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Occupation times of spectrally negative Lévy processes with applications

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Abstract

In this paper, we compute the Laplace transform of occupation times (of the negative half-line) of spectrally negative Lévy processes. Our results are extensions of known results for standard Brownian motion and jump-diffusion processes. The results are expressed in terms of the so-called scale functions of the spectrally negative Lévy process and its Laplace exponent. Applications to insurance risk models are also presented.

Keywords: Occupation time; Spectrally negative Lévy processes; Fluctuation theory; Scale functions; Ruin theory

1. Introduction and results

Let $X = (X_t)_{t \geq 0}$ be a spectrally negative Lévy process, that is, a Lévy process with no positive jumps. The law of X such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. As the Lévy process X has no positive jumps, its Laplace transform exists, and is given by

$$\mathbb{E}[\mathrm{e}^{\theta X_t}] = \mathrm{e}^{t\psi(\theta)},$$

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for $\theta, t \geq 0$, where

$$\psi(\theta) = \gamma\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{0} (e^{\theta z} - 1 - \theta z \mathbb{I}_{(-1,0)}(z))\Pi(dz),$$

for $\gamma \in \mathbb{R}$ and $\sigma \geq 0$. Also, Π is a σ-finite measure on $(-\infty, 0)$ such that

$$\int_{-\infty}^{0} (1 \wedge z^2)\Pi(dz) < \infty. \tag{1}$$

The measure Π is called the Lévy measure of X, while (γ, σ, Π) is referred to as the Lévy triplet of X. Note that $\mathbb{E}[X_1] = \psi'(0+)$.

For an arbitrary spectrally negative Lévy process, the Laplace exponent ψ is strictly convex, and $\lim_{\theta\to\infty} \psi(\theta) = \infty$. Thus, there exists a function $\Phi: [0, \infty) \to [0, \infty)$ defined by $\Phi(\theta) = \sup\{\xi \geq 0 \mid \psi(\xi) = \theta\}$ (its right-inverse) and such that

$$\psi(\Phi(\theta)) = \theta, \quad \theta \geq 0.$$

We first examine the total occupation time of the negative half-line $(-\infty, 0)$.

Theorem 1. *If $\psi'(0+) > 0$, then, for $\lambda \geq 0$,*

$$\mathbb{E}[e^{-\lambda \int_0^\infty \mathbb{I}_{\{X_s \leq 0\}} ds}] = \psi'(0+)\frac{\Phi(\lambda)}{\lambda}, \tag{2}$$

where $\Phi(\lambda)/\lambda$ is to be understood in the limiting sense when $\lambda = 0$.

We now recall the definition of the q-scale function $W^{(q)}$. For $q \geq 0$, the q-scale function of the process X is defined as the function with Laplace transform on $[0, \infty)$ given by

$$\int_0^\infty e^{-\theta z} W^{(q)}(z)dz = \frac{1}{\psi(\theta) - q}, \quad \text{for } \theta > \Phi(q),$$

and such that $W^{(q)}(x) = 0$ for $x < 0$. This function is unique, continuous, positive, and strictly increasing. We write $W = W^{(0)}$ when $q = 0$. We have that $W^{(q)}$ is differentiable except for at most countably many points; see Lemma 8.2 in [14]. Moreover, $W^{(q)}$ is continuously differentiable if X has paths of unbounded variation or if the tail of the Lévy measure, i.e., the function $x \mapsto \Pi(-\infty, x)$ on $(-\infty, 0)$, is continuous. Further, $W^{(q)}$ is twice continuously differentiable on $(0, \infty)$ if $\sigma > 0$. For more details on the smoothness properties of the q-scale function, see [4]. We will also use the functions $\{\overline{W}^{(q)}; q \geq 0\}$ and $\{Z^{(q)}; q \geq 0\}$, defined by

$$\overline{W}^{(q)}(x) = \int_0^x W^{(q)}(z)dz$$

and

$$Z^{(q)}(x) = 1 + q\overline{W}^{(q)}(x).$$

We can now state the following corollary to Theorem 1.

Corollary 1. *If $\psi'(0+) > 0$, then, for $\lambda > 0$ and $x \geq 0$,*

$$\mathbb{E}_x[e^{-\lambda \int_0^\infty \mathbb{I}_{\{X_s \leq 0\}} ds}] = \psi'(0+)\Phi(\lambda) \int_0^\infty e^{-\Phi(\lambda)z} W(x+z)dz.$$

Note that, when $x = 0$, by the definition of the scale function, we have

$$\int_0^\infty e^{-\Phi(\lambda)z} W(z)dz = \frac{1}{\psi(\Phi(\lambda))} = \frac{1}{\lambda},$$

therefore recovering Theorem 1 as a special case.

These two results generalize the work done in [23], where the sum of a compound Poisson process and a Brownian motion is analyzed (see, e.g., Eq. (4.9) in that paper). Using ruin theory terminology, the authors study the duration of negative surplus, also called the *time in red*, in such an insurance risk model; their work is itself an extension of [8] in the pure compound Poisson case. Related results can also be found in [15,22].

We now examine the occupation time of $(-\infty, 0)$ until a negative level $-b$ is crossed for the first time. Let τ_{-b}^- be the first passage time below $-b$ of X:

$$\tau_{-b}^- = \inf\{t > 0 : X_t < -b\}.$$

Theorem 2. *If $\psi'(0+) \geq 0$, then, for $\lambda \geq 0$,*

$$\mathbb{E}[e^{-\lambda \int_0^{\tau_{-b}^-} \mathbb{I}_{\{X_s \leq 0\}} ds}] = \frac{\psi'(0+) + \frac{\sigma^2}{2} \frac{A_1^{(\lambda)}(b)}{W^{(\lambda)}(b)} + \int_{-\infty}^{0-} A_2^{(\lambda)}(x) \int_0^\infty \Pi(dx - y)dy}{\psi'(0+) + \frac{\sigma^2}{2} \frac{W^{(\lambda)\prime}(b)}{W^{(\lambda)}(b)} + \int_{-\infty}^{0-} A_3^{(\lambda)}(x) \int_0^\infty \Pi(dx - y)dy}, \quad (3)$$

where

$$A_1^{(\lambda)}(b) = Z^{(\lambda)}(b) W^{(\lambda)\prime}(b) - \lambda (W^{(\lambda)}(b))^2,$$

$$A_2^{(\lambda)}(x) = Z^{(\lambda)}(x + b) - Z^{(\lambda)}(b) \frac{W^{(\lambda)}(x + b)}{W^{(\lambda)}(b)}$$

and

$$A_3^{(\lambda)}(x) = 1 - \frac{W^{(\lambda)}(x + b)}{W^{(\lambda)}(b)}.$$

As a special case, we recover the corresponding result for standard Brownian motion; see, e.g., [10,11].

Our proofs use fluctuation identities for spectrally negative Lévy processes and, as a consequence, our results are expressed in terms of the so-called scale functions of the spectrally negative Lévy process (see, e.g., [1,2]) and its Laplace exponent. Only elementary arguments are needed. To the authors' knowledge, the literature seems rather scarce on the relationship between scale functions and certain occupation times of a general spectrally negative Lévy process.

The current work has been partly motivated by the study of an insurance risk model with implementation delays. Insurance risk models use stochastic processes to describe the surplus of an insurance company. In risk models of a Parisian nature, an implementation delay in the recognition of an insurer's capital insufficiency is applied. More precisely, it is assumed that ruin occurs as soon as an excursion below a critical level is longer than a deterministic time; such models have been studied only very recently (see [7,6,17,18]), while the idea has been borrowed from finance and Parisian barrier options (see [5]). Of more interest in our context is the work by the same authors: in [17], instead of a deterministic delay, an exponentially distributed grace period is used in the definition of the Parisian ruin. It turns out that the probability of ruin in this model is strongly related to the occupation time of the underlying process. The reader is invited to consult Section 6.2 to obtain further details on this connection.

The rest of the paper is organized as follows. In the next section, we recall the relevant notions and results on scale functions and fluctuation identities. Then, in Sections 3–5, we prove Theorem 2, Theorem 1, and Corollary 1, respectively. Section 6 presents applications to the case of a Brownian motion with drift and insurance risk models.

2. Scale functions and fluctuation identities

We recall some of the properties of the q-scale function $W^{(q)}$ and its use in fluctuation theory. Let $d = \gamma - \int_{-1}^{0} z\Pi(\mathrm{d}z)$. The initial values of $W^{(q)}$ and $W^{(q)\prime}$ are known to be

$$W^{(q)}(0+) = \begin{cases} 1/d & \text{when } \sigma = 0 \text{ and } \displaystyle\int_{-1}^{0} z\Pi(\mathrm{d}z) < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$W^{(q)\prime}(0+) = \begin{cases} 2/\sigma^2 & \text{when } \sigma > 0, \\ (\Pi(-\infty,0)+q)/d^2 & \text{when } \sigma = 0 \text{ and } \Pi(-\infty,0) < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Now, define

$$\tau_0^- = \inf\{t > 0 \colon X_t < 0\},$$

and, for $a > 0$,

$$\tau_a^+ = \inf\{t > 0 \colon X_t > a\},$$

with the convention $\inf \emptyset = \infty$. It is well known (see, e.g., [14]) that, for $x \le a$,

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_a^+}; \tau_a^+ < \tau_0^-] = \frac{W^{(q)}(x)}{W^{(q)}(a)},$$

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_0^-}; \tau_0^- < \tau_a^+] = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)},$$

and

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_a^+}; \tau_a^+ < \infty] = \mathrm{e}^{-\Phi(q)(a-x)}.$$

If $\psi'(0+) \ge 0$, then $\mathbb{P}_x\{\tau_a^+ < \infty\} = 1$, and therefore

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_a^+}] = \mathrm{e}^{-\Phi(q)(a-x)}.$$

Also, we have that

$$\mathbb{P}_x\{\tau_0^- < \infty, X_{\tau_0^-} \in \mathrm{d}z\} = \frac{\sigma^2}{2}[W'(x) - \Phi(0)W(x)]\delta_0(\mathrm{d}z) + \int_0^\infty \Pi(\mathrm{d}z - y)\{\mathrm{e}^{-\Phi(0)y}W(x) - W(x-y)\}\mathrm{d}y, \tag{4}$$

where δ_0 is the Dirac measure at 0. The first term of this measure corresponds to the case when $X_{\tau_0^-} = 0$, a behaviour called *creeping*.

In this paper, $\int_{-\infty}^{a-}$ and $\int_{(-\infty,a)}$ have the same meaning, while $\int_{-\infty}^{a+}$ and $\int_{(-\infty,a]}$ have the same meaning. Using the distribution in Eq. (4), together with the fact that $\mathbb{P}_x\{\tau_0^- < \infty\} =$

$1 - \psi'(0+)W(x)$ if $\psi'(0+) \geq 0$ (in which case $\Phi(0) = 0$), we obtain

$$1 = \psi'(0+)W(x) + \frac{\sigma^2}{2}W'(x) + \int_{-\infty}^{0-}\int_0^{\infty} \Pi(\mathrm{d}z - y)\{W(x) - W(x-y)\}\mathrm{d}y. \tag{5}$$

For more details on spectrally negative Lévy processes and fluctuation identities, the reader is referred to [14]. Further information, examples, and numerical techniques related to the computation of scale functions can be found in [4,9,13,16,21].

3. Proof of Theorem 2

The main idea of the proof consists in defining a quantity underestimating and overestimating the occupation time

$$\int_0^{\tau_{-b}^-} \mathbb{I}_{\{X_s \leq 0\}}\mathrm{d}s.$$

This respectively leads to an upper and a lower bound to its Laplace transform. Subsequently, by taking an appropriate limit, we show that the two bounds converge to the expression on the right-hand side of (3).

First, we provide a lower bound to this Laplace transform by overestimating the occupation time. To this end, we consider a clock which starts at time 0 and stops when level ϵ is attained or when level $-b$ is crossed. Then, if level ϵ was attained first, every time we go below 0, we restart the clock and subsequently stop it when we get back to ϵ (without going below $-b$); let $L^{\epsilon,b}$ be the Laplace transform of this overestimating quantity of the occupation time when the process X sits at level ϵ at time 0. Hence, by the strong Markov property of X, we have

$$\mathbb{E}[\mathrm{e}^{-\lambda\int_0^{\tau_{-b}^-}\mathbb{I}_{\{X_s\leq 0\}}\mathrm{d}s}] \geq \mathbb{E}[\mathrm{e}^{-\lambda\tau_\epsilon^+}; \tau_\epsilon^+ < \tau_{-b}^-]L^{\epsilon,b} = \frac{W^{(\lambda)}(b)}{W^{(\lambda)}(b+\epsilon)}L^{\epsilon,b}.$$

Using the strong Markov property and the spatial homogeneity of X, we get

$$\begin{aligned} L^{\epsilon,b} &= \mathbb{P}_\epsilon\{\tau_0^- = \infty\} + \int_{-\infty}^{0+} \mathbb{P}_\epsilon\{\tau_0^- < \infty, X_{\tau_0^-} \in \mathrm{d}x\}\{\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_{-b}^-}; \tau_{-b}^- < \tau_\epsilon^+] \\ &\quad + \mathbb{E}_x[\mathrm{e}^{-\lambda\tau_\epsilon^+}; \tau_\epsilon^+ < \tau_{-b}^-]L^{\epsilon,b}\}. \end{aligned}$$

Note that, when $x < -b$,

$$\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_{-b}^-}; \tau_{-b}^- < \tau_\epsilon^+] = 1$$

and

$$\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_\epsilon^+}; \tau_\epsilon^+ < \tau_{-b}^-] = 0.$$

Consequently,

$$\begin{aligned} L^{\epsilon,b} &= \frac{\mathbb{P}_\epsilon\{\tau_0^- = \infty\} + \int_{-\infty}^{0+}\mathbb{P}_\epsilon\{\tau_0^- < \infty, X_{\tau_0^-} \in \mathrm{d}x\}\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_{-b}^-}; \tau_{-b}^- < \tau_\epsilon^+]}{1 - \int_{-\infty}^{0+}\mathbb{P}_\epsilon\{\tau_0^- < \infty, X_{\tau_0^-} \in \mathrm{d}x\}\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_\epsilon^+}; \tau_\epsilon^+ < \tau_{-b}^-]} \\ &= \frac{L_1^{\epsilon,b} + L_2^{\epsilon,b}}{L_3^{\epsilon,b}}, \end{aligned}$$

where, using some of the fluctuation identities in Section 2,

$$L_1^{\epsilon,b} = \psi'(0+)W(\epsilon) + \frac{\sigma^2}{2}W'(\epsilon)\left[Z^{(\lambda)}(b) - Z^{(\lambda)}(b+\epsilon)\frac{W^{(\lambda)}(b)}{W^{(\lambda)}(b+\epsilon)}\right],$$

$$L_2^{\epsilon,b} = \int_{-\infty}^{0-}\left[Z^{(\lambda)}(x+b) - Z^{(\lambda)}(b+\epsilon)\frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b+\epsilon)}\right] \times \int_0^\infty \Pi(\mathrm{d}x-y)\{W(\epsilon)-W(\epsilon-y)\}\mathrm{d}y,$$

and

$$L_3^{\epsilon,b} = 1 - \frac{\sigma^2}{2}W'(\epsilon)\frac{W^{(\lambda)}(b)}{W^{(\lambda)}(b+\epsilon)} - \int_{-\infty}^{0-}\frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b+\epsilon)}\int_0^\infty \Pi(\mathrm{d}x-y)\{W(\epsilon)-W(\epsilon-y)\}\mathrm{d}y.$$

Using Eq. (5), we get

$$L_3^{\epsilon,b} = \psi'(0+)W(\epsilon) + \frac{\sigma^2}{2}\frac{W'(\epsilon)}{W^{(\lambda)}(b+\epsilon)}[W^{(\lambda)}(b+\epsilon) - W^{(\lambda)}(b)] + \int_{-\infty}^{0-}\left[1 - \frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b+\epsilon)}\right]\int_0^\infty \Pi(\mathrm{d}x-y)\{W(\epsilon)-W(\epsilon-y)\}\mathrm{d}y.$$

Alternatively, we now develop a scheme to underestimate the occupation time in question. Every time we go below $-\epsilon$ ($-\epsilon > -b$), we start the clock and stop it when we get back to 0 (without going below $-b$); let $U^{\epsilon,b}$ be the Laplace transform of this underestimating quantity of the occupation time when $X_0 = 0$. Hence, by the strong Markov property of X, we also have

$$\mathbb{E}[\mathrm{e}^{-\lambda\int_0^{\tau_{-b}^-}\mathbb{I}_{\{X_s\le 0\}}\mathrm{d}s}] \le U^{\epsilon,b}.$$

As above, using the strong Markov property and the spatial homogeneity of X, we can write

$$U^{\epsilon,b} = \mathbb{P}_\epsilon\{\tau_0^- = \infty\} + \int_{-\infty}^{0+}\mathbb{P}_\epsilon\{\tau_0^- < \infty, X_{\tau_0^-}\in \mathrm{d}x\}\{\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_{-b+\epsilon}^-};\tau_{-b+\epsilon}^- < \tau_\epsilon^+] + \mathbb{E}_x[\mathrm{e}^{-\lambda\tau_\epsilon^+};\tau_\epsilon^+ < \tau_{-b+\epsilon}^-]U^{\epsilon,b}\},$$

and then

$$U^{\epsilon,b} = \frac{U_1^{\epsilon,b} + U_2^{\epsilon,b}}{U_3^{\epsilon,b}},$$

where

$$U_1^{\epsilon,b} = \psi'(0+)W(\epsilon) + \frac{\sigma^2}{2}W'(\epsilon)\left[Z^{(\lambda)}(b-\epsilon) - Z^{(\lambda)}(b)\frac{W^{(\lambda)}(b-\epsilon)}{W^{(\lambda)}(b)}\right],$$

$$U_2^{\epsilon,b} = \int_{-\infty}^{0-}\left[Z^{(\lambda)}(x+b-\epsilon) - Z^{(\lambda)}(b)\frac{W^{(\lambda)}(x+b-\epsilon)}{W^{(\lambda)}(b)}\right] \times \int_0^{\infty} \Pi(\mathrm{d}x-y)\{W(\epsilon)-W(\epsilon-y)\}\mathrm{d}y,$$

and

$$U_3^{\epsilon,b} = 1 - \frac{\sigma^2}{2}W'(\epsilon)\frac{W^{(\lambda)}(b-\epsilon)}{W^{(\lambda)}(b)} - \int_{-\infty}^{0-}\frac{W^{(\lambda)}(x+b-\epsilon)}{W^{(\lambda)}(b)}\int_0^{\infty}\Pi(\mathrm{d}x-y)\{W(\epsilon)-W(\epsilon-y)\}\mathrm{d}y.$$

As for $L_3^{\epsilon,b}$, using Eq. (5), we get

$$U_3^{\epsilon,b} = \psi'(0+)W(\epsilon) + \frac{\sigma^2}{2}\frac{W'(\epsilon)}{W^{(\lambda)}(b)}[W^{(\lambda)}(b) - W^{(\lambda)}(b-\epsilon)] + \int_{-\infty}^{0-}\left[1-\frac{W^{(\lambda)}(x+b-\epsilon)}{W^{(\lambda)}(b)}\right]\int_0^{\infty}\Pi(\mathrm{d}x-y)\{W(\epsilon)-W(\epsilon-y)\}\mathrm{d}y.$$

3.1. *Proof if* $\sigma > 0$

First, we assume that X has a Brownian component; that is, $\sigma > 0$. In this case, $W(0) = 0$ and $W^{(\lambda)\prime}(0+) < \infty$. Then,

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{L_1^{\epsilon,b}}{\epsilon} &= \lim_{\epsilon\to 0}\psi'(0+)\frac{W(\epsilon)}{\epsilon} + \frac{\sigma^2}{2}\frac{W'(\epsilon)}{W^{(\lambda)}(b+\epsilon)} \\
&\quad\times\left[\frac{Z^{(\lambda)}(b)W^{(\lambda)}(b+\epsilon) - Z^{(\lambda)}(b+\epsilon)W^{(\lambda)}(b)}{\epsilon}\right] \\
&= \psi'(0+)W'(0+) + \frac{\sigma^2}{2}\frac{W'(0+)}{W^{(\lambda)}(b)}[Z^{(\lambda)}(b)W^{(\lambda)\prime}(b) - \lambda(W^{(\lambda)}(b))^2],
\end{aligned}$$

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{L_2^{\epsilon,b}}{\epsilon} &= \lim_{\epsilon\to 0}\int_{-\infty}^{0-}\left[Z^{(\lambda)}(x+b) - Z^{(\lambda)}(b+\epsilon)\frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b+\epsilon)}\right] \\
&\quad\times\int_{0+}^{\infty}\Pi(\mathrm{d}x-y)\left\{\frac{W(\epsilon)}{\epsilon} - \frac{W(\epsilon-y)}{\epsilon}\right\}\mathrm{d}y \\
&= W'(0+)\int_{-\infty}^{0-}\left[Z^{(\lambda)}(x+b) - Z^{(\lambda)}(b)\frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b)}\right]\int_{0+}^{\infty}\Pi(\mathrm{d}x-y)\mathrm{d}y
\end{aligned}$$

and, similarly,

$$\lim_{\epsilon\to 0}\frac{L_3^{\epsilon,b}}{\epsilon} = \psi'(0+)\lim_{\epsilon\to 0}\frac{W(\epsilon)}{\epsilon} + \frac{\sigma^2}{2}\lim_{\epsilon\to 0}\frac{W'(\epsilon)}{W^{(\lambda)}(b+\epsilon)}\left[\frac{W^{(\lambda)}(b+\epsilon) - W^{(\lambda)}(b)}{\epsilon}\right] + \lim_{\epsilon\to 0}\int_{-\infty}^{0-}\left[1-\frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b+\epsilon)}\right]\int_0^{\infty}\Pi(\mathrm{d}x-y)\left\{\frac{W(\epsilon)}{\epsilon} - \frac{W(\epsilon-y)}{\epsilon}\right\}\mathrm{d}y$$

$$
\begin{aligned}
&= \psi'(0+)W'(0+) + \frac{\sigma^2}{2}W'(0+)\frac{W^{(\lambda)\prime}(b)}{W^{(\lambda)}(b)} + W'(0+) \\
&\quad \times \int_{-\infty}^{0-}\left[1 - \frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b)}\right]\int_0^{\infty} \Pi(\mathrm{d}x - y)\mathrm{d}y.
\end{aligned}
$$

3.2. Proof if $\sigma = 0$

If X does not have a Brownian component, we adapt the previous method as follows:

$$
\frac{L_1^{\epsilon,b}}{W(\epsilon)} = \psi'(0+), \quad \text{for all } \epsilon > 0,
$$

$$
\begin{aligned}
\lim_{\epsilon\to 0}\frac{L_2^{\epsilon,b}}{W(\epsilon)} &= \lim_{\epsilon\to 0}\int_{-\infty}^{0-}\left[Z^{(\lambda)}(x+b) - Z^{(\lambda)}(b+\epsilon)\frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b+\epsilon)}\right] \\
&\quad \times \int_0^{\infty}\Pi(\mathrm{d}x - y)\left\{\frac{W(\epsilon)}{W(\epsilon)} - \frac{W(\epsilon - y)}{W(\epsilon)}\right\}\mathrm{d}y \\
&= \int_{-\infty}^{0-}\left[Z^{(\lambda)}(x+b) - Z^{(\lambda)}(b)\frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b)}\right]\int_0^{\infty}\Pi(\mathrm{d}x - y)\mathrm{d}y
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
\lim_{\epsilon\to 0}\frac{L_3^{\epsilon,b}}{W(\epsilon)} &= \psi'(0+) + \lim_{\epsilon\to 0}\int_{-\infty}^{0-}\left[1 - \frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b+\epsilon)}\right] \\
&\quad \times \int_0^{\infty}\Pi(\mathrm{d}x - y)\left\{\frac{W(\epsilon)}{W(\epsilon)} - \frac{W(\epsilon - y)}{W(\epsilon)}\right\}\mathrm{d}y \\
&= \psi'(0+) + \int_{-\infty}^{0-}\left[1 - \frac{W^{(\lambda)}(x+b)}{W^{(\lambda)}(b)}\right]\int_0^{\infty}\Pi(\mathrm{d}x - y)\mathrm{d}y.
\end{aligned}
$$

3.3. Conclusion of the proof

In all cases, i.e., when X has or does not have a Brownian component, the limiting results for $U^{\epsilon,b}$ can be obtained in a similar fashion. In fact, it can easily be proved that, for each $i = 1, 2, 3$,

$$
\lim_{\epsilon\to 0}\frac{U_i^{\epsilon,b}}{L_i^{\epsilon,b}} = 1,
$$

and then

$$
\lim_{\epsilon\to 0}\frac{W^{(\lambda)}(b)}{W^{(\lambda)}(b+\epsilon)}L^{\epsilon,b} = \mathbb{E}[\mathrm{e}^{-\lambda\int_0^{\tau_{-b}^-}\mathbb{I}_{\{X_s\le 0\}}\mathrm{d}s}] = \lim_{\epsilon\to 0}U^{\epsilon,b}.
$$

The details are left to the reader.

Remark 3.1. Note that in the above proof Lebesgue's dominated convergence theorem has been used twice. For example, we used implicitly that the integral

$$
\int_{-\infty}^{0-}A_3^{(\lambda)}(x)\int_0^{\infty}\Pi(\mathrm{d}x - y)\mathrm{d}y = \int_0^{\infty}\int_{-\infty}^{-y-}A_3^{(\lambda)}(x+y)\Pi(\mathrm{d}x)\mathrm{d}y,
$$

where $A_3^{(\lambda)}(z) = 1 - W^{(\lambda)}(z+b)/W^{(\lambda)}(b)$, is finite. First, recall that, if $\psi'(0+) > -\infty$ then, for $\epsilon > 0$, $\int_{-\infty}^{-\epsilon}(-z)\Pi(\mathrm{d}z) < \infty$ (see, e.g., [19]). Using integration by parts, one can easily show that, for $\epsilon > 0$, $\int_{-\infty}^{-\epsilon}\Pi(-\infty, z)\mathrm{d}z < \infty$ and, in particular, when X has paths of bounded variation, $\int_{-\infty}^{0}\Pi(-\infty, z)\mathrm{d}z < \infty$; see also p. 275 of [14].

We assume that X has paths of unbounded variation. Since, for $x < -y < 0$, $0 \le A_3^{(\lambda)}(x+y) \le A_3^{(\lambda)}(x) \le 1$, we clearly have that

$$\left|\int_{b/2}^{\infty}\int_{-\infty}^{-y-} A_3^{(\lambda)}(x+y)\Pi(\mathrm{d}x)\mathrm{d}y\right| \le \int_{b/2}^{\infty}\Pi(-\infty, -y)\mathrm{d}y < \infty.$$

Also,

$$\begin{aligned}
&\left|\int_{0}^{b/2}\int_{-\infty}^{-y-} A_3^{(\lambda)}(x+y)\Pi(\mathrm{d}x)\mathrm{d}y\right| \\
&\quad \le \int_{-\infty}^{-b/2}\int_{0}^{b/2}\mathrm{d}y|A_3^{(\lambda)}(x)|\Pi(\mathrm{d}x) + \int_{-b/2}^{0-}\int_{0}^{-x}\mathrm{d}y|A_3^{(\lambda)}(x)|\Pi(\mathrm{d}x) \\
&\quad \le (b/2)\Pi(-\infty, -b/2) + \int_{-b/2}^{0-}(-x)(1 - W^{(\lambda)}(x+b)/W^{(\lambda)}(b))\Pi(\mathrm{d}x) \\
&\quad < \infty,
\end{aligned}$$

since

$$\int_{-b/2}^{0-}(-x)(W^{(\lambda)}(b) - W^{(\lambda)}(x+b))\Pi(\mathrm{d}x) = \int_{-b/2}^{0-} x^2 W^{(\lambda)\prime}(\xi_x + b)\Pi(\mathrm{d}x) < \infty,$$

where $\xi_x \in (-x, 0) \subset (-b/2, 0)$; the latter equality follows from Taylor's expansion. Since $W^{(\lambda)\prime}$ is continuous when X has paths of unbounded variation, the finiteness of this integral follows from (1). When X has paths of bounded variation, the proof is easier, and is left to the reader.

The proof of the finiteness of the integral involving $A_2^{(\lambda)}$ is similar.

4. Proof of Theorem 1

We use the same general idea as in the proof of Theorem 2. Define $L^{\epsilon} = L^{\epsilon,\infty}$. In this case, using the strong Markov property twice, we obtain

$$\begin{aligned}
L^{\epsilon} &= \mathbb{P}_{\epsilon}\{\tau_0^- = \infty\} + L^{\epsilon}\mathbb{E}[\mathrm{e}^{-\lambda\tau_{\epsilon}^+}]\int_{-\infty}^{0+}\mathbb{P}_{\epsilon}\{\tau_0^- < \infty, X_{\tau_0^-} \in \mathrm{d}x\}\mathbb{E}_x[\mathrm{e}^{-\lambda\tau_0^+}] \\
&= \psi'(0+)W(\epsilon) + L^{\epsilon}\mathrm{e}^{-\Phi(\lambda)\epsilon}\int_{-\infty}^{0+}\mathbb{P}_{\epsilon}\{\tau_0^- < \infty, X_{\tau_0^-} \in \mathrm{d}x\}\mathrm{e}^{\Phi(\lambda)x}.
\end{aligned}$$

Since, for $r > 0$,

$$\mathbb{E}_x[\mathrm{e}^{rX_{\tau_0^-}}; \tau_0^- < \infty] = \mathrm{e}^{rx} - \psi(r)\mathrm{e}^{rx}\int_0^x \mathrm{e}^{-rz}W(z)\mathrm{d}z - \frac{\psi(r)}{r}W(x), \tag{6}$$

we can write

$$\int_{-\infty}^{0+}\mathbb{P}_{\epsilon}\{\tau_0^- < \infty, X_{\tau_0^-} \in \mathrm{d}x\}\mathrm{e}^{\Phi(\lambda)x} = \mathbb{E}_{\epsilon}[\mathrm{e}^{\Phi(\lambda)X_{\tau_0^-}}; \tau_0^- < \infty]$$

$$= \mathrm{e}^{\Phi(\lambda)\epsilon} - \lambda \mathrm{e}^{\Phi(\lambda)\epsilon} \int_0^{\epsilon} \mathrm{e}^{-\Phi(\lambda)x} W(x)\mathrm{d}x - \frac{\lambda}{\Phi(\lambda)} W(\epsilon).$$

It is easily shown that

$$\lim_{\epsilon \to 0} \frac{\int_0^{\epsilon} \mathrm{e}^{-\Phi(\lambda)x} W(x)\mathrm{d}x}{W(\epsilon)} = 0.$$

Indeed, if $W(0+) > 0$, this is trivial; and, if $W(0+) = 0$, one can use L'Hôpital's rule. Consequently,

$$\lim_{\epsilon \to 0} L^{\epsilon} = \lim_{\epsilon \to 0} \frac{\psi'(0+)W(\epsilon)}{1 - \left[1 - \lambda \int_0^{\epsilon} \mathrm{e}^{-\Phi(\lambda)x} W(x)\mathrm{d}x - \frac{\lambda}{\Phi(\lambda)} \mathrm{e}^{-\Phi(\lambda)\epsilon} W(\epsilon)\right]} = \psi'(0+)\frac{\Phi(\lambda)}{\lambda}.$$

Similarly, if we define $U^{\epsilon} = U^{\epsilon,\infty}$, one can show that

$$\lim_{\epsilon \to 0} U^{\epsilon} = \psi'(0+)\frac{\Phi(\lambda)}{\lambda},$$

and the result immediately follows.

Remark 4.1. Here is (the sketch of) another proof for Theorem 1, as suggested by Ronnie Loeffen. Sparre Andersen's identity (see Lemma VI. 15 in [1]) says that, for all $t > 0$,

$$A_t \overset{\text{law}}{=} \overline{G}_t,$$

where $A_t = \int_0^t \mathbb{I}_{\{X_s \geq 0\}} \mathrm{d}s$ and $\overline{G}_t = \sup\{s < t \colon X_t = \overline{X}_t\}$, and where $\overline{X}_t = \sup_{s \leq t} X_s$. Therefore, if $\mathbf{e}_p$ is an independent exponential random variable with mean $1/p$, then, for $\lambda \geq 0$,

$$\mathbb{E}[\mathrm{e}^{-\lambda \int_0^{\mathbf{e}_p} \mathbb{I}_{\{X_s \leq 0\}} \mathrm{d}s}] = \mathbb{E}[\mathrm{e}^{-\lambda(\mathbf{e}_p - A_{\mathbf{e}_p})}] = \mathbb{E}[\mathrm{e}^{-\lambda(\mathbf{e}_p - \overline{G}_{\mathbf{e}_p})}].$$

By the Wiener–Hopf factorization, the term on the right-hand side is equal to

$$\frac{p\Phi(\lambda + p)}{\Phi(p)(\lambda + p)}$$

(see Theorem VII. 4 in [1]). Taking the limits when p goes to zero on both sides yields the result.

5. Proof of Corollary 1

Using the strong Markov property of X twice, using some of the fluctuation identities in Section 2, and then using Theorem 1, we have

$$\begin{aligned}
\mathbb{E}_x[\mathrm{e}^{-\lambda \int_0^{\infty} \mathbb{I}_{\{X_s \leq 0\}} \mathrm{d}s}] &= \mathbb{P}_x\{\tau_0^- = \infty\} + \mathbb{E}_x[\mathrm{e}^{-\lambda \int_0^{\infty} \mathbb{I}_{\{X_s \leq 0\}} \mathrm{d}s}; \tau_0^- < \infty] \\
&= \psi'(0+)W(x) + \psi'(0+)\frac{\Phi(\lambda)}{\lambda} \int_{-\infty}^{0+} \mathbb{P}_x\{\tau_0^- < \infty, X_{\tau_0^-} \in \mathrm{d}z\} \\
&\quad \times \mathbb{E}_z[\mathrm{e}^{-\lambda \tau_0^+}; \tau_0^+ < \infty] \\
&= \psi'(0+)W(x) + \psi'(0+)\frac{\Phi(\lambda)}{\lambda} \int_{-\infty}^{0+} \mathbb{P}_x\{\tau_0^- < \infty, X_{\tau_0^-} \in \mathrm{d}z\} \mathrm{e}^{\Phi(\lambda)z} \\
&= \psi'(0+)W(x) + \psi'(0+)\frac{\Phi(\lambda)}{\lambda} \mathbb{E}_x[\mathrm{e}^{\Phi(\lambda)X_{\tau_0^-}}; \tau_0^- < \infty].
\end{aligned}$$

Using the result in Eq. (6) once more, we get

$$\mathbb{E}_x[\mathrm{e}^{-\lambda \int_0^\infty \mathbb{I}_{\{X_s \leq 0\}} \mathrm{d}s}] = \psi'(0+) \frac{\Phi(\lambda)}{\lambda} \left\{ \mathrm{e}^{\Phi(\lambda)x} \left(1 - \lambda \int_0^x \mathrm{e}^{-\Phi(\lambda)z} W(z) \mathrm{d}z \right) \right\}$$
$$= \psi'(0+) \frac{\Phi(\lambda)}{\lambda} \left\{ \lambda \int_0^\infty \mathrm{e}^{-\Phi(\lambda)z} W(x+z) \mathrm{d}z \right\},$$

where in the last step a change of variables and an integration by parts were undertaken. The result follows.

6. Applications

6.1. Brownian motion with drift

For Brownian motion with drift, i.e., if X is of the form $X_t = mt + \sigma B_t$ with $m = \psi'(0+) \geq 0$ and $\sigma > 0$, where B is a standard Brownian motion, then $\Pi \equiv 0$, and, using Theorem 2, we have

$$\mathbb{E}[\mathrm{e}^{-\lambda \int_0^{\tau_{-b}^-} \mathbb{I}_{\{X_s \leq 0\}} \mathrm{d}s}] = \frac{mW^{(\lambda)}(b) + (\sigma^2/2)[Z^{(\lambda)}(b)W^{(\lambda)\prime}(b) - \lambda(W^{(\lambda)}(b))^2]}{mW^{(\lambda)}(b) + (\sigma^2/2)W^{(\lambda)\prime}(b)},$$

where, as one can easily verify from the definition,

$$W^{(\lambda)}(x) = \frac{2}{\sqrt{m^2 + 2\lambda\sigma^2}} \mathrm{e}^{-(m/\sigma^2)x} \sinh((x/\sigma^2)\sqrt{m^2 + 2\lambda\sigma^2}),$$

for $x > 0$.

Then, for standard Brownian motion, in which case $m = \psi'(0+) = 0$ and $\sigma = 1$, we have, from Theorem 2, that

$$\mathbb{E}[\mathrm{e}^{-\lambda \int_0^{\tau_{-b}^-} \mathbb{I}_{\{X_s \leq 0\}} \mathrm{d}s}] = \frac{Z^{(\lambda)}(b)W^{(\lambda)\prime}(b) - \lambda(W^{(\lambda)}(b))^2}{W^{(\lambda)\prime}(b)},$$

where

$$W^{(\lambda)}(x) = \sqrt{\frac{2}{\lambda}} \sinh(x\sqrt{2\lambda}),$$

and then

$$Z^{(\lambda)}(x) = \cosh(x\sqrt{2\lambda}).$$

Recalling that $\cosh^2(x) - \sinh^2(x) = 1$, we get that

$$\mathbb{E}[\mathrm{e}^{-\lambda \int_0^{\tau_{-b}^-} \mathbb{I}_{\{X_s \leq 0\}} \mathrm{d}s}] = \frac{2\cosh^2(b\sqrt{2\lambda}) - \lambda(2/\lambda)\sinh^2(b\sqrt{2\lambda})}{2\cosh(b\sqrt{2\lambda})}$$
$$= \frac{1}{\cosh(b\sqrt{2\lambda})},$$

therefore recovering Proposition 4.12 of [11] (see also [10]).

6.2. Insurance risk models

Classical insurance risk models describe the surplus process of an insurance company using a compound Poisson process or a Brownian motion with drift, i.e., special cases of spectrally negative Lévy processes. In those models, it is usually assumed that the *net profit condition* holds; this condition ensures that ruin will not occur almost surely. For a Lévy insurance risk process, i.e., a spectrally negative Lévy process with non-monotone paths, this amounts to $\mathbb{E}[X_1] = \psi'(0+) > 0$, and it is also equivalent to $\lim_{t\to\infty} X_t = \infty$ almost surely. An interpretation of Lévy insurance risk models for the surplus modelling of large insurance companies is for instance given in [12].

Classically, the probability of ruin has been the most studied risk measure to evaluate the quality of a company. More recently, the analysis of the duration of the negative surplus, in other words the occupation time of the negative half-line, has also been considered in the compound Poisson case [8] and then in a jump-diffusion model [23]; see also [3]. Recall that Theorem 1 extends Eq. (4.9) in [23], where the sum of a compound Poisson process and a Brownian motion is considered.

We now want to provide another link between Theorem 1 and insurance risk models. In [17], a *new* definition of the time to ruin is proposed. In that paper, each excursion of the surplus process X below 0 is accompanied by an independent copy of an independent (of X) and exponentially distributed random variable $\mathbf{e}_d$ with mean $1/d$; we will refer to it as the implementation clock. If the duration of a given excursion below 0 is less than that of its associated implementation clock, then ruin does not occur. More precisely, we assume that ruin occurs at the first time τ_d that an implementation clock rings before the end of its corresponding excursion below 0. It is worth pointing out that the time to ruin τ_d is easily defined when the Lévy insurance risk processes X has sample paths of bounded variation.

Therefore, in [17], the case of a surplus process of bounded variation is considered. In that model, one can show that the probability of ruin in this Parisian risk model with exponential implementation delays can be expressed as follows:

$$\mathbb{P}\{\tau_d < \infty\} = 1 - \mathbb{E}[\mathrm{e}^{-d\int_0^\infty \mathbb{I}_{\{X_s \le 0\}}\mathrm{d}s}],$$

if the exponential clock has mean $1/d$. Indeed, for each excursion of length T, the probability to survive, i.e., the probability that the exponential random variable associated with it is larger than T, is equal to $\exp\{-d\,T\}$. Using the independence assumption between the clocks and summing up over all the excursions (there are countably many of them), we get

$$\mathbb{E}[\mathrm{e}^{-d\int_0^\infty \mathbb{I}_{\{X_s \le 0\}}\mathrm{d}s}].$$

Using Theorem 1, we recover the corresponding expression in [17]; that is,

$$\mathbb{P}\{\tau_d < \infty\} = 1 - \psi'(0+)\frac{\Phi(d)}{d},$$

when the net profit condition $\psi'(0+) > 0$ is verified.

More generally, using Itô's excursion theory for spectrally negative Lévy processes, this Parisian risk model with exponential implementation delays can also be defined when the underlying surplus process has paths of unbounded variation. It suffices to *mark* the Poisson point process of excursions away from zero (see [20] for a definition of the corresponding excursion process) with independent copies of the generic random variable $\mathbf{e}_d$, similar to the proof of Theorem 6.16 in [14]; for an excursion away from zero starting above zero, this time spent above zero is simply ignored. As a consequence, Theorem 1 provides a generalization of the probability of ruin in a general Lévy insurance risk model with exponential implementation delays. Finally, note that the probability of ruin in a general Lévy insurance risk model with deterministic implementation delays has recently been computed; see [18].

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