The Effects of Exposure to Non-Canonical Equations on Children's Understanding of the Equal Sign

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ABSTRACT

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on Children's Understanding of the Equal Sign

Eva Sokol

It has been established that many children perceive the equal sign as an operational symbol rather than a relational one, and as a result, they have trouble solving non-canonical equations. In the present study, I investigated whether exposure to non-canonical equations would result in students' ability to solve such equations and acquire a relational view of the equal sign. Fifty three (N = 53) second- and third-grade students were tested for their ability to solve non-canonical problems and whether they viewed the equal sign as a relational symbol before and after an exposure intervention, during which students practiced solving different types of equations. In one condition, the students practiced solving canonical equations; in a second condition, the students practiced solving non-canonical part-whole equations; and in a third condition, they practiced solving non-canonical identity equations. The students were required to solve the problems in their condition without receiving any type of feedback on their work or any explanations on the meaning of the equal sign. All the students improved significantly on their ability to solve non-canonical problems regardless of the type of problem to which they were exposed during the intervention. The students did not, however, improve on their ability to generate relational definitions of the equal sign. The analysis also revealed a significant effect of problem type. Students were more successful on canonical problems than non-canonical ones. As well, students were more successful at solving identity problems than part-whole problems. The study's limitations and pedagogical implications are discussed.

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Chapter 1: Statement of the Problem

The National Council of Teachers of Mathematics (NCTM, 2000) stated the following:

> […] Algebra is more than moving symbols around. Students need to understand the concepts of algebra, the structures and principles that govern the manipulation of the symbols, and how the symbols themselves can be used for recording ideas and gaining insights into situations (NCTM, 2000, p. 37).

Scholars have argued that students should learn arithmetic in a way that provides a basis for learning algebra. The way that most students learn arithmetic, however, is not conducive to building a foundation for learning algebraic concepts later (Carpenter, Franke, & Levi, 2003). In elementary school, students do not learn algebra *per se*, but can be introduced to algebraic concepts through arithmetic in their mathematics class. The problem is that when students are not encouraged to think algebraically, algebra in high school is introduced as a completely separate course. Acknowledging students' mathematical thinking early, that is, drawing the link between arithmetic and algebra in elementary school, can provide students with a strong foundation for learning algebra in secondary school.

Mathematics concepts need to develop over a period of time, and because there usually is no link between arithmetic and algebra in their early years, students do not get the chance to develop a solid understanding of important mathematical concepts that are at the foundation of algebra. Moreover, researchers speculated that students are introduced to arithmetic concepts in a way that is not favourable to acquiring the skills to master algebraic thinking. For example, Carpenter et al. (2003) stated that elementary school students are capable of conceptualizing mathematical concepts as representing relationships – an important concept for successfully learning algebra - but they do not get the opportunity to think of symbols as representing relationships because of how these concepts are presented in school. For this reason, changes to mathematics curriculum and instruction have dominated discussions on how to improve mathematics education and best prepare students for higher education.

The equal sign is a symbol that specifies a relationship between the quantities on both sides of an equation. Viewing it as such means understanding the equal sign as a relational symbol. A large amount of research demonstrated that having misconceptions about the equal sign throughout elementary school can have a large negative effect on learning algebra (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; Byers & Herscovics 1977). Baroody and Ginsburg (1983) presented the problem:

> For instance, if 'equals' is not viewed as a relational sign, as a bridge between numerically equivalent expressions, algebra solution strategies (such as adding identical elements to each side of an equation to simplify the expression on one side) may not be meaningful and may simply be learned by rote. (p. 200)

Despite such findings, research has demonstrated that students are indeed able to acquire a relational understanding of the equal sign (Baroody & Ginsburg 1983; Rittle-Johnson & Alibali, 1999; Schliemann, Carraher, & Brizuela, 2007). Carpenter et al. (2003), for example, maintained that young children's intuitive notions of numbers allow them to understand important concepts in mathematics, such as equivalence. In other words, students have the ability to conceptualize the concept of equivalence, but are not given many opportunities to apply its relational meaning when solving problems. Although many experts have investigated ways in which they can help students understand this concept and use the equal sign correctly, the problem still persists.

Some researchers believed that using a different language when using the equal sign with students would result in their ability to interpret the equal sign as a relational symbol (Baroody & Ginsburg, 1983; Li, Ding, Capraro, & Capraro, 2008; Rittle-Johnson & Alibali, 1999). For example, it seems that in some cultures, such as Chinese for instance, students hear language that promotes the relational view of the equal sign (e.g., "the numbers on the left are equal to the numbers on the right of the equal sign"), which then assists them to learn to use the equal sign as a relational symbol. On the other hand, in cultures where mathematics instructors use operational language when referring to the equal sign (e.g., "adding the numbers on the left side of the equal sign *gives* the number on the right side of the equal sign"), students interpret the equal sign as an operational symbol (Li et al., 2008). Most North American textbooks and teacher guidebooks do not promote relational language, and this may explain why students do not learn to view the equal sign as relational.

Some researchers believe that students can acquire the relational meaning of the equal sign if its meaning is explicitly taught to them (Rittle-Johnson & Alibali, 1999). This type of intervention requires access to human resources (i.e., instructors), as well as the time to work with each student independently. Other researchers believe that using manipulatives (i.e., concrete objects) can elicit relational thinking when it comes to the equal sign (Seo & Ginsburg, 2003; Sherman & Bisanz, 2008), and can be used in both whole class and small group settings. Thus, it is clear that directly addressing the problem, and exposing children to the correct concept of the equal sign, can help them acquire relational thinking. The use of these various techniques and manipulatives, although effective, require access to human resources, materials, and time. I believe that it would be beneficial to find a means to help students think relationally using fewer instructional resources.

Still other researchers have provided evidence that children can acquire a relational understanding just by virtue of being exposed to specific forms of equations. McNeil et al. (2006), for example, showed that some forms of equations appear to elicit relational thinking. The researchers concluded that exposing students to operations on both sides of the equal sign is correlated with students' relational thinking, but the results of the study do not allow one to conclude that simply presenting students with different types of equations results in relational thinking. This is because of the design of the study, as McNeil et al. (2006) did not have a pre- and post-assessment to test for relational thinking.

Taken together, the research shows that children can think relationally and that language, instruction, and curricular materials can elicit relational thinking. Little is known, however, about the isolated effects of exposing children, in a classroom context, to various types of equations, on their relational thinking. (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Sherman & Bisanz, 2009). Thus, the objective of this study is to determine whether exposure to non-standard equations (i.e., equations not of the "standard" form $a + b = c$), free of teachers' verbalizations on the equal sign or any concrete tools, will promote students' interpretation of the equal sign as a relational symbol. Although it is observed that without instruction, elementary school students of all ages have difficulty understanding that the equal sign is a relational symbol (Seo & Ginsburg, 2003), it is unknown how students of different ages will respond to different types of exposure to the equal sign. Thus, grade level will be tested as a moderating factor in this study.

Chapter 2: Literature Review

Definitions

One of the first concepts that children are expected to acquire in mathematics is the concept of **equivalence,** which is represented symbolically by the equal sign ("=") (Baroody & Ginsburg, 1983). Kieran (1981) explained that "equivalence, in its broadest sense, implies a relation in which one item can be replaced by another" (for example, replacing 2 + 2 with 4 because 2 + 2 equals 4). Alibali (1999) stated that "mathematical equivalence [...] is the principle that the two sides of an equation represent the same quantity" (p. 128). Similarly, Baroody and Ginsburg (1983) and Carpenter et al. (2003) supported this definition and indicated that the equal sign is a **relational symbol** because it denotes a relationship of "sameness." Finally, Sherman and Bisanz (2009) explain that **thinking relationally** about the equal sign means that the student understands that both sides of the equal sign are equivalent. Yet, many students interpret the equal sign as an indicator to carry out a command or perform an action such as "adds up" or "produces"; in other words, they hold an **operational view** (Alibali 1999; Baroody & Ginsburg, 1983; Sherman & Bisanz, 2009).

Students' understanding of equivalence is often tested by presenting them with fill-in-the-blank equations such as $a + b + c = \_\_ + c$, where to get the correct answer, students have to ensure that both sides have the same quantity. Most equations, which are considered standard or **canonical,** are of the form $a + b = c$. In canonical equations, the operations are always on the left side of the equal sign and only one number and no operations are presented on the right side of the equal sign. These equations are considered canonical because they are the most common examples that appear in textbooks and teacher guidebooks (McNeil et al., 2006; Sherman & Bisanz, 2009). Equations of different forms, such as $a + b = c + d$, are considered to be non-standard, or **non-canonical.** Sherman and Bisanz (2009) categorized a variety of non-canonical equations. For example, they referred to equations of the form $a + b = c + d$ as **part-whole equations**. Other types of non-canonical equations are of the form $a + b = a + b$, and $a = a$, which Sherman and Bisanz (2009) categorized as **identity** equations.

Scholars have speculated that students hold misconceptions about the equal sign because of the exposure they receive to different types of equations. Sherman and Bisanz (2009) argued that students are exposed to different types of problems in their textbooks, on the board in the classroom, and in their assignments. For the purposes of the current study, **exposure** will be defined as the problems students see in materials and in the classroom in absence of teachers' verbalizations about the meaning of the equal sign.

Many students have trouble with the mathematical symbol of equivalence because they tend to focus on the symbol at a superficial level, which leads to the execution of procedures that are not usually connected to the concept of equivalence. Hiebert and Lefevre (1986) explain that **conceptual knowledge** is "knowledge that is rich in relationships" (p. 3), and in this context, means that understanding the concept of equivalence involves understanding the relationship between the concepts of sameness and concepts of quantity. The authors define **procedural knowledge** as "rules or procedures for solving mathematical problems" (Hiebert & Lefevre, 1986, p. 7). Thus, learning procedures entails knowing the steps to solve a particular mathematics problem (Hiebert & Carpenter, 1992). In the current context, for example, a procedure for solving 8 + 4 = ___ + 5 may entail adding the numbers on the right and subtracting 5 from the sum to find the number that fits in the blank.

Statistics on Children's Performance on Equivalence Problems

Falkner, Levi, and Carpenter (1999) investigated whether elementary school students were able to solve a non-canonical equation: they asked all the mathematics teachers in one elementary school to give the following open-number sentence to their students to solve:

What number would you put in the box to make this a true number sentence: 8 + 4 = ☐ + 5*?*

The authors discovered that fewer than 10% of the students, in any given grade, gave the correct response and used the equal sign as a relational symbol. Additionally, they found that performance did not improve with age and that in some cases, the older students' results were slightly worse than those of the younger students. Specifically, only 5% of students in the first and second grade filled in the blank correctly; 9% in the third and fourth grades got the right answer; and only 2 % of those in the fifth and sixth grades were able to do so. This study revealed that students hold misconceptions about what the equal sign represents.

In order to further understand students' difficulty with the equal sign, Li et al. (2008) tested samples of sixth-grade United States students and sixth-grade Chinese students. They found a significant difference in the two groups' interpretations of the equal sign, with only 28% of United States sixth grade students able to correctly solve non-canonical, open-number sentences compared to 98% of Chinese students. These results were consistent with other findings that sought to determine whether non-canonical equations proved challenging to US students. For example, Rittle-Johnson and Alibali (1999) found that only 31% of fourth grade students were able to solve non-canonical equivalence problems, and in another study, Knuth, Stephens, McNeil, and Alibali (2006) found that only 32% of sixth grade students acquired a relational meaning of the equal sign.

Children's Difficulties with the Equal Sign

Sherman and Bisanz (2009) put forward the idea that understanding a concept such as equivalence requires the student to understand that the "=" symbol represents a combination of the concept of equivalence and a procedure that would make both sides of the equal sign equal.

The researchers examined second graders' ability to solve non-canonical equations in symbolic and non-symbolic contexts. Students in one condition were required to solve non-canonical equations in a symbolic context (only using numbers and mathematical symbols such as + and =). Students in a second condition were asked to solve non-canonical equations in a non-symbolic context by using only wooden blocks and bins as manipulatives. The researchers presented the symbolic and non-symbolic equations to each participant individually. Once the equations were solved, the researchers asked each student to give a definition of the equal sign. Sherman and Bisanz (2009) found that students in the symbolic condition solved fewer problems correctly than students in the non-symbolic condition. They concluded that students in the non-symbolic condition approached problem solving with relational thinking (assessed by coding the students' definitions), and that students in the symbolic condition were more likely to display operational thinking.

Sherman and Bisanz (2009) concluded that when children are able to recognize that the equal sign is a relational symbol, they are capable of thinking relationally in symbolic contexts. Otherwise said, students need to learn the relational meaning of the symbol in order to be able to solve algebraic equations correctly. Unfortunately, many students learn to manipulate the equal sign without attaching conceptual understanding to it. As a result, they learn procedures without understanding the concepts that are associated with them. Moreover, practicing procedures does not always lead to a student's understanding of what the symbol represents. The procedure is then often associated with the symbol devoid of conceptual rationales. As a result, this lack of understanding will impede the acquisition of algebraic skills (Carpenter et al., 2003). Furthermore, Kaput (2008) mentioned that students need to connect the equal sign to the concept of relational equivalence, and need to be able to see that the symbol represents this concept in all contexts. Making an explicit link between the notion of equivalence and the equal sign (i.e., the symbol) will allow student to properly unpack the symbol and access its relational meaning in a variety of symbolic contexts.

Algebra and arithmetic are often presented as mutually exclusive in school, and as a result, students sometimes bring misconceptions from arithmetic to algebra, which makes learning algebra even more difficult. For example, a study by McNeil and Alibali (2005) demonstrated that when individuals are primed to think in an operational way, they tend to make more mistakes when solving algebraic equations. The researchers tested undergraduate students by priming them, through canonical equations and operational words (e.g. “totals,” “makes”), to think operationally about the equal sign. They were then asked to solve a series of non-canonical equations. The results indicated that the students who were primed to think operationally were likely to make errors by adding all the number on the left of the equal sign, and ignoring all the numbers on the right, except for the one immediately after it. On the other hand, students who were not primed to think operationally made significantly fewer errors on the non-canonical equations. Thus, the results of this study demonstrate that misconceptions about the symbols, and the frequency with which students are primed to think operationally, can lead to difficulties in algebra.

The nature of students’ misconceptions about the equal sign has, for the most part, been documented through case study methodologies (Carpenter et al., 2003; Knuth et al, 2006; McNeil & Alibali, 2005; Mevarech & Yitschak, 1983). For example, in one study, Carpenter et al. (2003) labelled and described a number of student misconceptions. They called one misconception, “the answer comes next” (p. 10). Baroody and Ginsburg (1983) stated that students with this misconception tend to see the equal sign in terms of performing an action such as "adds up to or produces." For example, when presented with 8 + 4 = ___ + 5, students would place the sum of 8 and 4 (i.e., 12) in the blank. Another misconception is called "use all the numbers" (Carpenter et al., 2003, p. 11). Students who have this misconception take into account all the numbers, but do not pay attention to where the symbols appear in the equation, and simply add all the numbers presented. For example, when presented with 8 + 4 = __ + 5, students would place the sum of 8, 4, and 5 (i.e., 17) in the blank. A third misconception is called "extend the problem" (Carpenter et al., 2003, p. 11). Students who have this misconception add the numbers on the left of the equal sign, place that number in the blank to the right of the equal sign. They then add the numbers to the right of the equal sign and place an additional "=" to the right, with the new sum placed after it. For example, when presented with 8 + 4 = __ + 5, students first add 8 and 4, write 12 in the blank, add another equal sign to the right of 12 + 5 together with the new sum (i.e., 8 + 4 = 12 + 5 = 17). In other words, students with this misconception carry out an operation on the numbers to the left side of the equal sign and then extend the operation to the numbers on the right side of the equal sign (Carpenter et al., 2003).

McNeil and Alibali (2004) speculated that such misconceptions are formed as a result of inaccurate addition schemas. They explained that when students apply the addition schema to canonical equations, they get the correct answer, and thus the operator view of the equal sign is wrongfully reinforced. Such misconceptions will govern the belief that addends are always on the left of the equal sign and the solutions are always on the right of it (McNeil, 2008; McNeil & Alibali, 2002, 2004). McNeil (2008) tested second and third grade students on their ability to solve non-canonical equations. The author administered a pretest to determine the students' ability to solve non-canonical arithmetic addition problems. They were then randomly assigned to one of two conditions. In the operational context condition, the students were exposed to four lessons on the concept of equivalence through worksheets that presented canonical equations involving addition designed to activate children's operational schema. In the non-operational condition, students were exposed to four lessons on the concept of equivalence through worksheets that presented non-canonical equivalence statements (e.g., 28 = 28, 1 foot = 12 inches) and therefore were not exposed to the operational view. The lessons required the students to solve an equation, see the correct answer along with an explanation of the concept of equivalence and see three incorrect equations along with an explanation of why they were wrong. The students then wrote a postest that required them to solve twelve non-canonical equations. McNeil (2008) found that students solved fewer equations correctly on the postest when they were exposed to the operational context than students who were not exposed to the operational context. They concluded that students made errors when solving the non-canonical equations on the postest because their addition schemas were activated beforehand.

Another explanation for children's difficulty with the equal sign stems from the symbolic context in which students are asked to interpret the equal sign. In the study described above, Sherman and Bisanz (2009) explored whether students are able to understand the concept of equivalence outside of the context of the symbol (i.e., without the equal sign). Sherman and Bisanz (2009) found that children solved equations differently depending on the context. Students who solved equivalence problems in a non-symbolic context generally were more successful than those who solved the problems in a strictly symbolic context. Moreover, students in the non-symbolic condition were more likely to provide relational explanations than those in the symbolic condition. The authors explained these findings by concluding that students' inability to solve non-canonical equations, or attribute a relational meaning to the equal sign, stems from their inability to link the equal sign symbol to the concept of equivalence. Therefore, presenting students the concept of equivalence through manipulatives appears to elicit a relational understanding of what the equal sign represents.

In another study, Alibali (1999) demonstrated that students are capable of algebraic thinking when using a balance and manipulatives, rather than solving symbolic equations with an unknown. She investigated 178 elementary school students and their ability to solve equations relationally using a seesaw type balance. Children were randomly assigned to one of five conditions. In all the conditions, the children were asked to solve three part-whole equations (e.g., 6 + 3 + 7 = ___ + 7). In one condition, the students received feedback on their problem solving, but received no instruction. In three other conditions, students received both feedback and three different types of instruction: in the *symbolic* condition, the investigator explained that the equal sign represents an equality between both sides; in the *analogy* condition, the investigator used a balance when explaining the equations and the students were required to make both sides of the balance the same to solve the problem; and finally, in the *procedure* condition, the investigator demonstrated the procedure in a symbolic context with no balance or manipulatives for solving the equation. In the control group, students did not receive feedback or instruction while solving the equations, and simply had to provide the answers.

The students were then tested individually on their ability to think relationally by solving six symbolic equations on the board in front of the experimenter. Using a paper and pencil pretest and posttest that required students to solve non-canonical problems presented symbolically to measure relational understanding, Alibali (1999) found that more children in the analogy condition, compared to the other instructional conditions, correctly solved the problems measuring relational understanding. That is, the students were able to think relationally about the equal sign, but only when they used manipulatives to solve the problems. It appears that students are able to access the relational understanding of the equal sign, but still have some difficulty doing so, especially in symbolic contexts. This demonstrates that students are capable of understanding the concept of equivalence, but that traditional teaching often does not promote its correct understanding.

Children's Ability to Think Relationally Regarding the Equal Sign

Many teachers believe that because older students often struggle with algebra, younger students are not capable of understanding algebraic concepts (Kaput, 2008; Schliemann et al., 2007). Schliemann et al. (2007) explored this issue by testing students' ability to understand the concept of equivalence. The researchers assigned 7 to 11 year old students ($N = 120$) to groups in which they were exposed to the equal sign in different contexts. Then the students met with the experimenter individually. There were four contexts in which the students were required to judge whether two sides of the equation were equivalent: (a) equivalence of weights on a balance, (b) equivalence between quantities of objects without a balance, (c) equivalence between quantities in a problem presented verbally, and (d) equivalence between two sides of an equation presented symbolically. In each context, students were given activities in which they determined whether the equality would remain if the same or different changes were made to each of the compared amounts. The students were also asked to justify their answers. The researchers found that even first grade students, in all four conditions, have the ability to understand that adding or subtracting the same amount on both sides of the equal sign maintains the relationship of equivalence. Because students at all ages represented in the study were able to think relationally, their work also suggests that students' difficulty with the concept of equivalence is not contingent upon their age.

Baroody and Ginsburg (1983) investigated whether young students were able to acquire the relational meaning of the equal sign. The investigators used the Wynroth Mathematics curriculum (Wynroth, 1975) with 45 first, second-, and third- graders over a period of seven months. The Wynroth curriculum emphasizes the relational view of the equal sign through exposure and instruction; the equal sign is defined as "the same as," emphasizing a relational meaning as opposed to "the answer is," which implies an operational view. The curriculum also includes various forms of non-canonical equations to help emphasize equivalence in a variety of symbolic contexts, rather than focus on presenting the equal sign in canonical contexts only. The authors then asked the students to assess certain types of equations as acceptable or unacceptable, and also to provide a definition of the equal sign. Based on the definitions and explanations provided by the students, Baroody and Ginsburg found that those who were only introduced to the Wynroth curriculum, and did not have previous experience with equivalence in a different curriculum, were better able to access the relational meaning of the equal sign. The results of the study demonstrated that students are able to acquire the relational definition of the equal sign when they are explicitly taught its meaning. Baroody and Ginsburg (1983) were also able to conclude that there are no maturational limitations that may impede elementary students' ability to perceive the equal sign as representing a relationship. Together, their results implied that appropriate mathematics instruction and curricular materials can promote a relational view of the equal sign regardless of age.

Other studies provide additional evidence that students can interpret the equal sign as a relational symbol if they are given instruction even in the form of a brief lesson. Rittle-Johnson and Alibali (1999) assessed a sample of 89 fourth- and firth- grade students on their ability to understand concepts and procedures associated with mathematical equivalence. All the students were screened for relational understanding prior to the study. Students who were unable to solve non-canonical equivalence problems correctly were then assigned to one of three instructional groups: conceptual instruction, procedural instruction, and no instruction as the control group. Children in the conceptual instruction group were presented with equivalence problems and were provided with a conceptual explanation of equivalence. In the procedural instruction condition, children were provided with a procedure for solving for the unknown in the problem. Children in the control condition did not receive any instruction on how to solve for the unknown and were simply asked to solve the problems. Children in the instruction groups improved on their ability to solve non-canonical equivalence problems more than the children in the control group. Moreover, children in the conceptual instruction condition improved more than the students in the procedural instruction condition.

In another study, Watchhorn, Osana, Sherman, Taha, and Bisanz (2011) demonstrated that several types of instruction on equivalence can elicit changes in students' understanding of the equal sign. Second- and fourth-grade students were assigned to one of four instructional groups. The students in the first group received conceptual instruction without manipulatives. The second group received conceptual instruction with manipulatives. The third group received procedural instruction without manipulatives, and the fourth group received procedural instruction with manipulatives. Other students were assigned to a control group and did not receive any instruction on the equal sign; they received regular classroom instruction and were not given any instruction on equivalence. All students in the instructional groups outperformed the students in the control group on their ability to solve non-canonical equations. Similar results were found at both grade levels. These results suggest that as long as students received instruction on equivalence, they were able to access relational understanding when solving problems.

Classroom Instruction on the Equal Sign

It has been established that students are indeed capable of conceptualizing the idea of equivalence, but the reason they do not do well on measures of relational thinking is because of their failure to link the equal sign (the symbol) to the concept of equivalence. Carpenter et al. (2003) argued that an explanation for this is how the mathematics curriculum is traditionally implemented in the classroom. Along the same lines, Knuth et al. (2005) argued that students are introduced to the equal sign at the beginning of their mathematics instruction (i.e., as early as Kindergarten), but that teachers spend very little time explicitly teaching the meaning of the symbol because they generally assume that students perceive it as signifying a relation. Knuth et al. (2005) speculated that because of such assumptions about students' knowledge, teachers do not provide them with enough explicit opportunities to develop a relational understanding of the equal sign, despite of how often the sign appears in the school mathematics curriculum.

Seo and Ginsburg (2003) examined how the equal sign was presented by a second grade teacher in her classroom. The researchers observed her twice a week over a period of six months and took extensive field notes and recorded classroom conversations. By analyzing their observations, the authors showed that the particular teacher in this study believed in the importance of developing a relational understanding of the equal sign and as such tried to provide students with various situations in which the equal sign could be viewed as relational. Seo and Ginsburg (2003) also found that when the teacher primed the students to think in a relational way, she was able to elicit their relational understanding of the symbol in non-canonical equations. Moreover, the teacher mostly used the word "equals" when talking about the equal sign instead of the word “gives.” As well, she used the word “equals” in a variety of everyday situations. For example, when the teacher talked about comparing the number of letters in children’s names, she used the word “equals” (such as, “there is an equal number of people who have five letters and six letters in their first names” (p. 117).

Seo and Ginsburg (2003) also found that even though the teacher was able to elicit students’ relational interpretations of the equal sign in specific contexts, there were some instances when the teacher tended to promote an operational view. For example, when the teacher set up symbolic problem-solving activities that encouraged students to solve a series of problems, the students were required to provide the “right” answer, thereby reinforcing the idea of the equal sign as an operational symbol. These types of repetitive exercises seemed to promote an operational view in the students because the teacher did not encourage them to use non-canonical representations in these situations.

Ding and Li (2006) investigated teachers’ presentations of the equal sign in the classroom. Their results documented that US teachers did not pay much attention to students’ errors when writing two different values on each side of the equal sign (e.g., $3/3 \times 2 = 6/8$). Moreover, it was found that teachers also made similar errors (e.g., $360/4 = 90 \times 3 = 270$) by writing such equations on the board. Thus, by promoting incorrect uses of the equal sign, teachers may be discouraging the relational meaning of the equal sign in their students.

It is unclear to what extent teachers use teacher manuals in their teaching, but the materials to which teachers are exposed may “trickle down” to the classroom. In other words, the materials that teachers use may affect what they present to the students in the classroom. After demonstrating that Chinese teacher guidebooks presented the equal sign in ways that reflected relational thinking relative to guidebooks in the United States that did not, Ding and Li (2006) tested samples of American and Chinese sixth-grade students on their ability to solve non-canonical equivalence problems. They found that only 28% of American students were able to accurately solve non-canonical problems compared to almost 99% of Chinese students who were able to correctly solve the same problems. The authors concluded that what teachers see and read in their guidebooks affects their classroom practice, which in turn impacts what students learn and how they perform on equivalence problems.

A further investigation by Li et al. (2008) revealed that American teacher guidebooks often suggest to use the equal sign with words such as "makes" for addition problems and "leaves" for subtraction. The researchers noted that this language may help students obtain the correct answer in addition and subtraction problems, but may also lead to misconceptions about the equal sign. This language indicates that these problems require an operation rather than a relation, and becomes problematic when students are presented with non-canonical equations. Thus, despite not having directly tested the relationship between the language teachers use and students' relational understanding, the researchers noted that it is possible that the language that teachers use with their students may activate students' operational thinking. Moreover, one American teacher guidebook examined by Li et al. (2008) presented the teachers with the meaning and definitions of four signs (i.e., the signs for addition, subtraction, multiplication, and division), but failed to define and provide a meaning for the equal sign itself. Not only was the equal sign not defined, it was also misused in the same guidebook, where it was presented as a substitute for the word "are." It seems, therefore, that students' difficulty with the equal sign might stem, in part, from the language that teachers are encouraged to use.

In addition, in Baroody and Ginsburg's (1983) study, the Wynroth curriculum uses the words "the same number as," rather than "gives" or "the answer is," when referring to the equal sign. For example, teachers who use this curriculum tend to ask students to solve “three plus five is the same number as” rather than “three plus five gives.” In other words, the language used in the Wynroth curriculum when referring to the equal sign reflects a relational view, not an operational one, and as a result helps students access the relational interpretation of the equal sign.

Other activities that teachers give to their students may further reinforce students’ misconceptions of the equal sign as an operator. For example, Baroody and Ginsburg (1983) suggested that rote exercises, such as practicing solving canonical equations repeatedly and without reflecting on the mathematics involved, may reinforce students’ views of the equal sign as an operational symbol. If students are only taught to provide answers to a given type of equation, they will not be able to see the meaning of the equal sign in other contexts (Baroody & Ginsburg, 1983; Kaput, 2008).

Effects of Exposure on Learning in Mathematics

Even though there are many factors that can explain students’ mathematics achievement, curricula and textbooks have a significant impact on the exposure students receive to mathematical representations, and thus, to their performance (Reys, Lindquist, Lambdin, Smith, & Suydam, 2003; Reys, Reys, & Ch´avez, 2004; Schmidt et al., 2001). As the evidence suggests, textbooks often determine the nature of teachers’ practice in the mathematics classroom. Moreover, textbooks expose students to the content in the absence of a teacher.

Although not specifically in the context of equivalence, Canobi (2009) found that students can gain conceptual understanding of mathematical principles by being exposed to problems that are sequenced such that target concepts are highlighted. The participants were 72 elementary school students who underwent a pretest through an interview process for having conceptual understanding of addition and subtraction. In the practice phase, which was the experimental intervention, the students were required to complete nine 10-minute worksheets on which were problems that were presented in a conceptually sequenced manner (e.g., by placing 6 + 3 directly after 3 + 6, the concept of commutativity is highlighted). By exposing the students to problems in a conceptually sequenced manner, Canobi (2009) hypothesized that the children would notice conceptual relationships between problems as they worked through them. After the practice phase, the students were individually interviewed on their ability to identify the target concepts. Canobi (2009) found that children's understanding of the concepts improved after having been exposed to conceptually sequenced practice problems compared to the other condition, in which students were exposed to randomly sequenced problems.

With respect to the equal sign, several researchers have indicated that what textbooks present to students likely affect their understanding of the symbol (McNeil et al., 2006; Seo & Ginsburg, 2003). In a study described earlier, Baroody and Ginsburg (1983) exposed second- and third-grade students to the Wynroth curriculum over a period of seven months. After implementing the curriculum, the authors assessed students' ability to judge the truth of a number of equations presented symbolically. Some of the equations were familiar to the students (e.g., equations they had seen previously), other equations were not familiar to them (e.g., 7 + 6 = XIII), and still other equations were incorrect (e.g., 4 + 2 = 42). In individual interviews, the students were asked which of the equations they considered acceptable and which were not. Most students accepted all the equations to which they had been exposed through textbooks and worksheets in the Wynroth curriculum. The researchers concluded that students are able to develop a relational view of the equal sign if teachers and textbooks provide exposure to non-canonical contexts. In other words, if students are exposed to materials that contain instances of non-canonical equations, they will more likely be able to think relationally when solving problems.

Textbook analyses have provided additional support for the effects of exposure on students' relational thinking. Li et al. (2008) examined Chinese mathematics preservice teacher textbooks and compared them to comparable comparable materials in the United States. The American textbooks analyzed in their study accounted for roughly 78% of those used in American university teacher preparation programs. The Chinese preservice teacher textbooks were five teacher guidebooks and preparation books. The authors examined every page of each book to identify the types of lessons and activities as well as recommendations for teachers on how to use the equal sign. The researchers placed all instances of the equal sign in two categories: "operations equals answer" context (canonical form) and non-canonical context.

The authors noticed that American preservice textbooks did not offer alternatives to canonical equations when presenting the equal sign. The investigators also observed that the lesson plans in the Chinese textbooks emphasized the equal sign with respect to other relational symbols, such as "greater than" and "less than" (i.e., > and <). Also, Chinese instructional materials explicitly encouraged teachers to present the equal sign in a variety of arrangements (i.e., canonical and non-canonical). Moreover, the investigators found that educators who followed the Chinese teacher guidebook did not allow students to use the equal sign if the two values on each side were not equal (e.g., 2 + 3 = 5 + 1)[1]. It appears then, that Chinese preservice teacher guidebooks recommend to teachers that they use more varied contexts when presenting the equal sign (Li et al., 2008). Thus, coupled with the finding showing Chinese students' facility

[1] When students are solving a series of computations, they have been found to use the equal sign as indications of different steps in the procedure (Carpenter, Franke & Levi, 2003). For example, when solving 2 + 3 + 1 = ___, students often write 2 + 3 = 5 + 1 = 6.

with non-canonical equations (Ding & Li, 2006), it appears that how students are exposed to the equal sign may have an impact on their relational thinking.

Seo and Ginsburg (2003) also examined mathematics curriculum materials by analyzing two student textbooks used by one elementary school teacher. The authors coded the instances of contexts in which the equal sign appeared in the two textbooks. They found that the most frequent type of number sentence was of canonical form and that the equal sign rarely appeared without an accompanying operation (i.e., 8 = 8). Moreover, they extended their investigation to middle school mathematics textbooks and found that those, too, failed to support the relational meaning of the equal sign because it presented the students with only canonical equations. Thus, it seems that textbooks in the United States do not tend to expose children to the equal sign as a relational symbol.

McNeil et al. (2006) conducted a textbook analysis of four middle-school textbook series (Grades 6 to 8): *Saxon Math* (Hake & Saxon, 2004, as cited in McNeil et al., 2006), *Prentice Hall Mathematics* (Charles, Branch-Boyd, Illingworth, Mills, & Reeves, 2004, as cited in McNeil et al., 2006), *Connected Mathematics* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998, as cited in McNeil et al., 2006), and *Mathematics in Context* (Romberg et al., 1998, as cited in McNeil et al., 2006). The investigators examined the instances of the equal sign and the contexts in which it appeared in the four textbooks. The analysis revealed that the textbooks presented the equal sign in contexts where there is an operation on the left hand-side and one number on the right hand-side of the equal sign. They observed that the equal sign was rarely presented in a context that included operations both to its left and to its right. Non-canonical equations accounted for only 5% to 9% of the equations in any given textbook. McNeil et al. (2006) speculated that such a narrow representation of the equal sign (canonical forms only) might reinforce students' operational interpretations. In short, these findings suggest that the ways in which students are exposed to the equal sign in their mathematics textbooks is not conducive to a relational interpretation.

In addition, McNeil et al. (2006) conducted an experiment to supplement their textbook analysis. The researchers randomly assigned elementary school and middle school students to three conditions: one in which they were presented with equations of the form operations equals answer (e.g., 3 + 4 = 7), the second group was presented with equations with operations on right side (e.g., 7= 3 + 4), and the third group saw identity equations (e.g., 7 = 7). The students were asked to identify what the "=" symbol was and what it represented. The researchers found that non-canonical equations, including identity equations, tend to elicit relational thinking. McNeil et al. (2006) then presented the same students with equations in one of two contexts: operations on the right side of the equal sign (e.g., 7 = 3 + 4) and operations on both sides of the equal sign (e.g., 3 + 4 = 2 + 5). The students were again expected to identify what the "=" symbol meant and what it represented in each of these contexts. The researchers concluded that exposing students to operations on both sides of the equal sign elicits their relational view of the equal sign in that context.

In another study, McNeil, Fyfe, Petersen, Dunwiddie and Brletic-Shipley (2011) investigated whether being presented with non-canonical forms of equations would help students construct a better understanding of equivalence. The researchers randomly assigned one hundred 7 and 8 year old students to one of three conditions. In the "traditional practice" condition, students were presented with canonical problems (e.g., 9 + 8 = ___); in the "non-traditional practice" condition, students were presented with non-canonical problems that had an operation on the right side of the equal sign (e.g., ___ = 9 + 8); and in the "no extra practice" condition, students did not receive any practice and continued with their regular classroom instruction.

Students in the practice conditions met individually with a tutor in three separate sessions, during which they practiced solving addition problems through games. The games involved cards and dice, flashcards, and a computer game. The tutor first demonstrated how to play each game (i.e., provide the correct answer). Between the sessions the students practiced solving addition problems that were in line with the problems given in their respective conditions as part of their homework assignment. After each assignment was handed in, the tutor and the students corrected the problems together. After the three sessions, the students met with another experimenter who assessed their understanding of equivalence. Students' understanding of equivalence was assessed with the use of a paper-and-pencil test that included a series of non-canonical problems to solve. They were also asked to provide verbal definitions of the equal sign. McNeil et al. (2011) found that students who practiced solving non-canonical problems developed a better understanding of equivalence than students who practiced solving canonical problems, as demonstrated by higher scores on the posttest. This is an important study because it showed that students can improve on their ability to understand equivalence and see the equal sign as a relational symbol if they practice solving non-canonical problems. But the study did not show the effects of exposure without any teacher or tutor interventions. This is the gap identified in the present study.

Present Study

The research reviewed suggests that the equations to which most North American elementary-aged students are exposed are not conducive to acquiring a relational understanding of the equal sign. Instead, students acquire a procedural or operational understanding of the equal sign, seemingly as a result of this lack of exposure. It is often observed that young students interpret canonical addition equations in terms of what the operation requires them to do, rather than in terms of what quantitative relationship the equal sign refers to (Falkner, Levi, & Carpenter, 1999). If students lack a relational understanding of the equal sign, they will likely have difficulty solving algebraic equations at higher grade levels (Falkner et al., 1999). Furthermore, it seems that students do not always apply their knowledge of the relational meaning of the equal sign in meaningful ways in novel situations. They simply apply routine computational procedures, often with errors, such as providing the “answer” after the equal sign in non-canonical problems (e.g., McNeil, 2004).

Researchers have been exploring ways in which educational materials can be designed and used by teachers to help students acquire a relational understanding of the equal sign. Little research exists, however, on the effects of exposure to different types of symbolic equations in absence of teacher intervention on students’ relational understanding. The objective of the present study is to investigate whether exposure to a particular form of equation (i.e., non-canonical) is more favorable than exposure to other forms (i.e., canonical) on students’ relational understanding of the equal sign. In this study, exposure is defined as the active engagement in solving symbolic equations in the absence of any verbal statements by the teacher regarding the meaning of the equal sign.

The effects of exposure were tested experimentally by asking students to complete the Equivalence Test before and after being exposed to a particular type or equation in the classroom through paper-and-pencil worksheets. The Equivalence Test is a measure of students’ ability to solve non-canonical problems.The students were also asked to complete the Definitions task before and after the exposure intervention, which required them to write their definitions of the equal sign. The exposure intervention required the participants to solve a number of open-number sentences over three 15-minute sessions in their respective classrooms.

Thus, the study aimed to answer the following questions:

1. Does exposure to non-canonical equations, (i.e., identity ($a = a$) and part-whole, ($a + b = c + d$)) result in students' improved ability to solve non-canonical equations compared to exposure to canonical equations?
2. What are the specific effects of exposure to non-canonical equations? Specifically,
    a. does exposure to non-canonical Part-Whole problems result in an improved ability to solve non-canonical Identity problems?
    b. does exposure to non-canonical Identity problems result in an improved ability to solve non-canonical Part-Whole problems?
3. Does exposure to non-canonical equations result in students' improved relational thinking about the equal sign?

The literature on exposure to mathematical representations allowed me to predict that exposure to non-canonical equations would result in an improved ability to solve non-canonical problems, as well as a greater relational understanding of the equal sign, compared to exposure to canonical equations. I also predicted that exposure to part-whole equations would result in the ability to solve non-canonical equations than exposure to identity equations (McNeil & Alibali, 2005; McNeil et al., 2006). Moreover, it is unknown how students of different ages respond to different types of exposure to the equal sign. Thus, grade level was tested as a moderating factor in this study.

Chapter 3: Method

The objective of this study was to investigate whether exposure to a particular form of equation in absence of instruction was more favourable than exposure to other forms for students' relational understanding of the equal sign and their ability to solve non-canonical equations. In particular, I investigated the relative effects of exposure to the equal sign presented in different non-canonical forms in second- and third-grade classrooms. The non-canonical forms were part-whole ($a + b = c + d$) and the identity (a = a).

Participants

The participants were 93 second-grade and third-grade students (44 students in second-grade and 49 students in third-grade) from 4 different classrooms in each of 2 different private schools in the Montreal area. This resulted in a total of 8 classrooms: half in second grade and half in third grade. Fifty-three students formed the final sample because of their baseline difficulty understanding the equal sign as a relational symbol. Ten students were in the second grade in school 1, 17 students were in the second grade in school 2, 8 students were in the third grade in school 1, and 18 students were in the third grade in school 2.

Design

This study used pretest-posttest experimental design, which is graphically depicted in Figure 1. The participants were first given Pretest 1 to measure their ability to solve non-canonical equations and their relational understanding of the equal sign. Following the procedure used by Watchorn (2011), students who were successful on 6 or more of the Equivalence Test problems on Pretest 1 were considered to have an ability to solve non-canonical problems and a relational understanding of the equal sign. These students were excluded from the study, resulting in 53 students in the final sample. The 53 participants were then randomly assigned to one of the three experimental conditions within each classroom. The conditions were canonical, non-canonical part-whole, and non-canonical identity. Two weeks later, they were given a second pretest (Pretest 2). The second pretest was used to measure any changes in students' ability to solve non-canonical problems from Pretest 1 before undergoing the exposure intervention. This pretest was added as a substitute for a control group, which was not feasible with the resources available for this study.

The exposure intervention required the students, in all conditions, to solve a series of open-number sentences presented symbolically on worksheets. The worksheets differed by condition. The Canonical Worksheets included canonical fill-in-the-blank problems, the Non-Canonical Part-Whole Worksheets had only non-canonical part-whole fill-in-the-blank problems, and the Non-Canonical Identity Worksheets had only identity fill-in-the-blank problems. After the intervention, the participants were given the Equivalence Test and the Definitions task to measure their ability to solve non-canonical equations and their relational understanding of the equal sign, respectively.

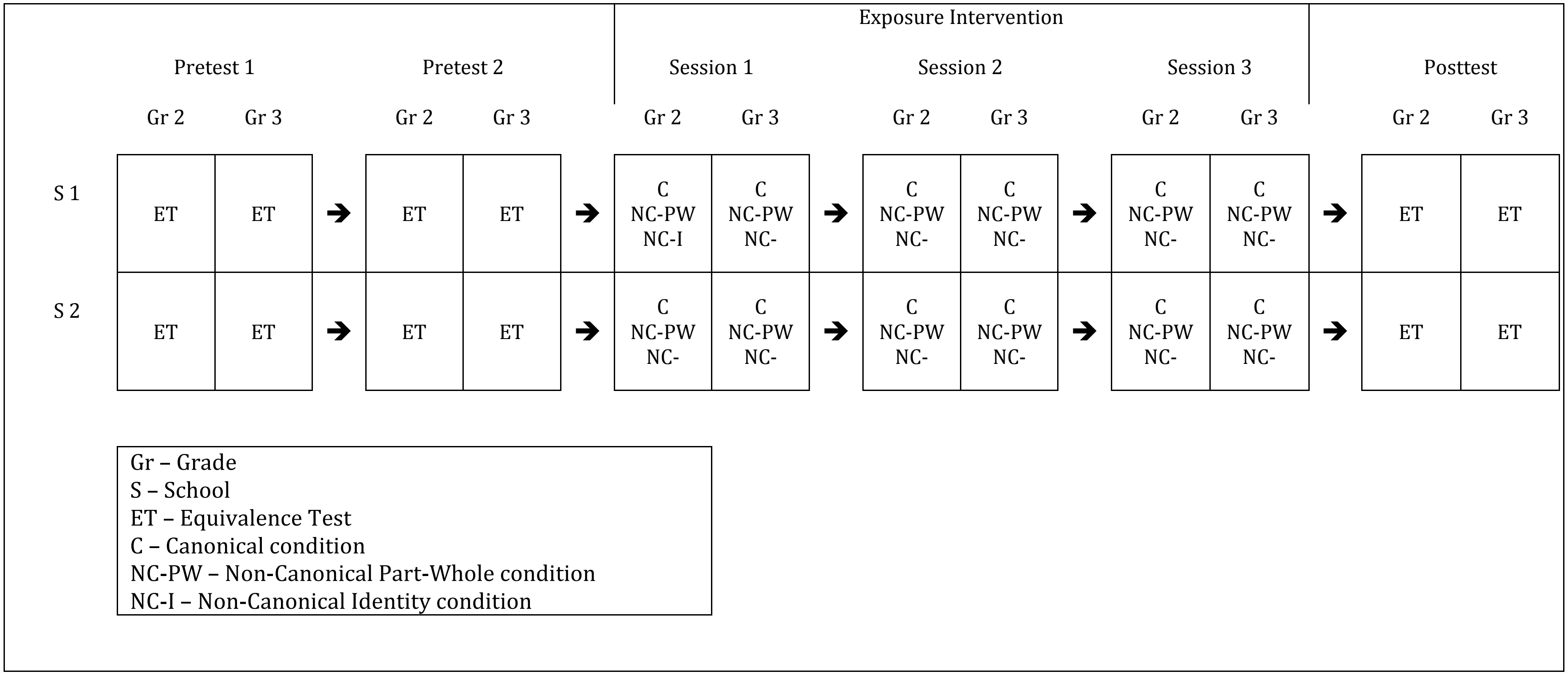

Figure 1. Study design.

Measures

Equivalence Test. The Equivalence Test (Watchorn, 2011) is a measure of students' ability to solve non-canonical equivalence problems. It is presented in Appendix A. The test consisted of equivalence problems presented symbolically and was administered as a paper-pencil test that each student completed independently at his or her desk. The test included 16 symbolic open-number addition sentences, 6 of which were part-whole problems, 6 were identity problems, and 4 were canonical problems. The canonical problems were added in order to allow the students to feel successful on the test, since I anticipated that some of them would have become discouraged by the non-canonical problems.

On each item, the student was required to choose the correct answer from a list of four possible answers. Each choice corresponded to a specific misconception that children have been shown to hold regarding the equal sign. Thus, the choices for each item corresponds to: (a) the correct answer; (b) the answer that would be obtained by adding up all the numbers to the left of equal sign; (c) the answer that would be obtained by adding all the numbers on both sides of the equal sign; and (d) an answer that is smaller in value than the correct answer. The order of the possible answers was different in each item, and no two contiguous items had the correct answer in the same position.

In each problem, there was a blank on the right side of the equal sign. The numbers in the open-number sentences were all single-digit, and the numbers in the answer choices were both single- and double-digit. There were 6 problems with 2 operations on the left side of the equal sign and 8 problems with 1 operation on the left side of the equal sign. There were 8 problems with 1 operation on the right side of the equal sign, and 6 problems with just a blank on the right side of the equal sign.

The test was scored by allocating 1 point for each correct answer and 0 points for each incorrect answer. The number of points was summed to obtain a score for each participant. The maximum score was 16 and the minimum score was 0. Scores were converted to percents.

Definitions task. Appended to the Equivalence Test was the Definitions task, designed to measure students' relational thinking. The Definitions task can be found in Appendix B. This task presented the student with a canonical equation (2 + 7 = 9) and the students were required to generate a definition of the equal sign. Borrowing the rubric created by Adrien, Osana, Watchorn, and Bisanz (2012), students' definitions of the equal sign were coded as either (a) Relational, (b) Operational, or (c) Combined. The scoring rubric for the Definitions task is included in Appendix D. The definitions were coded as Relational when students explained that both sides of the equal sign had to be the same. The definitions were coded as Operational when students describe the equal sign as an operator using explanations such as "gives" or "the answer comes next." Finally, definitions were coded as Combined when students provided a definition that included both relational and operational elements. The students received 2 points for Relational definitions, 1 point for Combined definitions, and no points for Operational ones.

Exposure Intervention. The exposure intervention was delivered in the three experimental conditions in the students' classrooms. In the canonical condition, the students were required to solve a series of canonical open-number sentences on a paper-and-pencil worksheet (e.g., 3 + 4 = ___). In the non-canonical identity condition, the students were required to solve a series of identity open-number sentences on a paper-and-pencil worksheet (e.g., 7 = ___). In the non-canonical part-whole condition, the students were required to solve a series of open-number sentences with operations on both sides of the equal sign (e.g., 5 + 2 = 3 + ___).

In each condition, the children were given three worksheets, each delivered on a separate day. The worksheets for each condition are presented in Appendix C. At the top of each worksheet were four examples that served as models for how to solve the open-number sentences, two of which were correct and two of which were incorrect (a sample worksheet is presented in Figure 2). The correctly solved open-number sentences were marked with a "√" to indicate the right answer. The incorrectly solved open-number sentences were marked with an "X." Below the models on the same page were 10 equations that either I or a the research assistant asked the students to solve. The instructions to the students were: "Please look at the examples on the top of the pages. Look at the ones that are right and the ones that are wrong. Please fill in the blanks in the rest of the examples. When you are done, I will come to collect your paper."

The students were then allowed 10 minutes to complete the worksheets independently at their desks. The students received the worksheets at the end of their mathematics class or during homeroom period.

Canonical Context Worksheet: Intervention 1

Name: ______________________

Look at these examples: 5 + 3 = _

1.	**7 + 1 = 8** ✓	
2.	**2 + 3 = 5** ✓	
3.	**8 + 2 = 9** ✗	**8 + 2 = 10** ✓
4.	**4 + 5 = 6** ✗	**4 + 5 = 9** ✓

Fill in the blank:

1 + 3 = _   7 + 3 = _

9 + 1 = _   3 + 5 = _

4 + 2 = _   8 + 1 = _

2 + 7 = _   4 + 3 = _

6 + 2 = _   1 + 6 = _

Figure 2. Sample worksheet for the canonical condition.

Procedures

During part of a mathematics class three weeks before the intervention began, all participants wrote the Equivalence Test as a first pretest measure. There were three versions of the Equivalence Test: version A, version B, and version C. All three versions contained the same problems, but placed randomly in different orders to control for order effects. Students who received version A of the Equivalence Test at Pretest 1 received version B of the test at Pretest 2, and received version C of the test at Posttest. Students who received version B of the Equivalence Test at Pretest 1, received version A or version C at Pretest 2 and Posttest, respectively. At each testing, a third of the students in each class received version A of the Equivalence Test, another third received version B, and the last third received version C.

Worksheets of the three different versions were randomly distributed to the students in the classroom. I gave oral instructions to the students to circle the correct answer for each item on the test, and they were given 15 minutes to complete the test. I timed the students with a stopwatch and asked them to stop after 15 minutes.

Once all the participants had completed Pretest 1, I scored the tests to determine which students had answered 6 or more questions correctly. Students from each classroom who did not meet this criterion were then randomly assigned to one of the three experimental conditions for the exposure task (i.e., identity, part-whole, and canonical condition). All the students within each classroom were placed on a numbered list, and a random number generator was used to assign the students to the different conditions. The students' names were written on the appropriate condition worksheets for each of the three sessions of the Exposure Intervention.

Two weeks later, the participants completed Pretest 2. One week after completing the second pretest, participants received the first session of the Exposure Intervention. Each student received a worksheet that matched the condition to which he or she had been assigned. While the worksheets were being distributed, the students were asked to keep their papers face down on their desks. Once all the worksheets were distributed, the students were instructed to turn them over and begin working. While the students were filling in the worksheets, the teachers remained in their respective classrooms. The teachers said nothing to the students during the time they worked on the worksheets, and were instructed not to talk about the equal sign during the Exposure Intervention at any point.

One week after the first exposure worksheet, the participants received the second exposure worksheet, and in the third week, they received the third. The same procedures were used for all three exposure sessions. One week after the third exposure session, the participants were given the Equivalence Test as a posttest measure in one of their mathematics classes. The same procedures as for the pretests were used.

Chapter 4: Results

To address the first research question regarding the effects of the intervention, inferential tests on the performance of second- and third-grade students on the non-canonical problems on the Equivalence Test across the three time points (first pretest, second pretest, and posttest administered after the intervention) were performed. A second test was conducted to address the second research question on the effects of both condition and problem type across the three time points. Finally, to address the third research question regarding students' relational thinking as a function of exposure type, an analysis on students' verbal definitions of the equal sign was conducted to determine whether students' interpretations of the equal sign had changed over time as a function of condition. Each set of results will be described in turn below.

The Effects of Condition on Students' Ability to Solve Non-Canonical Equations

The means and standard deviations of the students' percent scores on the non-canonical problems on the Equivalence Test at both pretests (Pretest 1 and Pretest 2) and Posttest are presented as a function of condition and grade in Table 1. A 3 (condition) x 3 (time) x 2 (grade) mixed ANOVA was conducted, with condition (canonical, non-canonical identity, non-canonical part-whole) and grade (second, third) as the between-group factors and time (Pretest 1, Pretest 2, Posttest) as the within-group factor. Scores on the non-canonical problems on the Equivalence Test, in percent, was used as the dependent measure.

Table 1

Students' Mean Scores and (Standard Deviations) on Non-Canonical Problems on the Equivalence Test

Grade	Condition								
	Canonical			Identity			Part-Whole		
	Pretest 1	Pretest 2	Posttest	Pretest 1	Pretest 2	Posttest	Pretest 1	Pretest 2	Posttest
2	.14 (.10)	.23 (.32)	.22 (.19)	.18 (.08)	.25 (.18)	.43 (.32)	.10 (.09)	.23 (.09)	.30(.26)
3	.13 (.07)	.25 (.19)	.38 (.34)	.14 (.08)	.25 (.25)	.45 (.30)	.12 (.08)	.40 (.41)	.62 (.42)

Results revealed a main effect of time, $F(2, 88) = 21.13$, $p < .001$. The means indicate that all students improved on their ability to solve the non-canonical problems on the Equivalence Test regardless of the condition or grade they were in. The means are graphed in Figure 3.

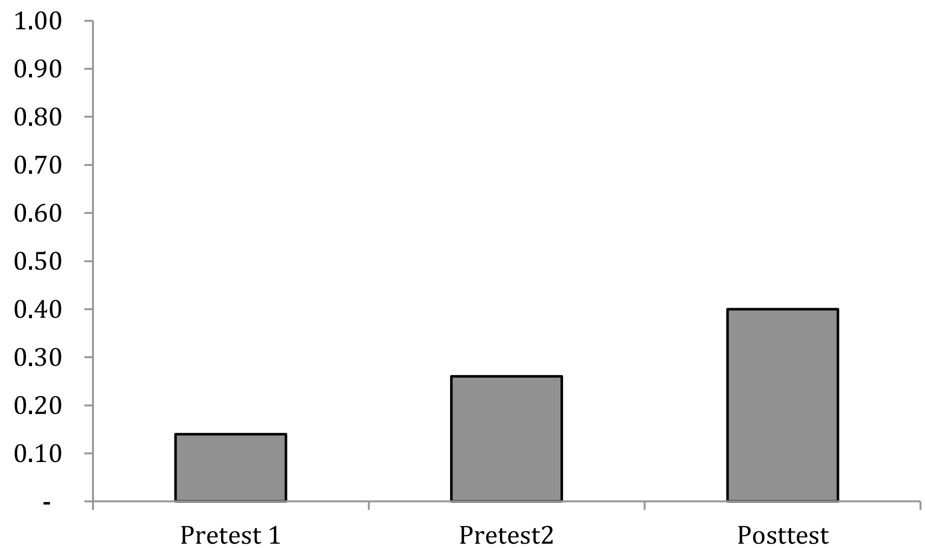

Figure 3. Students' mean scores on non-canonical equations on the Equivalence Test, across condition and grade level.

Post hoc analyses with Bonferroni corrections were conducted to compare students' ability to solve non-canonical problems on the Equivalence Test at each of the three time points (Pretest 1, Pretest 2, and Posttest). The analyses indicated that the mean Posttest score was significantly higher than the mean score at Pretest 1 ($p < .001$). The mean score at Posttest was also significantly higher than the mean score at Pretest 2 ($p < .001$), and the mean score at Pretest 2 was also significantly higher than the mean score at Pretest 1 ($p < .001$).

No other interactions or main effects were found, including main effects and interactions involving grade level. Therefore, grade was omitted as a factor from subsequent analyses, and all student data were analyzed with these two grade levels collapsed.

Effects of Condition and Problem Type on Student Performance Over Time

The means and standard deviations of the students' percent scores on the Equivalence Test at pretests (Pretest 1 and Pretest 2) and Posttest are presented as a function of condition and problem type in Table 2. To address the second research question regarding the effects of condition and problem type on students' performance, a 3 (condition) x 3 (time) x 3 (problem type) mixed ANOVA was conducted, with condition (canonical, non-canonical identity, non-canonical part-whole) as the between group factor, and time (Pretest 1, Pretest 2, Posttest) and problem type (canonical, identity, part-whole) as the within-group factors. The percent scores on each problem type on the Equivalence Test was used as the dependent variable in the analysis.

Table 2

Students' Mean Scores and (Standard Deviations) on All Problem Types on the Equivalence Test

Problem Type	Condition								
	Canonical			Identity			Part-Whole		
	Pretest 1	Pretest 2	Posttest	Pretest 1	Pretest 2	Posttest	Pretest 1	Pretest 2	Posttest
Canonical	.87 (.24)	.88 (.19)	.85 (.16)	.88 (.21).	.91 (.20)	.86 (.19)	.91 (.15)	.90 (.22)	.84 (.22)
NC - Identity	.26 (.16)	.37 (.30)	.38 (.25)	.29 (.13)	.38 (.21)	.64 (.28)	.21 (.16)	.46 (.23)	.54 (.30)
NC – Part-Whole	.01 (.05)	.10 (.28)	.18 (.31)	.03 (.06)	.12 (.24)	.23 (.38)	.01 (.04)	.13 (.33)	.33 (.43)

Note: NC = Non - Canonical

The results of the ANOVA revealed a main effect of time, F (2, 94) = 13.41, $p < .001$, and a main effect of problem type, F (2, 94) = 388.59, $p < .001$. The main effect of time indicated that students improved on their performance across time regardless of condition or problem type. Post hoc analyses with Bonferroni corrections indicated that the mean posttest score (M = .51, SD = .25) across conditions was significantly higher than the mean score at pretest 1 (M = .32, SD = .09, $p < .001$). The mean score at posttest across conditions was significantly higher than the mean score at pretest 2 (M = .42, SD = .19, $p < .001$), and the mean scores at pretest 2 was also significantly higher than the mean score at pretest 1 ($p < .001$).

The main effect of problem type indicated that students scored differently on the different types of problems on the Equivalence Test regardless of time and the condition they were in. Post hoc tests with Bonferroni corrections revealed that the students' scores on Canonical problems (M = .88, SD = .14) were significantly higher than Identity problems (M = .40, SD = .16, $p < .001$) and Part-Whole problems (M = .13, SD = .20, $p < .001$). Also, scores on Identity problems were significantly higher than Part-Whole problems ($p < .001$).

In addition, the ANOVA revealed a time x problem type interaction, F (4, 188) = 11.61, $p < .001$. This interaction means that students scored differently across time depending on the problem type. The means are graphed in Figure 4.

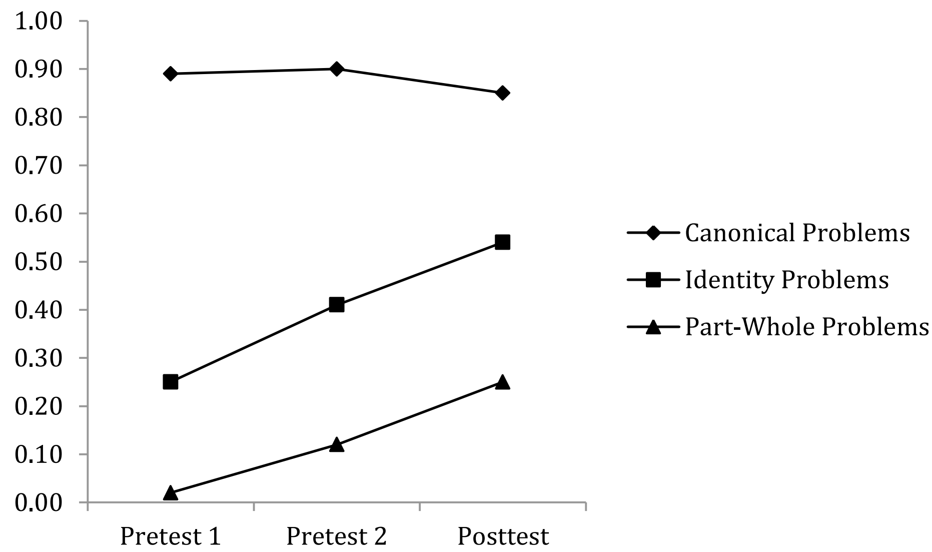

Figure 4. Students' means of solving canonical, identity, and part-whole equations on the Equivalence Test across time.

Tests of simple effects revealed that there was no significant difference in mean performance on Canonical problems between any two time points. In contrast, performance on Part-whole problems improved from Pretest 1 to Pretest 2 ($p < .05$), and from Pretest 2 to Posttest ($p < .05$), regardless of condition. The same pattern of improvement was found for Identity problems (both *p*s < .01). In addition, a significant difference between each pair of means was found at each time point, with performance on Canonical problems higher than each of Identity and Part-whole problems (all *p*s < .001), and performance on Identity higher than Part-whole problems (all $p < .001$).

No other main effects or interactions were found.

Effects of Exposure on Students' Relational Understanding of the Equal Sign

The means and standard deviations of the scores on the Definitions task are presented as a function of condition and time in Table 3. The sample used for this analysis only included 46 participants, because 7 students did not write an answer on the Definitions task. To address the third research question regarding the effects of exposure on students' relational understanding of the equal sign, a 3 (condition) x 3 (time) mixed ANOVA was conducted, with condition (Canonical, Non-canonical Identity, Non-canonical Part-Whole) as the between group factor and time (pretest 1, pretest 2, posttest) as the within-group factor. The scores on the Definitions task were used as the dependent variable in this analysis. The results of the ANOVA revealed no main effects or interactions.

Table 3

Students' Mean Scores and (Standard Deviations) on the Definitions Task

Condition	Time		
	Pretest 1	Pretest 2	Posttest
Canonical	.17 (.58)	.33 (.78)	.33 (.78)
Identity	.28 (.67)	.22 (.65)	.33 (.77)
Part-Whole	.24 (.66)	.12 (.49)	.12 (.49)

Note: Maximum score = 2

In addition, a 2 x 2 chi-square analysis was conducted to test for a relationship between students' scores on the Definitions task (i.e., 0 or 2) at Pretest 1 and the scores on the Definitions task (i.e., 0 or 2) at Posttest. The frequency counts and proportions are presented in Table 4. The "Combined" definition (i.e., score of 1) was excluded from the analysis because it only occurred twice at Pretest 1, and did not occur at all at Posttest.

Table 4

Frequencies of Scores on Definitions Task at Pretest 1 and at Posttest

	Posttest Score	
Pretest 1 Score	0	2
0	39 (97.5%)	2 (33.3%)
2	1 (2.5%)	4 (66.7%)
Total	40 (100%)	6 (100%)

Results indicated that students' scores on the Definitions task at the first pretest were significantly related to their scores on the Definitions task at Posttest, $\chi^2(1) = 22.17$, $p <$ .001.More specifically, 97.5 % of the 40 students who gave an operational definition on the Posttest also gave an operational definition at Pretest 1.In contrast, significantly fewer students (33.3%) of those who provided a relational definition at Posttest provided an operational one at Pretest 1. In addition, 66.7% of students who gave a relational definition at Posttest also gave a relational definition at Pretest 1.

Chapter 5: Discussion

Summary and Explanation of Results

The goal of the present study was to investigate whether exposure to non-canonical equations is more helpful than exposure to canonical equations for students to (a) accurately solve non-canonical problems, and (b) acquire a relational understanding of the equal sign. Exposure in the present study referred to the problems students see in their materials and in the classroom in absence of teachers' verbalizations about the meaning of the equal sign.

In addition, I wanted to investigate whether a particular type of non-canonical equation (i.e., equations in identity or part-whole formats) would result in improvement in the other type. More specifically, I was interested in whether exposure to non-canonical part-whole equations would result in improved ability to solve non-canonical identity equations, and vice versa. Otherwise said, I investigated whether students' increased performance on solving certain types of non-canonical equations would transfer to other types of non-canonical problems.

The results of the experiment revealed that all students improved on their ability to solve the non-canonical problems on the test of equivalence used in this study, regardless of the types of equations to which they were exposed or the grade they were in. Those students who were exposed to non-canonical problems did not improve more than the students in the canonical condition. This was contrary to my initial prediction that students who are exposed to non-canonical problems would be better able to solve non-canonical equations on the Posttest compared to students who were exposed to canonical problems only. These findings indicate that the context of the exposure intervention (i.e., canonical, non-canonical identity, and non-canonical part-whole) did not make a difference in students' abilities to solve non-canonical equations.

The fact that students improved on their ability to solve non-canonical equations at Pretest 2 was surprising because the students had not yet undergone their exposure intervention by the time that Pretest 2 was administered. I speculate that students' improved performance on non-canonical problems from Pretest 1 to Pretest 2 can be attributed to the exposure to the non-canonical problems on the assessment measure. Each student was exposed to all three problem types during Pretest 1. The effect of exposure on the assessment measure can be supported by the fact that there was an observed improvement in students' ability to solve non-canonical problems at Pretest 2 compared to Pretest 1, and there was no intervention between those two time points. Thus, I speculate that the assessment measure in itself constituted exposure to non-canonical equations for students in all three conditions.

Another explanation for the improved performance between the first two assessments could be because of the variety of problems included in the assessment measure. As previously demonstrated in Seo and Ginsburg's study (1983), when students saw the equal sign in a variety of contexts in the Wynroth curriculum (1975), they learned to view the concept of the equal sign as a relational one. The assessment measure used in this study exposed students to a variety of canonical and non-canonical equations. Thus, the variety by itself could account for students' improved ability to solve non-canonical equations on subsequent administrations.

A third possible explanation for the improvement from Pretest 1 to Pretest 2 is maturation. In other words, students could have improved just because of development over time. I maintain, however, that my own data refute this explanation as no grade effects were found. More specifically, all the students, regardless of grade level, responded similarly to the items on the assessment measure at each of the three time points and in each condition. In addition, this explanation is not supported by the literature, which also demonstrates that throughout elementary and sometimes continued through high school, students have difficulty solving non-canonical equations without instruction focused directly on the meaning of the equal sign (Knuth, Stephens, McNeil & Alibali, 2006; Falkner, Levi, and Carpenter, 1999; Rittle-Johnson & Alibali, 1999). Thus, it does not appear that the meaning of the equal sign is acquired through processes of cognitive development alone, particularly given that the tests in the present study were administered only two weeks apart.

A final explanation for the improvement may be because the problems on the assessment measure were the same at both pretests and at posttest. There may have been a test effect – that is, students may have learned how to solve these particular problems. In most studies about students' understanding of equivalence and the equal sign, however, researchers have used the same problems in their assessment measures. Because these researchers have assumed that knowledge about the equal sign can be assessed using the same instruments at different time points, I used a similar methodology, but further research is necessary to determine whether this limits the internal validity of the study

In addition, the results revealed that students scored differently on the different types of problems on the test of equivalence used in this study, regardless of the condition they were in, averaged across all three time points. The students scored highest on canonical problems compared to each type of non-canonical problem, which is consistent with research that demonstrates students' inherent difficulty with non-canonical problems (Sherman & Bisanz, 2009), but also shows that their difficulty solving non-canonical problems was not due to the arithmetic involved.

The students also scored higher on Identity problems than on Part-Whole problems regardless of condition or time. This finding is similar to Sherman and Bisanz (2009), who found that students were more successful at solving Identity problems than part-whole problems. The authors explained that identity problems can be solved by making both sides of the equal sign "look" the same. In other words, students can detect that the same numbers appear on both sides of the equal sign, and can fill in the blank without having to use any arithmetic (e.g., **3** + 2 = **3** + ___). On the other hand, the authors argue, when working on part-whole problems, students are less likely to solve such problems correctly without the arithmetic necessary to calculate how to make both sides of the equation the same.

In addition, the study revealed that students' performance over time was moderated by problem type. More specifically, students' scores were near perfect on canonical problems at Pretest 1, and significantly higher than the scores on part-whole and identity problems at the same time point. As such, there was little room for improvement on canonical problems and in effect, the results indicated no gains over time on these problems. In contrast, students' performance on identity and part-whole problems improved significantly from Pretest 1 to Pretest 2 and from the second pretest to posttest.

Finally, the results of the experiment also revealed that students' conceptions of the equal sign as assessed by their written definitions did not change over time nor were they different as a function of condition. The data revealed that students who did not have a relational understanding of the equal sign at Pretest 1 did not demonstrate the relational meaning of the equal sign at Pretest 2 nor at Posttest. Moreover, the results demonstrated that most students who understood the equal sign as a relational symbol at Posttest had provided relational definitions on the first pretest. In other words, if students did not have a relational understanding of the equal sign at the beginning, they did not acquire one as a result of either the intervention or exposure to different types of equations on the test of equivalence. Thus, it seems that students improved on their ability to solve non-canonical equations, but did not seem to have acquired a relational understanding of the equal sign. This points to the discrepancy between being able to solve non-canonical open-number sentences and being able to articulate a relational understanding of the equal sign.

Limitations

Because improvement in students' ability to solve non-canonical equations was identified from Pretest 1 to Pretest 2, and that students' ability to solve non-canonical problems improved across all conditions, I can conclude that the exposure intervention had no effect and that students improved as a result of the exposure to the problems on the assessment measure. Thus, the assessment measure itself was a confounding variable because it included all three types of equations that were in the three conditions combined. Because of the measure, the students in the canonical condition were exposed to equations they I had not intended to expose them to (i.e., non-canonical equations). Further research using a different assessment method that would not expose students to different non-canonical equations would better test the effects of the exposure intervention.

Moreover, the sample used in this study only included students from private schools. There was a large number of students who were excluded from the study at the first pretest because they surpassed the threshold for inclusion in the study (i.e., they were able to correctly solve 6 or more non-canonical problems on the test). In fact, the number of students excluded from the study (43%) was larger than the reported rates of students unable to solve non-canonical problems in several previous studies (e.g., Carpenter, Franke & Levi, 2003; McNeil & Alibali, 2005). The rate in the present study may be explained by the increased time and resources that teachers typically have in private schools to teach students about mathematical concepts and symbols. To be able to generalize the results of the present study to the general population, therefore, it would be important to look at a larger sample that would also include students from public schools and from different socio-economic levels in the population.

Another suggested improvement to the study would be to run the exposure intervention for more than three sessions. Even though I observed an improvement in students' ability to solve non-canonical equations from Pretest 1 to Pretest 2 and from Pretest 2 to Posttest, I argue that it is still not sufficient improvement because the rate of success on non-canonical problems was still only at 40% at Posttest. Perhaps more exposure to non-canonical problems would further increase scores.

Furthermore, based on the results of the study, I cannot conclude that students' relational thinking had improved. This may indicate a weakness of the Definitions task used in this study as an assessment measure for relational thinking. Because the Definitions task only allowed students to give written definitions of the equal sign, I could not probe the students to better understand their answers. Therefore, I had to speculate and interpret their written responses, which limited my ability to assess whether they had relational thinking. Moreover, given their ages, the students made several spelling mistakes, which increased the difficulty in interpreting their written responses. An assessment measure administered in an individual interview, for example, which would allow me to probe their reasoning, would be better suited to assessing students' relational thinking.

Contributions and Implications

The present study was the first direct test of exposure to non-canonical equations and the equal sign in the absence of teacher intervention. Although other researchers have investigated the effects of exposure to non-canonical equations, their studies involved the students receiving feedback, and in some cases even lessons, that directly addressed the meaning of the equal sign in different contexts. For example, NcNeil et al. (2011) investigated students' understanding of equivalence after they solved a series of canonical and non-canonical equations, but the students received one-on-one instruction from the instructor, as well as feedback on their answers. In other studies, such as the one conducted by Rittle-Johnson and Alibali (1999), the authors found that students can learn to solve symbolic non-canonical equations accurately when they receive instruction, but the data do not speak to exposure without instructor intervention, which has been identified by other researchers, such as Ding and Li (2006) and McNeil et al. (2006), as a factor predictive of students' ability to solve non-canonical equations. The present study investigated whether actively engaging in solving symbolic equations, in the absence of any verbal statements by the teacher regarding the meaning of the equal sign, would help students acquire a relational meaning of the equal sign and more accurately solve non-canonical equations.

The finding that all students can improve their ability to solve non-canonical equations even without teacher intervention has important pedagogical implications. The present study provides teachers and mathematics instructors with a possible approach to helping students solve non-canonical problems in the early grades. More specifically, the findings suggest that if teachers present students with a greater number of non-canonical equations, then they can learn to accept those equations as appropriate and learn to solve non-canonical equations more accurately. Even though it is unclear whether students who are exposed to non-canonical equations will begin to see the equal sign as a relational symbol, the findings do suggest that exposure can impact their ability to solve such equations. This being said, more research is necessary to demonstrate that exposure can positively impact students' conceptions of the equal sign.

Furthermore, the findings of the present study have implications on material development for mathematics classroom activities and lessons. As seen in previous studies regarding the types of equations to which students in different cultures are exposed (McNeil. et al., 2006; Li et al., 2008), students in cultures where textbooks and teacher manuals include representations of the equal sign in non-canonical contexts are more likely to perceive the equal sign as a relational symbol. The findings of the present study suggest that when students can see the equal sign in non-canonical contexts and practice solving them, they also improve on their ability to solve non-canonical equations. This idea indicates that if student textbooks can present the equal sign more often in non-canonical contexts, students may also learn to solve such equations as a result. In addition, researchers (Ding & Li, 2006) have speculated that if teacher guidebooks include more non-canonical examples, the teachers will be more likely to present the equal sign in non-canonical contexts to their students. Thus, helping students become more aware of non-canonical equations, through classroom and textbook exposure, can assist them to learn to accept and solve non-canonical equations.

I believe that the strength of the present study is that it offers a way to help students to solve non-canonical equations while taking relatively less time from teachers than would one-on-one instruction or intensive mathematical discussions in the classroom. One-on-one instruction, for example, would require the teacher to make time to meet with each student individually to teach the relational meaning of the equal sign and how to solve non-canonical equations. Mathematical discussions in the classroom are also resource intensive because they require the teacher to ensure that each student understands the concepts discussed. Rather, exposing students to non-canonical equations through mathematics textbooks and classroom materials does not require the teacher to intervene in such intensive ways. As such, teachers would not need to compromise the time they allocate for the different topics in the mathematics curriculum. Teachers would be able to continue their instruction on the curriculum without having to allocate considerably more time to instruction on the equal sign.

Conclusion

Even though the exposure intervention likely had no effect on students' ability to solve non-canonical equations, I still believe that the findings indicate, albeit indirectly, that exposure to non-canonical equations improves students' ability to solve them. The students were exposed to both canonical and non-canonical equations during both pretest sessions. The fact that they showed improvement on their ability to solve non-canonical equations on the second pretest may be explained by the exposure they received on the assessment measure itself. This explanation would also mean that even very brief exposure -- to only 12 non-canonical problems -- would account for the improved performance. Although further research is needed, the results of the study appear to suggest that if teachers and textbooks can more frequently expose students to non-canonical equations, the students will learn to solve such equations with more accuracy.

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Appendix A

Equivalence Test

Version A

Name: ______________________________

(Please circle the correct answer)

7 + 5 + 1 = ___			
6	20	13	75

4 + 5 + 6 = ___ + 2			
15	13	17	19

2 + 3 = 2 + ___			
5	7	8	3

5 + 6 + 4 = ___+ 4			
15	11	19	9

4 + 5 = 3 + ___			
6	4	9	12

6 + 4 = ___			
10	14	6	64

5 + 6 = ___ + 6			
11	5	17	3

5 + 3 + 7 = 4 + ___			
15	11	19	6

4 + 5 = ___			
45	9	4	14

5 + 3 + 4 = 5 + ___			
2	17	7	12

6 + 7 = ___ + 5			
18	13	4	8

9 = ___			
19	15	5	9

5 + 3 = ___ + 3			
11	5	8	15

4 + 6 = 4 + ___			
6	10	14	9

7 + 8 + 1 = ___			
78	16	9	15

7 = ___			
2	17	7	9

Version B

Name: ______________________

(Please circle the correct answer)

4 + 5 = ___			
45	9	4	14

5 + 6 = ___ + 6			
11	5	17	3

6 + 4 = ___			
10	14	6	64

5 + 3 + 4 = 5 + ___			
2	17	7	12

7 + 8 + 1 = ___			
78	16	9	15

2 + 3 = 2 + ___			
5	7	8	3

9 = ___			
19	15	5	9

4 + 6 = 4 + ___			
6	10	14	9

6 + 7 = ___ + 5			
18	13	4	8

5 + 3 = ___ + 3			
11	5	8	15

7 + 5 + 1 = ___			
6	20	13	75

5 + 6 + 4 = ___ + 4			
15	11	19	9

5 + 3 + 7 = 4 + ___			
15	11	19	6

4 + 5 + 6 = ___ + 2			
15	13	17	9

4 + 5 = 3 + ___			
6	4	9	12

7 = ___			
2	17	7	9

Version C

Name: ___________________________________

(Please circle the correct answer)

7 + 5 + 1 = ___			
6	20	13	75

2 + 3 = 2 + ___			
5	7	8	3

9 = ___			
19	15	5	9

5 + 6 + 4 = ___+ 4			
15	11	19	9

5 + 3 = ___ + 3			
11	5	8	15

5 + 6 = ___ + 6			
11	5	17	3

7 = ___			
2	17	7	9

5 + 3 + 7 = 4 + ___			
15	11	19	6

4 + 5 = 3 + ___			
6	4	9	12

5 + 3 + 4 = 5 + ___			
2	17	7	12

4 + 5 + 6 = ___ + 2			
15	13	17	19

6 + 7 = ___ + 5			
18	13	4	8

4 + 6 = 4 + ___			
6	10	14	9

6 + 4 = ___			
10	14	6	64

7 + 8 + 1 = ___			
78	16	9	15

4 + 5 = ___			
45	9	4	14

Appendix B

Definitions Task

Definitions Task

2 + 7 = 9

↑

Tell me what this math symbol means.

You can write your answer here:

Appendix C

Exposure Intervention: Worksheets

Canonical Context Worksheet: Intervention 1

Name: ___________________________________

Look at these examples: 5 + 3 = ✎ __

1.	**7 + 1 = <u>8</u> ✓**	
2.	**2 + 3 = <u>5</u> ✓**	
3.	**8 + 2 = <u>9</u> ✗**	**8 + 2 = <u>10</u> ✓**
4.	**4 + 5 = <u>6</u> ✗**	**4 + 5 = <u>9</u> ✓**

Fill in the blank:

1 + 3 = __

7 + 3 = __

9 + 1 = __

3 + 5 = __

4 + 2 = __

8 + 1 = __

2 + 7 = __

4 + 3 = __

6 + 2 = __

1 + 6 = __

Canonical Context Worksheet: Intervention 2

Name: ___________________________

Look at these examples: 5 + 3 = ✎ __

1.	**6 + 1 = _7_ ✔**	
2.	**4 + 2 = _6_ ✔**	
3.	**7 + 2 = _3_ ✘**	**7 + 2 = _9_ ✔**
4.	**4 + 3 = _6_ ✘**	**4 + 3 = _7_ ✔**

Fill in the blank:

2 + 3 = __

6 + 1 = __

8 + 1 = __

2 + 5 = __

4 + 4 = __

8 + 2 = __

1 + 7 = __

1 + 3 = __

6 + 4 = __

2 + 6 = __

Canonical Context Worksheet: Intervention 3

Name: ___________________________________

Look at these examples: 5 + 3 = ✎ __

1.	3 + 1 = 4 ✓	
2.	2 + 4 = 6 ✓	
3.	3 + 5 = 9 ✗	3 + 5 = 8 ✓
4.	3 + 3 = 4 ✗	3 + 3 = 6 ✓

Fill in the blank:

6 + 3 = __

7 + 2 = __

4 + 1 = __

3 + 2 = __

1 + 3 = __

8 + 2 = __

2 + 2 = __

7 + 3 = __

5 + 2 = __

1 + 8 = __

Non- Canonical Identity Context Worksheet: Intervention 1

Name: ____________________

Look at these examples: 5 = ✎ __

1.	**7 = 7 ✓**	
2.	**2 + 3 = 2 + 3 ✓**	
3.	**8 = 9 ✗**	**8 = 8 ✓**
4.	**4 + 6 = 4 + 3 ✗**	**4 + 6 = 4 + 6 ✓**

Fill in the blank:

1 = __

7 = __

9 = __

3 + 5 = 3 + __

4 + 1 = 4 + __

3 = __

2 = __

7 + 1 = 7 + __

6 + 2 = 6 + __

1 + 6 = 1 + __

Non- Canonical Identity Context Worksheet: Intervention 2

Name: ______________________________

Look at these examples: 5 = ✎ __

1.	**6 = 6 ✔**	
2.	**2 + 1 = 2 + 1 ✔**	
3.	**9 = 7 ✘**	**9 = 9 ✔**
4.	**2 + 6 = 2 + 3 ✘**	**2 + 6 = 2 + 6 ✔**

Fill in the blank:

3 = __

7 + 1 = 7 + __

4 = __

2 + 5 = 2 + __

3 + 1 = 3 + __

8 = __

5 = __

1 + 1 = 1 + __

6 + 4 = 6 + __

3 + 7 = 3 + __

Non- Canonical Identity Context Worksheet: Intervention 3

Name: ___________________________________

Look at these examples: 5 = ✎ __

1.	**1 = <u>1</u> ✓**	
2.	**3 + 6 = 3 + <u>6</u> ✓**	
3.	**3 = <u>9</u> ✗**	**3 = <u>3</u> ✓**
4.	**2 + 5 = 2 + <u>7</u> ✗**	**2 + 5 = 2 + <u>5</u> ✓**

Fill in the blank:

6 = __

2 = __

8 = __

3 + 1 = 3 + __

7 + 2 = 7 + __

9 = __

3 = __

8 + 1 = 8 + __

5 + 2 = 5 + __

4 + 6 = 4 + __

Non- Canonical Part - Whole Context Worksheet: Intervention 1

Name: ____________________________

Look at these examples: 5 + 2 = ✎ __ + 1

1.	**7 + 2 = 3 + 6 ✓**	
2.	**2 + 3 = 4 + 1 ✓**	
3.	**8 + 1 = 3 + 4 ✗**	**8 + 1 = 5 + 4 ✓**
4.	**4 + 6 = 2 + 3 ✗**	**4 + 6 = 2 + 8 ✓**

Fill in the blank:

1 + 3 = __ + 2

7 + 1 = __ + 6

9 + 1 = __ + 4

3 + 5 = 7 + __

3 + 3 = 4 + __

3 + 2 = __ + 4

2 + 5 = __ + 6

7 + 3 = 5 + __

6 + 2 = 3 + __

1 + 6 = 3 + __

Non- Canonical Part - Whole Context Worksheet: Intervention 2

Name: ___________________________________

Look at these examples: 5 + 2 = ✎ __ + 1

1.	**7 + 1 = 2 + 6 ✓**	
2.	**3 + 3 = 4 + 2 ✓**	
3.	**8 + 2 = 3 + 4 ✗**	**8 + 2 = 6 + 4 ✓**
4.	**4 + 1 = 2 + 2 ✗**	**4 + 1 = 2 + 3 ✓**

Fill in the blank:

1 + 7 = __ + 5 7 + 3 = __ + 6

6 + 1 = __ + 4 3 + 6 = 7 + __

3 + 5 = 4 + __ 6 + 2 = __ + 4

5 + 5 = __ + 6 6 + 3 = 5 + __

4 + 2 = 3 + __ 1 + 5 = 3 + __

Non- Canonical Part - Whole Context Worksheet: Intervention 3

Name: ___________________________________

Look at these examples: 5 + 2 = ✎ __ + 1

1.	**6 + 2 = <u>4</u> + 4 ✓**	
2.	**7 + 3 = 4 + <u>6</u> ✓**	
3.	**7 + 1 = <u>3</u> + 4 ✗**	**7 + 1 = <u>4</u> + 4 ✓**
4.	**4 + 5 = 2 + <u>3</u> ✗**	**4 + 5 = 2 + <u>7</u> ✓**

Fill in the blank:

2 + 3 = __ + 1

8 + 1 = __ + 6

7 + 1 = __ + 6

3 + 6 = 7 + __

6 + 3 = 4 + __

3 + 5 = __ + 4

5 + 5 = __ + 6

7 + 3 = 5 + __

Appendix D

Definitions Task Rubric

Coding Rubric for Students' Definitions of the Equal Sign in Equivalence Test

Code	Description	Example
Relational	When the student provides a relational definition of the equal sign	"both sides have the same amount," " what is on the left side is the same as on the right sides", if the students draws a balance.
Operational	When the student provides an operational definition of the equal sign.	When the student uses words such as: "the answer", "total", "adds up to", "makes", "gives".
Combined	When the student provides both and operational and relational definition.	"It can mean that the answer comes next or tha it is the same amount on both sides"
No Answer	When the student does not provide an answer	
Equation	When the student writes down the same equation as in the question or another equation instead of the definition	" 2 + 7 = 9" "1 + 4 = 5"
Equal sign symbol	When the student writes the equal sign symbol instead of the definition.	"="
Equal	When the student writes in words the equal sign or words relation to equal.	"equal sign", "it means equals", "what it equals"