Centro-affine normal flows and their applications

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ABSTRACT

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A family of geometric flows, p centro-affine normal flows, is studied and several applications of this family to convex geometry are presented. In Chapter 2, asymptotic behavior of the centro-affine normal flows is studied in the class of smooth, origin-symmetric convex bodies in $\mathbb{R}^2$. It is proved that the area preserving centro-affine normal flow evolve any smooth, origin-symmetric convex body to the unit disk in a finite time in the Hausdorff distance, module $SL(2)$. In Chapter 3, an application of centro-affine normal flow is given to the L_{-2} Minkowski problem. It is proved there that every even, smooth, positive function can be approached by a sequence of functions for which the L_{-2} Minkowski problem is solvable. In Chapter 4, another application of the centro-affine normal flows is given to the stability of the p-affine isoperimetric inequalities, $p \geq 1$. In Chapter 5, we end our study of the p centro-affine normal flows in dimension two by classifying compact, origin-symmetric, ancient solutions to these flows for $1 \leq p < 4$. In particular, we classify origin-symmetric, compact ancient solutions of the planar affine normal flow. In the last chapter, we study the long time behavior of the p centro-affine normal flows in $\mathbb{R}^n$ for $n > 2$ and $1 \leq p < \frac{n}{n-2}$.

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Introduction

The affine normal flow is a widely recognized evolution equation for hypersurfaces in which each point of the hypersurface moves with velocity given by the affine normal vector. This evolution equation is the simplest affine invariant flow in differential geometry and it arises naturally if one considers families of δ-convex floating bodies of a convex body [10], [97]. On the applicability aspect, the affine normal flow appears in image processing as a fundamental smoothing tool [88, 89, 91]. It also provides a nice proof of the Blaschke-Santaló inequality for smooth convex hypersurfaces and, respectively, for the classical affine isoperimetric inequality, both due to Andrews [7]. The affine normal evolution has also been implicitly deployed by Stancu in [97, 98] for a breakthrough towards the homothety conjecture for convex floating bodies by Schütt-Werner [93]. In this thesis, we consider a class of extensions of the affine normal flow in centro-affine differential geometry, namely the centro-affine normal flow introduced by Stancu [100], and we investigate its asymptotic behavior and applications to P.D.E. and convex geometry. The centro-affine normal flows are natural generalizations of the affine normal flow in a way which will be explained below. It is essential to say, the term *centro* in centro-affine differential geometry emphasizes that, contrary to affine differential geometry or classical differential geometry, Euclidean translations of an object in the ambient space are not allowed.

The setting of this thesis is the n-dimensional Euclidean space, $\mathbb{R}^n$. A compact convex subset of $\mathbb{R}^n$ with non-empty interior is called a *convex body*. Let K be a strictly convex body, having origin in its interior, and smoothly embedded in $\mathbb{R}^n$. Let

$$x_K : \mathbb{S}^{n-1} \to \mathbb{R}^n,$$

be the Gauss parametrization of ∂K, the boundary of K. Therefore, $x_K(z)$ takes $z \in \mathbb{S}^{n-1}$ into the point on the ∂K of the outer normal z. The support function of K is defined by

$$s_K(z) := \langle x_K(z), z \rangle,$$

for each $z \in \mathbb{S}^{n-1}$. We denote the matrix of the radii of curvature of ∂K by $\mathfrak{r} = [\mathfrak{r}_{ij}]_{1\leq i,j\leq n-1}$, the entries of $\mathfrak{r}$ are considered as functions on the unit sphere. And they are related to the support function by

$$\mathfrak{r}_{ij} := \bar{\nabla}_i \bar{\nabla}_j s + s\bar{g}_{ij},$$

where $\bar{g}_{ij}$ is the standard metric on $\mathbb{S}^{n-1}$ and $\bar{\nabla}$ is the standard Levi-Civita connection of $\mathbb{S}^{n-1}$. We denote the Gauss curvature of ∂K by $\mathcal{K}$ and remark that, as a function on the unit sphere, it is related to the support function of the convex body by

$$\frac{1}{\mathcal{K}} = S_{n-1} := \det_{\bar{g}}(\bar{\nabla}_i \bar{\nabla}_j s + \bar{g}_{ij} s) := \frac{\det \mathfrak{r}_{ij}}{\det \bar{g}_{ij}}.$$

Furthermore, the affine support function of K, denoted by σ, is defined pointwise by

$$\sigma = \frac{s}{\mathcal{K}^{1/(n+1)}}.$$

This thesis is concerned with the motion of hypersurfaces under centro-affine curvature driven evolution equations. The centro-affine normal flow is an $SL(n)$ invariant flow which at each point moves the the boundary of a convex body with velocity given by a power of centro-affine curvature in direction of the centro-affine normal vector:

We consider a smooth n-dimensional convex body K_0 whose boundary hypersurface, ∂K_0, is smoothly embedded in $\mathbb{R}^n$. Let $x_{K_0} : \mathbb{S}^{n-1} \to \mathbb{R}^n$ be the initial embedding and let $x : \mathbb{S}^{n-1} \times [0, T) \to \mathbb{R}^n$ be the one parameter family defined by

$$\frac{\partial}{\partial t} x(z,t) := -\mathcal{K}_0^{\alpha}(z,t)\mathcal{N}_0(z,t), \quad x(z,0) = x_{K_0}(z).$$

where α is a positive real power which will be made explicit soon, $\mathcal{K}_0 := \sigma^{-(n+1)}$ is, interestingly, an $SL(n)$ invariant called centro-affine curvature and, finally,

$$\mathcal{N}_0 = \mathcal{K}_0{}^{-\frac{1}{n+1}}(z) \left(\mathcal{K}^{\frac{1}{n+1}}(z)\, z + \bar{\nabla}(\mathcal{K}^{\frac{1}{n+1}}(z)) \right)$$

is the centro-affine normal, both as functions of z. As the name centro-affine suggests, solutions to the flow are $SL(n)$ invariant while Euclidean translations of an initial convex body will lead to different solutions, since translations affect the support function of the convex body which appears in the speed of the centro-affine normal flow.

We will now give a description of the centro-affine normal flow in terms of powers of the Gauss curvature and support function. For every fixed convex body K_0 whose interior contains the origin, and whose boundary is of class C^2 with strictly positive Gauss curvature, the above mentioned flow is equivalent to

$$\frac{\partial}{\partial t} x(z,t) := -\left(\frac{\mathcal{K}}{s^{n+1}}\right)^{\frac{p}{p+n} - \frac{1}{n+1}} (z,t)\ \mathcal{K}^{\frac{1}{n+1}}(z,t)\, z, \quad x(z,0) = x_{K_0}(z),$$

for a fixed $p \geq 1$. For a fixed p, we call the flow, p-flow, or alternatively, p centro-affine normal flow. The flow was defined by Stancu in [100] for the purpose of finding new global centro-affine invariants of smooth convex bodies in which a certain class of existing invariants arose naturally. Only the short time existence to the flow was then needed. Moreover, several interesting isoperimetric type inequalities were obtained via short time existence of the flow [100], and this p-flow approach led to a geometric interpretation of the L_ϕ surface area recently introduced by Ludwig and Reitzner [67]. See recent work of Stancu [101] for more applications of the p-flow.

In this thesis, we choose to work with the latter definition; the flow's definition as a time-dependent anisotropic flow by powers of the Gauss curvature and support function and we will resort to the affine differential geometry only for certain technical steps.

The case $p = 1$, the well-known affine normal flow, was already addressed by Andrews [7] for all dimensions, by Sapiro and Tannebaum [90] for convex planar curves, and by Angenent, Sapiro, and Tannebaum [13] for non-convex curves. In [90], it was proved that the flow evolves every initial strictly convex curve, not necessarily symmetric, until it shrinks to an *elliptical* point. Andrews, in [7], investigated completely this case for hypersurfaces and showed that the normalized flow evolves every initial strictly convex hypersurface exponentially fast, in the C^∞ topology, to an ellipsoid. He also proves, in [10], that every convex initial bounded open set shrinks to a point in finite time under the affine normal flow. In [13], the authors prove convergence to a point under the affine normal flow starting from C^2 planar curves, not necessarily convex, despite the fact that affine differential geometry is not defined for non-convex curves or hypersurfaces. In another direction, interesting results for the affine normal flow have been obtained in [64] by Loftin and Tsui regarding ancient solutions, existence and regularity of solutions on non-compact strictly convex hypersurfaces. It is necessary pointing out that the case $p = 1$, in contrast to the case $p > 1$, is the only instance when the centro-affine normal flow is a flow by a power of Gauss curvature no longer anisotropic. Moreover, the main difference between $p = 1$ and the other cases is that, for $p > 1$ the solution to the flow is not invariant under Euclidean translations. The translation invariancy of a flow is a main ingredient to prove the convergence to a point [5, 7, 8, 10]. We overcome these issues, and other difficulties in the study of the asymptotic behavior of this flow, Chapters 2 and 6, by restricting it to the class of origin-symmetric convex bodies and implementing p-affine isoperimetric inequalities developed by Lutwak [69]. This approach emphasizes the usefulness of the p-affine surface area and p-affine isoperimetric inequalities which have also been successfully employed by Lutwak and Oliker in [70] for obtaining regularity of the solutions to a generalization of Minkowski problem. See [29, 71, 72, 73, 75, 76, 102, 105] for more applications of these invaluable tools.

In this thesis, we study the long time behavior of the centro-affine normal flow, Chapters 2 and 6, and we study the ancient solutions to the flow in dimension two, Chapter 5. We present several applications of the centro-affine normal flow in two dimensional Euclidean space, Chapters 2, 3 and Chapter 4.

Let K be a compact, origin-symmetric, strictly convex body, smoothly embedded in $\mathbb{R}^n$. We denote the space of such convex bodies by $\mathcal{K}_{sym}$.

Let K and L be two origin-symmetric convex bodies in $\mathbb{R}^n$ with respective support functions s_K and s_L. Then the Hausdorff distance between K and L is defined by

$$d_{\mathcal{H}}(K, L) = \max_{\mathbb{S}^{n-1}} |s_K - s_L|.$$

In Chapter 2, the long time behavior of the centro-affine normal flow is settled in dimension two. The following theorem is proved there.

Theorem A. *[52] Let $p > 1$ be a real number. Let $x_{K_0} : \mathbb{S}^1 \to \mathbb{R}^2$ be an embedding of*

$K_0 \in \mathcal{K}_{sym}$. Then there exists a unique solution $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ of centro-affine normal flow with initial data x_{K_0}. The solution remains smooth and strictly convex on $[0, T)$ for a finite time T and it converges to the origin of the plane. The rescaled convex bodies, with the fixed area of π, converge in the Hausdorff metric to the unit disk modulo $SL(2)$.

In Chapter 3, we give an application of the centro-affine normal flow to the L_{-2} Minkowski problem. The L_{-2} Minkowski problem is a member of a large family problems, namely the L_p Minkowski problems. For $p = 1$, the L_1 Minkowski problem is the well-known classical Minkowski problem. In differential geometry, the classical Minkowski problem concerns the existence, uniqueness, and regularity of closed convex hypersurfaces whose Gauss curvature is prescribed as function of the normals. More generally, the Minkowski problem asks what are the necessary and sufficient conditions on a Borel measure on $\mathbb{S}^{n-1}$ to guarantee that it is the surface area measure of a convex body in $\mathbb{R}^n$. If the measure μ has a smooth density Φ with respect to the Lebesgue measure of the unit sphere $\mathbb{S}^{n-1}$, the Minkowski problem is equivalent to the study of the following partial differential equation on the unit sphere

$$\det(\bar{\nabla}^2 s + \mathrm{Id}\, s) = \Phi,$$

in an orthonormal frame on $\mathbb{S}^{n-1}$. Note that for a smooth convex body K with support function s, the quantity $\det(\bar{\nabla}^2 s + \mathrm{Id}\, s)$ is the reciprocal of the Gauss curvature of the boundary of K. The answer to the existence and uniqueness of the Minkowski problem is as follows. If the support of μ is not contained in a great subsphere of $\mathbb{S}^{n-1}$, and it satisfies

$$\int_{\mathbb{S}^{n-1}} z d\mu(z) = 0,$$

then it is the surface area of a convex body, and the solution is unique up to a translation. Minkowski himself solved the problem in the category of polyhedrons. A.D. Alexandrov and others solved the problem in general, however, without any information about the regularity of the (unique) convex hypersurface. Around 1953, L. Nirenberg (in dimension three) and A.V. Pogorelov (in all dimensions) solved the regularity problem in the smooth category independently. For references, one can see works by Minkowski [78, 79], Alexandrov [2, 3, 4], Fenchel and Jessen [34], A.V. Pogorelov [84, 85, 86], Lewy [62, 63], Nirenberg [81], Calabi [24], Cheng and Yau [26], Caffarelli et al. [23], and others.

In his seminal work [68, 69], Lutwak extended the Brunn-Minkowski theory to the Brunn-Minkowski-Firey theory, which made possible impressive new results in convex geometry [71, 72, 73, 76], stochastic geometry [38, 39], differential geometry and differential equations [29, 52, 70, 102, 103, 104, 105]. In the L_p Brunn-Minkowski-Firey theory, Lutwak introduced the notion of the L_p surface area. Therefore, it is natural to ask what are the necessary and sufficient conditions on a Borel measure on $\mathbb{S}^{n-1}$ which guarantee that it is the L_p surface area measure of a convex body. For $p \geq 1$, and an even measure, existence and uniqueness of the convex body was established by Lutwak [68]. If the measure μ has a smooth density Φ with respect to the Lebesgue measure of the unit sphere $\mathbb{S}^{n-1}$, the L_p problem is equivalent to the study of solutions to the following Monge-Ampère type equation on the unit sphere

$$s^{1-p} \det(\bar{\nabla}^2 s + \mathrm{Id}\, s) = \Phi,$$

where $\bar{\nabla}$ is the covariant derivative on $\mathbb{S}^{n-1}$ endowed with an orthonormal frame. Notice that for $p = 1$ this is the classical Minkowski problem. Solutions to many cases of these generalized problems followed later by J. Ai, K.S. Chou, B. Andrews, K.J. Böröczky, W. Chen, M. Gage, B. Guan, P. Guan, C.S. Lin, X.N. Ma, J. Li, Y.Y. Li, Y. Huang, Q.P. Lu, M. Jiang, E. Lutwak, V. Oliker, D. Yang, G. Zhang, A. Stancu, and V. Umanskiy [1, 8, 12, 21, 25, 27, 35, 36, 40, 41, 42, 43, 44, 46, 58, 70, 71, 74, 95, 96, 99, 107]. The progress in studying L_p Minkowski problems has been extremely fruitful and resulted in many applications to functional inequalities [29, 71, 72, 73, 74, 76]. This unified theory relates many problems that, previously, were not connected. Notice also that, for constant data Φ, many L_p problems were treated as self-similar solutions of geometric flows [5, 7, 8, 12, 35, 36] and others.

The cases $p = -n$ and $p = 0$ are quite special and more difficult. The even case $p = 0$ has been recently solved by Böröczky, Lutwak, Yang, and Zhang [21]. Many challenges remain for the problem with $p < 1$ and, particularly, for negative p. The above partial differential equation with $p \in [-2, 0]$ and $n = 2$ has been studied by Chen [25] and more recently by Jiang [58] for Φ not necessarily positive. For $p \leq -2$, some existence results were obtained by Dou and Zhu including generalizing the result obtained by Jiang in the case $p = -2$, [33]. Investigations of the L_{-n} Minkowski problem have been restricted mostly to the even L_{-n} Minkowski problem, e.q., the problem in which it is assumed that μ has the same values on antipodal Borel sets, [1, 25, 27, 58]. The technical difficulty lies again in the fact that Euclidean translations of solutions to the L_{-n} problem are no longer solutions to the problem. There are only a few works on the L_{-2} Minkowski problem [1, 25, 33, 57, 58, 107] and almost no results, except for a Kazdan-Warner type obstruction, on the L_{-n} Minkowski problem for $n > 2$, [27] and an existence result for rotationally symmetric data by J. Lu and X.J. Wang, [65].

If a measure μ has a smooth density Φ with respect to the Lebesgue measure of the unit sphere $\mathbb{S}^1$, the L_{-2} Minkowski is equivalent to the study of **positive** solutions to the following ordinary differential equation on the unit sphere

$$s^3(s_{\theta\theta} + s) = \Phi.$$

The even L_{-2} Minkowski problem considers those Φ such that $\Phi(z) = \Phi(-z)$ for all $z \in \mathbb{S}^1$. In convex geometry, we are interested in positive solutions of $s^3(s_{\theta\theta} + s) = \Phi$. The reason is that a positive solution to this equation corresponds to the existence of a convex body with support function s and with affine support function Φ. In Chapter 3 the following theorem is proved.

Theorem B. *[54] Given an even, smooth function $\Phi : \mathbb{S}^1 \to \mathbb{R}^+$, there exists a family of convex bodies $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{K}_{sym}$ such that*

$$\lim_{n\to\infty} \sup_{\mathbb{S}^1} \left|\left|s^3(s_{\theta\theta} + s) - \Phi\right|\right| = 0.$$

Furthermore, if Φ is $\frac{\pi}{k}$ periodic for $k \geq 2$, this family of convex bodies is uniformly bounded and it converges in the C^∞ norm to a smooth convex body whose support function satisfies $s^3(s_{\theta\theta} + s) = \Phi$.

In a different direction, let us recall that the well-known isoperimetric inequality says that, amongst all convex bodies in $\mathbb{R}^n$ of given volume, it is precisely the Euclidean balls that have minimum surface area. A proof of the isoperimetric inequality is a simple application of the Brunn-Minkowski inequality. In affine differential geometry, the celebrated affine isoperimetric inequality states that amongst all convex bodies in $\mathbb{R}^n$ of given volume, it is precisely the ellipsoids that have maximal affine surface area where the latter will be defined shortly. Another implication of Lutwak's Brunn-Minkowski-Firey theory was the extension of the notion of the affine surface area to p-affine surface areas, for $p > 1$. Subsequently, the notion of p-affine surface areas for $0 < p < 1$ has been introduced by D. Hug [47], for $-n < p < 0$ by Meyer and Werner [77] and for all $p \neq -n$ by C. Schütt and Werner in [94]. Later, in [66, 67] it was observed by Ludwig that p-affine surface areas, $p \neq -n$, belong to a larger family, called ϕ-affine surface area. For $p \geq 1$, the p-affine surface area of a convex body is related to the volume of the convex body by the p-affine isoperimetric inequality. For $p = 1$, this is the well-known affine isoperimetric inequality due to Blaschke with the equality case characterized in the class of convex bodies with C^2 boundary [17]. The characterization of the equality in general is due to Petty [83]. The p-affine isoperimetric inequality, for $p > 1$, was proved by Lutwak [69], including characterizing the equality case. The equality in the p-affine isoperimetric inequality is achieved only for ellipsoids centered at the origin. The p-affine isoperimetric inequality, for $p < 1,\ p \neq -n$ was proved by Werner and Ye [108]. Their inequalities for $p < -n$ depend on the constant arising from the inverse Blaschke-Santaló inequality.

For $p \geq 1$, the p-affine surface area of K is defined by

$$\Omega_p(K) = \int_{\mathbb{S}^{n-1}} \frac{s}{\mathcal{K}} \left(\frac{\mathcal{K}}{s^{n+1}} \right)^{\frac{p}{n+p}} d\mu_{\mathbb{S}^{n-1}},$$

where $\mu_{\mathbb{S}^{n-1}}$ is the standard Lebesgue measure on $\mathbb{S}^{n-1}$. The p-affine surface area of a convex body is bounded by the volume via the p-affine isoperimetric inequality. If the centroid of K is at the origin then

$$\left(\frac{\Omega_p^{n+p}(K)}{n^{n+p} V^{n-p}(K)} \right)^{\frac{1}{p}} \leq \omega_n^2,$$

with the equality case only for ellipsoids centered at the origin [69]. Here $V(K)$ is the volume of K defined by $V(K) = \dfrac{1}{n} \displaystyle\int_{\mathbb{S}^{n-1}} \frac{s}{\mathcal{K}} d\mu_{\mathbb{S}^{n-1}}$ and ω_n is the volume of the unit ball of $\mathbb{R}^n$. We call the quantity $\left(\dfrac{\Omega_p^{n+p}(K)}{n^{n+p} V^{n-p}(K)} \right)^{1/p}$ the p-affine isoperimetric ratio of K. In dimension two, it is more appropriate to replace $V(K)$ by $A(K)$ as the area of K.

Another important application of the centro-affine normal flow is given in Chapter 4 in connection to the *stability* of the p-affine isoperimetric inequality.

Let Φ be a real valued function on convex bodies. Given a geometric inequality $\Phi(K) \geq 0$, for every convex body K, in which the equality case is obtained only for a certain family of convex bodies, denoted by $\mathcal{F}$, a *stability* version of Φ concerns the following question. Find a positive constant ε_0, and a positive function f, such that the following holds: If for some $0 < \varepsilon \leq \varepsilon_0$ we have

$$\Phi(K) \leq \varepsilon,$$

then there exists a convex body in $\mathcal{F}$, denoted by L, such that

$$d(K, L) \leq f(\varepsilon),$$

where $d(\cdot, \cdot)$ is an appropriate norm in the context of the geometric inequality. Here f obeys the rule $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ (see the beautiful survey of H. Groemer [37]).

Stability of the p-affine isoperimetric inequality concerns the following question. If the p-affine isoperimetric ratio of a convex body is close enough to its maximal value, is it true that the convex body is close enough in an appropriate norm to an ellipsoid?

A stability version of the affine isoperimetric inequality was presented by K.J. Böröczky, in $\mathbb{R}^n$ for $n \geq 3$, [19]. He proved that if K is a convex body in $\mathbb{R}^n$ such that its affine surface is ε-close to the one of an ellipsoid, for a fixed $\varepsilon \in (0, \frac{1}{2})$, then K is "close" to the unit ball in the Banach-Mazur distance. Here "close" is an approximation of order $\varepsilon^{\frac{1}{6n}} |\log \varepsilon|^{\frac{1}{6n}}$. Later in [15], the order of approximation was improved to $\varepsilon^{\frac{1}{3(n+1)}}|\log\varepsilon|^{\frac{4}{3(n+1)}}$. The case $n=2$ was not addressed either in [15] or in [19].

In Chapter 4, we settle the stability of the p-affine isoperimetric inequality in $\mathbb{R}^2$ in the class of origin-symmetric convex bodies. We prove that:

Theorem C. *[53] Let $p \geq 1$. There exists an $\varepsilon_p > 0$, depending on p, such that the following holds. Let K be an origin-symmetric convex body with area π. If for an $0 < \varepsilon < \varepsilon_p$*

$$\left(\frac{\Omega_p^{2+p}(K)}{2^{2+p}A^{2-p}(K)}\right)^{\frac{1}{p}} > \pi^2(1-\varepsilon),$$

then there exist a disk $\mathcal{D}$, an ellipse $\mathcal{E}$ and a special linear transformation T such that

$$\mathcal{E} \subseteq TK \subseteq \left(1 + \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{2c_2}{c_1^2}\varepsilon^{\frac{3}{10}}\right)\mathcal{D},$$

and

$$d_{\mathcal{H}}\left(\mathcal{E}, \left(1 + \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{2c_2}{c_1^2}\varepsilon^{\frac{3}{10}}\right)\mathcal{D}\right) < C_p\varepsilon^{\frac{3}{10}},$$

for a universal constants c_1, c_2 and C_p.

In particular,

$$d_{\mathcal{H}}(TK, \mathcal{E}) < C_p\varepsilon^{\frac{3}{10}}.$$

In Chapter 5, we end our study of the centro-affine normal flow in dimension two by studying the ancient solutions to the flow. The problem of classifying ancient solutions to a given geometric flow has been always an interesting problem. Ancient solutions are solutions that exist on a time interval $(-\infty, T)$, for a positive finite time T. Classifications has been done completely for the curve shortening flow [31] and for the Ricci flow on surfaces [30]. It has been shown in [30] that an ancient solution of the Ricci flow on $\mathbb{S}^2$ must be either round sphere or the King-Rosenau sausage model.

See also an exposition by B. Chow, [28], on a formula of Daskalopoulos, Hamilton and Sesum used in [30]. In Chapter 5, we classify compact, origin-symmetric, ancient solutions to the p centro-affine flow for $p \in [1, 4)$. We classify ancient solutions to the centro-affine normal flows in dimension two in the class of origin-symmetric convex bodies. Pertaining to the affine normal flow, the classification has been done in all dimensions except in dimension two [64]. We demonstrate:

Theorem D. *[51] Let $1 \leq p < 4$. The only compact, origin-symmetric, ancient solutions to the p centro-affine normal flow are homothetic ellipses.*

In Chapter 6, we study the asymptotic behavior of the centro-affine normal flow in higher dimension and we prove that:

Theorem E. *[55] Assume $n > 2$. Let $1 \leq p < \frac{n}{n-2}$ be a real number. Let $x_{K_0} : \mathbb{S}^{n-1} \to \mathbb{R}^n$ be a smooth, strictly convex embedding of $K_0 \in \mathcal{K}_{sym}$. Then there exists a unique solution $x : \mathbb{S}^{n-1} \times [0, T) \to \mathbb{R}^n$ of the p centro-affine normal flow with initial data x_{K_0}. The rescaled convex bodies, having the fixed volumes of the one of the unit sphere, converge sequentially in the C^∞ topology to the unit ball, modulo $SL(n)$. Furthermore, when $p = 1$ the assumption of K_0 being origin-symmetric is not necessary.*

The work in Chapter 6 is joint with Stancu. In that Chapter, we develop a new technique to obtain regularity of the evolving convex bodies under the volume preserving centro-affine normal flows. To derive the lower bound bound on the Gauss curvature of the evolving bodies, we consider the evolution of the dual convex body and we apply Tso's technique to the speed of the dual p-flow. This procedure *avoids* the need for a Harnack inequality, or displacement bounds to obtain higher order regularities.

Let K be a convex body having the origin in its interior. The dual convex body to K denoted by K° is defined by $K^\circ = \{y \in \mathbb{R}^n \mid x \cdot y \leq 1,\ \forall x \in K\}$. It was proved in [100] that, if $\{K_t\}_{[0,T)}$ evolves by the p centro-affine normal flow, then $\{K_t^\circ\}_{[0,T)}$ is a solution of the following evolution equation, the expanding p-flow (alternatively called the dual p-flow):

$$\partial_t s^\circ = s^\circ \left(\frac{\mathcal{K}^\circ}{s^{\circ n+1}}\right)^{-\frac{p}{n+p}}.$$

This important observation is the key to obtain the regularity estimates in Chapter 6.

Chapter 1

Frequently used facts

In this chapter, we recall several definitions from affine differential geometry and a lemma on the stability of the centro-affine curvature which will be necessary in subsequent chapters. Furthermore, we state John's Inclusion and a generalized Hölder inequality. In the end, we state the uniqueness, short time existence and the Containment Principle for the solutions to the centro-affine normal flows.

We first will recall several definitions from planar affine differential geometry. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a smoothly embedded, strictly convex curve with the curve parameter θ. Define $\mathfrak{g}(\theta) := [\gamma_\theta, \gamma_{\theta\theta}]^{1/3}$, where, for two vectors u, v in $\mathbb{R}^2$, $[u, v]$ denotes the determinant of the matrix with rows u and v. The affine arc-length is then given by

$$\mathfrak{s}(\theta) := \int_0^\theta \mathfrak{g}(\xi) d\xi. \tag{1.0.1}$$

Furthermore, the affine tangent vector $\mathfrak{t}$, the affine normal vector $\mathfrak{n}$, and the affine curvature are defined, in this order, as follows:

$$\mathfrak{t} := \gamma_{\mathfrak{s}}, \quad \mathfrak{n} := \gamma_{\mathfrak{s}\mathfrak{s}}, \quad \mu := [\gamma_{\mathfrak{s}\mathfrak{s}}, \gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}}].$$

In the affine coordinate $\mathfrak{s}$, the following relations hold:

$$
\begin{aligned}
[\gamma_{\mathfrak{s}}, \gamma_{\mathfrak{s}\mathfrak{s}}] &= 1, \\
[\gamma_{\mathfrak{s}}, \gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}}] &= 0, \\
[\gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}\mathfrak{s}}, \gamma_{\mathfrak{s}}] &= \mu.
\end{aligned}
\tag{1.0.2}
$$

Moreover, it can be easily verified that $\mathcal{K}_0 = \frac{[\gamma_\theta, \gamma_{\theta\theta}]}{[\gamma, \gamma_\theta]^3} = \frac{[\gamma_{\mathfrak{s}}, \gamma_{\mathfrak{s}\mathfrak{s}}]}{[\gamma, \gamma_{\mathfrak{s}}]^3}$. Since $[\gamma_{\mathfrak{s}}, \gamma_{\mathfrak{s}\mathfrak{s}}] = 1$, we conclude that $\mathcal{K}_0 = \sigma^{-3} = \frac{1}{[\gamma, \gamma_{\mathfrak{s}}]^3}$.

Let K be a smooth convex body. We can write the area and the p-affine perimeter of K in terms of affine invariant quantities:

$$A(K) = \frac{1}{2} \int_{\partial K} \sigma d\mathfrak{s}$$

and

$$\Omega_p(K) = \int_{\partial K} \sigma^{1-\frac{3p}{p+2}} d\mathfrak{s}.$$

Let K be a convex body having the origin in its interior. The dual convex body associated to K with respect to the origin, denoted by K°, is defined by

$$K^\circ = \{y \in \mathbb{R}^2 \mid x \cdot y \leq 1,\ \forall x \in K\}.$$

The area of K°, denoted by $A^\circ = A(K^\circ)$ can also be represented in terms of affine invariant quantities:

$$A^\circ = \frac{1}{2} \int_{\mathbb{S}^1} \frac{1}{s^2} d\theta = \frac{1}{2} \int_{\partial K} \frac{1}{\sigma^2} d\mathfrak{s}.$$

By the Blaschke-Santaló inequality we can bound the area product:

$$A(K) A^\circ(K) \leq \pi^2,$$

with equality obtained only for origin centered ellipses [87].

Lemma 1 (Stability of the centro-affine curvature)**.** *[56] Suppose that K is a convex body in $\mathcal{K}_{sym}$. If $m \leq \mathcal{K}_0 \leq M$ for some positive numbers m and M, then there exist two ellipses $\mathcal{E}_{in}$ and $\mathcal{E}_{out}$ such that $\mathcal{E}_{in} \subseteq K \subseteq \mathcal{E}_{out}$ and*

$$\mathcal{K}_0(\mathcal{E}_{in}) = M, \quad \mathcal{K}_0(\mathcal{E}_{out}) = m.$$

Proof. We present here the argument for the inner ellipse, the case of the outer one being similar.

Recall that

$$\mathcal{K}_0 = \frac{[\dot{\gamma}, \ddot{\gamma}]}{[\gamma, \dot{\gamma}]^3},$$

where $t \mapsto \gamma(t)$ is any counter-clockwise parametrization of the boundary curve. For an ellipse, this is a constant inverse proportional to the square of its area. So, we have to prove that the maximum-area ellipse contained in K has $\mathcal{K}_0 \leq M$. Let $\mathcal{E}_{in}$ be the maximum-area ellipse contained in K. Since the problem is centro-affine invariant, we may assume that $\mathcal{E}_{in}$ is the unit circle. We will prove that $M \geq 1$. The result will then follow by shrinking the circle $\mathcal{E}_{in}$ until its centro-affine curvature is exactly M and re-denoting it, for simplicity, the same way.

Considering the points where ∂K touches $\mathcal{E}_{in}$ one easily sees that, there are at least four intersection points between ∂K and $\mathcal{E}_{in}$, otherwise $\mathcal{E}_{in}$ could be made larger. Thus, at least two of the intervals on the circle corresponding to the polar angle of the intersection points are not greater than $\pi/2$. In fact, due to the symmetry of K, there exist at least two diametrically opposite such intervals. Choose coordinates so that one of the intersection points is $(1, 0)$ and another intersection point is of the form $(\cos\theta, \sin\theta)$ for some $0 < \theta \leq \pi/2$. Observe that the arc of ∂K between these touch points is contained in the square $[0, 1] \times [0, 1]$.

Parameterize ∂K by the spanned area, i.e. by a curve $p \mapsto \gamma(p)$ such that $[\gamma, \dot{\gamma}] = 1$.

Therefore, we have $[\gamma, \ddot{\gamma}] = 0$, hence $\ddot{\gamma}(p) = -\mathcal{K}_0(p)\gamma(p)$, for all p, where $\mathcal{K}_0(p)$ is precisely the centro-affine curvature along the boundary of K. Let $\gamma(p) = (x(p), y(p))$, then $\ddot{x}(p) = -\mathcal{K}_0(p)x(p)$ and $\ddot{y}(p) = -\mathcal{K}_0(p)y(p)$. Suppose that $M = \sup \mathcal{K}_0(p) < 1$. Since $x(0) = 1$, $\dot{x}(0) = 0$, $y(0) = 0$ and $\dot{y}(0) = 1$, a standard comparison theorem for equations of the form $\ddot{x} = -a^2 x$ implies that $x(p) > \cos p$ and $y(p) > \sin p$ for all $p \in (0, \pi/2]$. Therefore, $x(p)^2 + y(p)^2 > 1$ for all $p \in (0, \pi/2]$. This means that γ leaves the square $[0,1] \times [0,1]$ before it has a chance to touch the circle again, contradicting our assumption. □

We recall the following frequently used fact in both convex geometry and analysis of PDEs which is due to Fritz John, 1948.

Theorem 2 (John's Inclusion)**.** *[60] Suppose that K is a convex body in $\mathbb{R}^n$, then there is a unique ellipse $\mathcal{E}_J$ of maximal volume contained in K. Furthermore, if K is origin symmetric, then*

$$\mathcal{E}_J \subseteq K \subseteq \sqrt{n}\mathcal{E}_J.$$

We state the following generalized Hölder inequality from [8].

Theorem 3 (A generalized Hölder inequality)**.** *[8] If M is a compact manifold with a volume form $d\omega$, g is a continues function on M and F is a decreasing real, positive function, then*

$$\frac{\int_M gF(g)d\omega}{\int_M F(g)d\omega} \leq \frac{\int_M g d\omega}{\int_M d\omega}.$$

If F is strictly decreasing, then equality occurs if and only if g is constant.

Proof. The proof follows from Fubini's theorem.

$$\begin{aligned}
&\int_M gF(g)d\omega \int_M d\omega - \int_M F(g)d\omega \int_M gd\omega \\
&\qquad = \int_{M\times M} [g(x)F(g(x)) - F(g(y))g(x)]dxdy \\
&\qquad = \frac{1}{2}\int_{M\times M} (g(x)-g(y))(F(g(x)) - F(g(y)))dxdy \\
&\qquad = \int_{g(x)>g(y)} (g(x)-g(y))(F(g(x)) - F(g(y)))dxdy \le 0.
\end{aligned}$$

□

Let us recall the centro-affine normal flow in $\mathbb{R}^n$.

$$\frac{\partial}{\partial t}x := -\left(\frac{\mathcal{K}}{s^{n+1}}\right)^{\frac{p}{p+2}-\frac{1}{n+1}} \mathcal{K}^{\frac{1}{n+1}}\, z, \quad x(\cdot,0) = x_{K_0}(\cdot), \quad x(\cdot,t) = x_{K_t}(\cdot) \tag{1.0.3}$$

for a fixed $p \ge 1$.

We state the following two propositions in connection to (1.0.3) which were proved in [100].

Proposition 4 (Short-time Existence and Uniqueness)**.** *[100] Let K_0 be a convex body belonging to $\mathcal{K}_{sym}$ and let $p \ge 1$. Then there exists a time $T > 0$ for which equation (1.0.3) has a unique solution starting from K_0.*

Proposition 5 (Containment Principle)**.** *[100] If K_{in} and K_{out} are the two convex bodies in $\mathcal{K}_{sym}$ such that $K_{int} \subset K_{out}$, and $p \ge 1$, then $K_{in}(t) \subseteq K_{out}(t)$ for as long as the solutions $K_{in}(t)$ and $K_{out}(t)$ of (1.0.3) (with given initial data $K_{in}(0) = K_{in}$, $K_{out}(0) = K_{out}$) exist in $\mathcal{K}_{sym}$.*

Chapter 2

Centro-affine normal flows on origin-symmetric convex curves

The contents of this chapter are taken from the paper "Centro-affine curvature flows on origin-symmetric convex curves" [52]. The paper will appear in the journal *Transactions of the American Mathematical Society*.

We consider two types of p centro-affine normal flows on smooth, centrally symmetric, closed convex planar curves, p-contracting and p-expanding. Here p is an arbitrary real number greater than 1. We show that, under p-contracting flows, the evolving curves shrink to a point in finite time and the only homothetic solutions of the flow are ellipses centered at the origin. Furthermore, the normalized curves with enclosed area π converge, in the Hausdorff metric, to the unit circle modulo $SL(2)$. As a p-expanding flow is, in a certain way, dual to a contracting one, we prove that, under p-expanding flows, curves expand to infinity in finite time, while the only homothetic solutions of the flow are ellipses centered at the origin. If the curves are normalized as to enclose constant area π, they display the same asymptotic behavior as the first type flow and converge, in the Hausdorff metric, and up to $SL(2)$ transformations, to the unit circle. At the end, we present a new proof of p-affine isoperimetric inequality, $p \geq 1$, for smooth, origin-symmetric convex bodies in $\mathbb{R}^2$.

2.1 Introduction

Let K be a compact convex body, having the origin in its interior, and smoothly embedded in $\mathbb{R}^2$. The support function of ∂K is defined by

$$s_{\partial K}(z) := \langle x_K(z), z \rangle,$$

for each $z = (\cos\theta, \sin\theta) \in \mathbb{S}^1$. We denote the curvature of ∂K by κ and, furthermore, the radius of curvature of the curve ∂K by $\mathfrak{r}$, viewed now as functions on $[0, 2\pi]$ identified with the unit circle. They are related to the support function by

$$\frac{1}{\kappa}(\theta) = \mathfrak{r}(\theta) := \frac{\partial^2}{\partial\theta^2} s(\theta) + s(\theta),$$

where θ is the angle parameter on $\mathbb{S}^1$ as above.

In this chapter, we study the long time behavior of the flow

$$\frac{\partial}{\partial t} x := -s \left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}} z, \quad x(\cdot, 0) = x_{K_0}(\cdot), \quad x(\cdot, t) = x_{K_t}(\cdot). \tag{2.1.1}$$

starting the flow from a convex body $K_0 \in \mathcal{K}_{sym}$. We will resort to the affine differential setting for a technical step in the study of the normalized evolution equation corresponding to (2.1.1).

Notice that the solution of (2.1.1) remains in $\mathcal{K}_{sym}$, as s and κ are *symmetric* in the sense

$$\forall\theta : \; s(\theta + \pi) = s(\theta), \quad \kappa(\theta + \pi) = \kappa(\theta).$$

Here and thereafter, we identify $z = (\cos\theta, \sin\theta)$ with the normal angle θ itself. We will give a proof of the fact that $K_t \in \mathcal{K}_{sym}$ as long as the flow exists in Lemma 6.

We can rewrite the evolution equation (2.1.1) as a scalar parabolic equation for the support functions on the unit circle:

$$s : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^+$$

$$\frac{\partial}{\partial t} s = -s^{1-\frac{3p}{p+2}} \mathfrak{r}^{-\frac{p}{p+2}}, \ s(\cdot, 0) = s_{\partial K}(\cdot), \ s(\cdot, t) = s_{\partial K_t}(\cdot), \tag{2.1.2}$$

leading, in general, to an anisotropic planar evolution. As in [8], it can be shown that there is a one-to-one correspondence between the solutions of (2.1.1) and those of (2.1.2).

In [100], the following expanding p centro-affine normal flow was defined in connection to (2.1.1)

$$\frac{\partial}{\partial t} x := s \left(\frac{\kappa}{s^3}\right)^{-\frac{p}{p+2}} z, \quad x(\cdot, 0) = x_{K_0^\circ}(\cdot), \quad x(\cdot, t) = x_{K_t^\circ}(\cdot). \tag{2.1.3}$$

It is easy to check as K_t, evolves according to (2.1.1), then K_t° evolves according to (2.1.3). Equivalently, the support function of ∂K_t°, $s_{\partial K_t^\circ}$, evolves according to

$$\frac{\partial}{\partial t} s = s^{1+\frac{3p}{p+2}} \mathfrak{r}^{\frac{p}{p+2}}, \quad p \geq 1 \tag{2.1.4}$$

with initial condition $s(\cdot, 0) = s_{K_0^\circ}(\cdot)$, see Lemma 24.

At a point p of ∂K, the centro-affine curvature mentioned earlier is inversely proportional to the square of the area of the centered osculating ellipse at p. The centro-affine curvature is thus constant along ellipses centered at the origin which are, therefore, evolving homothetically by (2.1.2), respectively (2.1.4). Coupled with the fact that these flows increase the product $A(K) \cdot A(K^\circ)$ which is known to reach the maximum for ellipses centered at the origin (Santaló inequality) and the applications of p-flow stated above, it was natural to investigate the asymptotic behavior of the flows which a priori suggests convergence to ellipses. While this was the first objective of the chapter, in the process we obtained sharp affine isoperimetric type inequalities. The latter is related to the p-affine surface area introduced by Lutwak in [69] which has been the subject of intense research since then, see [67] for a recent, outreaching work which motivates even the present work. Finally, to the best of our knowledge, this is the first study of an anisotropic curvature flow with time-dependent weight. We regard as weight, as well as anisotropic factor, a power of the support function of the evolving body. In this chapter, we prove the following two theorems:

Theorem (Main Theorem A)**.** *Let $p > 1$. Let $x_{K_0} : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth, strictly convex embedding of $K_0 \in \mathcal{K}_{sym}$. Then there exists a unique solution $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ of equation (2.1.1) with initial data x_{K_0}. The solution remains smooth and strictly convex on $[0, T)$ for a finite time T and it converges to the origin of the plane. The rescaled curves given by the embeddings $\sqrt{\frac{\pi}{A_t}}x(\theta, t)$ converge in the Hausdorff metric to the unit circle modulo $SL(2)$.*

Theorem (Main Theorem A′)**.** *Let $p > 1$. Let $x_{K_0} : \mathbb{S}^1 \to \mathbb{R}^2$ be a smooth, strictly convex embedding of $K_0 \in \mathcal{K}_{sym}$. Then there exists a unique solution $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ of equation (2.1.3) with initial data x_{K_0}. The solution remains smooth and strictly convex on $[0, T)$ for a finite time T and it expands in all directions to infinity. The rescaled curves given by the embeddings $\sqrt{\frac{\pi}{A_t}}x(\theta, t)$ converge in the Hausdorff metric to the unit circle modulo $SL(2)$.*

This chapter is structured as follows. The next section focuses on the p contracting centro-affine normal flow. We show that the evolving curves shrink to a point in finite time. To study the convergence of solutions, we resort to the affine differential geometry in the third section. In this section, we will obtain a sharp affine isoperimetric inequality along the flow. In the fourth section, we obtain a crucial result about the constant asymptotic value of the centro-affine curvature of any solution. It is here where we conclude the convergence of solutions to a circle modulo $SL(2)$. In the fifth section, we present the relation between the contracting and the expanding flows. Consequently, we deduce an analogous asymptotic behavior for the p expanding centro-affine normal flow. Finally, in last section, we present a new proof of p-affine isoperimetric inequality, $p \geq 1$, for smooth, origin-symmetric convex bodies in $\mathbb{R}^2$.

2.2 Convergence to a point and homothetic solutions

This section is devoted to the contracting p centro-affine normal flow. In what follows, by evolving curves we mean the curves that enclose the evolving convex bodies in $\mathcal{K}_{sym}$. We start by proving that $\{K_t\}$ remain in $\mathcal{K}_{sym}$ as long as the flow exists.

Lemma 6. *Let $\{K_t\}_t$ be a solution of (2.1.1) where $K_0 \in \mathcal{K}_{sym}$. Then $K_t \in \mathcal{K}_{sym}$ as long as the flow exists.*

Proof. Notice that both $-x(\cdot+\pi, t)$ and $x(\cdot, t)$ satisfy (2.1.1) with initial data $-x(\cdot+\pi, 0)$ and $x(\cdot, 0)$, respectively. At time $t = 0$ we have $-x(\cdot+\pi, 0) = x(\cdot, 0)$. Therefore, by Proposition 4, we conclude that $-x(\cdot+\pi, t) = x(\cdot, t)$ as long as the flow exists. □

The following evolution equations can be derived by a direct computation.

Lemma 7. *Under the flow (2.1.2), one has*

$$\frac{\partial}{\partial t}\mathfrak{r} = -\frac{\partial^2}{\partial \theta^2}\left(s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right) - s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}, \tag{2.2.1}$$

$$\frac{d}{dt}A(t) = -\Omega_p(t), \tag{2.2.2}$$

where $A(t) := A(K_t) = \frac{1}{2}\int_{\mathbb{S}^1} \frac{s}{\kappa}\, d\theta$ *is the area of* K_t*, and* $\Omega_p(t) := \Omega_p(K_t) = \int_{\mathbb{S}^1} \frac{s}{\kappa}\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}} d\theta$ *is the p-affine length of* ∂K_t.

In trying to prove the convergence of the evolving curves to a point, the main obstacle was that, except for the case $p = 1$, we could not find a uniform lower bound on the curvature of evolving curves. However, we could show, with several fruitful consequences, that there exists an entire family of increasing quantities related to the speed of the flow, $s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}$.

Proposition 8. *For* $1 \le q \le \frac{2p}{p+1}$*, or* $q = 0$*, the flow (2.1.2) increases in time*

$$\min_{\theta\in\mathbb{S}^1}\left(s^q\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)(\theta,t).$$

Proof. Using the evolution equations (2.1.2) and (3.2.1), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right) &= \left(\frac{\partial}{\partial t}s^{q-\frac{3p}{p+2}}\right)\mathfrak{r}^{-\frac{p}{p+2}} + s^{q-\frac{3p}{p+2}}\frac{\partial}{\partial t}\mathfrak{r}^{-\frac{p}{p+2}} \\ &= -\left(q-\frac{3p}{p+2}\right)s^{q-\frac{3p}{p+2}-1}\mathfrak{r}^{-\frac{p}{p+2}}\left(s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right) \\ &\quad+\frac{p}{p+2}\mathfrak{r}^{-\frac{p}{p+2}-1}s^{q-\frac{3p}{p+2}}\left[\left(s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta\theta}+s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right] \\ &= \left(\frac{3p}{p+2}-q\right)s^{q-\frac{6p}{p+2}}\mathfrak{r}^{-\frac{2p}{p+2}}+\frac{p}{p+2}s^{q-\frac{6p}{p+2}+1}\mathfrak{r}^{-\frac{2p}{p+2}-1} \\ &\quad+\frac{p}{p+2}\mathfrak{r}^{-\frac{p}{p+2}-1}s^{q-\frac{3p}{p+2}}\left(s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta\theta}.\end{aligned} \tag{2.2.3}$$

To apply the parabolic maximum principle, we need to bound the right-hand side of (2.2.3) from below.

$$\begin{aligned}\left(s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta\theta} &= \left(s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}s^{1-q}\right)_{\theta\theta} \\ &= s^{1-q}\left(s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta\theta}+s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\left(s^{1-q}\right)_{\theta\theta} \\ &\quad+2\left(s^{1-q}\right)_{\theta}\left(s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta}.\end{aligned} \tag{2.2.4}$$

At the point where the minimum of $s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}$ occurs, we have

$$\left(s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta\theta} \geq 0,$$

and

$$\left(s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta} = 0.$$

Therefore, by equation (2.2.4), we obtain that, at that point,

$$\begin{aligned}\left(s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta\theta} &\geq s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\left(s^{1-q}\right)_{\theta\theta}\\ &= s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\left[(1-q)\,s^{-q}s_{\theta\theta}-(1-q)\,q\left(s^{-1-q}\right)s_{\theta}^{2}\right]\\ &\geq s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\left[(1-q)\,s^{-q}s_{\theta\theta}\right] \qquad (2.2.5)\\ &= s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\left[(1-q)\,s^{-q}(\mathfrak{r}-s)\right]\\ &= (1-q)s^{-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}+1}-(1-q)s^{-\frac{3p}{p+2}+1}\mathfrak{r}^{-\frac{p}{p+2}},\end{aligned}$$

where, to pass from the second to the third line, we assumed that either $q = 0$ or $q \geq 1$. Combining (2.2.3), (2.2.4) and (2.2.5), at the point where the minimum of $s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}$ occurs, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(s^{q-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right) \geq &\left(\frac{3p}{p+2}-q\right)s^{q-\frac{6p}{p+2}}\mathfrak{r}^{-\frac{2p}{p+2}}+\frac{p}{p+2}s^{q-\frac{6p}{p+2}+1}\mathfrak{r}^{-\frac{2p}{p+2}-1}\\ &+\frac{p(1-q)}{p+2}s^{q-\frac{6p}{p+2}}\mathfrak{r}^{-\frac{2p}{p+2}}-\frac{p(1-q)}{p+2}s^{q-\frac{6p}{p+2}+1}\mathfrak{r}^{-\frac{2p}{p+2}-1}\\ =&\left(\frac{3p}{p+2}-q+\frac{p(1-q)}{p+2}\right)s^{q-\frac{6p}{p+2}}\mathfrak{r}^{-\frac{2p}{p+2}}+\frac{pq}{p+2}s^{q-\frac{6p}{p+2}+1}\mathfrak{r}^{-\frac{2p}{p+2}-1}.\end{aligned}$$

Since

$$\frac{3p}{p+2}-q+\frac{p(1-q)}{p+2}$$

is non-negative for $q \leq \frac{2p}{p+1}$, the claim follows. $\square$

Consequently, we have:

Corollary 9. *Convexity of the evolving curves is preserved as long as the flow exists.*

Proof. By Proposition 8, setting $q = 0$, we have, as long as the flow exists,

$$\min_{\theta\in\mathbb{S}^1}\left(\frac{\kappa}{s^3}\right)(\theta,t) \geq \min_{\theta\in\mathbb{S}^1}\left(\frac{\kappa}{s^3}\right)(\theta,0).$$

This inequality implies

$$\kappa(\theta,t) \geq s^3(\theta,t)\min_{\theta\in\mathbb{S}^1}\left(\frac{\kappa}{s^3}\right)(\theta,0) > 0,$$

which is precisely the claim of the corollary. □

Following an idea from [106], we consider the evolution of the function $\frac{s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}}{s-\rho}$, for some appropriate ρ, to obtain an upper bound on the speed of the flow as long as the inradius of the evolving curve is uniformly bounded from below.

Lemma 10. *If there exists an $r > 0$ such that $s \geq r$ on $[0,T)$, then κ is uniformly bounded from above on $[0,T)$.*

Proof. Define $\Psi(x,t) := \frac{s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}}{s-\rho}$, where $\rho = \frac{1}{2}r$. For convenience, we set $\alpha := 1 - \frac{3p}{p+2}$ and $\beta := -\frac{p}{p+2}$. At the point where the maximum of Ψ occurs, we have

$$\Psi_\theta = 0, \quad \Psi_{\theta\theta} \leq 0.$$

Hence, we obtain

$$\left(s^\alpha\mathfrak{r}^\beta\right)_{\theta\theta} + s^\alpha\mathfrak{r}^\beta \leq -\frac{\rho s^\alpha\mathfrak{r}^\beta - s^\alpha\mathfrak{r}^{\beta+1}}{s-\rho}. \tag{2.2.6}$$

We calculate

$$\frac{\partial\Psi}{\partial t} = \frac{s^\alpha}{s-\rho}\frac{\partial\mathfrak{r}^\beta}{\partial t} + \frac{\mathfrak{r}^\beta}{s-\rho}\frac{\partial s^\alpha}{\partial t} - \frac{s^\alpha\mathfrak{r}^\beta}{(s-\rho)^2}\frac{\partial s}{\partial t}.$$

Without loss of generality, we can assume that $\frac{\partial}{\partial t}\Psi \geq 0$. Using Lemma 7, and inequality (2.2.6), we infer that, at the point where the maximum of Ψ is reached, we have

$$0 \leq \frac{\partial}{\partial t}\Psi \leq \frac{1}{s-\rho}\left[-\beta s^{\alpha}\mathfrak{r}^{\beta-1}\left(-\frac{\rho s^{\alpha}\mathfrak{r}^{\beta} - s^{\alpha}\mathfrak{r}^{\beta+1}}{s-\rho}\right) + \mathfrak{r}^{\beta}\frac{\partial}{\partial t}s^{\alpha} + \frac{(s^{2\alpha}\mathfrak{r}^{2\beta})}{s-\rho}\right].$$

This last inequality gives

$$\beta\rho\kappa - \beta - \alpha + \alpha\rho\frac{1}{s} + 1 \geq 0.$$

Neglecting the non-positive term $\alpha\rho\frac{1}{s}$, we obtain

$$\beta\rho\kappa - \beta - \alpha + 1 \geq 0.$$

Notice that $\alpha + \beta - 1 = -\frac{4p}{p+2}$, therefore $0 \leq \kappa \leq \frac{4}{\rho}$, consequently, implying the lemma. □

Lemma 11. *Let T be the maximal time of existence of the solution to the flow (2.1.2) with a fixed initial body $K_0 \in \mathcal{K}_{sym}$. Then T is finite and the area of K_t, $A(t)$, tends to zero as t approaches T.*

Proof. Suppose that S_0 is a circle which, at time zero, encloses K_0. It is clear that, by applying the p-flow to S_0, the evolving circles S_t converge to a point in finite time. By Proposition 5, K_t remains in the closure of S_t, therefore T must be finite. Suppose now that $A(t)$ does not tend to zero. Then we must have $s \geq r$, for some $r > 0$ on $[0, T)$. By Corollary 37, and Lemma 10, the curvature of the solution remains bounded on [0,T) from below and above. Consequently the evolution equation (2.1.2) is uniformly parabolic on [0,T), and bounds on higher derivatives of the support function follows by [61] and Schauder theory. Hence, we can extend the solution after time T, contradicting its definition. □

Lemma 12. *Assume $1 \leq l < 2$. Then every solution of the flow (2.1.2) satisfies* $\lim_{t\to T}\Omega_l(t) = 0$.

Proof. From the p-affine isoperimetric inequality in $\mathbb{R}^2$, [69], we have

$$0 \leq \Omega_l^{2+l}(t) \leq 2^{2+l}\pi^{2l}A^{2-l}(t),$$

for every $l \geq 1$. Therefore, the result is a direct consequence of Lemma 38. $\square$

Proposition 13. *Let $L(t)$ be the Euclidean length of ∂K_t as K_t is evolving under (2.1.2). If $p \geq 1$, then* $\lim_{t\to T} L(t) = 0$.

Proof. We first seek an l with the following simultaneous properties:

1. $1 \leq l < 2$,
2. $1 \leq \frac{p}{p+2}\frac{l+2}{l} \leq \frac{2p}{p+1}$.

Notice that, by Lemma 12, the condition (1) implies $\lim_{t\to T}\Omega_l(t) = 0$. The condition (2) implies that

$$\min_{\theta\in\mathbb{S}^1}\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{l}{l+2}}\right)(\theta,t)$$

is increasing. Indeed

$$\left(\min_{\theta\in\mathbb{S}^1} s\left(\frac{\kappa}{s^3}\right)^{\frac{l}{l+2}}(\theta,t)\right)^{\frac{p}{p+2}\frac{l+2}{l}} = \min_{\theta\in\mathbb{S}^1}\left(s^{\frac{p}{p+2}\frac{l+2}{l}}\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)(\theta,t),$$

therefore the claim follows from Proposition 8.

We now proceed to prove the existence of such an l. Solving $\frac{p}{p+2}\frac{l+2}{l} \leq \frac{2p}{p+1}$ implies $l \geq \frac{2p+2}{p+3}$. Let

$$l := \frac{2p+2}{p+3}$$

and notice that it satisfies both conditions (1) and (2).

We further remark that

$$\min_{\theta\in\mathbb{S}^1}\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{l}{2+l}}\right)(\theta,t)\int_{\mathbb{S}^1}\frac{1}{\kappa}\,d\theta \leq \Omega_l(t) = \int_{\mathbb{S}^1}\frac{s}{\kappa}\left(\frac{\kappa}{s^3}\right)^{\frac{l}{2+l}}d\theta. \tag{2.2.7}$$

Thus, by taking the limit as $t \to T$ on both sides of inequality (3.2.4), we obtain

$$\lim_{t\to T} L(t) = \lim_{t\to T} \int_{\mathbb{S}^1} \frac{1}{\kappa} d\theta = 0.$$

$\square$

Proposition 14. *Centered ellipses are the only homothetic solutions to (2.1.2).*

Proof. Denote by $A^\circ(t) := A(K_t^\circ)$ and observe that $A(t)A^\circ(t)$ is scale-invariant. Therefore, for homothetic solutions this area product remains constant along the flow. Moreover, Proposition 2.2 in [100] states, in a larger generality, that, as long as the flow exists, the p centro-affine normal flow does not decrease the area product $A(t)A^\circ(t)$ and it remains constant if and only if the evolving curves are ellipses centered at the origin. The result follows now from the existence of solutions until the extinction time of evolving convex bodies which are origin-symmetric with the center of symmetry placed at the origin.

Alternatively, one can argue that having a homothetic solution to (2.1.2) is equivalent to $\frac{\kappa}{s^3}$ being constant along the boundary of K_t. Then Petty's lemma, [83], shows that the latter is equivalent to K_t being an ellipse centered at the origin. $\square$

2.3 Affine differential setting

In what follows, we work in the affine setting to obtain a sharp affine isoperimetric inequality along the p-flow, Theorem 17.

Let $K_0 \in \mathcal{K}_{sym}$. We consider a family $\{K_t\}_t \in \mathcal{K}_{sym}$, and their associated smooth embeddings $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$, which are evolving according to

$$\frac{\partial}{\partial t} x := \sigma^{1-\frac{3p}{p+2}} \mathfrak{n}, \quad x(\cdot, 0) = x_{K_0}(\cdot), \quad x(\cdot, t) = x_{K_t}(\cdot) \tag{2.3.1}$$

for a fixed $p \geq 1$. Observe that, up to a time-dependant diffeomorphism, the flow defined in (2.3.1) is equivalent to the flow defined by (2.1.1).

Lemma 15. *Let $\gamma(t) := \partial K_t$ be the boundary of a convex body K_t evolving under the flow (2.3.1). Then the following evolution equations hold:*

1. $\frac{\partial}{\partial t}\mathfrak{g} = -\frac{2}{3}\mathfrak{g}\sigma^{1-\frac{3p}{p+2}}\mu + \frac{1}{3}\mathfrak{g}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}},$
2. $\frac{\partial}{\partial t}\mathfrak{t} = \left[-\frac{1}{3}\sigma^{1-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right]\mathfrak{t} + \left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\mathfrak{n},$
3. $\frac{\partial}{\partial t}\sigma = \sigma^{1-\frac{3p}{p+2}}\left[-\frac{4}{3} + \left(\frac{p}{p+2}+1\right)\left(1-\frac{3p}{p+2}\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma} + \frac{p}{p+2}\sigma_{\mathfrak{s}\mathfrak{s}}\right],$
4. $\frac{d}{dt}\Omega_p(t) = \frac{2(p-2)}{p+2}\int_\gamma \sigma^{1-\frac{6p}{p+2}}d\mathfrak{s} + \frac{18p^2}{(p+2)^3}\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}.$

Proof. To prove the lemma, we will use repeatedly equations (1.0.2) without further mention.

$$\frac{\partial}{\partial t}\mathfrak{g}^3 = \frac{\partial}{\partial t}[\gamma_\theta, \gamma_{\theta\theta}] = [\frac{\partial}{\partial t}\gamma_\theta, \gamma_{\theta\theta}] + [\gamma_\theta, \frac{\partial}{\partial t}\gamma_{\theta\theta}].$$

We have

$$\begin{aligned}\left[\frac{\partial}{\partial t}\gamma_\theta, \gamma_{\theta\theta}\right] &= \left[\frac{\partial}{\partial \theta}\left(\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right), \gamma_{\theta\theta}\right] \\ &= \left[\mathfrak{g}\frac{\partial}{\partial \mathfrak{s}}\left(\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right), \gamma_{\theta\theta}\right] \\ &= \mathfrak{g}\left[\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\gamma_{\mathfrak{s}\mathfrak{s}} + \sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}}, \gamma_{\theta\theta}\right].\end{aligned}$$

Since $\frac{\partial^2}{\partial\theta^2} = \mathfrak{g}\mathfrak{g}_{\mathfrak{s}}\frac{\partial}{\partial \mathfrak{s}} + \mathfrak{g}^2\frac{\partial^2}{\partial \mathfrak{s}^2}$, we further have $\gamma_{\theta\theta} = \mathfrak{g}^2\gamma_{\mathfrak{s}\mathfrak{s}} + \mathfrak{g}\mathfrak{g}_{\mathfrak{s}}\gamma_{\mathfrak{s}}$. Thus

$$\begin{aligned}\left[\frac{\partial}{\partial t}\gamma_\theta, \gamma_{\theta\theta}\right] &= \mathfrak{g}\left[\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\gamma_{\mathfrak{s}\mathfrak{s}} + \sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}}, \mathfrak{g}^2\gamma_{\mathfrak{s}\mathfrak{s}} + \mathfrak{g}\mathfrak{g}_{\mathfrak{s}}\gamma_{\mathfrak{s}}\right] \\ &= -\mathfrak{g}^2\mathfrak{g}_{\mathfrak{s}}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}} - \mathfrak{g}^3\sigma^{1-\frac{3p}{p+2}}\mu.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left[\gamma_\theta, \frac{\partial}{\partial t}\gamma_{\theta\theta}\right] &= \left[\mathfrak{g}\gamma_\mathfrak{s}, \frac{\partial^2}{\partial\theta^2}\left(\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right)\right] \\ &= \left[\mathfrak{g}\gamma_\mathfrak{s}, \mathfrak{g}\mathfrak{g}_\mathfrak{s}\frac{\partial}{\partial\mathfrak{s}}\left(\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right) + \mathfrak{g}^2\frac{\partial^2}{\partial\mathfrak{s}^2}\left(\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right)\right] \\ &= \mathfrak{g}^2\mathfrak{g}_\mathfrak{s}\left(\sigma^{1-\frac{3p}{p+2}}\right)_\mathfrak{s} + \mathfrak{g}^3\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}} - \mathfrak{g}^3\sigma^{1-\frac{3p}{p+2}}\mu.\end{aligned}$$

Hence, we conclude that

$$\frac{\partial}{\partial t}\mathfrak{g}^3 = \mathfrak{g}^3\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}} - 2\mathfrak{g}^3\sigma^{1-\frac{3p}{p+2}}\mu,$$

which verifies our first claim.

To prove the second claim, we observe that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial\mathfrak{s}} = \frac{\partial}{\partial\mathfrak{s}}\frac{\partial}{\partial t} - \frac{1}{\mathfrak{g}}\frac{\partial\mathfrak{g}}{\partial t}\frac{\partial}{\partial\mathfrak{s}}. \tag{2.3.2}$$

By (2.3.2), we get

$$\begin{aligned}\frac{\partial}{\partial t}\mathfrak{t} &= \frac{\partial}{\partial t}\frac{\partial}{\partial\mathfrak{s}}\gamma \\ &= \frac{\partial}{\partial\mathfrak{s}}\left(\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right) + \left(\frac{2}{3}\sigma^{1-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right)\mathfrak{t} \\ &= \left(\sigma^{1-\frac{3p}{p+2}}\right)_\mathfrak{s}\mathfrak{n} + \sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}} + \left(\frac{2}{3}\sigma^{1-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right)\mathfrak{t}.\end{aligned}$$

We notice that $\gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}} = -\mu\gamma_\mathfrak{s}$ ending the proof of the second claim.

We now proceed to prove the third claim with

$$\frac{\partial}{\partial t}\sigma = \frac{\partial}{\partial t}[\gamma, \gamma_\mathfrak{s}] = \left[\frac{\partial}{\partial t}\gamma, \gamma_\mathfrak{s}\right] + \left[\gamma, \frac{\partial}{\partial t}\gamma_\mathfrak{s}\right].$$

By the evolution equation (2.3.1), the evolution equation for $\mathfrak{t}$, and the identities

$\sigma = [\gamma, \gamma_{\mathfrak{s}}]$ and $\sigma_{\mathfrak{s}} = [\gamma, \gamma_{\mathfrak{s}\mathfrak{s}}]$, we get that

$$\begin{aligned}
\frac{\partial}{\partial t}\sigma &= \left[\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}, \gamma_{\mathfrak{s}}\right] + \left[\gamma, \left(-\frac{1}{3}\sigma^{1-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right)\gamma_{\mathfrak{s}} + \left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\gamma_{\mathfrak{s}\mathfrak{s}}\right] \\
&= -\sigma^{1-\frac{3p}{p+2}} - \frac{1}{3}\sigma^{2-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\sigma + \left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\sigma_{\mathfrak{s}} \\
&= -\sigma^{1-\frac{3p}{p+2}} - \frac{1}{3}\sigma^{2-\frac{3p}{p+2}}\mu + \left(-\frac{1}{3} + \frac{p}{p+2}\right)\sigma^{1-\frac{3p}{p+2}}\sigma_{\mathfrak{s}\mathfrak{s}} \\
&\quad + \left(1 - \frac{3p}{p+2}\right)\frac{p}{p+2}\sigma^{-\frac{3p}{p+2}}\sigma_{\mathfrak{s}}^2 + \left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\sigma_{\mathfrak{s}}.
\end{aligned}$$

Observe that $\sigma_{\mathfrak{s}\mathfrak{s}} + \sigma\mu = 1$, and apply it to the second and third term of last sum, to obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\sigma &= -\frac{4}{3}\sigma^{1-\frac{3p}{p+2}} + \frac{p}{p+2}\sigma^{1-\frac{3p}{p+2}}\sigma_{\mathfrak{s}\mathfrak{s}} + \frac{p}{p+2}\left(1 - \frac{3p}{p+2}\right)\sigma^{-\frac{3p}{p+2}}\sigma_{\mathfrak{s}}^2 + \left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\sigma_{\mathfrak{s}} \\
&= -\frac{4}{3}\sigma^{1-\frac{3p}{p+2}} + \left(\frac{p}{p+2} + 1\right)\left(1 - \frac{3p}{p+2}\right)\sigma^{-\frac{3p}{p+2}}\sigma_{\mathfrak{s}}^2 + \frac{p}{p+2}\left(\sigma^{1-\frac{3p}{p+2}}\right)\sigma_{\mathfrak{s}\mathfrak{s}},
\end{aligned}$$

as claimed.

For the last claim of the lemma, consider

$$\begin{aligned}
\frac{d}{dt}\Omega_p(t) &= \frac{\partial}{\partial t}\int_\gamma \sigma^{1-\frac{3p}{p+2}}d\mathfrak{s} \\
&= \int_\gamma \frac{\partial}{\partial t}\left(\sigma^{1-\frac{3p}{p+2}}\right)d\mathfrak{s} + \int_\gamma \sigma^{1-\frac{3p}{p+2}}\frac{\partial}{\partial t}d\mathfrak{s}.
\end{aligned}$$

Using the previous part (3) of the lemma, and integration by parts, we obtain

$$\begin{aligned}
&\int_\gamma \frac{\partial}{\partial t}\left(\sigma^{1-\frac{3p}{p+2}}\right)d\mathfrak{s} \\
&= \left(\frac{4p}{p+2} - \frac{4}{3}\right)\int_\gamma \sigma^{1-\frac{6p}{p+2}}d\mathfrak{s} + \left(\frac{p}{p+2} + 1\right)\left(1 - \frac{3p}{p+2}\right)^2\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s} \\
&+ \frac{p}{p+2}\left(1 - \frac{3p}{p+2}\right)\int_\gamma \sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}\mathfrak{s}}d\mathfrak{s} \\
&= \left(\frac{4p}{p+2} - \frac{4}{3}\right)\int_\gamma \sigma^{1-\frac{6p}{p+2}}d\mathfrak{s} + \left(\frac{p}{p+2} + 1\right)\left(1 - \frac{3p}{p+2}\right)^2\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s} \\
&+ \frac{p}{p+2}\left(1 - \frac{3p}{p+2}\right)\left(\frac{6p}{p+2} - 1\right)\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}.
\end{aligned}$$

On the other hand, (1.0.1) gives

$$d\mathfrak{s} = \mathfrak{g} d\theta.$$

Thus, by part (1) we get

$$\frac{\partial}{\partial t} d\mathfrak{s} = \left[-\frac{2}{3}\sigma^{1-\frac{3p}{p+2}}\mu + \frac{1}{3}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}} \right] d\mathfrak{s}.$$

This implies

$$\begin{aligned}
\int_\gamma \sigma^{1-\frac{3p}{p+2}} \frac{\partial}{\partial t} d\mathfrak{s} &= \int_\gamma \sigma^{1-\frac{3p}{p+2}} \left[-\frac{2}{3}\sigma^{1-\frac{3p}{p+2}}\mu + \frac{1}{3}\left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}} \right] d\mathfrak{s} \\
&= -\frac{2}{3}\int_\gamma \sigma^{1-\frac{6p}{p+2}}(1-\sigma_{\mathfrak{s}\mathfrak{s}}) d\mathfrak{s} - \frac{1}{3}\left(1-\frac{3p}{p+2}\right)^2 \int_\gamma \sigma^{-\frac{6p}{p+2}} \sigma_{\mathfrak{s}}^2 d\mathfrak{s} \\
&= \left[\frac{2}{3}\left(\frac{6p}{p+2}-1\right) - \frac{1}{3}\left(1-\frac{3p}{p+2}\right)^2 \right] \int_\gamma \sigma^{-\frac{6p}{p+2}} \sigma_{\mathfrak{s}}^2 d\mathfrak{s} \\
&- \frac{2}{3}\int_\gamma \sigma^{1-\frac{6p}{p+2}} d\mathfrak{s}.
\end{aligned}$$

Setting

$$\begin{aligned}
Q := \left(\frac{p}{p+2}+1\right)\left(1-\frac{3p}{p+2}\right)^2 - \frac{p}{p+2}\left(\frac{3p}{p+2}-1\right)\left(\frac{6p}{p+2}-1\right) \\
+ \frac{2}{3}\left(\frac{6p}{p+2}-1\right) - \frac{1}{3}\left(1-\frac{3p}{p+2}\right)^2 = \frac{18p^2}{(p+2)^3},
\end{aligned}$$

and combining the above equations, we finally acquire that

$$\frac{d}{dt}\Omega_p(t) = \frac{2(p-2)}{p+2}\int_\gamma \sigma^{1-\frac{6p}{p+2}} d\mathfrak{s} + Q\int_\gamma \sigma^{-\frac{6p}{p+2}} \sigma_{\mathfrak{s}}^2 d\mathfrak{s}.$$

□

Now, we proceed to strengthen inequality (2.3.4). Let K and L be two convex bodies with support functions s and h, respectively. Then the mixed volume of K and L is defined by

$$V[s,h] = \int_{\mathbb{S}^1} s\mathfrak{r}[h] d\theta = \int_{\mathbb{S}^1} h\mathfrak{r}[s] d\theta.$$

By Minkowski's mixed volume inequality [92], we have

$$V^2[h,s] \geq V[s,s]V[h,h]. \tag{2.3.3}$$

More interestingly, inequality (2.3.3) still holds if h is an arbitrary function in $C^2(\mathbb{S}^1)$. Indeed, assuming that h is not the support function of some convex body, for a large positive constant c, the sum $h+cs$ is a support function and we obtain, due to the linearity of mixed volumes,

$$0 \leq V^2[h+cs,s] - V[h+cs,h+cs]V[s,s] = V^2[h,s] - V[h,h]V[s,s].$$

The following proposition, stated here only for $n=2$, is proved in [100] for all dimensions. Using our method in this section, we prove a stronger version of the planar inequality in Theorem 17.

Proposition 16. *Let $p \geq 1$, as K_t evolves under (2.1.2). Then we have*

$$\frac{d}{dt}\Omega_p(t) \geq \frac{p-2}{p+2}\frac{\Omega_p^2(t)}{A(t)}, \tag{2.3.4}$$

with equality if and only if K_t is an origin centered ellipse.

Theorem 17. *The following strong affine isoperimetric inequalities hold.*
If $1 \leq p \leq 2$, then

$$\frac{d}{dt}\Omega_p(t) \geq \frac{p-2}{p+2}\frac{\Omega_p^2}{A} + \frac{18(p-1)p^2}{(p+2)^3}\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}, \tag{2.3.5}$$

while, if $p \geq 2$, we then have

$$\frac{d}{dt}\Omega_p(t) \geq \frac{p-2}{p+2}\frac{\Omega_p^2}{A} + \frac{18p^2}{(p+2)^3}\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}. \tag{2.3.6}$$

Proof. To prove the second statement, we notice that Hölder's inequality gives

$$\int_\gamma \sigma^{1-\frac{6p}{p+2}} d\mathfrak{s} = \int_{\mathbb{S}^1} \frac{s}{\kappa}\left(\frac{\kappa}{s^3}\right)^{\frac{2p}{p+2}} d\theta \geq \frac{\Omega_p^2}{2A}.$$

Thus, part (4) of Lemma 15 implies that the affine isoperimetric inequality for $p \geq 2$. We now proceed to prove the first inequality. By Minkowski's mixed volume inequality (2.3.3), we have

$$V\left[s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}, s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right] \leq \frac{V\left[s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}, s\right]^2}{V[s,s]}. \tag{2.3.7}$$

Notice that the right hand side of the inequality is precisely $\frac{\Omega_p^2}{2A}$. Using the identity, see [9],

$$\begin{aligned}\frac{\mathfrak{r}\left[s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right]}{\mathfrak{r}[s]} &= \frac{s}{\kappa^{\frac{1}{3}}}\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\mu + \left(\frac{s}{\kappa^{\frac{1}{3}}}\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}} \\ &= \sigma^{1-\frac{3p}{p+2}}\mu + \left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}},\end{aligned}$$

we can rewrite the left hand side of (2.3.7) as follows:

$$\begin{aligned}V\left[s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}, s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right] &= \int_{\mathbb{S}^1} s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}} \mathfrak{r}\left[s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right] d\theta \\ &= \int_\gamma \frac{s}{\kappa^{\frac{1}{3}}}\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}} \frac{\mathfrak{r}\left[s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right]}{\mathfrak{r}} d\mathfrak{s} \\ &= \int_\gamma \sigma^{1-\frac{3p}{p+2}}\left(\sigma^{1-\frac{3p}{p+2}}\mu + \left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right) d\mathfrak{s} \\ &= \int_\gamma \sigma^{2-\frac{6p}{p+2}}\mu d\mathfrak{s} - \left(1-\frac{3p}{p+2}\right)^2 \int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}. \end{aligned} \tag{2.3.8}$$

Hence, combining equation (2.3.8) and inequality (2.3.7), we conclude that

$$\int_\gamma \sigma^{2-\frac{6p}{p+2}}\mu d\mathfrak{s} \leq \frac{\Omega_p^2}{2A} + \left(1-\frac{3p}{p+2}\right)^2 \int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}. \tag{2.3.9}$$

Inequality (2.3.9) is a special case of the affine-geometric Wirtinger inequality, Lemma 6, [9]. To finish the proof, notice also that

$$\begin{aligned}\frac{d}{dt}\Omega_p(t) &= \frac{2(p-2)}{p+2}\int_\gamma \sigma^{1-\frac{6p}{p+2}}d\mathfrak{s} + \frac{18p^2}{(p+2)^3}\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}\\ &= \frac{2(p-2)}{p+2}\int_\gamma \sigma^{2-\frac{6p}{p+2}}\left(\frac{1}{\sigma}-\frac{\sigma_{\mathfrak{s}\mathfrak{s}}}{\sigma}\right)d\mathfrak{s} + \frac{2(p-2)}{p+2}\int_\gamma \sigma^{2-\frac{6p}{p+2}}\frac{\sigma_{\mathfrak{s}\mathfrak{s}}}{\sigma}d\mathfrak{s}\\ &+ \frac{18p^2}{(p+2)^3}\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}\\ &= \left[\frac{2(p-2)}{p+2}\left(\frac{6p}{p+2}-1\right)+\frac{18p^2}{(p+2)^3}\right]\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}\\ &+ \frac{2(p-2)}{p+2}\int_\gamma \sigma^{2-\frac{6p}{p+2}}\mu d\mathfrak{s},\end{aligned}$$

which, by inequality (2.3.9), implies

$$\begin{aligned}&\frac{d}{dt}\Omega_p(t)\\ &\geq \left[\frac{2(p-2)}{p+2}\left(1-\frac{3p}{p+2}\right)^2+\frac{2(p-2)}{p+2}\left(\frac{6p}{p+2}-1\right)+\frac{18p^2}{(p+2)^3}\right]\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}\\ &+\frac{p-2}{p+2}\frac{\Omega_p^2}{A}\\ &= \frac{p-2}{p+2}\frac{\Omega_p^2}{A}+\frac{18(p-1)p^2}{(p+2)^3}\int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}.\end{aligned}$$

□

Lemma 18. *The p-affine isoperimetric ratio,* $\frac{\Omega_p^{2+p}(t)}{A^{2-p}(t)}$*, is non-decreasing along the flow (2.1.2) and remains constant if and only if* K_t *is an origin centered ellipse.*

Proof.

$$\begin{aligned}\frac{d}{dt}\frac{\Omega_p^{2+p}(t)}{A^{2-p}(t)} &= \frac{(2+p)\Omega_p^{p+1}(t)A^{2-p}(t)\frac{d}{dt}\Omega_p(t)+(2-p)A^{1-p}(t)\Omega_p^{3+p}(t)}{A^{2(2-p)}(t)} \qquad (2.3.10)\\ &= \frac{\Omega_p^{1+p}(t)}{A^{2-p}(t)}\left((2+p)\frac{d}{dt}\Omega_p(t)-(p-2)\frac{\Omega_p^2(t)}{A(t)}\right)\geq 0,\end{aligned}$$

where we used inequality (2.3.4) on the last line. □

Corollary 19. *If K_t evolves by (2.1.2) with extinction time T, the following limit holds:*

$$\liminf_{t\to T} \frac{\Omega_p^p(t)}{A^{1-p}(t)}\left[(2+p)\frac{d}{dt}\Omega_p(t)-(p-2)\frac{\Omega_p^2(t)}{A(t)}\right]=0. \tag{2.3.11}$$

Proof. By equations (2.2.2) and (2.3.10),

$$\frac{d}{dt}\frac{\Omega_p^{2+p}(t)}{A^{2-p}(t)}=-\frac{d}{dt}\ln(A(t))\left(\frac{\Omega_p^p(t)}{A^{1-p}(t)}\left[(2+p)\frac{d}{dt}\Omega_p(t)-(p-2)\frac{\Omega_p^2(t)}{A(t)}\right]\right).$$

If

$$\frac{\Omega_p^p(t)}{A^{1-p}(t)}\left[(2+p)\frac{d}{dt}\Omega_p(t)-(p-2)\frac{\Omega_p^2(t)}{A(t)}\right]\geq\varepsilon$$

in a neighborhood of T, then

$$\frac{d}{dt}\frac{\Omega_p^{2+p}(t)}{A^{2-p}(t)}\geq-\varepsilon\frac{d}{dt}\ln(A(t)).$$

Thus

$$\frac{\Omega_p^{2+p}}{A^{2-p}}(t)\geq\frac{\Omega_p^{2+p}}{A^{2-p}}(t_1)+\varepsilon\ln(A(t_1))-\varepsilon\ln(A(t)),$$

and the right hand side goes to infinity as $A(t)$ goes to zero. This contradicts that the left hand side is bounded from above by the p-affine isoperimetric inequality. □

2.4 Normalized flow

In this section, we study the normalized flows corresponding to the evolution described by (2.1.2). We consider the conventional rescaling such that the area enclosed by the normalized curves is π by taking

$$\tilde{s}_t:=\sqrt{\frac{\pi}{A(t)}}\,s_t,\quad \tilde{\kappa}_t:=\sqrt{\frac{A(t)}{\pi}}\,\kappa_t.$$

One can also define a new time parameter

$$\tau=\int_0^t\left(\frac{\pi}{A(K_t)(\xi)}\right)^{\frac{2p}{p+2}}d\xi$$

and can easily verify that

$$\frac{\partial}{\partial \tau}\,\tilde{s} = -\tilde{s}\left(\frac{\tilde{\kappa}}{\tilde{s}^3}\right)^{\frac{p}{p+2}} + \frac{\tilde{s}}{2\pi}\,\tilde{\Omega}_p, \tag{2.4.1}$$

where $\tilde{\Omega}_p$ stands for the p-affine length of $\partial\tilde{K}_t$ having support function $\tilde{s}_t$. More precisely,

$$\tilde{\Omega}_p(\tau) := \Omega_p(\tilde{K}_\tau) = \int_{\mathbb{S}^1} \frac{\tilde{s}}{\tilde{\kappa}}\left(\frac{\tilde{\kappa}}{\tilde{s}^3}\right)^{\frac{p}{p+2}} d\theta.$$

However, even in the normalized case, we prefer to work on the finite time interval $[0, T)$.

Corollary 20. *Let p be a real number, $p > 1$, and let $\{t_k\}_k$ be the sequence of times realizing the limit (2.3.11) in Corollary 19. Then along the normalized contracting p-flow, we have*

$$\lim_{t_k \to T} \tilde{\sigma}(t_k) = 1.$$

Proof. Since

$$\frac{1}{\left(\frac{3p}{p+2}-1\right)^2}\int_\gamma \left(\sigma^{1-\frac{3p}{p+2}}\right)^2_{\mathfrak{s}} d\mathfrak{s} = \int_\gamma \sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s},$$

by Theorem 17 and Corollary 19, we have

$$0 = \lim_{t_k \to T}\frac{\Omega_p^p}{A^{1-p}}\left[\frac{d}{dt}\Omega_p(t) - \frac{p-2}{p+2}\frac{\Omega_p^2}{A}\right] \geq \lim_{t_k\to T}\frac{\Omega_p^p}{A^{1-p}}\left(\phi(p)\int_\gamma\left(\sigma^{1-\frac{3p}{p+2}}\right)^2_{\mathfrak{s}} d\mathfrak{s}\right) \geq 0,$$

where $\phi(p) := \begin{cases} \frac{9p^2}{2(p+2)(p-1)}, & \text{if } 1 < p \leq 2 \\ \frac{9p^2}{2(p+2)(p-1)^2}, & \text{if } p \geq 2. \end{cases}$

As, by Lemma 18, the p-affine length $\tilde{\Omega}_p$ is increasing along the normalized flow, we conclude that, for each $p > 1$,

$$\lim_{t_k \to T}\int_{\tilde{\gamma}}\left(\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)^2_{\tilde{\mathfrak{s}}} d\tilde{\mathfrak{s}} = 0.$$

We notice that for any $\theta_1, \theta_2 \in \mathbb{S}^1$

$$\left|\int_{\theta_1}^{\theta_2} \left(\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)_\theta d\theta\right| \leq$$

$$\int_{\mathbb{S}^1} \left|\left(\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)_\theta\right| d\theta = \int_{\tilde{\gamma}} \left|\left(\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)_{\tilde{\mathfrak{s}}}\right| d\tilde{\mathfrak{s}} \leq \left(\int_{\tilde{\gamma}} \left(\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}}\right)^{1/2} \tilde{\Omega}_1^{1/2}.$$

Take θ_1 and θ_2 be two points where $\tilde{\sigma}$ reaches its extremal values. It is known that, for a smooth, simple curve with enclosed area π, $\min_{\mathbb{S}^1} \sigma \leq 1$ and $\max_{\mathbb{S}^1} \sigma \geq 1$, see Lemma 10 in [9]. Hence, as $\tilde{\Omega}_1$ is bounded from above by the classical affine isoperimetric inequality [69], we infer that $\lim_{t_k \to T} \tilde{\sigma}(t_k) = 1$. □

Theorem 21. *Suppose that $\tilde{s}_t$ is a solution of the normalized flow (2.4.1) for some initial convex body in $\mathcal{K}_{sym}$ and that $\{t_k\}_k$ is the sequence of times realizing the limit (2.3.11) in Corollary 19. Then there exist two families of centered ellipses $\{\mathcal{E}_{in}(t_k)\}$, $\{\mathcal{E}_{out}(t_k)\}$ such that*

$$\mathcal{E}_{in}(t_k) \subseteq \tilde{K}_{t_k} \subseteq \mathcal{E}_{out}(t_k). \tag{2.4.2}$$

Furthermore, the sequence $\{\partial\tilde{K}_{t_k}\}$ converges, in the Hausdorff metric, to the unit circle modulo $SL(2)$.

Proof. By Corollary 20, we have

$$\lim_{t_k \to T} \left(\frac{\tilde{\kappa}}{\tilde{s}^3}\right)(\theta, t_k) = 1. \tag{2.4.3}$$

Thus, the first half of the claim follows from Lemma 1.

Now we proceed to prove the second half of the claim. Evidently we can find an appropriate family of special linear transformations $\{L_{t_k}\}_{t_k}$ such that $L_{t_k}(\mathcal{E}_{out}(t_k))$ is a circle at each time t_k. Each such area preserving linear transformation L_{t_k} minimizes the Euclidean length of the ellipse $\mathcal{E}_{out}(t_k)$ at time t_k.

Thus, the constructions of $\mathcal{E}_{out}(t_k)$ and $\mathcal{E}_{in}(t_k)$ imply

$$\lim_{t_k \to T} L_{t_k}(\mathcal{E}_{out}(t_k)) = \lim_{t_k \to T} L_{t_k}(\mathcal{E}_{in}(t_k)) = \mathbb{S}^1$$

in the Hausdorff metric:

Recall, from Lemma 1, that

$$\min_{\theta \in \mathbb{S}^1} \left(\frac{\tilde{\kappa}}{\tilde{s}^3} \right) (\theta, t_k) = \frac{\kappa}{s^3}(\mathcal{E}_{out}(t_k)).$$

Since $\frac{\kappa}{s^3}$ is invariant under $SL(2)$, we have $\frac{\kappa}{s^3}(\mathcal{E}_{out}(t_k)) = \frac{\kappa}{s^3}(L_{t_k}(\mathcal{E}_{out}(t_k)))$, therefore $\lim\limits_{t_k \to T} \frac{\kappa}{s^3}(L_{t_k}(\mathcal{E}_{out}(t_k))) = 1$. This implies $\lim_{t_k \to T} L_{t_k}(\mathcal{E}_{out}(t_k)) = \mathbb{S}^1$ in the Hausdorff metric. Similarly, from the choice of $\mathcal{E}_{in}(t_k)$ in Lemma 1, we have

$$\max_{\theta \in \mathbb{S}^1} \left(\frac{\tilde{\kappa}}{\tilde{s}^3} \right) (\theta, t_k) = \frac{\kappa}{s^3}(\mathcal{E}_{in}(t_k)),$$

therefore $\lim\limits_{t_k \to T} \frac{\kappa}{s^3}(L_{t_k}(\mathcal{E}_{in}(t_k))) = 1$. This implies $\lim_{t_k \to T} A(L_{t_k}(\mathcal{E}_{in}(t_k))) = \pi$. As $L_{t_k}(\mathcal{E}_{in}(t_k)) \subseteq L_{t_k}(\mathcal{E}_{out}(t_k))$, we conclude that $\lim_{t_k \to T} L_{t_k}(\mathcal{E}_{in}(t_k)) = \mathbb{S}^1$ in the Hausdorff metric.

Now, applying $\{L_{t_k}\}_{t_k}$ to the inclusions (2.4.2), we obtain that the sequence $\{L_{t_k}(\tilde{K}_{t_k})\}_k$ converges to the unit disk in the Hausdorff metric. □

Corollary 22. *Along the flow (2.4.1) with an arbitrary initial condition in $\mathcal{K}_{sym}$, we have*

$$\lim_{t \to T} A(K_t)A(K_t^\circ) = \pi^2$$

for $p > 1$.

Proof. Recall that the area product $A(K_t)A(K_t^\circ)$ is invariant under the general linear group, $GL(2)$, and increasing along each p-flow, unless the boundaries of the evolving convex bodies are centered ellipses. Moreover, as the convex bodies are origin-symmetric with the center of symmetry at the origin, the Santaló inequality gives $A(t)A^{\circ}(t) \leq \pi^2$ with equality if and only if the boundary curves are ellipses centered at the origin, [87]. Consequently, Theorem 21 implies the claim. $\square$

Now Theorem 2 immediately implies that, if K is an origin symmetric convex body such that whose volumes is ω_n, volume of the unit ball in $\mathbb{R}^n$, then there is a linear transformation L such that $r^+(LK) \leq \sqrt{n}$ and $r_-(LK) \geq \frac{1}{\sqrt{n}}$, where $r^+(LK)$ and $r_-(LK)$ is the inradius and circumradius of LK respectively. Now, we are ready to prove one of the main theorems:

Theorem 23. *Let $p > 1$. Suppose $\tilde{K}_t$ is a solution of the normalized flow (2.4.1) for some initial convex body in $\mathcal{K}_{sym}$. Then there exists a family of special linear transformations $\{L_t\}_{t\in[0,T)} \subset SL(2)$ such that the sequence $\{L_t(\partial\tilde{K}_t)\}_t$ converges to $\mathbb{S}^1$ in the Hausdorff metric.*

Proof. At each time t, we apply a special linear transformation L_t such that the Euclidean length of $\partial\tilde{K}_t$ is minimized. Let $\{t_i\}_i$ be a sequence of times converging to T. John's Inclusion or Proposition 8 of [9] implies the compactness of the set of convex bodies $L_{t_i}(\tilde{K}_{t_i})$. By Corollary 22 and Blaschke Selection Theorem, each subsequence of $\{L_{t_i}(\partial\tilde{K}_{t_i})\}$ has a subsequence $\{L_{t_{i_j}}(\partial\tilde{K}_{t_{i_j}})\}$ such that the sequence $\{L_{t_{i_j}}(\partial\tilde{K}_{t_{i_j}})\}$ converges, in the Hausdorff metric, to an ellipse of enclosed area π. Thus, the length minimization condition rules out the degeneracy of the limit ellipse and, in fact, it implies that the sequence $\{L_{t_{i_j}}(\partial\tilde{K}_{t_{i_j}})\}$ converges to the unit circle in the Hausdorff topology. $\square$

2.5 Expanding p-flow

Lemma 24. *As K_t evolves by the centro-affine normal flow (2.1.2), its dual K_t° evolves, up to a diffeomorphism , under the flow*

$$\frac{\partial}{\partial t}s = s\left(\frac{\kappa}{s^3}\right)^{-\frac{p}{p+2}}, \; s(\cdot,t) = s_{\partial K_t^\circ}(\cdot), \; s(\cdot,0) = s_{\partial K_0^\circ}(.). \tag{2.5.1}$$

Proof. The proof of Lemma 24 is given in [100], but, for completeness, we'll present it here. Recall that $A(K^\circ) = \frac{1}{2}\int_{\mathbb{S}^1}\frac{1}{s^2}d\theta$ and that, under sufficient regularity assumptions on ∂K which are satisfied here, $\Omega_q(K) = \Omega_{\frac{4}{q}}(K^\circ)$ for each $q \neq -n$, in which case the q-affine length is not defined. Therefore, as K_t evolves by the centro-affine normal flow (2.1.2), the volume of the dual body K_t° changes by

$$\frac{d}{dt}A^\circ(t) = \Omega^\circ_{-\frac{p}{p+1}}(t),$$

where the notation stands for $\Omega_{-\frac{p}{p+1}}(K_t^\circ)$. Compared with the rate of change of the area of a convex body L whose boundary is deformed by a normal vector field with speed v, which is $\frac{d}{dt}A(L) = \int_{\mathbb{S}^1} v\frac{1}{\kappa_L}d\theta$, we infer that while K_t evolves, up to a time-dependant diffeomorphism, by (2.1.2), its dual K_t° evolves, up to a time-dependant diffeomorphism, by (2.5.1).

□

Similar to Propositions 4 and 5 of [100], we have

Proposition 25. *Let K_0 be a convex body belonging to $\mathcal{K}_{sym}$ and let $p \geq 1$. Then there exists a time $T > 0$ for which equation (2.5.1) has a unique solution starting from K_0.*

Proposition 26 (Containment Principle)**.** *If K_{in} and K_{out} are the two convex bodies in $\mathcal{K}_{sym}$ such that $K_{int} \subset K_{out}$, and $p \geq 1$, then $K_{in}(t) \subseteq K_{out}(t)$ for as long as the* *solutions $K_{in}(t)$ and $K_{out}(t)$ (with given initial data $K_{in}(0) = K_{in}$, $K_{out}(0) = K_{out}$) of (2.5.1) exist in $\mathcal{K}_{sym}$.*

Similar to Lemma 6 we have

Lemma 27. *Let $\{K_t\}_t$ be a solution of (2.5.1) where $K_0 \in \mathcal{K}_{sym}$. Then $K_t \in \mathcal{K}_{sym}$ as long as the flow exists.*

Combining Proposition 13, Lemma 24, Proportions 25 and 26 we obtain:

Proposition 28. *Suppose K_t is a family of convex bodies such that it evolves under the flow*

$$\frac{\partial}{\partial t}s(\cdot,t) = s\left(\frac{\kappa}{s^3}\right)^{-\frac{p}{p+2}}(\cdot,t)$$

with $p \geq 1$. Then

$$\forall\,\theta: \quad \lim_{t\to T} s(\theta,t) = \infty.$$

Proposition 29. *Ellipses centered at the origin are the only homothetic solutions to (2.5.1).*

Proof. The proof follows from the duality between the two flows and Proposition 14. □

Furthermore, we obtain:

Theorem 30. *Let $p > 1$. Suppose $\tilde{K}_t$ is a solution of the normalized flow derived from (2.5.1) for some initial convex body in $\mathcal{K}_{sym}$. Then there exists a family of linear transformations $\{L_t\}_{t\in[0,T)} \subset SL(2)$ such that the convex bodies $L_t(\partial\tilde{K}_t)$ converge to $\mathbb{S}^1$ in the Hausdorff metric.*

Proof. Let $\{L_t\}_{t\in[0,T)}$ be the family of length minimizing special linear transformations that we defined in the proof of Theorem 23. Since $(L_t(\tilde{K}_t))^\circ = L_t^{-t}(\tilde{K}_t^\circ)$, where L_t^{-t} is the inverse transpose of L_t, the claim follows. □

2.6 A proof of the p-affine isoperimetric inequality

In this section, we provide a new proof of the p-affine isoperimetric inequality, $p \geq 1$, for a convex body $K \in \mathcal{K}_{sym}$. Since our proofs of Theorems 23, and 30 are dependant on the p-affine isoperimetric inequality, we cannot apply our results on the p- centro-affine normal flows to obtain the p-affine isoperimetric inequality. Instead, we employ the affine normal flow to reach our goal, see [10].

We state the following general evolution equation for Ω_l under the contracting p-flow for each $l \in \mathbb{R}$:

$$\frac{d}{dt}\Omega_l(t) = \frac{2(l-2)}{l+2}\int_\gamma \sigma^{1-\frac{3p}{p+2}-\frac{3l}{l+2}}d\mathfrak{s} + \frac{18pl}{(l+2)^2(p+2)}\int_\gamma \sigma^{-\frac{3p}{p+2}-\frac{3l}{l+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}. \tag{2.6.1}$$

The proof of this equation is similar to the one of part four of Lemma 15.

Lemma 31. *The following sharp affine isoperimetric inequalities hold along the affine normal flow.*

If $1 \leq l \leq 2$, *then*

$$\frac{d}{dt}\Omega_l(t) \geq \frac{l-2}{l+2}\frac{\Omega_l\Omega_1}{A} + \frac{2(l-1)(4l^2+3l+2)}{(l+2)^3}\int_\gamma \sigma^{-1-\frac{3l}{l+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s},$$

while, if $l \geq 2$, *we then have*

$$\frac{d}{dt}\Omega_l(t) \geq \frac{l-2}{l+2}\frac{\Omega_l\Omega_1}{A} + \frac{6l}{(l+2)^2}\int_\gamma \sigma^{-1-\frac{3l}{l+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s},$$

Proof. Define $d\omega = \sigma d\mathfrak{s}$, $g = \sigma$ and $F(x) := x^{-\frac{3l}{l+2}}$ in Theorem 3. Furthermore, observe that, for a convex body K in $\mathbb{R}^2$, we have $2A = \int_{\partial K} \sigma d\mathfrak{s}$. This implies

$$\int_{\partial K} \sigma^{-\frac{3l}{l+2}} d\mathfrak{s} \geq \frac{\Omega_l \Omega_1}{2A},$$

hence the second claim follows by this last inequality and the evolution equation (2.6.1) for $p = 1$. To prove the first inequality, one can proceed similarly as in the proof of inequality (2.3.5), and use the affine-geometric Wirtinger inequality developed by Andrews, Lemma 6, [9].

□

Lemma 32. *Let $l \geq 1$, then the l-affine isoperimetric ratio, $\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}$ is non-decreasing along the affine normal flow and remains constant if and only if K_t is an origin centered ellipse.*

Proof.

$$\begin{aligned} \frac{d}{dt}\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)} &= \frac{(2+l)\Omega_l^{l+1}(t)A^{2-l}(t)\frac{d}{dt}\Omega_l + (2-l)A^{1-l}(t)\Omega_l^{2+l}(t)\Omega_p(t)}{A^{2(2-l)}(t)} \\ &= \frac{\Omega_l^{l+1}(t)}{A^{2-l}(t)}\left((2+l)\frac{d}{dt}\Omega_l - (l-2)\frac{\Omega_l(t)\Omega_1(t)}{A(t)}\right) \geq 0, \end{aligned}$$

where we used Lemma 31 on the last line. □

Theorem 33. *Let $l \geq 1$, then the following l-affine isoperimetric inequality holds for a convex body $K \in \mathcal{K}_{sym}$.*

$$\frac{\Omega_l^{2+l}(K)}{A^{2-l}(K)} \leq 2^{l+2}\pi^{2l},$$

Moreover, equality holds only for centered ellipses at origin.

Proof. The claim is an immediate consequence of the weak convergence of the solutions of the normalized affine normal flow to a centered ellipse and Lemma 32. □

Chapter 3

A flow approach to the L_{-2} Minkowski problem

The contents of this chapter are taken from the paper "A flow approach to the L_{-2} Minkowski problem"[54]. The paper is published in the journal *Advances in Applied Mathematics 50 (2013), pp. 445-464.*

We prove that the set of smooth, π-periodic, positive functions on the unit sphere for which the planar L_{-2} Minkowski problem is solvable is dense in the set of all smooth, π-periodic, positive functions on the unit sphere with respect to the L^{∞} norm. Furthermore, we obtain a necessary condition on the solvability of the even L_{-2} Minkowski problem. At the end, we prove uniqueness of the solutions up to an affine linear transformation.

3.1 Introduction

In this chapter, we address the smooth even case of the L_{-2} Minkowski problem. The main result obtained states that, although the L_{-2} Minkowski problem is not always solvable, we can always find functions that approximately solve the problem with any desired accuracy. We prove that:

Theorem 34 (Main Theorem B)**.** *Given an even, smooth function* $\Phi : \mathbb{S}^1 \to \mathbb{R}^+$*, there exists a family of smooth convex bodies* $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{K}_{sym}$*, such that*

$$\lim_{n\to\infty} \sup_{\mathbb{S}^1} \left|\left| \frac{s}{\kappa^{1/3}} - \Phi \right|\right| = 0.$$

Furthermore, if Φ *is* $\frac{\pi}{k}$ *periodic for* $k \geq 2$*, this family of convex bodies is uniformly bounded and it converges in the* C^∞ *norm to a smooth convex body whose support function satisfies* $s(s_{\theta\theta} + s)^{1/3} = \Phi$*.*

To prove our result, we will exploit a weighted centro-affine normal flow: Let $K_0 \in \mathcal{K}_{sym}$. We consider a family $\{K_t\}_t \in \mathcal{K}_{sym}$, and their associated smooth embeddings $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$, which are evolving according to the p-weighted centro affine flow namely,

$$\frac{\partial}{\partial t} x := -\Psi(z) s \left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}} z, \quad x(\cdot, 0) = x_{K_0}(\cdot), \quad x(\cdot, t) = x_{K_t}(\cdot) \tag{3.1.1}$$

for a fixed $p \in (1, 2)$. Here $\Psi : \mathbb{S}^1 \to \mathbb{R}^+$ is a smooth, even function. Short time existence for the flow follows from the theory of parabolic partial differential equations.

3.2 Convergence to a point

In this section, we prove that every solution of (3.1.1) starting from a smooth, symmetric convex body converges to a point in a finite time.

The following evolution equations can be derived by a direct computation.

Lemma 35. *Under the flow (3.1.1), one has*

$$\frac{\partial}{\partial t} \mathfrak{r} = -\frac{\partial^2}{\partial \theta^2} \left(\Psi s^{1-\frac{3p}{p+2}} \mathfrak{r}^{-\frac{p}{p+2}} \right) - \Psi s^{1-\frac{3p}{p+2}} \mathfrak{r}^{-\frac{p}{p+2}}, \tag{3.2.1}$$

and

$$\frac{d}{dt}A = -\int_{\mathbb{S}^1} \Psi \frac{s}{\kappa}\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}} d\theta. \tag{3.2.2}$$

Proposition 36. *The flow (3.1.1) increases in time the quantity*

$$\min_{\theta\in\mathbb{S}^1}\left(\Psi s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)(\theta,t).$$

Proof. Using the evolution equations (3.1.1) and (3.2.1), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(\Psi s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right) &= \Psi\left[\left(\frac{\partial}{\partial t}s^{1-\frac{3p}{p+2}}\right)\mathfrak{r}^{-\frac{p}{p+2}} + s^{1-\frac{3p}{p+2}}\frac{\partial}{\partial t}\mathfrak{r}^{-\frac{p}{p+2}}\right]\\ &= -\left(1-\frac{3p}{p+2}\right)\Psi s^{-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\left(\Psi s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)\\ &+\frac{p}{p+2}\Psi\mathfrak{r}^{-\frac{p}{p+2}-1}s^{1-\frac{3p}{p+2}}\left[\left(\Psi s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta\theta}+\Psi s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right]\\ &= \left(\frac{3p}{p+2}-1\right)\Psi^2 s^{1-\frac{6p}{p+2}}\mathfrak{r}^{-\frac{2p}{p+2}} + \frac{p}{p+2}\Psi^2 s^{2-\frac{6p}{p+2}}\mathfrak{r}^{-\frac{2p}{p+2}-1}\\ &+\frac{p}{p+2}\Psi\mathfrak{r}^{-\frac{p}{p+2}-1}s^{1-\frac{3p}{p+2}}\left(\Psi s^{1-\frac{3p}{p+2}}\mathfrak{r}^{-\frac{p}{p+2}}\right)_{\theta\theta}.\end{aligned} \tag{3.2.3}$$

Applying the parabolic maximum principle proves the claim. □

Consequently, we have:

Corollary 37. *The convexity of the evolving curves is preserved as long as the flow exists.*

Proof. By Proposition 36, we have, as long as the flow exists, that

$$\min_{\theta\in\mathbb{S}^1}\Psi s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}(\theta,t) \geq \min_{\theta\in\mathbb{S}^1}\Psi s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}(\theta,0).$$

From this, we conclude that κ remain strictly positive. □

Lemma 38. *For every solution to the flow (3.1.1), the area of $K(t)$, $A(t)$, converge to zero in a finite time T'.*

Proof. By (3.2.2), we have

$$\frac{d}{dt}A = -\int_{\mathbb{S}^1} \Psi \frac{s}{\kappa}\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}} d\theta \leq -\delta \int_{\mathbb{S}^1} \frac{1}{\kappa} = -\delta L,$$

where we used Proposition 36, and $\delta := \min_{\mathbb{S}^1} \Psi s \left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}} (\theta, 0)$. On the other hand, by the isoperimetric inequality, we have $L \geq \sqrt{4\pi A}$. Therefore, we obtain that

$$\frac{d}{dt}A \leq -\delta\sqrt{4\pi A}.$$

This last inequality implies

$$\frac{d}{dt}\sqrt{A} \leq -\frac{\delta\sqrt{4\pi}}{2}$$

from which we conclude the statement of the lemma. □

Lemma 39. *Every solution of the flow (3.1.1) satisfies* $\lim_{t\to T'} \Omega_p(t) = 0$.

Proof. From the p-affine isoperimetric inequality in $\mathbb{R}^2$, [69], we have

$$0 \leq \Omega_p^{2+p}(t) \leq 2^{2+p}\pi^{2p}A^{2-p}(t),$$

for each $p \geq 1$.

Therefore, the result is a direct consequence of Lemma 38. We recall that we consider the flow (3.1.1) for $1 < p < 2$. □

Proposition 40. *Let* $L(t)$ *be the length of* ∂K_t *as* K_t *evolves under (3.1.1). Then* $\lim_{t\to T'} L(t) = 0$.

Proof. We observe that

$$\min_{\theta\in\mathbb{S}^1}\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{2+p}}\right)(\theta, t)\int_{\mathbb{S}^1} \frac{1}{\kappa}\, d\theta \leq \Omega_p(t) = \int_{\mathbb{S}^1} \frac{s}{\kappa}\left(\frac{\kappa}{s^3}\right)^{\frac{p}{2+p}} d\theta. \tag{3.2.4}$$

Thus, by taking the limit as $t \to T'$ on both sides of inequality (3.2.4), and considering Proposition 36, we obtain

$$\lim_{t \to T'} L(t) = \lim_{t \to T'} \int_{\mathbb{S}^1} \frac{1}{\kappa} d\theta = 0.$$

$\square$

Following an idea from [106], we consider the evolution of a test function to obtain an upper bound on the speed of the flow as long as the inradius of the evolving curve is uniformly bounded from below.

Lemma 41. *If there exists an $r > 0$ such that $s \geq r$ on $[0, T)$, then κ is uniformly bounded from above on $[0, T)$.*

Proof. Define $Y(x,t) := \frac{\Psi s^{1-\frac{3p}{p+2}} \mathfrak{r}^{-\frac{p}{p+2}}}{s-\rho}$, where $\rho = \frac{1}{2} r$. For convenience, we set $\alpha := 1 - \frac{3p}{p+2}$ and $\beta := -\frac{p}{p+2}$. At the point where the maximum of Y occurs, we have

$$Y_\theta = 0, \quad Y_{\theta\theta} \leq 0.$$

Thus, we obtain

$$\left(\Psi s^\alpha \mathfrak{r}^\beta\right)_{\theta\theta} + \Psi s^\alpha \mathfrak{r}^\beta \leq -\Psi \left(\frac{\rho s^\alpha \mathfrak{r}^\beta - s^\alpha \mathfrak{r}^{\beta+1}}{s - \rho} \right). \tag{3.2.5}$$

Calculating

$$\frac{\partial}{\partial t} Y = \Psi \left(\frac{s^\alpha}{s-\rho} \frac{\partial \mathfrak{r}^\beta}{\partial t} + \frac{\mathfrak{r}^\beta}{s-\rho} \frac{\partial s^\alpha}{\partial t} - \frac{s^\alpha \mathfrak{r}^\beta}{(s-\rho)^2} \frac{\partial s}{\partial t} \right),$$

and using equation (3.2.1), and inequality (3.2.5), we infer that, at the point where the maximum of Y is reached, we have

$$0 \leq \frac{\partial}{\partial t} Y \leq \frac{\Psi}{s-\rho} \left[\beta s^\alpha \mathfrak{r}^{\beta-1} \left(\frac{\rho s^\alpha \mathfrak{r}^\beta - s^\alpha \mathfrak{r}^{\beta+1}}{s-\rho} \right) - \alpha \mathfrak{r}^{2\beta} s^{2\alpha-1} + \frac{s^{2\alpha} \mathfrak{r}^{2\beta}}{s-\rho} \right].$$

This last inequality gives

$$\beta\rho\kappa - \beta - \alpha + \alpha\rho\frac{1}{s} + 1 \geq 0.$$

Neglecting the non-positive term $\alpha\rho\frac{1}{s}$, we obtain

$$\beta\rho\kappa - \beta - \alpha + 1 \geq 0.$$

Notice that $\alpha + \beta - 1 = -\frac{4p}{p+2}$, therefore $0 \leq \kappa \leq \frac{4}{\rho}$, consequently, implying the lemma. □

Lemma 42. *Let T be the maximal time of existence of the solution to the flow (3.1.1) with a fixed initial body $K_0 \in \mathcal{K}_{sym}$, then $T = T'$.*

Proof. From Proposition 40, we know that $T \leq T'$. Therefore, if $T < T'$, we conclude that $A(t)$ has a uniform lower bound which implies that the inradius of the evolving curve is uniformly bounded from below by a constant. Now, Corollary 37 guarantees a uniform lower bound on the curvature of the evolving curve in the time interval $[0, T)$. On the other hand, Lemma 41 implies a uniform upper bound on the curvature of the evolving curve. Thus, the evolution equation (3.1.1) is uniformly parabolic on $[0, T)$, and bounds on higher derivatives of the support function follows by [61] and Schauder theory. Hence, we can extend the solution after time T, contradicting its definition. □

Therefore, we have proved:

Theorem 43. *Let T be the maximal time of existence of the solution to the flow (3.1.1) with a fixed initial body $K_0 \in \mathcal{K}_{sym}$, then the sequence $\{K_t\}_t$ converges to the origin.*

3.3 Affine differential setting

In what follows, we find it more appropriate to work in the affine setting. Let $K_0 \in \mathcal{K}_{sym}$. We consider the family $\{K_t\}_t \in \mathcal{K}_{sym}$, and their associated smooth embeddings $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$, which are evolving according to

$$\frac{\partial}{\partial t}x := \Psi\sigma^{1-\frac{3p}{p+2}}\mathfrak{n}, \quad x(\cdot, 0) = x_{K_0}(\cdot), \quad x(\cdot, t) = x_{K_t}(\cdot) \tag{3.3.1}$$

for a fixed $1 < p < 2$. Observe that up to a time-dependent diffeomorphism the flow defined in (3.3.1) is equivalent to the flow defined by (3.1.1).

In terms of affine invariant quantities, the area and the weighted p-affine length of K are

$$A(K) = \frac{1}{2}\int_\gamma \sigma d\mathfrak{s}, \quad \Omega_p^\Psi(K) := \int_\gamma \Psi\sigma^{1-\frac{3p}{p+2}}d\mathfrak{s},$$

where here and thereafter γ is the boundary curve of K.

Lemma 44. *Let us define e to be the Euclidean arc-length and $\gamma^t := \partial K_t$ be the boundary of a convex body K_t evolving under the flow (3.3.1). Then the following evolution equations hold:*

1. $\frac{\partial}{\partial t}z = \kappa^{\frac{2}{3}}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}} x_e,$
2. $\frac{\partial}{\partial t}\Psi = \Psi_{\mathfrak{s}}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}},$
3. $\frac{d}{dt}A = -\Omega_p^\Psi,$
4. $\frac{\partial}{\partial t}\mathfrak{g} = \left(-\frac{2}{3}\Psi\sigma^{1-\frac{3p}{p+2}}\mu + \frac{1}{3}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right)\mathfrak{g},$
5. $\frac{\partial}{\partial t}\mathfrak{t} = \left(-\frac{1}{3}\Psi\sigma^{1-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right)\mathfrak{t} + \left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\mathfrak{n},$
6. $\frac{\partial}{\partial t}\sigma = -\frac{4}{3}\sigma^{1-\frac{3p}{p+2}}\Psi + \frac{1}{3}\sigma^{1-\frac{3p}{p+2}}\Psi\sigma_{\mathfrak{s}\mathfrak{s}} - \frac{1}{3}\left(\sigma^{1-\frac{3p}{p+2}}\Psi\right)_{\mathfrak{s}\mathfrak{s}}\sigma + \left(\sigma^{1-\frac{3p}{p+2}}\Psi\right)_{\mathfrak{s}}\sigma_{\mathfrak{s}},$

and we have

$$\frac{d}{dt}\Omega_p^\Psi = \frac{2(p-2)}{p+2}\int_\gamma \Psi^2\sigma^{1-\frac{6p}{p+2}}d\mathfrak{s} + \frac{18p^2}{(p+2)^3}\int_\gamma \Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s} \\ - \frac{2}{p+2}\int_\gamma \Psi_{\mathfrak{s}}^2\sigma^{2-\frac{6p}{p+2}}d\mathfrak{s} - \frac{12p}{(p+2)^2}\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}. \tag{3.3.2}$$

Proof. To prove the lemma, we will use repeatedly equations (1.0.2) without further mention. Recall that $d\mathfrak{s} = \kappa^{\frac{1}{3}}de = \mathfrak{r}^{\frac{2}{3}}d\theta$.

Proof of (1): Since $\frac{\partial}{\partial t}z$ is a tangent vector

$$\begin{aligned} \frac{\partial}{\partial t}z &= \langle \frac{\partial}{\partial t}z, x_e\rangle x_e \\ &= -\langle z, \frac{\partial^2}{\partial e\partial t}x\rangle x_e \\ &= -\langle z, \frac{\partial}{\partial e}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\mathfrak{n}\right)\rangle x_e \\ &= -\langle z, \mathfrak{n}\rangle \frac{\partial}{\partial e}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)x_e \\ &= \kappa^{\frac{2}{3}}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}} x_e. \end{aligned}$$

Proof of (2): By the evolution equation (1), we have

$$\begin{aligned} \frac{\partial}{\partial t}\Psi(e) &= \frac{\partial}{\partial t}\Psi(e(z)) \\ &= \langle \Psi_\theta x_e, \frac{\partial}{\partial t}z\rangle \\ &= \kappa^{\frac{2}{3}}\Psi_\theta\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}, \end{aligned}$$

where we identified θ with z and we used $\langle x_e, x_e\rangle = 1$.

We shall elaborate on our computation presented for the proof of (2). Let $\phi : [0, 2\pi] \to \mathbb{R}$ be a smooth function. Let $\vec{n}_t$ and $\vec{t}_t$ be the normal vector and tangent vector of a curve $\mathcal{E}_t$, respectively. Suppose that $\mathcal{E}_t$ locally is the zero level set of a function f_t on $\mathbb{R}^2$. Hence, we have $\vec{n}_t = \frac{\nabla f_t}{||\nabla f_t||}$. Define $f_t^1 = \frac{\partial f_t}{\partial x_1}$ and $f_t^2 = \frac{\partial f_t}{\partial x_2}$. Identifying $\mathbb{S}^1$ with

$[0, 2\pi]$ we get $\phi(\vec{n}_t) = \phi\left(\arctan \frac{f_t^2}{f_t^1}\right)$. It is easy to see that

$$\frac{\partial}{\partial t}\phi(\vec{n}_t) = \phi_\theta \frac{f_t^1 \frac{\partial}{\partial t}(f_t^2) - f_t^2 \frac{\partial}{\partial t}(f_t^1)}{\sqrt{(f_t^1)^2 + (f_t^2)^2}}.$$

Now observe that the first coordinate of $\frac{\partial}{\partial t}\vec{n}_t$ is

$$\frac{\frac{\partial}{\partial t}(f_t^1)}{\sqrt{(f_t^1)^2 + (f_t^2)^2}} - \frac{f_t^1(f_t^1 \frac{\partial}{\partial t}(f_t^1) + f_t^2 \frac{\partial}{\partial t}(f_t^2))}{((f_t^1)^2 + (f_t^2)^2)^{3/2}}$$

and the second coordinate is

$$\frac{\frac{\partial}{\partial t}(f_t^2)}{\sqrt{(f_t^1)^2 + (f_t^2)^2}} - \frac{f_t^2(f_t^1 \frac{\partial}{\partial t}(f_t^1) + f_t^2 \frac{\partial}{\partial t}(f_t^2))}{((f_t^1)^2 + (f_t^2)^2)^{3/2}}.$$

Furthermore

$$\vec{t}_t = \left(\frac{-f_t^2}{\sqrt{(f_t^1)^2 + (f_t^2)^2}}, \frac{f_t^1}{\sqrt{(f_t^1)^2 + (f_t^2)^2}} \right).$$

Thus, we conclude that

$$\frac{\partial}{\partial t}\phi(\vec{n}_t) = \phi_\theta \langle \vec{t}_t, \frac{\partial}{\partial t}\vec{n}_t \rangle = \langle \phi_\theta(\mathcal{E}_t)_e, \frac{\partial}{\partial t}\vec{n}_t \rangle.$$

Proof of (3): Notice that (3) has been proved in Lemma 3.2.1.

Proof of (4):

$$\frac{\partial}{\partial t}\mathfrak{g}^3 = \frac{\partial}{\partial t}\left[\gamma_\theta, \gamma_{\theta\theta}\right] = \left[\frac{\partial}{\partial t}\gamma_\theta, \gamma_{\theta\theta}\right] + \left[\gamma_\theta, \frac{\partial}{\partial t}\gamma_{\theta\theta}\right].$$

We have

$$\begin{aligned}
\left[\frac{\partial}{\partial t}\gamma_\theta, \gamma_{\theta\theta}\right] &= \left[\frac{\partial}{\partial \theta}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right), \gamma_{\theta\theta}\right] \\
&= \left[\mathfrak{g}\frac{\partial}{\partial \mathfrak{s}}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right), \gamma_{\theta\theta}\right] \\
&= \mathfrak{g}\left[\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\gamma_{\mathfrak{s}\mathfrak{s}} + \Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}}, \gamma_{\theta\theta}\right].
\end{aligned}$$

Since $\dfrac{\partial^2}{\partial\theta^2} = \mathfrak{g}\mathfrak{g}_{\mathfrak{s}}\dfrac{\partial}{\partial\mathfrak{s}} + \mathfrak{g}^2\dfrac{\partial^2}{\partial\mathfrak{s}^2}$, we further have $\gamma_{\theta\theta} = \mathfrak{g}^2\gamma_{\mathfrak{s}\mathfrak{s}} + \mathfrak{g}\mathfrak{g}_{\mathfrak{s}}\gamma_{\mathfrak{s}}$ and, therefore

$$\begin{aligned}\left[\frac{\partial}{\partial t}\gamma_\theta, \gamma_{\theta\theta}\right] &= \mathfrak{g}\left[\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\gamma_{\mathfrak{s}\mathfrak{s}} + \Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}}, \mathfrak{g}^2\gamma_{\mathfrak{s}\mathfrak{s}} + \mathfrak{g}\mathfrak{g}_{\mathfrak{s}}\gamma_{\mathfrak{s}}\right]\\ &= -\mathfrak{g}^2\mathfrak{g}_{\mathfrak{s}}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}} - \mathfrak{g}^3\Psi\sigma^{1-\frac{3p}{p+2}}\mu.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\left[\gamma_\theta, \frac{\partial}{\partial t}\gamma_{\theta\theta}\right] &= \left[\mathfrak{g}\gamma_{\mathfrak{s}}, \frac{\partial^2}{\partial\theta^2}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right)\right]\\ &= \left[\mathfrak{g}\gamma_{\mathfrak{s}}, \mathfrak{g}\mathfrak{g}_{\mathfrak{s}}\frac{\partial}{\partial\mathfrak{s}}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right) + \mathfrak{g}^2\frac{\partial^2}{\partial\mathfrak{s}^2}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right)\right]\\ &= \mathfrak{g}^2\mathfrak{g}_{\mathfrak{s}}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}} + \mathfrak{g}^3\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}} - \mathfrak{g}^3\Psi\sigma^{1-\frac{3p}{p+2}}\mu.\end{aligned}$$

Hence, we conclude that

$$\frac{\partial}{\partial t}\mathfrak{g}^3 = \mathfrak{g}^3\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}} - 2\mathfrak{g}^3\Psi\sigma^{1-\frac{3p}{p+2}}\mu,$$

which verifies our fourth claim.

Proof of (5): To prove the fifth claim, we observe that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial\mathfrak{s}} = \frac{\partial}{\partial\mathfrak{s}}\frac{\partial}{\partial t} - \frac{1}{\mathfrak{g}}\frac{\partial\mathfrak{g}}{\partial t}\frac{\partial}{\partial\mathfrak{s}}. \tag{3.3.3}$$

By (3.3.3), we get

$$\begin{aligned}\frac{\partial}{\partial t}\mathfrak{t} &= \frac{\partial}{\partial t}\frac{\partial}{\partial\mathfrak{s}}\gamma\\ &= \frac{\partial}{\partial\mathfrak{s}}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}\right) + \left(\frac{2}{3}\Psi\sigma^{1-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right)\mathfrak{t}\\ &= \left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\mathfrak{n} + \Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}} + \left(\frac{2}{3}\Psi\sigma^{1-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right)\mathfrak{t}.\end{aligned}$$

We notice that $\gamma_{\mathfrak{s}\mathfrak{s}\mathfrak{s}} = -\mu\gamma_{\mathfrak{s}}$ ending the proof of (5).

Proof of (6): We now proceed to prove the sixth claim with

$$\frac{\partial}{\partial t}\sigma = \frac{\partial}{\partial t}\left[\gamma, \gamma_{\mathfrak{s}}\right] = \left[\frac{\partial}{\partial t}\gamma, \gamma_{\mathfrak{s}}\right] + \left[\gamma, \frac{\partial}{\partial t}\gamma_{\mathfrak{s}}\right].$$

By the evolution equation (3.3.1), the evolution equation for $\mathfrak{t}$, and the identities $\sigma = [\gamma, \gamma_{\mathfrak{s}}]$ and $\sigma_{\mathfrak{s}} = [\gamma, \gamma_{\mathfrak{s}\mathfrak{s}}]$, we get that

$$\begin{aligned}\frac{\partial}{\partial t}\sigma &= \left[\gamma, \left(-\frac{1}{3}\Psi\sigma^{1-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right)\gamma_{\mathfrak{s}} + \left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\gamma_{\mathfrak{s}\mathfrak{s}}\right] \\ &\quad + \left[\Psi\sigma^{1-\frac{3p}{p+2}}\gamma_{\mathfrak{s}\mathfrak{s}}, \gamma_{\mathfrak{s}}\right] \\ &= -\Psi\sigma^{1-\frac{3p}{p+2}} - \frac{1}{3}\Psi\sigma^{2-\frac{3p}{p+2}}\mu - \frac{1}{3}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\sigma + \left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\sigma_{\mathfrak{s}} \\ &= -\frac{4}{3}\Psi\sigma^{1-\frac{3p}{p+2}} + \frac{1}{3}\Psi\sigma^{1-\frac{3p}{p+2}}\sigma_{\mathfrak{s}\mathfrak{s}} - \frac{1}{3}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\sigma + \left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\sigma_{\mathfrak{s}}\end{aligned}$$

where we used $\sigma_{\mathfrak{s}\mathfrak{s}} + \sigma\mu = 1$ on the second line.

Proof of (3.3.2): The proof follows directly from (2), (4), (6) and arranging similar terms.

$$\frac{d}{dt}\Omega_p^{\Psi} = \underbrace{\int_{\gamma}\left(\frac{\partial}{\partial t}\Psi\right)\sigma^{1-\frac{3p}{p+2}}d\mathfrak{s}}_{I} + \underbrace{\int_{\gamma}\Psi\left(\frac{\partial}{\partial t}\sigma^{1-\frac{3p}{p+2}}\right)d\mathfrak{s}}_{II} + \underbrace{\int_{\gamma}\Psi\sigma^{1-\frac{3p}{p+2}}\frac{\partial}{\partial t}d\mathfrak{s}}_{III}.$$

We use (2) to compute I:

$$\begin{aligned}I &= \int_{\gamma}\left(\Psi_{\mathfrak{s}}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}\right)\sigma^{1-\frac{3p}{p+2}}d\mathfrak{s} \\ &= \int_{\gamma}\Psi_{\mathfrak{s}}^2\sigma^{2-\frac{6p}{p+2}}d\mathfrak{s} + \left(1 - \frac{3p}{p+2}\right)\int_{\gamma}\Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}.\end{aligned}$$

To simplify II, we deploy (6) and integration by parts:

$$II = \left(1-\frac{3p}{p+2}\right)\int_\gamma \Psi\sigma^{-\frac{3p}{p+2}}\left[-\frac{4}{3}\sigma^{1-\frac{3p}{p+2}}\Psi+\frac{1}{3}\sigma^{1-\frac{3p}{p+2}}\Psi\sigma_{\mathfrak{ss}}-\frac{1}{3}\left(\sigma^{1-\frac{3p}{p+2}}\Psi\right)_{\mathfrak{ss}}\sigma\right]d\mathfrak{s}$$

$$+\left(1-\frac{3p}{p+2}\right)\int_\gamma \Psi\sigma^{-\frac{3p}{p+2}}\left(\sigma^{1-\frac{3p}{p+2}}\Psi\right)_{\mathfrak{s}}\sigma_{\mathfrak{s}}d\mathfrak{s}$$

$$=\left(\frac{4p}{p+2}-\frac{4}{3}\right)\int_\gamma \Psi^2\sigma^{1-\frac{6p}{p+2}}d\mathfrak{s}+2\left(\frac{p}{p+2}-\frac{1}{3}\right)\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}$$

$$-\left(\frac{p}{p+2}-\frac{1}{3}\right)\left(\frac{6p}{p+2}-1\right)\int_\gamma \Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2d\mathfrak{s}-\left(\frac{p}{p+2}-\frac{1}{3}\right)\int_\gamma\left(\sigma^{1-\frac{3p}{p+2}}\Psi\right)_{\mathfrak{s}}^2d\mathfrak{s}$$

$$+\left(1-\frac{3p}{p+2}\right)\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}+\left(1-\frac{3p}{p+2}\right)^2\int_\gamma \Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2d\mathfrak{s}$$

$$=\left(\frac{4p}{p+2}-\frac{4}{3}\right)\int_\gamma \Psi^2\sigma^{1-\frac{6p}{p+2}}d\mathfrak{s}+2\left(\frac{p}{p+2}-\frac{1}{3}\right)\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}$$

$$-\left(\frac{p}{p+2}-\frac{1}{3}\right)\left(\frac{6p}{p+2}-1\right)\int_\gamma \Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2d\mathfrak{s}$$

$$+\left(\frac{1}{3}-\frac{p}{p+2}\right)\left(1-\frac{3p}{p+2}\right)^2\int_\gamma \Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2d\mathfrak{s}$$

$$+2\left(\frac{1}{3}-\frac{p}{p+2}\right)\left(1-\frac{3p}{p+2}\right)\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}+\left(\frac{1}{3}-\frac{p}{p+2}\right)\int_\gamma \Psi_{\mathfrak{s}}^2\sigma^{2-\frac{6p}{p+2}}d\mathfrak{s}$$

$$+\left(1-\frac{3p}{p+2}\right)\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}+\left(1-\frac{3p}{p+2}\right)^2\int_\gamma \Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2d\mathfrak{s}.$$

To calculate III, we use (4), integration by parts and the identity $\sigma_{\mathfrak{s}\mathfrak{s}}+\sigma\mu=1$:

$$\begin{aligned}
III &= \int_\gamma \Psi\sigma^{1-\frac{3p}{p+2}}\left(-\frac{2}{3}\Psi\sigma^{1-\frac{3p}{p+2}}\mu+\frac{1}{3}\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}\mathfrak{s}}\right)d\mathfrak{s}\\
&= -\frac{2}{3}\int_\gamma \Psi^2\sigma^{2-\frac{6p}{p+2}}\mu d\mathfrak{s}-\frac{1}{3}\int_\gamma\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}\\
&= -\frac{2}{3}\int_\gamma \Psi^2\sigma^{1-\frac{6p}{p+2}}(1-\sigma_{\mathfrak{s}\mathfrak{s}})d\mathfrak{s}-\frac{1}{3}\int_\gamma\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}\\
&= -\frac{2}{3}\int_\gamma \Psi^2\sigma^{1-\frac{6p}{p+2}}d\mathfrak{s}-\frac{4}{3}\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}+\left(\frac{4p}{p+2}-\frac{2}{3}\right)\int_\gamma\Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2d\mathfrak{s}\\
&-\frac{1}{3}\int_\gamma\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}\\
&= -\frac{2}{3}\int_\gamma \Psi^2\sigma^{1-\frac{6p}{p+2}}d\mathfrak{s}-\frac{4}{3}\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}+\left(\frac{4p}{p+2}-\frac{2}{3}\right)\int_\gamma\Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2d\mathfrak{s}\\
&-\frac{1}{3}\left(1-\frac{3p}{p+2}\right)^2\int_\gamma\Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2d\mathfrak{s}-\frac{2}{3}\left(1-\frac{3p}{p+2}\right)\int_\gamma\Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}\\
&-\frac{1}{3}\int_\gamma\Psi_{\mathfrak{s}}^2\sigma^{2-\frac{6p}{p+2}}d\mathfrak{s}.
\end{aligned}$$

Adding up I, II and III, we obtain equation (3.3.2). □

Lemma 45. *The weighted p-affine isoperimetric ratio,* $\frac{\Omega_p^\Psi}{A^{\frac{2-p}{2+p}}}$*, is non-decreasing along the flow (3.3.1) and remains constant if and only if* K_t *is a homothetic solution to the flow.*

Proof. Using equation $\sigma_{\mathfrak{s}\mathfrak{s}}+\sigma\mu=1$ which relates the affine curvature μ to the affine support function, we rewrite the first term in (3.3.2) as follows:

$$\frac{2(p-2)}{p+2}\int_\gamma\Psi^2\sigma^{1-\frac{6p}{p+2}}d\mathfrak{s}=\frac{2(p-2)}{p+2}\int_\gamma\Psi^2\sigma^{2-\frac{6p}{p+2}}\mu d\mathfrak{s}+\frac{2(p-2)}{p+2}\int_\gamma\Psi^2\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}\mathfrak{s}}d\mathfrak{s}. \tag{3.3.4}$$

On the other hand, by the geometric affine-Wirtinger inequality Lemma 6, [9], we have

$$\int_\gamma\Psi^2\sigma^{2-\frac{6p}{p+2}}\mu d\mathfrak{s}\le\frac{1}{2A}\left(\int_\gamma\Psi\sigma^{1-\frac{3p}{p+2}}d\mathfrak{s}\right)^2+\int_\gamma\left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}^2d\mathfrak{s}. \tag{3.3.5}$$

Therefore, by equation (3.3.4), we get

$$\int_\gamma \Psi^2\sigma^{1-\frac{6p}{p+2}}d\mathfrak{s} \le \frac{1}{2A}\left(\int_\gamma \Psi\sigma^{1-\frac{3p}{p+2}}d\mathfrak{s}\right)^2 + \int_\gamma \left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} + \int_\gamma \Psi^2\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}\mathfrak{s}}d\mathfrak{s}. \tag{3.3.6}$$

We also have

$$\int_\gamma \left(\Psi\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} = \left(1-\frac{3p}{p+2}\right)^2\int_\gamma \Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s} + \int_\gamma \Psi_{\mathfrak{s}}^2\sigma^{2-\frac{6p}{p+2}}d\mathfrak{s}$$
$$+2\left(1-\frac{3p}{p+2}\right)\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}, \tag{3.3.7}$$

and

$$\int_\gamma \Psi^2\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}\mathfrak{s}}d\mathfrak{s} = -2\int_\gamma \Psi_{\mathfrak{s}}\Psi\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s} + \left(\frac{6p}{p+2}-1\right)\int_\gamma \Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}. \tag{3.3.8}$$

Hence, by combining equation (3.3.2), inequality (3.3.6), equations (3.3.7), (3.3.8) and collecting similar terms, we obtain

$$\frac{d}{dt}\Omega_p^\Psi \ge \left(\frac{p-2}{p+2}\right)\frac{1}{A}\left(\int_\gamma \Psi\sigma^{1-\frac{3p}{p+2}}d\mathfrak{s}\right)^2 + \frac{18p^2(p-1)}{(p+2)^3}\int_\gamma \Psi^2\sigma^{-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}$$
$$+\frac{2(p-1)}{p+2}\int_\gamma \Psi_{\mathfrak{s}}^2\sigma^{2-\frac{6p}{p+2}}d\mathfrak{s} - \frac{12p(p-1)}{(p+2)^2}\int_\gamma \Psi\Psi_{\mathfrak{s}}\sigma^{1-\frac{6p}{p+2}}\sigma_{\mathfrak{s}}d\mathfrak{s}.$$

Now, we observe that the last three terms in the previous inequality can be grouped in a term that is almost a perfect square:

$$\frac{d}{dt}\Omega_p^\Psi \ge \left(\frac{p-2}{p+2}\right)\frac{1}{A}\left(\int_\gamma \Psi\sigma^{1-\frac{3p}{p+2}}d\mathfrak{s}\right)^2$$
$$+\frac{9p^2}{2(p^2+p-2)}\int_\gamma \left(\Psi^{\frac{2(p-1)}{3p}}\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}^2 \Psi^{\frac{2(p+2)}{3p}}d\mathfrak{s}.$$

To finish the proof, we notice that by (3) in Lemma 44 and the previous inequality, we have

$$\frac{d}{dt}\frac{\Omega_p^\Psi}{A^{\frac{2-p}{2+p}}}(t) = \frac{1}{A^{\frac{2-p}{2+p}}(t)}\left(\frac{d}{dt}\Omega_p^\Psi - \frac{p-2}{p+2}\frac{(\Omega_p^\Psi)^2}{A}\right)(t) \ge 0.$$

□

Lemma 46. *If K_t evolves by (3.3.1), the following limit holds as t approaches the extinction time T:*

$$\liminf_{t\to T}\frac{(\Omega_p^\Psi)^p}{A^{1-p}}\left[\frac{d}{dt}\Omega_p^\Psi-\frac{p-2}{p+2}\frac{(\Omega_p^\Psi)^2}{A}\right]=0. \tag{3.3.9}$$

Proof. We have

$$\frac{d}{dt}\frac{(\Omega_p^\Psi)^{2+p}}{A^{2-p}}(t)=-\frac{d}{dt}\ln(A(t))\left[\frac{(\Omega_p^\Psi)^p}{A^{1-p}}\left((2+p)\frac{d}{dt}\Omega_p^\Psi-(p-2)\frac{(\Omega_p^\Psi)^2}{A}\right)(t)\right].$$

If

$$\frac{(\Omega_p^\Psi)^p}{A^{1-p}}\left[(2+p)\frac{d}{dt}\Omega_p^\Psi-(p-2)\frac{(\Omega_p^\Psi)^2}{A}\right]\geq\varepsilon$$

in a neighborhood of T, then

$$\frac{d}{dt}\frac{(\Omega_p^\Psi)^{2+p}}{A^{2-p}}(t)\geq-\varepsilon\frac{d}{dt}\ln(A(t)).$$

Thus

$$\frac{(\Omega_p^\Psi)^{2+p}}{A^{2-p}}(t)\geq\frac{d}{dt}\frac{(\Omega_p^\Psi)^{2+p}}{A^{2-p}}(t_1)+\varepsilon\ln(A(t_1))-\varepsilon\ln(A(t)),$$

the right hand side goes to infinity as $A(t)$ goes to zero. This contradicts the p-affine isoperimetric inequality which states that the left hand side is bounded from above. □

3.4 Normalized flow

In this section, we study the asymptotic behavior of the evolving curves under a normalized flow corresponding to the evolution described by (3.1.1). We consider the conventional rescaling such that the area enclosed by the normalized curves is π by taking

$$\tilde{s}_t:=\sqrt{\frac{\pi}{A(t)}}\,s_t,\quad \tilde{\kappa}_t:=\sqrt{\frac{A(t)}{\pi}}\,\kappa_t.$$

One can also define a new time parameter

$$\tau = \int_0^t \left(\frac{\pi}{A(K_t)(\xi)}\right)^{\frac{2p}{p+2}} d\xi$$

and can easily verify that

$$\frac{\partial}{\partial \tau}\tilde{s} = -\Psi\tilde{s}\left(\frac{\tilde{\kappa}}{\tilde{s}^3}\right)^{\frac{p}{p+2}} + \frac{\Psi\tilde{s}}{2\pi}\tilde{\Omega}_p^{\Psi}, \tag{3.4.1}$$

where $\tilde{\Omega}_p^{\Psi}$ stands for the weighted p-affine length of $\partial\tilde{K}_t$ having support function $\tilde{s}_t$. More precisely,

$$\tilde{\Omega}_p^{\Psi}(\tau) := \Omega_p^{\Psi}(\tilde{K}_\tau) = \int_{\mathbb{S}^1} \Psi\frac{\tilde{s}}{\tilde{\kappa}}\left(\frac{\tilde{\kappa}}{\tilde{s}^3}\right)^{\frac{p}{p+2}} d\theta.$$

However, even in the normalized case, we prefer to work on the finite time interval $[0, T)$.

Corollary 47. *Let $\{t_k\}_k$ be the sequence of times realizing the limit (3.3.9) in Lemma 46. Then there exists a constant $c > 0$ such that along the normalized p-flow, we have*

$$\lim_{t_k \to T} \Psi^{\frac{2(p-1)}{3p}}\sigma^{1-\frac{3p}{p+2}}(t_k) = c.$$

Proof. By Lemma 46, we have

$$0 = \lim_{t_k\to T}\frac{(\Omega_p^{\Psi})^p}{A^{1-p}}\left[\frac{d}{dt}\Omega_p^{\Psi} - \frac{p-2}{p+2}\frac{(\Omega_p^{\Psi})^2}{A}\right] \geq \lim_{t_k\to T}\frac{c_p(\Omega_p^{\Psi})^p}{A^{1-p}}\int_\gamma \left(\Psi^{\frac{2(p-1)}{3p}}\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}}^2 \Psi^{\frac{2(p+2)}{3p}} d\mathfrak{s},$$

where $c_p := \frac{9p^2}{2(p+2)(p-1)}$. As by Lemma 45, the normalized weighted p-affine length $\tilde{\Omega}_p^{\Psi}$ is increasing along the normalized flow and Ψ has a lower bound, we conclude that

$$\lim_{t_k\to T}\int_{\tilde{\gamma}}\left(\Psi^{\frac{2(p-1)}{3p}}\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}} = 0.$$

We notice that, for each $\theta_1, \theta_2 \in \mathbb{S}^1$,

$$\left|\int_{\theta_1}^{\theta_2}\left(\Psi^{\frac{2(p-1)}{3p}}\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)_\theta d\theta\right| \leq \int_{\mathbb{S}^1}\left|\left(\Psi^{\frac{2(p-1)}{3p}}\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)_\theta\right| d\theta$$
$$= \int_{\tilde{\gamma}}\left|\left(\Psi^{\frac{2(p-1)}{3p}}\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)_{\tilde{\mathfrak{s}}}\right| d\tilde{\mathfrak{s}}$$
$$\leq \left(\int_{\tilde{\gamma}}\left(\Psi^{\frac{2(p-1)}{3p}}\tilde{\sigma}^{1-\frac{3p}{p+2}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}}\right)^{1/2}\tilde{\Omega}_1^{1/2}.$$

Take θ_1 and θ_2 be two points where $\Psi^{\frac{2(p-1)}{3p}}\tilde{\sigma}^{1-\frac{3p}{p+2}}$ reaches its extremal values. It is known that, for a smooth, simple curve with enclosed area π, $\min_{\mathbb{S}^1}\sigma \leq 1$ and $\max_{\mathbb{S}^1}\sigma \geq 1$, see Lemma 10 in [9]. Hence, as $\tilde{\Omega}_1$ is bounded from above by the classical affine isoperimetric inequality [69], we infer that

$$\lim_{t_k\to T}\Psi^{\frac{2(p-1)}{3p}}\tilde{\sigma}^{1-\frac{3p}{p+2}}(t_k) = c,$$

for some constant c. □

The following lemma will be needed in the proof of the main theorem.

Lemma 48. *Let s be the support function of a $\frac{\pi}{k}$ $(k \geq 2)$ periodic, smooth convex curve γ of enclosed area π. Then there exist uniform lower and upper bounds on s depending only on k.*

Proof. We write the cosine series of s, $s(\theta) = s_0 + \sum_{n=1}^{\infty} s_n\cos(2nk\theta)$. From this, we conclude that we can represent the radius of curvature $\mathfrak{r}$ as follows:

$$\mathfrak{r} = s_0 + \sum_{n=1}^{\infty}(1-4n^2k^2)s_n\cos(2nk\theta) > 0.$$

We will use now the positivity of $\mathfrak{r}$ to find an estimate for the upper bound of $|s_n|$.

$$\int_{\mathbb{S}^1}\mathfrak{r}(1\pm\cos(2nk\theta))d\theta = 2\pi s_0 \pm \pi s_n(1-4n^2k^2),$$

thus we have

$$|s_n| \leq \frac{2s_0}{4n^2k^2-1}\ , \quad \forall n \geq 1. \tag{3.4.2}$$

To find an upper bound for s, we use the assumption that γ encloses an area of π and inequality (3.4.2).

$$\begin{aligned} 2\pi = \int_{\mathbb{S}^1} \mathfrak{r} s d\theta &= 2\pi s_0^2 - \pi \sum_{n=1}^{\infty} (4n^2k^2-1)s_n^2 \\ &\geq 2\pi \left(1 - 2\sum_{n=1}^{\infty} \frac{1}{4n^2k^2-1}\right) s_0^2 \\ &= 2\pi \left(\frac{\pi \cot(\frac{\pi}{2k})}{2k}\right) s_0^2 =: 2\pi \frac{1}{c_k} s_0^2. \end{aligned}$$

Hence, we have

$$s_0 \leq \sqrt{c_k}.$$

On the other hand, we have

$$s(\theta) = \frac{1}{2}\left(s(\theta) + s(\theta+\pi)\right) \leq \frac{1}{4} L(\gamma) = \frac{1}{2}\pi s_0 \leq \frac{\pi\sqrt{c_k}}{2}. \tag{3.4.3}$$

To find a lower bound for s, we use the assumption that γ encloses an area of π, inequality (3.4.3) and the maximal ellipsoid contained in the convex body enclosed by γ. Let J denote the maximal ellipsoid (also known as the John ellipsoid) contained in the convex body of boundary γ. Recall from Theorem 2 that

$$J \subset \gamma \subset \sqrt{2} J, \tag{3.4.4}$$

see [59]. Therefore, J encloses an area of, at least, $\frac{\pi}{2}$. Suppose that the major axis of J has length l_1 and the minor axis of J has length l_2. Hence,

$$\frac{\pi}{2} \leq \pi \frac{l_1 l_2}{4} = A(J). \tag{3.4.5}$$

On the other hand, by (3.4.4) and inequality (3.4.3), we know that $l_1 \leq \frac{\pi\sqrt{c_k}}{4}$. Now, as $l_1 l_2 > 2$ by (3.4.5), we conclude that $l_2 > \frac{2}{\pi\sqrt{c_k}}$. Once again using (3.4.4) implies

$$s(\theta) \geq \frac{l_2}{2} > \frac{1}{\pi\sqrt{c_k}}.$$

□

3.5 Proof of the main theorem

In this section, we present a proof of the main theorem.

Proof. Define $\Phi =: \Psi^{\frac{p+2}{3p}}$ in (3.1.1). Then an appropriate rescaling of the evolving convex bodies and Corollary 47 prove the first part of the claim. To prove the second part, we start the flow (3.1.1) with a curve whose support function is $\frac{\pi}{k}$ periodic; for example $s(\theta, 0) := 1+\varepsilon\cos(2k\theta)$ for $\varepsilon > 0$ small enough. Therefore, the solution to the evolution equation (3.1.1) remains $\frac{\pi}{k}$-periodic. Hence, by Lemma 48, $\tilde{s}$ is bounded. Therefore, Corollary 47 and the standard theory of parabolic equations imply the claim. □

We remark that the periodicity of Φ with period $\frac{\pi}{k}$, $k \geq 2$, was also considered in a different way by Chen [25] as a sufficient condition for the solvability of the L_{-2} Minkowski problem.

3.6 A necessary condition and the uniqueness of solutions

In this section, we obtain a necessary condition on the solvability of the even L_{-2} Minkowski problem, hence showing that the existence of solutions to the problem cannot occur for all π-periodic smooth functions Ψ. Moreover, we will use the initial set up of this section to discuss the uniqueness of solutions to the even L_{-2} Minkowski problem.

Theorem 49. *Let γ be a smooth, origin-symmetric curve. Assume that $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is the Gauss parametrization of γ. Then σ, the affine support function of γ, as a function on the unit circle, has at least eight critical points, i.e., points at which $\sigma_\theta = 0$.*

Proof. Define a curve $\Lambda : \mathbb{S}^1 \to \mathbb{R}$ by

$$\Lambda(\theta) := \left(\int_0^\theta \frac{\cos\alpha}{s^3(\alpha)}\, d\alpha, \int_0^\theta \frac{\sin\alpha}{s^3(\alpha)}\, d\alpha \right) =: (x, y).$$

As γ is origin symmetric, $s(\theta + \pi) = s(\theta)$ for all $\theta \in \mathbb{S}^1$. This implies $\Lambda(2\pi) = \Lambda(0) = \vec{o}$ and that Λ is a closed curve. For convenience set $\ ' := \frac{d}{d\theta}$. We compute the Euclidean curvature of Λ:

$$\kappa_\Lambda = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}} = s^3.$$

Hence, Λ is a closed convex curve. We now proceed to obtain the affine curvature of Λ using the following formula

$$\mu_\Lambda = \underbrace{\frac{x''y''' - x'''y''}{(x'y'' - x''y')^{5/3}}}_{i} \underbrace{- \frac{1}{2}\left[\frac{1}{(x'y'' - x''y')^{2/3}}\right]''}_{ii}.$$

We have

$$i = \frac{\left(\frac{\cos\theta}{s^3}\right)'\left(\frac{\sin\theta}{s^3}\right)'' - \left(\frac{\cos\theta}{s^3}\right)''\left(\frac{\sin\theta}{s^3}\right)'}{\left(\frac{\sin\theta}{s^3}\left(\frac{\cos\theta}{s^3}\right)' - \left(\frac{\sin\theta}{s^3}\right)'\frac{\cos\theta}{s^3}\right)^{5/3}} = s^2\left(3ss'' + 6s'^2 + s^2\right)$$

and

$$ii = -\frac{1}{2}\left(\frac{1}{\left(\frac{\sin\theta}{s^3}\left(\frac{\cos\theta}{s^3}\right)' - \left(\frac{\sin\theta}{s^3}\right)'\frac{\cos\theta}{s^3}\right)^{2/3}}\right)'' = -2s^2\left(ss'' + 3s'^2\right).$$

Adding up i and ii gives $\mu_\Lambda = s^3\left(s_{\theta\theta} + s\right) = \dfrac{s^3}{\kappa} = \sigma^3$. It is known, see for example [22], that a symmetric oval (symmetric with respect to its center) has at least eight extatic points, i.e., points where $\mu_\theta = 0$. Therefore, if we prove that Λ is symmetric with respect to an interior point then we can conclude that σ must have, at least, eight critical points. To this aim, we notice that

1. The curvature of the newly defined curve Λ has π-symmetry as its curvature equals the cube of the support function, s^3, of an origin-symmetric curve γ.

2. The even Minkowski problem says if the initial data is π-symmetric, then there is an origin-symmetric solution. For example, there is a curve, C, such that its curvature is s^3. Now recall that the solution to the Minkowski problem is unique up to a linear transformation. Therefore $\Lambda = C + \vec{a}$ where $\vec{a}$ is a vector. Furthermore, this implies that Λ is symmetric with respect to a point which is $\vec{a}$.

□

Corollary 50. *If the even L_{-2} Minkowski problem with smooth data Ψ has a solution, then Ψ must have eight, or more, critical points on $[0, 2\pi]$.*

Proposition 51. *Let γ_1 and γ_2 be two smooth, origin-symmetric curves with support functions s_1 and s_2, respectively. If $\sigma_{\gamma_1} \equiv \sigma_{\gamma_2} =: \Phi$, then there exists a special linear transformation, $T \in SL(2)$ such that*

$$\gamma_2 = T(\gamma_1).$$

Furthermore, identifying θ and $(\cos\theta, \sin\theta)$ we have

$$\Phi(\theta) = \Phi\left(\frac{T^{-t}(\cos\theta, \sin\theta)}{||T^{-t}(\cos\theta, \sin\theta)||}\right). \tag{3.6.1}$$

Proof. It is well-known that affine curvature determines a curve uniquely up to an equiaffine transformation of the plane, [82]. Define

$$\Lambda_1(\theta) := \left(\int_0^\theta \frac{\cos\alpha}{s_1^3(\alpha)}\, d\alpha, \int_0^\theta \frac{\sin\alpha}{s_1^3(\alpha)}\, d\alpha\right), \Lambda_2(\theta) := \left(\int_0^\theta \frac{\cos\alpha}{s_2^3(\alpha)}\, d\alpha, \int_0^\theta \frac{\sin\alpha}{s_2^3(\alpha)}\, d\alpha\right).$$

Since $\mu_{\Lambda_1} \equiv \mu_{\Lambda_1}$, there exists a special linear transformation $T \in SL(2)$ such that

$$\Lambda_2 = T(\Lambda_1).$$

Let $\vec{\mathbf{n}}_{\Lambda_1}$ and $\vec{\mathbf{n}}_{\Lambda_2}$ denote the unit normal to, respectively, Λ_1 and Λ_2. Therefore, for each $x \in \Lambda_1$

$$\kappa_{\Lambda_1}(x) = ||T^{-t}(\vec{\mathbf{n}}_{\Lambda_1}(x))||^3 \kappa_{\Lambda_2}(T(x)).$$

On the other hand, using $\kappa_{\Lambda_1}(x) = s_1^3(x)$ and $\kappa_{\Lambda_2}(T(x)) = s_2^3(T(x))$, we obtain that

$$s_1^3(x) = ||T^{-t}(\vec{\mathbf{n}}_{\Lambda_1}(x))||^3 s_2^3(T(x)). \tag{3.6.2}$$

To prove the corollary, we need to rewrite the equation (3.6.2) on the unit sphere. Toward this goal, we observe the following relation between $\vec{\mathbf{n}}_{\Lambda_1}$ and $\vec{\mathbf{n}}_{\Lambda_2}$:

$$\vec{\mathbf{n}}_{\Lambda_2} = \frac{T^{-t}(\vec{\mathbf{n}}_{\Lambda_1})}{||T^{-t}(\vec{\mathbf{n}}_{\Lambda_1})||}.$$

Thus, $s_1(\vec{\mathbf{n}}_1) = ||T^{-t}(\vec{\mathbf{n}}_1)||s_2(\vec{\mathbf{n}}_2)$, where $\vec{\mathbf{n}}_1, \vec{\mathbf{n}}_2 \in \mathbb{S}^1$ and $\vec{\mathbf{n}}_2 = \frac{T^{-t}(\vec{\mathbf{n}}_1)}{||T^{-t}(\vec{\mathbf{n}}_1)||}$. This completes the proof of the first part. The proof of equation (3.6.1) also follows from the above observations. □

Remark:

Suppose that the even L_{-2} Minkowski problem is solvable for Φ. Equivalently, there exists a curve γ such that $\frac{s^3}{\kappa} = \Phi$. If $\Phi(\theta) = \Phi\left(\frac{T^{-t}(\cos\theta, \sin\theta)}{||T^{-t}(\cos\theta, \sin\theta)||}\right)$, then it is easy to show that $T(\gamma)$ also solves the even L_{-2} Minkowski problem corresponding to Φ. This fact and the previous corollary imply that every curve in

$$\left\{T(\gamma);\ T \in SL(2) \text{ and } \Phi(\theta) = \Phi\left(\frac{T^{-t}(\cos\theta, \sin\theta)}{||T^{-t}(\cos\theta, \sin\theta)||}\right)\right\}$$

solves $\frac{s^3}{\kappa} = \Phi$ and these are all the possible solutions.

3.7 Conclusions

We will recall first some results of Ai, Chou, and Wei [1], who employed a different sufficiency condition in their study of the L_{-2} problem.

Let $\Phi : \mathbb{S}^1 \to \mathbb{R}$ be a smooth positive function. Define

$$B(x, \Phi) := \int_0^\pi \frac{\Phi(x+t) - \Phi(x) - 2^{-1}\Phi'(x)\sin(2t)}{\sin^2 t} dt.$$

If $B(x) \neq 0$ at each critical point of Φ, then we say Φ is B-nondegenerate.

Theorem A [1] Assume that Φ is a positive, B-nondegenerate, C^2 function of period π. Then there exists a constant C which depends only on Φ such that

$$0 < C^{-1} \leq u(x) \leq C, \quad \text{and} \quad ||u||_{H^1(\mathbb{S}^1)} \leq C,$$

for every solution u of the L_{-2} Minkowski corresponding to Φ.

Theorem B [1] Assume that Φ is a positive, B-nondegenerate, C^2 function of period π. Then L_{-2} Minkowski problem with data Φ is solvable if the winding number of the map

$$x \mapsto (-B(x), \Phi'(x)), \quad x \in [0, \pi)$$

around the origin is not equal to -1.

Lemma 1.5. (Kazdan-Warner type obstruction) [1] For every solution u of the L_{-2} Minkowski problem corresponding to Φ, we have

$$\int_0^\pi \frac{\Phi'(x)\alpha(x)}{u^2(x)} dx = 0,$$

where α is in the set $\{1, \cos 2x, \sin 2x\}$.

Define $C := \{\Phi \in C^\infty_{even}(\mathbb{S}^1); \Phi > 0 \text{ and } \exists u : u'' + u = \frac{\Phi}{u^3}\}$. Then by our main theorem, S is dense in $D := \{\Phi \in C^\infty_{even}(\mathbb{S}^1), \Phi > 0\}$ with respect to the L^∞ norm. By Corollary 49, or the Kazdan-Warner type obstruction, if Φ is only π periodic, then it is possible that the corresponding L_{-2} is not solvable. A simple example is provided by $\Phi(\theta) = 2 + \cos(2\theta)$. For each non-solvable Φ, Theorem A and Theorem 34 imply that there exists a family of convex bodies such that their corresponding affine support functions are B-degenerate while approaching Φ in the L^∞ norm. Therefore, the B-non degeneracy of Φ is *not* a necessary condition for the existence of a solution to the L_{-2} Minkowski problem. Moreover, notice that by a result of Guggenheimer, [45], the above lemma also implies that Ψ_θ has, at least eight zeroes in $[0, 2\pi]$, hence assuming that Φ has eight critical points is not a sufficient condition as the above Kazdan-Warner type obstruction is not a sufficient condition.

Chapter 4

On the stability of the p-affine isoperimetric inequality

The contents of this chapter are taken from the paper "On the stability of the p-affine isoperimetric inequality" [53]. The paper will appear in *Journal of Geometric analysis*.

Employing the affine normal flow, we prove a stability version of the p-affine isoperimetric inequality, $p \geq 1$, in $\mathbb{R}^2$ in the class of origin-symmetric convex bodies. That is, if K is an origin-symmetric convex body in $\mathbb{R}^2$ such that it has area π and its p-affine perimeter is close enough to the one of an ellipse with the same area, then, after applying a special linear transformation, K is close to an ellipse in the Hausdorff distance.

4.1 Introduction

Versions of stability have been investigated for several important inequalities, including a stability version of the Brunn-Minkowski inequality due to Diskant [32], stability of the Rogers-Shephard inequality by Böröczky [18], stability of the Blaschke-Santaló inequality and the affine isoperimetric inequality in $\mathbb{R}^n$ for $n \geq 3$ by Böröczky [19], stability of the reverse Blaschke-Santaló inequality by Böröczky and Hug [20], stability of the Prékopa-Leindler inequality by Ball, and Böröczky [14], stability of a volume ratio by Hug, and Schneider [49], and recently stability of the functional forms of the Blaschke-Santaló inequality by Barthe, Böröczky and Fradelizi [16].

As we stated in the introduction of the thesis, a version of stability of the affine isoperimetric inequality was proved in [15] and [19] in $\mathbb{R}^n$ for $n \geq 3$, but the stability problem was not settled in $\mathbb{R}^2$. In this chapter, we prove a version of stability of the p-affine isoperimetric inequality, $p \geq 1$, in the class of origin-symmetric convex bodies in $\mathbb{R}^2$. The technique presented here to deal with stability is new as it approaches the problem from the perspective of geometric flows and ODEs. However, the interaction between convex geometry and geometric flows is not new. There are several important contributions of geometric flows to convex geometry. A proof of the affine isoperimetric inequality by Andrews using the affine normal flow [7], necessary and sufficient conditions for the existence of a solution to the discrete L_0-Minkowski problem, using discrete weighted curve shortening flow, by Stancu [95, 96, 99], and independently by Andrews [11], and a proof of the p-affine isoperimetric inequality in the class of origin-symmetric convex bodies in $\mathbb{R}^2$ using the affine normal [52]. See [97, 98, 100, 101] for more applications of flows. In particular, a newly defined family of centro-affine p-flows and their applications to centro-affine differential geometry by Stancu [100, 101].

Recall that, for $p \geq 1$, the p-affine perimeter of K in $\mathbb{R}^2$, having the origin in its interior, is defined by

$$\Omega_p(K) = \int_{\mathbb{S}^1} \frac{s}{\kappa} \left(\frac{\kappa}{s^3}\right)^{\frac{p}{2+p}} d\theta,$$

and the p-affine perimeter of a convex body is bounded by the the area via the p-affine isoperimetric inequality. If the centroid of K is at the origin, then

$$\left(\frac{\Omega_p^{2+p}(K)}{2^{2+p}A^{2-p}(K)}\right)^{\frac{1}{p}} \leq \pi^2,$$

with the equality case only for ellipses centered at the origin. Furthermore, recall that the quantity $\left(\frac{\Omega_p^{2+p}(K)}{2^{2+p}A^{2-p}(K)}\right)^{1/p}$ is called the p-affine isoperimetric ratio of K.

The stability problem that we study in this chapter is invariant under centro-affine transformations. Therefore, without loss of generality, using Theorem 2, we can assume that $c_1 \leq s_K \leq c_2$ for universal constants c_1 and c_2. These constants depends only on $A(K)$.

Theorem 52 (Main Theorem C)**.** *Let K be an origin-symmetric convex body in $\mathbb{R}^2$ with area π. Let $p \geq 1$. There exists an $\varepsilon_p > 0$, depending on p, such that the following holds. If, for an $0 < \varepsilon < \varepsilon_p$,*

$$\left(\frac{\Omega_p^{2+p}(K)}{2^{2+p}A^{2-p}(K)}\right)^{\frac{1}{p}} > \pi^2(1-\varepsilon),$$

then there exist a disk $\mathcal{D}$, an ellipse $\mathcal{E}$ and a special linear transformation T such that

$$\mathcal{E} \subseteq TK \subseteq \left(1 + \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{2c_2}{c_1^2}\varepsilon^{\frac{3}{10}}\right)\mathcal{D},$$

and

$$d_{\mathcal{H}}\left(\mathcal{E}, \left(1 + \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{2c_2}{c_1^2}\varepsilon^{\frac{3}{10}}\right)\mathcal{D}\right) < C_p\varepsilon^{\frac{3}{10}},$$

for a universal constant C_p.

In particular,

$$d_{\mathcal{H}}(TK, \mathcal{E}) < C_p\varepsilon^{\frac{3}{10}}.$$

To prove this theorem, we will implement the affine normal flow on curves. We use only results on the short time behavior of this flow. Let $K_0 := K \in \mathcal{K}_{sym}$. We consider a family $\{K_t\} \in \mathcal{K}_{sym}$, and their associated smooth embeddings $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$, which are evolving according to the affine flow, namely,

$$\partial_t x := -\kappa^{\frac{1}{3}} z, \quad x(\cdot, 0) = x_{K_0}(\cdot), \quad x(\cdot, t) = x_{K_t}(\cdot). \tag{4.1.1}$$

We point out here that we need only to prove the statement of our main theorem for smooth convex bodies. The reason is the instantaneous smoothing property of the affine normal flow [10] and monotonicity of the p-affine isoperimetric inequality along the affine normal flow [52]. Notice that thanks by a theorem of A.D. Alexandrov (see P.M. Gruber [39], page 22), the boundary of a convex body is twice differentiable in a generalized sense at almost everywhere with respect to its Hausdorff measure. Therefore, a generalized notion of Gauss curvature is available for convex bodies which are not necessary smooth. This in turn implies that the formula above of the p-affine perimeter can be extended by

$$\Omega_p(K) = \int_{\partial K} s \left(\frac{\kappa}{s^3} \right)^{\frac{p}{2+p}} dx,$$

for every convex body K in $\mathbb{R}^2$, having the origin in its interior such that when K is smooth these two definitions coincide for all $p \geq 1$.

4.2 Stability of the p-affine isoperimetric inequality

Throughout this section we assume that $K_0 = K$ is smooth and $A(K) = \pi$.

4.2.1 Preliminaries

We list several lemmas and a theorem necessary for our proof of the main theorem. We first recall two lemmas proved in [52] and Chapter 2, Lemmas 7 and 31.

Lemma 53 (Evolution equation of the area)**.** *[52] As $\{K_t\}$ evolve by evolution equation (4.1.1), then $A(K_t)$ evolves by $\frac{d}{dt}A(K_t) = -\Omega_1(K_t)$. In particular, $A(K_t)$ is decreasing.*

Lemma 54 (Ω_p along the affine normal flow)**.** *[52] The following affine isoperimetric inequalities hold along the affine normal flow.*
If $1 \le p \le 2$, then

$$\frac{d}{dt}\Omega_p(K_t) \ge \frac{p-2}{p+2}\frac{\Omega_p(K_t)\Omega_1(K_t)}{A(K_t)} + \frac{2(p-1)(4p^2+3p+2)}{(p+2)^3}\int_{\partial K_t} \sigma^{-1-\frac{3p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s},$$

while, if $p \ge 2$, we then have

$$\frac{d}{dt}\Omega_p(K_t) \ge \frac{p-2}{p+2}\frac{\Omega_p(K_t)\Omega_1(K_t)}{A(K_t)} + \frac{6p}{(p+2)^2}\int_{\partial K_t} \sigma^{-1-\frac{3p}{p+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s}.$$

Here, $\mathfrak{s}$ is the affine arc-length of the evolving boundary curve ∂K_t.

The affine support function is constant for an origin-centered ellipse and the relation between its value and the area of the ellipse is as follows. For an ellipse, $\mathcal{E}$, denote its constant affine support function by $\sigma_{\mathcal{E}}$. We have

$$\sigma_{\mathcal{E}} = \left(\frac{A(\mathcal{E})}{\pi}\right)^{2/3}.$$

We also restate Lemma 1 in a slightly different way for the affine support function.

Lemma 55 (Stability of the affine support function)**.** *Suppose that K is a convex body in $\mathcal{K}_{sym}$. If $m \le \sigma \le M$ for some positive numbers m and M, then there exist* *two ellipses* $\mathcal{E}_{in}$ *and* $\mathcal{E}_{out}$ *such that* $\mathcal{E}_{in} \subseteq K \subseteq \mathcal{E}_{out}$ *and*

$$\left(\frac{A(\mathcal{E}_{in})}{\pi}\right)^{2/3} = m, \quad \left(\frac{A(\mathcal{E}_{out})}{\pi}\right)^{2/3} = M.$$

Lemma 56. *Let* K *be an origin-symmetric, smooth convex body with area* π*. Then*

$$\min_{\partial K} \sigma \leq 1 \leq \max_{\partial K} \sigma.$$

Proof. The claim follows from Lemma 55: If $\min_{\mathbb{S}^1} \sigma > 1$, then there is an ellipse $\mathcal{E}_{in}$ which is contained in K and satisfies $\left(\frac{A(\mathcal{E}_{in})}{\pi}\right)^{2/3} > 1$. This implies that $A(\mathcal{E}_{in}) > \pi$. Similarly, if $\max_{\mathbb{S}^1} \sigma < 1$, then there is an ellipse $\mathcal{E}_{out}$ which contains K and has the area $A(\mathcal{E}_{out}) < \pi$. In both cases we reach a contradiction as the area of K is π. □

We state the following important Theorem 5 from [10].

Theorem 57 (Controlling Hausdorff distance I)**.** *[10] Let* $\{K_t\}$ *be a smooth, strictly convex solution of evolution equation (4.1.1). Then*

$$s(z,t) \geq s(z,0) - \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{c_2}{c_1} t^{\frac{3}{4}},$$

for $t \in \left(0, \frac{3}{4} c_1^{\frac{4}{3}}\right)$*. In particular,*

$$K \subseteq K_t + \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{c_2}{c_1} t^{\frac{3}{4}} B_1$$

for $t \in \left(0, \frac{3}{4} c_1^{\frac{4}{3}}\right)$.

Let $\mathcal{E}$ be an ellipse. We denote its semi-minor and semi-major axes by $a_{\mathcal{E}}$ and $b_{\mathcal{E}}$, respectively. We also need the following simple lemma.

Lemma 58 (Controlling Hausdorff distance II)**.** *Let* $\mathcal{E}$ *be an ellipse centered at the origin of the plane such that* $\mathcal{E} \subseteq B_R$*. Then we have*

$$d_{\mathcal{H}}(\mathcal{E}, B_R) \leq \frac{A(B_R) - A(\mathcal{E})}{\pi \left(\frac{A(B_R)}{\pi}\right)^{\frac{1}{2}}}.$$

Proof. We have

$$\begin{aligned} d_{\mathcal{H}}(\mathcal{E}, B_R) &\le R - a_{\mathcal{E}} \\ &= \left(\frac{A(B_R)}{\pi}\right)^{\frac{1}{2}} - \frac{A(\mathcal{E})}{\pi b_{\mathcal{E}}} \\ &\le \left(\frac{A(B_R)}{\pi}\right)^{\frac{1}{2}} - \frac{A(\mathcal{E})}{\pi R} = \frac{A(B_R) - A(\mathcal{E})}{\pi \left(\frac{A(B_R)}{\pi}\right)^{\frac{1}{2}}}. \end{aligned}$$

The proof is complete. □

4.2.2 Proof of the main theorem

In this section, we present a proof of the stability of the p-affine isoperimetric inequality.

Proof. Let $p > 1$ and $0 < \varepsilon_p < \frac{1}{2}$. The upper bound on ε_p will be determined later at the end of this section. Assume that

$$\left(\frac{\Omega_p^{2+p}(K)}{2^{2+p}A^{2-p}(K)}\right)^{\frac{1}{p}} > \pi^2(1 - \varepsilon_p). \tag{4.2.1}$$

Then from Lemma 54 and Lemma 53, it follows that

$$\begin{aligned} \frac{d}{dt}\left(\frac{\Omega_p^{2+p}(K_t)}{A^{2-p}(K_t)}\right)^{\frac{1}{p}} &= \frac{1}{p}\left(\frac{\Omega_p^{2+p}(K_t)}{A^{2-p}(K_t)}\right)^{\frac{1}{p}-1} \frac{d}{dt}\left(\frac{\Omega_p^{2+p}(K_t)}{A^{2-p}(K_t)}\right) \\ &\ge d_p \left(\frac{\Omega_p^{2+p}(K_t)}{A^{2-p}(K_t)}\right)^{\frac{1}{p}-1} \frac{\Omega_p^{p+1}(K_t)}{A^{2-p}(K_t)} \int_{\partial K_t} \left(\sigma^{\frac{1}{2}-\frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} \\ &= \frac{d_p}{\Omega_p(K_t)}\left(\frac{\Omega_p^{2+p}(K_t)}{A^{2-p}(K_t)}\right)^{\frac{1}{p}} \int_{\partial K_t} \left(\sigma^{\frac{1}{2}-\frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} \end{aligned} \tag{4.2.2}$$

where d_p is defined as follows

$$d_p := \begin{cases} \frac{2(4p^2+3p+2)}{p(p-1)}, & \text{if } 1 < p \le 2, \\ \frac{6(p+2)}{(p-1)^2}, & \text{if } p \ge 2. \end{cases}$$

We integrate both sides of the inequality (4.2.2) on the time interval $[0,\delta]$ with respect to dt.

$$\begin{aligned}\int_0^\delta \frac{d}{dt}\left(\frac{\Omega_p^{2+p}(K_t)}{A^{2-p}(K_t)}\right)^{\frac{1}{p}} dt &\geq \int_0^\delta \frac{d_p}{\Omega_p(K_t)}\left(\frac{\Omega_p^{2+p}(K_t)}{A^{2-p}(K_t)}\right)^{\frac{1}{p}} \int_{\partial K_t}\left(\sigma^{\frac{1}{2}-\frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}dt \\ &\geq \int_0^\delta \min_{t\in[0,\delta]}\left(\frac{d_p}{\Omega_p(K_t)}\left(\frac{\Omega_p^{2+p}(K_t)}{A^{2-p}(K_t)}\right)^{\frac{1}{p}} \int_{\partial K_t}\left(\sigma^{\frac{1}{2}-\frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}\right) dt \\ &= \frac{d_p\delta}{\Omega_p(K_{t_*})}\left(\frac{\Omega_p^{2+p}(K_{t_*})}{A^{2-p}(K_{t_*})}\right)^{\frac{1}{p}} \int_{\partial K_{t_*}}\left(\sigma^{\frac{1}{2}-\frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}\end{aligned}$$

where t_* is the time when $\min\limits_{t\in[0,\delta]} \frac{1}{\Omega_p(K_t)}\left(\frac{\Omega_p^{2+p}(K_t)}{A^{2-p}(K_t)}\right)^{\frac{1}{p}} \int_{\partial K_t}\left(\sigma^{\frac{1}{2}-\frac{3p}{2(p+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}$ is achieved. Therefore, by the Hölder inequality, we find

$$2^{\frac{p+2}{p}}\pi^2\varepsilon_p \geq \frac{d_p\delta}{\Omega_1(K_{t_*})\Omega_p(K_{t_*})}\left(\frac{\Omega_p^{2+p}(K_{t_*})}{A^{2-p}(K_{t_*})}\right)^{\frac{1}{p}}\left(\sigma_M^{\frac{1}{2}-\frac{3p}{2(p+2)}}(t_*) - \sigma_m^{\frac{1}{2}-\frac{3p}{2(p+2)}}(t_*)\right)^2.$$

Here, $\sigma_M^{\frac{1}{2}-\frac{3p}{2(p+2)}}(t_*)$ and $\sigma_m^{\frac{1}{2}-\frac{3p}{2(p+2)}}(t_*)$ are, respectively, the maximum and the minimum of $\sigma^{\frac{1}{2}-\frac{3p}{2(p+2)}}$ on ∂K_{t_*}. It follows that

$$\sqrt{\frac{2^{\frac{p+2}{p}}\pi^2\Omega_1(K_{t_*})\Omega_p(K_{t_*})\varepsilon_p}{d_p\delta}} \geq \left(\frac{\Omega_p^{2+p}(K_{t_*})}{A^{2-p}(K_{t_*})}\right)^{\frac{1}{2p}}\left(\sigma_M^{\frac{1}{2}-\frac{3p}{2(p+2)}}(t_*) - \sigma_m^{\frac{1}{2}-\frac{3p}{2(p+2)}}(t_*)\right). \tag{4.2.3}$$

To bound $\Omega_1(K_{t_*})\Omega_p(K_{t_*})$ from above, we need to consider two cases. Let $1 < p \leq 2$. By Lemma 53, we have $A(K_{t_*}) \leq A(K) = \pi$. Therefore, by the affine isoperimetric inequality and the p-affine isoperimetric inequality, we infer that

$$\Omega_1(K_{t_*}) \leq 2\pi^{\frac{2}{3}}A^{\frac{1}{3}}(K_{t_*}) \leq 2\pi,$$

$$\Omega_p(K_{t_*}) \leq 2\pi^{\frac{2p}{p+2}}A^{\frac{2-p}{p+2}}(K_{t_*}) \leq 2\pi,$$

and thus $\Omega_1(K_{t_*})\Omega_p(K_{t_*}) \leq 4\pi^2$.

Now we proceed to deal with the case $p > 2$. Recall from the evolution equation of the area, Lemma 53, that

$$\frac{d}{dt}A(K_t) = -\Omega_1(K_t) \geq -2\pi,$$

hence

$$A(K_\delta) \geq A(K) - 2\pi\delta = \pi(1-2\delta).$$

If $\delta < \frac{1}{4}$ then $A(K_\delta) > \frac{\pi}{2}$. In particular, this yields that $A(K_{t_*}) > \frac{\pi}{2}$. This observation combined with the p-affine isoperimetric inequality imply that

$$\Omega_p(K_{t_*}) \leq 2\pi^{\frac{2p}{p+2}} A^{\frac{2-p}{p+2}}(K_{t_*}) \leq 2^{\frac{2p}{p+2}}\pi.$$

As $\Omega_1(K_{t_*}) \leq 2\pi$, we get $\Omega_1(K_{t_*})\Omega_p(K_{t_*}) \leq 2^{\frac{2p+2}{p+2}}\pi^2 < 4\pi^2$. Consequently, assuming $\delta < \frac{1}{4}$, together with inequalities (4.2.1) and (4.2.3), yields

$$\left(\sigma_M^{\frac{1}{2}-\frac{3p}{2(p+2)}}(t_*) - \sigma_m^{\frac{1}{2}-\frac{3p}{2(p+2)}}(t_*)\right) \leq \frac{2\sqrt{2}\pi}{\sqrt{d_p}}\sqrt{\frac{\varepsilon_p}{\delta}}.$$

Define $d'_p := \frac{2\sqrt{2}\pi}{\sqrt{d_p}}$. Multiplying K_{t_*} by a factor λ, depending on δ, where $\lambda \geq 1$, we can have $A(\lambda K_{t_*}) = \pi$. Note that $\lim\limits_{\delta\to 0}\lambda = 1$. In particular, by this assumption and Lemma 56, we have

$$1 \in \left[\lambda^{\frac{(1-p)}{3(p+2)}}\sigma_m^{\frac{1-p}{2+p}}(t_*), \lambda^{\frac{(1-p)}{3(p+2)}}\sigma_M^{\frac{1-p}{2+p}}(t_*)\right].$$

As a result,

$$\lambda^{\frac{4(1-p)}{3(p+2)}}\sigma_M^{\frac{1-p}{2+p}}(t_*) \leq d'_p\sqrt{\frac{\varepsilon_p}{\delta}} + 1,$$

and

$$\lambda^{\frac{4(1-p)}{3(p+2)}}\sigma_m^{\frac{1-p}{2+p}}(t_*) \geq 1 - d'_p\sqrt{\frac{\varepsilon_p}{\delta}}.$$

Let us assume for now that

$$1 - d'_p\sqrt{\frac{\varepsilon_p}{\delta}} > 0. \tag{4.2.4}$$

Consequently,

$$\frac{1}{\left(1+d_p'\sqrt{\frac{\varepsilon_p}{\delta}}\right)^{\frac{p+2}{p-1}}} \leq \lambda^{\frac{4}{3}}\sigma(t_*) \leq \frac{1}{\left(1-d_p'\sqrt{\frac{\varepsilon_p}{\delta}}\right)^{\frac{p+2}{p-1}}}.$$

From the last inequality and Lemma 55 we deduce that there exist two ellipses, denoted by $\mathcal{E}_{in}$ and $\mathcal{E}_{out}$, such that

$$\mathcal{E}_{in} \subseteq K_{t_*} \subseteq \mathcal{E}_{out}, \tag{4.2.5}$$

and

$$\left(\frac{A(\mathcal{E}_{out})}{\pi}\right)^{2/3} = \frac{\lambda^{-\frac{4}{3}}}{\left(1-d_p'\sqrt{\frac{\varepsilon_p}{\delta}}\right)^{\frac{p+2}{p-1}}}, \quad \left(\frac{A(\mathcal{E}_{in})}{\pi}\right)^{2/3} = \frac{\lambda^{-\frac{4}{3}}}{\left(1+d_p'\sqrt{\frac{\varepsilon_p}{\delta}}\right)^{\frac{p+2}{p-1}}}.$$

On the other hand, let us assume that $\delta < \frac{3}{4}c_1^{\frac{4}{3}}$, then by Theorem 57

$$K_{t_*} \subseteq K \subseteq K_{t_*} + \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{c_2}{c_1} t_*^{\frac{3}{4}} B_1 \subseteq K_{t_*} + \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{c_2}{c_1} \delta^{\frac{3}{4}} B_1. \tag{4.2.6}$$

Combining relations (4.2.5) and (4.2.6), we find

$$\mathcal{E}_{in} \subseteq K \subseteq \mathcal{E}_{out} + \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{c_2}{c_1} \delta^{\frac{3}{4}} B_1.$$

Set $\delta := \varepsilon_p^{\frac{\beta}{2+\beta}}$, for a positive β, in the previous inequality. For $\varepsilon_p < \left(\frac{3}{4}c_1^{\frac{4}{3}}\right)^{\frac{2+\beta}{\beta}}$, we have

$$\mathcal{E}_{in} \subseteq K \subseteq \mathcal{E}_{out} + \left(\frac{4}{3}\right)^{\frac{3}{4}} \frac{c_2}{c_1} \varepsilon_p^{\frac{3\beta}{4(2+\beta)}} B_1, \tag{4.2.7}$$

and

$$\left(\frac{A(\mathcal{E}_{out})}{\pi}\right)^{2/3} = \frac{\lambda^{-\frac{4}{3}}}{\left(1-d_p'\varepsilon_p^{\frac{1}{2+\beta}}\right)^{\frac{p+2}{p-1}}}, \quad \left(\frac{A(\mathcal{E}_{in})}{\pi}\right)^{2/3} = \frac{\lambda^{-\frac{4}{3}}}{\left(1+d_p'\varepsilon_p^{\frac{1}{2+\beta}}\right)^{\frac{p+2}{p-1}}}. \tag{4.2.8}$$

We now get back to the assumption (4.2.4). If we choose $\varepsilon_p < \left(\frac{1}{d'_p}\right)^{2+\beta}$, then

$$1 - d'_p \varepsilon_p^{\frac{1}{2+\beta}} > 0.$$

On the other hand, $\delta := \varepsilon_p^{\frac{\beta}{2+\beta}} < \frac{1}{4}$ and $\varepsilon_p < \left(\frac{3}{4}c_1^{\frac{4}{3}}\right)^{\frac{2+\beta}{\beta}}$. Therefore, choosing

$$\varepsilon_p < \min\left\{\left(\frac{1}{4}\right)^{\frac{1+\beta}{\beta}}, \left(\frac{1}{d'_p}\right)^{2+\beta}, \left(\frac{3}{4}c_1^{\frac{4}{3}}\right)^{\frac{2+\beta}{\beta}}\right\}$$

guarantees that both assumptions (4.2.4) and (4.2.6) hold.

Recall that $B_{c_1} \subseteq K$. Therefore by the Containment Principle, $B_{c_1/2} \subseteq K_t$ for $t \in [0, \eta]$, for an η independent of K. Precisely, $\eta = \frac{3}{4}c_1^{\frac{4}{3}}\left(1 - \left(\frac{1}{2}\right)^{\frac{4}{3}}\right)$ is the time that B_{c_1} shrinks to $B_{c_1/2}$ under the affine normal flow. If we choose $\delta = \varepsilon_p^{\frac{\beta}{2+\beta}} < \eta$, then from (4.2.5) we get

$$B_{c_1/2} \subseteq K_{t_*} \subseteq \mathcal{E}_{out}.$$

From this we conclude that, if

$$\varepsilon_p < \min\left\{\left(\frac{1}{4}\right)^{\frac{1+\beta}{\beta}}, \left(\frac{1}{d'_p}\right)^{2+\beta}, \left(\frac{3}{4}c_1^{\frac{4}{3}}\right)^{\frac{2+\beta}{\beta}}\left(1 - \left(\frac{1}{2}\right)^{\frac{4}{3}}\right)^{\frac{(2+\beta)}{\beta}}\right\},$$

then

$$\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{c_2}{c_1}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}B_1 = \left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}B_{c_1/2} \subseteq \left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\mathcal{E}_{out}.$$

By (4.2.7), we find

$$\mathcal{E}_{in} \subseteq K \subseteq \left(1 + \left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\right)\mathcal{E}_{out}. \tag{4.2.9}$$

We apply a special linear transformation, $T \in SL(2)$, such that $T\mathcal{E}_{out}$ is a disk. Consequently, by relation (4.2.9) we get

$$T\mathcal{E}_{in} \subseteq TK \subseteq \left(1 + \left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\right)T\mathcal{E}_{out}. \tag{4.2.10}$$

Now, from the facts that $\mathcal{E}_{in} \subseteq \mathcal{E}_{out}$, area is invariant under special linear transformations, Lemma 58, and identities in (4.2.8), we have

$$d_{\mathcal{H}}\left(T\mathcal{E}_{in}, \left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\right)T\mathcal{E}_{out}\right) \leq \frac{\left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\right)^2 A(\mathcal{E}_{out}) - A(\mathcal{E}_{in})}{\left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\right)\pi\left(\frac{A(\mathcal{E}_{out})}{\pi}\right)^{\frac{1}{2}}}.$$

Therefore $d_{\mathcal{H}}\left(T\mathcal{E}_{in}, \left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\right)T\mathcal{E}_{out}\right)$ is bounded by

$$\frac{1}{\lambda}\frac{\left(1-d_p'\varepsilon_p^{\frac{1}{2+\beta}}\right)^{\frac{3(p+2)}{4(p-1)}}}{\left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\right)}\left(\frac{\left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\right)^2}{\left(1-d_p'\varepsilon_p^{\frac{1}{2+\beta}}\right)^{\frac{3(p+2)}{2(p-1)}}} - \frac{1}{\left(1+d_p'\varepsilon_p^{\frac{1}{2+\beta}}\right)^{\frac{3(p+2)}{2(p-1)}}}\right)$$

$$\leq \frac{\left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3\beta}{4(2+\beta)}}\right)^2}{\left(1-d_p'\varepsilon_p^{\frac{1}{2+\beta}}\right)^{\frac{3(p+2)}{2(p-1)}}} - \frac{1}{\left(1+d_p'\varepsilon_p^{\frac{1}{2+\beta}}\right)^{\frac{3(p+2)}{2(p-1)}}}.$$

To optimize the Hausdorff distance, we set $\beta = \frac{4}{3}$. Observe that

$$\lim_{\varepsilon_p\to 0^+}\frac{\frac{\left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}\varepsilon_p^{\frac{3}{10}}\right)^2}{\left(1-d_p'\varepsilon_p^{\frac{3}{10}}\right)^{\frac{3(p+2)}{2(p-1)}}} - \frac{1}{\left(1+d_p'\varepsilon_p^{\frac{3}{10}}\right)^{\frac{3(p+2)}{2(p-1)}}}}{\varepsilon_p^{\frac{3}{10}}} = 2\left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}+\frac{3(p+2)}{2(p-1)}d_p'\right).$$

Define $\mathcal{E} := T\mathcal{E}_{in}$, $\mathcal{D} := T\mathcal{E}_{out}$ and $C_p := 3\left(1+\left(\frac{4}{3}\right)^{\frac{3}{4}}\frac{2c_2}{c_1^2}+\frac{3(p+2)}{2(p-1)}d_p'\right)$. Therefore, choosing ε_p small enough implies the claim for $p>1$.

To complete the proof of the main theorem, we need to address the case $p=1$. We notice that if

$$\frac{\Omega_1^3(K)}{8A(K)} > \pi^2(1-\varepsilon),$$

then for every $p>1$

$$\left(\frac{\Omega_p^{2+p}(K)}{2^{2+p}A^{2-p}(K)}\right)^{\frac{1}{p}} > \pi^2(1-\varepsilon).$$

This is because the function $p \mapsto \left(\frac{\Omega_p^{2+p}(K)}{2^{2+p}A^{2-p}(K)}\right)^{\frac{1}{p}}$ is increasing, [69]. Hence, to prove the stability of the affine isoperimetric inequality, we can continue the argument for the stability of the p-affine isoperimetric inequality, for example with $p = 2$. In this case $C_1 = C_2$. The proof is now complete. $\square$

Chapter 5

Harnack inequality and ancient solutions

The contents of this chapter are taken from the manuscript ”Centro-affine normal flows: Harnack inequality and ancient solutions” [51].

We obtain a Harnack inequality for the planar p centro-affine normal flows on curves and we classify compact, origin-symmetric, ancient solutions to this family of flows if $1 \leq p < 4$. In particular, we classify origin-symmetric, compact ancient solutions to the planar affine normal flow.

5.1 Introduction

Let $K_0 \in \mathcal{K}_{sym}$. We consider a family $\{K_t\}_t \in \mathcal{K}_{sym}$, and their associated smooth embeddings $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$, which are evolving according to the p centro-affine normal flow, namely,

$$\partial_t x := -\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}-\frac{1}{3}} \kappa^{\frac{1}{3}}\, z, \quad x(\cdot, 0) = x_{K_0}(\cdot), \quad x(\cdot, t) = x_{K_t}(\cdot) \tag{5.1.1}$$

for a fixed $p \geq 1$.

In this chapter, we prove the following proposition and theorem:

Proposition (Harnack inequality)**.** *Let $p \geq 1$. Along the flow (5.1.1) we have*

$$\partial_t \left(s^{1-\frac{3p}{p+2}} \mathfrak{r}^{-\frac{p}{p+2}} t^{\frac{p}{2p+2}} \right) \geq 0,$$

on $\mathbb{S}^1 \times (0, T)$.

Theorem (Theorem D)**.** *Let $1 \leq p < 4$. The only compact, origin-symmetric, ancient solutions to the p-flow are homothetic ellipses.*

In the next section, we obtain the Harnack inequality for the p-flow. In the third section, as an application, we classify compact, ancient solutions to this family of flows in the class $\mathcal{K}_{sym}$, if $1 \leq p < 4$. In particular, we classify compact, origin-symmetric, ancient solutions to the planar affine normal flow.

5.2 Harnack inequality

In this section, we follow [6] to obtain the Harnack inequality.

Proof. For simplicity, we set $\alpha = -\frac{p}{p+2}$. To prove the proposition, using the parabolic maximum principle, we prove that the quantity defined by

$$\mathcal{R} := t\mathcal{P} - \frac{\alpha}{\alpha - 1} s^{1+3\alpha} \mathfrak{r}^{\alpha} \tag{5.2.1}$$

remains negative as long as the flow exists. Here $\mathcal{P}$ is defined as follows

$$\mathcal{P} := \partial_t \left(-s^{1+3\alpha} \mathfrak{r}^{\alpha} \right).$$

Let us restate Lemma 7 again.

Lemma 59. *[52] Along the p-flow, (5.1.1), we have*

- $\partial_t s = -s^{1+3\alpha}\mathfrak{r}^{\alpha}$,
- $\partial_t \mathfrak{r} = -\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}\right].$

Using the evolution equations of s and $\mathfrak{r}$ we find

$$\begin{aligned}\mathcal{P} &= (1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha} + \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}\right] \\ &:= (1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha} + \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{Q}. \end{aligned} \tag{5.2.2}$$

Lemma 60. *We have the following evolution equation for $\mathcal{P}$ as long as the flow exists.*

$$\begin{aligned}\partial_t\mathcal{P} = &-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right] + \left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right]s^{1+9\alpha}\mathfrak{r}^{3\alpha} \\ &+ \left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right]s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}}.\end{aligned}$$

Proof. We repeatedly use the evolution equation of s and $\mathfrak{r}$.

$$\begin{aligned}&\partial_t\mathcal{P} \\ &= -(1+3\alpha)(1+6\alpha)s^{1+9\alpha}\mathfrak{r}^{3\alpha} - 2\alpha(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha-1}\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}\right] \\ &- \alpha(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha-1}\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}\right] \\ &- \alpha(\alpha-1)s^{1+3\alpha}\mathfrak{r}^{\alpha-2}\left[\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta} + s^{1+3\alpha}\mathfrak{r}^{\alpha}\right]^2 - \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right] \\ &= -(1+3\alpha)(1+6\alpha)s^{1+9\alpha}\mathfrak{r}^{3\alpha} - 3\alpha(1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha-1}\mathcal{Q} \\ &- \alpha(\alpha-1)s^{1+3\alpha}\mathfrak{r}^{\alpha-2}\mathcal{Q}^2 - \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right].\end{aligned}$$

By the definition of $\mathcal{Q}$, (5.2.2):

$$\mathcal{Q}^2 = \frac{\mathcal{P}^2}{\alpha^2 s^{2+6\alpha}\mathfrak{r}^{2\alpha-2}} - \frac{2(3\alpha+1)}{\alpha^2}\frac{\mathcal{P}\mathfrak{r}^2}{s} + \frac{(3\alpha+1)^2}{\alpha^2}s^{6\alpha}\mathfrak{r}^{2\alpha+2}$$

and

$$\mathcal{Q} = \frac{\mathcal{P} - (1+3\alpha)s^{1+6\alpha}\mathfrak{r}^{2\alpha}}{\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}}.$$

Substituting these expressions into the evolution equation of $\mathcal{P}$ we find that

$$\partial_t \mathcal{P} = -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right] + \left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right] s^{1+9\alpha}\mathfrak{r}^{3\alpha}$$
$$- 3(3\alpha+1)s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P}.$$

This completes the proof of Lemma 60. □

We now proceed to find the evolution equation of $\mathcal{R}$ which is defined by (5.2.1). First note that

$$-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} = -t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P}_{\theta\theta} + \frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta}.$$

Therefore, by Lemma 60 and identity (5.2.2)

$$\begin{aligned}
&\partial_t \mathcal{R} \\
&= -t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left[\mathcal{P}_{\theta\theta}+\mathcal{P}\right] + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right] s^{1+9\alpha}\mathfrak{r}^{3\alpha} \\
&+ t\left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right] s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \mathcal{P} + \frac{\alpha}{\alpha-1}\mathcal{P} \\
&- \alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} + t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P}_{\theta\theta} - \frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right)_{\theta\theta} \\
&+ \frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right) - \frac{\alpha^2}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right) \\
&+ \frac{\alpha(3\alpha+1)}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+6\alpha}\mathfrak{r}^{2\alpha}\right) - \frac{\alpha(3\alpha+1)}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\left(s^{1+6\alpha}\mathfrak{r}^{2\alpha}\right) \\
&= -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right] s^{1+9\alpha}\mathfrak{r}^{3\alpha} \\
&+ t\left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right] s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \mathcal{P} \\
&+ \frac{\alpha}{\alpha-1}\mathcal{P} - \frac{\alpha}{\alpha-1}\mathcal{P} - t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P} + \frac{\alpha^2}{\alpha-1}s^{2+6\alpha}\mathfrak{r}^{2\alpha-1} + \frac{\alpha(3\alpha+1)}{\alpha-1}s^{2+9\alpha}\mathfrak{r}^{3\alpha-1}
\end{aligned}$$

$$
\begin{aligned}
&= -\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{R}_{\theta\theta} + t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right] s^{1+9\alpha}\mathfrak{r}^{3\alpha} \\
&+ t\left[-3(3\alpha+1) + \frac{2(\alpha-1)(3\alpha+1)}{\alpha}\right] s^{3\alpha}\mathfrak{r}^{\alpha}\mathcal{P} - t\frac{\alpha-1}{\alpha}\frac{\mathcal{P}^2}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \mathcal{P} \\
&- t\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1}\mathcal{P} + \frac{\alpha^2}{\alpha-1} s^{2+6\alpha}\mathfrak{r}^{2\alpha-1} + \frac{\alpha(3\alpha+1)}{\alpha-1} s^{2+9\alpha}\mathfrak{r}^{3\alpha-1}.
\end{aligned}
$$

To make the last computation useful, in the last expression, using the definition of $\mathcal{R}$, identity (5.2.1), we replace $t\mathcal{P}$ by $\mathcal{R} + \frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha}$. Therefore, at the point where the maximum of $\mathcal{R}$ is achieved, we have

$$
\begin{aligned}
&\partial_t \mathcal{R} \\
&\leq \mathcal{R}\left[-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha} - 3(3\alpha+1)\right] s^{3\alpha}\mathfrak{r}^{\alpha}\right] \\
&+ \frac{\alpha}{\alpha-1}\left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha} - 3(3\alpha+1)\right] s^{2+6\alpha}\mathfrak{r}^{2\alpha} + \frac{\alpha(3\alpha+1)}{\alpha-1} s^{2+9\alpha}\mathfrak{r}^{3\alpha-1} \\
&+ t\left[(3\alpha+1)(3\alpha+2) - \frac{(\alpha-1)(3\alpha+1)^2}{\alpha}\right] s^{1+9\alpha}\mathfrak{r}^{3\alpha} \\
&\leq \mathcal{R}\left[-\alpha s^{1+3\alpha}\mathfrak{r}^{\alpha-1} - \frac{\alpha-1}{\alpha}\frac{\mathcal{P}}{s^{1+3\alpha}\mathfrak{r}^{\alpha}} + \left[\frac{2(\alpha-1)(3\alpha+1)}{\alpha} - 3(3\alpha+1)\right] s^{3\alpha}\mathfrak{r}^{\alpha}\right].
\end{aligned}
$$

To obtain the last inequality, we used the fact that terms on the second and third line are negative for $p \geq 1$. Hence by the parabolic maximum principle, we have $\mathcal{R} = t\mathcal{P} - \frac{\alpha}{\alpha-1}s^{1+3\alpha}\mathfrak{r}^{\alpha} \leq 0$. Since at the time zero we have $\mathcal{R} \leq 0$. Negativity of $\mathcal{R}$ is equivalent to

$$
\partial_t \ln\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}\right) \geq \frac{\alpha}{1-\alpha}\frac{1}{t},
$$

for $t > 0$. From this, we conclude that

$$
\partial_t\left(s^{1+3\alpha}\mathfrak{r}^{\alpha}t^{\frac{\alpha}{\alpha-1}}\right) \geq 0
$$

for $t > 0$. The proof of the main proposition is complete. □

By the Harnack inequality, every solution to the flow (5.1.1) satisfies

$$\partial_t\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)+\frac{p}{2t(p+1)}\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)\geq 0. \tag{5.2.3}$$

Therefore, we have the following corollary.

Corollary 61. *Every ancient solution to the flow (5.1.1) satisfies*

$$\partial_t\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)\geq 0.$$

Proof. We let the flow start from a fixed time $t_0<0$. Then the inequality (5.2.3) becomes

$$\partial_t\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)+\frac{p}{2(t-t_0)(p+1)}\left(s\left(\frac{\kappa}{s^3}\right)^{\frac{p}{p+2}}\right)\geq 0.$$

Now letting t_0 go to $-\infty$ proves the claim. □

Corollary 62. *Every ancient solution to the flow (5.1.1) satisfies:*

$$\partial_t\left(\frac{\kappa^{\frac{1}{3}}}{s}\right)\geq 0.$$

Proof. The support function, s, is decreasing on the time interval $(-\infty,T)$. The claim now follows from the previous corollary. □

5.3 Affine differential setting

Let us outline our argument presented in the rest of this chapter. The Harnack inequality is an important ingredient in our argument: We first convert Corollary 62 stated in the Gauss parametrization ($\mathcal{G}$) to a corollary stated in the Euclidean parametrization ($\mathcal{E}$). By using the new corollary, the evolution equation of the affine support function, monotonicity of the l-affine isoperimetric ratio, for $l\geq 2$, we obtain the asymptotic value of the affine support function as t approaches negative infinity. Then using a standard ODE comparison theorem, Lemma 1, we prove that the area product converges to its maximum value, which is achieved only for ellipses, as t converges to negative infinity. We proved in Chapter 2 that the area product converges to its maximum value as t converges to the extinction time T, Corollary 22. On the other hand, the area product is a monotone quantity (Proposition 2.2, [100]). Therefore, the area product must be constant along the p-flow which in turn implies that the only origin-symmetric, ancient solutions are ellipses. To carry out the outlined strategy, we resort to affine differential geometry.

Let $K_0 \in \mathcal{K}_{sym}$. Consider a family $\{K_t\}_t \in \mathcal{K}_{sym}$, and their associated smooth embeddings $x : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$, which are evolving according to (5.1.1). Then, up to a time-dependant diffeomorphism, $\{K_t\}_t$ evolves according to

$$\frac{\partial}{\partial t}x := \sigma^{1-\frac{3p}{p+2}}\mathfrak{n}, \quad x(\cdot, 0) = x_{K_0}(\cdot), \quad x(\cdot, t) = x_{K_t}(\cdot). \tag{5.3.1}$$

Therefore, classification of compact, origin-symmetric ancient solutions to (5.1.1) is equivalent to the classification of compact, origin-symmetric ancient solutions to (5.3.1). In what follows, our reference flow is the evolution equation (5.3.1).

Notice that, as a family of convex bodies evolve according to the evolution equation (5.3.1), then in the Gauss parametrization their support functions and curvatures evolve according to Lemma 59. Consequently, as $\{K_t\}_t$ evolve according to the evolution equation (5.3.1) we have, in the Gauss parametrization, that $\partial_t \left(\frac{\kappa^{1/3}}{s}\right) \geq 0$, by Corollary 62.

It can be easily verified that the evolution equation of a geometric quantity Q in the Euclidean parametrization and in the Gauss parametrization along the flow

(5.3.1) are related by

$$(\partial_t Q)_{\mathcal{G}} = (\partial_t Q)_{\mathcal{E}} - Q_{\mathfrak{s}} \left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}},$$

see Lemma 2.3, [50]. In particular, we have

$$\begin{aligned} 0 \geq (\partial_t \sigma)_{\mathcal{G}} &= (\partial_t \sigma)_{\mathcal{E}} - \sigma_{\mathfrak{s}} \left(\sigma^{1-\frac{3p}{p+2}}\right)_{\mathfrak{s}} \\ &= (\partial_t \sigma)_{\mathcal{E}} + \left(\frac{3p}{p+2} - 1\right) \sigma_{\mathfrak{s}}^2 \sigma^{-\frac{3p}{p+2}}. \end{aligned}$$

Therefore, the affine support function is non-increasing along the flow (5.3.1). Furthermore, we have proved

Corollary 63. *Every ancient solution to the flow (5.1.1) satisfies:*

$$(\partial_t \sigma)_{\mathcal{E}} \leq -\left(\frac{3p}{p+2} - 1\right) \sigma_{\mathfrak{s}}^2 \sigma^{-\frac{3p}{p+2}}.$$

For the rest of this chapter, we work in the Euclidean parametrization and for simplicity, we drop the subscript $\mathcal{E}$. The next two Lemmas were proved in [52] and also in Chapter 2.

Lemma 64. *[52] Let $\gamma_t := \partial K_t$ be the boundary of a convex body K_t evolving under the flow (5.3.1). Then the following evolution equations hold:*

1. $\frac{\partial}{\partial t}\sigma = \sigma^{1-\frac{3p}{p+2}} \left(-\frac{4}{3} + \left(\frac{p}{p+2} + 1\right)\left(1 - \frac{3p}{p+2}\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma} + \frac{p}{p+2}\sigma_{\mathfrak{s}\mathfrak{s}}\right),$
2. $\frac{d}{dt}A = -\Omega_p.$

Lemma 65. *[52] The following evolution equation for Ω_l under the p-flow for each $l \geq 2$ and $p \geq 1$ holds:*

$$\frac{d}{dt}\Omega_l(t) = \frac{2(l-2)}{l+2}\int_{\gamma_t} \sigma^{1-\frac{3p}{p+2}-\frac{3l}{l+2}} d\mathfrak{s} + \frac{18pl}{(l+2)^2(p+2)}\int_{\gamma_t} \sigma^{-\frac{3p}{p+2}-\frac{3l}{l+2}} \sigma_{\mathfrak{s}}^2 d\mathfrak{s}. \quad (5.3.2)$$

Let us denote $\max_{\gamma_t} \sigma$ and $\min_{\gamma_t} \sigma$ by σ_M and σ_m, respectively.

Lemma 66. *There is a constant c, depending on p, such that*

$$\frac{\sigma_M}{\sigma_m} \le c$$

on the time interval $(-\infty, T)$.

Proof. By Corollary 63 and part (1) of Lemma 64, we have

$$\begin{aligned} -\left(\frac{3p}{p+2} - 1\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma^2} \ge \frac{\partial_t \sigma}{\sigma^{2-\frac{3p}{p+2}}} \\ = -\frac{4}{3\sigma} + \left(\frac{p}{p+2} + 1\right)\left(1 - \frac{3p}{p+2}\right)\frac{\sigma_{\mathfrak{s}}^2}{\sigma^2} + \frac{p}{p+2}\frac{\sigma_{\mathfrak{s}\mathfrak{s}}}{\sigma}. \end{aligned} \tag{5.3.3}$$

Integrating the inequality (5.3.3) against $d\mathfrak{s}$, we obtain

$$\begin{aligned} \frac{4}{3}\int_{\gamma_t} \frac{1}{\sigma} d\mathfrak{s} &\ge \frac{p}{p+2}\left(2 - \frac{3p}{p+2}\right)\int_{\gamma_t} \frac{\sigma_{\mathfrak{s}}^2}{\sigma^2} d\mathfrak{s} \\ &= \frac{p}{p+2}\left(2 - \frac{3p}{p+2}\right)\int_{\gamma_t} (\ln(\sigma))_{\mathfrak{s}}^2 d\mathfrak{s}. \end{aligned} \tag{5.3.4}$$

Set $d_p := \frac{p}{p+2}\left(2 - \frac{3p}{p+2}\right)$. Thus d_p is positive if

$$1 \le p < 4.$$

Applying the Hölder inequality to the left-hand side and right-hand side of inequality (5.3.4), we get

$$\frac{\left(\int |\left(\ln \sigma\right)_{\mathfrak{s}}| d\mathfrak{s}\right)^2}{\Omega_1} \le d_p' A^{\circ\ \frac{1}{2}} \Omega_1^{\frac{1}{2}} = d_p' \frac{A^{\frac{1}{2}} A^{\circ\ \frac{1}{2}} \Omega_1^{\frac{1}{2}}}{A^{\frac{1}{2}}},$$

for a new positive constant d_p'. Here we used the identities $\int_{\gamma_t} \frac{1}{\sigma^2} d\mathfrak{s} = 2A^\circ$ and $\int_{\gamma_t} d\mathfrak{s} = \Omega_1$. Now using the Blaschke-Santaló inequality, $AA^\circ \le \pi^2$, we have

$$\left(\ln \frac{\sigma_M}{\sigma_m}\right)^2 \le d_p'' \left(\frac{\Omega_1^3}{A}\right)^{\frac{1}{2}},$$

for a new constant d''_p. Observe that the affine isoperimetric ratio, $\frac{\Omega_1^3}{A}$, is bounded by the affine isoperimetric inequality. Therefore, we find that

$$\frac{\sigma_M}{\sigma_m} \leq c \tag{5.3.5}$$

on the time interval $(-\infty, T)$, for some positive universal constant c, depending only on p. □

Let $\{K_t\}_t$ be a solution of the flow (5.3.1). Then the family of convex bodies, $\{\tilde{K}_t\}_t$, defined by

$$\tilde{K}_t := \sqrt{\frac{\pi}{A(K_t)}} K_t$$

is called a normalized solution to the p-flow, equivalently a solution whose area is fixed and is equal to π.

We denote every quantity associated to the normalized solution with an over-tilde. For example the support function, curvature, and the affine support function of $\tilde{K}$ are denoted by $\tilde{s}$, $\tilde{\kappa}$, and $\tilde{\sigma}$, respectively.

Lemma 67. *There is a constant c, depending on p, such that*

$$\frac{\tilde{\sigma}_M}{\tilde{\sigma}_m} \leq c. \tag{5.3.6}$$

for every normalized solution to the p-flow on the time interval $(-\infty, T)$.

Proof. The estimate (5.3.5) is scaling invariant. Therefore, the same estimate holds for the normalized solution. □

For the rest of this section, we assume that $l \geq 2$.

Lemma 68. *Along the p-flow we have*

$$\frac{d}{dt}\Omega_l(t) \geq \frac{l-2}{l+2}\frac{\Omega_l \Omega_p}{A} + \frac{18pl}{(l+2)^2(p+2)} \int_{\gamma_t} \sigma^{-\frac{3p}{p+2}-\frac{3l}{l+2}} \sigma_{\mathfrak{s}}^2 d\mathfrak{s}.$$

Proof. Define $d\omega = \sigma d\mathfrak{s}$, $g = \sigma$ and $F(x) := x^{-\frac{3l}{l+2}}$ in Theorem 3. Furthermore, recall that for a convex body K in $\mathbb{R}^2$ we have $2A = \int_{\partial K} \sigma d\mathfrak{s}$. This implies

$$\int_{\partial K} \sigma^{1-\frac{3p}{p+2}-\frac{3l}{l+2}} d\mathfrak{s} \geq \frac{\Omega_l \Omega_p}{2A}.$$

The claim now follows by this last inequality and the evolution equation (5.3.2). □

Lemma 69. *The reciprocal of the l-affine isoperimetric ratio, $\dfrac{A^{2-l}(t)}{\Omega_l^{2+l}(t)}$, is non-decreasing along the p-flow.*

Proof.

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right)^{-1} \\
&\quad = -\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right)^{-2} \frac{d}{dt}\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right) \\
&\quad = -\left(\frac{\Omega_l^{l+2}(t)}{A^{2-l}(t)}\right)^{-2} \left(\frac{(2+l)\Omega_l^{l+1}(t)A^{2-l}(t)\frac{d}{dt}\Omega_l + (2-l)A^{1-l}(t)\Omega_l^{2+l}(t)\Omega_p(t)}{A^{2(2-l)}(t)}\right) \\
&\quad = -\frac{\Omega_p(t)}{A(t)}\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\left((2+l)\frac{d}{dt}\Omega_l - (l-2)\frac{\Omega_l(t)\Omega_p(t)}{A(t)}\right) \\
&\quad \leq -\frac{18pl}{(l+2)(p+2)}\frac{\Omega_p(t)}{A(t)}\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t} \sigma^{-\frac{3p}{p+2}-\frac{3l}{l+2}}\sigma_{\mathfrak{s}}^2 d\mathfrak{s},
\end{aligned}$$

where we used Lemma 68 on the last inequality. □

In the rest of this chapter, we assume that $1 \leq p < 4$.

Corollary 70. *There exists a constant $b_{l,p} > 0$ depending on l and p such that*

$$\frac{A^{2-l}(t)}{\Omega_l^{2+l}(t)} < b_{l,p}$$

on $(-\infty, T)$.

Proof. Note that

$$\frac{A^{2-l}(t)}{\Omega_l^{2+l}(t)} = \frac{\left(\frac{1}{2}\int_{\partial\gamma_t}\sigma d\mathfrak{s}\right)^{2-l}}{\left(\int_{\partial\gamma_t}\sigma^{1-\frac{3l}{l+2}}d\mathfrak{s}\right)^{2+l}}$$

is a $GL(2)$ invariant quantity. Therefore, we need only to prove the claim after applying appropriate special linear transformations to the normalized solution of the p-flow. By the estimate (5.3.6) and the facts that $\max_{\mathbb{S}^1}\tilde{\sigma} \geq 1$ and $\min_{\mathbb{S}^1}\tilde{\sigma} \leq 1$, Lemma 56 in Chapter 4, we have

$$\frac{1}{c} \leq \sigma = \frac{\tilde{s}}{\tilde{\kappa}^{\frac{1}{3}}} \leq c. \tag{5.3.7}$$

Observe that $\tilde{\sigma}$ is an $SL(2)$ invariant quantity. Therefore, the previous estimate holds even after applying an arbitrary special linear transformation. After applying a length minimizing special linear transformation at each time t to the normalized solution of the p-flow, by John's lemma, the support functions have uniform lower and upper bounds on the time interval $(-\infty, T)$. Therefore, by inequalities (5.3.7), the curvature is uniformly bounded from below and above on the time interval $(-\infty, T)$. Now the claim follows as $d\mathfrak{s} = \mathfrak{r}^{2/3}$. □

By the computation carried out in the proof of Lemma 69, we find

$$\frac{d}{dt}\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right)^{-1} \leq c_{l,p}\frac{d}{dt}\ln(A(t))\left[\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t}\left(\sigma^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}\right],$$

where $c_{l,p} := \dfrac{18pl}{(l+2)(p+2)\left(1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}\right)^2}$. Since $\frac{d}{dt}A(t) = -\Omega_p(t)$. This inequality will be used in the proof of the next corollary.

Corollary 71. *If K_t evolves by (5.3.1), the following limit holds:*

$$\liminf_{t\to-\infty}\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t}\left(\sigma^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} = 0. \tag{5.3.8}$$

Proof. Suppose the contrary. There exists an $\varepsilon > 0$ small, such that

$$\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t}\left(\sigma^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s} \geq \frac{\varepsilon}{c_{l,p}}$$

in a neighborhood of $-\infty$, say on $(-\infty, -N)$ for large enough N. Then

$$\frac{d}{dt}\left(\frac{\Omega_l^{2+l}(t)}{A^{2-l}(t)}\right)^{-1} \leq \varepsilon\frac{d}{dt}\ln(A(t)).$$

Thus, by Corollary 70 we get

$$\begin{aligned}\left(\frac{\Omega_l^{2+l}}{A^{2-l}}\right)^{-1}(t) &\leq \left(\frac{\Omega_l^{2+l}}{A^{2-l}}\right)^{-1}(t_1) + \varepsilon\ln(A(t)) - \varepsilon\ln(A(t_1))\\ &\leq b_{l,p} + \varepsilon\ln(A(t)) - \varepsilon\ln(A(t_1)).\end{aligned}$$

We obtain a contradiction by letting t_1 approach $-\infty$: Since $\lim\limits_{t\to-\infty} A(t_1) = +\infty$, the right hand side becomes negative. □

Corollary 72. *There is a sequence of times $\{t_k\}_{k\in\mathbb{N}}$ such that, as the numbers t_k converge to $-\infty$, we have*

$$\lim_{t_k\to-\infty}\tilde{\sigma}(t_k) = 1.$$

Proof. Notice that the quantity $\left(\frac{A^{3-l}(t)}{\Omega_p(t)\Omega_l^{l+3}(t)}\right)\int_{\gamma_t}\left(\sigma^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\mathfrak{s}}^2 d\mathfrak{s}$ is scaling invariant and $\frac{\tilde{A}^{3-l}(t)}{\tilde{\Omega}_p(t)\tilde{\Omega}_l^{l+3}(t)}$ is bounded from below by the l-affine isoperimetric inequality and by the p-affine isoperimetric inequality. Hence, Corollary 71 implies that there exists a sequence of times $\{t_k\}_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} t_k = -\infty$ and

$$\lim_{t_k\to-\infty}\int_{\tilde{\gamma}_{t_k}}\left(\tilde{\sigma}^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}} = 0.$$

On the other hand, by the Hölder inequality

$$\frac{\left(\tilde{\sigma}_M^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}} - \tilde{\sigma}_m^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)^2}{\tilde{\Omega}_1} \leq \int_{\tilde{\gamma}_{t_k}}\left(\tilde{\sigma}^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)_{\tilde{\mathfrak{s}}}^2 d\tilde{\mathfrak{s}}.$$

Therefore, by the boundedness of $\tilde{\Omega}_1$ from above, and by the affine isoperimetric inequality, we find that

$$\lim_{t_k\to-\infty}\left(\tilde{\sigma}_M^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}-\tilde{\sigma}_m^{1-\frac{3p}{2(p+2)}-\frac{3l}{2(l+2)}}\right)^2=0.$$

Since $\tilde{\sigma}_m \leq 1$ and $\tilde{\sigma}_M \geq 1$, the claim follows. □

Lemma 73 (Monotonicity of the area product)**.** *[100] The area product, $A(t)A^{\circ}(t)$, is strictly increasing along the p-flow unless K_t is an ellipse centered at the origin.*

Corollary 74. *For every compact ancient solution to the p-flow, we have*

$$\lim_{t\to-\infty} A(t)A^{\circ}(t)=\pi^2.$$

Proof. We first show that

$$\lim_{t_k\to-\infty} A(t)A^{\circ}(t)=\pi^2.$$

This is a consequence of Corollary 72 and Lemma 1:

By Corollary 72, we have

$$\lim_{t_k\to-\infty}\tilde{\sigma}(t_k)=1. \tag{5.3.9}$$

Thus, by Lemma 1 there exist two families of origin-centered ellipses $\{\mathcal{E}_{in}(t_k)\}$, $\{\mathcal{E}_{out}(t_k)\}$ such that

$$\mathcal{E}_{in}(t_k)\subseteq \tilde{K}_{t_k}\subseteq \mathcal{E}_{out}(t_k) \tag{5.3.10}$$

and

$$\lim_{t_k\to-\infty}\sigma(\mathcal{E}_{in}(t_k))=\lim_{t_k\to-\infty}\sigma(\mathcal{E}_{out}(t_k))=1.$$

Evidently, we can find an appropriate family of special linear transformations $\{L_{t_k}\}_k$ such that $L_{t_k}(\mathcal{E}_{out}(k))$ is a circle at each time t_k. Each such area preserving linear transformation L_{t_k} minimizes the Euclidean length of the ellipse $\mathcal{E}_{out}(t_k)$ at time t_k. Thus, the construction of $\mathcal{E}_{out}(t_k)$ and $\mathcal{E}_{in}(t_k)$ implies

$$\lim_{t_k \to -\infty} L_{t_k}(\mathcal{E}_{out}(t_k)) = \lim_{t_k \to -\infty} L_{t_k}(\mathcal{E}_{in}(t_k)) = \mathbb{S}^1$$

in the Hausdorff metric:

Since σ is invariant under $SL(2)$, we have $\sigma(L_{t_k}(\mathcal{E}_{out}(t_k))) = \sigma(\mathcal{E}_{out}(t_k)))$, therefore $\lim_{t_k \to -\infty} \sigma(L_{t_k}(\mathcal{E}_{out}(t_k))) = 1$. This implies $\lim_{t_k \to -\infty} L_{t_k}(\mathcal{E}_{out}(t_k)) = \mathbb{S}^1$ in the Hausdorff metric. Similarly, we have $\lim_{t_k \to -\infty} \sigma(L_{t_k}(\mathcal{E}_{in}(t_k)))) = 1$. This implies that

$$\lim_{t_k \to -\infty} A(L_{t_k}(\mathcal{E}_{in}(t_k))) = \pi.$$

As $L_{t_k}(\mathcal{E}_{in}(t_k)) \subseteq L_{t_k}(\mathcal{E}_{out}(t_k))$, we conclude that $\lim_{t_k \to -\infty} L_{t_k}(\mathcal{E}_{in}(t_k)) = \mathbb{S}^1$ in the Hausdorff metric.

Now, applying $\{L_{t_k}\}_k$ to the inclusions (5.3.10), we obtain that the sequence $\{L_{t_k}(\tilde{K}_{t_k})\}_k$ converges to the unit disk in the Hausdorff metric. So

$$\lim_{t_k \to -\infty} A(t)A^{\circ}(t) = \pi^2.$$

Now monotonicity of the area product, $A(t)A^{\circ}(t)$, stated in Lemma 73, finishes the proof. □

5.4 Proof of the main theorem

We have now gathered all the necessary ingredients to prove our main theorem.

Theorem (Ancient solutions)**.** *Let $1 \leq p < 4$. The only compact, origin-symmetric, ancient solutions to the p-flow are homothetic ellipses.*

Proof. We have proved that $\lim_{t_k \to T} A(t)A^\circ(t) = \pi^2$, Corollary 22 in Chapter 2, and $\lim_{t_k \to -\infty} A(t)A^\circ(t) = \pi^2$, Corollary 74. Therefore $A(t)A^\circ(t)$ achieves the same value at both ends of the interval $(-\infty, T)$. Since $A(t)A^\circ(t)$ is monotone we conclude that the area product is constant on $(-\infty, T)$. Now, Lemma 73 implies that K_t must be an ellipse centered at the origin of the plane for all $t \in (-\infty, T)$. $\square$

Chapter 6

Volume preserving centro-affine normal flows

The contents of this chapter are taken from the paper "Volume preserving centro-affine normal flows" [55]. The paper is co-authored with A. Stancu. The paper will appear in the journal *Communications in analysis and Geometry*.

We study the long time behavior of the volume preserving p-flow in $\mathbb{R}^n$ for $1 \leq p < \frac{n}{n-2}$. We prove that every solution to the volume preserving p-flow converges sequentially to the unit ball in the C^∞ topology, modulo the group of special linear transformations.

6.1 Introduction

Let $p \geq 1$ be a fixed real number and let $K_0 \in \mathcal{K}_{sym}$. We consider a family of convex bodies $\{K_t\}_t \in \mathcal{K}_{sym}$, and their associated smooth embeddings $x : \mathbb{S}^n \times [0, T) \to \mathbb{R}^n$, which are evolving according to the p centro-affine normal flow, namely,

$$\frac{\partial}{\partial t}x := -s\left(\frac{\mathcal{K}}{s^{n+1}}\right)^{\frac{p}{n+p}} z, \quad x(\cdot, 0) = x_{K_0}(\cdot), \quad x(\cdot, t) = x_{K_t}(\cdot). \tag{6.1.1}$$

In this chapter, we study the asymptotic behavior of this flow by applying the techniques of [10]. As we have seen, the long time behavior of the flow in $\mathbb{R}^2$ was studied in Chapter 2 using tools of affine differential geometry. It was proved there that the volume preserving p-flow with $p \geq 1$ evolves convex bodies in $\mathcal{K}_{sym}$ to the unit disk in Hausdorff distance, modulo $SL(2)$. Further applications to the L_{-2} Minkowski problem and to the stability of the p-affine isoperimetric inequality were given in Chapter 3 and Chapter 4, respectively.

In this chapter, we prove that:

Theorem (Main Theorem E)**.** *Let $1 \leq p < \frac{n}{n-2}$ be a real number. Let $x_{K_0} : \mathbb{S}^{n-1} \to \mathbb{R}^n$ be a smooth, strictly convex embedding of $K_0 \in \mathcal{K}_{sym}$. Then there exists a unique solution $x : \mathbb{S}^{n-1} \times [0, T) \to \mathbb{R}^n$ of equation (6.1.1) with initial data x_{K_0}. The rescaled convex bodies given by the embeddings $\left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{\frac{1}{n}} x_{K_t}$ converge sequentially in the C^∞ topology to the unit ball, modulo $SL(n)$. Furthermore, when $p = 1$ the assumption of K_0 being origin-symmetric is not necessary.*

6.2 Uniform lower and upper bounds on the principal curvatures

We will start this section by proving that, under uniform lower and upper bounds on the support function of the evolving convex body, we have uniform lower and upper bounds on the Gauss curvature which depend only on the dimension n, the value of p, the bounds on the support function, and time. To obtain the upper bound on the Gauss curvature, we apply a standard technique of Tso [106]. To derive the lower bound bound on the Gauss curvature, we consider the evolution of the dual convex body and we apply Tso's technique to the speed of the dual p-flow. This procedure avoids the need for a Harnack estimate, or displacement bounds.

As the convex bodies $\{K_t\}_t$ evolve by (6.1.1), their support functions satisfy the partial differential equation

$$\partial_t s = -s\left(\frac{\mathcal{K}}{s^{n+1}}\right)^{\frac{p}{n+p}}, \tag{6.2.1}$$

see also [100]. The short time existence and uniqueness of solutions for a smooth and strictly convex initial hypersurface follow from the strict parabolicity of the equation and it was shown in [100]. We will use this latter evolution equation to describe the flow throughout the rest of the chapter.

The proofs of the two lemmas pertaining to lower and upper bounds of the Gauss curvature of the evolving convex bodies have similar outline yet with some differences. For completeness, we will present both proofs. First, we present two elementary lemmas.

Lemma 75. *Let $F : \mathbb{S}^{n-1} \times [0, T') \to \mathbb{R}$ be a positive function that satisfies*

$$\partial_t F \geq c' F^{-\alpha} - c$$

for positive constants c, c' and α. Then there are positive constants c_1 and c_2 such that

$$\frac{1}{F} \leq c_1 t^{-\frac{1}{1+\alpha}} + c_2.$$

Proof. We have

$$\partial_t F^{1+\alpha} \geq (1+\alpha)(c' - cF^{\alpha}).$$

Suppose that $F^{\alpha} \leq \frac{c'}{2c}$. So

$$F^{1+\alpha} \geq (1+\alpha)\frac{c'}{2}t.$$

From this, we obtain that $F \geq \frac{1}{c_1} t^{\frac{1}{1+\alpha}}$ if $F^\alpha \leq \frac{c'}{2c}$. We conclude that

$$F \geq \min\left\{\frac{1}{c_1} t^{\frac{1}{1+\alpha}}, \left(\frac{c'}{2c}\right)^{\frac{1}{\alpha}} =: \frac{1}{c_2}\right\} \geq \frac{1}{c_1 t^{-\frac{1}{1+\alpha}} + c_2}.$$

□

Lemma 76. *Let $F : \mathbb{S}^{n-1} \times [0, T') \to \mathbb{R}$ be a positive function that satisfies*

$$\partial_t F \leq -c' F^\alpha + c$$

for positive constants c, c' and $\alpha > 1$. Then there are positive constants c_1 and c_2 such that

$$F \leq c_1 t^{\frac{1}{1-\alpha}} + c_2.$$

Proof. Since $\alpha > 1$, we have

$$\partial_t F^{1-\alpha} \geq c'(\alpha - 1) + \frac{c}{F^\alpha}(1 - \alpha).$$

Suppose $F^\alpha(\cdot, t) > \frac{2c}{c'}$. Therefore, from the last inequality we get

$$\partial_t F^{1-\alpha} \geq (\alpha - 1)\frac{c'}{2}.$$

This implies that

$$F^{1-\alpha}(\cdot, t) - F^{1-\alpha}(\cdot, 0) \geq (\alpha - 1)\frac{c'}{2} t.$$

Thus, we get

$$F(\cdot, t) \leq \left((\alpha - 1)\frac{c'}{2}\right)^{\frac{1}{1-\alpha}} t^{\frac{1}{1-\alpha}}.$$

Observe that we obtained $F(\cdot, t) \leq \left((\alpha - 1)\frac{c'}{2}\right)^{\frac{1}{1-\alpha}} t^{\frac{1}{1-\alpha}}$ provided that $F^\alpha(\cdot, t) > \frac{2c}{c'}$. Hence

$$\begin{aligned} F(\cdot, t) &\leq \max\left\{\left((\alpha - 1)\frac{c'}{2}\right)^{\frac{1}{1-\alpha}} t^{\frac{1}{1-\alpha}}, \left(\frac{2c}{c'}\right)^{\frac{1}{\alpha}}\right\} \\ &\leq \left((\alpha - 1)\frac{c'}{2}\right)^{\frac{1}{1-\alpha}} t^{\frac{1}{1-\alpha}} + \left(\frac{2c}{c'}\right)^{\frac{1}{\alpha}} =: c_1 t^{\frac{1}{1-\alpha}} + c_2. \end{aligned}$$

□

Given a convex body K, the inner radius of K, $r_-(K)$, is the radius of the largest ball contained in K; the outer radius of K, $r_+(K)$, is the radius of the smallest ball containing K. Notice that for each centrally symmetric convex body, the smallest and largest ball as above will be centered at the origin.

Lemma 77 (Upper bound on the Gauss curvature)**.** *For every smooth, strictly convex solution $\{K_t\}_{[0,t_0]}$ of the evolution equation (6.2.1) with $0 < R_- \le r_-(K_t) \le r_+(K_t) \le R_+ < +\infty$ for $t \in [0, t_0]$, we have*

$$\mathcal{K}^{\frac{p}{n+p}} \le \left(C + C' t^{-\frac{(n-1)p}{n(p+1)}}\right),$$

where C and C' are constants depending on n, p, R_- and R_+.

Proof. Let $\alpha := 1 - \frac{(n+1)p}{n+p}$ and $\beta := -\frac{p}{n+p}$. Consider the function

$$\Psi = \frac{s^\alpha S_{n-1}^\beta}{s - R_-/2},$$

where S_{n-1} stands for the $(n-1)$-th symmetric polynomial in the radii of curvature as a function on the sphere $\mathbb{S}^{n-1}$. Using the parabolic maximum principle, we will show that Ψ is bounded from above by a function of n, p, R_-, R_+ and time. At the point where the maximum of Ψ occurs, we have

$$0 = \bar\nabla_i \Psi = \bar\nabla_i \left(\frac{s^\alpha S_{n-1}^\beta}{s - R_-/2}\right) \quad \text{and} \quad \bar\nabla_i \bar\nabla_j \Psi \le 0.$$

Hence, we obtain

$$\frac{\bar\nabla_i (s^\alpha S_{n-1}^\beta)}{s - R_-/2} = \frac{(s^\alpha S_{n-1}^\beta) \bar\nabla_i s}{(s - R_-/2)^2},$$

and consequently

$$\bar\nabla_i \bar\nabla_j \left(s^\alpha S_{n-1}^\beta\right) + \bar g_{ij} \left(s^\alpha S_{n-1}^\beta\right) \le \frac{s^\alpha S_{n-1}^\beta \mathfrak{r}_{ij} - R_-/2 s^\alpha S_{n-1}^\beta \bar g_{ij}}{s - R_-/2}. \tag{6.2.2}$$

We calculate

$$\partial_t\Psi = -\frac{\beta s^\alpha S_{n-1}^{\beta-1}}{s-R_-/2}(\dot{S}_{n-1})_{ij}\left[\bar{\nabla}_i\bar{\nabla}_j\left(s^\alpha S_{n-1}^\beta\right)+\bar{g}_{ij}\left(s^\alpha S_{n-1}^\beta\right)\right] + \frac{S_{n-1}^\beta}{s-R_-/2}\partial_t s^\alpha + \frac{s^{2\alpha}S_{n-1}^{2\beta}}{(s-R_-/2)^2},$$

where $(\dot{S}_{n-1})_{ij} := \dfrac{\partial S_{n-1}}{\partial \mathfrak{r}_{ij}}$ is the derivative of the S_{n-1} with respect to the entry $\mathfrak{r}_{ij}$ of the radii of curvature matrix. By Theorem 1, page 102 of [80], applied to the top symmetric polynomial, we have $\dot{S} := \dfrac{\partial S}{\partial a_{ij}}$ is a positive definite bilinear form as long as ∂K has positive Gauss curvature at all points.

Using inequality (6.2.2) we infer that, at the point where the maximum of Ψ is reached, we have

$$\partial_t\Psi \le \Psi^2\left(-(n-1)\beta-\alpha+1+\frac{\beta R_-}{2}\mathcal{H}\right). \tag{6.2.3}$$

We can control the mean curvature $\mathcal{H}$ from below by a positive power of Ψ. First note that $\mathcal{H} \ge \frac{n-1}{S_{n-1}^{\frac{1}{n-1}}}$. Therefore

$$\begin{aligned}\mathcal{H} &\ge (n-1)\left(\frac{s-R_-/2}{s^\alpha S_{n-1}^\beta}\right)^{\frac{1}{(n-1)\beta}}\left(\frac{s^\alpha}{s-R_-/2}\right)^{\frac{1}{(n-1)\beta}}\\ &\ge (n-1)\Psi^{-\frac{1}{(n-1)\beta}}\left(\frac{R_+^\alpha}{R_- - R_-/2}\right)^{\frac{1}{(n-1)\beta}}.\end{aligned}$$

Therefore, we can rewrite the inequality (6.2.3) as follows

$$\begin{aligned}\partial_t\Psi &\le \Psi^2\left(-(n-1)\beta-\alpha+1+\frac{(n-1)\beta R_-}{2}\Psi^{-\frac{1}{(n-1)\beta}}\left(\frac{R_+^\alpha}{R_-/2}\right)^{\frac{1}{(n-1)\beta}}\right)\\ &= -\Psi^2\left(C'(n,p,R_-,R_+)\Psi^{\frac{n+p}{(n-1)p}}-C(n,p)\right),\end{aligned}$$

for positive constants $C(n,p)$ and $C'(n,p,C,R_-,R_+)$. From this last inequality and Lemma 75 for $F=\frac{1}{\Psi}$ and $\alpha=\frac{n+p}{(n-1)p}$, it follows that

$$\Psi \le \max\left\{C(n,p,R_-,R_+), C'(n,p,R_-,R_+)t^{-\frac{(n-1)p}{n(p+1)}}\right\}$$

for new constants C and C'. The corresponding claim for $\mathcal{K}$ follows. $\square$

Pertaining to the flow by powers of the Gauss curvature, a powerful technique to obtain a uniform lower bound on the Gauss curvature of the evolving convex body is using a Harnack inequality and a lower displacement bound [10]. The lower displacement bound controls how much the support of the evolving body decreases depending on time. The displacement bound is obtained by looking at how appropriate barriers, usually balls, with appropriate centers, move along the flow, combined with a containment principle. Here, we introduce a new technique to obtain a uniform lower bound on the Gauss curvature along the flow. We look at the geometric flow that evolves the dual convex body, the dual p-flow.

Let K° denote the dual body associated to K with respect to the origin

$$K^\circ = \{y \in \mathbb{R}^n \mid x \cdot y \leq 1,\ \forall x \in K\}.$$

We will use further the following lemma proved in [100].

Lemma 78 (The dual p-flow)**.** *[100] Let $\{K_t\}_{[0,T)}$ be a smooth, strictly convex solution of the evolution equation (6.2.1). Then $\{K_t^\circ\}_{[0,T)}$ is a solution of the following evolution equation, the expanding p-flow (alternatively called the dual p-flow):*

$$\partial_t s^\circ = s^\circ \left(\frac{\mathcal{K}^\circ}{s^{\circ n+1}} \right)^{-\frac{p}{n+p}}.$$

It is in the next lemma that we need to restrict to the case $p < \frac{n}{n-2}$.

Lemma 79 (Lower bound on the Gauss curvature)**.** *Let $1 \leq p < \frac{n}{n-2}$. Assume that $\{K_t\}_{[0,t_0]}$ is a smooth, strictly convex solution of equation (6.2.1) with $0 < R_- \leq r_-(K_t) \leq r_+(K_t) \leq R_+ < +\infty$ for $t \in [0, t_0]$. Then*

$$\mathcal{K}^{\frac{p}{n+p}} \geq \frac{1}{C + C' t^{\frac{(n-1)p}{(n-2)p-n}}},$$

where C and C' are constants depending on n, p, R_- and R_+.

Proof. Recall from Lemma 78 that $\partial_t s^\circ = s^\circ \left(\frac{\mathcal{K}^\circ}{s^{\circ n+1}}\right)^{-\frac{p}{n+p}}$. We define $\alpha := 1 + \frac{(n+1)p}{n+p}$ and $\beta := \frac{p}{n+p}$. Therefore, the dual flow takes the following form $\partial_t s^\circ = s^{\circ\alpha} S_{n-1}^{\circ\beta}$. Since $R_- \le r_-(K_t) \le r_+(K_t) \le R_+$, we have

$$\frac{1}{R_+} \le r_-(K_t^\circ) \le r_+(K_t^\circ) \le \frac{1}{R_-}.$$

Define

$$R_-^\circ := \frac{1}{R_+}, \quad R_+^\circ := \frac{1}{R_-}$$

and consider the function

$$\Phi = \frac{s^{\circ\alpha} S_{n-1}^{\circ\beta}}{2R_+^\circ - s^\circ}.$$

The subsequent computation is carried out at the point where the minimum of Φ occurs:

$$0 = \bar{\nabla}_i \Phi = \bar{\nabla}_i \left(\frac{s^{\circ\alpha} S_{n-1}^{\circ\beta}}{2R_+^\circ - s^\circ}\right) \quad \text{and} \quad \bar{\nabla}_i \bar{\nabla}_j \Phi \ge 0,$$

hence we obtain

$$\frac{\bar{\nabla}_i (s^{\circ\alpha} S_{n-1}^{\circ\beta})}{2R_+^\circ - s^\circ} = -\frac{s^{\circ\alpha} S_{n-1}^{\circ\beta} \bar{\nabla}_i s^\circ}{(2R_+^\circ - s^\circ)^2}$$

and

$$\bar{\nabla}_i \bar{\nabla}_j \left(s^{\circ\alpha} S_{n-1}^{\circ\beta}\right) + \bar{g}_{ij} \left(s^{\circ\alpha} S_{n-1}^{\circ\beta}\right) \ge \frac{-s^{\circ\alpha} S_{n-1}^{\circ\beta} \mathfrak{r}_{ij}^\circ + 2R_+^\circ s^{\circ\alpha} S_{n-1}^{\circ\beta} \bar{g}_{ij}}{2R_+^\circ - s^\circ}. \tag{6.2.4}$$

Calculating

$$\partial_t \Phi = \frac{\beta s^{\circ\alpha} S_{n-1}^{\circ\beta-1}}{2R_+^\circ - s^\circ} (\dot{S}_{n-1}^\circ)_{ij} \left[\bar{\nabla}_i \bar{\nabla}_j \left(s^{\circ\alpha} S_{n-1}^{\circ\beta}\right) + \bar{g}_{ij} \left(s^{\circ\alpha} S_{n-1}^{\circ\beta}\right)\right] + \frac{S_{n-1}^{\circ\beta}}{2R_+^\circ - s^\circ} \partial_t s^{\circ\alpha}$$
$$+ \frac{s^{\circ 2\alpha} S_{n-1}^{\circ 2\beta}}{(2R_+^\circ - s^\circ)^2}.$$

and applying inequality (6.2.4), we conclude that

$$\partial_t \Phi \geq \Phi^2 \left(1 - (n-1)\beta - \alpha + 2\beta R_+^\circ \mathcal{H}^\circ\right). \tag{6.2.5}$$

We now estimate the mean curvature $\mathcal{H}^\circ$ from below by a negative power of Φ. As in the proof of the previous lemma, we have

$$\begin{aligned}\mathcal{H}^\circ &\geq (n-1)\left(\frac{2R_+^\circ - s^\circ}{s^{\circ\alpha} S_{n-1}^{\circ\beta}}\right)^{\frac{1}{(n-1)\beta}} \left(\frac{s^{\circ\alpha}}{2R_+^\circ - s^\circ}\right)^{\frac{1}{(n-1)\beta}} \\ &\geq (n-1)\Phi^{-\frac{1}{(n-1)\beta}} \left(\frac{R_-^{\circ\alpha}}{2R_+^\circ - R_-^\circ}\right)^{\frac{1}{(n-1)\beta}}.\end{aligned}$$

Consequently, inequality (6.2.5) can be rewritten as follows

$$\begin{aligned}\partial_t \Phi &\geq \Phi^2 \left(1 - (n-1)\beta - \alpha + 2(n-1)\beta R_+^\circ \Phi^{-\frac{1}{(n-1)\beta}} \left(\frac{R_-^{\circ\alpha}}{2R_+^\circ - R_-^\circ}\right)^{\frac{1}{(n-1)\beta}}\right) \\ &= \Phi^2 \left(-C(n,p) + C'(n,p,R_-^\circ,R_+^\circ)\Phi^{-\frac{n+p}{(n-1)p}}\right),\end{aligned}$$

for positive constants $C(n,p)$ and $C'(n,p,R_-^\circ,R_+^\circ)$.

Hence

$$\partial_t \left(\frac{1}{\Phi}\right) \leq -C'(n,p,R_-^\circ,R_+^\circ)\left(\frac{1}{\Phi}\right)^{\frac{n+p}{(n-1)p}} + C(n,p).$$

From this last inequality and Lemma 76 for $F = \frac{1}{\Phi}$ and $\alpha = \frac{n+p}{(n-1)p}$, it follows that

$$\frac{1}{\Phi} \leq \max\left\{C, C' t^{\frac{(n-1)p}{(n-2)p-n}}\right\}$$

for new constants C and C'. Equivalently, we have a bound for Φ from below.

Therefore, we have bounded from above $\mathcal{K}^\circ$ in terms of n, p, R_-, R_+ and time. To complete the proof, we recall the following fact: for every $x \in \partial K$, there exists an $x^\circ \in \partial K^\circ$ such that

$$\left(\frac{\mathcal{K}}{s^{n+1}}\right)(x)\left(\frac{\mathcal{K}^\circ}{s^{\circ n+1}}\right)(x^\circ) = 1,$$

where x and x° are related by $\langle x, x^\circ \rangle = 1$, with $\langle \, , \rangle$ the inner product in $\mathbb{R}^n$. A proof of this identity in the smooth setting is simple. A proof in a more general non-smooth setting can be found in [48], see also [67]. For a non-smooth convex body, one needs to replace $\mathcal{K}$ with the generalized Gauss curvature, the identity then holds almost everywhere with respect to the Hausdorff measure on ∂K.
By the above identity, we conclude that $\mathcal{K}$ is bounded from below by constants depending on n, p, R_-, R_+ and time. □

We point out that in concluding the long time existence of solutions, and asymptotic behavior, to the flow in $\mathbb{R}^2$, the following two lemmas in this section are not necessary.

Lemma 80 (Lower bound on the principal curvatures)**.** *Assume that* $n > 1$. *Let* $\{K_t\}_{[0,t_0]}$ *be a smooth, strictly convex solution of equation (6.2.1) with* $0 < R_- \leq r_-(K_t) \leq r_+(K_t) \leq R_+ < +\infty$ *and suppose that*

$$C_1 \leq S_{n-1} \leq C_2$$

for all $t \in [0, t_0]$. *Then there exists constants* C *and* C' *depending on* n, p, R_-, R_+, C_1 *and* C_2 *such that*

$$\frac{1}{\kappa_i} \leq C + C' t^{-(n-2)},$$

for all $t \in [0, t_0]$.

Proof. We first compute the evolution equation of $\mathfrak{r}_{ij} = \bar{\nabla}_i \bar{\nabla}_j s + s \bar{g}_{ij}$. Set $\alpha :=$

$-1 + \frac{(n+1)p}{n+p}$ and $\beta := \frac{p}{n+p}$.

$$\begin{aligned}\partial_t \mathfrak{r}_{ij} &= \beta s^{-\alpha} S_{n-1}^{-(1+\beta)} (\dot{S}_{n-1})_{kl} \bar{\nabla}_k \bar{\nabla}_l \mathfrak{r}_{ij} - \beta(\beta+1) s^{-\alpha} S_{n-1}^{-(2+\beta)} \bar{\nabla}_i S_{n-1} \bar{\nabla}_j S_{n-1} \\ &\quad + \beta s^{-\alpha} S_{n-1}^{-(1+\beta)} (\ddot{S}_{n-1})_{kl;mn} \bar{\nabla}_i \mathfrak{r}_{kl} \bar{\nabla}_j \mathfrak{r}_{mn} \\ &\quad + ((n-1)\beta - 1) s^{-\alpha} S_{n-1}^{-\beta} \bar{g}_{ij} - \beta s^{-\alpha} S_{n-1}^{-(1+\beta)} (\dot{S}_{n-1})_{kl} \mathfrak{r}_{ij} \bar{g}_{kl} \\ &\quad + S_{n-1}^{-\beta} \bar{\nabla}_i \bar{\nabla}_j s^{-\alpha} + \beta S_{n-1}^{-(1+\beta)} \bar{\nabla}_i s^{-\alpha} \bar{\nabla}_j S_{n-1} + \beta S_{n-1}^{-(1+\beta)} \bar{\nabla}_j s^{-\alpha} \bar{\nabla}_i S_{n-1}.\end{aligned}$$

a *Estimating the term on the first line:* The first term on the first line is an essential good term viewed as an elliptic operator which is non-positive at the point and direction where a maximum eigenvalue occurs. The second term is an essential good negative term.

b *Estimating the term on the second line:* Concavity of $S_{n-1}^{\frac{1}{n-1}}$, see again [80], gives

$$\left[(\ddot{S}_{n-1})_{kl;mn} - \frac{n-2}{(n-1)S_{n-1}} (\dot{S}_{n-1})_{kl} (\dot{S}_{n-1})_{mn} \right] \bar{\nabla}_i \mathfrak{r}_{kl} \bar{\nabla}_j \mathfrak{r}_{mn} \leq 0. \tag{6.2.6}$$

c *Estimating the terms on the last line:*

$$\begin{aligned}\bar{\nabla}_i \bar{\nabla}_j s^{-\alpha} &= -\alpha \frac{\bar{\nabla}_i \bar{\nabla}_j s}{s^{\alpha+1}} + \alpha(\alpha+1) \frac{\bar{\nabla}_i s \bar{\nabla}_j s}{s^{\alpha+2}} \\ &= -\alpha \frac{(\mathfrak{r}_{ij} - \bar{g}_{ij} s)}{s^{\alpha+1}} + \alpha(\alpha+1) \frac{\bar{\nabla}_i s \bar{\nabla}_j s}{s^{\alpha+2}}.\end{aligned}$$

This gives

$$S_{n-1}^{-\beta} \bar{\nabla}_i \bar{\nabla}_j s^{-\alpha} \leq -\alpha C S_{n-1}^{-\beta} \mathfrak{r}_{ij} + \alpha C' S_{n-1}^{-\beta} \bar{g}_{ij} \leq \alpha C'' S_{n-1}^{-\beta} \bar{g}_{ij}, \tag{6.2.7}$$

where we used boundedness of $\bar{\nabla}_i s$ from above and the assumptions of the lemma. Note that $|x|^2 = s^2 + |\bar{\nabla} s|^2$. Therefore, as s is bounded, $|\bar{\nabla} s|$ must also be bounded. Here we used $|\cdot|$ for the Euclidean norm in $\mathbb{R}^n$

The other term on the last line can be estimated by Young's inequality:

$$|\bar{\nabla}_i s^{-\alpha} \bar{\nabla}_j S_{n-1}| \leq C\varepsilon |\bar{\nabla}_j S_{n-1}|^2 + C\varepsilon^{-1}. \tag{6.2.8}$$

Combining inequality (6.2.6) and equations (6.2.7) and (6.2.8), for ε small enough, we have

$$\partial_t \mathfrak{r}_{ij} \leq \beta s^{-\alpha} S_{n-1}^{-(1+\beta)} (\dot{S}_{n-1})_{kl} \bar{\nabla}_k \bar{\nabla}_l \mathfrak{r}_{ij} + C' S_{n-1}^{-\beta} \bar{g}_{ij} - \beta C S_{n-1}^{-(1+\beta)} (\dot{S}_{n-1})_{kl} \mathfrak{r}_{ij} \bar{g}_{kl}.$$

Therefore, the maximum of the hypersurface's $\mathfrak{r}_{ij}$, as a function of time, satisfies

$$\partial_t (\mathfrak{r}_{ij})_{\max} \leq C S_{n-1}^{-\beta} \left(\bar{g}_{ij} - C S_{n-1}^{-\frac{n-1}{n-2}} (\mathfrak{r}_{ij})_{\max}^{\frac{n-1}{n-2}} \right).$$

This implies that, for $(\mathfrak{r}_{ij})_{\max}$ very large, the quantity in parentheses is negative, while S_n is bounded away from zero, hence the behavior of $(\mathfrak{r}_{ij})_{\max}$ when large is modeled by the differential inequality

$$\frac{d(\mathfrak{r}_{ij})_{\max}}{dt} \leq -C (\mathfrak{r}_{ij})_{\max}^{\frac{n-1}{n-2}}.$$

We thus conclude that

$$\max_{i,j} (\mathfrak{r}_{ij})_{\max} \leq C + C' t^{-(n-2)},$$

for some positive constants C, C'. As, for every real symmetric matrix A, its highest eigenvalue is $\lambda_{max}(A) = \sup_{u \in \mathbb{R}^n,\ \|u\|=1} |\langle u, Au \rangle|$, we obtain the upper bound on the highest radius of curvature of the form $C + C' t^{-(n-2)}$, where the constants have been redenoted the same for simplicity. □

Lemma 81 (Lower and upper bounds on the principal curvatures)**.** *Assume that $n > 2$. Let $\{K_t\}_{[0,t_0]}$ be a smooth, strictly convex solution of equation (6.2.1) with $0 < R_- \leq r_-(K_t) \leq r_+(K_t) \leq R_+ < +\infty$ and*

$$C_1 \leq S_{n-1} \leq C_2$$

for all $t \in [0, t_0]$. Then there exists constants C_3, C_4, C_5 and C_6, depending on n, p, R_-, R_+, C_1 and C_2, such that $\forall t \in [0, t_0]$,

$$\frac{1}{C_3 + C_4 t^{-(n-2)}} \leq \kappa_i \leq \left(C_5 + C_6 t^{-(n-2)} \right)^{n-2}.$$

Proof. The lower bound on the principal curvatures has been, in fact, established in Lemma 80. Consequently, we also obtain now the upper bound as the product of the principal curvatures is bounded from above. Suppose that $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n$, then

$$C_1^{-1} \geq \mathcal{K} = \Pi_{i=1}^{n-1}\kappa_i = \kappa_1 \cdot \Pi_{i=1}^{n-2}\kappa_i \geq \kappa_1(C_3 + C_4 t^{-(n-2)})^{-(n-2)}.$$

□

Theorem 82. *Let $1 \leq p < \frac{n}{n-2}$ be a real number. Let $x_{K_0} : \mathbb{S}^{n-1} \to \mathbb{R}^n$ be a smooth, strictly convex embedding of $K_0 \in \mathcal{K}_{sym}$. Then there exists a unique solution $x : \mathbb{S}^{n-1} \times [0, T) \to \mathbb{R}^n$ of equation (6.1.1) with initial data x_{K_0}, for a maximal finite T, such that $\lim_{t \to T} V(K_t) = 0$.*

Proof. Suppose that S_0 is a sphere which, at time zero, encloses K_0. It is clear that, by applying the p-flow to S_0, the evolving spheres S_t converge to a point in finite time. By a comparison principle, K_t remains in the closure of S_t. Therefore T must be finite. Suppose now that $V(K_t)$ does not tend to zero. Then we must have $s \geq r$, for some $r > 0$ on $[0, T)$. By Lemmas 77, 79, 80 and 81 the principal curvatures of the solution remains uniformly bounded on $[0, T)$ from below and above. Consequently, the evolution equation (6.1.1) is uniformly parabolic on $[0, T)$, and bounds on higher derivatives of the support function follow by [61] and Schauder theory. Hence, we can extend the solution after time T, contradicting the definition of T . □

6.3 Convergence of the volume preserving p-flow

In this section, we will conclude the proof of the main theorem.

The following result follows directly from the inequality of Proposition 4.2 in [100].

Theorem (Monotonicity of p-affine isoperimetric ratio)**.** *Let $\{K_t\}_{[0,T)}$ be a smooth, strictly convex solution of equation (6.2.1). Then the p-affine isoperimetric ratio, $\dfrac{\Omega_p^{n+p}(K_t)}{V^{n-p}(K_t)}$, is non-decreasing along the p-flow. The monotonicity is strict unless K_t is an ellipsoid centered at the origin.*

Finally, we restate Theorem 2 in the following suitable form.

Theorem (John's lemma)**.** *[60] Let K be a convex body in $\mathbb{R}^n$. Then there exist absolute constants c and C, and an affine linear transformation $L \in SL(n)$, such that*

$$c \leq \left(\frac{V(\mathbb{B}^n)}{V(K)}\right)^{\frac{1}{n}} s_{L(K)} \leq C,$$

where $\mathbb{B}^n$ denotes the unit ball in $\mathbb{R}^n$.

Let now $x : \mathbb{S}^{n-1} \times [0,T] \to \mathbb{R}^n$ be a solution of equation (6.1.1). Then for each $\lambda > 0$, notice that x_λ defined by $x_\lambda : \mathbb{S}^{n-1} \times \left[0, \lambda^{\frac{2np}{n+p}} T\right] \to \mathbb{R}^n$ with

$$x_\lambda(\theta,t) = \lambda x\left(\theta, \lambda^{-\frac{2np}{n+p}} t\right)$$

is also a solution of evolution equation (6.1.1).

Proof of the main theorem: We follow the procedure in [10]. Fix $t \in [0,T)$. Define $\tilde{s}$ a solution of (6.2.1), by the rescaling property, as follows

$$\tilde{s}(z,\tau) = \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{\frac{1}{n}} s\left(z, t + \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{-\frac{2p}{n+p}} \tau\right).$$

Here, $\tilde{s}(\cdot,0)$ is the support function of $L_t K_t$ where $L_t \in SL(n)$ is obtained from John's lemma applied to the convex body K_t. Therefore

$$c \leq \tilde{s}(z,0) \leq C.$$

Let $\mathbb{B}_r$ denote the ball of radius r centered at the origin. Thus $\mathbb{B}_c$ is contained in the convex body associated with the support function $\tilde{s}(\cdot, 0)$. The containment principle, see for example Proposition 2.2 in [100], insures that $\mathbb{B}_{c/2}$ will be contained in the convex body associated with the support function $\tilde{s}(\cdot, \tau)$, for $\tau \in [0, \delta]$, where δ is the time that $\mathbb{B}_c$ becomes $\mathbb{B}_{c/2}$ under the p-flow. This time can be found explicitly as the evolution of a ball of radius ρ centered at the origin is $\rho_t = -\rho^{(n-(2n-1)p)/(n+p)}$. Therefore, $\tilde{s}(z, \tau)$ exists on the time interval $\left[0, \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{\frac{2p}{n+p}} (T - t)\right)$. In particular, we have $\left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{\frac{2p}{n+p}} (T - t) \geq \delta$ for every $t \in [0, T)$. Now Lemmas 77, 79, 80 and 81 imply that there are uniform lower and upper bounds on the principal curvatures and on the speed of the flow on the time interval $[\delta/2, \delta]$. Therefore, by [61], we conclude that there are uniform bounds on higher derivatives of the curvature. Consequently, all quantities related to the original solution that are both scaling invariant and invariant under $SL(n)$ satisfy uniform bounds on the time interval $\left[t + \frac{C}{2} V(K_t)^{\frac{2p}{n+p}}, t + CV(K_t)^{\frac{2p}{n+p}}\right]$. Since t is arbitrary and C is an absolute constant, we have uniform bounds on the time interval $[T/2, T)$. It means that all the affine invariant quantities of the normalized solution to the p-flow are uniformly bounded on the time interval $[T/2, T)$. We point out here that if $n = 1$, only Lemma 77 and Lemma 79 are needed to derive such uniform bounds on the time interval $[T/2, T)$. We shall elaborate on the argument we presented above. Observe that

$$\tilde{s}(z, \tau) = \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{\frac{1}{n}} s\left(z, t + \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{-\frac{2p}{n+p}} \tau\right)$$

has uniform C^k bounds for all $t \in [0, T)$ and $\tau \in [\frac{\delta}{2}, \delta]$. Furthermore, the volume of the convex body corresponding to $\tilde{s}(z, \tau)$ is bounded from below by $V(\mathbb{B}_{c/2})$. Thus, the ratio

$$\frac{V(K_t)}{V\left(K_{t+\left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{-\frac{2p}{n+p}}\tau}\right)}$$

is bounded by $\frac{V(\mathbb{B}^n)}{V(\mathbb{B}^n_{c/2})}$ for all $t \in [0, T)$ and $\tau \in [\frac{\delta}{2}, \delta]$. This implies that after multiplying a controlled constant depending on τ, we have uniform C^k bounds for

$$\left(\frac{V(\mathbb{B}^n)}{V\left(K_{t+\left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{-\frac{2p}{n+p}}\tau}\right)}\right)^{\frac{1}{n}} s\left(z, t + \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{-\frac{2p}{n+p}}\tau\right)$$

for all $t \in [0, T)$ and $\tau \in [\frac{\delta}{2}, \delta]$. Next, we show that for every $t_* \in [\frac{T}{2}, T)$ we can find $t \in [0, T)$ such that

$$t_* = t + \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{-\frac{2p}{n+p}} \frac{\delta}{2}.$$

Define $f(t) = t + \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{-\frac{2p}{n+p}} \frac{\delta}{2} - t_*$ on $[0, T)$. This is a continuous function. We have $f(T) = T - t_* > 0$. Therefore, if we show that $f(0) < 0$ the claim follows. We have

$$f(0) = \left(\frac{V(\mathbb{B}^n)}{V(K_0)}\right)^{-\frac{2p}{n+p}} \frac{\delta}{2} - t_*.$$

Notice that $\left(\frac{V(\mathbb{B}^n)}{V(K_0)}\right)^{-\frac{2p}{n+p}} \frac{\delta}{2} \leq \frac{T}{2}$ so $f(0) < 0$. Therefore, we have proved that for every $\bar{t} \in [T, \frac{T}{2})$, there is a $t \in [0, T)$ with $\bar{t} = t + \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{-\frac{2p}{n+p}} \frac{\delta}{2}$ and there is a special linear transformation $L_{\bar{t}}$, obtained from John's lemma applied to K_t, such that the family $\{\left(\frac{V(\mathbb{B}^n)}{V(K_{\bar{t}})}\right)^{\frac{1}{n}} L_{\bar{t}} K_{\bar{t}}\}_{\bar{t}\in[T/2,T)}$ has uniform C^k bounds.

Consequently, there is a sequence of times $\{t_k\}_{k\in\mathbb{N}}$ such that t_k approaches T and the sequence $\{L_{t_k}(K_{t_k})\}_k$ converges in the C^∞ topology to a convex body $\tilde{K}_T$. Now monotonicity of the p-affine isoperimetric ratio and Theorem 82 with, a similar argument as in [10], implies that $\tilde{K}_T$ must be an ellipsoid. Therefore

$$\lim_{t_k \to T} \frac{\Omega_p^{n+p}(K_{t_k})}{V^{n-p}(K_{t_k})} = n^{n+p}\omega_n^{2p},$$

and again by monotonicity of the p-affine isoperimetric ratio

$$\lim_{t \to T} \frac{\Omega_p^{n+p}(K_t)}{V^{n-p}(K_t)} = n^{n+p}\omega_n^{2p}.$$

From the equality case in the p-affine isoperimetric inequality [69], it follows that, modulo $SL(n)$,

$$\lim_{t \to T} \left(\frac{V(\mathbb{B}^n)}{V(K_t)}\right)^{\frac{1}{n}} K_t = \mathbb{B}^n$$

sequentially in the C^∞ topology.

Finally, notice that when $p = 1$ the flow is translation invariant. Therefore, after an appropriate translation of the initial convex body, it is guaranteed that the origin of the plane always belongs to the interior of the evolving convex body for all time. This in turn implies that the evolution equation for the dual convex with respect to the origin is still valid and thus the above argument is also applicable in this case.

Chapter 7

Conclusions

In this work, we studied the p centro-affine normal flows and we presented several applications of this family of flows to convex geometry and PDEs.

In Chapters 2 and 6, we studied the long time behavior of the p-flows. We studied whether a normalized solution to the p-flow starting from a smooth, origin-symmetric initial convex body converges, in an appropriate norm, to a smooth shape. We gave a positive answer to this question in the C^∞-norm, modulo the group of special linear transformations. Precisely, we proved that the volume preserving p-flow evolves smooth, origin-symmetric convex bodies in $\mathbb{R}^n$ to the unit ball, in the C^∞-norm, modulo $SL(n)$, provided that $1 \leq p < \frac{n}{n-2}$. We used two different techniques to obtain the above mentioned result. In Chapter 2, we based our argument on our calculations carried out in affine differential setting. We obtained sharp affine isoperimetric type inequalities from which we deduced the asymptotic value of the affine supports of the evolving convex bodies. In Chapter 6, to study the long time behavior, a novel method was developed by involving dual convex bodies. The techniques avoided use of Harnack's estimate and displacement bounds. An interesting question related to the long time behavior of the p-flow is as follows:

• What can we say about the asymptotic behavior of the normalized p-flow if, $p \geq \frac{n}{n-2}$ and $n > 2$?

In Chapter 3, we touched upon the L_{-2} Minkowski problem. Using techniques developed in Chapter 2, we proved that the set of smooth, π-periodic, positive functions on the unit sphere for which the planar L_{-2} Minkowski problem is solvable is dense in the set of all smooth, π-periodic, positive functions on the unit sphere with respect to the L^{∞} norm. It is an interesting question that

• How can we use the weighted p-flow to obtain a necessary or a sufficient condition for the existence of a solution to the L_{-2} Minkowski problem?

In Chapter 4, we proved a version of stability of the p-affine isoperimetric inequality, in the class of origin-symmetric convex bodies, for $p \geq 1$, by using the affine normal flow. There are several interesting questions in this regard. One question of high interest to us is as follows. Letting p go to ∞, it is easy to see that the p-affine isoperimetric ratio converges to the volume product, the product of the volume of a convex body and the volume of its dual convex body. As we stated earlier, the volume product is controlled by the Blaschke-Santaló inequality.

• Is it possible to use the p-flow to obtain a version of stability of the Blaschke-Santaló inequality, even in the class of origin-symmetric convex bodies?

In Chapter, 5, we classified ancient solutions to the planar p-flow provided $1 \leq p < 4$, in the class of origin-symmetric convex bodies. We proved that the only compact, origin-symmetric, ancient solutions to the p-flow are homothetic ellipses, if $1 \leq p < 4$. This in particular, states that in the class of origin-symmetric convex bodies, the only ancient solutions to the planar affine normal flow are homothetic ellipses. In this regard, I am interested in the answer of the following question.

- What are the ancient solutions to the p-flow in higher dimensions?

In conclusion, we studied several aspects of the p centro-affine normal flows. But there are still interesting questions which can be topics of future projects.

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