# On Low Dimensional Galilei Groups and Their Applications

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## ABSTRACT

**On low dimensional Galilei groups and their applications**

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This thesis consists of two main parts. The first part focuses on the (1+1) and (2+1) dimensional Galilei groups and their applications to signal analysis and noncommutative quantum mechanics. Various groups used in the current literature of signal analysis and image processing turn out to possess deep connections with the (1+1)-dimensional Galilei group, which, on the other hand, is the physical kinematical symmetry group of a non-relativistic system in one spatial and one time dimensions. To study this remarkable representation theoretic similarity of structures is one of the many goals of this thesis. The (1+1)-affine Galilei group, a 2-fold noncentral extension of the Galilei group, is precisely responsible for the above-mentioned bridging. Wigner functions associated with another extension of (1+1) affine Galilei group are computed and their support properties are subsequently discussed along with a comparative study of those related to various centrally extended (1+1)-Galilei groups. The remainder of the first part of the thesis is devoted to the study of the (2+1)-Galilei group and its relationship to non-commutative quantum mechanics (NCQM). We show that a certain triple central extension of the abelian group of translations in $\mathbb{R}^4$ can

be considered to be the *defining group* of NCQM in the same spirit as the Weyl-Heisenberg group is considered for the case of standard 2-dimensional quantum mechanics. The representations associated with various gauges studied in NCQM along with those of standard QM are all found to be sitting inside the unitary dual of the *triply extended group of translations in* $\mathbb{R}^4$.

The second part of the thesis, which concerns an entirely separate problem, involves a study of Poisson brackets between traces of monodromy matrices computed along free homotopy classes of loops on a compact Riemann surface $\Sigma$. We consider a 3-manifold $\Sigma \times \mathbb{R}$ with the connection 1-forms taking their values in the Lie algebra $\mathcal{G}$ associated to the structure Lie group $G$ of a principal $G$-bundle defined on the base manifold $\Sigma \times \mathbb{R}$. First we apply the Hamiltonian formulation of the Chern-Simons theory to compute the Atiyah-Bott brackets between the relevant $\mathcal{G}$-valued relevant gauge connections. The quotient of this infinite dimensional space of flat connections by the action of gauge transformations is what one calls the *moduli space of flat connections*. Traces of monodromies computed along the free homotopy classes of loops on $\Sigma$ are the underlying gauge invariant observables. We compute Poisson brackets between observables of this sort by applying the Hamiltonian formalism of soliton theory for various real structure Lie groups, e.g. $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $U(n)$, $SU(n)$ and $Sp(2n,\mathbb{R})$. The formulae, thus obtained, are found to be in exact agreement with the ones computed by Goldman in [29].

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## DEDICATION

*To Abba and Amma*

TABLE OF CONTENTS

List of Tables

LIST OF FIGURES

# List of Acronyms

## Abbreviations

UIR Unitary irreducible representation

PUR Projective unitary representation

NCQM Noncommutative quantum mechanics

## Notations

$\mathcal{G}_{\mathrm{s}}$ (1+1)-Galilei-Schödinger group

$\mathbb{S}$ Reduced Shearlet group

$G_{\mathrm{Gal}}^{\mathrm{ext}}$ 2-fold centrally extended (2+1)-Galilei group

$G_{\mathrm{Gal}}$ (2+1)-Galilei group

$\overline{\overline{G_T}}$ Double central extension of group of translations in $\mathbb{R}^4$

$\overline{\overline{\overline{G_T}}}$ Triple central extension of group of translations in $\mathbb{R}^4$

$\mathfrak{G}_{\mathrm{Gal}}^{\mathrm{ext}}$ 2-fold centrally extended (2+1)-Galilei Lie algebra

$G_{\mathrm{NC}}$ Defining group of noncommutative quantum mechanics

$G_{\mathrm{H}}^{p}$ Extended Heisenberg groups

$G_{+}^{\mathrm{aff}}$ Wavelet group

$\mathcal{G}^{\mathrm{M}}$ (1+1)-Quantum Galilei group

$\mathcal{G}_0$ (1+1)-Galilei group

| | |
|---|---|
| $\mathcal{G}_{\text{aff}}$ | (1+1)-affine Galilei group |
| $\mathcal{G}_{\text{aff}}^{\text{M}}$ | Extended (1+1)-affine Galilei group |
| $G_{\text{H}}$ | 2-dimensional Heisenberg group |
| $G_{\text{SW}}$ | Connected Stockwell group |
| $G_{\text{WH}}$ | 2-dimensional Weyl-Heisenberg group |
| $G_T$ | Group of translations in $\mathbb{R}^4$ |

# Chapter 1
# Introduction

Low dimensional Galilei groups exhibit much richer algebraic structures compared to their relativistic counterparts the Poincaré groups. Bargmann in [10] and Wigner in [49] pointed out that the Poincaré group, in $(d+1)$ dimensions with $d \geq 2$, does not admit a nontrivial central extension. On the other hand, the (1+1)-dimensional Poincaré group admits nontrivial central extensions as explored by Lévy-Leblond in [36] (page 75) and by Bargmann in [10] (page 37). In the non-relativistic setting, Lévy-Leblond (see [37]), showed that Galilei group, being a physical kinematical symmetry group for a system of particles in (3+1) dimensions, admits a nontrivial central extension. The (1+1) and (2+1)-dimensional cases are even more interesting. Both the Galilei groups $\mathcal{G}_0$ in (1+1)-dimensions and $G_{\text{Gal}}$ in (2+1)-dimensions admit two inequivalent central extensions. For the one dimensional case, the relevant extensions are interpreted as mass and force (see [41] and [36]), while for two dimensional case, they are interpreted as mass and spin (see [15]). It is also known that the trivial multipliers (see the preliminaries to follow for definitions) of Poincaré group, in (d+1) dimensions with $d \geq 2$, can be contracted to nontrivial local multipliers of Galilei group in the non-relativistic limit, a fact that has been discussed in [1]. In this thesis, we shall consider a 1-dimensional subspace of second cohomology group $H^2(\mathcal{G}_0, \mathbb{R})$ of $\mathcal{G}_0$ associated with only the central extension signifying mass of a non-relativistic particle and denote the centrally extended group by $\mathcal{G}^m$. Various central and non-central extensions of $\mathcal{G}_0$ will also be studied in this thesis in order to study its relationship with different groups of signal analysis and image processing. A certain non-central extension, of $\mathcal{G}_0$ by the abelian group $\mathbb{R}^2$, called *affine Galilei group* and denoted by $\mathcal{G}_{\text{aff}}$ is constructed in the sequel. Using the same line of arguments, presented for (3+1) dimensions in [8], one is led to conclude that the straightforward central extension of $\mathcal{G}_{\text{aff}}$ fails to generate the mass of the underlying non-relativistic system. One, therefore, chooses a different course to obtain a non-central extension $\mathcal{G}_{\text{aff}}^m$, parametrized by $m$. An elegant technique has been outlined in [35] to find Wigner functions (see preliminaries for definition) followed by the computations of relevant coadjoint orbits. Since $\mathcal{G}_{\text{aff}}^m$ admits similar semidirect product structure, we apply the above-mentioned technique to find its Wigner functions. A more general technique is outlined in [5] which we apply to do the same for $\mathcal{G}^m$. Subsequently, we discuss the support properties of the relevant Wigner functions in view of [35].

Numerous articles were written on *Noncommutative quantum mechanics* (NCQM) of late. In current literature (see [42] for example), one starts with a non-commutative configuration space and assumes that a certain set of commutation relations hold between the respective positions and momenta coordinates. In a more general setting, one deforms both the position-position and momentum-momentum commutators by requiring them to be non-vanishing along with the standard quantum mechanical position-momentum commutators. One then defines the Hilbert space of Hilbert-Schmidt operators acting on the underlying non-commutative space as the state space. Using this line of treatment, one then proves the resolution of identity and obtain the relevant coherent states (see [11] for a detailed account). In this thesis, we start with centrally extended (2+1) dimensional Galilei group $G_{\text{Gal}}^{\text{ext}}$ and then consider a particle constrained to move on a 2-dimensional plane subject to the symmetry of $G_{\text{Gal}}^{\text{ext}}$. We then compute the coherent states emanating from $G_{\text{Gal}}^{\text{ext}}$ and subsequently quantize the underlying phase-space variables using these coherent states. The resulting commutation relations, among these quantized operators on $L^2(\mathbb{R}^2)$ with respect to the Lebesgue measure, are found to be exactly the ones postulated in many existing literature of NCQM. In order to capture the more general picture of NCQM where the relevant momentum operators also fail to commute, we consider a certain triple central extension of abelian group of translations $G_{\text{NC}}$ in $\mathbb{R}^4$. What turns out at the end is that $G_{\text{NC}}$ could be regarded as the defining group of 2-dimensional NCQM in the same sense as Weyl-Heisenberg group defines standard 2-dimensional QM. Various gauges associated with NCQM have been studied in [24]. In this thesis, we prove that the representations, associated with these gauges and the ones for 2-dimensional standard QM, can all be obtained from the unitary dual of $G_{\text{NC}}$. Also, the group of transformations preserving the set of 2-dimensional non-commutative quantum mechanical commutation relations is obtained to be isomorphic to $Sp(4, \mathbb{R})$. An interesting family of biorthogonal polynomials giving rise to deformed complex Hermite polynomials are explored in [9] and [6]. The representations associated with these deformed complex Hermite polynomials are also found to be sitting inside the unitary dual of $G_{\text{NC}}$.

The thesis consists of a separate part which is independent of low dimensional Galilei group that we have been considering by far. In his seminal paper [29], Goldman discovered a remarkable Lie algebra structure among the free homotopy classes $\mathbb{Z}\hat{\pi}$ of oriented loops immersed in an oriented closed surface. He then considered the conjugacy classes of representations of these free homotopy classes of loops, i.e. $\text{Hom}(\pi, G)/G$, where G is any Lie group. In [29], an explicit homomorphism $\rho : \mathbb{Z}\hat{\pi} \to C^{\infty}(\text{Hom}(\pi, G)/G)$ is established. In this thesis, we model space-time as $\Sigma \times \mathbb{R}$ with $\Sigma$ being a compact Riemann surface without any boundary and then write down the Chern-Simons action on this 3-manifold. The relevant connection 1-forms take their values on the Lie algebra $\mathcal{G}$ of the structure Lie group $G$ of the underlying principal G-bundle. The time component of the gauge connections are gauged out using additional gauge freedoms. Curvature of the gauge connections are found to be zero. The infinite dimensional space of these flat connections is endowed with a natural symplectic structure (see [2]). The Poisson brackets between the relevant gauge connections are then computed. The quotient of the space of flat connections by action of gauge transformations is a finite dimensional space. And it is a well-known fact that this moduli space of flat connections can be identified with $\text{Hom}(\pi, G)/G$. Its symplectic structure has been investigated by Goldman in [28]. In [29], he computed the Lie brackets between invariant functions belonging to $C^\infty(\mathrm{Hom}(\pi, G)/G)$ using the already mentioned homomorphism $\rho$. In the setting of this thesis, we compute the Poisson bracket between Wilson lines, i.e. the traces of monodromy matrices along loops in $\mathbb{Z}\hat{\pi}$ using the formalism of the Hamiltonian method of soliton theory [26]. The Poisson brackets, thus computed, are seen to coincide with the ones computed by Goldman in [29].

In the following sections, we provide background materials required in the subsequent chapters.

## 1.1 Central extension and group cohomology

An elegant account of group multipliers and projective representations with their relations to quantum mechanics can be found in [48]. In this thesis, we are going to follow very closely the treatment by Bargmann in his classic paper [10]. Let $G$ be any locally compact second countable group (lcsc), $G'$ its connected component and $G^*$ its universal covering group. Also, let $\mathcal{K}$ be an lcsc abelian group. We shall be interested in *$\mathcal{K}$-local exponents* of $G'$ in any neighbourhood $\mathfrak{A}$, say that of the identity $e$. A $\mathcal{K}$-local exponent of $G'$ in $\mathfrak{A}$ is a continuous function $\xi : \mathfrak{A} \times \mathfrak{A} \to \mathcal{K}$, satisfying the following properties

$$\begin{aligned} &\xi(g'', g') + \xi(g''g', g) = \xi(g'', g'g) + \xi(g', g) \\ &\xi(g, e) = 0 = \xi(e, g),\ \xi(g, g^{-1}) = \xi(g^{-1}, g), \end{aligned} \tag{1.1}$$

with $g'g$, $g''g'$ and $g^{-1}$ all belonging to the neighbourhood $\mathfrak{A}$ of identity $e$.

Two local exponents $\xi$ and $\xi'$ defined on $\mathfrak{A}$ and $\mathfrak{A}'$, respectively, will be considered equivalent if the following holds on $\mathfrak{A}_1 = \mathfrak{A} \cap \mathfrak{A}'$:

$$\xi'(g', g) = \xi(g', g) + \Delta_{g',g}(\zeta), \tag{1.2}$$

where $\Delta_{g',g}(\zeta)$, in terms of the continuous function $\zeta : \mathfrak{A}_1 \to \mathcal{K}$ reads

$$\Delta_{g',g}(\zeta) = \zeta(g') + \zeta(g) - \zeta(g'g). \tag{1.3}$$

Now, the local exponents of $G'$ in $\mathfrak{A}$ can be extended to the whole of $G'$ if one demands that $G'$ be simply connected (see page 3 in [10]). In this case, the local exponents are called simply $\mathcal{K}$-*exponents*. In case if $G'$ is not simply connected, then one works with its universal covering group $G^*$ and find all its $\mathcal{K}$-*exponents*.

In what follows next, $G$ will be assumed to be a connected, simply connected Lie group and $\mathcal{K}$ to be the abelian Lie group $\mathbb{R}$. Now, if we want to translate the additive language of exponents to a multiplicative one, we define the $\mathbb{U}(1)$-*local factors* as $m(g, g') = e^{i\xi(g,g')}$. Under the equivalence relation given by (1.2), the $\mathbb{U}(1)$-factors of $G$ form a vector space known as its second cohomology group $H^2(G, \mathbb{U}(1))$. Using the definition of projective representation (see page 248 in [48]), one obtains for mappings $U : g \mapsto U(g)$ of the underlying Lie group into the unitary operators of a separable Hilbert space $\mathcal{H}$:

$$U(g)U(g') = e^{i\xi(g,g')}U(gg'), \tag{1.4}$$

with any $g, g' \in G$. Now based on (1.4), one can define ordinary representations $\mathbf{U}$, of the *centrally extended group* $\overline{G}$ with a generic element $(\theta, g)$ and group multiplication given by $(\theta, g)(\theta', g') = (\theta + \theta' + \xi(g, g'), gg')$, as

$$\mathbf{U}(\theta, g) = e^{i\theta} U(g). \tag{1.5}$$

Following (1.4), one obtains

$$\begin{aligned}
\mathbf{U}(\theta, g)\mathbf{U}(\theta', g') &= e^{i(\theta+\theta')} U(g) U(g') \\
&= e^{i[\theta+\theta'+\xi(g,g')]} U(gg') \\
&= \mathbf{U}(\theta + \theta' + \xi(g, g'), gg') \\
&= \mathbf{U}((\theta, g)(\theta', g')). \qquad (1.6)
\end{aligned}$$

Thus, we arrive at ordinary representations $\mathbf{U}$ of the centrally extended Lie group $\overline{G}$ starting from the projective or ray representations $U$ of the Lie group $G$.

### 1.2 Coadjoint orbits and Wigner functions

Let $G$ be a Lie group with semi-direct product structure given by $G = \mathbb{R}^n \rtimes H$ where $H$ is a closed subgroup of $GL(n, \mathbb{R})$. Also, let $\mathfrak{g}$ and $\mathfrak{g}^*$ be its Lie algebra and dual Lie algebra, respectively. Now, $G$ has a natural coadjoint action on $\mathfrak{g}^*$. It is well-known that the underlying coadjoint orbits $\mathcal{O}^*$ are symplectic leaves foliated inside $\mathfrak{g}^*$. We are interested in situations where at least one of $\mathcal{O}^*$-s is open and free in $\mathfrak{g}^*$. This amounts to say that the action of $H$ on $\hat{\mathbb{R}}^n$ (dual of $\mathbb{R}^n$), i.e. the dual orbit $\hat{\mathcal{O}}_{\vec{k}^T} = \{\vec{k}^T\mathbf{h} | \mathbf{h} \in H\}$, for some $\vec{k}^T \in \hat{\mathbb{R}}^n$, is required to be open free in $\hat{\mathbb{R}}^n$. And the coadjoint orbits are simply cotangent bundles on the dual orbits. Let us fix a vector $(\vec{0}^T, \vec{k}^T) \in \mathbb{R}^{2n}$ once and for all. Also, denote a generic element of coadjoint orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}^T)}$ by $(\vec{\gamma}_q^T, \vec{\gamma}_p^T)$. Since, $\mathcal{O}^*_{(\vec{0}^T,\vec{k}^T)} = T^*\hat{\mathcal{O}}_{\vec{k}^T}$, the coordinates denoted by $\vec{\gamma}_q^T$ are related to the tangent space while the ones denoted by $\vec{\gamma}_p^T$ are related to the base manifold (dual orbit) of the cotangent bundle. Now let $d\nu(\vec{k}^T) = c(\vec{k}^T)d\vec{k}^T$ be the invariant measure on $\hat{\mathcal{O}}_{\vec{k}^T}$ where $c$ is the Duflo-Moore operator [35]. And, let $d\Omega_{\vec{k}^T}$ be the invariant measure defined on the coadjoint orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}^T)}$. We then define the *Wigner map* for $G$ as

$$W : \mathcal{B}_2(\mathfrak{H}) \to L^2(\mathcal{O}^*_{(\vec{0}^T,\vec{k}^T)}, d\Omega_{\vec{k}^T}), \tag{1.7}$$

where the Hilbert space of Hilbert-Schmidt operators on $\mathfrak{H} = L^2(\hat{\mathcal{O}}_{\vec{k}^T}, d\nu(\vec{k}^T))$ is denoted as $\mathcal{B}_2(\mathfrak{H})$. The general formula for $W$ is given in [35]. In the context of extended (1+1)-affine Galilei group $\mathcal{G}^m_{\text{aff}}$, it is explicitly computed in chapter 3, following the computations of all necessary ingredients.

The technique outlined in [35] can not be applied to groups which do not admit any open free orbit, e.g. centrally extended (1+1)-Galilei group $\mathcal{G}^m$. The general technique for computing Wigner map for any Lie group with type-I regular representation is given in [5]. Let us briefly discuss what it means by a Wigner map in this more general context. Let the dual orbits $\hat{\mathcal{O}}_\sigma$ (need not be open free) are parametrized by $\sigma \in \hat{G}$, the space of equivalence classes of unitary irreducible representations of $G$. One can then define Plancherel measure $d\nu_G(\sigma)$ in this parameter space. Given the Hilbert spaces $\mathfrak{H}_\sigma = L^2(\hat{\mathcal{O}}_\sigma, d\mu)$ with $d\mu$ being some invariant measure on the dual orbit $\hat{\mathcal{O}}_\sigma$, we can form the direct integral Hilbert space of Hilbert-Schmidt operators on $L^2(\hat{\mathcal{O}}_\sigma, d\mu)$ given as

$$\mathcal{B}_2^\oplus = \int_{\hat{G}}^\oplus \mathcal{B}_2(L^2(\hat{\mathcal{O}}_\sigma, d\mu))d\nu_G(\sigma). \tag{1.8}$$

Let us now turn our attention to the coadjoint orbits $\mathcal{O}_\lambda$ foliated inside the dual Lie algebra $\mathfrak{g}^*$. Note that the foliation is indexed by a continuous parameter $\lambda$ taking its value in an index set $J$. The associated Lebesgue measure $dX^*$ in $\mathfrak{g}^*$ disintegrates in the following way

$$dX^* = \sigma_\lambda(X^*_\lambda)d\kappa(\lambda)d\Omega_\lambda(X^*_\lambda), \quad X^*_\lambda \in \mathcal{O}_\lambda, \tag{1.9}$$

where $\sigma_\lambda$ is a positive density defined on the coadjoint orbit $\mathcal{O}_\lambda$ and $d\Omega_\lambda$ is the (coad)-invariant measure on $\mathcal{O}_\lambda$. The measure $d\kappa(\lambda)$ is associated with the foliation parameter $\lambda$. All these measures and density functions are explicitly computed in the context of centrally extended (1+1)-Galilei group in chapter 3. We now define the direct integral Hilbert space $\mathfrak{H}^\sharp$ as

$$\mathfrak{H}^\sharp = \int_J^\oplus L^2(\mathcal{O}_\lambda, d\Omega_\lambda)d\kappa(\lambda). \tag{1.10}$$

With the notations introduced above, the Wigner map associated with $G$ is defined as a map between two direct integral Hilbert spaces given by

$$W : \mathcal{B}_2^\oplus \to \mathfrak{H}^\sharp. \tag{1.11}$$

The explicit formula for finding $W$ for any such Lie group $G$, admitting type-I regular representation, is given in [5] (page 22). In chapter 3, we make direct use of this very useful formula to compute the Wigner function for centrally extended (1+1)-Galilei group.

In this thesis, the manuscripts associated with Low dimensional Galilei groups and their relevant applications are added as separate chapters. In particular, chapter 2 and chapter 3 reproduce the contents of manuscripts [20] and [18], respectively. Chapter 4 and 5 are devoted to the study of (2+1) Galilei group and its application to NCQM. These chapters reflect the contents of manuscripts [21] and [19], respectively. Finally, in chapter 6 an independent problem is considered. This chapter reproduces the materials of manuscript [17].

# Chapter 2
# All the Groups of Signal Analysis from the $(1+1)$-affine Galilei Group

The contents of this chapter are taken from the article titled "All the Groups of Signal Analysis from the $(1+1)$-affine Galilei Group" [20]. Here, we study the relationship between the $(1+1)$-affine Galilei group and four groups of interest in signal analysis and image processing, viz., the wavelet or the affine group of the line, the Weyl-Heisenberg, the shearlet and the Stockwell groups. We show how all these groups can be obtained either directly as subgroups of the affine Galilei group, or as subgroups of central extensions of a subgroup of the affine Galilei group, namely the Galilei-Schrödinger group. We also study this at the level of unitary representations of the groups on Hilbert spaces.

## 2.1 Introduction

There are a number of groups that are used in the current literature, on signal analysis and image processing, to construct signal transforms, as functions representing the signals over convenient parameter spaces. Of these, the most commonly used are the wavelet group, i.e., the affine group of the real line $\mathbb{R}$, the Heisenberg and the Weyl-Heisenberg groups and the more recently introduced Stockwell and shearlet groups. Another set of groups, which are extensions of the Heisenberg group by one-parameter dilations, were introduced in [43]. These include the shearlet group as a special case and hence are also relevant for constructing signal transforms. As the name suggests, the wavelet group [4, 23, 47] is used to build the well-known continuous wavelet transform while the shearlet transform, using the shearlet group [22], is applicable to situations where the signal to be analyzed has undergone shearing transformations. The Weyl, or equivalently, the Weyl-Heisenberg group leads to the windowed Fourier transform, useful in time-frequency analysis [4, 33, 23], while the Stockwell transform [13, 40, 45] combines features of both the wavelet and time-frequency transforms. The Stockwell group is closely related to the wavelet group and indeed, as an interesting result we show here that it is just a trivial central extension of the wavelet group. (Of course, the wavelet group has no non-trivial central extensions.) This fact also has the implication that the unitary irreducible representations of the Stockwell group are square-integrable over a homogeneous space (the space consisting of the affine group parameters), a fact studied in [40].

The matrix representations of these various groups are as follows. A generic element of the Heisenberg group is given by a $3 \times 3$ matrix,

$$g = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R} , \tag{2.1}$$

while its one-parameter family of extensions obtained in [43] have the form

$$g = \begin{bmatrix} e^{\sigma} & ve^{\frac{\sigma}{p+1}} & a \\ 0 & e^{\frac{\sigma}{p+1}} & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad -1 < p \leq 1\,, \qquad a, b, v, \sigma \in \mathbb{R}\,. \tag{2.2}$$

with the shearlet group, which is a special case ($p = 1$), being of the type,

$$g = \begin{bmatrix} \mu & \nu\sqrt{\mu} & \alpha \\ 0 & \sqrt{\mu} & \beta \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mu > 0,\ \nu, \alpha, \beta \in \mathbb{R}\,. \tag{2.3}$$

The connected affine or wavelet group is given by $2 \times 2$ matrices of the form

$$g = \begin{bmatrix} d & t \\ 0 & 1 \end{bmatrix}, \qquad d > 0\,, t \in \mathbb{R}\,, \tag{2.4}$$

and finally, the Stockwell group can be represented by a $4 \times 4$ matrix,

$$g = \begin{bmatrix} 1 & \gamma\delta & 0 & \theta \\ 0 & \gamma & 0 & 1-\gamma \\ 0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \gamma > 0,\ \delta, \theta \in \mathbb{R}\,. \tag{2.5}$$

The question naturally arises as to whether there exists a matrix group which contains all the above groups as subgroups. It is also noteworthy that all these groups consist of upper triangular matrices.

The purpose of this chapter is firstly, to answer the above question., i.e., we show how all these groups can be obtained as subgroups of various extensions of the Galilei group in $(1+1)$-dimensions. This group is a physical kinematical group, which incorporates the symmetry of non-relativistic motion in a $(1+1)$-dimensional space-time. More precisely, we shall first extend this group by space and time dilations to obtain the $(1+1)$-affine Galilei group, which will then be shown to contain all the above groups as subgroups, except the Stockwell group. This last group which, as we mentioned earlier, is a trivial central extension of the wavelet group, will be obtained as a subgroup of a trivial central extension of the Galilei-Schrödinger group, which itself is a subgroup of the affine Galilei group. As a second and related problem we study how unitary irreducible representations of the affine Galilei and the various centrally extended Galilei-Schrödinger group decompose when restricted to the above subgroups. This would shed light on how signal transforms related to the bigger groups decompose into linear combinations of transforms based on the smaller subgroups. Physically this could correspond to situations where certain parameters of a more detailed transform are averaged over or ignored.

Before closing this section we might mention that extensions of the Galilei group and its Lie algebra have been studied in many other physical contexts, see for example [25] and references cited therein.

## 2.2 Extension to the affine Galilei group

We start with the (1+1)-Galilei group $\mathcal{G}_0$ which, as we said, is the kinematical group of a non-relativistic space-time of $(1+1)$-dimensions. This is a three parameter group, an element of which we shall denote by $(b, a, v)$. The parameters $b$, $a$, and $v$ stand for time translation, space translation and the Galilean or velocity boost, respectively. Under the action of this group, a space-time point $(x, t)$ transforms in the following manner

$$\begin{aligned} x &\mapsto x + vt + a \\ t &\mapsto t + b \end{aligned}$$

The group element $g = (b, a, v)$ can be faithfully represented by a $3 \times 3$ upper triangular matrix,

$$g = \begin{pmatrix} 1 & b & a \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} , \tag{2.6}$$

so that matrix multiplication captures the group composition law. This group, also known as the *Heisenberg group* in the mathematical and signal analysis literature, is a central extension of the group of translations of $\mathbb{R}^2$ (translations in time and velocity). The exponent giving this extension is

$$\xi_{\text{H}}(\mathbf{x}, \mathbf{x}') = bv' , \tag{2.7}$$

where, $\mathbf{x} = (b, v),\ \mathbf{x}' = (b', v')$. In the physical literature one usually works with another central extension of $\mathbb{R}^2$, the resulting group being referred to as the *Weyl-Heisenberg group*. This latter group is constructed using an exponent which is projectively equivalent to (2.7). We shall come back to this point later.

In discussing and constructing central extensions, we shall follow Bargmann's treatment in [10]. Given a connected and simply connected Lie group $G$, the local exponents $\xi$ giving its central extensions are functions $\xi : G \times G \to \mathbb{R}$, obeying the following properties:

$$\xi(g'', g') + \xi(g''g', g) = \xi(g'', g'g) + \xi(g', g)$$

$$\xi(g, e) = 0 = \xi(e, g),\ \xi(g, g^{-1}) = \xi(g^{-1}, g).$$

We call the central extension trivial when the corresponding local exponent is simply a *coboundary* term, in other words, when there exists a continuous function $\zeta : G \to \mathbb{R}$ such that the following holds

$$\xi(g', g) = \xi_{cob}(g', g) := \zeta(g') + \zeta(g) - \zeta(g'g).$$

Two local exponents $\xi$ and $\xi'$ are *equivalent* if they differ by a coboundary term, i.e. $\xi'(g', g) = \xi(g', g) + \xi_{cob}(g', g)$. A local exponent which is itself a coboundary is said to be trivial and the corresponding extension of the group is called a trivial extension. Such an extension is isomorphic to the direct product group $\mathbb{U}(1) \times G$. Exponentiating the inequivalent local exponents yields the $\mathbb{U}(1)$ local factors or the familiar group multipliers, and the set of all such inequivalent multipliers form the well known second cohomology group $H^2(G, \mathbb{U}(1))$ of $G$.

Next we construct a different kind of an extension of the Galilei group $\mathcal{G}_0$ itself, by forming its semidirect product with $\mathcal{D}_2$, the two-dimensional dilation group, i.e., we introduce two dilations (of space and time). The resulting group $\mathcal{G}_0 \rtimes \mathcal{D}_2$ will be denoted $\mathcal{G}_{\text{aff}}$. If the space and time dilations are given by $\sigma$ and $\tau$, respectively, and a generic group element of $\mathcal{G}_{\text{aff}}$ is written $(b, a, v, \sigma, \tau)$, then the corresponding group composition law reads

$$
\begin{aligned}
&(b, a, v, \sigma, \tau)(b', a', v', \sigma', \tau') \\
&\qquad = (b + e^{\tau} b', a + e^{\tau} b' v + e^{\sigma} a', v + e^{\sigma - \tau} v', \sigma + \sigma', \tau + \tau') \ . \qquad (2.8)
\end{aligned}
$$

We shall refer to $\mathcal{G}_{\text{aff}}$ as the *affine Galilei group*. It has the matrix representation

$$
(b, a, v, \sigma, \tau)_{\text{aff}} = \begin{bmatrix} e^{\sigma} & v e^{\tau} & a \\ 0 & e^{\tau} & b \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.9)
$$

### 2.3 From affine Galilei to extended Heisenberg, shearlet and wavelet groups

In this section, starting from the affine Galilei group $\mathcal{G}_{\text{aff}}$, we first derive the family of extensions $G^p_{\text{H}}$ of the Heisenberg group, originally obtained in [43]. Following this, we shall show how the reduced shearlet group, constructed in [22] is in fact one of the above groups. Finally, we shall obtain the wavelet group as another subgroup of the affine Galilei group.

In subsequent sections, using the matrix representations of two central extensions (one of them being a trivial extension) of the Galilei-Schrödinger group $\mathcal{G}_s$, we shall demonstrate that the Weyl-Heisenberg group and the connected Stockwell group are subgroups of these centrally extended groups. In other words, we shall have shown that all the groups of interest in time-frequency analysis and signal processing are obtainable from a single group, the affine Galilei $\mathcal{G}_{\text{aff}}$.

### 2.3.1 Extended Heisenberg group $G_{\text{H}}^p$ as subgroup of affine Galilei group $\mathcal{G}_{\text{aff}}$

Let us construct a family of subgroups of the the affine Galilei group $\mathcal{G}_{\text{aff}} = \mathcal{G}_0 \rtimes D_2$ by restricting the two dilations $\sigma$ and $\tau$ to lie on a line $\tau = m\sigma$, where $m$ is a constant. The special case where $m = 2$ is called the Galilei-Schrödinger group [8]. We shall come back to this group later.

Consider first the the family of (non-isomorphic) extensions $G_{\text{H}}^p$ of the Heisenberg group, worked out in [43]. This family of groups is parametrized by a real number $p$, where $-1 < p \leq 1$. The corresponding group law reads

$$(b, a, v, \sigma)(b', a', v', \sigma') = (b + e^{\frac{\sigma}{p+1}} b', a + e^{\sigma} a' + e^{\frac{\sigma}{p+1}} v b', e^{\frac{p\sigma}{p+1}} v' + v, \sigma + \sigma'). \quad (2.10)$$

The matrix representation of the above family of Lie groups, referred to in ([43]) as the *extended Heisenberg groups*, is easily seen to be

$$(b, a, v, \sigma)^p_{\mathrm{H}} = \begin{bmatrix} e^{\sigma} & v e^{\frac{\sigma}{p+1}} & a \\ 0 & e^{\frac{\sigma}{p+1}} & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad -1 < p \leq 1. \tag{2.11}$$

Comparing with (2.9), we immediately see that the groups $G^p_{\mathrm{H}}$ are subgroups of the (1+1) affine Galilei group $\mathcal{G}_{\mathrm{aff}}$ of the type where the two dilations are restricted to the line $\tau = m\sigma$, with $m = \frac{1}{p+1}$.

### 2.3.2 Reduced shearlet group as subgroup of the affine Galilei group $\mathcal{G}_{\mathrm{aff}}$

The reduced shearlet group $\mathbb{S}$, as described in [22], has a generic element,

$$s = (\mu, \nu, \alpha, \beta), \quad \mu \in \mathbb{R}^+, \ \nu \in \mathbb{R} \text{ and } (\alpha, \beta) \in \mathbb{R}^2,$$

with the multiplication law

$$\begin{aligned} &(\mu_1, \nu_1, \alpha_1, \beta_1)(\mu_2, \nu_2, \alpha_2, \beta_2) \\ &\quad = (\mu_1\mu_2, \nu_1 + \nu_2\sqrt{\mu_1}, \alpha_1 + \mu_1\alpha_2 + \nu_1\sqrt{\mu_1}\beta_2, \beta_1 + \sqrt{\mu_1}\beta_2) \ . \end{aligned} \tag{2.12}$$

The matrix representation for the group $\mathbb{S}$ is as follows

$$(\mu, \nu, \alpha, \beta) = \begin{bmatrix} \mu & \nu\sqrt{\mu} & \alpha \\ 0 & \sqrt{\mu} & \beta \\ 0 & 0 & 1 \end{bmatrix} \tag{2.13}$$

Comparing with (2.11), we see that this group corresponds to the special case $p = 1$, i.e., $m = \dfrac{1}{2}$,

$$(b, a, v, \sigma)_{\mathbb{S}} := (b, a, v, \sigma)_{\mathrm{H}}^{p=1} = \begin{bmatrix} e^{\sigma} & ve^{\frac{\sigma}{2}} & a \\ 0 & e^{\frac{\sigma}{2}} & b \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.14}$$

and the explicit identification

$$\begin{aligned} e^{\sigma} &\longrightarrow \mu \\ v &\longrightarrow \nu \\ a &\longrightarrow \alpha \\ b &\longrightarrow \beta\,. \end{aligned}$$

Thus, the reduced shearlet group $\mathbb{S}$ is a member of the family of extensions $G_{\mathrm{H}}^{p}$ of Heisenberg group (with $p = 1$) and hence also a subgroup of the $(1 + 1)$-affine Galilei group. $\mathcal{G}_{\text{aff}}$.

### 2.3.3 Wavelet group as subgroup of the affine Galilei group $\mathcal{G}_{\text{aff}}$

The connected affine group or the wavelet group is a two-parameter group $G_{+}^{\text{aff}}$ which consists of transformations on $\mathbb{R}$ given by

$$x \mapsto dx + t, \tag{2.15}$$

where $x \in \mathbb{R}$, $d > 0$ and $t \in \mathbb{R}$. Here $d$ and $t$ can be regarded as the dilation and translation parameters, respectively. The group law for this group is given by

$$(d_1, t_1)(d_2, t_2) = (d_1 d_2, d_1 t_2 + t_1) \tag{2.16}$$

The matrix representation of $G_+^{\text{aff}}$, compatible with the above group law, is given by

$$(d, t) = \begin{bmatrix} d & t \\ 0 & 1 \end{bmatrix} \tag{2.17}$$

In the matrix (2.14) of the reduced shearlet group if we set $b = v = 0$, we are left with

$$s\,|_{\text{Wavelet}} = \begin{bmatrix} e^{\sigma} & 0 & a \\ 0 & e^{\frac{\sigma}{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.18}$$

which is a $3 \times 3$ faithful matrix representation of $G_+^{\text{aff}}$ with the following identification

$$\begin{aligned} d &\longrightarrow e^{\sigma} \\ t &\longrightarrow a\,, \end{aligned}$$

i.e., we have obtained the wavelet group as a subgroup of the reduced shearlet group $\mathbb{S}$ and hence of the affine Galilei group $\mathcal{G}_{\text{aff}}$.

Thus, so far we have obtained all the groups mentioned in Section 2.1, except for the Stockwell group, as subgroups of the affine Galilei group. Although we shall later obtain the Stockwell group as a subgroup of a trivial central extension of the Galilei-Schrödinger group, which is itself a subgroup of the affine Galilei group, we might mention already here that we could obtain the Stockwell group also as a trivial central extension of the wavelet group. In this sense, we could have started with a trivial extension of the affine Galilei group and obtained all the groups mentioned in Section 2.1 essentially as subgroups of it.

### 2.4 Extensions of the affine Galilei and related groups

The Galilei group $\mathcal{G}_0$, has a non-trivial central extension [37], and in fact, there is only one such extension, up to projective equivalence. This extension, which we describe below, incorporates the quantum kinematics of a physical system in a space-time of $(1+1)$-dimensions.

Let $M$ be a non-zero, positive real number; the local exponent $\xi : \mathcal{G}_0 \times \mathcal{G}_0 \to \mathbb{R}$, giving the extension in question is:

$$\xi(g, g') = M[va' + \frac{1}{2}b'v^2] \ , \tag{2.19}$$

where $g \equiv (b, a, v)$ and $g' \equiv (b', a', v')$ are elements of $\mathcal{G}_0$. We denote this extended group by $\mathcal{G}^M$; writing a generic element of $\mathcal{G}^M$ as $(\theta, b, a, v)$, the group multiplication law reads,

$$\begin{aligned}&(\theta, b, a, v)(\theta', b', a', v')\\ &\quad = (\theta + \theta' + M[va' + \frac{1}{2}b'v^2], b + b', a + a' + vb', v + v')\end{aligned} \tag{2.20}$$

We shall refer to $\mathcal{G}^M$ as the *quantum Galilei group.*

### 2.4.1 Non-central extension of affine Galilei group

The group $\mathcal{G}_{\text{aff}}$ does not have non-trivial central extensions. Consequently, it cannot be used in quantum mechanics, since a trivial extension fails to generate mass [8]. From a physical point of view, it is therefore more meaningful to take the quantum Galilei group $\mathcal{G}^M$ and to form its semidirect product with $\mathcal{D}_2$. This way, we arrive at $\mathcal{G}^M_{\text{aff}} = \mathcal{G}^M \rtimes \mathcal{D}_2$, which is a *non-central extension* of the affine Galilei group. For simplicity we will call this group the *extended affine Galilei group*. Denoting a generic group element of this group by $(\theta, b, a, v, \sigma, \tau)$, the group multiplication law reads

$$\begin{aligned}
&(\theta, b, a, v, \sigma, \tau)(\theta', b', a', v', \sigma', \tau') \\
&\quad = (\theta + e^{2\sigma-\tau}\theta' + M[e^{\sigma}va' + \frac{1}{2}e^{\tau}v^2b'], b + e^{\tau}b', a + e^{\tau}b'v + e^{\sigma}a', v + e^{\sigma-\tau}v', \\
&\qquad \sigma + \sigma', \tau + \tau')
\end{aligned} \tag{2.21}$$

The matrix representation of an element of $\mathcal{G}^M_{\text{aff}}$, consistent with the above multiplication rule is

$$(\theta, b, a, v, \sigma, \tau)^M_{\text{aff}} = \begin{bmatrix} e^{\sigma} & ve^{\tau} & 0 & a \\ 0 & e^{\tau} & 0 & b \\ Mve^{\sigma} & \frac{1}{2}Mv^2e^{\tau} & e^{2\sigma-\tau} & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.22}$$

As mentioned in [37], all the multipliers for the (1+1) dimensional quantum Galilei group $\mathcal{G}^M$ are equivalent, i.e., there is only one equivalence class in the multiplier group of the $(1+1)$-dimensional Galilei group $\mathcal{G}_0$. In other words $H^2(\mathcal{G}_0, \mathbb{U}(1))$ is just one dimensional. It is noteworthy in this context that equation (2.22) is a matrix representation of $\mathcal{G}_{\text{aff}}^M$ provided that the multiplier we choose, from the one dimensional group $H^2(\mathcal{G}_0, \mathbb{U}(1))$ to obtain $\mathcal{G}^M$ during the two step construction of $\mathcal{G}_{\text{aff}}^M$, has the form $e^{i\xi(g_1,g_2)}$, with $\xi$ given by equation (2.19). Choosing another, though equivalent, multiplier will alter the form of the matrix (2.22).

### 2.4.2 Galilei-Schrödinger group: central extensions

Let us consider the particular case of the subgroup of $\mathcal{G}_{\text{aff}}$ when $\tau = 2\sigma$, i.e., $m = 2$ (or $p = -\dfrac{1}{2}$ in (2.11)). We denote the resulting one-dimensional dilation group by $\mathcal{D}_s$ and the corresponding subgroup of $\mathcal{G}_{\text{aff}}$ by $\mathcal{G}_s$, so that $\mathcal{G}_s = \mathcal{G}_0 \rtimes \mathcal{D}_s$. In the literature, this group is known as the *Galilei-Schrödinger group* [8]. It is easy to construct a central extension, denoted $\mathcal{G}_s^M$, of $\mathcal{G}_s$ by $\mathbb{U}(1)$, using a local exponent $\xi : \mathcal{G}_s \times \mathcal{G}_s \to \mathbb{R}$, or equivalently, using the multiplier $\exp i\xi : \mathcal{G}_s \times \mathcal{G}_s \to \mathbb{U}(1)$. We mention in this context that since we prefer working with addition rather than multiplication, we shall henceforth talk in terms of exponents rather than multipliers.

We proceed to construct two extensions of the Galilei-Schrödinger group, using two equivalent multipliers, and a third extension using a trivial or exact multiplier. To do that we first note that the group multiplication law for $\mathcal{G}_s$ is given by

$$(b, a, v, \sigma)(b', a', v', \sigma') = (b + e^{2\sigma}b', a + e^{\sigma}a' + e^{2\sigma}vb', v + e^{-\sigma}v', \sigma + \sigma') \qquad (2.23)$$

where a generic element of the group is denoted as $(b, a, v, \sigma)$. Now using the exponent

$$\xi((b, a, v, \sigma); (b', a', v', \sigma')) = M[ve^{\sigma}a' + \frac{1}{2}v^2 e^{2\sigma} b'] \ , \tag{2.24}$$

we obtain a central extension $\mathcal{G}_s^M$ of $\mathcal{G}_s$ by $\mathbb{U}(1)$. The group law for the centrally extended group $\mathcal{G}_s^M$ therefore reads

$$\begin{aligned}
&(\theta, b, a, v, \sigma)(\theta', b', a', v', \sigma') \\
&\quad = (\theta + \theta' + M[ve^{\sigma}a' + \frac{1}{2}v^2 e^{2\sigma} b'], b + e^{2\sigma} b', a + e^{2\sigma} v b' + e^{\sigma} a', v + e^{-\sigma} v', \\
&\qquad \sigma + \sigma') \ ,
\end{aligned} \tag{2.25}$$

which is consistent with the matrix representation,

$$(\theta, b, a, v, \sigma)_s^M = \begin{bmatrix} e^{\sigma} & ve^{2\sigma} & 0 & a \\ 0 & e^{2\sigma} & 0 & b \\ Mve^{\sigma} & \frac{1}{2}Mv^2 e^{2\sigma} & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2.26}$$

Comparing (2.22) and (2.26) we easily see that $\mathcal{G}_s^M \subset \mathcal{G}_{\text{aff}}^M$, which is clear since we have just set $\tau = 2\sigma$. It ought to be noted here, that in going from $\mathcal{G}_0$ to $\mathcal{G}_s^M$, two extensions were involved: first we extended $\mathcal{G}_0$ to the Galilei-Schrödinger group $\mathcal{G}_s$, by taking the semidirect product of the former with the dilation group $\mathcal{D}_s$, and then doing a central extension of this enlarged group. We could equivalently have reversed the process, i.e., first done a central extension of $\mathcal{G}_0$ to obtain the quantum Galilei group $\mathcal{G}^M$ and then taken a semi-direct of this group with $\mathcal{D}_s$ to again arrive at $\mathcal{G}_s^M$. In other words, in this case the two procedures commute.

Next consider a second local exponent, $\xi_1 : \mathcal{G}_s \times \mathcal{G}_s \to \mathbb{R}$ given by

$$\xi_1((b,a,v,\sigma);(b',a',v',\sigma')) = \frac{M}{2}[-vv'b'e^{\sigma} + va'e^{\sigma} - av'e^{-\sigma}] \,. \tag{2.27}$$

This exponent is easily seen to be equivalent equivalent to $\xi$, given in (2.24). Indeed, the difference of the above two exponents,

$$\begin{aligned} \xi - \xi_1 &= \frac{M}{2}[v^2e^{2\sigma}b' + vv'b'e^{\sigma} + va'e^{\sigma} + v'ae^{-\sigma}] \\ &= \frac{M}{2}(a + e^{2\sigma}vb' + e^{\sigma}a')(v + e^{-\sigma}v') - \frac{M}{2}av - \frac{M}{2}a'v' \end{aligned} \tag{2.28}$$

is a trivial exponent. In other words (2.28) can be rewritten in terms of the continuous function $\zeta_M : \mathcal{G}_s \to \mathbb{R}$,

$$\xi - \xi_1 = \zeta_M((b,a,v,\sigma)(b',a',v',\sigma')) - \zeta_M(b,a,v,\sigma) - \zeta_M(b',a',v',\sigma'), \tag{2.29}$$

where $\zeta_M(b,a,v,\sigma) = \frac{M}{2}av$.

Let $\mathcal{G}_s^{M\prime}$ denote the central extension of $\mathcal{G}_s$ by $\mathbb{U}(1)$ with respect to the exponent $\xi_1$ given by equation (2.27). The group multiplication law for $\mathcal{G}_s^{M\prime}$ reads

$$\begin{aligned} &(\theta,b,a,v,\sigma)(\theta',b',a',v',\sigma') \\ &\quad = (\theta + \theta' + \frac{M}{2}[-vv'b'e^{\sigma} + va'e^{\sigma} - av'e^{-\sigma}], b + e^{2\sigma}b', a + e^{\sigma}a' + e^{2\sigma}vb', \\ &\qquad\qquad v + e^{-\sigma}v', \sigma + \sigma') \end{aligned} \tag{2.30}$$

The matrix representation for $\mathcal{G}_s^{M\prime}$, compatible with the group law, (2.30) is

$$(\theta, b, a, v, \sigma)_s^{M\prime} = \begin{bmatrix} e^{\sigma} & -e^{-\sigma}b & 0 & a - vb \\ 0 & e^{-\sigma} & 0 & -v \\ \frac{1}{2}Mve^{\sigma} & \frac{1}{2}Mae^{-\sigma} & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2.31}$$

Finally, we extend the Galilei-Schrodinger group $\mathcal{G}_s$ centrally by $\mathbb{U}(1)$ with respect to the trivial exponent $\xi_2 : \mathcal{G}_s \times \mathcal{G}_s \to \mathbb{R}$ given by

$$\xi_2((b, a, v, \sigma); (b', a', v', \sigma')) = ae^{-\sigma}(1 - e^{-\sigma'}) - e^{\sigma-\sigma'}vb' . \tag{2.32}$$

We call this extension $\mathcal{G}_s^T$. Again, it is straightforward to verify that the exponent given in (2.32) is indeed trivial, since it can be rewritten in terms of the continuous function $\zeta_T : \mathcal{G}_s \to \mathbb{R}$,

$$\begin{aligned} &\xi_2((b, a, v, \sigma); (b', a', v', \sigma')) \\ &\quad = \zeta_T(b, a, v, \sigma) + \zeta_T(b', a', v', \sigma') - \zeta_T((b, a, v, \sigma)(b', a', v', \sigma')) , \end{aligned}$$

where $\zeta_T(b, a, v, \sigma) = ae^{-\sigma}$. Thus, the group law for the trivially extended Galilei-Schrodinger group $\mathcal{G}_s^T$ reads

$$\begin{aligned} &(\theta, b, a, v, \sigma)(\theta', b', a', v', \sigma') \\ &\quad = (\theta + \theta' + [ae^{-\sigma}(1 - e^{-\sigma'}) - e^{\sigma-\sigma'}vb'], b + e^{2\sigma}b', a + e^{\sigma}a' + e^{2\sigma}vb', \\ &\qquad v + e^{-\sigma}v', \sigma + \sigma') \end{aligned} \tag{2.33}$$

The matrix representation of $\mathcal{G}_s^T$ compatible with the above group law is given by

$$(\theta, b, a, v, \sigma)_s^T = \begin{bmatrix} 1 & ae^{-\sigma} & -e^{\sigma}v & \theta \\ 0 & e^{-\sigma} & 0 & 1 - e^{-\sigma} \\ 0 & -e^{-\sigma}b & e^{\sigma} & e^{-\sigma}b \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.34}$$

### 2.5 From Galilei-Schrödinger to Weyl-Heisenberg and Stockwell groups

In this section we obtain the Weyl-Heisenberg and Stockwell groups as subgroups of the centrally extended Galilei-Schrödinger groups. We shall also re-derive the Heisenberg group, which by construction was a subgroup of the affine Galilei group $\mathcal{G}_{\text{aff}}$, this time as a subgroup of one of the central extensions of the Galilei-Schrödinger group.

#### 2.5.1 Heisenberg and Weyl-Heisenberg groups as subgroups of centrally extended Galilei-Schrödinger groups

As mentioned in Section 2.2, the Heisenberg group is identical to the $(1 + 1)$-Galilei group $\mathcal{G}_0$, which means that it is trivially a subgroup of the affine Galilei group $\mathcal{G}_{\text{aff}}$. Moreover, the Heisenberg group is a central extension of the two-dimensional translation group of the plane, via the local exponent $\xi_{\text{H}}$ in (2.7). As also indicated earlier, in the physical literature one uses a different, but projectively equivalent, exponent $\xi_{\text{WH}}$ (see (2.41) below) to do this extension, the resulting group being called the *Weyl-Heisenberg group*. Thus, although the Heisenberg and the Weyl-Heisenberg groups are projectively equivalent, we shall continue to differentiate between them in this chapter. We now proceed to obtain these groups as subgroups of central extensions of the Galilei-Schrödinger group. Changing notations a bit let $(q, p)$ denote a point in the plane $\mathbb{R}^2$.

In constructing the Heisenberg group $G_{\text{H}}$ one uses the local exponent,

$$\xi_{\text{H}}((q,p);(q',p')) = pq' \; . \tag{2.35}$$

Writing a general element of this group as

$$g = (\theta, q, p), \quad \theta \in \mathbb{R}, \quad (q,p) \in \mathbb{R}^2,$$

the group multiplication law reads

$$(\theta, q, p)(\theta', q', p') = (\theta + \theta' + pq', q + q', p + p'), \tag{2.36}$$

with the matrix representation being

$$(\theta, q, p)_{\text{H}} = \begin{bmatrix} 1 & p & \theta \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} . \tag{2.37}$$

Now we form the subgroup $\mathcal{G}_s^M|_{\text{H}}$ of the centrally extended Galilei-Schrodinger group $\mathcal{G}_s^M$ by setting $b = \sigma = 0$, $\theta \in \mathbb{R}$ and $(a, v) \in \mathbb{R}^2$. The matrix representation of $\mathcal{G}_s^M|_{\text{H}}$ then has the form (see (2.26)):

$$(\theta, 0, a, v, 0)_s^M := (\theta, a, v)_s^M \mid_{\mathrm{H}} = \begin{bmatrix} 1 & v & 0 & a \\ 0 & 1 & 0 & 0 \\ Mv & \frac{1}{2}Mv^2 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.38}$$

which under the identification

$$\begin{aligned} Mv &\longrightarrow p \\ a &\longrightarrow q \\ \theta &\longrightarrow \theta \end{aligned} \tag{2.39}$$

reduces to

$$(\theta, q, p)_s^M \mid_{\mathrm{H}} = \begin{bmatrix} 1 & \frac{p}{M} & 0 & q \\ 0 & 1 & 0 & 0 \\ p & \frac{p^2}{2M} & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.40}$$

Here we assume that the mass term $M$ is never zero. The above $4 \times 4$ matrix is a faithful representation of the Heisenberg group $G_{\mathrm{H}}$, compatible with the group law (2.36).

Thus, the Heisenberg group constructed using the $\xi_{\mathrm{H}}$ in (2.35), can also be obtained as a subgroup of the nontrivial central extension $\mathcal{G}_s^M$ of the Galilei-Schrödinger group.

To obtain the Weyl-Heisenberg group in a similar manner, consider the local exponent

$$\xi_{\text{WH}}((q,p);(q',p')) = \frac{1}{2}(pq' - p'q)\ . \tag{2.41}$$

It is straightforward to verify that this exponent is equivalent to $\xi_{\text{H}}$ in (2.35). Indeed,

$$\begin{aligned}
\xi_{\text{H}} - \xi_{\text{WH}} &= pq' - \frac{1}{2}(pq' - p'q) \\
&= \frac{1}{2}pq' + \frac{1}{2}p'q \\
&= \frac{1}{2}(p+p')(q+q') - \frac{1}{2}pq - \frac{1}{2}p'q' \\
&= \zeta((q,p);(q',p')) - \zeta(q,p) - \zeta(q',p')\ ,
\end{aligned}$$

where $\zeta$ is a real valued continuous function defined on the group of translations of $\mathbb{R}^2$, and hence $\xi_{\text{H}} - \xi_{\text{WH}}$ is a trivial exponent. Using the exponent $\xi_{\text{WH}}$ we extend the group of translations of $\mathbb{R}^2$ to form the Weyl-Heisenberg group $G_{\text{WH}}$, which then obeys the following group law:

$$(\theta,q,p)(\theta',q',p') = (\theta + \theta' + \frac{1}{2}(pq' - p'q), q+q', p+p') \tag{2.42}$$

The matrix representation compatible with the above group law can be written as

$$(\theta,q,p)_{\text{WH}} = \begin{bmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 0 & -p \\ \frac{1}{2}p & \frac{1}{2}q & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix}\ . \tag{2.43}$$

Forming now the subgroup $\mathcal{G}_s^{M\prime}|_{\text{WH}}$ of the centrally extended Galilei-Schrödinger group $\mathcal{G}_s^{M\prime}$, obtained by setting $b = \sigma = 0$, $\theta \in \mathbb{R}$ and $(a, v) \in \mathbb{R}^2$ (see (2.31)), we get for its matrix representation

$$(\theta, 0, a, v, 0)_s^{M\prime} := (\theta, a, v)_s^{M\prime}\,|_{\text{WH}} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & -v \\ \frac{1}{2}Mv & \frac{1}{2}Ma & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2.44}$$

Making again the identification (2.39), this becomes

$$(\theta, q, p)_s^{M\prime}\,|_{\text{WH}} = \begin{bmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 0 & -\frac{p}{M} \\ \frac{1}{2}p & \frac{1}{2}Mq & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2.45}$$

Here we assume once more that the mass term $M$ is not zero. While the above matrix is not exactly of the same form as the one given in (2.43), it does reproduce the group multiplication rule (2.42). Moreover, the two matrix representations are equivalent, via the intertwining matrix

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{M} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ,$$

i.e., we have

$$S\ (\theta, q, p)_{\rm WH} S^{-1} = (\theta, q, p)_s^{M\prime}\ |_{\rm WH}\ .$$

In this way we have shown that the Weyl-Heisenberg group $G_{\rm WH}$ is a subgroup of the non-trivial central extension $\mathcal{G}_s^{M\prime}$ of Galilei-Schrödinger group.

### 2.5.2 Connected Stockwell group as subgroup of the trivial central extension $\mathcal{G}_s^T$ of the Galilei-Schródinger group

The connected Stockwell group $G_{\rm SW}$ (see [13, 40] for definition and properties) can be seen as a trivial central extension of a group $G'_{\rm aff}$, isomorphic to the connected affine group $G_+^{\rm aff}$ (see (2.17)). Given a group element $(\gamma, \delta) \in \mathbb{R}^{>0} \times \mathbb{R}$, we define the group law for $G'_{\rm aff}$ by

$$(\gamma_1, \delta_1)(\gamma_2, \delta_2) = (\gamma_1\gamma_2, \delta_1 + \frac{1}{\gamma_1}\delta_2) \tag{2.46}$$

Comparing with (2.16), we identify the group homomorphism $f : G_+^{\rm aff} \longrightarrow G'_{\rm aff}$

$$f(\gamma, \delta) = (\frac{1}{\gamma}, \delta)\ . \tag{2.47}$$

Let us extend the group $G'_{\rm aff}$ centrally using the exponent

$$\begin{aligned}\xi_s((\gamma_1, \delta_1); (\gamma_2, \delta_2)) &= \gamma_1\delta_1(1 - \gamma_2) \\ &= \gamma_1\delta_1 + \gamma_2\delta_2 - (\gamma_1\gamma_2)(\delta_1 + \frac{\delta_2}{\gamma_1})\ . \end{aligned} \tag{2.48}$$

This is in fact a trivial exponent since it can be written in terms of the continuous function $\zeta_s : G'_{\text{aff}} \to \mathbb{R}$:

$$\xi_s((\gamma_1, \delta_1); (\gamma_2, \delta_2)) = \zeta_s(\gamma_1, \delta_1) + \zeta_s(\gamma_2, \delta_2) - \zeta_s((\gamma_1, \delta_1)(\gamma_2, \delta_2)) \ , \tag{2.49}$$

where $\zeta_s(\gamma, \delta) = \gamma\delta$. The group so extended obeys the multiplication rule

$$(\theta_1, \gamma_1, \delta_1)(\theta_2, \gamma_2, \delta_2) = (\theta_1 + \theta_2 + [\gamma_1\delta_1(1 - \gamma_2)], \gamma_1\gamma_2, \delta_1 + \frac{1}{\gamma_1}\delta_2) \ , \tag{2.50}$$

which is the product rule for elements of the Stockwell group $G_{\text{sw}}$ [13]. This proves that the Stockwell group is a trivial central extension of the wavelet or affine group. The matrix representation of a group element of $G_{\text{sw}}$ is seen to be

$$(\theta, \gamma, \delta)_{\text{sw}} = \begin{bmatrix} 1 & \gamma\delta & \theta \\ 0 & \gamma & 1 - \gamma \\ 0 & 0 & 1 \end{bmatrix} \ . \tag{2.51}$$

We now show that this group can also be obtained as a subgroup of the trivially extended Galilei-Schrödinger group $\mathcal{G}_s^T$ (see ((2.32) - (2.34)). Indeed, comparing (2.32) to (2.48) it is clear that the former exponent reduces to he latter if $v$ is set equal to zero. Next, setting

$v = b = 0$ in $\mathcal{G}_s^T$ we see that (2.34) reduces to

$$(\theta, 0, a, 0, \sigma)_{\text{s}}^T := (\theta, a, \sigma)_{\text{s}}^T \mid_{\text{SW}} = \begin{bmatrix} 1 & ae^{-\sigma} & 0 & \theta \\ 0 & e^{-\sigma} & 0 & 1 - e^{-\sigma} \\ 0 & 0 & e^{\sigma} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2.52}$$

The identification

$$e^{-\sigma} \longrightarrow \gamma$$

$$a \longrightarrow \delta$$

$$\theta \longrightarrow \theta$$

and subsequent elimination of the redundant third row and column is then seen to yield the matrix (2.51).

We can conveniently depict all these various extensions and reductions to subgroups by means of the diagram 2–1.

## 2.6 Decomposition of UIRs of the affine Galilei group and central extensions of the Galilei-Schrödinger group restricted to various subgroups

The general procedure for building signal transforms, starting from a group $G$ is first to define functions over the group using matrix elements of unitary irreducible representations. Provided these functions possess certain desirable properties which, among others, enable one to reconstruct the signal, they can be used as transforms describing the signal. In

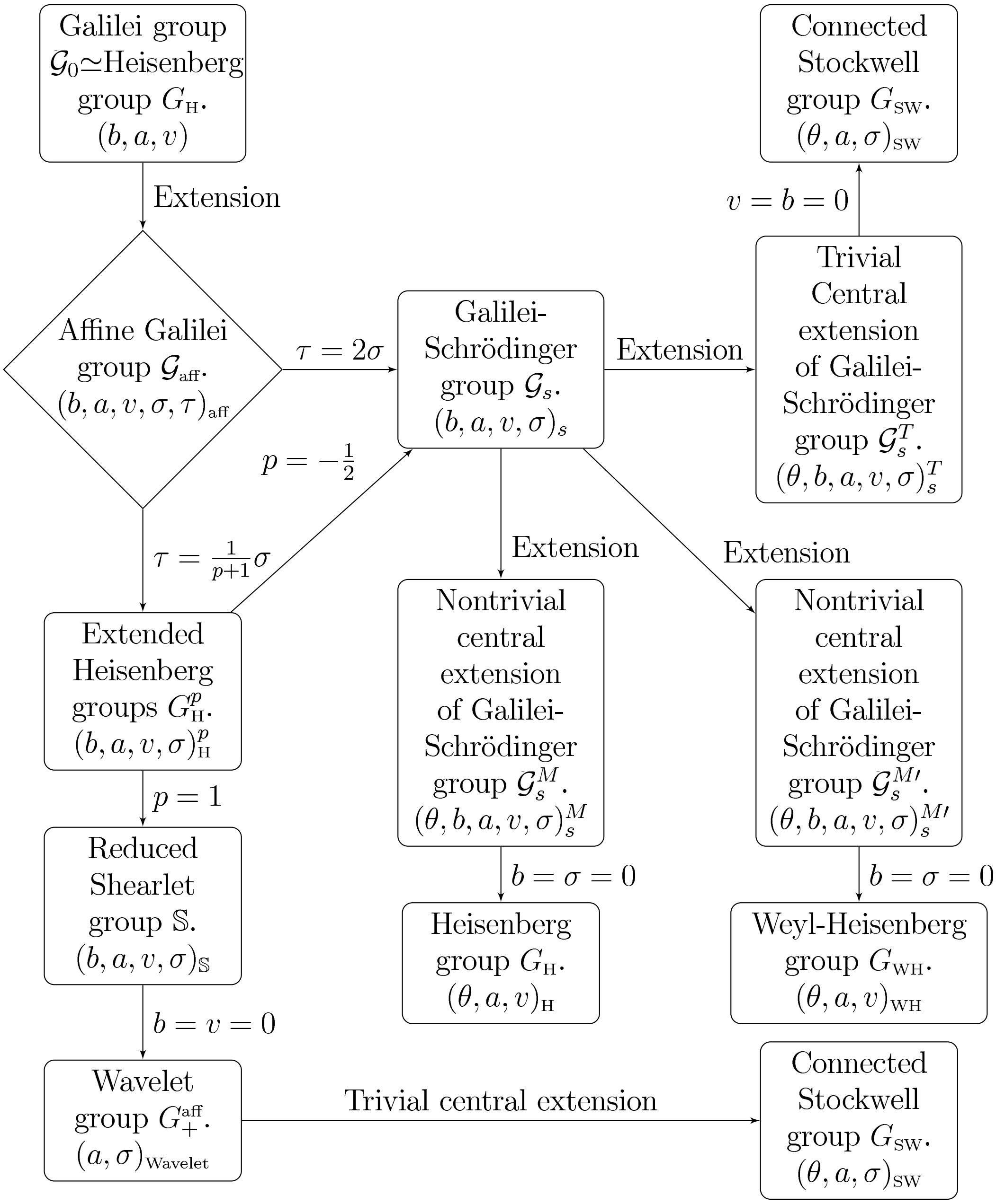



Figure 2–1: Flowchart showing the passage from the (1+1)-affine Galilei group to the various groups of signal analysis.

other words, the signal transforms are functions which encode the properties of the signal in terms of the group parameters. It is therefore of interest to construct unitary irreducible representations of the various groups discussed in the previous sections and to see how representations of the smaller subgroups, relevant to signal analysis, sit inside representations of the bigger groups.

The affine Galilei group $\mathcal{G}_{\text{aff}}$ was defined in Section 2.2, following which in Section 2.3 we studied its restriction to various subgroups of interest. In this section we shall first construct unitary irreducible representations of the affine Galilei group and then study their restrictions to the reduced shearlet and wavelet subgroups.

In later subsections we will find the UIRs of the two central extensions of the Galilei-Schrödinger and look at their restrictions to the Heisenberg group $G_{\text{H}}$ and the connected Stockwell group $G_{\text{SW}}$.

### 2.6.1 UIRs of affine Galilei group restricted to the reduced shearlet group

The group law and matrix representation of the affine Galilei group $\mathcal{G}_{\text{aff}}$ was given in (2.8) and (2.9). From the matrix representation, we easily infer the semidirect product structure, $\mathcal{G}_{\text{aff}} = \mathcal{T} \rtimes \mathcal{V}$, where $\mathcal{T}$ is an abelian subgroup, with generic element $(b, a)$ and $\mathcal{V}$ is the subgroup generated by the elements $(v, \sigma, \tau)$. Now, the action of $(v, \sigma, \tau)$ on the element $(b, a)$ as determined by (2.8) is seen to be

$$(v, \sigma, \tau)(b, a) = (e^{\tau} b, e^{\tau} vb + e^{\sigma} a) \tag{2.53}$$

We also have

$$(v,\sigma,\tau)^{-1}(b,a)=(e^{-\tau}b,e^{-\sigma}(a-vb))\ . \tag{2.54}$$

Now let $(E,p)$ denote a generic element of $\mathcal{T}^*$, the dual of $\mathcal{T}$, and the corresponding character by

$$< (E,p)\mid (b,a) >= e^{i(Eb+pa)}$$

The action of $(v,\sigma,\tau)\in\mathcal{V}$ on $(E,p)\in\mathcal{T}^*$ is then defined by

$$\begin{aligned}
&< (v,\sigma,\tau)(E,p)\mid (b,a) > \\
&\quad =< (E,p)\mid (v,\sigma,\tau)^{-1}(b,a) > \\
&\quad =< (E,p)\mid (e^{-\tau}b,e^{-\sigma}(a-vb)) > \\
&\quad = e^{i[(e^{-\tau}E-e^{-\sigma}pv)b+e^{-\sigma}pa]}\ ,
\end{aligned} \tag{2.55}$$

from which we easily find the dual action $(E,p)\longrightarrow(\bar{E},\bar{p})$,

$$\begin{aligned}
\bar{E} &= e^{-\tau}E-e^{-\sigma}pv \\
\bar{p} &= e^{-\sigma}p
\end{aligned} \tag{2.56}$$

which we can now use to compute the dual orbits. We see that the sign of $p$ is an invariant for the same orbit while $E$ takes on all real values independently. In other words, the orbits are $(i)$ the two open half planes $\mathbb{R}\times\mathbb{R}^{\gtrless 0}$, one corresponding to all positive values of $p$ and the other corresponding to negative values, $(ii)$ the two half lines $\mathbb{R}^{\gtrless 0}$, with $p=0,\ E\gtrless 0$, and $(iii)$ the degenerate orbit $E=p=0$. Note that none of these orbits are open-free (in the sense of [12]). Now using (2.54) and (2.56) we obtain

$$(v, \sigma, \tau)^{-1}(E, p) = (E', p') = (e^{\tau}(E + pv), e^{\sigma}p) \tag{2.57}$$

From this it follows that

$$dE'\ dp' = e^{\sigma+\tau}\ dE\ dp\ , \qquad \text{on} \qquad \mathbb{R} \times \mathbb{R}^{\gtrless 0}\ , \tag{2.58}$$

and

$$dE' = e^{\tau}\ dE\ , \qquad \text{on} \qquad \mathbb{R}^{\gtrless 0}\ . \tag{2.59}$$

Using the Mackey's theory of induced representations [38, 39], we obtain four unitary irreducible representations of $\mathcal{G}_{\text{aff}}$, corresponding to the above four orbits. We denote the representations corresponding to the two half-planar orbits $\mathbb{R} \times \mathbb{R}^{\gtrless 0}$ by $U^{\pm}_{\text{aff}}$, defined on $L^2(\mathbb{R} \times \mathbb{R}^{\pm}, dE\ dp)$, and the representations on the half lines $\mathbb{R}^{\gtrless 0}$, on $L^2(\mathbb{R}^{\pm}, dE)$, by $V^{\pm}_{\text{aff}}$. The representations are easily computed to be

$$(U^{\pm}_{\text{aff}}(b, a, v, \sigma, \tau)\hat{\psi})(E, p) = e^{\frac{\sigma+\tau}{2}} e^{i(Eb+pa)} \hat{\psi}(e^{\tau}(E + pv), e^{\sigma}p)\ , \quad p \gtrless 0\ , \tag{2.60}$$

and

$$(V^{\pm}_{\text{aff}}(b, a, v, \sigma, \tau)\hat{\psi})(E) = e^{\frac{\tau}{2}} e^{iEb} \hat{\psi}(e^{\tau}E)\ , \quad E \gtrless 0\ . \tag{2.61}$$

Note that the last two representations are non-trivial only on the subgroup of $\mathcal{G}_{\text{aff}}$ with $a = v = \sigma = 0$, i.e., the affine or wavelet group defined by the two remaining parameters $b, \tau$, and in fact, constitute the two unitary irreducible representations of that group. As is well known, these two representations of the affine group are square integrable and give rise to wavelet transforms.

We saw in Section 2.3.2 that the (reduced) shearlet group $\mathbb{S}$ is the subgroup of $\mathcal{G}_{\text{aff}}$ corresponding to $\tau = \dfrac{1}{2}\sigma$. Restricting $U^{\pm}_{\text{aff}}$ in (2.60) to this subgroup we get

$$(U^{\pm}_{\text{aff}} \mid_{\mathbb{S}} (b, a, v, \sigma)\hat{\psi})(E, p) = e^{\frac{3\sigma}{4}} e^{i(Eb+pa)} \hat{\psi}(e^{\frac{\sigma}{2}}(E + pv), e^{\sigma} p) \ , \quad p \gtrless 0 \ . \tag{2.62}$$

A quick examination of (2.56) shows that $\mathbb{R} \times \mathbb{R}^{\gtrless 0}$ are both open free orbits of $\mathbb{S}$. Also, as representations of the (reduced) shearlet group the two representations (2.60) are irreducible and hence square-integrable. Indeed, these are the representations used to build the shearlet transforms.

### 2.6.2 UIRs of affine Galilei group $\mathcal{G}_{\text{aff}}$ restricted to the wavelet group

We saw in Section 2.3.3 that the wavelet or affine group $G^{\text{aff}}_{+}$ could be obtained from the shearlet group as the subgroup with $b = v = 0$, or directly from the affine galilei group $\mathcal{G}_{\text{aff}}$ as the subgroup with $b = v = \tau = 0$.

Setting $b = v = \tau = 0$ in the representations $U^{\pm}_{\text{aff}}$ in (2.60) we obtain

$$(U^{\pm}_{\text{aff}} \mid_{\text{Wavelet}} (0, a, 0, \sigma, 0)\hat{\psi})(E, p) = e^{\frac{\sigma}{2}} e^{ipa} \hat{\psi}(E, e^{\sigma} p) \tag{2.63}$$

as representations of the wavelet group $G^{\text{aff}}_{+}$ on $L^2(\mathbb{R} \times \mathbb{R}^{\pm}, dE\ dp)$. However, these representations are not irreducible. Indeed, noting that

$$L^2(\mathbb{R} \times \mathbb{R}^{\pm}, dE\ dp) \simeq L^2(\mathbb{R}, dE) \otimes L^2(\mathbb{R}^{\pm}, dp),$$

the representations (2.63) are immediately seen to be of the form

$$U_{\text{aff}}^{\pm}\mid_{\text{Wavelet}} = I \otimes U_{\text{Wavelet}}^{\pm}\ , \tag{2.64}$$

where $I$ is the identity operator on $L^2(\mathbb{R}, dE)$ and $U_{\text{Wavelet}}^{\pm}$ are the two unitary irreducible representations of $G_+^{\text{aff}}$ on $L^2(\mathbb{R}^{\pm}, dp)$, given by

$$(U_{\text{Wavelet}}^{\pm}(a,\sigma)\hat{\psi})(p) = e^{\frac{\sigma}{2}} e^{ipa}\hat{\psi}(e^{\sigma}p)\ . \tag{2.65}$$

A decomposition of (2.64) into irreducibles is easily done. Indeed, let $\{\hat{\phi}_n\}_{n=0}^{\infty}$ be an orthonormal basis of $L^2(\mathbb{R}, dE)$ and $\mathfrak{H}_n$ the one-dimensional subspaces spanned by $\hat{\phi}_n,\ n = 0, 1, 2, \ldots, \infty$, so that $L^2(\mathbb{R}, dE) = \oplus_{n=0}^{\infty}\mathfrak{H}_n$. It is then immediately clear that

$$U_{\text{aff}}^{\pm}\mid_{\text{Wavelet}} (0, a, 0, \sigma, 0) = \oplus_{n=0}^{\infty}\, U_{\text{Wavelet}}^{\pm,\, n}(a, \sigma)\ , \tag{2.66}$$

where $U_{\text{Wavelet}}^{\pm,\, n}$ is an irreducible representation of $G_+^{\text{aff}}$ which is simply a direct product of the trivial representation of the wavelet group on $\mathfrak{H}_n$ with the irreducible representation $U_{\text{Wavelet}}^{\pm}$ on $L^2(\mathbb{R}^{\pm}, dp)$ given in (2.65). This decomposition also implies, that the shearlet transform, when restricted to the parameters of the wavelet group, decomposes into an infinite sum of wavelet transforms.

### 2.6.3 UIRs of centrally extended Galilei-Schrödinger group $\mathcal{G}_s^M$ restricted to the Heisenberg group $G_{\text{H}}$

The group law for the centrally extended Galilei-Schrödinger group $\mathcal{G}_s^M$, formed using the exponent $\xi$ in (2.24), is given by (2.25) and the corresponding matrix representation by (2.26). From the matrix representation one can deduce the semidirect product structure $\mathcal{G}_s^M = \mathcal{T} \rtimes \mathcal{V}$ where $\mathcal{T}$ is an abelian subgroup with generic element $(\theta, b, a)$ and $\mathcal{V}$ a semi-simple group consisting of the elements $(v, \sigma)$. Note that that $\mathcal{V}$ is just the affine or wavelet group which also has a semidirect product structure, since

$$(v_1, \sigma_1)(v_2, \sigma_2) = (v_1 + e^{-\sigma_1} v_2, \sigma_1 + \sigma_2).$$

Now let $(q, E, p)$ denote a generic element of $\mathcal{T}^*$, the dual of $\mathcal{T}$, and consider the character

$$< (q, E, p) \mid (\theta, b, a) >= e^{i(q\theta + Eb + pa)} \ .$$

The action of the subgroup $\mathcal{V}$ on the abelian subgroup $\mathcal{T}$ follows from (2.25)

$$(v, \sigma)(\theta, b, a) = (\theta + M[ve^{\sigma} a + \frac{1}{2} e^{2\sigma} v^2 b], be^{2\sigma}, e^{\sigma} a + e^{2\sigma} vb) \ . \tag{2.67}$$

Now the action of $(v, \sigma) \in \mathcal{V}$ on $(q, E, p) \in \mathcal{T}^*$ is defined by

$$\begin{aligned} & < (v, \sigma)(q, E, p) \mid (\theta, b, a) > \\ & \quad =< (q, E, p) \mid (v, \sigma)^{-1} (\theta, b, a) > \\ & \quad =< (q, E, p) \mid (\theta + M[-va + \frac{1}{2} v^2 b], e^{-2\sigma} b, e^{-\sigma}(a - vb)) > \\ & \quad = e^{i[q\theta + (e^{-2\sigma} E - e^{-\sigma} pv + \frac{1}{2} qMv^2) b + (e^{-\sigma} p - qMv) a]} \end{aligned} \tag{2.68}$$

Thus dual orbit elements $(\bar{q}, \bar{E}, \bar{p})$ corresponding to a fixed value of $(q, E, p)$ are given by

$$\bar{q} = q$$

$$\begin{aligned} \bar{E} &= e^{-2\sigma}E - e^{-\sigma}pv + \frac{1}{2}qMv^2 \\ \bar{p} &= e^{-\sigma}p - qMv \ , \end{aligned} \tag{2.69}$$

so that,

$$\bar{E} - \frac{\bar{p}^2}{2\bar{q}M} = e^{-2\sigma}(E - \frac{p^2}{2qM}) \tag{2.70}$$

where we assume that $q \neq 0$. Since $q$ remains invariant under the transformation (2.69), we take $\bar{q} = q = \kappa$. We thus get two dual orbits, the interior and exterior of the parabola given by $E - \frac{p^2}{2\kappa M} = 0$, lying on the two-dimensional plane determined by $q = \kappa$ in the $\bar{q}$-$\bar{E}$-$\bar{p}$ space. The parabola $E - \frac{p^2}{2\kappa M} = 0$ itself determines an orbit and there are additional orbits when $q = 0$. Here we shall only consider the first two orbits, i.e., the interior and exterior of the parabola, for each non-zero $\kappa \in \mathbb{R}$. Let us introduce the new variables

$$\begin{aligned} p &= k_1 \\ E - \frac{p^2}{2\kappa M} &= k_2 \end{aligned} \tag{2.71}$$

Then, for fixed value of $q = \kappa$, the coordinates $(k_1, k_2)$ are easily seen to transform as

$$\begin{aligned} \bar{k}_1 &= e^{-\sigma}k_1 - \kappa Mv \\ \bar{k}_2 &= e^{-2\sigma}k_2 \end{aligned} \tag{2.72}$$

In these new coordinates,

$$(v, \sigma)(q, k_1, k_2) = (q, e^{-\sigma}k_1 - qMv, e^{-2\sigma}k_2) \ ,$$

and

$$(v,\sigma)^{-1}(k_1,k_2) = (e^{\sigma}(k_1+\kappa Mv), e^{2\sigma}k_2) := (k_1', k_2')\ , \tag{2.73}$$

so that,

$$\begin{aligned} k_1' &= e^{\sigma}(k_1+\kappa Mv) \\ k_2' &= e^{2\sigma}k_2\ . \end{aligned}$$

Therefore we obtain

$$dk_1'\ dk_2' = e^{3\sigma}dk_1\ dk_2 \tag{2.74}$$

Using again the method of induced representations, we arrive at the two UIRs of $\mathcal{G}_s^M$ defined on the two Hilbert spaces $L^2(\mathbb{R}\times\mathbb{R}^{\pm}, dk_1\ dk_2)$, for each non-zero value of $q=\kappa$,

$$(U_{\pm}^{\kappa}(\theta,b,a,v,\sigma)\hat{\psi})(k_1,k_2) = e^{\frac{3\sigma}{2}}e^{i(\kappa\theta+k_1a+\{k_2+\frac{(k_1)^2}{2\kappa M}\}b)}\hat{\psi}(e^{\sigma}(k_1+\kappa Mv), e^{2\sigma}k_2)\ . \tag{2.75}$$

Let us now go back to the Heisenberg group $G_{\text{H}}$, as discussed in Section 2.5.1 and construct its unitary irreducible representations, following similar techniques. From the matrix representation in (2.37) we infer the semidirect product structure,

$$G_{\text{H}} = \mathcal{T} \rtimes \mathcal{A}$$

where $(\theta,q)$ constitute elements of the abelian subgroup $\mathcal{T}$ and $p$ is an element of the subgroup $\mathcal{A}$. Now $p\in\mathcal{A}$ acts on $(\theta,q)\in\mathcal{T}$ in the following manner

$$p(\theta,q) = (\theta+pq, q) \tag{2.76}$$

We now denote by $(s,t)$ a geneirc element of $\mathcal{T}^*$, the dual of the abelian subgroup $\mathcal{T}$. Let us take the character

$$< (s,t) \mid (\theta, q) >= e^{i(s\theta + tq)};$$

then

$$\begin{aligned} < p(s,t) \mid (\theta,q) > &= < (\bar{s},\bar{t}) \mid (\theta,q) > \\ &= e^{i(\bar{s}\theta+\bar{t}q)} \\ &= < (s,t) \mid p^{-1}(\theta,q) > \\ &= < (s,t) \mid (\theta - pq, q) > \\ &= e^{i[s\theta+(t-sp)q]} \end{aligned} \tag{2.77}$$

For fixed $(s,t)$ the coordinates of its orbit orbits under the action of $\mathcal{A}$ are

$$\begin{aligned} \bar{s} &= s \\ \bar{t} &= t - sp \end{aligned} \tag{2.78}$$

Thus, the dual orbits are a family of parallel straight lines, one for each value of $s$ and $dt$ is the invariant measure on the orbit. Once again, using Mackey's theory of induced representation we obtain the UIR, corresponding to each dual orbit, i.e., for each fixed value of $s$:

$$(U^s_{\mathrm{H}}(\theta,q,p)\hat{\psi})(t) = e^{is\theta}e^{itq}\hat{\psi}(t+sp) \ , \tag{2.79}$$

on the Hilbert space $L^2(\mathbb{R}, \ dt)$.

Now the restriction of the UIR (2.75) of the centrally extended Galilei-Schrödinger group $\mathcal{G}_s^M$ to the Heisenberg group $G_{\text{H}}$ is seen to be

$$(U_{\pm}^{\kappa}\mid_{\text{H}}(\theta, 0, a, v, 0)\hat{\psi})(k_1, k_2) = e^{i(\kappa\theta + k_1 a)}\hat{\psi}(k_1 + \kappa M v, k_2) \tag{2.80}$$

Thus,

$$U_{\pm}^{\kappa}\mid_{\text{H}} = U_{\text{H}}^{\kappa} \otimes I_{\pm} \tag{2.81}$$

where $U_{\text{H}}^{\kappa}$ is the unitary irreducible representation of the Heisenberg group on $L^2(\mathbb{R}, dk_1)$ and $I_{\pm}$ are the identity operators on $L^2(\mathbb{R}^{\pm}, dk_2)$. Once again we can decompose this representation as an infinite direct sum of irreducibles,

$$U_{\pm}^{\kappa}\mid_{\text{H}} = \oplus_{n=0}^{\infty} U_{\kappa}^{\pm,\,n} \; .$$

just as in (2.66). Here each $U_{\kappa}^{\pm,\,n}$ is a copy of the UIR (2.79) with $s = \kappa$ on the Hilbert space $L^2(\mathbb{R}, dk_1)$ times a trivial representation on a one dimensional subspace of $L^2(\mathbb{R}^{\pm},\ dk_2)$.

We also recall that in Section 2.5.1 we obtained the Weyl-Heisenberg group $G_{\text{WH}}$ as a subgroup of the centrally extended Galilei-Schrödinger group $\mathcal{G}_s^{M\prime}$. We could just as well have obtained similar representations of $G_{\text{WH}}$ and their decomposition into irreducibles from the UIR's of $\mathcal{G}_s^{M\prime}$.

### 2.6.4 UIRs of cenrally extended (trivial) Galilei-Schrödinger group $\mathcal{G}_s^T$ restricted to connected Stockwell group

In Section 2.4.2 we had introduced the Galilei-Schrödinger group $\mathcal{G}_s$, by setting $\tau = 2\sigma$ in the affine Galilei group (see (2.9)). Later we obtained a central extension of it using the trivial exponent $\xi_2$ in (2.32). Here we shall obtain UIRs of this centrally extended group by first finding unitary irreducible representations of $\mathcal{G}_s$ itself. The matrix representation of $\mathcal{G}_s$ is found by substituting $\tau = 2\sigma$ in (2.9):

$$(b, a, v, \sigma)_s = \begin{bmatrix} e^{\sigma} & ve^{2\sigma} & a \\ 0 & e^{2\sigma} & b \\ 0 & 0 & 1 \end{bmatrix} . \tag{2.82}$$

From this follows the semi-direct product structure, $\mathcal{G}_s = \mathcal{T} \rtimes \mathcal{V}$ where the abelian subgroup $\mathcal{T}$ consists of elements $(b, a)$ and the subgroup $\mathcal{V}$ consists of the elements $(v, \sigma)$.

Now let $(E, p)$ denote a generic element of $\mathcal{T}^*$, the dual to $\mathcal{T}$, and consider the corresponding character

$$< (E, p); (b, a) >= e^{i(Eb+pa)} .$$

The action of the subgroup $\mathcal{V}$ on the abelian subgroup $\mathcal{T}$ can be immediately read off. We find,

$$(v, \sigma)^{-1}(b, a) = (e^{-2\sigma}b, e^{-\sigma}(a + vb)) ,$$

and the action of $(v, \sigma) \in \mathcal{V}$ on $(E, p) \in \mathcal{T}^*$:

$$< (v, \sigma)(E, p); (b, a) >= e^{i[(e^{-2\sigma}E+e^{-\sigma}pv)b+e^{-\sigma}pa]}$$

Thus, writing

$$(v, \sigma)^{-1}(E, p) = (E', p')$$

we get the equations for the dual orbit, corresponding to $(E, p)$

$$\begin{aligned} E' &= e^{2\sigma}(E + pv) \\ p' &= pe^{\sigma} \end{aligned} \tag{2.83}$$

We shall only consider orbits for which $p \neq 0$. Making a change of variables $(E, p) \mapsto (t = \frac{E}{p^2}, p)$, the orbit equations become

$$\begin{aligned} t' &= t + \frac{v}{p} \\ p' &= pe^{\sigma} \end{aligned} \tag{2.84}$$

Thus we get two orbits in the $t$-$p$ space, namely, the two disjoint open half planes ($p \gtrless 0$). Also,

$$dt'\, dp' = e^{\sigma}\, dt\, dp \tag{2.85}$$

Again, following the standard Mackey construction we get the following two unitary irreducible representations of the ordinary Galilei-Schrödinger group, corresponding to these two orbits $\mathbb{R} \times \mathbb{R}^{\pm}$ in the $t$-$p$ space:

$$(U^{\pm}(b, a, v, \sigma)\hat{\psi})(t, p) = e^{i(tp^2 b + pa)} e^{\frac{\sigma}{2}} \hat{\psi}(t + \frac{v}{p}, e^{\sigma} p) \tag{2.86}$$

The representations are carried by the Hilbert spaces $L^2(\mathbb{R} \times \mathbb{R}^{\pm},\ dt\ dp)$, respectively.

In Section 2.4.2 the trivial exponent $\xi_2$ was shown to arise from the continuous function $\zeta_T : \mathcal{G}_s \to \mathbb{R}$ given by

$$\zeta_T(g) = ae^{-\sigma} \ . \tag{2.87}$$

where $g \equiv (b, a, v, \sigma)$ is a generic element of $\mathcal{G}_s$. In terms of this continuous function it follows immediately that $\tilde{U}^{\pm}(g) = e^{i\zeta_T(g)} U^{\pm}(g)$ are projective representations of the Galilei-Schrodinger group $\mathcal{G}_s$. In other words, $U_s^{T,\pm}(\theta, b, a, v, \sigma) := e^{i\theta}\tilde{U}^{\pm}(b, a, v, \sigma)$ are unitary irreducible representations of the trivial central extension $\mathcal{G}_s^T$ of the Galilei-Schrödinger group.

Next the UIRs $U_s^{T,\pm}$ restricted to the connected Stockwell group have the form

$$(U_s^{T,\pm}\mid_{\text{sw}} (\theta, 0, a, 0, \sigma)\hat{\psi})(t, p) = e^{i(\theta + ae^{-\sigma})} e^{ipa} e^{\frac{\sigma}{2}} \hat{\psi}(t, e^{\sigma} p) \tag{2.88}$$

Thus,

$$U_s^{T,\pm}\mid_{\text{sw}} = I \otimes U_{\text{sw}}^{\pm} \tag{2.89}$$

where $I$ is the identity operator on $L^2(\mathbb{R}, dt)$ and $U_{\text{sw}}^{\pm}$ are UIRs of the connected Stockwell group on $L^2(\mathbb{R}^{\pm},\ dt)$. The representation (2.89) again decomposes in the usual manner into an infinite direct sum of irreducibles.

We remark here that the UIRs of the Stockwell group $G_{\text{sw}}$ are not square-integrable (over the whole group). However, since taking $\theta = 0$ in (2.88) yields a projective representation of the affine group, the two non-trivial representaions of which are both square-integrable, this fact can be exploited to arrive at square-integrability over the homogeneous space $G_{\text{sw}}/\Theta$, where $\Theta$ is the phase subgroup. This is exactly the sense in which square-integrability for representations of the Stockwell group has been defined in [40] and is in accordance with the theory of square-integrability modulo subgroups (see, for example [4]).

To summarize, in this chapter, we studied the structures of various groups of interest in signal analysis and image processing. We also studied structures of various groups obtained from (1+1) Galilei group using central and non-central extensions. The structural similarities of these two sets of groups were exhibited in diagram 2–1. The diamond shaped box, representing (1+1)-affine Galilei group $\mathcal{G}_{\text{aff}}$ in this diagram, plays a significant role in analysing these structural similarities. The next chapter is devoted to the study of a certain non-central extension of $\mathcal{G}_{\text{aff}}$, its Wigner function and a comparative study of this function with those associated with centrally extended (1+1) Galilei group $\mathcal{G}^m$.

# Chapter 3
# Coadjoint Orbits and Wigner functions of (1+1)-Extended Affine Galilei Group and Galilei Group

The contents of this chapter are taken from the article titled "Coadjoint Orbits and Wigner functions of (1+1)-Extended Affine Galilei Group and Galilei Group" [18]. Here, we study the coadjoint orbits of the noncentrally extended (1+1)-affine Galilei group and compute the relevant Wigner functions built on them explicitly. We consider the centrally extended (1+1)-Galilei group and study its coadjoint orbits in the second half of the chapter. We also compute the Wigner functions built on the corresponding coadjoint orbits subsequently. Finally, a comparative study of the structure of the coadjoint orbits and corresponding Wigner functions between the extended (1+1)-affine Galilei group and the centrally extended (1+1)-Galilei group is presented along with possible physical interpretations.

## 3.1 Introduction

(1+1)- Galilei group $\mathcal{G}_0$ is the Kinematical group of non relativistic spacetime of dimension (1+1). In [8] an extension of (n+1)-Galilei group by the two dimensional dilation group $\mathcal{D}_2$ (independent space and time dilations) has been considered for $n \geqslant 3$. The resulting extended group is referred to as the (n+1)-affine Galilei group in the literature. We follow the similar construction to obtain (1+1)-affine Galilei group $\mathcal{G}_{\text{aff}}$.

This group has profound significance in signal analysis and image processing [20]. But this group does not seem to have any quantum mechanical feature associated with it. In order to have a well defined quantum mechanical feature we have to consider the projective representation of the underlying group. In other words, we have to find a nontrivial central extension of the given group and consider the true reprsentations of the centrally (nontrivial) extended group. But as in the higher dimensional case (3 or more) [8], one could show that the straightforward central extension of $\mathcal{G}_{\text{aff}}$ fails to generate the mass of the nonrelativistic spinless particle under the stated symmetry. This problematic feature was remedied by the two step construction of a noncentral extension of (1+1)-affine Galilei group $\mathcal{G}_{\text{aff}}$. First taking the central extension of the (1+1)-Galilei group $\mathcal{G}_0$, and then taking the semidirect product of the resulting extended group $\mathcal{G}^m$ with the two dimensional dilation group $\mathcal{D}_2$. In this way, we arrive at the group $\mathcal{G}_{\text{aff}}^m = \mathcal{G}^m \rtimes \mathcal{D}_2$. It is to be noted that the group so obtained is a noncentral extension of the (1+1)-affine Galilei group $\mathcal{G}_{\text{aff}}$.

### 3.2 Wigner functions of (1+1)-extended affine Galilei group

The group $\mathcal{G}_{\text{aff}}^m$ is defined by the following continuous transformation

$$\begin{aligned} x &\mapsto e^{\sigma}x + e^{\tau}vt + a \\ t &\mapsto e^{\tau}t + b \end{aligned}$$

where in addition to the parameters $(b, a, v)$ of the (1+1) dimensional Galilei group $\mathcal{G}_0$ we have two more parameters $\sigma, \tau \in \mathbb{R}$ representing independent space and time dilations respectively. So a generic element $g$ of the (1+1) dimensional affine Galilei group is represented as $(b, a, v, \sigma, \tau)$.

On the other hand, a generic element of (1+1) dimensional extended Galilei group $\mathcal{G}_{\text{aff}}^m$ is represented as $(\theta, b, a, v, \sigma, \tau)$ obeying the following group composition law

$$
\begin{aligned}
&(\theta, b, a, v, \sigma, \tau)(\theta', b', a', v', \sigma', \tau') \\
&\qquad = \theta + e^{2\sigma-\tau}\theta' + m[e^{\sigma}va' + \frac{1}{2}e^{\tau}v^2 b'], b + e^{\tau}b', a + e^{\tau}b'v + e^{\sigma}a', v + e^{\sigma-\tau}v', \\
&\qquad\qquad \sigma + \sigma', \tau + \tau')
\end{aligned}
$$

In this section we will study various coadjoint orbits of $\mathcal{G}_{\text{aff}}^m$ and develop the required tools to compute the Wigner functions built on them.

### 3.2.1 Dual orbits of (1+1)-extended affine Galilei group

An element $(\theta, b, a, v, \sigma, \tau)$ of $\mathcal{G}_{\text{aff}}^m$ can be represented by the following matrix

$$
(\theta, b, a, v, \sigma, \tau) = \begin{bmatrix} e^{\sigma} & ve^{\tau} & 0 & a \\ 0 & e^{\tau} & 0 & b \\ mve^{\sigma} & \frac{1}{2}mv^2e^{\tau} & e^{2\sigma-\tau} & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.1}
$$

where the group multiplication now reduces to matrix multiplication for the matrices given by (3.1). It can easily be seen that $\mathcal{T} = (\theta, b, a, 0, 0, 0) \sim \mathbb{R}^3$ and $\mathcal{V} = (0, 0, 0, v, 0, 0) \sim \mathbb{R}$ are abelian subgroups of $\mathcal{G}_{\text{aff}}^m$. In terms of these two abelian subgroups, $\mathcal{G}_{\text{aff}}^m$ can be written as $\mathcal{G}_{\text{aff}}^m = \mathcal{T} \rtimes (\mathcal{V} \rtimes \mathbb{R}^2)$. Now we proceed to find the dual orbits of $\mathcal{G}_{\text{aff}}^m$ under $H = \mathcal{V} \rtimes \mathbb{R}^2$ in

$\mathcal{T}^* \sim \hat{\mathbb{R}}^3$. If we denote by $(q, E, p)$ a generic element of $\mathcal{T}^*$ then the action of $(v, \sigma, \tau) \in H$ on $(q, E, p)$ is found to be

$$(v, \sigma, \tau)(q, E, p) = (e^{\tau-2\sigma}q, e^{-\tau}E + e^{-\sigma}pv + \frac{1}{2}qme^{\tau-2\sigma}v^2, e^{-\sigma}p + e^{\tau-2\sigma}qmv). \quad (3.2)$$

The set of all possible triples $(q', E', p') \in \mathbb{R}^3$ such that $(v, \sigma, \tau)(q, E, p) = (q', E', p')$, form the dual orbit due to the element $(q, E, p)$ under $H$ in $\mathbb{R}^3$. So we have to solve the following system of equations for $(q', E', p')$

$$\begin{aligned} q' &= e^{\tau-2\sigma}q \\ E' &= e^{-\tau}E + e^{-\sigma}pv + \frac{1}{2}qme^{\tau-2\sigma}v^2 \\ p' &= e^{-\sigma}p + e^{\tau-2\sigma}qmv \end{aligned} \quad (3.3)$$

From (3.3), for nonzero values of $q$, it follows immediately that

$$\begin{aligned} e^{2\sigma}\frac{q'}{q} &= e^{\tau} \\ E' - \frac{p'^2}{2q'm} &= e^{-\tau}(E - \frac{p^2}{2mq}) \end{aligned}$$

which in turn reflects the fact that the signs of both $q$ and $E - \frac{p^2}{2qm}$ are invariants on the same orbit. For different values of $q$, $p$, $E$, and $E - \frac{p^2}{2qm}$ we have eleven possible orbits as outlined in the following table.

The first four orbits $\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2, \hat{\mathcal{O}}_3$, and $\hat{\mathcal{O}}_4$ of Table 1 listed above are three dimensional regions depicted in Figure 3–1. $\hat{\mathcal{O}}_5$ and $\hat{\mathcal{O}}_6$ are the two dimensional surfaces described in Figure 3–2, while $\hat{\mathcal{O}}_7$ and $\hat{\mathcal{O}}_8$ are the two half-planes $\mathbb{R} \times \mathbb{R}^{>0}$ $(p > 0)$ and $\mathbb{R} \times \mathbb{R}^{<0}$ $(p < 0)$

Table 3–1: All possible orbits of $\mathcal{G}_{\text{aff}}^m$ in $\mathbb{R}^3$ under $H = \mathcal{V} \rtimes \mathbb{R}^2$

| Orbits | $q$ | $p$ | $E$ | $E - \frac{p^2}{2qm}$ |
|---|---|---|---|---|
| $\hat{\mathcal{O}}_1$ | $> 0$ | $-$ | $-$ | $> 0$ |
| $\hat{\mathcal{O}}_2$ | $> 0$ | $-$ | $-$ | $< 0$ |
| $\hat{\mathcal{O}}_3$ | $< 0$ | $-$ | $-$ | $< 0$ |
| $\hat{\mathcal{O}}_4$ | $< 0$ | $-$ | $-$ | $> 0$ |
| $\hat{\mathcal{O}}_5$ | $> 0$ | $-$ | $-$ | $= 0$ |
| $\hat{\mathcal{O}}_6$ | $< 0$ | $-$ | $-$ | $= 0$ |
| $\hat{\mathcal{O}}_7$ | $= 0$ | $> 0$ | $\in \mathbb{R}$ | $-$ |
| $\hat{\mathcal{O}}_8$ | $= 0$ | $< 0$ | $\in \mathbb{R}$ | $-$ |
| $\hat{\mathcal{O}}_9$ | $= 0$ | $= 0$ | $> 0$ | $-$ |
| $\hat{\mathcal{O}}_{10}$ | $= 0$ | $= 0$ | $< 0$ | $-$ |
| $\hat{\mathcal{O}}_{11}$ | $= 0$ | $= 0$ | $= 0$ | $-$ |

respectively due to $q = 0$. Now, $\hat{\mathcal{O}}_9$ and $\hat{\mathcal{O}}_{10}$ represent the positive $E$-axis ($\mathbb{R}^{>0}$) and the negative $E$-axis ($\mathbb{R}^{<0}$) respectively. Finally, $\hat{\mathcal{O}}_{11}$ stands for the origin ($q = p = E = 0$). It is interesting to see that the 3 dimensional non-degenerate orbits are disjoint and separated from one another lying in the same half regions (determined by $q > 0$ or $q < 0$) by the degenerate orbits $\hat{\mathcal{O}}_5$ and $\hat{\mathcal{O}}_6$ of Table 3–1. Also, two of them lying in opposite half regions are separated by the two dimensional plane corresponding to $q = 0$ ($\cup_{i=7}^{11}\hat{\mathcal{O}}_i$). Now, it is obvious that the first four orbits are open sets in $\mathbb{R}^3$. And it is easily verified using equation (3) that the set of all $(v, \sigma, \tau) \in \mathbb{R}^3$ such that $(v, \sigma, \tau)(q, E, p) = (q, E, p)$ is trivial, i.e, the element (0,0,0), which in turn implies that the stabilizer subgroup is trivial. So, the first four orbits are indeed open free and the two-dimensional surfaces in Figure 3–2 and the $q = 0$ plane separate these open free orbits.

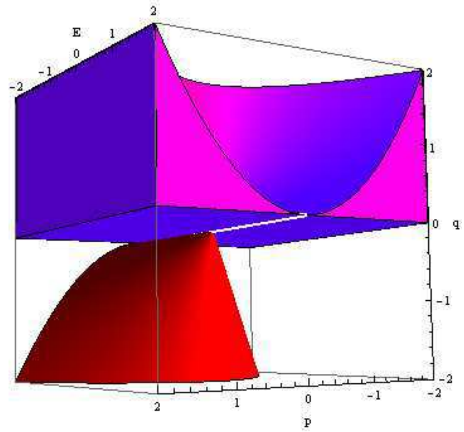



Figure 3–1: The four open free orbits of (1+1) dimensional extended affine Galilei group: the hollow region in the upper half-region ($q > 0$) represents $\hat{\mathcal{O}}_1$, and the filled region in the same half-region represents $\hat{\mathcal{O}}_2$. Similarly the filled region in the lower half-region ($q < 0$) represents $\hat{\mathcal{O}}_3$ and the corresponding hollow region there represents $\hat{\mathcal{O}}_4$.

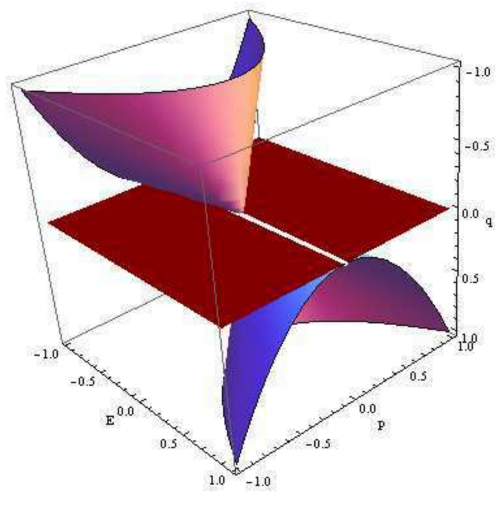



Figure 3–2: The degenerate orbits of (1+1) dimensional affine Galilei group: the two dimensional surface in the upper half-region ($q > 0$) represents $\hat{\mathcal{O}}_5$ and the one underneath ($q < 0$) represents $\hat{\mathcal{O}}_6$. Also, the plane $q = 0$ is the disjoint union of the other degenerate orbits $\hat{\mathcal{O}}_7, \hat{\mathcal{O}}_8, \hat{\mathcal{O}}_9, \hat{\mathcal{O}}_{10}$, and $\hat{\mathcal{O}}_{11}$.

### 3.2.2 Haar measures for the (1+1)-extended affine Galilei group and the corresponding modular function

The group $\mathcal{G}_{\text{aff}}^m$ is non-unimodular, as shown in the following lemma.

**Lemma 3.2.1.** *$\mathcal{G}_{aff}^m$ is non-unimodular and $e^{-4\sigma+\tau}$ is the corresponding modular function. The right invariant Haar measure is simply the Lebesgue measure defined on the underlying group manifold.*

*Proof.* We take a fixed group element $g_0$ and let it act on another element $g$ from the left to obtain the following

$$g_0 g = \begin{bmatrix} e^{\sigma'} & v'e^{\tau'} & 0 & a' \\ 0 & e^{\tau'} & 0 & b' \\ mv'e^{\sigma'} & \frac{1}{2}mv'^2e^{\tau'} & e^{2\sigma'-\tau'} & \theta' \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$\sigma' = \sigma + \sigma_0,\ \tau' = \tau + \tau_0,\ v' = ve^{\sigma_0-\tau_0} + v_0,\ a' = e^{\sigma_0}a + v_0e^{\tau_0}b + a_0, \quad (3.4)$$

$$b' = e^{\tau_0}b + b_0\ \theta' = mv_0e^{\sigma_0}a + \frac{1}{2}mv_0^2e^{\tau_0}b + e^{2\sigma_0-\tau_0}\theta + \theta_0.$$

Therefore, under the left action of a fixed group element $g_0 \equiv (\theta_0, b_0, a_0, v_0, \sigma_0, \tau_0)$, a generic group element $g \in \mathcal{G}_{\text{aff}}^m$ transforms as

$$(\theta, b, a, v, \sigma, \tau) \mapsto (\theta', b', a', v', \sigma', \tau').$$

obeying the system (3.4). Now it follows that

$$d\sigma' = d\sigma,\ d\tau' = d\tau,\ dv' = e^{\sigma_0-\tau_0}dv,\ da' = e^{\sigma_0}da + v_0e^{\tau_0}db,$$

$$db' = e^{\tau_0}db,\ d\theta' = mv_0e^{\sigma_0}da + \frac{1}{2}mv_0^2e^{\tau_0}db + e^{2\sigma_0-\tau_0}d\theta.$$

We can easily see from the above computation that

$$e^{-4\sigma'+\tau'}dv' \wedge da' \wedge db' \wedge d\theta' \wedge d\sigma' \wedge d\tau' = e^{-4\sigma+\tau}dv \wedge da \wedge db \wedge d\theta \wedge d\sigma \wedge d\tau \quad (3.5)$$

Therefore, (3.5) suggests that $e^{-4\sigma+\tau}dv \wedge da \wedge db \wedge d\theta \wedge d\sigma \wedge d\tau$ is the left invariant Haar measure for the (1+1) dimensional extended affine Galilei group.

Now we act with $g_0$ on $g$ from the right:

$$gg_0 = \begin{bmatrix} e^{\sigma'} & v'e^{\tau'} & 0 & a' \\ 0 & e^{\tau'} & 0 & b' \\ mv'e^{\sigma'} & \frac{1}{2}mv'^2e^{\tau'} & e^{2\sigma'-\tau'} & \theta' \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\sigma' = \sigma + \sigma_0,\ \tau' = \tau + \tau_0,\ v' = v + v_0e^{\sigma-\tau},$$

$$a' = e^{\sigma}a_0 + ve^{\tau}b_0 + a,\ b' = e^{\tau}b_0 + b,\ \theta' = mve^{\sigma}a_0 + \frac{1}{2}mv^2e^{\tau}b_0 + e^{2\sigma-\tau}\theta_0 + \theta.$$

So the Lebesgue measures along the group parameters transform in the following manner

$$d\sigma' = d\sigma,\ d\tau' = d\tau,\ dv' = dv + v_0e^{\sigma-\tau}d\sigma - v_0e^{\sigma-\tau}d\tau,$$

$$da' = a_0 e^{\sigma} d\sigma + b_0 e^{\tau} dv + b_0 v e^{\tau} d\tau + da,\ db' = b_0 e^{\tau} d\tau + db,$$

$$d\theta' = m(a_0 e^{\sigma} + b_0 v e^{\tau})dv + (m a_0 v e^{\sigma} + 2\theta_0 e^{2\sigma-\tau})d\sigma + (\frac{1}{2} m b_0 v^2 e^{\tau} - \theta_0 e^{2\sigma-\tau})d\tau + d\theta.$$

It follows immediately that

$$dv' \wedge da' \wedge db' \wedge d\theta' \wedge d\sigma' \wedge d\tau' = dv \wedge da \wedge db \wedge d\theta \wedge d\sigma \wedge d\tau \tag{3.6}$$

Therefore the right invariant Haar measure turns out to be the usual lebesgue measure on the underlying group manifold, which is just $dv \wedge da \wedge db \wedge d\theta \wedge d\sigma \wedge d\tau$. Now (3.5) and (3.6) together imply that the group $\mathcal{G}_{\text{aff}}^m$ is non-unimodular and $e^{-4\sigma+\tau}$ is the required modular function for the underlying group. □

### 3.2.3 Lie algebraic aspects and the coadjoint action matrix of (1+1) dimensional extended affine Galilei group

We first observe that our problem of the (1+1) dimensional extended affine Galilei group fits exactly into the framework of semidirect product groups discused in [35]. We are doing so because our ultimate goal is to construct Wigner map for $\mathcal{G}_{\text{aff}}^m$. Following the matrix representation (3.1) of a generic group element $g \equiv (\theta, b, a, v, \sigma, \tau)$ of $\mathcal{G}_{\text{aff}}^m$ we see that indeed $\mathcal{G}_{\text{aff}}^m = \mathbb{R}^3 \rtimes H$, where $H = \left\{ \begin{bmatrix} e^{\sigma} & ve^{\tau} & 0 \\ 0 & e^{\tau} & 0 \\ mve^{\sigma} & \frac{1}{2}mv^2 e^{\tau} & e^{2\sigma-\tau} \end{bmatrix} | v, \sigma, \tau \in \mathbb{R} \right\}$ is a closed subgroup of $GL(3, \mathbb{R})$. And $(\theta, b, a) \in \mathbb{R}^3$.

From the above discussion we see that $g$ can be written as $(\vec{x}, h)$ where $\vec{x} = \begin{pmatrix} a \\ b \\ \theta \end{pmatrix} \in \mathbb{R}^3$

and $h = \begin{bmatrix} e^{\sigma} & ve^{\tau} & 0 \\ 0 & e^{\tau} & 0 \\ mve^{\sigma} & \frac{1}{2}mv^2 e^{\tau} & e^{2\sigma-\tau} \end{bmatrix} \in H$. And the semidirect product is given by

$$(\vec{x}_1, h_1)(\vec{x}_2, h_2) = (\vec{x}_1 + h_1\vec{x}_2, h_1 h_2).$$

which can be verified by matrix multiplication using (3.1). With the help of the above notations, $g \in \mathcal{G}_{\text{aff}}^m$ can also be expressed in the following block form

$$g = \begin{bmatrix} h & \vec{x} \\ \vec{0}^T & 1 \end{bmatrix}.$$

Now, let us denote by $\mathfrak{g}$ and $\mathfrak{h}$, the lie algebras of the lie groups $\mathcal{G}_{\text{aff}}^m$ and $H$ respectively, where the dimension of $\mathfrak{g}$ is six while that of $\mathfrak{h}$ is three.

Now, we assume that $d\mu_G$ and $d\mu_r$ are the left and right invariant Haar measures for $\mathcal{G}_{\text{aff}}^m$ respectively with $\Delta_G$ and $\Delta_H$ being the corresponding modular functions. Then [35]

$$d\mu_G(\vec{x}, h) = \Delta_G(\vec{x}, h) d\mu_r(\vec{x}, h) = \frac{\Delta_H(h)}{|\text{det} h|} d\mu_r(\vec{x}, h)$$

But in section 3.2.2, we have already found the following measures and the corresponding modular function

$$
\begin{aligned}
d\mu_G(\vec{x}, h) &= e^{-4\sigma+\tau} dv\ da\ db\ d\theta\ d\sigma\ d\tau \\
d\mu_r(\vec{x}, h) &= dv\ da\ db\ d\theta\ d\sigma\ d\tau \\
\Delta_G(\vec{x}, h) &= e^{-4\sigma+\tau}
\end{aligned}
$$

Also, $|\mathrm{det} h| = e^{3\sigma}$. And therefore, we get

$$\Delta_H(h) = e^{-\sigma+\tau}. \tag{3.7}$$

Now, we proceed to find the generators for the group $\mathcal{G}_{\text{aff}}^m$ explcitly and subsequently find a generic group element of $\mathfrak{g}$. Following are the six generators $D_s, K, D_T, X, T, \Theta$ corresponding to $\sigma, v, \tau, a, b$ and $\theta$ respectively,

$$
D_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad
K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad
D_T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad
T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad
\Theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

The commutation relations between the above generators are listed below

$$
\begin{aligned}
&[K, D_T] = K,\ [D_s, D_T] = 0,\ [K, X] = m\Theta,\ [X, T] = 0,\ [K, T] = X \\
&[\Theta, T] = 0,\ [\Theta, K] = 0,\ [\Theta, X] = 0,\ [\Theta, D_T] = \Theta,\ [\Theta, D_s] = -2\Theta \\
&[K, D_s] = -K,\ [T, D_T] = -T,\ [X, D_s] = -X,\ [X, D_T] = 0,\ [T, D_s] = 0.
\end{aligned}
\tag{3.8}
$$

Now a generic algebra element $Y$ can be written as

$$
Y = x_1 D_s + x_2 K + x_3 D_T + x_4 X + x_5 T + x_6 \Theta. \tag{3.9}
$$

In matrix notation,

$$
Y = \begin{bmatrix} x_1 & x_2 & 0 & x_4 \\ 0 & x_3 & 0 & x_5 \\ mx_2 & 0 & 2x_1 - x_3 & x_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.10}
$$

$Y \in \mathfrak{g}$ can conveniently be written as

$$
Y = \begin{bmatrix} X_q & \vec{x_p} \\ \vec{0}^T & 0 \end{bmatrix},
$$

where we let

$$
X_q = \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & x_3 & 0 \\ mx_2 & 0 & 2x_1 - x_3 \end{bmatrix}, \tag{3.11}
$$

and

$$\vec{x_p} = \begin{pmatrix} x_4 \\ \\ x_5 \\ \\ x_6 \end{pmatrix}.$$

We also let

$$\vec{x_q} = \begin{pmatrix} x_1 \\ \\ x_2 \\ \\ x_3 \end{pmatrix},$$

so that the six dimensional column vector $\begin{pmatrix} \vec{x_q} \\ \\ \vec{x_p} \end{pmatrix}$ represents a generic algebra element. How a generic group element $(\vec{x}, h)$ acts on such a six dimensional vector is encoded in the so-called adjoint action matrix while the action of a group element on a dual algebra element (in this case six component row vector) is encapsulated in the coadjoint action matrix. Now we will find the coadjoint action matrix for the group $\mathcal{G}^m_{\text{aff}}$ explicitly. The inverse group element is given by

$$g^{-1} = \begin{bmatrix} e^{-\sigma} & -ve^{-\sigma} & 0 & e^{-\sigma}(vb - a) \\ \\ 0 & e^{-\tau} & 0 & -be^{-\tau} \\ \\ -mve^{\tau-2\sigma} & \frac{1}{2}mv^2 e^{\tau-2\sigma} & e^{\tau-2\sigma} & e^{\tau-2\sigma}(-\theta + mva - \frac{1}{2}mv^2 b) \\ \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.12}$$

The adjoint action of the underlying group on a generic lie algebra element is defined as

$$Ad_g Y = gYg^{-1}.$$

And hence

$$\begin{aligned} Ad_{g^{-1}}Y &= g^{-1}Yg \\ &= \begin{bmatrix} x'_1 & x'_2 & 0 & x'_4 \\ 0 & x'_3 & 0 & x'_5 \\ mx'_2 & 0 & 2x'_1 - x'_3 & x'_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

with

$$x'_1 = x_1,\ x'_2 = ve^{\tau-\sigma}x_1 + e^{\tau-\sigma}x_2 - ve^{\tau-\sigma}x_3,\ x'_3 = x_3,$$

$$x'_4 = e^{-\sigma}ax_1 + e^{-\sigma}bx_2 - e^{-\sigma}vbx_3 + e^{-\sigma}x_4 - ve^{-\sigma}x_5,\ x'_5 = be^{-\tau}x_3 + e^{-\tau}x_5,$$

$$x'_6 = e^{\tau-2\sigma}(2\theta - mva)x_1 + e^{\tau-2\sigma}(ma - mvb)x_2 + e^{\tau-2\sigma}(\frac{1}{2}mv^2b - \theta)x_3$$

$$-mve^{\tau-2\sigma}x_4 + \frac{1}{2}mv^2e^{\tau-2\sigma}x_5 + e^{\tau-2\sigma}x_6.$$

We, therefore, obtain

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \\ x'_5 \\ x'_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ ve^{\tau-\sigma} & e^{\tau-\sigma} & -ve^{\tau-\sigma} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ ae^{-\sigma} & be^{-\sigma} & -vbe^{-\sigma} & e^{-\sigma} & -ve^{-\sigma} & 0 \\ 0 & 0 & be^{-\tau} & 0 & e^{-\tau} & 0 \\ e^{\tau-2\sigma}(2\theta - mva) & e^{\tau-2\sigma}(ma - mvb) & e^{\tau-2\sigma}(\frac{1}{2}mv^2b - \theta) & -mve^{\tau-2\sigma} & \frac{1}{2}mv^2e^{\tau-2\sigma} & e^{\tau-2\sigma} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.$$

So, the coadjoint action matrix for the (1+1) dimensional extended affine Galilei group is given by the following six by six matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ ve^{\tau-\sigma} & e^{\tau-\sigma} & -ve^{\tau-\sigma} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ ae^{-\sigma} & be^{-\sigma} & -vbe^{-\sigma} & e^{-\sigma} & -ve^{-\sigma} & 0 \\ 0 & 0 & be^{-\tau} & 0 & e^{-\tau} & 0 \\ e^{\tau-2\sigma}(2\theta - mva) & e^{\tau-2\sigma}(ma - mvb) & e^{\tau-2\sigma}(\frac{1}{2}mv^2b - \theta) & -mve^{\tau-2\sigma} & \frac{1}{2}mv^2e^{\tau-2\sigma} & e^{\tau-2\sigma} \end{bmatrix}.$$

Now, let us have a closer look at the relevant coadjoint orbits. We already know that the coadjoint orbits are just the cotangent bundles on the dual orbits. So in our case, the non degenerate coadjoint orbits are given by $\mathcal{O}_1^*, \mathcal{O}_2^*, \mathcal{O}_3^*$, and $\mathcal{O}_4^*$, where

$$\begin{aligned} \mathcal{O}_1^* &= T^*\hat{\mathcal{O}}_1 \\ \mathcal{O}_2^* &= T^*\hat{\mathcal{O}}_2 \\ \mathcal{O}_3^* &= T^*\hat{\mathcal{O}}_3 \qquad (3.13) \\ \mathcal{O}_4^* &= T^*\hat{\mathcal{O}}_4. \end{aligned}$$

And the details about $\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2, \hat{\mathcal{O}}_3, \hat{\mathcal{O}}_4$ are outlined in Table 3–1.

Now, let us have a look at how the coadjoint orbits in (3.13) follow directly from the above coadjoint action matrix. Let the group act on a dual algebra element $(0, 0, 0, 0, k_1, k_2)$ via the coadjoint action matrix to give another element of the dual algebra which basically lies in one of the coadjoint orbits. Here it is assumed that $(k_1, k_2) \in \mathbb{R}^2 \setminus (0, 0)$. The resulting element $r$ of a coadjoint orbit is given by

$$r = [e^{\tau-2\sigma}(2\theta - mva)k_2,\ e^{\tau-2\sigma}(ma - mvb)k_2,\ k_1be^{-\tau} + e^{\tau-2\sigma}(\frac{1}{2}mv^2b - \theta)k_2,$$
$$-mvk_2e^{\tau-2\sigma},\ k_1e^{-\tau} + \frac{1}{2}mv^2k_2e^{\tau-2\sigma},\ k_2e^{\tau-2\sigma}].$$

The last three coordinates of the above vector are basically the coordinates of a point lying in either of the following three dimensional manifolds $\hat{\mathcal{O}}_1, \hat{\mathcal{O}}_2, \hat{\mathcal{O}}_3$, and $\hat{\mathcal{O}}_4$ depending on the sign of $k_1$ and $k_2$ as we can see from a change of variables

$$\begin{aligned}\hat{k}_1 &= -mvk_2e^{\tau-2\sigma}\\ \hat{k}_2 &= k_2e^{\tau-2\sigma}\\ \hat{k}_3 &= k_1e^{-\tau}.\end{aligned}$$

Further to this, we also see that as $\theta, b, a, v, \sigma, \tau$ run through $\mathbb{R}$ independently, the first three components of $r$ also run through $\mathbb{R}$ independently. Denoting them as $k_1^*, k_2^*$, and $k_3^*$, respectively, we can hence write $r$ in the following manner

$$r = \begin{bmatrix} k_1^* & k_2^* & k_3^* & \hat{k}_1 & \frac{(\hat{k}_1)^2}{2m\hat{k}_2} + \hat{k}_3 & \hat{k}_2 \end{bmatrix}. \tag{3.14}$$

Here, $k_1^*, k_2^*, k_3^*$ are the vector components relating to the fibre part (cotangent space) of the cotangent bundle while $\hat{k}_1, \hat{k}_2, \hat{k}_3$ are those corresponding to the base manifold (the dual orbit in question). As the sign of $k_1$ and that of $k_2$ determine the fact which coadjoint orbit we are in, we would like to denote the underlying non degenerate coadjoint orbits by $\mathcal{O}^*_{k_1,k_2}$. For example, we can take a fixed pair $(k_1, k_2) \in \mathbb{R}^2 \setminus (0, 0)$ such that $k_1, k_2 > 0$ and act the codjoint action matrix on an algebra element $(0,0,0,0,k_1,k_2)$ to get the coadjoint orbit $\mathcal{O}_1^*$ as each of the group parameters $\theta, b, a, v, \sigma$ and $\tau$ vary in $\mathbb{R}$. We can write the above fact as $\mathcal{O}^*_{\vec{K}_j^T} = T^*\hat{\mathcal{O}}_{\vec{K}_j^T}$ where $\vec{K}_j^T = (0,k_1,k_2)$. We can conveniently choose an ordered pair $(1,1)$ for the value of $(k_1,k_2)$ to generate the coadjoint orbit $\mathcal{O}_1^*$, i.e. $\mathcal{O}_1^* = \mathcal{O}^*_{\vec{K}_1^T} = T^*\hat{\mathcal{O}}_{\vec{K}_1^T}$, where $\vec{K}_1^T = (0,1,1)$. We can go on to find the other coadjoint orbits by suitably choosing an ordered pair for the value of $(k_1,k_2)$. Following is the table describing how we obtain different coadjoint orbits due to different signs of the non-zero components of the element $\begin{bmatrix} 0 & 0 & 0 & 0 & k_1 & k_2 \end{bmatrix}$ lying in the dual algebra along with the corresponding representative vectors in $\hat{\mathbb{R}}^3$. The coadjoint orbits in the following table are also expressed in terms of the dual orbits described in Table 3–1.

Table 3–2: Classification of the coadjoint orbits $\mathcal{O}^*_{\vec{K}_j^T}$ depending on the signs of the components of the vector $\vec{K}_j^T = (0,k_1,k_2)$.

| $\mathcal{O}^*_{\vec{K}_j^T}$ | $k_1$ | $k_2$ | representative vector $\vec{K}_j^T$ |
|---|---|---|---|
| $\mathcal{O}^*_{\vec{K}_1^T} := \mathcal{O}_1^* = T^*\hat{\mathcal{O}}_1$ | $> 0$ | $> 0$ | $\vec{K}_1^T = (0,1,1)$ |
| $\mathcal{O}^*_{\vec{K}_2^T} := \mathcal{O}_2^* = T^*\hat{\mathcal{O}}_2$ | $< 0$ | $> 0$ | $\vec{K}_2^T = (0,-1,1)$ |
| $\mathcal{O}^*_{\vec{K}_3^T} := \mathcal{O}_3^* = T^*\hat{\mathcal{O}}_3$ | $< 0$ | $< 0$ | $\vec{K}_3^T = (0,-1,-1)$ |
| $\mathcal{O}^*_{\vec{K}_4^T} := \mathcal{O}_4^* = T^*\hat{\mathcal{O}}_4$ | $> 0$ | $< 0$ | $\vec{K}_4^T = (0,1,-1)$ |

### 3.2.4 Necessary ingredients to cook up the Wigner function for (1+1) dimensional extended affine Galilei group

We have alreday noted that $\hat{\mathcal{O}}_{\vec{K}_1^T}, \hat{\mathcal{O}}_{\vec{K}_2^T}, \hat{\mathcal{O}}_{\vec{K}_3^T}$ and $\hat{\mathcal{O}}_{\vec{K}_4^T}$ are the only four non degenerate orbits for $\mathcal{G}^m_{\text{aff}}$ with $\cup_{j=1}^4 \hat{\mathcal{O}}_{\vec{K}_j^T}$ being dense in $\hat{\mathbb{R}}^3$. And also, $\cup_{j=1}^4 T^*\hat{\mathcal{O}}_{\vec{K}_j^T}$ is dense in $\mathbb{R}^6$. In addition to these, the orbits are open free. To be precise, each of the above four dual orbits is an open free H-orbit. We will find an unitarily inequivalent irreducible and square integrable representation due to each such open free orbit. These representations exhaust unitary irreducible representations for the underlying group exactly. And the quasi-regular representation of $\mathcal{G}^m_{\text{aff}} = \mathbb{R}^3 \rtimes H$ turns out to be just a direct sum of these four irreducible representations. We speak about quasi-regular representation beacause the underlying Hilbert space is no longer $L^2(\mathcal{G}^m_{\text{aff}}, d\mu_G(\vec{x}, h))$, rather it is $\mathfrak{H} = L^2(\mathcal{G}^m_{\text{aff}}/H \simeq \mathbb{R}^3, d\theta\ db\ da)$. It is convenient to work in the Fourier transformed space $\hat{\mathfrak{H}} = L^2(\hat{\mathbb{R}}^3, dq\ dE\ dp)$. And the quasi-regular representations in this Fourier-transformed space $\hat{\mathfrak{H}}$ is unitarily equivalent to those defined on the Hilbert space $\mathfrak{H}$. To be precise, the unitary operators $\hat{U}(\vec{x}, h)$ acts on the square integrable functions living in the Hilbert space $\hat{\mathfrak{H}} = L^2(\hat{\mathbb{R}}^3, dq\ dE\ dp)$ in the following manner [35]

$$\begin{aligned}
&(\hat{U}(\vec{x},h)\hat{f})(\begin{bmatrix} p & E & q \end{bmatrix}) \\
&\quad = e^{\frac{3\sigma}{2}} e^{i(q\theta+Eb+pa)} \hat{f}\left( \begin{bmatrix} p & E & q \end{bmatrix} \begin{bmatrix} e^{\sigma} & ve^{\tau} & 0 \\ 0 & e^{\tau} & 0 \\ mve^{\sigma} & \frac{1}{2}mv^2e^{\tau} & e^{2\sigma-\tau} \end{bmatrix} \right) \\
&\quad = e^{\frac{3\sigma}{2}} e^{i(q\theta+Eb+pa)} \hat{f}(\begin{bmatrix} e^{\sigma}(p+qmv) & e^{\tau}(pv+E+\frac{1}{2}qmv^2)e^{\tau} & qe^{2\sigma-\tau} \end{bmatrix}).
\end{aligned} \tag{3.15}$$

If we set $\hat{\mathfrak{H}}_j = L^2(\hat{\mathcal{O}}_{\vec{K}^T_j}, dq\ \ dE\ \ dp)$, we see that each of these spaces is an invariant subspace of $\hat{U}$. And $\hat{U}_j$, the restriction of $\hat{U}$ to the Hilbert space $\hat{\mathfrak{H}}_j$ is irreducible. In other words, we have the following direct sum decomposition of $\hat{U}$

$$\hat{U}(\vec{x}, h) = \oplus_{j=1}^{4} \hat{U}_j(\vec{x}, h). \tag{3.16}$$

And $\hat{\mathfrak{H}}$, the representation space of $\hat{U}(\vec{x}, h)$ decomposes in the following way

$$\hat{\mathfrak{H}} = \oplus_1^4 \hat{\mathfrak{H}}_j. \tag{3.17}$$

Now, that we are done with the business of representations, we move onto computing the Duflo-Moore operator in question. The Duflo-Moore operator pertaining to the open free orbit $\hat{\mathcal{O}}_{\vec{K}_j^T}$ is defined to be [35]

$$(C_j \hat{f})(\vec{k}^T) = (2\pi)^{\frac{n}{2}} [c_j(\vec{k}^T)]^{\frac{1}{2}} \hat{f}(\vec{k}^T) \tag{3.18}$$

where $\hat{f} \in L^2(\hat{\mathcal{O}}_{\vec{K}_j^T}, d\vec{k}^T)$ and $c_j : \hat{\mathcal{O}}_{\vec{K}_j^T} \to \mathbb{R}^+$ is a positive Lebesgue measurable function. Let us compute this measurable function explicitly for $\hat{\mathcal{O}}_{\vec{K}_1^T}$. It is to be noted that this orbit is basically the dual orbit outlined as $\hat{\mathcal{O}}_1$ in Table 3–1. We would have had the measurable functions $c$ to be constant, if $\mathcal{G}_{\text{aff}}^m$ were unimodular. But we will see now that because of the non unimodularity of the (1+1) dimensional extended affine Galilei group we have the function $c_1$ to be unbounded above. We first take an arbitrary element $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \in$ $\hat{\mathcal{O}}_{\vec{K}_1^T}$(because $1^2 - \frac{0^2}{2(m)(1)} > 0$) and let the $3 \times 3$ matrix $h$ introduced in section 3.2.3 act on this element

$$
\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}
\begin{bmatrix} e^{\sigma} & ve^{\tau} & 0 \\ 0 & e^{\tau} & 0 \\ mve^{\sigma} & \frac{1}{2}mv^2e^{\tau} & e^{2\sigma-\tau} \end{bmatrix}
= \begin{bmatrix} mve^{\sigma} & e^{\tau}+\frac{1}{2}mv^2e^{\tau} & e^{2\sigma-\tau} \end{bmatrix} \tag{3.19}
$$

Let us further introduce the following change of variables

$$
\begin{aligned}
mve^{\sigma} &= k_1' \\
e^{2\sigma-\tau} &= k_2' \\
e^{\tau} &= k_3'.
\end{aligned} \tag{3.20}
$$

With the above change of variables the right side of (3.19) takes the form $\begin{bmatrix} k_1' & k_3'+\frac{(k_1')^2}{2mk_2'} & k_2' \end{bmatrix}$. As $k_2'$ and $k_3'$ are always positive, the vector represented by the last matrix definitely lives in $\hat{\mathcal{O}}_{\vec{K}_1^T}$. We, therefore, constructed a homeomorphism from $H$, the underlying closed subspace of $GL(3,\mathbb{R})$ to the dual orbit $\hat{\mathcal{O}}_{\vec{K}_1^T}$. The corresponding lebesgue measures have the following transformation

$$
dk_1'\, dk_2'\, dk_3' = 2me^{3\sigma} dv\, d\sigma\, d\tau. \tag{3.21}
$$

Next, we transfer the left invariant Haar measure $d\mu_H$ from $GL(3,\mathbb{R})$ to $\hat{\mathcal{O}}_{\vec{K}_1^T}$ under the above mentioned homeomorphism.

Now, the left invariant Haar measure on $H$ can be computed to be [4]

$$d\mu_H(h) = e^{-\sigma+\tau} dv\ d\sigma\ d\tau. \tag{3.22}$$

Therefore, we have the following result

$$\begin{aligned} d\mu_H(h) &= e^{-\sigma+\tau} dv\ d\sigma\ d\tau \\ &= \frac{e^{-4\sigma+\tau}}{2m} dk_1'\ dk_2'\ dk_3' \\ &= \frac{1}{2m|k_2'|^2|k_3'|} dk_1'\ dk_2'\ dk_3'. \end{aligned} \tag{3.23}$$

From which follows the expression for the function $c_1$

$$c_1(\begin{bmatrix} k_1' & k_3' + \frac{(k_1')^2}{2mk_2'} & k_2' \end{bmatrix}) = \frac{1}{2m|k_2'|^2|k_3'|}. \tag{3.24}$$

Now the Duflo-Moore operator $C_1$ corresponding to the open free orbit $\hat{\mathcal{O}}_{\vec{K}_1^T}$ immediately follows from (3.18)

$$(C_1\hat{f})(\begin{bmatrix} k_1' & k_3' + \frac{(k_1')^2}{2mk_2'} & k_2' \end{bmatrix}) = \frac{(2\pi)^{\frac{3}{2}}}{(2m)^{\frac{1}{2}}|k_2'||k_3'|^{\frac{1}{2}}} \hat{f}(\begin{bmatrix} k_1' & k_3' + \frac{(k_1')^2}{2mk_2'} & k_2' \end{bmatrix}). \tag{3.25}$$

Now, we want to find the adjoint representation of $\mathfrak{h}$, the lie algebra of $H$. A generic element $X_q \in \mathfrak{h}$ is given by (3.11). The generators $K, D_s, D_T$ as mentioned in Section 3.2.3 form a basis for $\mathfrak{h}$. The corresponding commutation relations along with adjoint representations for the bases are given by

$$[K, K] = 0,\ \ [K, D_s] = -K,\ \ [K, D_T] = K, \tag{3.26}$$

leading to

$$\mathrm{ad}K = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{3.27}$$

Also,

$$[D_s, K] = K, \;\; [D_s, D_s] = 0, \;\; [D_s, D_T] = 0, \tag{3.28}$$

giving

$$\mathrm{ad}D_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.29}$$

Again,

$$[D_T, K] = -K, \;\; [D_T, D_s] = 0, \;\; [D_T, D_T] = 0, \tag{3.30}$$

which in turn gives

$$\mathrm{ad}D_T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.31}$$

Using (3.27), (3.29), and (3.31), we can express ad $\frac{X_q}{2}$, given a generic algebra element $X_q \in \mathfrak{h}$ with $X_q = x_2 K + x_1 D_s + x_3 D_T$, as

$$\mathrm{ad}\frac{X_q}{2} \;\; = \;\; \frac{x_2}{2}\mathrm{ad}K + \frac{x_1}{2}\mathrm{ad}D_s + \frac{x_3}{2}\mathrm{ad}D_T$$

$$= \begin{bmatrix} \frac{x_1-x_3}{2} & 0 & 0 \\ -\frac{x_2}{2} & 0 & 0 \\ \frac{x_2}{2} & 0 & 0 \end{bmatrix}. \tag{3.32}$$

Now, given an $n \times n$ matrix $A$, we define for notational convenience, sinch$A$ as [5]

$$\text{sinch}A = \mathbb{I}_n + \frac{1}{3!}A^2 + \frac{1}{5!}A^4 + \frac{1}{7!}A^6 + \dots \tag{3.33}$$

Using (3.32) and (3.33), we immediately obtain

$$\text{sinch}\ (\text{ad}\ \frac{X_q}{2}) = \begin{bmatrix} \text{sinch}\ (\frac{x_1-x_3}{2}) & 0 & 0 \\ -\frac{x_2}{x_1-x_3}\text{sinch}\ (\frac{x_1-x_3}{2}) + \frac{x_2}{x_1-x_3} & 1 & 0 \\ \frac{x_2}{x_1-x_3}\text{sinch}\ (\frac{x_1-x_3}{2}) - \frac{x_2}{x_1-x_3} & 0 & 1 \end{bmatrix}. \tag{3.34}$$

Therefore we have,

$$\det\ (\text{sinch ad}\ \frac{X_q}{2}) = \text{sinch}\ (\frac{x_1-x_3}{2}). \tag{3.35}$$

We also compute sinch $\frac{X_q}{2}$ and $\frac{1}{\text{sinch}\ \frac{X_q}{2}}$, which are as follows

$$\text{sinch}\left(\frac{X_q}{2}\right)$$

$$= \begin{bmatrix} \text{sinch}(\frac{x_1}{2}) & \frac{x_2}{x_1-x_3}[\text{sinch}(\frac{x_1}{2}) - \text{sinch}(\frac{x_3}{2})] & 0 \\ 0 & \text{sinch}(\frac{x_3}{2}) & 0 \\ \frac{mx_2}{x_1-x_3}[\text{sinch}(x_1 - \frac{x_3}{2}) - \text{sinch}\frac{x_1}{2}] & \frac{mx_2^2}{2(x_1-x_3)^2}[\text{sinch}(\frac{x_3}{2}) + \text{sinch}(x_1 - \frac{x_3}{2}) - 2\text{sinch}(\frac{x_1}{2})] & \text{sinch}(x_1 - \frac{x_3}{2}) \end{bmatrix}.$$

And

$$\frac{1}{\text{sinch}\left(\frac{X_q}{2}\right)}$$
$$= \begin{bmatrix} \frac{1}{\text{sinch}(\frac{x_1}{2})} & \frac{x_2}{x_1-x_3}[\frac{1}{\text{sinch}(\frac{x_1}{2})} - \frac{1}{\text{sinch}(\frac{x_3}{2})}] & 0 \\ 0 & \frac{1}{\text{sinch}(\frac{x_3}{2})} & 0 \\ \frac{mx_2}{x_1-x_3}[\frac{1}{\text{sinch}(x_1-\frac{x_3}{2})} - \frac{1}{\text{sinch}(\frac{x_1}{2})}] & \frac{mx_2^2}{2(x_1-x_3)^2}[\frac{1}{\text{sinch}(\frac{x_3}{2})} + \frac{1}{\text{sinch}(x_1-\frac{x_3}{2})} - \frac{2}{\text{sinch}(\frac{x_1}{2})}] & \frac{1}{\text{sinch}(x_1-\frac{x_3}{2})} \end{bmatrix}. \tag{3.36}$$

A generic lie algebra element $X_q$ of $\mathfrak{h}$ was given by

$$X_q = \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & x_3 & 0 \\ mx_2 & 0 & 2x_1 - x_3 \end{bmatrix}.$$

The entries of $X_q$ are all expressed in terms of $x_1, x_2$ and $x_3$. Now we denote the domain of integration by $\mathcal{D}$ where $\mathcal{D} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 \neq x_3\}$. The reason we exclude the points given by $x_1 = x_3$ is that the nonzero offdiagonal entries of the matrix sinch $\frac{X_q}{2}$ and those of $\frac{1}{\text{sinch}\,(\frac{X_q}{2})}$ all blow up at that point as can easily be verified from (3.36) and (3.36). Also if we take $X_q \in \mathfrak{h}$ such that $(x_1, x_2, x_3) \in \mathcal{D}$, then the exponential map taking the Lie algebra elements to the underlying group manifold is definitely a bijection onto a dense set of the Lie group $H$.

### 3.2.5 Wigner function for the (1+1)-extended affine Galilei group and the domain for the corresponding function

Now, that we have all the essential ingredients, to compute the Wigner function, at our disposal, we go ahead and do it. We will focus on the open free orbit $\hat{\mathcal{O}}_{\vec{K}_1^T}$. We assume both $k_1$ and $k_2$ to be equal to 1 in (3.14) as we are considering the cotangent bundle on the open free orbit $\hat{\mathcal{O}}_{\vec{K}_1^T}$. Finally, suppressing the hats in (3.14), we find that a point on the corresponding six dimensional coadjoint orbit $\mathcal{O}_1^*$ has coordinates $(k_1^*, k_2^*, k_3^*, k_1, k_3 + \frac{(k_1)^2}{2mk_2}, k_2)$ where $k_1^*, k_2^*, k_3^*$ and $k_1$ vary freely in $\mathbb{R}$ while $k_2$ and $k_3$ can assume values only from the positive real axis ($\mathbb{R}^+$). The first three coordinates $(k_1^*, k_2^*, k_3^*)$ correspond to the cotangent space of the underlying cotangent bundle. On the other hand, the last three coordinates correspond to the base manifold of the cotangent bundle, i.e. the open free orbit in question. At this point, we denote the vectors $\begin{bmatrix} k_1^* & k_2^* & k_3^* \end{bmatrix}$ and $\begin{bmatrix} k_1 & k_3 + \frac{k_1^2}{2mk_2} & k_2 \end{bmatrix}$ as $\vec{\gamma}_q^T$ and $\vec{\gamma}_p^T$ respectively.

Now if we denote the Hilbert space of Hilbert-Schmidt operators on $\mathfrak{H} = L^2(\hat{\mathcal{O}}_{\vec{K}_1^T}, \frac{1}{2m|k_2|^2|k_3|}dk_1\ dk_2\ dk_3)$ as $\mathcal{B}_2(\mathfrak{H})$, then the Wigner function due to the corresponding coadjoint orbit is essentially a map

$$W : \mathcal{B}_2(\mathfrak{H}) \to L^2(\mathcal{O}_1^*, \frac{dk_1^*\ dk_2^*\ dk_3^*\ dk_1\ dk_2\ dk_3}{2m|k_2|^2|k_3|}).$$

Since the orbit under study is an open free one, the Wigner function due to the corresponding coadjoint orbit is given by the following formula [35]

$$W(\hat{\phi},\hat{\psi}|(\vec{\gamma}_q^T,\vec{\gamma}_p^T)) = \int_{N_{0q}} d\vec{x}_q e^{-i\vec{\gamma}_q^T\vec{x}_q}\overline{\hat{\psi}\left(\vec{\gamma}_p^T\frac{e^{\frac{X_q}{2}}}{\text{sinch}\frac{X_q}{2}}\right)}\hat{\phi}\left(\vec{\gamma}_p^T\frac{e^{-\frac{X_q}{2}}}{\text{sinch}\frac{X_q}{2}}\right) \times c\left(\vec{\gamma}_p^T\frac{1}{\text{sinch}\frac{X_q}{2}}\right)^{-\frac{1}{2}} c(\vec{\gamma}_p^T)^{-\frac{1}{2}}\left|\frac{\det\left(\text{sinch ad}\frac{X_q}{2}\right)}{\det\left(\text{sinch }\frac{X_q}{2}\right)}\right|^{\frac{1}{2}} \tag{3.37}$$

Now, we apply (3.24), (3.34), (3.35), (3.36) and (3.36) to compute the following

$$c\left(\vec{\gamma}_p^T\frac{1}{\text{sinch}\frac{X_q}{2}}\right)^{-\frac{1}{2}} c(\vec{\gamma}_p^T)^{-\frac{1}{2}}\left|\frac{\det\left(\text{sinch ad}\frac{X_q}{2}\right)}{\det\left(\text{sinch }\frac{X_q}{2}\right)}\right|^{\frac{1}{2}} = \frac{2m|k_2|^2}{\text{sinch }(x_1-\frac{x_3}{2})}\left|\frac{k_3(k_3+r)}{\text{sinch }(\frac{x_3}{2})} - \frac{k_3 r\text{ sinch }(x_1-\frac{x_3}{2})}{\text{sinch}^2(\frac{x_1}{2})}\right|^{\frac{1}{2}} \times\left[\frac{\text{sinch }(\frac{x_1-x_3}{2})}{\text{sinch }(\frac{x_1}{2})\text{ sinch }(\frac{x_3}{2})\text{ sinch }(x_1-\frac{x_3}{2})}\right]^{\frac{1}{2}} \tag{3.38}$$

where $r=\frac{1}{2mk_2}\left(k_1-\frac{mk_2x_2}{x_1-x_3}\right)^2$.

Now, using (3.38) into (3.37), we compute the Wigner function corresponding to one of the coadjoint orbits $\mathcal{O}_1^*$ for (1+1) dimensional extended affine Galilei group explicitly

$$W_1(\hat{\phi},\hat{\psi}|k_1^*,k_2^*,k_3^*,k_1,k_3+\frac{k_1^2}{2mk_2},k_2) = 2m|k_2|^2\int_{\mathcal{D}} dx_1\ dx_2\ dx_3\ e^{-i(x_1k_1^*+x_2k_2^*+x_3k_3^*)}\overline{\hat{\psi}\left(\vec{\gamma}_p^T\frac{e^{\frac{X_q}{2}}}{\text{sinch}\frac{X_q}{2}}\right)}\hat{\phi}\left(\vec{\gamma}_p^T\frac{e^{-\frac{X_q}{2}}}{\text{sinch}\frac{X_q}{2}}\right) \times\left[\frac{\text{sinch}(\frac{x_1-x_3}{2})}{\text{sinch}(\frac{x_1}{2})\text{ sinch}(\frac{x_3}{2})\text{ sinch}^3(x_1-\frac{x_3}{2})}\right]^{\frac{1}{2}}\left|\frac{k_3(k_3+r)}{\text{sinch}(\frac{x_3}{2})}-\frac{k_3r\text{ sinch }(x_1-\frac{x_3}{2})}{\text{sinch}^2(\frac{x_1}{2})}\right|^{\frac{1}{2}} \tag{3.39}$$

In (3.39) $\vec{\gamma}_p^T = \begin{bmatrix} k_1 & k_3 + \frac{k_1^2}{2mk_2} & k_2 \end{bmatrix}$, where $k_1 \in \mathbb{R}$, $k_2 > 0$ and $k_3 > 0$. Also,

$$e^{X_q} = \begin{bmatrix} e^{x_1} & \frac{x_2(e^{x_1}-e^{x_3})}{x_1-x_3} & 0 \\ 0 & e^{x_3} & 0 \\ \frac{mx_2[e^{(2x_1-x_3)}-e^{x_1}]}{x_1-x_3} & \frac{mx_2^2[e^{x_3}-2e^{x_1}+e^{(2x_1-x_3)}]}{2(x_1-x_3)^2} & e^{2x_1-x_3} \end{bmatrix}.$$

As already mentioned, $r = \frac{1}{2mk_2}(k_1 - \frac{mk_2x_2}{x_1-x_3})^2$. And $\frac{1}{\text{sinch}\,(\frac{X_q}{2})}$ is given by (3.36).

Proceeding in the same manner, we can also find the Wigner functions corresponding to the other three coadjoint obits for $\mathcal{G}_{\text{aff}}^m$. Now we will discuss the domain of all four Wigner functions corresponding to various coadjoint orbits of the underlying group.

If we have a look at the most general expression of Wigner function $W_\lambda$ given by (3.37), we immediately see that whether or not this function is supported on the corresponding coadjoint orbit $\mathcal{O}_\lambda^*$ is entirely determined by the fact if the argument of $\hat{\phi}$ and that of $\hat{\psi}$, i.e. $\vec{\gamma}_p^T \frac{e^{\frac{X_q}{2}}}{\text{sinch}\,\frac{X_q}{2}}$ and $\vec{\gamma}_p^T \frac{e^{-\frac{X_q}{2}}}{\text{sinch}\,\frac{X_q}{2}}$ always stay inside the dual orbit $\hat{\mathcal{O}}_\lambda$ or not, which in turn implies that we have to ensure that the vector $\vec{\gamma}_p^T \in \hat{\mathcal{O}}_\lambda$ remains stable under the "sinch$''$ map to have the Wigner function supported on its coadjoint orbit.

Now, the polynomial function $\Delta$ as introduced in [35] reduces, for the (1+1) dimensional affine Galilei group case, to

$$\Delta(\begin{pmatrix} p & E & q \end{pmatrix}) = \det \begin{bmatrix} \begin{pmatrix} p & E & q \end{pmatrix} K \\ \begin{pmatrix} p & E & q \end{pmatrix} D_s \\ \begin{pmatrix} p & E & q \end{pmatrix} D_T \end{bmatrix}. \tag{3.40}$$

Here, $\begin{pmatrix} p & E & q \end{pmatrix} \in \mathbb{R}^3$, where the dual orbits are all embedded. And as we already know,

$$K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ m & 0 & 0 \end{bmatrix}, \ D_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \ D_T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.41}$$

(3.40) now reduces to

$$\begin{aligned} \Delta(\begin{pmatrix} p & E & q \end{pmatrix}) &= \det \begin{bmatrix} qm & p & 0 \\ p & 0 & 2q \\ 0 & E & -q \end{bmatrix} \\ &= q(p^2 - 2mqE) \end{aligned} \tag{3.42}$$

So, in terms of this new polynomial function $\Delta$, we can construct a table for the non degenerate dual orbits, using Table 3–1, which is as follows

We can easily see from the above table that the sign of $\Delta$ changes as we move back and forth between $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$. The same is true for $\hat{\mathcal{O}}_3$ and $\hat{\mathcal{O}}_4$. Now, we take an arbitrary

Table 3–3: Classification of orbits of $\mathcal{G}^m_{\text{aff}}$ in $\mathbb{R}^3$ under $H = \mathcal{V} \rtimes \mathbb{R}^2$

| Orbits | $q$ | $\Delta(\begin{pmatrix} p & E & q \end{pmatrix})$ | $E - p$ relation |
|---|---|---|---|
| $\hat{\mathcal{O}}_1$ | $> 0$ | $< 0$ | $E > \frac{p^2}{2mq}$ |
| $\hat{\mathcal{O}}_2$ | $> 0$ | $> 0$ | $E < \frac{p^2}{2mq}$ |
| $\hat{\mathcal{O}}_3$ | $< 0$ | $< 0$ | $E < \frac{p^2}{2mq}$ |
| $\hat{\mathcal{O}}_4$ | $< 0$ | $> 0$ | $E > \frac{p^2}{2mq}$ |

element $(p_0,\ E_0,\ q_0)$ from one of the nondegenerate dual orbits and then we act $\frac{1}{\text{sinch}\,\frac{X_q}{2}}$ on it to obtain the following vector in $\mathbb{R}^3$,

$$
\begin{aligned}
v = \Bigg( & \frac{p_0}{\text{sinch}(\frac{x_1}{2})} + \frac{m x_2 q_0}{x_1 - x_3}\left[\frac{1}{\text{sinch}(x_1 - \frac{x_3}{2})} - \frac{1}{\text{sinch}(\frac{x_1}{2})}\right], \\
& \frac{p_0 x_2}{x_1 - x_3}\left[\frac{1}{\text{sinch}(\frac{x_1}{2})} - \frac{1}{\text{sinch}(\frac{x_3}{2})}\right] + \frac{E_0}{\text{sinch}(\frac{x_3}{2})} + \frac{m x_2^2 q_0}{2(x_1 - x_3)^2} \\
& \times \left[\frac{1}{\text{sinch}(x_1 - \frac{x_3}{2})} + \frac{1}{\text{sinch}(\frac{x_3}{2})} - \frac{2}{\text{sinch}\,(\frac{x_1}{2})}\right], \frac{q_0}{\text{sinch}(x_1 - \frac{x_3}{2})}\Bigg)
\end{aligned}
$$

We observe that the sign of $q_0$ is invariant under the above transformation. In other words, if we start with a point lying in $\hat{\mathcal{O}}_1$ it can leak into $\hat{\mathcal{O}}_2$ at best. It can never leak down to $\hat{\mathcal{O}}_3$ or to $\hat{\mathcal{O}}_4$ through the $q = 0$ plane. Similarly, if we start with a point in $\hat{\mathcal{O}}_3$ we can end up with a point in $\hat{\mathcal{O}}_4$ under the action of the "$\frac{1}{\text{sinch}}$" map. But the point can never go across the $q = 0$ plane to reach either to $\hat{\mathcal{O}}_1$ or to $\hat{\mathcal{O}}_2$. Next, we compute the polynomial function $\Delta$ given by (3.42) at the point given by the vector $v$ explicitly.

$\Delta(v)$

$$= \det \begin{bmatrix} mq_0 D & p_0 L + \frac{m x_2 q_0}{x_1 - x_3}(D-L) & 0 \\ p_0 L + \frac{m x_2 q_0}{x_1 - x_3}(D-L) & 0 & 2q_0 D \\ 0 & \frac{p_0 x_2}{x_1 - x_3}(L-T) + E_0 T + \frac{m x_2^2 q_0}{2(x_1-x_3)^2}(D+T-2L) & -q_0 D \end{bmatrix} \tag{3.43}$$

where we have assumed the following

$$\begin{aligned} \frac{1}{\text{sinch}\,(\frac{x_1}{2})} &= L \\ \frac{1}{\text{sinch}\,(\frac{x_3}{2})} &= T \\ \frac{1}{\text{sinch}\,(x_1 - \frac{x_3}{2})} &= D, \end{aligned} \tag{3.44}$$

Computing the determinant given by the right side of (3.43) we have,

$$\begin{aligned} \Delta(v) = m^2 q_0^3 D(L^2 - DT)\frac{x_2^2}{(x_1-x_3)^2} - 2mp_0 q_0^2 D(L^2 - DT)\frac{x_2}{x_1 - x_3} \\ + q_0 D(p_0^2 L^2 - 2mq_0 E_0 DT). \end{aligned} \tag{3.45}$$

Setting $\Delta(v)$ to be zero and letting $\frac{x_2}{x_1 - x_3} = s$ we get

$$\begin{aligned} & m^2 q_0^3 (L^2 - DT)\left(s^2 - \frac{2p_0}{mq_0}s\right) + q_0(p_0^2 L^2 - 2mq_0 E_0 DT) = 0 \\ \Rightarrow \quad & m^2 q_0^3 (L^2 - DT)(s - \frac{p_0}{q_0})^2 = q_0 p_0^2 m (L^2 - DT) - q_0(p_0^2 L^2 - 2mq_0 E_0 DT) \\ \Rightarrow \quad & \left(s - \frac{p_0}{q_0}\right)^2 = \frac{DT(2mq_0 E_0 - p_0^2)}{m^2 q_0^2 (L^2 - DT)}. \end{aligned} \tag{3.46}$$

It can be verified that $L^2 - DT > 0$. Also, $DT > 0$. Therefore, what (3.46) tells us is that a point $(p_0,\ E_0,\ q_0)$ in one of the non degenerate dual orbits can only be unstable under the action of "$\frac{1}{\text{sinch}}$" map iff $p_0^2 - 2mq_0E_0 < 0$. But

$$\begin{aligned} & p_0^2 - 2mq_0E_0 < 0 \\ & \Rightarrow (p_0,\ E_0,\ q_0) \in \hat{\mathcal{O}}_1 \text{ or } (p_0,\ E_0,\ q_0) \in \hat{\mathcal{O}}_4, \end{aligned} \tag{3.47}$$

which can easily be seen from Table 3–3. We also find that the points in $\hat{\mathcal{O}}_2$ and those in $\hat{\mathcal{O}}_3$ are all stable under the "$\frac{1}{\text{sinch}}$" map, i.e. they do not leave the corresponding orbits under the action of that map.

The Wigner function corresponding to the coadjoint orbit $\mathcal{O}^*_\lambda$, as a function of $\vec{\gamma}_q^T \in \hat{\mathbb{R}}^3$ can be thought of as the Fourier transform of a function $F(X_q)$

$$W_{\vec{\omega}_0^T}(\hat{\phi}, \hat{\psi}|\mathcal{O}^*_\lambda) = \int_{\mathcal{D}} d\vec{x}_q\ e^{-i\vec{\gamma}_q^T\vec{x}_q} F(X_q), \tag{3.48}$$

where

$$\begin{aligned} F(X_q) &= \overline{\hat{\psi}\left(\vec{\omega}_0^T \frac{e^{\frac{X_q}{2}}}{\text{sinch}\frac{X_q}{2}}\right)}\hat{\phi}\left(\vec{\omega}_0^T \frac{e^{-\frac{X_q}{2}}}{\text{sinch}\frac{X_q}{2}}\right) \\ &\times\ c\left(\vec{\omega}_0^T \frac{1}{\text{sinch}\frac{X_q}{2}}\right)^{-\frac{1}{2}} c(\vec{\omega}_0^T)^{-\frac{1}{2}} \left|\frac{\det\left(\text{sinch ad}\frac{X_q}{2}\right)}{\det\left(\text{sinch }\frac{X_q}{2}\right)}\right|^{\frac{1}{2}}. \end{aligned} \tag{3.49}$$

Also, $\hat{\phi}, \hat{\psi} \in L^2(\hat{\mathcal{O}}_\lambda)$ and $\vec{\omega}_0^T$ is any point from one of the four disjoint nondegenerate dual orbits.

Now, for $\lambda = 2$, if $W_{\vec{\omega}_0^T}(\hat{\phi}, \hat{\psi}|\mathcal{O}_2^*)$ were supported on $\mathcal{O}_2^*$, then $F(X_q)$ would have been identically zero if we chose $\vec{\omega}_0^T \notin \hat{\mathcal{O}}_2$. But we have already seen that if we take $\vec{\omega}_0^T \in \hat{\mathcal{O}}_3$ or $\hat{\mathcal{O}}_4$, $\vec{\omega}_0^T \frac{1}{\text{sinch} \frac{X_q}{2}}$ can never be in $\hat{\mathcal{O}}_2$. On the other hand, (3.47) tells us that if $\vec{\omega}_0^T \in \hat{\mathcal{O}}_1$ we can end up with $\vec{\omega}_0^T \frac{1}{\text{sinch} \frac{X_q}{2}} \in \hat{\mathcal{O}}_2$. In other words, $F(X_q)$ can assume nonzero values when $\vec{\omega}_0^T \in \hat{\mathcal{O}}_1$. Therefore the Wigner function corresponding to $\mathcal{O}_2^*$ will have its support spread on both the coadjoint orbits $\mathcal{O}_1^*$ and $\mathcal{O}_2^*$.

Now, we consider $\lambda = 1$ in equation (45). So, $\hat{\phi}, \hat{\psi} \in L^2(\hat{\mathcal{O}}_1)$. Now, the question that we are going to address is whether we can have $\vec{\omega}_0^T \notin \hat{\mathcal{O}}_1$ such that $\vec{\omega}_0^T \frac{1}{\text{sinch} \frac{X_q}{2}} \in \hat{\mathcal{O}}_1$. It is obvious from our previous discussion that we can not have such a point in $\hat{\mathbb{R}}^3$. Again, for an element $\vec{\omega}_0^T \in \hat{\mathcal{O}}_1$ and $\vec{\omega}_0^T \frac{1}{\text{sinch} \frac{X_q}{2}} \notin \hat{\mathcal{O}}_1$ we have $F(X_q)$ to be identically zero as $\hat{\phi}, \hat{\psi} \in L^2(\hat{\mathcal{O}}_1)$ by assumption. Therefore, we find the support of $W_{\vec{\omega}_0^T}(\hat{\phi}, \hat{\psi}|\mathcal{O}_1^*)$ always lying inside $\mathcal{O}_1^*$.

Using exactly the same arguments we find that the Wigner function $W_{\vec{\omega}_0^T}(\hat{\phi}, \hat{\psi}|\mathcal{O}_4^*)$ is supported inside $\mathcal{O}_4^*$ while the support of $W_{\vec{\omega}_0^T}(\hat{\phi}, \hat{\psi}|\mathcal{O}_3^*)$ is spread out on both the coadjoint orbits $\mathcal{O}_3^*$ and $\mathcal{O}_4^*$.

We, therefore, conclude that the Wigner functions corresponding to the two coadjoint orbits $\mathcal{O}_1^*$ and $\mathcal{O}_4^*$ will have their supports concentrated inside $\mathcal{O}_1^*$ and $\mathcal{O}_4^*$ respectively. It is to be noted that the zero level sets of the polynomial function $\Delta$ introduced earlier in this section, restricted to $\mathcal{O}_1^*$ and $\mathcal{O}_4^*$ are not decomposable into hyperplanes. These two zero-level sets are the two dimensional surfaces in Figure 3–2 (above and below the plane

$q = 0$) correspoding to the degenerate orbits $\hat{\mathcal{O}}_5$ and $\hat{\mathcal{O}}_6$ respectively in Table 3–1. And hence we have verified that the converse of the following theorem due to A. E. Krasowska and S. T. Ali [35] is not true.

**Theorem 3.2.1.** *[35] Let $G$ be a semi-direct product group $\mathbb{R}^n \rtimes H$, such that $H$ acts on $\hat{\mathbb{R}}^n$ with open free orbits $\{\hat{\mathcal{O}}_i^m\}_{i=1}$. If an orbit $\hat{\mathcal{O}}_i$ is a dihedral cone (i.e. if the zero level set of the polynomial function $\Delta$, restricted to it, can be decomposed into hyperplanes) then the Wigner function $W_{\hat{\mathcal{O}}_i}$ has support concentrated on the corresponding coadjoint orbit $\mathcal{O}_i^* = \mathbb{R}^n \times \hat{\mathbb{O}}_i$.*

The sufficient condition for the Wigner function to be supported on one of its coadjoint orbits is that the corresponding dual orbit be a dihedral cone. However, it is not a necessary condition for the Wigner function to have its support inside one of its coadjoint orbits as we can see from the example of (1+1) dimensional extended affine Galilei group.

## 3.3 Wigner function for the (1+1)-centrally extended Galilei group

We extend the (1+1)-Galilei group $\mathcal{G}_0$ centrally using the canonical exponent $\xi(g, g')$ given by

$$\xi_1(g_1, g_2) = m(v_1 a_2 + \frac{1}{2} v_1^2 b_2), \tag{3.50}$$

where $g \equiv (b_1, a_1, v_1)$ and $g_2 \equiv (b_2, a_2, v_2)$ are elements of $\mathcal{G}_0$. The centrally extended Galilei group $\mathcal{G}^m$ then obeys the following group law

$$(\theta_1, b_1, a_1, v_1)(\theta_2, b_2, a_2, v_2)$$

$$= (\theta_1 + \theta_2 + m[v_1 a_2 + \frac{1}{2} v_1^2 b_2], b_1 + b_2, a_1 + a_2 + v_1 b_2, v_1 + v_2). \qquad (3.51)$$

This group $\mathcal{G}^m$ has been called the quantum Galilei group in [20]. Our first goal in this section would be to study this quantum Galilei group $\mathcal{G}^m$ in detail and subsequently to find its coadjoint orbits. As will turn out later in this section that by using the standard procedures [5], one fails to compute the correct Wigner function for $\mathcal{G}^m$ built on the relevant coadjoint orbits. In order to remedy this problem we consider a new exponent $\xi_2$ [37] of the (1+1)-Galilei group $\mathcal{G}_0$ which is equivalent to $\xi_1$ and is given by (3.50),

$$\xi_2(g_1, g_2) = \frac{1}{2} m(-v_1 v_2 b_2 + v_1 a_2 - v_2 a_1). \qquad (3.52)$$

Next, we will extend $\mathcal{G}_0$ centrally using the exponent $\xi_2$ given by (3.52) and denote the resulting group as $\mathcal{G}^{m\prime}$. The group composition law for $\mathcal{G}^{m\prime}$ is as follows

$$(\theta_1, b_1, a_1, v_1)(\theta_2, b_2, a_2, v_2)$$
$$= (\theta_1 + \theta_2 + \frac{1}{2} m(-v_1 v_2 b_2 + v_1 a_2 - v_2 a_1), b_1 + b_2, a_1 + a_2 + v_1 b_2, v_1 + v_2) \qquad (3.53)$$

We will then find the coadjoint action matrix for $\mathcal{G}^{m\prime}$ which will turn out to be exactly the same as to be found for the quantum Galilei group $\mathcal{G}^m$. In other words, the geometry of the coadjoint orbits remains unchanged. And we will arrive at the correct form of Wigner functions using this nontrivial central extension of the (1+1)-Galilei group as we will explore by the end of this section.

### 3.3.1 Dual orbits and the induced representation for the quantum Galilei group $\mathcal{G}^m$

The group element for the (1+1)-centrally extended Galilei group $\mathcal{G}^m$ or the quantum Galilei group is found to be the following

$$g = \begin{bmatrix} 1 & v & 0 & a \\ 0 & 1 & 0 & b \\ mv & \frac{1}{2}mv^2 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.54}$$

which we here denote as $(\theta, b, a, v)$. And the corresponding group multiplication is given by (3.51).

The inverse group element is given by

$$(\theta, b, a, v)^{-1} = (-\theta - \frac{1}{2}mv^2 b + mva, -b, vb - a, -v).$$

In matrix notation

$$(\theta, b, a, v)^{-1} = \begin{bmatrix} 1 & -v & 0 & vb - a \\ 0 & 1 & 0 & -b \\ -mv & \frac{1}{2}mv^2 & 1 & -\theta - \frac{1}{2}mv^2 b + mva \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The (1+1) dimensional extended Galilei group $\mathcal{G}^m$ or the quantum Galilei group can be viewed as $(\Theta \times \mathcal{T} \times \mathcal{S}) \rtimes \mathcal{V}$. The action of a pure Galilian boost $v \in \mathcal{V}$ on the abelian subgroup $(\theta, b, a) \in (\Theta \times \mathcal{T} \times \mathcal{S})$ is computed in the following way

$$(\theta_1, b_1, a_1, v_1)(\theta_2, b_2, a_2, v_2)$$

$$= ((\theta_1, b_1, a_1) + v_1(\theta_2, b_2, a_2), v_1 + v_2).$$

But according to the group multiplication law, given at the beginning the last expression should equal $(\theta_1 + \theta_2 + \frac{1}{2}mv_1^2 b_2 + mv_1 a_2, b_1 + b_2, a_1 + a_2 + v_1 b_2, v_1 + v_2)$, from which it follows that

$$v(\theta, b, a) = (\theta + \frac{1}{2}mv^2 b + mva, b, a + vb)$$

Let us assume that the dual of the abelian subgroup $\Theta \times \mathcal{T} \times \mathcal{S}$ is parametrized by $\gamma$, $E$ and $p$ where $E, p \in \mathbb{R}$ and $\gamma \in \mathbb{R} \setminus \{0\}$. The $\gamma = 0$ case will be handled separately. Now the dual pairing reads

$$\chi_{\gamma,E,p}(\theta, b, a) = \exp[i(\gamma\theta + Eb + pa)]$$

The dual action of $v$ on the character group can be defined by the following relation

$$\langle (v)\chi_{\gamma,E,p}; (\theta, b, a)\rangle = \langle \chi_{\gamma,E,p}; v^{-1}(\theta, b, a)\rangle$$

But it is seen that $v^{-1} = -v$. From which it follows immediately that

$$v^{-1}(\theta, b, a) = (-v)(\theta, b, a)$$

$$= (\theta + \frac{1}{2}mv^2 b - mva, b, a - vb).$$

Now,

$$\chi_{\gamma,E,p}(\theta + \frac{1}{2}mv^2 b - mva, b, a - vb)$$
$$= \exp i[\gamma\theta + \frac{m\gamma v^2}{2}b - m\gamma va + Eb + p(a - vb)]$$

And, therefore it follows that

$$\chi_{\gamma',E',p'}(\theta, b, a) = \mathrm{e}^{i[\gamma'\theta + E'b + p'a]}$$
$$= \mathrm{e}^{i[\gamma\theta + (\frac{m\gamma v^2}{2} + E - pv)b + (p - m\gamma v)a]}$$

So, under the dual group action the variables parameterizing the character group transforms according to the following equations

$$\begin{aligned} \gamma' &= \gamma, \\ E' &= \frac{m\gamma v^2}{2} + E - pv, \\ p' &= p - m\gamma v. \end{aligned} \tag{3.55}$$

We also find $E$ and $p$ to be constrained by an equation which follows from the following computation

$$\begin{aligned} p'^2 &= p^2 - 2m\gamma pv + m^2\gamma^2 v^2 \\ \frac{p'^2}{2m\gamma} &= \frac{p^2}{2m\gamma} - pv + \frac{1}{2}m\gamma v^2. \end{aligned}$$

Using the transformation rule for $E$, we immidiately have,

$$\frac{p'^2}{2m\gamma} = \frac{p^2}{2m\gamma} + E' - E,$$

which we can rewrite as

$$E - \frac{p^2}{2m\gamma} = E' - \frac{p'^2}{2m\gamma} = E_0$$

i.e,

$$E' = E_0 + \frac{p'^2}{2m\gamma}$$

So, the dual action on the character group can conveniently be written as

$$(v)\chi_{\gamma,E,p} = \chi_{\gamma',E',p'} = \chi_{\gamma,\frac{p'^2}{2m\gamma}+E_0,p'} \tag{3.56}$$

For a fixed value of $\gamma$ and that of $E_0$ the orbit is represented by a parabola parallel to the $E'p'$ plane and perpendicular to the $\gamma$-axis. As $\gamma$ varies over $\mathbb{R} \setminus \{0\}$ the parabola changes its shape continuously. Now, for $\gamma = 0$, the dual orbits are computed separately by putting $\gamma$ to be zero in (3.55). The corresponding orbits turn out to be simply one dimensional lines parallel to $E'$-axis lying in the $\gamma = 0$-plane. On the other hand, for $\gamma \neq 0$, the parabolas derived earlier tend to shrink down to lines parallel to $E'$-axis as $\gamma \to 0$. At the other extreme, the parabolas tend to widen with the increase of $|\gamma|$ and are almost lines parallel to the $p'$-axis when $|\gamma|$ is large enough. We will just consider the parabolic orbits arising from $\gamma \neq 0$ and $(E,p) \in \mathbb{R}^2$ in this work since the contribution of the $\gamma = 0$-plane in the representation level will be extremely meager.

Having found the dual orbits for the (1+1)-extended Galilei group or the quantum Galilei group $\mathcal{G}^m$ in (1+1)-dimension, we now proceed to find all the unitary irreducible representations of this group using the method suggested by Mackey.

We have already found that each dual orbit $\mathcal{O}_{\gamma,E_0}$ for the underlying Lie group is parametrized by two numbers $\gamma, E_0$. Now, we choose a representative $\chi_{\gamma,E_0,0}$ from each orbit $\mathcal{O}_{\gamma,E_0}$ for distinct ordered pairs of $(E_0,\gamma) \in \mathbb{R} \times \mathbb{R}^*$. Mackey's inducing construction suggests that to each dual orbit there corresponds a unitarily inequivalent irreducible representation. In other words, for each ordered pair $(E_0,\gamma)$, we shall obtain a unitary irreducible representation $U^{E_0,\gamma}$ and for any two different ordered pairs the representations will be unitarily inequivalent. First, it is evident using (3.55) that the stabilizer subgroup of $\mathcal{V}$ which leaves a particular character group element, say $\chi_{\gamma,E,p}$ dual to $(\theta,b,a) \in \Theta \times \mathcal{T} \times S$, stable, is the trivial identity element (0) of $\mathcal{V}$.

$$((v)\chi_{\gamma,E,p} = \chi_{\gamma,E,p}) \Longrightarrow v = 0,$$

we, therefore, find

$$\mathcal{O}_{\gamma,E_0} \simeq \mathcal{V}/\{\text{Identity element}\} \simeq \hat{\mathbb{R}}.$$

Now, according to the general theory, the irreducible representations of the (1+1) dimensional extended Galilei group $\mathcal{G}^m$, i.e. $(\Theta \times \mathcal{T} \times S) \rtimes \mathcal{V}$ can now be obtained from the UIR's $V_{\gamma,E_0}$ of the subgroup $(\Theta \times \mathcal{T} \times S)$, where

$$V_{\gamma,E_0}(\theta,b,a) \quad = \quad \chi_{\gamma,E_0,0}(\theta,b,a)$$

$$= \exp i[\gamma\theta + E_0 b]. \tag{3.57}$$

Because we are looking for UIR's due to a fixed orbit, we keep both $\gamma$ and $E_0$ fixed in (3.57). Now,

$$\mathcal{V}/\{\text{Id}\} \simeq \mathcal{G}^m/(\Theta \times \mathcal{T} \times S) \simeq \hat{\mathbb{R}}.$$

We define the section $\lambda : \mathcal{G}^m/(\Theta \times \mathcal{T} \times S) \simeq \hat{\mathbb{R}} \to \mathcal{G}^m$ by

$$\lambda(k) = (0, 0, 0, \frac{k}{m})$$

Therefore we have,

$$\begin{aligned}
g^{-1}\lambda(k) &= (-\theta - \frac{1}{2}mv^2 b + mva, -b, vb - a, -v)(0, 0, 0, \frac{k}{m}) \\
&= (-\theta - \frac{1}{2}mv^2 b + mva, -b, vb - a, \frac{k}{m} - v) \\
&= (0, 0, 0, \frac{k}{m} - v)(-\theta - \frac{k^2 b}{2m} + ka, -b, \frac{k}{m}b - a, 0),
\end{aligned}$$

which gives the cocycle $h : \mathcal{G}^m \times \hat{\mathbb{R}} \to \Theta \times \mathcal{T} \times S$ with

$$h(g^{-1}, k) = (-\theta - \frac{k^2 b}{2m} + ka, -b, \frac{k}{m}b - a, 0).$$

Therefore,

$$h(g^{-1}, k)^{-1} = (\theta + \frac{k^2 b}{2m} - ka, b, a - \frac{k}{m}b, 0).$$

Now,

$$V_{\gamma,E_0}(h(g^{-1}, k)^{-1}) = V_{\gamma,E_0}(\theta + \frac{k^2 b}{2m} - ka, b, a - \frac{k}{m}b, 0)$$

$$= \quad \exp i[\gamma(\theta + \frac{k^2 b}{2m} - ka) + E_0 b]$$

Therefore, we obtain the representation $U^{\gamma,E_0}$ of $\mathcal{G}^m$ induced from the UIR $V_{\gamma,E_0}$ of the subgroup $\Theta \times \mathcal{T} \times S$. This representation acts on the Hilbert space $L^2(\hat{\mathbb{R}}, dk)$ in the following way

$$(\hat{U}^{\gamma,E_0}(\theta, b, a, v)\hat{\phi})(k) = \exp i[\gamma(\theta + \frac{k^2 b}{2m} - ka) + E_0 b]\hat{\phi}(k - mv), \tag{3.58}$$

for all $\hat{\phi} \in L^2(\hat{\mathbb{R}}, dk)$. We see that each $(E_0, \gamma) \in \mathbb{R} \times \mathbb{R}^*$ gives rise to a unitary irreducible representation $\hat{U}^{\gamma,E_0}$ of $\mathcal{G}^m$. Further to this, if we take two distinct points $(E_0, \gamma)$ and $(E_0', \gamma')$ in the representation space $\mathbb{R} \times \mathbb{R}^*$ and label the corresponding UIR's as $\hat{U}^{\gamma,E_0}$ and $\hat{U}^{\gamma',E_0'}$ we observe that there does not exist a bounded linear operator $V$ on $L^2(\hat{\mathbb{R}}, dk)$ such that for all $(\theta, b, a, v) \in \mathcal{G}^m$ the following holds

$$V\hat{U}^{\gamma,E_0}(\theta, b, a, v)V^* = \hat{U}^{\gamma',E_0'}(\theta, b, a, v)$$

which implies that the UIR's pertaining to different orbits given by (3.58) are unitarily inequivalent.

### 3.3.2 Coadjoint orbits of $\mathcal{G}^m$ and comparison with the dual orbits found from the Mackey construction

The (1+1) dimensional extended Galilei group $\mathcal{G}^m$ or the quantum Galilei group is a real Lie group. Let us find the corresponding Lie algebra. We denote the group generators with $K, X, T, \Theta$ corresponding to the group parameters $v, a, b, \theta$ respectively. Using the expression for a generic group element in matrix form from (3.54) the group generators are found to be the following

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \Theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

And the corresponding commutation relations are given by

$$[K,X] = m\Theta, \ [X,T] = 0, \ [K,T] = X, \ [\Theta,T] = 0, \ [\Theta,K] = 0, \ [\Theta,X] = 0.$$

A general element of the lie algebra is given by

$$\begin{aligned} Y &= a_1K + a_2X + a_3T + a_4\Theta \\ &= \begin{bmatrix} 0 & a_1 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ ma_1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Now, the adjoint action of the lie group on a generic lie algebra element is as follows

$$\begin{aligned} Ad_gY &= gYg^{-1} \\ &= \begin{bmatrix} 0 & 0 & 0 & -ba_1 + a_2 + va_3 \\ 0 & 0 & 0 & a_3 \\ ma_1 & 0 & 0 & -ma_1 + mva_2 + \frac{1}{2}mv^2a_3 + a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Also, the coadjoint action of the group on the dual algebra is defined by the following relation

$$\langle Ad^{\#}_g(Y^*);Y\rangle = \langle Y^*;Ad_{g^{-1}}(Y)\rangle \tag{3.59}$$

$Ad_{g^{-1}}(Y)$ is found to be

$$Ad_{g^{-1}}(Y) = \begin{bmatrix} 0 & a_1 & 0 & ba_1 + a_2 - va_3 \\ 0 & 0 & 0 & a_3 \\ ma_1 & 0 & 0 & m(a-vb)a_1 - mva_2 + \frac{1}{2}mv^2a_3 + a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, we make the following identifications

$$Y = \begin{bmatrix} 0 & a_1 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ ma_1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$Ad_{g^{-1}}(Y) = \begin{bmatrix} 0 & a_1' & 0 & a_2' \\ 0 & 0 & 0 & a_3' \\ ma_1' & 0 & 0 & a_4' \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} a_1' \\ a_2' \\ a_3' \\ a_4' \end{bmatrix},$$

where,

$$a_1' = a_1,\ a_2' = ba_1 + a_2 - va_3,$$

$$a_3' = a_3,\ a_4' = m(a - vb)a_1 - mva_2 + \frac{1}{2}mv^2a_3 + a_4.$$

And

$$Y^* \to \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}.$$

With the above identifications, we have,

$$\begin{aligned}
\langle Y^*; Ad_{g^{-1}}(Y)\rangle &= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \\ a_3' \\ a_4' \end{bmatrix} \\
&= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & -v & 0 \\ 0 & 0 & 1 & 0 \\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \\
&= \langle Ad_g^{\#}(Y^*); Y\rangle.
\end{aligned}$$

Therefore, the coadjoint action of the underlying Lie group on the generic dual algebra element $Y^*$ is identified with the following matrix

$$M(g^{-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & -v & 0 \\ 0 & 0 & 1 & 0 \\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1 \end{bmatrix} \tag{3.60}$$

Now, we explicitly calculate the coadjoint orbits under the group action on the dual algebra elements $\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}$.

$$\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ b & 1 & -v & 0\\ 0 & 0 & 1 & 0\\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ b & 1 & -v & 0\\ 0 & 0 & 1 & 0\\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1\end{bmatrix} = \begin{bmatrix}b & 1 & -v & 0\end{bmatrix},$$

$$\begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ b & 1 & -v & 0\\ 0 & 0 & 1 & 0\\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix},$$

And

$$\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ b & 1 & -v & 0\\ 0 & 0 & 1 & 0\\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1\end{bmatrix} = \begin{bmatrix}m(a-vb) & -mv & \frac{1}{2}mv^2 & 1\end{bmatrix}.$$

The first and the third orbit are just points in $\mathbb{R}^4$, while the second one traces a two dimensional plane in $\mathbb{R}^4$ as $b, v$ keep on varying on the real line. The fourth orbit can be regarded as the cotangent bundle on a parabola which is again homeomorphic to $\mathbb{R}^2$. So we are basically obtaining two kinds of orbits, namely, one zero dimensional (point) and the other being two dimensional (plane). Now, if we take the dual algebra-element $\begin{bmatrix} 0 & 0 & k_1 & k_2 \end{bmatrix}$ and compute the corresponding coadjoint orbits for $k_1, k_2$, each varying on the real line subject to the condition that they are not both zero, we get a dense subspace in $\mathbb{R}^4$. Let us have a closer look at how this dense subspace looks like.

$$\begin{bmatrix} 0 & 0 & k_1 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & -v & 0 \\ 0 & 0 & 1 & 0 \\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} mk_2(a-vb) & -mvk_2 & k_1 + \frac{1}{2}mv^2k_2 & k_2 \end{bmatrix} \tag{3.61}$$

where $a, v, b$ varies independently on the real line and $k_1, k_2$ also varies independently subject to the condition that they are not both zero. Let $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$ be the basis vectors generating $\mathbb{R}^4$ and also denote the corresponding orthogonal axes by $X, Y, Z, W$. An arbitrary element in $\mathbb{R}^4$ is denoted as $\begin{bmatrix} x & y & z & w \end{bmatrix}$ where $x, y, z,$ and $w$ coordinatise the components along $X, Y, Z,$ and $W$ axes respectively. With this picture in mind, $\mathbb{R}^4$ can be considered as a stack of parallel

$\mathbb{R}^3$- hyperplanes othogonal to $W$-axis. For points in such an $\mathbb{R}^3$- hyperplane $x, y,$ and $z$-coordinates take their values independently on the respective axes while the value of the $w$ coordinate is kept fixed. And, there is a unique $\mathbb{R}^3$-hyperplane which is orthogonal to the $W$-axis and passes through the origin. Any point on this hyperplane is designated by $\begin{bmatrix} x & y & z & 0 \end{bmatrix}$. For the sake of definiteness, we denote this particular $\mathbb{R}^3$-hyperplane by $\mathbb{R}^3_0$.

Now, if we put $k_2$ to be zero in (3.61), we immediately see that the orbit reduces to points consisting of $\begin{bmatrix} 0 & 0 & k_1 & 0 \end{bmatrix}$ for $k_1 \in \mathbb{R} \setminus \{0\}$ which is just the $Z$-axis $\setminus \{0\}$, abbreviated as $Z^*$-axis. Therefore, in view of (3.61), the total orbit space due to dual algebra-elements in the form of $\begin{bmatrix} 0 & 0 & k_1 & k_2 \end{bmatrix}$ where $k_1$ and $k_2$ are not both zero (punctured $k_1$-$k_2$ plane), denoted as $\mathcal{O}^*_{k_1,k_2}$ turns out to be $[(\mathbb{R}^4 \setminus \mathbb{R}^3_0) \cup (Z^*\text{-axis})]$.

Next, we consider dual-algebra elements having the form of $\begin{bmatrix} 0 & k_3 & 0 & 0 \end{bmatrix}$ where $k_3 \in \mathbb{R} \setminus \{0\}$ and compute the corresponding coadjoint orbit space due to the elements of this form

$$\begin{bmatrix} 0 & k_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & -v & 0 \\ 0 & 0 & 1 & 0 \\ m(a - vb) & -mv & \frac{1}{2}mv^2 & 1 \end{bmatrix} = \begin{bmatrix} bk_3 & k_3 & -vk_3 & 0 \end{bmatrix} \tag{3.62}$$

Equation (3.62) determines the fact that the corresponding coadjoint orbit space (the union of all the coadjoint orbits for different nonzero values of $k_3$) denoted as $\mathcal{O}^*_{k_3}$ becomes $[\mathbb{R}^3_0 \setminus (X\text{-}Z \text{ plane})]$.

Now, we consider dual algebra elements of the form $\begin{bmatrix} k_4 & 0 & k_5 & 0 \end{bmatrix}$, where $k_4$ and $k_5$ are not both zero. The coadjoint action gives

$$\begin{bmatrix} k_4 & 0 & k_5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & -v & 0 \\ 0 & 0 & 1 & 0 \\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1 \end{bmatrix} = \begin{bmatrix} k_4 & 0 & k_5 & 0 \end{bmatrix} \tag{3.63}$$

From (3.63), we easily see that the total coadjoint orbit space due to elements of the form $\begin{bmatrix} k_4 & 0 & k_5 & 0 \end{bmatrix}$ denoted as $\mathcal{O}^*_{k_4,k_5}$,where $k_4$ and $k_5$ are not both zero is found to be just the $[X\text{-}Z \text{ plane} \setminus \{0\}]$ which we denote as $(X\text{-}Z \text{ plane})^*$.

Finally, the coadjoint orbit due to the dual-algebra element $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ denoted as $\mathcal{O}^*_0$ is just the origin of $\mathbb{R}^4$, i.e, $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$. The union of the above coadjoint orbits constitutes $\mathbb{R}^4$:

$$\begin{aligned} &\mathcal{O}^*_{k_1,k_2} \cup \mathcal{O}^*_{k_3} \cup \mathcal{O}^*_{k_4,k_5} \cup \mathcal{O}^*_0 \\ &\quad = [(\mathbb{R}^4 \setminus \mathbb{R}^3_0) \cup (Z^*\text{-axis})] \cup [\mathbb{R}^3_0 \setminus (X\text{-}Z \text{ plane})] \cup (X\text{-}Z \text{ plane})^* \cup \{0\} \\ &\quad = \mathbb{R}^4. \end{aligned} \tag{3.64}$$

Now, we consider a subset of $\mathcal{O}^*_{k_1,k_2}$ given in (3.64), say $\mathcal{O}^{*\prime}_{k_1,k_2}$, where instead of puncturing $k_1$-$k_2$ plane we just throw the $k_1$-axis off the $k_1$-$k_2$ plane. In other words, we are interested in the coadjoint orbits due to dual algebra-elements of the form $\begin{bmatrix} 0 & 0 & k_1 & k_2 \end{bmatrix}$ where $k_2$ is never zero. We can then rewrite (3.61) as

$$\begin{aligned}
&\begin{bmatrix} 0 & 0 & k_1 & k_2 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & -v & 0 \\ 0 & 0 & 1 & 0 \\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} mk_2(a-vb) & -mvk_2 & k_1 + \frac{1}{2}mv^2k_2 & k_2 \end{bmatrix} \\
&= \begin{bmatrix} mk_2(a-vb) & -mvk_2 & k_1 + \frac{1}{2mk_2}(-mvk_2)^2 & k_2 \end{bmatrix},
\end{aligned} \tag{3.65}$$

where (3.65) definitely makes sense because $k_2 \neq 0$. Now, we observe that the dual orbits of the underlying Lie group (Mackey orbits) are given by triples of the form $\begin{bmatrix} p' & E_0 + \frac{p'^2}{2m\gamma} & \gamma \end{bmatrix}$ in $p'E'\gamma$ space ($\mathbb{R}^3$), $\gamma$ being unequal to zero. We have already seen in Section 3.3.1 that each $(E_0, \gamma) \in \mathbb{R} \times \mathbb{R}^*$ gives rise to a parabolic dual orbit in $p'E'\gamma$ space, characterized by $\mathcal{O}_{\gamma,E_0}$. The triple $\begin{bmatrix} p' & E_0 + \frac{p'^2}{2m\gamma} & \gamma \end{bmatrix}$ coincides with the last three components of (3.65) under the following identification

$$\begin{aligned}
-mvk_2 &\leftrightarrow p' \\
k_1 &\leftrightarrow E_0. \\
k_2 &\leftrightarrow \gamma
\end{aligned} \tag{3.66}$$

Equation (3.65) represents the cotangent bundle on the family of parabolas given by

$$\begin{bmatrix} -mvk_2 & k_1 + \frac{1}{2mk_2}(-mvk_2)^2 & k_2 \end{bmatrix},$$

where $v_1, k_1 \in \mathbb{R}$ and $k_2 \in \mathbb{R} \setminus \{0\}$ and $\begin{bmatrix} -mvk_2 & k_1 + \frac{1}{2mk_2}(-mvk_2)^2 & k_2 \end{bmatrix}$ represents exactly the Mackey orbits under the identification given in (3.66).

We, therefore, obtain the correspondence of dual orbits of (1+1) dimensional Extended Galilei group with its coadjoint orbits which is encapsulated in the following equation

$$T^*\mathcal{O}_{\gamma,E_0} = \mathcal{O}^{*\prime}_{k_1,k_2} \tag{3.67}$$

### 3.3.3 Invariant measure on $\mathcal{G}^m$ and the Kirillov 2-forms on its nontrivial coadjoint orbits

Let a group element $g$ given in (3.54) be acted upon by a fixed group element $g_0$ from the left to yield the following

$$\begin{aligned} g_0 g &= \begin{bmatrix} 1 & v_0 & 0 & a_0 \\ 0 & 1 & 0 & b_0 \\ mv_0 & \frac{1}{2}mv_0^2 & 1 & \theta_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & v & 0 & a \\ 0 & 1 & 0 & b \\ mv & \frac{1}{2}mv^2 & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & v+v_0 & 0 & a+bv_0+a_0 \\ 0 & 1 & 0 & b+b_0 \\ m(v+v_0) & mvv_0 + \frac{1}{2}m(v^2+v_0^2) & 1 & \theta+\theta_0+mv_0a+\frac{1}{2}mv_0^2 b \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Under the left action of the group the measure $dv \wedge da \wedge db \wedge d\theta$ transforms as

$$\begin{aligned}
dv' \wedge da' \wedge db' \wedge d\theta' &= dv \wedge (da + v_0 db) \wedge db \wedge (d\theta + mv_0 da + \frac{1}{2}mv_0^2 db) \\
&= (dv \wedge da + v_0 dv \wedge db) \wedge db \wedge (d\theta + mv_0 da + \frac{1}{2}mv_0^2 db) \\
&= (dv \wedge da \wedge db) \wedge (d\theta + mv_0 da + \frac{1}{2}mv_0^2 db) \\
&= dv \wedge da \wedge db \wedge d\theta.
\end{aligned}$$

Therefore, it follows immediately that $dv \wedge da \wedge db \wedge d\theta$ is a left invariant Haar measure for the underlying Lie group.

Now, we act $g_0$ on $g$ from the right to obtain the following

$$gg_0 = \begin{bmatrix} 1 & v_0 + v & 0 & a + a_0 + vb_0 \\ 0 & 1 & 0 & b + b_0 \\ m(v + v_0) & mvv_0 + \frac{1}{2}m(v^2 + v_0^2) & 1 & \theta + \theta_0 + mva_0 + \frac{1}{2}mv^2 b_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Under this right action of the group, the measure $dv \wedge da \wedge db \wedge d\theta$ transforms according to

$$\begin{aligned}
dv' \wedge da' \wedge db' \wedge d\theta' &= dv \wedge (da + b_0 dv) \wedge db \wedge (ma_0 dv + mvb_0 dv + d\theta) \\
&= (dv \wedge da \wedge db) \wedge (ma_0 dv + mvb_0 dv + d\theta) \\
&= dv \wedge da \wedge db \wedge d\theta.
\end{aligned}$$

Therefore, $dv \wedge da \wedge db \wedge d\theta$ tuns out to be both a left and right invariant, i.e. an invariant Haar measure for the (1+1)- dimensional Extended Galilei group.

Now, let us take a fixed dual algebra-element $\begin{bmatrix} 0 & k_0 & 0 & 0 \end{bmatrix}$ and find its coadjoint orbits.

$$\begin{bmatrix} 0 & k_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & -v & 0 \\ 0 & 0 & 1 & 0 \\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1 \end{bmatrix} = \begin{bmatrix} bk_0 & k_0 & -vk_0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} a_1 & k_0 & -a_2 & 0 \end{bmatrix},$$

where $k_0 \neq 0$ is fixed, $a, v, b \in \mathbb{R}$ and $a_1 = bk_0$ and $a_2 = vk_0$. The coadjoint orbit is $\mathbb{R}^2$, parameterized by two independent variables $a_1$ and $a_2$. Now if we take a fixed group element and act it on this coadjoint orbit we have,

$$\begin{bmatrix} a_1 & k_0 & -a_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_0 & 1 & -v_0 & 0 \\ 0 & 0 & 1 & 0 \\ m(a_0-vb_0) & -mv_0 & \frac{1}{2}mv_0^2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} a_1 + b_0k_0 & k_0 & -k_0v_0 - a_2 & 0 \end{bmatrix}$$
$$:= \begin{bmatrix} a_1' & k_0 & -a_2' & 0 \end{bmatrix}.$$

We observe that the coadjoint orbit here is stable under the coadjoint action of a fixed group element. Next thing to see is that if we define a two-form on this coadjoint orbit as $da_1 \wedge da_2$

where $a_1$ and $a_2$ have been defined as above, we immediately find it to be invariant under the coadjoint action:

$$da_1' \wedge da_2' = da_1 \wedge da_2.$$

This is the well-known Kirillov 2-form for the coadjoint orbit under study.

Now, we study the same for the other non-trivial coadjoint orbit of the (1+1) dimensional extended Galilei group. The dual algebra element under consideration is now $\begin{bmatrix} 0 & 0 & 0 & k_0 \end{bmatrix}$. Its coadjoint orbit is given by

$$\begin{aligned}
&\begin{bmatrix} 0 & 0 & 0 & k_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & -v & 0 \\ 0 & 0 & 1 & 0 \\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} mk_0(a-vb) & -mvk_0 & \frac{1}{2}mv^2k_0 & k_0 \end{bmatrix} \\
&= \begin{bmatrix} a_1 & a_2 & \frac{a_2^2}{2mk_0} & k_0 \end{bmatrix},
\end{aligned}$$

where $k_0 \neq 0$ is fixed, $a, v, b \in \mathbb{R}$, and $a_1 = mk_0(a - vb)$ and $a_2 = -mvk_0$. Let us find how this two dimensional coadjoint orbit behaves under the coadjoint action of a fixed group element.

$$\begin{bmatrix} a_1 & a_2 & \frac{a_2^2}{2mk_0} & k_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_0 & 1 & -v_0 & 0 \\ 0 & 0 & 1 & 0 \\ m(a_0 - v_0 b_0) & -mv_0 & \frac{1}{2}mv_0^2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + b_0 a_2 + mk_0(a_0 - v_0 b_0) & a_2 - k_0 m v_0 & -a_2 v_0 + \frac{a_2^2}{2mk_0} + \frac{1}{2}mv_0^2 k_0 & k_0 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + b_0 a_2 + mk_0(a_0 - v_0 b_0) & a_2 - k_0 m v_0 & \frac{(a_2 - k_0 m v_0)^2}{2mk_0} & k_0 \end{bmatrix}$$

$$:= \begin{bmatrix} a_1' & a_2' & \frac{(a_2')^2}{2mk_0} & k_0 \end{bmatrix}.$$

From the above computation, we observe that this two dimensional coadjoint orbit (cotangent bundle of a parabola) is also stable under the coadjoint action. And, under the aforementioned definition of $a_1, a_2, a_1'$, and $a_2'$, it turns out that $da_1 \wedge da_2$ is the invariant Kirillov two-form on the coadjoint orbit under study

$$\begin{aligned} da_1' \wedge da_2' &= (da_1 + b_0 da_2) \wedge da_2 \\ &= da_1 \wedge da_2. \end{aligned}$$

### 3.3.4 Duflo-Moore operator and Plancherel measure for $\mathcal{G}^m$

We start with the following orthogonality condition [5]

$$\int_G \{ \int_{\hat{G}} \text{tr}([U^{\gamma, E_0}(x)^* A^1(\gamma, E_0) C_{\gamma, E_0}^{-1}]) d\nu_G(\gamma, E_0)$$
$$\times \int_{\hat{G}} \text{tr}([U^{\gamma', E_0'}(x)^* A^2(\gamma', E_0') C_{\gamma', E_0'}^{-1}]) d\nu_G(\gamma', E_0') \} d\mu(x) = \langle, A^1 | A^2 \rangle_{\mathcal{B}_2^{\oplus}} \quad (3.68)$$

where $\mathcal{B}_2$ is the underlying Hilbert space of Hilbert-Schmidt operators defined on the representation space (indexed by a certain set of parameters) of the unitary irreducible representations of the given group. And these Hilbert spaces generally vary as we keep varying the corresponding parameters determining the relevant representation spaces. In the present scenario, we take $A^1(\gamma, E_0) = A^2(\gamma, E_0) = |\psi_{\gamma,E_0}\rangle\langle\phi_{\gamma,E_0}|$ and the Plancherel measure as $d\nu_G(\gamma, E_0) = \rho(\gamma, E_0)dE_0 d\gamma$, so that we have

$$\begin{aligned}
\langle A^1|A^1\rangle_{\mathcal{B}_2^{\oplus}} &= \int_{(E_0,\gamma)\in\mathbb{R}\times\mathbb{R}^*} \text{tr}[A^1(\gamma, E_0)^* A^1(\gamma, E_0)]\rho(\gamma, E_0)d\gamma dE_0 \\
&= \int_{(E_0,\gamma)\in\mathbb{R}\times\mathbb{R}^*} \text{tr}[|\phi_{\gamma,E_0}\rangle\langle\psi_{\gamma,E_0}|\psi_{\gamma,E_0}\rangle\langle\phi_{\gamma,E_0}|]\rho(\gamma, E_0)d\gamma dE_0 \\
&= \int_{(E_0,\gamma)\in\mathbb{R}\times\mathbb{R}^*} \|\psi_{\gamma,E_0}\|^2\|\phi_{\gamma,E_0}\|^2\rho(\gamma, E_0)d\gamma dE_0. \qquad (3.69)
\end{aligned}$$

Now, subject to $A^1(\gamma, E_0) = A^2(\gamma, E_0) = |\psi_{\gamma,E_0}\rangle\langle\phi_{\gamma,E_0}|$ and the fact that the underlying group is unimodular so that the celebrated Duflo-Moore operator $C_{\gamma,E_0}$ is just a multiple of the identity operator acting on the Hilbert space $L^2(\hat{\mathbb{R}}, dk)$, the left side of (3.68) in the Fourier transformed space reads (from now on we will compute things in mommentum-space representation which is tractable compared to one in configuration space)

$$\begin{aligned}
&\frac{1}{N^2}\int_{\mathbb{R}^4}[\int_{\mathbb{R}\times\mathbb{R}^*}\overline{\langle\hat{\psi}_{\gamma,E_0}|\hat{U}^{\gamma,E_0}(\theta,b,a,v)\hat{\phi}_{\gamma,E_0}\rangle}\rho(\gamma,E_0)dE_0d\gamma \\
&\qquad\times\int_{\mathbb{R}\times\mathbb{R}^*}\langle\hat{\psi}_{\gamma',E_0'}|\hat{U}^{\gamma',E_0'}(\theta,b,a,v)\hat{\phi}_{\gamma',E_0'}\rangle\rho(\gamma',E_0')dE_0'd\gamma'] \; d\theta\; db\; da\; dv \\
&= \frac{1}{N^2}\int_{(E_0,\gamma)}\int_{(E_0',\gamma')}[\int_{\mathbb{R}^4}\{\int_{k\in\mathbb{R}}\int_{k'\in\mathbb{R}} e^{-i(\gamma-\gamma')\theta}e^{i(\gamma k-\gamma' k')a}e^{\frac{-i}{2m}(\gamma k^2-\gamma' k'^2)b} \\
&\qquad\times e^{-i(E_0-E_0')b}\overline{\hat{\phi}_{\gamma,E_0}(k-mv)}\hat{\psi}_{\gamma,E_0}(k)\overline{\hat{\psi}_{\gamma',E_0'}(k')}\hat{\phi}_{\gamma',E_0'}(k'-mv)\; dk\; dk'\}
\end{aligned}$$

$$\times d\theta\ db\ da\ dv]\rho(\gamma, E_0)\rho(\gamma', E_0')(dE_0\ d\gamma)(dE_0'\ d\gamma')$$

$$= \frac{2\pi}{N^2}\int_{(E_0,\gamma)}\int_{(E_0',\gamma')}\delta(\gamma-\gamma')[\int_{\mathbb{R}^3}\{\int_k\int_{k'} e^{i(\gamma k-\gamma' k')a}e^{-\frac{i}{2m}(\gamma k^2-\gamma' k'^2)b}e^{-i(E_0-E_0')b}$$

$$\times\overline{\hat{\phi}_{\gamma,E_0}(k-mv)}\hat{\psi}_{\gamma,E_0}(k)\overline{\hat{\psi}_{\gamma',E_0'}(k')}\hat{\phi}_{\gamma',E_0'}(k'-mv)\ dk\ dk'\}db\ da\ dv]$$

$$\times\rho(\gamma, E_0)\rho(\gamma', E_0')(dE_0\ d\gamma)(dE_0'\ d\gamma')$$

$$= \frac{2\pi}{N^2}\int_{(E_0,\gamma)}\int_{E_0'\in\mathbb{R}}[\int_{\mathbb{R}^3}\{\int_k\int_{k'} e^{i\gamma(k-k')a}e^{-\frac{i\gamma}{2m}(k^2-k'^2)b}e^{-i(E_0-E_0')b}$$

$$\times\overline{\hat{\phi}_{\gamma,E_0}(k-mv)}\hat{\psi}_{\gamma,E_0}(k)\overline{\hat{\psi}_{\gamma,E_0'}(k')}\hat{\phi}_{\gamma,E_0'}(k'-mv)\ dk\ dk'\}db\ da\ dv]$$

$$\times\rho(\gamma, E_0)\rho(\gamma, E_0')dE_0\ d\gamma\ dE_0'$$

$$= \frac{(2\pi)^2}{N^2}\int_{(E_0,\gamma)}\int_{E_0'\in\mathbb{R}}[\int_{\mathbb{R}^2}\{\int_k\int_{k'} \frac{\delta(k-k')}{|\gamma|}e^{-\frac{i\gamma}{2m}(k^2-k'^2)b}e^{-i(E_0-E_0')b}$$

$$\times\overline{\hat{\phi}_{\gamma,E_0}(k-mv)}\hat{\psi}_{\gamma,E_0}(k)\overline{\hat{\psi}_{\gamma,E_0'}(k')}\hat{\phi}_{\gamma,E_0'}(k'-mv)\ dk\ dk'\}db\ dv]$$

$$\times\rho(\gamma, E_0)\rho(\gamma, E_0')dE_0 d\gamma dE_0'$$

$$= \frac{(2\pi)^2}{N^2}\int_{(E_0,\gamma)}\int_{E_0'\in\mathbb{R}}[\int_{\mathbb{R}^2}\{\int_k e^{-i(E_0-E_0')b}\overline{\hat{\phi}_{\gamma,E_0}(k-mv)}\hat{\psi}_{\gamma,E_0}(k)\overline{\hat{\psi}_{\gamma,E_0'}(k)}$$

$$\times\hat{\phi}_{\gamma,E_0'}(k-mv)\ dk\}db\ dv]\frac{\rho(\gamma,E_0)}{|\gamma|}\rho(\gamma, E_0')dE_0\ d\gamma\ dE_0'$$

$$= \frac{(2\pi)^3}{N^2}\int_{(E_0,\gamma)}\int_{E_0'\in\mathbb{R}}[\int_{v\in\mathbb{R}}\{\int_k \delta(E_0-E_0')\overline{\hat{\phi}_{\gamma,E_0}(k-mv)}\hat{\psi}_{\gamma,E_0}(k)\overline{\hat{\psi}_{\gamma,E_0'}(k)}$$

$$\times\hat{\phi}_{\gamma,E_0'}(k-mv)\ dk\}dv]\frac{\rho(\gamma,E_0)}{|\gamma|}\rho(\gamma, E_0')dE_0 d\gamma dE_0'$$

$$= \frac{(2\pi)^3}{N^2}\int_{(E_0,\gamma)}[\int_{v\in\mathbb{R}}\{\int_k \overline{\hat{\phi}_{\gamma,E_0}(k-mv)}\hat{\psi}_{\gamma,E_0}(k)\overline{\hat{\psi}_{\gamma,E_0}(k)}\hat{\phi}_{\gamma,E_0}(k-mv)\ dk\}$$

$$\times dv]\frac{[\rho(\gamma,E_0)]^2}{|\gamma|}d\gamma dE_0$$

$$= \frac{(2\pi)^3}{N^2}\int_{(E_0,\gamma)}\frac{[\rho(\gamma,E_0)]^2}{|\gamma|}d\gamma dE_0[\int_{v\in\mathbb{R}}\{\int_k \overline{\hat{\phi}_{\gamma,E_0}(k-mv)}\hat{\psi}_{\gamma,E_0}(k)\overline{\hat{\psi}_{\gamma,E_0}(k)}$$

$$\times\hat{\phi}_{\gamma,E_0}(k-mv)\;dk\}dv]$$

$$= \frac{(2\pi)^3}{N^2}\int_{(E_0,\gamma)}\frac{[\rho(\gamma,E_0)]^2}{|\gamma|}d\gamma dE_0[\frac{1}{m}\int_{k'\in\mathbb{R}}\{\int_{t\in\mathbb{R}}\overline{\hat{\phi}_{\gamma,E_0}(t)}\hat{\psi}_{\gamma,E_0}(k')\overline{\hat{\psi}_{\gamma,E_0}(k')}$$

$$\times\hat{\phi}_{\gamma,E_0}(t)\;dt\}dk']$$

$$= \frac{(2\pi)^3}{N^2}\int_{(E_0,\gamma)}\frac{[\rho(\gamma,E_0)]^2}{m|\gamma|}\|\hat{\phi}\|^2\|\hat{\psi}\|^2 d\gamma dE_0.$$

Now, in view of (3.68) and (3.69), we finally obtain (in momentum space)

$$\rho(\gamma,E_0) \;=\; m|\gamma| \tag{3.70}$$

$$N \;=\; (2\pi)^{3/2}, \tag{3.71}$$

which are the Plancherel measure and the Duflo-Moore operator, respectively, for the quantum Galilei group case.

### 3.3.5 Computation that leads to the fact that the canonical exponent $\xi_1$ used to construct $\mathcal{G}^m$ is no good to compute the correct Wigner function

The most general expression for Wigner function is given by the following expression [5]

$$\begin{aligned} W(A|X^*_\lambda) = \frac{[\sigma_\lambda(X^*_\lambda)]^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}}\int_{N_0} e^{-i\langle X^*_\lambda;X\rangle} \\ \times[\int_{\hat{G}}\text{tr}(U_\sigma(e^{-X})[A(\sigma)C_\sigma^{-1}])[m(X)]^{\frac{1}{2}}d\nu_G(\sigma)]dX. \end{aligned} \tag{3.72}$$

The first exponential term in (3.72) is given by the following expression

$$\exp i(k_1^* v - k_2^* a + \{k_1 + \frac{(k_2^*)^2}{2mk_2}\}b + k_2\theta).$$

And the densities $\sigma$ and $m$ for the (1+1) dimensional Galilei group turn out to be simply 1. Here, we are interested in the coadjoint orbits $\mathcal{O}^*_{k_1,k_2}$ due to dual algebra elements $\begin{bmatrix} 0 & 0 & k_1 & k_2 \end{bmatrix}$. The induced representation of $\mathcal{G}^m$ was found to be

$$(\hat{U}^{\gamma,E_0}(\theta,b,a,v)\hat{\phi})(k) = e^{i[\gamma(\theta+\frac{k^2b}{2m}-ka)+E_0b]}\hat{\phi}(k-mv).$$

It follows immediately that

$$\begin{aligned}
(\hat{U}^{\gamma,E_0}(e^{-Y})\hat{\phi})(k) &= (\hat{U}^{\gamma,E_0}(-\theta,-b,a,-v)^{-1}\hat{\phi})(k) \\
&= (\hat{U}^{\gamma,E_0}(\theta+\frac{1}{2}mv^2b-mva,b,vb-a,v)\hat{\phi})(k) \\
&= e^{i\gamma\theta+\frac{i}{2}\gamma mv^2b-i\gamma mva+i\gamma\frac{k^2b}{2m}-i\gamma kvb+i\gamma ka+iE_0b}\hat{\phi}(k-mv). \qquad (3.73)
\end{aligned}$$

The Duflo-Moore operator was found to be just $(2\pi)^{3/2}$. So $C^{-1}$ in (3.72) is just $\frac{1}{(2\pi)^{3/2}}$. Also, in this case $n=4$. Now combining (3.73) with the exponential term mentioned at the start and putting them in (3.72), we find the following

$$\begin{aligned}
&W(|\phi_{\gamma,E_0}\rangle\langle\psi_{\gamma,E_0}|;k_1^*,k_2^*;k_1,k_2) \\
&= \frac{m}{(2\pi)^3\sqrt{2\pi}}\int_\theta\int_b\int_a\int_v e^{i(k_1^*v-k_2^*a+\{k_1+\frac{(k_2^*)^2}{2mk_2}\}b+k_2\theta)} \\
&\times[\int_\gamma\int_{E_0}\int_k e^{i\gamma\theta-i\gamma kvb+i\gamma ka+\frac{i}{2}\gamma mv^2b-i\gamma mva+i\gamma\frac{k^2b}{2m}+iE_0b}\overline{\hat{\psi}_{\gamma,E_0}(k)}\hat{\phi}_{\gamma,E_0}(k-mv) \\
&\times|\gamma|dk\; dE_0\; d\gamma]dv\; da\; db\; d\theta \\
&= \frac{m|k_2|}{(2\pi)^2\sqrt{2\pi}}\int_{E_0}[\int_b\int_a\int_v\int_k e^{ik_1^*v-ik_2^*a}e^{i\{k_1+\frac{(k_2^*)^2}{2mk_2}\}b} \\
&\times\overline{\hat{\psi}_{-k_2,E_0}(k)}e^{ik_2kvb-ik_2ka-\frac{i}{2}k_2mv^2b+ik_2mva-ik_2\frac{k^2b}{2m}+iE_0b}\hat{\phi}_{-k_2,E_0}(k-mv)
\end{aligned}$$

$$\times dk\ dv\ da\ db]dE_0$$

$$= \frac{m|k_2|}{(2\pi)^2\sqrt{2\pi}} \int_{E_0}\int_b\int_k\int_v [\int_a e^{-ik_2^* a - ik_2 ka + ik_2 mva} da] e^{ik_1^* v} e^{i\{k_1 + \frac{(k_2^*)^2}{2mk_2}\}b}$$

$$\times \overline{\hat{\psi}_{-k_2,E_0}(k)} e^{ik_2 kvb - \frac{i}{2}k_2 mv^2 b - ik_2\frac{k^2 b}{2m} + iE_0 b} \hat{\phi}_{-k_2,E_0}(k - mv)\ dv\ dk\ db\ dE_0$$

$$= \frac{m|k_2|}{2\pi\sqrt{2\pi}} \int_{E_0}\int_b\int_k\int_v \delta(k_2 k + k_2^* - k_2 mv) e^{ik_1^* v} e^{i\{k_1 + \frac{(k_2^*)^2}{2mk_2}\}b}$$

$$\times \overline{\hat{\psi}_{-k_2,E_0}(k)} e^{ik_2 kvb - \frac{i}{2}k_2 mv^2 b - ik_2\frac{k^2 b}{2m} + iE_0 b} \hat{\phi}_{-k_2,E_0}(k - mv)\ dv\ dk\ db\ dE_0$$

$$= \frac{m}{2\pi\sqrt{2\pi}} \int_{E_0}\int_b\int_v e^{ik_1^* v} e^{ik_1 b + iE_0 b} \overline{\hat{\psi}_{-k_2,E_0}(-\frac{k_2^*}{k_2} + mv)} \hat{\phi}_{-k_2,E_0}(-\frac{k_2^*}{k_2})$$

$$\times db\ dv\ dE_0$$

$$= \frac{m}{\sqrt{2\pi}} \int_v e^{ik_1^* v} \overline{\hat{\psi}_{-k_2,-k_1}(-\frac{k_2^*}{k_2} + mv)} \hat{\phi}_{-k_2,-k_1}(-\frac{k_2^*}{k_2}) dv. \tag{3.74}$$

Thus, we do not arrive at the correct Wigner function using the canonical exponent $\xi_1$.

### 3.3.6 Coadjoint action matrix and UIRs of the (1+1)-centrally extended Galilei group $\mathcal{G}^{m\prime}$

The geometry of the coadjoint orbits for the quantum Galilei group $\mathcal{G}^m$ is encoded in the coadjoint action matrix given by (3.60). Now, if we extend the (1+1)-Galilei group $\mathcal{G}_0$ using the exponent $\xi_2$ (see (3.52)) to yield $\mathcal{G}^{m\prime}$, the geometry of the coadjoint orbits of $\mathcal{G}^{m\prime}$ remains unaltered, when compared to those of $\mathcal{G}^m$. Now to verify that we will compute the coadjoint action matrix for $\mathcal{G}^{m\prime}$ explicitly.

A generic lie algebra element is denoted as $Y = a_1 K + a_2 X + a_3 T + a_4 \Theta$ where $K, X, T, \Theta$ are the lie algebra generators corresponding to boost, space translation, time translation and the central extension respectively. They are given by the following matrices

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{1}{2}m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}m & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \Theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

And the corresponding commutation relations are exactly the one that we obtained using the canonical exponent $\xi_1$

$$[K, X] = m\Theta, \ [K, T] = X, \ [X, T] = 0$$

$$[\Theta, T] = 0, \ [\Theta, K] = 0, \ [\Theta, X] = 0.$$

So, in terms of the above generators a generic lie algebra element reads

$$Y = \begin{bmatrix} 0 & a_3 & 0 & a_2 \\ 0 & 0 & 0 & -a_1 \\ \frac{1}{2}ma_1 & \frac{1}{2}ma_2 & 0 & a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.75}$$

But a generic group element $(\theta, b, a, v)$ is given by the following matrix

$$(\theta, b, a, v) = \begin{bmatrix} 1 & b & 0 & a - vb \\ 0 & 1 & 0 & -v \\ \frac{1}{2}mv & \frac{1}{2}ma & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.76}$$

obeying the group multiplication rule given by (3.53).

Also, $(\theta, b, a, v)^{-1} = (-\theta, -b, vb - a, -v)$. The matrix representation of an inverse group element follows

$$(\theta, b, a, v)^{-1} = \begin{bmatrix} 1 & -b & 0 & -a \\ 0 & 1 & 0 & v \\ \frac{1}{2}mv & \frac{1}{2}m(vb-a) & 1 & -\theta \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.77}$$

Now, given the fact that a generic group element is given by equation (3.76) and its inverse by (3.77) and that the adjoint action of a group element on the lie algebra element is defined by $\mathrm{Ad}_g Y = gYg^{-1}$, we have

$$\begin{aligned} &\mathrm{Ad}_{g^{-1}}(Y) \\ &= \begin{bmatrix} 0 & a_3 & 0 & ba_1 + a_2 - a_3 v \\ 0 & 0 & 0 & -a_1 \\ \frac{1}{2}ma_1 & \frac{1}{2}m(ba_1 + a_2 - va_3) & 0 & ma_1(a - vb) - mva_2 + \frac{1}{2}ma_3v^2 + a_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

From which the coadjoint action matrix for $\mathcal{G}^{m\prime}$ follows as

$$M(g^{-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ b & 1 & -v & 0 \\ 0 & 0 & 1 & 0 \\ m(a-vb) & -mv & \frac{1}{2}mv^2 & 1 \end{bmatrix}. \tag{3.78}$$

The above matrix is exactly the same as found in (3.60).

### 3.3.7 Computation of the Wigner function for $\mathcal{G}^{m\prime}$

The unitary irreducible representations of the (1+1)-centrally extended Galilei group $\mathcal{G}^m$ or the quantum Galilei group was computed in (3.58). The central extension procedure was carried out by the canonical exponent introduced in (3.50). The other exponent $\xi_2$ yielding the centrally extended group $\mathcal{G}^{m\prime}$ is mentioned in (3.52). The two exponents introduced are equivalent in the sense of [10]. In other words, the difference between the two exponents is a trivial one, which can be written by means of the following continuous function

$$\zeta_T(b, a, v) = \frac{mva}{2}, \tag{3.79}$$

in the following way

$$\begin{aligned} &\xi_1(b_1, a_1, v_1) - \xi_2(b_2, a_2, v_2) \\ &\quad = \frac{1}{2}mv_1a_2 + \frac{1}{2}mv_2a_1 + \frac{1}{2}mv_1^2b_2 + \frac{1}{2}mv_1v_2b_2 \\ &\quad = \zeta_T((b_1, a_1, v_1)(b_2, a_2, v_2)) - \zeta_T(b_1, a_1, v_1) - \zeta_T(b_2, a_2, v_2), \end{aligned}$$

which entails the fact that $\hat{U}'^{\gamma,E_0} := e^{\frac{i\gamma mva}{2}}\hat{U}^{\gamma,E_0}$ would be a projective representation of the (1+1)-Galilei group $\mathcal{G}_0$, where $\gamma$ is introduced for dimensional consistency and for keeping track with (3.58). In the language of ordinary representations, we can state that $\hat{U}'^{\gamma,E_0}$, so obtained, is a unitary irreducible representation of the (1+1)-centrally extended Galilei group $\mathcal{G}^{m\prime}$. The fact that irreducibility is preserved during the whole process of arriving at a unitary representation (e.g., $\hat{U}'^{\gamma,E_0}$) of the central extension of the given group (e.g., $\mathcal{G}_0$) with resepect to a certain multiplier (e.g., using the canonical exponent $\xi_1$) from the known UIR (e.g., $\hat{U}^{\gamma,E_0}$) of a central extension of the same group corresponding to another multiplier (e.g., the one due to $\xi_2$) by means of projecting and lifting it in several steps is described in [39].

Therefore, the UIRs of the (1+1)-centrally extended Galilei group $\mathcal{G}^{m\prime}$ acting on $L^2(\hat{\mathbb{R}}, dk)$ is given by

$$(\hat{U}'^{\gamma,E_0}(\theta, b, a, v)\hat{\phi})(k) = \exp i[\gamma(\theta - ka + \frac{mva}{2} + \frac{k^2 b}{2m}) + E_0 b]\hat{\phi}(k - mv). \tag{3.80}$$

Now, following the steps as mentioned in section 3.3.4, we can obtain the Duflo-Moore operator and the Plancherel measure for $\mathcal{G}^{m\prime}$ which are as follows

$$\rho(\gamma, E_0) = m|\gamma| \tag{3.81}$$

$$N = (2\pi)^{3/2}. \tag{3.82}$$

These are exactly the same as obtained for the other central extension of the (1+1)-Galilei group.

Now, we are well-equipped to compute the Wigner function for the centrally extended group $\mathcal{G}^{m\prime}$. If $Y$ ia a generic Lie algebra element given by $y = a_1K - a_2X + a_3T + a_4\Theta$, it follows from (3.80) (under the identification $a_1 \leftrightarrow -v, a_2 \leftrightarrow -a, a_3 \leftrightarrow -b, a_4 \leftrightarrow -\theta$) that

$$
\begin{aligned}
(\hat{U}^{\gamma,E_0}(e^Y)\hat{\phi})(k) &= (\hat{U}^{\gamma,E_0}(-\theta,-b,a,-v)\hat{\phi})(k) \\
&= e^{i[\gamma\{-\theta-ka-\frac{mva}{2}-\frac{k^2b}{2m}\}-E_0b]}\hat{\phi}(k+mv)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(\hat{U}^{\gamma,E_0}(e^{-Y})\hat{\phi})(k) &= (\hat{U}^{\gamma,E_0}(-\theta,-b,a,-v)^{-1}\hat{\phi})(k) \\
&= (\hat{U}^{\gamma,E_0}(\theta,b,vb-a,v)\hat{\phi})(k) \\
&= e^{i\gamma\theta-i\gamma kvb+i\gamma ka+\frac{i}{2}\gamma mv^2b-\frac{i}{2}\gamma mva+i\gamma\frac{k^2b}{2m}+iE_0b}\hat{\phi}(k-mv).
\end{aligned}
$$

We are going to use the most general expression for the Wigner function given in (3.72) to compute the Wigner function of $\mathcal{G}^{m\prime}$ like we used to compute that of $\mathcal{G}^m$.

Here also, both the densities $\sigma$ and $m$ are simply 1. So, the Wigner function for the the (1+1)-centrally extended Galilei group $\mathcal{G}^{m\prime}$ reads

$$
\begin{aligned}
&W(|\phi_{\gamma,E_0}\rangle\langle\psi_{\gamma,E_0}|;k_1^*,k_2^*;k_1,k_2) \\
&\quad = \frac{m}{(2\pi)^3\sqrt{2\pi}}\int_\theta\int_b\int_a\int_v e^{i(k_1^*v-k_2^*a+\{k_1+\frac{(k_2^*)^2}{2mk_2}\}b+k_2\theta)}
\end{aligned}
$$

$$\times [\int_\gamma \int_{E_0} \int_k e^{i\gamma\theta - i\gamma kvb + i\gamma ka + \frac{i}{2}\gamma mv^2 b - \frac{i}{2}\gamma mva + i\gamma \frac{k^2 b}{2m} + iE_0 b} \overline{\hat{\psi}_{\gamma,E_0}(k)} \hat{\phi}_{\gamma,E_0}(k - mv)$$

$$\times |\gamma| dk\, dE_0\, d\gamma] dvdadbd\theta$$

$$= \frac{m|k_2|}{(2\pi)^2\sqrt{2\pi}} \int_{E_0} [\int_b \int_a \int_v \int_k e^{ik_1^* v - ik_2^* a} e^{i\{k_1 + \frac{(k_2^*)^2}{2mk_2}\}b} \overline{\hat{\psi}_{-k_2,E_0}(k)}$$

$$\times e^{ik_2 kvb - ik_2 ka - \frac{i}{2}k_2 mv^2 b + \frac{ik_2 mva}{2} - ik_2 \frac{k^2 b}{2m} + iE_0 b} \hat{\phi}_{-k_2,E_0}(k - mv)\, dk\, dv\, da\, db] dE_0$$

$$= \frac{m|k_2|}{(2\pi)^2\sqrt{2\pi}} \int_{E_0} \int_b \int_k \int_v [\int_a e^{-ik_2^* a - ik_2 ka + \frac{ik_2 mva}{2}} da] e^{ik_1^* v} e^{i\{k_1 + \frac{(k_2^*)^2}{2mk_2}\}b} \overline{\hat{\psi}_{-k_2,E_0}(k)}$$

$$\times e^{ik_2 kvb - \frac{i}{2}k_2 mv^2 b - ik_2 \frac{k^2 b}{2m} + iE_0 b} \hat{\phi}_{-k_2,E_0}(k - mv)\, dv\, dk\, db\, dE_0$$

$$= \frac{m|k_2|}{2\pi\sqrt{2\pi}} \int_{E_0} \int_b \int_k \int_v \delta(k_2 k + k_2^* - \frac{k_2 mv}{2}) e^{ik_1^* v} e^{i\{k_1 + \frac{(k_2^*)^2}{2mk_2}\}b}$$

$$\times \overline{\hat{\psi}_{-k_2,E_0}(k)} e^{ik_2 kvb - \frac{i}{2}k_2 mv^2 b - ik_2 \frac{k^2 b}{2m} + iE_0 b} \hat{\phi}_{-k_2,E_0}(k - mv)\, dv\, dk\, db\, dE_0$$

$$= \frac{m}{2\pi\sqrt{2\pi}} \int_{E_0} \int_b \int_v e^{ik_1^* v} e^{i\{k_1 + \frac{(k_2^*)^2}{2mk_2}\}b} \overline{\hat{\psi}_{-k_2,E_0}(-\frac{k_2^*}{k_2} + \frac{mv}{2})}$$

$$\times \hat{\phi}_{-k_2,E_0}(-\frac{k_2^*}{k_2} + \frac{mv}{2} - mv) e^{ik_2 vb(-\frac{k_2^*}{k_2} + \frac{mv}{2}) - \frac{i}{2}k_2 mv^2 b - \frac{ik_2 b}{2m}(-\frac{k_2^*}{k_2} + \frac{mv}{2})^2 + iE_0 b}$$

$$\times dv\, db\, dE_0$$

$$= \frac{m}{2\pi\sqrt{2\pi}} \int_{E_0} \int_b \int_v e^{ik_1^* v} e^{ik_1 b - \frac{i}{2}k_2^* vb - \frac{i}{8}k_2 mv^2 b + iE_0 b} \overline{\hat{\psi}_{-k_2,E_0}(-\frac{k_2^*}{k_2} + \frac{mv}{2})}$$

$$\times \hat{\phi}_{-k_2,E_0}(-\frac{k_2^*}{k_2} - \frac{mv}{2})\, db\, dv\, dE_0$$

$$= \frac{m}{2\pi\sqrt{2\pi}} \int_{E_0} \int_v e^{ik_1^* v} [\int_b e^{-i(-k_1 + \frac{1}{2}k_2^* v + \frac{1}{8}k_2 mv^2 - E_0)b} db] \overline{\hat{\psi}_{-k_2,E_0}(-\frac{k_2^*}{k_2} + \frac{mv}{2})}$$

$$\times \hat{\phi}_{-k_2,E_0}(-\frac{k_2^*}{k_2} - \frac{mv}{2}) dv dE_0 \tag{3.83a}$$

$$= \frac{m}{\sqrt{2\pi}} \int_{E_0} \int_v e^{ik_1^* v} \delta(E_0 + k_1 - \frac{k_2^* v}{2} - \frac{k_2 mv^2}{8}) \overline{\hat{\psi}_{-k_2,E_0}(-\frac{k_2^*}{k_2} + \frac{mv}{2})}$$

$$\times \hat{\phi}_{-k_2,E_0}(-\frac{k_2^*}{k_2} - \frac{mv}{2}) dv\, dE_0$$

$$= \frac{m}{\sqrt{2\pi}} \int_v e^{ik_1^* v} \overline{\hat{\psi}_{-k_2, -k_1 + \frac{k_2^* v}{2} + \frac{k_2 m v^2}{8}} (-\frac{k_2^*}{k_2} + \frac{mv}{2})}$$

$$\times \hat{\phi}_{-k_2, -k_1 + \frac{k_2^* v}{2} + \frac{k_2 m v^2}{8}} (-\frac{k_2^*}{k_2} - \frac{mv}{2}) dv. \tag{3.83b}$$

Equation (3.83b) gives the Wigner function for $\mathcal{G}^{m\prime}$ due to a two dimensional coadjoint orbit embedded in $\mathbb{R}^4$, the dual algbera space. But the associated Wigner functions are no longer supported on the corresponding coadjoint orbits. Rather the support is spread out through a whole family of coadjoint orbits as will be discussed in some detail in the following section.

### 3.4 Summary of the main results and physical interpretation

In the first half of the chapter, we studied the (1+1)-extended affine Gaillei group $\mathcal{G}^m_{\text{aff}}$, the quantum version of the (1+1)-affine Galilei group $\mathcal{G}_{\text{aff}}$. The four nondegenerate coadjoint orbits of $\mathcal{G}^m_{\text{aff}}$ were all found to be six dimensional spaces embedded in $\mathbb{R}^6$, the underlying dual algebra space. These were basically cotangent bundles over the four open free 3-dimensional Mackey orbits described in Figure 3–1. Hence, the coadjoint orbits of $\mathcal{G}^m_{\text{aff}}$ are also open free in $\mathbb{R}^6$. Open free orbits have nice structural implications in representation theory. And the Wigner functions associated with each such coadjoint orbit were found employing the standard techniques developed in [35]. Finally, the domains of the four Wigner functions were studied. Two of the functions were found to be supported inside the relavant coadjoint orbits while the other two were found to have their support spread out through the $\mathbb{R}^6$ half hyperplane to which they belong.

Now, we look for a possible physical interpretation of the results obtained for the (1+1)-extended affine Galilei group $\mathcal{G}^m_{\text{aff}}$. The two representation-space parameters in this context were $q$ and $E_0 := E - \frac{p^2}{2mq}$. Different signs of $q$ and $E_0$ determine which coadjoint orbit we are in or which UIR we are talking about, as outlined in Table 3–1. The mass term $m$ stands for mass scale or mass unit [8], while $qm$ stands for the physical mass which changes under the action of the dilation parameters ($\sigma$ and $\tau$). In other words, each unitary irreducible representation of $\mathcal{G}^m_{\text{aff}}$ represents a nonrelativistic spinless particle of variable mass ($qm$) and internal energy ($E_0$). And the Wigner function of $\mathcal{G}^m_{\text{aff}}$ associated with a nonrelativistic spinless particle of variable positive mass and changing positive internal energy was found to be supported inside its coadjoint orbit. The Wigner function due to a nonrelativistic spinless particle of varible negative mass and changing positive internal energy was also found to be supported inside its coadjoint orbit. It turns out that requiring symmetry under the affine Galilei group $\mathcal{G}^m_{\text{aff}}$ leads to no physically interesting phenomenon. But mathematically the coadjoint orbits were nicely structured and two of the relevant Wigner functions were found to be supported inside the corresponding coadjoint orbits.

Since the requirement of symmetry under dilation parameters led to no physically interesting object, in the later half of the chapter we demanded only Galilean invariance and hence worked with the quantum Galilei group $\mathcal{G}^m$ and its variant $\mathcal{G}^{m\prime}$ (where the extension was executed using an equivalent multiplier).

The Wigner function for the (1+1)-extended Galilei group $\mathcal{G}^{m\prime}$ that we found in (3.83b), is basically a map between two direct integral Hilbert spaces given by

$$\begin{aligned} W : &\int_{(\gamma,E_0)\in\mathbb{R}^*\times\mathbb{R}}^{\oplus} \mathcal{B}_2(L^2(\hat{\mathcal{O}}_{\gamma,E_0}, dk))m|\gamma|d\gamma dE_0 \\ &\rightarrow \int_{(k_1,k_2)\in\mathbb{R}\times\mathbb{R}^*}^{\oplus} L^2(\mathcal{O}^*_{k_2,k_1}, dk_1^* dk_2^*)dk_1 dk_2. \end{aligned}$$

For

$$\begin{aligned} \phi_{k_2,k_1}, \psi_{k_2,k_1} &\in L^2(\mathcal{O}^*_{k_2,k_1}, dk_1^* dk_2^*) \\ \text{and } |\phi_{\gamma,E_0}\rangle\langle\psi_{\gamma,E_0}| &\in \mathcal{B}_2^{\oplus}(L^2(\hat{\mathcal{O}}_{\gamma,E_0}, dk)), \end{aligned}$$

where $\mathcal{B}_2$ denotes the space of Hilbert-Schmidt operators and $\mathcal{B}_2^{\oplus}$ represents the direct integral Hilbert space of measurable Hilbert-Schmidt operator fields. The Wigner function found in (3.83b) was restricted to the coadjoint orbit $\mathcal{O}^*_{k_2,k_1}$. But the final expression for the Wigner function reveals the fact that it is no longer supported on that single coadjoint orbit. Rather it has its support concentrated on the collection of coadjoint orbits exhausting $\mathbb{R}^3$ hyperplanes perpendicular to the fourth axis W. For brevity, we ask the reader to go back to section 3.3.2, where we explained the geometry of the relevant coadjoint orbits. In particular, we have a continuous collection of Wigner functions being supported in each such hyperplane characterized by the constant $W = k_2$. This hyperplane can be called a "support plane". In other words, to each hyperplane (corresponding to a fixed value of $k_2 \in \mathbb{R}\setminus\{0\}$ and $k_1$ assuming all real values in $\mathbb{R}$), we attach all such Wigner functions each pertaining to a non-relativistic spinless particle due to a fixed value of $\gamma$ (the analog of $q$, i.e. the mass scale in the extended affine case) and a definite real internal energy, having the states to be square integrable functions on the coadjoint orbits exhausting the $\mathbb{R}^3$ hyperpalne in question. In the language of representations we say that, the Wigner function restricted to the coadjoint orbit $\mathcal{O}^*_{k_2,k_1} = T^*\hat{\mathcal{O}}_{k_2,k_1}$ gets its contribution from representations associated to all the parabolic orbits corresponding to the $\mathbb{R}^2$ plane given by $\gamma = \gamma' = k_2$ in the $\gamma' E' p'$ space. The corresponding coadjoint orbits exactly exhaust the $\mathbb{R}^3$-hyperplane given by $W = k_2$ embedded in $\mathbb{R}^4$. Therefore, each Wigner map is associated with representations $U^{\gamma,E_0}$ due to a fixed value of $\gamma$ but all possible real values of $E_0$.

It was also found in Section 3.3.2 that it is important to choose the appropriate multiplier to find the correct Wigner function for the relevant group. The appropriateness is defined by the fact that the multiplier should reduce to one given by the Weyl-Heisenberg group under proper substitution. In this sense, $\xi_2$ defined by (3.52) was found to be appropriate in the context of the (1+1)-Galilei group.

Next, we ask the question whether under suitable conditions the Wigner function for $\mathcal{G}^{m\prime}$ given by equation (3.83b) reduces to the standard one due to the Weyl-Heisenberg group. One thing that marks a distinction between the above two groups is that the Galilei group has a time translation parameter while the Weyl-Heisenberg group does not, the second distinguishing characteristic being that the irreducible unitary representations of the former are parameterized by two constants namely $\gamma$ and $E_0$ while those of the latter are parameterized only by a single constant $\gamma$. With the above two considerations, we insert two Delta like functions $\delta(b)$ and $\delta(E_0)$ following the substitution of $k_1$ to be zero in equation (3.83a) to derive the following

$$\begin{aligned}
&W(|\phi_{k_2}\rangle\langle\psi_{k_2}|; k_1^*, k_2^*; k_2) \\
&= \frac{m}{2\pi\sqrt{2\pi}} \int_{E_0} \int_v e^{ik_1^* v} [\int_b e^{-i(\frac{k_2^* v}{2} - \frac{1}{8}k_2 m v^2 - k_1 - E_0)b} \delta(b) db] \\
&\times \overline{\hat{\psi}_{k_2,E_0}(-\frac{k_2^*}{k_2} + \frac{mv}{2})} \hat{\phi}_{k_2,E_0}(-\frac{k_2^*}{k_2} - \frac{mv}{2}) \delta(E_0) dv dE_0 \\
&= \frac{m}{2\pi\sqrt{2\pi}} \int_{E_0} \int_v e^{ik_1^* v} \delta(E_0) \overline{\hat{\psi}_{k_2,E_0}(-\frac{k_2^*}{k_2} + \frac{mv}{2})} \hat{\phi}_{k_2,E_0}(-\frac{k_2^*}{k_2} - \frac{mv}{2}) dv\, dE_0 \\
&= \frac{m}{2\pi\sqrt{2\pi}} \int_v e^{ik_1^* v} \overline{\hat{\psi}_{k_2,0}(-\frac{k_2^*}{k_2} + \frac{mv}{2})} \hat{\phi}_{k_2,0}(-\frac{k_2^*}{k_2} - \frac{mv}{2}) dv. \qquad (3.84)
\end{aligned}$$

The corresponding coadjoint orbits are foliated along one of the two othogonal axes of $\mathbb{R}^4$, along which the coadjoint orbits of the (1+1)-centrally extended Galilei group were foliated (see Section 3.3.2, where the geometry of its coadjoint orbits is discussed at length). The measurable fields of the Hilbert-Schmidt operators are now indexed only by $\gamma$. The total orbit space (union of the disjoint coadjoint orbits in question) is no longer a dense subspace of $\mathbb{R}^4$. It is just a set of measure zero (lower dimensional space) instead. But it fills out an $\mathbb{R}^3$ hyperplane embedded in $\mathbb{R}^4$. Each coadjoint orbit here is a cotangent bundle of a parabola characterized by the constant $k_2$, which is homeomorphic to $\mathbb{R}^2$, the coadjoint orbit for the Weyl-Heisenberg group. So, we find that the Wigner function for the (1+1) dimensional extended Galilei group, computed using an appropriate multiplier, reduces to the one for the Weyl- Heisenberg group under proper substitution.

The coadjoint action matrix for the (1+1)-centrally extended Galilei group turned out to be independent of the choice of multipliers (belonging to the same equivalence class) required to do the extension. In other words, we always end up with the same coadjoint action matrix no matter what multiplier we choose from the same equivalence class of the second cohomology group $H^2(\mathcal{G}_0, \mathbb{U}(1))$.

It is interesting to observe that the coadjoint orbits of the (1+1)-extended affine Galilei group $\mathcal{G}^m_{\text{aff}}$ (see Figure 3–1) disintegrate into a continuous family of parabolas, each of which is parameterized by a specific value of the ordered pair $(\gamma, E_0)$. These parabolas are just the dual orbits of the (1+1)-centrally extended Galilei group $\mathcal{G}^m$ or the quantum Galilei group. This disintegration of the dual orbits resolves the difficulty of a nonrelativistic spinless particle possessing variable mass, but as a price of the remedy, the beauty of the structure of the open free orbits gets lost. None of the Wigner functions associated with the massive nonrelativistic spinless particles under the symmetry of the Galilei group remains supported inside the corresponding phase space. In this sense, to earn physically meaningful object we had to sacrifice the associated mathematically elegant structure.

So far, in chapter 2 and 3, we considered various central and non-central extensions of (1+1) Galilei group. Their intimate relationships with the Lie groups used in signal analysis and image processing were explored. The Wigner functions associated with several of these extensions were subsequently computed along with a comparative study between the respective functions. In terms of its algebraic structure, the (2+1) Galilei group is even more interesting. The next chapter is devoted to the study of centrally extended (2+1) Galilei group and its role in two-dimensional noncommutative quantum mechanics.

# Chapter 4
# The Symmetry Groups of Noncommutative Quantum Mechanics and Coherent State Quantization

The contents of this chapter are taken from the article titled "The Symmetry Groups of Noncommutative Quantum Mechanics and Coherent State Quantization" [21]. Here, we explore the group theoretical underpinning of noncommutative quantum mechanics for a system moving on the two-dimensional plane. We show that the pertinent groups for the system are the two-fold central extension of the Galilei group in $(2 + 1)$-space-time dimensions and the two-fold extension of the group of translations of $\mathbb{R}^4$. This latter group is just the standard Weyl-Heisenberg group of standard quantum mechanics with an additional central extension. We also look at a further extension of this group and discuss its significance to noncommutative quantum mechanics. We build unitary irreducible representations of these various groups and construct the associated families of coherent states. A coherent state quantization of the underlying phase space is then carried out, which is shown to lead to exactly the same commutation relations as usually postulated for this model of noncommutative quantum mechanics.

## 4.1 Introduction

Noncommutative quantum mechanics is a much frequented topic of research these days. The expectation here is that a modification, or rather an extension, of standard quantum mechanics is needed to model physical space-time at very short distances. In this chapter, we restrict ourselves to the version of non-commutative quantum mechanics which describes a quantum system with two degrees of freedom and in which, in addition to having the usual canonical commutation relations, one also imposes an additional non-commutativity between the two position coordinates, i.e.,

$$[Q_i, P_j] = i\hbar\delta_{ij}I, \quad i, j = 1, 2, \qquad [Q_1, Q_2] = i\vartheta I, \tag{4.1}$$

Here the $Q_i, P_j$ are the quantum mechanical position and momentum observables, respectively, and $\vartheta$ is a (small, positive) parameter which measures the additionally introduced noncommutativity between the observables of the two spatial coordinates. The limit $\vartheta = 0$ then corresponds to standard (two-dimensional) quantum mechanics. The literature, even on this rather focused and simple model, is already extensive. We refer to [24, 42] and the many references cited therein for a review of the background and motivation behind the model. One could continue with this game of increasing noncommutativity between the observables by augmenting the above system by an additional commutator between the two momentum operators:

$$[P_i, P_j] = i\gamma\delta_{ij}I \ , \qquad i, j = 1, 2 \ , \tag{4.2}$$

where $\gamma$ is yet another positive parameter. Physically, such a commutator would signal, for example, the presence of a magnetic field in the system [24].

The purpose of this chapter is two-fold. First, we undertake a group theoretical analysis of the above sets of commutation relations, in order to find the groups behind noncommutative quantum mechanics, in the same way as a centrally extended Galilei group [37] or the Weyl-Heisenberg group underlies ordinary quantum mechanics. The second objective of this chapter is to arrive at the commutation relations (4.1) by the method of coherent state quantization (see, for example, [7] for a discussion of this method). This will involve constructing appropriate families of coherent states, emanating from the groups underlying noncommutative quantum mechanics, using standard techniques (see, for example, [4]). It will turn out however, that the coherent states that we shall be using here are very different from the ones introduced in [42], in that ours come from the representations of the related groups and satisfy standard resolutions of the identity condition.

## 4.2 Noncommutative quantum mechanics in the two-plane and the (2+1)-Galilei group

The (2+1)-Galilei group $G_{\text{Gal}}$ is a six-parameter Lie group. It is the kinematical group of a classical, non-relativistic space-time having two spatial and one time dimensions. It consists of translations of time and space, rotations in the two dimensional space and velocity boosts. As is well-known [37], non-relativistic quantum mechanics can be seen as arising from representations of central extensions of the Galilei group. We will thus be concerned here with the centrally extended (2+1)-Galilei group. The Lie algebra $\mathfrak{G}_{\text{Gal}}$ of the group $G_{\text{Gal}}$

has a three dimensional vector space of central extensions. This *extended algebra* has the following Lie bracket structure (see, for example, [14, 15]),

$$\begin{aligned} [M, N_i] &= \epsilon_{ij} N_j & \qquad [M, P_i] &= \epsilon_{ij} P_j \\ [H, P_i] &= 0 & \qquad [M, H] &= \mathfrak{h} \\ [N_i, N_j] &= \epsilon_{ij} \mathfrak{d} & \qquad [P_i, P_j] &= 0 \\ [N_i, P_j] &= \delta_{ij} \mathfrak{m} & \qquad [N_i, H] &= P_i, \end{aligned} \tag{4.3}$$

($i, j = 1, 2$ and $\epsilon_{ij}$ is the totally antisymmetric tensor with $\epsilon_{12} = -\epsilon_{21}$). The three central extensions are characterized by the three central generators $\mathfrak{h}, \mathfrak{d}$ and $\mathfrak{m}$ (they commute with each other and all the other generators). The $P_i$ generate space translations, $N_i$ velocity boosts, $H$ time translations and $M$ is the generator of angular momentum. Passing to the group level, the universal covering group $\widetilde{G}_{\text{Gal}}$, of $G_{\text{Gal}}$, has three central extensions, as expected. However, $G_{\text{Gal}}$ itself has only two central extensions (i.e., $\mathfrak{h} = 0$, identically [14]). We shall denote this 2-fold centrally extended $(2+1)$-Galilei group by $G_{\text{Gal}}^{\text{ext}}$ and its Lie algebra by $\mathfrak{G}_{\text{Gal}}^{\text{ext}}$.

A generic element of $G_{\text{Gal}}^{\text{ext}}$ may be written as $g = (\theta, \phi, R, b, \mathbf{v}, \mathbf{a}) = (\theta, \phi, r)$, where $\theta, \phi \in \mathbb{R}$, are phase terms corresponding to the two central extensions, $b \in \mathbb{R}$ a time-translation, $R$ is a $2 \times 2$ rotation matrix, $\mathbf{v} \in \mathbb{R}^2$ a 2-velocity boost, $\mathbf{a} \in \mathbb{R}^2$ a 2-dimensional space translation and $r = (R, b, \mathbf{v}, \mathbf{a})$. The two central extensions are given by two cocycles, $\xi^1_m$ and $\xi^2_\lambda$, depending on the two real parameters $m$ and $\lambda$. Explicitly, these are,

$$
\begin{aligned}
\xi^1_m(r;\ r') &= e^{\frac{im}{2}(\mathbf{a}\cdot R\mathbf{v}' - \mathbf{v}\cdot R\mathbf{a}' + b'\mathbf{v}\cdot R\mathbf{v}')}, \\
\xi^2_\lambda(r;\ r') &= e^{\frac{i\lambda}{2}(\mathbf{v}\wedge R\mathbf{v}')}\ , \quad \text{where} \quad \mathbf{q}\wedge\mathbf{p} = q_1p_2 - q_2p_1\ ,
\end{aligned} \tag{4.4}
$$

$(\mathbf{q} = (q_1, q_2),\ \mathbf{p} = (p_1, p_2))$.

The group multiplication rule is given by

$$
\begin{aligned}
gg' &= (\theta, \phi, R, b, \mathbf{v}, \mathbf{a})(\theta', \phi', R', b', \mathbf{v}', \mathbf{a}') \\
&= (\theta + \theta' + \xi^1_m(r;\ r'),\ \phi + \phi' + \xi^2_\lambda(r;\ r'), \\
&\qquad RR', b + b', \mathbf{v} + R\mathbf{v}', \mathbf{a} + R\mathbf{a}' + \mathbf{v}b')\ .
\end{aligned} \tag{4.5}
$$

The projective unitary irreducible representations (PURs) of $G^{\text{ext}}_{\text{Gal}}$, from which we can obtain its unitary irreducible representations, have all been computed in, e.g., [15]. In this chapter we shall only consider the case where $m \neq 0$ and $\lambda \neq 0$. These representations, realized on the Hilbert space $L^2(\mathbb{R}^2, d\mathbf{k})$ (see (4.12) below), are characterized by ordered pairs $(m, \vartheta)$ of reals and by the number $s$, expressed as an integral multiple of $\frac{\hbar}{2}$. Here, $m$ is to be interpreted as the mass of the nonrelativistic system under study, while $\lambda$ will be seen to be related to the parameter $\vartheta$ appearing in (4.1). The quantity $s$ is the eigenvalue of the intrinsic angular momentum opearator $S$ (representating rotations in the rest-frame).

The physical significance of these quantities have been studied extensively in [14, 15, 37] and [46].

Recall that we should take $\mathfrak{h} = 0$ in (4.3), to get the Lie algebra $\mathfrak{G}_{\text{Gal}}^{\text{ext}}$. In the representation Hilbert space of the PURs of the group $G_{\text{Gal}}^{\text{ext}}$, the basis elements of the algebra are realized as self-adjoint operators, the two central elements appearing as multiples of the identity operator. Thus, the operator representation of $\mathfrak{G}_{\text{Gal}}^{\text{ext}}$ looks like

$$
\begin{aligned}
[\hat{M}, \hat{N}_i] &= i\epsilon_{ij}\hat{N}_j & [\hat{M}, \hat{P}_i] &= i\epsilon_{ij}\hat{P}_j \\
[\hat{H}, \hat{P}_i] &= 0 & [\hat{M}, \hat{H}] &= 0 \\
[\hat{N}_i, \hat{N}_j] &= i\epsilon_{ij}\lambda\hat{I} & [\hat{P}_i, \hat{P}_j] &= 0 \\
[\hat{N}_i, \hat{P}_j] &= i\delta_{ij}m\hat{I} & [\hat{N}_i, \hat{H}] &= i\hat{P}_i\ .
\end{aligned}
\tag{4.6}
$$

Here the operators $\hat{N}_i$ generate velocity shifts. The other operators $\hat{P}_i$, $\hat{M}$, $\hat{H}$, and $\hat{I}$ are just the linear momentum, angular momentum, energy and the identity operators, respectively, acting on $L^2(\mathbb{R}^2, d\mathbf{k})$, the representation space of the PUIRs of $G_{\text{Gal}}^{\text{ext}}$.

Consider next the so-called two-dimensional *noncommutative Weyl-Heisenberg group*, or the group of *noncommutative quantum mechanics*. The group generators are the operators $Q_i, P_j$ and $I$, obeying the commutation relations (4.1). The resulting algebra of operators is also referred to as the the *noncommutative two-oscillator algebra.* Realized on the Hilbert space $L^2(\mathbb{R}^2, d\mathbf{x})$ (coordinate representation) these operators can be brought into the form

$$\tilde{Q}_1 = x + \frac{i\vartheta}{2}\frac{\partial}{\partial y} \qquad \tilde{Q}_2 = y - \frac{i\vartheta}{2}\frac{\partial}{\partial x}$$
$$\tilde{P}_1 = -i\hbar\frac{\partial}{\partial x} \qquad \tilde{P}_2 = -i\hbar\frac{\partial}{\partial y}. \tag{4.7}$$

If we add to this set the the Hamiltonian (corresponding to a mass $m$)

$$\tilde{H} = -\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}), \tag{4.8}$$

the angular momentum operator,

$$\tilde{M} = -i\hbar\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial x}\right). \tag{4.9}$$

and furthermore, define $\tilde{N}_i = m\tilde{Q}_i,\ i = 1, 2$, then the resulting set of seven operators is easily seen to obey the commutation relations

$$[\tilde{M}, \tilde{N}_i] = i\hbar\epsilon_{ij}\tilde{N}_j \qquad [\tilde{M}, \tilde{P}_i] = i\hbar\epsilon_{ij}\tilde{P}_j$$
$$[\tilde{H}, \tilde{P}_i] = 0 \qquad [\tilde{M}, \tilde{H}] = 0$$
$$[\tilde{N}_i, \tilde{N}_j] = i\epsilon_{ij}m^2\vartheta\tilde{I} \qquad [\tilde{P}_i, \tilde{P}_j] = 0$$
$$[\tilde{N}_i, \tilde{P}_j] = i\hbar\delta_{ij}m\tilde{I} \qquad [\tilde{N}_i, \tilde{H}] = i\hbar\tilde{P}_i, \tag{4.10}$$

Taking $\hbar = 1$ and writing $\lambda = m^2\vartheta$ this becomes exactly the same set of commutation relations as that in (4.6) of the Lie algebra $\mathfrak{G}^{\text{ext}}_{\text{Gal}}$. of the extended Galilei group. This tells us that the kinematical group of non-relativistic, noncommutative quantum mechanics is the

(2+1)-Galilei $G_{\text{Gal}}^{\text{ext}}$, with two extensions, a fact which has already been noted and exploited in [30].

At this point we note that in terms of $Q_1, Q_2$ and $P_1, P_2$, the usual quantum mechanical position and momentum operators defined on $L^2(\mathbb{R}^2, dx\ dy)$, the noncommutative position operators $\tilde{Q}_i$ can be written as

$$\begin{aligned} \tilde{Q}_1 &= Q_1 - \frac{\vartheta}{2\hbar} P_2 \\ \tilde{Q}_2 &= Q_2 + \frac{\vartheta}{2\hbar} P_1. \end{aligned} \tag{4.11}$$

The above transformation is linear and invertible and may be thought of as giving a non-canonical transformation on the underlying phase space. Since $\tilde{Q}_i = Q_i \Leftrightarrow \vartheta = 0$, the noncommutativity of the two-plane is lost if the parameter $\vartheta$ is turned off. However, from the group theoretical discussion above we see that the noncommutativity of the two spatial coordinates should not just be looked upon as a result of this non-canonical transformation. Rather, it is also the two-fold central extension of the (2+1)-Galilei group, governing nonrelativistic mechanics, which is responsible for it. The extent to which the two spatial coordinates fail to be commutative is encoded in the representation parameters of the underlying group, namely, $\vartheta$. It is noteworthy in this context that had we centrally extended the (2+1)-Galilei group using only the cocycle $\xi_m^1$ in (4.4) (i.e., set $\vartheta$=0), we would have just obtained standard quantum mechanics. In this sense we claim that the group underlying noncommutative quantum mechanics, as governed by the commutation relations (4.7), is the doubly centrally extended $(2+1)$-Galilei group. (It is also worth mentioning in this context that the noncommuting position operators $\hat{Q}_i$, arising from the (2+1)-centrally extended Galilei group, also describe the position of the center of mass of the underlying non-relativistic system (see [46]).)

### 4.3 Quantization using coherent states associated to non-commutative quantum mechanics

In this section we first write down the unitary irreducible representations of the extended Galilei group $G_{\text{Gal}}^{\text{ext}}$. Next we construct coherent states for these representations, which we identify as being the *coherent states of noncommutative quantum mechanics.* We then carry out a quantization of the underlying phase space using these coherent states, obtaining thereby the operators $Q_i, P_i$ (see (4.1)) of non-commutative quantum mechanics. In the literature other coherent states have been defined for noncommutative quantum mechanics – see, for example, [42]. These latter coherent states are basically the one-dimensional projection operators, $|z\rangle\langle z|,\ z \in \mathbb{C}$, where $|z\rangle$ is the well-known canonical coherent state, familiar from quantum mechanics (see, for example, [4]). These coherent states have been shown to satisfy a sort of an "operator resolution of the identity" and have been used to study localization properties of systems obeying noncommutative quantum mechanics. By contrast, the coherent states which we obtain (see (4.16) below), using the representations of the group $G_{\text{Gal}}^{\text{ext}}$, i.e., the kinematical group of noncommutative quantum mechanics, satisfy a standard resolution of the identity (see (4.17)). We shall also discuss the relationship of these coherent states to the canonical coherent states (in this case arising from the Weyl-Heisenberg group), for two degrees of freedom, and the fact that these latter can be recovered from the coherent states (4.16) of noncommutative quantum mechanics in the limit of $\vartheta = 0$.

### 4.3.1 UIRs of the group $G^{\text{ext}}_{\text{Gal}}$

The unitary irreducible representations of the extended Galilei group $G^{\text{ext}}_{\text{Gal}}$ can be obtained from its projective unitary irreducible representations worked out in [15]. We take, as mentioned earlier, both extension parameters $m$ and $\lambda$ to be non-zero. The representation space is $L^2(\mathbb{R}^2, d\mathbf{k})$ (momentum space representation). Denoting the unitary representation operators by $\hat{U}_{m,\lambda}$, we have,

$$
\begin{aligned}
&(\hat{U}_{m,\lambda}(\theta,\phi,R,b,\mathbf{v},\mathbf{a})\hat{f})(\mathbf{k}) \\
&\qquad = e^{i(\theta+\phi)}e^{i[\mathbf{a}\cdot(\mathbf{k}-\frac{1}{2}m\mathbf{v})+\frac{b}{2m}\mathbf{k}\cdot\mathbf{k}+\frac{\lambda}{2m}\mathbf{v}\wedge\mathbf{k}]}s(R)\hat{f}(R^{-1}(\mathbf{k}-m\mathbf{v})),
\end{aligned} \tag{4.12}
$$

for any $\hat{f} \in L^2(\hat{\mathbb{R}}^2, d\mathbf{k})$. Here, $s$ denotes the irreducible representation of the rotation group in the rest frame (spin). It is useful to Fourier transform the above representation to get its configuration space version (on $L^2(\mathbb{R}^2, d\mathbf{x})$). A straightforward computation, using Fourier transforms, leads to:

**Lemma 4.3.1.** *The unitary irreducible representations of $G^{ext}_{Gal}$ in the (two-dimensional) configuration space are given by*

$$
\begin{aligned}
&(U_{m,\lambda}(\theta,\phi,R,b,\mathbf{v},\mathbf{a})f)(\mathbf{x}) \\
&\qquad = e^{i(\theta+\phi)}e^{im(\mathbf{x}+\frac{1}{2}\mathbf{a})\cdot\mathbf{v}}e^{-i\frac{b}{2m}\nabla^2}s(R)f\left(R^{-1}\left(\mathbf{x}+\mathbf{a}-\frac{\lambda}{2m}J\mathbf{v}\right)\right),
\end{aligned} \tag{4.13}
$$

*where* $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, *$J$ is the $2 \times 2$ skew-symmetric matrix,* $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, *and* $f \in L^2(\mathbb{R}^2, d\mathbf{x})$.

### 4.3.2 Coherent states of the centrally extended (2+1)-Galilei group

It is easy to see from (4.13) that the representation $U_{m,\lambda}$ is *not square-integrable.* This means that there is no non-zero vector $\eta$ in the representation space for which the function $f_\eta(g) = \langle \eta \mid U_{m,\lambda}(g)\eta \rangle$ has finite $L^2$-norm, i.e., for all non-zero $\eta \in L^2(\mathbb{R}^2, d\mathbf{x})$,

$$\int_{G_{\text{Gal}}^{\text{ext}}} |f_\eta(g)|^2 \, d\mu(g) = \infty \ ,$$

$d\mu$ being the Haar measure.

On the other hand, the group composition law (4.5) reflects the fact that the subgroup $H := \Theta \times \Phi \times SO(2) \times \mathcal{T}$, with generic group elements $(\theta, \phi, R, b)$, is an abelian subgroup of $G_{\text{Gal}}^{\text{ext}}$. The left coset space $X := G_{\text{Gal}}^{\text{ext}}/H$ is easily seen to be homeomorphic to $\mathbb{R}^4$, corresponding to to the left coset decomposition,

$$(\theta, \phi, R, b, \mathbf{v}, \mathbf{a}) = (0, 0, \mathbb{I}_2, 0, \mathbf{v}, \mathbf{a})(\theta, \phi, R, \mathbf{0}, \mathbf{0}) \ , \quad (\mathbb{I}_2 = 2 \times 2 \text{ unit matrix}).$$

Writing $\mathbf{q}$ for $\mathbf{a}$ and replacing $\mathbf{v}$ by $\mathbf{p} := m\mathbf{v}$, we identify $X$ with the phase space of the quantum system corresponding to the UIR $\hat{U}_{m,\lambda}$ and write its elements as $(\mathbf{q}, \mathbf{p})$. The homogeneous space carries an invariant measure under the natural action of $G_{\text{Gal}}^{\text{ext}}$, which in these coordinates is just the Lebesgue measure $d\mathbf{q}\ d\mathbf{p}$ on $\mathbb{R}^4$. Also we define a section

$\beta : X \longmapsto G^{\text{ext}}_{\text{Gal}}$,

$$\beta(\mathbf{q}, \mathbf{p}) = (0, 0, \mathbb{I}_2, 0, \frac{\mathbf{p}}{m}, \mathbf{q}). \tag{4.14}$$

We show next that the representation $U_{m,\lambda}$ is square-integrable mod$(\beta, H)$ in the sense of [4] and hence construct coherent states on the homogeneous space (phase space) $X$. Let $\chi \in L^2(\mathbb{R}^2, d\mathbf{x})$ be a fixed vector. At a later stage (see Theorem 4.3.2) we shall need to impose a symmetry condition on this vector, but at the moment we leave it arbitrary. For each phase space point $(\mathbf{q}, \mathbf{p})$ define the vector,

$$\chi_{\mathbf{q},\mathbf{p}} = U_{m,\lambda}(\beta(\mathbf{q}, \mathbf{p}))\chi \ . \tag{4.15}$$

so that from (4.13) and (4.14),

$$\chi_{\mathbf{q},\mathbf{p}}(\mathbf{x}) = e^{i(\mathbf{x}+\frac{1}{2}\mathbf{q})\cdot\mathbf{p}}\chi\left(\mathbf{x} + \mathbf{q} - \frac{\lambda}{2m^2}J\mathbf{p}\right). \tag{4.16}$$

**Lemma 4.3.2.** *For all $f, g \in L^2(\mathbb{R}^2)$, the vectors $\chi_{\mathbf{q},\mathbf{p}}$ satisfy the square integrability condition*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \langle f \mid \chi_{\mathbf{q},\mathbf{p}}\rangle\langle\chi_{\mathbf{q},\mathbf{p}} \mid g\rangle \, d\mathbf{q}\, d\mathbf{p} = (2\pi)^2\|\chi\|^2\langle f \mid g\rangle. \tag{4.17}$$

The proof is given in Appendix A. Additionally, in the course of the proof we have also established that the operator integral

$$T = \int_{\mathbb{R}^2\times\mathbb{R}^2} |\chi_{\mathbf{q},\mathbf{p}}\rangle\langle\chi_{\mathbf{q},\mathbf{p}}| \, d\mathbf{q}\, d\mathbf{p} \ ,$$

in (A.3) converges weakly to $T = 2\pi \, \|\chi\|^2 \, I$. Let us now define the vectors

$$\eta = \frac{1}{\sqrt{2\pi}\|\chi\|}\chi, \quad \text{and} \quad \eta_{\mathbf{q},\mathbf{p}} = U(\beta(\mathbf{q},\mathbf{p}))\eta \; , \quad (\mathbf{q},\mathbf{p}) \in X \; . \tag{4.18}$$

Then, as a consequence of the above lemma, we have proved the following theorem.

**Theorem 4.3.1.** *The representation $U_{m,\lambda}$ in (4.13), of the extended Galilei group $G_{Gal}^{ext}$, is square integrable mod $(\beta, H)$ and the vectors $\eta_{\mathbf{q},\mathbf{p}}$ in (4.18) form a set of coherent states defined on the homogeneous space $X = G_{Gal}^{ext}/H$, satisfying the resolution of the identity*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} |\eta_{\mathbf{q},\mathbf{p}}\rangle\langle\eta_{\mathbf{q},\mathbf{p}}| \; d\mathbf{q} \; d\mathbf{p} = I \; , \tag{4.19}$$

*on $L^2(\mathbb{R}^2, d\mathbf{x})$.*

Note that

$$\eta_{\mathbf{q},\mathbf{p}}(\mathbf{x}) = e^{i(\mathbf{x}+\frac{1}{2}\mathbf{q})\cdot\mathbf{p}} \; \eta\left(\mathbf{x} + \mathbf{q} - \frac{\lambda}{2m^2}J\mathbf{p}\right) \; . \tag{4.20}$$

We shall consider these coherent states to be the ones associated with non-commutative quantum mechanics. Note that writing $\vartheta = \dfrac{\lambda}{m^2}$ as before, and letting $\vartheta \to 0$, we recover the standard or *canonical coherent states* of ordinary quantum mechanics, if $\eta$ is chosen to be the gaussian wave function. Since this also corresponds to setting $\lambda = 0$, it is consistent with constructing the coherent states of the (2+1)-Galilei group with one central extension (using only the first of the two cocycles in (4.4), with mass parameter $m$).

Let us emphasize again that the coherent states (4.20) are rooted in the underlying symmetry group of noncommutative quantum mechanics and they are very different from the ones introduced, for example, in [42] and often used in the literature. These latter coherent states are defined as $|z) = |z\rangle\langle z|,\ z \in \mathbb{C}$, where $|z\rangle$ is the usual canonical coherent state of ordinary quantum mechanics. If $\mathfrak{H}$ denotes the Hilbert space of a one dimensional quantum system moving on the line, then $|z)$, is an element of the space $\mathcal{B}_2(\mathfrak{H})$ of Hilbert-Schmidt operators on $\mathfrak{H}$ and this space is then taken to be the state space of noncommutative quantum mechanics. The coherent states $|z)$ satisfy a resolution of the identity which is also of a very different nature from (4.19). On $\mathcal{B}_2(\mathfrak{H})$ the algebra of operators in (4.7) is realized by the operators $\widehat{\mathbf{Q}}_i, \widehat{\mathbf{P}}_i,\ i = 1, 2$. These have the actions

$$\begin{aligned} \widehat{\mathbf{Q}}_1 X &= QX, & \widehat{\mathbf{Q}}_2 X &= \vartheta P X, \\ \widehat{\mathbf{P}}_1 X &= \hbar[P, X], & \widehat{\mathbf{P}}_2 X &= -\frac{\hbar}{\vartheta}[Q, X], \end{aligned} \tag{4.21}$$

on elements $X$ of $\mathcal{B}_2(\mathfrak{H})$. The $Q$ and $P$ are two operators on $\mathfrak{H}$, satisfying the commutation relation $[Q, P] = iI_{\mathfrak{H}}$. The state space with which we are working here and on which the operators (4.7) are realized, is $L^2(\mathbb{R}^2, d\mathbf{x})$. It is not hard to see that the unitary *Wigner map*, $\mathcal{W} : \mathcal{B}_2(\mathfrak{H}) \longrightarrow L^2(\mathbb{R}^2, d\mathbf{x})$, given by

$$(\mathcal{W}X)(x, y) = \frac{1}{\sqrt{2\pi}}\ \mathrm{Tr}[e^{-i(xQ+yP)}X]\ , \tag{4.22}$$

transforms the set $\{\widehat{\mathbf{Q}}_i, \widehat{\mathbf{P}}_i\}$ to the set $\{\tilde{Q}_i, \tilde{P}_i\}$ in (4.7). In other words, the formulation of noncommutative quantum mechanics on these two state spaces are completely equivalent.

### 4.3.3 Coherent state quantization on phase space leading to the noncommutative plane

It has been already noted that we are identifying the homogeneous space $X = G^{\text{ext}}_{\text{Gal}}/H$ with the phase space of the system. We shall now carry out a coherent state quantization of functions on this phase space, using the above coherent states of the extended Galilei group. It will turn out that such a quantization of the phase space variables of position and momentum will lead precisely to the operators (4.7).

Recall that given a (sufficiently well behaved) function $f(\mathbf{q}, \mathbf{p})$, its quantized version $\hat{\mathcal{O}}_f$, obtained via coherent state quantization, is the operator (on $L^2(\mathbb{R}^2, d\mathbf{x})$) given by the prescription (see, for example, [7] ),

$$\hat{\mathcal{O}}_f = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(\mathbf{q}, \mathbf{p})|\eta_{\mathbf{q},\mathbf{p}}\rangle\langle\eta_{\mathbf{q},\mathbf{p}}|\; d\mathbf{q}\; d\mathbf{p} \tag{4.23}$$

provided this operator is well-defined (again the integral being weakly defined). The operators $\hat{\mathcal{O}}_f$ act on a $g \in L^2(\mathbb{R}^2, d\mathbf{x})$ in the following manner

$$(\hat{\mathcal{O}}_f g)(\mathbf{x}) = \int_{\mathbb{R}^2\times\mathbb{R}^2} f(\mathbf{q}, \mathbf{p})\eta_{\mathbf{q},\mathbf{p}}(\mathbf{x}) \left[\int_{\mathbb{R}^2} \overline{\eta_{\mathbf{q},\mathbf{p}}(\mathbf{x}')} g(\mathbf{x}')d\mathbf{x}'\right]\; d\mathbf{q}\; d\mathbf{p}\; . \tag{4.24}$$

If we now take the function $f$ to be one of the coordinate functions, $f(\mathbf{q}, \mathbf{p}) = q_i,\; i = 1, 2$, or one of the momentum functions, $f(\mathbf{q}, \mathbf{p}) = p_i,\; i = 1, 2$, then the following theorem shows that the resulting quantized operators $\hat{\mathcal{O}}_{q_i}$ and $\hat{\mathcal{O}}_{p_i}$ are exactly the ones given in (4.1) for noncommutative quantum mechanics (with $\hbar = 1$) or the ones in (4.7), for the generators of the UIRs of $G^{\text{ext}}_{\text{Gal}}$ or of the noncommutative Weyl-Heisenberg group.

**Theorem 4.3.2.** *Let $\eta$ be a smooth function which satisfies the rotational invariance condition, $\eta(\mathbf{x}) = \eta(\|\mathbf{x}\|)$, for all $\mathbf{x} \in \mathbb{R}^2$. Then, the operators $\hat{\mathcal{O}}_{q_i}, \hat{\mathcal{O}}_{p_i},\ i = 1, 2$, obtained by a quantization of the phase space functions $q_i, p_i,\ i = 1, 2$, using the coherent states (4.18) of the (2+1)-centrally extended Galilei group, $G^{ext}_{Gal}$, are given by*

$$(\hat{\mathcal{O}}_{q_1} g)(\mathbf{x}) = \left(x_1 + \frac{i\lambda}{2m^2}\frac{\partial}{\partial x_2}\right) g(\mathbf{x}) \qquad (\hat{\mathcal{O}}_{q_2} g)(\mathbf{x}) = \left(x_2 - \frac{i\lambda}{2m^2}\frac{\partial}{\partial x_1}\right) g(\mathbf{x})$$

$$(\hat{\mathcal{O}}_{p_1} g)(\mathbf{x}) = -i\frac{\partial}{\partial x_1} g(\mathbf{x}) \qquad (\hat{\mathcal{O}}_{p_2} g)(\mathbf{x}) = -i\frac{\partial}{\partial x_2} g(\mathbf{x}), \tag{4.25}$$

*for $g \in L^2(\mathbb{R}^2, d\mathbf{x})$, in the domain of these operators.*

In (4.25) if we make the substitution $\vartheta = \dfrac{\lambda}{m^2}$, we get the operators (4.7) and the commutation relations of non-commutative quantum mechanics (with $\hbar = 1$):

$$[\hat{\mathcal{O}}_{q_1}, \hat{\mathcal{O}}_{q_2}] = i\vartheta I, \qquad [\hat{\mathcal{O}}_{q_i}, \hat{\mathcal{O}}_{p_j}] = i\delta_{ij} I, \qquad [\hat{\mathcal{O}}_{p_i}, \hat{\mathcal{O}}_{p_j}] = 0. \tag{4.26}$$

Moreover, it ought to be emphasized here that the rotational invariance of $\eta$, in the sense that $\eta(\mathbf{x}) = \eta(\|\mathbf{x}\|)$ was essential in deriving (4.25).

Two final remarks, before leaving this section, are in order. First, the operators $Q_i, P_i$, appearing in (4.11), together with the identity operator $I$, generate a representation of the Lie algebra of the Wey-Heisenberg group. Thus, it would seem that the operators $\tilde{Q}_i, \tilde{P}_i$ are just a different basis in this same algebra. However, this only appears to be so at the representation level, in which both central elements of the extended Galilei group are mapped to the identity operator. The two sets of operators, $Q_i, P_i$ and $\tilde{Q}_i, \tilde{P}_i$, in fact refer to the Lie algebras of two different groups namely, the $(2+1)$-Galilei groups with one and two extensions, respectively. Moreover, the set of commutation relations (4.1), governing noncommutative quantum mechanics, is not unitary equivalent to that of standard quantum mechanics (where $\vartheta = 0$). In the following section we look at extensions of the Weyl-Heisenberg group which will throw more light on this issue. As a second point, we note that the first commutation relation, between $\hat{\mathcal{O}}_{q_1}$ and $\hat{\mathcal{O}}_{q_2}$ in (4.26) above, also implies that the two dimensional plane $\mathbb{R}^2$ becomes noncommutative as a result of quantization.

## 4.4 Central extensions of the abelian group of translations in $\mathbb{R}^4$ and noncommutative quantum mechanics

We start out with the abelian group of translations $G_T$ in $\mathbb{R}^4$, a generic element of which, denoted $(\mathbf{q}, \mathbf{p})$, obeys the group composition rule

$$(\mathbf{q}, \mathbf{p})(\mathbf{q}', \mathbf{p}') = (\mathbf{q} + \mathbf{q}', \mathbf{p} + \mathbf{p}') \ . \tag{4.27}$$

At the level of the Lie algebra, all the generators commute with each other. In order to arrive at quantum mechanics out of this abelian Lie group, and to go further to obtain non-commutative quantum mechanics, we need to centrally extend this group of translations by some other abelian group, say by $\mathbb{R}$. In this section we will first discuss the double central extension of $G_T$ and see that the double central extension by $\mathbb{R}$ yields the commutation relations (4.1) of noncommutative quantum mechanics. We will, next go a step further and extend $G_T$ triply by $\mathbb{R}$. The Lie algebra basis will be found to satisfy the additional commutation relation (4.2) between the momentum operators. We start by recalling some facts about central extensions, following closely the treatment of Bargmann in [10].

Given a connected and simply connected Lie group $G$, the local exponents $\xi$ giving its central extensions are functions $\xi : G \times G \to \mathbb{R}$, obeying the following properties:

$$\xi(g'', g') + \xi(g''g', g) = \xi(g'', g'g) + \xi(g', g) \tag{4.28}$$

$$\xi(g, e) = 0 = \xi(e, g),\ \xi(g, g^{-1}) = \xi(g^{-1}, g). \tag{4.29}$$

We call the central extension trivial when the corresponding local exponent is simply a *coboundary* term, in other words, when there exists a continuous function $\zeta : G \to \mathbb{R}$ such that

$$\xi(g', g) = \xi_{cob}(g', g) := \zeta(g') + \zeta(g) - \zeta(g'g). \tag{4.30}$$

Two local exponents $\xi$ and $\xi'$ are *equivalent* if they differ by a coboundary term, i.e. $\xi'(g', g) = \xi(g', g) + \xi_{cob}(g', g)$. A local exponent which is itself a coboundary is said to be trivial and the corresponding extension of the group is called a trivial extension. Such an extension is isomorphic to the direct product group $\mathbb{U}(1) \times G$. Exponentiating the inequivalent local exponents yields the $\mathbb{U}(1)$ local factors or the familiar group multipliers, and the set of all such inequivalent multipliers form the well known second cohomology group $H^2(G, \mathbb{U}(1))$ of $G$.

**Theorem 4.4.1.** *The three real valued functions $\xi$, $\xi'$ and $\xi''$ on $G_T \times G_T$ given by*

$$\xi((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) = \frac{1}{2}[q_1 p_1' + q_2 p_2' - p_1 q_1' - p_2 q_2'], \tag{4.31}$$

$$\xi'((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) = \frac{1}{2}[p_1 p_2' - p_2 p_1'], \tag{4.32}$$

$$\xi''((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) = \frac{1}{2}[q_1 q_2' - q_2 q_1'], \tag{4.33}$$

*are inequivalent local exponents for the group, $G_T$, of translations in $\mathbb{R}^4$ in the sense of (4.30).*

The proof is given in the Appendix A.

### 4.4.1 Double central extension of $G_T$

In this section, we study the doubly (centrally) extended group $\overline{\overline{G_T}}$ where the extension is achieved by means of the two multipliers $\xi$ and $\xi'$ enumerated in Theorem 4.4.1. The group composition rule for the extended group $\overline{\overline{G_T}}$ reads

$$\begin{aligned}&(\theta, \phi, \mathbf{q}, \mathbf{p})(\theta', \phi', \mathbf{q}', \mathbf{p}') \\ &\qquad = (\theta + \theta' + \frac{\alpha}{2}[\langle \mathbf{q}, \mathbf{p}' \rangle - \langle \mathbf{p}, \mathbf{q}' \rangle], \phi + \phi' + \frac{\beta}{2}[\mathbf{p} \wedge \mathbf{p}'], \mathbf{q} + \mathbf{q}', \mathbf{p} + \mathbf{p}'), \end{aligned} \tag{4.34}$$

where $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$. Also, $\langle ., . \rangle$ and $\wedge$ are defined as $\langle \mathbf{q}, \mathbf{p} \rangle := q_1 p_1 + q_2 p_2$ and $\mathbf{q} \wedge \mathbf{p} := q_1 p_2 - q_2 p_1$ respectively.

A matrix representation for the group $\overline{\overline{G_T}}$ obeying the group multiplication rule (4.34) is given by the following $7 \times 7$ upper triangular matrix

$$(\theta, \phi, \mathbf{q}, \mathbf{p})_{\alpha,\beta} = \begin{bmatrix} 1 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{4.35}$$

Let us denote the generators of the Lie group $\overline{\overline{G_T}}$, or equivalently the basis of the associated Lie algebra, $\overline{\overline{\mathcal{G}_T}}$ by $\Theta, \Phi, Q_1, Q_2, P_1$ and $P_2$. These generate the one-parameter subgroups corresponding to the group parameters $\theta, \phi, p_1, p_2, q_1$ and $q_2$, respectively. The bilinear Lie brackets between the basis elements of $\overline{\overline{\mathcal{G}_T}}$ are given by

$$\begin{aligned} &[P_i, Q_j] = \alpha\delta_{i,j}\Theta,\ [Q_1, Q_2] = \beta\Phi, \quad [P_1, P_2] = 0, \quad [P_i, \Theta] = 0, \\ &[Q_i, \Theta] = 0, \quad [P_i, \Phi] = 0, \quad [Q_i, \Phi] = 0, \quad [\Theta, \Phi] = 0, \quad i, j = 1, 2\ . \end{aligned} \tag{4.36}$$

It is easily seen from (4.36) that $\Theta$ and $\Phi$ form the center of the algebra $\overline{\overline{\mathcal{G}_T}}$. It is also noteworthy that, unlike in standard quantum mechanics, the two generators of space translation, $Q_1, Q_2$, no longer commute, the noncommutativity of these two generators being controlled by the central extension parameter $\beta$. It is in this context that it is reasonable to call the Lie group $\overline{\overline{G_T}}$ the *noncommutative Weyl-Heisenberg group* and the corresponding Lie algebra the *noncommutative Weyl-Heisenberg algebra*.

We now proceed to find a unitary irreducible representation of $\overline{\overline{G_T}}$. From the matrix representation (4.35) we see that $\overline{\overline{G_T}}$ is a nilpotent Lie group. Hence, it is convenient to apply the orbit method of Kirillov (see [34]) for finding the unitary dual of the group.

Switching to a slightly different notation, for computational convenience, we replace the group parameters $p_1, p_2, q_1, q_2, \theta$ and $\phi$ by $a_1, a_2, a_3, a_4, a_5$ and $a_6$, respectively. then a generic group element $g(a_1, a_2, a_3, a_4, a_5, a_6)$ is represented by the following matrix (compare with (4.35)):

$$g(a_1, a_2, a_3, a_4, a_5, a_6) = \begin{bmatrix} 1 & 0 & -\frac{\alpha}{2}a_1 & -\frac{\alpha}{2}a_2 & \frac{\alpha}{2}a_3 & \frac{\alpha}{2}a_4 & a_5 \\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}a_2 & \frac{\beta}{2}a_1 & a_6 \\ 0 & 0 & 1 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.37}$$

If $X_1, X_2, X_3, X_4, X_5$ and $X_6$ stand for the respective group generators, a generic Lie algebra element can be written as $X = x^1X_1 + x^2X_2 + x^3X_3 + x^4X_4 + x^5X_5 + x^6X_6$. In matrix notation, $X$ can be read off as

$$X = \begin{bmatrix} 0 & 0 & -\frac{\alpha}{2}x^1 & -\frac{\alpha}{2}x^2 & \frac{\alpha}{2}x^3 & \frac{\alpha}{2}x^4 & x^5 \\ 0 & 0 & 0 & 0 & -\frac{\beta}{2}x^2 & \frac{\beta}{2}x^1 & x^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.38}$$

An element $F \in \left(\overline{\overline{\mathcal{G}_T}}\right)^*$ with coordinates $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ is now represented by the following $7 \times 7$ lower triangular matrix

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_5 & X_6 & X_3 & X_4 & X_1 & X_2 & 0 \end{bmatrix}, \tag{4.39}$$

with the dual pairing being given as $\langle F, X\rangle = \mathrm{tr}(FX) = \sum_{i=1}^{6} x^i X_i$. Hence the coadjoint action $K$ of the underlying group $G_T$ on the dual Lie algebra $\left(\overline{\overline{\mathcal{G}_T}}\right)^*$ can be computed as

$$g(a_1, a_2, a_3, a_4, a_5, a_6)\ F\ g(a_1, a_2, a_3, a_4, a_5, a_6)^{-1} = \begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ X_5' & X_6' & X_3' & X_4' & X_1' & X_2' & * \end{bmatrix}, \tag{4.40}$$

with

$$\begin{aligned} &X_1' = X_1 - \frac{\alpha}{2}a_3X_5 + \frac{\beta}{2}a_2X_6, \quad X_2' = X_2 - \frac{\alpha}{2}a_4X_5 - \frac{\beta}{2}a_1X_6, \\ &X_3' = X_3 + \frac{\alpha}{2}a_1X_5, \quad X_4' = X_4 + \frac{\alpha}{2}a_2X_5, \quad X_5' = X_5, \quad X_6' = X_6. \end{aligned} \tag{4.41}$$

The required coadjoint action $K$ of the group on the dual algebra is therefore given by

$$\begin{aligned} &Kg(a_1, a_2, a_3, a_4, a_5, a_6)(X_1,\ X_2, X_3, X_4, X_5, X_6) \\ &\quad = (X_1 - \frac{\alpha}{2}a_3X_5 + \frac{\beta}{2}a_2X_6,\ X_2 - \frac{\alpha}{2}a_4X_5 - \frac{\beta}{2}a_1X_6,\ X_3 + \frac{\alpha}{2}a_1X_5, \\ &\quad\quad X_4 + \frac{\alpha}{2}a_2X_5,\ X_5, X_6). \end{aligned} \tag{4.42}$$

The entries denoted by $*$'s in (4.40) are some nonzero values that we are not interested in. From (4.42) one observes that the two coordinates $X_5$ and $X_6$ remain unchanged under the coadjoint action. This is expected since they correspond to the center of the underlying algebra. The only two *polynomial invariants* in this case are just $P(F) = X_5$ and $Q(F) = X_6$. The coadjoint orbits are given by the set $S_{\rho,\sigma}$, for some fixed real numbers $\rho, \sigma$, with

$$S_{\rho,\sigma} = \{F \in \left(\overline{\overline{\mathcal{G}_T}}\right)^* \mid P(F) = \rho,\ Q(F) = \sigma\}. \tag{4.43}$$

Now, the first four coordinates of the vector on the right hand side of (4.42) can be made zero by a suitable choice of the group parameters $a_1, a_2, a_3, a_4, a_5$ and $a_6$. Therefore, for nonzero values of $\rho$ and $\sigma$ in (4.43), the vector $(0, 0, 0, 0, \rho, \sigma)$ will lie in a coadjoint orbit $S_{\rho,\sigma}$ of codimension 2. Since the dual algebra space is six dimensional, i.e., the coadjoint orbit in question is 4 dimensional and it passes through the point $(0, 0, 0, 0, \rho, \sigma)$.

We next have to find the subalgebra, of correct dimension, subordinate to $F$ (see (4.39)). If we work with this appropiate polarizing subalgebra and solve the master equation (see [34]), the representation we end up with will be irreducible and unitary. The correct dimension of the polarizing subalgebra in this context turns out to be $\frac{2+6}{2} = 4$. The maximal abelian subalgebra $\mathfrak{h}$ of the underlying algebra $\overline{\overline{\mathcal{G}_T}}$ serves as the appropiate poralizing subalgebra in this case, i.e. $\mathfrak{h}$ is the maximal subalgebra with $F|_{[\mathfrak{h},\mathfrak{h}]} = 0$. A generic element of $\mathfrak{h}$ can be obtained from (4.38) just by putting $x^1 = x^2 = 0$ in there. A generic element of the corresponding abelian subgroup $H$ can be represented by the following matrix

$$h(\theta, \phi, \mathbf{q}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & 0 & 0 & \phi \\ 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.44}$$

We now choose a section $\delta : S = H\backslash\overline{\overline{G_T}} \to \overline{\overline{G_T}}$ with $\delta(\mathbf{s}) = \delta(s_1, s_2)$ being given by the following $7 \times 7$ matrix

$$\delta(s_1, s_2) = \begin{bmatrix} 1 & 0 & -\frac{\alpha}{2}s_1 & -\frac{\alpha}{2}s_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}s_2 & \frac{\beta}{2}s_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & s_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & s_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.45}$$

With all the relevant matrices at our disposal, we move on to solving the master equation, which in this case involves solving the matrix equation

$$
\begin{aligned}
&\begin{pmatrix} 1 & 0 & -\frac{\alpha}{2}s_1 & -\frac{\alpha}{2}s_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}s_2 & \frac{\beta}{2}s_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & s_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & s_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -\frac{\alpha}{2}(p_1+s_1) & -\frac{\alpha}{2}(p_2+s_2) & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta-\frac{\alpha}{2}q_1s_1-\frac{\alpha}{2}q_2s_2 \\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}(p_2+s_2) & \frac{\beta}{2}(p_1+s_1) & \phi-\frac{\beta}{2}p_1s_2+\frac{\beta}{2}p_2s_1 \\ 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & p_1+s_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & p_2+s_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4.46) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{\alpha}{2}A & \frac{\alpha}{2}B & C \\ 0 & 1 & 0 & 0 & 0 & 0 & D \\ 0 & 0 & 1 & 0 & 0 & 0 & A \\ 0 & 0 & 0 & 1 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & -\frac{\alpha}{2}E & -\frac{\alpha}{2}F & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}F & \frac{\beta}{2}E & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & E \\ 0 & 0 & 0 & 0 & 0 & 1 & F \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -\frac{\alpha}{2}E & -\frac{\alpha}{2}F & \frac{\alpha}{2}A & \frac{\alpha}{2}B & C+\frac{\alpha}{2}BF+\frac{\alpha}{2}AE \\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}F & \frac{\beta}{2}E & D \\ 0 & 0 & 1 & 0 & 0 & 0 & A \\ 0 & 0 & 0 & 1 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & 1 & 0 & E \\ 0 & 0 & 0 & 0 & 0 & 1 & F \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (4.47)
\end{aligned}
$$

for the unknowns $A, B, C, D, E$ and $F$. Comparing (4.46) with (4.47), one gets

$$
\begin{aligned}
&A = q_1, \quad B = q_2, \quad E = p_1 + s_1, \quad F = p_2 + s_2, \\
&C = \theta - \alpha\langle \mathbf{q}, \mathbf{s} + \frac{1}{2}\mathbf{p}\rangle, \quad D = \phi - \frac{\beta}{2}\mathbf{p} \wedge \mathbf{s}.
\end{aligned} \tag{4.48}
$$

We recall that the coadjoint orbit vector, associated to which we found the polarizing algebra, was of the form $(0, 0, 0, 0, \rho, \sigma)$. In view of (4.48), we therefore have the following theorem

**Theorem 4.4.2.** *The noncommutative Weyl-Heisenberg group* $\overline{\overline{G_T}}$ *admits a unitary irreducible representation realized on* $L^2(\mathbb{R}^2, d\mathbf{s})$ *by the operators* $U(\theta, \phi, \mathbf{q}, \mathbf{p})$*:*

$$
(U(\theta, \phi, \mathbf{q}, \mathbf{p})f)(\mathbf{s}) = \exp i\left(\theta + \phi - \alpha\langle \mathbf{q}, \mathbf{s} + \frac{1}{2}\mathbf{p}\rangle - \frac{\beta}{2}\mathbf{p} \wedge \mathbf{s}\right) f(\mathbf{s} + \mathbf{p}), \tag{4.49}
$$

*where* $f \in L^2(\mathbb{R}^2, d\mathbf{s})$.

From the one-parameter unitary groups $U(\theta, 0, 0, 0, 0, 0),\ U(0, 0, q_1, 0, 0, 0)$, etc, we obtain the their self-adjoint generators (on $L^2(\mathbb{R}^2, d\mathbf{s})$), $\hat{\Theta}$, $\hat{\Phi}$, $\hat{P}_1$, $\hat{P}_2$, $\hat{Q}_1$ and $\hat{Q}_2$, using the general formula

$$\hat{X}_\phi = i\frac{dU(\phi)}{d\phi}\bigg|_{\phi=0}.$$

Thus, we have the following Hilbert space representation of the noncentral group generators

$$\begin{aligned}\hat{P}_1 &= \alpha s_1, \quad \hat{Q}_1 = \frac{\beta}{2}s_2 + i\frac{\partial}{\partial s_1},\\ \hat{P}_2 &= \alpha s_2, \quad \hat{Q}_2 = -\frac{\beta}{2}s_1 + i\frac{\partial}{\partial s_2},\end{aligned} \tag{4.50}$$

while the two central generators $\hat{\Theta}$ and $\hat{\Phi}$ are both mapped to the Identity operator $\mathbb{I}_{\mathfrak{H}}$ of $\mathfrak{H} = L^2(\mathbb{R}^2, d\mathbf{s})$. An inverse Fourier transformation leads to the expressions, (on the coordinate Hilbert space $L^2(\mathbb{R}^2, d\mathbf{x})$)

$$\begin{aligned}\hat{P}_1 &= -i\alpha\frac{\partial}{\partial x}, \quad \hat{P}_2 = -i\alpha\frac{\partial}{\partial y},\\ \hat{Q}_1 &= x - \frac{i\beta}{2}\frac{\partial}{\partial y}, \quad \hat{Q}_2 = y + \frac{i\beta}{2}\frac{\partial}{\partial x}.\end{aligned} \tag{4.51}$$

which coincide with (4.7) if we identify $\alpha$ with $\hbar$ and $-\beta$ with $\vartheta$.

The commutation relations are now

$$[\hat{Q}_i, \hat{P}_j] = i\alpha\delta_{i,j}\mathbb{I}_{\mathfrak{H}}, \quad [\hat{Q}_1, \hat{Q}_2] = -i\beta\mathbb{I}_{\mathfrak{H}}, \quad [\hat{P}_1, \hat{P}_2] = 0. \tag{4.52}$$

If we now set $\alpha = \hbar$ and $-\beta = \vartheta$, we again retrieve the commutation relations (4.1) of noncommutative quantum mechanics. This means, that as in the case of the Galilei group, an additional central extension of the Weyl-Heisenberg group leads to non-commutative quantum mechanics.

### 4.4.2 Triple central extension of $G_T$

In this section we study the triple central extension of $G_T$ by $\mathbb{R}$ and compute a unitary irreducible representation of the extended group $\overline{\overline{\overline{G_T}}}$. We will make use of all the three local exponents $\xi$, $\xi'$ and $\xi''$ enumerated in Theorem 4.4.1 to do this triple extension. The group composition rule for the resulting triply extended Lie group $\overline{\overline{\overline{G_T}}}$ then reads

$$\begin{aligned}
&(\theta, \phi, \psi, \mathbf{q}, \mathbf{p})(\theta', \phi', \psi', \mathbf{q}', \mathbf{p}') \\
&\quad = (\theta + \theta' + \frac{\alpha}{2}[\langle \mathbf{q}, \mathbf{p}' \rangle - \langle \mathbf{p}, \mathbf{q}' \rangle], \phi + \phi' + \frac{\beta}{2}[\mathbf{p} \wedge \mathbf{p}'], \psi + \psi' + \frac{\gamma}{2}[\mathbf{q} \wedge \mathbf{q}'] \\
&\qquad , \mathbf{q} + \mathbf{q}', \mathbf{p} + \mathbf{p}').
\end{aligned} \tag{4.53}$$

The matrix representation of $\overline{\overline{\overline{G_T}}}$, consistent with the above group law, is then seen to be

$$(\theta, \phi, \psi, \mathbf{q}, \mathbf{p})_{\alpha,\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & -\frac{\gamma}{2}q_2 & \frac{\gamma}{2}q_1 & 0 & 0 & \psi \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.54}$$

Let us denote the Lie algebra of $\overline{\overline{\overline{G_T}}}$ by $\overline{\overline{\overline{\mathcal{G}_T}}}$. Denoting the basis elements of $\overline{\overline{\overline{\mathcal{G}_T}}}$ by $\Theta, \Phi, \Psi, Q_1, Q_2, P_1$ and $P_2$, corresponding to the group parameters $\theta, \phi, \psi, p_1, p_2, q_1$ and $q_2$, respectively, we have the following Lie bracket relations between them

$$\begin{aligned} &[P_i, Q_j] = \alpha\delta_{i,j}\Theta, \quad [Q_1, Q_2] = \beta\Phi, \quad [P_1, P_2] = \gamma\Psi, \quad [P_i, \Theta] = 0, \\ &[Q_i, \Theta] = 0, \quad [P_i, \Phi] = 0, \quad [Q_i, \Phi] = 0, \quad [P_i, \Psi] = 0, \\ &[Q_i, \Psi] = 0, \quad [\Theta, \Phi] = 0, \quad [\Phi, \Psi] = 0, \quad [\Theta, \Psi] = 0, \quad i, j = 1, 2\ . \end{aligned} \tag{4.55}$$

In addition to the two central elements $\Theta$ and $\Phi$ appearing in the double extension case (see (4.36)), we have a third central element $\Psi$ in (4.55), which makes the two generators $P_1$ and $P_2$ noncommutative as well, with the noncommutativity being controlled by the extension parameter $\gamma$. We shall call this centrally extended Lie group $\overline{\overline{\overline{G_T}}}$ the *triply extended* *group of translations* and the corresponding Lie algebra $\overline{\overline{\overline{\mathcal{G}_T}}}$ the *triply extended algebra of translations*.

It remains now to find a unitary irreducible representation of the group $\overline{\overline{\overline{G_T}}}$. In doing so we will be following exactly the same course as for the UIR of the $\overline{\overline{G_T}}$ in Section 4.4.1. Since $\overline{\overline{\overline{G_T}}}$ is also a nilpotent Lie group, (see (4.54)), we again apply the orbit method of Kirillov.

We again change notations and replace the group parameters $p_1, p_2, q_1,\ q_2, \theta, \phi$ and $\psi$ by $a_1, a_2, a_3, a_4, a_5, a_6$ and $a_7$, respectively. Then, a generic group element has the matrix representation

$$g(a_1,a_2,a_3,a_4,a_5,a_6,a_7) = \begin{bmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}a_1 & -\frac{\alpha}{2}a_2 & \frac{\alpha}{2}a_3 & \frac{\alpha}{2}a_4 & a_5 \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}a_2 & \frac{\beta}{2}a_1 & a_6 \\ 0 & 0 & 1 & -\frac{\gamma}{2}a_4 & \frac{\gamma}{2}a_3 & 0 & 0 & a_7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (4.56)$$

Denoting by $X_1, X_2, X_3, X_4, X_5, X_6$ and $X_7$, the respective group generators, and writing a a generic Lie algebra element as $X = x^1X_1 + x^2X_2 + x^3X_3 + x^4X_4 + x^5X_5 + x^6X_6 + x^7X_7$, we have the matrix

$$X = \begin{bmatrix} 0 & 0 & 0 & -\frac{\alpha}{2}x^1 & -\frac{\alpha}{2}x^2 & \frac{\alpha}{2}x^3 & \frac{\alpha}{2}x^4 & x^5 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\beta}{2}x^2 & \frac{\beta}{2}x^1 & x^6 \\ 0 & 0 & 0 & -\frac{\gamma}{2}x^4 & \frac{\gamma}{2}x^3 & 0 & 0 & x^7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.57}$$

We represent an element $F \in \left(\overline{\overline{\overline{\mathcal{G}_T}}}\right)^*$ with the following $8 \times 8$ lower tringular matrix

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{\alpha}X_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{\alpha}X_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_5 & X_6 & X_7 & X_3 & X_4 & 0 & 0 & 0 \end{bmatrix}, \tag{4.58}$$

where the dual pairing is given by $\langle F, X\rangle = \text{tr}(FX) = \sum_{i=1}^{7} x^i X_i$. Therefore, the coadjoint action of the underlying Lie group $\overline{\overline{\overline{G_T}}}$ on the corresponding dual Lie algebra $\left(\overline{\overline{\overline{\mathcal{G}_T}}}\right)^*$ follows from the following computation

$$g(a_1, a_2, a_3, a_4, a_5, a_6, a_7)Fg(a_1, a_2, a_3, a_4, a_5, a_6, a_7)^{-1}$$
$$= \begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ -\frac{2}{\alpha}X_1' & * & * & * & * & * & * & * \\ -\frac{2}{\alpha}X_2' & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ X_5' & X_6' & X_7' & X_3' & X_4' & * & * & * \end{bmatrix}, \tag{4.59}$$

with

$$\begin{aligned} &X_1' = -\frac{2}{\alpha}X_1 + a_3X_5, \quad X_2' = -\frac{2}{\alpha}X_2 + a_4X_5, \quad X_3' = \frac{\alpha}{2}a_1X_5 + \frac{\gamma}{2}a_4X_7 + X_3, \\ &X_4' = \frac{\alpha}{2}a_2X_5 - \frac{\gamma}{2}a_3X_7 + X_4, \quad X_5' = X_5, \quad X_6' = X_6, \quad X_7' = X_7. \end{aligned} \tag{4.60}$$

The required coadjoint action of the group on the dual algebra is therefore given by

$$\begin{aligned} &Kg(a_1, a_2, a_3, a_4, a_5, a_6, a_7)(X_1, X_2, X_3, X_4, X_5, X_6, X_7) \\ &\quad = (-\frac{2}{\alpha}X_1 + a_3X_5, -\frac{2}{\alpha}X_2 + a_4X_5, \frac{\alpha}{2}a_1X_5 + \frac{\gamma}{2}a_4X_7 + X_3 \\ &\qquad , \frac{\alpha}{2}a_2X_5 - \frac{\gamma}{2}a_3X_7 + X_4, X_5, X_6, X_7). \end{aligned} \tag{4.61}$$

The nonzero entries denoted by $*$'s in (5.7) are of no interest to us. From (4.61), one observes that the three dual algebra coordinates $X_5, X_6$ and $X_7$ remain unaltered under the coadjoint action of the underlying group element, coming as they do from the center of the Lie algebra. We therefore have three polynomial invariants in our theory given by $P(F) = X_5$, $Q(F) = X_6$ and $R(F) = X_7$. The coadjoint orbits in this case are given by the sets $S_{\rho,\sigma,\tau}$ with

$$S_{\rho,\sigma,\tau} = \{F \in \left(\overline{\overline{\overline{\mathcal{G}_T}}}\right)^* \mid P(F) = \rho,\ Q(F) = \sigma,\ R(F) = \tau\}. \tag{4.62}$$

It is also obvious from (4.61) that by choosing $a_1$, $a_2$, $a_3$ and $a_4$ in a suitable manner, we can make all of $X_1'$, $X_2'$, $X_3'$ and $X_4'$ vanishing at the same time. Therefore, for nonzero values of $\rho$, $\sigma$ and $\tau$, the vector $(0, 0, 0, 0, \rho, \sigma, \tau)$ will always lie in the coadjoint orbit $S_{\rho,\sigma,\tau}$ of codimension 3. In other words, the underlying coadjoint orbit $S_{\rho,\sigma,\tau}$ turns out to be $4$ dimensional which passes through the point $(0, 0, 0, 0, \rho, \sigma, \tau)$ of the dual algebra space.

We now have to find the maximal subalgebra subordinate to $F$ given by (4.58). This maximal subalgebra or the polarizing subalgebra turns out to be of the correct dimension $\frac{3+7}{2} = 5$ and hence, the representation for $\overline{\overline{\overline{G_T}}}$ that we end up with using the orbit method will be irreducible and unitary. As in the case of $\overline{\overline{G_T}}$, the maximal abelian subalgebra $\mathfrak{h}$ of the Lie algebra $\overline{\overline{\overline{\mathcal{G}_T}}}$ serves as the polarizing subalgebra. A generic element of the corresponding abelian subgroup $H$ can be represented by the following $8 \times 8$ matrix

$$h(\theta, \phi, \psi, p_1, q_2) = \begin{bmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}p_1 & 0 & 0 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & -\frac{\gamma}{2}q_2 & 0 & 0 & 0 & \psi \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.63}$$

Then the section $\delta : H\backslash\overline{\overline{\overline{G_T}}} \to \overline{\overline{\overline{G_T}}}$ will be represented by the following matrix

$$\delta(r_1, s_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{\alpha}{2}s_2 & \frac{\alpha}{2}r_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\gamma}{2}r_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & r_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & s_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.64}$$

Thus, we again have to solve the master equation,

$$
\begin{pmatrix}
1&0&0&0&-\frac{\alpha}{2}s_2&\frac{\alpha}{2}r_1&0&0\\
0&1&0&0&0&-\frac{\beta}{2}s_2&0&0\\
0&0&1&0&\frac{\gamma}{2}r_1&0&0&0\\
0&0&0&1&0&0&0&r_1\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&s_2\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix}
1&0&0&-\frac{\alpha}{2}p_1&-\frac{\alpha}{2}p_2&\frac{\alpha}{2}q_1&\frac{\alpha}{2}q_2&\theta\\
0&1&0&0&0&-\frac{\beta}{2}p_2&\frac{\beta}{2}p_1&\phi\\
0&0&1&-\frac{\gamma}{2}q_2&\frac{\gamma}{2}q_1&0&0&\psi\\
0&0&0&1&0&0&0&q_1\\
0&0&0&0&1&0&0&q_2\\
0&0&0&0&0&1&0&p_1\\
0&0&0&0&0&0&1&p_2\\
0&0&0&0&0&0&0&1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1&0&0&-\frac{\alpha}{2}p_1&-\frac{\alpha}{2}(p_2+s_2)&\frac{\alpha}{2}(q_1+r_1)&\frac{\alpha}{2}q_2&\theta-\frac{\alpha}{2}q_2s_2+\frac{\alpha}{2}p_1r_1\\
0&1&0&0&0&-\frac{\beta}{2}(p_2+s_2)&\frac{\beta}{2}p_1&\phi-\frac{\beta}{2}p_1s_2\\
0&0&1&-\frac{\gamma}{2}q_2&\frac{\gamma}{2}(q_1+r_1)&0&0&\psi+\frac{\gamma}{2}q_2r_1\\
0&0&0&1&0&0&0&q_1+r_1\\
0&0&0&0&1&0&0&q_2\\
0&0&0&0&0&1&0&p_1\\
0&0&0&0&0&0&1&p_2+s_2\\
0&0&0&0&0&0&0&1
\end{pmatrix} \tag{4.65}
$$

$$
= \begin{pmatrix}
1&0&0&-\frac{\alpha}{2}A&0&0&\frac{\alpha}{2}B&C\\
0&1&0&0&0&0&\frac{\beta}{2}A&D\\
0&0&1&-\frac{\gamma}{2}B&0&0&0&E\\
0&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&B\\
0&0&0&0&0&1&0&A\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1
\end{pmatrix}
\begin{pmatrix}
1&0&0&0&-\frac{\alpha}{2}F&\frac{\alpha}{2}G&0&0\\
0&1&0&0&0&-\frac{\beta}{2}F&0&0\\
0&0&1&0&\frac{\gamma}{2}G&0&0&0\\
0&0&0&1&0&0&0&G\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&F\\
0&0&0&0&0&0&0&1
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1&0&0&-\frac{\alpha}{2}A&-\frac{\alpha}{2}F&\frac{\alpha}{2}G&\frac{\alpha}{2}B&C+\frac{\alpha}{2}BF-\frac{\alpha}{2}GA\\
0&1&0&0&0&-\frac{\beta}{2}F&\frac{\beta}{2}A&D+\frac{\beta}{2}AF\\
0&0&1&-\frac{\gamma}{2}B&\frac{\gamma}{2}G&0&0&E-\frac{\gamma}{2}BG\\
0&0&0&1&0&0&0&G\\
0&0&0&0&1&0&0&B\\
0&0&0&0&0&1&0&A\\
0&0&0&0&0&0&1&F\\
0&0&0&0&0&0&0&1
\end{pmatrix}. \tag{4.66}
$$

The unknowns $A, B, C, D, E, F$ and $G$ can easily be computed by comparing (4.65) with (4.66). We get

$$
\begin{aligned}
&A = p_1,\ B = q_2,\ G = r_1 + q_1,\ F = s_2 + p_2,\\
&C = \theta - \alpha q_2 s_2 + \alpha p_1 r_1 + \frac{\alpha}{2}q_1 p_1 - \frac{\alpha}{2}q_2 p_2,\\
&D = \phi - \beta p_1 s_2 - \frac{\beta}{2}p_1 p_2,\ E = \psi + \gamma q_2 r_1 + \frac{\gamma}{2}q_1 q_2.
\end{aligned} \tag{4.67}
$$

Now, the dual algebra vector lying in the underlying four dimensional coadjoint orbit was found to be $(0,0,0,0,\rho,\sigma,\tau)$. In light of (4.67), we therefore have the following theorem

**Theorem 4.4.3.** *The triply extended group of translations $\overline{\overline{\overline{G_T}}}$ admits a unitary irreducible representation realized on $L^2(\mathbb{R}^2)$. The explicit form of the representation is given by*

$$
\begin{aligned}
&(U(\theta, \phi, \psi, q_1, q_2, p_1, p_2)f)(r_1, s_2) \\
&\quad = e^{i(\theta - \alpha q_2 s_2 + \alpha p_1 r_1 + \frac{\alpha}{2} q_1 p_1 - \frac{\alpha}{2} q_2 p_2)} e^{i(\phi - \beta p_1 s_2 - \frac{\beta}{2} p_1 p_2)} \\
&\quad \times e^{i(\psi + \gamma q_2 r_1 + \frac{\gamma}{2} q_2 q_1)} f(r_1 + q_1, s_2 + p_2),
\end{aligned} \tag{4.68}
$$

*where $f \in L^2(\mathbb{R}^2, dr_1 ds_2)$.*

Now, let us take the Fourier transform of (4.68) with respect to the first coordinate $r_1$ and call the transformed coordinate $s_1$. The noncentral generators of $\overline{\overline{\overline{G_T}}}$ can be represented by self adjoint operators defined on $L^2(\mathbb{R}^2, d\mathbf{s})$ in the following manner

$$
\begin{aligned}
&\hat{P}_1 = -s_1, \quad \hat{Q}_1 = \beta s_2 - i\alpha \frac{\partial}{\partial s_1}, \\
&\hat{P}_2 = \alpha s_2 - i\gamma \frac{\partial}{\partial s_1}, \quad \hat{Q}_2 = i \frac{\partial}{\partial s_2},
\end{aligned} \tag{4.69}
$$

while the three central elements $\Theta, \Phi$ and $\Psi$ of the corresponding Lie algebra $\overline{\overline{\overline{\mathcal{G}_T}}}$ are all mapped to the identity operator $\mathbb{I}_{\mathfrak{H}}$ of the uderlying Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^2, d\mathbf{s})$. The corresponding commutation relations now read

$$
[\hat{Q}_i, \hat{P}_j] = i\alpha \delta_{i,j} \mathbb{I}_{\mathfrak{H}}, \quad [\hat{Q}_1, \hat{Q}_2] = -i\beta \mathbb{I}_{\mathfrak{H}}, \quad [\hat{P}_1, \hat{P}_2] = -i\gamma \mathbb{I}_{\mathfrak{H}}. \tag{4.70}
$$

Once again, if we write $\alpha = \hbar, \ -\beta = \vartheta$ and replace $\gamma$ by $-\gamma$ we recover (4.1) together with (4.2), the additional central extension making the two momentum operators noncommuting.

In this chapter, we presented a rigorous treatment of two-dimensional noncommutative quantum mechanics (NCQM) from group and repreentation theoretic point of views. In particular, we showed that noncommutative quantum mechanics, associated with position noncommutativity only, can be essentially regarded as coherent state quantization of phase space variables of two-dimensional non-relativistic system constrained by the symmetry of (2+1) centrally extended Galilei group. Towards the end of the chapter, we encountered double and triple central extensions of abelian group of translations in $\mathbb{R}^4$. In fact, the latter central extension turns out to play a very crucial role in two-dimensional NCQM as is about to be explored in the following chapter.

# Chapter 5
# Triply Extended Group of Translations of $\mathbb{R}^4$ as Defining Group of NCQM: relation to various gauges

The contents of this chapter are taken from the article titled "Triply Extended Group of Translations of $\mathbb{R}^4$ as Defining Group of NCQM: relation to various gauges" [19]. The triply extended group of translations of $\mathbb{R}^4$ has been encountered in the context of two-dimensional noncommutative quantum mechanics (NCQM) in [21]. In this chapter, we revisit the coadjoint orbit structure and various irreducible representations of the group associated with them. The two irreducible representations corresponding to the Landau and symmetric gauges are found to arise from the underlying defining group. The group structure of the transformations, preserving the commutation relations of NCQM, has been studied along with specific examples. Finally, the relationship of a certain family of UIRs of the underlying defining group with a family of deformed complex Hermite polynomials has been explored.

## 5.1 Introduction

The Weyl-Heisenberg group, whose generators in a unitary irreducible representation on a Hilbert space give the position and momentum operators of quantum mechanics, can be thought of as being the defining group of standard non-relativistic quantum mechanics. The analog of this group, in the setting of a two-dimensional noncommutative quantum system (i.e., a system in which the two operators of position are also non-commuting), was explored in [21]. There the possibility of an additional non-commutativity (that of the two momentum operators as well) was also considered. In the literature (see, for example, [24]), two different gauges and their physical interpretations have been pointed out, connected with this latter non-commutativity (of the momenta). We shall show in this chapter of the thesis that the irreducible representations of the resulting commutation relations, postulated there arise indeed from the irreducible unitary representations of the triply extended group of translations in $\mathbb{R}^4$, which we shall denote as $G_{\text{NC}}$ from now on. (In [21] the notation $\overline{\overline{\overline{G}}}_T$ had been used for this group). In this sense it is this group which is the defining group of noncommutative quantum mechanics. Indeed, as will be shown in the sequel, the different unitary irreducible representations of it describe all the possible types of non-commutativities presently considered in the literature.

In this chapter, we give a complete description of all the unitary irreducible representations of the group $G_{\text{NC}}$ and its Lie algebra $\mathfrak{g}_{\text{NC}}$, following the classification of the underlying coadjoint orbits. The unitary irreducible representations of the 2 dimensional Weyl-Heisenberg group are found to be sitting inside the unitary dual of $G_{\text{NC}}$. We compute the unitary irreducible representations, associated with the *Landau gauge* and *symmetric gauges* of $G_{\text{NC}}$, explicitly. The transformation group, preserving the commutation relations of noncommutative quantum mechanics, is studied along with an example related to the matrix of transformation between the UIRs of $\mathfrak{g}_{\text{NC}}$ in the Landau and symmetric gauges.

Finally, we obtain a family of coadjoint orbits in $\mathfrak{g}^*_{\text{NC}}$ that gives rise to the representations associated to the deformed complex Hermite polynomials studied at length in ([9], [6]).

## 5.2 Coadjoint orbits and UIRs of $G_{\text{NC}}$

The phase space of a free classical system, moving in two spatial dimensions, is the four dimensional abelian group of translations of $\mathbb{R}^4$. Let us denote a general element of this group by $(\mathbf{q}, \mathbf{p})$, in terms of the two-vectors of position and momentum, respectively. A generic element of the triple central extension $G_{\text{NC}}$ of this abelian group will be denoted by $(\theta, \phi, \psi, \mathbf{q}, \mathbf{p})$. The group composition law for $G_{\text{NC}}$ reads (see [21])

$$
\begin{aligned}
&(\theta, \phi, \psi, \mathbf{q}, \mathbf{p})(\theta', \phi', \psi', \mathbf{q}', \mathbf{p}') \\
&\quad = (\theta + \theta' + \frac{\alpha}{2}[\langle \mathbf{q}, \mathbf{p}' \rangle - \langle \mathbf{p}, \mathbf{q}' \rangle], \phi + \phi' + \frac{\beta}{2}[\mathbf{p} \wedge \mathbf{p}'], \psi + \psi' + \frac{\gamma}{2}[\mathbf{q} \wedge \mathbf{q}'] \\
&\quad\ \ , \mathbf{q} + \mathbf{q}', \mathbf{p} + \mathbf{p}'),
\end{aligned} \tag{5.1}
$$

where $\alpha$, $\beta$ and $\gamma$ stand for the extension parameters corresponding to $\theta$, $\phi$, and $\psi$, respectively.

A matrix realization of $G_{\rm NC}$ is as follows

$$(\theta, \phi, \psi, \mathbf{q}, \mathbf{p})_{\alpha,\beta,\gamma} = \begin{bmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & -\frac{\gamma}{2}q_2 & \frac{\gamma}{2}q_1 & 0 & 0 & \psi \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{5.2}$$

Let us denote the Lie algebra of $G_{\rm NC}$ by $\mathfrak{g}_{\rm NC}$. If we denote the basis elements of $\mathfrak{g}_{\rm NC}$ by $\Theta, \Phi, \Psi, Q_1, Q_2, P_1$ and $P_2$, corresponding to the on-parameter subgroups generated by the group parameters $\theta, \phi, \psi, p_1, p_2, q_1$ and $q_2$, respectively, we end up with the following Lie bracket relations between them

$$\begin{aligned} &[P_i, Q_j] = \alpha\delta_{i,j}\Theta, \quad [Q_1, Q_2] = \beta\Phi, \quad [P_1, P_2] = \gamma\Psi, \quad [P_i, \Theta] = 0, \\ &[Q_i, \Theta] = 0, \quad [P_i, \Phi] = 0, \quad [Q_i, \Phi] = 0, \quad [P_i, \Psi] = 0, \\ &[Q_i, \Psi] = 0, \quad [\Theta, \Phi] = 0, \quad [\Phi, \Psi] = 0, \quad [\Theta, \Psi] = 0, \quad i, j = 1, 2 \, . \end{aligned} \tag{5.3}$$

For the sake of later convenience, we switch to a different notation by replacing the group parameters $p_1$, $p_2$, $q_1$, $q_2$, $\theta$, $\phi$ and $\psi$ with $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$ and $a_7$, respectively. The respective group generators are denoted $X_1$, $X_2$, $X_3$, $X_4$, $X_5$, $X_6$ and $X_7$. Writing a generic Lie algebra element $X$ as a linear combination of the above basis elements yields

$$X = x^1 X_1 + x^2 X_2 + x^3 X_3 + x^4 X_4 + x^5 X_5 + x^6 X_6 + x^7 X_7,$$

while the corresponding matrix realization reads

$$X = \begin{bmatrix} 0 & 0 & 0 & -\frac{\alpha}{2}x^1 & -\frac{\alpha}{2}x^2 & \frac{\alpha}{2}x^3 & \frac{\alpha}{2}x^4 & x^5 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\beta}{2}x^2 & \frac{\beta}{2}x^1 & x^6 \\ 0 & 0 & 0 & -\frac{\gamma}{2}x^4 & \frac{\gamma}{2}x^3 & 0 & 0 & x^7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.4}$$

Under the above mentioned change in notation, a generic group element of $G_{\text{NC}}$ is now represented by the following matrix:

$$g(a_1,a_2,a_3,a_4,a_5,a_6,a_7) = \begin{bmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}a_1 & -\frac{\alpha}{2}a_2 & \frac{\alpha}{2}a_3 & \frac{\alpha}{2}a_4 & a_5 \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}a_2 & \frac{\beta}{2}a_1 & a_6 \\ 0 & 0 & 1 & -\frac{\gamma}{2}a_4 & \frac{\gamma}{2}a_3 & 0 & 0 & a_7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.5}$$

If we denote the dual lie algebra by $\mathfrak{g}^*_{\text{NC}}$, then an element $F$ of it can conveniently be represented as the following lower triangular matrix:

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_5 & X_6 & X_7 & X_3 & X_4 & X_1 & X_2 & 0 \end{bmatrix}, \tag{5.6}$$

with the dual pairing being given by $\langle F, X\rangle = \text{tr}(FX) = \sum_{i=1}^{7} x^i X_i$. We, therefore, note that

$$g(a_1,a_2,a_3,a_4,a_5,a_6,a_7)Fg(a_1,a_2,a_3,a_4,a_5,a_6,a_7)^{-1}$$
$$= \begin{bmatrix} * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * \\ X_5' & X_6' & X_7' & X_3' & X_4' & X_1' & X_2' & * \end{bmatrix}, \tag{5.7}$$

with

$$\begin{aligned} &X_1' = X_1 - \frac{\alpha}{2}a_3X_5 + \frac{\beta}{2}a_2X_6, \quad X_2' = X_2 - \frac{\alpha}{2}a_4X_5 - \frac{\beta}{2}a_1X_6, \\ &X_3' = X_3 + \frac{\gamma}{2}a_4X_7 + \frac{\alpha}{2}a_1X_5, \quad X_4' = X_4 - \frac{\gamma}{2}a_3X_7 + \frac{\alpha}{2}a_2X_5, \\ &X_5' = X_5, \quad X_6' = X_6, \quad X_7' = X_7. \end{aligned} \tag{5.8}$$

The entries, denoted by $*$'s in (5.7), are of no interest for the present computations. Thus we arrive at the required coadjoint action of the group on the dual algebra, given by

$$Kg(a_1,a_2,a_3,a_4,a_5,a_6,a_7)(X_1,X_2,X_3,X_4,X_5,X_6,X_7)$$
$$= (X_1 - \frac{\alpha}{2}a_3X_5 + \frac{\beta}{2}a_2X_6, \quad X_2 - \frac{\alpha}{2}a_4X_5 - \frac{\beta}{2}a_1X_6$$

$$, X_3 + \frac{\gamma}{2} a_4 X_7 + \frac{\alpha}{2} a_1 X_5,\ X_4 - \frac{\gamma}{2} a_3 X_7 + \frac{\alpha}{2} a_2 X_5,\ X_5,\ X_6,\ X_7). \quad (5.9)$$

From (5.9), we immediately see that $X_5$, $X_6$ and $X_7$ belonging to the center of $\mathfrak{g}_{\text{NC}}$ remain invariant under the coadjoint action of $G_{\text{NC}}$, as expected. These three invariant coordinates on the right side of (5.9) refer to the $G_{\text{NC}}$-invariant polynomial functions on $\mathfrak{g}^*_{\text{NC}}$ related to $X_5, X_6, X_7$, respectively, all belonging to to the center $\mathcal{Z}(\mathfrak{g}_{\text{NC}})$. We therefore have three polynomial invariants in the present setting given by $P(F) = X_5$, $Q(F) = X_6$ and $R(F) = X_7$. Now the coadjoint orbits can be denoted using the sets $S_{\rho,\sigma,\tau}$ with

$$S_{\rho,\sigma,\tau} = \{F \in \mathfrak{g}_{\text{NC}}{}^* \mid P(F) = \rho,\ Q(F) = \sigma,\ R(F) = \tau\}. \quad (5.10)$$

Let us pause briefly and study the geometry of the relevant coadjoint orbits before computing all unitary irreducible representations of $G_{\text{NC}}$.

The triple $(\rho, \sigma, \tau)$ solely determines the geometry of the underlying coadjoint orbit. For all three parameters $\rho$, $\sigma$ and $\tau$ assuming non-zero values, the vector $(0, 0, 0, 0, \rho, \sigma, \tau)$ will always lie in the coadjoint orbit $S_{\rho,\sigma,\tau}$ of codimension 3. In other words, the underlying coadjoint orbit $S_{\rho,\sigma,\tau}$ turns out to be $4$ dimensional which passes through the point $(0, 0, 0, 0, \rho, \sigma, \tau)$ of the dual algebra space $\mathfrak{g}^*_{\text{NC}}$. A generic element of $S_{\rho,\sigma,\tau}$ can be written as $(k_1, k_2, k_3, k_4, \rho, \sigma, \tau)$, where $(k_1, k_2, k_3, k_4)$ takes values in $\mathbb{R}^4$. These nonintersecting four dimensional coadjoint orbits (one for each choice of values of the parameters $\rho, \sigma, \tau$) are sitting inside the 7-dimensional dual Lie algebra $\mathfrak{g}^*_{\text{NC}}$ in the following way. $\mathbb{R}^7$ can be regarded as a continuum of nonintersecting $\mathbb{R}^4$ spaces going through each point of an

$\mathbb{R}^3$ space embedded in $\mathbb{R}^7$. Let us denote a generic point of the embedded $\mathbb{R}^3$ space by $(0,0,0,0,\rho,\sigma,\tau)$. Restricting $\rho$, $\sigma$, and $\tau$ to nonzero real values, we obtain a disconnected toplogical space with $8$ connected components which we denote as $\mathbb{R}^3_0$. An $\mathbb{R}^4$ coadjoint orbit $\mathcal{O}_4^{\rho,\sigma,\tau}$ passes through a point $(0,0,0,0,\rho,\sigma,\tau) \in \mathfrak{g}^*_{\text{NC}}$ for nonzero $\rho$, $\sigma$, and $\tau$, as we have already noted.

Now we consider the rest of the points $(0,0,0,0,\rho,\sigma,\tau)$ of the underlying $\mathbb{R}^3$ space embedded in $\mathbb{R}^7$ and denote this set $\mathbb{R}^3 \setminus \mathbb{R}^3_0$ by $\mathbb{R}^3_1$. Let us subdivide the points belonging to $\mathbb{R}^3_1$ into the following classes

- The points $\mathbb{S}_{\rho,\sigma}$ on the $\rho - \sigma$ plane ($\tau = 0$) with nonzero real values of both $\rho$ and $\sigma$, e.g. $(0,0,0,0,\rho,\sigma,0)$.
- The points $\mathbb{S}_{\rho,\tau}$ on the $\rho - \tau$ plane ($\sigma = 0$) with nonzero real values of both $\rho$ and $\tau$, e.g. $(0,0,0,0,0,\rho,0,\tau)$.
- The points $\mathbb{S}_{\sigma,\tau}$ on the $\sigma - \tau$ plane ($\rho = 0$) with nonzero real values of both $\sigma$ and $\tau$, e.g. $(0,0,0,0,0,\sigma,\tau)$.
- The disconnected set of points $\mathbb{L}_\rho$ on a line with $\sigma$ and $\tau$ both being zero and $\rho$ being nonzero, e.g. $(0,0,0,0,\rho,0,0)$.
- The disconnected set of points $\mathbb{L}_\sigma$ on a line with $\rho$ and $\tau$ both being zero and $\sigma$ being nonzero, e.g. $(0,0,0,0,0,\sigma,0)$.
- The disconnected set of points $\mathbb{L}_\tau$ on a line with $\rho$ and $\sigma$ both being zero and $\tau$ being nonzero, e.g. $(0,0,0,0,0,0,\tau)$.

- The origin $O$ of the underlying dual algebra space $\mathbb{R}^7$ with $\rho = \sigma = \tau = 0$. The coordinate of $O$ is just $(0, 0, 0, 0, 0, 0, 0)$.

From the coadjoint action given by (5.9), one finds all the coadjoint orbits associated with the above enumerated points in the embedded $\mathbb{R}^3$ space. These coadjoint orbits are listed below

- ◦ $\mathbb{R}_4$ spaces $\mathcal{O}_4^{\rho,\sigma,0}$ each of which passes through a point lying in a disconnected topological space $\mathbb{S}_{\rho,\sigma}$, with $\rho \neq 0$ and $\sigma \neq 0$ rendering to its disconnectedness in the usual Euclidean topology. A generic point on such an orbit (fixed $\rho$ and $\sigma$) is given by $(k_1, k_2, k_3, k_4, \rho, \sigma, 0)$ with each of $k_1$, $k_2$, $k_3$, and $k_4$ assuming real values.
- ◦ $\mathbb{R}_4$ spaces $\mathcal{O}_4^{\rho,0,\tau}$ each of which passes through a point lying in the disconnected set $\mathbb{S}_{\rho,\tau}$, where disconnectedness refers to one in the usual Euclidean topology with each of $\rho$ and $\tau$ being nonzero. A generic point on such an orbit (fixed $\rho$ and $\tau$) is given by $(k_1, k_2, k_3, k_4, \rho, 0, \tau)$ with each of $k_1$, $k_2$, $k_3$, and $k_4$ assuming real values.
- ◦ $\mathbb{R}_4$ spaces $\mathcal{O}_4^{0,\sigma,\tau}$ each of which passes through a point lying in the disconnected set $\mathbb{S}_{\sigma,\tau}$, the disconnectedness in the usual Euclidean topology being attributed to $\sigma \neq 0$ and $\tau \neq 0$. A generic point on such an orbit (fixed $\sigma$ and $\tau$) is given by $(k_1, k_2, k_3, k_4, 0, \sigma, \tau)$ with each of $k_1$, $k_2$, $k_3$, and $k_4$ assuming real values.
- ◦ $\mathbb{R}_4$ spaces $\mathcal{O}_4^{\rho,0,0}$ each of which passes through a point lying in the disconnected set $\mathbb{L}_\rho$ ($\rho \neq 0$ contributes to its disconnectedness in the usual Euclidean topology). A

generic point on such an orbit (fixed $\rho$) is given by $(k_1, k_2, k_3, k_4, \rho, 0, 0)$ with each of $k_1$, $k_2$, $k_3$, and $k_4$ assuming real values.

- $\mathbb{R}_2$-plane ${}^{c_3,c_4}\mathcal{O}_2^{0,\sigma,0}$ due to a fixed ordered pair $(c_3, c_4)$. Such a plane lies in the $\mathbb{R}^4$ space that passes through each point of $\mathbb{L}_\sigma$, where $\mathbb{L}_\sigma$ is the punctured line with $\sigma \neq 0$. A generic point on such an orbit (fixed $\sigma$) is given by $(k_1, k_2, c_3, c_4, 0, \sigma, 0)$ with both $k_1$ and $k_2$ assuming real values.
- $\mathbb{R}_2$ plane ${}^{c_1,c_2}\mathcal{O}_2^{0,0,\tau}$ due to a fixed ordered pair $(c_1, c_2)$. The plane lies in the $\mathbb{R}^4$ space that passes through each point of the punctured line $\mathbb{L}_\tau$ with $\tau \neq 0$. A generic point on such an orbit (fixed $\tau$) is given by $(c_1, c_2, k_3, k_4, 0, 0, \tau)$ with both $k_3$ and $k_4$ assuming real values.
- $0$-dimensional point ${}^{c_1,c_2,c_3,c_4}\mathcal{O}_0^{0,0,0}$ due to a fixed ordered quadruple $(c_1, c_2, c_3, c_4)$. Such a point lies in the $\mathbb{R}^4$ space that passes through the origin $O$. The corresponding zero dimensional orbit is denoted as $(c_1, c_2, c_3, c_4, 0, 0, 0)$.

We are now all set to resume our computations on finding the unitary irreducible representations of $G_{\text{NC}}$. From the method of orbits (see [34]), we know that the unitary irreducible representations of the connected simply connected nilpotent Lie group $G_{\text{NC}}$ are in 1-1 correspondence with its coadjoint orbits. The UIRs corresponding to the $4$ dimensional orbits have functional dimension $2$, i.e. the representation space is $L^2(\mathbb{R}^2)$ with respect to the usual Lebesgue measure. Let us compute the UIRs $U^{\rho}_{\sigma,\tau}$, $U^{\rho}_{\sigma,0}$, $U^{\rho}_{0,\tau}$, $U^{0}_{\sigma,\tau}$, and $U^{\rho}_{0,0}$ corresponding to the coadjoint orbits $\mathcal{O}_4^{\rho,\sigma,\tau}$, $\mathcal{O}_4^{\rho,\sigma,0}$, $\mathcal{O}_4^{\rho,0,\tau}$, $\mathcal{O}_4^{0,\sigma,\tau}$, and $\mathcal{O}_4^{\rho,0,0}$, respectively.

The most crucial part for the remaining task is to find the polarizing subalgebra, i.e. a maximal subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\text{NC}}$ which is subordinate to $F \in \mathfrak{g}^*_{\text{NC}}$ with representation given by (5.6). In other words, $\mathfrak{h}$ must satisfy $F|_{[\mathfrak{h},\mathfrak{h}]} = 0$. For the $4$ dimensional orbits, the polarizing subalgebra has to have dimension equal to $\frac{7+3}{2}$, i.e. $5$. The maximal abelian subalgebra of $\mathfrak{g}_{\text{NC}}$ serves as the polarizing subalgebra in this case. A generic element $h$ of the corresponding abelian subgroup $H \subset G_{\text{NC}}$ has the following matrix representation:

$$h(\theta,\phi,\psi,p_1,q_2) = \begin{bmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}p_1 & 0 & 0 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & -\frac{\gamma}{2}q_2 & 0 & 0 & 0 & \psi \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (5.11)$$

Let us consider the following section $\delta : H\backslash G_{\text{NC}} \to G_{\text{NC}}$. The matrix representation of the section $\delta$ then reads

$$\delta(r_1, s_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{\alpha}{2}s_2 & \frac{\alpha}{2}r_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\gamma}{2}r_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & r_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & s_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.12}$$

All we have to do now is to solve the master equation,

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{\alpha}{2}s_2 & \frac{\alpha}{2}r_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}s_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\gamma}{2}r_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & r_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & s_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & -\frac{\gamma}{2}q_2 & \frac{\gamma}{2}q_1 & 0 & 0 & \psi \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}(p_2+s_2) & \frac{\alpha}{2}(q_1+r_1) & \frac{\alpha}{2}q_2 & \theta-\frac{\alpha}{2}q_2s_2+\frac{\alpha}{2}p_1r_1 \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}(p_2+s_2) & \frac{\beta}{2}p_1 & \phi-\frac{\beta}{2}p_1s_2 \\ 0 & 0 & 1 & -\frac{\gamma}{2}q_2 & \frac{\gamma}{2}(q_1+r_1) & 0 & 0 & \psi+\frac{\gamma}{2}q_2r_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1+r_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & p_2+s_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
&= \begin{pmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}A & 0 & 0 & \frac{\alpha}{2}B & C \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{\beta}{2}A & D \\ 0 & 0 & 1 & -\frac{\gamma}{2}B & 0 & 0 & 0 & E \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & A \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{\alpha}{2}F & \frac{\alpha}{2}G & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}F & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\gamma}{2}G & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & G \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & F \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}A & -\frac{\alpha}{2}F & \frac{\alpha}{2}G & \frac{\alpha}{2}B & C+\frac{\alpha}{2}BF-\frac{\alpha}{2}GA \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}F & \frac{\beta}{2}A & D+\frac{\beta}{2}AF \\ 0 & 0 & 1 & -\frac{\gamma}{2}B & \frac{\gamma}{2}G & 0 & 0 & E-\frac{\gamma}{2}BG \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & G \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & A \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & F \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned} \tag{5.14}$$

The unknowns $A, B, C, D, E, F$ and $G$ can easily be computed by comparing (5.13) with (5.14). We get

$$\begin{aligned}
&A = p_1,\ B = q_2,\ G = r_1 + q_1,\ F = s_2 + p_2, \\
&C = \theta - \alpha q_2 s_2 + \alpha p_1 r_1 + \frac{\alpha}{2} q_1 p_1 - \frac{\alpha}{2} q_2 p_2, \\
&D = \phi - \beta p_1 s_2 - \frac{\beta}{2} p_1 p_2,\ E = \psi + \gamma q_2 r_1 + \frac{\gamma}{2} q_1 q_2.
\end{aligned} \tag{5.15}$$

Now, the dual algebra vector lying in the $4$ dimensional coadjoint orbit $\mathcal{O}_4^{\rho,\sigma,\tau}$ was found to be $(0, 0, 0, 0, \rho, \sigma, \tau)$. Using (5.15), a family of representations $U^{\rho}_{\sigma,\tau}$ associated with these coadjoint orbits follow immediately

$$\begin{aligned}
&(U^{\rho}_{\sigma,\tau}(\theta, \phi, \psi, q_1, q_2, p_1, p_2) f)(r_1, s_2) \\
&\quad = e^{i\rho(\theta - \alpha q_2 s_2 + \alpha p_1 r_1 + \frac{\alpha}{2} q_1 p_1 - \frac{\alpha}{2} q_2 p_2)} e^{i\sigma(\phi - \beta p_1 s_2 - \frac{\beta}{2} p_1 p_2)} \\
&\quad \times e^{i\tau(\psi + \gamma q_2 r_1 + \frac{\gamma}{2} q_2 q_1)} f(r_1 + q_1, s_2 + p_2),
\end{aligned} \tag{5.16}$$

where none of $\rho$, $\sigma$ and $\tau$ are zero and $f \in L^2(\mathbb{R}^2, dr_1 ds_2)$.

The required dimension of the polarizing subalgebra $\mathfrak{h}$ due to the other coadjoint orbits is also $5$. And hence, the polarizing subalgebra that was used to compute the UIRs associated with the orbits $\mathcal{O}_4^{\rho,\sigma,\tau}$, also serves for the other $4$ dimensional orbits of $G_{\text{NC}}$. Therefore, the results, obtained in (5.15), apply to all other $4$ dimensional coadjoint orbits, as well.

Knowing that the 4-dimensional orbit $\mathcal{O}^{\rho}_{\sigma,0}$ passes through the point $(0,0,0,0,\rho,\sigma,0)$, we can easily obtain the corresponding family of UIRs:

$$
\begin{aligned}
&(U^{\rho}_{\sigma,0}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)f)(r_1,s_2)\\
&\quad = e^{i\rho(\theta-\alpha q_2 s_2+\alpha p_1 r_1+\frac{\alpha}{2}q_1p_1-\frac{\alpha}{2}q_2p_2)}e^{i\sigma(\phi-\beta p_1 s_2-\frac{\beta}{2}p_1p_2)}f(r_1+q_1,s_2+p_2),
\end{aligned}
\tag{5.17}
$$

where $f \in L^2(\mathbb{R}^2, dr_1ds_2)$.

Now, the orbit $\mathcal{O}^{\rho}_{0,\tau}$ was found to pass through the point $(0,0,0,0,\rho,0,\tau)$ of the dual algebra space $\mathfrak{g}^*_{\text{NC}}$. Therefore, the continuous family of UIRs corresponding to these 4-dimensional coadjoint orbits follows as

$$
\begin{aligned}
&(U^{\rho}_{0,\tau}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)f)(r_1,s_2)\\
&\quad = e^{i\rho(\theta-\alpha q_2 s_2+\alpha p_1 r_1+\frac{\alpha}{2}q_1p_1-\frac{\alpha}{2}q_2p_2)}e^{i\tau(\psi+\gamma q_2 r_1+\frac{\gamma}{2}q_2q_1)}f(r_1+q_1,s_2+p_2),
\end{aligned}
\tag{5.18}
$$

where $f \in L^2(\mathbb{R}^2, dr_1ds_2)$.

Also, $\mathcal{O}^{0,\sigma,\tau}_4$, being a $4$ dimensional coadjoint orbit, passes through the point $(0,0,0,0,0,\sigma,\tau) \in \mathfrak{g}^*_{\text{NC}}$. Therefore, the corresponding family of UIRs is given by

$$
\begin{aligned}
&(U^{0}_{\sigma,\tau}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)f)(r_1,s_2)\\
&\quad = e^{i\sigma(\phi-\beta p_1 s_2-\frac{\beta}{2}p_1p_2)}e^{i\tau(\psi+\gamma q_2 r_1+\frac{\gamma}{2}q_2q_1)}f(r_1+q_1,s_2+p_2),
\end{aligned}
\tag{5.19}
$$

where both $\sigma$ and $\tau$ are nonzero and $f \in L^2(\mathbb{R}^2, dr_1ds_2)$.

The only remaining $4$ dimensional coadjoint orbit is $\mathcal{O}^{\rho}_{0,0}$ which passes through the point $(0,0,0,0,\rho,0,0) \in \mathfrak{g}^*_{\text{NC}}$. The family of unitary irreducible representations associated with these orbits are found to be

$$(U^{\rho}_{0,0}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)f)(r_1,s_2)$$
$$= e^{i\rho(\theta-\alpha q_2 s_2+\alpha p_1 r_1+\frac{\alpha}{2}q_1p_1-\frac{\alpha}{2}q_2p_2)}f(r_1+q_1,s_2+p_2), \tag{5.20}$$

where $\rho$ is nonzero and $f \in L^2(\mathbb{R}^2, dr_1 ds_2)$.

There are two $2$-dimensional coadjoint orbits in the present setting. The orbits ${}^{c_3,c_4}\mathcal{O}^{0,\sigma,0}_2$ and ${}^{c_1,c_2}\mathcal{O}^{0,0,\tau}_2$ pass through the points $(0,0,c_3,c_4,0,\sigma,0)$ and $(c_1,c_2,0,0,0,0,\tau)$ of $\mathfrak{g}^*_{\text{NC}}$, respectively. But the required dimension of the polarizing subalgebra is no longer $5$. It is now $\frac{7+5}{2}=6$.

In case of the coadjoint orbit ${}^{c_1,c_2}\mathcal{O}_2^{0,0,\tau}$, a generic element of the polarizing subalgebra $\mathfrak{h}$ is given by

$$\begin{bmatrix} 0 & 0 & 0 & -\frac{\alpha}{2}x^1 & -\frac{\alpha}{2}x^2 & \frac{\alpha}{2}x^3 & 0 & x^5 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\beta}{2}x^2 & \frac{\beta}{2}x^1 & x^6 \\ 0 & 0 & 0 & 0 & \frac{\gamma}{2}x^3 & 0 & 0 & x^7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

One can easily verify that under the above choice of polarizing subalgebra the following holds

$$F|_{[\mathfrak{h},\mathfrak{h}]} = 0, \tag{5.21}$$

where the matrix representation of a dual algebra element $F$ is given by (5.6). Therefore, an element of the corresponding subgroup $H \subset G_{\text{NC}}$ (note that this subgroup is no longer abelian) is as follows

$$h(\theta,\phi,\psi,p_1,p_2,q_1) = \begin{bmatrix} 1 & 0 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & 0 & \theta \\ 0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & 0 & \frac{\gamma}{2}q_1 & 0 & 0 & \psi \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.22}$$

We now consider the following section $\delta : H\backslash G_{\text{NC}} \to G_{\text{NC}}$ given by

$$\delta(r) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha}{2}r & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\gamma}{2}r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.23}$$

Then the corresponding master equation leads to

$$
\begin{aligned}
&\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha}{2}r & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -\frac{\gamma}{2}r & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & r\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta\\
0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi\\
0 & 0 & 1 & -\frac{\gamma}{2}q_2 & \frac{\gamma}{2}q_1 & 0 & 0 & \psi\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & p_2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}\\
&= \begin{pmatrix}
1 & 0 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}(q_2+r) & \theta+\frac{\alpha}{2}p_2 r\\
0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi\\
0 & 0 & 1 & -\frac{\gamma}{2}(q_2+r) & \frac{\gamma}{2}q_1 & 0 & 0 & \psi-\frac{\gamma}{2}q_1 r\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2+r\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & p_2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned}
\tag{5.24}
$$

$$
\begin{aligned}
&= \begin{pmatrix}
1 & 0 & 0 & -\frac{\alpha}{2}A & -\frac{\alpha}{2}B & \frac{\alpha}{2}C & 0 & D\\
0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}B & \frac{\beta}{2}A & E\\
0 & 0 & 1 & 0 & \frac{\gamma}{2}C & 0 & 0 & F\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & C\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & A\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & B\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha}{2}G & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -\frac{\gamma}{2}G & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & G\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}\\
&= \begin{pmatrix}
1 & 0 & 0 & -\frac{\alpha}{2}A & -\frac{\alpha}{2}B & \frac{\alpha}{2}C & \frac{\alpha}{2}G & D-\frac{\alpha}{2}BG\\
0 & 1 & 0 & 0 & 0 & -\frac{\beta}{2}B & \frac{\beta}{2}A & E\\
0 & 0 & 1 & -\frac{\gamma}{2}G & \frac{\gamma}{2}C & 0 & 0 & F+\frac{\gamma}{2}CG\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & C\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & G\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & A\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & B\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\end{aligned}
\tag{5.25}
$$

The unknowns $A, B, C, D, E, F$ and $G$ can easily be computed by comparing (5.24) with (5.25). We get

$$
\begin{aligned}
&A = p_1,\ B = p_2,\ C = q_1,\ G = q_2 + r,\\
&D = \theta + \alpha p_2 r + \frac{\alpha}{2}p_2 q_2,\\
&E = \phi,\ F = \psi - \gamma q_1 r - \frac{\gamma}{2}q_1 q_2.
\end{aligned}
\tag{5.26}
$$

Now, the dual algebra vector lying in the 2 dimensional coadjoint orbit ${}^{c_1,c_2}\mathcal{O}_2^{0,0,\tau}$ was found to be $(c_1, c_2, 0, 0, 0, 0, \tau)$. Using (5.26), a family of representations $U_{0,0,\tau}^{c_1,c_2}$ associated with these coadjoint orbits, for fixed $c_1$ and $c_2$, follow immediately

$$
(U_{0,0,\tau}^{c_1,c_2}(\theta, \phi, \psi, q_1, q_2, p_1, p_2)f)(r)
$$

$$= e^{ic_1p_1+ic_2p_2}e^{i\tau(\psi-\gamma q_1r-\frac{\gamma}{2}q_1q_2)}f(r+q_2), \tag{5.27}$$

where $\tau$ is nonzero and $f \in L^2(\mathbb{R}, dr)$.

Following exactly the same computations of the 2 dimensional coadjoint orbits ${}^{c_1,c_2}\mathcal{O}_2^{0,0,\tau}$ except for a different choice of 6 dimensional polarizing subalgebra, one can derive the UIRs associated with the remaining 2 dimensional coadjoint orbits ${}^{c_3,c_4}\mathcal{O}_2^{0,\sigma,0}$ with a fixed ordered pair $(c_3, c_4)$,

$$(U_{0,\sigma,0}^{c_3,c_4}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)f)(s)$$
$$= e^{ic_3q_1+ic_4q_2}e^{i\sigma(\phi-\beta p_1s-\frac{\beta}{2}p_1p_2)}f(s+p_2), \tag{5.28}$$

where $\sigma$ is nonzero and $f \in L^2(\mathbb{R}, ds)$.

There is only one zero dimensional orbit for $\mathfrak{g}_{\text{NC}}$. The zero dimensional coadjoint orbit ${}^{c_1,c_2,c_3,c_4}\mathcal{O}_0^{0,0,0}$ passes through the point $(c_1, c_2, c_3, c_4, 0, 0, 0)$ of the dual Lie algebra $\mathfrak{g}_{\text{NC}}^*$. And hence, follows the associated family of 1 dimensional representations,

$$U_{0,0,0}^{c_1,c_2,c_3,c_4}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)$$
$$= e^{ic_1p_1+ic_2p_2+ic_3q_1+ic_4q_2}. \tag{5.29}$$

### 5.3 Representation of the Lie algebra $\mathfrak{g}_{\text{NC}}$

The basis elements for the Lie algebra $\mathfrak{g}_{\text{NC}}$ are $Q_1$, $Q_2$, $P_1$, $P_2$, $\Theta$, $\Phi$, and $\Psi$, where the last three elements form the 3 dimensional center of the underlying Lie algebra. One has to represent these basis elements as appropriate operators on the corresponding group representation space, (see section 5.2). We compute the various unitary irreducible group representations restricted to one-parameter subgroups and thereby find the Hilbert space operators associated with the respective group parameters using the following equation:

$$\hat{X}_\eta = -iC\frac{dU(\eta)}{d\eta}\,|_{\eta=0}, \tag{5.30}$$

where $\eta$ is one of the group parameters of $G_{\text{NC}}$ and $C$ is a constant fixed by the corresponding UIR with appropriate dimension.

We consider the following cases

### 5.3.1 Case $\rho \neq 0, \sigma \neq 0, \tau \neq 0$.

A family of unitary irreducible representations $U^{\rho}_{\sigma,\tau}$ associated with the $4$ dimensional coadjoint orbits $\mathcal{O}_4^{\rho,\sigma,\tau}$ were found (see (5.16)) in section 5.2. These representations are labeled by nonzero real values of $\rho$, $\sigma$, and $\tau$. Let us now consider a unitary operator $T$ on $L^2(\mathbb{R}^2, dr_1 ds_2)$ given by

$$(Tf)(r_1, s_2) = f(r_1, -s_2), \tag{5.31}$$

with $f \in L^2(\mathbb{R}^2, dr_1 ds_2)$. The inverse $T^{-1}$ turns out immediately to be equal to $T$.

Then a very straightforward computation shows that

$$T^{-1}U^{\rho}_{\sigma,\tau}T = \tilde{U}^{\rho}_{\sigma,\tau}, \tag{5.32}$$

with $f$ lying in $L^2(\mathbb{R}^2, dr_1 ds_2)$ and $\tilde{U}^{\rho}_{\sigma,\tau}$ given as

$$(\tilde{U}^{\rho}_{\sigma,\tau}(\theta, \phi, \psi, q_1, q_2, p_1, p_2)f)(r_1, s_2)$$

$$= e^{i\rho(\theta+\alpha q_2 s_2+\alpha p_1 r_1+\frac{\alpha}{2}q_1 p_1-\frac{\alpha}{2}q_2 p_2)}e^{i\sigma(\phi+\beta p_1 s_2-\frac{\beta}{2}p_1 p_2)}$$

$$\times e^{i\tau(\psi+\gamma q_2 r_1+\frac{\gamma}{2}q_2 q_1)}f(r_1+q_1, s_2-p_2). \tag{5.33}$$

Let us now take the inverse Fourier transform of (5.33) with respect to second coordinate $s_2$ and call it $r_2$. Then using (5.30) with $C = \frac{1}{\rho\alpha}$ the noncentral elements of $\mathfrak{g}_{\text{NC}}$ can be represented as the following operators on $L^2(\mathbb{R}^2, dr_1 dr_2)$:

$$\begin{aligned} \hat{Q}_1 &= r_1 + i\vartheta\frac{\partial}{\partial r_2}, \qquad \hat{Q}_2 = r_2, \\ \hat{P}_1 &= -i\hbar\frac{\partial}{\partial r_1}, \qquad \hat{P}_2 = -\frac{\mathcal{B}}{\hbar}r_1 - i\hbar\frac{\partial}{\partial r_2}, \end{aligned} \tag{5.34}$$

with the following identification:

$$\hbar = \frac{1}{\rho\alpha}, \quad \vartheta = -\frac{\sigma\beta}{(\rho\alpha)^2}, \text{ and } \mathcal{B} = -\frac{\tau\gamma}{(\rho\alpha)^2}. \tag{5.35}$$

$B$, here, can be interpreted as the constant magnetic field coupled to the standard quantum mechanical system. The commutators between the Hilbert space operators (5.34) now read

$$\begin{aligned} &[\hat{Q}_1, \hat{P}_1] = [\hat{Q}_2, \hat{P}_2] = i\hbar\mathbb{I}, \quad [\hat{Q}_1, \hat{Q}_2] = i\vartheta\mathbb{I}, \\ &[\hat{P}_1, \hat{P}_2] = i\mathcal{B}\mathbb{I}, \quad [\hat{Q}_1, \hat{P}_2] = [\hat{Q}_2, P_1] = 0, \end{aligned} \tag{5.36}$$

$\mathbb{I}$ being the identity operator on $L^2(\mathbb{R}^2, dr_1 dr_2)$. Physically, (5.34) – (5.36) correspond to the so-called *Landau gauge* for the $\hat{Q}$'s (see [24] and articles cited therein for a detailed account).

### 5.3.2 Case $\rho \neq 0, \sigma \neq 0, \tau = 0.$

Let us consider the group representations $U^{\rho}_{\sigma,0}$, pertaining to the $4$ dimensional coadjoint orbits $\mathcal{O}_4^{\rho,\sigma,0}$ and given by equation (5.17). We can find a unitary operator on $L^2(\mathbb{R}^2, dr_1 ds_2)$ to obtain a unitary irreducible representation which will be equivalent to the one given by (5.17). One then has to take the inverse Fourier transform of the equivalent representation thus obtained with respect to the second coordinate. The pertinent representation of the algebra then reads off using (5.30) with $C = \frac{1}{\rho\alpha}$

$$\begin{aligned}
&\hat{Q}_1 = r_1 + i\vartheta\frac{\partial}{\partial r_2}, \quad \hat{Q}_2 = r_2, \\
&\hat{P}_1 = -i\hbar\frac{\partial}{\partial r_1}, \quad \hat{P}_2 = -i\hbar\frac{\partial}{\partial r_2},
\end{aligned} \tag{5.37}$$

with the same identification given by (5.35). And the commutators between the corresponding Hilbert space operators read

$$\begin{aligned}
&[\hat{Q}_1, \hat{P}_1] = [\hat{Q}_2, \hat{P}_2] = i\hbar\mathbb{I}, \quad [\hat{Q}_1, \hat{Q}_2] = i\vartheta\mathbb{I}, \\
&[\hat{P}_1, \hat{P}_2] = 0, \quad [\hat{Q}_1, \hat{P}_2] = [\hat{Q}_2, P_1] = 0.
\end{aligned} \tag{5.38}$$

Note that here $\mathcal{B} = -\frac{\tau\gamma}{(\rho\alpha)^2} = 0$. Physically, it refers to the same system (5.36) with the magnetic field turned off.

### 5.3.3 Case $\rho \neq 0, \sigma = 0, \tau \neq 0.$

A continuous family of unitary irreducible representations was found for the group $G_{\text{NC}}$ in (5.18) arising from the coadjoint orbits $\mathcal{O}_4^{\rho,0,\tau}$. We can now carry out a procedure similar to the one adopted in Sec (5.3.1) to find a unitary irreducible representation equivalent to

(5.18). The corresponding representation of the Lie algebra $\mathfrak{g}_{\text{NC}}$ then reads

$$
\begin{aligned}
&\hat{Q}_1 = r_1, \quad \hat{Q}_2 = r_2, \\
&\hat{P}_1 = -i\hbar\frac{\partial}{\partial r_1}, \quad \hat{P}_2 = -\frac{\mathcal{B}}{\hbar}r_1 - i\hbar\frac{\partial}{\partial r_2},
\end{aligned}
\tag{5.39}
$$

where $\hbar$ and $\mathcal{B}$ follow from (5.35). Here also we have used $C = \frac{1}{\rho\alpha}$ in (5.30) to compute the relevant noncentral generators. And the corresponding commutators are given as

$$
\begin{aligned}
&[\hat{Q}_1, \hat{P}_1] = [\hat{Q}_2, \hat{P}_2] = i\hbar\mathbb{I}, \quad [\hat{Q}_1, \hat{Q}_2] = 0, \\
&[\hat{P}_1, \hat{P}_2] = i\mathcal{B}\mathbb{I}, \quad [\hat{Q}_1, \hat{P}_2] = [\hat{Q}_2, P_1] = 0.
\end{aligned}
\tag{5.40}
$$

Physically, (5.40) just represents a *Landau system* in the presence of a constant magnetic field $\mathcal{B}$.

### 5.3.4 Case $\rho \neq 0, \sigma = 0, \tau = 0$.

In this case as well, we obtain an irreducible representation of $G_{\text{NC}}$, unitarily equivalent to $U^{\rho}_{0,0}$ given by (5.20). The associated $4$ dimensional coadjoint orbits were $\mathcal{O}^{\rho,0,0}_4$. Now the representation of $\mathfrak{g}_{\text{NC}}$ on $L^2(\mathbb{R}^2, dr_1dr_2)$ reads off

$$
\begin{aligned}
&\hat{Q}_1 = r_1, \quad \hat{Q}_2 = r_2, \\
&\hat{P}_1 = -i\hbar\frac{\partial}{\partial r_1}, \quad \hat{P}_2 = -i\hbar\frac{\partial}{\partial r_2},
\end{aligned}
\tag{5.41}
$$

with the canonical commutation relations of standard quantum mechanics given by

$$
\begin{aligned}
&[\hat{Q}_1, \hat{P}_1] = [\hat{Q}_2, \hat{P}_2] = i\hbar\mathbb{I}, \quad [\hat{Q}_1, \hat{Q}_2] = 0, \\
&[\hat{P}_1, \hat{P}_2] = 0, \quad [\hat{Q}_1, \hat{P}_2] = [\hat{Q}_2, P_1] = 0.
\end{aligned}
\tag{5.42}
$$

*The unitary irreducible representation of standard quantum mechanics is sitting inside the unitary dual of the triply extended group $G_{\text{NC}}$ of translations in $\mathbb{R}^4$!*

### 5.3.5 Case $\rho = 0, \sigma \neq 0, \tau \neq 0$.

A family of unitary irreducible representations of $G_{\text{NC}}$ equivalent to $U^0_{\sigma,\tau}$ (see 5.19), corresponding to the $4$ dimensional coadjoint orbits $\mathcal{O}_4^{0,\sigma,\tau}$, has the following Lie algebra representation on $L^2(\mathbb{R}^2, dr_1 dr_2)$:

$$\begin{aligned}
\hat{Q}_1 &= i\kappa_1 \frac{\partial}{\partial r_2}, \quad \hat{Q}_2 = r_2, \\
\hat{P}_1 &= -i\frac{\partial}{\partial r_1}, \quad \hat{P}_2 = -\kappa_2 r_1,
\end{aligned} \tag{5.43}$$

with $\kappa_1 = -\sigma\beta$ and $\kappa_2 = -\tau\gamma$. The corresponding commutators read

$$\begin{aligned}
&[\hat{Q}_1, \hat{P}_1] = [\hat{Q}_2, \hat{P}_2] = 0, \quad [\hat{Q}_1, \hat{Q}_2] = i\kappa_1 \mathbb{I}, \\
&[\hat{P}_1, \hat{P}_2] = i\kappa_2 \mathbb{I}, \quad [\hat{Q}_1, \hat{P}_2] = [\hat{Q}_2, P_1] = 0.
\end{aligned} \tag{5.44}$$

(5.44) could be considered to represent an uncoupled system of two noncommutative planes. Referring back to (5.19), the $\hat{Q}_1$, $\hat{Q}_2$, $\hat{P}_1$, and $\hat{P}_2$ in (5.43)are just the generators corresponding to abstract group parameters $p_1$, $p_2$, $q_1$, and $q_2$, respectively that represent translations in $\mathbb{R}^4$. They are not to be treated as position or momentum variables as they were in all four preceding cases and hence they are taken to be all dimensionless. The absence of $\hbar$ in the representation (5.43) of the noncentral generators also indicates the uncoupledness between the two underlying noncommutative planes.

### 5.3.6 Case $\rho = 0, \sigma = 0, \tau \neq 0$.

This situation is very much similar to that of (5.3.5) except that we have a single noncommutative plane instead of two. The 2 dimensional coadjoint orbits ${}^{c_1,c_2}\mathcal{O}_2^{0,0,\tau}$ gave rise to the family of UIRs $U_{0,0,\tau}^{c_1,c_2}$ as described in (5.27). The corresponding Lie algebra representation on $L^2(\mathbb{R}, dr)$ reads

$$\begin{aligned} &\hat{Q}_1 = c_1\mathbb{I}, \;\; \hat{Q}_2 = c_2\mathbb{I}, \\ &\hat{P}_1 = \kappa_2 r, \;\; \hat{P}_2 = -i\frac{\partial}{\partial r}, \end{aligned} \tag{5.45}$$

while the corresponding commutators are given by

$$\begin{aligned} &[\hat{Q}_1, \hat{P}_1] = [\hat{Q}_2, \hat{P}_2] = 0, \;\; [\hat{Q}_1, \hat{Q}_2] = 0, \\ &[\hat{P}_1, \hat{P}_2] = i\kappa_2\mathbb{I}, \;\; [\hat{Q}_1, \hat{P}_2] = [\hat{Q}_2, P_1] = 0. \end{aligned} \tag{5.46}$$

Physically, (5.46) refers to one of the two uncoupled noncommutative planes (see (5.43)), the noncommutativity of which is measured by the dimensionless quantity $\kappa_2 = -\tau\gamma$.

### 5.3.7 Case $\rho = 0, \sigma \neq 0, \tau = 0$.

The UIRs $U_{0,\sigma,0}^{c_3,c_4}$, given by (5.28), were found to be associated with the 2 dimensional coadjoint orbits ${}^{c_3,c_4}\mathcal{O}_2^{0,\sigma,0}$. We introduce the following operator of involution on $L^2(\mathbb{R}, ds)$:

$$Tf(s) = f(-s), \tag{5.47}$$

with $f \in L^2(\mathbb{R}, ds)$. We then find a representation $\tilde{U}^{c_3,c_4}_{0,\sigma,0}$ unitarily equivalent to the one given by (5.28), i.e. $T^{-1}U^{c_3,c_4}_{0,\sigma,0}T = \tilde{U}^{c_3,c_4}_{0,\sigma,0}$, with $\tilde{U}^{c_3,c_4}_{0,\sigma,0}$ given by

$$
\begin{aligned}
&(\tilde{U}^{c_3,c_4}_{0,\sigma,0}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)f)(s) \\
&\qquad = e^{ic_3q_1+ic_4q_2}e^{i\sigma(\phi+\beta p_1 s-\frac{\beta}{2}p_1p_2)}f(s-p_2),
\end{aligned}
\tag{5.48}
$$

where $f \in L^2(\mathbb{R}, ds)$. The corresponding representation of $\mathfrak{g}_{\text{NC}}$ on the same Hilbert space now reads

$$
\begin{aligned}
&\hat{Q}_1 = -\alpha_1 s, \quad \hat{Q}_2 = i\frac{\partial}{\partial s}, \\
&\hat{P}_1 = c_3\mathbb{I}, \quad \hat{P}_2 = c_4\mathbb{I}.
\end{aligned}
\tag{5.49}
$$

The corresponding commutators read off immediately

$$
\begin{aligned}
&[\hat{Q}_1,\hat{P}_1] = [\hat{Q}_2,\hat{P}_2] = 0, \quad [\hat{Q}_1,\hat{Q}_2] = i\kappa_1\mathbb{I}, \\
&[\hat{P}_1,\hat{P}_2] = 0, \quad [\hat{Q}_1,\hat{P}_2] = [\hat{Q}_2,P_1] = 0.
\end{aligned}
\tag{5.50}
$$

Physically, (5.50) corresponds to just one of the two uncoupled noncommutative planes (see (5.43)) whose noncommutativity is controlled by the dimensionless parameter $\kappa_1 = -\sigma\beta$.

### 5.3.8 Case $\rho = 0, \sigma = 0, \tau = 0$.

The $0$ dimensional coadjoint orbits ${}^{c_1,c_2,c_3,c_4}\mathcal{O}^{0,0,0}_0$, admitting $1$ dimensional group representations given by (5.29), have the trivial algebra representation where all the basis elements of $\mathfrak{g}_{\text{NC}}$ are mapped to the scalar multiples of identity. The corresponding commutators are the same as those of the abelian group of translations in $\mathbb{R}^4$.

This concludes the classification of all the families of unitary irreducible representations of $\mathfrak{g}_{\text{NC}}$ on appropriate Hilbert spaces. It is noteworthy that all possible representations of NCQM, as postulated in the multitude of existing physical literatures (see, for example, [24]), and the unitary irreducible representation of the Weyl-Heisenberg group for a quantum mechanical system of two degrees of freedom are all obtainable from the unitary dual of the *triply extended group of translations in* $\mathbb{R}^4$.

## 5.4 Various gauges of noncommutative quantum mechanics and their relation to $G_{\text{NC}}$

Let us go back to the representation $\tilde{U}^{\rho}_{\sigma,\tau}$ of $G_{\text{NC}}$, given in (5.32), and the associated generators (5.34), obeying the commutation relations (5.36). As is well known (as shown for example in [24]), there are other possible realizations of the operators $\hat{Q}_i, \hat{P}_i$, which also obey the same commutation relations, which can in many cases be related to the choice of a gauge in the following sense: the commutation relation $[\hat{P}_1, \hat{P}_2] = i\mathcal{B}\mathbb{I}$ signals the presence of a constant magnetic field in the system. This field can be obtained in the usual way through a vector potential. A change of gauge for this potential does not affect the physics of the system. At the quantum mechanical level a change of gauge affects the exact realization of the operators $\hat{Q}_i, \hat{P}_i$, without altering the commutation relation $[\hat{P}_1, \hat{P}_2] = i\mathcal{B}\mathbb{I}$. Furthermore, the differently realized generators would then lift up to unitarily equivalent representations of $G_{\text{NC}}$. As mentioned in Section 5.3.1, the realization given in (5.34) corresponds to the Landau gauge. We look now at a second possible gauge, the so-called *symmetric gauge,* which is also often studied in the literature (see [24] for a detailed discussion on this topic).

We have the following theorem

**Theorem 5.4.1.** *The nilpotent Lie group $G_{NC}$, obeying the group law (5.1), admits a unitary irreducible representation $\mathcal{U}_{sym}$ given by*

$$\begin{aligned}
&(\mathcal{U}_{sym}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)f)(r_1,r_2)\\
&\quad = e^{i(\theta+\phi+\psi)}e^{i\left[\alpha p_1r_1+\alpha p_2r_2-\frac{\alpha(\alpha-\sqrt{\alpha^2-\beta\gamma})}{\beta}(q_1r_2-q_2r_1)+\frac{\sqrt{\alpha^2-\beta\gamma}}{2}(p_1q_1+p_2q_2)\right]}\\
&\quad \times f\left(r_1-\frac{\beta}{2\alpha}p_2+\frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2\alpha}q_1,r_2+\frac{\beta}{2\alpha}p_1+\frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2\alpha}q_2\right),
\end{aligned} \tag{5.51}$$

*with $f\in L^2(\mathbb{R}^2,dr_1dr_2)$. This representation is unitarily equivalent to $\tilde{U}^{\rho}_{\sigma,\tau}$.*

The proof is given in the Appendix B.

We choose $\alpha=\frac{1}{\hbar}$ in (5.51). One can verify that this choice is dimensionally consistent by looking at (5.1) or (5.51). Hence, we take $C=\frac{1}{\alpha}$ in (5.30) and obtain the corresponding unitary irreducible representation of the noncentral elements of $\mathfrak{g}_{NC}$ on $L^2(\mathbb{R}^2,dr_1dr_2)$

which is as follows

$$\begin{aligned}
\hat{Q}_1 &= r_1 + \frac{i\vartheta}{2}\frac{\partial}{\partial r_2}, \\
\hat{Q}_2 &= r_2 - \frac{i\vartheta}{2}\frac{\partial}{\partial r_1}, \\
\hat{P}_1 &= \frac{(\hbar - \sqrt{\hbar^2 - \mathcal{B}\vartheta})}{\vartheta} r_2 - \frac{i(\hbar + \sqrt{\hbar^2 - \mathcal{B}\vartheta})}{2}\frac{\partial}{\partial r_1}, \\
\hat{P}_2 &= \frac{(\sqrt{\hbar^2 - \mathcal{B}\vartheta} - \hbar)}{\vartheta} r_1 - \frac{i(\hbar + \sqrt{\hbar^2 - \mathcal{B}\vartheta})}{2}\frac{\partial}{\partial r_2},
\end{aligned} \tag{5.52}$$

and the central elements of the algebra are all mapped to the scalar multiple of identity of the underlying Hilbert space. Here, in addition to taking $\alpha = \frac{1}{\hbar}$, we have chosen $\beta = -\frac{\vartheta}{\hbar^2}$ and $\gamma = -\frac{\mathcal{B}}{\hbar^2}$. The representation (5.52) is easily seen to satisfy the set of commutation relations given by (5.36). As indicated earlier, this representation is due to the choice of the *symmetric gauge* (see [24] for details) for the underlying vector potential.

Similarly, other unitarily equivalent realizations of the commutation relations (5.36) may be obtained by using other gauge equivalent vector potentials. It is also clear from (5.34) and (5.52) that the two sets of operators $\hat{Q}_i, \hat{P}_i,\ i = 1, 2$, appearing in those two sets of equations are related by a linear transformation. It is therefore natural to ask what is the largest set of such transformations which would leave the commutation relations (5.36) invariant. This question is answered in the following section.

## 5.5 Group of transformations preserving the commutation relations of noncommutative quantum mechanics

It is a well-known fact that in classical mechanics the set of transformations which preserve the canonical Poisson brackets between the phase space variables $p_i$ and $q_j$ in $\mathbb{R}^{2n}$, form the Lie group $Sp(2n, \mathbb{R})$. In standard quantum mechanics the canonical commutation relations are also invariant under this same group. For the noncommutative system of two degrees of freedom, the phase space is $\mathbb{R}^4$ and the transformations between two different sets of $\{\hat{Q}_i, \hat{P}_i\},\ i = 1, 2$, obeying the same commutation relations (5.36), also form a group that is isomorphic to $Sp(4, \mathbb{R})$, as will follow from the following considerations.

Consider two sets of phase space variables $\hat{Q}_1, \hat{P}_2, \hat{Q}_2, \hat{P}_1$ and $\hat{Q}'_1, \hat{P}'_2, \hat{Q}'_2, \hat{P}'_1$ in $\mathbb{R}^4$ satisfying the commutation relations (5.36). Let $\mathbb{M}$ be a $4 \times 4$ matrix, with real entries, for which

$$\begin{bmatrix} \hat{Q}'_1 \\ \hat{P}'_2 \\ \hat{Q}'_2 \\ \hat{P}'_1 \end{bmatrix} = \mathbb{M} \begin{bmatrix} \hat{Q}_1 \\ \hat{P}_2 \\ \hat{Q}_2 \\ \hat{P}_1 \end{bmatrix} \tag{5.53}$$

We then have the following theorem:

**Theorem 5.5.1.** *The $4 \times 4$ real matrices $\mathbb{M}$ in (5.53), preserving the commutation relations (5.36) of a general non-commutative quantum system of two degrees of freedom, satisfy the condition*

$$\mathbb{M}\mathbb{Q}\mathbb{M}^T = \mathbb{Q}, \tag{5.54}$$

*where $\mathbb{Q}$ is the $4\times$ block off-diagonal matrix,*

$$\mathbb{Q} = \begin{bmatrix} 0 & Q \\ -Q^T & 0 \end{bmatrix}, \tag{5.55}$$

*with $2 \times 2$ matrix $Q$ given by the $2 \times 2$ matrix*

$$Q = \begin{bmatrix} -\frac{\vartheta}{\hbar} & -1 \\ \\ 1 & \frac{\mathcal{B}}{\hbar} \end{bmatrix}. \tag{5.56}$$

The proof is given in the Appendix B.

**Remark 5.5.1.** *A few remarks are in order. The converse of Theorem (5.5.1) is also true. As a result, (5.54) is a necessary and sufficient condition for the noncommutative commutation relations to be preserved. Also, the $2 \times 2$ matrix $Q$, given by (5.56), is required to be invertible, i.e. $\hbar^2 - \mathcal{B}\vartheta \neq 0$, a fact that has also been exploited in [24]. Finally, all $4 \times 4$ real matrices $\mathbb{M}$ , satisfying (5.54), can easily be verified to form a group under matrix multiplication. Actually, as shown below, these matrices form a real Lie group, hence forth denoted by $\mathfrak{S}(4, \mathbb{R})$.*

We have the following isomorphism of groups.

**Proposition 5.5.1.** *The $10$ dimensional real Lie group $\mathfrak{S}(4, \mathbb{R})$ is isomorphic to the simple Lie group $Sp(4, \mathbb{R})$. The isomorphism $f : \mathfrak{S}(4, \mathbb{R}) \to Sp(4, \mathbb{R})$, can be written as $f(\mathbb{M}) = \mathcal{U}^{-1}\mathbb{M}\mathcal{U}$, where $\mathcal{U}$ is the $4 \times 4$ invertible matrix:*

$$\mathcal{U} = \begin{bmatrix} -1 & \frac{\vartheta}{\hbar} & 0 & 0 \\ \\ \frac{\mathcal{B}}{\hbar} & -1 & 0 & 0 \\ \\ 0 & 0 & 1 & 0 \\ \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.57}$$

The proof is given in the Appendix B.

**Remark 5.5.2.** *The isomorphism $f$ in this context is what one expects to follow naturally because the relevant operators representing the noncentral generators of $G_{\text{NC}}$ can be expressed as linear combinations of those which generate the CCR of standard quantum mechanics on $L^2(\mathbb{R}^2, dr_1 dr_2)$. Also, $Sp(4,\mathbb{R})$ is the group of transformations that preserve the CCR for a system with $2$ degrees of freedom. The $4\times 4$ matrix $\mathcal{U}$ in (5.57) is actually the matrix of transformation between the standard quantum mechanical and noncommutative quantum mechanical (in this case, Landau gauge) representations as can be readily seen from (5.34). We could also have chosen $\mathcal{U}$ as the one arising from the symmetric gauge (5.52). Thus, the choice of $\mathcal{U}$ is evidently not unique.*

As a concrete example of $\mathbb{M}$, introduced in (5.53), let us consider the phase space variables associated with the Landau gauge (see 5.34) and symmetric gauge (see 5.52) with

$$\begin{bmatrix} \hat{Q}_1 \\ \hat{P}_2 \\ \hat{Q}_2 \\ \hat{P}_1 \end{bmatrix}^{\text{sym}} = \mathbb{M} \begin{bmatrix} \hat{Q}_1 \\ \hat{P}_2 \\ \hat{Q}_2 \\ \hat{P}_1 \end{bmatrix}^{\text{Landau}} \tag{5.58}$$

.

After some straightforward but rather lengthy computations, one arrives at

$$\mathbb{M} = \begin{bmatrix} 1 + \frac{\mathcal{B}\vartheta}{2(\hbar^2 - \mathcal{B}\vartheta)} & \frac{\vartheta\hbar}{2(\hbar^2 - \mathcal{B}\vartheta)} & 0 & 0 \\ \frac{\mathcal{B}(2\hbar\sqrt{\hbar^2 - \mathcal{B}\vartheta} - \mathcal{B}\vartheta)}{2(\hbar^2 - \mathcal{B}\vartheta)(\hbar + \sqrt{\hbar^2 - \mathcal{B}\vartheta})} & \frac{\hbar(3\sqrt{\hbar^2 - \mathcal{B}\vartheta} - \hbar)}{2(\hbar^2 - \mathcal{B}\vartheta)} & 0 & 0 \\ 0 & 0 & 1 & \frac{\vartheta}{2\hbar} \\ 0 & 0 & \frac{\hbar - \sqrt{\hbar^2 - \mathcal{B}\vartheta}}{\vartheta} & \frac{\hbar + \sqrt{\hbar^2 - \mathcal{B}\vartheta}}{2\hbar} \end{bmatrix} . \tag{5.59}$$

It can, then, be immediately verified that $\mathbb{M}$, given by (5.59), indeed satisfies (5.54).

## 5.6 Relationship with complex Hermite polynomials

We explore in this section a connection between a model of non-commutative quantum mechanics, governed by a certain restricted version of the commutation relations (5.36), and a family of *deformed complex Hermite polynomials.* We note first of all, that an irreducible representation of the commutation relations

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}\mathbb{I}, \quad i, j = 1, 2, \tag{5.60}$$

of a standard quantum mechanical system for two degrees of freedom, can be constructed on the Hilbert space $L^2(\mathbb{C}, e^{-|z|^2} \frac{dx\,dy}{\pi})$ as

$$a_1 = \partial_z, \qquad a_1^\dagger = z - \partial_{\overline{z}}, \qquad\qquad a_2 = \partial_{\overline{z}}, \qquad a_2^\dagger = \overline{z} - \partial_z \ . \tag{5.61}$$

We denote by $\mathbf{I}$ the constant function in $L^2(\mathbb{C}, e^{-|z|^2} \frac{dx\,dy}{\pi})$, which is equal to one everywhere. Then the vectors,

$$H_{n,k} = \frac{(a_1^\dagger)^n \, (a_2^\dagger)^k}{\sqrt{n!\, k!}}\mathbf{I} \, , \qquad n, k = 0, 1, 2, \ldots, \infty \tag{5.62}$$

form an orthonormal basis of $L^2(\mathbb{C}, e^{-|z|^2} \frac{dx\,dy}{\pi})$. It can be shown that

$$H_{n,k}(z, \overline{z}) = \frac{(-1)^{n+k}}{\sqrt{n!\,k!}}\; e^{|z|^2} \partial_z^n \partial_{\overline{z}}^k\; e^{-|z|^2}\;, \tag{5.63}$$

or explicitly,

$$H_{n,k}(z, \overline{z}) = \sqrt{n!\,k!} \sum_{j=0}^{n \curlyvee k} \frac{(-1)^j}{j!} \frac{(\overline{z})^{n-j}}{(n-j)!} \frac{z^{k-j}}{(k-j)!}\;, \tag{5.64}$$

where $n \curlyvee k$ denotes the smaller one of the two integers $n$ and $k$. The functions $H_{n,k}(z, \overline{z})$ are known in the literature (see, for example [27, 31, 32, 44]) as the *complex Hermite polynomials*. They form a basis in $L^2(\mathbb{C}, e^{-|z|^2} \frac{dx\,dy}{\pi})$ and satisfy the orthonormality condition

$$\int_{\mathbb{C}} \overline{H_{n,k}(z, \overline{z})} H_{m,l}(z, \overline{z})\; e^{-|z|^2}\; \frac{dx\;dy}{\pi} = \delta_{nm}\; \delta_{kl}\;. \tag{5.65}$$

From the way we have introduced them here, it is clear that these polynomials are the ones naturally associated to a standard quantum mechanical system of two degrees of freedom (or with two independent oscillators).

Consider now a non-commutative quantum system obeying the commutation relations (5.36) and let us assume that we are in the symmetric gauge (5.52). We define the *deformed creation and annihilation operators,* using the operators $\hat{Q}_i,\ \hat{P}_i,\ \ i = 1, 2$, in (5.52),

$$\begin{aligned} a_i^{\text{nc}\dagger} &= \sqrt{\frac{M\Omega}{2\hbar}} \left( \hat{Q}_i - \frac{i}{M\Omega} \hat{P}_i \right), \\ a_i^{\text{nc}} &= \sqrt{\frac{M\Omega}{2\hbar}} \left( \hat{Q}_i + \frac{i}{M\Omega} \hat{P}_i \right)\;, \quad i = 1, 2\;, \end{aligned} \tag{5.66}$$

where the $M$ and $\Omega$ are a mass and an angular frequency parameter, which can be adjusted later. These operators are seen obey the commutation relations

$$
\begin{aligned}
[a_i^{\text{nc}}, a_j^{\text{nc}\dagger}] &= \delta_{ij}\mathbb{I} + \frac{i\epsilon_{ij}}{2\hbar}M\Omega\left(\vartheta + \frac{\mathcal{B}}{M^2\Omega^2}\right)\mathbb{I} \\
[a_i^{\text{nc}}, a_j^{\text{nc}}] &= \frac{i\epsilon_{ij}}{2\hbar}M\Omega\left(\vartheta - \frac{\mathcal{B}}{M^2\Omega^2}\right)\mathbb{I},
\end{aligned}
\tag{5.67}
$$

where $i, j = 1, 2$ and $\epsilon_{ij}$ is the totally antisymmetric symbol. Since we want the two operators $a_i^{\text{nc}},\ i = 1, 2$, to still represent independent bosons, we impose the condition that the second commutator above be zero. This implies taking

$$
\vartheta = \frac{\mathcal{B}}{M^2\Omega^2}. \tag{5.68}
$$

and hence the other commutator now reads

$$
[a_i^{\text{nc}}, a_j^{\text{nc}\dagger}] = \delta_{ij}\mathbb{I} + \frac{i\epsilon_{ij}\vartheta M\Omega}{\hbar}\mathbb{I}, \tag{5.69}
$$

which still means that we are in the framework of noncommutative quantum mechanics, since $[a_1^{\text{nc}}, a_2^{\text{nc}\dagger}] \neq 0$.

We next introduce the standard creation and annihilation operators (obeying the commutation relations (5.60)), in terms of the usual position and momentum operators of quantum mechanics,

$$
\begin{aligned}
a_i^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(Q_i - \frac{i}{m\omega}P_i\right) \\
a_i &= \sqrt{\frac{m\omega}{2\hbar}}\left(Q_i + \frac{i}{m\omega}P_i\right), \quad i = 1, 2,
\end{aligned}
\tag{5.70}
$$

where

$$m\omega = \frac{2\hbar M\Omega}{\hbar + \sqrt{\hbar^2 - \mathcal{B}\vartheta}} = \frac{2\hbar\sqrt{\mathcal{B}}}{\sqrt{\vartheta}(\hbar + \sqrt{\hbar^2 - \mathcal{B}\vartheta})} \ . \tag{5.71}$$

A straightforward computation, using (5.52) then gives

$$\begin{aligned} a_1^{\mathrm{nc}\dagger} &= \sqrt{\nu}a_1^\dagger - i\sqrt{1-\nu}a_2^\dagger \\ a_2^{\mathrm{nc}\dagger} &= i\sqrt{1-\nu}a_1^\dagger + \sqrt{\nu}a_2^\dagger, \qquad \nu = \frac{\hbar + \sqrt{\hbar^2 - \mathcal{B}\vartheta}}{2\hbar} \ . \end{aligned} \tag{5.72}$$

It is now interesting and useful to realize the above operators using the complex representation (5.61) and to look at the *deformed complex polynomials*

$$H_{n,k}^{\mathrm{nc}} = \frac{(a_1^{\mathrm{nc}\dagger})^n \ (a_2^{\mathrm{nc}\dagger})^k}{\sqrt{n! \ k!}}\mathbf{I} \ , \qquad n,k = 0,1,2,\ldots,\infty \ , \tag{5.73}$$

in analogy with (5.62). These polynomials do not satisfy an orthogonality relation of the type (5.65). However, it has been shown in [9] that there exists a dual set of polynomials $\widetilde{H}_{n,k}^{\mathrm{nc}}$ for which one has the *biorthogonality relation*

$$\int_{\mathbb{C}} \overline{\widetilde{H}_{n,k}^{\mathrm{nc}}(z,\overline{z})} H_{m,l}^{\mathrm{nc}}(z,\overline{z}) \ e^{-|z|^2} \ \frac{dx \ dy}{\pi} = \delta_{nm} \ \delta_{kl} \ . \tag{5.74}$$

We can go further and define deformed creation and annihilation operators using an arbitrary $GL(2,\mathbb{C})$ matrix

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

in the manner

$$a_1^{g\dagger} = g_{11}a_1^\dagger + g_{21}a_2^\dagger, \qquad a_2^{g\dagger} = g_{12}a_1^\dagger + g_{22}a_2^\dagger. \tag{5.75}$$

and construct the corresponding deformed polynomials $H^g_{n,k}$. In this case the dual polynomials are obtained using the matrix $\widetilde{g} = (g^\dagger)^{-1}$. However, to relate these more general polynomials to noncommutative quantum mechanics one still has to impose the condition $[a^g_1, a^g_2] = 0$. Thus, generally, one gets a matrix of the form

$$g = \begin{bmatrix} re^{i\kappa} & \sqrt{1-r^2}e^{i\{\kappa+\epsilon(r)\}} \\ \sqrt{1-r^2}e^{i\delta} & -re^{i\{\delta-\epsilon(r)\}} \end{bmatrix}. \tag{5.76}$$

Here, $\epsilon(r)$, being a function of $r$, is given by

$$\epsilon(r) = \arcsin\left(\frac{\vartheta M\Omega}{2\hbar r\sqrt{1-r^2}}\right). \tag{5.77}$$

From (5.76), one finds that $0 < r \leq 1$. But the condition $-1 \leq \frac{\vartheta M\Omega}{2\hbar r\sqrt{1-r^2}} \leq 1$ puts further restrictions on $r$, requiring that

$$r \in \left[\sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\vartheta^2 M^2\Omega^2}{4\hbar^2}}}, \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\vartheta^2 M^2\Omega^2}{4\hbar^2}}}\right], \tag{5.78}$$

along with, $0 < \frac{\vartheta M\Omega}{\hbar} \leq 1$. Also, $\kappa \in [-\epsilon(r), 2\pi - \epsilon(r))$ and $\delta \in [\epsilon(r), 2\pi + \epsilon(r))$, where, $\epsilon(r) \in \left[\arcsin\left(\frac{\vartheta M\Omega}{\hbar}\right), \frac{\pi}{2}\right]$, as a transcendental function of $r$, varies according to (5.77).

To summarize, if we consider the operators $\hat{Q}_1$, $\hat{Q}_2$, $\hat{P}_1$ and $\hat{P}_2$, in the *symmetric gauge representation* (5.52) of the *triply extended algebra of translations* $\mathfrak{g}_{\text{NC}}$, to be the respective positions and momenta of the two bosons of the underlying coupled system and impose a constraint given by (5.68), then the resulting creation and annihilation operators are linear combinations of the canonical creation and annihilation operators via the invertible matrix

$$g_{\text{sym}} = \begin{bmatrix} \sqrt{\nu} & i\sqrt{1-\nu} \\ -i\sqrt{1-\nu} & \sqrt{\nu} \end{bmatrix}. \tag{5.79}$$

It is easy to see that $g_{\text{sym}}$, given by (5.79), is a special case of the matrix $g$ introduced in (5.76) with $r^2 = \nu$, $\kappa = 0$, $\delta = \frac{3\pi}{2}$, and $\epsilon = \frac{\pi}{2}$.

Before closing this section, we examine the geometric consequences of (5.68) from the representation theoretic point of view. To this end, (5.68) together with (5.35) imply

$$\tau = \frac{\beta M^2 \Omega^2}{\gamma}\sigma := K_g \sigma. \tag{5.80}$$

Also, together with (5.35), the inequality $0 < \frac{\vartheta M \Omega}{\hbar} \leq 1$, as has already been mentioned earlier in this section, puts severe restrictions both on $\sigma$ and $\rho$:

$$\sigma \in (-\infty, 0) \;\text{ and }\; \rho \geq -\sigma\left(\frac{\beta M \Omega}{\alpha}\right). \tag{5.81}$$

In other words, a family of $4$ dimensional coadjoint orbits $\mathcal{O}^{\rho,\sigma,K_g\sigma}$ (See section 5.2 for the notation) and the associated unitary irreducible representations of the *triply extended group of translations* $G_{\text{NC}}$ (see 5.33) are the ones that describe the coupled bosonic system under study. Here, $K_g = \frac{\beta M^2 \Omega^2}{\gamma}$ is a dimensionless coefficient. Also, $\rho$ and $\sigma$ can take values on the real line in accordance with (5.81).

The study of (1+1) and (2+1)-dimensional Galilei groups and their applications to *Signal analysis* and *Noncommutative quantum mechanics*, respectively, closes in this chapter.

In the following chapter, as a separate part of the thesis, we study Poisson structures associated with classical non-abelian gauge field theory using Hamiltonian formalism of the theory of Soliton.

# Chapter 6
# On Derivation of Goldman Bracket

The contents of this chapter are taken from the article titled "On Derivation of Goldman Bracket" [17]. Non abelian Gauge field theory on space-time, modeled as a noncompact 3-manifold $\Sigma \times \mathbb{R}$, with $\Sigma$ being a compact Riemann surface and time taking values in $\mathbb{R}$, has been considered in this chapter. The Atiyah-Bott brackets between the gauge fields have been computed in this infinite dimensional setting. Traces of monodromies of the gauge connections around free homotopy classes of closed loops on the underlying Riemann surface and the Poisson brackets between them are computed using the formalism originated from hamiltonian methods of Soliton theory. Finally, the brackets for real Lie groups $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $U(n)$, $SU(n)$ and $Sp(2n, \mathbb{R})$ are explicitly worked out.

## 6.1 Introduction

The purpose of this chapter of the thesis is to find the Poisson brackets between traces of monodromy matrices computed along free homotopy classes of loops on the Riemann surface $\Sigma$. Given the fundamental group $\pi$ of a closed oriented surface $S$ and a Lie group $G$, Goldman considered (see [29]) the space $\mathrm{Hom}(\pi, G)/G$ by taking the quotient of the action of $G$ on the analytic variety $\mathrm{Hom}(\pi, G)$ using conjugation. He studied the geometry of the symplectic structure of this quotient space using a family of functions, on $\mathrm{Hom}(\pi, G)/G$, which he called Invariant functions. He also shows that there is a Lie algebra structure of homotopy classes of oriented closed curves immersed in the surface. And then he establishes a Lie algebra homomorphism between these homotopy classes of loops and the Lie algebra of functions on $\mathrm{Hom}(\pi, G)/G$ under Poisson bracket ([29], page 267).

In the present setting, on the other hand, we start out with a 3-manifold $\Sigma \times \mathbb{R}$. We then consider the principal $G$-bundle over the base manifold $\Sigma \times \mathbb{R}$ with $G$ being a real Lie group. The gauge fields $A$-s take their values on the underlying Lie algebra $\mathcal{G}$. We write down the Chern-Simons action for the gauge fileds on this 3-manifold taking the gauge freedom into account and then compute the Atiyah-Bott brackets between the relavant connection 1 forms and the momenta conjugate to them. The curvature of the 1-forms is easily seen to be zero and hence we have an infinite dimensional space of flat connections. And it is known to us that the space of flat connections up to gauge transformations, i.e. the moduli space of flat connections is isomorphic to $\mathrm{Hom}(\pi(\Sigma), G)/G$. The Wilson loops, i.e. the traces of monodromy matrices computed along free homotopy classes of loops on $\Sigma$ are gauge invariant objects. The purpose of this chapter is to compute Poisson brackets between these monodromy matrices for various choices of real Lie groups. The cases for $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SU(n)$, $U(n)$ and $Sp(2n, \mathbb{R})$ are handled explicitly. Of them, the final expression for the Poisson bracket between $GL(n, \mathbb{R})$ monodromies, for two transversally intersecting oriented closed curves $\gamma_1$ and $\gamma_2$ on $\Sigma$, turn out to be rather simple given by

$$\{\mathrm{Tr}\, M_1, \mathrm{Tr}\, M_2\} = \frac{4}{k}\, \mathrm{Tr}\, M_{\gamma_1 \circ \gamma_2}, \tag{6.1}$$

while the one for a relatively difficult case of $Sp(2n, \mathbb{R})$ reads

$$\{\operatorname{Tr} M_1, \operatorname{Tr} M_2\} = \frac{2}{k}(\operatorname{Tr} M_{\gamma_1 \circ \gamma_2} - \operatorname{Tr} M_{\gamma_1 \circ \gamma_2^{-1}}). \tag{6.2}$$

The constant, $k$ in 6.1 and 6.2, arises from Chern-Simons action which depends on the topology of $G$. Also, $\gamma_1 \circ \gamma_2$ and $\gamma_1 \circ \gamma_2^{-1}$ represent deformed loops on $\Sigma$ with appropriate orientation. They are conveniently depicted in Figure 6–1 and 6–2. Finally, one finds that the Poisson bracket between traces of monodromy matrices, thus computed, coincides with the one computed by Goldman in [29].

### 6.2 Hamiltonian Chern-Simons theory

In this section, we discuss the preliminaries that lead to the Atiyah-Bott brackets between connection 1-forms. For the sake of completeness, we work out the well-known results of Hamiltonian Chern-Simons theory in detail.

We start out with the well-known Chern-Simons action functional on the 3-manifold $\Sigma \times \mathbb{R}$

$$S_{\text{CS}} = \frac{k}{4\pi} \int_{\Sigma \times \mathbb{R}} \operatorname{Tr} \, (A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{6.3}$$

Where $k$ is a constant depending only on the topology of the structure Lie group. The Lie algebra valued connection 1-forms on the principal G-bundle reads

$$A = A_z(z, \bar{z}, t) dz + A_{\bar{z}}(z, \bar{z}, t) d\bar{z} + A_0(z, \bar{z}, t) dt. \tag{6.4}$$

If the group generators for the abstract Lie group G are given by $t_a$-s, with $a = 1, ....., n$ and $n$ being the dimension of G as a smooth manifold, then each of $A_z$, $A_{\bar{z}}$ and $A_0$ reads off

$$A_i = \sum_{a=1}^{n} A_i^a t_a, \tag{6.5}$$

where $i$ in the above expression stands for any of $z$, $\bar{z}$ and $0$, i.e. the space-time labels. The $A_i^a$-s in (6.5) are just complex valued functions. Also, the structure constants for the underlying Lie algebra $\mathcal{G}$ are given by

$$[t_a, t_b] = \sum_{c=1}^{n} f_{ab}^c t_c. \tag{6.6}$$

Now, the action functional $S_{CS}$ in (6.3) reads

$$S_{\text{CS}} = k \int_{\Sigma\times\mathbb{R}} \left[ \sum_{i,j,k} \sum_{a=1}^{n} \epsilon^{ijk} A_i^a (\partial_j A_k^a - \partial_k A_j^a) + \sum_{a,b,c=1}^{n} A_0^a A_z^b A_{\bar{z}}^c f_{bc}^a \right] dt \wedge dz \wedge d\bar{z}, \tag{6.7}$$

where the superscripts in the connection 1-forms, i.e. a, b, c, are the algebra indices while the subscripts $i, j, k$ are space-time labels and $\epsilon^{ijk}$ is the totally anti-symmetric symbol. Quantization of the constant k as an integer multiple of $2\pi$ in (6.3) ([50]) leaves us with just a half integer and the multiplicative factor $\frac{1}{2}$ of the half integer gets absorbed into the integrand of (6.7) by requiring that we choose the group generators $t_a$-s such that the following holds

$$\frac{1}{2}\text{Tr}(t_a t_b) = f(a)\delta_{ab}, \tag{6.8}$$

with $f(a) = \pm 1$.

Now, varying the action functional in (6.7) with respect to the fields $A_0^a$ we obtain,

$$\frac{\partial A_{\bar{z}}^a}{\partial z} - \frac{\partial A_z^a}{\partial \bar{z}} + [A_z, A_{\bar{z}}]^a = 0. \tag{6.9}$$

Here,

$$[A_z, A_{\bar{z}}] = \sum_{a=1}^{n} [A_z, A_{\bar{z}}]^a t_a.$$

Similarly, by varying (6.7) with respect to $A_z^b$ and $A_{\bar{z}}^c$ we obtain,

$$\frac{\partial A_0^b}{\partial \bar{z}} - \frac{\partial A_{\bar{z}}^b}{\partial t} + [A_{\bar{z}}, A_0]^b = 0 \quad \text{and} \tag{6.10}$$

$$\frac{\partial A_z^c}{\partial t} - \frac{\partial A_0^c}{\partial z} + [A_0, A_z]^c = 0, \tag{6.11}$$

respectively. Now, (6.9), together with (6.10) and (6.11) imply the flatness condition for the gauge fields. In other words, the curvature $F$, in this setting, vanishes,

$$F = dA + A \wedge A = 0. \tag{6.12}$$

By far, we have not taken the gauge freedom into consideration to reduce the degrees of freedom of the underlying gauge fields. We do so now. First, we extract the component from the connection 1 form (6.4) that we want to gauge out, i.e. $A_0(z, \bar{z}, t)$. We then have

$$A = \mathbb{A} + A_0 dt. \tag{6.13}$$

Under gauge transformation, the gauge field $A$ transforms as

$$A \mapsto A' \;\; = \;\; gAg^{-1} + dgg^{-1}$$

$$= \quad g\mathbb{A}g^{-1} + gA_0g^{-1}dt + dgg^{-1}. \tag{6.14}$$

In view of (6.14), the solution of the following differential equation,

$$\frac{dg}{dt} = -gA_0 \tag{6.15}$$

will kill the time component $A_0$ in the connection 1-form and hence (6.14) will read

$$A' = g\mathbb{A}g^{-1} = g(A_z dz + A_{\bar{z}} d\bar{z})g^{-1}. \tag{6.16}$$

Note that $A_z$, $A_{\bar{z}}$, and $g$ in (6.16) are all matrices with entries being complex-valued functions of $z$, $\bar{z}$ and $t$. We have the following theorem

**Theorem 6.2.1.** *The gauge fixed Chern-Simons action, under the action of an element of the gauge group given by (6.15), is as follows*

$$\tilde{S}_{cs} = k \int_{\Sigma\times\mathbb{R}} \left[ \sum_{a=1}^{n} f(a)(A^a_{\bar{z}}\dot{A}^a_z - A^a_z\dot{A}^a_{\bar{z}}) \right] dt \wedge dz \wedge d\bar{z}, \tag{6.17}$$

*where $f(a)$ is just $\pm 1$, as given in (6.8).*

**Proof** .

If we choose $g$ as a solution of (6.15) and plug it in (6.14), the transformed gauge fields, then, read

$$A' = g\mathbb{A}g^{-1}. \tag{6.18}$$

Exterior derivative of the transformed gauge field then yields,

$$dA' = dg \wedge \mathbb{A}g^{-1} + gd\mathbb{A}g^{-1} - g\mathbb{A} \wedge dg^{-1}. \tag{6.19}$$

Therefore,

$$A' \wedge dA' = g\mathbb{A}g^{-1} \wedge gd\mathbb{A}g^{-1}. \tag{6.20}$$

Also,

$$A' \wedge A' \wedge A' = 0. \tag{6.21}$$

Using (6.20) and (6.21) in (6.3) and recalling $k$ being an integer multiple of $2\pi$, we obtain the gauge fixed expression for Chern-Simons action

$$\begin{aligned}
\tilde{S}_{\text{cs}} &= \frac{k}{2}\int_{\Sigma\times\mathbb{R}} \mathrm{Tr}(A' \wedge dA') \\
&= \frac{k}{2}\int_{\Sigma\times\mathbb{R}} \mathrm{Tr}\left(gA_z g^{-1}dz \wedge g\frac{\partial A_{\bar{z}}}{\partial t}g^{-1}dt \wedge d\bar{z} + gA_{\bar{z}}g^{-1}d\bar{z} \wedge g\frac{\partial A_z}{\partial t}g^{-1}dt \wedge dz\right) \\
&= \frac{k}{2}\int_{\Sigma\times\mathbb{R}} \mathrm{Tr}\left(A_z g^{-1}dz \wedge g\frac{\partial A_{\bar{z}}}{\partial t}dt \wedge d\bar{z} + A_{\bar{z}}g^{-1}d\bar{z} \wedge g\frac{\partial A_z}{\partial t}dt \wedge dz\right) \\
&= \frac{k}{2}\int_{\Sigma\times\mathbb{R}} \mathrm{Tr}\left(A_z\frac{\partial A_{\bar{z}}}{\partial t}dz \wedge dt \wedge d\bar{z} + A_{\bar{z}}\frac{\partial A_z}{\partial t}d\bar{z} \wedge dt \wedge dz\right) \\
&= \frac{k}{2}\int_{\Sigma\times\mathbb{R}} \mathrm{Tr}(A_{\bar{z}}\dot{A}_z - A_z\dot{A}_{\bar{z}})dt \wedge dz \wedge d\bar{z} \\
&= \frac{k}{2}\int_{\Sigma\times\mathbb{R}} \mathrm{Tr}\sum_{a,b=1}^{n}(A^a_{\bar{z}}\dot{A}^b_z t_a t_b - A^b_z\dot{A}^a_{\bar{z}}t_b t_a)dt \wedge dz \wedge d\bar{z} \\
&= k\int_{\Sigma\times\mathbb{R}} \left[\sum_{a=1}^{n} f(a)(A^a_{\bar{z}}\dot{A}^a_z - A^a_z\dot{A}^a_{\bar{z}})\right] dt \wedge dz \wedge d\bar{z}. 
\end{aligned} \tag{6.22}$$

□

Theorem(6.2.1) has several consequences. There are $2n$ independent gauge fields at each point $(z, \bar{z}, t)$ of the underlying 3-manifold $\Sigma \times \mathbb{R}$. They are given by $A^a_z(z, \bar{z}, t)$ and $A^b_{\bar{z}}(z, \bar{z}, t)$, where the superscript indices $a, b$ run from $1$ to $n$ and $z$ varies over space-time manifold. Hence, we have an infinite dimensional theory of non abelian gauge fields. We have the following corollary to Theorem (6.2.1),

**Corollary 6.2.1.** *The canonical momenta conjugate to $A_z^a$ and $A_{\bar{z}}^a$ are given by*

$$\Pi_{A_z^a} = kf(a)A_{\bar{z}}^a \quad \textit{and} \quad \Pi_{A_{\bar{z}}^a} = -kf(a)A_z^a. \tag{6.23}$$

*The Hamiltonian of the gauge fixed system (6.17) is zero. And the Poisson structure of the underlying infinite dimensional space of the non-abelian gauge fields is encoded in the Atiyah-Bott brackets between the gauge fields and the respective canonically conjugate momenta as given by (6.23). The Atiyah-Bott brackets are given by*

$$\{A_z^a, A_{\bar{z}'}^b\} = \frac{2f(a)}{k}\delta^{ab}\delta^{(2)}(z - z'). \tag{6.24}$$

*All other brackets are identically zero.*

**Proof** . The Lagrangian density for the gauge fixed system (6.17) follows as

$$\mathcal{L}_{\text{CS}} = k\sum_{a=1}^{n} f(a)(A_{\bar{z}}^a \dot{A}_z^a - A_z^a \dot{A}_{\bar{z}}^a). \tag{6.25}$$

From which immediately follow the canonically conjugate momenta

$$\begin{aligned} \Pi_{A_z^a} &= \frac{\partial \mathcal{L}_{\text{CS}}}{\partial \dot{A}_z^a} = kf(a)A_{\bar{z}}^a \\ \Pi_{A_{\bar{z}}^a} &= \frac{\partial \mathcal{L}_{\text{CS}}}{\partial \dot{A}_{\bar{z}}^a} = -kf(a)A_z^a. \end{aligned}$$

And hence, the Hamiltonian density reads

$$
\begin{aligned}
\mathcal{H}_{\text{CS}} &= \sum_{a=1}^{n} \Pi_{A_z^a} \dot{A}_z^a + \sum_{a=1}^{n} \Pi_{A_{\bar{z}}^a} \dot{A}_{\bar{z}}^a - \mathcal{L}_{\text{CS}} \\
&= k \sum_{a=1}^{n} f(a) (A_{\bar{z}}^a \dot{A}_z^a - A_z^a \dot{A}_{\bar{z}}^a) - \mathcal{L}_{\text{CS}} \\
&= 0.
\end{aligned}
$$

One can, now, easily compute the Poisson brackets between $A_z^a$ and $A_{\bar{z}}^b$ from the above data,

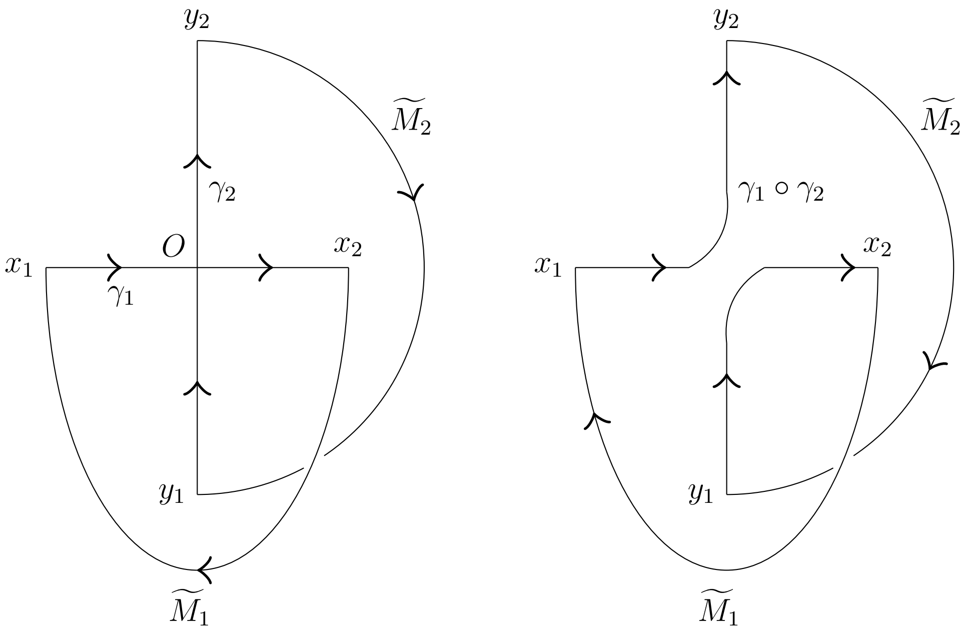
$$
\begin{aligned}
&\{A_z^a, A_{\bar{z}'}^b\} \\
&\quad = \frac{1}{k} \int_{\Sigma} \left[ \sum_{c=1}^{n} f(c) \left( \frac{\partial A_z^a}{\partial A_w^c} \frac{\partial A_{\bar{z}'}^b}{\partial A_{\bar{w}}^c} - \frac{\partial A_z^a}{\partial A_{\bar{w}}^c} \frac{\partial A_{\bar{z}'}^b}{\partial A_w^c} \right) \right. \\
&\qquad \left. + \sum_{d=1}^{n} f(d) \left( \frac{\partial A_z^a}{\partial A_w^d} \frac{\partial A_{\bar{z}'}^b}{\partial A_{\bar{w}}^d} - \frac{\partial A_z^a}{\partial A_{\bar{w}}^d} \frac{\partial A_{\bar{z}'}^b}{\partial A_w^d} \right) \right] dw \wedge d\bar{w} \\
&\quad = \frac{1}{k} \int_{\Sigma} \left[ \sum_{c=1}^{n} f(c) \delta^{ac} \delta^{cb} \left( \frac{\partial A_z^a}{\partial A_w^a} \frac{\partial A_{\bar{z}'}^b}{\partial A_{\bar{w}}^b} - \frac{\partial A_z^a}{\partial A_{\bar{w}}^a} \frac{\partial A_{\bar{z}'}^b}{\partial A_w^b} \right) \right. \\
&\qquad \left. + \sum_{d=1}^{n} f(d) \delta^{ad} \delta^{db} \left( \frac{\partial A_z^a}{\partial A_w^a} \frac{\partial A_{\bar{z}'}^b}{\partial A_{\bar{w}}^b} - \frac{\partial A_z^a}{\partial A_{\bar{w}}^a} \frac{\partial A_{\bar{z}'}^b}{\partial A_w^b} \right) \right] dw \wedge d\bar{w} \\
&\quad = \frac{f(a)}{k} \int_{\Sigma} \delta^{ab} \left[ \left( \frac{\partial A_z^a}{\partial A_w^a} \frac{\partial A_{\bar{z}'}^a}{\partial A_{\bar{w}}^a} - \frac{\partial A_z^a}{\partial A_{\bar{w}}^a} \frac{\partial A_{\bar{z}'}^a}{\partial A_w^a} \right) + \left( \frac{\partial A_z^a}{\partial A_w^a} \frac{\partial A_{\bar{z}'}^a}{\partial A_{\bar{w}}^a} - \frac{\partial A_z^a}{\partial A_{\bar{w}}^a} \frac{\partial A_{\bar{z}'}^a}{\partial A_w^a} \right) \right] dw \wedge d\bar{w} \\
&\quad = \frac{2f(a)}{k} \int_{\Sigma} \delta^{ab} \left( \frac{\partial A_z^a}{\partial A_w^a} \frac{\partial A_{\bar{z}'}^a}{\partial A_{\bar{w}}^a} - \frac{\partial A_z^a}{\partial A_{\bar{w}}^a} \frac{\partial A_{\bar{z}'}^a}{\partial A_w^a} \right) dw \wedge d\bar{w} \\
&\quad = \frac{2f(a)}{k} \delta^{ab} \delta^{(2)}(z - z'). \qquad (6.26)
\end{aligned}
$$

□

### 6.3 Moduli space of flat connections and Goldman brackets between Wilson lines

We introduced the infinite dimensional space of gauge fields and constructed Atiyah-Bott bracket between components of the gauge fields along the group generators. The infinite dimensional space, in question, is not easy to handle. In order to reduce the field theory to one with finitely many degrees of freedom, we consider the homotopy classes of the free loops, i.e. the conjugacy classes of the fundamental group of the underlying Riemann surface by the structure group. Traces of the monodromies, i.e. the so-called Wilson lines, are well defined gauge invariant observables. The Wilson lines computed along equivalence classes of loops on the Riemann surface, under study, are found to form a Lie algebra in terms of the intersection points between the loops considered [29]. Goldman considered an arbitrary Lie group satisfying fairly general conditions to be the space where the fundamental group of the given Riemann surface is represented. Then he computed brackets between invariant functions defined over equivalence classes of loops on the underlying Riemann surface.

We take any real Lie group $G$ to be the structure group of the principle $G$-bundle. The connection $1$ forms take their values in the associated real Lie algebra $\mathcal{G}$. We compute the bracket between the trace of monodromies along two homotopically inequivalent loops that intersect transversally at a single point. The generalization to many intersection points is pretty straight forward.




(a) Two intersecting loops.

(b) Deformed loop $\gamma_1 \circ \gamma_2$

Figure 6–1: Traces of monodromies are computed along two free loops that are homotopically inequivalent and intersect transversally at a single point. In the following subfigure, trace of monodromy along a single loop, deformed at the point of intersection, has been considered.

Also, since we are dealing with Topological field theory, the transversal point of intersection can be taken as an orthogonal one. Let $x_1x_2x_1$ and $y_1y_2y_1$ be two loops that intersect orthogonally at $O$ lying on the compact Riemann surface. We shall be denoting the loops $x_1x_2x_1$ and $y_1y_2y_1$ with $\gamma_1$ and $\gamma_2$, respectively. Also, the two parts $x_1Ox_2$ and $y_1Oy_2$ are taken to lie along $X$ and $Y$ axes, respectively, as shown in Figure 6–1. For the sake of notational convenience, monodromy along loop $\gamma_i$ will simply be denoted as $M_i$, where, $i = 1, 2$. $T(x_1, x_2)$ and $T(y_1, y_2)$ are the relevant transition matrices. $\widetilde{M}_1$ and $\widetilde{M}_2$ are the remaining contribution to monodromies $M_1$ and $M_2$, respectively. $\widetilde{M}_1$ and $\widetilde{M}_2$ Poisson commute with each other and with other transition matrices in question, since they are due to part of the loops far away from the intersection point $O$ and hence have nothing to do with each other. What stands out to be in the second subfigure of Figure 6–1 is the two loops, combined and deformed at the point of intersection. Note that the orientation of the loops are preserved under the deformation. Monodromy around this deformed loop $\gamma_1 \circ \gamma_2$ is denoted with $M_{\gamma_1 \circ \gamma_2}$. We, now, have

$$\begin{aligned} M_1 &= T(x_1, x_2)\widetilde{M}_1, \\ M_2 &= T(y_1, y_2)\widetilde{M}_2. \end{aligned} \tag{6.27}$$

$M_1$ and $M_2$ take their values in the structure Lie group $G$ of the principal G-bundle. Note that we can write the connection 1 form (gauge fixed) $A$ as

$$A = A_z(z, \bar{z})dz + A_{\bar{z}}(z, \bar{z})d\bar{z} = A_1(x, y)dx + A_2(x, y)dy. \tag{6.28}$$

Now in the light of (6.28), 1 forms, restricted to the real and imaginary axes, read

$$\begin{aligned} A(x, 0) &= A_1(x, 0)dx, \quad \text{and} \\ A(0, y) &= A_2(0, y)dy. \end{aligned} \tag{6.29}$$

respectively. now, in terms of real and imaginary parts of connection 1-forms, i.e. $A_1$ and $A_2$, the Atiyah-Bott brackets (6.24) computed in section (6.2) read

$$\{A_1^a(x, y), A_2^b(x', y')\} = \frac{2}{k} f(a)\delta^{ab}\delta(x - x')\delta(y - y'). \tag{6.30}$$

**Lemma 6.3.1.** *The fundamental Poisson brackets between $\mathcal{G}$ valued 1-forms are given by*

$$\{A_1(x, 0) \overset{\otimes}{,} A_2(0, y)\} = \frac{2}{k}\delta(x)\delta(y)\Gamma, \tag{6.31}$$

*where $\Gamma$ is the Casimir tensor for $\mathcal{G}$ given by*

$$\Gamma = \sum_{a=1}^{n} f(a)(t_a \otimes t_a). \tag{6.32}$$

**Proof** . The above Lemma is just a consequence of (6.30).

$$\begin{aligned}
\{A_1(x,0) \overset{\otimes}{,} A_2(0,y)\} &= \{\sum_{a=1}^{n} A_1^a(x,0)t_a \overset{\otimes}{,} \sum_{b=1}^{n} A_2^b(0,y)t_b\} \\
&= \sum_{a=1}^{n}\sum_{b=1}^{n} \{A_1^a(x,0), A_2^b(0,y)\}(t_a \otimes t_b) \\
&= \frac{2}{k}\delta(x)\delta(y)\sum_{a=1}^{n} f(a)(t_a \otimes t_a).
\end{aligned}$$

□

**Remark 6.3.1.** *We should emphasize in the context of Lemma (6.3.1) that the basis of the underlying Lie algebra is chosen in such a way that the trace form between the group generators is diagonalised in order to comply with what was used in the derivation of the Atiyah-Bott brackets (6.26) in section (6.2). Lemma (6.3.1) is independent of the representation of the Lie algebra, though. All it means is that the same representation has to be chosen during both the derivations of the Atiyah-Bott brackets and the fundamental Poisson brackets.*

Using Lemma (6.3.1), one obtains the Poisson bracket between transition matrices along two small paths of the given loops around the intersection point $O$ as described in Figure6–1,

**Lemma 6.3.2.** *Let $T(x_1, x_2)$ and $T(y_1, y_2)$ be the transition matrices pertaining to path $x_1Ox_2$ and $y_1Oy_2$ as indicated in Figure 6–1. The Poisson brackets between them is given by*

$$\{T(x_1, x_2) \overset{\otimes}{,} T(y_1, y_2)\} = \frac{2}{k}\left[T(x_1, 0) \otimes T(y_1, 0)\right] \Gamma \left[T(0, x_2) \otimes T(0, y_2)\right]. \tag{6.33}$$

*Where $\Gamma$ is the Casimir tensor given by (6.32). Also, $T(x_1, y_2)$ and $T(y_1, x_2)$, standing on the right side of (6.33), are computed along the deformed loop in Figure 6–1.*

**Proof** . The Poisson brackets between transition matrices in the context of Hamiltonian theory of Solitons are given in ([26], page 192). In our setting, this formula gives

$$\begin{aligned}
&\{T(x_1, x_2) \overset{\otimes}{,} T(y_1, y_2)\} \\
&\quad = \int_{x_1}^{x_2}\int_{y_1}^{y_2} [(T(x_1, x)) \otimes T(y_1, y)] \{A_1(x, 0) \overset{\otimes}{,} A_2(0, y)\} [(T(x, x_2)) \otimes T(y, y_2)]\, dx\, dy \\
&\quad = \frac{2}{k}\left[T(x_1, 0) \otimes T(y_1, 0)\right] \Gamma \left[T(0, x_2) \otimes T(0, y_2)\right].
\end{aligned} \tag{6.34}$$

□

**Lemma 6.3.3.** *The Poisson bracket between traces of monodromy matrices is as follows*

$$\{\mathrm{Tr}\, M_1, \mathrm{Tr}\, M_2\} = \frac{2}{k}\,\mathrm{Tr}_{12}[(T(0, x_2)\widetilde{M}_1 T(x_1, 0) \otimes T(0, y_2)\widetilde{M}_2 T(y_1, 0))\Gamma], \tag{6.35}$$

*where $M_1$ and $M_2$ are given by (6.27). In (6.35),* Tr *is trace in the vector space $\mathbb{R}^n$ while* $\mathrm{Tr}_{12}$ *is one in the tensor product space $\mathbb{R}^n \otimes \mathbb{R}^n$.*

**Proof** . Using (6.27), one obtains

$$\begin{aligned}
\{M_1 \overset{\otimes}{,} M_2\} &= \{T(x_1,x_2)\widetilde{M_1} \overset{\otimes}{,} T(y_1,y_2)\widetilde{M_2}\} \\
&= \{T(x_1,x_2) \overset{\otimes}{,} T(y_1,y_2)\widetilde{M_2}\}(\widetilde{M_1} \otimes I_2) \\
&\quad +[T(x_1,x_2) \otimes I_2]\{\widetilde{M_1} \overset{\otimes}{,} T(y_1,y_2)\widetilde{M_2}\} \\
&= \{T(x_1,x_2) \overset{\otimes}{,} T(y_1,y_2)\}(I_2 \otimes \widetilde{M_2})(\widetilde{M_1} \otimes I_2), \qquad (6.36)
\end{aligned}$$

where we have exploited the fact that $\widetilde{M_1}$ and $\widetilde{M_2}$ both Poisson commute with $T(x_1,x_2)$ and $T(y_1,y_2)$, and amongst themselves. Now using Lemma (6.3.2), one obtains

$$\begin{aligned}
&\{M_1 \overset{\otimes}{,} M_2\} \\
&\quad = \{T(x_1,x_2) \overset{\otimes}{,} T(y_1,y_2)\}(\widetilde{M_1} \otimes \widetilde{M_2}) \\
&\quad = \frac{2}{k}\,[T(x_1,0) \otimes T(y_1,0)]\,\Gamma\,[T(0,x_2) \otimes T(0,y_2)]\,(\widetilde{M_1} \otimes \widetilde{M_2}) \\
&\quad = \frac{2}{k}\,[T(x_1,0) \otimes T(y_1,0)]\,\Gamma[T(0,x_2)\widetilde{M_1} \otimes T(0,y_2)\widetilde{M_2}]. \qquad (6.37)
\end{aligned}$$

Taking trace on both sides of equation (6.37) and subsequently making use of the cyclic property of trace, one finally obtains

$$\{\operatorname{Tr} M_1, \operatorname{Tr} M_2\} = \frac{2}{k}\operatorname{Tr}_{12}[(T(0,x_2)\widetilde{M_1}T(x_1,0) \otimes T(0,y_2)\widetilde{M_2}T(y_1,0))\Gamma]. \qquad (6.38)$$

$\square$

## 6.4 Examples of Goldman brackets between Wilson loops for various real Lie groups

In the previous section, we obtained a general formula (6.35) for Poisson brackets between traces of monodromy matrices computed along free homotopy classes of loops on $\Sigma$. In this section, we shall derive explicit formulas of those Poisson brackets for $GL(n,\mathbb{R})$, $U(n)$, $SL(n,\mathbb{R})$, $SU(n)$ and $Sp(2n,\mathbb{R})$ monodromies. The Casimir tensor $\Gamma$, associated with the underlying Lie algebra $\mathcal{G}$, solely determines the underlying Poisson bracket between the respective Wilson loops in (6.35).

We, first, note that the generalized Gell-Mann matrices in $n$ dimensions read

$$\begin{aligned}
h_1^n &= \sqrt{\frac{2}{n}}\sum_{i=1}^{n} e_{ii},\\
h_k^n &= \sqrt{\frac{2}{k(k-1)}}\sum_{i=1}^{k-1} e_{ii} - \sqrt{2-\frac{2}{k}}e_{kk}, \quad \text{for } 1<k\le n,\\
f_{k,j}^n &= e_{kj}+e_{jk}, \quad \text{for } k<j,\\
f_{k,j}^n &= -i(e_{jk}-e_{kj}), \quad \text{for } k>j.
\end{aligned} \tag{6.39}$$

Here, $e_{jk}$ is an $n\times n$ matrix with $1$ in the $jk$-th entry and $0$ elsewhere.

We need a couple of preparatory Lemmas in order to prove the main results concerning poisson brackets between traces of monodromy matrices.

**Lemma 6.4.1.** *Given the matrices $h_1^n$ and $h_k^n$ as in (6.39), we have*

$$h_1^n\otimes h_1^n + \sum_{k=2}^{n} h_k^n\otimes h_k^n = 2e_{11}\otimes e_{11} + 2\sum_{k=2}^{n} e_{kk}\otimes e_{kk}. \tag{6.40}$$

**Proof** .

$$
\begin{aligned}
&\sum_{k=2}^{n} h_k^n \otimes h_k^n \\
&\quad = \sum_{k=2}^{n} \left[ \left( \sqrt{\frac{2}{k(k-1)}} \sum_{i=1}^{k-1} e_{ii} - \sqrt{2 - \frac{2}{k}} e_{kk} \right) \otimes \left( \sqrt{\frac{2}{k(k-1)}} \sum_{i=1}^{k-1} e_{ii} - \sqrt{2 - \frac{2}{k}} e_{kk} \right) \right] \\
&\quad = \sum_{k=2}^{n} \left[ \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \frac{2}{k} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \frac{2}{k} \sum_{i=1}^{k-1} e_{ii} \otimes e_{kk} \right. \\
&\qquad \left. - \frac{2}{k} \sum_{i=1}^{k-1} e_{kk} \otimes e_{ii} + \left( 2 - \frac{2}{k} \right) e_{kk} \otimes e_{kk} \right].
\end{aligned} \tag{6.41}
$$

Now, on the right side of (6.41), we compute

$$
\begin{aligned}
&\sum_{k=2}^{n} \left[ \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \frac{2}{k} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) \right] \\
&\quad = \sum_{k=2}^{n} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-2} e_{ii} \otimes \sum_{i=1}^{k-2} e_{ii} \right) \\
&\quad = \sum_{k=2}^{n} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \sum_{k=3}^{n+1} \frac{2}{k-1} \left[ \left( \sum_{i=1}^{k-1} e_{ii} - e_{k-1,k-1} \right) \right. \\
&\qquad \left. \otimes \left( \sum_{i=1}^{k-1} e_{ii} - e_{k-1,k-1} \right) \right] \\
&\quad = \sum_{k=2}^{n} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) - \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes \sum_{i=1}^{k-1} e_{ii} \right) \\
&\qquad + \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes e_{k-1,k-1} \right) + \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{k-1,k-1} \otimes e_{ii} \right) \\
&\qquad - \sum_{k=3}^{n+1} \frac{2}{k-1} (e_{k-1,k-1} \otimes e_{k-1,k-1}) \\
&\quad = 2e_{11} \otimes e_{11} - \frac{2}{n} \left( \sum_{i=1}^{n} e_{ii} \otimes \sum_{i=1}^{n} e_{ii} \right) + \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{ii} \otimes e_{k-1,k-1} \right) \\
&\qquad + \sum_{k=3}^{n+1} \frac{2}{k-1} \left( \sum_{i=1}^{k-1} e_{k-1,k-1} \otimes e_{ii} \right) - \sum_{k=3}^{n+1} \frac{2}{k-1} (e_{k-1,k-1} \otimes e_{k-1,k-1}) \\
&\quad = 2e_{11} \otimes e_{11} - \frac{2}{n} \left( \sum_{i=1}^{n} e_{ii} \otimes \sum_{i=1}^{n} e_{ii} \right) + \sum_{k=2}^{n} \frac{2}{k} \left( \sum_{i=1}^{k} e_{ii} \otimes e_{kk} \right)
\end{aligned}
$$

$$+\sum_{k=2}^{n}\frac{2}{k}\left(\sum_{i=1}^{k}e_{kk}\otimes e_{ii}\right)-\sum_{k=2}^{n}\frac{2}{k}(e_{kk}\otimes e_{kk})$$
$$=2e_{11}\otimes e_{11}-\frac{2}{n}\left(\sum_{i=1}^{n}e_{ii}\otimes\sum_{i=1}^{n}e_{ii}\right)+\sum_{k=2}^{n}\frac{2}{k}\left(\sum_{i=1}^{k-1}e_{ii}\otimes e_{kk}\right)+\sum_{k=2}^{n}\frac{2}{k}(e_{kk}\otimes e_{kk})$$
$$+\sum_{k=2}^{n}\frac{2}{k}\left(\sum_{i=1}^{k-1}e_{kk}\otimes e_{ii}\right)+\sum_{k=2}^{n}\frac{2}{k}(e_{kk}\otimes e_{kk})-\sum_{k=2}^{n}\frac{2}{k}(e_{kk}\otimes e_{kk})$$
$$=2e_{11}\otimes e_{11}-\frac{2}{n}\left(\sum_{i=1}^{n}e_{ii}\otimes\sum_{i=1}^{n}e_{ii}\right)+\sum_{k=2}^{n}\frac{2}{k}\left(\sum_{i=1}^{k-1}e_{ii}\otimes e_{kk}\right)+\sum_{k=2}^{n}\frac{2}{k}(e_{kk}\otimes e_{kk})$$
$$+\sum_{k=2}^{n}\frac{2}{k}\left(\sum_{i=1}^{k-1}e_{kk}\otimes e_{ii}\right). \tag{6.42}$$

Plugging (6.42) into (6.41), one obtains

$$\sum_{k=2}^{n}h_k^n\otimes h_k^n=2e_{11}\otimes e_{11}-\frac{2}{n}\left(\sum_{i=1}^{n}e_{ii}\otimes\sum_{i=1}^{n}e_{ii}\right)+2\sum_{k=2}^{n}e_{kk}\otimes e_{kk}, \tag{6.43}$$

which leads to

$$h_1^n\otimes h_1^n+\sum_{k=2}^{n}h_k^n\otimes h_k^n=2e_{11}\otimes e_{11}+2\sum_{k=2}^{n}e_{kk}\otimes e_{kk}.$$

□

**Lemma 6.4.2.** *Given $f_{k,j}^n$ for $k<j$ and $k>j$ as in (6.39), the tensor product between them is given by*

$$\sum_{k\neq j}f_{k,j}^n\otimes f_{k,j}^n=2\sum_{k\neq j}e_{jk}\otimes e_{kj}. \tag{6.44}$$

**Proof** .

$$\begin{aligned}\sum_{k<j}f_{kj}^n\otimes f_{kj}^n &= \sum_{k<j}(e_{kj}+e_{jk})\otimes(e_{kj}+e_{jk})\\ &= \sum_{k<j}e_{kj}\otimes e_{kj}+\sum_{k<j}e_{kj}\otimes e_{jk}+\sum_{k<j}e_{jk}\otimes e_{kj}+\sum_{k<j}e_{jk}\otimes e_{jk}.\end{aligned}$$

$$(6.45)$$

Also,

$$\begin{aligned}\sum_{k>j} f^n_{kj} \otimes f^n_{kj} &= \sum_{k>j}(-ie_{jk} + ie_{kj}) \otimes (-ie_{jk} + ie_{kj}) \\ &= -\sum_{k>j} e_{jk} \otimes e_{jk} + \sum_{k>j} e_{jk} \otimes e_{kj} + \sum_{k>j} e_{kj} \otimes e_{jk} - \sum_{k>j} e_{kj} \otimes e_{kj}.\end{aligned} \tag{6.46}$$

Now (6.45) together with (6.46) imply

$$\begin{aligned}\sum_{k\neq j} f^n_{k,j} \otimes f^n_{k,j} &= 2\sum_{k>j} e_{jk} \otimes e_{kj} + 2\sum_{k<j} e_{jk} \otimes e_{kj} \\ &= 2\sum_{k\neq j} e_{jk} \otimes e_{kj}.\end{aligned} \tag{6.47}$$

□

We are now all set to cook up the Casimir operator $\Gamma$, arising in (6.35) for $GL(n,\mathbb{R})$, $U(n)$, $SL(n,\mathbb{R})$, and $SU(n)$. In what follows next, the $n^2 \times n^2$ permutation matrix is denoted by $P$. Given two $n \times n$ matrices $A$ and $B$, $P$ enjoys the following properties

$$\begin{aligned}P(A \otimes B) = (B \otimes A)P \\ \mathrm{Tr}_{12}[(A \otimes B)P] = \mathrm{Tr}(AB).\end{aligned} \tag{6.48}$$

**Proposition 6.4.1.** *For $GL(n,\mathbb{R})$ and $U(n)$ to be the structure Lie group of the underlying principal $G$ bundle, the Casimir tensor in (6.35) reads*

$$\Gamma = 2P, \tag{6.49}$$

*where* $P = \sum\limits_{k,j=1}^{n} e_{jk} \otimes e_{kj}$ *is the Permutation matrix.*

**Proof** .

**Case** 1**:** $GL(n, \mathbb{R})$

The Lie algebra associated with $GL(n, \mathbb{R})$ is $\mathfrak{gl}(n, \mathbb{R})$, the vector space of all real $n \times n$ matrices. The dimension of this vector space is $n^2$. We choose the matrix $h_1^n$, $n-1$ matrices $h_k^n$ with $1 < k \leq n$, $\frac{n^2-n}{2}$ matrices $f_{k,j}^n$ with $k < j$, and another $\frac{n^2-n}{2}$ matrices $if_{k,j}^n$ with $k > j$ from (6.39) to form a basis of $\mathfrak{gl}(n, \mathbb{R})$. Also, in (6.8), associated with the preceding choice of generators for $GL(n, \mathbb{R})$, $f(a) = -1$ for $\frac{n^2-n}{2}$ basis elements $if_{k,j}^n$ with $k > j$. And for the rest of the $n^2$ basis elements $f(a) = 1$.

With the above choice of the basis of $\mathfrak{gl}(n, \mathbb{R})$, the Casimir tensor $\Gamma$ reads,

$$\begin{aligned}
\Gamma &= h_1^n \otimes h_1^n + \sum_{k=2}^{n} h_k^n \otimes h_k^n + \sum_{k<j} f_{k,j}^n \otimes f_{k,j}^n + \sum_{k>j} -(if_{k,j}^n \otimes if_{k,j}^n) \\
&= h_1^n \otimes h_1^n + \sum_{k=2}^{n} h_k^n \otimes h_k^n + \sum_{k<j} f_{k,j}^n \otimes f_{k,j}^n + \sum_{k>j} (f_{k,j}^n \otimes f_{k,j}^n). \qquad (6.50)
\end{aligned}$$

Using Lemma (6.4.1) together with Lemma (6.4.2) in (6.50), what one immediately obtains for $GL(n, \mathbb{R})$ is

$$\Gamma = 2 \sum_{k,j=1}^{n} e_{jk} \otimes e_{kj}. \qquad (6.51)$$

**Case** 2**:** $U(n)$

An appropriate choice of basis for the Lie algebra $\mathfrak{u}(n)$, in the context of (6.8), would be the $n^2$ skew-Hermitian matrices (see 6.39) $ih_1^n$, $ih_k^n$ for $1 < k \leq n$ and $if_{k,j}^n$ for $k \neq j$. In accordance with the choice of these generators of unitary group $U(n)$, $f(a) = -1$ in (6.8) for $a = 1, 2 \ldots, n^2$. The corresponding Casimir tensor $\Gamma$ reads off immediately

$$\begin{aligned}
\Gamma &= -(ih_1^n \otimes ih_1^n) + \sum_{k=2}^{n} -(ih_k^n \otimes ih_k^n) + \sum_{k<j} -(if_{k,j}^n \otimes if_{k,j}^n) + \sum_{k>j} -(if_{k,j}^n \otimes if_{k,j}^n) \\
&= h_1^n \otimes h_1^n + \sum_{k=2}^{n} h_k^n \otimes h_k^n + \sum_{k<j} f_{k,j}^n \otimes f_{k,j}^n + \sum_{k>j} f_{k,j}^n \otimes f_{k,j}^n \\
&= 2 \sum_{k,j=1}^{n} e_{jk} \otimes e_{kj}. \qquad (6.52)
\end{aligned}$$

Here, again, we use Lemma (6.4.1) and Lemma (6.4.2) to arrive at (6.52). □

Direct application of Proposition (6.4.1) in (6.35) and subsequent use of the properties of $P$, enumerated in (6.48), yield the formula of Poisson bracket between traces of monodromy matrices for $GL(n, \mathbb{R})$ and $U(n)$, as given in the following theorem

**Theorem 6.4.1.** *The poisson bracket (6.35) for $M_1$ and $M_2$ being either in $GL(n, \mathbb{R})$ or in $U(n)$ reads*

$$\{\operatorname{Tr} M_1, \operatorname{Tr} M_2\} = \frac{4}{k} \operatorname{Tr} M_{\gamma_1 \circ \gamma_2}, \qquad (6.53)$$

*where $M_{\gamma_1 \circ \gamma_2}$ is a $GL(n, \mathbb{R})$ or $U(n)$ monodromy computed along the deformed loop $\gamma_1 \circ \gamma_2$ in Figure 6–1.*

**Proposition 6.4.2.** *The Casimir tensor in (6.35) for the structure Lie group to be either $SL(n, \mathbb{R})$ or $SU(n)$ reads*

$$\Gamma = 2P - \frac{2}{n} \mathbb{I}, \qquad (6.54)$$

*with $P = \sum_{k,j=1}^{n} e_{jk} \otimes e_{kj}$ being the Permutation matrix and $\mathbb{I}$ being the $n^2 \times n^2$ identity matrix.*

**Proof** .

**Case** 1**:** $SL(n, \mathbb{R})$

The Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ consists of traceless $n \times n$ real matrices. We, therefore, choose $n-1$ matrices $h_k^n$ with $1 < k \leq n$, $\frac{n^2-n}{2}$ matrices $f_{kj}^n$ for $k \leq j$ and another $\frac{n^2-n}{2}$ real matrices $if_{kj}^n$ with $k > j$ from the ones enumerated in (6.39). As was in the case of $\mathfrak{gl}(n, \mathbb{R})$, $f(a) = -1$ in (6.8) holds only for the $SL(n, \mathbb{R})$ group generators $if_{kj}^n$. Therefore, the associated Casimir tensor reads

$$\begin{aligned} \Gamma &= \sum_{k=2}^{n} h_k^n \otimes h_k^n + \sum_{k<j} f_{k,j}^n \otimes f_{k,j}^n + \sum_{k>j} -(if_{k,j}^n \otimes if_{k,j}^n) \\ &= \sum_{k=2}^{n} h_k^n \otimes h_k^n + \sum_{k<j} f_{k,j}^n \otimes f_{k,j}^n + \sum_{k>j} f_{k,j}^n \otimes f_{k,j}^n \\ &= 2P - h_1^n \otimes h_1^n \\ &= 2P - \frac{2}{n}\mathbb{I}. \end{aligned} \tag{6.55}$$

**Case** 2**:** $SU(n)$

The real Lie algebra $\mathfrak{su}(n)$ consists of $n \times n$ tracless skew-Hermitian matrices. We choose, as a basis of $\mathfrak{su}(n)$, $n-1$ traceless skew-Hermitian matrices $ih_k^n$ with $1 < k \leq n$ and another $n^2 - n$ such matrices $if_{kj}^n$ for $k \neq j$ from the matrices enumerated in (6.39). Here, we only have $f(a) = -1$ in (6.8) for all such $(n^2-1)$ $SU(n)$ group generators. the corresponding Csimir tensor reads

$$\Gamma = \sum_{k=2}^{n} -(ih_k^n \otimes ih_k^n) + \sum_{k<j} -(if_{k,j}^n \otimes if_{k,j}^n) + \sum_{k>j} -(if_{k,j}^n \otimes if_{k,j}^n)$$

$$
\begin{aligned}
&= \sum_{k=2}^{n} h_k^n \otimes h_k^n + \sum_{k<j} f_{k,j}^n \otimes f_{k,j}^n + \sum_{k>j} f_{k,j}^n \otimes f_{k,j}^n \\
&= 2P - h_1^n \otimes h_1^n \\
&= 2P - \frac{2}{n}\mathbb{I}. \qquad (6.56)
\end{aligned}
$$

We have repeatedly used Lemma (6.4.1) and Lemma (6.4.2) in establishing (6.55) and (6.56). □

Following the use of Proposition (6.4.2) in (6.35) and subsequent use of properties of $P$ as given by (6.48), one obtains the Poisson bracket for $SL(n,\mathbb{R})$ and $SU(n)$ monodromies.

**Theorem 6.4.2.** *The Poisson bracket between traces of two $SL(n,\mathbb{R})$ or two $SU(n)$ monodromy matrices is given by*

$$
\{\operatorname{Tr} M_1, \operatorname{Tr} M_2\} = \frac{4}{k}\left(\operatorname{Tr} M_{\gamma_1\circ\gamma_2} - \frac{1}{n}\operatorname{Tr} M_1 \operatorname{Tr} M_2\right). \qquad (6.57)
$$

In course of proving Theorem (6.4.2), one also makes use of the identity $\operatorname{Tr}_{12}(A \otimes B) = \operatorname{Tr} A \operatorname{Tr} B$ for any two $n \times n$ matrices $A$ and $B$.

We shall now handle the case of the real Lie group $Sp(2n,\mathbb{R})$. It is being dealt separately since an appropriate choice of basis for the associated Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$, in the light of (6.8), is unrelated with the generalized Gell-Mann matrices enumerated in (6.39).

The Lie algebra $\mathfrak{sp}(2n,\mathbb{R})$ is an $n(2n+1)$ dimensional real vector space. An appropriate choice of basis, along with respective $f(a) = \pm 1$ for $a = 1, 2, \ldots, n(2n+1)$ in (6.8), is outlined in the following table

| $\mathbf{i, j, k}$ | **Basis elements** | $\mathbf{f(a)}$ | **No. of elements** |
|---|---|---|---|
| $1 \leq i < j \leq n$ | $\frac{1}{\sqrt{2}}(e_{i,j+n} + e_{j,i+n} + e_{j+n,i} + e_{i+n,j})$ | 1 | $\frac{n^2-n}{2}$ |
| $1 \leq i < j \leq n$ | $\frac{1}{\sqrt{2}}(e_{i,j+n} + e_{j,i+n} - e_{j+n,i} - e_{i+n,j})$ | -1 | $\frac{n^2-n}{2}$ |
| $1 \leq k \leq n$ | $e_{k,n+k} + e_{n+k,k}$ | 1 | n |
| $1 \leq k \leq n$ | $e_{k,n+k} - e_{n+k,k}$ | -1 | n |
| $1 \leq i < j \leq n$ | $\frac{1}{\sqrt{2}}(e_{ij} + e_{ji} - e_{i+n,j+n} - e_{j+n,i+n})$ | 1 | $\frac{n^2-n}{2}$ |
| $1 \leq i < j \leq n$ | $\frac{1}{\sqrt{2}}(e_{ij} - e_{ji} + e_{i+n,j+n} - e_{j+n,i+n})$ | -1 | $\frac{n^2-n}{2}$ |
| $1 \leq k \leq n$ | $e_{kk} - e_{k+n,k+n}$ | 1 | n |

Table 6–1: Appropriate choice of basis for $\mathfrak{sp}(2n, \mathbb{R})$

Now, the Casimir tensor for the structure Lie group $Sp(2n, \mathbb{R})$ is provided by the following proposition

**Proposition 6.4.3.** *The Casimir tensor* $\Gamma$ *in (6.35), for* $Sp(2n, \mathbb{R})$ *to be the structure Lie group of the underlying principal G-bundle, reads*

$$\Gamma = P + \chi, \tag{6.58}$$

*with* $\chi$ *given as*

$$\begin{aligned}\chi = \sum_{1 \leq i < j \leq n} &(e_{i,j+n} \otimes e_{i+n,j} + e_{j,i+n} \otimes e_{j+n,i} + e_{j+n,i} \otimes e_{j,i+n} + e_{i+n,j} \otimes e_{i,j+n} \\ &- e_{ij} \otimes e_{i+n,j+n} - e_{j+n,i+n} \otimes e_{ji} - e_{ji} \otimes e_{j+n,i+n} - e_{i+n,j+n} \otimes e_{ij}) \\ &+ \sum_{1 \leq k \leq n} (e_{k,n+k} \otimes e_{n+k,k} + e_{n+k,k} \otimes e_{k,n+k} - e_{kk} \otimes e_{k+n,k+n} - e_{k+n,k+n} \otimes e_{kk}).\end{aligned} \tag{6.59}$$

*We shall be calling* $\chi$ *as the defect matrix henceforth.*

**Proof** . In order to prove proposition (6.59), we first note that, for $a$ and $b$ to be two $n \times n$ matrices, the following holds

$$(a+b)\otimes(a+b)-(a-b)\otimes(a-b)=2(a\otimes b+b\otimes a). \tag{6.60}$$

Now, using the above fact, we compute for $1 \le i < j \le n$,

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(e_{i,j+n}+e_{j,i+n}+e_{j+n,i}+e_{i+n,j})\otimes\frac{1}{\sqrt{2}}(e_{i,j+n}+e_{j,i+n}+e_{j+n,i}+e_{i+n,j})\\
&-\frac{1}{\sqrt{2}}(e_{i,j+n}+e_{j,i+n}-e_{j+n,i}-e_{i+n,j})\otimes\frac{1}{\sqrt{2}}(e_{i,j+n}+e_{j,i+n}-e_{j+n,i}-e_{i+n,j})\\
&=e_{i,j+n}\otimes e_{j+n,i}+e_{i,j+n}\otimes e_{i+n,j}+e_{j,i+n}\otimes e_{j+n,i}+e_{j,i+n}\otimes e_{i+n,j}\\
&+e_{j+n,i}\otimes e_{i,j+n}+e_{j+n,i}\otimes e_{j,i+n}+e_{i+n,j}\otimes e_{i,j+n}+e_{i+n,j}\otimes e_{j,i+n}.
\end{aligned} \tag{6.61}$$

We also compute for $1 \le k \le n$,

$$\begin{aligned}
&(e_{k,n+k}+e_{n+k,k})\otimes(e_{k,n+k}+e_{n+k,k})-(e_{k,n+k}-e_{n+k,k})\otimes(e_{k,n+k}-e_{n+k,k})\\
&\qquad=2(e_{k,n+k}\otimes e_{n+k,k}+e_{n+k,k}\otimes e_{k,n+k}).
\end{aligned} \tag{6.62}$$

Again, considering another set of $n^2-n$ generators and applying (6.60), one obtains for $1 \le i < j \le n$,

$$\begin{aligned}
&\frac{1}{\sqrt{2}}(e_{ij}+e_{ji}-e_{i+n,j+n}-e_{j+n,i+n})\otimes\frac{1}{\sqrt{2}}(e_{ij}+e_{ji}-e_{i+n,j+n}-e_{j+n,i+n})\\
&-\frac{1}{\sqrt{2}}(e_{ij}-e_{ji}+e_{i+n,j+n}-e_{j+n,i+n})\otimes\frac{1}{\sqrt{2}}(e_{ij}-e_{ji}+e_{i+n,j+n}-e_{j+n,i+n})\\
&=e_{ij}\otimes e_{ji}-e_{ij}\otimes e_{i+n,j+n}-e_{j+n,i+n}\otimes e_{ji}+e_{j+n,i+n}\otimes e_{i+n,j+n}\\
&+e_{ji}\otimes e_{ij}-e_{ji}\otimes e_{j+n,i+n}-e_{i+n,j+n}\otimes e_{ij}+e_{i+n,j+n}\otimes e_{j+n,i+n}.
\end{aligned} \tag{6.63}$$

Finally, for $n$ diagonal generators of $Sp(2n,\mathbb{R})$, we obtain with $1 \leq k \leq n$,

$$(e_{kk} - e_{k+n,k+n}) \otimes (e_{kk} - e_{k+n,k+n})$$
$$= e_{kk} \otimes e_{kk} - e_{kk} \otimes e_{k+n,k+n} - e_{k+n,k+n} \otimes e_{kk} + e_{k+n,k+n} \otimes e_{k+n,k+n}. \quad (6.64)$$

Adding (6.61) with (6.63) and (6.62) with (6.64) followed by summing over $1 \leq i < j \leq n$ and $1 \leq k \leq n$, respectively and finally adding up the two summands, we obtain,

$$\begin{aligned}
\Gamma = \Bigg[ \sum_{1 \leq i < j \leq n} & (e_{i,j+n} \otimes e_{j+n,i} + e_{j,i+n} \otimes e_{i+n,j} + e_{j+n,i} \otimes e_{i,j+n} + e_{i+n,j} \otimes e_{j,i+n} \\
& + e_{ij} \otimes e_{ji} + e_{j+n,i+n} \otimes e_{i+n,j+n} + e_{ji} \otimes e_{ij} + e_{i+n,j+n} \otimes e_{j+n,i+n}) \\
& + \sum_{1 \leq k \leq n} (e_{k,n+k} \otimes e_{n+k,k} + e_{n+k,k} \otimes e_{k,n+k} + e_{kk} \otimes e_{kk} + e_{k+n,k+n} \otimes e_{k+n,k+n}) \Bigg] \\
& + \Bigg[ \sum_{1 \leq i < j \leq n} (e_{i,j+n} \otimes e_{i+n,j} + e_{j,i+n} \otimes e_{j+n,i} + e_{j+n,i} \otimes e_{j,i+n} + e_{i+n,j} \otimes e_{i,j+n} \\
& - e_{ij} \otimes e_{i+n,j+n} - e_{j+n,i+n} \otimes e_{ji} - e_{ji} \otimes e_{j+n,i+n} - e_{i+n,j+n} \otimes e_{ij}) \\
& + \sum_{1 \leq k \leq n} (e_{k,n+k} \otimes e_{n+k,k} + e_{n+k,k} \otimes e_{k,n+k} - e_{kk} \otimes e_{k+n,k+n} - e_{k+n,k+n} \otimes e_{kk}) \Bigg] \\
& = P + \chi. \qquad (6.65)
\end{aligned}$$

$\square$

We require the following Lemma to prove the main result regarding the Poisson bracket for $Sp(2n,\mathbb{R})$ monodromy matrices.

**Lemma 6.4.3.** *For $A, B \in Sp(2n, \mathbb{R})$, $\chi$ being the defect matrix as in Proposition (6.4.3) and $P$ being the permutation matrix, we have the following identity*

$$\mathrm{Tr}_{12}[(A \otimes B)\chi] = -\,\mathrm{Tr}(AB^{-1}). \tag{6.66}$$

**Proof** . Given the $2n \times 2n$ symplectic matrix $B$, its inverse is given by the following sets of equations:

For matrix entries with $1 \leq i < j \leq n$,

$$\begin{aligned}
&(B^{-1})_{ij} = B_{j+n,i+n}, \qquad &&(B^{-1})_{ji} = B_{i+n,j+n}, \qquad &&(B^{-1})_{i,j+n} = -B_{j,i+n} \\
&(B^{-1})_{j,i+n} = -B_{i,j+n}, \qquad &&(B^{-1})_{n+i,j} = -B_{j+n,i}, \qquad &&(B^{-1})_{j+n,i} = -B_{n+i,j} \\
&(B^{-1})_{i+n,j+n} = B_{ji}, \qquad &&(B^{-1})_{j+n,i+n} = B_{ij}.
\end{aligned} \tag{6.67}$$

And, for matrix entries with $1 \leq k \leq n$,

$$\begin{aligned}
&(B^{-1})_{kk} = B_{k+n,k+n}, \qquad &&(B^{-1})_{k,n+k} = -B_{k,n+k} \\
&(B^{-1})_{n+k,k} = -B_{n+k,k}, \qquad &&(B^{-1})_{k+n,k+n} = B_{kk}.
\end{aligned} \tag{6.68}$$

Now, using the explicit expression of the defect matrix $\chi$ given in (6.59) and that of the symplectic matrix $B^{-1}$ in (6.67) and (6.68), one obtains

$$\begin{aligned}
&\mathrm{Tr}_{12}[(A \otimes B)\chi] \\
&= \sum_{1 \leq i < j \leq n} (A_{j+n,i}B_{j,i+n} + A_{i+n,j}B_{i,j+n} + A_{i,j+n}B_{i+n,j} + A_{j,i+n}B_{j+n,i}) \\
&+ \sum_{1 \leq k \leq n} (A_{n+k,k}B_{k,n+k} + A_{k,n+k}B_{n+k,k}) - \sum_{1 \leq i < j \leq n} (A_{ji}B_{j+n,i+n} + A_{i+n,j+n}B_{ij})
\end{aligned}$$

$$- \sum_{1\leq i<j\leq n}(A_{ij}B_{i+n,j+n} + A_{j+n,i+n}B_{ji}) - \sum_{1\leq k\leq n}(A_{kk}B_{k+n,k+n} + A_{k+n,k+n}B_{kk})$$

$$= - \sum_{1\leq i<j\leq n}[A_{j+n,i}(B^{-1})_{i,j+n} + A_{i+n,j}(B^{-1})_{j,i+n} + A_{i,j+n}(B^{-1})_{j+n,i}$$

$$+ A_{j,i+n}(B^{-1})_{i+n,j}]$$

$$- \sum_{1\leq k\leq n}[A_{n+k,k}(B^{-1})_{k,n+k} + A_{k,n+k}(B^{-1})_{n+k,k}] - \sum_{1\leq i<j\leq n}[A_{ji}(B^{-1})_{ij}$$

$$+ A_{i+n,j+n}(B^{-1})_{j+n,i+n}]$$

$$- \sum_{1\leq i<j\leq n}[A_{ij}(B^{-1})_{ji} + A_{j+n,i+n}(B^{-1})_{i+n,j+n}] - \sum_{1\leq k\leq n}[A_{kk}(B^{-1})_{kk}$$

$$+ A_{k+n,k+n}(B^{-1})_{k+n,k+n}]$$

$$= - \operatorname{Tr}(AB^{-1}). \tag{6.69}$$

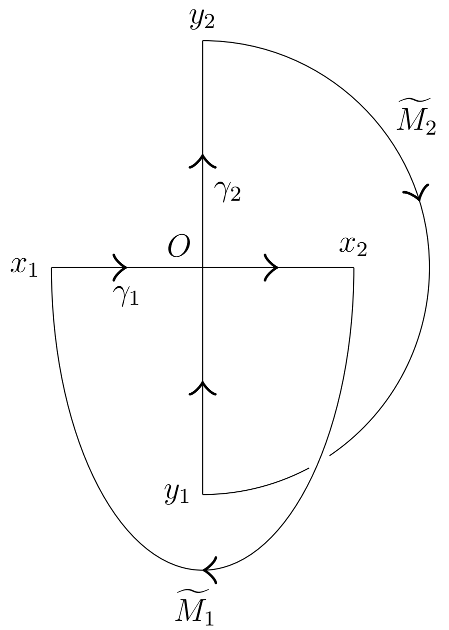
□

We now prove the main theorem concerning the Poisson bracket between traces of $Sp(2n, \mathbb{R})$ monodromies.

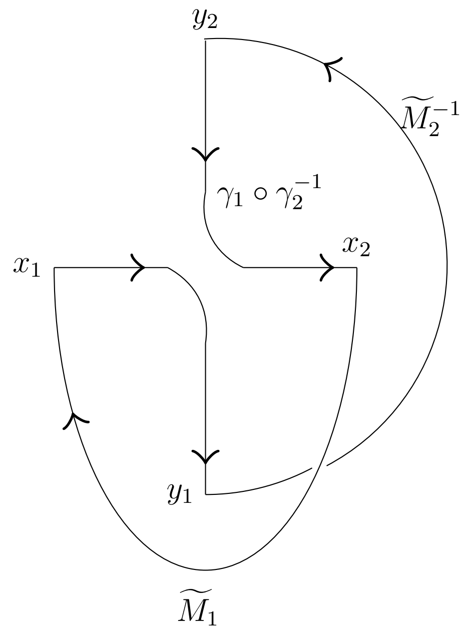
**Theorem 6.4.3.** *The Poisson bracket between traces of $Sp(2n, \mathbb{R})$ monodromy matrices $M_1$ and $M_2$ is given by*

$$\{\operatorname{Tr} M_1, \operatorname{Tr} M_2\} = \frac{2}{k}(\operatorname{Tr} M_{\gamma_1\circ\gamma_2} - \operatorname{Tr} M_{\gamma_1\circ\gamma_2^{-1}}), \tag{6.70}$$

*where $M_{\gamma_1\circ\gamma_2}$ is an $Sp(2n, \mathbb{R})$ monodromy, computed along the deformed loop $\gamma_1 \circ \gamma_2$, as shown in Figure (6–1) while the monodromy $M_{\gamma_1\circ\gamma_2^{-1}}$ is computed along the other deformed loop $\gamma_1 \circ \gamma_2^{-1}$, as described by Figure (6–2).*




(a) Two intersecting loops.




(b) Deformed loop $\gamma_1 \circ \gamma_2^{-1}$

Figure 6–2: Traces of monodromies are computed along two free loops that are homotopically inequivalent and intersect transversally at a single point. In the following subfigure, trace of monodromy along a single loop, deformed at the point of intersection, has been considered.

**Proof** . Plugging the Casimir tensor $\Gamma$ (see $6.58$) back in (6.35) and using the identity from lemma (6.4.3), one obtains

$$\begin{aligned}
&\{\operatorname{Tr} M_1, \operatorname{Tr} M_2\} \\
&= \frac{2}{k} \operatorname{Tr}_{12}[(T(0,x_2)\widetilde{M}_1 T(x_1,0) \otimes T(0,y_2)\widetilde{M}_2 T(y_1,0))(P+\chi)] \\
&= \frac{2}{k} \operatorname{Tr} M_{\gamma_1\circ\gamma_2} + \frac{2}{k} \operatorname{Tr}_{12}[(T(0,x_2)\widetilde{M}_1 T(x_1,0) \otimes T(0,y_2)\widetilde{M}_2 T(y_1,0))\chi] \\
&= \frac{2}{k} \operatorname{Tr} M_{\gamma_1\circ\gamma_2} - \frac{2}{k} \operatorname{Tr}[T(0,x_2)\widetilde{M}_1 T(x_1,0) T(0,y_1)\widetilde{M}_2^{-1} T(y_2,0)] \\
&= \frac{2}{k}(\operatorname{Tr} M_{\gamma_1\circ\gamma_2} - \operatorname{Tr} M_{\gamma_1\circ\gamma_2^{-1}}).
\end{aligned} \tag{6.71}$$

□

We note that (6.4.1), (6.4.2) and (6.70) coincide with Goldman's formula in ([29], page 266). Also, we computed the Poisson brackets for various real Lie groups for a single point of transversal intersection. The proof for many intersection points follow similarly.

## 6.5 Acknowledgements

The author gratefully acknowledges useful discussions with Prof. D. Korotkin.

# Chapter 7
# Conclusion and Future Directions

The main results of this thesis are related with Lie algebraic and representation theoretic methods.

We studied the structural similarities between extensions of the (1+1)-Galilei group and the groups frequently used in signal analysis and image processing. The fact that the various groups of signal analysis, enumerated in chapter 2, are all obtainable from the affine Galilei group shows a remarkable unity in their structures and consequently of their unitary irreducible representations. In the following chapter, we made a comparative study of the structures of their co-adjoint orbits and built Wigner functions on them. From the point of view of signal transforms, all this could lead to a deeper understanding of how signal transforms, defined over a larger set of parameters, reduce when a smaller set of parameters is used, with the original signal still being reconstructible from the smaller set.

In chapter 4, we have derived the commutation relations between the position and momentum operators of noncommuttive quantum mechanics by three different means: using the appropriate unitary irreducible representations of the centrally extended (2+1)-Galilei group $G_{\text{Gal}}^{\text{ext}}$, of the doubly extended group $\overline{\overline{G_T}}$, of translations of $\mathbb{R}^4$, and by a coherent state quantization of the classical phase space variables of position and momentum, using the coherent states of $G_{\text{Gal}}^{\text{ext}}$. It is not hard to see, from the expressions for the unitary representations (4.12) and (4.49), that the same commutation relations could also be obtained by a coherent state quantization, using the coherent states of $\overline{\overline{G_T}}$ (which could be similarly constructed). There is, as usual a positive operator valued (POV) measure naturally associated to the coherent states (4.18). Indeed, for any measurable set $\Delta$ of $\mathbb{R}^2$ (phase space), we can associate the positive operator

$$a(\Delta) = \int_{\Delta} |\eta_{\mathbf{q},\mathbf{p}}\rangle\langle\eta_{\mathbf{q},\mathbf{p}}| \; d\mathbf{q}\; d\mathbf{p}\;.$$

These define localization operators on phase space, whose marginals in $\mathbf{q}$ and $\mathbf{p}$ should then give localization operators in configuration and momentum spaces, respectively. For the canonical coherent states and standard quantum mechanics, such operators have been studied extensively, in e.g., [3, 16]. There, one understands these localization operators in an extended or unsharp sense. It would be interesting to do a similar study for the present case.

In chapter 5, we have shown that the *triply extended group of translations in* $\mathbb{R}^4$, $G_{\text{NC}}$ (note that $\overline{\overline{\overline{G_T}}}$ denotes the same Lie group in chapter 4), contains various representations, associated with different gauges of noncommutative quantum mechanics (see [24]), viz., the Landau and symmetric gauges, in its unitary dual. The unitary irreducible representations of standard quantum mechanics are also sitting inside its unitary dual. The representations associated with a coupled bosonic system, that give rise to certain deformed complex Hermite polynomials (see [6] and [9] for detail), are just a family of unitary irreducible representations of $G_{\text{NC}}$. The relevant coadjoint orbits of $G_{\text{NC}}$, sitting inside the 7-dimensional dual Lie algebra, have all been identified. The second cohomology group of the group of translations in $\mathbb{R}^4$ is a 6-dimensional vector space. In chapter 5, we considered $[\hat{Q}_1, \hat{P}_1]$, $[\hat{Q}_2, \hat{P}_2]$, $[\hat{Q}_1, \hat{Q}_2]$, and $[\hat{P}_1, \hat{P}_2]$ to be nonvanishing in general, as is done in NCQM. The strengths of the first two noncommutativity were chosen to be the same in order to preserve the structure of standard quantum mechanics. Along with this quantum mechanical noncommutativity, the position and momentum noncommutativity give rise to three independent central extensions of the abelian group of translations in $\mathbb{R}^4$. While the goal of chapter 5 was to study the role of this triply extended group $G_{\text{NC}}$ in NCQM, it would be interesting to study the other extensions of the group of translations in physically meaningful contexts, e.g. rotational invariance. Here, we restricted ourselves to 2 degrees of freedom meaning that we studied 2-dimensional NCQM from a group-theoretic point of view. But one could study possible extensions of the theory to quantum systems with additional degrees of freedom as well and look for a more general theory by constructing a more general version of $G_{\text{NC}}$.

In chapter 6, we considered a separate problem where space-time is modeled as a 3-manifold $\Sigma \times \mathbb{R}$. We considered the infinite dimensional field theory associated with connection 1-forms , taking their values in the Lie algebra $\mathcal{G}$ of the structure Lie group $G$ of the underlying principal $G$-bundle. Well-known Atiyah-Bott brackets, between the connection

1-forms and the momenta conjugate to them, were also computed starting from the Chern-Simons action defined on the 3-manifold. Time dependence in the 1-forms was gauged out using additional gauge freedom. The space of flat connections up to a gauge transformation is a finite dimensional space. The observables associated with this finite dimensional moduli space are just the Wilson lines computed along the free homotopy classes of loops on our Riemann surface. The brackets between these observables, i.e. brackets between the traces of $G$-valued monodromy matrices are known as the Goldman bracket. Making use of the Hamiltonian formalism of soliton theory, we computed Poisson brackets between traces of these monodromy matrices for the cases of $GL(n,\mathbb{R})$, $U(n)$, $SL(n,\mathbb{R})$, $SU(n)$ and $Sp(2n,\mathbb{R})$. We plan to apply similar algebraic formalism in order to find the brackets for the remaining cases of semi-simple Lie groups, say, $SU(p,q)$, $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$, in future.

## Appendix A
## Proofs related to Chapter 4

In this Appendix we collect together the proofs of some of the results quoted in chapter 4.

**Proof of Lemma 4.3.2**

We start out by taking two compactly supported and infinitely differentiable functions $f, g \in L^2(\mathbb{R}^2, d\mathbf{x})$. Then,

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \langle f \mid \chi_{\mathbf{q},\mathbf{p}}\rangle\langle \chi_{\mathbf{q},\mathbf{p}} \mid g\rangle \; d\mathbf{q}\; d\mathbf{p}$$
$$= \int_{\mathbb{R}^2\times\mathbb{R}^2} d\mathbf{q}\; d\mathbf{p} \left[\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\chi\left(\mathbf{x}+\mathbf{q}-\frac{\lambda}{2m^2}J\mathbf{p}\right)\overline{\chi\left(\mathbf{x}'+\mathbf{q}-\frac{\lambda}{2m^2}J\mathbf{p}\right)}\right.$$
$$\left.\times\overline{f(\mathbf{x})}g(\mathbf{x}')\; d\mathbf{x}\; d\mathbf{x}'\right] \tag{A.1}$$

Making the change of variables, $\mathbf{q} - \dfrac{\lambda}{2m^2}J\mathbf{p} = \mathbf{q}'$,

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \langle f \mid \chi_{\mathbf{q},\mathbf{p}}\rangle\langle \chi_{\mathbf{q},\mathbf{p}} \mid g\rangle \; d\mathbf{q}\; d\mathbf{p}$$
$$= \int_{\mathbb{R}^2\times\mathbb{R}^2} d\mathbf{q}'\; d\mathbf{p}\; \left[\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\chi(\mathbf{x}+\mathbf{q}')\overline{\chi(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')\overline{f(\mathbf{x})}\; d\mathbf{x}\; d\mathbf{x}'\right]$$
$$= (2\pi)^2\int_{\mathbb{R}^2} d\mathbf{q}'\left[\int_{\mathbb{R}^2\times\mathbb{R}^2} \delta(\mathbf{x}-\mathbf{x}')\chi(\mathbf{x}+\mathbf{q}')\overline{\chi(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')\overline{f(\mathbf{x})}\; d\mathbf{x}\; d\mathbf{x}'\right]$$
$$= (2\pi)^2\int_{\mathbb{R}^2} d\mathbf{q}'\; \left[\int_{\mathbb{R}^2} \chi(\mathbf{x}+\mathbf{q}')\overline{\chi(\mathbf{x}+\mathbf{q}')}g(\mathbf{x})\overline{f(\mathbf{x})}\; d\mathbf{x}\right]$$
$$= (2\pi)^2\|\chi\|^2\langle f \mid g\rangle, \tag{A.2}$$

the change in the order of integration and the introduction of the delta measure being easily justified in view of the compact supports and smoothness property of the functions $f$ and $g$. Thus, introducing the formal operator

$$T = \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\chi_{\mathbf{q},\mathbf{p}}\rangle\langle\chi_{\mathbf{q},\mathbf{p}}| \, d\mathbf{q} \, d\mathbf{p} \; , \tag{A.3}$$

we see that for functions $f, g$ of the chosen type,

$$\langle f|Tg\rangle = 2\pi\|\chi\|^2 \; \langle f|Ig\rangle \; ,$$

$I$ being the identity operator on $L^2(\mathbb{R}^2, d\mathbf{x})$. But since the compactly supported and infinitely differentiable functions are dense in $L^2(\mathbb{R}^2, d\mathbf{x})$, we use the continuity of the scalar product to extend the above equality to arbitrary pairs of functions $f, g$ in $L^2(\mathbb{R}^2, d\mathbf{x})$, thus proving the lemma. □

**Proof of Theorem 4.3.2**

We only work out the derivation of the first of the above equations, the others being obtained in similar ways. By (4.20) and (4.24)

$$\begin{aligned}(\hat{\mathcal{O}}_{q_1} g)(\mathbf{x}) & \\ &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} q_1 \left[ \int_{\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}} \; \eta \left( \mathbf{x} + \mathbf{q} - \frac{\lambda}{2m^2} J\mathbf{p} \right) \right. \\ &\left. \times \overline{\eta \left( \mathbf{x}' + \mathbf{q} - \frac{\lambda}{2m^2} J\mathbf{p} \right)} g(\mathbf{x}') \; d\mathbf{x}' \right] d\mathbf{q} \; d\mathbf{p} \; . \end{aligned} \tag{A.4}$$

Making the change of variables $\mathbf{q} - \frac{\lambda}{2m^2} J\mathbf{p} = \mathbf{q}'$, and noting the form of the skew-symmetric matrix $J$ from Lemma (4.3.1), we have

$$q_1' = q_1 + \frac{\lambda}{2m^2} p_2, \qquad q_2' = q_2 - \frac{\lambda}{2m^2} p_1 \ ,$$

using which (A.4) becomes

$$\begin{aligned}
&(\hat{\mathcal{O}}_{q_1} g)(\mathbf{x}) \\
&= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( q_1' - \frac{\lambda}{2m^2} p_2 \right) \left[ \int_{\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot \mathbf{p}} \ \eta(\mathbf{x}+\mathbf{q}') \overline{\eta(\mathbf{x}'+\mathbf{q}')} g(\mathbf{x}') \ d\mathbf{x}' \right] \ d\mathbf{q}' \ d\mathbf{p} \\
&= \int_{\mathbb{R}^2 \times \mathbb{R}^2} q_1' \ \left[ \int_{\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot \mathbf{p}} \eta(\mathbf{x}+\mathbf{q}') \overline{\eta(\mathbf{x}'+\mathbf{q}')} g(\mathbf{x}') \ d\mathbf{x}' \right] \ d\mathbf{q}' d\mathbf{p} \\
&- \frac{\lambda}{2m^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} p_2 \left[ \int_{\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot \mathbf{p}} \eta(\mathbf{x}+\mathbf{q}') \overline{\eta(\mathbf{x}'+\mathbf{q}')} g(\mathbf{x}') d\mathbf{x}' \right] \ d\mathbf{q}' \ d\mathbf{p} \ . \qquad \text{(A.5)}
\end{aligned}$$

Let us consider the first integral in (A.5). Assuming $\eta$ to be sufficiently smooth functions, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2 \times \mathbb{R}^2} q_1' \ \left[ \int_{\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot \mathbf{p}} \ \eta(\mathbf{x}+\mathbf{q}') \overline{\eta(\mathbf{x}'+\mathbf{q}')} g(\mathbf{x}') d\mathbf{x}' \right] \ d\mathbf{q}' \ d\mathbf{p} \\
&\quad = (2\pi)^2 \int_{\mathbb{R}^2} q_1' \left[ \int_{\mathbb{R}^2} \delta(\mathbf{x}-\mathbf{x}')] \ \eta(\mathbf{x}+\mathbf{q}') \overline{\eta(\mathbf{x}'+\mathbf{q}')} g(\mathbf{x}') d\mathbf{x}' \right] d\mathbf{q}' \\
&\quad = (2\pi)^2 \int_{\mathbb{R}^2} q_1' \ |\eta(\mathbf{x}+\mathbf{q}')|^2 g(\mathbf{x}) \ d\mathbf{q}' \qquad \text{(A.6)}
\end{aligned}$$

Making a second change of variables, $\mathbf{x} + \mathbf{q}' = -\mathbf{u}$, the last term in (A.6) becomes

$$\begin{aligned}
&(2\pi)^2 \int_{\mathbb{R}^2} q_1' |\eta(\mathbf{x}+\mathbf{q}')|^2 g(\mathbf{x}) \ d\mathbf{q}' \\
&\quad = (2\pi)^2 x_1 g(\mathbf{x}) \int_{\mathbb{R}^2} |\eta(\mathbf{u})|^2 \ du + (2\pi)^2 g(\mathbf{x}) \int_{\mathbb{R}^2} u_1 |\eta(\mathbf{u})|^2 g(\mathbf{x}) \ d\mathbf{u} \ . \qquad \text{(A.7)}
\end{aligned}$$

The second integral in the last line vanishes since, in view of the imposed symmetry, $\eta$ is an even function of $u_1$. Thus, noting the normalization of $\eta$ in (4.18),

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} q_1' \left[\int_{\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\, \eta(\mathbf{x}+\mathbf{q}')\overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')d\mathbf{x}'\right] d\mathbf{q}'\, d\mathbf{p} = x_1 g(\mathbf{x}). \tag{A.8}$$

Next, we observe that,

$$\begin{aligned}
&-i\frac{\partial}{\partial x_2}\left(e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\, \eta(\mathbf{x}+\mathbf{q}')\overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')\right)\\
&\quad = p_2 e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\, \overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')\eta(\mathbf{x}+\mathbf{q}')\\
&\quad + e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')\left(-i\frac{\partial}{\partial x_2}\right)(\eta(\mathbf{x}+\mathbf{q}')),
\end{aligned}$$

so that the second integral in (A.5) becomes

$$\begin{aligned}
&\frac{\lambda}{2m^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} p_2 \left[\int_{\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\, \eta(\mathbf{x}+\mathbf{q}')\overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')\, d\mathbf{x}'\right] d\mathbf{q}'\, d\mathbf{p}\\
&\quad = \frac{\lambda}{2m^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\left(-i\frac{\partial}{\partial x_2}\right)(e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\, \eta(\mathbf{x}+\mathbf{q}')\overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}'))\, d\mathbf{x}'\right] d\mathbf{q}'\, d\mathbf{p}\\
&\quad -\frac{\lambda}{2m^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\{e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\, \overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')\left(-i\frac{\partial}{\partial x_2}\right)\eta(\mathbf{x}+\mathbf{q}')\}\, d\mathbf{x}'\right] d\mathbf{q}'\, d\mathbf{p}.
\end{aligned} \tag{A.9}$$

Assuming the usual smoothness condition on $\eta$ and again introducing a delta-distribution in $\mathbf{x}, \mathbf{x}'$, the first integral on the right hand side of (A.9) gives

$$\begin{aligned}
&\frac{\lambda}{2m^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left(-i\frac{\partial}{\partial x_2}\right)\left[\left\{\int_{\mathbb{R}^2} e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\, \eta(\mathbf{x}+\mathbf{q}')\overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')d\mathbf{x}'\right\} d\mathbf{q}'\, d\mathbf{p}\right]\\
&\quad = \frac{(2\pi)^2\lambda}{2m^2}\int_{\mathbb{R}^2}\left(-i\frac{\partial}{\partial x_2}\right)\left[\left\{\int_{\mathbb{R}^2}\delta(\mathbf{x}-\mathbf{x}')\eta(\mathbf{x}+\mathbf{q}')\overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')d\mathbf{x}'\right\} d\mathbf{q}'\right]\\
&\quad = -\frac{i\lambda}{2m^2}\frac{\partial}{\partial x_2}g(\mathbf{x})\ .
\end{aligned} \tag{A.10}$$

Similarly, the second integral (A.9) yields

$$
\begin{aligned}
&-\frac{\lambda}{2m^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\{e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{p}}\,\overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')\left(-i\frac{\partial}{\partial x_2}\right)\eta(\mathbf{x}+\mathbf{q}')\}\,d\mathbf{x}'\right]\,d\mathbf{q}'\,d\mathbf{p}\\
&\quad=-\frac{\lambda}{2m^2}\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\{\delta(\mathbf{x}-\mathbf{x}')\overline{\eta(\mathbf{x}'+\mathbf{q}')}g(\mathbf{x}')\left(-i\frac{\partial}{\partial x_2}\right)\eta(\mathbf{x}+\mathbf{q}')\}\,d\mathbf{x}'\}\right]\,d\mathbf{q}'\\
&\quad=-\frac{\lambda}{2m^2}g(\mathbf{x})\int_{\mathbb{R}^2}\overline{\eta(\mathbf{x}+\mathbf{q}')}\left(-i\frac{\partial}{\partial x_2}\right)\eta(\mathbf{x}+\mathbf{q}')\,d\mathbf{q}'.
\end{aligned}
$$

Introducing another change of variables, $\mathbf{x}+\mathbf{q}'=\mathbf{u}$, this becomes

$$
\begin{aligned}
&-\frac{\lambda}{2m^2}g(\mathbf{x})\int_{\mathbb{R}^2}\overline{\eta(\mathbf{u})}\left(-i\frac{\partial}{\partial u_2}\right)\eta(\mathbf{u})\,d\mathbf{u}\\
&\quad=\frac{i\lambda}{2m^2}g(\mathbf{x})\int_{\mathbb{R}^2}\overline{\eta(\mathbf{u})}\frac{\partial}{\partial u_2}\eta(\mathbf{u})\,d\mathbf{u}=0,
\end{aligned}
\tag{A.11}
$$

the last equality following since, in view of the evenness of $\eta$, the derivative term, $\dfrac{\partial}{\partial u_2}\eta(\mathbf{u})$, is an odd function.

Thus finally, combining (A.11) with (A.5), (A.8), and (A.10), we obtain

$$
(\hat{\mathcal{O}}_{q_1}g)(\mathbf{x})=\left(x_1-\frac{i\lambda}{2m^2}\frac{\partial}{\partial x_2}\right)g(\mathbf{x}).
$$

□

**Proof of Theorem 4.4.1**

Using (4.28) and (4.29), it can easily be verified that $\xi$, $\xi'$, and $\xi''$ given in Proposition (4.4.1) are local exponents for the group of translations $G_T$ in $\mathbb{R}^4$.

It remains to prove the inequivalence of the given multipliers. Let us first prove the fact that $\xi_1 := \xi - \xi'$ is not trivial. Indeed we have,

$$
\begin{aligned}
&\xi_1((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) \\
&\quad = \xi((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) - \xi'((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) \\
&\quad = \frac{1}{2}q_1p_1' + \frac{1}{2}q_2p_2' + \frac{1}{2}p_2p_1' - \frac{1}{2}p_1q_1' - \frac{1}{2}p_2q_2' - \frac{1}{2}p_1p_2'. \qquad \text{(A.12)}
\end{aligned}
$$

Now from (4.30), it follows immediately that triviality of a multiplier $\eta$ for some abelian group in terms of a suitable continous function implies the fact that $\eta(g, g') = \eta(g', g)$ holds for any two group elements of the given abelian group. By contrapositivity, $\eta(g, g') \neq \eta(g', g)$ guarantees the nontriviality of the multiplier in question.

In other words, to prove the nontriviality of $\xi_1$, it suffices to show that

$$\xi_1((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) \neq \xi_1((q_1', q_2', p_1', p_2'), (q_1, q_2, p_1, p_2))$$

always holds. Indeed,

$$
\begin{aligned}
&\xi_1((q_1', q_2', p_1', p_2'), (q_1, q_2, p_1, p_2)) \\
&\quad = \frac{1}{2}q_1' + \frac{1}{2}q_2'p_2 + \frac{1}{2}p_2'p_1 - \frac{1}{2}p_1'q_1 - \frac{1}{2}p_2'q_2 - \frac{1}{2}p_1'p_2, \\
&\quad = -\xi_1((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')). \qquad \text{(A.13)}
\end{aligned}
$$

Let us now prove that $\xi_2 := \xi' - \xi''$ is nontrivial. We have,

$$\xi_2((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2'))$$

$$
\begin{aligned}
&= \xi'((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) - \xi''((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) \\
&= \frac{1}{2}[p_1 p_2' + q_1 q_2' - p_2 p_1' - q_2 q_1'] \\
&= -\xi_2((q_1', q_2', p_1', p_2'), (q_1, q_2, p_1, p_2)). \qquad \text{(A.14)}
\end{aligned}
$$

The above equation reflects the fact that $\xi_2$ is indeed nontrivial which in turn implies that $\xi'$ and $\xi''$ are inequivalent. Hence it follows that $\xi$, $\xi'$ and $\xi''$ are three inequivalent local exponents of $G_T$. $\square$

## Appendix B
## Proofs related to Chapter 5

In this Appendix we collect together the proofs of some of the results quoted in chapter 5.

### Proof of Theorem 5.4.1

We first prove that $\mathcal{U}_{\text{sym}}$, given by (5.51), is indeed a representation of $G_{\text{NC}}$.

$$\begin{aligned}
&(\mathcal{U}_{\text{sym}}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)\mathcal{U}_{\text{sym}}(\theta',\phi',\psi',q_1',q_2',p_1',p_2')f)(r_1,r_2)\\
&\quad=\mathcal{U}_{\text{sym}}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)(\mathcal{U}_{\text{sym}}(\theta',\phi',\psi',q_1',q_2',p_1',p_2')f)(r_1,r_2)\\
&\quad=(\mathcal{U}_{\text{sym}}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)g)(r_1,r_2),
\end{aligned} \tag{B.1}$$

where we have chosen $\mathcal{U}_{\text{sym}}(\theta',\phi',\psi',q_1',q_2',p_1',p_2')f=g$. Then (B.1) reads

$$\begin{aligned}
&(\mathcal{U}_{\text{sym}}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)g)(r_1,r_2)\\
&\quad=e^{i(\theta+\phi+\psi)}e^{i\left[\alpha p_1r_1+\alpha p_2r_2-\frac{\alpha(\alpha-\sqrt{\alpha^2-\beta\gamma})}{\beta}(q_1r_2-q_2r_1)+\frac{\sqrt{\alpha^2-\beta\gamma}}{2}(p_1q_1+p_2q_2)\right]}\\
&\quad\times g\left(r_1-\frac{\beta}{2\alpha}p_2+\frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2\alpha}q_1,r_2+\frac{\beta}{2\alpha}p_1+\frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2\alpha}q_2\right).
\end{aligned} \tag{B.2}$$

But

$$g\left(r_1-\frac{\beta}{2\alpha}p_2+\frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2\alpha}q_1,r_2+\frac{\beta}{2\alpha}p_1+\frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2\alpha}q_2\right)$$

$$
\begin{aligned}
&= (\mathcal{U}_{\text{sym}}(\theta', \phi', \psi', q_1', q_2', p_1', p_2')f)(r_1 - \frac{\beta}{2\alpha}p_2 + \frac{\alpha + \sqrt{\alpha^2 - \beta\gamma}}{2\alpha}q_1,\ r_2 + \frac{\beta}{2\alpha}p_1 \\
&\qquad\qquad + \frac{\alpha + \sqrt{\alpha^2 - \beta\gamma}}{2\alpha}q_2) \\
&= e^{i(\theta'+\phi'+\psi')} e^{i\left[\alpha p_1' r_1 - \frac{\beta}{2}p_1' p_2 + \frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2}p_1' q_1 + \alpha p_2' r_2 + \frac{\beta}{2}p_1 p_2' + \frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2}p_2' q_2\right]} \\
&\times e^{-i\left[\frac{\alpha(\alpha-\sqrt{\alpha^2-\beta\gamma})}{\beta}\left(q_1' r_2 + \frac{\beta}{2\alpha}q_1' p_1 + \frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2\alpha}q_1' q_2 - q_2' r_1 + \frac{\beta}{2\alpha}q_2' p_2 + \frac{\alpha+\sqrt{\alpha^2-\beta\gamma}}{2\alpha}q_2' q_1\right)\right]} \\
&\times e^{\frac{i\sqrt{\alpha^2-\beta\gamma}}{2}(p_1' q_1' + p_2' q_2')} f(r_1 - \frac{\beta}{2\alpha}(p_2 + p_2') + \frac{\alpha + \sqrt{\alpha^2 - \beta\gamma}}{2\alpha}(q_1 + q_1'),\ r_2 + \frac{\beta}{2\alpha}(p_1 + p_1') \\
&\qquad\qquad + \frac{\alpha + \sqrt{\alpha^2 - \beta\gamma}}{2\alpha}(q_2 + q_2')).
\end{aligned}
\tag{B.3}
$$

On the other hand, in view of the group law (5.1) of $G_{\text{NC}}$, we have

$$
\begin{aligned}
&(\mathcal{U}_{\text{sym}}((\theta, \phi, \psi, q_1, q_2, p_1, p_2)(\theta', \phi', \psi', q_1', q_2', p_1', p_2'))f)(r_1, r_2) \\
&= (\mathcal{U}_{\text{sym}}(\theta + \theta' + \frac{\alpha}{2}q_1 p_1' + \frac{\alpha}{2}q_2 p_2' - \frac{\alpha}{2}p_1 q_1' - \frac{\alpha}{2}p_2 q_2',\ \phi + \phi' + \frac{\beta}{2}p_1 p_2' \\
&- \frac{\beta}{2}p_2 p_1',\ \psi + \psi' + \frac{\gamma}{2}q_1 q_2' - \frac{\gamma}{2}q_2 q_1',\ q_1 + q_1',\ q_2 + q_2',\ p_1 + p_1',\ p_2 + p_2')f)(r_1, r_2) \\
&= e^{i(\theta+\theta'+\frac{\alpha}{2}q_1 p_1' + \frac{\alpha}{2}q_2 p_2' - \frac{\alpha}{2}p_1 q_1' - \frac{\alpha}{2}p_2 q_2')} e^{i(\phi+\phi'+\frac{\beta}{2}p_1 p_2' - \frac{\beta}{2}p_2 p_1')} e^{i(\psi+\psi'+\frac{\gamma}{2}q_1 q_2' - \frac{\gamma}{2}q_2 q_1')} \\
&\times e^{i\alpha\left[p_1 r_1 + p_1' r_1 + p_2 r_2 + p_2' r_2 - \frac{\alpha-\sqrt{\alpha^2-\beta\gamma}}{\beta}(q_1 r_2 + q_1' r_2 - q_2 r_1 - q_2' r_1)\right]} \\
&\times e^{i\frac{\sqrt{\alpha^2-\beta\gamma}}{2}\left[(p_1+p_1')(q_1+q_1') + (p_2+p_2')(q_2+q_2')\right]} f(r_1 - \frac{\beta}{2\alpha}(p_2 + p_2') \\
&+ \frac{\alpha + \sqrt{\alpha^2 - \beta\gamma}}{2\alpha}(q_1 + q_1'),\ r_2 + \frac{\beta}{2\alpha}(p_1 + p_1') + \frac{\alpha + \sqrt{\alpha^2 - \beta\gamma}}{2\alpha}(q_2 + q_2')).
\end{aligned}
\tag{B.4}
$$

Comparing (B.3) and (B.4), we obtain the following

$$
(\mathcal{U}_{\text{sym}}((\theta, \phi, \psi, q_1, q_2, p_1, p_2)(\theta', \phi', \psi', q_1', q_2', p_1', p_2'))f)(r_1, r_2)
$$

$$= (\mathcal{U}_{\text{sym}}(\theta, \phi, \psi, q_1, q_2, p_1, p_2)\mathcal{U}_{\text{sym}}(\theta', \phi', \psi', q_1', q_2', p_1', p_2')f)(r_1, r_2), \tag{B.5}$$

with $f \in L^2(\mathbb{R}^2, dr_1 dr_2)$. Hence it follows that $\mathcal{U}_{\text{sym}}$ is indeed a representation of the nilpotent Lie group $G_{\text{NC}}$.

The adjoint of $\mathcal{U}_{\text{sym}}$ now reads

$$\begin{aligned}
&(\mathcal{U}^*_{\text{sym}}(\theta, \phi, \psi, q_1, q_2, p_1, p_2)f)(r_1, r_2) \\
&\quad = e^{-i(\theta+\phi+\psi)} e^{-i\left[\alpha p_1 r_1 + \alpha p_2 r_2 - \frac{\alpha(\alpha - \sqrt{\alpha^2 - \beta\gamma})}{\beta}(q_1 r_2 - q_2 r_1) + \frac{\sqrt{\alpha^2-\beta\gamma}}{2}(p_1 q_1 + p_2 q_2)\right]} \\
&\quad \times f\left(r_1 + \frac{\beta}{2\alpha}p_2 - \frac{\alpha + \sqrt{\alpha^2 - \beta\gamma}}{2\alpha}q_1, r_2 - \frac{\beta}{2\alpha}p_1 - \frac{\alpha + \sqrt{\alpha^2 - \beta\gamma}}{2\alpha}q_2\right),
\end{aligned} \tag{B.6}$$

from which unitarity of $\mathcal{U}_{\text{sym}}$ follows immediately

$$(\mathcal{U}_{\text{sym}}\mathcal{U}^*_{\text{sym}}f)(r_1, r_2) = (\mathcal{U}^*_{\text{sym}}\mathcal{U}_{\text{sym}}f)(r_1, r_2) = f(r_1, r_2), \tag{B.7}$$

where $f \in L^2(\mathbb{R}^2, dr_1 dr_2)$.

It remains to prove the irreducibility of the unitary representation $\mathcal{U}_{\text{sym}}$ of the Lie group $G_{\text{NC}}$. Since, the representation of the corresponding Lie algebra $\mathfrak{g}_{\text{NC}}$ given by (5.52) is clearly irreducible and $G_{\text{NC}}$ is a connected, simply connected Lie group, the corresponding representation of the Lie group is also irreducible. But the equivalence classes of unitary irreducible representations of $G_{\text{NC}}$ are all obtained in Section 5.2. And the group representation complying with the commutation relations (5.36) is given by (5.16). Therefore, the unitary irreducible representation of $G_{\text{NC}}$, due to the choice of symmetric gauge of vector potential, has to be equivalent to one of the representations (5.16) computed in the Hilbert space $L^2(\mathbb{R}^2, dr_1 dr_2)$, for some nonzero value of $\rho$, $\sigma$, and $\tau$. Hence, the unitary representation (5.51) of $G_{\text{NC}}$ is also irreducible. $\square$

**Proof of Theorem 5.5.1**

We first write $\mathbb{M}$ in the block-form:

$$\mathbb{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & C_{22} & D_{21} & D_{22} \end{bmatrix} \tag{B.8}$$

(5.53) then yields

$$\begin{aligned} &\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{Q}_1 \\ \hat{P}_2 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \hat{Q}_2 \\ \hat{P}_1 \end{bmatrix} = \begin{bmatrix} \hat{Q}'_1 \\ \hat{P}'_2 \end{bmatrix} \\ \implies &\begin{bmatrix} A_{11}\hat{Q}_1 + A_{12}\hat{P}_2 + B_{11}\hat{Q}_2 + B_{12}\hat{P}_1 \\ A_{21}\hat{Q}_1 + A_{22}\hat{P}_2 + B_{21}\hat{Q}_2 + B_{22}\hat{P}_1 \end{bmatrix} = \begin{bmatrix} \hat{Q}'_1 \\ \hat{P}'_2 \end{bmatrix}. \end{aligned} \tag{B.9}$$

Similarly

$$\begin{aligned} &\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \hat{Q}_1 \\ \hat{P}_2 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \hat{Q}_2 \\ \hat{P}_1 \end{bmatrix} = \begin{bmatrix} \hat{Q}'_2 \\ \hat{P}'_1 \end{bmatrix} \\ \implies &\begin{bmatrix} C_{11}\hat{Q}_1 + C_{12}\hat{P}_2 + D_{11}\hat{Q}_2 + D_{12}\hat{P}_1 \\ C_{21}\hat{Q}_1 + C_{22}\hat{P}_2 + D_{21}\hat{Q}_2 + D_{22}\hat{P}_1 \end{bmatrix} = \begin{bmatrix} \hat{Q}'_2 \\ \hat{P}'_1 \end{bmatrix}. \end{aligned} \tag{B.10}$$

Using (B.9) and (B.10), one gets

$$
\begin{aligned}
&[\hat{Q}_1', \hat{P}_1'] \\
&= [A_{11}\hat{Q}_1 + A_{12}\hat{P}_2 + B_{11}\hat{Q}_2 + B_{12}\hat{P}_1, C_{21}\hat{Q}_1 + C_{22}\hat{P}_2 + D_{21}\hat{Q}_2 + D_{22}\hat{P}_1] \\
&= A_{11}D_{21}(i\vartheta\mathbb{I}) + A_{11}D_{22}(i\hbar\mathbb{I}) + A_{12}D_{21}(-i\hbar\mathbb{I}) + A_{12}D_{22}(-i\mathcal{B}\mathbb{I}) \\
&+B_{11}C_{21}(-i\vartheta\mathbb{I}) + B_{11}C_{22}(i\hbar\mathbb{I}) + B_{21}C_{21}(-i\hbar\mathbb{I}) + B_{12}C_{22}(i\mathcal{B}\mathbb{I}) \\
&= i\hbar(A_{11}D_{22} - A_{12}D_{21} + B_{11}C_{22} - B_{12}C_{21})\mathbb{I} + i\vartheta(A_{11}D_{21} - B_{11}C_{21})\mathbb{I} \\
&+i\mathcal{B}(B_{12}C_{22} - A_{12}D_{22})\mathbb{I}. \qquad \text{(B.11)}
\end{aligned}
$$

But we are given that $[\hat{Q}_1', \hat{P}_1'] = i\hbar\mathbb{I}$. Therefore, (B.11) reduces to

$$
\frac{\vartheta}{\hbar}(B_{11}C_{21}-A_{11}D_{21})+\frac{\mathcal{B}}{\hbar}(A_{12}D_{22}-B_{12}C_{22})+(A_{12}D_{21}+B_{12}C_{21}-A_{11}D_{22}-B_{11}C_{22}) = -1. \qquad \text{(B.12)}
$$

In exactly the same way one can go on to compute $[\hat{Q}'_2, \hat{P}'_2]$, $[\hat{Q}'_1, \hat{Q}'_2]$, $[\hat{P}'_1, \hat{P}'_2]$, $[\hat{Q}'_1, \hat{P}'_2]$, and $[\hat{Q}'_2, \hat{P}'_1]$ and thereby obtain the following set of equations:

$$\begin{aligned}
&\frac{\vartheta}{\hbar}(B_{21}C_{11} - A_{21}D_{11}) + \frac{\mathcal{B}}{\hbar}(A_{22}D_{12} - B_{22}C_{12}) \\
&\qquad + (B_{22}C_{11} + A_{22}D_{11} - B_{21}C_{12} - A_{21}D_{12}) = 1, \\
&(A_{12}D_{11} + B_{12}C_{11} - A_{11}D_{12} - B_{11}C_{12}) + \frac{\vartheta}{\hbar}(B_{11}C_{11} - A_{11}D_{11}) \\
&\qquad + \frac{\mathcal{B}}{\hbar}(A_{12}D_{12} - B_{12}C_{12}) = -\frac{\vartheta}{\hbar}, \\
&(B_{22}C_{21} + A_{22}D_{21} - B_{21}C_{22} - A_{21}D_{22}) + \frac{\mathcal{B}}{\hbar}(A_{22}D_{22} - B_{22}C_{22}) \\
&\qquad + \frac{\vartheta}{\hbar}(B_{21}C_{21} - A_{21}D_{21}) = \frac{\mathcal{B}}{\hbar}, \\
&\frac{\vartheta}{\hbar}(A_{11}B_{21} - A_{21}B_{11}) + \frac{\mathcal{B}}{\hbar}(A_{22}B_{12} - A_{12}B_{22}) \\
&\qquad + (A_{11}B_{22} + A_{22}B_{11} - A_{12}B_{21} - A_{21}B_{12}) = 0, \\
&\frac{\vartheta}{\hbar}(C_{11}D_{21} - C_{21}D_{11}) + \frac{\mathcal{B}}{\hbar}(D_{12}C_{22} - C_{12}D_{22}) \\
&\qquad + (C_{11}D_{22} - C_{12}D_{21} + D_{11}C_{22} - D_{12}C_{21}) = 0.
\end{aligned} \tag{B.13}$$

Now (B.12) and the set of relations enumerated in (B.13) can all be compactified into the following three matrix equations:

$$\begin{aligned}
AQB^T - BQ^TA^T &= 0, \\
CQD^T - DQ^TC^T &= 0, \\
AQD^T - BQ^TC^T &= Q,
\end{aligned} \tag{B.14}$$

where $Q$ is the $2 \times 2$ matrix given by (5.56).

The three matrix equations (B.14) can yet be incorporated in one single $4 \times 4$ matrix equation given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & Q \\ -Q^T & 0 \end{bmatrix} \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} = \begin{bmatrix} 0 & Q \\ -Q^T & 0 \end{bmatrix},$$

which boils down to

$$\mathbb{M}\mathbb{Q}\mathbb{M}^T = \mathbb{Q}.$$

$\square$

**Proof of Proposition 5.5.1**

An element $\mathbb{M}$ of $\mathfrak{S}(4, \mathbb{R})$ is a $4 \times 4$ real matrix satisfying (5.54). In other words, not all the elements of $\mathbb{M}$ are independent of each other. There are six distinct constraint equations (see (B.12) and (B.13)) between various entries of the underlying $4 \times 4$ matrix. The dimension of $\mathfrak{S}(4, \mathbb{R})$ is therefore $10$.

Note that if we had written $\begin{bmatrix} \hat{Q}_1 \\ \hat{Q}_2 \\ \hat{P}_1 \\ \hat{P}_2 \end{bmatrix}$ instead of $\begin{bmatrix} \hat{Q}_1 \\ \hat{P}_2 \\ Q_2 \\ P_1 \end{bmatrix}$ in (5.53), the real entries of $\mathbb{M}$ in (B.8) would have been reshuffled accordingly:

$$\begin{bmatrix} \hat{Q}'_1 \\ Q'_2 \\ \hat{P}'_1 \\ \hat{P}'_2 \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & B_{12} & A_{12} \\ C_{11} & D_{11} & D_{12} & C_{12} \\ C_{21} & D_{21} & D_{22} & C_{22} \\ A_{21} & B_{21} & B_{22} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{Q}_1 \\ \hat{Q}_2 \\ \hat{P}_1 \\ \hat{P}_2 \end{bmatrix}. \tag{B.15}$$

Therefore, in our notation the canonical skew-symmetric $4 \times 4$ matrix $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$ reads $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$. In what follows, we shall be denoting the latter matrix by $\mathbb{J}$. We obtain

$$\mathcal{U}\mathbb{J}\mathcal{U}^T = \mathbb{Q}, \tag{B.16}$$

where $\mathbb{Q}$ is given in Theorem (5.5.1) and the invertible matrix $\mathcal{U}$ is as follows

$$\mathcal{U} = \begin{bmatrix} -1 & \frac{\vartheta}{\hbar} & 0 & 0 \\ \frac{\mathcal{B}}{\hbar} & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

But, by definition, $\mathbb{M} \in \mathfrak{S}(4, \mathbb{R})$ satisfies (5.54). One then obtains

$$\begin{aligned} &\mathbb{M}\mathbb{Q}\mathbb{M}^T = \mathbb{Q} \\ &\implies \mathbb{M}\mathcal{U}\mathbb{J}\mathcal{U}^T\mathbb{M}^T = \mathcal{U}\mathbb{J}\mathcal{U}^T \\ &\implies (\mathcal{U}^{-1}\mathbb{M}\mathcal{U})\mathbb{J}\mathcal{U}^T\mathbb{M}^T(\mathcal{U}^{-1})^T = \mathbb{J} \\ &\implies (\mathcal{U}^{-1}\mathbb{M}\mathcal{U})\mathbb{J}(\mathcal{U}^{-1}\mathbb{M}\mathcal{U})^T = \mathbb{J}. \end{aligned} \tag{B.17}$$

In view of (B.17), one immediately finds that $f : \mathfrak{S}(4, \mathbb{R}) \rightarrow Sp(4, \mathbb{R})$ with $f(\mathbb{M}) = \mathcal{U}^{-1}\mathbb{M}\mathcal{U}$ is the required isomorphism.

Note that $\mathbb{Q}$ in (B.16), for both the choices of $\mathcal{U}$ as discussed in (5.5.2), can easily be verified to be unique up to a scalar multiple. Also, note that the invertibility of $Q$ has been tacitly exploited in establishing the isomorphism $f$. □

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