k, t_{k+1})$. From the Leibniz-Newton formula and by using appropriately dimensioned matrices $\tilde{N}_{li}$, $l = 1, 2, 3$, one can write

$$
\begin{aligned}
& 2 \left[x^T(t) \tilde{N}_{1i} + x^T(t-\tau(t)) \tilde{N}_{2i} + \dot{x}^T(t) \tilde{N}_{3i} \right] \times \\
& \left[\dot{x}(t) - \sum_{v_i=1}^{p_i} \xi_i^{(v_i)} A_i^{(v_i)} x(t) - \sum_{v_i=1}^{p_i} \xi_i^{(v_i)} B_i^{(v_i)} x(t-\tau(t)) \right] = 0
\end{aligned}
\tag{4.32}
$$

Moreover, according to (4.8) and for appropriately dimensioned matrices $\tilde{T}_{li}$, $l = 1, 2, 3$, the following equality holds

$$
\begin{aligned}
&2\left[x^T(t)\tilde{T}_{1i} + x^T(t-\tau(t))\tilde{T}_{2i} + \dot{x}^T(t)\tilde{T}_{3i}\right] \times \\
&\left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \sum_{u_{q(s)}=1}^{p_{q(s)}} \xi_{q(s)}^{(u_{q(s)})} A_{q(s)}^{(u_{q(s)})} x(s)\, ds \right. \\
&\left. \quad - \int_{t-\tau(t)}^{t} \sum_{u_{q(s)}=1}^{p_{q(s)}} \xi_{q(s)}^{(u_{q(s)})} B_{q(s)}^{(u_{q(s)})} x(s-\tau(s))\, ds \right] = 0
\end{aligned} \tag{4.33}
$$

where $q(s)$ is a piecewise constant function representing the active subsystem at $t = s$. On the other hand

$$
h\eta^T(t)\tilde{\Gamma}_i\eta(t) - \int_{t-h}^{t} \eta^T(t)\tilde{\Gamma}_i\eta(t)ds \geq 0 \tag{4.34}
$$

Adding (4.32), (4.33) and (4.34) to the right-hand side of (4.31) yields

$$
\begin{aligned}
\dot{V}_i + \alpha V_i \leq & \sum_{v_i=1}^{p_i} \eta^T(t)\xi_i^{(v_i)}\tilde{E}_i^{(v_i)}\eta(t) + h\eta^T(t)\tilde{\Gamma}_i\eta(t) \\
& - \int_{t-\tau(t)}^{t} \zeta^T(t,s) \sum_{u_{q(s)}=1}^{p_{q(s)}} \xi_{q(s)}^{(u_{q(s)})} \tilde{\Psi}_{i,q(s)}^{(u_{q(s)})} ds\zeta(t,s)
\end{aligned} \tag{4.35}
$$

where $\tilde{E}_i^{(v_i)}$ is defined in (4.30) and

$$
\tilde{\Psi}_{i,q(s)}^{(u_{q(s)})} = \begin{bmatrix} \tilde{\Gamma}_i & \tilde{T}_i \left[A_{q(s)}^{(u_{q(s)})} \quad B_{q(s)}^{(u_{q(s)})}\right] \\ * & \tilde{Z}_i e^{-\alpha h} \end{bmatrix}
$$

Now, similar to the statements of Lemma 4.1, if $\sum_{v_i=1}^{p_i} \xi_i^{(v_i)}\tilde{E}_i^{(v_i)} + h\tilde{\Gamma}_i < 0$, and $\tilde{\Psi}_{i,q(s)}^{(u_{q(s)})} \geq 0$, then $\dot{V}_i + \alpha V_i < -\epsilon||x(t)||^2$ for a sufficiently small ϵ ($i \in \bar{\mathbf{m}}$). These conditions are provided in (4.28), and guarantee that $V_i(x_t) \leq e^{\alpha(t_k - t)}V_i(x_{t_k^+}))$, $\forall t \in [t_k, t_{k+1})$. ∎

Theorem 4.1. *Assume the conditions of Lemma 4.3 hold. Then, the switched system (4.4) with the uncertainties (4.3) is exponentially stable for any switching signal with the dwell time T_d satisfying*

$$T_d \geq \frac{\ln \mu}{\alpha}$$

where $\mu \geq 1$ fulfills the following conditions

$$\max_{v_i \in \bar{\mathbf{p}}_i} \tilde{P}_i^{v_i} \leq \mu \min_{v_j \in \bar{\mathbf{p}}_j} \tilde{P}_j^{v_j}, \quad \max_{v_i \in \bar{\mathbf{p}}_i} \tilde{Q}_i^{v_i} \leq \mu \min_{v_j \in \bar{\mathbf{p}}_j} \tilde{Q}_j^{v_j}$$

$$\tilde{Z}_i \leq \mu \tilde{Z}_j, \ \forall i, j \in \bar{\mathbf{m}}$$

Proof: The proof is similar to that of Lemma 4.2, and is omitted here due to space restrictions. ■

Remark 4.3. *It is to be noted in (4.7) and the proof of Lemma 4.3, that the after-effect phenomenon (associated with delay) is taken into consideration in the stability analysis of the underlying switched system. This is carried out by using (4.27) (after ignoring the last term in its right side) and (4.33). The following corollary is a special case of Lemma 4.3, where the after-effect is not considered in the subsystems.*

Corollary 4.1. *The Switched time-delay system (4.4) with polytopic uncertainties (4.3) is exponentially stable with the average dwell time $T_d = \frac{\ln \mu}{\alpha}$, $\alpha > 0$, $\mu \geq 1$, if the following matrix inequalities are satisfied*

$$\tilde{N}_i = \begin{bmatrix} \tilde{N}_{1i}^T & \tilde{N}_{2i}^T & \tilde{N}_{3i}^T \end{bmatrix}^T$$

for all $v_i \in \bar{\mathbf{p}}_i$, $i \in \bar{\mathbf{m}}$, where

$$\tilde{E}_i^{(v_i)} = \begin{bmatrix} \tilde{E}_{11i}^{(v_i)} & \tilde{E}_{12i}^{(v_i)} & \tilde{E}_{13i}^{(v_i)} \\ * & \tilde{E}_{22i}^{(v_i)} & \tilde{E}_{23i}^{(v_i)} \\ * & * & \tilde{E}_{33i}^{(v_i)} \end{bmatrix} < 0 \tag{4.36}$$

$$\begin{aligned}
\tilde{E}_{11i}^{(v_i)} =&\tilde{Q}_i^{(v_i)} + \alpha \tilde{P}_i^{(v_i)} - \tilde{N}_{1i} A_i^{(v_i)} - \left(A_i^{(v_i)}\right)^T N_{1i}^T \\
\tilde{E}_{12i}^{(v_i)} =&- \tilde{N}_{1i} B_i^{(v_i)} - \left(A_i^{(v_i)}\right)^T \tilde{N}_{2i}^T \\
\tilde{E}_{13i}^{(v_i)} =&\tilde{P}_i^{(v_i)} + \tilde{N}_{1i} - \left(A_i^{(v_i)}\right)^T \tilde{N}_{3i}^T \\
\tilde{E}_{22i}^{(v_i)} =&- (1-d)\tilde{Q}_i^{(v_i)} e^{-\alpha h} - \tilde{N}_{2i} B_i^{(v_i)} - \left(B_i^{(v_i)}\right)^T \tilde{N}_{2i}^T \\
\tilde{E}_{23i}^{(v_i)} =&\tilde{N}_{2i} - \left(B_i^{(v_i)}\right)^T \tilde{N}_{3i}^T \\
\tilde{E}_{33i}^{(v_i)} =&\tilde{N}_{3i} + \tilde{N}_{3i}^T
\end{aligned}$$

$$\max_{v_i \in \bar{\mathbf{p}}_i} \tilde{P}_i^{v_i} \leq \mu \min_{v_j \in \bar{\mathbf{p}}_j} \tilde{P}_j^{v_j}, \quad \max_{v_i \in \bar{\mathbf{p}}_i} \tilde{Q}_i^{v_i} \leq \mu \min_{v_j \in \bar{\mathbf{p}}_j} \tilde{Q}_j^{v_j} \ \forall i, j \in \bar{\mathbf{m}}$$

Proof: Choosing the Lyapunov-Krasovskii functional

$$V_i = \sum_{v_i=1}^{p_i} x^T(t) \xi_i^{(v_i)} \tilde{P}_i^{(v_i)} x(t) + \sum_{v_i=1}^{p_i} \int_{t-\tau(t)}^{t} x^T(s) \xi_i^{(v_i)} \tilde{Q}_i^{(v_i)} e^{\alpha(s-t)} x(s) ds$$

the proof is similar to the proof of Lemmas 4.2 and 4.3, and is omitted here. ■

Remark 4.4. *It is to be noted that, the results presented in Corollary 4.1 can be extended for the stability of switched time-delay systems with arbitrary switching signal $\sigma(t)$. In other words, if $\bar{P}_i = \bar{P}_j$, $\bar{Q}_i = \bar{Q}_j$, $\forall i, j \in \bar{\mathbf{m}}$ (i.e., $\mu = 1$) and $\alpha = 0$ the stability conditions stated in Corollary 4.1 can be rewritten for the asymptotic stability of switched time-delay system (4.4) with polytopic uncertainties and in the presence of arbitrary switching signal as follows:*

Corollary 4.2. *The Switched time-delay system (4.4) with polytopic uncertainties (4.3) is asymptotically stable with arbitrary switching signal $\sigma(t)$ if the following matrix inequality is satisfied*

$$\tilde{N}_i = \begin{bmatrix} \tilde{N}_{1i}^T & \tilde{N}_{2i}^T & \tilde{N}_{3i}^T \end{bmatrix}^T$$

for all $v_i \in \bar{\mathbf{p}}_i$, $i \in \bar{\mathbf{m}}$, *where*

$$\tilde{E}_i^{(v_i)} = \begin{bmatrix} \tilde{E}_{11i}^{(v_i)} & \tilde{E}_{12i}^{(v_i)} & \tilde{E}_{13i}^{(v_i)} \\ * & \tilde{E}_{22i}^{(v_i)} & \tilde{E}_{23i}^{(v_i)} \\ * & * & \tilde{E}_{33i}^{(v_i)} \end{bmatrix} < 0 \tag{4.37}$$

$$\begin{aligned}
\tilde{E}_{11i}^{(v_i)} =& \tilde{Q}_i^{(v_i)} - \tilde{N}_{1i} A_i^{(v_i)} - \left(A_i^{(v_i)}\right)^T N_{1i}^T \\
\tilde{E}_{12i}^{(v_i)} =& - \tilde{N}_{1i} B_i^{(v_i)} - \left(A_i^{(v_i)}\right)^T \tilde{N}_{2i}^T \\
\tilde{E}_{13i}^{(v_i)} =& \tilde{P}_i^{(v_i)} + \tilde{N}_{1i} - \left(A_i^{(v_i)}\right)^T \tilde{N}_{3i}^T \\
\tilde{E}_{22i}^{(v_i)} =& - (1-d)\tilde{Q}_i^{(v_i)} - \tilde{N}_{2i} B_i^{(v_i)} - \left(B_i^{(v_i)}\right)^T \tilde{N}_{2i}^T \\
\tilde{E}_{23i}^{(v_i)} =& \tilde{N}_{2i} - \left(B_i^{(v_i)}\right)^T \tilde{N}_{3i}^T \\
\tilde{E}_{33i}^{(v_i)} =& \tilde{N}_{3i} + \tilde{N}_{3i}^T
\end{aligned}$$

Proof: The proof is similar to the proof of Lemma 4.3 and is omitted here due to space restrictions. ■

4.3.2 Stability of Uncertain Switched Neutral Time-Delay Systems

Consider the switched time-delay system (4.1). Assume in this case that the delay in the right side of the equation (4.1) is also fixed, and is equal to the delay in the left side of the equation, i.e. $\tau(t) = \bar{\tau}$, for any $t \geq 0$. Hence, the initial condition in this case is expressed by $x(t) = \phi(t), \quad t \in [-\bar{\tau}, 0\,]$

Similar to the previous case, the system matrices A_i, B_i, $i \in \bar{\mathbf{m}}$ are assumed to be subject to polytopic uncertainty defined in (4.3). It is desired to find conditions under which the switched neutral time-delay system (4.1) is stable. To this end, let the case of fixed matrices A_i, B_i, $i \in \bar{\mathbf{m}}$, be considered first.

Define the following positive-definite Lyapunov-Krasovskii functional

$$
\begin{aligned}
V_i =& [x(t) - Dx(t-\bar{\tau})]^T \bar{P}_i [x(t) - Dx(t-\bar{\tau})] \\
&+ \int_{t-\bar{\tau}}^{t} \begin{bmatrix} x(s) \\ x(s-\bar{\tau}) \end{bmatrix}^T \bar{Q}_i e^{\alpha(s-t)} \begin{bmatrix} x(s) \\ x(s-\bar{\tau}) \end{bmatrix} ds \\
&+ \int_{-\bar{\tau}}^{0} \int_{t+\theta}^{t} \begin{bmatrix} x(s) \\ x(s-\bar{\tau}) \end{bmatrix}^T \bar{Z}_i e^{\alpha(s-t)} \begin{bmatrix} x(s) \\ x(s-\bar{\tau}) \end{bmatrix} ds d\theta
\end{aligned} \tag{4.38}
$$

for the i-th subsystem, where $\bar{P}_i$, $\bar{Z}_i$ are symmetric positive-definite matrices, and $\bar{Q}_i$ is a symmetric positive-semidefinite matrix, $\forall i \in \bar{\mathbf{m}}$.

Lemma 4.4. *Consider the switched time-delay system (4.1) with polytopic uncertainties (4.3). Let $\sigma(t)$ be equal to i, for any $t \in [t_k, t_{k+1})$, $i \in \bar{\mathbf{m}}$. The Lyapunov-Krasovskii functional (4.38) is exponentially decaying in each time interval $[t_k, t_{k+1})$ (i.e. $V_i(x_t) \leq e^{\alpha(t_k - t)} V_i(x_{t_k^+})$, $\forall t \in [t_k, t_{k+1})$) if the following matrix inequalities are satisfied*

$$
\bar{E}_i + h\bar{\Gamma}_i < 0 \tag{4.39a}
$$

$$
\begin{bmatrix} \bar{\Gamma}_i & \bar{T}_i \begin{bmatrix} A_q & B_q \end{bmatrix} \\ * & \bar{Z}_i e^{-\alpha\bar{\tau}} \end{bmatrix} \geq 0 \tag{4.39b}
$$

for all $i, q \in \bar{\mathbf{m}}$, where

$$
\bar{T}_i = \begin{bmatrix} \bar{T}_{1i} \\ \vdots \\ \bar{T}_{4i} \end{bmatrix}, \quad \bar{N}_i = \begin{bmatrix} \bar{N}_{1i} \\ \vdots \\ \bar{N}_{4i} \end{bmatrix}
$$

$$
\bar{E}_i = \bar{\Phi}_i + \bar{T}_i \bar{\Pi}_i + \bar{\Pi}_i^T \bar{T}_i^T + \bar{N}_i \bar{\Lambda}_i + \bar{\Lambda}_i^T \bar{N}_i^T \tag{4.40a}
$$

$$\bar{\Phi}_i = \begin{bmatrix} \bar{\Phi}_{11i} & \bar{\Phi}_{12i} & 0 & \bar{\Phi}_{14i} \\ * & \bar{\Phi}_{22i} & \bar{\Phi}_{23i} & \bar{\Phi}_{24i} \\ * & * & 0 & 0 \\ * & * & * & \bar{\Phi}_{44i} \end{bmatrix}$$

$$\begin{aligned} \bar{\Phi}_{11i} =& \bar{Q}_{11i} + \alpha \bar{P}_i + \bar{\tau} \bar{Z}_{11i} \\ \bar{\Phi}_{12i} =& \bar{\tau} \bar{Z}_{12i} + \bar{Q}_{12i} + \alpha \bar{P}_i D, \quad \bar{\Phi}_{13i} = \bar{P}_i \\ \bar{\Phi}_{22i} =& -\bar{Q}_{11i} e^{-\alpha\bar{\tau}} + \bar{\tau} \bar{Z}_{22i} + \bar{Q}_{22i} + \alpha D^T \bar{P}_i D \\ \bar{\Phi}_{23i} =& D^T \bar{P}_i, \quad \bar{\Phi}_{24i} = -e^{-\alpha\bar{\tau}} \bar{Q}_{12i} \\ \bar{\Phi}_{44i} =& -e^{-\alpha\bar{\tau}} \bar{Q}_{22i} \end{aligned} \tag{4.40b}$$

$$\begin{aligned} \bar{\Pi} &= \begin{bmatrix} I & -(D+I) & 0 & D \end{bmatrix} \\ \bar{\Lambda}_i &= \begin{bmatrix} -A_i & -B_i & I & 0 \end{bmatrix} \end{aligned} \tag{4.40c}$$

$$\bar{\Gamma}_i = \begin{bmatrix} \bar{\Gamma}_{11i} & \bar{\Gamma}_{12i} & \bar{\Gamma}_{13i} & \bar{\Gamma}_{14i} \\ * & \bar{\Gamma}_{22i} & \bar{\Gamma}_{23i} & \bar{\Gamma}_{24i} \\ * & * & \bar{\Gamma}_{33i} & \bar{\Gamma}_{34i} \\ * & * & * & \bar{\Gamma}_{44i} \end{bmatrix} \geq 0 \tag{4.41a}$$

$$\bar{Z}_i = \begin{bmatrix} \bar{Z}_{11i} & \bar{Z}_{12i} \\ * & \bar{Z}_{22i} \end{bmatrix} \geq 0 \tag{4.41b}$$

Proof: Take the derivative of the Lyapunov-Krasovskii functional (4.38) along the trajectories of (4.1) with fixed matrices A_i, B_i to obtain

$$
\begin{aligned}
\dot{V}_i + \alpha V_i =& 2[x(t) - Dx(t-\bar{\tau})]^T \bar{P}_i \mathcal{E}(t,\bar{\tau}) \\
&+ \alpha[x(t) - Dx(t-\bar{\tau})]^T \bar{P}_i [x(t) - Dx(t-\bar{\tau})] \\
&+ \begin{bmatrix} x(t) \\ x(t-\bar{\tau}) \end{bmatrix}^T \bar{Q}_i \begin{bmatrix} x(t) \\ x(t-\bar{\tau}) \end{bmatrix} - \begin{bmatrix} x(t-\bar{\tau}) \\ x(t-2\bar{\tau}) \end{bmatrix}^T \bar{Q}_i e^{-\alpha\bar{\tau}} \begin{bmatrix} x(t-\bar{\tau}) \\ x(t-2\bar{\tau}) \end{bmatrix} \\
&+ \bar{\tau} \begin{bmatrix} x(t) \\ x(t-\bar{\tau}) \end{bmatrix}^T \bar{Z}_i \begin{bmatrix} x(t) \\ x(t-\bar{\tau}) \end{bmatrix} - \int_{t-\bar{\tau}}^{t} \begin{bmatrix} x(s) \\ x(s-\bar{\tau}) \end{bmatrix}^T \bar{Z}_i e^{-\alpha\bar{\tau}} \begin{bmatrix} x(s) \\ x(s-\bar{\tau}) \end{bmatrix} ds
\end{aligned} \tag{4.42}
$$

for any $t \in [t_k, t_{k+1})$, where $\mathcal{E}(t,\bar{\tau}) = \dot{x}(t) - D\dot{x}(t-\bar{\tau})$. Define now

$$
\bar{\eta}(t) = \begin{bmatrix} x^T(t) & x^T(t-\bar{\tau}) & x^T(t-2\bar{\tau}) & \mathcal{E}^T(t,\bar{\tau}) \end{bmatrix}^T
$$

From the Leibniz-Newton formula, and by using appropriately dimensioned matrices $\bar{N}_i$, $\bar{T}_i$, one can write

$$
2\bar{\eta}^T(t)\bar{N}_i \times \left[\mathcal{E}^T(t,\bar{\tau}) - A_i x(t) - B_i x(t-\bar{\tau})\right] = 0 \tag{4.43}
$$

$$
\begin{aligned}
2\bar{\eta}^T(t)\bar{T}_i \times [&x(t) - (D+I)x(t-\bar{\tau}) + Dx(t-2\bar{\tau}) \\
&- \int_{t-\bar{\tau}}^{t} A_{q(s)} x(s)\, ds - \int_{t-\bar{\tau}}^{t} B_{q(s)} x(s-\bar{\tau})\, ds \Big] = 0
\end{aligned} \tag{4.44}
$$

where $q(s)$ is a piecewise constant function representing the active subsystem's index at $t = s$. Add (4.43), (4.44) to the right-hand side of (4.42) to obtain

$$
\begin{aligned}
&\dot{V}_i + \alpha V_i \leq \bar{\eta}^T(t)\bar{E}_i\bar{\eta}(t) \\
&- \int_{t-\bar{\tau}}^{t} \left(2\bar{\eta}^T(t)\bar{T}_i[A_{q(s)} B_{q(s)}] \begin{bmatrix} x(s) \\ x(s-\bar{\tau}) \end{bmatrix} + \begin{bmatrix} x(s) \\ x(s-\bar{\tau}) \end{bmatrix}^T \bar{Z}_i e^{-\alpha\bar{\tau}} \begin{bmatrix} x(s) \\ x(s-\bar{\tau}) \end{bmatrix} \right) ds
\end{aligned} \tag{4.45}
$$

where $\bar{E}_i$ is defined in (4.40a). Note that using the Leibniz-Newton formula and (4.45) yields

$$\dot{V}_i + \alpha V_i \leq \bar{\eta}^T(t)\bar{E}_i\bar{\eta}(t) + \bar{\tau}\bar{\eta}^T(t)\bar{\Gamma}_i\bar{\eta}(t) - \int_{t-\bar{\tau}}^{t} \bar{\zeta}^T(t,s)\bar{\Psi}_i(q(s))\bar{\zeta}(t,s)ds \tag{4.46}$$

where

$$\bar{\zeta}(t,s) = [\bar{\eta}^T(t) \;\; x^T(s) \;\; x^T(s-\bar{\tau})]^T$$

$$\bar{\Psi}_i(q(s)) = \begin{bmatrix} \bar{\Gamma}_i & \bar{T}_i[\bar{A}_{q(s)}\bar{B}_{q(s)}] \\ * & \bar{Z}_i e^{-\alpha\bar{\tau}} \end{bmatrix}$$

If $\bar{E}_i + \bar{\tau}\bar{\Gamma}_i < 0$ and $\bar{\Psi}_i(q(s)) \geq 0$, then $\dot{V}_i + \alpha V_i < -\epsilon||x(t)||^2$ for a sufficiently small ϵ, which ensures that $V_i(x_t) \leq e^{\alpha(t_k - t)}V_i(x_{t_k^+}))$, $\forall t \in [t_k, t_{k+1})$. This completes the proof. ∎

In Lemma 4.4, the exponential decaying of Lyapunov-Krasovskii functional (4.38) for any switching time interval $[t_k, t_{k+1})$ (with the switching instant t_k) and for each subsystem of (4.1) is studied under the assumption that A_i, B_i, $i \in \bar{\mathbf{m}}$, are fixed matrices. The following lemma presents the exponential decaying conditions of Lyapunov-Krasovskii functional (4.38) for any switching time interval $[t_k, t_{k+1})$ and for each subsystem of (4.1) for the case where the matrices A_i, B_i, $i \in \bar{\mathbf{m}}$, are subject to polytopic uncertainties.

Lemma 4.5. *Consider the switched time-delay system (4.1) with polytopic uncertainties (4.3). Let $\sigma(t)$ be equal to i, for any $t \in [t_k, t_{k+1})$, $i \in \bar{\mathbf{m}}$. The Lyapunov-Krasovskii functional (4.38) is exponentially decaying in each time interval $[t_k, t_{k+1})$ (i.e. $V_i(x_t) \leq e^{\alpha(t_k - t)}V_i(x_{t_k^+})$, $\forall t \in [t_k, t_{k+1})$) if the following matrix inequalities are satisfied*

$$\tilde{\bar{E}}_i^{(v_i)} + h\tilde{\bar{\Gamma}}_i < 0 \tag{4.47a}$$

$$\begin{bmatrix} \tilde{\bar{\Gamma}}_i & \tilde{\bar{T}}_i \begin{bmatrix} A_j^{(v_j)} & B_j^{(v_j)} \end{bmatrix} \\ * & \tilde{\bar{Z}}_i e^{-\alpha\bar{\tau}} \end{bmatrix} \geq 0 \tag{4.47b}$$

where

$$\tilde{\bar{T}}_i = \begin{bmatrix} \tilde{\bar{T}}_{1i} \\ \vdots \\ \tilde{\bar{T}}_{4i} \end{bmatrix}, \tilde{\bar{N}}_i = \begin{bmatrix} \tilde{\bar{N}}_{1i} \\ \vdots \\ \tilde{\bar{N}}_{4i} \end{bmatrix}$$

$$\tilde{\bar{E}}_i^{(v_i)} = \tilde{\bar{\Phi}}_i^{(v_i)} + \tilde{\bar{T}}_i \tilde{\bar{\Pi}}_i + \tilde{\bar{\Pi}}_i^T \tilde{\bar{T}}_i^T + \tilde{\bar{N}}_i \tilde{\bar{\Lambda}}_i^{(v_i)} + \left(\tilde{\bar{\Lambda}}_i^{(v_i)}\right)^T \tilde{\bar{N}}_i^T \tag{4.48a}$$

$$\tilde{\bar{\Phi}}_i^{(v_i)} = \begin{bmatrix} \tilde{\bar{\Phi}}_{11i}^{(v_i)} & \tilde{\bar{\Phi}}_{12i}^{(v_i)} & 0 & \tilde{\bar{\Phi}}_{14i}^{(v_i)} \\ * & \tilde{\bar{\Phi}}_{22i}^{(v_i)} & \tilde{\bar{\Phi}}_{23i}^{(v_i)} & \tilde{\bar{\Phi}}_{24i}^{(v_i)} \\ * & * & 0 & 0 \\ * & * & * & \tilde{\bar{\Phi}}_{44i}^{(v_i)} \end{bmatrix}$$

$$\begin{aligned} \tilde{\bar{\Phi}}_{11i}^{(v_i)} =& \tilde{\bar{Q}}_{11i}^{(v_i)} + \alpha \tilde{\bar{P}}_i^{(v_i)} + \bar{\tau} \tilde{\bar{Z}}_{11i} \\ \tilde{\bar{\Phi}}_{12i}^{(v_i)} =& \bar{\tau} \tilde{\bar{Z}}_{12i} + \tilde{\bar{Q}}_{12i}^{(v_i)} + \alpha \tilde{\bar{P}}_i^{(v_i)} D, \quad \tilde{\bar{\Phi}}_{13i}^{(v_i)} = \tilde{\bar{P}}_i^{(v_i)} \\ \tilde{\bar{\Phi}}_{22i}^{(v_i)} =& - \tilde{\bar{Q}}_{11i}^{(v_i)} e^{-\alpha\bar{\tau}} + \bar{\tau} \tilde{\bar{Z}}_{22i} + \tilde{\bar{Q}}_{22i}^{(v_i)} + \alpha D^T \tilde{\bar{P}}_i^{(v_i)} D \\ \tilde{\bar{\Phi}}_{23i}^{(v_i)} =& D^T \tilde{\bar{P}}_i^{(v_i)}, \quad \tilde{\bar{\Phi}}_{24i}^{(v_i)} = -e^{-\alpha\bar{\tau}} \tilde{\bar{Q}}_{12i}^{(v_i)} \\ \tilde{\bar{\Phi}}_{44i}^{(v_i)} =& - e^{-\alpha\bar{\tau}} \tilde{\bar{Q}}_{22i}^{(v_i)} \end{aligned} \tag{4.48b}$$

$$\begin{aligned} \tilde{\bar{\Pi}} &= \begin{bmatrix} I & -(D+I) & 0 & D \end{bmatrix} \\ \tilde{\bar{\Lambda}}_i^{(v_i)} &= \begin{bmatrix} -A_i^{(v_i)} & -B_i^{(v_i)} & I & 0 \end{bmatrix} \end{aligned} \tag{4.48c}$$

$$\tilde{\bar{\Gamma}}_i = \begin{bmatrix} \tilde{\bar{\Gamma}}_{11i} & \tilde{\bar{\Gamma}}_{12i} & \tilde{\bar{\Gamma}}_{13i} & \tilde{\bar{\Gamma}}_{14i} \\ * & \tilde{\bar{\Gamma}}_{22i} & \tilde{\bar{\Gamma}}_{23i} & \tilde{\bar{\Gamma}}_{24i} \\ * & * & \tilde{\bar{\Gamma}}_{33i} & \tilde{\bar{\Gamma}}_{34i} \\ & * & * & \tilde{\bar{\Gamma}}_{44i} \end{bmatrix} \geq 0 \tag{4.49a}$$

$$\tilde{\bar{Z}}_i = \begin{bmatrix} \tilde{\bar{Z}}_{11i} & \tilde{\bar{Z}}_{12i} \\ * & \tilde{\bar{Z}}_{22i} \end{bmatrix} \geq 0 \tag{4.49b}$$

for all $v_i \in \bar{\mathbf{p}}_i$, *and* $i, j \in \bar{\mathbf{m}}$.

The following theorem completes the main contributions of this chapter by presenting sufficient conditions for the stability of the uncertain switched neutral time-delay system described by (4.1).

Theorem 4.2. *Assume the conditions of Lemma 4.5 hold. Then, the switched system (4.1) with uncertainties (4.3) is exponentially stable for any switching signal with the dwell time* T_d *satisfying*

$$T_d \geq \frac{\ln \mu}{\alpha}$$

where $\mu \geq 1$ *meets the following inequalities*

$$\max_{v_i \in \bar{\mathbf{p}}_i} \tilde{\bar{P}}_i^{v_i} \leq \mu \min_{v_j \in \bar{\mathbf{p}}_j} \tilde{\bar{P}}_j^{v_j}, \quad \max_{v_i \in \bar{\mathbf{p}}_i} \tilde{\bar{Q}}_i^{v_i} \leq \mu \min_{v_j \in \bar{\mathbf{p}}_j} \tilde{\bar{Q}}_j^{v_j},$$

$$Z_i \leq \mu Z_j, \ \forall i, j \in \bar{\mathbf{m}}$$

Proof: The proof is similar to the proof of Lemma 4.2, and is omitted here due to space restrictions. ■

The following Corollary provides conditions under which the switched neutral time-delay system (4.1) with polytopic uncertainties (4.3) be asymptotically stable with arbitrary switching signal $\sigma(t)$.

Corollary 4.3. *The Switched neutral time-delay system (4.1) with polytopic uncertainties (4.3) and arbitrary switching signal* $\sigma(t)$ *is asymptotically stable if the following matrix inequality is satisfied*

$$\tilde{\bar{E}}_i^{(v_i)} = \tilde{\bar{\Phi}}_i^{(v_i)} + \tilde{\bar{N}}_i \tilde{\bar{\Lambda}}_i^{(v_i)} + \left(\tilde{\bar{\Lambda}}_i^{(v_i)}\right)^T \tilde{\bar{N}}_i^T < 0$$

where

$$\tilde{\tilde{N}}_i = \begin{bmatrix} \tilde{\tilde{N}}_{1i}^T & \cdots & \tilde{\tilde{N}}_{4i}^T \end{bmatrix}^T$$

$$\tilde{\tilde{\Phi}}_i^{(v_i)} = \begin{bmatrix} \tilde{\tilde{\Phi}}_{11i}^{(v_i)} & \tilde{\tilde{\Phi}}_{12i}^{(v_i)} & 0 & \tilde{\tilde{\Phi}}_{14i}^{(v_i)} \\ * & \tilde{\tilde{\Phi}}_{22i}^{(v_i)} & \tilde{\tilde{\Phi}}_{23i}^{(v_i)} & \tilde{\tilde{\Phi}}_{24i}^{(v_i)} \\ * & * & 0 & 0 \\ * & * & * & \tilde{\tilde{\Phi}}_{44i}^{(v_i)} \end{bmatrix}$$

$$\tilde{\tilde{\Phi}}_{11i}^{(v_i)} = \tilde{\tilde{Q}}_{11i}^{(v_i)} + \bar{\tau}\tilde{\tilde{Z}}_{11i}$$

$$\tilde{\tilde{\Phi}}_{12i}^{(v_i)} = \bar{\tau}\tilde{\tilde{Z}}_{12i} + \tilde{\tilde{Q}}_{12i}^{(v_i)}, \quad \tilde{\tilde{\Phi}}_{13i}^{(v_i)} = \tilde{\tilde{P}}_i^{(v_i)}$$

$$\tilde{\tilde{\Phi}}_{22i}^{(v_i)} = -\tilde{\tilde{Q}}_{11i}^{(v_i)} + \bar{\tau}\tilde{\tilde{Z}}_{22i} + \tilde{\tilde{Q}}_{22i}^{(v_i)}$$

$$\tilde{\tilde{\Phi}}_{23i}^{(v_i)} = D^T\tilde{\tilde{P}}_i^{(v_i)}, \quad \tilde{\tilde{\Phi}}_{24i}^{(v_i)} = -\tilde{\tilde{Q}}_{12i}^{(v_i)}$$

$$\tilde{\tilde{\Phi}}_{44i}^{(v_i)} = -\tilde{\tilde{Q}}_{22i}^{(v_i)}$$

$$\bar{\Lambda}_i^{(v_i)} = \begin{bmatrix} -A_i^{(v_i)} & -B_i^{(v_i)} & I & 0 \end{bmatrix}$$

for all $v_i \in \bar{\mathbf{p}}_i$*,* $i \in \bar{\mathbf{m}}$*.*

4.4 Numerical Examples

In this section, three examples are presented to show the effectiveness of the results obtained in this chapter.

Example 4.1. *Consider system (4.4) with polytopic uncertainties of the form (4.3) described by the following matrices*

$$A_1^{(1)} = \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.4 \end{bmatrix}, A_1^{(2)} = \begin{bmatrix} -0.8 & 0 \\ 1 & -0.1 \end{bmatrix}$$

$$B_1^{(1)} = B_1^{(2)} = \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix}$$

$$A_2^{(1)} = \begin{bmatrix} -1 & -0.5 \\ -0.5 & -1 \end{bmatrix}, A_2^{(2)} = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}, A_2^{(3)} = \begin{bmatrix} 0.1 & 0 \\ 1 & 0.1 \end{bmatrix}$$

$$B_2^{(1)} = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, B_2^{(2)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B_2^{(3)} = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}$$

These matrices represent a switched system consisting of two uncertain subsystems. Solving the LMIs in Lemma 4.3, it is concluded that each subsystem is exponentially stable for $h = 0.3$. *Using Theorem 4.1, one can deduce that the switched system given above is exponentially stable for* $\mu = 4.3$, $\alpha = 0.15$ *and* $T_d = 40.4s$.

Example 4.2. *A mechanical rotational cutting process can be modeled as [52], [80]*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t - \tau(t)) + Cu(t) \\ x(t) &= \phi(t), \quad t \in [-h, 0\,] \end{aligned} \tag{4.50}$$

where the nominal system matrices are given by

$$A = \begin{bmatrix} -2 & 0.2 & 0.35 \\ -0.5 & 0.15 & -0.3 \\ 1 & -0.2 & -0.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.5 & 0.1 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix} \tag{4.51a}$$

$$C = \begin{bmatrix} 0.5 & -0.65 & 0.5 \\ -0.3 & 0.5 & -0.25 \end{bmatrix}^T, \quad \tau(t) = 0.2\sin(0.1t) + 0.2 \tag{4.51b}$$

It is assumed that the system actuators are subject to poor performance or failure, but at least one actuator remains operational at any time. The poor performance occurs when an actuator's output level changes due to mechanical problems. It is also supposed that the system matrix A *has polytopic uncertainties on its diagonal elements. With the above assumptions, the operation of the system can be modeled by two subsystems: the first subsystem is represented by the nominal matrices (4.51),* *and the second subsystem is subject to polytopic uncertainties. Let the uncertain subsystems be represented by the following matrices*

$$C_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ -0.3 & 0.5 & -0.25 \end{bmatrix}^T, \quad C_2^{(1)} = \begin{bmatrix} 0.5 & -0.65 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}^T$$

$$A_2 = A + \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix}, \quad |\delta_1| < 0.5, \quad |\delta_2| < 0.10, \quad |\delta_3| < 0.15 \tag{4.52}$$

Define

$$A_2^{(2)} = \begin{bmatrix} -1.5 & 0.2 & 0.35 \\ -0.5 & 0.25 & -0.3 \\ 1 & -0.2 & -0.1 \end{bmatrix}, \quad A_2^{(1)} = \begin{bmatrix} -2.5 & 0.2 & 0.35 \\ -0.5 & 0.05 & -0.3 \\ 1 & -0.2 & -0.4 \end{bmatrix}$$

Using the switched system formulation here, $\sigma(t)$ is 1 when there is no actuator failure in the system, and is 2 while the actuators are not functioning properly. Let the system failure be formulated in the framework of polytopic-type uncertainties as $C_2 = \xi_2^{(1)} C_2^{(1)} + \xi_2^{(2)} C_2^{(2)}$, $A_2 = \xi_2^{(1)} A_2^{(1)} + \xi_2^{(2)} A_2^{(2)}$, $\xi_2^{(1)} + \xi_2^{(2)} = 1$, $\xi_2^{(1)}, \xi_2^{(2)} > 0$. Let also $u(t) = Kx(t)$, where

$$K = \begin{bmatrix} -0.4 & 1 & -1 \\ -0.3 & -1 & 0.5 \end{bmatrix}$$

Now, using the results of Lemma 4.3 it can be shown that each subsystem of (4.50) is exponentially stable with $h = 0.4$. It is straightforward to verify that $\alpha = 0.05$ and $\mu = 12.74$ satisfy the conditions of Theorem 4.1. Consequently, T_d can be as small as $\ln \frac{\mu}{\alpha} = 50.90s$.

Let the initial condition of the system be $\phi(\theta) = [-2\sin(\theta) \;\; -\sin(3\theta) \;\; 2\sin(\theta)]^T$, $\forall \theta \in [-h, 0]$, and assume that the time interval between consecutive switching instants is 60s, which is greater than the dwell time condition stated above. Fig. 4.1 *depicts the state trajectories of the system, and shows that they all converge to zero; hence, the system is stable. This demonstrates the effectiveness of the results proposed in this work.*

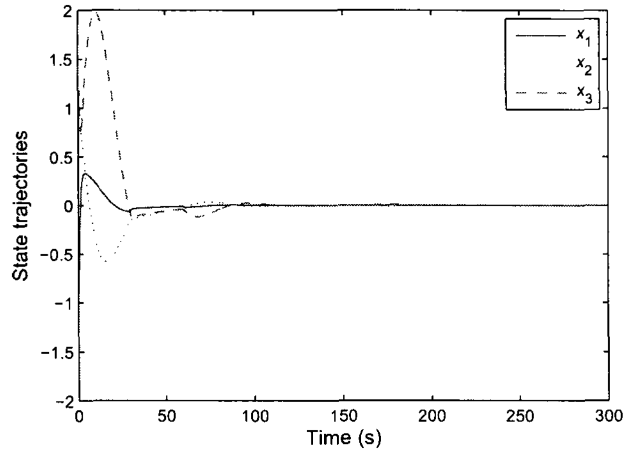


Figure 4.1: State trajectories of the perturbed switched time-delay system in Example 4.2 with sufficiently small perturbation.

Note that the LMIs presented in this work are infeasible for small perturbations of the system in Example 4.2 (beyond the bounds given in (4.52)). For example, by choosing $A_2 = A + \mathrm{diag}(\delta_1,\ \delta_2,\ \delta_3)$, *where* $|\delta_2| < 0.15$ *and* $\delta_1,\ \delta_3$ *are equal to their upper bounds in (4.52), Lemma 4.3 and Theorem 4.1 cannot guarantee the exponential stability of the system. In other words, the LMIs presented in Lemma 4.3 and Theorem 4.1 for the stability of the switched time-delay system become infeasible. State trajectories for the perturbations beyond the bounds given in (4.52) and the switching time 15s (which is less than the dwell time obtained earlier) are demonstrated in Fig. 4.2. These trajectories confirm that the underlying switched time-delay system can be unstable if the sufficient conditions provided for the stability of the system in this chapter are violated.*

Example 4.3. *Consider the switched neutral time-delay system (4.1), where* $\tau(t) = \bar{\tau}$ *and with polytopic uncertainties given by*

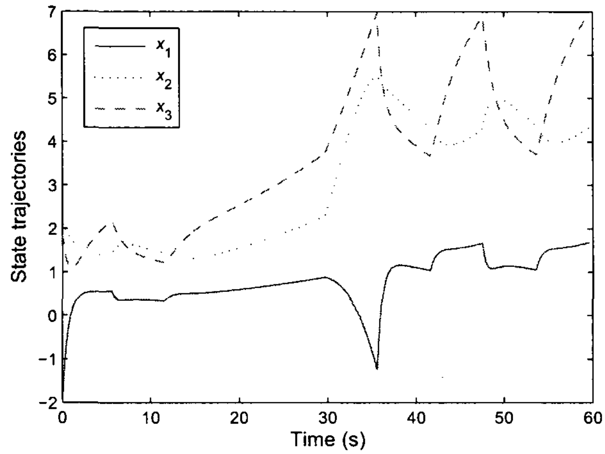


Figure 4.2: State trajectories of the perturbed switched time-delay system in Example 4.2 with large perturbation.

$$D = \begin{bmatrix} 0.2 & 0 \\ -0.1 & -0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A_2^{(1)} = \begin{bmatrix} -1.5 & 0 \\ 2 & -1.5 \end{bmatrix}, \quad A_2^{(2)} = \begin{bmatrix} -3 & 3 \\ 0 & -5 \end{bmatrix}$$

$$B_2^{(1)} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_2^{(2)} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

The above switched neutral time-delay system consists of two subsystems, where one subsystem is fixed, and the other one is subject to polytopic uncertainties. Using Lemma 4.5, it can be deduced that the uncertain subsystem is exponentially stable for $\bar{\tau} = 5.5$. It can also be inferred from Theorem 4.2 that the switched system is exponentially stable with $\mu = 1.99 \times 10^3$, $\alpha = 0.1$ and $T_d = 75.97s$. It is worth mentioning that for $\bar{\tau} = 0.5$ and $\alpha = 0.4$, the value of T_d would be 17.36s. In other words, to obtain faster exponential response (greater α) delay should be smaller.

Chapter 5

An Adaptive Regulation Method for a Class of Uncertain Time-Delay Systems

5.1 Introduction

Stability analysis for time-delay systems has attracted many researchers in recent years due to its applications in a wide range of real-world control systems [68]. Such applications include, for example, process control systems, communication networks and power systems, to name only a few [8], [106], [40].

Most of the published work on the stability of uncertain time-delay systems assume that upper bounds on the system uncertainties are available; e.g., see [106], [25], [48], [101], [87], [37], [96] and references therein. In [65], [13] a memoryless feedback is employed to obtain a control strategy that does not depend on the delay, but uses the upper bounds on the uncertainties to stabilize the time-delay system. In control applications, however, bounds on the system uncertainties may not be known *a priori*. Therefore, it is more desirable to develop an approach to stabilize time-delay systems without the requirement of uncertainties bounds.

Time-delay systems can be classified in two different categories: the ones expressed by retarded functional differential equations and the ones described by neutral functional differential equations. In retarded functional differential equations only state delay is considered, whereas neutral functional differential equations involve also derivatives of the state with delays (see [23], [40] for more details).

Various adaptive control schemes are presented in the literature to stabilize uncertain retarded time-delay systems; e.g. see [58], [94], [93], [107], [66]. In [58], a pre-routed switching approach is taken to stabilize a class of uncertain continuous-time systems with time delay, where the delay is assumed to be known. However, the corresponding approach needs the upper bounds on the uncertainties. In [94], [93], [107], [66], different adaptive control approach for the stabilization of uncertain retarded time-delay systems are presented, where the upper bounds on the uncertainties are assumed to be unknown. The above-mentioned adaptive robust controllers are employed to estimate the upper bounds of the uncertainties and subsequently stabilize the uncertain time-delay system. Various techniques are proposed in prior literature to design adaptive robust controllers to stabilize the uncertain retarded time-delay systems. However, there are very few results on the stabilization of neutral time-delay systems (due to their complexity) when no upper bounds on the uncertainties are available. Adaptive robust control schemes are proposed in [79] to stabilize uncertain neutral time-delay systems. However, the drawback of the work [79] is that it has chattering phenomena close the origin.

In this chapter, a novel adaptive state feedback control design technique is introduced to asymptotically stabilize an uncertain neutral time-delay system. The upper bounds of the uncertainties are assumed to be unknown. First, using some adaptation laws the upper bounds on the uncertainties are estimated. Then, using the updated parameters, a state feedback adaptive controller is introduced to robustly stabilize the uncertain neutral time-delay system. It is shown that the state of the resultant closed-loop system is asymptotically stable.

The remainder of this chapter is organized as follows. The problem formulation and some essential assumptions are given in Section 5.2. An adaptive robust control design technique is introduced in Section 5.3 as the main contribution of this work. Simulations are presented in Section 5.4 to demonstrate the efficacy of the proposed method.

5.2 Problem Formulation

Consider an uncertain neutral time-delay system described by the following differential equation

$$\dot{x}(t) - D\dot{x}(t-h) = (A_0 + \Delta A_0(t))x(t) + \Delta A_1(t)x(t-h) + Bu(t) \tag{5.1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector and $u(t) \in \mathbf{R}^m$ is the input vector. Moreover, $A_0 \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $D \in \mathbf{R}^{n\times n}$ are known constant matrices, and $\Delta A_0, \Delta A_1$ represent the system uncertainties and are assumed to be continuous with respect to their parameter. Furthermore, h is the time-delay which is positive, and is assumed to be constant. The initial condition for the system (5.1) is given by

$$x(t) = \phi(t), \qquad t \in [t_0 - h,\ t_0] \tag{5.2}$$

where $\phi(t)$ is a continuous function on the interval $[t_0 - h,\ t_0]$. It is to be noted that the model given in (5.1) is widely used in the literature (e.g., see [94], [93]).

To proceed further, the following assumptions are made.

Assumption 5.1. *All admissible uncertainties can be expressed as follows*

$$\Delta A_0(t) = BE_{A_0}(t) \tag{5.3a}$$

$$\Delta A_1(t) = BE_{A_1}(t) \tag{5.3b}$$

where $E_{A_0}(\cdot)$, $E_{A_1}(\cdot)$ *are unknown matrices with appropriate dimensions and the following unknown bounds*

$$||E_{A_0}(t)|| \leq \rho^*_{A_0} \tag{5.4a}$$

$$||E_{A_1}(t)|| \leq \rho^*_{A_1} \tag{5.4b}$$

In the above inequalities $||\cdot||$ *denotes the spectral norm, and* $\rho^*_{A_0}, \rho^*_{A_1}$ *are unknown positive constants.*

Assumption 5.2. *There exist symmetric positive-definite* $n \times n$ *matrices* P, R, Q, *a positive semi-definite matrix* $S \in \mathbf{R}^{n\times n}$, *and a strictly positive constant* η, *such that the following relations hold*

$$PA_0 + A_0^T P + Q + PA_0 DR^{-1} D^T A_0^T P + S + S^T - \eta PBB^T P < 0 \tag{5.5a}$$

$$2D^T SD - S + R \leq 0 \tag{5.5b}$$

Assumption 5.3. *[37] The martix* D *in (5.1) is such that* $||D|| < 1$.

Remark 5.1. *It is to be noted that by multiplying the left and right sides of the algebraic Riccati inequality (5.5a) by* P^{-1}, *and multiplying the left and right sides of (5.5b) by* S^{-1}, *one can write (5.5a) and (5.5b) in the form of LMIs as follows*

$$\begin{bmatrix} A_0X + XA_0^T + W - \eta BB^T & X & A_0D \\ * & -\frac{1}{2}Z & 0 \\ * & * & -H \end{bmatrix} < 0 \tag{5.6a}$$

$$\begin{bmatrix} -Z & Z & ZD^T \\ * & -H & 0 \\ * & * & -\frac{1}{2}Z \end{bmatrix} \leq 0 \tag{5.6b}$$

where the matrices in (5.5) and (5.6) are related by, $P = X^{-1}$, $Q = PWP$, $S = Z^{-1}$, *and* $R = H^{-1}$.

Throughout the remainder of this work, the symbols $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are used for the maximum and minimum eigenvalues of a matrix, respectively. The following definition will prove convenient in the development of the main results.

Definition 5.1. *The vector* Θ^* *and* Ψ *are defined as follows*

$$\Theta^* := [(\rho^*_{A_0})^2 \quad \rho^*_{A_0}\rho^*_{A_1} \quad (\rho^*_{A_1})^2]^T \tag{5.7a}$$

$$\Psi := [(\lambda^{-1}_{\min}(Q) + \lambda^{-1}_{\min}(R)||D||) \quad 2||D||\lambda^{-1}_{\min}(R) \quad \lambda^{-1}_{\min}(R)]^T \tag{5.7b}$$

where Q, R *are the symmetric positive-definite* $n \times n$ *matrices.*

In order to proceed further, the following well-known fact is borrowed from [90].

Fact 5.1. *[90] For any vectors or matrices* z, y *of appropriate dimensions and any symmetric positive-definite matrix* $M = M^T > 0$, *the following relations are satisfied:*

$$\begin{aligned} -z^Ty - y^Tz &\leq z^TMz + y^TM^{-1}y \\ z^Ty + y^Tz &\leq z^TMz + y^TM^{-1}y \end{aligned} \tag{5.8}$$

It is desired now to design a feedback controller to stabilize the uncertain time-delay system (5.1).

5.3 Adaptive Robust Control Design

The following controller is proposed to stabilize (5.1)

$$u(t) = -\frac{1}{2}\eta B^T P \mathcal{M}_h(t) - \frac{1}{2}\Theta^T(t)\Psi B^T P \mathcal{M}_h(t) \tag{5.9}$$

$$\mathcal{M}_h(t) = x(t) - Dx(t-h) \tag{5.10}$$

where $P \in \mathbf{R}^{n\times n}$ is a symmetric positive-definite matrix, Ψ is the regressor matrix defined in (5.7b), and η is a strictly positive constant. The function $\Theta(t)$ represents the estimate of the unknown parameter vector Θ^* defined in (5.7a), which is updated through the following adaptation law

$$\dot{\Theta}(t) = \Gamma\Psi||\mathcal{M}_h^T(t)PB||^2 \tag{5.11}$$

where Γ is any symmetric positive-definite 3×3 matrix.

Theorem 5.1. *The uncertain time-delay system (5.1) with the control input (5.9) is asymptotically stable provided the conditions of Assumptions 5.1-5.3 are satisfied.*

Proof: Applying the control input (5.9) to (5.1) leads to the following closed-loop system

$$\begin{aligned}\dot{x}(t) - D\dot{x}(t-h) = &(A_0 + \Delta A_0(t))x(t) + \Delta A_1(t)x(t-h)\\ &- \eta BB^T P\mathcal{M}_h(t) - \Theta^T(t)\Psi BB^T P\mathcal{M}_h(t)\end{aligned} \tag{5.12}$$

Choose the following positive-definite Lyapunov-Krasovskii functional candidate

$$\begin{aligned}V(x(t),\tilde{\Theta}(t)) =&[x(t) - Dx(t-h)]^T P[x(t) - Dx(t-h)]\\ &+ \int_{t-h}^{t} x^T(s)Sx(s)ds + \frac{1}{2}\tilde{\Theta}^T(t)\Gamma^{-1}\tilde{\Theta}(t)\end{aligned} \tag{5.13}$$

In (5.13), the matrices P, Γ are symmetric positive-definite and the matrix S is a positive semi-definite matrix. Define the parameter estimation error vector as

$$\tilde{\Theta}(t) := \Theta(t) - \Theta^* \tag{5.14}$$

(note that Θ^* is defined in (5.7a), and that $\Theta(t)$ is its estimate).

For $t \geq t_0$, the derivative of the Lyapunov-Krasovskii functional introduced in (5.13) is obtained as follows

$$\begin{aligned}\dot{V}(x(t),\tilde{\Theta}(t)) =& 2\left[x(t) - Dx(t-h)\right]^T P\left[\dot{x}(t) - D\dot{x}(t-h)\right] \\ &+ x^T(t)Sx(t) - x^T(t-h)Sx(t-h) + \tilde{\Theta}^T(t)\Gamma^{-1}\dot{\Theta}(t)\end{aligned} \tag{5.15}$$

Substituting (5.12) in (5.15) and using (5.3) (Assumption 5.1) yield that

$$\begin{aligned}\dot{V}(x(t),\tilde{\Theta}(t)) =& \mathcal{M}_h^T(t)(PA_0 + A_0^T P)\mathcal{M}_h(t) + 2\mathcal{M}_h^T(t)P\Delta A_0\mathcal{M}_h(t) \\ &+ 2\mathcal{M}_h^T(t)P\left[(A_0 + \Delta A_0)D + \Delta A_1\right]x(t-h) \\ &- (\eta + \Theta^T(t)\Psi)\mathcal{M}_h^T(t)PBB^T P\mathcal{M}_h(t) \\ &+ x^T(t)Sx(t) - x^T(t-h)Sx(t-h) + \tilde{\Theta}^T(t)\Gamma^{-1}\dot{\Theta}(t)\end{aligned} \tag{5.16}$$

Note from Fact 5.1 that, there exist symmetric positive-definite $n \times n$ matrices R, Q such that (5.16) can be rewritten as follows

$$
\begin{aligned}
&\dot{V}(x(t), \tilde{\Theta}(t)) \\
\leq& \mathcal{M}_h^T(t)(PA_0 + A_0^T P)\mathcal{M}_h(t) + 2\mathcal{M}_h^T(t)PBE_0(t)\mathcal{M}_h(t) \\
&+ \mathcal{M}_h^T(t)\left[P(A_0 + \Delta A_0)D + P\Delta A_1\right] R^{-1} \left[P(A_0 + \Delta A_0)D + P\Delta A_1\right]^T \mathcal{M}_h(t) \\
&+ x^T(t-h)Rx(t-h) - (\eta + \Theta^T(t)\Psi)\mathcal{M}_h^T(t)PBB^T P\mathcal{M}_h(t) \\
&+ 2\mathcal{M}_h^T(t)SDx(t-h) - 2\mathcal{M}_h^T(t)SDx(t-h) \\
&+ x^T(t)Sx(t) - x^T(t-h)Sx(t-h) + \tilde{\Theta}^T(t)\Gamma^{-1}\dot{\Theta}(t) \\
\leq& \mathcal{M}_h^T(t)(PA_0 + A_0^T P)\mathcal{M}_h(t) + 2\mathcal{M}_h^T(t)PBE_0(t)\mathcal{M}_h(t) \\
&+ \mathcal{M}_h^T(t)\left[P(A_0 + BE_0(t))D + PBE_1(t)\right] R^{-1} \\
&\qquad\qquad\qquad \times \left[P(A_0 + BE_0(t))D + PBE_1(t)\right]^T \mathcal{M}_h(t) \\
&+ \mathcal{M}_h^T(t)S\mathcal{M}_h(t) + x^T(t-h)Rx(t-h) - (\eta + \Theta^T(t)\Psi)\mathcal{M}_h^T(t)PBB^T P\mathcal{M}_h(t) \\
&+ x^T(t-h)D^T SDx(t-h) - 2x^T SDx(t-h) + 2x^T(t-h)D^T SDx(t-h) \\
&+ x^T(t)Sx(t) - x^T(t-h)Sx(t-h) + \tilde{\Theta}^T(t)\Gamma^{-1}\dot{\Theta}(t) \\
=& \mathcal{M}_h^T(t)(PA_0 + A_0^T P)\mathcal{M}_h(t) + 2\mathcal{M}_h^T(t)PBE_0(t)\mathcal{M}_h(t) \\
&+ \mathcal{M}_h^T(t)\left[P(A_0 + BE_0(t))D + PBE_1(t)\right] R^{-1} \\
&\qquad\qquad\qquad \times \left[P(A_0 + BE_0(t))D + PBE_1(t)\right]^T \mathcal{M}_h(t) \\
&+ \mathcal{M}_h^T(t)(S + S^T)\mathcal{M}_h(t) - (\eta + \Theta^T(t)\Psi)\mathcal{M}_h^T(t)PBB^T P\mathcal{M}_h(t) \\
&+ x^T(t-h)Rx(t-h) + 2x^T(t-h)D^T SDx(t-h) \\
&- x^T(t-h)Sx(t-h) + \tilde{\Theta}^T(t)\Gamma^{-1}\dot{\Theta}(t)
\end{aligned} \tag{5.17}
$$

Now, using (5.4) and Fact 5.1, there exists a positive-definite matrix Q such that the derivative of the Lyapunov-Krasovskii functional (5.15) in (5.17) becomes

$$
\begin{aligned}
&\dot{V}(x(t), \tilde{\Theta}(t)) \\
&\leq \mathcal{M}_h^T(t)(PA_0 + A_0^T P)\mathcal{M}_h(t) + \mathcal{M}_h^T(t)Q\mathcal{M}_h(t) \\
&\quad + (\rho_{A_0}^*)^2 \lambda_{\min}^{-1}(Q)\mathcal{M}_h^T(t)PBB^T P\mathcal{M}_h(t) + \mathcal{M}_h^T(t)(PA_0 D)R^{-1}D^T A_0^T P\mathcal{M}_h(t) \\
&\quad + \left[(\rho_{A_0}^*)^2 ||D||^2 + 2||D||\rho_{A_0}^*\rho_{A_1}^* + (\rho_{A_1}^*)^2\right] \lambda_{\min}^{-1}(R)\mathcal{M}_h^T(t)PBB^T P\mathcal{M}_h(t) \\
&\quad + \mathcal{M}_h^T(t)(S + S^T)\mathcal{M}_h(t) - (\eta + \Theta^T(t)\Psi)\mathcal{M}_h^T(t)PBB^T P\mathcal{M}_h(t) \\
&\quad + x^T(t-h)Rx(t-h) - x^T(t-h)Sx(t-h) \\
&\quad + 2x^T(t-h)D^T SDx(t-h) + \tilde{\Theta}^T(t)\Gamma^{-1}\dot{\Theta}(t) \\
&= \mathcal{M}_h^T(t)\left(PA_0 + A_0^T P + Q + (PA_0 D)R^{-1}(D^T A_0^T P) + S + S^T - \eta PBB^T P\right)\mathcal{M}_h(t) \\
&\quad + x^T(t-h)\left(2D^T SD - S + R\right)x(t-h) \\
&\quad + ((\Theta^*)^T - \Theta^T(t))\Psi\mathcal{M}_h^T(t)PBB^T P\mathcal{M}_h(t) + \tilde{\Theta}^T(t)\Gamma^{-1}\dot{\Theta}(t)
\end{aligned}
\tag{5.18}
$$

If the conditions of Assumption 5.2 are satisfied, then it can be shown that under the update law (5.11) the relation (5.18) is reduced to

$$
\dot{V}(x(t), \tilde{\Theta}(t)) \leq -\epsilon\mathcal{M}_h^T(t)\mathcal{M}_h(t) \tag{5.19}
$$

Therefore, it can be concluded that

$$
\lim_{t\to\infty} ||x(t) - Dx(t-h)|| = 0 \tag{5.20}
$$

Hence, it results from the inequality $||D|| < 1$ (Assumption 5.3) that $||x(t)||$ approaches zero as t goes to infinity [27], [37]; this implies the asymptotical stability of system (5.1). ■

5.4 Numerical Example

Example 5.1. *Consider the uncertain time-delay system (5.1) with the following parameters*

$$A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad \Delta A_0(t) = \begin{bmatrix} 0 & 0 \\ 1 + 2\sin(t) & 2(1 + 2\sin(t)) \end{bmatrix}$$

$$\Delta A_1(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 - 0.5\sin(t) \end{bmatrix}$$

Using the conditions of Assumption 5.2, the following are obtained

$$P = \begin{bmatrix} 0.0166 & 0.0222 \\ 0.0222 & 0.0382 \end{bmatrix}, \quad \eta = 385.31$$

Let the parameter $\Theta(t_0)$, gain matrix Γ, and initial condition ϕ be given by

$$\Theta(t_0) = [1.5 \ \ 1.5 \ \ 1.5]^T, \quad \Gamma = \text{diag}(100, \ 100, \ 100)$$

$$\phi(t) = [0.5\cos(t) \ \ 0.5\cos(t)]^T, \ t \in [t_0 - h, \ t_0], \ h = 0.5$$

On applying the proposed control law with the above values, Figs. 5.1-3 are obtained for the resultant closed-loop system. The state trajectories are given in Fig. 5.1, which show that using the proposed adaptive control law the system is asymptotically stable. Fig. 5.2, on the other hand, depicts the corresponding adaptive control input. It can be observed from this figure that the control effort approaches zero as the state variables converge to origin. In Fig. 5.3, the 2-norm of the adaptation parameter vector $\Theta(t)$ is depicted which shows the parameter estimates are convergent.

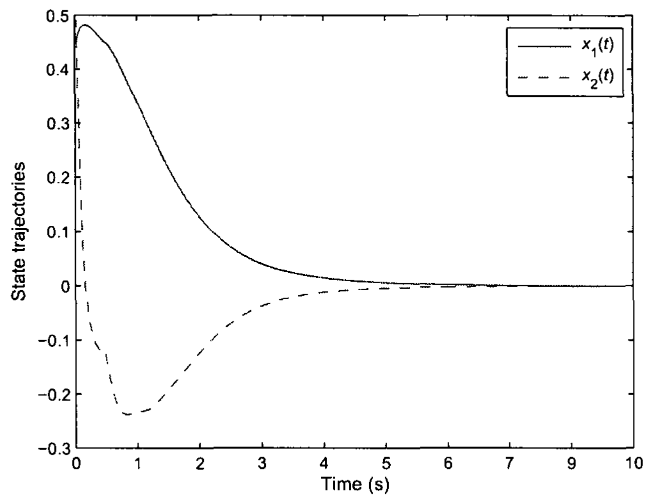


Figure 5.1: The state trajectories in Example 5.1 using the proposed control law.

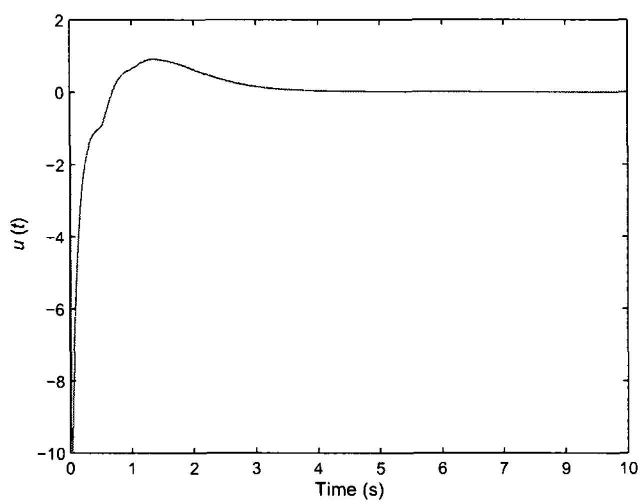


Figure 5.2: The adaptive control input in Example 5.1 obtained by using the proposed method.

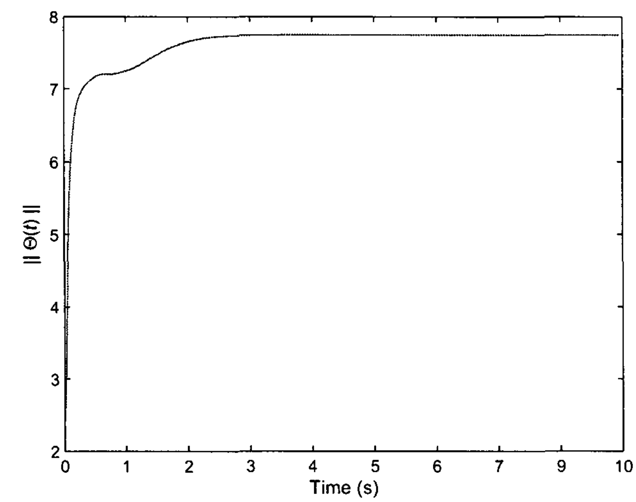


Figure 5.3: The 2-norm of the adaptation parameter vector $\Theta(t)$ obtained in Example 5.1.

Chapter 6

Conclusions

6.1 Summary

The results developed in this thesis can be summarized as follows.

In Chapter 2, robust stability of a class of piecewise affine (PWA) systems with time-varying delay is considered. It is assumed that the system is subject to bounded uncertainty. It is also assumed that the time delay is unknown and time-varying, but upper bounds on the magnitude of delay and its rate of variation exist. Sufficient conditions in the form of linear matrix inequalities (LMI) are derived for the robust stability of the system. Numerical examples are provided to demonstrate the usefulness of the proposed approach.

An adaptive switching control algorithm is developed in Chapter 3 for uncertain time-varying discrete-time systems with time-varying delay, in the presence of disturbance. A set of discrete-time controllers are designed with the property that at least one of them can stabilize the system. An adaptive switching algorithm is then established to find a stabilizing controller through fast model falsification. The proposed switching scheme is convergent and guarantees the stability of the closed-loop system. Simulation results elucidate the effectiveness of the method in stabilizing a highly uncertain time-delay system with a relatively large upper bound on delay.

Chapter 4 is concerned with the stability of switched time-delay systems with polytopic uncertainties and time-varying delay. First, a parameter-dependent Lyapunov functional is proposed. Then, using the Leibniz-Newton formula along with free weighting matrices, the uncertain parameters of the system are incorporated into the derivative of the Lyapunov functional. New sufficient delay-dependent stability criteria in the form of LMIs are subsequently obtained. The efficacy of the proposed method is demonstrated by numerical examples.

In Chapter 5, the problem of robust regulation for the class of neutral time-delay systems with uncertainties is investigated. It is supposed that the uncertainties in the system matrices have unknown bounds. By utilizing the estimate of the unknown parameters, an adaptive robust feedback controller is developed which guarantees the stability of the uncertain time-delay system. Simulations confirm the effectiveness of the approach.

6.2 Suggestions for Future Work

In what follows, some of the possible extensions to the results obtained in this thesis as well as some relevant problems for future study are presented.

- As a natural extension of the results of Chapters 2 and 4, it is always desirable to find less conservative conditions for the stability of uncertain time-delay PWA systems, as well as switched time-delay systems. It would be interesting to consider other Lyapunov-Krasovskii functionals for the stability analysis of switched time-delay systems to find new (and perhaps less conservative) stability conditions.

- Adaptive controller design for highly uncertain continuous systems has been well documented in the literature. However, stabilization of continuous time-delay systems using adaptive control techniques needs further investigation. In this thesis, the problem of adaptive controller design for uncertain discrete time-delay systems is studied. Extension of the results of Chapter 3 to continuous uncertain time-delay systems would be an interesting research topic with important applications.

- The adaptive control scheme proposed in Chapter 5 requires certain information about the time-delay in the feedback loop. However, such information is not typically available *a priori* in many applications. Hence, it would be very useful to design an adaptive controller for this type of system, which does not depend on the delay information.

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