Some support properties for a class of Λ-Fleming-Viot processes

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ABSTRACT

Some support properties for a class of Λ-Fleming-Viot processes

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Using Donnelly and Kurtz's lookdown construction, we prove that the Λ-Fleming-Viot process with underlying Brownian motion has a compact support at any fixed time provided that the associated Λ-coalescent comes down from infinity not too slowly. We also find both upper and lower bounds on Hausdorff dimension for the support at any fixed time. When the associated Λ-coalescent has a nontrivial Kingman component, the Hausdorff dimension for the support is exactly two at any fixed time.

For such a Λ-Fleming-Viot process, we further prove a one-sided modulus of continuity result for the ancestry process recovered from Donnelly and Kurtz's lookdown construction. As an application, we can prove that its support process also has the one-sided modulus of continuity (with modulus function $C\sqrt{t\log(1/t)}$) at any fixed time.

In addition, we obtain that the support process is compact simultaneously at all positive times, and given the initial compactness, its range is uniformly compact over time interval $[0, t)$ for all $t > 0$. Under a mild condition on the Λ-coalescence rates, we also find a uniform upper bound on Hausdorff dimension for the support and an upper bound on Hausdorff dimension for the range.

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Chapter 1

Introduction

The Fleming-Viot processes are probability-measure-valued Markov processes for mathematical population genetics. They arise as diffusion approximations for various Markov chain models and describe the evolution of relative frequencies for different types of individuals in a large population undergoing resampling together with possible mutation, selection and recombination.

Fleming and Viot (1979) first proposed the classical Fleming-Viot process to describe the frequencies of alleles in population genetic models. A survey of early work on the subject of Fleming-Viot processes can be found in Ethier and Kurtz (1993). We also refer to Dawson (1993) and Etheridge (2000, 2012) and references therein for a collection of results on Fleming-Viot processes. The study of Fleming-Viot processes and related population models has become an important and active field in probability theory. In this thesis, we study the support properties for generalized Fleming-Viot process.

We begin with introducing several population genetic models.

1.1 Several population genetic models

In population genetics, when evolution is treated as a random process, reproduction is the most basic source of randomness that leads to *genetic drift*. Namely, the distribution of genetic types in a population changes due to randomness in the individuals' reproduction. Wright (1931) and Fisher (1990) developed the earlier model of genetic drift, which is known as the Wright-Fisher model. We also refer to Cannings (1974, 1975) for the Cannings model and Moran (1958) for the Moran model that also capture the feature of genetic drift. For such models, looking forwards in time, the frequencies of alleles can be approximated by Markov processes taking values in the space of probability measures. Looking backwards in time, we can recover the genealogy of all individuals from the population.

We follow Birkner and Blath (2009a) and Etheridge (2012) to introduce several population genetic models. First of all, we briefly go over some concepts in population genetics. We consider a population in which every individual is equally likely to mate with every other and in which all individuals experience the same conditions. Such a population is *panmictic*. A population is *neutral* if the reproductive mechanism is the same for every individual. A *haploid* population means that each individual has a single copy of each chromosome (such as most bacteria) while a *diploid* population means that each individual has two copies of each chromosome (such as humans). For the haploid population, each individual has exactly one parent.

1.1.1 The neutral Cannings model

The neutral Cannings model for a panmictic, haploid population of fixed size $N \in \{1, 2, 3, \ldots\}$ is defined as follows. For nonoverlapping generations $t \in \{0, 1, 2, \ldots\}$, the individuals in generation t are labeled by $\{1, \ldots, N\}$. The generation $t+1$ is determined by an exchangeable random vector $\nu(t) \equiv (\nu_1(t), \ldots, \nu_N(t))$ with $\sum_{k=1}^{N} \nu_k(t) = N$, where $\nu_k(t)$ denotes the number of children for the kth individual in generation t.

For all the positive integers t, the vectors $\nu(t)$ are independent and identically distributed. Let σ be any permutation on $\{1, 2, \ldots, N\}$. For each fixed t, it follows from the neutrality that the vectors $(\nu_1(t), \ldots, \nu_N(t))$ and $\left(\nu_{\sigma(1)}(t), \ldots, \nu_{\sigma(N)}(t)\right)$ have the same distribution. Further, the family sizes $\nu_1(t)$, $\nu_2(t)$, $\ldots$, $\nu_N(t)$ are exchangeable. For convenience, we use the notation $\nu_i \equiv \nu_i(1)$ for $i \in \{1, 2, \ldots, N\}$.

Now we look at an example of the two-allele Cannings model. Let $\{\mathbb{a}, \mathbb{A}\}$ be the collection of alleles. Each individual has the same type as its parent. For each generation t, denote by $Y^N(t)$ the number of individuals which carry the $\mathbb{a}$-allele. Then $Y^N(t)$ is a finite Markov chain on $\{0, 1, \ldots, N\}$ as well as a martingale. Its dynamics can be represented as

$$Y^N(t+1) = \sum_{i=1}^{Y^N(t)} \nu_i(t). \tag{1.1.1}$$

Note that $Y^N(t)$ will almost surely be absorbed in either 0 or N. The probability that $Y^N(t)$ is absorbed in N equals to its initial frequency $Y^N(0)/N$.

1.1.2 The neutral Wright-Fisher model

In this subsection, we still consider a panmictic, haploid population of size N. In the neutral Wright-Fisher model, the population of N individuals evolves in discrete generations. For any $t \in \{0, 1, 2, \ldots\}$, each individual in generation $t+1$ randomly chooses its parent from those individuals in generation t, i.e., the generation $t+1$ is formed from generation t by taking i.i.d. samples of size N with replacement.

In fact, the Wright-Fisher model is a special case of the Cannings model in which $(\nu_1(t), \ldots, \nu_N(t))$ has the multinomial distribution with N trials and equal weights.

Now we look at an example of the two-allele Wright-Fisher model. Recall that $Y^N(t)$ is the number of individuals which carry the $\mathbb{a}$-allele. Its dynamics can be represented as

$$\mathbb{P}\left(Y^N(t+1) = k \middle| Y^N(t)\right) = \binom{N}{k} p_t^k (1-p_t)^{N-k},$$

where $p_t = Y^N(t)/N$ is the proportion of individuals with $\mathbb{a}$-allele at generation t.

1.1.3 The Moran model

A population of N individuals evolves according to the Moran model if during its reproduction events, at an exponential rate $\binom{N}{2}$ a pair of individuals is chosen uniformly at random from the population, one dies and the other splits into two.

Compared with the Wright-Fisher model, the generations of the Moran model overlap while the Wright-Fisher model evolves in discrete generations. Further, in the Moran model an individual can have either zero or two offspring while in the Wright-Fisher model an individual can have up to N offspring.

The Moran model is not a Cannings model. But it can be fit into the Cannings class if we choose $\nu(t)$ uniformly distributed on all the permutations of $(2, 0, 1, \ldots, 1)$.

1.1.4 Wright-Fisher diffusion as a limit of "many" Cannings models

For a population of large size, it is more convenient to consider a diffusion limit. In the Cannings model, let c_N be the probability that two individuals chosen randomly without replacement from some generation have a common ancestor in the previous generation. Then

$$c_N = \frac{\sum_{i=1}^{N} \mathbb{E}\left(\nu_i\left(\nu_i - 1\right)\right)}{N(N-1)} = \frac{\mathbb{E}\left(\nu_1\left(\nu_1 - 1\right)\right)}{N-1} = \frac{\mathrm{Var}\left(\nu_1\right)}{N-1}, \tag{1.1.2}$$

where we have used the property $\mathbb{E}(\nu_1) = 1$. The number of generations it takes for any two randomly chosen individuals back to their most recent common ancestor has a geometric distribution with success probability $p = c_N$. Consequently, the expected number of generations to get back to their most recent common ancestor is $1/c_N$, which determines a time scaling.

The probability that three randomly chosen individuals have a common ancestor in the previous generation is

$$\frac{\mathbb{E}\left(\nu_1\left(\nu_1 - 1\right)\left(\nu_1 - 2\right)\right)}{\left(N-1\right)\left(N-2\right)}.$$

If we measure time in units of $1/c_N$ and assume that

$$c_N \to 0 \text{ and } \frac{\mathbb{E}\left(\nu_1\left(\nu_1 - 1\right)\left(\nu_1 - 2\right)\right)}{N^2 c_N} \to 0 \text{ as } N \to \infty, \tag{1.1.3}$$

we exclude the possibility that more than two different individuals share a common ancestor in the previous generations in the ancestral lineages.

Two-allele Wright-Fisher diffusion

We come back to the two-allele Wright-Fisher model. $Y^N(t)$ is the number of individuals which carry the a-allele. Denote by

$$X^N(t) \equiv \frac{1}{N} Y^N\left(\lfloor t/c_N \rfloor\right) = \frac{1}{N} Y^N\left(\lfloor tN \rfloor\right), \quad t \geq 0,$$

where $\lfloor t/c_N \rfloor$ is the integer part of t/c_N. We also use $\nu_1 \overset{D}{=} \mathrm{Bin}\,(N, 1/N)$ to conclude that $c_N = 1/N$. If condition (1.1.3) holds, the process $\left(X^N(t)\right)_{t\geq 0}$ weakly converges to a Markov process $(X(t))_{t\geq 0}$ in $[0,1]$ with generator given by

$$\mathcal{L}f(x) = \frac{1}{2}x(1-x)\frac{\partial^2}{\partial x^2}f(x), \quad x \in [0,1], \quad f \in C^2\left([0,1]\right).$$

Multiple-allele Wright-Fisher diffusion

We assume now that there are k different alleles in the model. If condition (1.1.3) holds, we could extend the above-mentioned model to models with finitely many alleles. At generation $\lfloor t/c_N \rfloor$, the frequencies of all the alleles can be approximated by

$$X(t) \equiv (X_1(t), X_2(t), \ldots, X_k(t)) \in \left\{ (x_1, x_2, \ldots, x_k) : x_i \geq 0, \sum_i x_i = 1 \right\},$$

which is a diffusion with generator $\mathcal{L}^{(k)}$ such that for any $f \in C^2([0,1]^k)$,

$$\mathcal{L}^{(k)} f\left(x_1, x_2, \ldots, x_k\right) = \frac{1}{2} \sum_{i,j=1}^{k} x_i \left(\delta_{ij} - x_j\right) \frac{\partial^2}{\partial x_i \partial x_j} f\left(x_1, x_2, \ldots, x_k\right),$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

1.2 The classical Fleming-Viot process

In this subsection, we first introduce the classical mutationless Fleming-Viot process for population genetics. Then we add mutation to the classical Fleming-Viot process. Finally, we discuss the dual process for the classical Fleming-Viot process with resampling and mutation.

1.2.1 The classical mutationless Fleming-Viot process

In the population genetic models, a different approach is required when the type space contains infinitely many alleles. Let E be any locally compact metric space, which represents the collection of infinitely many alleles. Denote by $M_1(E)$ the space of probability measures on E equipped with the topology of weak convergence.

In the Cannings model with infinitely many alleles, let $\tilde{Y}^N(t,i)$ be the type of individual i in generation t. Denote by $Z^N(t)$ the empirical measure for alleles of all the individuals in generation t such that

$$Z^N(t) \equiv \frac{1}{N}\sum_{i=1}^{N} \delta_{\tilde{Y}^N(t,i)}.$$

If condition (1.1.3) holds and $Z^N(0) \to \mu \in M_1(E)$, the time-rescaled process $Z^N(\lfloor t/c_N \rfloor)$ converges in weak topology to the classical mutationless Fleming-Viot process $X(t)$ with generator given by

$$\mathcal{L}\Phi(\mu) \equiv \sum_{J\subseteq\{1,2,\ldots,n\},|J|=2} \int_E \cdots \int_E \left(\phi\left(x_1^J,\ldots,x_n^J\right) - \phi\left(x_1,\ldots,x_n\right)\right)\mu\left(dx_1\right)\cdots\mu\left(dx_n\right), \tag{1.2.1}$$

where the test function Φ is defined as

$$\Phi(\mu) \equiv \int_E \cdots \int_E \phi(x_1,\ldots,x_n)\mu(dx_1)\cdots\mu(dx_n), \tag{1.2.2}$$

n is any positive integer and $\phi : E^n \to \mathbb{R}$ is measurable and bounded, and for $(x_1,\ldots,x_n) \in E^n$ and $J \subseteq \{1,2,\ldots,n\}$, we put

$$x_i^J = x_{\min J} \text{ if } i \in J \text{ and } x_i^J = x_i \text{ if } i \notin J, i = 1,2,\ldots,n. \tag{1.2.3}$$

Intuitively, for any subset $K \subseteq E$, $X(t,K)$ represents the proportion of individuals with alleles in K at time t.

1.2.2 Adding mutation

Mutation is another important feature that changes the frequencies of alleles in the evolution. For convenience, we always use A to represent the mutation operator for the

Fleming-Viot process throughout the thesis. Let $\mathcal{B}(E)$ be the set of bounded functions on E. We assume that A generates a semigroup $(\mathcal{T}(t))$ on $\mathcal{B}(E)$ which is given by a transition function (P_t) such that for any $f \in \mathcal{B}(E)$,

$$\mathcal{T}(t)f(x) = \int_E f(y)P_t(x,dy).$$

For each $n \geq 1$, we define the semigroup $(\mathcal{T}_n(t))$ on $\mathcal{B}(E^n)$ such that for any $f \in \mathcal{B}(E^n)$,

$$\mathcal{T}_n(t)\, f(x_1,\ldots,x_n) = \int_E \cdots \int_E f(\xi_1,\ldots,\xi_n)\, P_t(x_1,d\xi_1)\cdots P_t(x_n,d\xi_n).$$

Let $A^{(n)}$ be the generator for $(\mathcal{T}_n(t))$ and $\mathcal{D}\left(A^{(n)}\right)$ be the domain of $A^{(n)}$. Clearly, $\mathcal{D}\left(A^{(n)}\right)$ is a subspace of $\mathcal{B}(E^n)$.

1.2.3 The classical Fleming-Viot process with resampling and mutation

The classical Fleming-Viot process with resampling and mutation is a probability-measure-valued Markov process with generator given by

$$\begin{aligned}\mathcal{L}\Phi(\mu) \equiv &\int_E \cdots \int_E A^{(n)}\phi(x_1,\ldots,x_n)\mu(dx_1)\cdots\mu(dx_n)\\ &+ \sum_{J\subseteq\{1,2,\ldots,n\},|J|=2} \int_E \cdots \int_E \left(\phi\left(x_1^J,\ldots,x_n^J\right) - \phi(x_1,\ldots,x_n)\right)\mu(dx_1)\cdots\mu(dx_n),\end{aligned}$$

where $\phi \in \mathcal{D}(A^{(n)})$.

1.2.4 The dual process of classical Fleming-Viot process

The classical Fleming-Viot process is dual to a function-valued process $(\zeta_t)_{t\geq 0}$ which takes its value in the space $\mathscr{C} \equiv \cup_{n=1}^{\infty}\mathcal{B}(E^n)$, whose evolution can be described as follows.

- Given $n \geq 2$ and $\zeta_t \in \mathcal{B}(E^n)$, ζ_t jumps from $\mathcal{B}(E^n)$ to $\mathcal{B}(E^{n-1})$ at an exponential rate $n(n-1)/2$.

- At the jump time, a pair of distinct integers (k,l) is chosen at random from $\{1,2,\ldots,n\}$. Set $J \equiv \{k,l\}$. Then $\zeta_t(x_1,x_2,\ldots,x_n)$ is replaced by $\zeta_t\left(x_1^J,x_2^J,\ldots,x_n^J\right)$, where x_i^J is defined by (1.2.3) for any $1 \leq i \leq n$.

- Between jump times, the process is deterministic with each of its coordinate function driven by the semigroup $(\mathcal{T}(t))$.

- No further jump happens after the process takes its value in $\mathcal{B}(E)$.

The moments of the Fleming-Viot process can be expressed by its dual process such that

$$\mathbb{E}_{X(0)} \langle X^n(t), \zeta_0 \rangle = \mathbb{E}_{(\zeta_0, n)} \langle X^p(0), \zeta_t \rangle ,$$

where we assume that $\zeta_0 \in \mathcal{B}(E^n)$ and $\zeta_t \in \mathcal{B}(E^p)$ at time t.

In fact, the dual process of classical Fleming-Viot process with resampling and mutation is governed by mutation semigroup and Kingman's coalescent. We will introduce the Kingman's coalescent in Subsection 2.1.1. The Fleming-Viot process can involve not only resampling and mutation, but also selection and recombination. We refer to Ethier and Kurtz (1993) for the Fleming-Viot process with resampling, mutation, selection and recombination. In this thesis, we only focus on the Fleming-Viot process with resampling and mutation.

1.3 Generalized Fleming-Viot processes

When the classical Fleming-Viot process only involves mutation and resampling, the moment dual of the classical Fleming-Viot process is a function-valued Markov process governed by Kingman's coalescent and mutation semigroup. During the past ten years, more general coalescent processes have been proposed and studied by many authors. For examples, the Λ-coalescent (cf. Pitman (1999), Möhle and Sagitov (2001), Sagitov (1999)) is a coalescent with possible multiple collisions and the Ξ-coalescent (cf. Schweinsberg (2000a), Sagitov (2003)) is a coalescent with possible simultaneous multiple collisions.

The moment dual of generalized Fleming-Viot process evolves in the same way as the classical Fleming-Viot process but with the Kingman's coalescent replaced by a general coalescent. For example, the Λ-Fleming-Viot process generalizes the classical Fleming-Viot process by replacing Kingman's coalescent with Λ-coalescent of multiple collisions.

Formally, the Λ-Fleming-Viot process is a Fleming-Viot process with general branching mechanism so that the total number of children of a parent can be comparable to the size of population. We refer to Birkner et al. (2005) for a connection between a mutationless Λ-Fleming-Viot process and a continuous state branching process. When the spatial motion of the particle is negated, namely, the mutation is 0, the generalized Fleming-Viot process has been studied by Bertoin and Le Gall (2003, 2005, 2006) and Birkner et al. (2005). Birkner et al. (2009) constructed the (Ξ, A)-Fleming-Viot process for parent independent mutation generator. Li et al. (2011) proved the existence of the (Ξ, A)-Fleming-Viot process for general mutation operator A. They further studied the reversibility and both the weak and strong uniqueness of solution to the associated partial differential equation. Feng et al. (2011) proved that the reversibility fails for a system of Fleming-Viot processes living on a countable number of colonies interacting with each other if both migration and mutation are nontrivial.

The support property is interesting in the study of measure-valued processes. For the Dawson-Watanabe superBrownian motion arising as high density limit of empirical measures for near critical branching Brownian motions, the modulus of continuity and the carrying dimensions have been studied systematically for its support process. We refer to Chapter 7 of Dawson (1992), Chapter 9 of Dawson (1993), Chapter III of Perkins (1999), Dawson and Perkins (1991) and references therein for a collection of these results. The proofs involve the historically cluster representation, the Palm distribution for the canonical measure and estimates obtained from PDE associated with the Laplace functional.

Perkins (1989) discussed the Hausdorff measure for the closed support of the superBrownian motion and proved that the closed supports are Lebesgue null sets for all positive times almost surely when $d \geq 2$. Dawson et al. (1989) obtained a one-sided modulus of continuity and an exact Hausdorff measure function of the range and closed support of superBrownian motion. Le Gall (1998) found an exact Hausdorff measure function for the range of superBrownian motion in dimension $d \geq 4$. Le Gall and Perkins (1995) further found an exact Hausdorff measure function for the support of two-dimensional superBrownian motion at a fixed time. Dhersin (1998) obtained the lower function for the support of superBrownian motion to describe the minimum speed at which the support of a superBrownian motion starting at the Dirac mass at 0 moves away from 0. Le Gall (2006) described the asymptotic behavior of the occupation measure of the unit ball for superBrownian motion starting from the Dirac measure at a distant point x and conditioned to hit the unit ball.

Ren (2004) provided the criteria for the compact support property and the compactness of the global support for superBrownian motion with spatially dependent branching rate. For superBrownian motion with general branching mechanism, Delmas (1999) discussed the path properties such as the Hausdorff dimensions for the supports and the estimations on hitting probabilities of small balls. It has also been proved in Delmas (1999) that in low dimensions the random measure of a super α-stable process with a general branching mechanism is absolutely continuous with respect to the Lebesgue measure.

However, the method for Dawson-Watanabe superBrownian motion does not always apply to Fleming-Viot process since the Fleming-Viot process is not infinitely divisible. Consequently, there are only a few results available for the support of Fleming-Viot processes. The earliest work on the compact support property for classical Fleming-Viot process is due to Dawson and Hochberg (1982) where they proved that at any fixed time $T > 0$ the classical Fleming-Viot process with underlying Brownian motion has a compact support and the support has a Hausdorff dimension not greater than two. Using non-standard techniques Reimers (1993) improved the above result by proving that the Hausdorff dimension for the support of classical Fleming-Viot process is at most two for all positive times simultaneously. Applying a generalized Perkins disintegration theorem, the support dimension was found in Ruscher (2009) for a Fleming-Viot-like process obtained from mass normalization and time change of superBrownian motion with stable branching. Blath (2009), Birkner and Blath (2009b) pointed out that the Λ-Fleming-Viot process with underlying Brownian motion does not have a compact support at any fixed time if the corresponding Λ-coalescent does not come down from infinity.

The idea of expressing the measure-valued process as the empirical measure of an exchangeable system of particles was firstly introduced by Dawson and Hochberg (1982), where the classical Fleming-Viot process on E can be obtained as the empirical measure of an E^∞-valued particle system. Donnelly and Kurtz (1996, 1999a,b) exploited this idea further by proposing the lookdown construction, which is a powerful tool to study various properties of the measure-valued stochastic process. Loosely speaking, the lookdown construction is a discrete representation for the measure-valued process. Such a discrete representation carries the genealogy of the measure-valued model and thus considerably simplifies the study of the measure-valued process. In a sense it plays the role of cluster representation for Dawson-Watanabe superprocess.

Donnelly and Kurtz (1996) established the lookdown construction of countably many particles embedded into the classical Fleming-Viot process. They used this representation to study various path properties of the classical Fleming-Viot process and showed the duality between classical Fleming-Viot process and Kingman's coalescent. This construction and the associated duality results have been extended to the Λ-Fleming-Viot process in Donnelly and Kurtz (1999b), where they proposed a modified lookdown construction, which gives an explicit connection between genealogical models and diffusion models in populations. The modified lookdown construction in Donnelly and Kurtz (1999b) also applied to a larger class of measure-valued models, including the neutral Fleming-Viot processes and the Dawson-Watanabe superprocesses. Via the modified lookdown construction, they found a simple representation of the Dawson-Perkins historical process and described various applications on conditioning, martingale property, limiting behavior and so on. Donnelly and Kurtz (1999a) proposed a discrete representation for the classical Fleming-Viot process with selection and recombination, where they used two ways to characterize the E^∞-valued system of particles. One is through solutions to an infinite system of ordinary stochastic differential equations and the other is via a martingale problem.

Birkner and Blath (2009a) further discussed the modified lookdown construction in Donnelly and Kurtz (1999b) for the Λ-Fleming-Viot process. They also described how to recover the Λ-coalescent from the modified lookdown construction. A Poisson point process construction of the Ξ-lookdown model can be found in Birkner et al. (2009) which extended the modified lookdown construction of Donnelly and Kurtz (1999b).

1.4 Main results of the thesis

The lookdown construction plays a crucial role in all of our major arguments. In the lookdown construction each particle is attached a "level" from the set $\{1, 2, \ldots\}$. The evolution of a particle at level n only depends on the evolution of the particles at lower levels. For any positive integer n, the first n levels can be embeded into the first $n+1$ levels. This projective property allows us to construct approximating particle systems, and their limit as $n \to \infty$ in the same probability space. In this thesis, we study the support properties for a class of Λ-Fleming-Viot processes.

The first part of the main results is introduced in Chapter 3, which is based on Liu and Zhou (2012). We extend the compact support property at fixed time for the classical Fleming-Viot process to a class of Λ-Fleming-Viot processes with the associated Λ-coalescents coming down from infinity. Applying Donnelly and Kurtz's lookdown construction, we adapt the idea of Dawson and Hochberg (1982) as follows.

Given any fixed time $T > 0$, we can represent the Λ-Fleming-Viot process at time T as limit of empirical measures of the exchangeable particle systems obtained via the lookdown construction. For a sequence of random times T_n converging increasingly to T, by the lookdown construction and the property of coming down from infinity there exist finitely many common ancestors at each time T_n for those particles at time T. Our assumption on the time it takes to come down from infinity allows us to estimate the number of common ancestors at time T_n. Then locations of the ancestors at time T_{n+1}

are determined by a collection of possibly dependent Brownian motions starting from the locations of ancestors at time T_n and stopping after time $T_{n+1} - T_n$. By the modulus of continuity for Brownian motion we can estimate the maximal dislocation of the ancestors at time T_{n+1} from those at time T_n. Choosing (T_n) properly and applying Borel-Cantelli lemma we can show that for m large enough the maximal dislocations between T_n and T_{n+1} for all $n \geq m$ are summable. Then all the particles at time T are situated in the union of finitely many closed balls centered at the ancestors' locations at time T_m respectively. The compact support property then follows.

As a byproduct of the estimates we can also find an upper bound on Hausdorff dimension for the support at time T. The moments of the Λ-Fleming-Viot process can be expressed in terms of a dual process involving Λ-coalescent and heat flow. By Frostman's lemma and a computation involving the second moment, we also find a lower bound on Hausdorff dimension for the support at time T. As a corollary, we conclude that when the associated Λ-coalescent has a nontrivial Kingman component, the Hausdorff dimension for the support is exactly two at any fixed positive time. These results generalize the previous results of Dawson and Hochberg (1982) on the classical Fleming-Viot process.

The second part of the main results is introduced in Chapters 4 & 5, which is based on Liu and Zhou (2013). This part is a refinement of the arguments in Liu and Zhou (2012). We mainly focus on discussing some further support properties for the class of Λ-Fleming-Viot processes in the previous part, such as the one-sided modulus of continuity, the uniform compactness of the support and range, and the upper bounds on Hausdorff dimensions for the support and range.

We outline our approach as follows. Given any finite interval $[0, T]$, we first divide it into small disjoint subintervals with step length $\Delta \equiv \Delta_n = 2^{-n}$. Given n, for each $0 \leq k \leq T2^n - 1$, choose a sequence of random times $\left(T_m^{n,k}\right)_m \subseteq [k2^{-n}, (k+1)\,2^{-n})$ increasingly convergent to $(k+1)\,2^{-n}$ as $m \to \infty$. The coming down from infinity property implies that there are finitely many ancestors at each time $T_m^{n,k}$ for those countably many particles at time $(k+1)\,2^{-n}$. The dislocations between those particles at time $(k+1)\,2^{-n}$

and their corresponding ancestors at time $k2^{-n}$ are determined by possibly dependent piecewise Brownian paths. Each segment of the piecewise Brownian path connects the locations of ancestors at times $T_m^{n,k}$ and $T_{m+1}^{n,k}$. We can estimate the maximal oscillation of each segment by the modulus of continuity for Brownian motion. The summation of the oscillations of all segments in each piecewise Brownian path dominates the maximal dislocation among all the particles at the endpoint and their respective ancestors at the beginning of each small subinterval.

Choosing $\left(T_m^{n,k}\right)_m$ properly and applying Borel-Cantelli lemma, we can find a uniform upper bound on the maximal dislocations among all the particles at the endpoint and their respective ancestors at the beginning of all these $\lfloor T2^n \rfloor$ small subintervals when n is large enough. Note that the whole endpoints of the subintervals are the collection of all the dyadic rationals in $[0, T]$, which is a dense subset of $[0, T]$. For any $0 \leq r < s \leq T$, the dislocations between the countably many particles at time s and their finite ancestors at time r can be approximated by the dislocations at dyadic rational times when s is close enough to r. In this way, we obtain our first result on the one-sided modulus of continuity for the ancestry process defined via the lookdown construction.

As an application, we can prove the one-sided modulus of continuity for the Λ-Fleming-Viot support process at any fixed time. These estimates for the modulus of continuity naturally result in finite covers for the support and range of the Λ-Fleming-Viot process to get the uniform compactness of the support and range.

Under an additional mild condition on the coalescence rates of the associated Λ-coalescent, we can find a sharper estimate for the number of ancestors in order to obtain more precise covers for the support and range, which leads to two results on the support dimensions. One is an uniform upper bound on Hausdorff dimension for the support. The other is an upper bound on Hausdorff dimension for the range.

There are overlaps between these two parts. To keep the integrity, we introduce all the results obtained in Liu and Zhou (2012, 2013).

1.5 Organization of the thesis

The thesis is organized as follows. In Chapter 1, we give a brief overview on population genetic models, the Fleming-Viot processes and the main results of the thesis.

In Chapter 2, we first introduce Kingman's coalescent, Λ-coalescent, Ξ-coalescent and their coming down from infinity property. Then we introduce the Λ-Fleming-Viot process. Further, we present the lookdown constructions for both the classical Fleming-Viot process and the Λ-Fleming-Viot process. The lookdown construction is key to our later arguments.

In Chapter 3, we prove the compact support property for a class of Λ-Fleming-Viot processes at fixed time with the associated Λ-coalescents coming down from infinity. In addition, we find both lower and upper bounds on Hausdorff dimension for their supports at fixed time.

In Chapter 4, we begin with showing the one-sided modulus of continuity for the ancestry process recovered from the lookdown construction. As an application of this result, we prove the one-sided modulus of continuity for the Λ-Fleming-Viot support process at any fixed time.

In Chapter 5, we first prove the uniform compactness of the support and range for the Λ-Fleming-Viot process. Then under an additional mild condition on the coalescence rates for the associated Λ-coalescent, we obtain the upper bounds on Hausdorff dimensions for the support and range.

In Chapter 6, we propose some topics for future research.

Chapter 2

Preliminary

In this chapter, we first introduce several classes of partition-valued coalescent processes: Kingman's coalescent, Λ-coalescent and Ξ-coalescent. Then we introduce the Λ-Fleming-Viot process. Further, we present Donnelly and Kurtz's lookdown constructions for both classical Fleming-Viot process and Λ-Fleming-Viot process. We also illustrate how to recover the genealogy and ancestry processes from the lookdown constructions.

2.1 Coalescents

We introduce some notations from Bertoin (2006). Put $[n] \equiv \{1, \ldots, n\}$ and $[\infty] \equiv \{1, 2, \ldots\}$. An ordered *partition* of $D \subset [\infty]$ is a countable collection $\pi \equiv \{\pi_i, i = 1, 2, \ldots\}$ of disjoint *blocks* such that $\cup_i \pi_i = D$ and $\min \pi_i < \min \pi_j$ for $i < j$. Then blocks in π are ordered by their least elements.

Denote by $\mathcal{P}_n$ the set of ordered partitions of $[n]$ and by $\mathcal{P}_\infty$ the set of ordered partitions of $[\infty]$. Write $\mathbf{0}_{[n]} \equiv \{\{1\}, \ldots, \{n\}\}$ for the partition of $[n]$ consisting of singletons and $\mathbf{0}_{[\infty]}$ for the partition of $[\infty]$ consisting of singletons. Given $n \in [\infty]$ and $\pi \in \mathcal{P}_\infty$, let $R_n(\pi) \in \mathcal{P}_n$ be the restriction of π to $[n]$.

Given $n \in [\infty]$, let $\Pi_n \equiv (\Pi_n(t))_{t\geq 0}$ be a $\mathcal{P}_n$-valued stochastic process with right-continuous step function paths such that $\Pi_n(t)$ is a refinement of $\Pi_n(s)$ for every $t < s$. Denote by $\Pi \equiv (\Pi(t))_{t\geq 0}$ a $\mathcal{P}_\infty$-valued stochastic process with right-continuous step function paths such that $\Pi(t)$ is a refinement of $\Pi(s)$ for every $t < s$.

2.1.1 Kingman's coalescent

For Kingman's coalescent, given that there are b blocks at present, each 2-tuple of blocks merges independently to form a single block at rate 1. Therefore, the transition rate for the Kingman's coalescent from b blocks to $b-1$ blocks is $b(b-1)/2$. Note that only *binary mergers* are allowed. Kingman (1982a,b) showed that there exists a $\mathcal{P}_\infty$-valued Markov process $\Pi \equiv (\Pi(t))_{t\geq 0}$ which is called *Kingman's coalescent*, and whose restriction to the first n positive integers is an *n-coalescent*. For all $m < n < \infty$, the coalescent process $R_m(\Pi_n(t))$ given $\Pi_n(0) = \pi_n$ has the same distribution as $\Pi_m(t)$ given $\Pi_m(0) = R_m(\pi_n)$.

2.1.2 Λ-coalescent

In this section, we first introduce the Λ-coalescent. Then we illustrate the coming down from infinity or staying infinite properties for the Λ-coalescent. Finally, we list some examples of Λ-coalescents and consider whether they come down from infinity or stay infinite.

Introduction on Λ-coalescent

Donnelly and Kurtz (1999b), Pitman (1999) and Sagitov (1999) independently generalized the Kingman's coalescent to the Λ-*coalescent*, which allows *multiple collisions*, i.e., more than two blocks may merge at a time. The Λ-coalescent is defined as a $\mathcal{P}_\infty$-valued Markov process $\Pi \equiv (\Pi(t))_{t\geq 0}$ such that for each $n \in [\infty]$, its restriction to $[n]$, $\Pi_n \equiv (\Pi_n(t))_{t\geq 0}$ is a $\mathcal{P}_n$-valued Markov process whose transition rates are described as follows: if there are currently b blocks in the partition, then each k-tuple of blocks $(2 \leq k \leq b)$ independently merges to form a single block at rate

$$\lambda_{b,k} = \int_{[0,1]} x^{k-2}(1-x)^{b-k}\Lambda(dx), \tag{2.1.1}$$

where Λ is a finite measure on $[0,1]$. It is clear that $\Lambda([0,1]) = \lambda_{2,2}$. For $i, j \in [\infty]$ with i and j in different blocks of Π. Let $\tau_{i,j}$ be the collision time of i and j, meaning the unique time t such that i and j belong to the same block of $\Pi(t)$ but different blocks of $\Pi(t-)$. Then $\tau_{i,j}$ has the exponential distribution with rate $\Lambda([0,1])$.

It is easy to check that the rates $(\lambda_{b,k})$ are consistent so that for all $2 \le k \le b$,

$$\lambda_{b,k} = \lambda_{b+1,k} + \lambda_{b+1,k+1}.$$

Consequently, for any $1 \le m < n \le \infty$, the coalescent process $R_m(\Pi_n(t))$ given $\Pi_n(0) = \pi_n$ has the same distribution as $\Pi_m(t)$ given $\Pi_m(0) = R_m(\pi_n)$.

With the transition rates determined by (2.1.1), there exists a one to one correspondence between Λ-coalescents and finite measures Λ on $[0,1]$.

For $n = 2, 3, \ldots$, denote by

$$\lambda_n \equiv \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \tag{2.1.2}$$

the total coalescence rate starting with n blocks. It is clear that $(\lambda_n)_{n\ge 2}$ is an increasing sequence, i.e., $\lambda_n \le \lambda_{n+1}$ for any $n \ge 2$. In addition, denote by

$$\gamma_n \equiv \sum_{k=2}^{n} (k-1) \binom{n}{k} \lambda_{n,k}$$

the rate at which the number of blocks decreases.

Coming down from infinity property for Λ-coalescent

For any $n \in [\infty]$, let $\#\Pi_n(t)$ be the number of blocks in the partition $\Pi_n(t)$ and $\#\Pi(t)$ be the number of blocks in the partition $\Pi(t)$. The Λ-coalescent *comes down from infinity* if

$$P(\#\Pi(t) < \infty) = 1$$

for all $t > 0$. It *stays infinite* if

$$P(\#\Pi(t) = \infty) = 1$$

for all $t > 0$.

Suppose that the measure Λ has no atom at 1. It is shown in Schweinsberg (2000b) that

- the Λ-coalescent comes down from infinity if and only if $\sum_{n=2}^{\infty} \gamma_n^{-1} < \infty$;
- the Λ-coalescent stays infinite if and only if $\sum_{n=2}^{\infty} \gamma_n^{-1} = \infty$.

Examples

- If $\Lambda = \delta_1$, the corresponding coalescent is called *star-shaped coalescent*. It is clear that $\lambda_{n,n} = 1$ for any $n \geq 2$ and $\lambda_{n,k} = 0$ for any $2 \leq k \leq n-1$. Consequently, $\lambda_n = 1$ and $\gamma_n = n-1$ for any $n \geq 2$. The star-shaped coalescent only allows all the blocks to merge into one single block after an exponential time with parameter 1. Thus it neither comes down from infinity nor stays infinite.

- If Λ is the uniform distribution on $[0,1]$, the corresponding coalescent is called *U-coalescent*. For any $2 \leq k \leq n$, we have
$$\lambda_{n,k} = \frac{(k-2)!\,(n-k)!}{(n-1)!} \quad \text{and} \quad \lambda_n = n-1.$$
For the U-coalescent, we can verify that $\gamma_n \leq n \ln n$ and
$$\sum_{n=2}^{\infty} \gamma_n^{-1} \geq \sum_{n=2}^{\infty} 1/(n \ln n) = \infty.$$
So, it stays infinite.

- If $\Lambda = \delta_0$, the corresponding coalescent degenerates to Kingman's coalescent. We have $\lambda_{n,2} = 1$ for any $n \geq 2$ and $\lambda_{n,k} = 0$ for any $3 \leq k \leq n$. Then $\lambda_n = \gamma_n = n(n-1)/2$ for any $n \geq 2$. Consequently, the corresponding coalescent comes down from infinity.

- We say that a Λ-coalescent has the *(c,ϵ,γ)-property*, if there exist constants $c > 0$ and $\epsilon, \gamma \in (0,1)$ such that the measure Λ restricted to $[0,\epsilon]$ is absolutely continuous with respect to Lebesgue measure and
$$\Lambda(dx) \geq c x^{-\gamma} dx \ \text{ for all } \ x \in [0,\epsilon].$$

The total coalescence rate satisfies $\lambda_n \geq C(c,\gamma,\epsilon)\, n^{1+\gamma}$, which will be proved in Lemma 3.9. It follows from

$$\sum_{n=2}^{\infty} \gamma_n^{-1} \leq \sum_{n=2}^{\infty} \lambda_n^{-1} < \infty$$

that the Λ-coalescent with the (c,ϵ,γ)-property comes down from infinity.

- For $\beta \in (0,2)$, the *Beta*$(2-\beta,\beta)$*-coalescent* is the Λ-coalescent with the finite measure Λ on $[0,1]$ denoted by

  $$\Lambda(dx) = \frac{\Gamma(2)}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta} (1-x)^{\beta-1}\, dx.$$

  It follows from (2.1.1) and (2.1.2) that for any $2 \leq k \leq n$,

  $$\lambda_{n,k} = \frac{\Gamma(k-\beta)\,\Gamma(n-k+\beta)}{\Gamma(2-\beta)\,\Gamma(\beta)\,(n-1)!}$$

  and

  $$\lambda_n = \sum_{k=2}^{n} \frac{n\Gamma(k-\beta)\,\Gamma(n-k+\beta)}{k!\,(n-k)!\Gamma(2-\beta)\,\Gamma(\beta)}.$$

  - If $\beta \in (0,1]$, the Beta$(2-\beta,\beta)$-coalescent stays infinite ;
  - If $\beta \in (1,2)$, the Beta$(2-\beta,\beta)$-coalescent has the $(c,\epsilon,\beta-1)$-property and comes down from infinity. Now λ_n has the same order as n^{β}.

  We refer to Example 15 in Schweinsberg (2000b) for the arguments on coming down from infinity or staying infinite properties for Beta$(2-\beta,\beta)$-coalescent.

2.1.3 Ξ-coalescent

Schweinsberg (2000a) introduced the Ξ*-coalescent* allowing *simultaneous multiple collisions*. We first recall the notion of *coagulation*. Given a partition $\pi \in \mathcal{P}_n$ for some n and $\pi' \in \mathcal{P}_k$ with $|\pi| \leq k$ where $|\pi|$ denotes the cardinality of π, the *coagulation* of π by π', denoted by Coag(π,π'), is defined as the following partition of $[n]$,

$$\pi'' \equiv \left\{ \pi''_j \equiv \cup_{i\in\pi'_j} \pi_i : j = 1,\ldots,|\pi'| \right\}.$$

Given a partition π with $|\pi| = b$ and a sequence of positive integers $s, k_1, \ldots, k_r$ such that $k_i \geq 2, i = 1, \ldots, r$ and $b = s + \sum_{i=1}^r k_i$, we say a partition π'' is obtained by a $(b; k_1, \ldots, k_r; s)$-*collision* of π if $\pi'' = \text{Coag}(\pi, \pi')$ for some partition π' such that

$$\{|\pi'_i| : i = 1, \ldots, |\pi'|\} = \{k_1, \ldots, k_r, k_{r+1}, \ldots, k_{r+s}\},$$

where $k_{r+1} = \cdots = k_{r+s} = 1$, i.e., π'' is a merger of the b blocks of π into $r + s$ blocks in which s blocks remain unchanged and the other r blocks contain $k_1, \ldots, k_r$ blocks from π.

The Ξ-coalescent is a $\mathcal{P}_\infty$-valued process $\Pi \equiv (\Pi(t))_{t\geq 0}$ starting from partition $\Pi(0) \in \mathcal{P}_\infty$ such that for any $n \in [\infty]$, its restriction to $[n]$, $\Pi_n \equiv (\Pi_n(t))_{t\geq 0}$ is a Markov chain and that given $\Pi_n(t)$ has b blocks, each $(b; k_1, \ldots, k_r; s)$-collision occurs at rate $\lambda_{b;k_1,\ldots,k_r;s}$.

For the Ξ-coalescent to be well defined, it is sufficient and necessary that there exists a finite measure $\Xi = \Xi_0 + \sigma^2 \delta_0$ on the infinite simplex

$$\Delta \equiv \left\{ \mathbf{x} = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1 \right\}$$

such that Ξ_0 with no atom at 0 represents the measure of multiple coagulation, δ_0 is a point mass at 0, σ^2 denotes the rate of binary coagulation and

$$\lambda_{b;k_1,\ldots,k_r;s} = \sigma^2 1_{\{r=1,k_1=2\}} + \beta_{b;k_1,\ldots,k_r;s},$$

where

$$\beta_{b;k_1,\ldots,k_r;s} \equiv \int_\Delta \sum_{l=0}^{s} \sum_{i_1 \neq \cdots \neq i_{r+l}} \binom{s}{l} x_{i_1}^{k_1} \cdots x_{i_r}^{k_r} x_{i_{r+1}} \cdots x_{i_{r+l}} \left(1 - \sum_{j=1}^{\infty} x_j\right)^{s-l} \frac{\Xi_0(d\mathbf{x})}{\sum_{j=1}^\infty x_j^2}$$

denotes the rate of simultaneous multiple coagulation. As a result, the coagulation rates satisfy the consistency condition

$$\lambda_{b;k_1,\ldots,k_r;s} = \sum_{m=1}^{r} \lambda_{b+1;k_1,\ldots,k_{m-1},k_m+1,k_{m+1},\ldots,k_r;s} + s\lambda_{b+1;k_1,\ldots,k_r,2;s-1} + \lambda_{b+1;k_1,\ldots,k_r;s+1}.$$

There exists a one to one correspondence between Ξ-coalescents and finite measures Ξ on the infinite simplex Δ.

An example of Ξ-coalescent is introduced in Sagitov (2003) as follows. Let the measure Ξ be the Poisson-Dirichlet distribution $\Pi_\theta(dx)$ with a positive parameter θ defined on the infinite simplex $\Delta^* = \{x \in \Delta : \sum_{i=1}^\infty x_i = 1\}$. The corresponding coalescence rates are

$$\lambda_{b;k_1,\ldots,k_r;s} = \frac{\theta^{r+s}}{\theta^{[b]}} \prod_{i=1}^{r} (k_i - 1)!$$

for any $r \geq 1$, $k_1, \ldots, k_r \geq 2$ and $s \geq 0$, where $b = k_1 + \cdots + k_r + s$ and

$$\theta^{[b]} = \theta(\theta + 1) \cdots (\theta + b - 1)$$

is the ascending factorial power.

For the Ξ-coalescent, we refer to Schweinsberg (2000a) for some sufficient conditions on coming down from infinity or staying infinite.

2.2 The Λ-Fleming-Viot process

In this section, we first introduce the Λ-Fleming-Viot process without mutation from the time-rescaled empirical measure process of the Cannings model beyond finite variance (cf. Birkner and Blath (2009a)). Then we introduce the Λ-Fleming-Viot process with mutation operator A.

2.2.1 The Cannings model beyond finite variance

The condition (1.1.3) for the Cannings model in Subsection 1.1.1 assumes that each family size ν_i is small compared with the total population size N. A generalization can be motivated by considering that occasionally a single family has a large family size compared with N. Eldon and Wakeley (2006) introduced a class of Cannings models where $\nu(t)$ is a (uniform) permutation of

$$(2, 0, 1, \ldots, 1) \quad \text{or} \quad (\lfloor \psi N \rfloor, 0, \ldots, 0, 1, \ldots, 1) \tag{2.2.1}$$

with probability $1 - N^{-\phi}$ and $N^{-\phi}$ respectively for some fixed parameter $\psi \in (0, 1]$ and $\phi > 0$.

We follow Birkner and Blath (2009a) to discuss this kind of Cannings models. For the simple case with type space $E = \{\mathbb{a}, \mathbb{A}\}$. Recall that $Y^N(t)$ is the number of individuals which carry the $\mathbb{a}$-allele. Consider the Markov chain (1.1.1) on the time scale $1/c_N$, where c_N is denoted by (1.1.2). If $c_N \to 0$ and there exists a probability measure F on $[0, 1]$ such that

$$\frac{N}{c_N}\mathbb{P}\left(\nu_1 > Nx\right) \to \int_{(x,1]} \frac{1}{y^2} F\left(dy\right) \tag{2.2.2}$$

for all $x \in (0, 1)$ with $F\left(\{x\}\right) = 0$ and

$$\frac{\mathbb{E}\left(\nu_1\left(\nu_1 - 1\right)\nu_2\left(\nu_2 - 1\right)\right)}{N^2 c_N} \to 0, \quad \text{as} \quad N \to \infty, \tag{2.2.3}$$

then the process $Y^N\left(\lfloor t/c_N \rfloor\right)/N$ weakly converges to a $[0, 1]$-valued Markov process $X(t)$ with generator given by

$$\begin{aligned}\mathcal{L}f(x) \equiv& \frac{F\left(\{0\}\right)}{2} x(1-x)\frac{d^2}{dx^2}f(x) \\ &+ \int_{(0,1]} \left(xf\left(z + (1-z)x\right) + (1-x)\, f\left((1-z)x\right) - f(x)\right) \frac{1}{z^2} F(dz)\end{aligned} \tag{2.2.4}$$

for $f \in C^2([0, 1])$.

Birkner and Blath (2009a) further proposed that for the situation of E with infinitely many alleles, the corresponding limiting measure-valued process converges to the Λ-Fleming-Viot process with generator given by

$$\begin{aligned}\mathcal{L}\Phi\left(\mu\right) \equiv& F\left(\{0\}\right) \sum_{J\subseteq\{1,2,\ldots,n\},|J|=2} \int_E \cdots \int_E \left(\phi\left(x_1^J, \ldots, x_n^J\right) - \phi\left(x_1, \ldots, x_n\right)\right) \mu\left(dx_1\right)\cdots\mu\left(dx_n\right) \\ &+ \int_{(0,1]} \int_E \left(\Phi\left((1-z)\,\mu + z\delta_x\right) - \Phi\left(\mu\right)\right) \mu(dx) \frac{F_0(dz)}{z^2}\end{aligned} \tag{2.2.5}$$

where $F_0 = F - F\left(\{0\}\right)\delta_0$ and the test function $\Phi(\mu)$ is defined by (1.2.2).

2.2.2 The (Λ, A)-Fleming-Viot process

Now we introduce the Λ-Fleming-Viot process with mutation operator A. For any locally compact metric space E, recall that $M_1(E)$ is the collection of probability measures on

E and $\mathcal{D}(A)$ is the domain of mutation operator A. Let $f_1,\ f_2, \ldots \in \mathcal{D}(A)$ be uniformly bounded functions that separate points in $M_1(E)$ in the sense that

$$\int_E f_k d\mu = \int_E f_k d\nu$$

for any positive integer k implies $\mu = \nu$. Let d be the metric on $M_1(E)$ such that

$$d(\mu, \nu) \equiv \sum_k \frac{1}{2^k} \left| \int_E f_k d\mu - \int_E f_k d\nu \right| \quad \text{for any } \mu,\ \nu \in M_1(E).$$

Similar to the arguments in Chapter 1 of Li (2011), we can prove that this metric is compatible with the weak convergence topology of $M_1(E)$.

Let $\Omega \equiv D([0,\infty), M_1(E))$ be furnished with the Skorohod topology. For any $t \geq 0$, define $X(t) : \Omega \to M_1(E)$ by $X(t,\omega) \equiv \omega(t)$ for $\omega \in \Omega$. Let $\mathcal{F}_t = \sigma(X(s) : 0 \leq s \leq t)$, $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$. Then $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, (X(t))_{t\geq 0})$ defines the canonical probability-measure-valued process.

Given any mutation operator A and the test function Φ defined by (1.2.2), we always assume that $\phi \in \mathcal{D}(A^{(n)})$.

Definition 2.1 *Let Λ be a finite measure on $[0,1]$. The (Λ, A)-Fleming-Viot process is a probability-measure-valued Markov process $(X(t))_{t\geq 0}$ with paths in $D([0,\infty), M_1(E))$, whose generator is given by*

$$\begin{aligned}\mathcal{L}\Phi(\mu) \equiv & r\int_E \cdots \int_E A^{(n)}\phi(x_1, \ldots, x_n)\mu(dx_1)\cdots\mu(dx_n) \\ & + \lambda_{n,|J|} \sum_{J\subseteq\{1,2,\ldots,n\}, |J|\geq 2} \int_E \cdots \int_E \left(\phi\left(x_1^J, \ldots, x_n^J\right) - \phi(x_1, \ldots, x_n)\right)\mu(dx_1)\cdots\mu(dx_n),\end{aligned}$$

where

$$\lambda_{n,k} = \int_{[0,1]} x^{k-2}(1-x)^{n-k}\Lambda(dx), \qquad 2 \leq k \leq n,$$

and r is the mutation rate. The generator can also be represented as:

$$\begin{aligned}\mathcal{L}\Phi(\mu) \ \equiv\ & r\int_E \cdots \int_E A^{(n)}\phi(x_1, \ldots, x_n)\mu(dx_1)\cdots\mu(dx_n) \qquad (2.2.6)\\ + \ & \Lambda(\{0\}) \sum_{J\subseteq\{1,2,\ldots,n\}, |J|=2} \int_E \cdots \int_E \left(\phi\left(x_1^J, \ldots, x_n^J\right) - \phi(x_1, \ldots, x_n)\right)\mu(dx_1)\cdots\mu(dx_n)\end{aligned}$$

$$+ \quad \int_{(0,1]} \int_E \left(\Phi(z\delta_x + (1-z)\mu) - \Phi(\mu)\right) \mu(dx) z^{-2} \Lambda(dz).$$

Note that the last two rows in the generator (2.2.6) is the same as the generator (2.2.5) if we take the finite measure $\Lambda(\{0\}) = F(\{0\})$ and $\Lambda(dx) = F(dx)$ for any $x \in (0, 1]$.

2.3 Lookdown constructions

Donnelly and Kurtz (1996, 1999a,b) introduced the lookdown construction of countably many particles whose empirical measure converges to the measure-valued process. To guarantee the existence of empirical measure, it is required that the distribution of the countably many particles is exchangeable. We begin with studying the definition of exchangeability.

Definition 2.2 *(Exchangeability) An exchangeable sequence of random variables is a finite or infinite sequence* $Z_1, Z_2, Z_3, \ldots$ *of random variables such that for any finite permutation* σ *of the indices* $1, 2, 3, \ldots$*, i.e., any permutation* σ *that leaves all but finitely many indices fixed, the joint probability distribution of the permuted sequence*

$$Z_{\sigma(1)}, Z_{\sigma(2)}, Z_{\sigma(3)}, \ldots$$

is the same as the joint probability distribution of the original sequence.

In this section, we present the lookdown constructions for both the classical Fleming-Viot process and the Λ-Fleming-Viot process. We also explain how to recover Kingman's coalescent and Λ-coalescent from the lookdown constructions. The lookdown construction allows all the particles to perform independent motions of Markov processes with Feller generator A.

In our work, we consider some support properties of Λ-Fleming-Viot process with underlying Brownian motion. Therefore, we assume that all the particles in the lookdown constructions move according to underlying spatial Brownian motions in $\mathbb{R}^d$. Equivalently, the mutation operator A for the Fleming-Viot process is the Brownian generator. Thus, the transition function (P_t) for the semigroup $(\mathcal{T}(t))$ is the heat flow, i.e., for any $f \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathcal{T}(t) f(x) = \int_{\mathbb{R}^d} f(y) (2\pi t)^{-d/2} e^{-\frac{|x-y|^2}{2t}} dy.$$

2.3.1 Lookdown construction for the classical Fleming-Viot process with underlying Brownian motion

Donnelly and Kurtz (1996) introduced the following lookdown construction governed by a collection of independent Poisson processes for the classical Fleming-Viot process.

Lookdown construction for the classical Fleming-Viot process

Let $\{\mathbf{N}_{ij} : 1 \leq i < j < \infty\}$ be a collection of independent Poisson processes with rate 1 and $\{\mathbf{B}_i(t) : i = 1, 2, \ldots\}$ be independent d-dimensional standard Brownian motions. Process $\mathbf{N}_{ij}$ determines the time at which the particle at level j looks down at level i. Then $\sum_{1 \leq i < j} \mathbf{N}_{ij}$ is the total number of lookdowns from the jth level. Set τ_{ijk} to be the kth jump time of $\mathbf{N}_{ij}$.

For any $x \in \mathbb{R}^d$, let $U \equiv (U(x,t))_{t \geq 0}$ be $\mathbb{R}^d$-valued Markov process with transition function P_t for Brownian motion and $U(x, 0) = x$. Define

$$\{U_{ijk} : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d, 1 \leq i < j, 1 \leq k < \infty\}$$

and $\{U_{i0} : i \geq 1\}$ as independent realizations of U.

Let $\{X_i(0) : i \geq 1\}$ be an exchangeable sequence of $\mathbb{R}^d$-valued random variables, independent of $\{U_{ijk}\}$, $\{U_{i0}\}$ and $\{\mathbf{N}_{ij}\}$. Define

$$\gamma_{ijk} \equiv \min\left\{\tau_{i'jk'} : i' < j, \tau_{i'jk'} > \tau_{ijk}\right\},$$

i.e., γ_{ijk} is the first jump time of $\mathbf{N}_j \equiv \sum_{i<j} \mathbf{N}_{ij}$ after τ_{ijk}. Let γ_{j0} be the first jump time of $\mathbf{N}_j$, then $\gamma_{j0} \equiv \min\{\tau_{ij1} : i < j\}$.

Finally, define

$$\begin{cases} X_j(t) = U_{j0}\left(X_j(0), t\right), & 0 \leq t < \gamma_{j0}, \\ X_j(t) = U_{ijk}\left(X_i(\tau_{ijk}), t - \tau_{ijk}\right), & \tau_{ijk} \leq t < \gamma_{ijk}. \end{cases}$$

Between jump times of the Poisson processes, all the particles perform independent Brownian motions. The spatial locations can be obtained as solutions to the following system of stochastic differential equations:

$$\begin{cases} X_1(t) = X_1(0) + \mathbf{B}_1(t), \\ X_j(t) = X_j(0) + \mathbf{B}_j(t) + \sum_{i=1}^{j-1} \int_0^t \left(X_i(s-) - X_j(s-)\right) d\mathbf{N}_{ij}(s), \quad \text{for } j \geq 2. \end{cases}$$

Theorem 2.3 (Donnelly and Kurtz (1996)) *For each $t > 0$, $X_1(t), X_2(t), \ldots$ are exchangeable. Let*

$$\hat{X}_n(t) \equiv \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)},$$

then there exists a probability-measure-valued process $(X(t))_{t\geq 0}$ such that for each $t > 0$,

$$X(t) \equiv \lim_{n\to\infty} \hat{X}_n(t)$$

almost surely and $(X(t))_{t\geq 0}$ is the classical Fleming-Viot process with underlying Brownian motion.

Kingman's coalescent in the lookdown construction

We can recover the Kingman's coalescent by following the lookdown construction backwards in time (cf. Etheridge (2000), Donnelly and Kurtz (1996), Ethier and Kurtz (1993)).

Assume that the lookdown construction is defined on time interval $(-\infty, \infty)$. Denote by

$$\mathbf{N}_j(a, b] \equiv \sum_{1\leq i<j} \mathbf{N}_{ij}(a, b]$$

for $1 < j \leq n$ (where $\mathbf{N}_{ij}(a, b]$ is the number of points in $\mathbf{N}_{ij}$ falling in the time interval $(a, b]$). Let $\gamma_j(s)$ be the time of the most recent lookdown from the jth level, i.e.,

$$\gamma_j(s) = \sup\left\{u : \mathbf{N}_j(u, s] > 0\right\}.$$

Let $\alpha_j\left(\gamma_j\left(s\right)\right)$ be the level i such that $\gamma_j(s) \in \mathbf{N}_{ij}$. Given j and s, for $t < s$, define

$$a_j\left(t,s\right) = \begin{cases} j, \quad \text{for} \quad \gamma_j(s) \leq t < s, \\ \alpha_j\left(\gamma_j(s)\right), \quad \text{for} \quad \gamma_{\alpha_j(\gamma_j(s))}\left(\gamma_j(s)\right) \leq t < \gamma_j(s), \end{cases} \tag{2.3.1}$$

and extend the definition $a_j(t,s)$ to all $t < s$ in the obvious manner.

Remark 2.4 *Looking backwards in time, for any $t < s$, $a_j(t,s)$ gives the level at time t of the particle with level j at time s. Intuitively, we can think the particle with level $a_j(t,s)$ at time t as the ancestor and the particle with level j at time s as its offspring. Denote by $X_{a_j(t,s)}(t-)$ the ancestor location and $a_j(t,s)$ the ancestor level.*

Let $\mathbb{T}_n(t,s) \equiv \{a_j(t,s) : j = 1,\ldots,n\}$, i.e., $\mathbb{T}_n(t,s)$ is the collection of ancestor levels at time t of the particles with the first n levels at time s. Let $|\mathbb{T}_n(t,s)|$ be the cardinality of $\mathbb{T}_n(t,s)$. For an arbitrary but fixed T and $s \geq 0$, set $D_n(s) \equiv |\mathbb{T}_n(T-s,T)|$ and define a partition $\Pi_n(s)$ on $\{1,\ldots,n\}$ by i and j belonging to the same block of $\Pi_n(s)$ if and only if $a_i(T-s,T) = a_j(T-s,T)$.

Theorem 2.5 (Donnelly and Kurtz (1996)) *The process $\left(\Pi_n(s)\right)_{s\geq 0}$ is an n-coalescent and the number of equivalence classes $\left(D_n(s)\right)_{s\geq 0}$ is a pure death Markov chain with transition rates:*

$$q_{i,j} = \begin{cases} \frac{i(i-1)}{2}, \quad \text{if } j = i-1, \\ 0, \quad \text{otherwise.} \end{cases}$$

Finally, we define a $\mathcal{P}_\infty$-valued process $\Pi \equiv \left(\Pi(s)\right)_{s\geq 0}$ such that i and j are in the same block if and only if $a_i(T-s,T) = a_j(T-s,T)$. Then Π is the Kingman's coalescent.

2.3.2 Modified lookdown construction for the Λ-Fleming-Viot process with underlying Brownian motion

Donnelly and Kurtz (1999b) further introduced a modified lookdown construction with the empirical measure processes converging to measure-valued stochastic processes, including both the neutral Fleming-Viot process and the Dawson-Watanabe process. Birkner and Blath (2009a) presented the modified lookdown construction in Donnelly and Kurtz (1999b) for the Λ-Fleming-Viot process.

In this subsection, we begin with introducing the general Moran model with Brownian mutation, whose time-rescaled empirical measure converges to the Λ- Fleming-Viot process with underlying Brownian motion. Then we present the modified lookdown construction of the model using countably many particles governed by a system of countable stochastic differential equations, whose empirical measure converges to the Λ-Fleming-Viot process with underlying Brownian motion. Further, we explain how to recover the Λ-coalescent and ancestry process from the modified lookdown construction. Finally, we give an assumption (Assumption I) and two conditions (Condition A & Condition B) that are sufficient for the Λ-coalescent to come down from infinity.

General Moran model with Brownian mutation

We refer to Birkner and Blath (2009a) for the general Moran model. Denote by

$$(Y_1(t), Y_2(t), \ldots, Y_N(t)) \in \left(\mathbb{R}^d\right)^N$$

the spatial locations of the N particles general Moran model with Brownian mutation. Given any finite measure Λ on $[0,1]$, note that

$$\Lambda \equiv a\delta_0 + \Lambda_0, \tag{2.3.2}$$

where $a\delta_0$ is the restriction of Λ to $\{0\}$.

Let $\mu_N(k)$ be a finite measure on $\{1, 2, \ldots, N-1\}$ such that

$$\mu_N(k) \equiv Na\mathbf{1}_{\{k=1\}} + \frac{1}{N}\int_{(0,1]} \binom{N}{k+1} x^{k-1}(1-x)^{N-k-1}\Lambda(dx)$$

for $k = 1, 2, \ldots, N-1$.

Let $\mathcal{B}_N$ be a Poisson point process on $[0,\infty) \times \{1, 2, \ldots, N-1\}$ with intensity measure $dt \otimes \mu_N$. When the event $(t,k) \in \mathcal{B}_N$ happens, k particles uniformly chosen from the N particles die and the number of possible choices is $\binom{N}{k}$. After that, one particle uniformly chosen from the $(N-k)$ remaining particles gives birth to k new particles that have the same type as their parent. When there is neither birth nor death event happening, all the particles perform independent Brownian motions in $\mathbb{R}^d$. Denote by $\mathtt{r}_N$ the mutation rate of each particle and assume that $N\mathtt{r}_N \to \mathtt{r} \in [0, \infty)$. Let

$$\hat{Y}_N(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta_{Y_i(t)}$$

be the empirical measure process for the N particles general Moran model. To guarantee the existence of limit empirical measure by de Finetti's theorem, we require that the initial values $Y_i(0)$, $i = 1, 2, \ldots, N$ to be exchangeable. Given the initial exchangeability, $Y_1(t), \ldots, Y_N(t)$ are also exchangeable for any $t > 0$.

Theorem 2.6 (Birkner and Blath (2009a)) *The time-rescaled empirical measure process weakly converges to a measure valued process, i.e.,*

$$\hat{Y}_N(Nt) \to X(t), \qquad \text{as } N \to \infty,$$

where $(X(t))_{t\geq 0}$ is the Λ-Fleming-Viot process with underlying Brownian motion.

Modified lookdown construction for the Λ-Fleming-Viot process

Following Birkner and Blath (2009a), we now give an introduction on the modified lookdown construction for the Λ-Fleming-Viot process with underlying Brownian motion. Let

$$(X_1(t), X_2(t), X_3(t), \ldots)$$

be an $(\mathbb{R}^d)^\infty$-valued random variable, where for any $i \in [\infty]$, $X_i(t)$ represents the spatial location of the particle at level i. We require the initial values $\{X_i(0), i \in [\infty]\}$ to be exchangeable random variables so that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(0)}$$

exists almost surely by de Finetti's theorem.

Let Λ be the finite measure associated to the Λ-coalescent. The reproduction in the particle system consists of two kinds of birth events: the events of single birth determined by measure $a\delta_0$ with $a = \Lambda(\{0\})$ and the events of multiple births determined by measure Λ restricted to $(0, 1]$ that is denoted by Λ_0.

To describe the evolution of the system during events of single birth, let $\{\mathbf{N}_{ij}(t) : 1 \leq i < j < \infty\}$ be independent Poisson processes with common rate a. At a jump time t of $\mathbf{N}_{ij}$, the particle at level j looks down at the particle at level i and assumes its location (therefore, particle at level i gives birth to a new particle). Values of particles at levels above j are shifted accordingly, i.e., for $\Delta\mathbf{N}_{ij}(t) = 1$, we have

$$X_k(t) = \begin{cases} X_k(t-), & \text{if } k < j, \\ X_i(t-), & \text{if } k = j, \\ X_{k-1}(t-), & \text{if } k > j. \end{cases} \tag{2.3.3}$$

See Figure 2.1 for the single birth event.

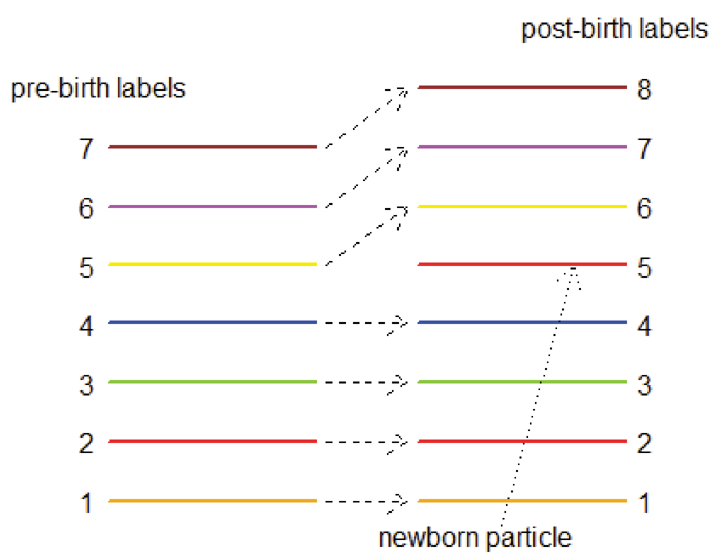

Figure 2.1: Relabeling after a lookdown event involving levels 2 and 5

For those events of multiple births we can construct an independent Poisson point process $\tilde{\mathbf{N}}$ on $\mathbb{R}^+ \times (0, 1]$ with intensity measure $dt \otimes x^{-2}\Lambda_0(dx)$. Let $\{U_{ij},\ i, j \in [\infty]\}$

be i.i.d. uniform $[0,1]$ random variables. Jump points (t_i, x_i) for $\tilde{\mathbf{N}}$ correspond to the multiple birth events. For $t \geq 0$ and $J \subset [n]$ with $|J| \geq 2$, define

$$\mathbf{N}_J^n(t) \equiv \sum_{i: t_i \leq t} \prod_{j \in J} \mathbf{1}_{\{U_{ij} \leq x_i\}} \prod_{j \in [n] \setminus J} \mathbf{1}_{\{U_{ij} > x_i\}}. \tag{2.3.4}$$

Then $\mathbf{N}_J^n(t)$ counts the number of birth events among the particles at levels $\{1, 2, \ldots, n\}$ such that exactly those at levels in J are involved up to time t. Intuitively, at a jump time t_i, a uniform coin is tossed independently for each level. All the particles at levels j with $U_{ij} \leq x_i$ participate in the lookdown event. More precisely, those particles involved jump to the location of the particle at the lowest level involved. The spatial locations of particles on the other levels, keeping their original order, are shifted upwards accordingly, i.e., if $t = t_i$ is the jump time and j is the lowest level involved, then

$$X_k(t) = \begin{cases} X_k(t-), \text{ for } k \leq j, \\ X_j(t-), \text{ for } k > j \text{ with } U_{ik} \leq x_i, \\ X_{k - J_t^k}(t-), \text{ otherwise,} \end{cases}$$

where $J_{t_i}^k \equiv \#\{m < k, U_{im} \leq x_i\} - 1$. We refer to Figure 2.2 for the multiple birth event.

Between jump times of the Poisson processes, particles at different levels move independently according to Brownian motions in $\mathbb{R}^d$.

Let $\{\mathbf{B}_i(t) : i = 1, 2, \ldots\}$ be a sequence of independent and standard d-dimensional Brownian motions. The particle on level 1 evolves according to Brownian motion, i.e.,

$$X_1(t) = X_1(0) + \mathbf{B}_1(t).$$

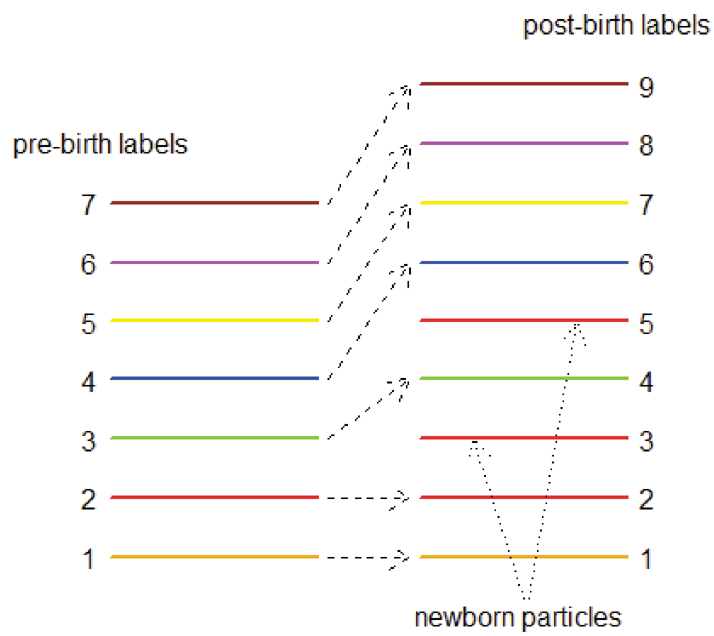

Figure 2.2: Relabeling after a lookdown event involving levels 2, 3 and 5

All the other levels above one can look down. For $n \geq 2$, define

$$\begin{aligned} X_n(t) =& X_n(0) + \mathbf{B}_n(t) + \sum_{1\leq i<j<n} \int_0^t \left(X_{n-1}(s-) - X_n(s-)\right) d\mathbf{N}_{ij}(s) \\ &+ \sum_{1\leq i<n} \int_0^t \left(X_i(s-) - X_n(s-)\right) d\mathbf{N}_{in}(s) \\ &+ \sum_{J\subset\{1,2,\ldots,n\}, n\in J} \int_0^t \left(X_{\min(J)}\left(s-\right) - X_n\left(s-\right)\right) d\mathbf{N}_J^n\left(s\right) \\ &+ \sum_{J\subset\{1,2,\ldots,n\}, n\notin J} \int_0^t \left(X_{n-J_s^n}\left(s-\right) - X_n\left(s-\right)\right) d\mathbf{N}_J^n\left(s\right). \end{aligned}$$

The first integral describes that the lookdown event involving levels i and j happens below level n; the second integral describes that level n looks down to level i; the third and fourth integrals describe multiple levels are involved in the lookdown event in a similar way.

We assume that the above-mentioned modified lookdown construction is carried out in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For each $t > 0$, $X_1(t), X_2(t), \ldots$ are known to be exchangeable random variables so that

$$X(t) \equiv \lim_{n\to\infty} X^{(n)}(t) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)}$$

exists almost surely by de Finetti's theorem and follows the probability law of the Λ-Fleming-Viot process with underlying Brownian motion. Further, we have that $X^{(n)}$ converges to X in the path space $D([0,\infty), M_1(\mathbb{R}^d))$ equipped with the Skorohod topology, where $M_1(\mathbb{R}^d)$ denotes the space of probability measures on $\mathbb{R}^d$ equipped with the topology of weak convergence. See Theorem 3.2 of Donnelly and Kurtz (1999b).

We refer to Figure 2.3 for an example of the modified lookdown construction of Λ-Fleming-Viot process, where three lookdown events are involved.

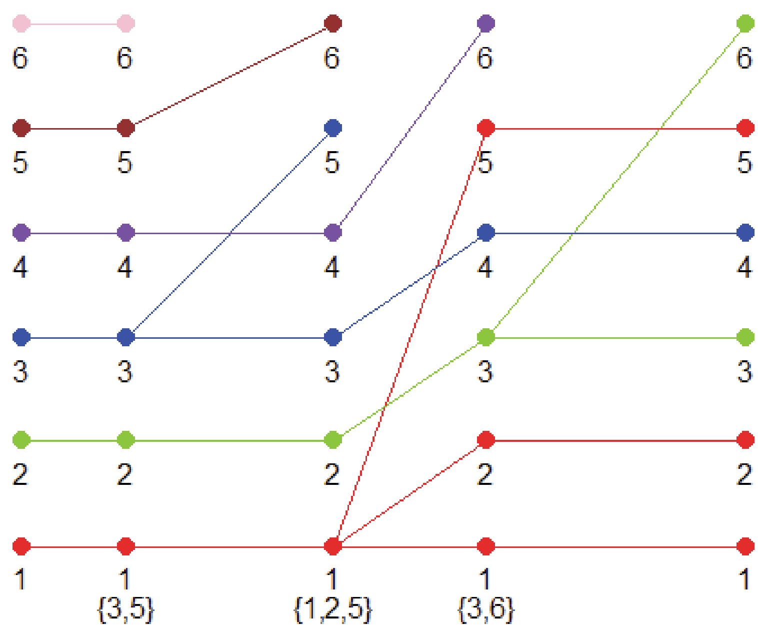

Figure 2.3: Modified lookdown construction for the Λ-Fleming-Viot process

Lemma 2.7 *With probability one, at any fixed $t \geq 0$, the spatial locations of the countably many particles in the modified lookdown construction satisfy*

$$\{X_1(t), X_2(t), X_3(t), \ldots\} \subseteq supp\, X(t)\ \mathbb{P} a.s.,$$

where we denote by $supp\,\mu$ the closed support for any measure μ.

Proof. In the modified lookdown construction, $(X_n(t))_{n\geq 1}$ are exchangeable at any time $t \geq 0$. By de Finetti's theorem (cf. Aldous (1985)) such a system is a mixture of i.i.d. sequence, i.e., given empirical measure

$$X(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)},$$

the random variables $\{X_i(t), i = 1, 2, \ldots\}$ are jointly distributed as i.i.d. samples from the directing measure $X(t)$. Therefore, $X_n(t) \in \operatorname{supp} X(t)$ a.s. for any $n \in [\infty]$. $\square$

Λ-coalescent in the modified lookdown construction

The birth events induce a family structure to the particle system so we can define the *genealogy process*. For any $s \geq 0$, $0 \leq t \leq s$ and $n \in [\infty]$, denote by $L_n^s(t)$ the ancestor's level at time t for the particle with level n at time s. Consequently, the genealogy process $L_n^s(t)$ satisfies the equation

$$\begin{aligned} L_n^s(t) = n &- \sum_{1\leq i<j<n} \int_{t-}^{s} \mathbf{1}_{\{L_n^s(u)>j\}} d\mathbf{N}_{ij}(u) \\ &- \sum_{1\leq i<j\leq n} \int_{t-}^{s} (j-i)\, \mathbf{1}_{\{L_n^s(u)=j\}} d\mathbf{N}_{ij}(u) \\ &- \sum_{J\subset[n]} \int_{t-}^{s} \left(L_n^s(u) - \min(J)\right) \mathbf{1}_{\{L_n^s(u)\in J\}} d\mathbf{N}_J^n(u) \\ &- \sum_{J\subset[n]} \int_{t-}^{s} \left(|J \cap \{1,\ldots,L_n^s(u)\}| - 1\right) \times \mathbf{1}_{\{L_n^s(u)>\min(J), L_n^s(u)\notin J\}} d\mathbf{N}_J^n(u)\,. \end{aligned}$$

For fixed $T > 0$ and $i \in [\infty]$, $L_i^T(T-t)$ gives the ancestor level at time $T-t$ of the particle with level i at time T and $X_{L_i^T(T-t)}((T-t)-)$ is the ancestor location.

Write $\left(\Pi^T(t)\right)_{0\leq t\leq T}$ for the $\mathcal{P}_\infty$-valued process such that i and j belong to the same block of $\Pi^T(t)$ if and only if $L_i^T(T-t) = L_j^T(T-t)$, i.e., i and j belong to the same block if and only if the two particles with levels i and j, respectively, at time T share a common ancestor at time $T-t$. The process $\left(\Pi^T(t)\right)_{0\leq t\leq T}$ turns out to have the same law as the Λ-coalescent running up to time T (cf. Donnelly and Kurtz (1999b) and Birkner and Blath (2009a)).

We refer to Figure 2.4 which recovers the genealogy process from the modified look-down construction in Figure 2.3.

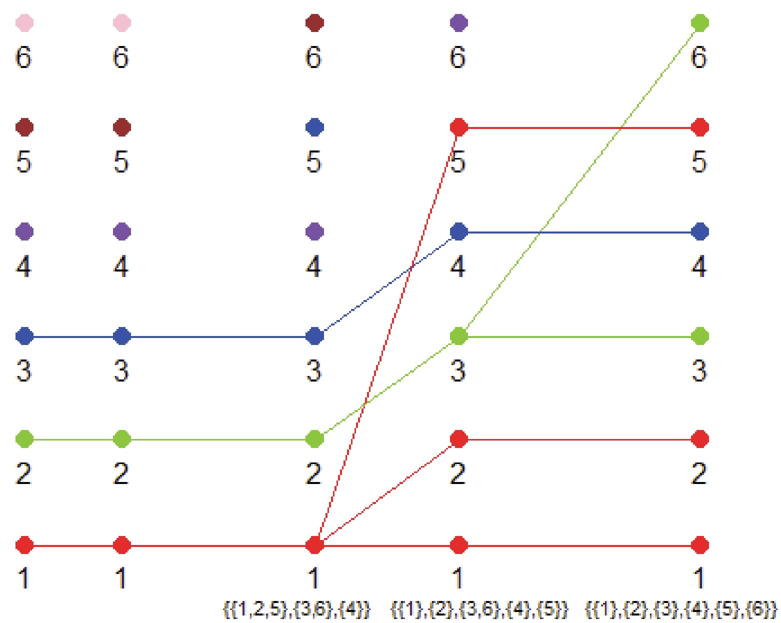

Figure 2.4: Genealogy process in the modified lookdown construction

The following lemma is an observation on the partition induced by the lookdown construction.

Lemma 2.8 *For fixed $T > 0$, let $\left(\Pi^T(t)\right)_{0\leq t\leq T}$ be the Λ-coalescent defined above from the modified lookdown construction. Then for any $0 \leq t \leq T$, we have*

$$L_j^T(T-t) = l \quad \textit{for any} \quad j \in \pi_l$$

where $1 \leq l \leq \#\Pi^T(t)$ and $\pi_l \equiv \pi_l(t)$ are the disjoint blocks of $\Pi^T(t)$ ordered by their least elements.

Proof. For any $1 \leq l \leq \#\Pi^T(t)$, the particles with levels in block π_l at time T have the same ancestor at time $T-t$. Let $i_l = \min \pi_l$.

It is trivial that $i_1 = 1$ and $L_{i_1}^T(T-t) = 1$, i.e.,

$$L_j^T(T-t) = 1 \text{ for any } j \in \pi_1.$$

Now we consider the case $l \geq 2$. No collision happens for the Λ-coalescent process $\left\{\Pi^T(\nu) : 0 \leq \nu \leq t\right\}$ between i_l and $\{1, 2, \ldots, i_l - 1\}$ since $i_l = \min \pi_l$. Then looking forwards in time, the ancestor of the particle with level i_l at time T never looks down to any lower levels during time interval $[T-t, T]$. As an increasing piecewise constant function, the ancestor level $(L_{i_l}^T(u))_{T-t \leq u \leq T}$ would increase only because of upward shift.

Let s be any fixed jump point of $(L_{i_l}^T(u))_{T-t \leq u \leq T}$. Then a lookdown event happens between levels $\left\{1, 2, \ldots, L_{i_l}^T(s-) - 1\right\}$. Let $J \subset \left\{1, 2, \ldots, L_{i_l}^T(s-) - 1\right\}$ be the levels involved. We have

$$L_{i_l}^T(s) - L_{i_l}^T(s-) = |J| - 1.$$

At time s, $\left\{L_j^T(s) : j \in [i_l]\right\}$ is the collection of ancestor levels of the particles with first i_l levels at time T. Its cardinality also increases by $|J| - 1$, i.e.,

$$\left|\left\{L_j^T(s) : j \in [i_l]\right\}\right| - \left|\left\{L_j^T(s-) : j \in [i_l]\right\}\right| = |J| - 1.$$

Note that the total number of jumps for $(L_{i_l}^T(u))_{T-t \leq u \leq T}$ is finite. It follows that

$$L_{i_l}^T(T) - L_{i_l}^T(T-t) = \left|\left\{L_j^T(T) : j \in [i_l]\right\}\right| - \left|\left\{L_j^T(T-t) : j \in [i_l]\right\}\right|. \tag{2.3.5}$$

Since $\{\pi_p, 1 \leq p \leq \#\Pi^T(t)\}$ are ordered by their least elements, then $1, 2, \ldots, i_l$ should be contained in the first l blocks and $\pi_p \cap \{1, 2, \ldots, i_l\} \neq \emptyset$ for $1 \leq p \leq l$.

Recall that $\Pi_{i_l}^T(t)$ is the restriction of $\Pi^T(t)$ to $\{1, 2, \ldots, i_l\}$. It implies that

$$\#\Pi_{i_l}^T(t) = l.$$

Thus

$$|\left\{L_j^T(T-t) : j \in [i_l]\right\}| = l.$$

Since

$$L_{i_l}^T(T) = i_l \text{ and } |\left\{L_j^T(T) : j \in [i_l]\right\}| = i_l,$$

applying (2.3.5), we have $L^T_{i_l}(T-t) = l$. All the particles in the same block have a common ancestor, therefore

$$L^T_j(T-t) = l \quad \text{for any} \quad j \in \pi_l.$$

$\square$

Ancestry process in the modified lookdown construction

For any $T > 0$, denote by

$$(X_{1,s}, X_{2,s}, X_{3,s}, \ldots)_{s \leq T}$$

the *ancestry process* with

$$X_{i,s}(t) \equiv X_{L^s_i(t)}(t-) \quad \text{for} \quad t \leq s. \tag{2.3.6}$$

Intuitively $X_{i,s}$ keeps track of locations for all the ancestors of the particle with level i at time s.

For any $s \geq 0$, we can recover the Λ-coalescent $\{\Pi^s(t) : 0 \leq t \leq s\}$ from the lookdown construction. For any $0 \leq r \leq s$, put

$$N^{r,s} \equiv \#\Pi^s(s-r) \tag{2.3.7}$$

and

$$\Pi^s(s-r) \equiv \{\pi_l : 1 \leq l \leq N^{r,s}\},$$

where $\pi_l \equiv \pi_l(r,s),\ 1 \leq l \leq N^{r,s}$ are all the disjoint blocks of $\Pi^s(s-r)$ ordered by their least elements.

Let $H(r,s)$ be the maximal dislocation between the countably many particles at time s and their respective ancestors at time r. Applying Lemma 2.8, we have

$$\begin{aligned} H(r,s) &\equiv \max_{1 \leq l \leq N^{r,s}} \max_{j \in \pi_l} \left| X_j(s) - X_{L^s_j(r)}(r-) \right| \\ &= \max_{1 \leq l \leq N^{r,s}} \max_{j \in \pi_l} \left| X_j(s) - X_l(r-) \right|. \end{aligned} \tag{2.3.8}$$

Sufficient Conditions for the Λ-coalescent to come down from infinity

For any $T > 0$, let $(\Pi^T(t))_{0\le t\le T}$ be the Λ-coalescent recovered from the lookdown construction with $\Pi^T(0) = \mathbf{0}_{[\infty]}$. Write $\Pi \equiv (\Pi(t))_{t\ge 0}$ for the unique (in law) Λ-coalescent such that $(\Pi(t))_{0\le t\le T}$ has the same distribution as $(\Pi^T(t))_{0\le t\le T}$. We call Π the Λ-coalescent associated to the Λ-Fleming-Viot process X. Π_n is the restriction of Π to $[n]$.

For any $n > m$, set

$$T_m^n \equiv \inf\left\{t \ge 0 : \#\Pi_n(t) \le m\right\}$$

and

$$T_m \equiv T_m^\infty \equiv \inf\left\{t \ge 0 : \#\Pi(t) \le m\right\} \tag{2.3.9}$$

with the convention $\inf \emptyset = \infty$. From the modified lookdown construction, it is clear that

$$T_m^n \le T_m^{n+1} \le T_m^{n+2} \le \cdots \le\uparrow T_m. \tag{2.3.10}$$

Assumption I: There exists a constant $\alpha > 0$ such that the associated Λ-coalescent Π satisfies

$$\limsup_{m\to\infty} m^\alpha \mathbb{E} T_m < \infty.$$

Note that under Assumption I, the associated Λ-coalescent comes down from infinity.

Condition A: There exists a constant $\alpha > 0$ such that the associated Λ-coalescent Π satisfies

$$\limsup_{m\to\infty} m^\alpha \sum_{b=m+1}^{\infty} {\lambda_b}^{-1} < \infty,$$

where λ_b is the total coalescence rate defined by (2.1.2).

For any $n > m$, the block counting process $(\#\Pi_n(t) \vee m)_{t\ge 0}$ is a Markov chain with initial value n and absorbing state m. For any $n \ge b > m$, let $(\mu_{b,k})_{m\le k\le b-1}$ be its transition rates such that

$$\begin{cases} \mu_{b,b-1} = \binom{b}{2}\lambda_{b,2}, \\ \mu_{b,b-2} = \binom{b}{3}\lambda_{b,3}, \\ \cdots\cdots \\ \mu_{b,m+1} = \binom{b}{b-m}\lambda_{b,b-m}, \\ \mu_{b,m} = \sum_{k=b-m+1}^{b}\binom{b}{k}\lambda_{b,k}. \end{cases}$$

The total transition rate is

$$\mu_b = \sum_{k=m}^{b-1}\mu_{b,k} = \sum_{k=2}^{b}\binom{b}{k}\lambda_{b,k} = \lambda_b.$$

For any $b > m$, let $\gamma_{b,m}$ be the total rate at which the block counting Markov chain starting at b is decreasing, i.e.,

$$\gamma_{b,m} = \begin{cases} \sum_{k=2}^{b-m}(k-1)\binom{b}{k}\lambda_{b,k} + \sum_{k=b-m+1}^{b}(b-m)\binom{b}{k}\lambda_{b,k}, & \text{if } \ b \geq m+2, \\ \sum_{k=2}^{b}\binom{b}{k}\lambda_{b,k}, & \text{if } \ b = m+1. \end{cases} \tag{2.3.11}$$

Lemma 2.9 *For any $m < b$, we have $\gamma_{b,m} \leq \gamma_{b+1,m}$.*

Proof. We first recall the consistency condition on the coalescence rates

$$\lambda_{b,k} = \lambda_{b+1,k} + \lambda_{b+1,k+1}. \tag{2.3.12}$$

According to the values of b and m, we consider the following three different cases respectively.

Case I: $b = m+1$. By the consistency condition (2.3.12) and the definition for $\gamma_{b,m}$ by (2.3.11), we have

$$\begin{aligned}\gamma_{b,m} &= \sum_{k=2}^{b}\binom{b}{k}\lambda_{b,k} = \sum_{k=2}^{b}\binom{b}{k}(\lambda_{b+1,k} + \lambda_{b+1,k+1}) \\ &= \binom{b}{2}\lambda_{b+1,2} + \sum_{k=3}^{b}\binom{b}{k}\lambda_{b+1,k} + \sum_{k=3}^{b}\binom{b}{k-1}\lambda_{b+1,k} + \lambda_{b+1,b+1} \\ &= \binom{b}{2}\lambda_{b+1,2} + \sum_{k=3}^{b}\left(\binom{b}{k} + \binom{b}{k-1}\right)\lambda_{b+1,k} + \lambda_{b+1,b+1}.\end{aligned}$$

By the identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}, \tag{2.3.13}$$

we then have

$$\gamma_{b,m} \leq \binom{b+1}{2}\lambda_{b+1,2} + \sum_{k=3}^{b} 2\binom{b+1}{k}\lambda_{b+1,k} + 2\lambda_{b+1,b+1} = \gamma_{b+1,m}.$$

Case II: $b = m + 2$. Similarly, it follows from the consistency condition (2.3.12) and the definition of $\gamma_{b,m}$ by (2.3.11) that

$$\begin{aligned}
\gamma_{b,m} =& \binom{b}{2}\lambda_{b,2} + \sum_{k=3}^{b} 2\binom{b}{k}\lambda_{b,k} \\
=& \binom{b}{2}\lambda_{b+1,2} + \binom{b}{2}\lambda_{b+1,3} + \sum_{k=3}^{b} 2\binom{b}{k}\left(\lambda_{b+1,k} + \lambda_{b+1,k+1}\right) \\
=& \binom{b}{2}\lambda_{b+1,2} + \binom{b}{2}\lambda_{b+1,3} + \sum_{k=3}^{b} 2\binom{b}{k}\lambda_{b+1,k} + \sum_{k=4}^{b} 2\binom{b}{k-1}\lambda_{b+1,k} + 2\lambda_{b+1,b+1} \\
\leq& \binom{b+1}{2}\lambda_{b+1,2} + 2\left(\binom{b}{2} + \binom{b}{3}\right)\lambda_{b+1,3} + \sum_{k=4}^{b} 3\left(\binom{b}{k} + \binom{b}{k-1}\right)\lambda_{b+1,k} + 3\lambda_{b+1,b+1}.
\end{aligned}$$

Applying the identity (2.3.13), we have

$$\begin{aligned}
\gamma_{b,m} \leq& \binom{b+1}{2}\lambda_{b+1,2} + 2\binom{b+1}{3}\lambda_{b+1,3} + \sum_{k=4}^{b} 3\binom{b+1}{k}\lambda_{b+1,k} + 3\lambda_{b+1,b+1} \\
=& \sum_{k=2}^{3}(k-1)\binom{b+1}{k}\lambda_{b+1,k} + \sum_{k=4}^{b+1} 3\binom{b+1}{k}\lambda_{b+1,k} \\
=& \gamma_{b+1,m}.
\end{aligned}$$

Case III: $b \geq m + 3$. The proof involves similar but longer arguments as the first two cases.

$$
\begin{aligned}
\gamma_{b,m} =& \sum_{k=2}^{b-m} (k-1) \binom{b}{k} \lambda_{b,k} + \sum_{k=b-m+1}^{b} (b-m) \binom{b}{k} \lambda_{b,k} \\
=& \sum_{k=2}^{b-m} (k-1) \binom{b}{k} (\lambda_{b+1,k} + \lambda_{b+1,k+1}) + \sum_{k=b-m+1}^{b} (b-m) \binom{b}{k} (\lambda_{b+1,k} + \lambda_{b+1,k+1}) \\
=& \sum_{k=2}^{b-m} (k-1) \binom{b}{k} \lambda_{b+1,k} + \sum_{k=b-m+1}^{b} (b-m) \binom{b}{k} \lambda_{b+1,k} \\
& + \sum_{k=3}^{b-m+1} (k-2) \binom{b}{k-1} \lambda_{b+1,k} + \sum_{k=b-m+2}^{b+1} (b-m) \binom{b}{k-1} \lambda_{b+1,k} \\
=& \binom{b}{2} \lambda_{b+1,2} + \sum_{k=3}^{b-m} \left((k-1) \binom{b}{k} + (k-2) \binom{b}{k-1} \right) \lambda_{b+1,k} \\
& + (b-m-1) \binom{b}{b-m} \lambda_{b+1,b+1-m} + (b-m) \binom{b}{b-m+1} \lambda_{b+1,b-m+1} \\
& + \sum_{k=b-m+2}^{b} (b-m) \left(\binom{b}{k-1} + \binom{b}{k} \right) \lambda_{b+1,k} + (b-m)\lambda_{b+1,b+1}.
\end{aligned}
$$

With Equation (2.3.13),

$$
\begin{aligned}
\gamma_{b,m} \leq& \binom{b}{2} \lambda_{b+1,2} + \sum_{k=3}^{b-m} (k-1) \binom{b+1}{k} \lambda_{b+1,k} + (b-m) \binom{b+1}{b+1-m} \lambda_{b+1,b+1-m} \\
& + \sum_{k=b-m+2}^{b} (b-m) \binom{b+1}{k} \lambda_{b+1,k} + (b-m)\lambda_{b+1,b+1} \\
\leq& \sum_{k=2}^{b+1-m} (k-1) \binom{b+1}{k} \lambda_{b+1,k} + \sum_{k=b+2-m}^{b+1} (b+1-m) \binom{b+1}{k} \lambda_{b+1,k} \\
=& \gamma_{b+1,m}.
\end{aligned}
$$

□

Condition B: There exists a constant $\alpha > 0$ such that

$$
\limsup_{m\to\infty} m^{\alpha} \sum_{b=m+1}^{\infty} {\gamma_{b,m}}^{-1} < \infty.
$$

Lemma 2.10 *Condition A implies Condition B which is sufficient for Assumption I.*

Proof. We first show

$$
\mathbb{E}T_m \leq \sum_{b=m+1}^{\infty} \gamma_{b,m}^{-1}
$$

by adapting the idea of Lemma 6 in Schweinsberg (2000b).

For any $n > m$ and $1 \le k \le n-m$, define

$$\mathcal{R}_0 = 0,$$
$$\mathcal{R}_k = \begin{cases} \inf\left\{t \ge 0 : \#\Pi_n(t) < \#\Pi_n(\mathcal{R}_{k-1})\right\}, & \text{if } \#\Pi_n(\mathcal{R}_{k-1}) > m, \\ \mathcal{R}_{k-1}, & \text{if } \#\Pi_n(\mathcal{R}_{k-1}) = m. \end{cases}$$

Note that $T_m^n = \mathcal{R}_{n-m}$. For $i = 0, 1, 2, \dots, n-m$, let $\mathcal{N}_i = \#\Pi_n(\mathcal{R}_i)$. For $i = 1, 2, \dots, n-m$, let $L_i = \mathcal{R}_i - \mathcal{R}_{i-1}$ and $\mathcal{J}_i = \mathcal{N}_{i-1} - \mathcal{N}_i$.

On the event $\{\mathcal{N}_{i-1} > m\}$, for any $n \ge b > m$, we have

$$P\left(\mathcal{J}_i = k-1 | \mathcal{N}_{i-1} = b\right) = \binom{b}{k}\frac{\lambda_{b,k}}{\lambda_b}$$

for $k = 2, 3, \dots, b-m$ and

$$P\left(\mathcal{J}_i = b-m | \mathcal{N}_{i-1} = b\right) = \sum_{k=b-m+1}^{b}\binom{b}{k}\frac{\lambda_{b,k}}{\lambda_b}.$$

Consequently, on the event $\{\mathcal{N}_{i-1} > m\}$, we have

$$\mathbb{E}\left(\mathcal{J}_i | \mathcal{N}_{i-1} = b\right) = \sum_{k=2}^{b-m}(k-1)\binom{b}{k}\frac{\lambda_{b,k}}{\lambda_b} + (b-m)\sum_{k=b-m+1}^{b}\binom{b}{k}\frac{\lambda_{b,k}}{\lambda_b} = \frac{\gamma_{b,m}}{\lambda_b}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}T_m^n = \mathbb{E}\mathcal{R}_{n-m} = \mathbb{E}\sum_{i=1}^{n-m} L_i = \sum_{i=1}^{n-m}\mathbb{E}\mathbb{E}\left(L_i|\mathcal{N}_{i-1}\right) = \sum_{i=1}^{n-m}\mathbb{E}\left(\lambda_{\mathcal{N}_{i-1}}^{-1}\mathbf{1}_{\{\mathcal{N}_{i-1}>m\}}\right) \\
= \sum_{i=1}^{n-m}\mathbb{E}\left(\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathbb{E}\left(\mathcal{J}_i|\mathcal{N}_{i-1}\right)\mathbf{1}_{\{\mathcal{N}_{i-1}>m\}}\right) \\
= \sum_{i=1}^{n-m}\mathbb{E}\mathbb{E}\left(\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathcal{J}_i\mathbf{1}_{\{\mathcal{N}_{i-1}>m\}}|\mathcal{N}_{i-1}\right).
\end{aligned}$$

Since $\mathcal{J}_i = 0$ on the event $\{\mathcal{N}_{i-1} = m\}$, we have

$$\begin{aligned}
\mathbb{E}T_m^n = \sum_{i=1}^{n-m}\mathbb{E}\left(\mathbb{E}\left(\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathcal{J}_i|\mathcal{N}_{i-1}\right)\right) = \sum_{i=1}^{n-m}\mathbb{E}\left(\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathcal{J}_i\right) \\
= \mathbb{E}\left(\sum_{i=1}^{n-m}\gamma_{\mathcal{N}_{i-1},m}^{-1}\mathcal{J}_i\right) = \mathbb{E}\left(\sum_{i=1}^{n-m}\sum_{j=0}^{\mathcal{J}_i-1}\gamma_{\mathcal{N}_{i-1},m}^{-1}\right).
\end{aligned}$$

Since $(\gamma_{b,m})_{b=m+1}^{\infty}$ is an increasing sequence by Lemma 2.9, it follows that

$$\mathbb{E}T_m^n \leq \mathbb{E}\left(\sum_{i=1}^{n-m}\sum_{j=0}^{\mathcal{J}_i-1}\gamma_{\mathcal{N}_{i-1}-j,m}^{-1}\right) = \mathbb{E}\left(\sum_{b=m+1}^{n}\gamma_{b,m}^{-1}\right) \leq \sum_{b=m+1}^{\infty}\gamma_{b,m}^{-1}.$$

By the Monotone Convergence Theorem, we have

$$\mathbb{E}T_m = \lim_{n\to\infty}\mathbb{E}T_m^n \leq \sum_{b=m+1}^{\infty}\gamma_{b,m}^{-1}.$$

Recalling the definitions of $\gamma_{b,m}$ by (2.3.11) and λ_b by (2.1.2), we have $\lambda_b \leq \gamma_{b,m}$ for any $b > m$. For any $\alpha > 0$, we have

$$m^{\alpha}\mathbb{E}T_m \leq m^{\alpha}\sum_{b=m+1}^{\infty}{\gamma_{b,m}}^{-1} \leq m^{\alpha}\sum_{b=m+1}^{\infty}{\lambda_b}^{-1}.$$

Therefore, Condition A implies Condition B which is sufficient for Assumption I. $\square$

Chapter 3

The compact support property for a class of Λ-Fleming-Viot processes

In this chapter, we proceed to prove the compact support property for a class of Λ-Fleming-Viot processes at fixed positive time given the associated coalescent processes coming down from infinity. We also find both upper and lower bounds on the Hausdorff dimension of the support at fixed positive time.

Intuitively, if the associated Λ-coalescent comes down from infinity, then for any fixed $T > 0$, the random variables $(X_1(T), X_2(T), \ldots)$ in the lookdown system are highly correlated. This is because the particles at time T are offspring of finitely many particles alive at an arbitrary time before T. Our approach is to group the countably many particles at time T into finitely many disjoint subclusters according to their respective ancestors at an earlier time. When this earlier time is close enough to T, the distances between the particles at time T and their respective ancestors have to be small. Then each subcluster is contained in a small neighborhood of its ancestor and all the neighborhoods are contained in a compact set. The compact support property thus follows. As a byproduct, we can obtain a cover for the support so as to get an upper bound on the Hausdorff dimension for the support at fixed time.

Throughout the thesis, we always write C or C with subscript for a positive constant and write $C(x)$ for a constant depending on x whose value might vary from place to place. Denote by X the Λ-Fleming-Viot process with underling Brownian motion. We also assume that the measure Λ has no mass at 1, i.e., $\Lambda(\{1\}) = 0$.

Recall the notion of *Hausdorff dimension*. Given any Borel set $K \subset \mathbb{R}^d$ and $\beta > 0$, $\eta > 0$, let

$$\mathbf{\Lambda}_{\eta}^{\beta}(K) \equiv \inf_{\{S_l\} \in \varphi_\eta} \sum_l \left(d\left(S_l\right)\right)^{\beta},$$

where $d\left(S_l\right)$ denotes the diameter of ball S_l in $\mathbb{R}^d$ and φ_η denotes the collection of η-*covers* of set K by balls, i.e.,

$$\varphi_\eta \equiv \left\{\{S_l\} \text{ is a cover of } K \text{ by balls with } d\left(S_l\right) < \eta \text{ for each } l\right\}.$$

The Hausdorff β-measure of K is defined by

$$\mathbf{\Lambda}^{\beta}(K) \equiv \lim_{\eta \to 0} \mathbf{\Lambda}_{\eta}^{\beta}(K). \tag{3.0.1}$$

The Hausdorff dimension of K is defined by

$$\dim K \equiv \inf\left\{\beta > 0 : \mathbf{\Lambda}^{\beta}(K) = 0\right\} \equiv \sup\left\{\beta > 0 : \mathbf{\Lambda}^{\beta}(K) = \infty\right\}.$$

Given $\eta > 0$, for any Borel set $K \subset \mathbb{R}^d$, let $\mathbb{B}(K, \eta)$ be its *closed* η*-neighborhood* such that

$$\mathbb{B}(K, \eta) \equiv \overline{\bigcup_{x \in K} \mathbb{B}(x, \eta)},$$

where $\mathbb{B}(x, \eta)$ is the closed ball centered at x with radius η.

3.1 An estimate on Brownian motion

Write

$$(\mathbf{B}(s))_{s \geq 0} \equiv (B_1(s), B_2(s), \ldots, B_d(s))_{s \geq 0}$$

for d-dimensional standard Brownian motion with initial value $\mathbf{0}$, where

$$(B_i(s))_{s \geq 0}, \quad i = 1, \ldots, d$$

are independent one-dimensional standard Brownian motions. For any vector $\mathbf{z} = (z_1, z_2, \dots, z_d) \in \mathbb{R}^d$, write $|\mathbf{z}| \equiv \sqrt{\sum_{i=1}^d z_i^2}$ as usual.

Lemma 3.1 *Given any $x > 0$ and d-dimensional standard Brownian motion $(\mathbf{B}(s))_{s\geq 0}$, we have*

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |\mathbf{B}(s)| > x\right) \leq \sqrt{\frac{8d^3t}{\pi}}\frac{1}{x}\exp\left(-\frac{x^2}{2dt}\right).$$

Proof. By *reflection principle*, it is easy to show that

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |\mathbf{B}(s)| > x\right) \leq 2d\mathbb{P}\left(|B_1(t)| > x/\sqrt{d}\right) \leq 4d\mathbb{P}\left(B_1(t) > x/\sqrt{d}\right).$$

Problem 9.22 of Karatzas and Shreve (1998) gave an estimate on one-dimensional Brownian motion such that for any $x > 0$,

$$\frac{x}{1+x^2}e^{-\frac{x^2}{2}} \leq \int_x^\infty e^{-\frac{u^2}{2}}du \leq \frac{1}{x}e^{-\frac{x^2}{2}}.$$

Consequently, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq s\leq t} |\mathbf{B}(s)| > x\right) &\leq 4d\mathbb{P}\left(\frac{B_1(t)}{\sqrt{t}} > \frac{x}{\sqrt{dt}}\right)\\ &\leq \sqrt{\frac{8d^3t}{\pi}}\frac{1}{x}\exp\left(-\frac{x^2}{2dt}\right).\end{aligned} \tag{3.1.1}$$

□

3.2 The compact support property for the Λ-Fleming-Viot process at a fixed time

In this section, we discuss the compact support property for the Λ-Fleming-Viot process with the associated coalescent satisfying Assumption I. Clearly, Assumption I is sufficient for the coalescent to come down from infinity.

Given $\alpha > 0$ in Assumption I, for any $k \in [\infty]$, set

$$N_k \equiv 2^{k/\alpha}k^{2/\alpha}.$$

Let $\left(\Pi^T(t)\right)_{0\le t\le T}$ be the Λ-coalescent recovered from the lookdown construction. T_m is defined by (2.3.9). For all $m \in [\infty]$, the number of ancestors at time $T - T_{N_m} \wedge T$ is equal to $\#\Pi^T(T_{N_m} \wedge T)$, which is almost surely finite by the coming down from infinity property.

Put

$$N_m^* \equiv \#\Pi^T\left(T_{N_m} \wedge T\right) \quad \text{and} \quad \Pi^T\left(T_{N_m} \wedge T\right) \equiv \left\{\pi_l : l = 1, \ldots, N_m^*\right\},$$

where $\{\pi_l \equiv \pi_l(m), 1 \le l \le N_m^*\}$ are all the disjoint blocks of $\Pi^T\left(T_{N_m} \wedge T\right)$ ordered by their least elements. Note that $L_j^T(T - T_{N_m} \wedge T) = l$ for any $j \in \pi_l$ by Lemma 2.8. The *maximal radius of subclusters* is equal to:

$$\begin{aligned} H\left(T - T_{N_m} \wedge T, T\right) &\equiv \max_{1\le l\le N_m^*} \max_{j\in\pi_l} \left| X_j(T) - X_{L_j^T(T-T_{N_m}\wedge T)}\left((T - T_{N_m} \wedge T)-\right)\right| \\ &= \max_{1\le l\le N_m^*} \max_{j\in\pi_l} \left| X_j(T) - X_l\left((T - T_{N_m} \wedge T)-\right)\right|. \end{aligned}$$

Lemma 3.2 *Under Assumption I, for any $\delta \in (0, 1/2)$, there exists a positive constant $C(\delta)$ such that almost surely,*

$$H\left(T - T_{N_m} \wedge T, T\right) \le C(\delta) 2^{-m\left(\frac{1}{2}-\delta\right)}$$

for m big enough.

Proof. For $k \in [\infty]$, define time interval $J_k = \left[T - T_{N_k} \wedge T, T - T_{N_{k+1}} \wedge T\right]$. Let $|J_k|$ be the length of interval J_k. Thus

$$|J_k| = \left(T - T_{N_{k+1}} \wedge T\right) - \left(T - T_{N_k} \wedge T\right) = T_{N_k} \wedge T - T_{N_{k+1}} \wedge T \le T_{N_k}.$$

Let D_k be the maximal dislocation over time interval J_k of all the Brownian motions involving the ancestors of the countably many particles at T, i.e.,

$$D_k \equiv \max_{1\le l\le N_k^*} \max_{j\in\pi_l} \left| X_{L_j^T(T-T_{N_{k+1}}\wedge T)}(T - T_{N_{k+1}} \wedge T) - X_l\left((T - T_{N_k} \wedge T)-\right)\right|. \tag{3.2.1}$$

For any fixed $1 \le l \le N_k^*$, the collection of ancestor levels

$$\left\{L_j^T(T - T_{N_{k+1}} \wedge T) : j \in \pi_l\right\}$$

has a finite cardinality because of the coming down from infinity property. Thus both maximums in (3.2.1) are taken over finite sets.

For the trivial case of $T_{N_{k+1}} \wedge T = T$, we have $|J_k| = 0$ and the dislocation of Brownian motion over J_k is equal to 0. Consequently,

$$P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}, |J_k| = 0\right) = 0.$$

In the case of $|J_k| > 0$, the total number of Brownian motions involved over J_k is no more than

$$N_{k+1} = 2^{(k+1)/\alpha}\,(k+1)^{2/\alpha}.$$

Thus we have

$$\begin{aligned}
&P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}\right)\\
=&P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}, |J_k| = 0\right) + P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}, |J_k| > 0\right)\\
=&P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}, 0 < |J_k| \le 2^{-k}\right) + P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}\middle||J_k| > 2^{-k}\right) \times P\left(|J_k| > 2^{-k}\right)\\
\le&N_{k+1} \times P\left(\sup_{0\le s\le 2^{-k}} |\mathbf{B}(s)| > 2^{-k\left(\frac{1}{2}-\delta\right)}\right) + P\left(|J_k| > 2^{-k}\right)\\
\equiv&I_1(k) + I_2(k).
\end{aligned}$$

By Lemma 3.1, there exist positive constants C_1 and C_2 such that

$$P\left(\sup_{0\le s\le 2^{-k}} |\mathbf{B}(s)| > 2^{-k\left(\frac{1}{2}-\delta\right)}\right) \le C_1 2^{-k\delta} \exp\left(-C_2 2^{2\delta k}\right).$$

Consequently,

$$\begin{aligned}
I_1(k) \le& 2^{\frac{k+1}{\alpha}}\,(k+1)^{\frac{2}{\alpha}}\, C_1 2^{-k\delta} \exp\left(-C_2 2^{2\delta k}\right)\\
\le& C_1 2^{\frac{4k}{\alpha}} \exp\left(-C_2 2^{2\delta k}\right).
\end{aligned}$$

It is clear that $\sum_k I_1(k) < \infty$.

Under Assumption I, there exists a constant C such that for k large enough

$$\mathbb{E}T_{N_k} \le C 2^{-k} k^{-2}.$$

Since $|J_k| \le T_{N_k}$, it follows from the Markov's inequality that for k large

$$I_2(k) \le P\left(T_{N_k} > 2^{-k}\right) \le 2^k \mathbb{E} T_{N_k} \le Ck^{-2}.$$

Therefore,

$$\sum_k P\left(D_k > 2^{-k\left(\frac{1}{2}-\delta\right)}\right) \le \sum_k I_1(k) + \sum_k I_2(k) < \infty.$$

Applying the Borel-Cantelli lemma, we have almost surely

$$D_k \le 2^{-k\left(\frac{1}{2}-\delta\right)}$$

for k large enough. Then $\mathbb{P}$-a.s.,

$$\begin{aligned} H\left(T - T_{N_m} \wedge T, T\right) &\le \sum_{k=m}^{\infty} D_k \le \sum_{k=m}^{\infty} 2^{-k\left(\frac{1}{2}-\delta\right)} \\ &= \frac{2^{-m\left(\frac{1}{2}-\delta\right)}}{1 - 2^{-\left(\frac{1}{2}-\delta\right)}} \equiv C(\delta) 2^{-m\left(\frac{1}{2}-\delta\right)} \end{aligned}$$

for m large enough. $\square$

Theorem 3.3 *Under Assumption I, for any $T > 0$, with probability one the random measure $X(T)$ has a compact support.*

Proof. For m large enough and for all $k \ge m$, by Lemma 3.2 we have

$$\begin{aligned} X_j\left(\left(T - T_{N_k}\right)-\right) &\subseteq \bigcup_{l=1}^{N_m^*} \mathbb{B}\left(X_l\left(\left(T - T_{N_m}\right)-\right), H\left(T - T_{N_m} \wedge T, T\right)\right) \\ &\subseteq \bigcup_{l=1}^{N_m^*} \mathbb{B}\left(X_l\left(\left(T - T_{N_m}\right)-\right), C(\delta) 2^{-m\left(\frac{1}{2}-\delta\right)}\right) \\ &\equiv \mathbf{B}, \end{aligned}$$

where $\mathbb{B}(x, r)$ is the closed ball centered at x with radius r.

For each $n \in [\infty]$, from the lookdown construction there exists a random variable $\delta_n > 0$ such that during the time interval $[T - \delta_n, T]$, the particle at level n never looks down to those particles at lower levels $\{1, 2, \ldots, n-1\}$. It follows from Lemma 2.8 that for any $j \in [n]$, $L_j^T(s) = j$ for all $s \in [T - \delta_n, T]$. Further, the sample path continuity for Brownian motion implies that

$$X_j(T) = X_j(T-) = \lim_{k \to \infty} X_j\left((T - T_{N_k}) -\right).$$

Therefore, $X_j(T)$ is a limit point for the compact set $\mathbf{B}$ and we have $X_j(T) \in \mathbf{B}$ for all j. Let

$$\hat{X}_n(T) \equiv \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(T)}.$$

By the lookdown construction for the Λ-Fleming-Viot process we have

$$X(T) = \lim_{n \to \infty} \hat{X}_n(T).$$

Clearly,

$$\operatorname{supp} \hat{X}_n(T) \subseteq \mathbf{B}$$

for all n, which implies that

$$\operatorname{supp} X(T) \subseteq \mathbf{B}.$$

Consequently, $\operatorname{supp} X(T)$ is compact. □

3.3 The upper and lower bounds on Hausdorff dimension for the support

In this section, we discuss the upper and lower bounds on Hausdorff dimension for $\operatorname{supp} X(T)$ at any fixed $T > 0$.

Theorem 3.4 *Under Assumption I, for any $T > 0$, we have $\mathbb{P}$-a.s.*

$$\mathit{dim\, supp}\, X(T) \leq 2/\alpha.$$

Proof. It follows from the proof of Theorem 3.3 that the collection of closed balls

$$\left\{\mathbb{B}\left(X_l\left((T - T_{N_m})-\right), C(\delta) 2^{-m\left(\frac{1}{2}-\delta\right)}\right) : l = 1, \ldots, N_m^*\right\}$$

is a cover of $\operatorname{supp} X(T)$ for m large enough.

For any $\epsilon > 0$, choose $\delta > 0$ small enough so that

$$\left(\frac{1}{2} - \delta\right)(2+\epsilon) > 1.$$

For all m big enough we also have $N_m^* \le N_m$. Then

$$\begin{aligned}
&\lim_{m\to\infty} N_m^* C(\delta)^{\frac{2+\epsilon}{\alpha}} 2^{-m\left(\frac{1}{2}-\delta\right)\frac{2+\epsilon}{\alpha}} \\
\le &\lim_{m\to\infty} 2^{\frac{m}{\alpha}} m^{\frac{2}{\alpha}} C(\delta)^{\frac{2+\epsilon}{\alpha}} 2^{-m\left(\frac{1}{2}-\delta\right)\frac{2+\epsilon}{\alpha}} \\
= &C(\delta)^{\frac{2+\epsilon}{\alpha}} \lim_{m\to\infty} m^{\frac{2}{\alpha}} 2^{-\frac{m}{\alpha}\left[\left(\frac{1}{2}-\delta\right)(2+\epsilon)-1\right]} \\
= &0
\end{aligned}$$

and we have

$$\dim \operatorname{supp} X(T) \le (2+\epsilon)/\alpha.$$

ϵ is arbitrary, so the Hausdorff dimension for $\operatorname{supp} X(T)$ is bounded from above by $2/\alpha$.

$\square$

The lemma below on a lower bound for the Hausdorff dimension can be found in Falconer (1985).

Lemma 3.5 *Let K be any Borel subset of $\mathbb{R}^n$. If there is a mass distribution μ, supported by K such that*

$$I_a(K) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|x-y|^a} \mu(dx)\,\mu(dy) < \infty,$$

then $\dim K \ge a$.

By adapting the approach of Proposition 6.14 in Etheridge (2000), we could also find a lower bound on Hausdorff dimension for $\operatorname{supp} X(T)$ at any fixed $T > 0$.

Theorem 3.6 *Let X be the Λ-Fleming-Viot process with underlying Brownian motion in $\mathbb{R}^d$ for $d \ge 2$. Then for any $T > 0$, $\mathbb{P}$-a.s.*

$$\mathit{dim\,supp}\, X(T) \ge 2.$$

Proof. The moments of the Λ-Fleming-Viot process can be expressed in terms of a dual process involving Λ-coalescent and heat flow, see Section 5.2 of Birkner et al. (2009) for such a dual process. For lack of multiple collisions, expression for the second moment of the Λ-Fleming-Viot process is the same as that for classical Fleming-Viot process given in Proposition 2.27 of Etheridge (2000). Then for any $\phi_1, \phi_2 \in C_b(\mathbb{R}^d)$, we have

$$\begin{aligned}\mathbb{E}\left[\langle X(T), \phi_1\rangle \langle X(T), \phi_2\rangle\right] =& e^{-rT}\langle X(0), P_T\phi_1\rangle \langle X(0), P_T\phi_2\rangle \\ &+ \left\langle X(0), \int_0^T re^{-rs}P_{T-s}\left(P_s\phi_1 P_s\phi_2\right)ds\right\rangle,\end{aligned}$$

where P_s is the heat flow and r is the total coalescence rate when the number of existing blocks is 2, i.e., $r = \lambda_2$.

Following arguments similar to Proposition 6.14 of Etheridge (2000), we can show that for any nonnegative function of the form $\psi(x, y)$,

$$\begin{aligned}&\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,y)\,X(T)(dx)X(T)(dy)\right] \\ =& e^{-rT}\int\cdots\int p(T,z,w)\,p\left(T,z',w'\right)\psi\left(w,w'\right)dwdw'X(0)(dz)X(0)(dz') \\ &+ r\int_0^T\int\cdots\int e^{-rs}p(T-s,z,w)\,p(s,w,y)\,p\left(s,w,y'\right)\psi\left(y,y'\right)dydy'dwX(0)(dz)ds,\end{aligned}$$

where $p(\cdot,\cdot,\cdot)$ denotes the heat kernel.

Without loss of generality, we assume that $X(0) = \delta_0$. Then

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,y)\,X(T)(dx)X(T)(dy)\right] \equiv I_1 + I_2,$$

where

$$I_1 = e^{-rT}\int\int p(T,0,w)\,p\left(T,0,w'\right)\psi\left(w,w'\right)dwdw'$$

and

$$I_2 = r\int_0^T\int\cdots\int e^{-rs}p(T-s,0,w)\,p(s,w,y)\,p\left(s,w,y'\right)\psi\left(y,y'\right)dydy'dwds.$$

Replace $\psi(x,y) = 1/|x-y|^a$. By Lemma 3.5, it suffices to show that for any $T > 0$ and $1 < a < 2$, both I_1 and I_2 are finite.

Note that

$$
\begin{aligned}
I_1 \le & \iint_{\left|w-w'\right|\ge 1} \frac{1}{(2\pi T)^d} e^{-\frac{|w|^2}{2T}} e^{-\frac{|w'|^2}{2T}} \frac{1}{\left|w-w'\right|^a} dw dw' \\
& + \iint_{\left|w-w'\right|<1} \frac{1}{(2\pi T)^d} e^{-\frac{|w|^2}{2T}} e^{-\frac{|w'|^2}{2T}} \frac{1}{\left|w-w'\right|^a} dw dw' \\
\equiv & I_{11} + I_{12}.
\end{aligned}
$$

It is easy to show that

$$
I_{11} \le \frac{1}{(2\pi T)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{|w|^2}{2T}} e^{-\frac{|w'|^2}{2T}} dw dw' < \infty \tag{3.3.1}
$$

and

$$
\begin{aligned}
I_{12} \le & \frac{1}{(2\pi T)^d} \int_{\mathbb{R}^d} e^{-\frac{|w'|^2}{2T}} dw' \int_{\left|w-w'\right|<1} \frac{1}{\left|w-w'\right|^a} dw \\
\le & \frac{1}{(2\pi T)^d} \int_{\mathbb{R}^d} e^{-\frac{|w'|^2}{2T}} dw' \, (2\pi)^{d-1} \int_0^1 r^{d-1-a} dr.
\end{aligned}
$$

Thus if $d-1-a > -1$, i.e., $a < d$, we have

$$
I_{12} \le \frac{(2\pi)^{d-1}}{(2\pi T)^d (d-a)} \int_{\mathbb{R}^d} e^{-\frac{|w'|^2}{2T}} dw' = \frac{(2\pi)^{d-1} (2T\pi)^{d/2}}{(2\pi T)^d (d-a)} < \infty. \tag{3.3.2}
$$

Now we are ready to estimate I_2. Note that

$$
|w-y|^2 + \left|w-y'\right|^2 = 2\left(\left|w - \frac{y+y'}{2}\right|^2\right) + \frac{1}{2}\left|y-y'\right|^2,
$$

so we have

$$
p(s,w,y)\, p\left(s,w,y'\right) = p\left(\frac{s}{2}, w, \frac{y+y'}{2}\right) p\left(2s, y, y'\right).
$$

Then

$$
\begin{aligned}
I_2 \le & r \int_0^T \int \cdots \int p(T-s,0,w)\, p\left(\frac{s}{2}, w, \frac{y+y'}{2}\right) p\left(2s,y,y'\right) \psi\left(y,y'\right) dw dy dy' ds \\
= & r \int_0^T ds \int_{\mathbb{R}^d} dy' \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} \frac{1}{(2\pi(T-s))^{d/2} (\pi s)^{d/2} (4\pi s)^{d/2}} e^{-\frac{|w|^2}{2(T-s)}} \\
& \times e^{-\frac{\left|w-\frac{y+y'}{2}\right|^2}{s}} e^{-\frac{|y-y'|^2}{4s}} \psi\left(y,y'\right) dw.
\end{aligned}
$$

Since

$$\frac{|w|^2}{2(T-s)}+\frac{|w-c|^2}{s}=\frac{\left|w-\frac{2(T-s)}{2T-s}c\right|^2}{\frac{2(T-s)s}{2T-s}}+\frac{|c|^2}{2T-s},$$

we have

$$\begin{aligned}
I_2 \le & r\int_0^T ds\int_{\mathbb{R}^d}dy'\int_{\mathbb{R}^d}dy\int_{\mathbb{R}^d}\frac{1}{(2\pi(T-s))^{d/2}(\pi s)^{d/2}(4\pi s)^{d/2}}e^{-\frac{|y-y'|^2}{4s}}\psi\left(y,y'\right)\\
&\times\exp\left(-\frac{\left|w-\frac{2(T-s)}{2T-s}\frac{y+y'}{2}\right|^2}{\frac{2(T-s)s}{2T-s}}\right)e^{-\frac{\left|\frac{y+y'}{2}\right|^2}{2T-s}}dw\\
=&r\int_0^T ds\int_{\mathbb{R}^d}dy'\int_{\mathbb{R}^d}\frac{\left(\frac{2(T-s)s\pi}{2T-s}\right)^{d/2}}{(2\pi(T-s))^{d/2}(\pi s)^{d/2}(4\pi s)^{d/2}}e^{-\frac{|y-y'|^2}{4s}}\psi\left(y,y'\right)e^{-\frac{\left|\frac{y+y'}{2}\right|^2}{2T-s}}dy\\
=&r\int_0^T ds\int_{\mathbb{R}^d}dy'\int_{\mathbb{R}^d}\frac{1}{(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}}e^{-\frac{|y-y'|^2}{4s}}e^{-\frac{|y+y'|^2}{4(2T-s)}}\frac{1}{|y-y'|^a}dy\\
=&r\int_0^T ds\iint_{|y-y'|\ge 1}\frac{1}{(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}}e^{-\frac{|y-y'|^2}{4s}}e^{-\frac{|y+y'|^2}{4(2T-s)}}\frac{1}{|y-y'|^a}dydy'\\
&+r\int_0^T ds\iint_{|y-y'|<1}\frac{1}{(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}}e^{-\frac{|y-y'|^2}{4s}}e^{-\frac{|y+y'|^2}{4(2T-s)}}\frac{1}{|y-y'|^a}dydy'\\
\equiv&I_{21}+I_{22}.
\end{aligned}$$

We continue to estimate I_{21} and I_{22}, respectively.

$$I_{21}\le r\int_0^T ds\int_{\mathbb{R}^d}dy'\int_{|y-y'|\ge 1}\frac{1}{(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}}e^{-\frac{|y-y'|^2}{4s}}e^{-\frac{|y+y'|^2}{4(2T-s)}}dy.$$

As we know

$$\frac{\left|y-y'\right|^2}{4s}+\frac{\left|y+y'\right|^2}{4(2T-s)}=\frac{\left|y-\frac{T-s}{T}y'\right|^2}{\frac{2s(2T-s)}{T}}+\frac{\left|y'\right|^2}{2T},$$

it follows that

$$\begin{aligned}
I_{21}\le&r\int_0^T ds\int_{\mathbb{R}^d}dy'\int_{\mathbb{R}^d}\frac{1}{(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}}e^{-\frac{\left|y'\right|^2}{2T}}e^{-\frac{\left|y-\frac{T-s}{T}y'\right|^2}{\frac{2s(2T-s)}{T}}}dy\\
=&r\int_0^T ds\int_{\mathbb{R}^d}\frac{2^{d/2}}{(4\pi)^{d/2}T^{d/2}}e^{-\frac{\left|y'\right|^2}{2T}}dy'\\
=&rT<\infty.
\end{aligned}\tag{3.3.3}$$

$$I_{22} = r\int_0^T ds \int_{\mathbb{R}^d} dy' \int_{|y-y'|<1} \frac{1}{(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}} e^{-\frac{|y-y'|^2}{4s}} e^{-\frac{\left|y+y'\right|^2}{4(2T-s)}} \frac{1}{|y-y'|^a} dy.$$

It follows from

$$\frac{\left|y-y'\right|^2}{8s} + \frac{\left|y+y'\right|^2}{4(2T-s)} = \frac{\left|y-\frac{2T-3s}{2T+s}y'\right|^2}{\frac{8s(2T-s)}{2T+s}} + \frac{\left|y'\right|^2}{2T+s}$$

that

$$\begin{aligned}
I_{22} =& r\int_0^T ds \int_{\mathbb{R}^d} dy' \int_{|y-y'|<1} \frac{1}{(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}} e^{-\frac{\left|y-y'\right|^2}{8s}} e^{-\frac{\left|y'\right|^2}{2T+s}} \\
&\times \exp\left(-\frac{\left|y-\frac{2T-3s}{2T+s}y'\right|^2}{\frac{8s(2T-s)}{2T+s}}\right)\frac{1}{|y-y'|^a}dy \\
\leq& r\int_0^T ds \int_{\mathbb{R}^d} dy' \int_{|y-y'|<1} \frac{1}{(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}} e^{-\frac{\left|y-y'\right|^2}{8s}} e^{-\frac{\left|y'\right|^2}{2T+s}} \frac{1}{|y-y'|^a}dy \\
\leq& r\int_0^T \frac{1}{(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}} ds \int_{\mathbb{R}^d} e^{-\frac{\left|y'\right|^2}{2T+s}} dy' \int_0^1 e^{-\frac{r^2}{8s}} \frac{r^{d-1}}{r^a}(2\pi)^{d-1}dr \\
\leq& r\int_0^T \frac{(2\pi)^{d-1}8^{(d-a)/2}s^{(d-a)/2}}{2(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}} ds \int_{\mathbb{R}^d} e^{-\frac{\left|y'\right|^2}{2T+s}} dy' \int_0^\infty e^{-u}u^{\frac{d-a}{2}-1}du.
\end{aligned}$$

With the condition that $a < (d \wedge 2)$, we have

$$\int_0^\infty e^{-u}u^{\frac{d-a}{2}-1}du = \Gamma\left(\frac{d-a}{2}\right).$$

Thus

$$\begin{aligned}
I_{22} \leq& \Gamma\left(\frac{d-a}{2}\right) r\int_0^T \frac{(2\pi)^{d-1}8^{(d-a)/2}s^{(d-a)/2}}{2(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}} ds \int_{\mathbb{R}^d} e^{-\frac{\left|y'\right|^2}{2T+s}} dy' \\
=& \Gamma\left(\frac{d-a}{2}\right) r\int_0^T \frac{(2\pi)^{d-1}8^{(d-a)/2}s^{(d-a)/2}(\pi(2T+s))^{d/2}}{2(2T-s)^{d/2}\pi^{d/2}(4\pi s)^{d/2}} ds \\
\leq& \frac{\Gamma\left(\frac{d-a}{2}\right) r (2\pi)^{d-1}8^{(d-a)/2}(3T\pi)^{d/2}}{2T^{d/2}\pi^{d/2}(4\pi)^{d/2}} \times \int_0^T s^{-a/2}ds \\
\leq& C(\pi,d,a,T,r)\int_0^T s^{-a/2}ds \\
=& C(\pi,d,a,T,r)\frac{T^{1-a/2}}{1-a/2} < \infty.
\end{aligned} \tag{3.3.4}$$

Combing (3.3.1), (3.3.2), (3.3.3) and (3.3.4), we have

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{|x-y|^a}X(T)(dx)X(T)(dy)\right] < \infty$$

for $1 < a < (d \wedge 2)$. Therefore, the Hausdorff dimension for $\operatorname{supp} X(T)$ is at least 2. $\square$

Corollary 3.7 *Suppose that $d \geq 2$ and $\Lambda(\{0\}) > 0$, i.e., the Λ-coalescent has a nontrivial Kingman component. Then at any fixed time $T > 0$, with probability one the Λ-Fleming-Viot process has a compact support of Hausdorff dimension two.*

Proof. Since $\Lambda(\{0\}) > 0$, the Λ-coalescent has a nontrivial Kingman component. Then

$$\lambda_b \geq \frac{\Lambda(\{0\})b(b-1)}{2}$$

and

$$\sum_{b=m+1}^{\infty} \frac{1}{\lambda_b} \leq \sum_{b=m+1}^{\infty} \frac{2}{\Lambda(\{0\})b(b-1)} = \frac{2}{\Lambda(\{0\})m},$$

i.e., Condition A holds with $\alpha = 1$, which is sufficient for Assumption I. The results follow from Theorems 3.4 and 3.6. $\square$

Remark 3.8 *Corollary 3.7 complements the result on Hausdorff dimension for the classical Fleming-Viot process in Dawson and Hochberg (1982).*

3.4 Examples

In this section, we give some examples of Λ-Fleming-Viot processes and further consider their support properties.

The Λ-Fleming-Viot process with its coalescent having the (c, ϵ, γ)-property

Lemma 3.9 *For $n \geq 2$, there exists a positive constant $C(c, \gamma, \epsilon)$ such that the total coalescence rate of the Λ-coalescent with the (c, ϵ, γ)-property satisfies*

$$\lambda_n \geq C(c, \gamma, \epsilon) n^{1+\gamma},$$

where

$$C(c, \gamma, \epsilon) = \frac{c\epsilon^{1-\gamma}}{2(1-\gamma)} \left(\frac{1}{3(2-\gamma)} \right)^{\gamma} e^{-\frac{\gamma^2}{2(1-\gamma)}}.$$

Proof. By the definition of λ_n, we have

$$\begin{aligned}
\lambda_n &= \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k} \geq \binom{n}{2} \lambda_{n,2} \\
&\geq c \binom{n}{2} \int_0^{\epsilon} x^{-\gamma} (1-x)^{n-2} dx \\
&= c \binom{n}{2} \int_0^{1} (y\epsilon)^{-\gamma} (1-y\epsilon)^{n-2} \epsilon dy \\
&\geq c \binom{n}{2} \epsilon^{1-\gamma} \int_0^{1} y^{-\gamma} (1-y)^{n-2} dy \\
&= \frac{c\epsilon^{1-\gamma} n}{2} \frac{(n-1)!\Gamma(1-\gamma)}{\Gamma(n-\gamma)} \\
&\equiv \frac{c\epsilon^{1-\gamma} n}{2(1-\gamma)} \times B,
\end{aligned}$$

where

$$B = \frac{n-1}{n-1-\gamma} \times \frac{n-2}{n-2-\gamma} \times \cdots \times \frac{3}{3-\gamma} \times \frac{2}{2-\gamma}.$$

It follows from the inequality $\ln(1+x) \geq x - x^2/2$ for $0 < x < 1$ that

$$\begin{aligned}
\ln B &= \sum_{l=2}^{n-1} \ln\left(\frac{l}{l-\gamma}\right) = \sum_{l=2}^{n-1} \ln\left(1 + \frac{\gamma}{l-\gamma}\right) \\
&\geq \sum_{l=2}^{n-1} \frac{\gamma}{l-\gamma} - \frac{\gamma^2}{2} \sum_{l=2}^{n-1} \frac{1}{(l-\gamma)^2} \\
&\geq \int_2^{n} \frac{\gamma}{x-\gamma} dx - \frac{\gamma^2}{2} \int_1^{n-1} \frac{1}{(x-\gamma)^2} dx \\
&= \gamma \ln \frac{n-\gamma}{2-\gamma} - \frac{\gamma^2}{2} \left(\frac{1}{1-\gamma} - \frac{1}{n-1-\gamma}\right) \\
&\geq \gamma \ln \frac{n-\gamma}{2-\gamma} - \frac{\gamma^2}{2(1-\gamma)}.
\end{aligned}$$

Consequently,

$$\lambda_n \geq \frac{c\epsilon^{1-\gamma} n}{2(1-\gamma)} \left(\frac{n-\gamma}{2-\gamma}\right)^{\gamma} e^{-\frac{\gamma^2}{2(1-\gamma)}}.$$

Since $\gamma \in (0,1)$, then $n - \gamma \geq n/3$ for any $n \geq 2$. Therefore,

$$\lambda_n \geq \frac{c\epsilon^{1-\gamma} n}{2(1-\gamma)} \left(\frac{n}{3(2-\gamma)}\right)^{\gamma} e^{-\frac{\gamma^2}{2(1-\gamma)}} \equiv C(c, \gamma, \epsilon) n^{1+\gamma}.$$

$\square$

Proposition 3.10 *Let X be any Λ-Fleming-Viot process with underlying Brownian motion in $\mathbb{R}^d$ for $d \geq 2$. If the associated Λ-coalescent has the (c, ϵ, γ)-property, then for any $T > 0$, with probability one the random measure X has a compact support at time T. Further,*

$$2 \leq \mathit{dim\,supp}\, X\,(T) \leq 2/\gamma.$$

Proof. It follows from Lemma 3.9 that

$$\begin{aligned}\sum_{k=m+1}^{\infty} \lambda_k^{-1} &\leq \sum_{k=m+1}^{\infty} \frac{1}{C\,(c,\gamma,\epsilon)\,k^{1+\gamma}} \\ &\leq \int_m^{\infty} \frac{1}{C\,(c,\gamma,\epsilon)\,x^{1+\gamma}} dx \\ &= \frac{1}{\gamma C\,(c,\gamma,\epsilon)\,m^{\gamma}},\end{aligned}$$

which implies Condition A holds with $\alpha = \gamma$. Since Condition A is sufficient for Assumption I, the results follow from Theorems 3.3, 3.4 and 3.6. $\square$

The Beta$(2-\beta,\beta)$-Fleming-Viot process

Proposition 3.11 *Suppose that $d \geq 2$. For any $T > 0$, with probability one the Beta$(2-\beta, \beta)$-Fleming-Viot process X with underlying Brownian motion in $\mathbb{R}^d$ has a compact support at time T if and only if $\beta \in (1,2)$. Further, for $\beta \in (1,2)$,*

$$2 \leq \mathit{dim\,supp}\, X\,(T) \leq 2/\left(\beta - 1\right).$$

Proof. For $\beta \in (0,1]$, the corresponding Beta$(2-\beta,\beta)$-coalescent does not come down from infinity.

For $\beta \in (1,2)$, then $\beta - 1 \in (0,1)$ and given $\epsilon \in (0,1)$, for all $x \in [0,\epsilon]$, we have

$$\begin{aligned}\Lambda(dx) &= \frac{\Gamma(2)}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta}\left(1-x\right)^{\beta-1} dx \\ &\geq \frac{\Gamma(2)\left(1-\epsilon\right)^{\beta-1}}{\Gamma(2-\beta)\Gamma(\beta)} x^{1-\beta} dx,\end{aligned}$$

which implies that the Beta$(2-\beta,\beta)$-coalescent has the $(c,\epsilon,\beta-1)$-property.

By Proposition 7.2 of Blath (2009) and Proposition 3.10, the Beta$(2-\beta,\beta)$-Fleming-Viot process has a compact support if and only if $\beta \in (1,2)$ and the Hausdorff dimension for its support is between 2 and $2/(\beta-1)$. $\square$

Remark 3.12 *Intuitively, since the Beta-coalescent comes down from infinity at a speed slower than Kingman's coalescent, the particles in the lookdown representation are less correlated. So we expect a higher Hausdorff dimension for the support of Beta-Fleming-Viot process with underlying Brownian motion.*

Remark 3.13 *By Proposition 3.11 the coming down from infinity property is equivalent to the compact support property for Beta*$(2-\beta,\beta)$*-Fleming-Viot processes, which suggests that the Assumption I is rather mild.*

Chapter 4

The modulus of continuity for Λ-Fleming-Viot support process

Intuitively, the modulus of continuity for Λ-Fleming-Viot support process tells us how fast the support process propagates. In this chapter, we first discuss the one-sided modulus of continuity for the ancestry process recovered from the modified lookdown construction of Λ-Fleming-Viot process. As an application, we also prove the one-sided modulus of continuity for the Λ-Fleming-Viot support process at any fixed positive time. Such a result has never been proved for Fleming-Viot processe before.

4.1 Modulus of continuity for the ancestry process

In this section, we first obtain some estimates on the Λ-coalescent and on the maximal dislocation of the particles from their respective ancestors. Then we prove the one-sided modulus of continuity for the ancestry process of the Λ-Fleming-Viot process.

Denote by $\lfloor x \rfloor$ the integer part of x for $x \in \mathbb{R}$. Given $T > 0$ and $\Delta > 0$, we can divide the interval $[0, T]$ into subintervals as follows:

$$[0,\Delta], [\Delta, 2\Delta], \ldots, [\lfloor T/\Delta - 1 \rfloor \Delta, \lfloor T/\Delta \rfloor \Delta], [\lfloor T/\Delta \rfloor \Delta, T].$$

Set $\Delta \equiv \Delta_n = 2^{-n}$. Let S_n^T be the collection of the endpoints of the first $\lfloor 2^n T \rfloor$ subintervals, i.e.,

$$S_n^T \equiv \left\{k2^{-n} : 0 \leq k \leq \lfloor 2^n T \rfloor\right\}.$$

Put

$$S^T \equiv \bigcup_{n\geq 1} S_n^T = \bigcup_{n\geq 1} \left\{k2^{-n} : 0 \leq k \leq \lfloor 2^n T \rfloor\right\}.$$

Clearly, given any $T > 0$, S^T is the collection of all the dyadic rationals in $[0, T]$. So S^T is a dense subset of $[0, T]$.

For any $n \in [\infty]$, let $\{\mathbb{A}_{n,k} : 1 \leq k \leq \lfloor 2^n T \rfloor\}$ be the collection of the first $\lfloor 2^n T \rfloor$ subintervals in the partition so that

$$\mathbb{A}_{n,k} \equiv \left[(k-1)2^{-n}, k2^{-n}\right].$$

For simplicity, we denote

$$N_{n,k} \equiv N^{(k-1)2^{-n}, k2^{-n}}$$

where $N^{r,s}$ is defined by (2.3.7) for any $0 \leq r \leq s$.

Also denote by $H_{n,k}$ the maximal dislocation over interval $\mathbb{A}_{n,k}$ of all the Brownian motions followed by the countably many particles alive at time $k2^{-n}$ and their respective ancestors at time $(k-1)\, 2^{-n}$, i.e.,

$$H_{n,k} \equiv H\left((k-1)\, 2^{-n}, k2^{-n}\right)$$

where $H(r, s)$ is defined by (2.3.8) for any $0 \leq r \leq s$.

For any positive integer m let

$$T_m^{n,k} \equiv \inf\left\{t \in [0, 2^{-n}] : \#\Pi^{k2^{-n}}(t) \leq m\right\}$$

with the convention $\inf \emptyset = 2^{-n}$.

Given any fixed $n \in [\infty]$ and $m \in [\infty]$, the random times $\{T_m^{n,k} : 1 \leq k \leq \lfloor 2^n T \rfloor\}$ follow the same distribution. Write $T_x^{n,k} \equiv T_{\lfloor x \rfloor}^{n,k}$ for any $x > 0$.

For any $0 < t < 1$, let

$$h(t) \equiv \sqrt{t \log(1/t)}. \tag{4.1.1}$$

Lemma 4.1 *Under Assumption I, we have $\mathbb{P}$-a.s.*

$$\max_{1\le k\le 2^n T} N_{n,k} < 8^{\frac{n}{\alpha}}$$

for n large enough.

Proof. Under Assumption I, there exists a positive constant C such that

$$\mathbb{E}T_m \le Cm^{-\alpha} \tag{4.1.2}$$

for m large enough.

Given n, for any $1 \le k \le 2^n T$, $T^{n,k}_{8^{n/\alpha}}$ follows the same distribution as $T_{8^{n/\alpha}} \wedge 2^{-n}$. Choosing $8^{n/\alpha}$ large enough, by (4.1.2) we have

$$\begin{aligned}
\mathbb{P}\left(\max_{1\le k\le 2^n T} N_{n,k} \ge 8^{\frac{n}{\alpha}}\right) &\le \sum_{1\le k\le 2^n T} \mathbb{P}\left(N_{n,k} \ge 8^{\frac{n}{\alpha}}\right)\\
&\le \sum_{1\le k\le 2^n T} \mathbb{P}\left(T^{n,k}_{8^{n/\alpha}} \ge 2^{-n}\right)\\
&\le 2^n T\mathbb{E}T^{n,k}_{8^{n/\alpha}}/2^{-n}\\
&\le CT2^{-n},
\end{aligned}$$

which is summable with respect to n. Applying Borel-Cantelli lemma, we then have $\mathbb{P}$-a.s.

$$\max_{1\le k\le 2^n T} N_{n,k} < 8^{\frac{n}{\alpha}}$$

for n large enough. □

Lemma 4.2 *Under Assumption I, for any $T > 0$, there exists a positive constant $C_4(d,\alpha)$ such that $\mathbb{P}$-a.s.*

$$\max_{1\le k\le 2^n T} H_{n,k} \le C_4(d,\alpha)h\left(2^{-n}\right)$$

for n large enough, where h is defined by (4.1.1).

Proof. Given any n and $1 \le k \le 2^n T$, we first divide each interval $\mathbb{A}_{n,k}$ into countably many random subintervals as follows:

$$J^{n,k}_0 \equiv \left[(k-1)2^{-n}, k2^{-n} - T^{n,k}_{8^{(n+1)/\alpha}}\right]$$

and

$$J_l^{n,k} \equiv \left[k2^{-n} - T^{n,k}_{8^{(n+l)/\alpha}}, k2^{-n} - T^{n,k}_{8^{(n+l+1)/\alpha}}\right]$$

for $l = 1, 2, 3, \ldots$. Consequently, the lengths of these countably many subintervals satisfy that

$$\left|J_0^{n,k}\right| \le 2^{-n} \quad \text{and} \quad \left|J_l^{n,k}\right| \le T^{n,k}_{8^{(n+l)/\alpha}} = T^{n,k}_{2^{(3n+3l)/\alpha}} \quad \text{for} \quad l = 1, 2, 3, \ldots.$$

The right endpoints of these subintervals $\left(b_l^{n,k}\right)_{l\ge 1} \equiv \left(k2^{-n} - T^{n,k}_{2^{(3n+3l)/\alpha}}\right)_{l\ge 1}$ consist of a sequence of random times converging increasingly to $k2^{-n}$. Set $b_0^{n,k} \equiv (k-1)\, 2^{-n}$ for convenience.

For $l = 0, 1, 2, \ldots$, let $D_l^{n,k}$ be the maximal dislocation over subinterval $J_l^{n,k}$ of all the Brownian motions involving the ancestors of the countably many particles alive at time $k2^{-n}$, i.e.,

$$D_l^{n,k} \equiv \max_{1\le i\le N^{b_l^{n,k}, k2^{-n}}} \max_{j\in\pi_i} \left| X_{L_j^{k2^{-n}}\left(b_{l+1}^{n,k}\right)}\left(b_{l+1}^{n,k}-\right) - X_i\left(b_l^{n,k}-\right)\right|, \tag{4.1.3}$$

where $\left\{\pi_i : 1 \le i \le N^{b_l^{n,k}, k2^{-n}}\right\}$ denotes the collection of all the disjoint blocks of partition $\Pi^{k2^{-n}}\left(k2^{-n} - b_l^{n,k}\right)$ ordered by their least elements.

By the lookdown construction and the coming down from infinity property, there exists a finite number of ancestors at each time $b_l^{n,k}$, $l = 0, 1, \ldots$ for those countably many particles alive at time $k2^{-n}$, i.e.,

$$\#\left\{L_j^{k2^{-n}}\left(b_l^{n,k}\right) : j \in [\infty]\right\} < \infty.$$

So both maximums in (4.1.3) are in fact taken over finite sets. Put

$$D^{n,k} \equiv \sum_{l=0}^{\infty} D_l^{n,k}.$$

For dimension d and constant α in Assumption I, let $C_1(d,\alpha)$ be a positive constant satisfying

$$C_1(d,\alpha) > \sqrt{2d\,(3/\alpha + 1)}.$$

Now we estimate the total maximal dislocation $D^{n,k}$ as follows. Let

$$I_n \equiv \mathbb{P}\left(\max_{1\le k\le 2^n T} D^{n,k} > \sum_{l=0}^{\infty} C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right).$$

Since $D^{n,k} = \sum_{l=0}^{\infty} D_l^{n,k}$, we have

$$\left\{ D^{n,k} > \sum_{l=0}^{\infty} C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right) \right\} \subseteq \bigcup_{l=0}^{\infty} \left\{ D_l^{n,k} > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right) \right\}.$$

Therefore,

$$I_n \leq \sum_{k=1}^{2^n T} \sum_{l=0}^{\infty} \mathbb{P}\left(D_l^{n,k} > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right) \right).$$

Under Assumption I, there exists a positive constant C such that for $\mathbf{N}$ large enough and for all $n > \mathbf{N}$, $\mathbb{E}T_{8^{n/\alpha}} \leq C8^{-n}$. For any $n > \mathbf{N}$, since $D_l^{n,k} = 0$ for l with interval length $\left|J_l^{n,k}\right| = 0$, we only need to consider the case of $\left|J_l^{n,k}\right| > 0$.

For $l = 0, 1, 2, \ldots$, the total number of Brownian motions over the subinterval $J_l^{n,k}$, which involve the finite ancestors of the countably many particles alive at $k2^{-n}$, is at most $8^{(n+l+1)/\alpha}$. Since $|J_0^{n,k}| \leq 2^{-n}$, we have

$$\mathbb{P}\left(D_0^{n,k} > C_1(d,\alpha)\, h\left(2^{-n}\right) \right) \leq 8^{\frac{n+1}{\alpha}} \mathbb{P}\left(\sup_{0 \leq s \leq 2^{-n}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-n}\right) \right).$$

For $l = 1, 2, \ldots$, we have

$$\begin{aligned} &\mathbb{P}\left(D_l^{n,k} > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right) \right) \\ \leq &\mathbb{P}\left(\left|J_l^{n,k}\right| > 2^{-(n+2l)} \right) + \mathbb{P}\left(D_l^{n,k} > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right), 0 < \left|J_l^{n,k}\right| \leq 2^{-(n+2l)} \right). \end{aligned}$$

Since $|J_l^{n,k}| \leq T^{n,k}_{2^{(3n+3l)/\alpha}}$, for any $n > \mathbf{N}$ the length of interval $J_l^{n,k}$ satisfies

$$\begin{aligned} \mathbb{P}\left(\left|J_l^{n,k}\right| > 2^{-(n+2l)} \right) \leq &\mathbb{P}\left(T^{n,k}_{2^{(3n+3l)/\alpha}} > 2^{-(n+2l)} \right) \\ \leq &2^{n+2l} \mathbb{E} T^{n,k}_{2^{(3n+3l)/\alpha}} \leq C2^{-(2n+l)}. \end{aligned}$$

We further have

$$\begin{aligned} &\mathbb{P}\left(D_l^{n,k} > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right) \right) \\ \leq &C2^{-(2n+l)} + 8^{\frac{n+l+1}{\alpha}} \mathbb{P}\left(\sup_{0 \leq s \leq 2^{-(n+2l)}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right) \right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
I_n \leq & 2^n T 8^{\frac{n+1}{\alpha}} \mathbb{P}\left(\sup_{0\leq s\leq 2^{-n}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-n}\right)\right) \\
& + 2^n T \sum_{l=1}^{\infty} \left(C 2^{-(2n+l)} + 8^{\frac{n+l+1}{\alpha}} \mathbb{P}\left(\sup_{0\leq s\leq 2^{-(n+2l)}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right)\right) \\
= & \sum_{l=1}^{\infty} CT 2^{-(n+l)} + 2^n T \sum_{l=0}^{\infty} 8^{\frac{n+l+1}{\alpha}} \mathbb{P}\left(\sup_{0\leq s\leq 2^{-(n+2l)}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right).
\end{aligned}
$$

It follows from Lemma 3.1 that

$$
\begin{aligned}
& \mathbb{P}\left(\sup_{0\leq s\leq 2^{-(n+2l)}} |\mathbf{B}(s)| > C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right)\right) \\
= & \mathbb{P}\left(\sup_{0\leq s\leq 2^{-(n+2l)}} |\mathbf{B}(s)| > C_1(d,\alpha)\sqrt{2^{-(n+2l)}(n+2l)\log 2}\right) \\
\leq & \frac{1}{C_1(d,\alpha)}\sqrt{\frac{8d^3}{\pi(n+2l)\log 2}} \exp\left(-\frac{C_1^2(d,\alpha)(n+2l)\log 2}{2d}\right) \\
\leq & \frac{1}{C_1(d,\alpha)}\sqrt{\frac{8d^3}{\pi\log 2}} 2^{-\frac{C_1^2(d,\alpha)(n+2l)}{2d}} \\
\equiv & C_2(d,\alpha) 2^{-\frac{C_1^2(d,\alpha)(n+2l)}{2d}}.
\end{aligned}
$$

Therefore, for any $n > \mathbf{N}$ we have

$$
\begin{aligned}
I_n & \leq CT2^{-n} + 2^n T \sum_{l=0}^{\infty} 8^{\frac{n+l+1}{\alpha}} C_2(d,\alpha) 2^{-\frac{C_1^2(d,\alpha)(n+2l)}{2d}} \\
& \leq CT2^{-n} + \sum_{l=0}^{\infty} TC_2(d,\alpha) 2^{-\left(\frac{C_1^2(d,\alpha)}{2d}-\frac{3}{\alpha}-1\right)n-\left(\frac{C_1^2(d,\alpha)}{d}-\frac{3}{\alpha}\right)l+\frac{3}{\alpha}}.
\end{aligned}
$$

Since $C_1(d,\alpha) > \sqrt{2d(3/\alpha+1)}$, there exists a positive constant

$$
C_3(d,\alpha) \equiv \sum_{l=0}^{\infty} C_2(d,\alpha) 2^{-\left(\frac{C_1^2(d,\alpha)}{d}-\frac{3}{\alpha}\right)l+\frac{3}{\alpha}}
$$

such that

$$
I_n \leq CT2^{-n} + TC_3(d,\alpha) 2^{-\left(\frac{C_1^2(d,\alpha)}{2d}-\frac{3}{\alpha}-1\right)n}. \tag{4.1.4}
$$

Both terms on the right hand side of (4.1.4) are summable with respect to n. Thus, $\sum_n I_n < \infty$, and it follows from the Borel-Cantelli lemma that $\mathbb{P}$-a.s.

$$\begin{aligned}\max_{1\le k\le 2^n T} D^{n,k} \le & \sum_{l=0}^{\infty} C_1(d,\alpha)\, h\left(2^{-(n+2l)}\right) \\ =& C_1(d,\alpha) \sum_{l=0}^{\infty} \sqrt{2^{-(n+2l)}(n+2l)\log 2} \\ \le& C_1(d,\alpha)\sqrt{2^{-n} n \log 2}\left(1+\sum_{l=1}^{\infty}\sqrt{2^{-2l+1} l}\right) \\ \equiv& C_4(d,\alpha)\sqrt{2^{-n} n\log 2}\end{aligned} \tag{4.1.5}$$

for n large enough.

By the lookdown construction and the arguments of Lemma 3.2, we have $H_{n,k} \le D^{n,k}$. Thus, $\mathbb{P}$-a.s.

$$\max_{1\le k\le 2^n T} H_{n,k} \le \max_{1\le k\le 2^n T} D^{n,k} \le C_4(d,\alpha) h\left(2^{-n}\right)$$

for n large enough. $\square$

The following Lemma follows from the lookdown construction for Λ-Fleming-Viot process.

Lemma 4.3 *For any r,t,s with $0 \le r \le t \le s$ we have*

$$H(r,s) \le H(r,t) + H(t,s)$$

with the convention $H(r,r) = H(s,s) \equiv 0$.

Theorem 4.4 *Under Assumption I and for any $T > 0$, there exist a positive random variable $\theta \equiv \theta(T,d,\alpha)$ and a constant $C \equiv C(d,\alpha)$ such that $\mathbb{P}$-a.s.*

$$\max_{\substack{r,s\in[0,T]\\ 0<s-r\le\theta}} H(r,s) \le C\sqrt{(s-r)\log\left(1/(s-r)\right)}. \tag{4.1.6}$$

Proof. We first show that

$$\max_{\substack{r,s\in S^T\\ 0<s-r\le\theta}} H(r,s) \le Ch(s-r). \tag{4.1.7}$$

By Lemma 4.2, given $T > 0$, there exist an event $\Omega_{T,d,\alpha}$ of probability one, and an integer-valued random variable $\mathbf{N}(T,d,\alpha)$ big enough such that $2^{-\mathbf{N}(T,d,\alpha)} \leq e^{-1}$ and

$$\max_{1\leq k\leq 2^n T} H_{n,k} \leq C_4(d,\alpha)h\left(2^{-n}\right), \quad n > \mathbf{N}(\omega,T,d,\alpha),\ \omega\in\Omega_{T,d,\alpha}. \tag{4.1.8}$$

Let $\theta \equiv \theta\left(\omega,T,d,\alpha\right) = 2^{-\mathbf{N}(\omega,T,d,\alpha)}$. For any $r,s\in S^T$ with $0 < s-r \leq 2^{-\mathbf{N}(\omega,T,d,\alpha)} = \theta$, there exists an $n \geq \mathbf{N}(\omega,T,d,\alpha)$ such that $2^{-(n+1)} < s-r \leq 2^{-n}$. Recall that

$$S_k^T = \left\{l2^{-k} : 0 \leq l \leq \lfloor 2^k T\rfloor\right\} \quad \text{and} \quad \overline{S^T} = \overline{\cup_{k\geq 1} S_k^T} = [0,T].$$

For any $k > n$, choose $s_k \in S_k^T$ such that $s_k \leq s$ and s_k is the largest such value. Then

$$s_k \uparrow s, \quad s_{k+1} = s_k + j_{k+1}2^{-(k+1)} \quad \text{with} \quad j_{k+1}\in\{0,1\}.$$

Since $s\in S^T$, then $(s_k)_{k>n}$ is a sequence with at most finite terms that are not equal to s. Applying (4.1.8), we have

$$H\left(s_k,s_{k+1}\right) \leq C_4(d,\alpha)j_{k+1}h\left(2^{-(k+1)}\right).$$

By Lemma 4.3,

$$\begin{aligned}
H\left(s_{n+1},s\right) \leq& \sum_{k=n+1}^{\infty} H\left(s_k,s_{k+1}\right)\\
\leq& \sum_{k=n+1}^{\infty} C_4(d,\alpha)j_{k+1}h\left(2^{-(k+1)}\right)\\
\leq& C_4(d,\alpha)\sum_{k=n+1}^{\infty}\sqrt{2^{-(k+1)}\left(k+1\right)\log 2}\\
=& C_4(d,\alpha)\sqrt{2^{-(n+1)}\left(n+1\right)\log 2}\sum_{k=n+1}^{\infty}\sqrt{2^{-(k-n)}\frac{k+1}{n+1}}\\
=& C_4(d,\alpha)\sqrt{2^{-(n+1)}\left(n+1\right)\log 2}\sum_{k=1}^{\infty}\sqrt{2^{-k}\left(1+\frac{k}{n+1}\right)}\\
\leq& C_4(d,\alpha)\sqrt{2^{-(n+1)}\left(n+1\right)\log 2}\sum_{k=1}^{\infty}\sqrt{2^{-k+1}k}\\
\equiv& C_5(d,\alpha)\sqrt{2^{-(n+1)}\left(n+1\right)\log 2},
\end{aligned} \tag{4.1.9}$$

where observe that only finitely many terms are nonzero in the summation on the right hand side of the first inequality.

For any $k > n$, choose $r_k \in S_k^T$ such that $r_k \geq r$ and r_k is the smallest such value. Then

$$r_k \downarrow r, \quad r_{k+1} = r_k - j_{k+1}^{'} 2^{-(k+1)} \quad \text{with} \quad {j'}_{k+1} \in \{0, 1\}.$$

Applying (4.1.8), we have

$$H\left(r_{k+1}, r_k\right) \leq C_4(d,\alpha) j_{k+1}^{'} h\left(2^{-(k+1)}\right).$$

Similar to (4.1.9), by Lemma 4.3 we have

$$\begin{aligned} H\left(r, r_{n+1}\right) \leq & \sum_{k=n+1}^{\infty} H\left(r_{k+1}, r_k\right) \\ \leq & \sum_{k=n+1}^{\infty} C_4(d,\alpha) j_{k+1}^{'} h\left(2^{-(k+1)}\right) \\ \leq & C_5(d,\alpha)\sqrt{2^{-(n+1)}\left(n+1\right)\log 2}. \end{aligned} \tag{4.1.10}$$

Since $2^{-(n+1)} < s - r \leq 2^{-n}$, we have $0 \leq s_{n+1} - r_{n+1} \leq i_{n+1} 2^{-(n+1)}$ with $i_{n+1} \in \{0, 1, 2\}$. Consequently,

$$\begin{aligned} H\left(r_{n+1}, s_{n+1}\right) \leq & 2C_4(d,\alpha) h\left(2^{-(n+1)}\right) \\ = & 2C_4(d,\alpha)\sqrt{2^{-(n+1)}\left(n+1\right)\log 2}. \end{aligned} \tag{4.1.11}$$

Combining (4.1.9), (4.1.10) and (4.1.11), we have

$$\begin{aligned} & \max_{\substack{r,s \in S^T \\ 0 < s - r \leq \theta}} H\left(r, s\right) \\ \leq & \max_{\substack{r,s \in S^T \\ 0 < s - r \leq \theta}} \left(H\left(r, r_{n+1}\right) + H\left(r_{n+1}, s_{n+1}\right) + H\left(s_{n+1}, s\right)\right) \\ \leq & \max_{\substack{r,s \in S^T \\ 0 < s - r \leq \theta}} \left(2C_4(d,\alpha)\sqrt{2^{-(n+1)}\left(n+1\right)\log 2} + 2C_5(d,\alpha)\sqrt{2^{-(n+1)}\left(n+1\right)\log 2}\right) \\ \leq & C(d,\alpha)\sqrt{2^{-(n+1)}\left(n+1\right)\log 2}, \end{aligned}$$

where $C(d,\alpha) \equiv 2C_4(d,\alpha) + 2C_5(d,\alpha)$.

Function h is increasing on $(0, e^{-1}]$. Since

$$2^{-(n+1)} < s - r \leq \theta \leq e^{-1},$$

we have

$$\max_{\substack{r,s \in S^T \\ 0<s-r\leq\theta}} H(r,s) \leq C(d,\alpha) h\left(2^{-(n+1)}\right) \leq C(d,\alpha) h(s-r). \tag{4.1.12}$$

Finally, for any $0 < r < s < T$ with $s - r < \theta/2$, find sequences $(r_m) \subseteq S^T$ and $(s_n) \subseteq S^T$ with $r_m \uparrow r$ and $s_n \downarrow s$. By the lookdown construction, for any $j \in [\infty]$,

$$\begin{aligned} &|X_j(s) - X_{L_j^s(r)}(r-)| \\ \leq &|X_j(s) - X_j(s_n)| + |X_j(s_n) - X_{L_j^{s_n}(r_m)}(r_m-)| \\ &+ |X_{L_j^{s_n}(r_m)}(r_m-) - X_{L_j^{s_n}(r)}(r-)| + |X_{L_j^{s_n}(r)}(r-) - X_{L_j^s(r)}(r-)|. \end{aligned} \tag{4.1.13}$$

Let both n and m be big enough such that $0 < s_n - r_m \leq \theta$. It follows from (4.1.12) that the second term on the right hand side of (4.1.13) is bounded from above by $C(d,\alpha)\, h(s_n - r_m)$. First fix n and let $m \to \infty$. The third term tends to 0 because $X_{L_j^{s_n}(\cdot)}(\cdot-)$ is continuous for any $j \in [\infty]$. Then letting $n \to \infty$, the first term tends to 0 because $X_j(\cdot)$ is right continuous for any $j \in [\infty]$. The last term is equal to 0 for large n since we could find s_n close enough to s such that there is no lookdown event involving levels $\{1, 2, \ldots, j\}$ during time interval $(s, s_n]$. Consequently,

$$\begin{aligned} &|X_j(s) - X_{L_j^s(r)}(r-)| \\ \leq &\lim_{n\to\infty} |X_j(s) - X_j(s_n)| + \lim_{n\to\infty}\lim_{m\to\infty} C(d,\alpha)\, h(s_n - r_m) \\ &+ \lim_{n\to\infty}\lim_{m\to\infty} |X_{L_j^{s_n}(r_m)}(r_m-) - X_{L_j^{s_n}(r)}(r-)| + \lim_{n\to\infty} |X_{L_j^{s_n}(r)}(r-) - X_{L_j^s(r)}(r-)| \\ = &C(d,\alpha)\, h(s-r). \end{aligned}$$

Then (4.1.6) follows. $\square$

4.2 Modulus of continuity for the Λ-Fleming-Viot support process at fixed time

In this section, we prove the one-sided modulus of continuity for the Λ-Fleming-Viot support process at any fixed time.

We will need the following observation on weak convergence.

Lemma 4.5 *If $\left\{(\nu_n)_{n\geq 1}, \nu\right\} \subseteq M_1\left(\mathbb{R}^d\right)$ and ν_n weakly converges to ν, then we have*

$$supp\,\nu \subseteq \cap_{m\geq 1}\overline{\cup_{n\geq m} supp\,\nu_n}.$$

Proof. Suppose that there exists an $x \in \mathbb{R}^d$ such that

$$x \in \operatorname{supp}\nu \cap \overline{\cup_{n\geq m}\operatorname{supp}\nu_n}^{\,c}$$

for some m. Since $\overline{\cup_{n\geq m}\operatorname{supp}\nu_n}^{\,c}$ is an open set, there exists a positive value δ such that $\{y : |y-x| < \delta\} \subseteq \overline{\cup_{n\geq m}\operatorname{supp}\nu_n}^{\,c}$. We can define a nonnegative and continuous function g satisfying $g > 0$ on $\{y : |y-x| < \delta/2\}$ and $g = 0$ on $\{y : |y-x| \geq \delta\}$. Then $\langle \nu_n, g\rangle = 0$ for any $n \geq m$ but $\langle \nu, g\rangle > 0$. Consequently, $\langle \nu_n, g\rangle \nrightarrow \langle \nu, g\rangle$, which contradicts the fact that ν_n weakly converges to ν. $\square$

Remark 4.6 *In Lemma 4.5, the complementary result*

$$supp\,\nu \supseteq \cap_{m\geq 1}\overline{\cup_{n\geq m} supp\,\nu_n}$$

is not always true. A counterexample is as follows. Let ν_n be a sequence of probability measures on $[0,1]$ with $\nu_n(\{0\}) = 1 - 1/n$ and density fuction $1/n$ on $(0,1]$. It is clear that $supp\,\nu_n = [0,1]$ and ν_n weakly converges to ν with $supp\,\nu = \{0\}$.

Theorem 4.7 *Under Assumption I and given any fixed $t \geq 0$, there exist a positive random variable $\theta \equiv \theta(t,d,\alpha)$ and a constant $C \equiv C(d,\alpha)$ such that for any Δt with $0 < \Delta t \leq \theta$ we have $\mathbb{P}$-a.s.*

$$supp\,X(t+\Delta t) \subseteq \mathbb{B}\left(supp\,X(t), C\sqrt{\Delta t \log(1/\Delta t)}\right). \tag{4.2.1}$$

Proof. Applying Theorem 4.4, there exist a positive random variable $\theta \equiv \theta(T, d, \alpha)$ and a constant $C \equiv C(d, \alpha)$ such that given any fixed $t \in [0, T)$, for any $r \in S^T \cap (t, t+\theta]$, we have $\mathbb{P}$ a.s.

$$H(t, r) \leq Ch(r - t),$$

which gives the upper bound for the maximal dislocation between the countably many particles at time r and their corresponding ancestor at time t. By Lemma 2.8, the ancestors at time t are exactly $\{X_1(t-), X_2(t-), \ldots, X_{N^{t,r}}(t-)\}$, so we have $\mathbb{P}$ a.s.

$$\{X_1(r), X_2(r), \ldots\} \subseteq \bigcup_{1 \leq i \leq N^{t,r}} \mathbb{B}(X_i(t-), Ch(r-t)).$$

For the given $t \in [0, T)$, $\mathbb{P}$ a.s.

$$X_j(t) = X_j(t-) \ \text{ for any } j \in [\infty],$$

where $X_j(0-) \equiv X_j(0)$, so for any $r \in S^T \cap (t, t+\theta]$, we have $\mathbb{P}$ a.s.

$$\{X_1(r), X_2(r), \ldots\} \subseteq \bigcup_{1 \leq i \leq N^{t,r}} \mathbb{B}(X_i(t), Ch(r-t)). \tag{4.2.2}$$

Apply Lemma 2.7, for the given $t \in [0, T)$, $\mathbb{P}$ a.s.

$$\{X_1(t), X_2(t), \ldots, X_{N^{t,r}}(t)\} \subseteq \operatorname{supp} X(t).$$

It follows from (4.2.2) that

$$\{X_1(r), X_2(r), \ldots\} \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(r-t)).$$

For all $r \in S^T \cap (t, t+\theta]$, we have $\mathbb{P}$-a.s.

$$X^{(n)}(r) \equiv \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(r)} \to X(r).$$

Clearly,

$$\operatorname{supp} X^{(n)}(r) \subseteq \{X_1(r), X_2(r), \ldots\} \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(r-t))$$

for all n, which implies

$$\operatorname{supp} X(r) \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(r-t)). \tag{4.2.3}$$

Then for any s satisfying $t < s \leq (t + \theta/2) \wedge T$, we can choose a sequence $(s_l)_{l\geq 1} \subseteq S^T \cap (t, t+\theta]$ such that $s_l \downarrow s$. By the right continuity property, we have $X(s_l) \to X(s)$ as $l \to \infty$. It follows from Lemma 4.5 that

$$\operatorname{supp} X(s) \subseteq \bigcap_{m\geq 1} \overline{\bigcup_{l\geq m} \operatorname{supp} X(s_l)}.$$

By (4.2.3), we have

$$\operatorname{supp} X(s_l) \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(s_l - t))$$

for all l. Consequently, for any $t < s \leq (t+\theta/2) \wedge T$,

$$\begin{aligned}
\operatorname{supp} X(s) &\subseteq \bigcap_{m\geq 1} \overline{\bigcup_{l\geq m} \mathbb{B}(\operatorname{supp} X(t), Ch(s_l - t))} \\
&= \bigcap_{m\geq 1} \mathbb{B}(\operatorname{supp} X(t), Ch(s_m - t)) \\
&= \mathbb{B}(\operatorname{supp} X(t), Ch(s-t)).
\end{aligned}$$

Therefore, given any fixed $t \geq 0$, there exist a positive random variable $\theta \equiv \theta(t, d, \alpha)$ and a constant $C \equiv C(d, \alpha)$ such that for any Δt with $0 < \Delta t \leq \theta$, $\mathbb{P}$-a.s.

$$\operatorname{supp} X(t + \Delta t) \subseteq \mathbb{B}(\operatorname{supp} X(t), Ch(\Delta t)) = \mathbb{B}\left(\operatorname{supp} X(t), C\sqrt{\Delta t \log(1/\Delta t)}\right).$$

□

Remark 4.8 *The constants $C \equiv C(d, \alpha)$ in Theorems 4.4 and 4.7 are the same. From the proofs of Lemma 4.2 and Theorems 4.4, 4.7, it is clear that*

$$\begin{aligned}
C(d,\alpha) &= 2C_4(d,\alpha) + 2C_5(d,\alpha) \\
&= 2C_4(d,\alpha) + 2C_4(d,\alpha)\sum_{k=1}^{\infty}\sqrt{2^{-k+1}k} \\
&= 2C_1(d,\alpha)\left(1 + \sum_{l=1}^{\infty}\sqrt{2^{-2l+1}l}\right)\left(1 + \sum_{k=1}^{\infty}\sqrt{2^{-k+1}k}\right),
\end{aligned}$$

where $C_1(d,\alpha)$ is any constant satisfying $C_1(d,\alpha) > \sqrt{2d(3/\alpha + 1)}$.

Remark 4.9 *It is well-known that Lévy's modulus of continuity theorem gives the result on the behavior of the modulus of continuity for Brownian motion as follows. For any* $0 < \delta < 1$, *let*

$$g(\delta) \equiv \sqrt{2\delta \log(1/\delta)}.$$

Given the standard Brownian motion $(\mathbf{B}(t))_{0 \leq t \leq 1}$, *we have*

$$\mathbb{P}\left(\limsup_{\delta \downarrow 0} \frac{1}{g(\delta)} \max_{\substack{0 \leq s < t \leq 1 \\ t-s \leq \delta}} |\mathbf{B}(t) - \mathbf{B}(s)| = 1\right) = 1.$$

In other words, with probability one, the sample path of Brownian motion have modulus of continuity with function $g(\delta)$ *for sufficiently small* $\delta > 0$. *We refer to Chapter 2 of Karatzas and Shreve (1998) for the proof.*

Further, we have the result on modulus of continuity for Brownian motion $(\mathbf{B}(t))_{t \geq 0}$ *at any fixed time* t *as follows:*

$$\mathbb{P}\left(\limsup_{\delta \downarrow 0} \frac{|\mathbf{B}(t+\delta) - \mathbf{B}(t)|}{\sqrt{2\delta \log\log(1/t)}} = 1\right) = 1.$$

Chapter 5

The uniform compactness and upper bounds on Hausdorff dimensions for the support and range of Λ-Fleming-Viot process

In this chapter, we first discuss the uniform compactness for the support and range of the Λ-Fleming-Viot process. Then we find a uniform upper bound on Hausdorff dimension for the support and an upper bound on Hausdorff dimension for the range. Finally, we introduce some corollaries and propositions related to other support properties for the Λ-Fleming-Viot process.

For any subset $\mathcal{I} \subset \mathbb{R} \cap [0, \infty)$, let

$$\mathcal{R}(\mathcal{I}) \equiv \overline{\cup_{t \in \mathcal{I}} \operatorname{supp} X(t)}$$

be the range of $\operatorname{supp} X$ on time interval $\mathcal{I}$.

5.1 Uniform compactness for the support and range of Λ-Fleming-Viot process

Given any Λ-Fleming-Viot process with the associated coalescent satisfying Assumption I, we prove the uniform compactness for its support and range in this section.

Theorem 5.1 *Under Assumption I, supp $X(t)$ is compact for all $t > 0$ $\mathbb{P}$-a.s.. Further, if supp $X(0)$ is compact, then $\mathcal{R}([0,t))$ is compact for all $t > 0$ $\mathbb{P}$-a.s..*

Proof. Under Assumption I, by Lemma 4.1 we have $\mathbb{P}$-a.s.

$$\max_{1\le k\le 2^n T} N_{n,k} < 8^{\frac{n}{\alpha}} \tag{5.1.1}$$

for n large enough.

Given any constants $0 < \sigma < T$, we first show that $\mathcal{R}([\sigma, T))$ is a.s. compact. Applying Theorem 4.4, there exist a positive random variable $\theta \equiv \theta(T, d, \alpha) > 0$ and a constant $C \equiv C(d,\alpha)$ such that $\mathbb{P}$-a.s.

$$\max_{\substack{r,s\in S^T\\ 0<s-r\le\theta}} H(r,s) \le Ch(s-r).$$

For the given σ, choose n big enough so that $2^{-n} \le \theta \wedge \sigma$ and (5.1.1) holds. For any $1 \le k \le 2^n T$ and $t \in S^T \cap [k2^{-n}, (k+1)2^{-n} \wedge T)$, we have

$$\begin{aligned} H\left((k-1)2^{-n}, t\right) &\le H\left((k-1)2^{-n}, k2^{-n}\right) + H\left(k2^{-n}, t\right) \\ &\le 2Ch\left(2^{-n}\right). \end{aligned}$$

It follows from the lookdown construction and Lemma 2.8 that

$$\operatorname{supp} X(t) \subseteq \bigcup_{1\le i\le N^{(k-1)2^{-n},t}} \mathbb{B}\left(X_i\left((k-1)2^{-n}-\right), 2Ch\left(2^{-n}\right)\right).$$

By (5.1.1) we have

$$N^{(k-1)2^{-n},t} \le N^{(k-1)2^{-n},k2^{-n}} = N_{n,k} < 8^{n/\alpha}.$$

Consequently,

$$\operatorname{supp} X\left(t\right) \subseteq \bigcup_{1\leq i<8^{n/\alpha}} \mathbb{B}\left(X_i\left(\left(k-1\right)2^{-n}-\right), 2Ch\left(2^{-n}\right)\right). \tag{5.1.2}$$

For general $t \in [k2^{-n}, (k+1)2^{-n} \wedge T)$. We can select a decreasing sequence

$$\left(t_l^{n,k}\right)_{l\geq 1} \subseteq S^T \cap [k2^{-n}, (k+1)2^{-n} \wedge T) \ \text{ satisfying } \ t_l^{n,k} \downarrow t.$$

Since the Λ-Fleming-Viot process X is right continuous, it follows from Lemma 4.5 that

$$\operatorname{supp} X\left(t\right) \subseteq \bigcap_{m\geq 1} \overline{\bigcup_{l\geq m} \operatorname{supp} X\left(t_l^{n,k}\right)}.$$

By (5.1.2), for any $t \in [k2^{-n}, (k+1)2^{-n} \wedge T)$, we have

$$\operatorname{supp} X\left(t\right) \subseteq \bigcup_{1\leq i<8^{n/\alpha}} \mathbb{B}\left(X_i\left(\left(k-1\right)2^{-n}-\right), 2Ch\left(2^{-n}\right)\right), \tag{5.1.3}$$

i.e., $\mathcal{R}\left([k2^{-n}, (k+1)2^{-n} \wedge T)\right)$ is contained in at most $\lfloor 8^{n/\alpha} \rfloor$ closed balls each of which has radius bounded from above by $2Ch\left(2^{-n}\right)$. Then

$$\begin{aligned}\mathcal{R}\left([\sigma, T)\right) &\subseteq \mathcal{R}\left(\left[2^{-n}, T\right)\right) \\ &\subseteq \bigcup_{1\leq k\leq 2^n T} \mathcal{R}\left(\left[k2^{-n}, \left(k+1\right)2^{-n} \wedge T\right)\right) \\ &\subseteq \bigcup_{1\leq k\leq 2^n T} \bigcup_{1\leq i<8^{n/\alpha}} \mathbb{B}\left(X_i\left(\left(k-1\right)2^{-n}-\right), 2Ch\left(2^{-n}\right)\right),\end{aligned} \tag{5.1.4}$$

where the right hand side is the union of finite closed balls. $\mathcal{R}\left([\sigma, T)\right)$ is contained in at most $\lfloor 2^n T \rfloor \times \lfloor 8^{n/\alpha} \rfloor$ closed and bounded balls. So $\mathcal{R}\left([\sigma, T)\right)$ is compact.

Consequently, the random measure $X(t)$ has compact support for all times $t \in [\sigma, T)$ simultaneously. Let $\sigma = 1/T$ and $T \to \infty$. Then the random measure $X\left(t\right)$ has compact support for all times $t \in (0, \infty)$ simultaneously.

Further, given that $\operatorname{supp} X(0)$ is compact, we can adapt the above-mentioned strategy to find a finite cover for $\mathcal{R}([0, T))$. Applying Theorem 4.7, for n large enough, we have

$$\mathcal{R}\left([0, 2^{-n})\right) = \overline{\bigcup_{t\in[0,2^{-n})} \operatorname{supp} X(t)} \subseteq \mathbb{B}\left(\operatorname{supp} X\left(0\right), Ch\left(2^{-n}\right)\right).$$

Then

$$
\begin{aligned}
&\mathcal{R}([0,T]) \\
&\subseteq \bigcup_{0\le k\le 2^nT} \mathcal{R}\left(\left[k2^{-n},(k+1)\,2^{-n}\wedge T\right)\right) \\
&\subseteq \mathbb{B}\left(\operatorname{supp} X(0), Ch\left(2^{-n}\right)\right) \bigcup \left(\bigcup_{1\le k\le 2^nT} \bigcup_{1\le i<8^{n/\alpha}} \mathbb{B}\left(X_i\left((k-1)\,2^{-n}-\right), 2Ch\left(2^{-n}\right)\right) \right),
\end{aligned}
$$

where the right hand side is compact given the compactness of $\operatorname{supp} X(0)$. So, $\mathcal{R}([0,T])$ is compact.

Note that $\mathcal{R}([0,T])$ is increasing with respect to T. Let $T\to\infty$. It is clear that $\mathcal{R}([0,t))$ is compact for all $t>0$ $\mathbb{P}$-a.s.. □

5.2 Upper bounds on Hausdorff dimensions for the support and range

In this section, we consider the upper bounds on Hausdorff dimensions for the support and range of the Λ-Fleming-Viot process under Condition A.

Given any Λ-coalescent $(\Pi(t))_{t\ge 0}$ with $\Pi(0)=\mathbf{0}_{[\infty]}$, recall that

$$T_m \equiv \inf\left\{t\ge 0 : \#\Pi(t)\le m\right\}$$

with the convention $\inf\emptyset=\infty$. $(\Pi_n(t))_{t\ge 0}$ is its restriction to $[n]$ with $\Pi_n(0)=\mathbf{0}_{[n]}$. For any $n\ge m$, we have

$$T_m^n \equiv \inf\left\{t\ge 0 : \#\Pi_n(t)\le m\right\}$$

with the convention $\inf\emptyset=\infty$.

For any $x>0$, write $T_x^n\equiv T_{\lfloor x\rfloor}^n$ and $T_x\equiv T_{\lfloor x\rfloor}$.

Let $(\hat{T}_n)_{n\ge 2}$ be independent random variables such that $\hat{T}_n$ has the same distribution as T_{n-1}^n.

Lemma 5.2 *For any $n > m$, T_m^n is stochastically less than $\sum_{i=m+1}^n \hat{T}_i$, i.e., for any $t > 0$,*

$$\mathbb{P}\left(T_m^n \geq t\right) \leq \mathbb{P}\left(\sum_{i=m+1}^{n} \hat{T}_i \geq t\right). \tag{5.2.1}$$

Proof. We use a coupling argument by defining an auxiliary $[n] \times [n]$-valued continuous time Markov chain (Y_1, Y_2) describing the following urn model. Intuitively, there are balls in an urn of color either white or black. Let $Y_1(t)$ and $Y_2(t)$ represent the number of white and black balls at time t, respectively.

After each independent exponential sampling time a random number of balls are taken out of the urn and then immediately replaced with certain white or black colored balls so that the total number of balls in the urn decreases exactly by one overall afterwards. More precisely, given that there are w white balls and b black balls in the urn, at rate $\lambda_{w+b,k}$ each group of k balls with $2 \leq k \leq w+b$ is independently removed. Suppose that w' white balls and $k - w'$ black balls have been chosen and removed at time t, we then immediately return $k-1$ balls to the urn so that among the returned balls, either one is white and all the others are black if $w' > 0$ or all of them are black if $w' = 0$. At such a sampling time t we define

$$\begin{cases} Y_1(t) = w - w' + 1 \ \text{ and } \ Y_2(t) = b + w' - 2 = w + b - 1 - Y_1(t), \ \text{ if } \ w' > 0; \\ Y_1(t) = w \ \text{ and } \ Y_2(t) = b - 1, \ \text{ if } \ w' = 0, \end{cases}$$

and the value of (Y_1, Y_2) keeps unchanged between the sampling times. The above-mentioned procedure continues until there is one white ball left in the urn. Suppose that there are n white balls and no black balls in the urn initially, i.e., $(Y_1(0), Y_2(0)) = (n, 0)$.

Observe that Y_1 follows the law of the Λ-coalescent starting with n-blocks and $(\hat{T}_i)_{i \leq n}$ has the same distribution as the inter-decreasing times for process $Y_1 + Y_2$. Plainly,

$$\inf\{t : Y_1(t) \leq m\} \leq \inf\{t : Y_1(t) + Y_2(t) \leq m\}.$$

Inequality (5.2.1) thus follows. □

The estimate in Lemma 4.1 is not enough to find the upper bound on the Hausdorff dimensions for the support and range. A sharper estimate is obtained in the following result under a stronger condition.

Lemma 5.3 *Suppose that Condition A holds. We have $\mathbb{P}$-a.s.*

$$\max_{1\le k\le 2^n T} N_{n,k} < 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}} \tag{5.2.2}$$

for n large enough.

Proof. Under Condition A, there exists a positive constant C such that for n large enough and for any $b > 2^{n/\alpha} n^{2/\alpha}$,

$$\lambda_b \ge (C\lfloor 2^{n/\alpha} n^{2/\alpha}\rfloor^{-\alpha})^{-1} > 2^{n+1} n. \tag{5.2.3}$$

Letting $n \to \infty$ in (5.2.1), for any $t > 0$ and $m \in [\infty]$ we have

$$\mathbb{P}\left(T_m \ge t\right) \le \mathbb{P}\left(\sum_{i>m} \hat{T}_i \ge t\right). \tag{5.2.4}$$

With estimate (5.2.4) we can find a sharper uniform upper bound for the maximal number of ancestors as follows:

$$\begin{aligned}
\mathbb{P}\left(\max_{1\le k\le 2^n T} N_{n,k} \ge 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right) \le& \mathbb{P}\left(\max_{1\le k\le 2^n T} T^{n,k}_{2^{n/\alpha} n^{2/\alpha}} \ge 2^{-n}\right) \\
\le& 2^n T \mathbb{P}\left(T_{2^{n/\alpha} n^{2/\alpha}} \ge 2^{-n}\right) \\
\le& 2^n T \mathbb{P}\left(\sum_{i>2^{n/\alpha} n^{2/\alpha}} \hat{T}_i \ge 2^{-n}\right) \\
\le& 2^n T e^{-n} \mathbb{E}\exp\left(\sum_{i>2^{n/\alpha} n^{2/\alpha}} 2^n n \hat{T}_i\right) \\
=& 2^n T e^{-n} \prod_{i>2^{n/\alpha} n^{2/\alpha}} \mathbb{E}\exp\left(2^n n \hat{T}_i\right),
\end{aligned}$$

where $\hat{T}_i$ follows an exponential distribution with parameter λ_i. It follows from (5.2.3) that when n is large enough, $\lambda_i > 2^n n$ for any $i > 2^{n/\alpha} n^{2/\alpha}$, which guarantees the existence of moment generating function for $\hat{T}_i$. As a result,

$$\mathbb{P}\left(\max_{1\le k\le 2^nT} N_{n,k} \ge 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right) \le 2^n T e^{-n} \prod_{i>2^{n/\alpha}n^{2/\alpha}} \frac{\lambda_i}{\lambda_i - n2^n}$$
$$\equiv 2^n T e^{-n} Q.$$

Then

$$\begin{aligned}\ln Q =& \sum_{i>2^{n/\alpha}n^{2/\alpha}} \ln\left(1+\frac{n2^n}{\lambda_i - n2^n}\right)\\ \le& \sum_{i>2^{n/\alpha}n^{2/\alpha}} \frac{n2^n}{\lambda_i - n2^n}\\ \le& n2^n \sum_{i>2^{n/\alpha}n^{2/\alpha}} \frac{1}{\lambda_i - \lambda_i/2}\\ \le& n2^{n+1} \sum_{i>2^{n/\alpha}n^{2/\alpha}} \frac{1}{\lambda_i}.\end{aligned}$$

We have by Condition A for n large enough,

$$\ln Q \le n2^{n+1} C\left(\lfloor 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\rfloor\right)^{-\alpha} \le n2^{n+1}C\left(2^{\frac{n}{\alpha}}n^{\frac{2}{\alpha}}/2\right)^{-\alpha} = 2^{\alpha+1}Cn^{-1}.$$

Then

$$\sum_n \mathbb{P}\left(\max_{1\le k\le 2^nT} N_{n,k} \ge 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right) < \infty,$$

which, by the Borel-Cantelli lemma, implies that $\mathbb{P}$-a.s.

$$\max_{1\le k\le 2^nT} N_{n,k} < 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}$$

for n large enough. □

Theorem 5.4 *Suppose that Condition A holds. Then*

$$dim\, supp\, X(t) \le 2/\alpha$$

for all $t > 0$ $\mathbb{P}$-a.s..

Proof. Given any $0 < \sigma < T$, we first consider the uniform upper bound on the Hausdorff dimension for $\operatorname{supp} X(t)$ at all times $t \in [\sigma, T)$. We adapt the same idea as the proof of Theorem 5.1 to find a cover for $\operatorname{supp} X(t)$ at any time $t \in [\sigma, T)$. Since we have a sharper estimate for $N_{n,k}$ under Condition A, for n large enough, (5.1.3) in the proof of Theorem 5.1 can be replaced by

$$\operatorname{supp} X(t) \subseteq \bigcup_{1 \leq i < 2^{n/\alpha} n^{2/\alpha}} \mathbb{B}\left(X_i\left((k-1)2^{-n}-\right), 2Ch\left(2^{-n}\right)\right)$$

for any $t \in [k2^{-n}, (k+1)2^{-n} \wedge T)$ and $1 \leq k \leq 2^n T$, i.e., for any $t \in [\sigma, T) \subseteq [2^{-n}, T)$, $\operatorname{supp} X(t)$ is contained in at most $\lfloor 2^{n/\alpha} n^{2/\alpha} \rfloor$ closed balls each of which has a radius bounded from above by $2Ch(2^{-n})$.

For any $\epsilon > 0$ we have

$$\begin{aligned}
\lim_{n\to\infty} \lfloor 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}} \rfloor \left(2Ch\left(2^{-n}\right)\right)^{\frac{2+\epsilon}{\alpha}} &\leq \lim_{n\to\infty} (2C)^{\frac{2+\epsilon}{\alpha}} 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}} \left(h\left(2^{-n}\right)\right)^{\frac{2+\epsilon}{\alpha}} \\
&= \lim_{n\to\infty} (2C)^{\frac{2+\epsilon}{\alpha}} (\log 2)^{\frac{2+\epsilon}{2\alpha}} 2^{-\frac{n\epsilon}{2\alpha}} n^{\frac{6+\epsilon}{2\alpha}} \\
&= 0.
\end{aligned}$$

Since ϵ is arbitrary, the Hausdorff dimension for $\operatorname{supp} X(t)$ is uniformly bounded from above by $2/\alpha$ at all times $t \in [\sigma, T)$.

Finally, let $\sigma \equiv 1/T$ and $T \to \infty$. The Hausdorff dimension for $\operatorname{supp} X(t)$ has uniform upper bound $2/\alpha$ at all positive times simultaneously. $\square$

Theorem 5.5 *Suppose that Condition A holds. Then for any $0 < \delta < T$,*

$$\dim \mathcal{R}([\delta, T)) \leq 2 + 2/\alpha \quad \mathbb{P}\text{-a.s.}.$$

Proof. Given any $0 < \delta < T$, we also follow the proof of Theorem 5.1 to find a finite cover for $\mathcal{R}([\delta, T))$. Choose n large enough such that $2^{-n} \leq \theta \wedge \delta$ and (5.2.2) holds. Similarly as (5.1.4) in the proof of Theorem 5.1, we have

$$\begin{aligned}
\mathcal{R}([\delta, T)) &\subseteq \mathcal{R}\left(\left[2^{-n}, T\right)\right) \\
&\subseteq \bigcup_{k=1}^{2^n T} \mathcal{R}\left(\left[k2^{-n}, (k+1)2^{-n} \wedge T\right)\right) \\
&\subseteq \bigcup_{1 \leq k \leq 2^n T} \bigcup_{1 \leq i < 2^{n/\alpha} n^{2/\alpha}} \mathbb{B}\left(X_i\left((k-1)2^{-n}-\right), 2Ch\left(2^{-n}\right)\right),
\end{aligned}$$

which implies that $\mathcal{R}([\delta, T))$ is contained in at most $\lfloor 2^n T \rfloor \times \lfloor 2^{n/\alpha} n^{2/\alpha} \rfloor$ closed balls, each of which has radius bounded from above by $2Ch(2^{-n})$.

For any $\epsilon > 0$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \lfloor 2^n T \rfloor \times \left\lfloor 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}} \right\rfloor \left(2Ch(2^{-n})\right)^{\frac{2}{\alpha}+2+\epsilon} \\
\leq & C(T, d, \alpha, \epsilon) \lim_{n\to\infty} 2^{-\frac{n\epsilon}{2}} n^{\frac{3}{\alpha}+1+\frac{\epsilon}{2}} = 0.
\end{aligned}$$

Since ϵ is arbitrary, the Hausdorff dimension for the range $\mathcal{R}([\delta, T))$ is bounded from above by $2/\alpha + 2$. □

5.3 Some Corollaries and Propositions

For $t > 0$, let

$$r(t) \equiv \inf \{R \geq 0 : \operatorname{supp} X(t) \subseteq \mathbb{B}(0, R)\}$$

and

$$S_t \equiv \cap_{n=1}^{\infty} \mathcal{R}([t, t + 1/n)).$$

It follows from Theorem 5.1 that $\mathcal{R}([t, t + 1/n))$ is compact for any $n \geq 1$ and $t > 0$.

Proposition 5.6 *Under Assumption I and for any $T > 0$, there exist a positive random variable $\theta \equiv \theta(T, d, \alpha) < 1$ and a constant $C \equiv C(d, \alpha)$ such that $\mathbb{P}$-a.s.*

$$supp\, X(t + \Delta t) \subseteq \mathbb{B}(S_t, Ch(\Delta t)) \tag{5.3.1}$$

for all $0 \leq t < t + \Delta t \leq T$ and $0 < \Delta t \leq \theta$.

Proof. Let $\{t_i\}$ be any dense subset of $[0, T]$. Combining the proofs for Theorem 4.4 and Theorem 4.7, there exist $\theta \equiv \theta(T, d, \alpha) < e^{-1}$ and $C \equiv C(d, \alpha)$ such that $\mathbb{P}$-a.s.

$$\operatorname{supp} X(t_i + \Delta t) \subseteq \mathbb{B}(\operatorname{supp} X(t_i), Ch(\Delta t))$$

for all i and $0 < \Delta t \leq \theta \wedge (T - t_i)$. Then for any $t \in [0, T)$, there exists a subsequence (t_{i_j}) with $t_{i_j} \downarrow t$ such that given n,

$$\begin{aligned}
\mathrm{supp}X(t+\Delta t) &= \mathrm{supp}X\left(t_{i_j} + \Delta t - \left(t_{i_j} - t\right)\right) \\
&\subseteq \mathbb{B}(\mathrm{supp}X(t_{i_j}), Ch(\Delta t)) \\
&\subseteq \mathbb{B}(\mathcal{R}\left([t, t+1/n)\right), Ch(\Delta t))
\end{aligned}$$

for $0 < \Delta t \leq \theta \wedge (T - t)$ and j large enough. To prove (5.3.1), we only need to show that for any $\delta > 0$,

$$\cap_{n=1}^{\infty} \mathbb{B}(\mathcal{R}\left([t, t+1/n)\right), \delta) \subseteq \mathbb{B}(S_t, \delta)$$

Without loss of generality, we assume that supp $X(0)$ is compact. Then $(\mathcal{R}([t, t+1/n)))_{n\geq 1}$ is decreasing and compact for any $t \in [0, T)$.

For any $x \in \cap_{n=1}^{\infty} \mathbb{B}(\mathcal{R}\left([t, t+1/n)\right), \delta)$, there exists $y_n \in \mathcal{R}\left([t, t+1/n)\right)$ such that $|x - y_n| \leq \delta$ for all $n \geq 1$. Since $\{y_n : n \geq 1\} \subseteq \mathcal{R}([t, t+1))$ which is compact, there exists a convergent subsequence $(y_{n_k})_{k\geq 1}$ of $(y_n)_{n\geq 1}$ such that y_{n_k} converges to y as $k \to \infty$. In addition,

$$|x - y| \leq \lim_{k\to\infty} \left(|x - y_{n_k}| + |y_{n_k} - y|\right) \leq \delta.$$

By the monotonicity and compactness of $(\mathcal{R}\left([t, t+1/n)\right))_{n\geq 1}$, it is clear that

$$y \in \cap_{n=1}^{\infty} \mathcal{R}\left([t, t+1/n)\right) \equiv S_t.$$

Consequently, we have $x \in \mathbb{B}(S_t, \delta)$. □

The next result is similar to Theorem 2.1 of Tribe (1989) on the support process of superBrownian motion; also see Theorem 9.3.2.3 of Dawson (1993). It follows immediately from Theorem 4.7.

Corollary 5.7 *Under Assumption I, there exists a constant $C > 0$ such that*

$$\mathbb{P}_{\delta_0}\left(\limsup_{t\downarrow 0} \frac{\sup_{0\leq u\leq t} r(u)}{\sqrt{t \log\left(1/t\right)}} \leq C\right) = 1.$$

Corollary 5.8 *Suppose that Condition A holds. For any $T > 0$, we have*

$$\mathbb{P}_{\delta_0}\left(dim\,\mathcal{R}\left([0,T]\right) \le 2 + 2/\alpha\right) = 1.$$

Proof. With initial value δ_0, applying Theorem 4.7, it is clear that almost surely

$$\mathcal{R}\left([0,2^{-n}]\right) \subseteq \mathbb{B}\left(0, Ch(2^{-n})\right)$$

for n large enough. From the proof of Theorem 5.5, we have

$$\begin{aligned}\mathcal{R}\left([0,T]\right) \subseteq & \mathcal{R}\left([0,2^{-n}]\right) \bigcup \mathcal{R}\left([2^{-n},T]\right) \\ \subseteq & \mathbb{B}\left(0, Ch(2^{-n})\right) \bigcup \bigcup_{1\le k\le 2^n T} \bigcup_{1\le i<2^{n/\alpha}n^{2/\alpha}} \mathbb{B}\left(X_i\left((k-1)\,2^{-n}-\right), 2Ch\left(2^{-n}\right)\right)\end{aligned}$$

for n large enough.

Therefore, $\mathcal{R}\left([0,T]\right)$ is contained in at most $\lfloor 2^n T\rfloor \times \lfloor 2^{n/\alpha} n^{2/\alpha}\rfloor + 1$ closed balls, each of which has radius bounded from above by $2Ch(2^{-n})$.

For any $\epsilon > 0$, we have

$$\lim_{n\to\infty}\left(\lfloor 2^n T\rfloor \times \left\lfloor 2^{\frac{n}{\alpha}} n^{\frac{2}{\alpha}}\right\rfloor + 1\right)\left(2Ch(2^{-n})\right)^{\frac{2}{\alpha}+2+\epsilon} = 0.$$

Since ϵ is arbitrary, the Hausdorff dimension for the range $\mathcal{R}\left([0,T]\right)$ is bounded from above by $2/\alpha + 2$. □

Lemma 5.9 (Falconer (1985) Lemma 6.3) *Let K be a compact subset of $\mathbb{R}^n$ with $\mathbf{\Lambda}^s(K) < \infty$, where $\mathbf{\Lambda}$ is defined by (3.0.1). Let μ be a mass distribution supported by K and let*

$$K_0 \equiv \left\{x \in K : \limsup_{r\to 0} \mu\left(\mathbb{B}\left(x,r\right)\right)/r^s = 0\right\}.$$

Then $\mu\left(K_0\right) = 0$.

The next result follows from Lemma 5.9 and the arguments of Theorem 5.4.

Proposition 5.10 *Suppose that Condition A holds. Then $\mathbb{P}$-a.s. for all $t > 0$ and $\epsilon > 0$ we have*

$$\limsup_{r\to 0+} \frac{X(t)(\mathbb{B}(x,r))}{r^{2/\alpha+\epsilon}} > 0$$

for $X(t)$ almost all x.

Proof. Theorem 5.4 implies that $\boldsymbol{\Lambda}^{2/\alpha+\epsilon}(\operatorname{supp} X(t)) = 0$ for any $\epsilon > 0$. Applying Lemma 5.9, if there exists some point $x_0 \in \operatorname{supp} X(t)$ such that

$$\limsup_{r\to 0} \mu\left(\mathbb{B}\left(x_0, r\right)\right)/r^{2/\alpha+\epsilon} = 0,$$

then $X(t)(\{x_0\}) = 0$ which contradicts $x_0 \in \operatorname{supp} X(t)$. Thus for any $x \in \operatorname{supp} X(t)$, we have

$$\limsup_{r\to 0+} \frac{X(t)(\mathbb{B}(x,r))}{r^{2/\alpha+\epsilon}} > 0.$$

$\square$

Proposition 5.11 *Let X be any Λ-Fleming-Viot process with $\Lambda(\{0\}) > 0$ and underlying Brownian motion in $\mathbb{R}^d$ for $d \geq 2$. Then given any fixed $t \geq 0$, with probability one the process $\operatorname{supp} X(t)$ has the one-sided modulus of continuity with respect to Ch, where $C \equiv C(d)$ is the constant determined in Theorem 4.7. Further, with probability one $\operatorname{supp} X(t)$ is compact for all $t > 0$ and if $\operatorname{supp} X(0)$ is compact, then $\mathcal{R}([0,t))$ is also compact for all $t > 0$. In addition, with probability one*

$$\dim \operatorname{supp} X(t) \leq 2$$

for all $t > 0$. Finally, given any $0 < \delta < T$, with probability one

$$\dim \mathcal{R}([\delta, T)) \leq 4.$$

Proof. Since $\Lambda(\{0\}) > 0$, the Λ-coalescent has a nontrivial Kingman component. Then

$$\lambda_b \geq \frac{\Lambda(\{0\})b(b-1)}{2}$$

and

$$\sum_{b=m+1}^{\infty} \frac{1}{\lambda_b} \leq \sum_{b=m+1}^{\infty} \frac{2}{\Lambda(\{0\})b(b-1)} = \frac{2}{\Lambda(\{0\})m},$$

i.e., Condition A holds with $\alpha = 1$. Therefore, the results follow from Lemma 2.10 and Theorems 4.7, 5.1, 5.4, 5.5. $\square$

Remark 5.12 *The uniform upper bound for the Hausdorff dimension of classical Fleming-Viot support process was first proved by Reimers (1993), where a non-standard construction of the classical Fleming-Viot process is used to establish this result.*

Proposition 5.13 *Let X be any Λ-Fleming-Viot process with underlying Brownian motion in $\mathbb{R}^d$ for $d \geq 2$. If the associated Λ-coalescent has the (c, ϵ, γ)-property, then given any fixed $t \geq 0$, with probability one the process $\mathrm{supp}\, X(t)$ has the one-sided modulus of continuity with respect to Ch, where $C \equiv C(d, \gamma)$ is the constant determined in Theorem 4.7. Further, with probability one $\mathrm{supp}\, X(t)$ is compact for all $t > 0$ and if $\mathrm{supp}\, X(0)$ is compact, then $\mathcal{R}([0, t))$ is also compact for all $t > 0$. In addition, with probability one*

$$\dim \mathrm{supp}\, X(t) \leq 2/\gamma$$

for all $t > 0$. Finally, given any $0 < \delta < T$, with probability one

$$\dim \mathcal{R}([\delta, T)) \leq 2 + 2/\gamma.$$

Proof. It follows from Lemma 3.9 that there exists a positive constant $C(c, \epsilon, \gamma)$ such that the total coalescence rate of the Λ-coalescent with the (c, ϵ, γ)-property satisfies

$$\lambda_n \geq C(c, \epsilon, \gamma) n^{1+\gamma}.$$

Then

$$\sum_{b=m+1}^{\infty} \frac{1}{\lambda_b} \leq \frac{1}{C(c, \epsilon, \gamma)} \int_m^{\infty} \frac{1}{x^{1+\gamma}} dx \leq \frac{1}{\gamma C(c, \epsilon, \gamma) m^{\gamma}},$$

i.e., Condition A holds with $\alpha = \gamma$. Consequently, the results follow from Lemma 2.10 and Theorems 4.7, 5.1, 5.4, 5.5. □

It is known that the Beta$(2 - \beta, \beta)$-coalescent stays infinite if $\beta \in (0, 1]$ and comes down from infinity if $\beta \in (1, 2)$. For $\beta \in (1, 2)$, given any $\epsilon \in (0, 1)$, the Beta$(2 - \beta, \beta)$-coalescent has the $(c, \epsilon, \beta - 1)$-property. Therefore, the conclusions of Proposition 5.13 hold with $\gamma = \beta - 1$.

Chapter 6

Future Research

We propose some topics for future research at the end of this thesis. Fleming-Viot processes and Dawson-Watanabe superprocesses are two fundamental classes of superprocesses. They have many similar properties. In the future, we want to generalize some of the available results on Dawson-Watanabe superprocesses to Fleming-Viot processes.

- For the class of Λ-Fleming-Viot processes $X \equiv (X(t))_{t\geq 0}$ in the thesis, we have already obtained the lower bound on Hausdorff dimension for $\operatorname{supp} X(T)$ at fixed $T > 0$ and the uniform upper bound on Hausdorff dimensions for $\operatorname{supp} X(t)$ at all $t > 0$. It seems that the upper bound is sharp. So, it would be interesting to find the exact Hausdorff dimension and the exact Hausdorff measure function for the support process at any fixed time, as well as the uniform lower bound on Hausdorff dimensions for $\operatorname{supp} X(t)$ at all $t > 0$.

- Now we assume that the Λ-Fleming-Viot process X with underlying Brownian motion starts at δ_0. For any $t \geq 0$, recall that

  $$r(t) \equiv \inf\{R \geq 0 : \operatorname{supp} X(t) \subseteq \mathbb{B}(0, R)\}$$

  is the maximal distance reached by the support of $X(t)$. Denote by

  $$R(t) \equiv \sup\left\{r(t^{'}) : 0 \leq t^{'} \leq t\right\}$$

the maximal distance reached by the support up to time t. In the future, we could apply Donnelly and Kurtz's lookdown construction for Fleming-Viot process to find the lower bound for $R(t)$ when t is small enough. Intuitively, the lower bound describes the minimum speed at which the support propagates away from its initial location 0. Similar results for superBrownian motion are obtained in Dhersin (1998).

- The so called support propagation happens for the superLévy process, which is the superprocess with underlying Lévy motion. See Evans and Perkins (1991) and Section 3.2 (Pages 200-207) of Perkins (1999). Intuitively, the support propagation means that the support of the superprocess would propagate instantaneously to any points to which the underlying spatial motion can jump. Throughout the thesis, we assume the Λ-Fleming-Viot process has underlying Brownian motion. In the future, we would like to know whether the support propagation also occurs for the Λ-Fleming-Viot process with underlying Lévy motion.

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