A GENERAL UPPER BOUND ON BROADCAST FUNCTION $B(N)$ USING KNÖDEL GRAPH

Sirma Çağil Altay

A thesis
in
The Department
of
Computer Science and Software Engineering

Presented in Partial Fulfillment of the Requirements
For the Degree of Master of Computer Science
Concordia University
Montréal, Québec, Canada

September 2013

CONCORDIA UNIVERSITY
School of Graduate Studies

This is to certify that the thesis prepared

By: **Sırma Çağıl Altay**

Entitled: **A general upper bound on broadcast function $B(n)$ using Knödel graph**

and submitted in partial fulfillment of the requirements for the degree of

Master of Computer Science

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____________________ Chair
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Dr. T. Fevens

____________________ Examiner
Dr. L. Narayanan

____________________ Supervisor
Dr. H. Harutyunyan

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Faculty of Engineering and Computer Science

Abstract

A general upper bound on broadcast function $B(n)$ using Knödel graph

Sırma Çağıl Altay

Broadcasting in a graph is the process of transmitting a message from one vertex, the *originator*, to all other vertices of the graph. We will consider the classical model in which an informed vertex can only inform one of its uninformed neighbours during each time unit. A *broadcast graph* on n vertices is a graph in which broadcasting can be completed in $\lceil \log_2 n \rceil$ time units from any originator. A *minimum broadcast graph* on n vertices is a broadcast graph that has the least possible number of edges, $B(n)$, over all broadcast graphs on n vertices.

This thesis enhances studies about broadcasting by applying a *vertex deletion method* to a specific graph topology, namely *Knödel graph*, in order to construct broadcast graphs on odd number of vertices. This construction provides an improved general upper bound on $B(n)$ for all odd n except when $n = 2^k - 1$.

Acknowledgments

I would like to express my deepest gratitude to my supervisor Prof. Hovhannes Harutyunyan above all for his sincerity, humanity and joyfulness. His visionary guidance, patience, knowledge and open-mindedness motivated me throughout my thesis. I will never forget all that he has done for me.

Dr. Nadia Hardy, who I would like to thank specially, introduced me to graph theory through a wonderful and colourful course.

I am deeply indebted to Dr. Ali Akgunduz, Dr. Lievin Devos, Inan Erdem and Halina Monkiewicz who were crucial in starting my thesis in Canada.

I am particularly grateful to my mother Tülay Akarsoy and my father Erol Altay who taught me above all to be original, creative, independent and unencumbered by preconceptions. I would also like to thank my sister Fırat Melek Altay, my grand mother Melek Akarsoy, my cousin Alkım Uyar Güsar, my aunt Nuray Uyar, her husband Hasan Uyar, my other aunts Güler Altay and Vildan Altay and my sweet nephew Tuna Güsar, who always supported me and encouraged me to follow my dreams.

I want to thank to Dr. Didem Dönmez Aksüt who supported me throughout my thesis by showing me that life is filled with a multitude of colors.

I would like to offer my special thanks to my fellow labmates Hayk Grigoryan, Puspal Bhabak, Louisa Harutyunyan and Mohsen Eftekhari who were always helpful and cheerful.

I could never have such a beautiful adventure without the support of my dear friends Joseph Ajay Philip, Talha Rahman, Amine Benasla, Jingang Li, Zhe Qiu, Alexandre Zahy Raslan, Shrikant Ambulkar, Gökhan Aslan, Rebecca Bernard, Deniz Leblanc Suna, Aude Verwaerde and Irina Malinovskaya who stood by me in the most difficult times and were a source of encouragement.

Foremost, I would like to express my gratitude to and thank those young and brave people of Gezi Park of Istanbul, Turkey who gave me the courage to continue working on my thesis with their struggle in June 2013 defending their city, their trees, parks, forests and most of all their freedom, by democratic means despite hostile actions and in some cases paying the price with the loss of their very precious lives. They will never be forgotten.

Contents

List of Figures

List of Tables

Chapter 1

Introduction and preliminaries

Broadcasting and gossiping were born as basic combinatorial problems. The *Gossiping* problem is defined by Knödel in [66] as follows: "Given n persons, each with a bit of information, wishing to distribute their information to one another in binary calls, each call taking a fixed time; how long must it take before each knows everything ?". *Broadcasting* is a similar problem where only one person has the totality of information that needs to be distributed to all others using binary calls. Although applicable to many domains such as social networks, forensic networks, databases, virus spreading, logistics etc., broadcasting and gossiping problems are mainly studied in the theoretical branch of parallel computing that deals with the dissemination of information in inter-connection networks.

An inter-connection network is represented by a set of inter-connected microprocessors having their own memories. This differs from shared memory multicomputers where processors communicate via a shared central memory. The inter-connection network permits a larger number of processors to communicate more efficiently. As opposed to Wireless Sensor Networks (WSNs), processors are considered stationary and their inter-connections are known in advance.

The inter-connection network is represented by a simple graph $G(V, E)$ where the set of vertices, V, represents processors (also called nodes) and the set of edges, E, their interconnections (also called links). In addition to broadcasting (*one-to-all* communication primitive) two other variants of gossiping (*all-to-all* communication primitive) problems have been studied: *routing* (*one-to-one* communication primitive) where an initial node should send its message to another specific node, and *multicasting* (*one-to-many* communication primitive) where an initial node should send its message to a specific subset of nodes.

The main focus of this thesis is on the broadcasting problem in interconnection networks which can be more formally defined as the problem of disseminating a piece of information from a certain node, the *originator*, to all other nodes [61]. Two nodes are defined as *neighbours* if there is an edge in graph G connecting their representative vertices. For all other graph theoretical definitions and concepts please refer to [15]. A node is *informed* if it contains the information to spread, hence at the beginning of the broadcasting process there is only one informed node, the originator. A *call* is the action of an informed node informing its neighbouring node. Broadcasting is performed through a series of calls from the set of informed nodes to the set of uninformed nodes and finishes when all nodes are informed.

The broadcasting problem has been studied under many models. Communication link models [62] are

- *one-way* mode also called telegraph or half-duplex, where a message can only be sent according to the direction of the connection [20]. A directed graph represents the system.

- *two-way* mode also called telephone or full-duplex, where a message can be sent in both directions regardless of the connection direction. An undirected graph represents the system.

Communication time between two nodes can be modelled as

- *linear* where the time needed to transmit and receive a message is linearly dependent on the size of message. This type of broadcasting is studied in [4, 32].

- *constant* where the time needed to transmit and receive a message is constant.

There are three main communication node models [31]:

- *processor-bound* also called 1-port or whispering, where an informed node can only inform one of its neighbours at a given time.

- *link-bound* also called n-port or shouting, where an informed node can inform all of its neighbours simultaneously.

- *k-broadcasting* also referred to as c-broadcasting, where an informed node can inform k of its neighbours simultaneously. Some articles related to this type of broadcasting are [45, 46, 49, 53, 54, 67].

As for broadcasting models, they can be classified into two categories

- *Global control* models where each node knows the state of all other nodes and can act intelligently to finish broadcasting in minimum time. *Neighbourhood broadcasting* is a example of global control model studied in [6, 14, 27, 28, 36, 37, 69, 70, 77, 80].

- *Local control* models where each node has only partial information about its neighbouring nodes and may not act intelligently (a node can try to send the information to an already informed node while it could inform an uninformed node). *Orderly broadcasting* [16], *restricted protocol broadcasting* and *messy broadcasting* (see [2, 13, 42, 43, 72]) are examples of local control models.

Further types of broadcasting models and constraints can be found in [78]. The focus of this thesis is on the often used *classical broadcast* model which represents a cluster of one-way, constant, 1-port and global control models. Hence, the constraints of the classical broadcast model are:

- The time is discrete.

- Each call involves only one informed node and one of its uninformed neighbours.

- Each call requires one unit of time.

- A node can participate in only one call per unit of time.

- In one unit of time, many calls can be performed in parallel.

A graph $G = (V, E)$ representing the inter-connection network under the above constraints is a simple undirected graph that does not contain any self-loops. Surveys on this type of broadcasting can be found in [31, 57, 60, 61].

In this thesis, all variables are integers and all logarithms are to the base 2 unless otherwise specified. Words "vertex" and "node", representing the same entity, are used interchangeably.

Given a graph $G = (V, E)$ and an originator $u \in V(G)$, the broadcast time of u which is represented by $b(u)$ or $b(u, G)$, is the minimum necessary time to finish broadcasting from u. The broadcast time of a graph $G = (V, E)$ is then defined as

$$b(G) = max\{b(u)|u \in V\}.$$

Given a graph $G = (V, E)$ on n vertices, while broadcasting from any originator, at each time unit the amount of informed vertices can at most double, hence

$$b(G) \geq \lceil \log n \rceil.$$

A *broadcast graph*, also referred to as a *minimal broadcast graph*, is a graph with a broadcast time of $\lceil \log n \rceil$.

There are two main research directions related to the broadcasting problem. The first one deals with finding the minimum possible broadcast time of different graphs and the strategies to achieve this. This problem proved to be hard. In fact finding the broadcast time of a given vertex in a random graph is shown to be NP-complete in [87]. Schindelhauer [84] provided results regarding the inapproximality of this problem. Various approximation and heuristic algorithms to determine the broadcast time of random graphs have been presented in [3, 5, 19, 20, 23, 33, 34, 35, 50, 55, 59, 68, 81, 83, 85]. Many other studies are devoted to determining the broadcast time of specific classes of graphs. Although the problem remains NP-complete even for some restricted classes like planar graphs and bounded degree graphs [63], there exist some topologies for which a polynomial time algorithm is achievable (see [63, 75, 87]). In [61, 75, 78] different graph topologies (path, cycle, complete tree, complete graph, hypercube, cube-connected cycles, butterfly network, shuffle-exchange network, Bruijn network, grid network, Knödel graphs) and their broadcast time properties are mentioned.

The second research direction in broadcasting is about designing broadcast graphs that have as few edges as possible for a given number of vertices n. A *minimum broadcast graph* (MBG), also referred to as an *optimal broadcast graph*, on n vertices is a broadcast graph that has the least possible number of edges over all broadcast graphs on n vertices. The number of edges of a minimum broadcast graph on n vertices is represented by $B(n)$. [21, 22, 26, 39, 64, 73, 76] provide insights on different classes of MBGs. A deeper review of this field of research is given in Chapter 2.

In this thesis we prove that for all $n = 2^k - 2x - 1$ where $1 \leq x \leq 2^{k-2} - 1$,

$$B(n) \leq \left\lceil \frac{n\lfloor \log n \rfloor}{2} \right\rceil.$$

It is known from [8, 32, 66] that for all even n,

$$B(n) \leq \left\lceil \frac{n\lfloor \log n \rfloor}{2} \right\rceil.$$

Our new upper bound expands this bound to all odd $n \neq 2^k - 1$. Although there exist tighter upper bounds for some restricted range of n, this new upper bound covers a wider range and still shows an improvement of about

$$\frac{n}{4} - \frac{\lfloor \log n \rfloor}{2}$$

over the best known general upper bound

$$\frac{n-1}{2}\lfloor \log n \rfloor + 2^{\lceil \log n \rceil - 2}$$

given by Harutyunyan [41] for any n. Note that for $n = 2^k - 1$ our upper bound is not valid [71]. In order to obtain our upper bound, we construct a new broadcast graph on $2^k - 2x - 1$ vertices by deleting a vertex and adding edges to a specific broadcast graph topology known as *Knödel graph* on $n = 2^k - 2x$ vertices. This construction is taken from [44] where it has been used on Knödel graphs on $2^k - 2$ vertices. We also apply a similar construction to Knödel graphs on 2^k vertices but the upper bound for $B(2^k - 1)$ obtained in this way is not better than the one given in [41]. Lemmas proved in Chapter 3 and Chapter 4 are helpful to our construction and represent broadcasting properties of Knödel graphs under a specific type of broadcast scheme, namely *dimensional broadcast scheme*.

The thesis is structured as follows. The first chapter provides a general introduction to the broadcasting problem and basic graph terminologies. Chapter 2 gives a detailed literature review on minimum broadcast graphs and broadcast function $B(n)$ and introduces Knödel graphs as well as dimensional broadcast schemes. Chapter 3 is dedicated to the study of Knödel graphs on $2^k - 2x$ vertices where $1 \leq x \leq 2^{k-2} - 1$ using dimensional broadcast schemes. In this chapter, applying a vertex deletion method, we give a new upper bound on $B(n)$ for all odd $n \neq 2^k - 1$. Chapter 4 presents an analysis of Knödel graphs on 2^k vertices using dimensional broadcast schemes. The last chapter concludes with possible directions for further research.

Chapter 2

Literature Review

2.1 Minimum broadcast graphs and $B(n)$

Finding $B(n)$ for any given number of vertices n proved to be difficult. Even today the values of $B(n)$ are known only for some small values and for some specific ranges of n. There are only two general values for $B(n)$:

- $B(2^k) = k2^{k-1}$ for all $k \geq 1$

  This value may be obtained using three families of non-isomorphic infinite minimum broadcast graphs ([26]):

  - hypercube of dimension k [21]
  - recursive circulant $G(2^k, 4)$ [79]
  - Knödel graph $W_{k,2^k}$ [66]

- $B(2^k - 2) = (k-1)(2^{k-1} - 1)$ for all $k \geq 2$

  This value may be obtained using Knödel graph $W_{k-1,2^k-2}$ [65].

The argument used to prove $B(2^k)$ and $B(2^k - 2)$ values relies on the minimum degree that each vertex should have in order for all vertices to be informed in $\lceil \log n \rceil$ time units where n is the total number of vertices. For example, we show that the degree of any vertex v is $deg(v) \geq k - 1$ in any broadcast graph on $2^k - 2$ vertices. Let $G = (V, E)$ be a broadcast graph on $n = 2^k - 2$ vertices. Broadcasting in G should finish in $\lceil \log n \rceil = k$ time units. Let $u \in V(G)$ be the originator of broadcasting. Assume that the degree of u is $deg(u) = k - 2$ and that all other vertices are of degree at least k. Figure 1 illustrates the broadcasting tree of a graph G where only edges participating in broadcasting is presented. It shows the best case broadcasting where labels on edges represent the time at which a specific neighbour of u is informed, and labels in triangles are the maximum number of vertices that can be informed during the remaining time.

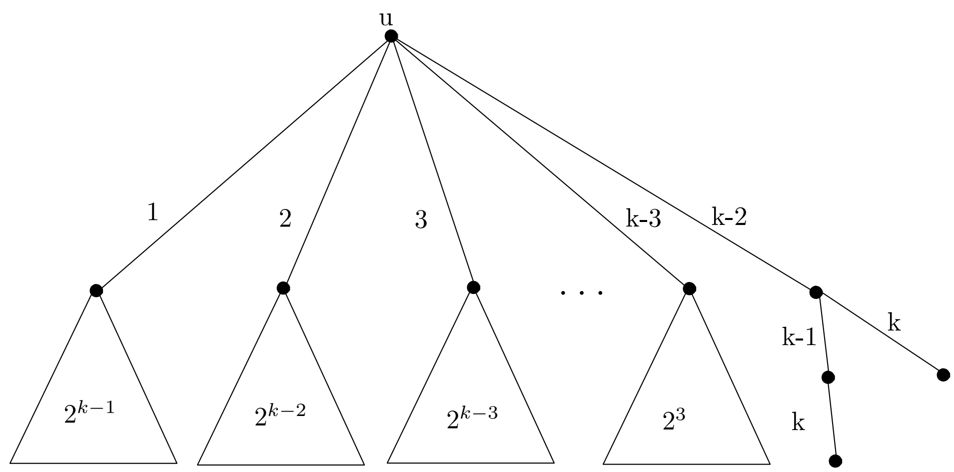

Figure 1: Vertices informed in k time units when the originator, vertex u, has degree $k-2$ and all other vertices are of degree at least k. Labels on edges represent the time at which a specific neighbour of u is informed, and labels in triangles are the maximum number of vertices that can be informed during the remaining time

It is observed that if $deg(u) = k-2$, the maximum number of vertices that can be informed in k time units is

$$2^{k-1} + 2^{k-2} + 2^{k-3} + \ldots + 2^3 + 2^2 = 2^k - 3.$$

So, if the originator is of degree $k-2$, one cannot inform $2^k - 2$ vertices in k time units, hence $deg(u)$ should be at least $k-1$. For G to be a broadcast graph this should be true for all originators, so for all $u \in V(G)$, $deg(u) \geq k-1$. Knödel graph $W_{k-1,2^k-2}$ is known to be a broadcast graph and is $(k-1)$-regular [29], this implies that $W_{k-1,2^k-2}$ is a minimum broadcast graph and its number of edges gives the value for $B(2^k-2)$. The same argument is also valid for the case of $B(2^k)$.

However, when $n = 2^k - 4$ broadcasting should finish in $\lceil \log n \rceil = k$ time units, and by the previous argument one can see that for all vertices to be informed in k time units, all vertices of the graph should have at least $k-2$ incident edges. As a case in point, Knödel graph $W_{k-1,2^k-4}$ is a broadcast graph but it is $(k-1)$-regular. Hence it cannot be a minimum broadcast graph on $n = 2^k - 4$ vertices using the previous argument. This observation is also valid for other well known broadcast topologies on $n = 2^k - 4$ vertices. Moreover, if a graph G is $(k-2)$-regular, then the maximum number of vertices that can be informed from any originator in k time units is less than $2^k - 4$. The basic argument is that if all vertices inform a new vertex at each time unit, then a total of 2^k vertices will be informed at the end of k time units; but the originator is of degree $k-2$ so the number of informed vertices will be 3 less than what could be informed in k time units. The neighbour of the originator that will be informed at time unit 1 is also $k-2$ regular so here again the number of informed vertices will be 3 less than what could be informed in k time units. This implies that the number of vertices that will be informed by a $(k-2)$-regular graph in k time units will be less than $2^k - 6$. Hence the main objective in finding $B(2^k - 4)$ is to construct an irregular broadcast graph defined for all k on $n = 2^k - 4$ vertices, that contains some vertices of degree $k-1$ and some others of degree $k-2$. Such a graph is not found yet.

The previous reasoning can be generalized by expressing the number of vertices as

$$n = 2^k - 2^m - j \quad \text{where} \quad 0 \leq m \leq k-2 \quad \text{and} \quad 0 \leq j \leq 2^m - 1.$$

In fact, if the originator is of degree $m-k$ and the rest of the vertices are of degree k, broadcasting in $\lceil \log n \rceil$ time units from this originator becomes feasible. Hence the following lower bound is obtained:
For all $n = 2^k - 2^m - j$ where $0 \leq m \leq k-2$ and $0 \leq j \leq 2^m - 1$,

$$B(n) \geq \frac{(k-m)n}{2}.$$

As observed earlier this bound is only reachable for broadcast graphs where $2^m + j = 0$ and $2^m + j = 2$. When $2^m + j$ becomes larger, broadcast graphs on $2^k - 2^m - j$ vertices cannot be $(k-m)$-regular and the corresponding $B(n)$ value diverges from the lower bound given above. This observation is formalized by Harutyunyan and Liestman [47] whose study on monotonicity of $B(n)$ showed in particular that $B(n)$ is non-decreasing for values of n in the interval

$$[2^{k-1} + 1, 2^{k-1} + 2^{k-3}].$$

A few papers presented lower bounds for $B(n)$ on more restricted ranges of n (see [37, 38, 71, 82]) but broadcast graphs reaching those bounds have not been constructed yet so the exact values for those $B(n)$ remain unknown. Another general property of $B(n)$ is due to Grigni and Peleg [38] who showed that $B(n) \in \Theta(nL(n))$, where $L(n)$ is the number of leading 1's in the binary representation of $n-1$.

Difficulties in finding general expressions for $B(n)$ directed researchers to find $B(n)$ for some specific values of n. Other than $B(2^k)$ and $B(2^k-2)$, all known $B(n)$ values are given in Table 1 with their references. Note that the smallest value of n for which $B(n)$ is unknown is 23. Figure 2 shows some minimum broadcast graphs for small values of n.

n	$B(n)$	Reference	n	$B(n)$	Reference
1	0	[22]	20	26	[74]
2	1	[22]	21	28	[74]
3	2	[22]	22	31	[74]
4	4	[22]	26	42	[82, 92]
5	5	[22]	27	44	[82]
6	6	[22]	28	48	[82]
7	8	[22]	29	52	[82]
8	12	[22]	30	60	[10]
9	10	[22]	31	65	[10]
10	12	[22]	32	80	[22]
11	13	[22]	58	121	[82]
12	15	[22]	59	124	[82]
13	18	[22]	60	130	[82]
14	21	[22]	61	136	[82]
15	24	[22]	62	155	[21]
16	32	[22]	63	162	[71]
17	22	[76]	127	389	[91]
18	23	[10, 90]	1023	4650	[86]
19	25	[10, 90]	4095	22680	[86]

Table 1: Known $B(n)$ values and their references except $B(2^k)$ and $B(2^k-2)$

Another direction of research is to design some broadcast graphs in order to find good upper bounds for $B(n)$. By definition, if G is a broadcast graph its number of edges directly becomes an upper bound for $B(n)$. Although a few papers presented direct construction of broadcast graphs for large n [10, 38], this type of construction proved to be difficult. The first general upper bound of $B(n)$ is given by Farley [21] in 1979:

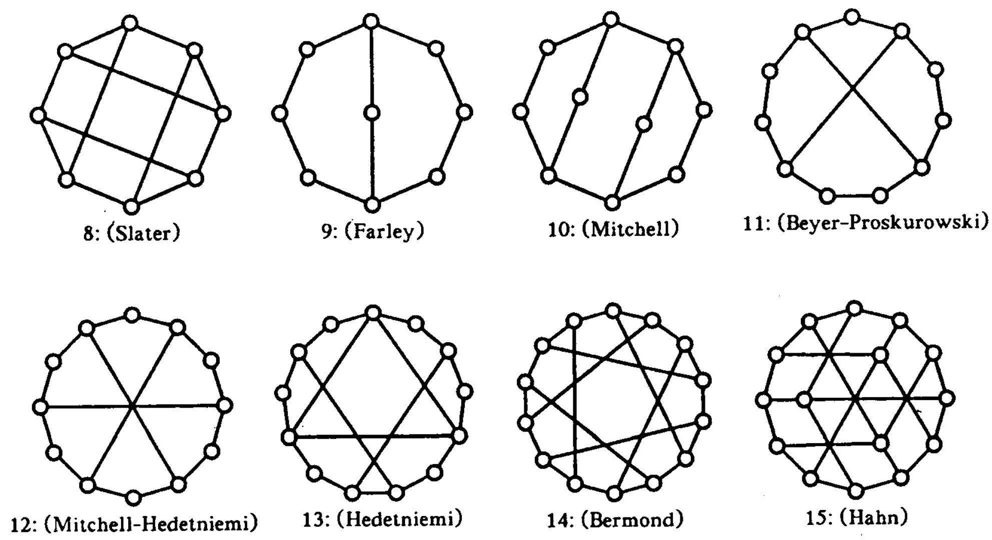

Figure 2: Minimum broadcast graphs for small number of vertices. The captions represent the number of vertices and the name of authors

For all n

$$B(n) \leq \frac{n\lceil \log n \rceil}{2}.$$

In 1999, Harutyunyan and Liestman [44] presented another general upper bound: For $n = 2^k - 2^m - j$ where $0 \leq m \leq k-2$ and $0 \leq j \leq 2^m - 1$,

$$B(n) \leq n(k-m+1) - 2^{k-m} - \frac{1}{2}(k-m)(3k+m-3) + 2m. \tag{1}$$

Later, Harutyunyan [41] improved the previously known general upper bounds providing the following upper bound:
For all n,

$$B(n) \leq \frac{n-1}{2}\lfloor \log n \rfloor + 2^{\lceil \log n \rceil - 2}. \tag{2}$$

These bounds are the only known upper bounds valid for all n. Another upper bound that is only valid for even n is a direct result of Knödel graphs defined in the next section. Since Knödel graphs are broadcast graphs, their number of edges is an upper bound for all even number of vertices (see [8, 32, 66]). This observation results in the following bound:
For all even n

$$B(n) \leq \left\lceil \frac{n\lfloor \log n \rfloor}{2} \right\rceil. \tag{3}$$

Several papers studied the construction of broadcast graphs by combining smaller broadcast graphs or minimum broadcast graphs with the aim of finding upper bounds for some restricted range of n (see [7, 9, 10, 11, 12, 17, 18, 21, 37, 39, 44, 48, 65, 88, 89]). It is worth to mention that most of the known broadcast graphs and minimum broadcast graphs being on even number of vertices, only a few methods were dedicated to construct new broadcast graphs on odd number of vertices. In 1999, Harutyunyan and Liestman [44] improved the previously found upper bounds for many n by applying compounding, vertex deletion, vertex merging and binomial tree methods to Knödel graphs and hypercubes. They proved the following upper bounds:
By [44] Corollary 2.4, for any $0 \leq r < 2^p$,

$$B(2^{k+p} - 2^{p+1} - r) \leq (2^{k+p-2} - 2^{p-1})(2k + p - 2). \tag{4}$$

By [44] Corollary 3.4, for any $1 \leq m < k$,

$$B(2^k - 2^m + 1) \leq 2^{k-1}(k - m/2). \tag{5}$$

By [44] Theorem 4.3,

$$B(2^p(2^k - 3)) \leq 2^p(\lceil (k-1)(2^k - 3)/2 \rceil + p(2^{k-2} - 1)). \tag{6}$$

Improvements obtained for specific values of n by Equations (4) to (6) are presented in [44] Section 6, Table 1 and Table 2. In addition Harutyunyan and Liestman [44] showed the following upper bounds:
For all $2^{m-1} + i2^{m-4} < n \leq 2^{m-1} + (i+1)2^{m-4}$,

$$\begin{aligned} &B(n) < 3n \text{ when } 0 \leq i \leq 3. \\ &B(n) < 4n \text{ when } 4 \leq i \leq 5. \\ &B(n) < 5n \text{ when } i = 6. \end{aligned} \tag{7}$$

These bounds are a direct result of the general bound of Equation (1) which also improves upper bounds given on previous papers when $n \geq 2^{20}$.

In this thesis, we apply a vertex deletion method to Knödel graphs. By this method which was first introduced in [44], we prove the following upper bound:
For all $n = 2^k - 2x - 1$ where $1 \leq x \leq 2^{k-2} - 1$,

$$B(n) \leq \left\lceil \frac{n\lfloor \log n \rfloor}{2} \right\rceil.$$

This result combined with the bound given in Equation (3) results in the following bound:
For all $n \neq 2^k - 1$

$$B(n) \leq \left\lceil \frac{n\lfloor \log n \rfloor}{2} \right\rceil.$$

For some specific ranges of n, the upper bounds such as the ones given in Equations (4) to (7) provide a better result than the above upper bound. However the above upper bound covers all n values except $n = 2^k - 1$ and shows an improvement of about

$$\frac{n}{4} - \frac{\lfloor \log n \rfloor}{2}$$

compared to the best known general upper bound that is given by Equation (2).

A vertex deletion method is also applied to Knödel graph on 2^k vertices, but the upper bound for $B(2^k - 1)$ obtained by this method is bigger than that previously found. In 1999, Harutyunyan and Liestman [44] found the following upper bound on $B(2^k - 1)$ using a compound method on hypercubes:

$$B(2^k - 1) \leq 2^{k-1}(k - 1/2).$$

In 2008, this result was slightly improved in [41] using a new version of the vertex addition method that was first introduced in [10]. The improved upper bound for $B(2^k - 1)$ is the following:

$$B(2^k - 1) \leq 2^{k-1}(k - 1/2) - (k - 1).$$

Later, this upper bound on $B(2^k - 1)$ was further improved in [48] with some constraints on k. This upper bound is the following:
If k is prime and $2^d \not\equiv 1 (mod\ k)$, where $d < k - 1$ is a divisor of $k - 1$ then

$$B(2^k - 1) \leq \frac{(k-1)(2^k - 2)}{2} + \frac{2^k - 2}{k} + k - 2.$$

This latest upper bound happens to be close to the best known lower bound proven by Labahn [71]:
For all $k \geq 2$,

$$B(2^k - 1) \geq \frac{(k-1)(2^k - 1)}{2} + \frac{2^k - 1}{2(k+1)}.$$

This lower bound shows that the upper bound of

$$\left\lceil \frac{n\lfloor \log n \rfloor}{2} \right\rceil$$

cannot be generalized to the case where $n = 2^k - 1$.

2.2 Knödel graphs

A *Knödel graph* represents a graph topology on an even number of vertices. Since their first use implicitly in a proof by Knödel in [66], Knödel graphs have been defined in various ways throughout the literature. In [8, 24, 25, 26, 29, 30, 32] their definition highlights their bipartite nature, in [18] they are presented as Cayley graphs on the dihedral group $D_{\frac{n}{2}}$ and in [1, 40, 41, 44, 51, 52, 56, 65] their edge set is defined through an equation linking different vertices. Isomorphism between graphs resulting from those definitions has been proved in [58] for all Knödel graphs and in [26] for the case where the vertex number is $n = 2^k - 2$ and maximum degree is $\Delta = \lfloor \log n \rfloor$. In this thesis the following definition of Knödel graphs is used. The vertices are labelled by numbers and edges are given by an equation. All numbers used in such equations refer to the labels of vertices and not to vertices themselves.

Definition 1 (Generalized Knödel graph $W_{\Delta,n}$)**.** Let $W_{\Delta,n}$ be a graph on even $n \geq 2$ vertices and of maximum degree $1 \leq \Delta \leq \lfloor \log n \rfloor$. $W_{\Delta,n}$ has the following vertex and edge set

$$V(W_{\Delta,n}) = 0 \ldots n-1,$$

$$E(W_{\Delta,n}) = \{(x,y) | x + y = 2^s - 1 \mod n \text{ for } 1 \leq s \leq \Delta\}.$$

Edges are of *dimension* s and vertices that they connect are *neighbours of dimension* s. Any given vertex has only one neighbour of each dimension.

[29] presents a good survey on generalized Knödel graphs. Generalized Knödel graph $W_{\Delta,n}$ is vertex-transitive, Δ-regular and has vertex-connectivity (see [29])

$$2\Delta/3 < \kappa(W_{\Delta,n}) \leq \Delta.$$

Morosan [78] studied the spectra of generalized Knödel graphs and gave an upper bound for the number of spanning trees in generalized Knödel graphs. A *Knödel graph* on n vertices is a generalized Knödel graph where the maximum degree Δ is equal to $\lfloor \log n \rfloor$, resulting in the following definition.

Definition 2 (Knödel graph KG_n [8, 44])**.** Let KG_n be a graph on even $n \geq 2$ vertices. KG_n has the following vertex and edge set

$$V(KG_n) = 0 \ldots n-1,$$

$$E(KG_n) = \{(x, y)|x + y = 2^s - 1 \mod n \text{ for } 1 \leq s \leq \lfloor \log n \rfloor\}.$$

Figure 3 shows an example of a Knödel graph on 12 vertices. The same graph has been presented twice; the right hand graph highlights the fact that Knödel graphs are bipartite.

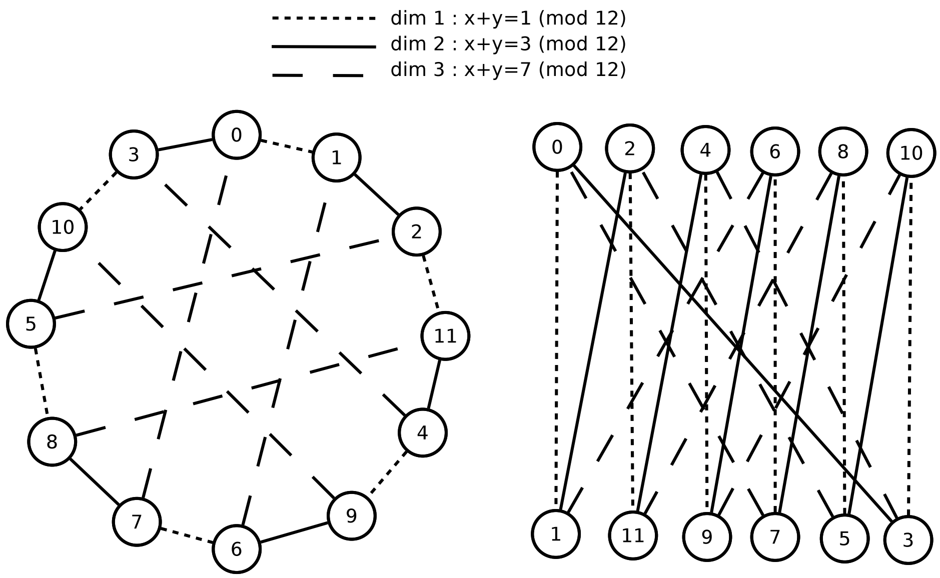

Figure 3: Two different ways of drawing a Knödel graph KG_n where $n = 12$

Knödel graphs KG_{2^k} and KG_{2^k-2x} where $x \neq 1$ are *not* edge-transitive, while KG_{2^k-2} is edge-transitive (see [29] (Proposition 6 and Remark 2)). Generalized Knödel graphs are *not* broadcast graphs for all Δ; for example when the maximum degree Δ is equal to 2, $W_{2,n}$ becomes a circle on n vertices and it is visible that the broadcasting from any originator in this graph cannot terminate in $\lceil \log n \rceil$ time units. However Knödel graphs KG_n ($W_{\lfloor \log n \rfloor, n}$) are broadcast graphs on any even number of vertices (see [29]). Although Knödel graph KG_n is a broadcast graph, it is *not* a minimum broadcast graph for all even n; that is, for some even n there exist broadcast graphs containing less edges than KG_n. Only KG_{2^k} ([26]) and KG_{2^k-2} ([18, 65]) are proved to be minimum broadcast graphs on 2^k and $2^k - 2$ vertices respectively for any k. For an exhaustive list of Knödel graphs and their broadcasting properties please refer to [29] (Table 1). As a result of their broadcasting properties, Knödel graphs, in particular KG_{2^k} and KG_{2^k-2}, have been used in many papers to construct other broadcast graphs, especially on an odd number of vertices (see [18, 25, 41, 44, 65]).

One of the challenging problems related to Knödel graphs consists of finding its diameter. In fact the diameter is known only for Knödel graph KG_{2^k} ($W_{k,2^k}$). Fertin et al. [30] proved that the diameter of KG_{2^k} is

$$D_k = \left\lceil \frac{k+2}{2} \right\rceil$$

for any $k \geq 2$. It is important to note that the diameter is unknown even for the Knödel graph KG_{2^k-2}.

In this thesis we consider only Knödel graphs KG_n ($W_{\lfloor \log n \rfloor, n}$). We would like to emphasize that KG_{2^k} is special by its recursive construction ([29] Proposition 3) and that KG_{2^k-2} exhibits some broadcasting properties that become invalid for KG_{2^k-2x} when $x \neq 1$.

2.3 Dimensional broadcast schemes

A *broadcast scheme* for an originator u, is a set of calls through which the broadcasting is performed starting from the only initially informed vertex, vertex u. A *dimensional broadcast scheme* ([8, 44]) is a sequence of dimensions of a graph that shows the order in which vertices will be informed during broadcasting. It can be used for any graph having a dimensional structure (e.g. hypercubes, Knödel graphs, etc.) We use the term *treating dimension* d when at a given time t all informed vertices inform their neighbours of d^{th} dimension. Following a dimensional broadcast scheme, at each time unit, one dimension of the sequence is treated in the given order (note that time may differ from the treated dimension, that is at time t we may not treat dimension t). A *valid broadcast scheme* ([8, 44]) for a graph G on n vertices, is a broadcast scheme that completes broadcasting in $\lceil \log n \rceil$ time units.

Ahlswede et al. [1](Section 6.2) analyzed some dimensional broadcast schemes applying them to Knödel graph KG_{2^k-2}. They implicitly showed that given a basic valid broadcast scheme, one can derive other valid broadcast schemes by applying cyclic shift or reversal operations. Bermond et al. [8] further analyzed valid broadcast schemes expanding the results of [1] to the case of Knödel graphs KG_{2^k-2x} where $x \neq 1$. Harutyunyan and Liestman [44] constructed new broadcast graphs based on Knödel graph KG_{2^k-2} by referring to lemmas from [1] and [8].

In [8], Bermond et al. showed the existence of at least one valid broadcast scheme for KG_{2^k-2x} where $1 \leq x \leq 2^{k-2} - 1$.

Lemma 1 ([8] Lemma 2.2). *The dimensional broadcast scheme formed by the sequence of dimensions*

$$1, 2, \ldots, k-1, 1$$

is a valid broadcast scheme for the Knödel graph KG_{2^k-2x}.

Figure 4 and Table 2 represent an example on how to apply the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ by choosing 0 as the originator to a Knödel graph on $n = 12$ vertices. Figure 4 represents the graph and Table 2 the result of broadcasting. Note that in Table 2 not all vertices inform an uninformed vertex at time 4, that is, there are some idle vertices at this time unit.

In [8] Bermond et al. proved the following theorem that is used to generate new valid dimensional broadcast schemes for KG_{2^k-2x} where $1 \leq x \leq 2^{k-2} - 1$, using the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$.

Theorem 1 ([8] Theorem 2.5). *Any combination of cyclic shifts and reversals of the sequence of dimensions*

$$1, 2, \ldots, k-1$$

followed by the copy of the first dimension obtained after those operations, gives a valid broadcast scheme for KG_{2^k-2x}.

For example,

$$5, \ldots, k-1, 1, \ldots, 4, 5$$

and

$$k-1, k-2, \ldots, 1, k-1$$

and

$$k-3, \ldots, 1, k-1, k-2, k-3$$

are valid broadcast schemes for KG_{2^k-2x}. For KG_{2^k-2x}, the dimensional broadcast schemes formed by any combination of cyclic shifts and reversals of the sequence of dimensions

$$1, 2, \ldots, k-1$$

followed by the copy of the first dimension obtained after those operations, are the only valid broadcast schemes known so far. This is the reason why all lemmas in Chapter 3 are proved using these dimensional broadcast schemes.

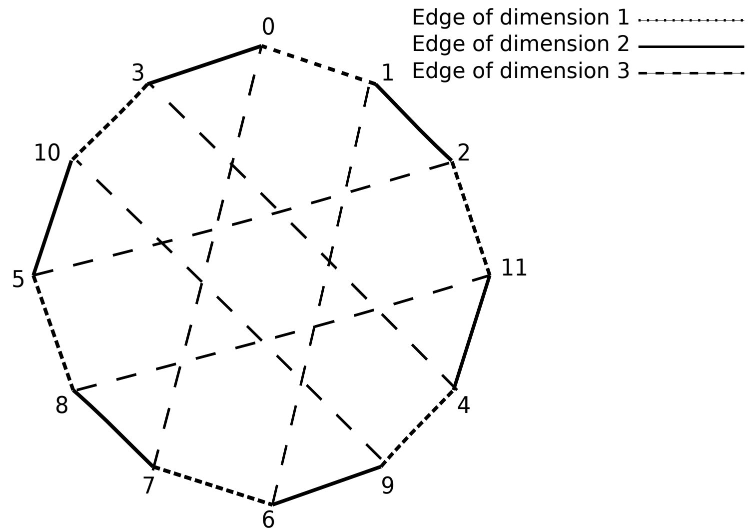

Figure 4: Example of KG_n where $n = 2^4 - 2 \times 2$. Vertices are labelled from 0 to 11 and each vertex has three incident edges that represent each dimension of the graph

Time Unit	**Dimension**	**Informed Vertices**
0	-	0
1	1	0 - $1^{(0)}$
2	2	0 - 1 - $3^{(0)}$ - $2^{(1)}$
3	3	0 - 1 - 3 - 2 - $7^{(0)}$ - $6^{(1)}$ - $5^{(2)}$ - $4^{(3)}$
4	1	0 - 1 - 3 - 2 - 7 - 6 - 5 - 4 - $10^{(3)}$ - $11^{(2)}$ - $8^{(5)}$ - $9^{(4)}$

Table 2: $1, 2, \ldots, k-1, 1$ from originator vertex 0 applied to the graph of Figure 4. The column entitled "Dimension" shows the treated dimension at each time unit. Superscripts on informed vertices show by which vertex they are informed

Chapter 3

A new upper bound on $B(n)$ for odd $n \neq 2^k - 1$

In this chapter, we give a new upper bound on $B(n)$ by applying a vertex deletion method to Knödel graphs on $2^k - 2x$ vertices where $1 \leq x \leq 2^{k-2} - 1$. Section 3.1 is dedicated to the study of dimensional broadcast schemes in Knödel graphs on $2^k - 2x$ vertices resulting in lemmas that are used in the proof of the new bound on $B(n)$. In Section 3.2 we present a construction by vertex deletion on Knödel graphs on $2^k - 2x$ vertices and give a new upper bound on $B(n)$ for all odd $n \neq 2^k - 1$.

3.1 Study of dimensional broadcast schemes in Knödel graphs on $2^k - 2x$ vertices

In this section, we present five lemmas about Knödel graphs on $2^k - 2x$ vertices. Lemma 2 and Lemma 3 are auxiliary lemmas that are used in the proofs of Lemma 4 and Lemma 5. Lemma 4, Lemma 5 and Lemma 6 are the main lemmas that are used in the proof of the new upper bound on $B(n)$ given in Section 3.2.

KG_{2^k-2x} denotes a Knödel graph where the number of vertices, $2^k - 2x$ with $1 \leq x \leq 2^{k-2} - 1$, is not an exact power of 2. KG_{2^k-2x} is $(k-1)$-regular and has a broadcast time of k time units (see [29]).

We study different properties of broadcasting in KG_{2^k-2x} through dimensional broadcast schemes. Dimensional broadcast schemes that are able to finish broadcasting in KG_{2^k-2x} from any originator in $\lceil \log n \rceil = k$ time units, are shown to follow some specific ordering of all $k-1$ dimensions of KG_{2^k-2x} with an extra last dimension that is a copy of the first dimension. In particular, by Lemma 1 (page 16), the dimensional broadcast scheme that is formed by the sequence of dimensions

$$1, 2, \ldots, k-1, 1$$

finishes the broadcasting in KG_{2^k-2x} in k time units. As KG_{2^k-2x} is vertex transitive [29], the originator is not important. By Theorem 1 (page 16), we can obtain other valid broadcast schemes for KG_{2^k-2x} by applying any number of cyclic shift and reversal operations to the sequence of dimensions

$$1, 2, \ldots, k-1$$

and adding the copy of the first dimension obtained after those operations.

Definition 3 (Idle vertex)**.** An *idle vertex* at time unit t is an informed vertex that either informs no other vertices or tries to inform an already informed vertex, during this time unit.

The concept of an *idle vertex* is important because it simplifies the proofs related to dimensional broadcasting. For example, let x and y be two vertices of a graph G and let there be a broadcast scheme such that x informs y at time unit t. We want to prove that y is informed by x only at time unit t, that is, no other vertex informs y before that time unit. If we prove that following this broadcast scheme there is no idle vertex during the first t time units, then we can say that y is only informed by x at time unit t. If any other vertex had informed y before time unit t, x would have been idle at time unit t.

In [1] Lemma 6.1, it is proved that in KG_{2^k-2}, from any originator, following any broadcast scheme formed by any combination of cyclic shifts and reversals of the sequence of dimensions

$$1, 2, \ldots, k-1$$

followed by the copy of the first dimension obtained after those operations, no informed vertex is idle during the first $k-1$ time units. This is not valid for KG_{2^k-2x} where $x \neq 1$. For example, when

$$n = 2^{14} - 2 \times 4095$$

using the broadcast scheme with the sequence of dimensions

$$13, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$$

from originator 0, the first idle vertex occurs at time 4. However in the case of KG_{2^k-2x} with $1 \leq x \leq 2^{k-2} - 1$, the previous lemma can be proved for two specific dimensional broadcast schemes,

$$1, 2, \ldots, k-1, 1$$

and

$$k-1, k-2, \ldots, 1, k-1.$$

.

Lemma 2. *In KG_{2^k-2x}, following the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$, from any originator, no informed vertex is idle during the first $k-1$ time units of broadcasting.*

Proof. Let $a \in V(KG_{2^k-2x})$ be the originator of the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ in KG_{2^k-2x}. In the first part of this proof, we give the general equations of vertices that are informed during the first $k-1$ time units using the dimensional broadcast scheme described above, from originator a. In the second part of this proof, we show that all vertices that are informed during the first $k-1$ time units are distinct.

In the rest of this proof all operations are modulo $n = 2^k - 2x$. Applying the dimensional broadcast scheme with the sequence of dimensions $1, 2, \ldots, k-1, 1$ from originator a, we obtain the following informed vertices during the first 3 time units.

Time 1: a informs $1-a$

Time 2: a informs $3-a$

$1-a$ informs $2+a$

Time 3: a informs $7-a$

$1-a$ informs $6+a$

$3-a$ informs $4+a$

$2+a$ informs $5-a$

It is easy to see that after $k-1$ time units, the informed vertices are

$$\begin{aligned}
&a, \\
&2^{k-1} - p - a \quad \text{for all odd } \ 1 \leq p \leq 2^{k-1} - 1, \\
&2^{k-1} - q + a \quad \text{for all even } \ 2 \leq q \leq 2^{k-1} - 2.
\end{aligned}$$

It remains to show that all of the above vertices are distinct. In order to achieve that we should compare the real labels of the vertices found by evaluating their equations in modulo $n = 2^k - 2x$. Note that for all $1 \leq x \leq 2^{k-2} - 1$,

$$2^{k-1} + 2 \leq n \leq 2^k - 2.$$

n is even, hence it does not change the parity of the previous equations.

Vertex a is of different parity than vertices of equation

$$2^{k-1} - p - a \quad \text{for all odd } \ 1 \leq p \leq 2^{k-1} - 1.$$

So vertex a can only be equal to vertices of equation

$$2^{k-1} - q + a \quad \text{for all even } \ 2 \leq q \leq 2^{k-1} - 2.$$

Note that for any even $2 \leq q \leq 2^{k-1} - 2$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2} - 1$,

$$1 \leq a \leq n-1 \quad \text{and} \quad 0 < 2^{k-1} - q + a < 2n.$$

For any even $2 \leq q \leq 2^{k-1} - 2$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2} - 1$,

$$a < 2 + a \leq 2^{k-1} - q + a,$$

and

$$2^{k-1} - q + a - n \leq a - 4 < a.$$

Hence vertex a is not equal to any vertex of equation $2^{k-1} - q + a$ and it is informed only once during the first $k-1$ time units.

The vertices of equation $2^{k-1} - p - a$ for all odd $1 \leq p \leq 2^{k-1} - 1$ are of different parity than the vertices of equation

$$2^{k-1} - q + a \quad \text{for all even } \ 2 \leq q \leq 2^{k-1} - 2.$$

So any vertex of equation $2^{k-1}-p-a$ can only be equal to the vertices of equation

$$2^{k-1}-p'-a \quad \text{for all odd } \; 1 \le p' \le 2^{k-1}-1 \quad \text{such that } \; p \neq p'.$$

Note that for any odd $1 \le p \le 2^{k-1}-1$, $0 \le a \le n-1$ and $1 \le x \le 2^{k-2}-1$,

$$-n < 2^{k-1}-p-a < n.$$

For any odd $1 \le p, p' \le 2^{k-1}-1$, $0 \le a \le n-1$ and $1 \le x \le 2^{k-2}-1$,

$$2^{k-1}-p-a \neq 2^{k-1}-p'-a \quad \text{and} \quad 2^{k-1}-p-a+n \neq 2^{k-1}-p'-a+n.$$

Moreover for any odd $1 \le p, p' \le 2^{k-1}-1$, $0 \le a \le n-1$ and $1 \le x \le 2^{k-2}-1$,

$$2^{k-1}-p-a \le 2^{k-1}-1-a < 2^{k-1}+3-a \le 2^{k-1}-p'-a+n.$$

Hence vertices of equation $2^{k-1}-p-a$ for all odd $1 \le p \le 2^{k-1}-1$ are informed only once during the first $k-1$ time units.

The vertices of equation $2^{k-1}-q+a$ for all even $2 \le q \le 2^{k-1}-2$ are of different parity than the vertices of equation

$$2^{k-1}-p-a \quad \text{for all odd } \; 1 \le p \le 2^{k-1}-1.$$

So any vertex of equation $2^{k-1}-q+a$ can only be equal to the vertices of equation

$$2^{k-1}-q'+a \quad \text{for all even } \; 2 \le q' \le 2^{k-1}-2 \quad \text{such that } \; q \neq q'.$$

Note that for any even $2 \le q \le 2^{k-1}-2$, $0 \le a \le n-1$ and $1 \le x \le 2^{k-2}-1$,

$$0 < 2^{k-1}-q+a < 2n.$$

For any even $2 \le q, q' \le 2^{k-1}-2$, $0 \le a \le n-1$ and $1 \le x \le 2^{k-2}-1$,

$$2^{k-1}-q+a \neq 2^{k-1}-q'+a \quad \text{and} \quad 2^{k-1}-q+a-n \neq 2^{k-1}-q'+a-n.$$

Moreover for any even $2 \le q, q' \le 2^{k-1}-2$, $0 \le a \le n-1$ and $1 \le x \le 2^{k-2}-1$,

$$2^{k-1}-q+a \ge 2+a > a-4 \ge 2^{k-1}-q'+a-n.$$

Hence vertices of equation $2^{k-1}-q+a$ for all even $2 \le q \le 2^{k-1}-2$ are informed only once during the first $k-1$ time units. $\square$

Lemma 3. *In KG_{2^k-2x}, following the dimensional broadcast scheme formed by the sequence of dimensions $k-1, k-2, \ldots, 1, k-1$, from any originator, no informed vertex is idle during the first $k-1$ time units of broadcasting.*

Proof. Let $a \in V(KG_{2^k-2x})$ be the originator of the dimensional broadcast scheme formed by the sequence of dimensions $k-1, k-2, \ldots, 1, k-1$ in KG_{2^k-2x}. In the first part of this proof, we give the general equations of vertices that are informed during the first $k-1$ time units using the dimensional broadcast scheme described above, from originator a. In the second part of this proof, we show that all vertices that are informed during the first $k-1$ time units are distinct.

In the rest of this proof all operations are modulo $n = 2^k - 2x$. Applying the dimensional broadcast scheme with the sequence of dimensions $k-1, k-2, \ldots, 1, k-1$ from originator a, we obtain the following informed vertices during the first 3 time units.

$$\begin{aligned}
\text{Time 1: } & a \text{ informs } 2^{k-1} - 1 - a \\
\text{Time 2: } & a \text{ informs } 2^{k-2} - 1 - a \\
& 2^{k-1} - 1 - a \text{ informs } -2^{k-2} + a \\
\text{Time 3: } & a \text{ informs } 2^{k-3} - 1 - a \\
& 2^{k-1} - 1 - a \text{ informs } -2^{k-2} - 2^{k-3} + a \\
& 2^{k-2} - 1 - a \text{ informs } -2^{k-3} + a \\
& -2^{k-2} + a \text{ informs } 2^{k-3} + 2^{k-2} - 1 - a
\end{aligned}$$

In the rest of this proof $b_i \in \{0, 1\}$ denotes a binary coefficient for any i. Let

$$B = \{b_{k-2}2^{k-2} + b_{k-3}2^{k-3} + \ldots + b_1 2^1 \mid (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\}.$$

It is easy to see that after $k-1$ time units, the informed vertices are

$$a \quad \text{and} \quad 2^{k-1} - 1 - a$$

and all the vertices

$$b - 1 - a \quad \text{and} \quad -b + a.$$

It remains to show that all of the above vertices are distinct. In order to achieve that we should compare the real labels of the vertices found by evaluating their equations in modulo $n = 2^k - 2x$. Note that for all $1 \leq x \leq 2^{k-2} - 1$,

$$2^{k-1} + 2 \leq n \leq 2^k - 2$$

and for all $b \in B$,

$$2 \leq b \leq 2^{k-1} - 2.$$

n is even, hence it does not change the parity of the previous equations.

Vertex a is of different parity than the vertex of equation

$$2^{k-1} - 1 - a$$

and the vertices of equation

$$b - 1 - a$$

for all $b \in B$. So vertex a can only be equal to the vertices of equation

$$-b + a$$

for all $b \in B$. Note that for any $b \in B$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2} - 1$,

$$1 \leq a \leq n-1 \quad \text{and} \quad -n < -b + a < n.$$

For any $b \in B$ and $0 \leq a \leq n-1$,

$$a \neq -b + a.$$

Moreover for any $b \in B$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2} - 1$,

$$a < 4 + a \leq -b + a + n.$$

Hence vertex a is informed only once during the first $k-1$ time units.

The vertices of equation $-b + a$ for all $b \in B$ are of different parity than the vertex of equation

$$2^{k-1} - 1 - a$$

and the vertices of equation

$$b - 1 - a$$

for all $b \in B$. So any vertex of equation $-b + a$ can only be equal to the vertices of equation

$$-b' + a$$

for all $b \neq b' \in B$. Note that for any $b \in B$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2} - 1$,

$$-n < -b + a < n.$$

For any $b \neq b' \in B$ and $0 \leq a \leq n-1$,

$$-b + a \neq -b' + a \quad \text{and} \quad -b + a + n \neq -b' + a + n.$$

Moreover for any $b \neq b' \in B$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2} - 1$,

$$-b + a \leq a - 2 < 4 + a \leq -b' + a + n.$$

Hence the vertices of equation $-b + a$ for all $b \in B$ are informed only once during the first $k - 1$ time units.

$\underline{\text{The vertex of equation } 2^{k-1} - 1 - a}$ is of different parity than vertex a and the vertices of equation $-b + a$ for all $b \in B$. So the vertex of equation $2^{k-1} - 1 - a$ can only be equal to the vertices of equation

$$b - 1 - a$$

for all $b \in B$. Note that for any $b \in B$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2} - 1$,

$$-n \leq 2^{k-1} - 1 - a < n \quad \text{and} \quad -n < b - 1 - a < n.$$

For any $b \in B$ and $0 \leq a \leq n-1$,

$$2^{k-1} - 1 - a > 2^{k-1} - 3 - a \geq b - 1 - a \quad \text{and} \quad 2^{k-1} - 1 - a + n > b - 1 - a + n.$$

Moreover for any $b \in B$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2} - 1$,

$$2^{k-1} - 1 - a < 2^{k-1} + 3 - a \leq b - 1 - a + n$$

and

$$2^{k-1} - 1 - a + n \geq 2^k + 1 - a > 2^{k-1} - 3 - a \geq b - 1 - a.$$

Hence the vertex of equation $2^{k-1} - 1 - a$ is informed only once during the first $k - 1$ time units.

$\underline{\text{The vertices of equation } b - 1 - a}$ for all $b \in B$ are of different parity than vertex a and the vertices of equation

$$-b + a$$

for all $b \in B$. So any vertex of equation $b - 1 - a$ can only be equal to the vertices of equation

$$b' - 1 - a$$

for all $b \neq b' \in B$. Note that for any $b \in B$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2}-1$,

$$-n < b - 1 + a < n.$$

For any $b \neq b' \in B$ and $0 \leq a \leq n-1$,

$$b - 1 - a \neq b' - 1 - a \quad \text{and} \quad b - 1 - a + n \neq b' - 1 - a + n.$$

Moreover for any $b \neq b' \in B$, $0 \leq a \leq n-1$ and $1 \leq x \leq 2^{k-2}-1$,

$$b - 1 - a \leq 2^{k-1} - 3 - a < 2^{k-1} + 3 - a \leq b' - 1 - a + n.$$

Hence the vertices of equation $b - 1 - a$ for all $b \in B$ are informed only once during the first $k-1$ time units. $\square$

The two previous lemmas, Lemma 2 and Lemma 3, are auxiliary lemmas that help us with the proof of incoming lemmas. The next three lemmas are the main lemmas that are used to prove that the graph we construct in Section 3.2 is a broadcast graph.

In Section 3.2, we construct a graph by deleting a vertex and its incident edges from KG_{2^k-2x} and by adding some other edges. As KG_{2^k-2x} is vertex transitive [29], we choose to delete the vertex of label 0. This new graph has $2^k - 2x - 1$ vertices. So to prove that this new graph is a broadcast graph, one should prove that for any originator there exists a broadcast scheme such that broadcasting can finish in

$$\lceil \log(2^k - 2x - 1) \rceil = k$$

time units. This also corresponds to the broadcast time of KG_{2^k-2x} under some dimensional broadcast schemes. The subsequent lemmas show that for any originator in KG_{2^k-2x}, it is possible to find a dimensional broadcast scheme that finishes broadcasting in k time units and informs vertex 0 either during the last time unit, k or at time unit $k-1$. So in KG_{2^k-2x}, from each originator, following a specific dimensional broadcast scheme, we encounter two situations. Either vertex 0 does not have time to inform any uninformed vertex, in which case deleting vertex 0 does not change anything to the rest of broadcasting and every vertex of the new graph is informed in k time units. Or vertex 0 is informed at time unit $k-1$ and has time to inform one uninformed vertex. In this case we see that using the extra edges added in the new graph, the vertex that was supposed to be informed by vertex 0 can be informed by another vertex and broadcasting in the new graph can still finish in k time units.

The following two lemmas provide the dimensional broadcast schemes that are used for all originators which are neighbours of vertex 0 in KG_{2^k-2x}. The last lemma shows the existence of dimensional broadcast schemes informing vertex 0 at time unit k or $k-1$, for all originators which are non-neighbours of vertex 0.

Note that if a vertex u is a neighbour of vertex 0 of i^{th} dimension with

$$1 \leq i \leq k-1$$

then its equation becomes

$$u = 2^i - 1 \bmod (2^k - 2x) \text{ where } 1 \leq x \leq 2^{k-2} - 1.$$

As $2^i - 1 < n$ for all $1 \leq i \leq k-1$, $\bmod (2^k - 2x)$ may be omitted.

Lemma 4. *Let $u \in V(KG_{2^k-2x})$ and $0 \in V(KG_{2^k-2x})$ be neighbours of dimension i for any $1 \leq i \leq k-1$. If vertex u is the originator of the dimensional broadcast scheme formed by $k-i-1$ right cyclic shifts of the sequence of dimensions $1, 2, \ldots, k-1$ followed by the copy of the first dimension obtained after cyclic shifts, then vertex 0 will be informed at time unit $k-1$ by vertex u (i.e. no other vertex will inform vertex 0 before that time unit).*

Proof. Let $u \in V(KG_{2^k-2x})$ be a neighbour of vertex 0 of dimension i. Then

$$u = 2^i - 1 \text{ for any } 1 \leq i \leq k-1.$$

Let vertex u be the originator of the dimensional broadcast scheme formed by

$$k-i-1$$

right cyclic shifts of the sequence of dimensions

$$1, 2, \ldots, k-1$$

followed by the copy of the first dimension obtained after cyclic shifts. We want to show that during the broadcasting, vertex 0 is informed at time unit $k-1$ by vertex u and that vertex 0 is not informed before time unit $k-1$. Cases where $u = 2^{k-1} - 1$ and $u = 2^i - 1$ with $1 \leq i \leq k-2$ are analyzed separately.

<u>If $u = 2^{k-1} - 1$</u>, $k-i-1$ right cyclic shifts of the sequence of dimensions

$$1, 2, \ldots, k-1$$

followed by the copy of the first dimension becomes the dimensional broadcast scheme formed by the sequence of dimensions

$$1, 2, \ldots, k-1, 1.$$

Vertex $u = 2^{k-1} - 1$ is connected to vertex 0 through dimension $k-1$, so at time $k-1$, when treating dimension $k-1$, vertex u informs vertex 0. It remains to show that vertex 0 is not informed before that time unit. By Lemma 2, following the dimensional broadcast scheme formed by the sequence of dimensions

$$1, 2 \ldots, k-1, 1$$

from any originator, during the first $k-1$ time units there are no idle vertices. So vertex 0 is not informed before time unit $k-1$; otherwise vertex u would have been idle at time unit $k-1$. So vertex 0 is only informed at time unit $k-1$ by vertex u.

If $u = 2^i - 1$ with $1 \leq i \leq k-2$, $k-i-1$ right cyclic shifts of the sequence of dimensions

$$1, 2, \ldots, k-1$$

followed by the copy of the first dimension becomes the dimensional broadcast scheme formed by the sequence of dimensions

$$i+1, i+2, \ldots, k-1, 1, \ldots, i, i+1.$$

Vertex $u = 2^i - 1$ is connected to vertex 0 through dimension i. Hence it informs vertex 0 at time unit $k-1$ when treating dimension i. It remains to show that vertex 0 is not informed before that time unit. To prove that, we first find the equations of all vertices informed during the first $k-2$ time units applying the dimensional broadcast scheme formed by the sequence of dimensions

$$i+1, i+2, \ldots, k-1, 1, \ldots, i, i+1$$

from originator $u = 2^i - 1$ and then show that 0 is not equal to any of them.

In the rest of this proof all operations are modulo $n = 2^k - 2x$. Applying the dimensional broadcast scheme formed by the sequence of dimensions

$$i+1, i+2, \ldots, k-1, 1, \ldots, i, i+1$$

from originator u, we obtain the following informed vertices during the first 3 time units.

Time 1: u informs $2^{i+1} - 1 - u$

Time 2: u informs $2^{i+2} - 1 - u$

$2^{i+1} - 1 - u$ informs $2^{i+1} + u$

Time 3: u informs $2^{i+3} - 1 - u$

$2^{i+1} - 1 - u$ informs $2^{i+2} + 2^{i+1} + u$

$2^{i+2} - 1 - u$ informs $2^{i+2} + u$

$2^{i+1} + u$ informs $2^{i+2} + 2^{i+1} - 1 - u$

In the rest of this proof $b_i \in \{0, 1\}$ denotes a binary coefficient for any i. Let

$$A = \{b_{i-2}2^{i-2} + \ldots + b_1 2^1 \mid 3 \leq i \leq k-2 \ \text{ and } \ (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\}$$

and

$$B = \{b_{k-2}2^{k-2} + \ldots + b_{i+1}2^{i+1} \mid 1 \leq i \leq k-3 \ \text{ and } \ (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\}.$$

It is easy to see that after $k-2$ time units, the informed vertices are

$$u \ \ , \ \ 2^{k-1} - 1 - u \ \ , \ \ 2^{i-1} - 1 - u \ \ , \ \ 2^{i-1} - 2^{k-1} + u$$

and all the vertices

$$b + u \ \ , \ \ b - 1 - u \ \ , \ \ 2^{i-1} - 1 - b - u \ \ , \ \ 2^{i-1} - b + u$$

and all the vertices

$$a + u \ \ , \ \ a - 1 - u \ \ , \ \ a + 2^{k-1} - 1 - u \ \ , \ \ a - 2^{k-1} + u$$

and all the vertices

$$a + b + u \ \ , \ \ a + b - 1 - u \ \ , \ \ a - 1 - b - u \ \ , \ \ a - b + u.$$

Note that A is not defined for $i = 1, 2$, so all vertices that contain a in their equation do not exist when $i = 1, 2$. Also B is not defined for $i = k-2$, so all vertices that contain b in their equation do not exist when $i = k-2$. Moreover the vertices of equations

$$2^{i-1} - 1 - u \ \ , \ \ 2^{i-1} - 2^{k-1} + 1 + u \ \ , \ \ 2^{i-1} - 1 - b - u \ \ , \ \ 2^{i-1} - b + u$$

do not exist if $i = 1$. Replacing u by $2^i - 1$, the informed vertices after $k-2$ time units become the vertices

$$2^i - 1 \quad , \quad 2^{k-1} - 2^i \quad , \quad -2^{i-1} \quad , \quad -2^{k-1} + 2^i + 2^{i-1} - 1$$

and all the vertices

$$b + 2^i - 1 \quad , \quad b - 2^i \quad , \quad -b - 2^{i-1} \quad , \quad -b + 2^i + 2^{i-1} - 1$$

and all the vertices

$$a + 2^i - 1 \quad , \quad a - 2^i \quad , \quad a + 2^{k-1} - 2^i \quad , \quad a - 2^{k-1} + 2^i - 1$$

and all the vertices

$$b + a + 2^i - 1 \quad , \quad b + a - 2^i \quad , \quad -b + a + 2^i - 1 \quad , \quad -b + a - 2^i$$

for all $b \in B$ and $a \in A$.

It remains to show that none of the above vertices are equal to 0. In order to achieve that we should compare the real labels of the vertices found by evaluating their equations in modulo $n = 2^k - 2x$. n is even, hence it does not change the parity of the previous equations. So vertices that can be equal to 0 in mod n are

$$2^{k-1} - 2^i \quad \text{and} \quad -2^{i-1}$$

and all the vertices

$$b - 2^i \quad , \quad -b - 2^{i-1} \quad , \quad a - 2^i \quad , \quad a + 2^{k-1} - 2^i \quad , \quad b + a - 2^i \quad , \quad -b + a - 2^i$$

for all $b \in B$ and $a \in A$. Now we compare all the above vertices to 0 in mod n with $n = 2^k - 2x$ where $1 \leq x \leq 2^{k-2} - 1$. Note that for all $1 \leq x \leq 2^{k-2} - 1$,

$$2^{k-1} + 2 \leq n \leq 2^k - 2.$$

- For all $1 \leq i \leq k-2$ and $1 \leq x \leq 2^{k-2} - 1$,

  $$0 < 2^{k-2} + 2^{k-3} + 2 \leq n - 2^{i-1}.$$

  Moreover the vertex of equation -2^{i-1} does not exist when $i = 1$, so the vertex of equation -2^{i-1} is not equal to 0.

- For all $1 \leq i \leq k-2$ and $1 \leq x \leq 2^{k-2}-1$,

$$0 < 2^{k-1} - 2^i < n,$$

hence the vertex of equation $2^{k-1} - 2^i$ is not equal to 0.

- The vertices of equation $b - 2^i$ and $-b - 2^{i-1}$ exist only when $1 \leq i \leq k-3$. The set B was defined as

$$B = \{b_{k-2}2^{k-2} + \ldots + b_{i+1}2^{i+1} \mid 1 \leq i \leq k-3 \\ \text{and } (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\}.$$

So for all $b \in B$,

$$2^{i+1} \leq b \leq 2^{k-1} - 2^{i+1}.$$

Hence for all $b \in B$ and $1 \leq x \leq 2^{k-2}-1$,

$$0 < 2^i \leq b - 2^i \leq 2^{k-1} - 2^{i+1} - 2^i < n,$$

moreover for all $b \in B$ and $1 \leq x \leq 2^{k-2}-1$,

$$-b - 2^{i-1} \leq -2^{i+1} - 2^{i-1} < 0 \quad \text{and} \quad 0 < 2 + 2^i + 2^{i-1} \leq n - b - 2^{i-1}.$$

So for any $b \in B$, the vertices of equation $b - 2^i$ and $-b - 2^{i-1}$ are not equal to 0.

- The vertices of equation $a - 2^i$ and $a + 2^{k-1} - 2^i$ exist only when $3 \leq i \leq k-2$. The set A was defined as

$$A = \{b_{i-2}2^{i-2} + \ldots + b_1 2^1 \mid 3 \leq i \leq k-2 \\ \text{and } (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\}.$$

So for all $a \in A$,

$$2 \leq a \leq 2^{i-1} - 2.$$

Hence for all $a \in A$ and $1 \leq x \leq 2^{k-2}-1$,

$$a - 2^i \leq -2^{i-1} - 2 < 0 \quad \text{and} \quad 0 < 2^{k-1} - 2^i + 4 \leq n + a - 2^i,$$

moreover for all $a \in A$ and $1 \leq x \leq 2^{k-2}-1$,

$$0 < 2 + 2^{k-1} - 2^i \leq a + 2^{k-1} - 2^i \leq 2^{k-1} - 2^{i-1} - 2 < n.$$

So for any $a \in A$, the vertices of equation $a - 2^i$ and $a + 2^{k-1} - 2^i$ are not equal to 0.

- The vertices of equation $b + a - 2^i$ and $-b + a - 2^i$ exist only when

$$3 \leq i \leq k - 3.$$

For all $a \in A$, $b \in B$ and $1 \leq x \leq 2^{k-2} - 1$,

$$0 < 2^i + 2 \leq b + a - 2^i \leq 2^{k-1} - 2^{i+1} - 2^{i-1} - 2 < n,$$

moreover for all $a \in A$, $b \in B$ and $1 \leq x \leq 2^{k-2} - 1$,

$$-b + a - 2^i \leq -2^{i+1} - 2^{i-1} - 2 < 0 \quad \text{and} \quad 0 < 2^i + 4 \leq n - b + a - 2^i.$$

So for any $a \in A$ and $b \in B$, the vertices of equation $b + a - 2^i$ and $-b + a - 2^i$ are not equal to 0.

Hence vertex 0 is informed at time unit $k - 1$ by vertex u. □

Lemma 5. *Let $u \in V(KG_{2^k-2x})$ and $0 \in V(KG_{2^k-2x})$ be neighbours of dimension i for any $1 \leq i \leq k - 1$. If vertex u is the originator of the dimensional broadcast scheme formed by $i - 1$ right cyclic shifts of the reversal of the sequence of dimensions*

$$1, 2, \ldots, k - 1$$

followed by the copy of the first dimension obtained after cyclic shifts, then vertex 0 will be informed at time unit $k - 1$ by vertex u (i.e. no other vertex will inform vertex 0 before that time unit).

Proof. Let $u \in V(KG_{2^k-2x})$ be a neighbour of vertex 0 of dimension i, then $u = 2^i - 1$ for any $1 \leq i \leq k - 1$. Let vertex u be the originator of the dimensional broadcast scheme formed by $i - 1$ right cyclic shifts of the reversal of the sequence of dimensions

$$1, 2, \ldots, k - 1$$

followed by the copy of the first dimension obtained after cyclic shifts. We want to show that during the broadcasting vertex 0 is informed at time unit $k - 1$ by vertex u and that vertex 0 is not informed before time unit $k - 1$. Cases where $u = 2^1 - 1$ and $u = 2^i - 1$ with $i \neq 1$ are analyzed separately.

<u>If $u = 2^1 - 1$</u>, $i - 1$ right cyclic shifts of the reversal of the sequence of dimensions

$$1, 2, \ldots, k - 1$$

followed by the copy of the first dimension becomes the dimensional broadcast scheme formed by the sequence of dimensions

$$k-1, k-2, \ldots, 1, k-1.$$

Vertex $u = 2^1 - 1$ is connected to vertex 0 through dimension 1, so at time unit $k-1$, when treating dimension 1, vertex u will inform vertex 0. It remains to show that vertex 0 is not informed before that time unit. By Lemma 3, following the dimensional broadcast scheme formed by the sequence of dimensions

$$k-1, k-2, \ldots, 1, k-1$$

from any originator, during the first $k-1$ time units there are no idle vertices. So vertex 0 is not informed before time unit $k-1$; otherwise vertex u would have been idle at time unit $k-1$. So vertex 0 is only informed at time unit $k-1$ by vertex u.

If $u = 2^i - 1$ with $2 \leq i \leq k-1$, $i-1$ right cyclic shifts of the reversal of the sequence of dimensions

$$1, 2, \ldots, k-1$$

followed by the copy of the first dimension becomes the dimensional broadcast scheme formed by the sequence of dimensions

$$i-1, i-2, \ldots, 1, k-1, k-2, \ldots, i, i-1.$$

Vertex $u = 2^i - 1$ is connected to vertex 0 through dimension i. Hence it will inform vertex 0 at time unit $k-1$ when treating dimension i. It remains to show that vertex 0 is not informed before that time unit. To prove that, we first find the equations of all vertices informed during the first $k-2$ time units by applying the dimensional broadcast scheme formed by the sequence of dimensions

$$i-1, i-2, \ldots, 1, k-1, k-2, \ldots, i, i-1$$

from originator $u = 2^i - 1$ and then show that 0 is not equal to any of them.

In the rest of this proof all operations are modulo $n = 2^k - 2x$. Applying the dimensional broadcast scheme formed by the sequence of dimensions

$$i-1, i-2, \ldots, 1, k-1, k-2, \ldots, i, i-1$$

from originator u, we obtain the following informed vertices during the first 3 time units.

$$\begin{aligned}
\text{Time 1: } & u \text{ informs } 2^{i-1}-1-u \\
\text{Time 2: } & u \text{ informs } 2^{i-2}-1-u \\
& 2^{i-1}-1-u \text{ informs } -2^{i-2}+u \\
\text{Time 3: } & u \text{ informs } 2^{i-3}-1-u \\
& 2^{i-1}-1-u \text{ informs } -2^{i-2}-2^{i-3}+u \\
& 2^{i-2}-1-u \text{ informs } -2^{i-3}+u \\
& -2^{i-2}+u \text{ informs } 2^{i-3}+2^{i-2}-1-u
\end{aligned}$$

In the rest of this proof $b_i \in \{0,1\}$ denotes a binary coefficient for any i. Let

$$A = \{b_{i-2}2^{i-2}+\ldots+b_1 2^1 \mid 3 \le i \le k-1 \ \text{ and } \ (b_{k-2},\ldots,b_{i+1}) \neq (0,\ldots,0)\}$$

and

$$B = \{b_{k-2}2^{k-2}+\ldots+b_{i+1}2^{i+1} \mid 2 \le i \le k-3 \ \text{ and } \ (b_{k-2},\ldots,b_{i+1}) \neq (0,\ldots,0)\}$$

It is easy to see that after $k-2$ time units, the informed vertices are

$$u \ , \ 2^{i-1}-1-u \ , \ 2^{k-1}-1-u \ , \ 2^{k-1}-2^{i-1}+u$$

and all the vertices

$$a-1-u \ , \ -a+u \ , \ 2^{k-1}-a+u \ , \ 2^{k-1}-1+a-u$$

and all the vertices

$$b-1-u \ , \ b-2^{i-1}+u \ , \ -b+u \ , \ -b+2^{i-1}+1+u$$

and all the vertices

$$b-a+u \ , \ b+a-1-u \ , \ -b+a-1-u \ , \ -b-a+u.$$

Note that A is not defined for $i=2$, so all vertices that contain a in their equation do not exist when $i=2$. Also B is not defined for $i=k-1,k-2$, so all vertices that contain b in their equation do not exist when $i = k-1, k-2$. Moreover the vertices of equations

$$2^{k-1} - 1 - u \quad , \quad 2^{k-1} - 2^{i-1} + u \quad , \quad 2^{k-1} - a + u \quad , \quad 2^{k-1} - 1 + a - u$$

do not exist if $i = k-1$. Replacing u by $2^i - 1$, the informed vertices after $k-2$ time units, become the vertices

$$2^i - 1 \quad , \quad -2^{i-1} \quad , \quad 2^{k-1} - 2^i \quad , \quad 2^{k-1} + 2^{i-1} - 1$$

and all the vertices

$$a - 2^i \quad , \quad -a - 2^i + 1 \quad , \quad 2^{k-1} + 2^i - a - 1 \quad , \quad 2^{k-1} - 2^i + a$$

and all the vertices

$$b - 2^i \quad , \quad b + 2^{i-1} - 1 \quad , \quad 2^i - 1 - b \quad , \quad 2^i + 2^{i-1} - b$$

and all the vertices

$$2^i - 1 + b - a \quad , \quad -2^i + b + a \quad , \quad -2^i - b + a \quad , \quad 2^i - 1 - b - a$$

for all $a \in A$ and $b \in B$.

It remains to show that none of the above vertices are equal to 0. In order to achieve that we should compare the real labels of those vertices found by evaluating their equations in modulo $n = 2^k - 2x$. n is even, hence it does not change the parity of the previous equations. So vertices that can be equal to 0 in mod n are

$$-2^{i-1} \quad \text{and} \quad 2^{k-1} - 2^i$$

and all the vertices

$$a - 2^i \quad , \quad 2^{k-1} - 2^i + a \quad , \quad b - 2^i \quad , \quad 2^i + 2^{i-1} - b \quad , \quad -2^i + b + a \quad , \quad -2^i - b + a$$

for all $b \in B$ and $a \in A$. Now we compare all the above vertices to 0 in mod n with $n = 2^k - 2x$ where $1 \leq x \leq 2^{k-2} - 1$. Note that for all $1 \leq x \leq 2^{k-2} - 1$,

$$2^{k-1} + 2 \leq n \leq 2^k - 2.$$

- For all $2 \le i \le k-2$ and $1 \le x \le 2^{k-2}-1$,

$$0 < 2^{k-2} \le 2^{k-1} - 2^i \le 2^{k-1} - 2 < n.$$

Moreover the vertex of equation $2^{k-1} - 2^i$ does not exist when $i = k-1$, so the vertex of equation $2^{k-1} - 2^i$ is not equal to 0.

- For all $2 \le i \le k-1$ and $1 \le x \le 2^{k-2}-1$,

$$-2^{i-1} < 0 \quad \text{and} \quad 0 < 2^{k-1} \le n - 2^{i-1},$$

hence the vertex of equation -2^{i-1} is not equal to 0.

- The vertices of equation $a - 2^i$ and $a + 2^{k-1} - 2^i$ exist only when $3 \le i \le k-1$. The set A was defined as

$$A = \{b_{i-2}2^{i-2} + \ldots + b_1 2^1 \mid 3 \le i \le k-1 \quad \text{and} \quad (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\}.$$

So for all $a \in A$,

$$2 \le a \le 2^{i-1} - 2.$$

Hence for all $a \in A$ and $1 \le x \le 2^{k-2}-1$,

$$a - 2^i \le -2^{i-1} - 2 < 0 \quad \text{and} \quad 0 < 2^{k-1} - 2^i + 4 \le n + a - 2^i,$$

moreover for all $a \in A$ and $1 \le x \le 2^{k-2}-1$,

$$0 < 2 + 2^{k-1} - 2^i \le a + 2^{k-1} - 2^i \le 2^{k-1} - 2^{i-1} - 2 < n.$$

So for any $a \in A$, the vertices of equation $a - 2^i$ and $a + 2^{k-1} - 2^i$ are not equal to 0.

- The vertices of equation $b - 2^i$ and $2^i + 2^{i-1} - b$ exist only when $2 \le i \le k-3$. The set B was defined as

$$B = \{b_{k-2}2^{k-2} + \ldots + b_{i+1}2^{i+1} \mid 2 \le i \le k-3 \quad \text{and} \quad (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\}.$$

So for all $b \in B$,

$$2^{i+1} \le b \le 2^{k-1} - 2^{i+1}.$$

Hence for all $b \in B$ and $1 \leq x \leq 2^{k-2} - 1$,

$$0 < 2^i \leq b - 2^i \leq 2^{k-1} - 2^{i+1} - 2^i < n,$$

moreover for all $b \in B$ and $1 \leq x \leq 2^{k-2} - 1$,

$$2^i + 2^{i-1} - b \leq -2^{i-1} < 0 \quad \text{and} \quad 0 < 2^{i+1} + 2^i + 2^{i-1} + 2 \leq n + 2^i + 2^{i-1} - b.$$

So for any $b \in B$, the vertices of equation $b - 2^i$ and $2^i + 2^{i-1} - b$ are not equal to 0.

- The vertices of equation $b + a - 2^i$ and $-b + a - 2^i$ exist only when

$$3 \leq i \leq k - 3.$$

For all $a \in A$, $b \in B$ and $1 \leq x \leq 2^{k-2} - 1$,

$$0 < 2^i + 2 \leq b + a - 2^i \leq 2^{k-1} - 2^{i+1} - 2^{i-1} - 2 < n,$$

moreover for all $a \in A$, $b \in B$ and $1 \leq x \leq 2^{k-2} - 1$,

$$-b + a - 2^i \leq -2^{i+1} - 2^{i-1} - 2 < 0 \quad \text{and} \quad 0 < 2^i + 4 \leq n - b + a - 2^i.$$

So for any $a \in A$ and $b \in B$, the vertices of equation $b + a - 2^i$ and $-b + a - 2^i$ are not equal to 0.

Hence vertex 0 is informed at time unit $k - 1$ by vertex u. $\square$

The following lemma is used in Section 3.2 to prove that there exists a valid broadcast scheme for any non-neighbour originator of vertex 0 that informs vertex 0 at time unit k or $k - 1$. In the case where vertex 0 is informed at time unit $k - 1$, it is idle at time unit k hence it does not inform a new vertex.

In [1] Lemma 6.2, it is proved that if

$$a, b \in V(KGN_{2^k-2}) \quad \text{and} \quad (a, b) \notin E(KGN_{2^k-2}),$$

then there is a valid broadcast scheme for originator a such that the message reaches vertex b at the last time unit.

Given a Knödel graph on $n = 2^k - 2x$ vertices where $x \neq 1$, the previous lemma is not always valid. When x increases, circles of small length appear on KG_{2^k-2x},

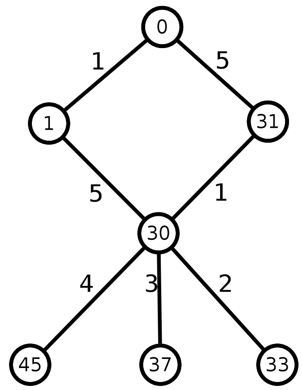

Figure 5: Subgraph of KG_{60} where $n = 2^6 - 2 \times 2$

forcing some vertices to be informed before the last time unit using any valid broadcast scheme found so far. Figure 5 is an example of a subgraph of such a Knödel graph where

$$n = 2^6 - 2 \times 2 = 60.$$

The numbers in nodes represent vertex labels and the numbers on edges represent dimensions. Let vertex 0 be the originator of a broadcast scheme in this Knödel graph. If dimension 1 is used before dimension 5 during the first $k-1$ time units, vertex 30 will be informed through vertex 1 before the last time unit k; and, if dimension 5 is used before dimension 1, during the first $k-1$ time units, vertex 30 will be informed through vertex 31 before the last time unit k. In all valid broadcast schemes known so far (any combination of cyclic shifts and reversals of the sequence of dimensions

$$1, 2, \ldots, k-1, 1,$$

followed by the copy of the first dimension obtained after those operations) every dimension of the graph is used once during the first $k-1$ time units. So for KG_{2^k-2x} where $x \neq 1$ the following lemma is used.

Lemma 6. *Let KG_{2^k-2x} be a Knödel graph where $1 \leq x \leq 2^{k-2} - 1$. If*

$$a, b \in V(KG_{2^k-2x}) \quad \textit{and} \quad (a,b) \notin E(KG_{2^k-2x}),$$

then either

- *there is a valid broadcast scheme for originator a in KG_{2^k-2x} such that the message reaches vertex b at time unit k, that is, at the last time unit or,*

- *there is a valid broadcast scheme for originator a in KG_{2^k-2x} such that the message reaches vertex b at time unit $k-1$ and vertex b is idle at time unit k.*

Proof. Let KG_{2^k-2x} be a Knödel graph where $1 \leq x \leq 2^{k-2}-1$ and let

$$a, b \in V(KG_{2^k-2x}) \quad \text{and} \quad (a,b) \notin E(KG_{2^k-2x}).$$

When $x = 1$ there is a valid broadcast scheme for originator a such that the message reaches vertex b at the last time unit (see [1] Lemma 6.2). In the remainder of this proof,

$$2 \leq x \leq 2^{k-2} - 1.$$

KG_{2^k-2x} is vertex transitive [29], so if a valid broadcast scheme fulfils the above requirements for originator 0, the same scheme can be applied to other originators and the same result is obtained. For example, assume that we found some dimensional broadcast schemes for originator 0 such that when we apply all of them, all non-neighbours of vertex 0 are either informed at time unit k or at time unit $k-1$ but are idle at time unit k. Now pinpoint all the vertices that are non-neighbours of vertex 0 and delete the labels of all vertices. Take a random vertex label $1 \leq u \leq 2^k - 2x - 1$ and rename the vertex that was labelled 0 by u. Rename all other vertices following the dimensions and the Knödel graph equation. The new vertex labelling corresponds to a Knödel graph. Moreover, all edges and dimensions of the graph are kept the same. Hence choosing vertex u as originator and applying the same broadcast schemes that were found for originator 0, we can inform all pinpointed vertices in k or in $k-1$ time units. The vertices that are informed at time unit $k-1$ will still be idle at time unit k. And all the pinpointed vertices correspond to all vertices that are non-neighbours of vertex u. Hence without loss of generality in this proof, vertex 0 is the originator. We next find a set of valid broadcast schemes such that following at least one broadcast scheme of this set, any vertex that is non-neighbour of vertex 0 is either informed only at time unit k or, at time unit $k-1$ but is idle at time unit k.

In the first part, we find the vertices informed only at time unit k or, at time unit $k-1$ but idle at time unit k by applying the dimensional broadcast scheme formed by the sequence of dimensions

$$1, 2, \ldots, k-1, 1$$

from originator 0. In the second part, we find the vertices informed only at time unit k by applying the dimensional broadcast scheme formed by the sequence of dimensions

$$k-1, k-2, \ldots, 1, k-1$$

from originator 0. In the third part, we find the vertices informed only at time unit k by applying the dimensional broadcast schemes formed by the sequence of dimensions

$$i+1, i+2, \ldots, k-1, 1, \ldots, i, i+1$$

for all $1 \leq i \leq k-2$ from originator 0. Finally, we show that the set formed by the vertices found in the first, second and third part is equal to the set of vertices that are not neighbours of vertex 0.

In the rest of this proof all operations are modulo $n = 2^k - 2x$.

First we apply the dimensional broadcast scheme formed by the sequence of dimensions

$$\underline{(1, 2, \ldots, k-1, 1)}$$

from originator 0 and find all vertices informed at time unit k or $k-1$ but idle at time unit k. By Lemma 1 (page 16), this dimensional broadcast scheme, from any originator, finishes informing all vertices of KG_{2^k-2x} in $\lceil \log(2^k - 2x) \rceil = k$ time units. To find vertices that are only informed at time unit k, we find all vertices informed during the first $k-1$ time units and all the remaining vertices of KG_{2^k-2x} form the set of vertices informed only at time unit k. This set is

$$U_k = \{\text{vertices informed at time unit } k \quad \text{applying the sequence of dimensions } 1, \ldots, k-1, 1\}.$$

To find all vertices informed only at time unit $k-1$, we first find the set of vertices informed after $k-2$ time units and subtract it from the set of vertices informed after $k-1$ time units. Next, we check which of these vertices are idle at time unit k. In order to achieve this, we find the vertices informed at time unit k by vertices informed only at time unit $k-1$ and then find the ones that were already informed before time unit k by comparing them to the set of vertices informed after $k-1$ time units. The set of vertices informed at time unit $k-1$ and idle at time unit k is

$$U_{k-1} = \{\text{vertices informed at time unit } k-1 \text{ and idle at time unit } k \quad \text{applying the sequence of dimensions } 1, \ldots, k-1, 1\}.$$

Now we find the vertices informed at time unit k. It is easy to see that, using the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ from originator 0, after $k-1$ time units, the informed vertices are

$$0$$

and all the vertices

$$2^{k-1}-1, \quad 2^{k-1}-3, \quad \ldots, \quad 1$$

and all the vertices

$$2^{k-1}-2, \quad 2^{k-1}-4, \quad \ldots, \quad 2.$$

Note that, for all $1 \leq x \leq 2^{k-2}-1$, all the above values are between 0 and $n-1$, so they represent the actual labels of the vertices. These vertices are all the vertices between 0 and $2^{k-1}-1$. By Lemma 1 (page 16), the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ finishes broadcasting in KG_{2^k-2x} in $\lceil \log(2^k-2x) \rceil = k$ time units from any originator. Hence all vertices with labels bigger than $2^{k-1}-1$ are informed during the last time unit. So the set of vertices informed at time unit k is

$$U_k = \{u \in V(KG_{2^k-2x}) \mid 2^{k-1} \leq u \leq 2^k - 2x - 1\}.$$

The next step is to find the vertices informed only at time unit $k-1$ and idle at time unit k. For that, first we find the vertices informed at time unit $k-1$ by figuring out the set of vertices informed during the first $k-2$ time units and subtracting this set from the set of vertices informed during the first $k-1$ time units. Applying the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ from originator 0, it is easy to see that after $k-2$ time units, the informed vertices are

$$0$$

and all the vertices

$$2^{k-2}-1, \quad 2^{k-2}-3, \quad \ldots, \quad 1$$

and all the vertices

$$2^{k-2}-2, \quad 2^{k-2}-4, \quad \ldots, \quad 2.$$

Note that, for all $1 \leq x \leq 2^{k-2}-1$, all the above values are between 0 and $n-1$, so they represent the actual labels of the vertices. Those vertices are all the vertices between 0 and $2^{k-2}-1$. The difference between the vertices informed after $k-1$ time units and $k-2$ time units provides the vertices informed only at time unit $k-1$. Hence the vertices informed at time unit $k-1$ are all $v \in V(KG_{2^k-2x})$ such that

$$2^{k-2} \leq v \leq 2^{k-1}-1.$$

We should check which of those vertices are idle at time unit k (i.e. tries to inform an already informed vertex). Let u be a vertex informed at time unit k by any vertex $2^{k-2} \leq v \leq 2^{k-1}-1$. The dimension used during the last time unit of the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ is the dimension 1, so for any $2^{k-2} \leq v \leq 2^{k-1}-1$ and $2 \leq x \leq 2^{k-2}-1$,

$$u = 2^1 - 1 - v = n + 1 - v = 2^k - 2x + 1 - v.$$

Hence the vertices informed at time unit k by the vertices informed at time unit $k-1$ are all $u \in V(KG_{2^k-2x})$ such that

$$2^{k-1} - 2x + 2 \leq u \leq 2^{k-1} + 2^{k-2} - 2x + 1.$$

Now we need to find which of those vertices were already informed before time unit k. We know that all vertices informed after $k-1$ time units are all the vertices between 0 and $2^{k-1}-1$. Moreover for all $2 \leq x \leq 2^{k-2}-1$,

$$0 < 2^{k-1} - 2x + 2 \leq 2^{k-1} - 1 \leq 2^{k-1} + 2^{k-2} - 2x + 1.$$

So the vertices that are informed at time unit k by the vertices informed at time unit $k-1$, and that were already informed before time unit k are all $u \in V(KG_{2^k-2x})$ such that

$$2^{k-1} - 2x + 2 \leq u \leq 2^{k-1} - 1.$$

The vertices that inform any of the above vertices at time unit k are the idle vertices at that time unit. Now we need to find the subset of vertices that are informed at time unit $k-1$ such that at time unit k they inform a vertex $2^{k-1} - 2x + 2 \leq u \leq 2^{k-1} - 1$. Let v' be a vertex that informs $2^{k-1} - 2x + 2 \leq u \leq 2^{k-1} - 1$ at time unit k, then we know that for all $2 \leq x \leq 2^{k-2} - 1$,

$$v' = 2^k - 2x + 1 - u.$$

Hence for all $2 \leq x \leq 2^{k-2} - 1$, all vertices that are informed at time unit $k-1$ and idle at time unit k are

$$2^{k-2} - 2x + 2 \leq v' \leq 2^{k-1} - 1.$$

So the set of vertices informed at time unit $k-1$ and idle at time unit k is

$$U_{k-1} = \{u \in V(KG_{2^k - 2x}) \mid 2^{k-2} - 2x + 2 \leq u \leq 2^{k-1} - 1\}.$$

In conclusion, if we apply the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ from originator 0, the set of vertices that are either informed during time unit k or informed at time unit $k-1$ but idle at time unit k is

$$U_k \cup U_{k-1} = \{u \in V(KG_{2^k - 2x}) \mid 2^{k-1} - 2x + 2 \leq u \leq 2^k - 2x - 1\}.$$

In the rest of the proof, we show that there exist other valid dimensional broadcast schemes for originator 0 such that all vertices that are non-neighbours of vertex 0 in the range $[0, 2^{k-1} - 2x + 1]$ are informed at time unit k.

In the second part of the proof, we apply the dimensional broadcast scheme formed by the sequence of dimensions $\underline{(k-1, k-2, \ldots, 1, k-1)}$ from originator 0 and find all vertices

$$0 \leq v \leq 2^{k-1} - 2x + 1$$

informed at time unit k. This dimensional broadcast scheme is the reversal of the sequence of dimensions $1, 2, \ldots, k-1$ followed by the copy of the first dimension obtained after the reversal operation. By Theorem 1 (page 16), it is a valid broadcast scheme and finishes informing all vertices of $KG_{2^k - 2x}$ in $\lceil \log(2^k - 2x) \rceil = k$ time units from any originator. To find vertices that are only informed at time unit k, we find all vertices informed during the first $k-1$ time units and all the remaining vertices of $KG_{2^k - 2x}$ form the set of vertices informed only at time unit k. This set is

$$\begin{aligned} V_k = \{&\text{vertices in the range } [0, 2^{k-1} - 2x + 1] \text{ informed at time unit } k \\ &\qquad\qquad \text{applying the sequence of dimensions } k-1, \ldots, 1, k-1\}. \end{aligned}$$

In the rest of this proof $b_i \in \{0, 1\}$ denotes a binary coefficient for any i. Let

$$A = \{b_{k-2} 2^{k-2} + \ldots + b_1 2^1 \mid (b_{k-2}, \ldots, b_1) \neq (0, \ldots, 0)\}$$

It is easy to see that, using the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ from originator 0, after $k-1$ time units, the informed vertices are

$$0 \quad \text{and} \quad 2^{k-1} - 1$$

and all the vertices

$$a - 1 \quad \text{and} \quad -a$$

for all $a \in A$. The dimensional broadcast scheme formed by the sequence of dimensions $(k-1, k-2, \ldots, 1, k-1)$ is a valid broadcast scheme so all of the remaining vertices are informed during the last time unit, k. In order to find the range of vertices informed at time unit k, we need to find the range of vertices informed after $k-1$ time units in modulo $n = 2^k - 2x$. n is even, hence it does not change the parity of the previous equations. A represents all even vertices in the range $[2, 2^{k-1} - 2]$, hence the vertices of equation $a - 1$ represent all odd vertices in the range

$$[1, 2^{k-1} - 3]$$

and the vertices of equation $-a = n - a$ represent all even vertices in the range

$$[2^{k-1} - 2x + 2, 2^k - 2x - 2].$$

For $x > 2$, the intersection of the above intervals is not null. So the range of the vertices informed after $k-1$ time units can be analyzed by dividing the interval $[0, 2^k - 2x - 1]$ into four distinct intervals. Hence the vertices informed after $k-1$ time units are

$0 \quad \text{and} \quad 2^{k-1} - 1$,
and all odd vertices in the range $[1, 2^{k-1} - 2x + 1]$,
and all vertices in the range $[2^{k-1} - 2x + 2, 2^{k-1} - 3]$,
and all even vertices in the range $[2^{k-1} - 2, 2^k - 2x - 2]$.

Note that in the case where $x = 2$, the interval $[2^{k-1} - 2x + 2, 2^{k-1} - 3]$ is undefined, hence the vertices that are in this range do not exist. We can see that all remaining vertices, that is, vertices informed only at time unit k are all the even vertices in the range

$$[2, 2^{k-1} - 2x]$$

and all the odd vertices in the range

$$[2^{k-1} - 1, 2^k - 2x - 3].$$

We are looking for vertices informed at time unit k in the range

$$[0, 2^{k-1} - 2x + 1].$$

For all $2 \leq x \leq 2^{k-2} - 1$,

$$2^{k-1} - 2x < 2^{k-1} - 2x + 1$$

and for all $2 \leq x \leq 2^{k-2} - 1$,

$$2^{k-1} - 1 > 2^{k-1} - 2x + 1.$$

So the set of vertices informed by the dimensional broadcast scheme formed by the sequence of dimensions $k-1, k-2, \ldots, 1, k-1$ from originator 0, at time unit k in the range $[0, 2^{k-1} - 2x + 1]$ is

$$V_k = \{u \in V(KG_{2^k - 2x}) \mid u \text{ is even and } 2 \leq u \leq 2^{k-1} - 2x\}.$$

In conclusion of this second part, if we apply the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ and the dimensional broadcast scheme formed by the sequence of dimensions $k-1, k-2, \ldots, 1, k-1$ from originator 0, the set of vertices that are either informed during time unit k or informed at time unit $k-1$ but idle at time unit k is

$$\begin{aligned} U_k \cup U_{k-1} \cup V_k = \{u \in V(KG_{2^k - 2x}) \mid \text{even } u \in [2, 2^{k-1} - 2x] \\ \text{and } u \in [2^{k-1} - 2x + 2, 2^k - 2x - 1]\}. \end{aligned}$$

In the rest of the proof, we show that there exist other valid dimensional broadcast schemes for originator 0 such that all odd non-neighbours of vertex 0 in the range $[1, 2^{k-1} - 2x + 1]$ are informed at time unit k. Since 1 and 3 are neighbours of vertex 0, this range can be further reduced to $[5, 2^{k-1} - 2x + 1]$.

In the third part of the proof, we apply all dimensional broadcast schemes formed by the sequence of dimensions $\underline{(i+1, \ldots, k, 1, \ldots, i, i+1)}$ for all $2 \leq i \leq k-2$ from originator 0 and find all odd vertices

$$5 \leq v \leq 2^{k-1} - 2x + 1$$

that are informed at time unit k. In order to accomplish that, we first find the equation as a function of i of some vertices that are only informed at time unit k, then we show that by changing the value of i between 2 and $k-2$ those vertices are equal to nearly all odd vertices in the range $[5, 2^{k-1} - 2x + 1]$. The set of those vertices is

$$W_k = \{ \text{ odd vertices in the range } [5, 2^{k-1} - 2x + 1] \text{ informed at time unit } k$$
$$\text{applying the sequence of dimensions } i{+}1, \ldots, k, 1, \ldots, i, i{+}1 \text{ for all } 2 \leq i \leq k{-}2\}.$$

Finally we show that the missing vertices are all neighbours of vertex 0. The dimensional broadcast schemes formed by the sequence of dimensions

$$i+1, \ldots, k, 1, \ldots, i, i+1$$

correspond to $k-i-1$ right cyclic shift of the sequence of dimensions $1, 2, \ldots, k-1$ followed by the copy of the first dimension obtained after cyclic shifts. By Theorem 1 (page 16), they are valid broadcast schemes and they finish informing all vertices of KG_{2^k-2x} in $\lceil \log(2^k - 2x) \rceil = k$ time units from any originator. We first give the equation of all vertices informed in the first $k-1$ time units. In the rest of this proof $b_i \in \{0, 1\}$ denotes a binary coefficient for any i. Let

$$B = \{b_{i-1}2^{i-1} + \ldots + b_1 2^1 \mid 2 \leq i \leq k-2 \text{ and } (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\}$$

and

$$C = \{b_{k-2}2^{k-2} + \ldots + b_{i+1}2^{i+1} \mid 1 \leq i \leq k-3 \text{ and } (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\}.$$

It is easy to see that applying the dimensional broadcast scheme formed by the sequence of dimensions $i+1, \ldots, k, 1, \ldots, i, i+1$ from originator 0, after $k-1$ time units, the informed vertices are

$$0 \ , \ 2^{k-1} - 1 \ , \ 2^i - 1 \ , \ 2^i - 2^{k-1}$$

and all the vertices

$$c \ , \ c-1 \ , \ 2^i - 1 - c \ , \ 2^i - c$$

and all the vertices

$$b \ , \ b-1 \ , \ b + 2^{k-1} - 1 \ , \ b - 2^{k-1}$$

and all the vertices

$$b+c \quad , \quad b+c-1 \quad , \quad b-c-1 \quad , \quad b-c$$

for all $b \in B$ and $c \in C$. Note that C is not defined for $i = k-2$, so all vertices that contain c in their equation do not exist when $i = k-2$. Of all the vertices informed during the first $k-1$ time units, we pick a particular group of vertices, vertices of equation b for all $b \in B$. Next we show that the vertices informed at time unit k by the vertices of equation b are not informed before that time unit. Then we show that by changing the value of variable i between 2 and $k-2$, the vertices informed only at time unit k by the vertices of equation b are equal to all non-neighbours of 0 in the range $[5, 2^{k-1} - 2x + 1]$. At time unit k, while treating dimension $i+1$, for all $b \in B$, the vertices of equation b will inform the vertices of equation

$$2^{i+1} - 1 - b.$$

In order to prove that the vertices of equation $2^{i+1} - 1 - b$ for all $b \in B$ are informed only at the time unit k, we need to compare them with all the vertices informed during the first $k-1$ time units in modulo $n = 2^k - 2x$. n is even, hence it does not change the parity of the expression of vertices. The vertices of equation $2^{i+1} - 1 - b$ are odd vertices and they can be equal only to other odd vertices. Hence the vertices informed during the first $k-1$ time units that can be equal to the vertices of equation $2^{i+1} - 1 - b$ are

$$2^{k-1} - 1 \quad , \quad 2^i - 1$$

and all the vertices

$$c-1 \quad , \quad 2^i - 1 - c \quad , \quad b-1 \quad , \quad b + 2^{k-1} - 1 \quad , \quad b+c-1 \quad \text{and} \quad b-c-1,$$

for all $b \in B$ and $c \in C$. Note that for all $2 \leq x \leq 2^{k-2} - 1$,

$$2^{k-1} + 2 \leq n \leq 2^k - 4,$$

for all $c \in C$,

$$2^{i+1} \leq c \leq 2^{k-1} - 2^{i+1},$$

and for all $b \in B$,

$$2 \leq b \leq 2^i - 2.$$

Hence for all $b \in B$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

$$0 < 2^i + 1 \leq 2^{i+1} - 1 - b \leq 2^{i+1} - 3 < n.$$

Now we compare all odd vertices informed in the first $k-1$ time units to the vertices of equation $2^{i+1}-1-b$.

- For all $b \in B$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

  $$0 < 2^{i+1} - 1 - b \leq 2^{i+1} - 3 < 2^{k-1} - 1 < n.$$

  So the vertex of equation $2^{k-1}-1$ is not equal to any of the vertices of equation $2^{i+1}-1-b$ for any $b \in B$.

- For all $b \in B$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

  $$0 < 2^{i+1} - 1 - b \leq 2^i - 1 < n.$$

  So the vertex of equation $2^i - 1$ is not equal to any of the vertices of equation

  $$2^{i+1} - 1 - b \text{ for any } b \in B.$$

- For all $b \in B$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

  $$0 < 1 \leq b - 1 \leq 2^i - 3 < 2^{i+1} - 1 - b < n.$$

  So the vertices of equation $b-1$ are not equal to the vertices of equation $2^{i+1}-1-b$ for any $b \in B$.

- For all $b \in B$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

  $$0 < 2^{i+1} - 1 - b \leq 2^{i+1} - 3 < 2^{k-1} + 1 \leq b + 2^{k-1} - 1.$$

  Also for all $b \in B$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

  $$b + 2^{k-1} - 1 - n \leq 2^i - 5 < 2^i + 1 \leq 2^{i+1} - 1 - b.$$

  So the vertices of equation $b + 2^{k-1} - 1$ are not equal to the vertices of equation $2^{i+1}-1-b$ for any $b \in B$.

- For all $b \in B$, $c \in C$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

$$0 < 2^{i+1} - 1 - b \leq 2^{i+1} - 3 < 2^{i+1} - 1 \leq c - 1 \leq 2^{k-1} - 2^{i+1} < n.$$

  So the vertices of equation $c-1$ are not equal to the vertices of equation $2^{i+1}-1-b$ for any $b \in B$ and $c \in C$.

- For all $c \in C$ and $2 \leq i \leq k-2$,

$$2^{i} - 1 - c \leq -2^{i} - 1 < 0.$$

  Also for all $b \in B$, $c \in C$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

$$2^{i+1} - 1 - b \leq 2^{i+1} - 3 < 2^{i+1} + 2^{i} + 1 \leq 2^{i} - 1 - c + n.$$

  So the vertices of equation $2^{i}-1-c$ are not equal to the vertices of equation $2^{i+1}-1-b$ for any $b \in B$ and $c \in C$.

- For all $b \in B$, $c \in C$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

$$0 < 2^{i+1} - 1 - b \leq 2^{i+1} - 3 < 2^{i+1} + 1 \leq b + c - 1 \leq 2^{k-1} - 2^{i} - 2 < n.$$

  So the vertices of equation $b+c-1$ are not equal to the vertices of equation $2^{i+1}-1-b$ for any $b \in B$ and $c \in C$.

- For all $b \in B$, $c \in C$ and $2 \leq i \leq k-2$,

$$b - c - 1 \leq -2^{i} - 3 < 0.$$

  Also for all $b \in B$, $c \in C$, $2 \leq i \leq k-2$ and $2 \leq x \leq 2^{k-2}-1$,

$$2^{i+1} - 1 - b \leq 2^{i+1} - 3 < 2^{i+1} + 3 \leq b - c - 1 + n.$$

  So the vertices of equation $b-c-1$ are not equal to the vertices of equation $2^{i+1}-1-b$ for any $b \in B$ and $c \in C$.

Hence we proved that for all $b \in B$ and $2 \leq i \leq k-2$, the vertices of equation

$$2^{i+1} - 1 - b$$

are informed only during the time unit k. The set B is defined as

$$B = \{b_{i-1}2^{i-1} + \ldots + b_1 2^1 \mid 2 \leq i \leq k-2 \ \text{ and } \ (b_{k-2}, \ldots, b_{i+1}) \neq (0, \ldots, 0)\},$$

hence it represents all even vertices between 2 and $2^i - 2$ for all $2 \leq i \leq k-2$. So the vertices of equation

$$2^{i+1} - 1 - b$$

correspond to all odd numbers between $2^i + 1$ and $2^{i+1} - 3$ for all $2 \leq i \leq k-2$. Now, changing the value of i between 2 and $k-2$, we can find all the odd vertices informed during the last time unit. For example, for $i = 2$, the dimensional broadcast scheme formed by the sequence of dimensions

$$(3, 4, ..., k-1, 1, 2, 3)$$

applied from originator 0 informs the vertex labelled 5 during the last time unit; and for $i = 3$, the dimensional broadcast scheme formed by the sequence of dimensions

$$(4, 5, ..., k-1, 1, 2, 3, 4)$$

applied from originator 0 informs the odd vertices between 9 and 13 during the last time unit etc. Hence, for each odd vertex between 5 and $2^{k-1} - 3$ except the odd vertices in the interval

$$[2^{i+1} - 2, 2^{i+1}] \text{ for all } i \in [2, k-2],$$

there is at least one dimensional broadcast scheme formed by the sequence of dimensions $i+1, \ldots, k, 1, \ldots, i, i+1$ for any $2 \leq i \leq k-2$ that will inform it only at the time unit k. The only odd number in the interval $[2^{i+1} - 2, 2^{i+1}]$ is $2^{i+1} - 1$ and by definition it is a neighbour of vertex 0 for any $2 \leq i \leq k-2$. Note that,

$$2^{k-1} - 3 \geq 2^{k-1} - 2x + 1 \text{ for } x \geq 2.$$

So the set of odd vertices in the range $[5, 2^{k-1} - 2x + 1]$ informed at time unit k applying the dimensional broadcast scheme formed by the sequence of dimensions $i+1, \ldots, k, 1, \ldots, i, i+1$ for all $2 \leq i \leq k-2$ is

$$W_k = \{u \in V(KG_{2^k - 2x}) \mid u \text{ is not a neighbour of vertex 0, is odd and } 5 \leq u \leq 2^{k-1} - 2x + 1\}.$$

In conclusion, if we apply the dimensional broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$ and the dimensional broadcast scheme formed by the sequence of dimensions $k-1, k-2, \ldots, 1, k-1$ and the dimensional broadcast schemes formed by the sequence of dimensions $i+1, \ldots, k, 1, \ldots, i, i+1$ for all $2 \leq i \leq k-2$ from originator 0, the set of vertices that are either informed during time unit k or informed at time unit $k-1$ but idle at time unit k is

$$U_k \cup U_{k-1} \cup V_k \cup W_k = \{u \in V(KG_{2^k-2x}) \mid u \text{ is not a neighbour of } 0 \text{ and } u \in [0, 2^k - 2x - 1]\}.$$

□

3.2 A broadcast graph construction by vertex deletion from Knödel graphs on $2^k - 2x$ vertices

In this section, we study the construction of a new broadcast graph by deleting an arbitrary vertex from a Knödel graph on $2^k - 2x$ vertices. In the special case where $n = 2^k - 2$ and k *is odd*, Ahlswede et al. [1, Property 6.3], showed how to construct a new broadcast graph by deleting a vertex. Harutyunyan and Liestman [44, Theorem 4.1] modified this construction to extend it to the case where k *is even*. Our proof follows the main idea of [44] generalizing the given construction for arbitrary k and x.

Let KG_{2^k-2x} be a Knödel graph where $k \geq 2$. As KG_{2^k-2x} is vertex transitive [29], we chose to delete vertex 0. HN_{2^k-2x-1} is constructed from KG_{2^k-2x} as follows:

- Delete vertex 0 and all of its incident edges,
- Add edge $(2^i - 1, 2^{i-1} - 1)$ for all even $i \in [2..k-1]$,
- Add edge $(2^{k-1} - 1, 1)$ if k is even.

Figure 6 and Figure 7 illustrate the construction of HN_{2^k-2x-1} from KG_{2^k-2x} by focusing on subgraphs containing newly added edges (bold lines) and deleted edges (dashed lines). Figure 6 represents the case when k is odd and Figure 7 the case when k is even.

Note that KG_{2^k-2x} is $(k-1)$-regular [29] so when k is odd (respectively even) vertex 0 has an even (respectively odd) number of neighbours.

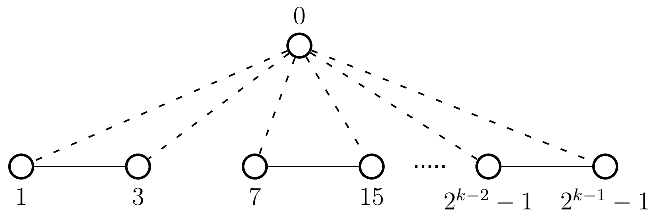

Figure 6: Subgraph of HN_{2^k-2x-1} representing added and deleted edges when k is odd

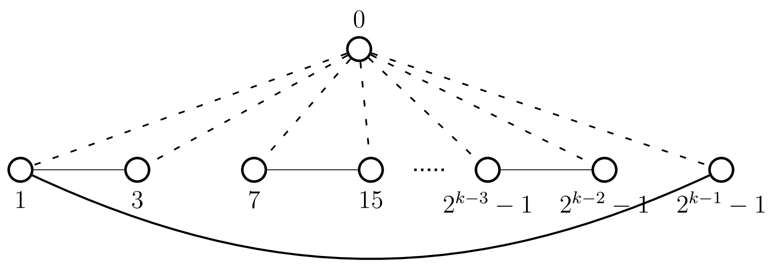

Figure 7: Subgraph of HN_{2^k-2x-1} representing added and deleted edges when k is even

Theorem 2. *HN_{2^k-2x-1} is a broadcast graph and*

$$B(2^k-2x-1) \leq \left\lceil \frac{(2^k-2x-1)(k-1)}{2} \right\rceil$$

where $k \geq 2$ and $1 \leq x \leq 2^{k-2}-1$.

Proof. If we prove that HN_{2^k-2x-1} is a broadcast graph, its number of edges becomes an upper bound for $B(2^k-2x-1)$. For all $k \geq 2$ and $1 \leq x \leq 2^{k-2}-1$, HN_{2^k-2x-1} has 2^k-2x-1 vertices. When k is odd, HN_{2^k-2x-1} is $(k-1)$-regular and has $\frac{1}{2}(2^k-2x-1)(k-1)$ edges. When k is even, HN_{2^k-2x-1} has 2^k-2x-2 vertices of degree $k-1$ and one vertex, namely vertex 1, of degree k; so it has $\frac{1}{2}(2^k-2x-1)(k-1)+1$ edges. Hence for all $k \leq 2$,

$$E(HN_{2^k-2x-1}) = \left\lceil \frac{(2^k-2x-1)(k-1)}{2} \right\rceil.$$

Next we prove that HN_{2^k-2x-1} is a broadcast graph. HN_{2^k-2x-1} has 2^k-2x-1 vertices and for all $k \geq 2$ and $1 \leq x \leq 2^{k-2}-1$,

$$\lceil \log(2^k-2x-2) \rceil = k.$$

So for HN_{2^k-2x-1} to be a broadcast graph, for each originator we should find a particular broadcast scheme that is able to finish broadcasting in HN_{2^k-2x-1} in k time units. This also corresponds to the broadcast time of KG_{2^k-2x}. Let $u \in V(HN_{2^k-2x-1})$ be the originator of broadcasting in HN_{2^k-2x-1}. So vertex u is also a vertex of KG_{2^k-2x}. We study the case where vertex u is a neighbour of vertex 0 in KG_{2^k-2x} and the case where vertex u is not a neighbour of vertex 0 in KG_{2^k-2x} separately.

If vertex u is not a neighbour of vertex 0 in KG_{2^k-2x}, then suppose that vertex u is the originator of a broadcast scheme in KG_{2^k-2x}. As $(u, 0) \notin E(KG_{2^k-2x})$, by Lemma 6 (page 38), there is a valid broadcast scheme for originator u such that either the message reaches vertex 0 at the last time unit k, or the message reaches vertex 0 at time unit $k-1$ but vertex 0 is idle at time unit k (i.e. the vertex it tries to inform during the last time unit is already informed before that time unit). In both cases, if we delete the call to vertex 0 from this broadcast scheme, the only uninformed vertex will be vertex 0. Since vertex 0 is deleted in HN_{2^k-2x-1}, following this broadcast scheme without the call for vertex 0, from originator u, all vertices of HN_{2^k-2x-1} are informed in k time units.

If vertex u and vertex 0 are neighbours of dimension i in KG_{2^k-2x}, then the equation of vertex u becomes $2^i - 1$ where $1 \leq i \leq k-1$. We provide a different valid broadcast schemes for different values of i.

<u>If i is even</u>, then we can take $k \geq 3$. Let vertex u be the originator in KG_{2^k-2x} of the dimensional broadcast scheme formed by $i-1$ right cyclic shifts of the reversal of the sequence of dimensions $1, 2, \ldots, k-1$ followed by the copy of the first dimension obtained after cyclic shifts. As $i \geq 2$, this broadcast scheme corresponds to the dimensional broadcast scheme formed by the sequence of dimensions

$$i-1, i-2, \ldots, 1, k-1, k-2, \ldots, i, i-1.$$

By Theorem 1 (page 16), this broadcast scheme is a valid broadcast scheme for KG_{2^k-2x} and it finishes informing all vertices of KG_{2^k-2x} in

$$\lceil \log(2^k - 2x) \rceil = k \text{ time units}$$

from any originator. By Lemma 5 (page 32), following this broadcast scheme in KG_{2^k-2x}, vertex 0 is informed at time unit $k-1$ by vertex u and it informs the vertex of label $2^{i-1} - 1$ at time unit k. KG_{2^k-2x} and HN_{2^k-2x-1} differ only by edges incident to vertex 0. Hence using this broadcast scheme in HN_{2^k-2x-1}, after $k-2$ time units, informed vertices in HN_{2^k-2x-1} and KG_{2^k-2x} are the same. As i is even, by construction vertex u is connected to the vertex of label $2^{i-1}-1$ in HN_{2^k-2x-1}. So in the broadcast scheme formed by the sequence of dimensions $i-1, i-2, \ldots, 1, k-1, k-2, \ldots, i, i-1$, replacing the call from vertex u to vertex 0 at time unit $k-1$ and the call from vertex 0 to the vertex of label $2^{i-1}-1$ at time unit k by a call from vertex u to the vertex of label $2^{i-1}-1$ at time unit $k-1$, we obtain a valid broadcast scheme for HN_{2^k-2x-1}.

<u>If i is odd and $i \neq k-1$</u>, let vertex u be the originator in KG_{2^k-2x} of the dimensional broadcast scheme formed by $k-i-1$ right cyclic shifts of the sequence of dimensions $1, 2, \ldots, k-1$ followed by the copy of the first dimension obtained after cyclic shifts. As $i \leq k-2$, this broadcast scheme corresponds to the dimensional broadcast scheme formed by the sequence of dimensions

$$i+1, i+2, \ldots, k-1, 1, \ldots, i, i+1.$$

By Theorem 1 (page 16), this broadcast scheme is a valid broadcast scheme for KG_{2^k-2x} and it finishes informing all vertices of KG_{2^k-2x} in

$$\lceil \log(2^k - 2x) \rceil = k \text{ time units}$$

from any originator. By Lemma 4 (page 27), following this broadcast scheme in KG_{2^k-2x}, vertex 0 is informed at time unit $k-1$ by vertex u and it informs the vertex of label $2^{i+1}-1$ at time unit k. KG_{2^k-2x} and HN_{2^k-2x-1} differ only by edges incident to vertex 0. Hence using this broadcast scheme in HN_{2^k-2x-1}, after $k-2$ time units, informed vertices in HN_{2^k-2x-1} and KG_{2^k-2x} are the same. As $i+1$ is even, by construction the vertex of label $2^{i+1}-1$ is connected to vertex u in HN_{2^k-2x-1}. So in the broadcast scheme formed by the sequence of dimensions $i+1, i+2, \ldots, k-1, 1, \ldots, i, i+1$, replacing the call from vertex u to vertex 0 at time unit $k-1$ and the call from vertex 0 to the vertex of label $2^{i+1}-1$ at time unit k by a call from vertex u to the vertex of label $2^{i+1}-1$ at time unit $k-1$, we obtain a valid broadcast scheme for HN_{2^k-2x-1}.

<u>If i is odd and $i = k-1$</u>, let vertex u be the originator in KG_{2^k-2x} of the dimensional broadcast scheme formed by the sequence of dimensions

$$1, 2, \ldots, k-1, 1.$$

By Lemma 1 (page 16), this broadcast scheme is a valid broadcast scheme for KG_{2^k-2x} and it finishes informing all vertices of KG_{2^k-2x} in

$$\lceil \log(2^k - 2x) \rceil = k \text{ time units}$$

from any originator. By Lemma 4 (page 27), following this broadcast scheme in KG_{2^k-2x}, vertex 0 is informed at time unit $k-1$ by vertex u and it informs the vertex 1 at time unit k. KG_{2^k-2x} and HN_{2^k-2x-1} differ only by edges incident to vertex 0. Hence using this broadcast scheme in HN_{2^k-2x-1}, after $k-2$ time units, informed vertices in HN_{2^k-2x-1} and KG_{2^k-2x} are the same. By construction, vertex u is connected to vertex 1 in HN_{2^k-2x-1}. So in the broadcast scheme formed by the sequence of dimensions $1, 2, \ldots, k-1, 1$, replacing the call from vertex u to vertex 0 at time unit $k-1$ and the call from vertex 0 to vertex 1 at time unit k by a call from vertex u to vertex 1 at time unit $k-1$, we obtain a valid broadcast scheme for HN_{2^k-2x-1}. $\square$

Theorem 2 implies that for all odd $n \neq 2^k - 1$ where $k \geq 2$,

$$B(n) \leq \left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil.$$

We also know that by counting the number of edges of Knödel graphs the following bound is obtained ([8, 32, 66]) for all even n,

$$B(n) \leq \left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil.$$

Hence combining those bounds we obtain the following corollary.

Corollary 1. *For all $n \neq 2^k - 1$ where $k \geq 2$*

$$B(n) \leq \left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil.$$

Chapter 4

Study of dimensional broadcast schemes in Knödel graphs on 2^k vertices

In this chapter, we present some lemmas about Knödel graphs on 2^k vertices using dimensional broadcast schemes. These lemmas help us with the construction of a new broadcast graph by deleting an arbitrary vertex from a Knödel graph on 2^k vertices. Note that this construction does not improve the already known upper bound for $B(2^k - 1)$ given in [44].

We would like to emphasize that due to its geometry, lemmas about Knödel graphs on 2^k vertices are stronger and more general than the ones for Knödel graphs on $2^k - 2x$ vertices where $1 \leq x \leq 2^{k-2} - 1$. Also, a Knödel graph on 2^k vertices is quite similar to a hypercube in terms of the recursive construction that is described in the following section.

In this section, KG_{2^k} denotes a Knödel graph where the number of vertices, 2^k, is an exact power of 2. KG_{2^k} is k-regular and has a broadcast time of k time units (see [29]).

Proposition 1 ([29] Proposition 3). *It is possible to recursively construct KG_{2^k} starting from KG_2, by taking two copies of $KG_{2^{k-1}}$ and linking the vertices of each copy by a perfect matching. Dimension i in $KG_{2^{k-1}}$ will become dimension $i+1$ in KG_{2^k} for every $i \in [1, k-1]$, hence it suffices to add dimension 1. The dimension 1 represents the perfect matching between the copies of $KG_{2^{k-1}}$.*

Figure 8 shows the construction described in Proposition 1.

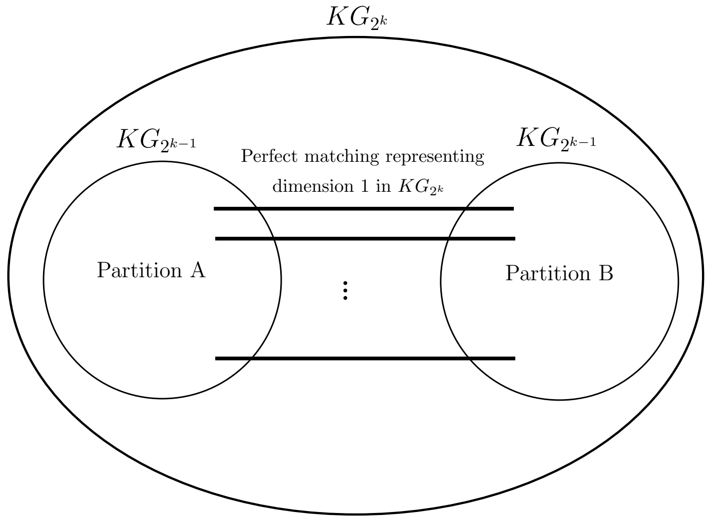

Figure 8: KG_{2^k} recursively built using two copies of $KG_{2^{k-1}}$

The previous construction is only valid for dimension 1, that is, if we delete all edges of dimension i with $i \neq 1$ from KG_{2^k}, we do not obtain two separate graphs. This fact highlights that Knödel graphs are not edge transitive. Although the given recursive construction of KG_{2^k} is very similar to that of a hypercube, Fertin and Raspaud [26] proved that these two topologies are not isomorphic when their dimension is bigger than 4. Fertin and Raspaud [26] also proved that KG_{2^k} is non-isomorphic to recursive circulant graph $G(2^k, 4)$ for $k \geq 4$.

Lemma 7. *Any dimensional broadcast scheme formed by a permutation of the dimensions of KG_{2^k} is a valid broadcast scheme for KG_{2^k} for any originator.*

Proof. Let $d_1, \ldots, d_k$ be the set of dimensions of KG_{2^k}. We prove that any permutation of this set forms a broadcast scheme for KG_{2^k} that terminates broadcasting in $\lceil \log 2^k \rceil = k$ time units. Following is a proof by induction on k.

Basis: KG_{2^2} is formed by four vertices and dimensions 1 and 2. Dimensional broadcast schemes formed by the sequence of dimensions $2, 1$ and $1, 2$ are valid broadcast schemes for KG_{2^2} for any originator.

Inductive hypothesis: Any dimensional broadcast scheme formed by any permutation of the dimensions of $KG_{2^{k-i}}$ with $i > 0$ is a valid broadcast scheme for $KG_{2^{k-i}}$ for any originator.

Rank k: KG_{2^k} is vertex transitive [29], so we assume that the originator of broadcasting is an arbitrary vertex u. As described in Proposition 1 and Figure 9, KG_{2^k} is formed by two copies of $KG_{2^{k-1}}$ connected through dimension 1 by a perfect matching. Vertex u is in one of the two copies of $KG_{2^{k-1}}$.

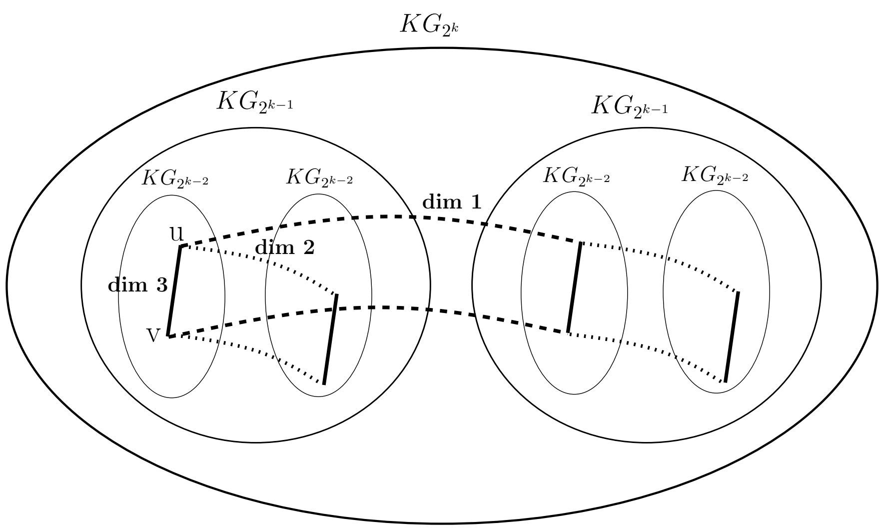

Figure 9: Geometry of KG_{2^k}

Let $d_1, \ldots, d_{i-1}, d_i, d_{i+1}, \ldots, d_k$ be a dimensional broadcast scheme formed by a sequence of dimensions of KG_{2^k} where d_i represents dimension 1. We apply this dimensional broadcast scheme in KG_{2^k} from originator u. Until the dimension d_i is treated, broadcasting continues in the copy of $KG_{2^{k-1}}$ that contains vertex u. When treating the dimension d_i, all informed vertices inform their mirrors in the other copy of $KG_{2^{k-1}}$. For the rest, broadcasting continues in parallel in both copies of $KG_{2^{k-1}}$. By inductive hypothesis, the dimensional broadcast scheme formed by the sequence of dimensions $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_k$ is a valid broadcast scheme for $KG_{2^{k-1}}$ that finishes broadcasting in $k-1$ time units. Note that the previous sequence of dimensions corresponds to the sequence

$$d_1 - 1, \ldots, d_{i-1} - 1, d_{i+1} - 1, \ldots, d_k - 1$$

when written in terms of dimensions of $KG_{2^{k-1}}$. So at time unit k broadcasting in both $KG_{2^{k-1}}$ finishes and all vertices of KG_{2^k} are informed.

□

Lemma 8. *If $a, b \in V(KG_{2^k})$ then there is a valid broadcast scheme for originator a in KG_{2^k} such that the message reaches vertex b at the last time unit, k.*

Proof. Let $a, b \in V(KG_{2^k})$. In the following proof, cases where vertex a and vertex b are neighbours and non-neighbours are analyzed separately. One key concept that is used throughout this proof is the concept of the *idle vertex*. An *idle vertex* at time unit t is an informed vertex that either informs no other vertices or tries to inform an already informed vertex, during this time unit.

If $(a, b) \in E(KG_{2^k})$, then let vertex a and vertex b be connected through dimension i. By Lemma 7, there exists a valid dimensional broadcast scheme for KG_{2^k} for any originator, such that the dimension i is the last dimension to be treated. Let vertex a be the originator of such a dimensional broadcast scheme in KG_{2^k}. Following this broadcast scheme, vertex a informs vertex b at the last time unit. It remains to show that no other vertex informs vertex b before the last time unit. In KG_{2^k}, following any valid dimensional broadcast scheme from any originator, no informed vertex is idle. That is due to the fact that any valid dimensional broadcast scheme for KG_{2^k} finishes broadcasting in k time units and KG_{2^k} contains 2^k vertices. It follows that if even one vertex is idle, broadcasting cannot finish in time. Hence in KG_{2^k}, no other vertex can inform vertex b before the last time unit, otherwise vertex a would have been idle.

If $(a, b) \notin E(KG_{2^k})$, then let $i - 1$ be the largest integer such that a copy of $KG_{2^{k-i+1}}$ contains both vertex a and vertex b. This copy of $KG_{2^{k-i+1}}$ cannot be KG_2 otherwise vertex a and vertex b would have been neighbours. So this copy of $KG_{2^{k-i+1}}$ contains two copies of $KG_{2^{k-i}}$, one of them containing vertex a and the other one vertex b. Let D be a dimensional broadcast scheme for KG_{2^k} that starts with the sequence of dimensions $i + 1, i + 2, \ldots, k$ followed by some other dimensions and finishes with dimension i. Each dimension of KG_{2^k} appears only once in D. Let vertex a be the originator of D in KG_{2^k}. By Lemma 7 the dimensional broadcast scheme formed by the sequence of dimensions $i + 1, \ldots, k$ is a valid broadcast scheme for $KG_{2^{k-i}}$ for any originator, so while treating the first $k - i$ dimensions of D, $i + 1, \ldots, k$, all vertices inside the copy of $KG_{2^{k-i}}$ containing vertex a are informed. As no dimension smaller than $i + 1$ is treated, the vertices inside that copy of $KG_{2^{k-i}}$ are the only ones informed. During the last time unit, when treating dimension i, all vertices of this copy of $KG_{2^{k-i}}$ will inform all vertices of the copy of $KG_{2^{k-i}}$ containing vertex b. It remains to show that no other vertex informs vertex b before the last time unit. By Lemma 7 the dimensional broadcast scheme D is a valid broadcast scheme for KG_{2^k} for any originator. In KG_{2^k} following any valid dimensional broadcast scheme from any originator, no informed vertex is idle. So before the last time unit, no other vertex can inform any vertex of the copy of $KG_{2^{k-i}}$ containing vertex b; otherwise at least one vertex of the copy of $KG_{2^{k-i}}$ containing vertex a would have been idle. So following the dimensional broadcast scheme D, vertex b is informed during the last time unit. □

Proposition 2. *Any graph resulting from the deletion of all vertices of one copy of $KG_{2^{k-i}}$ with $0 < i < k$ from KG_{2^k} is a broadcast graph on $2^k - 2^{k-i}$ vertices.*

Proof. Let G be a graph obtained by deleting all vertices of any $KG_{2^{k-i}}$ with $0 < i < k$ and their incident edges from KG_{2^k}. We consider the case where $i = 1$ and

$$2 \leq i \leq k - 1$$

separately. One key concept that is used throughout this proof is the concept of the *idle vertex*. An *idle vertex* at time unit t is an informed vertex that either informs no other vertices or tries to inform an already informed vertex, during this time unit.

If $i = 1$, we obtain G by deleting all vertices and their incident edges of one of the two copies of $KG_{2^{k-1}}$ forming KG_{2^k}. So the graph G becomes $KG_{2^{k-1}}$ which is a broadcast graph on 2^{k-1} vertices [29].

If $2 \leq i \leq k - 1$, we obtain G by deleting all vertices and their incident edges of one of the 2^i copies of $KG_{2^{k-i}}$ forming KG_{2^k}. Hence the total number of vertices of G is $2^k - 2^{k-i}$ and for G to be a broadcast graph, broadcasting should finish in

$$\lceil \log(2^k - 2^{k-i}) \rceil = k$$

time units from any originator. Let $u \in V(G)$ be the originator of a broadcast scheme in KG_{2^k}. By definition vertex u is in one of the copies of $KG_{2^{k-i}}$ that is not deleted from KG_{2^k}. We study two different cases separately. In the first case, the copy of $KG_{2^{k-i}}$ that contains vertex u is a neighbour of the deleted copy of $KG_{2^{k-i}}$, so they are in the same copy of $KG_{2^{k-i+1}}$ and their vertices are connected by a perfect matching following dimension i. Figure 10 shows the subgraph of KG_{2^k} that represents the configuration decribed above.

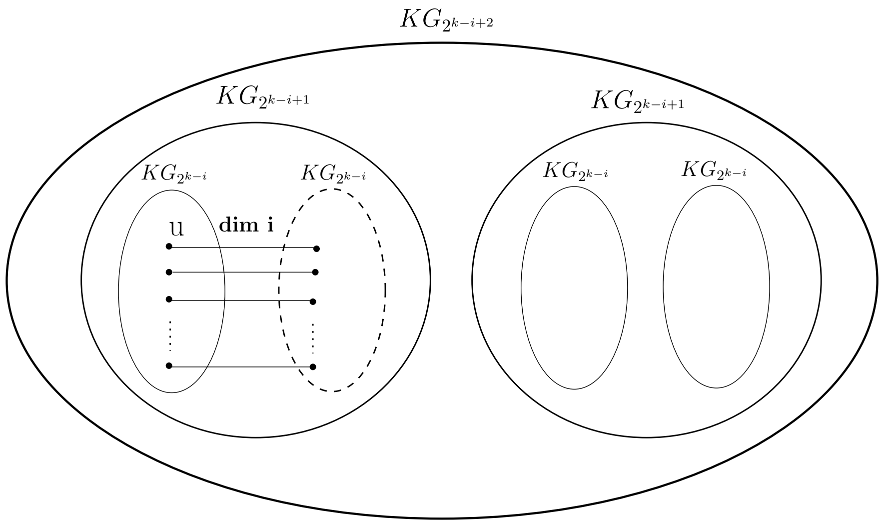

Figure 10: Subgraph of KG_{2^k} where the vertices of the copy of $KG_{2^{k-i}}$ that contains vertex u are connected to the vertices of the copy of $KG_{2^{k-i}}$ deleted in G by a perfect matching through dimension 1 (dashed)

In the second case, the copy of $KG_{2^{k-i}}$ that contains vertex u is not a neighbour of the deleted copy of $KG_{2^{k-i}}$, so they are in different copies of $KG_{2^{k-i+1}}$ and they are not connected by a perfect matching following dimension i. In this case we consider the largest integer j such that a copy of $KG_{2^{k-j}}$ contains both copies of $KG_{2^{k-i+1}}$ and we build a valid broadcast scheme using this dimension j. Note that $1 \leq j < i - 1$. Figure 11 shows the subgraph of KG_{2^k} that represents the configuration described above.

Note that in the subgraph of KG_{2^k} of Figure 11 only 8 copies of $KG_{2^{k-i+1}}$ are shown, but $KG_{2^{k-j}}$ contains 2^{i-j-1} copies of $KG_{2^{k-i+1}}$. Following is the analysis of previously described cases.

- *If the vertices of the copy of $KG_{2^{k-i}}$ that contains vertex u are connected to the vertices of the copy of $KG_{2^{k-i}}$ that is deleted in G, through dimension i in KG_{2^k} (Figure 10).* Let D be a dimensional broadcast scheme for KG_{2^k} that starts with dimensional sequence $i+1, i+2, \ldots, k$ followed by some other dimensions and finishes with dimension i. Each dimension of KG_{2^k} appears only once in D. Let vertex u be the originator of D in KG_{2^k}. By Lemma 7, the dimensional broadcast scheme formed by the sequence of dimensions $i+1, \ldots, k$ is a valid broadcast scheme for $KG_{2^{k-i}}$ for any originator. Hence if we apply D from originator u in KG_{2^k}, while treating the first $k-i$ dimensions of D, $i+1, \ldots, k$,

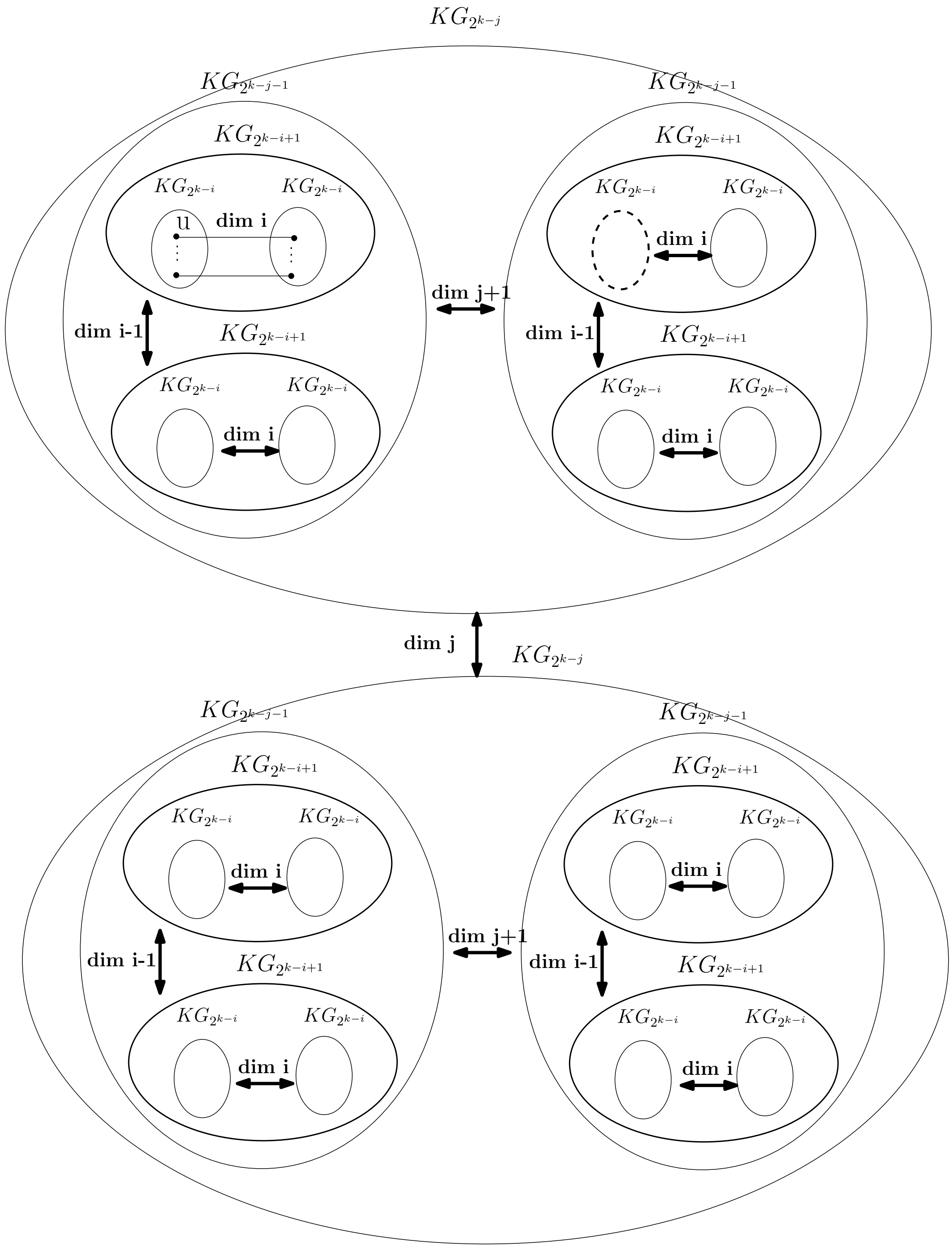

Figure 11: Subgraph of KG_{2^k} where the copy of $KG_{2^{k-i}}$ that contains vertex u is not in the same $KG_{2^{k-i+1}}$ as the copy of $KG_{2^{k-i}}$ deleted in G (dashed)

all vertices inside the copy of $KG_{2^{k-i}}$ containing vertex u are informed. As no dimension smaller than $i+1$ is treated, the vertices inside that copy of $KG_{2^{k-i}}$ are the only ones informed. At the last time unit, when treating dimension i, all vertices of this copy of $KG_{2^{k-i}}$ will inform all vertices of the copy of $KG_{2^{k-i}}$ that is deleted in G. It remains to show that no other vertex informs the vertices of the copy of $KG_{2^{k-i}}$ that is deleted in G, before the last time unit. By Lemma 7, D is a valid broadcast scheme for KG_{2^k} for any originator. In KG_{2^k} following any valid dimensional broadcast scheme from any originator, no informed vertex is idle. So before the last time unit, no other vertex can inform any vertex of the copy of $KG_{2^{k-i}}$ that is deleted in G; otherwise the vertices of the copy of $KG_{2^{k-i}}$ that contains vertex u would have been idle. So following D, the vertices of the copy of $KG_{2^{k-i}}$ that is deleted in G, are informed during the last time unit. Hence applying D from originator u, broadcasting in G finishes in k time units and D is a valid broadcast scheme for G.

- *If the vertices of the copy of $KG_{2^{k-i}}$ that contains vertex u are not connected to the vertices of the copy of $KG_{2^{k-i}}$ that is deleted in G, through dimension i in KG_{2^k} (Figure 11).* Let j be the largest integer such that a copy of $KG_{2^{k-j}}$ contains both, the copy of $KG_{2^{k-i+1}}$ that contains vertex u and the copy of $KG_{2^{k-i+1}}$ that is deleted in G. Note that $1 \leq j < i-1$ and that the copy of $KG_{2^{k-i+1}}$ that contains vertex u and the copy of $KG_{2^{k-i+1}}$ that contains the copy of $KG_{2^{k-i}}$ that is deleted in G are in different copies of $KG_{2^{k-j-1}}$ connected through dimension $j+1$. Let D' be a dimensional broadcast scheme for KG_{2^k} that starts with the sequence of dimensions $j+2, j+3, \ldots, k$ followed by some other dimensions and finishes with dimension $j+1$. Each dimension of KG_{2^k} appears only once in D'. Let vertex u be the originator of D' in KG_{2^k}. By Lemma 7, the dimensional broadcast scheme formed by the sequence of dimensions $j+2, \ldots, k$ is a valid broadcast scheme for $KG_{2^{k-j-1}}$ for any originator. Hence if we apply D' from originator u in KG_{2^k}, while treating the first $k-j-1$ dimensions of D', $j+2, \ldots, k$, all vertices inside the copy of $KG_{2^{k-j-1}}$ containing the copy of $KG_{2^{k-i+1}}$ that contains vertex u are informed. As no dimension smaller than $j+2$ is treated, the vertices inside that copy of $KG_{2^{k-j-1}}$ are the only ones informed. During the last time unit, when treating dimension $j+1$, all vertices of this copy of $KG_{2^{k-j-1}}$ will inform all vertices of the copy of $KG_{2^{k-j-1}}$ containing the copy of $KG_{2^{k-i+1}}$ that is deleted in G. It remains to show that no other vertex informs the vertices of the copy of $KG_{2^{k-i+1}}$ that is deleted in G, before the last time unit. By Lemma 7, D' is a valid broadcast scheme for KG_{2^k} for any originator. In KG_{2^k} following any valid dimensional broadcast scheme from any originator, no informed vertex is idle. So before the last time unit, no other vertex can inform any vertex of the copy of $KG_{2^{k-i+1}}$ that is deleted in G; otherwise some of the vertices of the copy of $KG_{2^{k-j-1}}$ containing vertex u would have been idle. So following D', the vertices of the copy of $KG_{2^{k-i+1}}$ that is deleted in G are informed during the last time unit. Hence applying D' from originator u, broadcasting in G finishes in k time units and D' is a valid broadcast scheme for G.

□

Theorem 3. *Let HP_{2^k-1} be a graph constructed from KG_{2^k} by deleting an arbitrary vertex and all of its incident edges. Then HP_{2^k-1} is a broadcast graph and*

$$B(2^k - 1) \leq k(2^{k-1} - 1).$$

Proof. HP_{2^k-1} has $2^k - 1$ vertices. $2^k - 1 - k$ of these vertices are of degree k and k of these vertices are of degree $k-1$, hence HP_{2^k-1} has $k(2^{k-1}-1)$ edges. If HP_{2^k-1} is a broadcast graph, its number of edges becomes an upper bound for $B(2^k - 1)$. The following is the proof that HP_{2^k-1} is a broadcast graph.

Let $v \in V(KG_{2^k})$ be the deleted vertex from KG_{2^k} and $u \in V(HP_{2^k-1})$ be the originator of a broadcast scheme in HP_{2^k-1}. By Lemma 8, there is a valid broadcast scheme for originator u in KG_{2^k} such that vertex v is informed during the last time unit. Until the last time unit, no edge incident to vertex v is used and vertex v informs no other vertex. So this broadcast scheme without the call for vertex v at time unit k, completes broadcasting in HP_{2^k-1} in k time units and

$$\lceil \log n \rceil = \lceil \log(2^k - 1) \rceil = k.$$

□

Note that HP_{2^k-1} is not a minimum broadcast graph and the best known upper bound on $B(2^k - 1)$ is

$$B(2^k - 1) \leq 2^{k-1}(k - \frac{1}{2}) - (k-1) \quad ([41]) \quad .$$

This upper bound is tighter than the upper bound given in Theorem 3,

$$B(2^k - 1) \leq k(2^{k-1} - 1).$$

Moreover Labahn [71] proved that for all $k \geq 2$

$$B(2^k - 1) \geq \frac{(k-1)(2^k - 1)}{2} + \frac{2^k - 1}{2(k+1)}.$$

This lower bound is greater than the upper bound that we try to generalize which is

$$\left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil = \left\lceil \frac{(2^k - 1)(k-1)}{2} \right\rceil.$$

Hence the general upper bound

$$\left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil$$

cannot be expanded to the case where $n = 2^k$.

Chapter 5

Conclusion and future work

In this thesis we proved that for all $n \neq 2^k - 1$,

$$B(n) \leq \left\lceil \frac{n\lfloor \log n \rfloor}{2} \right\rceil.$$

For $n = 2^k - 1$, this upper bound is smaller than the lower bound given by Labahn [71], hence it cannot be gneralized to the case where $n = 2^k - 1$. Although there exist better upper bounds for some restricted ranges of n, our upper bound covers a wider range of integers and shows an improvement of about

$$\frac{n}{4} - \frac{\lfloor \log n \rfloor}{2}$$

compared to the general upper bound

$$B(n) \leq \frac{n-1}{2}\lfloor \log n \rfloor + 2^{\lceil \log n \rceil - 2}$$

given by Harutyunyan [41] for any n.

In order to obtain this upper bound, a construction introduced in [44] has been used to create a new broadcast graph on $2^k - 2x - 1$ vertices by deleting a vertex and adding edges to the Knödel graph on $n = 2^k - 2x$ vertices. A similar construction has been applied to the Knödel graph on 2^k vertices but the upper bound on $B(2^k - 1)$ obtained in this way is not better than the one given in [44].

To help with the previously described construction, we generalized some known lemmas for KG_{2^k-2} to lemmas for KG_{2^k-2x} where $x \neq 1$, using specific dimensional broadcast schemes. In particular we proved in Lemma 6 the existence of valid dimensional broadcast schemes for any originator that informs all its non-neighbouring vertices either at time unit $\lceil \log(2^k - 2) \rceil$ or at time unit $\lceil \log(2^k - 2) \rceil - 1$ leaving the vertices idle at time unit $\lceil \log(2^k - 2) \rceil$.

As for Knödel graphs KG_{2^k}, we proved that a dimensional broadcast scheme formed by a permutation of the dimensions of KG_{2^k} forms a valid broadcast scheme for this graph. We created new graphs by deleting one or multiple vertices from KG_{2^k} and used different dimensional broadcast schemes formed by permutations of the dimensions of KG_{2^k} to prove that the newly created graphs are broadcast graphs.

However not all dimensional broadcast schemes formed by a permutation of dimensions of KG_{2^k-2x} followed by the copy of the first dimension, is a valid broadcast scheme for KG_{2^k-2x} where $1 \leq x \leq 2^{k-2} - 1$. This is also underlined by the fact that not all possible permutations can be obtained by applying cyclic shift and reversal operations to known valid broadcast schemes.

Figuring out different valid broadcast schemes for a given graph is important as it leads the way to the construction of new broadcast graphs. In the case of Knödel graphs there can be some basic operations, other than cyclic shift or reversal, to be applied to a valid broadcast scheme in order to obtain a new one.

As for generalized Knödel graphs $W_{\Delta,n}$ where maximum degree $\Delta \neq \lfloor \log n \rfloor$, one interesting topic is to find the minimum Δ for the graph to be a broadcast graph. A way to achieve this is to find valid broadcast schemes for $W_{\Delta,n}$. Current studies concentrate on $W_{\Delta,n}$ where $\Delta = \lfloor \log n \rfloor - 1$ as it differs from KG_n by one dimension and has a higher probability to be a broadcast graph than the rest of the generalized Knödel graphs. Some types of broadcast schemes to be considered are for example, dimensional broadcast schemes where dimensions alternate between odd and even numbers, or dimensional broadcast schemes where one dimension is repeated several times, or broadcast schemes where each node informs a neighbour of different dimensions.

As mentioned in Chapter 2, another challenging research direction is to find the diameter of Knödel graphs. The diameter is known only for KG_{2^k} and is

$$D_k = \left\lceil \frac{k+2}{2} \right\rceil$$

for any $k \geq 2$ [30]. The diameter of Knödel graph KG_{2^k-2x} is expected to be at most the same. Current research efforts focus on finding upper bounds for the diameter of generalized Knödel graphs $W_{\Delta,n}$ especially when $\Delta = \lceil \log n \rceil$.

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