ON SOME MEASURE THEORY TEXTBOOKS AND THEIR USE BY SOME PROFESSORS IN GRADUATE-LEVEL COURSES

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A thesis
in
The Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
For the Degree of Master of Mathematics
Concordia University
Montréal, Québec, Canada

September 2013

Concordia University

School of Graduate Studies

This is to certify that the thesis prepared

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Entitled: **On some measure theory textbooks and their use by some professors in graduate-level courses**

and submitted in partial fulfillment of the requirements for the degree of

Master of Arts (Mathematics)

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Abstract

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Felix Sidokhine

Previous research has reported on students' uses of mathematics textbooks at pre-university and undergraduate levels. Also, some research has been done on textbooks as standalone objects, looking at their format and didactic and mathematical discourses. However, very few researchers have investigated instructors' uses of textbooks. In this thesis we do so at the graduate level; in particular, we investigate one instructor's account of his use of a textbook and two mathematics professors' views of which textbook they would use and how if they were to teach a Measure Theory course. We chose measure theory because of its important role as a foundation for much of what is modern analysis, a branch of mathematics with many applications such as electronics, signal processing and even probability and statistics. We chose the graduate level because textbooks we assume them to have an important role in the teaching and learning of graduate mathematics courses; this is based on our own classroom experiences as little research has been done so far in this direction. In the first part of our research, we draw on Eco's notion of model reader and characterized the target instructor audience of four measure theory textbooks. We also analyzed the mathematical knowledge that these textbooks contain in light of a review of the history of the development of measure theory. Finally, we analyzed the textbooks as teaching instruments from the perspective of Sierpinska's notion of apodictic vs. liberal textbooks. In the second part of our research, we interviewed three university professors in order to understand their beliefs about textbooks, mathematics and learning. In particular, we sketched different types of instructors' profiles and have been able to match them with textbooks whose use in their teaching activity would likely be most effective.

Acknowledgments

I would like to thank Dr. Nadia Hardy for her patience, help and advice during the writing of this thesis. I would also like to thank Dr. Anna Sierpinska for her comments, suggestions and guidance during the preparation of the final submission.

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Introduction

The motivation for this research comes primarily from personal experience as a student of measure theory, where it felt like measure theory textbooks were not being used to their full potential and the supplementary textbooks (textbooks recommended as supplementary reading in the course outline) seemed very different from the required textbook. This led us to wonder about the relationship between (measure theory) textbooks and instructors. Since measure theory is a relatively large topic our research emphasizes on advanced topics in measure theory in the context of Lebesgue measure on real spaces. Moreover, due to the lack of existing tools and research this thesis is more of an exploratory study rather than an in-depth treatise of a unique aspect of measure theory textbooks. When looking at textbooks as standalone objects, we wanted to understand in what sense these textbooks are different from one another and what these differences mean in terms of the knowledge they foster. We also wanted to understand what professors' beliefs about the use of textbooks are and how they use textbooks at the graduate level.

Based on these considerations, we formulated the following two research questions that we address in this thesis: (1) how do the mathematical content and didactic organization of measure theory textbooks differ from one another? And (2) what are professors' beliefs about the use of textbooks? In analysing the collected data, we found that professors' beliefs about mathematics and about learning play an important role in the use they envision for textbooks. We hypothesize that these beliefs and the use they envision for textbooks in the teaching and learning activity puts them in correspondence with *instructor model readers* (the reader for whom the author wrote the text; Eco, 1979) of particular textbooks.

We addressed the question of mathematical content and didactic organization by studying the contents of four measure theory textbooks. We looked at the content and how each textbook constructed measure theory. For example, we looked whether they followed the historical development of the topic or chose an alternative path such as a purely axiomatic approach via Caratheodory's definition. When possible we tried to find biographic information about the textbooks' authors to understand to what mathematical school they belonged and how they used measure theory, hence allowing us to infer about the reasons behind their choice of a particular approach. Finally we examined the problem sets and tried to understand what their role is in a given textbook; for example some problems filled in blanks left in the theory sections of a chapter, others were designed to test a reader's ability to reproduce arguments from previously seen proofs.

Based on the information we collected from the textbooks we reconstructed the *instructor model reader*, and classify the textbooks as *open* or *closed* (addressing the possible interpretations the textbook offers with respect to a topic; cf. Eco, 1979) and as *liberal* or *apodictic* (addressing the level of freedom a textbook affords an instructor who uses it to prepare his or her lectures; cf. Sierpinska, 1997).

Our analysis of textbooks showed us that different textbooks can have radically different instructor model readers. This in turn imposes certain constraints on how such textbooks can be used in the teaching practice. Depending on what beliefs instructors have about teaching activities and the general mathematical content and the model reader of a textbook, the text can either complement an instructor's teaching practice or conflict with it.

The second research question was addressed by creating a semi-structured interview lasting between 30 and 45 minutes. The interviews contained two parts. In the first part, the questions were designed to reveal the instructor's beliefs and intentions when using measure theory textbooks for planning and preparing his or her lessons. The second part was designed to reveal the instructor's beliefs about mathematics and about learning. The results of the interviews allowed us to reconstruct the empirical reader the instructor is when he or she reads a measure theory textbook. We then used this information to verify our hypothesis about instructors and model readers by comparing the participant's choice of a textbook as empirical readers and the textbook whose model reader was closest to such empirical reader.

Besides our belief that the above research questions are intrinsically interesting in the context of mathematics education, we argue that from a pragmatic point of view they underline the need for instructors to actively question and evaluate their use of textbooks, which play a central role in the learning process at the graduate level. In particular, our results suggest that instructors could better exploit their use of textbooks for teaching if they are aware of their teaching style and beliefs about mathematics and about learning, and understand how these relate to the mathematical and didactic organization of textbooks.

This thesis is structured as follows. In the next section (1.1) we present some general information about textbooks and organization of a graduate-level measure theory course in various Canadian institutions. In chapter 2 we present a literature review to position our research with respect to existing treatises. Then we present a history of the development of measure theory (chapter 3) which became a useful tool when trying to understand the different approaches to the subject exhibited by various textbooks. This is followed by a presentation of the data collected from textbooks, its classification as open or closed and as liberal or apodictic, and our extrapolation of the model reader (chapter 4). In chapter 5, we present the data we collected from the interviews with three mathematics professors and present our findings in terms of empirical readers whom we contrast against the model readers identified in chapter 4. Finally, we present and discuss our conclusions and propose venues for future research (chapter 6). Transcripts of the interviews appear in the appendix.

1.1 Educational Context

In this research, we examined the beliefs that university professors may have about textbooks used in a measure theory course offered at the graduate level. Three tenured professors from a university that I will call University U were interviewed; their views are analyzed and discussed in chapter 5. In this section, I describe the educational context at University U with respect to the measure theory course.

At University U there is academic freedom, thus, professors can, if they want, prepare the outline of the course they have to teach. Available data, however, shows that the outline of the measure theory course has remained mostly fixed in the last 4 years (if not longer); this includes list of topics, assessment style, and textbook. This may be due to an academic consideration or social norms. The academic consideration may be that a consensus has been reached within the institution as to what knowledge a student needs to acquire about measure theory in order to be able to follow future courses given at University U. It may be, however, that a professor newly appointed to teach measure theory believes that the outline he or she receives from his or her predecessor is the norm and hesitates to change it; relinquishing academic freedom to abide the institutional norms.

Other "logistic factors" may also play a role; for example, the university has various agreements with different publishers which all carry different texts in their portfolios and hence is limited in its choice of texts. Also, the choice of a text may be subject to considering which ones are available in the university's library, price, purchasing availability, etc.

The outline of the measure theory course offered at University U expects the professor to cover Lebesgue measure and integral on the real line, Lebesgue spaces (L^p spaces) and an introduction to abstract measure. The document does not contain specifics about the material that makes up the topics.

1.1.1 The students

The expected audience are "advanced undergraduate" and graduate students. The university's calendar states that an undergraduate student is eligible to take the course upon successful completion of the courses: Advanced Calculus, Analysis II and Real Analysis. Hence a typical attendee of lectures in measure theory generally possesses a strong background in calculus, analysis and topology. According to the calendar, we can also infer that it is very likely the undergraduate audience has previously attended courses in complex analysis and group theory.

Assuming that the graduate student previously completed all the courses required for a B.Sc. in mathematics, according to the calendar, we can infer that the student possesses the same knowledge as the advanced undergraduate but have had the time to "maturate" his or her knowledge; the student might also have a larger background in abstract algebra, more experience with proofs and formal mathematics.

1.1.2 The textbook

The outline of the course specifies an official textbook and recommended texts. At University U, the textbook is *Real Analysis* by H.L. Royden and it has been this one for at least the last 4 years (available data shows that this was also the text 6 years ago). As additional resources, the following textbooks are listed: *Real Analysis, Measure Theory and Integration*, by E.M. Stein and R. Shakarchi and *Functional Analysis*, by E.M. Stein and R. Shakarchi. All these textbooks are available at the university's library.

The outline does not state an expected use of the textbook by either the professor or the student. Anecdotal data gathered from students who took the course at University U and a look at the outlines of some of the other courses offered by the department suggest that professors expect students to read the material that will be covered prior to the lectures and use it as a reference text when working on problems.

1.1.3 The course at other universities

An examination of outlines from other four comparable institutions suggests that the content offered at University U is standard. The core of the measure theory course in these other institutions is Lebesgue measure and integral on the real line, Lebesgue (L^p) spaces and an introduction to abstract measure. The following table 1 shows the texts found in each of the outlines:

Institution	Required Text	Supplementary Text
University U	*Real Analysis*, 3rd edition, H.L. Royden	*Real Analysis, Measure Theory and Integration*, E.M Stein and R. Shakarchi
University A	*Real Analysis, Measure Theory and Integration, and Hilbert Spaces*, E.M Stein and R. Shakarchi	N/A
University B	*Real Analysis*, 4th edition, H.L. Royden	N/A
University C	*Measure Theory and Integration (Graduate Studies in Mathematics)*, Michael E. Taylor	Various including: *Measure and Integral: An introduction to Real Analysis* by Wheeden and Zygmund. Not including: H.L. Royden
University D	N/A	Various including: *Real Analysis*, H.L. Royden (4th Edition)

Table 1: Measure theory texts in different Canadian universities

From this table we can see that *Real Analysis* by H.L. Royden is a generally accepted text within four out of five universities, with only its role changing from required to supplementary text.

We chose to include *Real Analysis* by H.L. Royden in our analysis due to its popularity in the academic community; on the same grounds we included *Measure and Integral: An Introduction to Real Analysis* by Antoni Zygmund. Our decision to include the other two textbooks (*Elements of the Theory of Functions and Functional Analysis* by A.N. Kolmogorov and *Real and Complex Analysis* by Walter Rudin) was motivated by informal discussions we had with other members of our mathematics department who expressed their belief that Kolmogorov's text is one of the "classical treatises of measure theory (and analysis in general), whereas Rudin's has an established reputation among scholars to be a very creative and graduate-student oriented approach to measure theory.

Chapter 2

Literature Review

In order to situate this thesis in the landscape of mathematics education, we consulted the scarce previous research on mathematics textbooks. Rezat (2006) published an article of expository character where he gave a short overview of the research done on textbooks (at the high-school,college and undergraduate levels) and proposed a model of textbook use. He focused on describing the various dimensions of textbook-use in the educational process through a three dimensional polygon whose vertices are the student, the instructor, the textbook and mathematical knowledge. Each side of this polygon is a triangle isolating a particular interaction, for example "student - textbook - mathematical knowledge".

Rezat's model is a convenient symbolic and visual representation of the elements one finds in the learning and teaching processes. In his model textbooks are described as artefacts due to their nature in the learning process; since textbooks are man-made tools and are not direct products of the teaching and learning activities. Rezat's model is also practical when trying to understand other research and the position this thesis with respect to them. Prior research on textbooks was focused on the "student - textbook - mathematical knowledge" (e.g., Mesa, 2004; Raman, 2004; Weinberg, 2009; Lithner, 2004) and "textbook - instructor - student" (e.g., Sierpinska, 1997) triangles, whereas this thesis is focused on the "instructor - textbook - mathematical knowledge" triangle.

Rezat also assumes based on prior-research that instructors have a very specific way of using textbooks in their teaching activity: to prepare lectures and assign readings and assignments. In this thesis, we try to understand /textithow instructors use textbooks; that is to say, if instructors use textbooks to prepare their lectures, how do they do so, which parts of the textbook they choose, do they use the didactic discourse of the text, etc., and if they use the textbook to assign homework, which problems do they choose and why.

2.1 Rezat's Model

Before over-viewing prior research, we explain how the triangles Rezat proposed should be interpreted. For example the "student - textbook - mathematical knowledge" triangle represents the textbook's mediation of the interactions between the student and mathematical knowledge. In the same way the triangle "student - instructor- textbook" represents the instructor's mediation of the interactions between the student and the textbook.

For example, let us take the "student - instructor - textbook" triangle. By "mediation of the interactions" we mean that the instructor guides (explicitly or implicitly) a student's relationship with the textbook. Examples of such "mediation" would be assigning particular pages to read or selected problems, or even telling the student how to read and what to look for in his or her reading.

In this thesis we are looking the "instructor - textbook - mathematical knowledge" dimension where the textbook mediates the interactions between the instructor and mathematical knowledge. A textbook mediates these interactions in a very concrete way: it forces the instructor to adopt a specific point of view (i.e., definitions, interpretations, etc.) towards a piece of mathematical knowledge (for example fractions), which may or not be consistent with his personal mathematical beliefs. In the case of fractions, an instructor may have a mathematical belief that they are the field of fractions generated by a ring, whereas the textbook is forcing him to interpret them as portions of a pizza.

This is not equivalent to the "instructor - textbook - student" triangle, where the textbook is mediating the interactions between the instructor and the student. In the case of high-school, college and undergraduate studies, Sierpinska (1997), Keitel (as cited in Rezat, 2006) and McNamara et al. (as cited in Weinberg & Wiesner, 2010) have found that the textbook may be shaping the instructor-student interactions not only in terms of mathematical knowledge but also pedagogical practices (such as classroom activities) and reading strategies instructors pass on to students.

Our research is not addressing the "instructor - textbook - student" triangle for several reasons. First, as noted by McNamara (as cited in Weinberg and Wiesner, 2010) classroom practices are very controlled and standardized at the high-school, college and even undergraduate levels by more than textbooks, which is not the case at the graduate level. For example, Mesa (2004) refers to the existence of "teacher's guides' " for high school and college level, which are not mathematics textbooks but rather books that tell instructors how to perform their entire teaching activity (almost to the point of scripting an entire class like a theatre play). Such instructor's guides do not exist in graduate level literature. Secondly, the institutional constraints greatly differ between graduate level courses and lower level courses. For example, at the graduate level instructors have freedom in choosing textbooks, whereas at the high-school this choice is made (at least in Quebec) by the Ministry of Education, and at the college and undergraduate level this choice often depends on the multi-section nature of the courses and is the outcome of an agreement between several parties.

2.2 Textbooks and students

Much of the research previously cited studied exclusively the interactions between textbooks and students. Sometimes, this research did not capture an entire triangle described by Rezat, but rather a very particular sub-activity in a triangle (such would be the case of research focused on how students perform tasks found in textbooks (e.g., Mesa, 2004; Lithner, 2000; Lithner 2004).

2.2.1 Students as readers

Weinberg and Weisner (2010) studied K-12 students' interactions with calculus textbooks through reader-oriented theory. Their method mirrors our use of Eco's and Sierpinska's models as Weinberg and Weisner propose the notions of intended reader, implied reader and empirical reader. The intended reader being the reader that is foreseen by the author, the implied reader being the embodiment competencies, codes and behaviours required from a reader in order to be able to respond in a way that is meaningful and accurate relative to the response the author sought to evoke, and the empirical reader being any reader that picks up the textbook.

In our research we merged the intended and implied reader together into the model reader for the main reason that the intended reader described by Weinberg and Weisner is redundant in the context of graduate-level courses. Both graduate students and instructors have a much greater mathematical maturity than their college or undergraduate counterparts, and are highly proficient in terms of understanding symbols and mathematical language.

Weinberg and Weisner identified two reader models: text-centered and reader-centered. In the case of the text-centered model, a reader is not dealing with the content generatively (i.e., he is not trying to construct knowledge) but is scanning for particular words, symbols, and examples. An example of text-centered reading would be a student preparing for a quiz about limits of rational functions who, when reading, skips the theory section that explains why certain operations are valid and jumps directly to concrete examples without ever returning to the theory. In this sense, the student is not generating mathematical knowledge as he or she skips to the procedural aspect of the material without knowing why the procedure is correct or how the procedure was derived. The reader-centered model is not the opposite of text-centered but rather a complement of it where the students tries to construct mathematical knowledge and is not simply scanning for particular words or examples in his reading.

Weinberg and Weisner concluded that the success of a students' reading depends on the degree of closeness between the intended and the empirical reader as well as the adoption of reader-centered reading models by students. They identified that instructors often endorse text-centered reading models by emphasizing "specialized vocabulary and symbols, frequent graphical representations, condensed syntax and the overall layout of texts". (Weinberg and Weisner, 2010, pg. 60)

The last element regarding the "overall layout of texts" furthers the cause for research about the interactions between textbooks and instructors. Particularly, in graduate studies, the goal is to engage students in critical thinking and creative use of the material, hence instructors should be endorsing a reader-centered reading model. However, as we can see from Weinberg and Weisner's research, textbooks do have an influence in the possibility of endorsing a reader-centered reading, furthering the need for research on textbooks and graduate level courses.

Weinberg (2009) studied "textbook tensions", a term he used to describe the challenges students face when working with textbooks. He identified that undergraduate students face tensions between understanding and examples as well as tensions between "colloquial and technical language". Tensions between understanding and examples are found in situations when students confuse their ability to successfully reproduce examples found in textbooks (as a procedural activity) with understanding the theory. For example they would make the false assumption that their ability of taking the derivatives of polynomial and transcendental functions implies that they have gained a conceptual understanding.

The tensions between "colloquial and technical language" occurs when authors use colloquial language in a mathematical context without explicitly defining it. Examples of such colloquial language found in calculus textbooks would be references to words such as "breaks", "jumps" and "holes". Using reader-response criticism Weinberg linked the tensions students experience to the fact that these students were not acting as implied readers of certain textbooks. An example of such a tension occurs when students adopt a text-centered reading model whereas the textbook expects him[1] to adopt a reader-centered reading model.

In this thesis we did not address this issue as we assumed that graduate students and instructors know the connections between colloquial and technical language.

2.2.2 Students and problem-solving

The problem-solving activity has received extensive coverage in mathematics education (see Lithner, 2000; Mesa, 2004).

Lithner (2000) studied the relationship between mathematical reasoning and task solving in the context of undergraduate mathematics. Brousseau (as cited by Lithner, 2000) identified that the most important distinction between school mathematics and professional use of mathematics is founded by the didactic contract where the student does not always have to be certain that his or her result is correct.

Brousseau's argument is likely to have interesting implications in the context of graduate-level mathematics, where students are expected to behave as professional users of mathematics, yet are still subject to a didactic contract (e.g., written examinations and assignments), which justifies the need for research about graduate level didactic practices and textbooks.

In particular, Lithner found that when students engage in problem-solving their activity is dominated by plausible reasoning and established experiences.

According to Lithner, plausible reasoning is met if it is founded on the mathematical properties of the components involved in the reasoning and is meant to guide towards what is probably the truth. An example of plausible reasoning is proof by a special case, such as a maximization problem where one may choose to use derivatives to find a maximal point without formally proving his ideas (i.e., justify why finding the point at which the derivative is equal to 0 is a maximal point) but will use the property of the mathematical object as the reason for his choice (i.e., will explicitly state that at the top the slope is 0).

Established experiences reasoning happens when one is founding his or her arguments on notions and procedures on his or her past experiences in the learning environment and is meant to guide towards what is probably the truth. An example of established experience reasoning would be automatically taking derivatives to find maximums of a function without appealing to the mathematical properties but by making a statement such as "that is how this was done in class for such problems".

[1] In this thesis, the use of masculine pronouns is a generic sense to refer to a person regardless of that person's gender.

Lithner found that students will often begin their problem-solving activity by appealing to established experiences before engaging in plausible reasoning. He found that this approach is problematic because many problems require students to engage in plausible reasoning (i.e., discuss mathematical properties of objects found in a problem) before engaging into an established experiences reasoning (during which they may neglect the mathematical properties of the objects in the problem). He identified that often students founded their control mechanism (i.e., self-assessing if a problem is solved correctly) on familiarity rather than on mathematical properties and that students did not perceive plausible reasoning as a main tool when working on mathematical tasks.

Lithner's analysis leads to wonder if textbooks are responsible for the development of plausible reasoning and established experiences. In particular, we would assume that a fair amount of established experiences are created in the classroom and while reading textbooks (as supported by Rezat (2006)). Furthermore, classroom experiences are mediated by an instructor, who in turn, according to Rezat, bases his or her lectures and examples on material found in textbooks. Then, textbooks might play a relevant role in shaping students reasoning and problem solving strategies, and hence their relationship with instructors deserves attention.

2.3 Instructors, students and problem-solving

Sierpinska (1997) studied the "instructor- textbook - student" relationship in the context of a college-level linear algebra course. Unfortunately for us, her instructors were tutors and not instructors. Her research gives insight on how textbooks mediate the interactions between students and tutors in the context of the problem-solving activity.

2.3.1 Formatting

Sierpinska's paper used the notion of *formatting*, a concept introduced by Jerome S. Bruner (1915). However, Bruner's research focused on child psychology and learning processes (see for example Bruner & Ratner, 1977). Bruner described formats as a regulating mechanism for the exchanges between the student and the instructor (Bruner, 1990).

In her research on textbooks, Sierpinska identified that textbook - student interactions were subject to two types of formatting: formatting of interpretation and formatting of use. Formats of interpretation are aimed at shaping a student's interpretation of a particular piece of knowledge. In textbooks common strategies used to achieve such goal can be as simple as omission of other possible interpretations or a written paragraph which describes other possible interpretations, which is concluded with a "do not use these" clause.

Formatting of use is shaping how a textbook should be used. Strategies to achieve this goal are found typically in sections in the preface of the textbook which tell the reader how he is expected to use the textbook. We adapted these considerations to instructors without any major modifications (except for the case of "formatting interpretations", since instructors often have a very solid knowledge of the topic in the textbook and their personal interpretations can hardly be formatted).

Sieprinska used Eco's model reader and open/closed frameworks in her research, as well as her own model of apodictic and liberal textbooks. Her article focused on understanding how formatting was a mechanism for stabilizing (negotiating) the meaning of mathematical statements between students, tutors and textbooks.

For example, one of tutors (PI) considered that learning methods and techniques instead of definitions and verifying properties of objects was more important to the learning process, whereas another tutor (PIII) was putting substantial effort into negotiating meanings of definitions and theorems. The interaction between PIII and her student was dominated by argumentation and verification of hypothesis, whereas the interactions of PI were dominated by oral revisions of definitions and techniques.

Sieprinska then compared the formatting done by the textbook (i.e., the textbook itself) and the formatting done by the use of the textbook (i.e., the teaching practice of the tutor) only to establish a paradoxical consequence: the textbook does not reach its didactic goal if it is followed to the letter. This was best seen when two students worked on a traffic problem: Sandy 3 (a student of PIII) did the part that was relevant to the text's didactic goal (he modeled the problem and put the conditions in matrix format) whereas Endy 1 (a student of PI) simply reproduced the arguments he found in the textbook.

This underlines the need for more research about textbooks and instructors since as we can see from Sierpinska's research, the same textbook in the hands of different tutors and students resulted in a very different problem-solving activity. At the graduate-level this is an important moment because students are expected to exhibit the behaviour of Sandy 3, but this behaviour was attained not through textbook-use exclusively but also by an intervention of PIII by steering Sandy 3 into a successful reading possibly more investigative than the contents of the textbook itself.

2.4 Conclusions

Researchers seem to agree that textbooks have a fairly large influence on learning and teaching practices at all level of education. However, only few studies looked at instructors' uses of textbooks and none (to our knowledge) considered the graduate level as we do in this thesis.

Weinberg and Weisner (2010) showed that the successful use of textbooks by students depends on the degree of closeness between the implied reader and the empirical reader (the student). They also identified that instructors are playing a role in guiding students' adoption of a particular reading-model (text-centered vs. reader-centered). For us, this justifies the need to have an in-depth study of the relationship between instructors and textbooks.

Sierpinska's (1997) article makes the best case for this thesis. Her findings that two different tutors using the same textbooks relayed completely different messages to their students reinforces the need for more research on textbooks and instructor uses.

Chapter 3

History

In this chapter, we try to understand the origins and evolution of the mathematical content found in modern measure theory textbooks as well as identify the key transition moments in which mathematical concepts underwent reformulation and for what goals or reasons. In the context of this study, such information becomes relevant because it might help revealing the raison-d'etre of the textbooks' mathematical and didactic organization and its didactic goals.

3.1 Antiquity: Geometry and Measure Theory

Ancient cultures such as the Babylonian, Egyptian and Greek societies are best known in mathematics as contributors to geometry (Cooke, 2005). In fact, many problems that were eventually reclassified as algebraic were originally formulated in the language of geometry; such is the example of the trisection of angles or squaring the circle.

One of the main preoccupations of all the antiquity mathematicians were computational problems pertaining to geometric figures. Their main interest laid in devising methods to evaluate areas and volumes of different geometric shapes they encountered in everyday life, such as squares, circles, parallelepipeds and pyramids. These activities gave many interesting results such as the discovery of π and Pythagoras' theorem but most importantly for this thesis, they also set the foundation for the theory of integration, which was the reason for the development of measure theory.

The first integration can be attributed to Democritus (460 B.C.E - 370 B.C.E), a pre-Socratic philosopher known for the atomic hypothesis, which arguably influenced his mathematical thoughts. Democritus developed a unique approach to the problem of computing the volume of cones: he proposed to solve the problem by studying a pyramid and varying its number of sides, arguing that as he increased the number of sides, the base of the pyramid would “get closer and closer” to a circle, and hence the solid would “get closer and closer” to the original cone; an idea consistent with his interpretation of the world as collections of tiny elements (Struik, 1948).

Democritus’ method makes use of what is known today as “infinitesimals”, and he encountered resistance to his ideas among his contemporaries. Some of the more clever arguments came from Zeno of Elea (490 B.C.E - 430 B.C.E) who went further than philosophical argumentation and demonstrated paradoxes arising from accepting infinitesimals, which he summarized in the problem of Achilles and the Tortoise. It cannot be claimed that Zeno’s arguments were without merit: the paradoxes he formulated did not result from a poor understanding of infinitesimals, but rather a lack of solid foundation, a problem that plagued infinitesimals throughout their entire existence in mathematics.

Nonetheless, Antiphon the Sophist (480 B.C.E - 411 B.C.E) and Eudoxus of Cnidus (410 B.C.E - 355 B.C.E) eventually proposed a method analogous to Democritus’ which is known today as the Method of Exhaustion. Given a shape, they would circumscribe geometric figures into it whose area or volume could easily be computed. Their idea was picked-up and partially improved by Archimedes of Syracuse (287 B.C.E - 212 B.C.E). Archimedes’ improvement was not an alteration of the method itself, but rather a mathematical approach to the problem of infinitesimals. Instead of approaching the problem from a purely philosophical point of view, he examined the infinite series $\sum_{k=1}^{\infty} \frac{1}{4^k}$ and proposed a rigorous proof that it equals $\frac{1}{3}$. Archimedes’ statement and proof, rewritten in modern symbols, can be summarized as: Given a series of areas $A, B, C, D, ..., Z$, of which A is the greatest, and each is equal to four times the next in order, then $A + B + C + D + ... + Z + \frac{1}{3}Z = \frac{4}{3}A$ The proof is based on the following calculations:

$$B + C + ... + Z + \frac{B}{3} + \frac{C}{3} + ... + \frac{Z}{3} = \frac{4B}{3} + \frac{4C}{3} + ... + \frac{4Z}{3} = \frac{1}{3}(A + B + ... + Y) \tag{1}$$

However we also have:

$$\frac{B}{3} + \frac{C}{3} + ... + \frac{Y}{3} = \frac{1}{3}(B + C + ... + Y) \tag{2}$$

Subtracting this last equation from the one above it gives us:

$$B + C + ... + Z + \frac{Z}{3} = \frac{1}{3}A \tag{3}$$

And finally adding A to both sides yields:

$$A + B + C + ... + Z + \frac{Z}{3} = \frac{4}{3}A \tag{4}$$

(Heath, 2002). He then re-used Atiphon's and Edoxus' methods to compute the area enclosed by a parabola and a straight line while arguing the validity of the result by examining the series $\sum_{k=1}^{\infty} \frac{1}{4^k}$, generated by his use of triangles as the inscribed figures (Calinger, 1999).

Archimedes' approach marked a first rigorous approach to integration, but his results and arguments hardly invalidate Zeno's objections and these would resurface again in history with different flavors. Zeno's paradoxes were a great discovery on their own right as they represent critical elements of mathematics: not everything that is obvious or intuitive is necessarily true. Without necessarily intending to do so, he underlined a crucial problem with the use of infinitesimals that divided mathematicians and philosophers for centuries.

3.2 The 17$^{\text{th}}$ Century Calculus

After the fall or Greek and Roman societies Europe was plunged into the middle ages. This period was known to be unfavourable to scientific progress in general as the Catholic Church prevented the widespread of scientific ideas, locking away the ancient Greek manuscripts. Only around the 1400s would mathematics resurface again due to its importance in maritime navigation.

Integration was revived in the 17th century when scientists began more in-depth studies of Calculus for purely technical reasons. One was its utility in predicting the positions of stars in maritime navigation; the second was its use by upcoming physicist Isaac Newton (1642 - 1727) as a tool to compare quantities that did not behave in a simple linear way such as velocity and acceleration. Moreover, Rene Descartes (1596 - 1650) introduction of the Cartesian plane led to a reformulation of many geometric problems into analytical ones and proved a valuable tool when representing functions visually. Areas under curves now had concrete physical interpretations describing velocity or acceleration and motivated the study of integration, while instantaneous rates of change motivated the study of differentiation.

In 1635 Bonaventura Cavalieri (1598 - 1647) proposed to interpret curves as the result of a sketch left by a moving point and generalized this idea to areas by claiming they were made up of moving lines. Cavalieri's integration method was called "the method of indivisibles". This labeling came from his claim that if one considers a curve as the sum of its points, then these points are the "indivisibles", and analogously the same terminology described the relationship between lines and areas (Struik, 1948).

Cavalieri started by studying a simple figure, a triangle. He first built a bounding rectangle around it, whose area he knew. He proceeded to divide the base into segments of equal length and built rectangles on them that would have the height corresponding to the intersection of the rectangles height line with the triangle's hypotenuse. For example, given a triangle with a base of 6 units in length and 5 units in height, he would construct a large rectangle around it, and then at every 1 unit a smaller rectangle with the height growing by 1 unit every time. The ratio of the computed area to the area of the rectangle is $\frac{1}{2}$, thus consistent with the result from Euclidean geometry.

The next step was Cavallieri's use of indivisibles to imagine that the number of these inscribed rectangles was infinite. His argument was that as he reduced the length of the rectangle's base, the rectangles slowly became small enough to simply be lines, the jagged steps they formed would eventually simply become a straight line, and the region that the rectangles occupy would simply become the shape whose area he was trying to compute. Since his method did not contradict the known result about the area of triangles, Cavallieri proceeded to use it to compute the area under many curves, for example the parabola.

In 1665, Sir John Wallis (1616 - 1703), a geometry professor at Oxford familiar with Cavalieri's method decided to study the relationship between a function f and the area-function under the curve determined by f. Wallis' work resulted in his discovery of the Fundamental Theorem of Calculus for polynomials. Using Cavalieri's method, he showed that given a function $f(x) = ax^n$, its area function is $F(x) = \frac{a}{n+1}x^{n+1}$, including the validity of the formula for negative and fractional exponents. He then proceeded to prove that for curves defined by polynomials, the area can be computed by applying his area-function law to the individual terms. His law is found today in the appendix of all Calculus textbooks (Hooper, 1958).

From a mathematical perspective, Wallis' and Cavallieri's works are more technical achievements than deep scientific discoveries. Neither examined the implications caused by their assumptions about "indivisibles". However, Cavallieri did go a step further than his predecessors in that to come to his conclusions he tested his methods on shapes with known areas. However, nor Wallis or Cavallieri identified the relationship between differentiation and integration, a link that was discovered independently by Isaac Newton and Gottfried Leibniz (1646 - 1716) during the mid-1600s.

While Newton and Leibniz's debate occupies an important part in the history of Calculus, for the purpose of this thesis it is much more meaningful to focus on the main differences between the approaches to integration of these two great minds. Perhaps the most important distinction between Newton and Leibniz lies in their personal views on integration and the concept of infinitesimals. Both mathematicians proposed improved versions of Wallis' and Cavallieri's results. As a physicist, Newton interpreted integration as "calculating a momentary rate of change and then extrapolating the total area". Through creative applications of the binominal theorem to infinitesimals he formulated the Fundamental Theorem of Calculus, unifying integration and differentiation. However Newton's reliance on empirical arguments from his physical observations prevented him from giving any mathematically rigorous justifications of his results. Just like Zeno centuries ago, Netwon felt unease in using infinitesimals, which he considered paradoxical as he could not observe them in the physics experiments which served as the foundation for his Calculus. Newton finally settled his internal conflict by changing terminology and labelled rates of generated change "fluxions", which he denoted with a dotted letter. He published this new version in the text "De Quadratura Curvarum" in 1676, which he considered satisfactory as now all the arguments relied on his results from the study of motion (Boyer, 1949).

Leibniz was a lot less pragmatic than Newton. Since he was a philosopher, just like Democritus, he was solely preoccupied whether his theories contradicted his philosophical frameworks. He accepted infinitesimals passively since he felt their existence was implied by his "Principle of Continuity", which stated that "any change passes through some intermediate change and there is an actual infinity of things" (Burnham, 2001). He was also more simplistic in formulating concepts such as derivatives, which he did not consider to be intrinsically tied to physical processes: for example he claimed that the tangent to a curve was simply the ratio between the abscissa and ordinate of a curve. Much the same way he viewed integrals as sums of the ordinates for infinitesimal intervals in the abscissa.

Both Newton and Leibniz's approaches have strengths and weakness. One cannot discard Newton's adverse attitude toward infinitesimals as absurd, in fact much of Riemann's work on integration and measure theory no longer uses infinitesimals as the foundation of integration. The problem of using infinitesimals was exposed by George Berkeley in "The Analyst" published in 1734. Despite the aggressiveness of his text, most agree that the text shows a deep and important reflection expected of mathematicians: "Berkeley's criticisms of the rigor of the calculus were witty, unkind, and - with respect to the mathematical practices he was criticizing- essentially correct" (Grabiner, 1997).

3.3 The 19$^{\text{th}}$ Century: The Riemann Integral and Early Mathematical Rigor

During the 18$^{\text{th}}$ century Calculus was not marked by any major reforms but rather an abundance of technical results. During this period Brook Taylor (1685 - 1731) developed the Taylor series expansion for functions, Leonard Euler (1707 - 1783) developed various techniques to solve differential equations and established results about certain series (in particular Basel's problem of representing π using infinite series) and various other results. It is during the 19$^{\text{th}}$ century that solid foundations were laid for Calculus and mathematical analysis. This process began with the French mathematician Augustin-Louis Cauchy (1789 - 1857).

Cauchy's work on sequences and series was the first rigorous approach to limits and convergence, hence partially justifying the earlier methods of integration such as Cavallieri's. Historians disagree whether Cauchy was truly using the $\epsilon - \delta$ approach (called the epsilonistic approach) and whether his arguments about infinitesimals were any different from Newton's (Laszczyk, Katz, & Sherry, 2012). In particular, the definition of the concept of limit that we attribute today to him,

$$\lim_{x \to c} f(x) = L \to \forall \epsilon > 0 \exists \delta : \forall x (0 < |x - c| < \delta \to |f(x) - L| < \epsilon), \tag{5}$$

is nowhere to be found in his work. Nonetheless, Cauchy published the first textbook in analysis, called "Cours d'Analyse" (Cauchy, 1821), which he designed for his students at Ecole Polytechnique. The text is regarded by some historians as the first rigorous treatise of analysis and much of its content is still used today (Grabiner, 1981). The epsilonistic definition of limit might have been given by Weierstrass (1815 - 1897) rather than Cauchy himself. However, since Cauchy's work was instrumental to the appearance of this definition, history has decided to attribute him the honors.

Cauchy's definitions eliminated the ambiguities and logical paradoxes introduced by infinitesimals. Calculus was from this point reformulated in the language of limits, which now had a rigorous mathematical foundation. Integration was rigorously studied and described by Bernhard Riemann (1826 - 1866). Riemann's integral largely followed the legacy left behind by Cavallieri, Newton and Leibniz.

Given a function $f(x)$ defined on the interval $[a, b]$, Riemann proposed to partition the interval into $P = \{[x_0, x_1) \cup [x_1, x_2) \cup ... \cup [x_{n-1}, x_n]\}$, where $x_0 = a$ and $x_n = b$. He then defined what we now call the upper and lower Riemann sums:

$$U(f, P) = \sum_{k=0}^{n-1} \sup{(f)}_{[x_k, x_{k+1})}(x_{k+1} - x_k) \tag{6}$$

$$L(f, P) = \sum_{k=0}^{n-1} \inf{(f)}_{[x_k, x_{k+1})}(x_{k+1} - x_k) \tag{7}$$

Riemann then noted that partitions could be made finer and finer, and defined the function $f(x)$ to be Riemann-integrable if as the partitions get finer and finer the upper and lower sums converge to the same value (or their difference converges to 0). We shall omit the questions regarding the independence in the choice of partition since this fact is not the cause or source of the deficiencies of the Riemann integral.

Aside from using upper and lower sums to determine the integrability of a function, Riemann derived several conditions under which a function f was integrable without the need to study the sums:

- If f is continuous on the interval, then it is integrable;
- If f has only a finite number of discontinuities and is bounded on the interval, then it is integrable;
- If f is monotone on the interval then it is integrable.

All known elementary functions satisfy one of these conditions, and only fairly strange functions such as the Dirichlet function fail these criterions, hence making the Riemann integral generally suitable.

Riemann's integral was also the first treatment of integration which circumvented infinitesimals. One could argue that this is merely a stylistic statement since "making partitions finer and finer" is not particularly different from infinitesimal approaches. I would argue that it is fundamentally different since there is no inference about the infinitely small in Riemann's theory, but rather a clever application of Cauchy's theory of limits, whose language allowed to avoid contradictions with what is called the "Axiom of Archimedes".

However, the versatility of the Riemann integral was very limited. Mathematicians of his era were already contemplating sequences and series whose terms were functions and the resulting limit also a function. But series and sequences of functions, contrary to numerical sequences and series, have two types of convergence: point-wise and uniform. Riemann's definition of the integral did not permit to commute the integral and limit operators freely; this was only possible if the sequence of functions under the integral operator was uniformly convergent hence greatly restricting its area of applicability. An example of such a series is the trigonometric series introduced by Joseph Fourier (1768 - 1830) used to decompose periodic functions into a sum of sines and cosines. Such series do not always converge uniformly (particularly for the case of decomposition of functions belonging to L^2; Lennart Carleson (1928 - present) has shown such series converge "point-wise almost everywhere" (Carleson, 1966)). Moreover Riemann's integral could not be used on spaces other than $\mathbb{R}^n$ while mathematicians of the time, for example Maurice Frechet (1878 - 1973), were already working with functions on abstract spaces.

3.4 Set Theory and Topology

Set theory would eventually play an important role in integration as mathematicians of the period began working with abstract spaces and objects, which could only be described by set theory. Moreover as they started seeing the short shortcomings of Riemann's integral, the new approaches in domains of set theory and topology would play an essential role in the development of measure theory.

Set theory was developed by Georg Cantor (1845 - 1918) in the context of a study on transcendental numbers. Joseph Liouville (1809 - 1882) had previously constructed a transcendental number in 1884 (known as the Liouville constant). While the mathematical community accepted that the collection of such numbers was infinite, they did not address the question of whether this collection was equinumerous with the natural numbers.

In 1874 Cantor published the article "On a Property of the Collection of All Real Algebraic Numbers" in which he established that the collection of transcendental numbers is not equinumerous with the collection of positive integers and provided the first rigorous proof that there exists more than one type of infinity (Dauben, 1990). Even though this article earned him the wrath of the Catholic church and fellow mathematician Leopold Kronecker, Cantor proceeded to publish a series of monographs where he laid rigorous foundations for set theory carefully defining each and every object used, proposed the notion of mappings and extensively studied their different properties.

Both set theory and topology (concerned with the most basic properties of space) were extremely important to the development of measure theory. First, both are powerful abstraction tools since they allow breaking free from concrete objects, rather addressing abstract collections. Second, they greatly benefit our understanding about functions and their properties. In topology a real function is simply a one-to-one mapping whose range are the real numbers. We can study what happens to collections of variables, what properties this collection possessed originally, and what properties it acquired or lost under a specific mapping. Measure theory, as we shall see, is a shift from considering functions on isolated points or intervals to studying how functions act on sets and their properties. Moreover, Cantor's work on infinities and Cauchy's study of infinitesimals with its eventual abandon in favor of the delta-epsilon formulation permitted Henri Lebesgue (1875 - 1941) to construct a model of the integral which is independent of the constraints of the original model of the real line. Considering functions as mappings of sets (or subsets of topological spaces) allowed him to see that using the range of a function in the integration process is just as valid as using the domain. Moreover, Cantor's result about countable and uncountable infinities, and the real numbers being uncountable, led Lebesgue to wonder about the real input of countable sets in integration and the consequence of these infinities for analysis.

3.5 Henri Lebesgue and Measure Theory

Measure theory was born in Lebesgue's doctoral thesis Integral, Length, Area in 1902. Lebesgue was undoubtedly familiar with the earlier work on measure by Camille Jordan (1838 - 1922). Jordan's definition of measure of a set was very close to Lebesgue's as he considered what he called the inner measure (m_*) and outer measure (m^*) of sets. Given a set B, Jordan defined the following two quantities:

- The inner measure: $m_*(B) = \sup_{S' \subset B} m(S')$
- The outer measure: $m^*(B) = \inf_{B \subset S'} m(S')$

Where S and S' are simple sets. In Jordan's framework a simple set was a set that could be expressed as a finite union of rectangles. If both quantities were equal, then such a set B was said to be Jordan measurable and the value of either outer or inner measure simply taken as its measure.

Unfortunately Jordan's work did not succeed in generalizing the class of integrable functions, and the class of sets that were measurable using his definition was fairly small. In fact, a (bounded) set is Jordan-measurable, if and only if its characteristic function is Riemann integrable. Hence, the Jordan measurable class did not extend the class of Riemann integrable functions (in fact, the Riemann integral can be constructed from Jordan measure much like Lebesgue's integral can be constructed from Lebesgue measure (Frink, 1933), making the Jordan measurable functions and Riemann-integrable functions equivalent). The issue was that Jordan's elementary sets were too restrictive (he only permitted rectangles) hence leading to the same conditions as those required for Riemann integration.

Lebesgue proposed his theory of integration as a tool to circumvent the limitations of the Riemann integral. In particular he addressed the issue of interchanging the limit and integral operators by building a model in which point-wise convergence is sufficient. (As stated previously, a widespread use of Fourier series, which describe many different physical processes such as those associated with electricity, required a toolbox that allowed working only with point-wise convergence.)

Lebesgue decided that instead of studying a function's domain to find a fundamental unit of area he would study the range. Intuitively, Lebesgue described his integral as follows, in a letter to fellow mathematician Paul Montel (1876 - 1975): "I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral." (Siegmund-Schultze, 2008). This description is a fairly good intuitive justification since looking at the range introduces a freedom of rearranging the values of a function in a more convenient fashion. This is very well demonstrated when integrating the Dirichlet function ($1_{\mathbb{Q}}$). To explicitly show this, let the interval $(0,1) = (0,1)_{\mathbb{Q}} \cup A$, where $(0,1)_{\mathbb{Q}}$ is the set of all rational numbers in $(0,1)$ and A are all numbers in $(0,1)$ that are not rational. Let $f(x) = 1_{\mathbb{Q}}(x)$ Then f on $(0,1)$ takes the value 1, and 0 on A. Since $\int_0^1 f(x)dx = \int_{(0,1)_{\mathbb{Q}}} f(x)dx + \int_A f(x)dx = 0 + 0 = 0$.

However, since Cauchy, mathematicians demanded rigorous proofs and definitions (Grabiner, 1981). Therefore, in order to justify his integral, Lebesgue introduced the notion of measure. Measure is a generalization of concepts such as length, area and volume to a much larger class of sets than intervals, rectangles or parallelepipeds. This implies that the class of sets that could be used to approximate the area under curves was now much larger as well, and no longer restricted to rectangles as was the case in Riemann integration. Lebesgue defined inner and outer measures on real spaces for a set B exactly as Jordan did, with the sole difference that the sets that covered (or inscribed) B need not be "simple sets" and needed not be finite unions.

The next building block of the integral were simple functions (i.e., functions taking only a finite number of values) which he used along with measurable sets to define the integral. Without loss of generality, consider the non-negative function $f : D \to \mathbb{R}$, where that D is measurable and the pre-image of any measurabale set is measurable. Then the integral of f on D is defined as:

$$\int_D f(x)dx = \inf_{\phi(x) \geq f(x)} \int_D \phi(x)dx \tag{8}$$

where $\phi(x)$ are simple functions. Since simple functions can be expressed through the indicator function as $\sum_k a_k 1_(S_k)$, where S_k is the set on which the function takes the value a_k, we have the following: $\int_D \phi(x)dx = \sum_k a_k \mu(S_k \cap D)$. In the general case of a function f, one would have to break the function into its non-negative part f^+ and its negative part f^- and study each of the integrals individually.

Like Riemann, Lebesgue was compelled to provide conditions under which a function would be integrable according to his theory, which he summed up in one short sentence: a function that is measurable, is integrable (a measurable function being a function whose domain is measurable and such that pre-images of measurable sets are measurable sets). Proofs of this statement can be found in any measure theory text. This duality can be seen in much of measure theory through indicator functions (which are also used in simple functions). If we study the indicator function of a set S, we have:

$$\int 1_s d\mu = \mu(S) \tag{9}$$

In 1918, Percy John Daniell (1889 - 1946) proposed a different construction of the Lebesgue integral known as the Daniell integral. Daniell did not seek to construct a different theory because he was unsatisfied with Lebesgue's results, but because before one could pass to integration it was necessary to build (or learn) a workable measure theory which he felt was somewhat impractical if one wished to quickly generalize the theory to higher dimensional spaces (Daniell, 1918). Daniell followed Hilbert's ideas of axiomatization of mathematics and constructed an axiomatic theory of integration. From his construction measure can be defined through indicator functions and is equivalent to the Lebesgue measure.

Lebesgue's theory of integration was now dominant in mathematics because it was less restrictive than Riemann's theory. For example, in Lebesgue's construction sets of measure 0 did not contribute to the integral while in Riemann's construction these sets potentially created problems. One way to visualize this difference is to think of a function with a countable number of discontinuities. Riemann's integral is powerless on such functions, while Lebesgue's integral "merges" all these discontinuities into one place and shows that they are not meaningful in the integration process. Similarly, since Lebesgue's theory was designed to relax the conditions on interchanging the limit and the integration, it was now possible to take certain integrals without necessarily computing the limit function to which a sequence was only point-wise convergent. Functions defined on manifolds are also outside the reach of Riemann's theory since its method of partitioning requires by definition a real space, while Lebesgue's theory only requires us to introduce a measure on the manifold. Overall, Lebesgue's theory is very intuitive. Its use of words such as "almost everywhere" (to describe a property that holds everywhere except on a set of measure 0) coincides with our belief that very small inputs are unlikely to influence an outcome. Almost continuous functions became indistinguishable from continuous ones, and most of the subsets of the real line used in analysis were found to be measurable, leading some mathematicians, including Lebesgue himself, to speculate that any set was measurable (Struik, 1948).

This last statement was eventually refuted by Giuseppe Vitali (1875 - 1932) who used the axiom of choice to construct an example of a non-measurable set - thus infuriating Lebesgue (Hooper, 1958). Another "flaw" in the theory was the incompatibility with improper Riemann integrals. While Lebesgue's work was shown to reproduce all the results of proper Riemann integrals, in cases such as $\int_a^{+\infty} f(x)dx$ both integrals could not always be proven equal. This deficiency comes from the need to break a function into its non-negative and negative parts in order to integrate. In particular Apostol links the existence of an improper Lebesgue integral of a function f with the existence of two improper Riemann integrals (Apostol's Theorem): $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} |f(x)|dx$ (Apostol, 1974).

This deficiency is best seen in the example $f(x) = \frac{\sin(x)}{x}$ on the interval $(0, +\infty)$. In this case, f^+ and f^- are both infinite between any finite endpoints $(0, a)$, rendering it non-integrable. Alternatively, by using Apostol's theorem (stated above), the improper Riemann integral of $\int_0^{+\infty} \frac{\sin(x)}{x} dx$ does not exist hence eliminating the possibility of an improper Lebesgue integral.

3.6 Measure theory in the present

Today, the Lebesgue integral has become a standard topic in the university mathematics curriculum. Most of Lebesgue's original theory and ideas have remained untouched, except for some modernization and precisions done by later researchers. For example, Lebesgue's conjecture about all sets on the real line being measurable turned out to be "semi-incorrect"; it has been proven that without the axiom of choice all subsets of the real numbers are measurable (Herrlich, 2006). The use of inner measure to construct measures has been largely abandoned in favour of a definition using exclusively outer measures proposed by Constantin Caratheodory (1873 - 1950) because it is much more practical when working with abstract spaces because it only requires the definition of the outer measure versus having to define the inner measure as well. However, this definition does not intuitively follow from Lebesgue's work since it is disconnected from the geometric principles. For example it no longer discusses the closeness between objects, whereas Lebesgue's definition does.

Measure theory quickly started to propagate into all branches of mathematics which somehow used analysis. For example, in statistics, measure theory allowed to unify the discrete and continuous probability distributions, with different probability spaces requiring the introduction of different measures (Billingsley, 1979). In abstract algebra, particularly in the study of continuous groups, Alfred Haar (1885 - 1933) introduced the Haar measure which was immediately used by John von Neumann (1903 - 1957) to solve Hilbert's 5^{th} problem in the case of compact groups (von Neumann, 1933). Measure theory was now studied not simply for its earlier value in integration theory, but also as a discipline on its own. In the case of some problems, researchers even went on to generalize measure by dropping conditions such as non-negativity, later called a signed measure, best know through the Hahn decomposition theorem (Billingsley, 1979).

Lebesgue's work on integrals also spurred a general interest in the mathematics community to develop more theories of integration each addressing an individual deficiency of Lebesgue's original construction or endowing it with additional properties. Two of such examples are the Henstock Integral and the Khinchin Integral. The Henstock integral, created by Ralph Henstock (1923 - 2007), addressed the problem of improper integrals while preserving the advantages of the Lebesgue integral (Bartle, 2001). The drawback of Henstock's construction is its inability to generalize to spaces other than the real line due to its founding principles. The Khinchin Integral is a much more complex structure which is a generalization of both Riemann and Lebesgue integrals and is built on the idea of integrating derivatives and uses concepts such as approximate derivatives and generalized absolutely continuous functions (Gordon, 1994).

Lebesgue's work resulted in a better understanding of integration by the mathematical community. Some of its benefits were directly mathematical, providing researchers with new methods to tackle various problems. Another more indirect benefit of this understanding is the generalization of the integral known as functional integration, where the domain of integration is a space of functions

(Kleinert, 2004). For example, Norbert Wiener (1894 - 1964) introduced the Wiener measure in physics, through his study of Brownian motion, to assign and to compute the probability of a particle's random path. Wiener's path integral was hence based on the concept of measure while Richard Feynman (1918 - 1988) proposed a path integral without appealing to measure theory.

This overview will eventually give us an insight about how authors of various textbooks decided to interpret measure theory. In particular it gives us a good idea about the reasons why some authors choose to follow Lebesgue's original theory, while others prefer to start directly with Caratheodory. Moreover it gives us a hint which approach is more prone to producing new mathematical knowledge since Lebesgue's work is a direct descendent of Jordan's measure theory, hinting that such an approach favours associative thinking (i.e. looking for analogies between different objects), whereas Caratheodory's approach feels more like an exercise in generalization of existing concepts rather than the production of new knowledge.

Chapter 4

Frameworks

As stated in the introduction, the goal of this thesis was to understand how measure theory textbooks are different from one another, and what professors' beliefs about the use of textbooks, mathematics and learning are. The research therefore contains two dimensions: one is a study of textbooks themselves, and the other are interviews conducted with three mathematics professors which focused on their textbook use.

To account for the differences in the mathematical and didactic organization of the textbooks in the context of these being used by professors to design their courses or prepare lectures, we turned to Eco's work on *open* and *closed* texts, along with his theory of *model readers*. We also discuss the textbooks in the context of Sieprinska's model of *apodictic* and *liberal* textbooks. To analyze the interviews we proposed our own classification of empirical readers based on characteristics exhibited by the participants.

4.1 Textbooks' strategies of formatting their interpretation: Open and Closed texts

Sometime in the late 60s and early 70s, the focus of Eco's work shifted from considering the process through which messages are generated to considering the addressees' interpretations of the message. This shift was not smoothly received by scholars in the semiotics community; "the idea of taking into account the role of the addressee looked like a disturbing intrusion, disquietingly jeopardizing the notion of semiotic texture to be analyzed in itself and for the sake of itself." (Eco's (1967) response to Claude Lévi-Strauss critique). It was in this context that Eco introduced the notions *open* and *closed* text. In his framework, he characterized an *open* text as "a work of art that actively involves the addressee in its production", and a *closed text* as "one that holds the addressee at bay and seeks to evoke a limited and predetermined response." (Eco, 1979) In an open text, the reader's interpretations are part of the generative process of the text. (Bondanella, 1997)

The question of *interpretation* led Eco to the study of *misinterpretation* and *aberrant interpretation*; later on, he would address the question of *limits of interpretations* in a more systematic manner.

A few observations are in order when one tries to use these notions in research, in general, and in the case of textbooks, in particular.

In a literary piece of work, the notion of interpretation is strongly related to the notion of *model reader* as devised in the author's mind; the reader envisioned by the author as he or she generated the text or in other words, a reader who interprets the text as the author intended. In a didactic piece of work such as mathematics textbooks we have to account for the interpretations intended by the author and for how these relate to the mathematics topic the textbook is supposed to present.

Open texts are supposed "to invite their *model readers* to reproduce their own processes of deconstruction by a plurality of free interpretative choices." (Eco, 1979) In a textbook, however, freedom of interpretation is restricted by mathematical meaning. On the other hand, closed texts should be free of misinterpretation allowing only for the one the author intended. But mathematical textbooks seldom present a variety of approaches to a mathematical concept. Mathematical objects, however, can (and should) be interpreted in different ways - understanding different representations and different, equivalent definitions and theorems is essential in understanding a mathematical concept. One may argue that it might be difficult to present a variety of definitions and examples in an elementary textbook. We argue that the case of graduate textbooks, such as the ones we are concerned with in this thesis, is different; readers of these texts are expected to have the background that would allow them to understand different approaches to the same topic, how these relate to each other and how they contribute to the understanding of the mathematical concept at stake.

These considerations led us to adapt Eco's notions of open and closed text to the study of textbooks. In this adaptation, an *open-text textbook* is a textbook that seeks to present many possible interpretations of the same subject, which might do, for example, by including a discussion of equivalent definitions or examples. *Closed-text textbooks* are textbooks that restrict themselves to a single possible *interpretation*, preventing others by various strategies such as omitting equivalent definitions. Other strategies may include a discussion of other possible interpretations only to discard them at the end of the paragraph. Therefore, the notions of closed- and open-text characterize certain strategies for formatting interpretation.

Textbooks containing few equivalent definitions and little discussion about mathematical objects would be, in this context, closed textbooks since they are less likely to provide the reader with a message that would allow him or her to construct a variety of interpretations from which the reader can draw the most suitable one for a particular situation. Similarly, textbooks containing many definitions and fair amounts of discussion about mathematical objects would be open textbooks.

4.2 The model reader and the empirical reader

The notions of *model reader* and *empirical reader* are core elements of this thesis. Eco's definition of the *model reader* is the reader foreseen by the author who is able to cooperate in the text's actualization in a specific manner, and who is also "able to deal interpretively with the text in the same way as the author deals generatively" (Eco, 1979).

The *empirical reader* is described as the "concrete subject of acts of [textual] cooperation"; he "deduces a model image of something that has been previously verified as an act of utterance and which is textually present as an utterance." (Guillemette & Cossette, 2013). In short, the empirical reader is anyone who reads the text (pragmatically).

Once again, a few observations are in order when one tries to use these notions in research, in general, and in the case of textbooks, in particular.

In his early work Eco took his own works as the basis for his research, giving him an unfair advantage since he was playing both the author's and researcher's roles. For a researcher who's not the author of the text, however, characterizing the model reader might be a challenge. In order to identify these instructor model readers we had to reconstruct them from the textbooks themselves, since they are not documented and inquiring with the authors is impossible. In this thesis, we do so by studying the preface sections, content and its organization, and the historical context in which the text was written.

Furthermore, in the case of textbooks and of institutionalized education, we argue that two model readers exist: a *student model reader* and an *instructor model reader* (which implies the existence of two "classes" of empirical readers). Our argument is based on the well accepted assumption that the textbook plays a role in teaching and learning as part of a student-textbook-instructor triumvirate, where the instructor mediates the textbook's message ((Rezat, 2006), (Keitel, Otte,

& Seeger, 1980), (Sierpinska, 1997)). These considerations are important because Eco's research assumed the reader-text relationship to be free of any mediating mechanisms other than the text itself.

Our analysis of textbooks in chapter 4 allows us to extrapolate the instructor model reader (or "an" instructor model reader). The analysis of the interviews, in chapter 5, allows us to characterized different empirical instructor-readers. Comparing the two allows us to understand whether real instructors are actually using the textbook as described by earlier researchers and whether such use influences their textbook choice. Also it allows us to better understand the nature of the conflicts that plague an instructor-textbook relationship.

4.3 Apodictic and liberal textbooks

In her study of the interactions between students, textbooks and tutors, Sierpinska (1997) defines the notions of *apodictic* and *liberal* textbooks. In particular, Sierpinska discusses liberal and apodictic as formatting strategies employed by authors with respect to textbook use. *Apodictic* texts are restrictive in their possible use, while *liberal* texts leave a "wide margin of liberty" to the student" (Sierpinska, 1997). Examples of *apodictic* textbooks are constructivist and programmed textbooks. A programmed text contains, in addition to instructional material (what to learn), directions on how to learn: how to combine visual or aural apprehension of material (reading or listening) with verification of the assimilation of knowledge and skills (Cherviakova, 1979). A programmed textbook also indicates how to find and eliminate discrepancies between the projected level of assimilation of knowledge and the level actually achieved. According to Cherviakova, a programmed text performs some of the functions attributed to a teacher: it serves as a source of information, organizes the instructional process, monitors the degree of assimilation of the material, regulates the rate of study, gives necessary explanations, and prevents mistakes. The student's work can be checked immediately through the answers provided in the textbook. If the student answered questions correctly, he or she may proceed to the next section; if the answers are incorrect, then the programmed textbook may attempt to elucidate the typical mistakes made by the student, and present a remediation strategy. A constructivist text is similar to the programmed text and may exhibit many of the same patterns. Sierpinska characterizes constructivist texts as those that use a problem-solving approach to construct knowledge: a student learns the material by completing a series of tasks which give him or her the opportunity to construct knowledge necessary for solving the subsequent batch of problems.

Liberal textbooks are also rare, but we speculate this is due to a very fine fragmentation and/or compartmentalization of mathematical knowledge. A liberal textbook on integration would discuss the many different approaches to integration. For example it would present the integral as a geometric problem, as an analytic problem, an operator problem and maybe even as a general physical problem. It would expect the reader to understand the implications of these approaches and in what situations each should be used.

It is important to note that formatting of interpretation and formatting of use are not completely independent. Sometimes when a particular formatting strategy is aimed at interpretation it can have consequences for use, and vice-versa. For example, an open-text textbook implicitly achieves (but not always) a "use" as reference text for many possible situations, whereas closed-text textbooks do not.

Sierpinska's article addressed the question of "how can students and tutors use [a particular] textbook". In this thesis, we look at instructors' uses of textbooks, instead. We conjecture that in the case of instructors, the textbook takes on a utilitarian character: they use the textbook to prepare lectures, to assign problems, sometimes maybe even to prepare exams.

Because of our interest for instructors' uses of textbooks, we redefine' the notions of *liberal* and *apodictic* textbooks. We do so by considering the amount of freedom a textbook gives an instructor who uses it to prepare his or her lectures. We define a *liberal* textbook as one which gives the instructor a considerable' amount of liberty when preparing his or her lectures in the sense of mathematical and didactic approaches. In particular, a liberal textbook will contain many equivalent definitions, and won't be imposing a particular development of a subject, letting the teacher construct a development he or she feels appropriate. An *apodictic* textbook is one which restricts the instructor's use to one single approach to the mathematics and didactic presentation of the material.

This may imply that the notions of liberal and apodictic are relative to that of empirical reader since different instructor-readers seek different levels of liberty: some may consider a text as giving them a lot of liberty, while others may feel restricted by the same text.

4.4 Practice and production problems

Another important element when working with textbooks is trying to understand their problem sets because they often provide insight about how the text is organized and whether the author purposely left out certain results in order to give them as exercises. This provides us with information about the instructor model reader (e.g. if a lot of important results are left as exercises to the students, one can speculate the author assumes the instructor to have very little input in the learning activity of the students in the classroom, but rather through assignments). There is also an impact of the apodictic/liberal character of the textbook, since perhaps a particular use by an instructor would require him to solve a lot of the problems in the classrooms leaving the problem pool for assignments dry. After examining several problem sections we propose two classes of problems: *practice problems* and *production problems*. If we consider an object from measure theory, for example a measurable set or the measure mapping, the end result obtained when solving a practice problem is not a new result about such object. This does not mean that some of the steps involved in the solution do not lead to production of new knowledge about measure theory objects. Production problems are problems whose end result is a new result about a measure theory object, relative to the student's knowledge about this object prior to solving the problem.

An example of a practice problem would be: Show that the interval (a, b) is an uncountable set. An example of a production problem would be: Show that the Lebesgue measure (m) on $\mathbb{R}^n$ is invariant with respect to the group $SU(n)$ (i.e. given that A is measurable, $m(A) < \infty$, $M \in SU(n)$, and B is the result of M acting on A, then $m(A) = m(B)$).

4.5 A framework for analyzing the interviews

The interviews were originally conducted to identify an instructor's text preference and the reasons behind it. However, they became much more valuable because the answers turned out to be a reflection on how the participant-instructor sees himself or herself conducting the teaching activity when it has to be done in the context of using textbooks. The answers also provided us with information about their teaching style, their beliefs about mathematics, their expectations from incoming students, and their beliefs on how students come to understand the material.

We used Grounded Theory Methodology, which involves "generating theory and doing social research [as] two parts of the same process" (Strauss & Corbin, 1994), to analyze the interviews.

We began by collecting data through the interviews. We then proceeded to code the participants' responses. These were then organized in concepts which served to construct the characterizations of empirical readers found below.

The three characterizations given below account for how instructors intend to use a textbook for teaching, their beliefs about mathematics and about learning, respectively. The characterizations are based on some of the patterns we have observed in teachers: some are more creative, others less. Much the same way as some will follow the textbook to the letter while others will use the textbook only as a problem goldmine.

Creative vs. Passive reader: This distinction is based on how the instructor uses the textbook in relation to his or her lectures. The creative instructor-reader uses the textbook as a tool to supplement his or her lectures. Often, he or she uses its skeletal structure without necessarily reproducing its arguments. The passive instructor-reader relies on the textbook to construct his or her lectures. He or she uses not only the skeletal structure, but also the arguments and proofs from the text.

Constructivist vs. Sophist instructor: This distinction is based on the instructor's beliefs about students' learning and on how these beliefs impact his or her teaching practice. The constructivist instructor is one that believes that one learns the material through the process of solving problems. Therefore, he or she sees the problem sets and exercises as the core of the teaching activity. The sophist instructor is one that believes that one learns the material through passive reception'. Therefore, he sees lectures and readings as the core of the teaching activity; the problems are supplementary and simply reinforce the above activities.

Intuitivist vs. Hilbertist instructor: This distinction is based on the instructor's perceptions of the mathematics content he or she is teaching and on how these perceptions impact his or her teaching practice. The intuitivist instructor is one that favors the use of intuitive arguments in his or her lectures, not only because he or she believes that these are important to the learning process but because this is how he or she relates to mathematics concepts. This instructor often wants to include visual representations, analogies with physical objects and any relevant parallels that would justify the mathematical concepts. The Hilbertist instructor does not care about including intuitive arguments or graphical representations even though he or she may occasionally make them unintentionally. He or she believes that only the accuracy and rigor of the mathematical content matter in mathematics, in general, and in the learning process, in particular.

The interviews form a link between our theoretical considerations on model readers and apodictic and liberal textbooks. With our experimental data we can compare the model reader we extrapolated from the textbooks and the empirical readers (the instructors we interviewed). Moreover, by studying the organization of the material in different textbooks we were able to classify them as apodictic or liberal.

4.6 Overview of how the frameworks are used in the analyses

In this thesis, we have two dimensions: a theoretical study of four measure theory textbooks, their content and didactic organization, and an extrapolation of their model readers; and an empirical section consisting of a series of interviews with three mathematics professors about textbooks use in measure theory.

The lenses we proposed, combined, allow us to address our research questions. The analyses of textbooks as open/closed and liberal/apodictic serve to account for the differences we find in measure theory textbooks - differences that are of mathematical and didactic natures. The analysis of a textbook's model reader allows us to understand what characteristics an instructor using such textbook in his teaching practice should posses (competences, abilities, teaching style, etc...) in order not to come into conflict with the textbook. The analysis of the interviews gave us a picture of what are professors' beliefs about the use of textbooks, about mathematics and about learning. We classified the interviewed professors according to the profiles presented in the previous section. We then conjectured a preferred textbook for each participant (based on the results obtained from the textbook analyses) and compared them to their final textbook choice (i.e., we played the model readers we extrapolated from the textbooks against the empirical readers who were the participants).

Chapter 5

Analysis of Textbooks

This chapter presents a study of four measure theory texts commonly used in the teaching of measure theory in university courses, which we shall examine using the ideas and concepts outlined in the frameworks chapter (chapter 4). This analysis comes in two parts. First, based on the content of the textbooks we extrapolate the instructor model reader the author envisioned. Secondly, we try to understand how different empirical readers can use the texts and whether these textbooks are liberal or apodictic and open or closed.

According to the course outlines collected from randomly chosen Canadian universities (this was done by making a quick search in Google with the keywords “measure theory”, “outline”, “Canada”), the most commonly used text in measure theory is Real Analysis by Royden. As discussed in the introduction, we chose to include three other texts in our analysis: Zygmund and Wheeden’s, Kolmogorov and Fomin’s, and Rudin’s. We included Zygmund and Wheeden’s due to its popularity in the academic community. Our decision to include Kolmogorov’s and Rudin’s textbooks was motivated by informal discussions we had with other members of our mathematics department who expressed their belief that Kolmogorov’s text is one of the “classical treatises of measure theory (and analysis in general), whereas Rudin has an established reputation among scholars to be a very creative and graduate-student oriented approach to measure theory.

The following table presents each text with some biographic information about the authors.

Title	Author(s)	Year Published	Country of Origin (Text/Author)
Real Analysis	H.L. Royden - Ph.D. (Harvard 1951) - Harmonic Functions on Open Riemann Surfaces - Further research was on Riemann surfaces.	1st Edition 1963, 3rd Edition 1988	Text: U.S.A Author: U.S.A
Measure and Integral: An Introduction to Real Analysis	A. Zygmund - Ph.D. (Warsaw University 1923) - Area of Interest: Harmonic Functions. R. Wheeden - Ph.D. (University of Chicago 1965) - On Trigonometric Series Associated with Hypersingular Integrals	1st Edition 1977	Text: U.S.A A. Zygmund : Poland & U.S.A R. Wheeden: U.S.A
Real and Complex Analysis	W. Rudin - Ph.D. (Duke University 1949) - Uniqueness Theory for Laplace Series - Research Interests: Harmonic and Complex Analysis	1st Edition 1966 3rd Edition 1987	Text: U.S.A W. Rudin: U.S.A
Elements of the Theory of Functions and Functional Analysis	A. Kolmogorov - Ph.D. (Moscow State University 1925) - Research Interests: Probability, Topology, Logic, etc S. Fomin - Ph.D. (Steklov Institute of Mathematics 1942) - Research Interest: unknown.	1st Edition 1957	Text: U.S.S.R A. Kolmogorov : U.S.S.R S. Fomin: U.S.S.R

Table 2: Biographical Information of Different Textbooks

The following four sections present an analysis of each of the four textbooks. In each case we construct a table containing the text's choice of definitions for outer measure, measurable set, an evaluation of the techniques used (i.e., set-theoretic proof, real line structure based proof, topological proof, etc...) and discuss the didactic and mathematical layers in the text, similarly to the research done by Sierpinska (1997). We then investigate the set of problems proposed by each text and classify them as practice problems or production problems. Finally, we use the analysis and the text's history, when available, to extrapolate the instructor model reader.

5.1 Elements of the Theory of Functions and Functional Analysis (A. Kolmogorov)

Object	**Definition/Method/Emphasis**
Elementary set	P_k, rectangles, declared measurable a-priori in view of geometric considerations.
Outer measure (μ^*)	$\mu^*(A) = \inf\{\sum m(P_k) \mid A \subset \cup P_k\}$, where the lower bound is taken over all coverings of A by countable collections of rectangles.
Inner measure (μ_*)	$\mu_*(A) = 1 - \mu^*(E - A)$
Measurable set	A set A is measurable if $\mu^*(A) = \mu_*(A)$
Other measures	Jordan measure and its extensions
Properties of collections of measurable sets	Studied by considering the collection of sets under two operations and demonstrating that they define a ring. Emphasis on the algebraic properties, particularly with respect to operations such as union and intersections, large sections devoted to Borel sets.
Non-measurable set	Small section, text assumes the axiom of choice but does not explicitly state it.
Measurable Function	A function f is measurable if $f^{-1}(A) \in S_n$ for every Borel set A on the real line.

Table 3: Main concepts of Kolmogorov's textbook

From the above table we can see that the text tries to preserve the historic development of measure theory since it uses many of the arguments found in Lebesgue's original work, particularly the concepts of inner and outer measure. The presence of the Jordan measure and its discussion may be interpreted as an attempt to underline the importance of studying different possibilities and not restrict oneself to using, for example, only finite unions of sets. Overall, the text "constructs" the concepts "step-by-step": it does not assume that the reader knows what elementary sets are or how a collection of sets forms a ring.

The advantage of studying the algebraic properties of measurable sets permitted the author to achieve a certain level of generalization: while constructing measure theory on the real line, he progressively considers the set theoretic structure of this space in his arguments. This progression feels natural since the generalization is achieved by studying the behavior of concrete objects and not simply passing to abstraction. The section on measurable functions is short and concise. It recycles many of the results on measurable sets to show that certain types of functions are measurable.

The reader is presented a fairly short section on non-measurable sets, with no justification for its presence.

The text itself is organized in chapters containing sections devoted to the study of particular objects or properties. Each section is concluded with a set of problems. The chapter on measurable sets contains a total of 26 problems, out of which 8 are production problems and 18 are practice problems. We provide two examples below.

An example of a practice problem found in Kolmogorov's text is found on page 14, chapter 5:

> "Let F be the Cantor set constructed on $[0, 1]$ [...] Show that μ_ϕ, the Lebesgue-Stieltjes measure generated by ϕ on the set $[0, 1]$, is a singular function"

The end result is not about the measure mapping's properties but about its behavior on a very specific set, making it more of practical application of measure theory to an analysis problem.

On the other hand, the following is an example of a production problem found on page 15, chapter 5:

> "Derive Lebesgue's criterion for measurability. A set $A \subseteq E$ is measurable if and only if for every $\epsilon > 0$ there exists G open (relative to E), and F closed such that $F \subset A \subset G$ and $\mu(G - F) < \epsilon$"

The role of this problem is to present the reader with new knowledge because it explores arbitrary sets, not specifically constructed ones. In particular, it provides the reader with a new way of identifying measurable sets in topological spaces on which measure is defined.

5.1.1 The formatting of use

The formatting of the textbook suggests that the author tries to keep the mathematical and didactic contents connected. When the author introduces a new concept he provides the reader with analogies, historical information, and sometimes even a preview of the purpose this material will have later in the text. For example he provides the reader with geometric motivations for the introduction of measure by presenting the similarities between area, length, volume and integral. The author also explains that the material is not self-contained and can be applied in other branches of mathematics. The symbolic notation is not used abusively and is often supplemented with explanations to clarify purely mathematical statements.

The text's restrictive strategy is to be rigorous and to remain as mathematically accurate as possible. The problems do not contain the bulk of the material nor is their solution necessary in order to move from chapter to chapter as it would be in the case of programmed texts.

The above considerations place this textbook in the open textbooks category since it is clearly seeking to evoke many interpretations and approaches to the material.

5.1.2 Historical information

Historically, the text is based on the lectures delivered by Kolmogorov with the final draft prepared by Fomin (formerly Kolmogorov's student). We do not know whether Kolmogorov himself was using a text in his lectures or not. It is also important to note, that the original Russian text did not contain exercises, these were added later by H. Kamel during the translation to English with the goal to "increase the usefulness of the text, test the reader's understanding and introduce and extend certain topics that were omitted in the original print" (Kolmogorov & Fomin, 1961). A plausible conjecture regarding the addition of the problem set based on the text's history was simply to give it the format expected in western textbooks which are always written in a "material followed by problems" fashion. "[The problem sets were added to] increase the usefulness of the text, test the reader's understanding " could have meant to increase the text usefulness in institutionalized education, giving within an institution a way to test students' understanding of the text" (it's unlikely that the idea was to allow students to test their own understanding as correct solutions are not presented). Kamel's final claim "[The problem sets were added to] introduce and extend certain topics that were omitted in the original print" is again related to institutions and not student readers. Only an institution has the authority to decide if something is "missing" or insufficiently covered in a text.

5.1.3 Liberal vs apodictic

In view of the historical information and analysis of the different dimensions, the textbook is of liberal character. It can be used in many different ways by instructors. Due to its open character (presentation of many possible interpretations of measure theory) an instructor can construct a course which develops the subject based on his mathematical beliefs about the subject. The instructor does not need to supplement Kolmogorov's text since the topics are treated very rigorously and there are no inconsistencies or blanks to fill with respect to the mathematical discourse. However the textbook cannot be used in a modular way since a lot of the chapters have subtle links among each other making it impossible to skip some of them without encountering nasty surprises later on.

Particularly its balance between intuitive and abstract arguments contributes enormously to its liberal character. If someone decides to read only the abstract contents and finds himself struggling he can always return to the intuitive arguments with concrete examples to understand the material better. This balance permits the instructor to use the textbook effectively in the teaching activity.

5.1.4 The model reader

At first glance the instructor model reader of Kolmogorov's textbook is relatively simple. He is only required to know the basic foundations of measure theory such as topology and real analysis. The text documents all the elements and explains well the connections between them.

However, it is unlikely that this textbook used as-is would be effective in the classroom. First the amount of material inside it exceeds the classroom hours (we don't think that it is possible to teach the entire measure theory section of the textbook in one semester), hence the instructor model reader is quite experienced and knows how to use the textbook in a way to stay within the allocated class time; however this particular discussion is outside the scope of this research.

Let us call Kolmogorov's instructor model reader, MR_K. Based on all the information we collected MR_K is an instructor who wishes to stay close to the historical path to measure theory as long as possible, while still wishing to expose the purely algebraic interpretation and definitions whenever possible. He is an instructor who does not see measure theory as something self-contained, but rather wishes his students to go and apply the material to other branches of mathematics (since Kolmogorov provides such discourse in his textbook). He is not overly concerned with problem sets, since they do not contain the bulk of the material, most likely using them as a verification mechanism to test the understanding of the material by his students.

5.2 Real Analysis (H.L. Royden)

Object	**Definition/Method/Emphasis**
Elementary set	Implicitly hinted to be intervals of the real line.
Outer measure (μ^*)	$m^*(A) = \inf_{A \subseteq \cup I_n} \sum l(I_n)$
Inner measure (μ_*)	Not used
Measurable set	A set E is measurable if for each set A we have: $m^*(A) = m^*(A \cap E) + m^*(A \cap cE)$.
Other measures	None are presented
Properties of collections of measurable sets	The claims regarding the algebraic structure of collections of measurable sets are present in the text, but the emphasis seems to be on the proofs and demonstrating how to derive these results through the definition of measurable sets. The results are not recycled or re-used in the section.
Non-measurable set	Small section, text assumes the axiom of choice but does not explicitly state it.
Measurable Function	Given as a list of possibilities under the assumption of a measurable domain. One of the possibilities: if for any α, the set $\{x : f(x) > \alpha\}$ is measurable, then f is measurable. (the remaining possibilities are equivalent to this statement).

Table 4: Main concepts of Royden's textbook

From the above table, we can see a net divergence from Lebesgue's original model of measure theory. Royden has removed inner measure and substituted the "classical" definition of a measurable set with Caratheodory's generalized definition. The text is full of hidden assumptions, for example the use of intervals as elementary sets. Its focus is on very specific facts about measurable sets rather than trying to give a rigorous platform for measure theory. The algebraic properties of collections of measurable sets are stated, but not extensively explored or used: they occupy lonely places in a sea of formulas and proofs. The text's didactic discourse is often placed inside the mathematical discourse which makes it very hard to follow. The author rarely supplements proofs with additional explanations or motivations. Overall the text is written in a declarative manner. It states facts without necessarily discussing or constructing them from previous knowledge.

The text itself is organized in chapters containing sections devoted to the study of particular objects or properties. Each section is concluded with a set of problems. The chapter on measurable sets contains a total of 28 problems, out of which 11 are production problems and 17 are practice problems. We provide two examples below.

A good example of a practice problem can be found on page 64, chapter 3. It asks the reader to prove a proposition previously stated in the section: Proposition 15. Let E be a given set, then the following 4 statements are equivalent:

i E is measurable

ii Given $\epsilon > 0$ there is an open set $O \supset E$ with $m^*(O - E) < \epsilon$

iii Given $\epsilon > 0$ there is a closed set $F \subset E$ with $m^*(E - F) < \epsilon$

iv There is a G in G_δ with $E \subset G$, $m^*(G - E) = 0$

v There is an F in F_σ with $F \subset E$, $m^*(E - F) = 0$

If $m^*(E)$ is finite, the above statements are equivalent to:

vi Given $\epsilon > 0$, there is a finite union U of open intervals such that $m^*(U \Delta E) < \epsilon$.

Problem 13:

"Prove proposition 15. [Hints:

a Show that $m^*E < \infty$, (i) $\rightarrow$ (ii) $\iff$ (vi)

b Use (a) to show that for arbitrary sets E, (i) $\rightarrow$ (ii) $\rightarrow$ (iv) $\rightarrow$ (i)

c Use (b) to show that (i) $\rightarrow$ (iii) $\rightarrow$ (v) $\rightarrow$ (i).]"

At first examination this qualifies as a problem for a programmed text, making the reader follow a series of steps in order to construct knowledge. However, upon closer examination it is a directed exercise aimed at proving a statement that already exists in the text and acknowledged as true by the author. To get some evidence for this claim let us examine solely (a):

The following fact is given in this section: for any set E and $\epsilon > 0$ one can find a set O_ϵ such that $E \subset O_\epsilon$ and $m^*(O_\epsilon) \leq m^*(E) + \epsilon$.

Since E is measurable, we can take our arbitrary set A in Caratheodory's definition as the set O_ϵ:

$$m^*(O_\epsilon) = m^*(O_\epsilon \cap E) + m^*(O_\epsilon \cap cE) \tag{10}$$

Since $E \subset O_\epsilon, O_\epsilon \cap E = E$. Also, $O_\epsilon \cap cE = O_\epsilon - E$. Using these two facts and the above valuation, we have:

$$m^*(E) + m^*(O_\epsilon - E) \leq m^*(E) + \epsilon \tag{11}$$

This concludes the proof because we can choose epsilon arbitrarily. The proof of (ii) $\iff$ (vi) is obtained through a series of set theoretic manipulations. Nowhere in the proof was any new knowledge about the measure of sets discovered.

A good example of a production problem can be found on page 71 chapter 3:

"Let f be measurable and B a Borel set. Then $f^{-1}(B)$ is a measurable set"

A first examination leads to speculate that the problem is simply a test of the reader's ability to work with measurable functions. However, it contains a hidden piece of knowledge which is a much more versatile definition of measurable functions, other than the one presented in the text. Unfortunately the problem does not state these goals explicitly. The reader who does not solve the problem will not be aware of this alternative definition.

5.2.1 The formatting of use

The text does not seem to employ any restrictive strategy toward the instructor except preventing him for using geometric arguments and other intuitive methods due to his choice of Caratheodory's definition. This strategy may have been Royden's assumption that presenting abstract definitions with large areas of applicability would allow the reader to "see the true nature of the mathematical object at stake. Furthermore, the textbook is strongly trying to mimic programmed textbooks. It requires one to follow the entire textbook blindly assuming it will produce knowledge about measure theory. The author chose a very strict "theorem proof style and rarely interludes to link results. Many of these important links are left as problems but this fact is never stated. It is closed as it does not seek to evoke any interpretations other than an abstract picture of measure theory.

5.2.2 Historical information

Historically, Royden's text has origins similar to Kolmogorov's. According to the preface, the text is based on the course "Theory of Functions of a Real Variable" that Royden was teaching occasionally at Stanford University. Whether Royden was using a text when giving his lectures or not remains speculation (but there is evidence to support that he was not, for example, in the preface to his text, he claims: "[My thanks to] Herman Rubin, who provided counter-examples to many of the theorems the first time I taught the course"(Royden, 1988). Royden's text has been subjected to two revisions. Its second edition was intended to "cover the basic material that every graduate student should know in the classical theory of functions of a real variable and in measure and integration theory". He claims his treatment of measure theory and integration was based on his pedagogical experience. He claims that the second edition was also adjusted to reflect comments he received from colleagues and students, but unfortunately he is not explicit as to what these comments were. Unlike Kamel, the translator of Kolmogorov's text, he does not elaborate why he decided to include problem sets or what goals he had in mind for them. The third edition was mostly modified in sections unrelated to Lebesgue measure theory, with the exception of the addition of new problems (why this was necessary or done is once more un-explained).

5.2.3 Liberal vs apodictic

Royden's textbook is apodictic because any omission leads to a collapse of the material. As seen from the above analysis, the lack of didactic discourse about the relationship between different results makes it impossible to predict whether dropping even a minor sentence will lead to a catastrophe. The same can be said about substituting existing material with alternate approaches a result might be that the problems at the end of each section are useless.

The textbook has a very narrow use in the hands of instructors because it has a very narrow and closed view of measure theory. An instructor with more geometric beliefs about measure theory cannot use such it since it does not permit him or her to do so without coming in conflict with the rest of the material.

5.2.4 The model reader

Let us call the instructor model reader of Royden's textbook MR_R. Based on the above analysis, the instructor model reader of Royden's textbook is an instructor who has a strong preference for an axiomatic approach to mathematics and prefers the "definition-theorem-proof" approach packed with formal symbolism rather than intuitive explanations. Royden's textbook has many gaps in terms of didactic discourse and hence expects an instructor who can fill them in on his or her own.

In the context of institutional constraints this textbook is quite convenient since its sections are relatively short and it is possible to cover all the measure theory topics in one semester.

5.3 Real and Complex Analysis (W. Rudin)

Object	**Definition/Method/Emphasis**
Elementary set	None stated.
Outer measure (μ^*)	Not used
Inner measure (μ_*)	Not used
Measurable set	If $\mathfrak{M}$ is a σ-algebra in X, then X is called a measurable space, and the members of $\mathfrak{M}$ are called the measurable sets in X.
Other measures	None are presented
Properties of collections of measurable sets	Emphasis on the link between topological spaces and measure theory.
Non-measurable set	Omitted.
Measurable Function	If X is a measurable space, Y is a topological space and f is a mapping of X into Y, then f is said to be measurable provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

Table 5: Main concepts of Rudin's textbook

Rudin's text is a radical change from the organization found in Royden's and Kolmogorov's manuscripts. The didactic discourse offers historical background about measure theory and the motivation of its study, particularly emphasizing its role in integration. Unlike the previous two authors, Rudin dismisses the need to examine Lebesgue's theory on the real line, and instead develops his theory in topological spaces. He approaches the task carefully and makes sure that every concept he chooses to use is defined before-hand, often including didactic discourse in his arguments. His text is similar to Kolmogorov's since the emphasis is once more on the algebraic properties (set-theoretic algebra) of measurable sets. His definition of measurable sets is radically different from Royden and Kolmogorov: he simply characterizes measurability through belonging to a particular collection of sets which form a sigma-algebra (Rudin, 1986).

Unfortunately none of the problems proposed in Rudin's text pertain to measurable sets or measure mappings. This situation is explained by his "unorthodox" approach to the subject. His definition through membership rather than constructing a mapping impairs him from providing problems that explore the properties of measure as a mapping on sets. Furthermore, the production problems are presented (as in Royden's) without explaining what concepts the student should be deriving when engaging in the problem-solving activity.

5.3.1 The formatting of use

Rudin's textbook reminds scholarly articles in its format. The text feels like a literary work rather than a textbook. It has interludes in the mathematical layer but overall addresses the reader in the way it is seen in scholarly articles. Whenever the text of the textbook is closed or open is hard to judge. It is perhaps open in the context of what is understood as "modern mathematics" (where mathematics is described entirely through set theory). However if we consider the textbook in the context of a first encounter with measure theory, it is closed since it develops a single, abstract, path to the topic. Therefore, the formatting strategy behind Rudin's text is clear: it's an attempt to force the reader to follow it step by step by blending everything into a package that only makes sense when delivered in its entirety, abandoning the modularity in favour of giving it a "modern mathematics" flavour.

5.3.2 Historical information

The history behind Rudin's text is not as well-known as it was for the previous two texts. In particular he is recognized in the mathematics community for his monographs and 3 textbooks. A review of the text by Victor Shapiro in the Bulletin of the American Mathematical Society notes that this text received extensive attention from the mathematical community. Shapiro claims the text excels in two areas. According to him, the choice of topics serves as a superior introduction into much of what is current mathematical analysis (Shapiro, 1968). Second, the text blends both concrete and abstract viewpoints on analysis. He praises the self-containment of the text with respect to many of the non-trivial theorems of analysis. Shapiro also provides insight on how he used the text before publishing the review. His first pedagogical experience was using the text in a first-year graduate level real analysis course and the second was a second-year special course on harmonic analysis and PDEs. He claims to have been able to construct the special course entirely from Rudin's text. In the first-year course, he felt the text avoided the pitfalls of other texts containing large preliminary sections on logic and set-theory (Shapiro, 1968). He warns the instructor, however, that choosing Rudin's text will require him to fill-in many details omitted in the text's proofs. Ultimately, Shapiro claims Rudin's text can be used for both first and second-year graduate courses. The main pitfall of the text, he claims, are the problems which, according to him, lack coordination, with some even being outright wrong.

5.3.3 Liberal vs apodictic

Rudin's strategy of formatting the reader's use of the text is apodictic because it cannot be supplemented. It only makes sense as a whole package and prevents the use of any other arguments or mathematical interpretations by remaining self-sufficient.

Although Shapiro (1968) states that Rudin's text can be used in different courses, he also issues many warnings. These issues are the level of the instructor, the level of the audience he or she is to deliver his lectures to, and the familiarity of the audience with some of the concepts used in the text. Shapiro claims he constructed a course based on the text, but unfortunately does not provide insight whether he used the text's structure to deliver the material. If he did, he might have been severely impaired in his ability to provide intuitive arguments regarding measurable sets. Rudin's text feels like a sequel to his earlier text Principles of Mathematical Analysis, which we conjecture he considered a pre-requisite to this text.

Another interesting distinction between Rudin's and the previous two texts is his omission of non-measurable sets. However, this omission is logical in his text: non-measurable sets can only be constructed and do not naturally arise from a "member in the collection" definition which uses topological characterizations.

5.3.4 The model reader

Let us call the instructor model reader of Rudin's text MR_W. Based on the above analysis, MR_W should be more radical in his beliefs about mathematics, preferring modern approaches exclusively through set theory and topological characterizations. MR_W is not concerned about the intuitiveness of the material; this instructor only cares about mathematical rigor. Considering that the problem sets do not seem to be intrinsically connected with the theoretical contents of the chapters, we conjecture that MR_W does not see measure theory as a subject of interest in itself, rather a tool to be used in other problems.

5.4 Measure and Integral: An Introduction to Real Analysis (A. Zygmund)

Object	Definition/Method/Emphasis
Elementary set	Intervals and "volumes" : $\sigma(S) = \sum_{I_k \in S} (I_k)$.
Outer measure (μ^*)	$\lvert E\rvert_e = \inf \sigma(S)$ over all covers S of E.
Inner measure (μ_*)	Not used
Measurable set	A subset E of $\mathbb{R}^n$ is measurable if for any $\epsilon > 0$ there exists an open set G such that $\lvert G - E\rvert_e < \epsilon$.
Other measures	None are presented
Properties of collections of measurable sets	Various results regarding the connection between measurable sets and various classes of sets of the real line (Borel, Topological, etc).
Non-measurable set	Small section, uses the axiom of choice in Zermelo's form.
Measurable Function	f on a set E is a measurable function if for any finite a the set $\{x \in E : f(x) > a\}$ is measurable.

Table 6: Main contents of Zygmund's textbook

From the above table we can conclude that the text attempts to stay close to the geometric origins of measure theory. Zygmund uses the topological structure of the real line in many of his results (including his definition of a measurable set). Most of the theorems (and there is a quite large amount of them) are given with proofs and there is a fair amount of didactic discourse linking the different results. Many of the proofs are similar in their arguments, showing an attempt by the author to emphasize the techniques that a reader should consider when faced with a problem found in the text. It does not assume any knowledge other than the structure of different collections of sets of the real line. The author includes a section where he shows some equivalent definitions of measurable sets including Caratheodory's.

The text itself is organized in chapters containing sections devoted to the study of particular objects or properties. The problems are given in bulk at the end of each chapter. The chapter on measurable sets contains a total of 47 problems, out of which 13 are production problems and 34 are practice problems. We provide two examples below.

A good example of practice problems can be found on page 48, chapter 3:

> Prove (3.29), where (3.29) is the following theorem: Suppose that $|E|_e < +\infty$. Then E is measurable if and only if given $\epsilon > 0$, $E = (S \cup N_1) - N_2$, where S is a finite union of non-overlapping intervals and $|N_1|_e, |N_2|_e < \epsilon$.

If E can be expressed as stated by theorem (3.29), then it is measurable because S is measurable (since it is a finite union of non-overlapping intervals) and both N_1, N_2 are measurable. Their measurability follows from the fact that we can always find a closed set F contained in them such that $\forall \epsilon > 0$, $|N - F| < |N| < \epsilon$. This concludes the first part of the proof.

Let $\epsilon > 0$. Let $S' = \bigcup_{k=1}^{m} I_k$ a finite union of nonoverlapping intervals; N_1' and N_2 such that $E = (S' \cup N_1') - N_2$ and $|N_1'|_e, |N_2|_e < \epsilon/4$. Then $E = [\bigcup_{k=1}^{m} I_k^\circ \cup (\bigcup_{k=1}^{m} \partial(I_k) \cup N_1')] - N_2$. Let $S = \bigcup_{k=1}^{m} I_k^\circ$ and $N_1 = \bigcup_{k=1}^{m} \partial(I_k) \cup N_1'$. Therefore S is open and by subadditivity and monotony of the outer meausure $|N_1|_e < \epsilon/4$. Also $E = (S \cup N_1) - N_2$. Since the outer measure is defined as an inf, there exist a countable collection of open intervals $\{J_k\}_{k\in\Delta}$ that cover N_1 and satisfies $\sum_{k\in\Delta} |J_k| < |N_1|_e + \epsilon/4 < \epsilon/2$. Let $O = \bigcup_{k\in\Delta} J_k$. Then O is open, $N_1 \subseteq O$ and by the subadditivity and monotony of the outer meausure $|O \setminus N_1|_e \leq |O| \leq \sum_{k\in\Delta} |J_k| < \epsilon/2$.

On the other hand, is possible to show that $(S \cup N_1) - E = (S \cup N_1) \cap N_2 \subseteq N_2$, and therefore $|(S \cup N_1) - E|_e \leq |N_2|_e < \epsilon/4 < \epsilon/2$.

Finally, $E \subseteq S \cup N_1 \subseteq S \cup O$, wich is open, and by the subadditivity and monotony of the outer measure we get:

$$
\begin{aligned}
|(S \cup O) - E|_e &= |[S \cup (N_1 \cup (O \setminus N_1))] - E|_e \\
&= |[(S \cup N_1) \cup (O \setminus N_1)] - E|_e \\
&\leq |(S \cup N_1) - E|_e + |(O \setminus N_1) - E|_e \\
&\leq |(S \cup N_1) - E|_e + |O \setminus N_1|_e \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

Another practice problem is found on page 48, chapter 3:

"Carry out the details of the construction of a non-measurable subset of $\mathbb{R}^n$ where $n > 1$"

While I shall omit the proof for the above problem, these problems are practice problems because they do not require much more than set-theoretic constructions for their solutions and their end result is already assumed true therefore it is hard to see what information the reader has gained from this process other than practicing his set-theory skills.

A good example of a production problem is found on page 48, chapter 3:

"If E_1 and E_2 are measurable subsets of $\mathbb{R}$, show that $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1||E_2|$".

This problem shows the reader that a space resulting from a direct product preserves many of the properties of the original space.

5.4.1 The formatting of use

Zygmund's formatting strategy is to stay close to the original ideas behind Lebesgue measure. While he does avoid inner measure, he still leaves it as an exercise in the problem sets. He presents concrete examples and includes pictorial representations of some of the material.

His textbook is open because it does not try to restrict the reader to unique definitions: an entire section of the textbook is devoted to showing equivalent definitions of measurability. Moreover, these definitions are not only stated but also proved with a lot of care and explanations.

The problem sets do not contain the bulk of the material, but do contain certain important results. Skipping them, however, does not prevent a reader from understanding future material found in the textbook.

5.4.2 Historical information

The history behind Zygmund's text is a mystery. In their preface they do however explain the motivations behind their choice of format. They believed that "considering special cases leads to better understanding of general situations" and that everyone "learns by repetition" (Zygmund & Wheeden, 1977).

5.4.3 Liberal vs apodictic

The textbook is liberal because it invites the instructor to explore the material with the students. It can be used in its entirety as it is well balanced in terms of mathematical and didactic discourses.

It is not restrictive, often even citing other textbooks when presenting certain results. The problem sets are at the discretion of the instructor as they do not influence the outcome of the reading. The presentation of many different approaches makes it accessible to instructors with different beliefs about measure theory.

5.4.4 The model reader

Let us call the instructor reader of Zygmund's textbook MR_Z. MR_Z is an instructor who wishes to remain close to the original treatise of measure theory while trying to make it as accessible as possible by removing some of the more technical elements. MR_Z seeks to evoke many different interpretations of the material in his students and likes to supplement the results with examples whenever possible. MR_Z also believes that it is beneficial to include graphical representations whenever possible. MR_Z is also interested in presenting a larger picture of measure theory since some of the problems are applications of measure theory to other branches of mathematics.

5.5 Remarks and Analysis

The four texts were published in a twenty year period: between 1957 and 1977. It seems a reasonable hypothesis that the four texts are the results of the authors' efforts to formalize their course notes. The differences in approach might respond to their belonging to different schools of thought, to their pedagogical beliefs, to the ways they used measure theory and analysis in their research, and quite likely, to all these three aspects together.

This short overview leads one to conjecture that the format of measure theory texts evolved over this short period of time. For example, Kolmogorov's text is likely to be the most accurate historically since it recycles all of Lebesgue's original argument and only adds new information. Kolmogorov's text is an exhibition of almost all the mathematical knowledge established at his time about measure theory. One could even debate that this is an "encyclopedia of measure theory" and turned into a textbook by a publisher, particularly since the original manuscript contained no problem sets.

Royden's textbook chronologically follows Kolmogorov's and exhibits a different approach. Instead of extensively studying the measure map it uses Caratheodory's abstract definition with the real line as its realization. Caratheodory's definition is practical because it allows circumventing a lot of results necessary in the original model for a "functional" measure theory. This in turn comes at a price: an instructor opting for this text loses the possibility of using intuitive arguments to justify the material.

Rudin's text, as Shapiro's review indicates, is an attempt at a radical approach to the topic. The textbook feels like a university-level stab at the "New Math" program of the 1960s. As stated in its goals "New Math" claimed that if "the axiomatic foundations of mathematics were introduced to children, they could easily cope with the theorems of the mathematical system later." (Kline, 1973). One could adapt this sentence into: "if the study of analysis via other disciplines was introduced to students, they could easily cope with the problems they will encounter later as professional mathematicians." Shapiro described the text exactly in the same way but with different words: "it reflects much of what is modern analysis". Hence, we could speculate that it tries to study measure theory through the unification of all mathematical disciplines involved and through an axiomatic approach (no measure map is actually ever constructed but only stated and the emphasis seems to be on integration).

Finally, the most recent treatise is Zygmund's textbook. Zygmund's book is a convenient balance between the rigor of Kolmogorov's and the modularity of Royden's. Everything is in the textbook, and the instructor is relatively free to construct a course which he feels appropriate.

All four texts contain problem sets. Zygmund argues in favor of such formatting and for the inclusion of a large number of practice problems in his claim "everyone learns by repetition" while other authors did not express opinions on this matter simply including problem sets as some formatting standard they are obliged to follow.

However, the inclusion of problem sets suffers from certain deficiencies. Even Zygmund's text, whose problems often include explanations as to why they are placed in the text, sometimes omits information regarding their purpose. This is especially puzzling in the case of the problems about the Cantor set and the non-measurable sets, which are reduced to a minimum in the other texts. Possibly, Zygmund assumes that the instructor will supplement the problem set with certain explanations, or simply leaves the instructor the luxury of choosing from the pool of problems (Zygmund's text contains the largest number of problems with respect to the other texts).

There is also much divergence on Zygmund's claim that "considering special cases [leads] to better understanding of general situations". Rudin disagrees with this simply because his text makes the emphasis on general situations, to the point of not distinguishing between functions of the real and complex variables (Shapiro, 1968). Royden attempts to reconcile the difference, by including more general definitions (e.g., Caratheodory's) while studying special cases such as measure on the real line. Kolmogorov avoids the conflict by simply constructing the knowledge through its historical path, hence not choosing to be specifically abstract or concrete.

The most paradoxical element in all four textbooks is the non-measurable set. Royden, Zygmund and Kolmogorov all include it as a section in their textbooks but fail to provide a mathematical or didactic, coherent explanation for their decision. They all make a very good case for the study of measure theory in their introduction chapters, but non-measurable sets seem to be outside of their reach. This may be the result of two problems that plague this set. As seen in Rudin's textbook, this set has no chance to naturally arise if the algebraic path to measure theory (at least on $\mathbb{R}^n$) is taken, it has to be "forced" onto the real line by a clever application of the axiom of choice. The second is its actual contribution and value to the goals of measure theory. This can be argued, but if we assume for the sake of the discussion that the goal of measure theory is integration, then our problem is not constructing non-measurable functions but identifying them.

The physical features of the texts are also quite interesting. Zygmund's text is a fairly small book focused on measure and the integral while the three other texts are thick, quasi-encyclopedic texts on analysis in general. As Shapiro's experience shows, these texts are meant to have a large area of applicability in the teaching process: a single text can be used to teach multiple courses.

Chapter 6

Interview Analysis

In chapter 5, we have studied different textbooks and identified the model readers pre-supposed by these textbooks. The next step in our study was to see whether some examples of such model readers could be identified within academic institutions as empirical readers. To accomplish this goal, we recruited three instructors at University U who agreed to participate in a semi-structured interview. Our selection process was based on their familiarity with the topic, their use of the measure theory in their professional activities and their experience teaching measure theory. Unfortunately, only one of the professors we interviewed has taught measure theory in the past. The interviews were conducted orally and lasted between 45 and 60 minutes. Below is a list of some of the questions that were used in the process:

- Have you ever taught measure theory and if yes, which textbook did you use?
- Which textbook would you use and why?
- How closely would you follow the textbook?
- Given the following definitions, which would you prefer to use in the classroom and why?
- Do you think that graphical representations help students in the learning process?
- Do you think that the inclusion of narratives and supplementary explanations in proofs is beneficial to the students?

The fourth item above refers to two definitions that the participants were shown in the course of the interviews. One was taken from Royden's text and the other from Zygmund's. To show the definitions to the participants, the definitions were retyped so to hide any indication of where they came from in order to eliminate bias. Such bias would occur if they liked some sections of a particular textbook and would be willing to sacrifice some of their beliefs with respect to the section on measurable sets since the textbook is, at least in their opinion, overall to their taste. The full definitions that we presented are shown in the appendix. We hoped these questions would allow us to gain an insight into how professors would hypothetically use a textbook in their teaching activity.

6.1 Participant A

Participant A has never taught measure theory, however he[1] does have experience teaching probability and statistics courses and feels qualified to teach measure theory. In his choice of text, participant A emphasized the fact that Measure Theory and Probability by Adams and Guillemin presented constructions of measure and gave practical examples of how these constructions are applied to probability. Moreover, he stressed that this text gives *more* motivation to the study of measure theory.

When asked about the closeness between his lectures and the text, he expressed that he would do "an interpretation of the material" by perhaps substituting or amending certain explanations found in the text with his own but still stressed that the main theorems and concepts would be drawn from the text without alterations.

Participant A generally expressed the opinion that students are more at ease with concrete objects than abstract mathematical structures. According to him, the intuitive definitions and explanations should not be neglected, but at the same time it is important to teach how to translate from them to non-intuitive objects and theories.

After being presented with several definitions and theorems, participant A concluded that if he was mandated to teach measure on real spaces he would not use Caratheodory's abstract definition in favour of the open set definition found in Zygmund, and would turn to Caratheodory's approach once (or if) he reached the sections on abstract spaces. He felt Caratheodory's approach is not constructive and it doesn't provide any insight with respect to computing measures.

Another interesting insight was participant A's justification for the inclusion of non-measurable sets. He felt that their existence justified the study of measure theory because "[...] if you don't warn them that there are ugly sets which thankfully can be negligible when compared to the nice sets then we don't think they see the purpose of studying something complicated for it." However he remained firm on the idea that the construction of a non-measurable set needs to give the reader a good intuition about what is going on, and that overall it was the existence of this set, and not so

[1] In this thesis, all three participants are treated as completely anonymous - as part of this anonymity, they are addressed in the masculine form without any connotation to their actual gender.

much the technique used in the construction that best justifies its presence.

When inquired whether he had a preference for the definition of measurable functions based on the definition of measurable sets, participant A claimed that he would prefer a definition that shows the closeness of measurable functions to continuous functions, to draw a parallel between the behaviour of measurable sets and measurable functions with respect to well-known objects.

Participant A also supported the inclusion of pictorial representations in texts and narratives in proofs.

Interesting also was his claim that the culture of writing textbooks has changed, based on different books he has seen over a period of 5 years. He felt that they are now written with the modularity of teaching a class in mind, even including diagrams of the topics that should be included in the context of different courses. He preferred this writing style to the one found, according to him, in texts written 50 years ago.

Textbook use	Beliefs about learning	Beliefs about mathematics
Uses the textbook to prepare lecture but would prefer to interpret the material rather than simply reproducing it.	Students learn better from concrete examples rather than abstract models. They should be exposed to multiple definitions and the approaches should be constructive.	Mathematics is not compartmentalized, all topics are connected between each other. Mathematics is about doing something creative with your knowledge.

Table 7: Summary of Participant A's answers

Based on the answers, Participant A is a *creative reader*. This can be seen in his choice of describing the teaching activity as "doing an interpretation of the material" as opposed to merely following the textbook. He does not feel obliged to honor the textbook entirely and accepts that he may have to do alterations to the text in order to produce a more meaningful teaching activity. Participant A is closer to a *sophist instructor* type as he did not mention problem-sets as the main mechanism in the learning process - also because he talks about what he has to deliver to the students but says nothing about what the students have to do in turn with what they "receive". He was more focused on the future use of the material claiming that this course is only meaningful if the students "go and do something creative with it". He is also closer to the *intuitivist instructor* type as he acknowledges the importance of being mathematically coherent, but prefers the presence of intuitive arguments, concrete examples and some connections with other disciplines where measure theory acts as a powerful instrument rather than exposing the material as something within itself.

6.2 Participant B

Participant B has never taught measure theory but has taught courses that required him to give preliminaries of measure theory. He did not exhibit any particular preference for any text, except for Rudin's Real and Complex Analysis, a choice he justified by citing the following reasons: it was the only book he read completely as a student, it is thin, it is written by a great analyst of the century, its material and problems are nicely organized and there are no logical mistakes.

Participant B is averse to following textbooks, claiming that a book is for a student's home reading and that it is "stupid to reproduce proofs found in texts". He also chose Caratheodory's definition of measurable sets over Zygmund's definition because that's the one he was taught and knew, while he felt a need to prove Zygmund's definition is consistent.

He exhibited no beliefs when it came to a student's understanding of either definition (Caratheodory's or the one that appears in Zygmund's text). He stressed that these questions on student understanding depended on the level of mathematical maturity of the presumed audience for the given course. In general, he expressed his feeling that it would be more appropriate to leave the proof of certain theorems and definitions as exercises to the students. He did not elaborate as to why he would choose such practice, but based on his further answers we can see that he believes that the act of learning, at least from his point of view, is something that should be only the concern of the students, rather than of the instructor. Regarding the learning process itself, participant B stated that (mathematical) knowledge comes mostly from communicating with fellow professors, attending lectures and reading serious monographs and scientific papers. According to him, serious people simply do not use these "special" textbooks.

Participant B also claimed that the core of a general measure theory course was to pass as quickly as possible to Lebesgue integration, which according to him was the only (measure theory) tool that a professional mathematician needs. He based this claim on his career as professional mathematician saying that "[...] since [studying this as an undergraduate student] I have never used it".

When discussing non-measurable sets, he felt that it was the technique used to construct such set that was important. In his words: "[...] I was taught [the construction of a non-measurable set] 30 years ago and I still remember this. The idea is so bright [that] it should be taught."

When inquired about the definition of a measurable function and the influence of the definition of a measurable set, he claimed that this would depend on the space where the function is defined and did not exhibit any specific preference. He also shunned pictorial representations because measure theory discusses pathological objects, claiming that the construction of Lebesgue's theory is not of geometric character.

When discussing the presentation of proofs in textbooks, he said that he would prefer narratives explaining what the author plans on doing and a general structure of the proof. He also felt that for future mathematicians all of the material learned from measure theory textbooks was nonsense.

Textbook use	Beliefs about learning	Beliefs about mathematics
Does not use them, claims serious mathematicians do not use them either.	Students should construct their own knowledge from communicating with fellows, reading scholarly articles, etc... Textbooks do not play an essential role. He would like to give multiple definitions.	Mathematics is about rigor and coming up with your own proofs and ideas. Sees it as a community activity (mathematics community).

Table 8: Summary of Participant B's answers

Participant B is a *creative reader* since he claims outright that he does not use textbooks for more than problem sets and skeletal structures. He prefers looking for ingenious proofs as well as fabricating them when possible. He is closer to the *constructivist instructor* type as he claims that students have to build the knowledge on their own from interacting with colleagues, coming up with their own proofs and reading peer-reviewed papers. For him the textbook should not be the source of mathematical knowledge.

He is also a *Hilbertist instructor* because he seems to only care about the mathematical validity of arguments rather than their intuitiveness. This is best exhibited in his attitude when asked to choose definitions, as he claimed to have no preference as long as the definition was mathematically accurate. Moreover he did not care about intuitive arguments or visual representations which he finds inappropriate for measure theory.

6.3 Participant C

Participant C had a lot of experience teaching measure theory. He has alternated between two texts: Rudin's Real and Complex Analysis for graduate courses and Royden's Real Analysis for the undergraduate sections. He did not choose Royden's text himself, this was done previously by another faculty member and he decided to keep using it because he could not find another text he really liked. He felt that an important aspect of the textbook was the problem sets.

He claimed that if he was satisfied with the textbook he would follow it pretty closely, but he would use his own examples and only keep the organization of the text such as the main ideas and theorems. He stressed the importance of having a good pool of challenging problems which he could assign.

When proposed the different definitions of measurability he claimed that this choice would be dictated by the development of the material as whole. According to him picking a definition is impossible without knowing the prior development that leads up to it in the text. He also claimed that every measure theory textbook has a particular development and you really need to pick one and follow it closely in order to avoid ambiguities. He also shunned showing multiple definitions because, in his opinion, it would be confusing for the students. He stressed the importance of Caratheodory's definition because it had a much larger area of applicability than any other accepted definition.

Participant C also felt that a lot of the theorems should have been left as exercises because according to him it is the problem-solving activity that leads to understanding.

When discussing non-measurable sets, he found himself a little puzzled because he had not devoted much attention to it in the past simply because "we always do it". After a short reflection he concluded that on one hand it would justify why we bother with the constructions of measure theory, and on the other because it was a standard equivalence class construction used in mathematics, which according to him is a very important idea.

When inquired about the relationship between the definitions of a measurable function and a measurable set, he said that the two were entirely independent.

In regard to narratives in proofs, he felt these were important because a lot of it is learning how to write mathematics, something he felt was missing when he was reading students' assignments or papers.

Other notable mentions included that he felt that students were overly attached to textbooks. He claimed with amusement, that while undergraduate students are often ignoring books, he would rather see that behaviour in graduate students, who according to him are often trying to memorize entire textbooks. He also had organizational objections regarding the divisions of undergraduate and graduate level course; in particular he did not like the fact that the measure theory was now a mixed course, as it is hard to find a textbook that would be adequate to both levels.

Textbook use	Beliefs about learning	Beliefs about mathematics
Take the main body of the material so students can easily follow the course.	Students learn from solving problems. They should be only given unique definitions.	It is not only important to be mathematical accurate but also express your ideas (writing) properly and clearly.

Table 9: Summary of Participant C's answers

Participant C is a *creative reader* since he also tries to avoid straight-forward reproduction of material found in textbooks. While he acknowledges keeping the main blocks intact he does prefer to come up with his own examples and explanations. He is a *constructivist instructor* because he firmly believes students learn mathematics through the problem-solving activity. Finally, he is an *intuitivist instructor* as he supports the presentation of pictorial representations and generally favors more explanatory proofs than ones packed with obscure mathematical formulas.

6.4 Analysis and Discussion

The interviews provide us with much insight regarding the relationship between texts and instructors. First, we can observe a connection between the instructor model readers extrapolated in chapter 5 and the real readers found here. In order to illustrate our point, I present a table with the model readers of the textbooks we examined containing their key characteristics and a table with the participants' characterizations as readers. By comparing these two tables we can construct a table that puts the participants in a correspondence with model readers of the textbooks we examined:

MR	Main characteristics
MR_K	stays close to historical path, uses intuitive arguments, many different approaches, problem sets are optional, mathematical rigor, likes discussing the material and its possible applications
MR_R	only mathematical accuracy counts, theorem-proof style, restricts himself to a unique definition, does not discuss the material or the applications
MR_W	radical approaches, wants to be modern and abstract, does not care to be intuitive, very goal oriented (i.e. measure theory for the purpose of integration)
MR_Z	remains closer to historical path, many different approaches/definitions, includes graphic representations when possible, problem sets demonstrate possible applications of measure theory to other branches of mathematics

Table 10: Model readers from textbooks and their main characteristics

Participant	Profile
Participant A	Creative - Sophist - Intuitivist
Participant B	Creative - Constructivist - Hillbertist
Participant C	Creative - Constructivist - Intuitivist

Table 11: Profiles of participants

Participant	MR	Textbook
Participant A	MR_Z	Zygmund's textbook
Participant B	MR_W	Rudin's textbook
Participant C	MR_R	Royden's textbook

Table 12: Correspondence between participants and MRs

The correspondences were established based on the answers in the interviews and our characterizations of instructor-types. For example, a creative instructor is likely to prefer open-text textbooks, whereas a passive instructor is likely to prefer closed-text textbooks. The same way, a constructivist instructor is likely to prefer liberal textbooks, whereas the sophist is likely to be more at ease with apodictic textbooks. Finally, intuitivist are likely to prefer open-text textbooks while Hillbertists are likely to prefer closed-text textbooks. The reverse is also true, for example Royden's model reader (MR_R) seems to put emphasis on the mathematical accuracy and rigor, which makes him a Hillbertist, whereas Zygmund's model reader (MR_Z) seems to put the emphasis on the variety of interpretations and connections with the physical world, making him a intuitivist instructor.

In what follows, we discuss some other important findings derived from the participants' answers. These include the belief "the institution won't fail me" and their understanding of the role of non-measurable sets in the teaching of measure theory.

The "institution won't fail me" belief is composed of two-pieces: one is the participants' reference to texts they originally learned from, and the second their passive acceptance (i.e., no strong rejection) of the institutionally chosen text. For example, participants A and B represented instructors with no previous experience in teaching measure theory. While they did have some preference towards certain texts they generally fell back on their experiences as students. Since they must have passed the course successfully as students using a given text, they may perhaps link their original success with the success of their hypothetical students. Another possible explanation could be simply due to a prolonged semi-successful use of a given text (semi-successful implying the success rate of passing a course using such text in the past years is an acceptable percentage): instructors accept using it since there is no immediate evidence the text is unsatisfactory in the context of the teaching process. The second piece is the implicit hierarchy of the institution. While the instructors have academic freedom they are also subject to an inner hierarchy where there is an implicit need to respect decisions made by previous faculty members because of considerations such as experience and seniority. This was mentioned by participant C who told us that Royden's text was already in use at University U prior to him joining the faculty. Also, participants A and B mentioned that they wouldn't mind using Royden's if that was the textbook typically used in the institution.

Regarding the role of non-measurable sets, both participants A and C claimed that these justify the need for the construction of measure theory to a student, which seems to be a didactic justification. However, mathematically it seems absurd to give such an importance to measurable sets due to their artificial nature. First, measure theory itself only requires ZF (Zermelo-Frankel) logic, and does not rely on the axiom of choice; hence it is mathematically incorrect to claim that the ZFC (ZF amended with the axiom of choice) must hold in order for measure theory to be constructed. For that matter, even constructing the real numbers through Dedekind cuts on the rationals or Cauchy sequences only requires ZF. By claiming that the existence of a non-measurable set justifies the rigor and mechanics of measure theory, an instructor is introducing the false belief that measure theory requires ZFC. Second, although Lebesgue's original motivation was to solve the limit/integral problem, in the further development of measure theory the focus was more on justifying why certain results obtained by the Riemann integral are actually true (e.g., the ones pertaining to Fourier series integration), and on extending the integration process to abstract spaces. Evidence of this focus are the Daniell integral and the Haar integral: Daniell partially provides a "usable" (i.e., procedural) version of Lebesgue integral for the real line, and Haar provides a realization of Lebesgue's idea on continuous groups. Thinking of non-measurable sets as the motivation for constructing measure theory doesn't seem accurate. The non-measurable set described by Vitali is more of a curiosity, much like Tarski's balls, than an essential concept in measure theory.

Participant B has a different perspective perhaps more consistent with the use of measure theory in today's mathematics research: he only wants to teach the amount of measure theory necessary to pass to integration (he only sees interesting to include non-measurable sets in the curriculum because of the technique used in constructing them and not because of what they mean to measure theory).

On a different note, participant C discussed problems of time constraints and organizational issues of a course, which leads to suspect that some instructors may simply not have the time to familiarize themselves with all the available didactic literature for their course.

The research completed here shows that an instructor would need to be self-aware of what model reader he is most close to in order for a textbook to be used to its best potential.

Chapter 7

Conclusions

Upon completion of this research we have come to understand a little better what are, in essence, the differences we had intuitively and a priori observed in measure theory textbooks. These differences are of didactic nature - constraining the possible interpretations and uses for the text in the teaching practice - and of mathematical nature - following different paths to develop the concepts of measure theory. Part of these differences may be explained as the result of the schools of thought to which the authors belonged, historical context, and their beliefs about didactics and mathematics.

A reflection of these differences was observed in the interviews as the three participants exhibited different beliefs about the use of textbooks, about measure theory, and about learning.

7.1 Instructors' uses of textbooks at the graduate level

Regarding instructors' "practical" use of textbooks, we found that Rezat's claim that instructors follow them closely in preparing their lectures and for assigning readings/assignments (Rezat, 2006) is debatable at the graduate level. The interviews have shown that the use of textbooks in the teaching activity at the graduate level is much more diverse than it is at the elementary and college levels (as reported in the research literature; c.f. Weinberg, 2009; Weinberg & Wiesner, 2010; Mesa 2004). This was best seen with participant B who simply does not see textbooks as essential or useful to his teaching activity. Moreover participant C emphasized the problem that was pointed out by Keitel: the textbook becomes the object of learning; something Keitel believes to be detrimental to the learning process (Keitel et al., 1980). In the same vein, Rezat's remark that textbooks actually represent mathematical knowledge was refuted by participant B who claimed that mathematical knowledge is everything but textbooks (e.g., discussions with fellows, scholarly articles, seminars, etc.).

In our interviews, we found that the three participants are creative readers, thus not simply following the textbook step by step. The ways in which they use the textbook seems to be affected by their beliefs about learning (two of them strongly believing in problem solving as the root for learning - constructivists - while the other focusing more on delivering' knowledge - a sophist) and their beliefs about mathematics (two of them giving an important place to intuitive arguments - intuitivists - while the other believing mostly in the axiomatic nature of mathematics - a Hilbertist).

While this thesis did not include students, it is nonetheless connected to Weinberg's findings about implied and empirical readers. With respect to K-12 students, Weinberg (2010) identified that the degree of success of a textbook as a pedagogical tool depends on the extent to which the implied reader matches the empirical reader. In our research, we have identified that this statement is likely to hold true also in the case of instructors since we did identify that each instructor, based on his mathematical beliefs and teaching practices, was the implied reader (in our case the term is model reader) of a particular textbook but not all textbooks.

7.2 The non-measurable set

In our study of textbooks we found that textbooks approach differently (if at all) the notion of non-measurable set. We tried to understand the source of this difference by looking at the textbooks and discussing with our participants. The textbooks did not provide many answers; they all included the topic without elaborating why. This attitude was mirrored in the interviews as all the participants were unable, at first, to make a strong case for the inclusion of this topic. Their initial answers seemed to convey the belief that it is more of a didactic convention: we introduce in our teaching pathological examples without necessarily questioning what they mean in mathematical terms (apart from the fact of being pathological). Upon reflection, their explanations for including non-measurable sets in their teaching didn't seem to be consistent with the role that these sets play(ed) in (the development of) measure theory.

This issue may be a result of the dual nature of measure theory as pointed out by our participants. If measure theory is studied as a self-contained topic, then non-measurable sets are undeniably interesting objects. However, if as participant B claimed, the goal of measure theory is to pass as quickly as possible to integration, or as participant A claimed, to go and do something with it, non-measurable sets are not a particularly interesting or useful object (non-integrable functions will often be the result of unboundedness rather than of a pre-image of a subset of its range being non-measurable sets).

7.3 Model readers vs empirical readers

Our research suggest that a correspondence between model readers of textbooks and empirical readers can be established. Such correspondence would “match” formats of use and interpretation with readers who are already predisposed to use the textbook and interpret its content in the ways allowed/imposed by its formatting strategies.

We speculate that instructors are subconsciously aware of what kind of readers they are. However, as pointed out by participant C, instructors often feel they cannot afford the time to browse through all the possible textbooks and really think about how they fit in their teaching practice. Moreover, the pressure of the institution to simultaneously deliver the course to graduate and undergraduate students only makes the situation messier.

Conflict between the instructor reader and the textbook seems to occur at two layers: mathematical and didactic. The mathematical disagreement is most acute when instructors who have beliefs about the purpose of measure theory end up with textbooks who do not share their beliefs. For example, participant B’s who believe that the teaching of measure theory should focus exclusively on integration with be in conflict with (most) textbooks as they typically devote half of their contents to measurable sets. An instructor such as participant A who believes that the purpose of measure theory is of a practical nature (e.g., its applications to probability) would conflict when using textbooks that present measure theory as a topic in and of itself.

The didactic conflict may occur when the textbook’s model reader has an implied teaching practice incompatible with the one of the actual instructor. For example, an instructor such as participant C who believes that students need to solve problems in order to learn a mathematical topic would experience conflict when using a textbook that doesn’t offer good sets of problems or doesn’t leave any of the important results as exercises (his claims). An instructor such as participant A who wants to incorporate intuitive arguments and applications would experience conflict when using a textbook that has a pure axiomatic nature.

The interviews have shown us that instructors generally wish to exercise more freedom than that given by textbooks yet often rely on them when preparing lectures. Their answers suggested that they may do so as a result of feeling the need to satisfy the conventions established within the institution which is often dominated by textbooks rather than lectures. Their feeling of being overloaded also impacts the time they give, if any, to looking for a textbook closer to their teaching practice.

Most importantly, the interviews have shown that instructors are not aware of their beliefs about the use of textbooks, about mathematics and about learning. Reflecting on these issues may smooth the interactions in the triangle student-instructor-textbook. Finally, we wonder why authors bother writing seven page introductions discussing the abilities of the student model reader and zero pages discussing the instructor model reader. After all, the authors (or at least the editors) are well-aware that their texts are not used only by students but are part of the student-teacher-book triangle. In our opinion, the inclusion of such a section would allow instructors to be a little more aware about how the textbook fits into their teaching practice than they are at the moment. It may be that the authors simply didn't think about it and assumed that teachers will adapt to the instructor model reader pre-supposed by the text (as they may often do at elementary and college levels). The interviews, however, showed clearly that this is not the case.

7.4 Future research

This research is only a peek into textbook use by instructors. It would have been interesting to correlate these results with general classroom success for various professors.

Another interesting study that can contribute to a better understanding of instructors' uses of textbooks would be to document instructors' preparation for teaching and its implementation in the classroom.

It is also useful to note that, as McNamara et al. (1996) (as cited in Weinberg & Wiesner, 2010) pointed out, reading models (text-centered v.s. reader-centered) depends on textbooks as much as they depend on instructors. These authors have identified that textbooks also play an important role in establishing a given reading practice independently of the instructor's intentions. Part of this is mirrored in our research as we have found that generally (but not always) instructors try to stay close to textbooks. However, this decision forces them to endorse the reading-model proposed by a textbook. However, a research about what reading-models instructors are endorsing and why at the graduate level would require us to empirically (by observing and interviewing readers) identify textbooks that are inherently endorsing text-centered or reader-centered reading models and study the differences between them. Such a research would involve students and textbooks, where we would first identify the reading-model endorsed by the textbook, and then have a test group of students to an unconstrained reading (i.e. without the interference of an instructor) and a second test group of students who would have an instructor interfere. After, we would try to compare the reading models endorsed by the students in both groups and identify how they came to be endorsed.

Our research made us notice that the didactic contract described by Brousseau (1984) (as cited by Lithner, 2000) in the case of high-school, college and undergraduate levels is not the same as the didactic contract between the teacher and the student at the graduate level. We believe it would be interesting to characterize the main properties of such didactic contract where the goal of education is to form professional mathematicians.

7.5 A final note

Finally, we note that measure theory today is often disconnected from its original formulation; while eastern European authors (Zygmund and Kolmogorov) did not abandon the Lebesgue approach, western authors whose textbooks are used in most North American universities (such as Rudin and Royden) favored Caratheodory's axiomatic approach. To us, this points to an excessive "algebraisation" of measure theory, as it was done with college-level calculus, which can be argued to have had the contrary results than those anticipated.

Appendix A

Definitions of measurable sets given to participants of the study

Definition text A: A set E in $\mathbb{R}^n$ is said to be measurable if for each set A we have $m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$

Definition text B: A subset E of $\mathbb{R}^n$ is said to be Lebesgue measurable or simply measurable if given $\epsilon > 0$ there exists an open set G such that $E \subset G$ and $|G - E|_e < \epsilon$.

Appendix B

Transcript from Interview with Participant A

I: Have you ever taught the measure theory course?

A: No

I: If today you were approached with the task of teaching this course what textbook would you use if you have any idea?

A: I would probably go with one of the texts I learned things from that I might of liked, well, I learned measure theory from the Royden book, not sure that would be my favorite book to use, I quite like this book that I acquired later that was called Measure Theory with Probability Applications (Adams and Gilman), so they do constructions of measure and stuff but they also give practical examples of how it's applied to probability.

I: Ok, so that would be your reason for using a textbook like that, because they're giving examples?

A: It gives a little more motivation as opposed to just straight theory without knowing why bother...

I: So if you were given the task and the freedom of choice in the text you would go with this text based on your previous answer that it's giving examples and that it's showing students what measure theory could be potentially used for?

A: Yes.

I: Were you actually aware that for the last couple of years the textbook that's been used within Concordia University for this course was actually Royden?

A: No. I'm aware that it's a textbook that's used in a lot of schools but I'm not aware that it was used at Concordia.

I: But would you accept using it or would you still rather go with another text?

A: I think I would accept using it not as the required book but what we call it... supplementary book... a reference book. I would rather teach from Measure Theory with Probability Applications.

I: So how closely would you follow the textbook when teaching the course?

A: I'd say if the book is good fairly closely. Fairly closely, meaning you know, maybe 15 percent of the time you make additions and make changes.

I: So for example if you are preparing a lesson plan and you have the book how would you use it?

A: Well I would do some interpretation of the material but basically... it's you know... I never teach literally from the book meaning that I don't read... I don't read/write what's written in the book. When I say I would follow the book it means that it would be easy for students to know which chapter would then be relevant to read. So you know, I give some slightly different explanation of what's going on in the book but the main theorems and you know main concepts would be in the book.

I: Given the following two definitions of measurable sets [page 1] on RN.

A: So I have two notations for one thing?

I: Yes. [Reading Moment]

A: On open set that contains it can be found can be found with small exterior measure... ok... yes!

I: Which one would you prefer using in the classroom?

A: I feel like this is a trick question because I know that A is what's most often used in books, whereas the second one (refers to B), it's... it's not bad. I think what I don't like about the second one is the absolute value sign, I don't like the notation very much, because it involves an absolute value sign, if it was something else... although I know... I'm looking at the difference of the sizes of two things... I feel like there's two components to this, there's the notation part and then there's the form of the definition part.

I: The form you mean the language that's being used?

A: No... the mathematical content, whenever you're using open sets or whenever you're using complements. I don't know.

I: Which one do you think students will understand better?

A: I wouldn't be surprised if it's B.

I: Why?

A: Because it's less abstract.

I: Do you think the notation plays any role for the students?

A: Well a little bit. I think that taking the difference of two sets is something that is more intuitve to read versus intersections. Intersections are not very intuitive concepts. Well you think they are because you can draw the sets, but you have to draw them in your head, whereas substraction is something we do very easily. So I think that part of the notation is somewhat more intutive.

I: So you would think it's text B (the definition students would understand better)?

A: I think B.

I: Do you think that it is important to show both definitions? Why?

A: I think it's always good to show 2.

I: It's always good to show 2?

A: Why not? Because part of the problem with mathematics is that at some point there are certain concepts that are used in mathematics that are not intuitive so if you don't know how to translate from intuitive to non-intuitive it's useful to sort of see...

I: This [page 2] is an excerpt from text B, given after the definition of a measurable set. Would you include this in your lesson plan?

A: I have a feeling that yes, because it's leading up to a theorem that if you want to work with abstract measures and abstract spaces you would probably need to have. So you know there is... you know. It seems to me that a few, sometimes you have things, theorems that you don't necessarily see immediately the purpose for, but they are used as part of a bigger theorem that you have to have so.

I: Well in the course outline a large focus is on the real space with a very small part at the end dedicated to abstract measure, the course does not include abstract integration.

A: Okay well if you're teaching measure theory on real spaces then you don't need this.

I: But if you had to teach at the end of the course something about abstract measure?

A: Well then you'd include it in the end. Well it's hard to say because I don't know what are the full list of topics for the course, right? So that makes it a bit more complicated. So my take is if you're doing measure theory in the real spaces, you don't need this, and I wouldn't include it, but if you are doing, although I have to say that theorem 3.29 seems nice because it's still talking about how you can write any measurable set as, in terms of intervals plus or minus some stuff you don't care about, or has you know really small measure. But if at the end you need to go back and teach abstract measure then you include the theorem when you need to teach abstract measure.

I: Okay so you wouldn't necessarily include it in the section about real spaces but you think it's place is at the end where you're going to teach abstract measure?

A: Yes.

I: Based on that do you think that this inclusion as it is here, following the definition of a measurable set, do you think it would be helpful for students?

A: I think the middle theorem might be helpful, 3.29

I: And why only 3.29 but not the whole excerpt?

A: Well i think if you're teaching things in R^n then 3.29 basically summarizes everything, I mean it's a bit of a cheat, because it does in a way include 3.28, so 3.29 is a combination of the two things in 3.28, but you know I wouldn't necessarily include the Caratheodory's theorem, because it's not something you will directly use, because you will never start doing things with respect to the exterior measure and I think 3.30 is not a constructive theorem, and it doesn't tell you anything really about how you should compute, I don't feel like it tells you how you should compute the measure of a set, you know, whereas I feel like 3.29 tells you that you have a set and you can decompose it into something whose measure it is very easy to compute because they are intervals and then you have plus/minus stuff that is, you know, you can throw away. So I feel like 3.29 is in some sense constructive whereas 3.30 really isn't.

I: Do you think there's any harm in not giving these characterizations to students?

A: Any of them?

I: Yes

A: I think there is a mild harm in the sense that they will think that every measurable set is an interval, so you know every measurable set is a borel set and I think you know the whole point of going through the trouble is to say that it's almost true but not quite true, so if you're going to go through something that's not just total variation measure with respect to a function that you can just pull out then you know there's, then you don't have to bother with all this stuff before, so if you are going to bother with all the stuff before then you might as well tell them well, you know, you sort of need that because there are some ugly sets, so if you don't warn them that there are ugly set which hopefully thankfully can be negligble when differenced from the nice sets then I don't think they see the purpose of having studied something complicated for it. I mean if you're going to throw the whole thing out without even the 3.29 then you might as well throw out the whole exterior measure.

I: Given the following two constructions of a non-measurable set [page 4], which one would you prefer to use in the classroom?

A: I think I like text B better.

I: Why?

A: I think it breaks down what's going on a bit better. It's hard to say it's sort of, I feel like it's the difference between constructing examples sort of step by step without telling people where you're heading which is what text A does and then B sort of says we're first going to prove a little helping lemma which you could say well it's kind of interesting in its own right, I don't know what it's going to be used for and then the theorem just gets straight to the point with kind of this, these are the key points, it sort of highlights the things you have to use and then leaves the technical stuff to the lemma as opposed to the first construction [ref A] which kind of goes step by step by step

I: Which one do you think would be easier for students to understand?

A: That's tough to say, depends on what you mean by the word understand. Sometimes the students if it would help them get an intuition about what's going on yes. But I think that a lot of students consider understanding can I follow from step A to step B.

I: Okay so if we were talking about understanding as getting the intuition?

A: I think B

I: And as simply following from step A to step B?

A: Possibly A But I feel like with subjects like this being able to reproduce is getting you nowhere. Nobody is going to ask you to reproduce a proof of something that's already proved, this is not undergraduate material right? The only purpose of learning this material is so you can use it to do something creative. Generally even in my undergraduate classes I wouldn't want them to be able to simply reproduce theorems as is.

I: From your perspective, is it important to show students the existence and/or construction of a nonmeasurable set at this point? Why?

A: Again, it's hard to say it depends on time constraints, where you know the full outline of topics needed to be covered, but I think you know let's say assigning it as part of a homework it is useful. I won't say that you would necessarily want to spend class time because there might be other things you need to cover but I think having, asking them to read this after class as supplementary stuff is important. The thing in math is if you think that everything is nice at some point you'll screw yourself. The point is in math you're supposed to be a little vigilant to the fact that sometimes things are not nice, so if you're not shown once every course that there's something that can be turkey, then you're lawled into to this notion that everything is differentiable, integrable and so on.

I: So do you think the real value is the construction or the existence aspect?

A: It's really in the existence. I: Which definition of a measurable set would you prefer using if your final goal is to talk about measurable functions ?

A: I don't know, I think still B.

I: Why?

A: Just less abstract.

I: From which definition do you think it would be easier for students to make the transition from measurable sets to measurable functions and why?

A: I think it's still B. B is a lot more about closeness of you know measurable sets to open sets which is sort of similar to the closeness of measurable functions to continuous functions, so I think if you're going to draw any parallel between what the sets do and what the functions do, it's much more useful to think of B than to think of A.

I: Do you think that the presence of pictorial representations helps students understand the material better and why?

A: I think yes but I think in some subjects it's probably hard to do it. I think whenever you can do it, when it's sort of meaningful and it's not just you know, to draw just to draw. If it's doing anything in terms of representing the material I think it's useful.

I: What kind of pictorial representations would you think would be useful in measure theory texts?

A: Tricky... I really don't know, because most of it, I don't think most of it would be that meaningful, I mean it's just, it's really hard to see what you could possibly draw. Okay like when you teach about what a Lebesgue integral is that makes sense because you sort of chop up the y-axis in terms of the range of the function, but when it comes to the construction of the measure itself, I don't know. I mean there's a couple of pictures you can do where you have sort of open sets above them, closed sets inside them, squishing it. I think there's really like one picture you could do. I don't know

I: Do you like when authors intervene in the proofs by explaining or do you prefer when they just present the mathematical results without any intervention?

A: I like interventions. I prefer a narrative.

I: Do you think it is useful to the students?

A: I think so, I think people like stories.

I: Do you think that sometimes it could lead to confusion?

A: I think it's confusing if you're not really trying to learn, because it's a distraction. It's true that it's a distraction so if you're not really trying to focus and trying to learn what the proof is, you're going to remember few words from that distraction, you're going to remember them incorrectly, but I don't see how eliminating that would make you do the proof any better, you know. I've seen people use narratives incorrectly because they're confused but those are the same exact people who can't use related concepts and I think for people who are actually trying to focus and understand as much as possible from this then I think the narrative is helpful because you can always do the A to B to C for yourself on the side but sometimes it's hard to see why you would bother.

I: How often do you think students consult the textbook?

A: I think they consult the text when they need to do the assignments. It depends, I mean that's the lower bound, I would say they sometimes consult the text when they're going over something that ha penned in class, sometimes, maybe half of the time, but usually when they're trying to do the assignments and they get stuck on something they probably will consult the text, it seems to have more gravitas, we're hoping for some wisdom or solution to the problem from the book that has not been presented in class.

I: How do you think students are using the textbook? Are they reading it before the lectures to prepare?

A: Almost no one reads before the lecture. I would say 5 percent that do that. I think predominantly it's assignments, I think they kind of scan for other examples, I'm really not sure that they want to re-read stuff that was done in class, some people do. I think the majority reads it for examples and trying to solve problems and I think 1/3 actually uses it to read the proofs and theorems. I think that sometimes students think they read the text but when you ask them about something they'll be like where is that? but you just told me you read the previous 5 pages and you know it's like a big theorem in the previous pages, so I think sometimes they think they read the text but what they do they actually scan for things.

I: Is there anything else, comments, concerns, criticism, that you have about the use of textbooks in other math courses you teach?

A: No not really. I think it's a shame that there are not more copies of some well-known or well-used books in the library, because I think given what the price of a textbook is it's hard for students to buy one let alone multiple books for a course.

I: Do you think lectures can replace texts?

A: I think to some extent yes, because some of them are written like texts. I think there's no reason why they aren't as good, they are a little more modular because you can take these chapters and swap them for other chapters, I wouldn't have a problem with that.

I: Do you think they could even be better than a text?

A: I think it depends whenever you're talking about books written 50 years ago or 20 years ago. I think the culture of writing books has changed really, at least from the ones I've consulted in the last 5 years. People do write them with the modularity of teaching a class in mind and they'll give you a diagram of you know if you want to do these topics you can follow this path. So it really depends. So sure, better than a text written 50 years ago.

Appendix C

Transcript from Interview with Participant B

I: Have you taught measure theory?

B: No. Well I taught functional analysis maybe I gave some preliminaries but very fast and not in detail, basically I made use of measure theory during my lectures in mathematical physics and functional analysis and special course I never taught.

I: If you had to teach measure theory what textbook would you use?

B: The one I used when I was a a student it is Vulih book, I'm not sure that it is translated, well it gathers the experience of I guess modern, 20 years ago of teaching in St-Petersbourgh university, I know this book, it is thin, the material is nicely organized it contains good set of problems.

I: Is it a Russian text?

B: It is Russian I guess there are translations, well Kolmogorov and Fomin for example it is classical analysis and it has two chapters on measure theory, again well it is [inaudible].

I: So would you want to use Vulich?

B: Not necessarily, Kolmogorov Fomin is equally good, by the way, Rudin is also excellent book, just principles of calculus, so Vulich, Kolomogorov Fomin and Rudin. Most probably I'll take Rudin, because he's in the library.

I: And why would you choose Rudin aside from the fact that it's in the library?

B: Well, it is the only book I read completely when I was a student. Well I used to know it by heart, now of course. Well it is thin, it is written by highly qualified mathematician, great analyst of the century, and well the stuff is nicely organized, nicely organized problems, no logical mistakes.

I: Logical mistakes?

B: Well because very often when you take textbook, you don't know, you might encounter a lot of inconsistencies, but here everything is crystal clear.

I: Inconsistencies?

B: Well some gaps, some gaps in theorems, sometimes they use something which is not yet introduced for example and they have to do some roundabout tricks, all that well, good mathematical textbook is a rare thing, if we look at these fat books they are selling to Concordia students, they have to go directly to the garbage.

I: Are you aware that for the last couple of years the textbook in use at this department is Royden? Would you accept using it?

B: Well I did not read it, why not maybe, I don't know this book.

I: How closely do you follow or would follow the textbook when teaching measure theory?

B: Not at all, I never follow the book. Basically they use the book for their home reading, but I find it stupid to reproduce the proofs one can find in the book, I use some alternative way of explanation well, I never follow the book.

I: Given the following two definitions of measurable sets [page 1] on RN, which one would you prefer using in the classroom?

B: Definitely the first one, this I was taught and this I know, well you can construct the sigma algebra.

I: And why would you want to use the first one?

B: Because I have to think at the moment, at the moment looking at the first I recognize the well known definition I was taught and I know... and the second, well the second, it's a problem to prove that, the set is measurable if...

I: Which do you think students will understand better and why?

B: Uhhh.... Well you see since they said open set, it should be topological space and when you construct sigma-algebra if it is standard construction of measurable sets, it's, I think it's about measurable in the sense of Borel, and this is general (points to 1st) so it depends on the way you are teaching, if you teach abstract measure theory then of course the first one.

I: Suppose that we were only in the Real space.

B: Should I teach only for R1 so borel sets, all that, and then,....

I: So which one in this context do you think students would understand best?

B: It depends, well in a sense the question is a little strange because it depends of course where you are in measure theory, in the abstract measure, the first one is the basic definition of a measurable set and the second is simply an exercise where you show that sigma algebra is borel algebra

I: Should we show both definitions?

B: Yes, more equivalent definitions you show, more they understand basically the course.

I: This [page 2] is an exert from the text B given after the definition of a measurable set.

B: Honestly, theorem of Caratheodory of course I studied but at the moment I don't remember what is it... Well I don't know will I include or won't it depends, well, no it looks special, at first sight it looks special basically what student need, they need only that amount of measure theory which is needed for integration so definition of measurable set and all that, measurable functions and then you jump to completeness of Lp-spaces, special course, well maybe, Egoroff theorem this is maybe needed, bounded integration, so Lebesgue theorem about integration, so Fubini's theorem, these are extremely important and all special facts on measure well I'm not sure they need it, basically you need the facts which are used in integration theory.

I: Do you think it would be helpful for students? Why?

B: Well as an exercise, as an assignment, juggling with abstract definition, well this measure theory is not that deep. Well once they understood the construction of sigma-algebra and got the idea of exterior measure well after that simple, simple problems. So as fast as it is possible to pass to integration here.

I: Do you think there's any harm in not giving these characterizations to students?

B: Yes of course. By the way working mathematician well, I can say that 99 /mathematician do not use this, well i was taught this way when I was second year university student, since then I never used, I've been working in mathematics, well I know what is a measurable function and Lebesgue integral, the only fact needed for a working mathematician is the passing to the limit in the integrals.

I: Given the following two constructions of a non-measurable set [page 4].

B: That is the rational twist on the circle yes, yes. So this construction should certainly be shown, every student in math should know.

I: So would you say the important part is the technique or the idea?

B: But it's so simple just tale take circle divide into classes, take one element in each class, I was taught 30 years ago I still remember this. The idea is so bright so why not. It should be taught of course.

I: Which one would you prefer to use in the classroom? Why?

B: The rational twist A and B, I will take the one with the rational twist.

I: But both have the twist...

B: Well of course if time permits you may show both and well, well the question of choice, well if I was teaching this year I'd take this example, next year for sure I'll take another one, that is just, because it's extremely boring to repeat the thing, so next year you change, you never repeat the same course again.

I: Which definition of a measurable set would you prefer using if your final goal is to talk about measurable functions (see page 7)? Why?

B: Integration theory, now, if it is, it is the course is void if after this course they know only the measure theory this abstract nonsense.

I: So if the final goal is to talk about measurable functions?

B: Well, you, well, the definition of measurable function uses the definition of measurable set, so preimage should be measurable, well you know, pre-images of intervals are measurable that is the definition and which definition you take it depends upon where your function lives, of course you may use the second one, why not, if your function lives in an abstract space, the the first one

I: Let's say we stay in the R^N space...

B: Well of course, will use Borel!

I: So the definition from text B?

B: Yes.

B: By the way there is the construction of measure theory starting from functions, you start from step functions and that is in Russian books of Shilov measure theory and integration and the measure theory is constructed using measurable functions, and the measurable functions are those that could be approximated by step functions, that is an independent way to prove this theory.

I: So would you be more interested in such an approach?

B: Well this year you use this one, next year another one.

I: Do you think that the presence of pictorial representations helps students understand the material better and why? What kind of pictorial representations would you think would be useful in texts?

B: Ummm... No, because, well because pictures and measure theory, well you can of course you can write this parallelipipid which are basic basic bricks to constructs Borel algebra and that's all because basically measure theory studies pathology, pathological pictures cannot be shown, that is the point, it is why people do not like and do not use it, because general general topology most topologists despise generally topology, they study pathological situations and we study nice geometric pictures and all that, same for measure theory. Real analysis, well strange people continue studying this, all my respect but I would not talk to them, I am not interested in what they are doing. It's not geometrical here, it's just construction of Lebesgue algebra is not geometric construction it's extremely tricky and well Lebesgue was a genius the idea was very nice but since then well century passed.

I: Do you like when authors intervene in the proofs by explaining or do you prefer when they just present the mathematical results without any intervention? Why?

B: Well, good author starts the proof with a big discussion of the idea, well say when I'm making the talk first 5 minutes I am explaining I will do this and that, the idea is explained by waving the hands and leave the all thenical difficulties for after.

I: So having such an introduction is a good idea?

B: Well depends, sometimes, the good proof is not found, the proof is long calculation and it is impossible to extract and show the idea.

I: But if it can be done?

B: Then of course the proof should be as short as possible, as non-technical as possible, and in measure theory basically all proofs are short, so.

I: Do you think it is useful to the students?

B: Well see depends who you are teaching. If future mathematicians then all this nonsense should be sent to the garbage. Of course you describe ideas all that but you look at the subject. Of course I can dance, sing, stand on my head, do anything they want. So it is standard course, it shouldn't be long, all these technicalities, well there are a lot of problems in mathematical Olympiads, for those guys who like solving problems already solved, tricky problems, that's ok, but in the research I'm not sure, maybe if you do probability theory, all that, well measure theory is used much much seriously, but for all other domains.

I: How often do you think students consult the textbook?

B: Normally they should read book... now I am teaching differential geometry I am explaining them ideas, sometimes time is lacking so well no time to explain details, so they read the book.

I: Do you think they read it?

B: Of course, textbooks basically closed subject, all the textbooks are more or less similar and they are all good.

I: But the question do you think they are reading or not, using?

B: So the best way to learn mathematics is to solve problems, so the best textbook is the textbook containing small pieces of theory and a long set of cleverly organized problems, so they should start form trivial to more complicated, so student has to solve them one after another.

I: So would you use text as source for problems?

B: Once again I'm rarely using texts, like if I have the book, I never repeat the proof given in the book on the blackboard, either I use a proof I found in another book, or my own, something like that.

I: Do you have any other comments about the use of textbooks in general from the courses that you taught by you or by students?

B: Well I can repeat what I said in the very beginning in my life I read only one book completely it was Rudin. I never read textbooks. If students are in mathematics, the education, you get education from communicating to other, with fellows, professors, listening to lectures, and the reading of serious monographs and scientific papers, and these special textbooks, well I'm not sure, because serious people don't use them.

Appendix D

Transcript from Interview with Participant C

I: Have you taught measure theory?

C:Yes, here at University U.

I: What textbook did you use?

C:Well I did choose it a few times I chose Rudin's Real and Complex analysis, but that was for a graduate level course, for the undergraduate course I've gone with the textbook that was chosen previously which is Royden but I tried to look for another book and I couldn't find one that I really liked.

I: And do you prefer Rudin to Royden, or what did you dislike about Royden?

C:Well there were 2 separate issues, one was the nature of the course because at some point we introduced a graduate level course that was strictly separate from the undergraduate level course and so I wanted a text that was more advanced and that would be good for students who already had the measure theory course at the undergraduate, this one is better for, it's better if you want to do abstract measure theory to understand measure in a more abstract context, and it has much better problems. I mean for me that's a big feature, it's the problems and Royden that's the weakest feature that I found and it's sort of gotten worst because now there's a fourth edition, the author of the fourth edition who is not Royden even though Royden's name is still on the book, I think Royden died, took it upon himself to solve all the problems or many of the problems in the third edition which probably the publisher encouraged him to make it obsolete, so now it's even worse I think. A lot of the problems that appeared in the third edition are now theorems in the text. So it's the problems mostly I have trouble with, you know the material I can present it the way I want I mean the students obviously look in the text as well but you know I present it the way I want in class but the, you know coming up with original exercises is hard so I prefer having a textbook that has a good selection of exercises.

I: How closely do you follow or would follow the textbook when teaching measure theory?

C:It depends on how happy I am with the textbook. I guess that if I was satisfied with the text I would follow it pretty closely.

I: Would you take examples from the text?

C:No, not necessarily, I mean examples are usually easy to come up with, often examples can be from the exercises so if there are good exercises you can assign those and that gives a good selection of examples, but no I don't usually take examples from the text, it's usually more the order of the material the main theorems, you know the development of the subject but you know I wouldn't follow every line of the text. I think that it's weird that someone would simply copy the text at that level of a course.

I: Given the following two definitions of measurable sets [page 1] on RN which one would you prefer using in the classroom?

C:Well I mean it depends again because it's sort of taken out of context the first one is clearly Caratheodory's definition which is based on having defined outer measure, the second one you have to know what this distance between two sets is, I guess there's a little e... Well it has to be defined previously so it really depends on what you've done. So I can't really say what I would do because I would have to see the whole development of the subject in terms of how, I mean certainly since I've used Royden so often I always use the first definition but you know if you're going to do it the other way, it's hard to just pick that definition you really have to see the whole development in the text of what it means.

I: Which do you think students will understand better?

C:Really depends on what they've seen before. I mean the important thing about measure theory that's why I always tell students that if we have a textbook they shouldn't really try to follow other textbooks a lot, measure theory textbooks are not like say real analysis textbooks, measure theory textbooks each one has a particular development, I mean there are some standard ones but it's very hard to follow if you kind of pick from different texts, you really pick one and you follow, that's why I said that I would follow the text quite closely because you don't want to introduce ambiguities. I:

Do you think that it is important to show both definitions? Why?

C:No no I wouldn't do that because I think that would be confusing. Of course once you've defined the measurability or outer measure even you can always show a regularity by showing that every set can be approximated by open sets in outer measures, so it's sort of your regularity is a corollary of the definition of outer measure rather than the definition. It just happens in the case of R^n that you have this regularity.

I: This [page 2] is an exert from the text B given after the definition of a measurable set.

C:So this text I guess goes in the completely opposite direction because they start with this approximation by open sets and then show the Caratheodory definition... it's the one that we use in Royden is completely the opposite, you first use the definition of Caratheodory to prove all these things about G-deltas and so on...

I: So if you had to use text B, would you include this in your lesson plan?

C:Yes.

I: Why

C:Well because it's a very important result it's that definition of Caratheodoty, even here it says this character cons... abstract spaces, that one of the main tools for constructing measure! You know measure theory, Lebesgue measure on the real line is not really a goal in itself because students are going to go on to their probability course and they're going to see other measure, they're going to go onto their dynamical systems course and they're going to see Hausdorff measure, so students have to know that there are other measures exist and can be constructed.

I: Do you think there's any harm in not giving these characterizations to students?

C:Yes because of the same reasons I just gave.

I: Given the following two constructions of a non-measurable set [page 4], which one would you prefer to use in the classroom? Why?

C:So i guess the first one I know because it looks like the one in Royden. I think it's basically the same... yeah the last part of this one this 3.39 is actually given in Royden as an exercise, I mean other than that I really don't have an opinion, the first one I'm more familiar with, but it's really the same proof it's just the way it's written. Second part left exercises but in this text proved which one students to follow since yes no difference... in order for students to follow advantage given missing... Well you see I think we have, it's like a disagreement about this, because what you're thinking is students reading the text and understanding sort of like someone would read an ancient text or something... what I consider students understanding the material is exactly not that, it's students reading the text, going to lectures and being able to solve problems and therefore having that corollary is a problem. You see you cannot understand mathematics until you do mathematics, mathematics is not literature, but in mathematics when you're doing mathematics you have to be able to solve problems you have to be able to do proofs, so no matter how many times you read a text and you memorize it, you can reproduce it on an exam that's worthless. What's worth it is being able to take what you learned and apply it to something else, this is why when I say that this corollary is given as an exercise, it's the existence of that exercise, it's that exercise that's going to help students understand what's written in the text because they will have to go back to the proof in the text and understand what it's doing and go and do the exercise. So measure theory is not calculus, it's not that you can do the exercises without just by calculating something, you have to go back to the text you have to process the text through your brain in order to do the exercise so that's why I can't say this is better because it actually does something that the students would have to do because reading what it does will not help them understand, i feel very strongly about this, because this idea that the text will help you understand but it's doing the exercises that really helps you understand.

I: From your perspective, is it important to show students the existence and/or construction of a nonmeasurable set at this point? Why?

C:Well we always do it, I don't know. It's true that students are sometimes puzzled by it because sometimes they haven't often seen applications of the axiom of choice so in that sense you could say that it's something kind of bizzare, but that's also a positive aspect, because A they see something unusual which is the axiom of choice, B it helps to justify why we have to bother with the definition of measurablity because otherwise if all sets would be measurable then what's the point, and another thing that's useful and I guess in both of these proofs this is featured, but I think that the proof in Royden emphasizes that a little more is the idea of taking the reals modulo the rationals, that's a construction that's kind of a standard construction not just in analysis but also in other areas of maths to take equivalence classes so you know when we did real analysis we always constructed reals as equivalence classes of Cauchy sequences, again an equivalence relation is kind of a very important idea so in that sense it helps to define that you know.

I: In your opinion, if we return to page 1, which definition of a measurable set would you prefer using if your final goal is to talk about measurable functions (see page 7)? Why?

C:Well I don't it really makes a difference because the definition of a measurable function does not depend on how you defined measure, once you have a measure you can have measurable functions so it really doesn't make a difference, it really doesn't depend on that.

I: Do you think that the presence of pictorial representations helps students understand the material better and why? What kind of pictorial representations would you think would be useful in texts?

C:Yeah, usually you don't, but it could be useful I mean yeah if you show some an open cover and stuff like that, you know things about the cantor set, you can use some pictures to illustrate that, I guess it could be now that they can be easily generated by computer you could do all kinds of nice things with that.

I: Do you like when authors intervene in the proofs by explaining or do you prefer when they just present the mathematical results without any intervention? Why?

C:I think it's good they explain yeah. I think it's good writing.

I: Do you think it is useful to students?

C:Yeah of course, because it's, a lot of it is also learning how to write mathematics and that writing part is what is often missing because they don't explain what they're doing and they think that a sequence of formulas is what's gonna constitute a proof but really you know it's the writing that's more important than the formulas in a way you know because if you're saying what you want to do even if you make a typo in your formula the proof can still be correct whereas when you have a sequence of formulas and there is some typo there, well you know go figure.

I: Do you give extra marks to those who write properly?

C:Yes. Well exams students are under time pressure so often the writing suffers but on homeworks I mean they have to write and explain what they are doing because it's not, it's not enough just to write a formula it's often you know especially if you don't know what the definition of anything is and things like that.

I: So would you rather see a paper with great writing or just mathematically correct?

C:Ummm well I think that provided that the mistakes were not fundamental, they were really just typos then yeah the student who explains probably understood better. Yeah I think so, because of course the mathematics has to be accurate and precise but you have to really know what you're doing and writing a sequence of formulas because what happens with formulas is very ambiguous, because suppose you have an inequality, well usually there is something applied to give you this inequality, you could think that the student knew that and that's how they did it but if they don't explain that's what they're doing then how do you know? Of course depending on the level of the mathematician for some some things will be trivial but for students this written part is the only indicator that would tell us that they know anything.

I: How often do you think students consult the textbook?

C:I don't know, I think for measure theory quite a lot, and that's why I was kind of very forceful before because sometimes maybe too much because they kind of, they're very dependent on the book and they think that reading the book, like I said memorizing it line by line is what they think they're going to need to do which is in a way a mistake because it's good to consult the textbook but it's trying to do the problems that's going to help you understand. So if you try a problem and you go back and you see what is relevant to that problem somehow your brain works differently than just memorizing. I think that in that level of course the students are often very dependent on the text, it's the opposite in the lower level courses.

I: Other comments and criticism about the use of texts or the structure of the course in particular about the mixing of grad and undergrad levels?

C:Yeah well it was like that before, when I came it was like that, and I was instrumental to try to get it separate, I don't it's, I see a a lot of incoming master's students who do have not heard measure theory previously, so in a way it's good for them to have the undergraduate course, the main problem is that it's in the spring that's my main objection. My objection is not so much that it's cross-listed because we made it optional for pure and applied students, but if we put it in the fall it has to be at the same time as 464, but that creates problems as organization because then for grad students it's hard to take it in the spring. The texts, I really see them as a great source for problems.

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