Sampled-data Networked Control Systems: A Lyapunov-Krasovskii Approach

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ABSTRACT

Sampled-data Networked Control Systems: A Lyapunov-Krasovskii Approach

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The main goal of this thesis is to develop computationally efficient methods for stability analysis and controller synthesis of sampled-data networked control systems. In sampled-data networked control systems, the sensory information and feedback signals are exchanged among different components of the system (sensors, actuators, and controllers) through a communication network. Stabilization of sampled-data networked control systems is a challenging problem since the introduction of multi-rate sample and holds, time-delays, and packet losses into the system degrades its performance and can lead to instability. A diverse range of systems with linear, piecewise affine (PWA), and nonlinear vector fields are studied in this thesis. PWA systems are a class of state-based switched systems with affine vector field in each mode. Stabilization of PWA networked control systems are even more challenging since they simultaneously involve switches due to the hybrid vector fields (state-based switching) and switches due to the sample and hold devices in the network (event-based switching).

The objectives of this thesis are: (a) to design controllers that guarantee exponential stability of the system for a desired sampling period; (b) to design observers that guarantee exponential convergence of the estimation error to the origin for a desired sampling period; and (c) given a controller, to find the maximum allowable network-induced delay that guarantees exponential stability of the sampled-data networked control system. Lyapunov-Krasovskii based approaches are used to propose sufficient stability and stabilization conditions for sampled-data networked control systems. Convex relaxation techniques are employed to cast the proposed stability

analysis and controller synthesis criteria in terms of linear matrix inequalities that can be solved efficiently.

To Zeinab, Masih, and Rashad

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I could not have finished this thesis without the unconditional love, support, and encouragement of my family. I am forever indebted to my parents, Zeinab and Masih,

and my brother, Rashad, who set examples of thirst for knowledge and dedication to work for me.

"The trouble is that the nature of research is forever to be doing something that we do not know how to do and, as soon as we have learned how to do it, to stop doing it and look for a new problem; this means that a researcher's mind is forever fixed on what has not been achieved–which, by the standards of the world, means being condemned to a life of perpetual discouragement. That this is not the way that we researchers perceive it is one of the great miracles of human creativity, and the primary reason that we love our work as much as we do."

Klopsteg memorial lecture:

"What science knows about violins–and what it does not know",

by Gabriel Weinreich, 1992

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LIST OF SYMBOLS

BMI	Bilinear matrix inequality
LKF	Lyapunov-Krasovskii functional
LMI	Linear matrix inequality
MASP	Maximum allowable sampling period
MAUP	Maximum allowable update period
PWA	Piecewise affine
PWL	Piecewise linear
$\mathrm{diag}(d_1, \ldots, d_m)$	Block diagonal matrix with diagonal entries $d_1, \ldots, d_m$
$\lambda_{\min}(.)$	The minimum eigenvalue of a matrix
$\lambda_{\max}(.)$	The maximum eigenvalue of a matrix
$\mathcal{X}$	State space
$\mathcal{R}_i$	Polytopic region in the state space
$\overline{\mathcal{R}}_i$	Closure of $\mathcal{R}_i$
$\otimes$	Kronecker product
$\star$	Symmetric entries of a symmetric matrix
0	The zero matrix of the appropriate dimension
I	The identity matrix of the appropriate dimension
$\mathbf{1}$	The column vector with all elements equal to 1
Z^T	Transpose of a matrix Z
$Z_1 > Z_2$ (or $Z_1 < Z_2$)	$Z_1 - Z_2$ is positive (or negative) definite
$Z_1 \geq Z_2$ (or $Z_1 \leq Z_2$)	$Z_1 - Z_2$ is positive (or negative) semi definite
$\succ$, $\prec$, $\succeq$, and $\preceq$	Elementwise inequalities
$\lvert . \rvert$	Norm of a vector

Chapter 1

Introduction

The methodology of this thesis is important to solve several problems in control systems where decision making subject to delay is present, including but not limited to teleoperated robotics (such as robotic surgery), centralized power grids, and control of the flow in canals. In such applications it is very difficult to determine what is the maximum allowable delay for receiving the sensing measurements so that the closed loop will be stable.

In networked control systems, sensory information and feedback signals are exchanged among different components of the system (i.e. sensors, actuators, and controllers) through a communication network. The reader is referred to [1–3] for applications of networked control systems to document printing, air vehicles and satellites, and to an inverted pendulum, respectively. As an example, in a modern long-range aircraft, there exist about 170 (Km) of signal wiring which account for almost 700 (Kg) of the weight of the aircraft [4]. Other than weight, the main drawbacks of wired communication links include connector/pin failures, cracked insulation issues, arc faults, and maintenance/upgrade difficulties [5]. The inherent benefits of wireless communication systems and the recent advancements in this field have led to a growing interest in wireless flight control systems (i.e. fly-by-wireless) [6]. However, the effects of non-ideal communication networks on stability and performance of the system become more prominent in the case of wireless communication networks [7] and motivate a thorough study of networked control systems.

Consider the networked control system illustrated in Fig. 1.1. The camera and the on-board inertial measurement unit (the sensors) send the sensory information to the computer (the controller) through wired and wireless (XBee modules) communication networks, respectively. The controller transmits the control signals through wireless communication to the microcontroller (Arduino) which in turn generates the

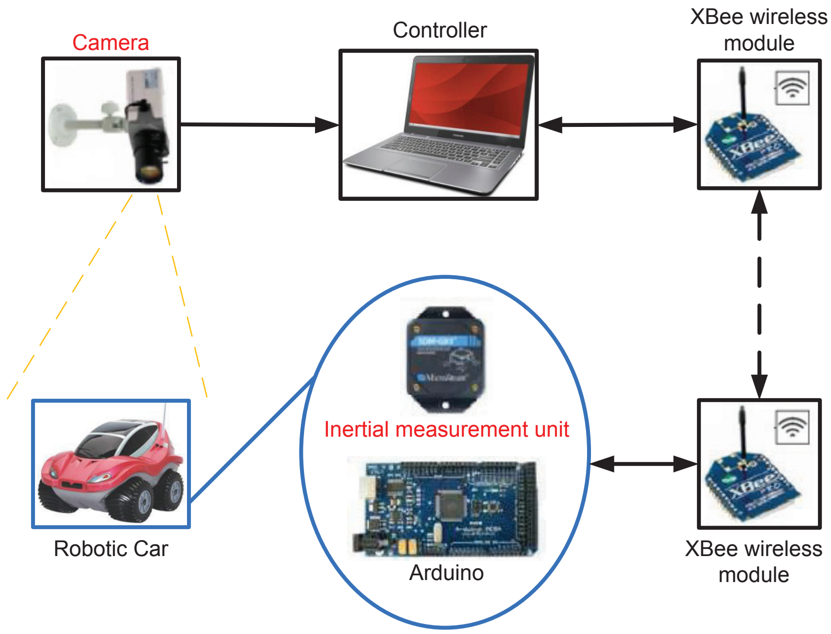


Figure 1.1: Networked control system: a robotic car example

actuating PWM[1] signals for the motors. The communication network introduces sample and holds, quantization, time-delays, data packet losses, and congestion into the system. From a control perspective, the addition of each of these phenomena can degrade the performance of the system and even lead to instability. The effects of the network on stability of the system are not always intuitive and neglecting them can have catastrophic consequences. Therefore, along with the advancements in wired and wireless communication networks, the study of networked control systems has attracted numerous researchers in the past decade (see [8–12] and the references therein).

Figure 1.2 illustrates the schematic diagram of a networked control system with a sensing block (with multiple samplers), an actuating block (with multiple zero order holds), and time delays and data packet dropouts in the communication links. In general, the samplers and the zero order holds work asynchronously. The sampling rates of the sensors and the update rates of the actuators can be different from each other, and can be uncertain and time-varying (e.g. sampling jitters [13, 14]). The sampler-controller delay and the controller-actuator delay are in general different, uncertain, and time-varying. The packet dropouts are modeled as a switch. When the switch is closed, data is transmitted through the network. When the switch is

[1]Pulse width modulation

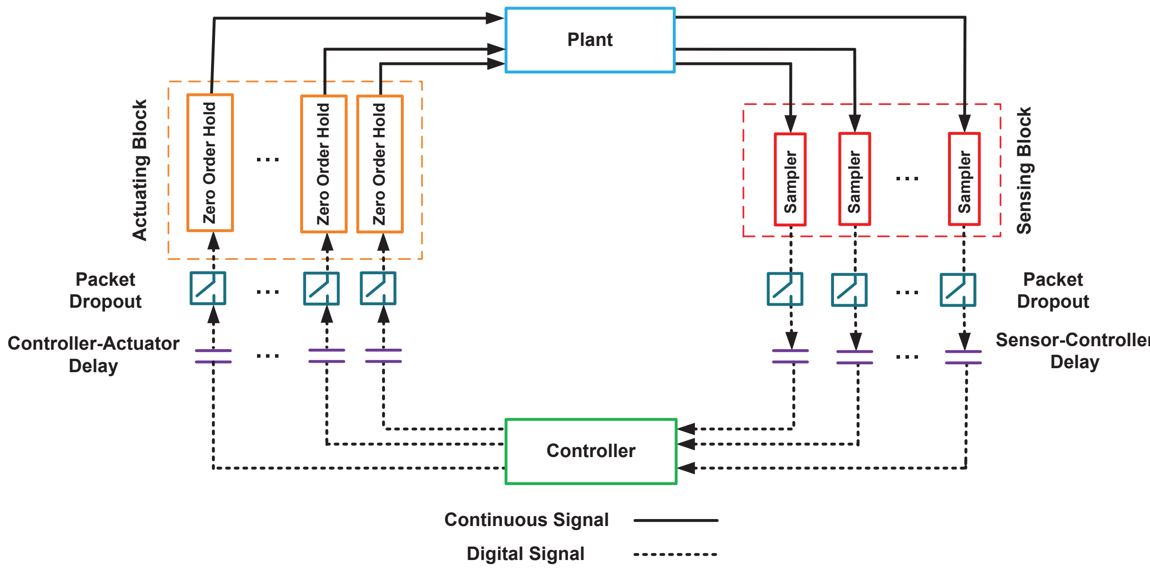


Figure 1.2: The schematic diagram of a networked control system

open, however, data is assumed to be dropped.

In this thesis, networked control systems are modeled as infinite dimensional time-delay systems where the vector field is a function of the current value of the state vector as well as its values in a past time interval. The sample and hold blocks and the data packet dropouts are modeled as time-varying delays in the control input (see Chapter 2). This approach is known as the *input delay modeling* in the literature [15] (see Subsection 1.2.3 for more details). We address sampled-data networked control systems with linear (Chapters 2-5, and 9), piecewise affine (PWA) (Chapters 6 and 7), and nonlinear (Chapter 8) vector fields.

The study of PWA systems is motivated by two factors. First, PWA and other switched linear systems are an important class of models that arise in many practical control applications (e.g. systems subject to saturation, hysteresis, and dead zones). Furthermore, the results will be used to explore nonlinear networked control systems, since PWA systems can approximate the nonlinearities that arise in the model. One of the objectives of this thesis is to bridge the gap between the relatively well studied linear networked control systems and the more complicated nonlinear networked control systems. The inherent nonlinear structure of PWA systems enables one to explore more realistic models of engineering applications.

1.1 Objectives

The main objective of this thesis is to propose computationally efficient stability and stabilization criteria for linear, PWA, and nonlinear sampled-data networked control systems. We address systems with multiple sampling rates, data packet losses, and time-delays. For stability analysis, the problem of finding a lower bound on the maximum allowable sampling period (MASP) that guarantees exponential stability will be cast as an optimization program in terms of linear matrix inequalities (LMIs). The resulting LMIs can be solved efficiently using available software packages [16, 17]. For controller synthesis, the problem of finding a state feedback controller that guarantees exponential stability for a desired MASP will be cast as a feasibility problem in terms of LMIs. Furthermore, as the dual of the sampled-data controller synthesis problem, algorithms will be provided for sampled-data observer design. Note that convex optimizations over LMIs are solvable in polynomial time. For example, the computational complexity of the solver SeDuMi [16] is in $\mathcal{O}(n^2m^{2.5} + m^{3.5})$, where n is the number of decision variables and m is the number of rows of the LMIs [18]. In the next section, we review the research contributions in sampled-data networked control systems.

1.2 Literature Review

Networked control systems have attracted numerous research contributions in the past decade. The special issues published on networked control systems in journals such as IEEE Control Systems Magazine (Feb. 2001), IEEE Transactions on Automatic Control (Sept. 2004), and Proceedings of the IEEE (Jan. 2007) are evidence to the growing interest in these systems. In networked control systems (as well as sampled-data systems and time-delay systems, as special cases of networked control systems), the vector field is defined as a function of the current and the past values of the state vector. *Retarded functional differential equations* [19, 20] are widely used as a framework for modeling, stability analysis, and controller synthesis of deterministic and stochastic networked control systems (see [19–21] and the references therein). The main objective of this section is to address the previous work on networked control systems. However, we begin the literature review by introducing PWA systems. Next, frequency-domain approaches to study of sampled-data networked control systems are presented. Then, time-domain approaches are presented and their advantages and weaknesses are discussed. Finally, the section ends with a few concluding remarks.

1.2.1 Piecewise affine systems

PWA systems are a class of state-based switched systems where the vector field is affine in each mode or region. PWA systems arise in many engineering problems (e.g. systems with saturation, deadband, and hysteresis). PWA systems have been used as a tool for approximating nonlinear systems for a few decades (see [22, 23] and the references therein). Stability analysis and controller synthesis of PWA systems have received an increasing number of contributions since the late nineties. The reader is referred to [24–29] for stability analysis, to [26, 29–32] for controller synthesis and to [23, 33] for observer based control and output feedback control of PWA systems in continuous-time. Reference [34] addresses stability and performance analysis for PWA systems. The Hamilton-Jacobi-Bellman equation and convex optimization methods are used to obtain lower and upper bounds for the optimal control cost function. A unified dissipativity approach for stability analysis of piecewise smooth (and PWA) systems with continuous and discontinuous vector fields is presented in [28]. Stability of PWA systems is addressed in [26, 29] using quadratic and piecewise quadratic Lyapunov functions. Furthermore, piecewise linear (PWL) state feedback controllers are designed for stabilizing PWA systems by solving an optimization problem in terms of LMIs in [29]. Reference [30] shows that PWA state feedback controller synthesis for PWA slab systems based on a quadratic Lyapunov function can be cast as a set of quasi-concave optimization problems analytically parameterized by a vector. Reference [32] uses this result to provide a set of sufficient conditions for PWA controller synthesis for PWA slab systems in terms of LMIs.

1.2.2 Frequency-domain approaches to sampled-data networked control systems

Frequency-domain approaches to sampled-data networked control systems are based on studying the characteristic quasipolynomial of the system. These approaches study the zero-crossing frequencies of the characteristic quasipolynomial using classical control theory techniques such as the Routh-Hurwitz criterion. More systematic approaches include the frequency-sweeping tests, the constant matrix tests, and the small gain theorem [20, 35]. For single input single output systems with known delay, Smith predictor [36] provides a controller synthesis technique. Ignoring the delay, it first designs a controller for the delay-free system. Next, it defines a new compensator such that the closed-loop transfer function of the system with delay is equivalent to the transfer function of the delay-free system. This procedure is sensitive to delay uncertainty and works only when the delay is perfectly known (see [36] and the references therein).

The advantages of the frequency-domain approaches are their "conceptual simplicity and computational ease [20]". Nevertheless, the frequency-domain approaches are not suitable for the case of uncertain networked control systems and systems with time-varying delays. While these approaches work well for the case of multiple commensurate delays[2], their extension to the case of incommensurate delays leads to conservative stability criteria. These criteria are usually a paraphrased version of the stability definition itself and cannot be implemented in an optimization software. Time-domain approaches to study of sampled-data networked control systems are presented in the following two subsections.

1.2.3 Time-domain approaches to sampled-data networked control systems

This subsection begins with the literature that focuses on sampled-data systems. Next, the research papers that consider more complicated network structures are presented.

Sampled-data systems

According to [37, 38], there are three main approaches to sampled-data controller synthesis. In the emulation approach, a continuous-time controller is designed based on the continuous-time plant, then approximated in discrete-time, and finally implemented via a sample and hold device. In this method, the controller can easily be designed based on performance specifications. The performance, however, is only guaranteed for sufficiently high sampling frequencies. In other words, the maximum allowable sampling period (MASP) should be sufficiently small. This results in a trade-off between performance and the cost of sensing equipment. In the second approach, the discrete-time controller is designed based on an approximate discretized model of the plant. The advantage of this approach is its simplicity at the cost of ignoring the inter-sample behaviour of the system. A common drawback of the first two approaches is that the "exact discrete-time models of continuous-time nonlinear processes are typically impossible to compute" [39, 40]. Finally, the direct sampled-data design approach is more mathematically involved because it addresses

[2]Where the delays are commensurate, the ratios of all delays are rational numbers

the continuous-time plant and the discrete-time control signal simultaneously. Its advantage, however, is that the approximation step in the other two approaches is obviated.

A general framework for the design of nonlinear controllers using the emulation approach is presented in [40]. First, a dissipation property is used to design a continuous-time controller. Next, the authors propose conditions that should be satisfied by the approximate discretized controller in order to preserve the dissipation property. Following the emulation approach, reference [41] addresses input-to-state stability of nonlinear systems with dynamic sampled-data controllers. A controller redesign scheme can later be used to improve the performance of the designed controller [42, 43].

For a discrete-time controller design based on an approximate discrete-time model of the plant, the reader is referred to [38, 39, 44] and the references therein. First, a parametrized family of approximate discrete-time models of the plant is developed. Next, a corresponding family of discrete-time controllers is designed for the approximate models. Reference [39] provides conditions to guarantee that the exact nonlinear sampled-data system is stable for sufficiently small modeling parameters and uniform samplings. As mentioned earlier, ignoring the inter-sample behaviour is a drawback of this approach. One way to address this issue is the lifting technique [37], where the closed-loop sampled-data system is modeled as a finite dimensional discrete-time system. The reader is referred to [45] for a study of sampled-data tracking problems and to [46] for H_∞ sampled-data control using the lifting technique.

The direct sampled-data design approach has recently gained an increasing interest in the literature of linear sampled-data systems (see [15, 47–49] and the references therein). In this approach, the sampled-data system is usually modelled as either a continuous-time system with a time-varying *input delay* [15, 47] or a *hybrid* (impulsive) system with jumps at the sampling instants [48]. Razumikhin or Krasovskii-type theorems [20] are then used to develop sufficient stability and stabilization conditions for the sampled-data system. These conditions are usually cast in terms of linear matrix inequalities (LMIs) which can be efficiently solved using software packages such as SeDuMi [16] and YALMIP [17]. While the Razumikhin-type theorems are based on classical Lyapunov *functions*, Krasovskii-type theorems use Lyapunov *functionals* and are known to be less conservative [9, 15, 20]. For direct sampled-data design of linear systems using the lifting technique the reader is referred to [37].

There are scarce references in the literature of nonlinear sampled-data systems where the *input delay* model (for static controllers) [41] or the *hybrid* model [50–52] of the system is studied. In all these references, however, a continuous-time controller is assumed to be available. In other words, the controller synthesis is performed while ignoring the sample and hold structure of the feedback. Therefore, similar to the emulation approach, references [41, 50–52] cannot be used to design controllers that provide a desired MASP.

Stability and stabilization of PWA sampled-data systems are challenging problems since the resulting hybrid systems simultaneously involve state-based switching (due to the PWA vector field) and event-based switching (due to the sampling). Given a PWA plant and a stabilizing continuous-time controller, references [53, 54] study the stability of the closed-loop PWA system in a sampled-data framework. Assuming uniform sampling intervals, reference [53] uses a quadratic Lyapunov function to provide sufficient conditions for convergence of the PWA sampled-data system to an invariant set containing the origin. Following the input-delay approach and using Krasovskii functionals, reference [54] addresses the same stability problem for the case of samplers with unknown nonuniform sampling intervals. References [55, 56] address optimal control of PWA sampled-data systems. However, the PWA sampled-data structure discussed in [55, 56] is different from the one in this thesis. In [55, 56], the switching is only event-based (i.e. occurs at the sampling instants), whereas in this thesis the switching is both state-based and event-based.

Similar approaches have been used in the literature to address networked control systems. This topic is studied in the next subsection.

Networked control systems

In a networked control system, a continuous-time plant is in feedback with a discrete-time emulation of a controller. The control signal is computed using state measurements that are sampled in intervals that are not necessarily uniform [3, 47, 48]. These signals go through a quantization process [57], and experience uncertain and time varying delays [58, 59], data packet dropouts, and congestion over the communication network. The main approaches for studying networked control systems include the lifting approach [37, 45, 60, 61], the impulsive model approach [1, 11, 48, 62], and the input delay approach [12, 15, 47, 63, 64]. These approaches were discussed in more detail in the previous subsection.

Most of the work in the literature focuses on only one aspect of networked control systems. There are papers, however, that study two or more features of an networked control system at the same time. Reference [2] studies H_∞ control of a class of uncertain stochastic networked control systems with both delays and packet dropouts.

Sufficient conditions are proposed to ensure exponential stability in mean square of the closed-loop system subject to a performance measure. The robust filtering problem is addressed in [65] for a class of discrete-time uncertain nonlinear networked systems with both multiple stochastic time-varying communication delays and multiple packet dropouts. A method for designing a linear full-order filter is proposed such that the estimation error converges to zero exponentially in the mean square sense while the disturbance rejection attenuation is constrained to a given level. Reference [66] studies the distributed finite-horizon filtering problem for a class of time-varying systems over lossy sensor networks with quantization errors and successive packet dropouts. Through available output measurements from a sensor and its neighbors (according to a given topology), a sufficient condition is established for the desired distributed finite-horizon filter to ensure that the prescribed average filtering performance constraint is satisfied.

The networked control system considered in [67] comprises a linear sampled-data controller and an uncertain, time varying delay. Two drawbacks of that model are that the sampling intervals are assumed to be constant and the delay is assumed to be upper bounded by the sampling period. A more general model of networked control systems is studied in [11, 12], where a linear sampled-data controller with uncertain sampling rates, the possibility of data packet dropouts, and an unknown, time varying delay are considered. While the stability theorems in [12] are less conservative than the corresponding theorems in [11], they are more computationally expensive as they involve solving two times as many LMIs. Moreover, due to the complexity of the LKF in [12], the number of LMIs increases even more if additional information on the time varying delay (e.g. a lower bound) is available.

Reference [68–71] address the stability problem of PWA systems with time-delays. The PWA networked control system in [68] constitutes of a time-varying delay in the linear term. Reference [69] considers a more general model where the delay also appears in the affine term of the vector field. Robust stability of PWA systems is also addressed. The authors provide sufficient Krasovskii-based criteria for asymptotic stability of PWA systems with time-delays and formulate them as a set of LMIs. The results are extended in [71] to PWA systems with structured uncertainties. Reference [70] uses the PWA time-delay framework to design a controller for an automotive all-wheel drive clutch system. The main drawbacks of these papers, however, is a restricting assumption on the derivative of the delay. In those papers the time derivative of the delay is assumed to be strictly less than one. While this assumption is fairly standard in the time-delay systems literature, the obtained results are not applicable to a general networked control framework with sample and hold blocks, where the derivative of the delay is equal to one.

1.2.4 Concluding remarks

Exponential stability and stabilization problems for linear networked control systems have received numerous research contributions. However, these problems are considered as open problems in the case of PWA and nonlinear networked control systems. We believe that PWA systems are powerful tools in order to bridge the gap between the relatively well studied linear networked control systems and the more complicated nonlinear networked control systems. Furthermore, stability and stabilization of networked control systems with multiple communication links (that inevitably experience samplings at different rates and different time-delays) are considered as open problems even in the case of linear systems. Motivated by these conclusions, the main contributions of the thesis are summarized in the following section.

1.3 Contributions

The main contributions of the thesis are as follows.

1. To propose sufficient stability criteria for exponential stability of linear, PWA, and (a class of) nonlinear single rate sampled-data networked control systems. PWA differential inclusions are used to address the stability analysis problem for a class of nonlinear systems. For the first time, sufficient conditions for exponential stability of PWA and nonlinear sampled-data systems are presented using a piecewise smooth Krasovskii functional. This decreases the conservativeness of the proposed sufficient conditions when compared with the use of smooth Krasovskii functionals. The stability criteria are used to formulate the problem of finding a lower bound on the MASP as an optimization program in terms of LMIs. These LMIs can be solved efficiently using available software packages.

   Importance: The developed theorems allow one to answer questions such as what should the sampling frequency of a sensor in a sampled-data networked control system be such that exponential stability is guaranteed? or what is the maximum number of consecutive data packet dropouts that does not lead to instability of the sampled-data networked control system? or what is the maximum delay in the communication link that does not result in instability of the sampled-data networked control system?

2. To propose sufficient stabilization criteria for exponential stability of linear, PWA, and (a class of) nonlinear single rate sampled-data networked control systems. The controller synthesis technique is based on the direct sampled-data design approach. To the best of the author's knowledge, the direct sampled-data design approach was not applied to PWA and nonlinear systems before. Using this approach the controller is designed with the MASP as an optimization parameter. The controller design problem usually leads to non-convex optimization problems. Therefore, convex relaxation techniques are used in the derivation of the sufficient Krasovskii-based stabilization criteria. The stabilization criteria are used to formulate the problem of finding a controller gain that guarantees exponential stability (for a desired MASP) as a feasibility program in terms of LMIs. These LMIs can be solved efficiently using available software packages.

   Importance: The consideration of the desired MASP as an optimization parameter guarantees that the designed controller satisfies the requirements dictated by the sensing equipment. The developed theorems allow one to answer questions such as how should the controller gains be modified in order to increase the MASP for each sensor and increase the allowable number of consecutive data packet dropouts in the communication link?

3. To propose sufficient conditions for design of linear multi-rate sampled-data observers. Given the MASP for each sensor, sufficient Krasovskii-based conditions are presented for design of linear observers. The designed observers guarantee exponential convergence of the estimation error to the origin. The sufficient conditions are cast as a set of LMIs that can be solved efficiently. Furthermore, given an observer gain, the problem of finding MASPs that guarantee exponential stability of the estimation error is also formulated as a convex optimization program in terms of LMIs.

   Importance: The continuous-time state estimation problem using asynchronous multi-rate discrete-time output measurements is a practically relevant problem. The developed theorems allow one to design observers that can be used in output feedback control of sampled-data networked control systems.

4. To propose stability and stabilization criteria for linear multi-rate sampled-data

systems. The proposed sampled-data scheme comprises a sensing block with several sensors and an actuating block with several actuators which is more general than previous work in the literature. For each sensor (or actuator), the problem of finding an upper bound on the lowest sampling frequency (or refresh rate) that guarantees exponential stability is cast as an optimization problem in terms of LMIs.

Importance: The proposed sampled-data scheme finds application in asynchronous multi-agent systems and systems with actuators that have relatively low update rates (e.g. solenoids, electric cylinders, electroactive polymers, and shape memory alloys).

5. To propose a sensor allocation strategy that guarantees exponential stability of linear multi-rate sampled-data systems. The state vector is partitioned and each part of the state vector is sampled by a dedicated sensor. The proposed Krasovskii-based sufficient stability conditions yield a partition of the state vector such that exponential stability is guaranteed. The problem of finding such a partition is cast as a mixed integer program subject to LMIs.

   Importance: The developed theorems allow one to answer questions such as which states should be sampled at a higher rate and which states should be sampled at a lower rate? or can increasing the sampling rate of a phenomenon in an already stable sampled-data system lead to instability? The answer to the latter question is not intuitive as will be shown in Chapter 4.

1.3.1 Publications

The main contributions of the thesis are documented in the following publications.

Journal papers

1. **M. Moarref** and L. Rodrigues, "Stability and stabilization of linear sampled-data systems with multi-rate samplers and time driven zero order holds", *submitted*.

2. **M. Moarref** and L. Rodrigues, "Sampled-data piecewise affine differential inclusions", *submitted*.

3. **M. Moarref** and L. Rodrigues, "On exponential stability of linear networked control systems", International Journal of Robust and Nonlinear Control, in

press.

4. **M. Moarref** and L. Rodrigues, "Asymptotic stability of sampled-data piecewise affine slab systems", Automatica, vol. 48, 2012, pp. 2874–2881.

Conference papers

1. **M. Moarref** and L. Rodrigues, "Observer design for linear multi-rate sampled-data systems", submitted to the American Control Conference, Portland, OR, 2014.

2. **M. Moarref** and L. Rodrigues, "A convex approach to stabilization of sampled-data piecewise affine slab systems", in proceedings of the 52^{nd} IEEE Conference on Decision and Control, Florence, Italy, 2013, pp. 4748–4753.

3. **M. Moarref** and L. Rodrigues, "Exponential stability and stabilization of linear multi-rate sampled-data systems", in proceedings of the American Control Conference, Washington, DC, 2013, pp. 158–163.

4. **M. Moarref** and L. Rodrigues, "Asymptotic stability of piecewise affine systems with sampled-data piecewise linear controllers", in proceedings of the 50^{th} IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, 2011, pp. 8315–8320.

1.4 Structure of the Thesis

The structure of the thesis is shown in Fig. 1.3. Chapter 2 addresses linear sampled-data systems and lays the foundation upon which more complex systems are addressed. These systems experience multi-rate samplings, have switched or nonlinear dynamics, and have non ideal communication links with delays and packet dropouts. Preliminary notions on functional spaces and LKFs are also provided in Chapter 2. In Chapter 3 linear multi-rate sampled-data systems are studied. The results of this chapter are extended in Chapter 4 where sensor allocation strategies are proposed which guarantee exponential stability of linear multi-rate sampled-data systems. The observer design problem for linear multi-rate sampled-data systems is covered in Chapter 5. Stability analysis and controller synthesis of PWA sampled-data systems are addressed in Chapters 6 and 7, respectively. The results of the latter two chapters are used in Chapter 8 to provide sufficient stability and stabilization criteria

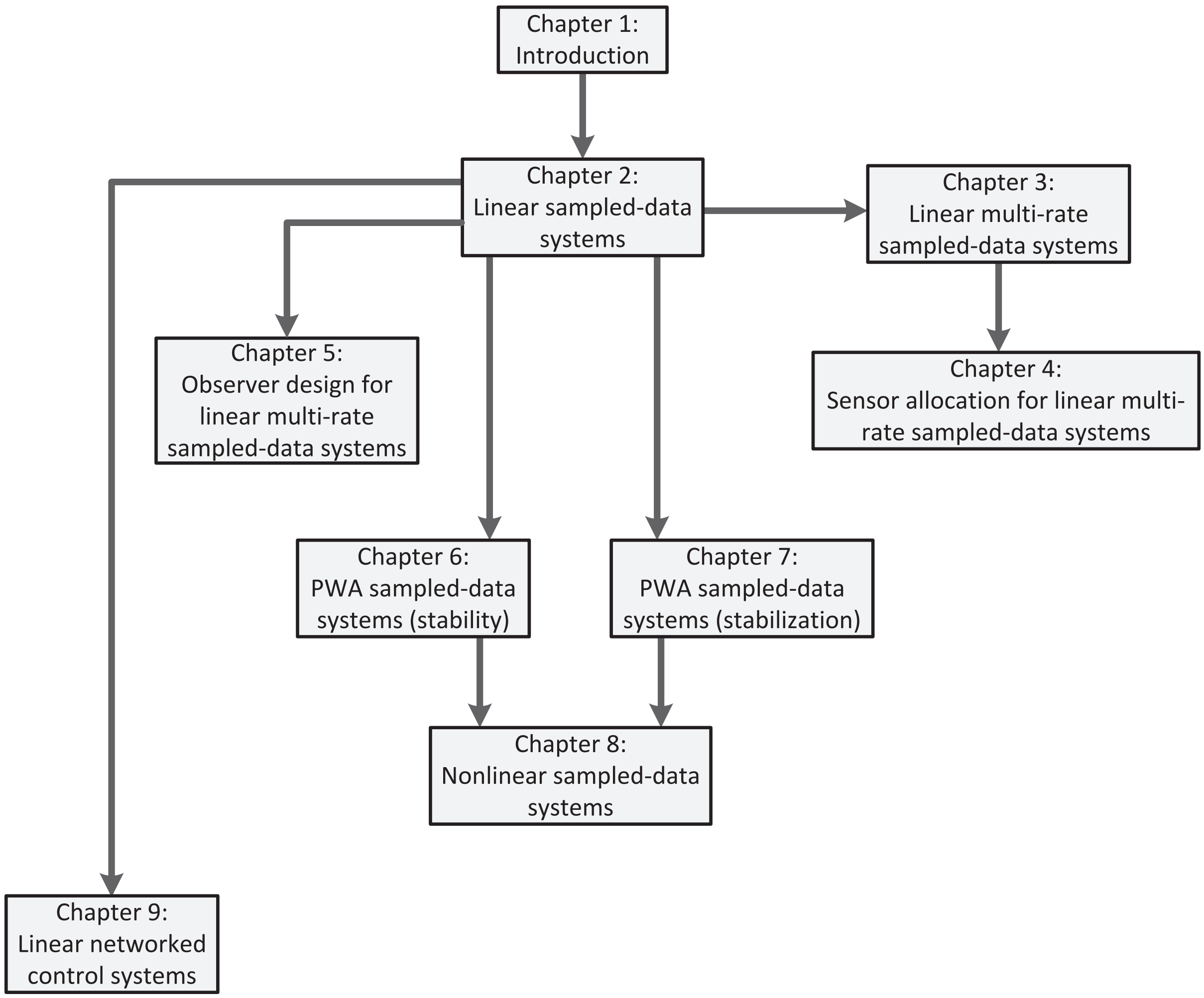


Figure 1.3: Structure of the thesis

for nonlinear sampled-data systems. Finally, Chapter 9 addresses linear networked control systems with sample and hold blocks, time-delays, and data packet losses. Concluding remarks are presented in Chapter 10.

Chapter 2

Linear Sampled-data Systems

The main objective of this chapter is to present the main ideas and the methodology that will be used throughout the thesis. To this end, we address stability analysis and controller synthesis of linear single rate sampled-data networked control systems which is the simplest network structure studied in this thesis. This chapter lays the foundation upon which more complex systems are addressed. These complex systems experience multi-rate samplings (Chapters 3-5), have switched (Chapters 6 and 7) or nonlinear (Chapter 8) dynamics, and have non ideal communication links with time-varying delays (Chapter 9). Preliminary definitions and notions about stability of functional differential equations and Lyapunov-Krasovskii functionals (LKFs) are also presented in this chapter. In terms of notation, throughout this thesis and where there is no confusion a vector $x(t)$ is simply written as x.

2.1 Introduction

Stability and stabilization of linear sampled-data systems has been the subject of numerous research [15, 37, 47–49, 61, 62, 64, 72]. In these systems a continuous-time linear plant is controlled by a linear controller which is located in the feedback loop between a sampler and a zero order hold. Furthermore, it is assumed that the communication link between the sampler and the controller experiences data packet dropouts (see Fig. 2.1). The main approaches for studying linear sampled-data systems include the lifting approach [37, 60, 61], the impulsive model approach [48, 62, 72], and the input delay approach [15, 47, 64].

In the lifting approach, the closed-loop sampled-data system is modeled as a finite dimensional discrete-time system. Lifting is used in studying systems with constant or uncertain sampling rates [49]. However, the lifting approach is not applicable to

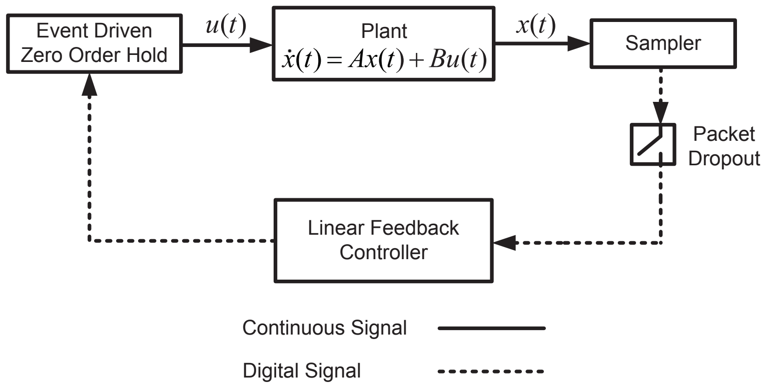


Figure 2.1: The schematic diagram of a linear single rate sampled-data networked control system.

systems with uncertain parameters. In the impulsive model approach, the closed-loop sampled-data system is modeled as an impulsive system which exhibits continuous state evolutions (described by ordinary differential equations) and instantaneous state jumps. In the input delay approach, the linear sampled-data system is modeled as a continuous-time system with a delayed control input. Both the impulsive model and input delay approaches use Razumikhin or Krasovskii-type [20] theorems to prove stability of sampled-data systems. While the Razumikhin-type theorems are based on classical Lyapunov *functions*, Krasovskii-type theorems use Lyapunov *functionals* and are known to be less conservative [9, 15, 20].

The evolution of LKFs over the past decade has yielded less conservative stability conditions. These conditions are usually cast in terms of LMIs which can efficiently be solved using software packages such as SeDuMi [16] and YALMIP [17]. In [15], neglecting the saw-tooth structure of the delay in sampled-data systems, robust stability and stabilization conditions were presented based on a time-delay model with bounded delay. Reference [72], addressed this issue by introducing functionals that took the saw-tooth property into account. The proposed LKF in [72], was further modified in [47, 64], where less conservative conditions were provided.

The main contribution of this chapter is to propose new sufficient conditions for exponential stability and stabilization of linear sampled-data systems. The new sufficient conditions are derived based on a modified LKF. The sufficient conditions compare favorably with other sufficient conditions available in the literature. The problem of finding a lower bound on the MASP that preserves exponential stability is formulated as an optimization program in terms of LMIs. The controller synthesis problem is cast as an optimization problem subject to LMIs with the MASP as a parameter. The results are also extended to the case of linear sampled-data systems with uncertain parameters.

The chapter is organized as follows. Problem formulation and preliminary notions on LKFs are provided in Section 2.2. The stability analysis and controller synthesis theorems are presented in Sections 2.3 and 2.4, respectively. The results are applied to benchmark examples in Section 2.5. Concluding remarks are presented in Section 2.6.

2.2 Problem Formulation and Preliminaries

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector, $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, and $u \in \mathbb{R}^{n_u}$ is the control input. Let a continuous-time linear controller for (2.1) be defined by

$$u(t) = Kx(t),$$

where $K \in \mathbb{R}^{n_u \times n_x}$. Assume that the state vector is measured at sampling instants t_n, $n \in \mathbb{N}$. Without loss of generality, by the index $n \in \mathbb{N}$, we denote only the instants t_n for which a data packet is not lost. Therefore, the control input can be rewritten as

$$u(t) = Kx(t_n), \text{ for } t \in [t_n, t_{n+1}). \tag{2.2}$$

The time elapsed since the last sampling instant is represented by a sawtooth function (see Fig. 2.2) defined as

$$\rho(t) = t - t_n, \text{ for } t \in [t_n, t_{n+1}), \tag{2.3}$$

and the largest sampling interval is denoted by

$$\tau = \sup_{n \in \mathbb{N}} (t_{n+1} - t_n). \tag{2.4}$$

Therefore,

$$\rho(t) < \tau. \tag{2.5}$$

Considering (2.3), the control signal (2.2) is rewritten as

$$u(t) = Kx(t - \rho(t)). \tag{2.6}$$

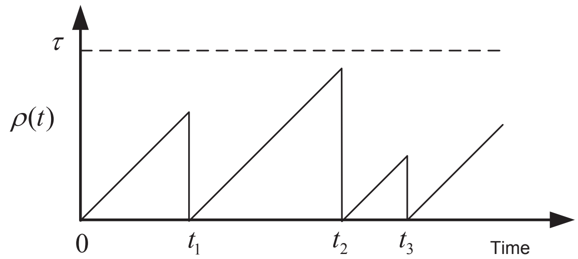


Figure 2.2: The sawtooth function $\rho(t)$.

The following assumption models the fact that two sampling instants cannot occur simultaneously in practice. It is used in the proof of the main results to rule out the occurrence of the Zeno phenomenon.

Assumption 2.1. *There exists $\epsilon > 0$ such that $t_{n+1} - t_n > \epsilon$ for any $n \in \mathbb{N}$.*

In this thesis, we follow the input delay approach to model sampled-data networked control systems as retarded functional differential equations [19, 20]. Preliminary notions about retarded functional differential equations are presented in the next subsection.

2.2.1 Stability in the Functional Space

Let $\mathcal{W}([-\tau, 0], \mathcal{X})$ be the space of absolutely continuous functions[1] with square integrable first-order derivatives mapping the interval $[-\tau, 0]$ to $\mathcal{X} \subseteq \mathbb{R}^n$. Consider the function $x_t \in \mathcal{W}$ defined as

$$x_t(r) = x(t + r), \quad -\tau \leq r \leq 0. \tag{2.7}$$

Similar to [47], we denote the norm of x_t by

$$||x_t||_{\mathcal{W}} = \max_{r \in [-\tau, 0]} |x_t(r)| + \left[\int_{-\tau}^{0} |\dot{x}_t(r)|^2 \,\mathrm{d}r \right]^{\frac{1}{2}}. \tag{2.8}$$

We will use the definition (2.8) for the norm of x_t throughout this thesis, unless stated otherwise. The general form of a retarded functional differential equation [19, 20] is

[1] [73] A function $g(x)$ is absolutely continuous if and only if g has a derivative g' almost everywhere, the derivative is Lebesgue integrable (i.e. $\int |g| \,\mathrm{d}\mu$ is finite, where μ is a measure), and $g(x) = g(a) + \int_a^x g'(t) \,\mathrm{d}t$ for all x on the compact interval $[a, b]$.

written as

$$\dot{x}(t) = f(t, x_t),$$

where $x(t) \in \mathcal{X}$ and $f : \mathbb{R} \times \mathcal{W} \to \mathbb{R}^n$ is a real valued vector defined on the product set of real numbers and absolutely continuous functions. In other words, the evolution of the state vector $x(t)$ is a function of time t and the state vector $x(s)$ at times $t - \tau \leq s \leq t$. In this thesis, sampled-data networked control systems are modeled as retarded functional differential equations.

Definition 2.1. *The solution $x(t)$ of the system $\dot{x} = f(t, x_t)$ is said to be locally uniformly exponentially stable with decay rate λ if there exist $\Omega \subseteq \mathcal{W}([-\tau, 0], \mathcal{X})$, $\delta > 0$, and $\lambda > 0$, such that for any initial condition $x_0 \in \Omega$, the solution $x(t)$ is defined in $\mathcal{X}$ for all $t \geq 0$ and satisfies*

$$|x(t)| \leq \delta e^{-\lambda t} ||x_0||_{\mathcal{W}}. \tag{2.9}$$

Moreover, if (2.9) is verified, the state space $\mathcal{X}$ is equal to $\mathbb{R}^{n_x}$, and $\Omega = \mathcal{W}([-\tau, 0], \mathbb{R}^{n_x})$, then the solution is globally uniformly exponentially stable.

The decay rate λ can be considered as a measure of the performance of the sampled-data networked control system. Next, a Krasovskii-type theorem is presented to prove stability of retarded functional differential equations. This theorem is adapted from similar theorems in [47, 48].

Theorem 2.1. *Given $\lambda > 0$, the solution $x(t)$ of the system $\dot{x} = f(t, x_t)$ is globally uniformly exponentially stable with a decay rate greater than $\lambda/2$ if there exists a functional $V(t, x_t)$, differentiable for all $t \neq t_n$, $n \in \mathbb{N}$, and a finite integer q, satisfying*

$$c_1 |x_t(0)|^2 \leq V(t, x_t) \leq c_2 ||x_t||^2_{\mathcal{W}}, \tag{2.10}$$

$$V(t_n, x_{t_n}) \leq V(t_n^-, x_{t_n^-}), \ \forall n \in \mathbb{N}, \tag{2.11}$$

$$\dot{V}(t, x_t) + \lambda V(t, x_t) < 0, \ \forall t \neq t_n, \ n \in \mathbb{N}, \tag{2.12}$$

$$0 < \epsilon < t_{n+q} - t_n, \ \forall n \in \mathbb{N}, \tag{2.13}$$

where c_1, c_2, and ϵ, are positive scalars and $V(t_n^-, x_{t_n^-}) = \lim_{t \nearrow t_n} V(t, x_t)$.

Proof. Solving (2.12) for $t \in (t_n, t_{n+1})$ and using (2.11) yields

$$V(t, x_t) \leq e^{-\lambda(t - t_n)} V(t_n, x_{t_n}) \leq e^{-\lambda(t - t_n)} V(t_n^-, x_{t_n^-}) \leq \ldots \leq e^{-\lambda t} V(0, x_0).$$

The Krasovskii functional V strictly decreases in intervals (t_n, t_{n+1}) that have a nonzero length (note that, according to Assumption 2.1, any interval (t_n, t_{n+1}), $n \in \mathbb{N}$, has a length of at least $\epsilon > 0$). Now, inequality (2.10) yields

$$|x(t)| = |x_t(0)| \leq \left(\frac{V(t, x_t)}{c_1}\right)^{\frac{1}{2}} \leq \left(\frac{e^{-\lambda t} V(0, x_0)}{c_1}\right)^{\frac{1}{2}} \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2} t} ||x_0||_{\mathcal{W}}.$$

Hence, the system is globally uniformly exponentially stable with a decay rate greater than $\lambda/2$ and an overshoot smaller than $\sqrt{c_2/c_1}$. Note that the Zeno phenomenon does not occur since, based on (2.13), for any time interval with a length smaller than ϵ, there exists a finite number of (at most q) instant t_n, $n \in \mathbb{N}$. □

We finish this section by presenting a definition that will be used in the proof of the main results of the thesis.

Definition 2.2. *[74] Consider a symmetric matrix Z partitioned as*

$$Z = \begin{bmatrix} Z_a & Z_b \\ Z_b^T & Z_c \end{bmatrix}.$$

If Z_c is invertible, the matrix $\mathbb{S} = Z_a - Z_b Z_c^{-1} Z_b^T$ is called the Schur complement of Z_c in Z and has the following properties

1. *$Z > 0$ if and only if $Z_c > 0$ and $\mathbb{S} > 0$,*

2. *If $Z_c > 0$, then $Z \geq 0$ if and only if $\mathbb{S} \geq 0$.*

The subject of LKFs is addressed in the next section.

2.2.2 Lyapunov-Krasovskii functionals

A candidate LKF $V(t, x_t)$ is a functional which penalizes the deviation of x_t from 0. The evolution of LKFs over the past decade has decreased the conservatism of sufficient Krasovskii-based stability conditions. These conditions are usually cast in terms of LMIs which can be solved efficiently using software packages such as SeDuMi [16] and YALMIP [17]. Let a candidate LKF be defined as

$$V(t, x_t) = V_1 + V_2 + V_3, \ t \in [t_n, t_{n+1}), \tag{2.14}$$

where V_1, V_2, and V_3 are presented in Table 2.1. In these functionals, P, R, and X are positive definite matrix variables to be computed by the software packages that solve

Table 2.1: LKF candidates $\forall\, t \in [t_n, t_{n+1})$

$V_1 = x^T(t)Px(t)$
$V_2 = (\tau - \rho)\displaystyle\int_{t-\rho}^{t} e^{\alpha(s-t)} \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix} R \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix}^T \mathrm{d}s$
$V_3 = (\tau - \rho)\left(x(t) - x(t_n)\right)^T X \left(x(t) - x(t_n)\right)$

LMI conditions. The function ρ is defined in (2.3) and α is a given positive scalar. The first component of the LKF, i.e. V_1, is the most common form of Lyapunov functions and penalizes the deviation of the state vector from the origin. Several variants of the last component, V_3, can be found in the literature [47, 48, 64, 72]. The functional V_3 penalizes the deviation of the current state vector from the sampled state vector. One of the contributions of this chapter is the introduction of the functional V_2. This functional penalizes the derivative of the state vector and the sampled state vector in the interval $[t-\rho, t]$ for $t \in [t_n, t_{n+1})$. Using the new functional V_2, the theorems in this chapter can provide less conservative sufficient stability criteria as will be shown in Section 2.5.

Bounds on the Lyapunov-Krasovskii functionals

In this subsection, lower and upper bounds on the LKF (2.14) will be computed. The bounds will be used in the proof of the main results of this chapter to ensure that inequality (2.10) is satisfied. The LKF candidate V_1 is a quadratic function and the matrix P is positive definite. Therefore,

$$\lambda_{\min}(P)|x(t)|^2 \leq V_1 \leq \lambda_{\max}(P)|x(t)|^2.$$

However, according to (2.7) and (2.8), we have $x(t) = x_t(0)$ and

$$|x(t)| \leq ||x_t||_{\mathcal{W}}. \tag{2.15}$$

Therefore,

$$\lambda_{\min}(P)|x_t(0)|^2 \leq V_1 \leq \lambda_{\max}(P)||x_t||^2_{\mathcal{W}}.$$

The LKF candidate V_2 is the integral of a quadratic function and the matrix R

is positive definite. Therefore, V_2 is non-negative at all times. For $s \in [t-\rho, t]$ and $\alpha > 0$, we can write

$$e^{\alpha(s-t)} \begin{bmatrix} \dot{x}^T(s) & x^T(t_n) \end{bmatrix} R \begin{bmatrix} \dot{x}^T(s) & x^T(t_n) \end{bmatrix}^T \leq \lambda_{\max}(R) \left| \begin{bmatrix} \dot{x}^T(s) & x^T(t_n) \end{bmatrix} \right|^2 = \lambda_{\max}(R) \left(|\dot{x}(s)|^2 + |x(t_n)|^2 \right).$$

Using (2.5), the definition of V_2 in Table 2.1 yields

$$V_2 \leq \tau \lambda_{\max}(R) \left(\int_{t-\rho}^{t} |\dot{x}(s)|^2 \, \mathrm{d}s + \int_{t-\rho}^{t} |x(t_n)|^2 \, \mathrm{d}s \right).$$

With a change of variables and using the definition of norm in (2.8), we can write

$$\int_{t-\rho}^{t} |\dot{x}(s)|^2 \, \mathrm{d}s = \int_{-\rho}^{0} |\dot{x}(t+r)|^2 \, \mathrm{d}r = \int_{-\rho}^{0} |\dot{x}_t(r)|^2 \, \mathrm{d}r \leq ||x_t||^2_{\mathcal{W}}.$$

Furthermore, $x(t_n)$ is constant between two sampling instants and considering (2.8), we have

$$|x(t_n)| \leq ||x_t||_{\mathcal{W}}. \tag{2.16}$$

Therefore, we can use (2.5) to write

$$V_2 \leq \tau \lambda_{\max}(R)(1+\tau)||x_t||^2_{\mathcal{W}}. \tag{2.17}$$

The LKF candidate V_3 is a quadratic function and the matrix X is positive definite. Therefore, V_3 is non-negative at all times. Next, inequalities (2.15) and (2.16) yield

$$V_3 \leq 4\tau \lambda_{\max}(X)||x_t||^2_{\mathcal{W}}.$$

The lower and upper bounds on the LKF candidates are summarized in Table 2.2.

Continuity of the Lyapunov-Krasovskii functionals

Here, we study the continuity properties of the LKF (2.14). The results will be used in the proof of the main results to ensure that inequality (2.11) is satisfied. The Lyapunov function V_1 is a quadratic function and continuous. The LKF candidate V_2 is continuous in the interval between two consecutive instants t_n, $n \in \mathbb{N}$. Furthermore, it is non-negative at t_n^-, where $t_n^- = \lim_{t \nearrow t_n} t$. However, V_2 vanishes at the instants t_n because the lower and upper limits of the integral become equal (according to (2.3); $\rho(t_n) = 0$). Therefore, the LKF candidate V_2 is non-increasing at instants t_n, $n \in \mathbb{N}$.

Table 2.2: Lower and upper bounds on the LKF candidates

LKF candidate	Lower bound	Upper bound
V_1	$\lambda_{\min}(P)\lvert x_t(0)\rvert^2$	$\lambda_{\max}(P)\lvert\lvert x_t\rvert\rvert^2_{\mathcal{W}}$
V_2	0	$\tau(1+\tau)\lambda_{\max}(R)\lvert\lvert x_t\rvert\rvert^2_{\mathcal{W}}$
V_3	0	$4\tau\lambda_{\max}(X)\lvert\lvert x_t\rvert\rvert^2_{\mathcal{W}}$

Table 2.3: Continuity properties of the LKF candidates

LKF candidate	Continuous	Discontinuous at t_n, $n \in \mathbb{N}$, but non-increasing
V_1	✓	-
V_2	-	✓
V_3	-	✓

The LKF candidate V_3 is continuous in the interval between two consecutive instants t_n, $n \in \mathbb{N}$. Furthermore, it is non-negative at t_n^-, $n \in \mathbb{N}$. Nonetheless, V_3 vanishes at the instants t_n because $x(t)|_{t=t_n} = x(t_n)$. Therefore, the LKF candidate V_3 is non-increasing at instants t_n, $n \in \mathbb{N}$. The continuity properties of the LKF candidates are summarized in Table 2.3.

Time derivative of Lyapunov-Krasovskii functionals

The stability and stabilization conditions are based on the time derivatives of the LKFs. The time derivative of the LKFs used in this chapter are summarized in Table 2.4. Computing the time derivatives of V_1 and V_3 are straightforward. Here, the time derivative of the LKF candidate V_2 is computed. Applying the Leibniz integral rule to V_2 and using (2.3) yields

$$\begin{aligned}\dot{V}_2 = &-\int_{t-\rho}^{t} e^{\alpha(s-t)} \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix} R \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix}^T \mathrm{d}s \\ &+ (\tau-\rho)\begin{bmatrix}\dot{x}^T & x^T(t_n)\end{bmatrix} R \begin{bmatrix}\dot{x}^T & x^T(t_n)\end{bmatrix}^T - \alpha V_2. \end{aligned} \quad (2.18)$$

For $R > 0$, $\alpha > 0$, $s \in [t-\rho, t]$, $0 \le \rho < \tau$, and an arbitrary time varying vector $h(t) \in \mathbb{R}^{2n_x}$ we can write

$$\begin{bmatrix}\begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix} & h^T\end{bmatrix} \begin{bmatrix} e^{\alpha(s-t)}R & -I \\ -I & e^{\alpha\tau}R^{-1}\end{bmatrix} \begin{bmatrix}\begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix} & h^T\end{bmatrix}^T \ge 0.$$

This inequality can be verified using Schur complement (see Definition 2.2). Therefore,

$$\begin{aligned} -e^{\alpha(s-t)}\begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix} R \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix}^T \le & h^T e^{\alpha\tau}R^{-1}h - \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix} h \\ & - h^T \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix}^T. \end{aligned}$$

Note that x is absolutely continuous and in the interval between $t-\rho$ and t, $t \in [t_n, t_{n+1})$, the vector $x(t_n)$ is constant. Therefore, integrating both sides from $t-\rho$ to t, with respect to s, yields

$$\begin{aligned} & -\int_{t-\rho}^{t} e^{\alpha(s-t)}\begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix} R \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix}^T \mathrm{d}s \\ & \le \rho h^T e^{\alpha\tau}R^{-1}h - \begin{bmatrix}x^T - x^T(t_n) & \rho x^T(t_n)\end{bmatrix} h - h^T \begin{bmatrix}x^T - x^T(t_n) & \rho x^T(t_n)\end{bmatrix}^T. \end{aligned} \tag{2.19}$$

Replacing (2.19) in (2.18), we get

$$\begin{aligned} \dot{V}_2 \le & \rho h^T e^{\alpha\tau}R^{-1}h - \begin{bmatrix}x^T - x^T(t_n) & \rho x^T(t_n)\end{bmatrix} h - h^T \begin{bmatrix}x^T - x^T(t_n) & \rho x^T(t_n)\end{bmatrix}^T \\ & + (\tau-\rho)\begin{bmatrix}\dot{x}^T & x^T(t_n)\end{bmatrix} R \begin{bmatrix}\dot{x}^T & x^T(t_n)\end{bmatrix}^T - \alpha V_2. \end{aligned} \tag{2.20}$$

The main results of this chapter are presented in the following two sections.

2.3 Stability Analysis

Assume that a linear controller is designed to stabilize the linear system (2.1) in continuous-time. In practice, however, the controller will be located between a sampler and a zero-order-hold in the feedback loop. In this section, our objective is to find a lower bound on the MASP that preserves exponential stability of the closed-loop linear system. We propose sufficient stability conditions in the form of LMIs that can be solved efficiently using available software.

Table 2.4: Time derivative of the LKF candidates for $t \in [t_n, t_{n+1})$

$\dot{V}_1 = \dot{x}^T P x + x^T P \dot{x}$
$\dot{V}_2 \leq \rho h^T e^{\alpha\tau} R^{-1} h - \begin{bmatrix} x^T - x^T(t_n) & \rho x^T(t_n) \end{bmatrix} h - h^T \begin{bmatrix} x^T - x^T(t_n) & \rho x^T(t_n) \end{bmatrix}^T + (\tau - \rho) \begin{bmatrix} \dot{x}^T & x^T(t_n) \end{bmatrix} R \begin{bmatrix} \dot{x}^T & x^T(t_n) \end{bmatrix}^T - \alpha V_2$
$\dot{V}_3 = -\left(x(t) - x(t_n)\right)^T X \left(x(t) - x(t_n)\right) + (\tau - \rho)\left(\dot{x}^T(t) X \left(x(t) - x(t_n)\right) + \left(x(t) - x(t_n)\right)^T X \dot{x}(t)\right)$

Theorem 2.2. *Consider the closed-loop linear sampled-data system defined in (2.1) and (2.2) with nonuniform sampling intervals smaller than $\tau > 0$. Given $\alpha > 0$, the system is globally uniformly exponentially stable, with a decay rate greater then $\alpha/2$, if there exist symmetric positive definite matrices P, R, and X, and a matrix N, with appropriate dimensions, satisfying*

$$\Psi + \tau M_1 < 0 \tag{2.21}$$

$$\begin{bmatrix} \Psi + \tau M_2 & \tau N \\ \tau N^T & -\tau e^{-\alpha\tau} R \end{bmatrix} < 0 \tag{2.22}$$

where

$$\begin{aligned}
\Psi = {} & \begin{bmatrix} A & BK \end{bmatrix}^T \begin{bmatrix} P & 0 \end{bmatrix} + \begin{bmatrix} P & 0 \end{bmatrix}^T \begin{bmatrix} A & BK \end{bmatrix} + \alpha \begin{bmatrix} I & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 \end{bmatrix} \\
& - \begin{bmatrix} I & -I \end{bmatrix}^T X \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix}^T N^T - N \begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix}, \\
M_1 = {} & \begin{bmatrix} A & BK \\ 0 & I \end{bmatrix}^T R \begin{bmatrix} A & BK \\ 0 & I \end{bmatrix} + \alpha \begin{bmatrix} I & -I \end{bmatrix}^T X \begin{bmatrix} I & -I \end{bmatrix} + \begin{bmatrix} A & BK \end{bmatrix}^T X \begin{bmatrix} I & -I \end{bmatrix} \\
& + \begin{bmatrix} I & -I \end{bmatrix}^T X \begin{bmatrix} A & BK \end{bmatrix}, \\
M_2 = {} & -\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}^T N^T - N \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.
\end{aligned}$$

Proof. Here, we prove that the LMIs in Theorem 2.2 are sufficient conditions for the inequalities in Theorem 2.1 to be satisfied. To this end, note that the linear sampled-data system defined in (2.1) and (2.2) can be written as a retarded functional differential equation $\dot{x} = f(t, x_t)$ with $f(t, x_t) = Ax(t) + BKx_t(-\rho)$. Consider the LKF candidate (2.14). Based on Table 2.2, it is straightforward to show that the LKF (2.14) satisfies inequality (2.10). According to Table 2.3, it is easy to see that the LKF (2.14) satisfies inequality (2.11). Next, we study $\dot{V}$ in the interval between two consecutive sampling instants. Recalling (2.1) and (2.2), we have

$$\dot{x}(t) = \begin{bmatrix} A & BK \end{bmatrix} \zeta(t),$$

where $\zeta(t) = \begin{bmatrix} x^T(t) & x^T(t_n) \end{bmatrix}^T$, $t \in [t_n, t_{n+1})$. The time derivative of V is composed of three terms that can be found in Table 2.4. Let $h(t) = N^T \zeta(t)$, where N is a matrix in $\mathbb{R}^{2n_x \times 2n_x}$. Therefore,

$$\begin{aligned}
\dot{V} + \alpha V = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \alpha(V_1 + V_2 + V_3) \\
\leq \zeta^T \Bigg(& \begin{bmatrix} A & BK \end{bmatrix}^T P \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix}^T P \begin{bmatrix} A & BK \end{bmatrix} + \alpha \begin{bmatrix} I & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 \end{bmatrix} \\
& + \rho N e^{\alpha\tau} R^{-1} N^T - \begin{bmatrix} I & -I \\ 0 & \rho I \end{bmatrix}^T N^T - N \begin{bmatrix} I & -I \\ 0 & \rho I \end{bmatrix} \\
& + (\tau - \rho) \begin{bmatrix} A & BK \\ 0 & I \end{bmatrix}^T R \begin{bmatrix} A & BK \\ 0 & I \end{bmatrix} + (\alpha(\tau - \rho) - 1) \begin{bmatrix} I & -I \end{bmatrix}^T X \begin{bmatrix} I & -I \end{bmatrix} \\
& + (\tau - \rho) \begin{bmatrix} A & BK \end{bmatrix}^T X \begin{bmatrix} I & -I \end{bmatrix} + (\tau - \rho) \begin{bmatrix} I & -I \end{bmatrix}^T X \begin{bmatrix} A & BK \end{bmatrix} \Bigg) \zeta.
\end{aligned} \tag{2.23}$$

Hence, for $\rho = 0$, LMI (2.21) implies $\dot{V} + \alpha V < 0$. Using Schur complement (see Definition 2.2), LMI (2.22) implies $\dot{V} + \alpha V < 0$ for $\rho = \tau$. Since (2.23) is affine in ρ, LMIs (2.21) and (2.22) are sufficient conditions for $\dot{V} + \alpha V < 0$ to hold for any $\rho \in (0, \tau)$, that is for any $t \in (t_n, t_{n+1})$. Therefore, inequality (2.12) in Theorem 2.1 is satisfied. Note that based on Assumption 2.1, inequality (2.13) holds for all sampling instants t_n, $n \in \mathbb{N}$, with $q = 1$. This finishes the proof. □

Remark 2.1. *In intuitive terms, relaxing Assumption 2.1 by letting the sampling intervals approach zero, yields $\rho(t) \to 0$ and $x(t) = x(t_n)$ for $t_n \leq t < t_{n+1}$. Therefore,*

V_2 and V_3 in (2.14) vanish and the LMIs of Theorem 2.2 reduce to

$$\begin{aligned} &P > 0, \\ &(A + BK)^T P + P(A + BK) < 0, \end{aligned}$$

that are the conditions for stability of continuous-time linear systems.

The following proposition, addresses robust stability of linear sampled-data systems with polytopic uncertainty in system parameters.

Proposition 2.1. *Suppose that the pair of system matrices $S = \begin{bmatrix} A & B \end{bmatrix}$ is unknown but satisfies the following condition*

$$S \in \left\{ \sum_{i=1}^{p} \alpha_i S_i, \ 0 \leq \alpha_i \leq 1, \ \sum_{i=1}^{p} \alpha_i = 1 \right\},$$

where $S_i = \begin{bmatrix} A_i & B_i \end{bmatrix}$, $i \in \{1, ..., p\}$, denote the vertices of a convex polytope. If the LMIs in Theorem 2.2 hold for each S_i, $i \in \{1, ..., p\}$, with the same variables P, R, and X, then the closed-loop linear sampled-data system, with variable sampling intervals smaller than τ, is globally uniformly exponentially stable.

Proof. Assume that the LMIs in Theorem 2.2 hold for each S_i, $i \in \{1, ..., p\}$, with the same matrix variables P, R, and X. Given the linear structure of the stability criteria in Theorem 2.2, it is guaranteed that the LMIs (2.21) and (2.22) hold for any matrix parameter lying in the convex hull of S_i, $i \in \{1, ..., p\}$. Therefore, the uncertain linear sampled-data system is globally uniformly exponentially stable. □

Remark 2.2. *Assume that the sampled-data system has a time-varying uncertain parameter that is lower and upper bounded and appears linearly in the vector field. Similar to the proof of Proposition 2.1, it can be proved that if the LMIs in Theorem 2.2 are satisfied at the lower and upper bounds of the uncertain parameter then the closed-loop system is exponentially stable.*

Based on Theorem 2.2, the problem of finding a lower bound on the MASP that preserves exponential stability is formulated as

Problem 2.1.

$$\begin{aligned} &\textit{maximize} \ \ \tau \\ &\textit{subject to} \ \ P > 0, \ R > 0, \ X > 0, \ (2.21) \ \textit{and} \ (2.22). \end{aligned}$$

In this chapter, the computed lower bound on the MASP that preserves exponential stability is denoted by $\tau_{\max}$.

2.4 Controller Synthesis

When the controller gain K is unknown, the LMIs in Theorem 2.2 turn into bilinear matrix inequalities that cannot be solved efficiently. The following theorem addresses this issue and provides sufficient conditions for controller synthesis that can be cast as LMIs.

Theorem 2.3. *Consider the closed-loop linear sampled-data system defined in (2.1) and (2.2) with nonuniform sampling intervals smaller than $\tau > 0$. Given $\alpha > 0$, there exists an exponentially stabilizing linear feedback gain $K = YQ^{-1}$ if there exist a symmetric positive definite matrix Q, matrices Y and N_s, with appropriate dimensions, and a positive scalar ϵ_X, satisfying*

$$\begin{bmatrix} \Psi_s + \tau M_{1_s} & \tau \begin{bmatrix} AQ & BY \\ 0 & Q \end{bmatrix}^T \\ \tau \begin{bmatrix} AQ & BY \\ 0 & Q \end{bmatrix} & -\tau \overline{Q} \end{bmatrix} < 0 \tag{2.24}$$

$$\begin{bmatrix} \Psi_s + \tau M_{2_s} & \tau N_s \\ \tau N_s^T & -\tau e^{-\alpha\tau} \overline{Q} \end{bmatrix} < 0 \tag{2.25}$$

where

$$\begin{aligned} \overline{Q} &= \mathrm{diag}(Q, Q), \qquad\qquad (2.26)\\ \Psi_s &= \begin{bmatrix} AQ & BY \end{bmatrix}^T \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} I & 0 \end{bmatrix}^T \begin{bmatrix} AQ & BY \end{bmatrix} + \alpha \begin{bmatrix} I & 0 \end{bmatrix}^T Q \begin{bmatrix} I & 0 \end{bmatrix} \\ &\quad - \epsilon_X \begin{bmatrix} I & -I \end{bmatrix}^T Q \begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix}^T N_s^T - N_s \begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix}, \\ M_{1_s} &= \alpha\epsilon_X \begin{bmatrix} I & -I \end{bmatrix}^T Q \begin{bmatrix} I & -I \end{bmatrix} + \epsilon_X \begin{bmatrix} AQ & BY \end{bmatrix}^T \begin{bmatrix} I & -I \end{bmatrix} + \epsilon_X \begin{bmatrix} I & -I \end{bmatrix}^T \begin{bmatrix} AQ & BY \end{bmatrix}, \\ M_{2_s} &= - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}^T N_s^T - N_s \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

Proof. Here, we prove that inequalities (2.24) and (2.25) are sufficient conditions for LMIs (2.21) and (2.22) to be satisfied. Suppose there exist matrices $Q > 0$, Y, and

N_s, with appropriate dimensions, and a positive scalar ϵ_X, satisfying the stabilization criteria in (2.24) and (2.25). Let

$$P = Q^{-1},\ X = \epsilon_X Q^{-1},\ R = \overline{Q}^{-1}, N = \overline{Q}^{-1} N_s \overline{Q}^{-1},\ K = YQ^{-1}, \tag{2.27}$$

where $\overline{Q}$ is defined in (2.26). Multiplying (2.24) and (2.25) from left and right by a block diagonal matrix of appropriate size, with Q^{-1} as the diagonal entries, and using Schur complement (see Definition 2.2) yields LMIs (2.21) and (2.22), with the change of variables (2.27). The proof is complete since for any set of matrix variables satisfying (2.24) and (2.25), there exists a set of matrix variables (2.27) satisfying the stability criteria in Theorem 2.2. □

Based on Theorem 2.3, the problem of designing a state feedback controller which provides a larger lower bound on the MASP that preserves exponential stability is formulated as

Problem 2.2.

$$\begin{aligned} &\textit{maximize} \ \ \tau \\ &\textit{subject to} \ \ Q > 0,\ \epsilon_X > 0,\ (2.24)\ \textit{and}\ (2.25). \end{aligned}$$

The controller gain is then computed as $K = YQ^{-1}$.

2.5 Numerical Examples

In this section, the conditions of Theorem 2.2 and 2.3 are applied to several benchmark examples and the results are compared with [47, 48].

Example 2.1. *[47, 48, 75] Consider the closed-loop linear sampled-data system defined in (2.1) and (2.2) with the following parameters*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},\ K = -\begin{bmatrix} 3.75 & 11.5 \end{bmatrix}.$$

The computed lower bound on the MASP that preserves exponential stability τ_{max} is compared with other research in the literature in Table 2.5. According to Table 2.5, the results of this chapter compare favorably with previous research.

Table 2.5: Comparison of the computed lower bound on the MASP that preserves exponential stability $\tau_{\max}$ (s)

	[48]	[47]	Theorem 2.2
Example 2.1	1.113	1.698	1.719
Example 2.2	0.447	0.591	0.705

Example 2.2. *[15, 48] Consider the closed-loop linear sampled-data system defined in (2.1) and (2.2) with the following parameters*

$$A = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1+g_2 \\ -1 \end{bmatrix}, \; |g_1| \leq 0.1, \; |g_2| \leq 0.3, \; K = -\begin{bmatrix} 2.6884 & 0.6649 \end{bmatrix}. \tag{2.28}$$

The computed lower bound on the MASP that preserves exponential stability τ_{max} is compared with other research in the literature in Table 2.5. According to Table 2.5, the results of this chapter compare favorably with previous research. Note that, based on Proposition 2.1, we simultaneously check the stability criteria in Theorem 2.2 for each combination of A_i and B_j, $i, j \in \{1, 2\}$, defined by

$$A_1 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & -1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 0.5 \\ 0.1 & -1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.7 \\ -1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1.3 \\ -1 \end{bmatrix}.$$

Solving Problem 2.2 we find a new controller gain $\overline{K} = -\begin{bmatrix} 2.3765 & 0.5911 \end{bmatrix}$. Solving Problem 2.1 with the new gain, one can see that the lower bound on the MASP that preserves exponential stability is increased to $\tau_{max} = 0.763$ (s). This shows that the new controller $\overline{K}$ is more robust to the sampling frequency than controller K defined in (2.28).

2.6 Conclusion

A modified LKF was proposed to present new stability and stabilization criteria for exponential stability of uncertain sampled-data linear systems. The problem of finding a lower bound on the MASP that preserves exponential stability was formulated as an optimization program in terms of LMIs. The controller synthesis problem was cast as an optimization problem subject to LMIs with the MASP as a parameter. The results of this chapter will be extended to linear sampled-data systems with multiple samplers and actuators in the following chapter.

Chapter 3

Linear Multi-rate Sampled-data Systems

The main objective of this chapter is to propose sufficient Krasovskii-based stability and stabilization criteria for linear sampled-data systems, with multi-rate samplers and time driven zero order holds, as a set of linear matrix inequalities (LMIs). In the sampled-data structure discussed in this chapter, a plant with linear dynamics is controlled by a linear controller which is located in the feedback loop between a sensing block and an actuating block. The sensing block comprises several sensors that sample at different rates and have non-uniform sampling intervals. The actuating block comprises several actuators that are updated asynchronously and are modeled as time driven zero order holds. For each sensor (or actuator), the problem of finding an upper bound on the lowest sampling frequency (or refresh rate) that guarantees exponential stability is cast as an optimization problem in terms of LMIs. It is shown through examples that choosing the right sensors with adequate sampling frequencies and the right actuators with adequate refresh rates has a considerable impact on controller design and stability of the closed-loop system.

3.1 Introduction

In sampled-data systems, a continuous-time plant is controlled by a discrete-time controller which is located in the feedback loop between a sensing block and an actuating block. Furthermore, it is assumed that the non-ideal communication links experience data packet dropouts (see Fig. 3.1). The sensing block gathers the sensory information through several sensors that work at potentially different sampling rates. One reason is that different phenomena (e.g. temperature, pressure, or voltage) are

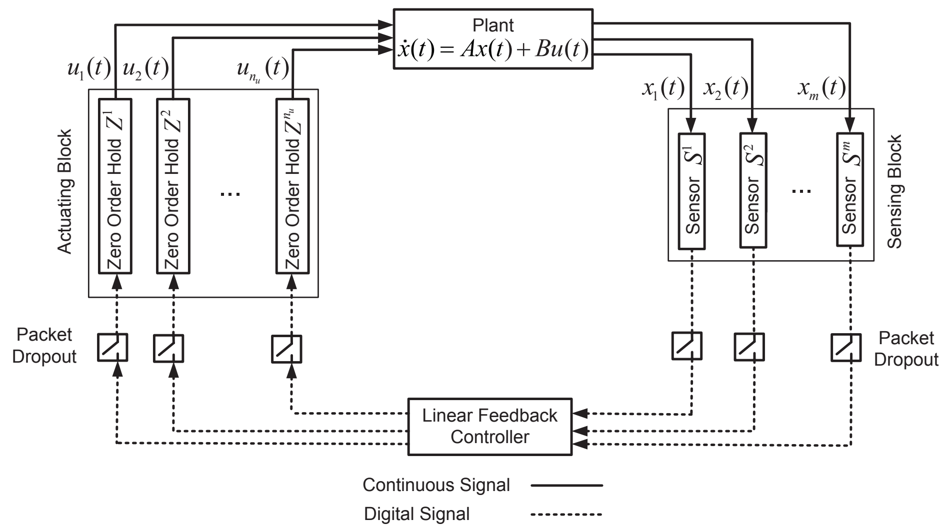


Figure 3.1: The schematic diagram of a linear multi-rate sampled-data system.

measured with different sensors that naturally work at different sampling rates. Second, different methods of sensing the same phenomenon can lead to different sampling frequencies (e.g. measuring an angle with a potentiometer, an encoder, or a camera through image processing). Even if the sensors are synchronized, the inevitable delays and packet losses in non-ideal communication links result in the sensory data arriving at the controller at different and non-uniform rates. The controller computes new control signals as soon as new data becomes available from any of the sensors. The actuating block applies the control signals to the plant through different actuators that potentially have different refresh rates. If the refresh rate of the actuators are high, the actuators can apply the control signal almost at the same time that the controller is updated. Therefore, such actuators can be modeled as event driven zero order holds where the events are the controller update instants. Electrostatic and piezoelectric actuators that work at frequencies around 1 (MHz) are examples of these actuators [76]. There are actuators, however, that have relatively low update rates. Actuators that work based on electroactive polymers and shape memory alloys as well as actuators such as electric cylinders and solenoids have a refresh rate of 10 (Hz) or less [76]. When using these actuators, the control signal computed by the controller is not instantly applied to the plant. In these cases, the delay in applying the control signal is not negligible and affects stability and performance of the closed-loop system. In this chapter, we focus on actuators with low refresh rates and model them as time driven zero order holds.

Stability analysis and controller synthesis of multi-rate sampled-data systems are practically relevant problems that have attracted researchers for several decades (see [77, 78] and the references therein). Reference [79] develops a frequency domain technique for a dual-rate sampled-data systems (with 2 : 1 and 4 : 1 sampling ratios) and applies it to the Space Shuttle flight control system. Reference [80] studies the controller synthesis problem for linear multi-rate sampled-data systems in discrete-time using pole placement. Necessary and sufficient conditions for reachability, controllability, and stabilizability of linear multi-rate sampled-data systems are presented in [81]. In [82], nest algebra is used to address H_2 and H_∞ control problems of multi-rate sampled-data systems. The H_∞ controller synthesis problem of asynchronous multi-rate sampled-data systems is addressed in [83]. However, the stabilization criteria are convex only when the sample and hold rates are synchronous. Following [84] and [83], a synthesis method for robust multi-rate track-following in hard disk drives is proposed in [85]. In [86], a control strategy is presented to retune a multi-rate PID controller in accordance with the delays detected in a networked control system. The construction of LKFs for coupled differential-difference equations with a constant delay in each sensing channel is addressed in [87]. The main drawback of these works is that they are restricted to sensors with uniform sampling intervals or communication links with constant delays. Moreover, the uniform sampling intervals or the constant delays are assumed to be commensurate, i.e. to have rational ratios. In practice, however, sampling intervals and delays are not always constant and known. For instance, in the servo control of brushless DC motors via Hall-effect sensors, the sampling intervals depend on the motor speed and are not predetermined [88]. According to [88], similar phenomenon occurs in applications such as hard disk drives and CD-ROM servo systems. Furthermore, all sensors are prone to uncertain non-uniform samplings due to non-ideal communication links with delays and packet losses [48].

In contrast, one of the contributions of this chapter is to address the multi-rate sampled-data problem with non-uniform sampling intervals (for the sensors) and update intervals (for the actuators). To the best of the authors' knowledge, the multi-rate sampled-data problem with non-uniform sampling and update intervals has not received many research contributions. Dual-rate sampled-data systems with non-uniform sampling and update intervals are studied in [89]. However, in [89] all the states are sampled *simultaneously* by the sensors and the control signals are applied *synchronously* to the plant via time driven zero order holds. It is called a dual-rate sampled-data structure because the actuators are updated at a different rate from the samplers. In this chapter, we address a more general problem where the states are sampled by dedicated sensors at *different rates* and the inputs are *asynchronously* applied to the plant through multiple time driven zero order holds.

The main contribution of this chapter is to present sufficient Krasovskii-based stability and stabilization criteria for linear sampled-data systems with multi-rate samplers and time driven zero order holds. Most importantly, the stability and stabilization criteria are cast as LMIs that can be solved efficiently using available optimization software such as SeDuMi [16] and YALMIP [17]. For each sensor (or actuator), the problem of finding an upper bound on the lowest sampling frequency (or refresh rate) that guarantees exponential stability is cast as an optimization problem in terms of LMIs. It is shown through examples that choosing the right sensors with adequate sampling frequencies has a considerable impact on controller design and stability of the closed-loop system.

The rest of this chapter is organized as follows. Section 3.2 is dedicated to problem statement and preliminary notions. Stability analysis and controller synthesis results are presented in Section 3.3 and Section 3.4, respectively. Numerical examples are provided in Section 3.5, followed by the concluding remarks in Section 3.6.

3.2 Problem Statement

Consider a stabilizable linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3.1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector, $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, and $u \in \mathbb{R}^{n_u}$ is the control input. The case of systems with polytopic uncertainty in system matrices A and B will also be addressed in Sections 3.3 and 3.4. Let a continuous-time stabilizing linear controller for (3.1) be defined by

$$u(t) = Kx(t), \tag{3.2}$$

where $K \in \mathbb{R}^{n_u \times n_x}$. In practice, the controller is located in the feedback loop between a sensing block and an actuating block (see Fig. 3.1). The sensing block comprises m sensors S^i, $i \in \{1, \dots, m\}$, where $m \leq n_x$. Each sensor S^i is dedicated to sampling one component of the state vector which we denote by x_i. Each of the m components x_i can possibly be a vector (e.g. a camera provides the position of an object in a two dimensional space). The state vector is then written as $x^T = \begin{bmatrix} x_1^T & \dots & x_m^T \end{bmatrix}$. The actuating block comprises n_u actuators. Each actuator is modeled as a zero order hold Z^j, $j \in \{1, \ldots, n_u\}$. The sensors and the zero order holds are time driven and asynchronous. Furthermore, the sampling frequency for the sensors and the refresh rate for the actuators are uncertain and non-uniform.

Assumption 3.1. *The sensor S^i, $i \in \{1, \ldots, m\}$, samples the i^{th} component of the state vector x_i at sampling instants s^i_k, where $0 < \epsilon_s < s^i_{k+1} - s^i_k < \tau^i_s$, $\forall\, k \in \mathbb{N}$.*

Assumption 3.2. *The zero order hold Z^j, $j \in \{1, \ldots, n_u\}$, is updated at instants z^j_k, where $0 < \epsilon_z < z^j_{k+1} - z^j_k < \tau^j_z$, $\forall\, k \in \mathbb{N}$.*

Without loss of generality, by the index $k \in \mathbb{N}$, we denote only the instants s^i_k and z^j_k for which a data packet is not lost. The positive constant ϵ_s (respectively ϵ_z) models the fact that a sensor (respectively an actuator) cannot measure a particular phenomenon (respectively be updated) twice at the same instant. The scalars τ^i_s, $i \in \{1, \ldots, m\}$, and τ^j_z, $j \in \{1, \ldots, n_u\}$, denote the longest interval between two consecutive samplings by the sensor S^i and the longest interval between two consecutive updates of the actuator Z^j, respectively. For each sensor S^i, $i \in \{1, \ldots, m\}$, the time elapsed since the sensor's last sampling instant is denoted by a sawtooth function $\rho^i_s(t)$ (see the top two plots in Fig. 3.2) defined as

$$\rho^i_s(t) = t - s^i_k, \ \forall\, t \in [s^i_k, s^i_{k+1}). \tag{3.3}$$

Similarly, the time elapsed since the last update of each zero order hold Z^j, $j \in \{1, \ldots, n_u\}$, is denoted by a sawtooth function $\rho^j_z(t)$ (see Fig. 3.2) defined as

$$\rho^j_z(t) = t - z^j_k, \ \forall\, t \in [z^j_k, z^j_{k+1}). \tag{3.4}$$

Therefore, equation (3.3) and Assumption 3.1 yield

$$0 \leq \rho^i_s < \tau^i_s, \ i \in \{1, \ldots, m\}, \tag{3.5}$$

and equation (3.4) and Assumption 3.2 yield

$$0 \leq \rho^j_z < \tau^j_z, \ j \in \{1, \ldots, n_u\}. \tag{3.6}$$

The instants at which (at least) one of the n_u zero order holds is updated constitute an increasing sequence in time, represented by $\{z_k\}$, $k \in \mathbb{N}$. Each instant z_k, $k \in \mathbb{N}$, is associated with (i.e. is equal to) at least one and at most n_u instants $z^j_{k_j}$, $k_j \in \mathbb{N}$, with different $j \in \{1, \ldots, n_u\}$ (see Assumption 3.2). The time elapsed since the last

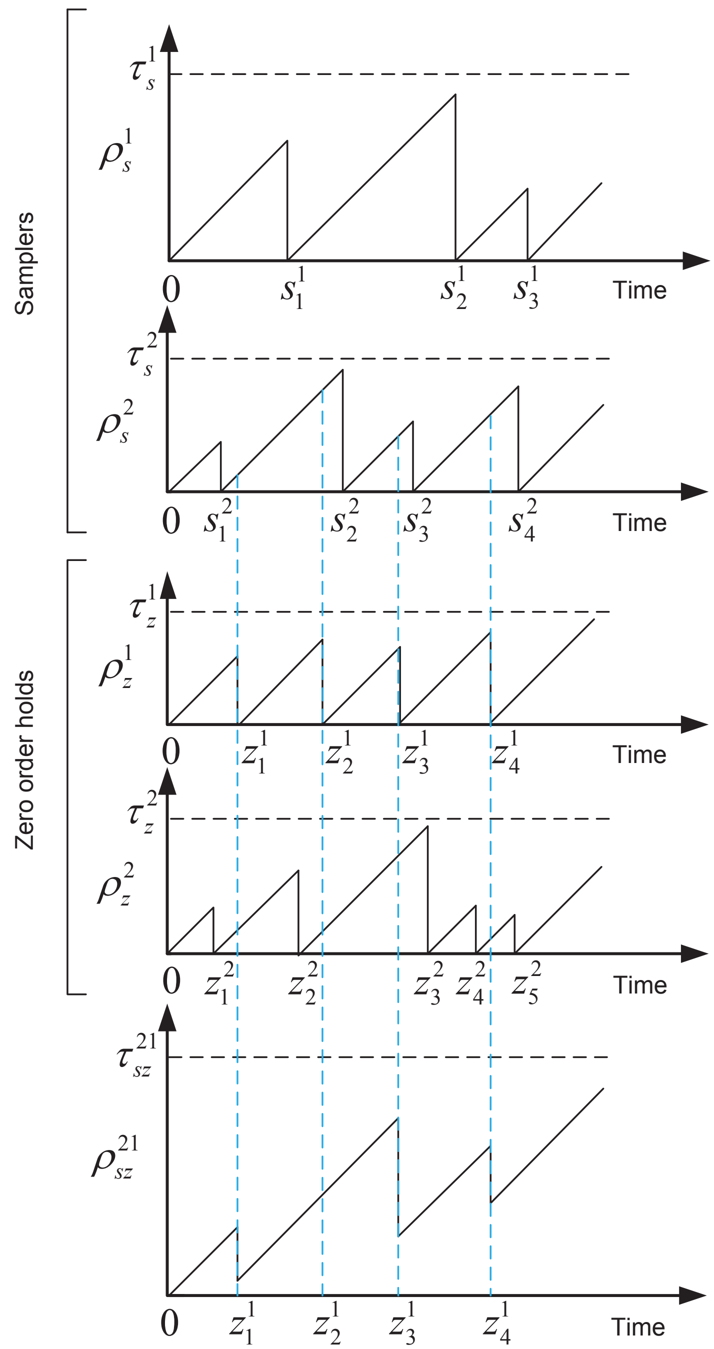


Figure 3.2: The sawtooth functions $\rho_s^1(t)$, $\rho_s^2(t)$, $\rho_z^1(t)$, $\rho_z^2(t)$, and $\rho_{sz}^{21}(t)$ in a multi-rate sampled-data structure with two sensors and two actuators.

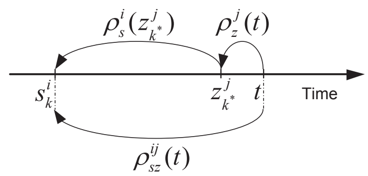


Figure 3.3: The function $\rho_{sz}^{ij}(t)$

update of any of the n_u zero order holds is denoted by $\rho_z(t)$, i.e.

$$\begin{aligned}\rho_z(t) &= t - z_k, \;\; \forall\, t \in [z_k, z_{k+1}) \\ &= \min_j \rho_z^j(t), \;\; j \in \{1, \ldots, n_u\}.\end{aligned} \tag{3.7}$$

Therefore, based on (3.6)

$$0 \leq \rho_z(t) < \tau_z, \tag{3.8}$$

where

$$\tau_z = \min_j \tau_z^j, \;\; j \in \{1, \ldots, n_u\}. \tag{3.9}$$

The controller is assumed to send new values to the zero order holds as soon as it receives new data from a sensor. The zero order holds are time driven, however, and are not updated instantly. The time elapsed since data acquired by sensor S^i was last used to update the zero order hold Z^j is denoted by $\rho_{sz}^{ij}(t)$, i.e.

$$\begin{aligned}\rho_{sz}^{ij}(t) &= t - z_{k^*}^j + \rho_s^i(z_{k^*}^j), \;\; \forall\, t \in [z_{k^*}^j, z_{k^*+1}^j), \\ &= \rho_z^j(t) + \rho_s^i(z_{k^*}^j), \;\; \forall\, t \in [z_{k^*}^j, z_{k^*+1}^j).\end{aligned} \tag{3.10}$$

Figure 3.3 illustrates the sawtooth function $\rho_{sz}^{ij}(t)$. For further clarification, Figure 3.2 shows the function $\rho_{sz}^{21}(t)$ in a multi-rate sampled-data structure with two sensors and two actuators. Note that at each instant z_k^j, $j \in \{1, \ldots, n_u\}$, the function $\rho_z^j(t)$ vanishes and the sawtooth function $\rho_{sz}^{ij}(t)$ jumps to $\rho_s^i(z_{k^*}^j)$. Therefore, unlike $\rho_s^i(t)$ and $\rho_z^j(t)$, the function $\rho_{sz}^{ij}(t)$ does not necessarily decrease to zero at its discontinuities. Based on (3.5) and (3.6), equation (3.10) yields

$$0 \leq \rho_{sz}^{ij} < \tau_z^j + \tau_s^i = \tau_{sz}^{ij}. \tag{3.11}$$

In our proposed multi-rate sampled-data structure, τ_{sz}^{ij} denotes the maximum allowable transfer interval from the sensor S^i to the zero order hold Z^j. The control signal at each input channel $j \in \{1, \dots, n_u\}$ is now computed as

$$u_j(t) = K_j \overline{x}_j(t),$$

where K_j represents the j^{th} row of K, i.e. $K^T = \begin{bmatrix} K_1^T & \dots & K_{n_u}^T \end{bmatrix}$, and

$$\overline{x}_j(t) = \begin{bmatrix} x_1^T(t - \rho_{sz}^{1j}(t)) & \dots & x_m^T(t - \rho_{sz}^{mj}(t)) \end{bmatrix}^T. \tag{3.12}$$

Therefore, the control signal (3.2) is rewritten as

$$u(t) = \overline{K} \widetilde{x}(t), \tag{3.13}$$

where

$$\overline{K} = \begin{bmatrix} K_1 & 0 & \dots & 0 \\ 0 & K_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & K_{n_u} \end{bmatrix} \tag{3.14}$$

and

$$\widetilde{x}(t) = \begin{bmatrix} \overline{x}_1^T(t) & \dots & \overline{x}_{n_u}^T(t) \end{bmatrix}^T. \tag{3.15}$$

Based on Assumption 3.2, there exists a lower bound on the length of the interval (z_k^j, z_{k+1}^j). The length of the interval (z_k, z_{k+1}), however, can approach zero because two zero order holds might possibly be updated at the same time. Nonetheless, the following statement is valid based on Assumption 3.2.

Lemma 3.1. *Let $T_z = \max_j \tau_z^j$, $j \in \{1, \dots, n_u\}$. For any interval spanning a time period longer than $(n_u + 1)T_z$, there exists at least one interval $(z_{k'}, z_{k'+1})$, $k' \in \mathbb{N}$, with a length larger than ϵ_z / n_u.*

Proof. Based on Assumption 3.2, each zero order hold is updated at least once in any interval spanning a time period of length T_z. Hence, any such interval contains at least n_u instants z_k^j, $k \in \mathbb{N}$, $j \in \{1, \dots, n_u\}$. Therefore, any interval spanning a time period longer than $(n_u + 1)T_z$ contains at least $(n_u + 1)n_u$ instants z_k^j, $k \in \mathbb{N}$, $j \in \{1, \dots, n_u\}$. Since each z_k, $k \in \mathbb{N}$, is associated with at most n_u instants $z_{k^*}^j$, $k^* \in \mathbb{N}$, $j \in \{1, \dots, n_u\}$, there exist at least $n_u + 1$ instants z_k, $k \in \mathbb{N}$, in any interval spanning a time period longer than $(n_u + 1)T_z$. Equivalently, any interval spanning a time period longer than $(n_u + 1)T_z$ contains at least one time interval (z_k, z_{k+n_u}), $k \in \mathbb{N}$. Since each z_k, $k \in \mathbb{N}$, is associated with at least one $z_{k^*}^j$, $k^* \in \mathbb{N}$, any interval (z_k, z_{k+n_u}) contains at least two instants $z^j_{k^*}$ and $z^j_{k^*+1}$, corresponding to the same zero order hold Z^j, $j \in \{1, \ldots, n_u\}$. Therefore, Assumption 3.2 guarantees that the interval (z_k, z_{k+n_u}) has a length greater than ϵ_z. Hence, the interval (z_k, z_{k+n_u}) contains at least one interval $(z_{k'}, z_{k'+1})$ with a length more than ϵ_z/n_u, for some k' verifying $k \le k' \le k + n_u - 1$. □

Lemma 3.1 is used in the proof of the main results to guarantee that an LKF candidate strictly decreases in intervals spanning a time period longer than $(n_u+1)T_z$ (see Theorem 2.1 for more details). The following lemma presents a useful property of Kronecker products.

Lemma 3.2. *Let H and ξ be a matrix and a vector, respectively, of the appropriate dimensions. Then $\mathbf{1} \otimes (H\xi) = (\mathbf{1} \otimes H)\xi$, where $\mathbf{1}$ denotes the column vector with all elements equal to* 1.

Proof. It is a well known fact (see e.g. [90] Lemma 4.3.1) that

$$(\mathcal{B}^T \otimes \mathcal{A}) \operatorname{vec}(\mathcal{C}) = \operatorname{vec}(\mathcal{A}\mathcal{C}\mathcal{B}), \tag{3.16}$$

where $\mathcal{A} \in \mathbb{R}^{m \times n}$, $\mathcal{B} \in \mathbb{R}^{p \times q}$, $\mathcal{C} \in \mathbb{R}^{n \times p}$, and vec(.) represents the vector of stacked columns of a matrix. Let $\mathcal{B}^T = \mathbf{1}$, $\mathcal{A} = H$, and $\mathcal{C} = \xi$. Hence, equation (3.16) yields

$$(\mathbf{1} \otimes H)\xi = (\mathbf{1} \otimes H) \operatorname{vec}(\xi) = \operatorname{vec}(H\xi\mathbf{1}^T) = \operatorname{vec}((H\xi)\mathbf{1}^T) = \mathbf{1} \otimes (H\xi),$$

where we used the fact that $H\xi$ is a vector in the last equality. □

The main results of this chapter will be presented in the next two sections.

3.3 Stability Analysis

In this section, we address stability analysis of linear multi-rate sampled-data systems. It is assumed that a stabilizing controller is already designed in continuous-time. Our objective is to find lower bounds on the MASPs (τ^i_s, $i \in \{1, \ldots, m\}$) and maximum allowable update periods (MAUPs) (τ^j_z, $j \in \{1, \ldots, n_u\}$) that preserve exponential stability. The controller synthesis problem for linear multi-rate sampled-data systems is addressed in Section 3.4. Let $\mathcal{W}([-T, 0], \mathbb{R}^{n_x})$ be the space of absolutely continuous functions mapping the interval $[-T, 0]$ to $\mathbb{R}^{n_x}$, where

$$T = \max_{i,j} \tau^{ij}_{sz}, \; i \in \{1, \ldots, m\}, \; j \in \{1, \ldots, n_u\}.$$

Based on (3.7), (3.10), and (3.11), note that

$$0 < \rho_z \le \rho_{sz}^{ij} < T. \tag{3.17}$$

We define the function $x_t \in \mathcal{W}$ as $x_t(r) = x(t+r), \ -T \le r \le 0$, and denote its norm by

$$||x_t||_{\mathcal{W}} = \sum_{i=1}^{m} \max_{r\in[-T,0]} |x_i(t+r)| + \left[\int_{-T}^{0} |\dot{x}_t(r)|^2 \, \mathrm{d}r\right]^{\frac{1}{2}}. \tag{3.18}$$

The following theorem provides a set of sufficient conditions for which the closed-loop trajectories of a linear multi-rate sampled-data system exponentially converge to the origin.

Theorem 3.1. *Consider the closed-loop linear multi-rate sampled-data system defined in (3.1) and (3.13) under Assumptions 3.1 and 3.2. The system is globally uniformly exponentially stable with a decay rate greater than* $\alpha/2$ *if there exist symmetric positive definite matrices* P, R, R_{ij}, R'_{ij}, $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n_u\}$, *and* X_1, *and matrices* N, $\overline{N}$, $\overline{N}'$, *and* $\overline{N}''$, *with appropriate dimensions, satisfying*

$$\begin{bmatrix} \begin{pmatrix} \Psi + \tau_z M_1 + (\boldsymbol{1}\otimes F)^T \\ \times \overline{\tau}(\overline{R}+\overline{R}')(\boldsymbol{1}\otimes F) \end{pmatrix} & \overline{N}\overline{\tau}\overline{E} & \overline{N}'\overline{\tau}\overline{E} \\ \overline{E}\overline{\tau}\overline{N}^T & -\overline{E}\overline{\tau}\overline{R} & 0 \\ \overline{E}\overline{\tau}\overline{N}'^T & 0 & -\overline{E}\overline{\tau}\overline{R}' \end{bmatrix} < 0 \tag{3.19}$$

$$\begin{bmatrix} \begin{pmatrix} \Psi + \tau_z M_2 + (\boldsymbol{1}\otimes F)^T \\ \times \overline{\tau}(\overline{R}+\overline{R}')(\boldsymbol{1}\otimes F) \end{pmatrix} & \overline{N}\overline{\tau}\overline{E} & \overline{N}'\overline{\tau}\overline{E} & \tau_z N & \tau_z \overline{N}'' \\ \overline{E}\overline{\tau}\overline{N}^T & -\overline{E}\overline{\tau}\overline{R} & 0 & 0 & 0 \\ \overline{E}\overline{\tau}\overline{N}'^T & 0 & -\overline{E}\overline{\tau}\overline{R}' & 0 & 0 \\ \tau_z N^T & 0 & 0 & -\tau_z \frac{R}{e^{\alpha\tau_z}} & 0 \\ \tau_z \overline{N}''^T & 0 & 0 & 0 & -\tau_z \frac{\overline{R}'}{e^{\alpha\tau_z}} \end{bmatrix} < 0 \tag{3.20}$$

where τ_z *is defined in (3.9) and*

$$\begin{aligned} X &= \begin{bmatrix} I & -I \end{bmatrix}^T X_1 \begin{bmatrix} I & -I \end{bmatrix}, & (3.21)\\ \overline{\tau}_j &= \mathrm{diag}(\tau_{sz}^{1j} I, \dots, \tau_{sz}^{mj} I), & (3.22)\\ \overline{\tau} &= \mathrm{diag}(\overline{\tau}_1, \dots, \overline{\tau}_{n_u}), & (3.23)\\ \overline{E}_j &= \mathrm{diag}(e^{\alpha\tau_{sz}^{1j}} I, \dots, e^{\alpha\tau_{sz}^{mj}} I), & (3.24) \end{aligned}$$

$$\overline{E} = \mathrm{diag}(\overline{E}_1, \ldots, \overline{E}_{n_u}), \tag{3.25}$$
$$\overline{R}_j = \mathrm{diag}(R_{1j}, \ldots, R_{mj}), \tag{3.26}$$
$$\overline{R} = \mathrm{diag}(\overline{R}_1, \ldots, \overline{R}_{n_u}), \tag{3.27}$$
$$\overline{R}'_j = \mathrm{diag}(R'_{1j}, \ldots, R'_{mj}), \tag{3.28}$$
$$\overline{R}' = \mathrm{diag}(\overline{R}'_1, \ldots, \overline{R}'_{n_u}), \tag{3.29}$$

$$F = \begin{bmatrix} A & B\overline{K} & 0 \end{bmatrix},$$

$$\begin{aligned}
\Psi =& F^T \begin{bmatrix} P & 0 & 0 \end{bmatrix} + \begin{bmatrix} P & 0 & 0 \end{bmatrix}^T F + \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} \\
&- \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T N^T - N \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \left(\boldsymbol{1} \otimes \begin{bmatrix} I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & I & 0 \end{bmatrix} \right)^T \overline{N}^T \\
&- \overline{N} \left(\boldsymbol{1} \otimes \begin{bmatrix} I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & I & 0 \end{bmatrix} \right) - \left(\boldsymbol{1} \otimes \begin{bmatrix} 0 & 0 & I \end{bmatrix} - \begin{bmatrix} 0 & I & 0 \end{bmatrix} \right)^T \overline{N}'^T \\
&- \overline{N}' \left(\boldsymbol{1} \otimes \begin{bmatrix} 0 & 0 & I \end{bmatrix} - \begin{bmatrix} 0 & I & 0 \end{bmatrix} \right) - \left(\boldsymbol{1} \otimes \begin{bmatrix} I & 0 & -I \end{bmatrix} \right)^T \overline{N}''^T \\
&- \overline{N}'' \left(\boldsymbol{1} \otimes \begin{bmatrix} I & 0 & -I \end{bmatrix} \right) - \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^T X \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
M_1 =& \begin{bmatrix} A & B\overline{K} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T R \begin{bmatrix} A & B\overline{K} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \alpha \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^T X \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\
&+ \begin{bmatrix} F \\ 0 \end{bmatrix}^T X \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^T X \begin{bmatrix} F \\ 0 \end{bmatrix},
\end{aligned}$$

$$M_2 = - \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T N^T - N \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Proof. Consider the LKF candidate

$$V(t, x_t) = \sum_{l=1}^{5} V_l, \ t \in [z_k, z_{k+1}), \tag{3.30}$$

where

$$V_1 = x^T(t) P x(t),$$

$$
\begin{aligned}
V_2 =&(\tau_z - \rho_z) \int_{t-\rho_z}^{t} e^{\alpha(s-t)} \begin{bmatrix} \dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k) \end{bmatrix} R \begin{bmatrix} \dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k) \end{bmatrix}^T \mathrm{d}s, \\
V_3 =& \sum_{j=1}^{n_u} \sum_{i=1}^{m} \int_{t-\rho_{sz}^{ij}}^{t} (\tau_{sz}^{ij} - t + s) e^{\alpha(s-t)} \dot{x}_i^T(s) R_{ij} \dot{x}_i(s) \, \mathrm{d}s, \\
V_4 =& \sum_{j=1}^{n_u} \sum_{i=1}^{m} \int_{t-\rho_{sz}^{ij}}^{t} (\tau_{sz}^{ij} - t + s) e^{\alpha(s-t)} \dot{x}_i^T(s) R'_{ij} \dot{x}_i(s) \, \mathrm{d}s, \\
V_5 =&(\tau_z - \rho_z) \begin{bmatrix} x^T(t) & x^T(z_k) \end{bmatrix} X \begin{bmatrix} x^T(t) & x^T(z_k) \end{bmatrix}^T ,
\end{aligned}
$$

where $\widetilde{x}(t)$ and X are defined in (3.15) and (3.21), respectively, P, R, R_{ij}, R'_{ij}, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n_u\}$, and X_1 are positive definite matrices, and $\alpha/2$ is the desired bound on the decay rate. The reason for defining two similar functionals V_3 and V_4 becomes clear later in the proof where we take different approaches to compute $\dot{V}_3$ and $\dot{V}_4$ (see (3.36) and (3.37)). This decreases the conservatism of the sufficient Krasovskii-based conditions.

In the rest of the proof, we show that the LMIs in Theorem 3.1 are sufficient conditions for the LKF (3.30) to satisfy the conditions of Theorem 2.1. To this end, a procedure similar to the one in Chapter 2 is followed. Considering (3.17) and (3.18), observe that

$$
\begin{aligned}
|x(t)| &\leq \sum_{i=1}^{m} |x_i(t+0)| \leq ||x_t||_{\mathcal{W}}, \\
|x(z_k)| &\leq \sum_{i=1}^{m} |x_i(t-\rho_z)| \leq ||x_t||_{\mathcal{W}}.
\end{aligned}
$$

Similarly, equations (3.12) and (3.15)-(3.18) yield

$$
\begin{aligned}
|\overline{x}_j(t)| &\leq \sum_{i=1}^{m} |x_i(t-\rho_{sz}^{ij})| \leq ||x_t||_{\mathcal{W}}, \\
|\widetilde{x}(t)| &= \left(\sum_{j=1}^{n_u} |\overline{x}_j(t)|^2 \right)^{1/2} \leq \sqrt{n_u} ||x_t||_{\mathcal{W}}.
\end{aligned}
$$

Using these inequalities it becomes straightforward to compute the lower and upper bounds on the LKF (3.30) (see Table 3.1). Therefore, the LKF (3.30) satisfies (2.10). Next, we show that the LKF (3.30) satisfies (2.11). The arguments for the functionals V_1, V_2, and V_5 are similar to the ones that were used in Chapter 2. The functionals V_3 and V_4 are the sum of non-negative integrals. Note that $\rho_s^i(z_{k+1}^j) - \rho_s^i(z_k^j) \leq z_{k+1}^j - z_k^j$

Table 3.1: Lower and upper bounds on the LKF candidates (3.30)

LKF candidate	Lower bound	Upper bound
V_1	$\lambda_{\min}(P)\lvert x_t(0)\rvert^2$	$\lambda_{\max}(P)\lvert\lvert x_t\rvert\rvert^2_{\mathcal{W}}$
V_2	0	$\tau_z\lambda_{\max}(R)(1+\tau_z(1+n_u))\lvert\lvert x_t\rvert\rvert^2_{\mathcal{W}}$
V_3	0	$\sum_{j=1}^{n_u}\sum_{i=1}^{m}\tau_{sz}^{ij}\lambda_{\max}(R_{ij})\lvert\lvert x_t\rvert\rvert^2_{\mathcal{W}}$
V_4	0	$\sum_{j=1}^{n_u}\sum_{i=1}^{m}\tau_{sz}^{ij}\lambda_{\max}(R'_{ij})\lvert\lvert x_t\rvert\rvert^2_{\mathcal{W}}$
V_5	0	$4\tau\lambda_{\max}(X_1)\lvert\lvert x_t\rvert\rvert^2_{\mathcal{W}}$

because according to (3.3) the time derivative of ρ_s^i is defined almost everywhere and is equal to one. Therefore, V_3 and V_4 are non-increasing at instants $t = z_k$ because the integrands are non-negative and based on (3.10) the lower limits of the integrals increase from $z_k^j - \rho_s^i(z_k^j)$ to $z_{k+1}^j - \rho_s^i(z_{k+1}^j)$. Therefore, the LKF is non-increasing at instants z_k and satisfies (2.11). Next, we study the time derivative of V in the interval $t \in (z_k, z_{k+1})$. The functional $\dot{V}$ is composed of five terms computed as follows. The time derivative of V_1 is

$$\dot{V}_1 = \dot{x}^T P x + x^T P \dot{x}. \tag{3.31}$$

From (3.7) we have $\dot{\rho}_z = 1$. Hence, applying the Leibniz integral rule to V_2 yields

$$\begin{aligned}\dot{V}_2 = &-\int_{t-\rho_z}^{t} e^{\alpha(s-t)} \begin{bmatrix}\dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k)\end{bmatrix} R \begin{bmatrix}\dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k)\end{bmatrix}^T \mathrm{d}s \\ &+ (\tau_z - \rho_z)\begin{bmatrix}\dot{x}^T & \widetilde{x}^T & x^T(z_k)\end{bmatrix} R \begin{bmatrix}\dot{x}^T & \widetilde{x}^T & x^T(z_k)\end{bmatrix}^T - \alpha V_2.\end{aligned} \tag{3.32}$$

Since R is positive definite, $\alpha > 0$, and $\rho_z < \tau_z$, for any $s \in [t-\rho_z, t]$ and any arbitrary time varying vector $h(t) \in \mathbb{R}^{(2+n_u)n_x}$, we can write

$$\begin{bmatrix}\begin{bmatrix}\dot{x}(s)\\ \widetilde{x}(s)\\ x(z_k)\end{bmatrix}\\ h\end{bmatrix}^T \begin{bmatrix} e^{\alpha(s-t)}R & -I \\ -I & e^{\alpha\tau_z}R^{-1}\end{bmatrix} \begin{bmatrix}\begin{bmatrix}\dot{x}(s)\\ \widetilde{x}(s)\\ x(z_k)\end{bmatrix}\\ h\end{bmatrix} \geq 0. \tag{3.33}$$

This inequality can be verified using Schur complement. Hence, for all $s \in [t-\rho_z, t]$,

$$
\begin{aligned}
&-e^{\alpha(s-t)} \begin{bmatrix}\dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k)\end{bmatrix} R \begin{bmatrix}\dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k)\end{bmatrix}^T \\
&\leq h^T e^{\alpha\tau_z} R^{-1} h - \begin{bmatrix}\dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k)\end{bmatrix} h - h^T \begin{bmatrix}\dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k)\end{bmatrix}^T.
\end{aligned}
$$

Note that for s varying between $t-\rho_z$ and t, the vectors $\widetilde{x}(s)$ and $x(z_k)$ are constant, and $x(s) = x_s(0) \in \mathcal{W}$ is absolutely continuous. Therefore, integrating both sides with respect to s yields

$$
\begin{aligned}
&-\int_{t-\rho_z}^{t} e^{\alpha(s-t)} \begin{bmatrix}\dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k)\end{bmatrix} R \begin{bmatrix}\dot{x}^T(s) & \widetilde{x}^T(s) & x^T(z_k)\end{bmatrix}^T \mathrm{d}s \\
&\quad\leq \rho_z h^T e^{\alpha\tau_z} R^{-1} h - \begin{bmatrix}x^T - x^T(t-\rho_z) & \rho_z\widetilde{x}^T & \rho_z x^T(z_k)\end{bmatrix} h \\
&\qquad - h^T \begin{bmatrix}x^T - x^T(t-\rho_z) & \rho_z\widetilde{x}^T & \rho_z x^T(z_k)\end{bmatrix}^T. \qquad (3.34)
\end{aligned}
$$

Based on (3.7), $t-\rho_z = z_k$. Hence, replacing (3.34) in (3.32) yields

$$
\begin{aligned}
\dot{V}_2 \leq& \rho_z h^T e^{\alpha\tau_z} R^{-1} h - \begin{bmatrix}x^T - x^T(z_k) & \rho_z\widetilde{x}^T & \rho_z x^T(z_k)\end{bmatrix} h \\
&- h^T \begin{bmatrix}x^T - x^T(z_k) & \rho_z\widetilde{x}^T & \rho_z x^T(z_k)\end{bmatrix}^T \\
&+ (\tau_z - \rho_z) \begin{bmatrix}\dot{x}^T & \widetilde{x}^T & x^T(z_k)\end{bmatrix} R \begin{bmatrix}\dot{x}^T & \widetilde{x}^T & x^T(z_k)\end{bmatrix}^T - \alpha V_2. \qquad (3.35)
\end{aligned}
$$

Similarly, we can write the following

$$
\begin{aligned}
\dot{V}_3 =& \sum_{j=1}^{n_u}\sum_{i=1}^{m}\Big(\tau_{sz}^{ij}\dot{x}_i^T R_{ij}\dot{x}_i - \int_{t-\rho_{sz}^{ij}}^{t} e^{\alpha(s-t)}\dot{x}_i^T(s) R_{ij}\dot{x}_i(s)\,\mathrm{d}s\Big) - \alpha V_3 \\
\leq& \sum_{j=1}^{n_u}\sum_{i=1}^{m}\Big(\tau_{sz}^{ij}\dot{x}_i^T R_{ij}\dot{x}_i + \rho_{sz}^{ij} h_{ij}^T e^{\alpha\tau_{sz}^{ij}} R_{ij}^{-1} h_{ij} - \left[x_i - x_i(t-\rho_{sz}^{ij})\right]^T h_{ij} \\
&\qquad - h_{ij}^T\left[x_i - x_i(t-\rho_{sz}^{ij})\right]\Big) - \alpha V_3, \qquad (3.36) \\
\dot{V}_4 =& \sum_{j=1}^{n_u}\sum_{i=1}^{m}\Big(\tau_{sz}^{ij}\dot{x}_i^T R'_{ij}\dot{x}_i - \int_{t-\rho_{sz}^{ij}}^{t-\rho_z} e^{\alpha(s-t)}\dot{x}_i^T(s) R'_{ij}\dot{x}_i(s)\,\mathrm{d}s \\
&\qquad - \int_{t-\rho_z}^{t} e^{\alpha(s-t)}\dot{x}_i^T(s) R'_{ij}\dot{x}_i(s)\,\mathrm{d}s\Big) - \alpha V_4 \\
\leq& \sum_{j=1}^{n_u}\sum_{i=1}^{m}\Big(\tau_{sz}^{ij}\dot{x}_i^T R'_{ij}\dot{x}_i + (\rho_{sz}^{ij} - \rho_z) h_{ij}'^T e^{\alpha\tau_{sz}^{ij}} R_{ij}'^{-1} h'_{ij} - \left[x_i(z_k) - x_i(t-\rho_{sz}^{ij})\right]^T h'_{ij} \\
&\qquad - h_{ij}'^T\left[x_i(z_k) - x_i(t-\rho_{sz}^{ij})\right] + \rho_z h_{ij}''^T e^{\alpha\tau_z} R_{ij}'^{-1} h''_{ij} - \left[x_i - x_i(z_k)\right]^T h''_{ij}
\end{aligned}
$$

$$- h_{ij}''^T \left[x_i - x_i(z_k)\right] \Big) - \alpha V_4, \tag{3.37}$$

where $h_{ij}(t)$, $h'_{ij}(t)$, and $h''_{ij}(t)$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n_u\}$, are arbitrary time-varying vectors of the appropriate dimension. Based on (3.10) and (3.11), $0 \leq \rho_{sz}^{ij} - \rho_z \leq \rho_{sz}^{ij} < \tau_{sz}^{ij}$. Hence, inequalities (3.36) and (3.37) can be rewritten in a more compact form as

$$\begin{aligned} \dot{V}_3 \leq & \sum_{j=1}^{n_u} \Big(\dot{x}^T \overline{\tau}_j \overline{R}_j \dot{x} + \overline{h}_j^T \overline{\tau}_j \overline{E}_j \overline{R}_j^{-1} \overline{h}_j - \left[x - \overline{x}_j\right]^T \overline{h}_j - \overline{h}_j^T \left[x - \overline{x}_j\right] \Big) - \alpha V_3 \\ & = (\mathbf{1} \otimes \dot{x})^T \overline{\tau} \overline{R} (\mathbf{1} \otimes \dot{x}) + \overline{h}^T \overline{\tau} \overline{E} \overline{R}^{-1} \overline{h} - \left[\mathbf{1} \otimes x - \widetilde{x}\right]^T \overline{h} - \overline{h}^T \left[\mathbf{1} \otimes x - \widetilde{x}\right] - \alpha V_3, \end{aligned} \tag{3.38}$$

$$\begin{aligned} \dot{V}_4 \leq & \sum_{j=1}^{n_u} \Big(\dot{x}^T \overline{\tau}_j \overline{R}'_j \dot{x} + \overline{h}_j'^T \overline{\tau}_j \overline{E}_j \overline{R}_j'^{-1} \overline{h}'_j - \left[x(z_k) - \overline{x}_j\right]^T \overline{h}'_j - \overline{h}_j'^T \left[x(z_k) - \overline{x}_j\right] \\ & + \rho_z \overline{h}_j''^T e^{\alpha \tau_z} \overline{R}_j'^{-1} \overline{h}''_j - \left[x - x(z_k)\right]^T \overline{h}''_j - \overline{h}_j''^T \left[x - x(z_k)\right] \Big) - \alpha V_4 \\ & = (\mathbf{1} \otimes \dot{x})^T \overline{\tau} \overline{R}' (\mathbf{1} \otimes \dot{x}) + \overline{h}'^T \overline{\tau} \overline{E} \overline{R}'^{-1} \overline{h}' - \left[\mathbf{1} \otimes x(z_k) - \widetilde{x}\right]^T \overline{h}' - \overline{h}'^T \left[\mathbf{1} \otimes x(z_k) - \widetilde{x}\right] \\ & + \rho_z \overline{h}''^T e^{\alpha \tau_z} \overline{R}'^{-1} \overline{h}'' - (\mathbf{1} \otimes \left[x - x(z_k)\right])^T \overline{h}'' - \overline{h}''^T (\mathbf{1} \otimes \left[x - x(z_k)\right]) - \alpha V_4, \end{aligned} \tag{3.39}$$

where $\overline{\tau}_j$, $\overline{\tau}$, $\overline{E}_j$, $\overline{E}$, $\overline{R}_j$, $\overline{R}$, $\overline{R}'_j$, $\overline{R}'$, $\overline{x}_j$, and $\widetilde{x}$ are defined in (3.22)-(3.29), (3.12), and (3.15), and

$$\begin{aligned} \overline{h}_j &= \begin{bmatrix} h_{1j}^T & \ldots & h_{mj}^T \end{bmatrix}^T \in \mathbb{R}^{n_x}, \ \overline{h}'_j = \begin{bmatrix} h_{1j}'^T & \ldots & h_{mj}'^T \end{bmatrix}^T \in \mathbb{R}^{n_x}, \\ \overline{h}''_j &= \begin{bmatrix} h_{1j}''^T & \ldots & h_{mj}''^T \end{bmatrix}^T \in \mathbb{R}^{n_x}, \ \overline{h} = \begin{bmatrix} \overline{h}_1^T & \ldots & \overline{h}_{n_u}^T \end{bmatrix}^T \in \mathbb{R}^{n_u n_x}, \\ \overline{h}' &= \begin{bmatrix} \overline{h}_1'^T & \ldots & \overline{h}_{n_u}'^T \end{bmatrix}^T \in \mathbb{R}^{n_u n_x}, \ \overline{h}'' = \begin{bmatrix} \overline{h}_1''^T & \ldots & \overline{h}_{n_u}''^T \end{bmatrix}^T \in \mathbb{R}^{n_u n_x}. \end{aligned}$$

The time derivative of V_5 is computed as

$$\begin{aligned} \dot{V}_5 = & - \begin{bmatrix} x^T & x^T(z_k) \end{bmatrix} X \begin{bmatrix} x^T & x^T(z_k) \end{bmatrix}^T + (\tau_z - \rho_z) \begin{bmatrix} \dot{x}^T & 0 \end{bmatrix} X \begin{bmatrix} x^T & x^T(z_k) \end{bmatrix}^T \\ & + (\tau_z - \rho_z) \begin{bmatrix} x^T & x^T(z_k) \end{bmatrix} X \begin{bmatrix} \dot{x}^T & 0 \end{bmatrix}^T. \end{aligned} \tag{3.40}$$

We now define an augmented vector $\zeta(t) \in \mathbb{R}^{(2+n_u)n_x}$ as

$$\zeta(t) = \begin{bmatrix} x^T(t) & \widetilde{x}^T(t) & x^T(z_k) \end{bmatrix}^T, \ t \in [z_k, z_{k+1}), \tag{3.41}$$

where $\widetilde{x}$ is defined in (3.15). Recalling (3.1) and (3.13), the closed-loop vector field can be written as

$$\dot{x}(t) = \begin{bmatrix} A & B\overline{K} & 0 \end{bmatrix} \zeta(t). \tag{3.42}$$

Replacing (3.42) in (3.31), (3.35), and (3.38)-(3.40), setting $h(t) = N^T\zeta(t)$, $\overline{h}(t) = \overline{N}^T\zeta(t)$, $\overline{h}'(t) = \overline{N}'^T\zeta(t)$, $\overline{h}''(t) = \overline{N}''^T\zeta(t)$, where N, $\overline{N}$, $\overline{N}'$, and $\overline{N}''$ are matrices of the appropriate dimensions, and using Lemma 3.2, yields

$$\begin{aligned}
\dot{V} + \alpha V = \ & \textstyle\sum_{l=1}^{5}(\dot{V}_l + \alpha V_l) \\
\leq \zeta^T \Bigg(& \begin{bmatrix} A & B\overline{K} & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} A & B\overline{K} & 0 \end{bmatrix} \\
& + \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} + \rho_z N e^{\alpha\tau_z} R^{-1} N^T - \begin{bmatrix} I & 0 & -I \\ 0 & \rho_z I & 0 \\ 0 & 0 & \rho_z I \end{bmatrix}^T N^T \\
& - N \begin{bmatrix} I & 0 & -I \\ 0 & \rho_z I & 0 \\ 0 & 0 & \rho_z I \end{bmatrix} + (\tau_z - \rho_z) \begin{bmatrix} A & B\overline{K} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T R \begin{bmatrix} A & B\overline{K} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\
& + \left(\mathbf{1} \otimes \begin{bmatrix} A & B\overline{K} & 0 \end{bmatrix}\right)^T \overline{\tau}(\overline{R} + \overline{R}') \left(\mathbf{1} \otimes \begin{bmatrix} A & B\overline{K} & 0 \end{bmatrix}\right) \\
& + \overline{N}\overline{\tau}\overline{E}\overline{R}^{-1}\overline{N}^T - \left(\mathbf{1} \otimes \begin{bmatrix} I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & I & 0 \end{bmatrix}\right)^T \overline{N}^T \\
& - \overline{N}\left(\mathbf{1} \otimes \begin{bmatrix} I & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & I & 0 \end{bmatrix}\right) + \overline{N}'\overline{\tau}\overline{E}\overline{R}'^{-1}\overline{N}'^T \\
& - \left(\mathbf{1} \otimes \begin{bmatrix} 0 & 0 & I \end{bmatrix} - \begin{bmatrix} 0 & I & 0 \end{bmatrix}\right)^T \overline{N}'^T \\
& - \overline{N}'\left(\mathbf{1} \otimes \begin{bmatrix} 0 & 0 & I \end{bmatrix} - \begin{bmatrix} 0 & I & 0 \end{bmatrix}\right) + \rho_z \overline{N}'' e^{\alpha\tau_z} \overline{R}'^{-1} \overline{N}''^T \\
& - \left(\mathbf{1} \otimes \begin{bmatrix} I & 0 & -I \end{bmatrix}\right)^T \overline{N}''^T - \overline{N}''\left(\mathbf{1} \otimes \begin{bmatrix} I & 0 & -I \end{bmatrix}\right) \\
& + \begin{bmatrix} 0 & I & 0 \end{bmatrix}^T (\overline{N}^T + \overline{N}'^T) + (\overline{N} + \overline{N}') \begin{bmatrix} 0 & I & 0 \end{bmatrix} \\
& + (\alpha(\tau_z - \rho_z) - 1) \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^T X \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\
& + (\tau_z - \rho_z) \begin{bmatrix} A & B\overline{K} & 0 \\ 0 & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\
& + (\tau_z - \rho_z) \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^T X \begin{bmatrix} A & B\overline{K} & 0 \\ 0 & 0 & 0 \end{bmatrix} \Bigg) \zeta.
\end{aligned} \tag{3.43}$$

Using Schur complement, for $\rho_z = 0$, LMI (3.19) implies

$$\dot{V} + \alpha V < 0. \tag{3.44}$$

Similarly, LMI (3.20) implies (3.44) is valid for $\rho_z = \tau_z$. Since (3.43) is affine in ρ_z, LMIs (3.19) and (3.20) are sufficient conditions for (3.44) to hold for any $\rho_z \in (0, \tau_z)$, i.e. for the interval between (z_k, z_{k+1}), $k \in \mathbb{N}$. Therefore, inequality (2.12) in Theorem 2.1 is satisfied. According to Assumption 3.2, for any time interval with a length smaller than ϵ_z, there exists a finite number of (at most n_u) instants z_k, $k \in \mathbb{N}$. Therefore, inequality (2.13) is satisfied with $q = n_u$. This finishes the proof. □

The following proposition addresses robust stability of linear multi-rate sampled-data systems with uncertain parameters.

Proposition 3.1. *Suppose that the pair of system matrices* $\Omega = \begin{bmatrix} A & B \end{bmatrix}$ *is unknown but satisfies the following condition*

$$\Omega \in \left\{ \sum_{l=1}^{p} \beta_l \Omega_l,\ 0 \leq \beta_l \leq 1,\ \sum_{l=1}^{p} \beta_l = 1 \right\}, \tag{3.45}$$

where $\Omega_l = \begin{bmatrix} A_l & B_l \end{bmatrix}$, $l \in \{1, ..., p\}$, *denote the vertices of a convex polytope. If the LMIs in Theorem 3.1 hold for each* Ω_l, $l \in \{1, ..., p\}$, *with the same variables* P, R, R_{ij}, R'_{ij}, $i \in \{1, ..., m\}$, $j \in \{1, ..., n_u\}$, *and* X_1, *then the closed-loop linear multi-rate sampled-data system described in (3.1) and (3.13) under Assumptions 3.1 and 3.2 is globally uniformly exponentially stable.*

Proof. The proof is similar to the proof of Proposition 2.1 and is hence omitted. □

Based on Theorem 3.1, the problem of finding a lower bound on the MASP τ_s^i or the MAUP τ_z^j such that exponential stability is preserved, can be formulated as a convex optimization problem in terms of LMIs. These LMIs can be solved efficiently using available optimization software such as SeDuMi [16] and YALMIP [17].

Problem 3.1.

$$\begin{aligned} &\textit{maximize} && \tau_s^i \text{ (or } \tau_z^j) \\ &\textit{subject to} && R_{ij} > 0,\ R'_{ij} > 0,\ i \in \{1, ..., m\},\ j \in \{1, ..., n_u\}, \\ & && P > 0,\ R > 0,\ X_1 > 0,\ (3.19) \textit{ and } (3.20). \end{aligned}$$

We denote the computed lower bound on the MASP τ_s^i (or the MAUP τ_z^j) that preserves exponential stability by $\tau_{s,\max}^i$ (or $\tau_{z,\max}^j$). The controller synthesis problem for linear multi-rate sampled-data systems is addressed in the next section.

3.4 Controller Synthesis

In this section, we address controller synthesis of linear multi-rate sampled-data systems. When the controller gain K is unknown, the LMIs in Theorem 3.1 turn into bilinear matrix inequalities and cannot be solved efficiently. The following theorem addresses this issue by providing sufficient conditions for the controller synthesis problem that can be cast as LMIs.

Theorem 3.2. *Consider the closed-loop linear multi-rate sampled-data system defined in (3.1) and (3.13) under Assumptions 3.1 and 3.2. There exists an exponentially stabilizing linear state feedback gain* $\overline{K} = \overline{YQ}^{-1}$*, if there exist symmetric positive definite matrices* Q, Q_{ij}, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n_u\}$, *matrices* Y, $\mathcal{N}$, $\overline{\mathcal{N}}$, $\overline{\mathcal{N}}'$, *and* $\overline{\mathcal{N}}''$, *with appropriate dimensions, and positive scalar* ϵ_X, *satisfying LMIs (3.46) and (3.47), where* τ_z, $\overline{\tau}$, *and* $\overline{E}$ *are defined in (3.9), (3.23), and (3.25), respectively, and*

$$\overline{Q}_j = \operatorname{diag}(Q_{1j}, \ldots, Q_{mj}), \tag{3.48}$$

$$\overline{Q} = \operatorname{diag}(\overline{Q}_1, \ldots, \overline{Q}_{n_u}), \tag{3.49}$$

$$\widetilde{Q} = \operatorname{diag}(Q, \overline{Q}, Q), \tag{3.50}$$

$$Y^T = \begin{bmatrix} Y_1^T & \ldots & Y_{n_u}^T \end{bmatrix},$$

$$\overline{Y} = \begin{bmatrix} Y_1 & 0 & \ldots & 0 \\ 0 & Y_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & Y_{n_u} \end{bmatrix}, \tag{3.51}$$

$$\begin{aligned}\Phi = &\begin{bmatrix} AQ & B\overline{Y} & 0 \end{bmatrix}^T \begin{bmatrix} I & 0 & 0 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T \begin{bmatrix} AQ & B\overline{Y} & 0 \end{bmatrix} \\ &+ \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T Q \begin{bmatrix} I & 0 & 0 \end{bmatrix} - \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \mathcal{N}^T - \mathcal{N} \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & I & 0 \end{bmatrix}^T (\overline{\mathcal{N}}^T + \overline{\mathcal{N}}'^T) + (\overline{\mathcal{N}} + \overline{\mathcal{N}}') \begin{bmatrix} 0 & I & 0 \end{bmatrix}\end{aligned}$$

$$
\begin{bmatrix}
\Phi+\tau_z\mathcal{M}_1 & \star & \star & \star & \star & \star & \star & \star \\
\overline{\mathcal{N}}^T-\mathbf{1}\otimes\begin{bmatrix}Q & 0 & 0\end{bmatrix} & -\overline{Q} & \star & \star & \star & \star & \star & \star \\
\overline{\mathcal{N}}'^T-\mathbf{1}\otimes\begin{bmatrix}0 & 0 & Q\end{bmatrix} & 0 & -\overline{Q} & \star & \star & \star & \star & \star \\
\overline{\mathcal{N}}''^T-\mathbf{1}\otimes\begin{bmatrix}Q & 0 & -Q\end{bmatrix} & 0 & 0 & -\overline{Q} & \star & \star & \star & \star \\
\overline{\tau}\left(\mathbf{1}\otimes\begin{bmatrix}AQ & B\overline{Y} & 0\end{bmatrix}\right) & 0 & 0 & 0 & \frac{-1}{2}\overline{\tau}\overline{Q} & \star & \star & \star \\
\tau_z\begin{bmatrix}AQ & B\overline{Y} & 0\\ 0 & \overline{Q} & 0\\ 0 & 0 & Q\end{bmatrix} & 0 & 0 & 0 & 0 & -\tau_z\widetilde{Q} & \star & \star \\
\overline{\tau}\overline{E\mathcal{N}}^T & 0 & 0 & 0 & 0 & 0 & -\overline{QE}\overline{\tau} & \star \\
\overline{\tau}\overline{E\mathcal{N}}'^T & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{QE}\overline{\tau}
\end{bmatrix} < 0 \tag{3.46}
$$

$$
\begin{bmatrix}
\Phi+\tau_z\mathcal{M}_2 & \star & \star & \star & \star & \star & \star & \star & \star \\
\overline{\mathcal{N}}^T-\mathbf{1}\otimes\begin{bmatrix}Q & 0 & 0\end{bmatrix} & -\overline{Q} & \star & \star & \star & \star & \star & \star & \star \\
\overline{\mathcal{N}}'^T-\mathbf{1}\otimes\begin{bmatrix}0 & 0 & Q\end{bmatrix} & 0 & -\overline{Q} & \star & \star & \star & \star & \star & \star \\
\overline{\mathcal{N}}''^T-\mathbf{1}\otimes\begin{bmatrix}Q & 0 & -Q\end{bmatrix} & 0 & 0 & -\overline{Q} & \star & \star & \star & \star & \star \\
\overline{\tau}\left(\mathbf{1}\otimes\begin{bmatrix}AQ & B\overline{Y} & 0\end{bmatrix}\right) & 0 & 0 & 0 & \frac{-1}{2}\overline{\tau}\overline{Q} & \star & \star & \star & \star \\
\overline{\tau}\overline{E\mathcal{N}}^T & 0 & 0 & 0 & 0 & -\overline{QE}\overline{\tau} & \star & \star & \star \\
\overline{\tau}\overline{E\mathcal{N}}'^T & 0 & 0 & 0 & 0 & 0 & -\overline{QE}\overline{\tau} & \star & \star \\
\tau_z\mathcal{N}^T & 0 & 0 & 0 & 0 & 0 & 0 & -e^{-\alpha\tau_z}\tau_z\widetilde{Q} & \star \\
\tau_z\overline{\mathcal{N}}''^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -e^{-\alpha\tau_z}\tau_z\overline{Q}
\end{bmatrix} < 0 \tag{3.47}
$$

$$
\begin{aligned}
&- \epsilon_X \begin{bmatrix} I & 0 & -I \end{bmatrix}^T Q \begin{bmatrix} I & 0 & -I \end{bmatrix}, \\
\mathcal{M}_1 = &\begin{bmatrix} \epsilon_X AQ & \epsilon_X B\overline{Y} & 0 \\ 0 & 0 & 0 \\ -\epsilon_X AQ & -\epsilon_X B\overline{Y} & 0 \end{bmatrix}^T + \begin{bmatrix} \epsilon_X AQ & \epsilon_X B\overline{Y} & 0 \\ 0 & 0 & 0 \\ -\epsilon_X AQ & -\epsilon_X B\overline{Y} & 0 \end{bmatrix} \\
&+ \alpha\epsilon_X \begin{bmatrix} I & 0 & -I \end{bmatrix}^T Q \begin{bmatrix} I & 0 & -I \end{bmatrix}, \\
\mathcal{M}_2 = &- \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T \mathcal{N}^T - \mathcal{N} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.
\end{aligned}
$$

Proof. Here, we prove that inequalities (3.46) and (3.47) are sufficient conditions for LMIs (3.19) and (3.20). Suppose there exist matrices $Q > 0$, $Q_{ij} > 0$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n_u\}$, matrices Y, $\mathcal{N}$, $\overline{\mathcal{N}}$, $\overline{\mathcal{N}}'$, and $\overline{\mathcal{N}}''$, with appropriate dimensions, and positive scalar ϵ_X, satisfying the stabilization criteria in (3.46) and (3.47). Let

$$
\begin{aligned}
&P = Q^{-1},\ X_1 = \epsilon_X Q^{-1},\ R = \widetilde{Q}^{-1},\ N = \widetilde{Q}^{-1}\mathcal{N}\widetilde{Q}^{-1},\ R_{ij} = R'_{ij} = Q_{ij}^{-1}, \\
&\overline{R}_j = \overline{R}'_j = \overline{Q}_j^{-1} = \mathrm{diag}(Q_{1j}^{-1}, \dots, Q_{mj}^{-1}),\ \overline{R} = \overline{R}' = \overline{Q}^{-1} = \mathrm{diag}(\overline{Q}_1^{-1}, \dots, \overline{Q}_{n_u}^{-1}), \\
&\overline{N} = \widetilde{Q}^{-1}\overline{\mathcal{N}}\overline{Q}^{-1},\ \overline{N}' = \widetilde{Q}^{-1}\overline{\mathcal{N}}'\overline{Q}^{-1},\ \overline{N}'' = \widetilde{Q}^{-1}\overline{\mathcal{N}}''\overline{Q}^{-1},\ \overline{K} = \overline{Y}\overline{Q}^{-1},
\end{aligned} \tag{3.52}
$$

where $i = \{1, \dots, m\}$, $j = \{1, \dots, n_u\}$, and $\overline{Q}$, $\widetilde{Q}$, and $\overline{Y}$ are defined in (3.49)-(3.51). Multiplying (3.46) from left and right by $\mathrm{diag}(\widetilde{Q}^{-1}, I)$ and using Schur complement yields

$$
\begin{bmatrix} \begin{pmatrix} \Psi + \tau_z M_1 + (\mathbf{1} \otimes F)^T \\ \times \overline{\tau}(\overline{R} + \overline{R}')(\mathbf{1} \otimes F) + \Upsilon \end{pmatrix} & \overline{N}\overline{\tau}\overline{E} & \overline{N}'\overline{\tau}\overline{E} \\ \overline{E}\overline{\tau}\overline{N}^T & -\overline{E}\overline{\tau}\overline{R} & 0 \\ \overline{E}\overline{\tau}\overline{N}'^T & 0 & -\overline{E}\overline{\tau}\overline{R}' \end{bmatrix} < 0, \tag{3.53}
$$

where Ψ, M_1, and F are defined in Theorem 3.1 with the change of variables (3.52) and

$$
\begin{aligned}
\Upsilon = &\overline{NQN}^T + \overline{N}'\overline{QN}'^T + \overline{N}''\overline{QN}''^T + \left(\mathbf{1} \otimes \begin{bmatrix} I & 0 & 0 \end{bmatrix}\right)^T \overline{Q}^{-1} \left(\mathbf{1} \otimes \begin{bmatrix} I & 0 & 0 \end{bmatrix}\right) \\
&+ \left(\mathbf{1} \otimes \begin{bmatrix} 0 & 0 & I \end{bmatrix}\right)^T \overline{Q}^{-1} \left(\mathbf{1} \otimes \begin{bmatrix} 0 & 0 & I \end{bmatrix}\right) \\
&+ \left(\mathbf{1} \otimes \begin{bmatrix} I & 0 & -I \end{bmatrix}\right)^T \overline{Q}^{-1} \left(\mathbf{1} \otimes \begin{bmatrix} I & 0 & -I \end{bmatrix}\right).
\end{aligned} \tag{3.54}
$$

Since $\overline{Q} > 0$, one can conclude that Υ is positive semi-definite. Therefore, comparing (3.53) and (3.19), it can be seen that inequality (3.53) implies LMI (3.19). Hence, LMI (3.46) is a sufficient condition for LMI (3.19). Similarly, multiplying LMI (3.47) from left and right by $\text{diag}(\widetilde{Q}^{-1}, I)$ and using Schur complement yields LMI (3.20) with the change of variables (3.52). The proof is complete since for any set of matrix variables satisfying inequalities (3.46) and (3.47), there exists a set of matrix variables (3.52) that satisfy the stability criteria in Theorem 3.1. □

Remark 3.1. *The stabilization criteria in Theorem 3.2 are sufficient conditions for the stability criteria in Theorem 3.1 and therefore are more conservative. However, they can be used to design linear controllers by solving a convex optimization program that can be solved efficiently using available software packages. Numerical examples will show the effectiveness of this approach (see Section 3.5).*

Proposition 3.2. *Let the pair of system matrices $\Omega = \begin{bmatrix} A & B \end{bmatrix}$ be unknown but satisfy the polytopic uncertainty condition (3.45). Assume that the LMIs in Theorem 3.2 hold for each Ω_l, $l \in \{1, ..., p\}$, with the same variables Q, Q_{ij}, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n_u\}$, Y, and ϵ_X. Then the closed-loop linear multi-rate sampled-data system described in (3.1) and (3.13) under Assumptions 3.1 and 3.2 is exponentially stabilized with the linear state feedback gain $\overline{K} = \overline{Y}\overline{Q}^{-1}$, where $\overline{Q}$ and $\overline{Y}$ are defined in (3.49) and (3.51), respectively.*

Proof. The proof is similar to the proof of Proposition 2.1 and is hence omitted. □

Based on Theorem 3.2, the problem of designing a state feedback controller that gives a larger lower bound on the MASP τ_s^i (or the MAUP τ_z^j), such that exponential stability is guaranteed, can be formulated as a convex optimization problem in terms of LMIs. These LMIs can be solved efficiently using available optimization software such as SeDuMi [16] and YALMIP [17].

Problem 3.2.

maximize τ_s^i (or τ_z^j)

subject to $Q > 0,\ Q_{ij} > 0,\ i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n_u\},\ \epsilon_X > 0,$ (3.46) *and* (3.47).

The controller gain $\overline{K}$ is then computed based on equation $\overline{K} = \overline{Y}\overline{Q}^{-1}$.

3.5 Numerical Examples

In this section, the main results of the chapter are applied to three examples of linear multi-rate sampled-data systems. The lower bound on the MASP is usually used in the literature as a criterion for comparing the conservativeness of stability theorems. The greater is the computed lower bound, the less conservative is the stability theorem. In the following examples, we use the same criterion to demonstrate the effectiveness of the proposed sufficient stability and stabilization conditions.

Example 3.1. *Consider the closed-loop system defined in (3.1) and (3.13) with the following matrix parameters*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \ K = -\begin{bmatrix} 3.75 & 11.5 \end{bmatrix}.$$

It is known (see e.g. [48]) that the MASP in a single-rate scenario for this example is 1.7 *(s) (in a single-rate scenario such as [48] all the elements of the state vector are sampled at the same sampling instants and the event-driven actuator is updated instantly). Now consider a multi-rate scenario where each of the two states of the system (i.e.* x_1 *and* x_2*) is sampled by a dedicated sensor (*S^1 *and* S^2*, respectively) at different unknown non-uniform sampling intervals. Furthermore, the control signal is applied via an actuator* Z^1 *whose update time is not synchronized with any of the two sensors. Assume that the sampling intervals of the sensor* S^1 *and the refresh rate of the actuator* Z^1 *have known upper bounds, i.e.* τ_s^1 *and* τ_z^1 *are fixed. Using Theorem 3.1, the computed lower bound on the MASP for sensor* S^2 *(*$\tau_{s,max}^2$*) that guarantees exponential stability is presented in Table 3.2. It can be seen that, in this example, the sampling intervals of sensor* S^1 *can be longer than the limit for the single-rate case if sensor* S^2 *performs samplings at a faster rate. In other words, we can decrease the controller's dependency on the data from the first sensor by increasing the sampling rate of the second sensor. According to Table 3.2, as the MAUP* τ_z^1 *increases the MASP* $\tau_{s,max}^2$ *decreases to compensate for the late updates of the actuator.*

As a special case, when the MAUP τ_z^1 *approaches zero, the control signal is applied to the system as soon as new data arrives from any of the sensors. Equivalently, the zero order holds can be assumed to be event-driven. This case was studied in [91]. As expected, the values computed for* $\tau_{s,max}^2$ *when* $\tau_z^1 = 0.0001$ *(s) are very close (absolute error* $= 0.01$*) to the values computed in [91].*

Table 3.2: The MASP $\tau^2_{s,\max}$ that guarantees exponential stability in a multi-rate scenario for Example 3.1 with $\alpha = 0.001$.

Known upper bounds	$\tau^2_{s,\max}$
$\tau^1_s = 2$ (s) and $\tau^1_z = 0.0001$ (s)	0.56 (s)
$\tau^1_s = 2$ (s) and $\tau^1_z = 0.1$ (s)	0.43 (s)
$\tau^1_s = 2$ (s) and $\tau^1_z = 0.4$ (s)	0.06 (s)
$\tau^1_s = 3$ (s) and $\tau^1_z = 0.0001$ (s)	0.16 (s)
$\tau^1_s = 3$ (s) and $\tau^1_z = 0.1$ (s)	0.04 (s)

Example 3.2. *Consider the closed-loop system defined in (3.1) and (3.13) with controller gain* $K = -\begin{bmatrix} 2.6884 & 0.6649 \end{bmatrix}$ *and the following uncertainty in matrix parameters taken from [15]*

$$A = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 + g_2 \\ -1 \end{bmatrix}, \ |g_1| \leq 0.1, \ |g_2| \leq 0.3.$$

Based on Proposition 3.1, the stability criteria should be simultaneously checked for each pair of matrices A_p *and* B_q, $p, q \in \{1, 2\}$, *defined by*

$$A_1 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & -1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 0.5 \\ 0.1 & -1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.7 & -1 \end{bmatrix}^T, \ B_2 = \begin{bmatrix} 1.3 & -1 \end{bmatrix}^T.$$

Consider a multi-rate scenario where each of the two states of the system (i.e. x_1 *and* x_2*) is sampled by a dedicated sensor (*S^1 *and* S^2*, respectively) at different unknown non-uniform sampling intervals. Furthermore, the control signal is applied via an actuator* Z^1 *whose update time is not synchronized with any of the two sensors. Assume that the sampling intervals of sensors* S^1 *and* S^2 *have known upper bounds, i.e.* τ^1_s *and* τ^2_s *are fixed. Using Theorem 3.1, the lower bound on the MAUP* τ^1_z *that guarantees exponential stability is presented in Table 3.3. As expected, when the MASPs of sensors* S^1 *and* S^2 *decrease, the computed lower bound on the MAUP* $\tau^1_{z,max}$ *increases. From an engineering point of view, choosing the right sensors with adequate sampling frequencies allows the engineer to select actuators with lower update rates.*

Example 3.3. *Consider the closed-loop system defined in (3.1) and (3.13) with the*

Table 3.3: The MAUP $\tau^1_{z,\max}$ that guarantees exponential stability in a multi-rate scenario for Example 3.2 with $\alpha = 0.001$

Known upper bounds	$\tau^1_{z,\max}$
$\tau^1_s = 0.15$ (s) and $\tau^2_s = 0.25$ (s)	0.07 (s)
$\tau^1_s = 0.10$ (s) and $\tau^2_s = 0.25$ (s)	0.11 (s)
$\tau^1_s = 0.05$ (s) and $\tau^2_s = 0.25$ (s)	0.15 (s)
$\tau^1_s = 0.05$ (s) and $\tau^2_s = 0.15$ (s)	0.17 (s)
$\tau^1_s = 0.05$ (s) and $\tau^2_s = 0.10$ (s)	0.18 (s)

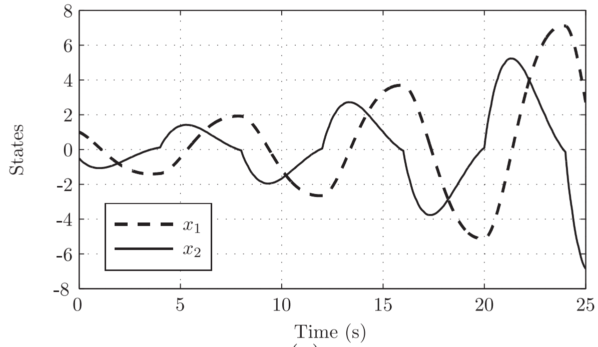
following matrix parameters taken from [47]

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; K = \begin{bmatrix} -1 & 1 \end{bmatrix}.$$

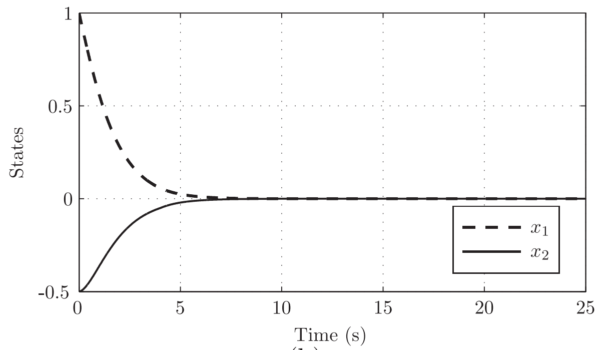
In a multi-rate scenario, suppose that the first and the second elements of the state vector are to be sampled by different sensors at unknown non-uniform sampling intervals smaller than $\tau^1_s = 4$ (s) and $\tau^2_s = 0.1$ (s), respectively. Furthermore, assume that the actuator is updated in non-uniform intervals smaller than $\tau^1_z = 1/3$ (s). Figure 3.4(a) shows that the system is unstable for the mentioned values of MASPs and MAUP. Using Theorem 3.2, with $\alpha = 0.001$ and $\epsilon_X = 1$, we find a new controller gain $K' = -\begin{bmatrix} 0.0101 & 0.0617 \end{bmatrix}$ that guarantees exponential stability of the closed-loop multi-rate system. Figure 3.4(b) illustrates the convergence of the states to the origin when using the controller gain K'.

3.6 Conclusion

Exponential stability and stabilization of linear sampled-data systems with multi-rate samplers and time driven zero order holds were addressed. Sufficient Krasovskii-based stability and stabilization criteria were proposed for linear sampled-data systems as a set of LMIs. For each sensor (or actuator), the problem of finding an upper bound on the lowest sampling frequency (or refresh rate) that guarantees exponential stability was cast as an optimization problem in terms of LMIs. It was shown through examples that choosing the right sensors with adequate sampling frequencies and the right actuators with adequate refresh rates has a considerable impact on controller design and stability of the closed-loop system. In the next chapter, we extend the results



(a)



(b)

Figure 3.4: The evolution of the states for the linear sampled-data system in Example 3.3. The result using controller gain K is shown in Fig. 3.4(a) and the result using controller gain K' is illustrated in Fig. 3.4(b).

of this chapter to propose sensor allocation strategies that guarantee exponential stability of linear multi-rate sampled-data systems.

Chapter 4

Sensor Allocation for Linear Multi-rate Sampled-data Systems

This chapter addresses sensor allocation with guaranteed exponential stability for linear multi-rate sampled-data systems. It is assumed that a continuous-time linear plant is exponentially stabilized by a continuous-time linear controller. Given sensors with incommensurate sampling rates, the objective is to allocate each state to a sensor such that the resulting multi-rate sampled-data system remains exponentially stable. To this end, we propose sufficient Krasovskii-based conditions to partition the state vector among sensors such that exponential stability of the closed-loop system is guaranteed. The problem of finding a partition that guarantees exponential stability is cast as a mixed integer program subject to LMIs.

4.1 Introduction

In multi-rate sampled-data systems, data are gathered through several sensors that work at different sampling rates. One reason is that different phenomena (e.g. temperature, pressure, or voltage) are measured with different sensors that work at different sampling rates. Second, different methods of sensing the same phenomenon can lead to different sampling frequencies (e.g. measuring an angle with a potentiometer, an encoder, or a camera through image processing). Finally, even if the sensors are synchronized, the inevitable delays and packet losses in non-ideal communication links result in the data arriving at the controller at different rates.

Stability analysis and controller synthesis of multi-rate sampled-data systems are practically relevant problems and have attracted researchers for several decades [77–83, 85, 88, 89]. A frequency domain technique for dual-rate sampled-data systems (with 2 : 1 and 4 : 1 sampling ratios) is developed in [79] and applied to the Space Shuttle flight control system. In [80], the controller synthesis problem for linear multi-rate sampled-data systems is studied in discrete-time using pole placement. Necessary and sufficient conditions for reachability, controllability, and stabilizability of linear multi-rate sampled-data systems are presented in [81]. Reference [82] uses nest algebra to address H_2 and H_∞ control problems for multi-rate sampled-data systems. In [85], a synthesis method for robust multi-rate track-following in hard disk drives is proposed and solved using LMIs. A common drawback of [77–82, 85] is that the sampling rate ratios of the sensors are assumed to be rational numbers (commensurate samplings). The commensurate sampling assumption does not hold in practice because the sampling interval of each sensor is usually not uniform. For instance, in servo control of brushless DC motors via Hall-effect sensors, the sampling intervals depend on the motor speed and are not uniform [88]. Delays and data packet losses also contribute to nonuniform sampling intervals. The H_∞ controller synthesis problem of systems with incommensurate samplings is addressed in [83]. However, the stabilization conditions are convex only when the sample and hold rates are commensurate. In [89] and Chapter 3, sufficient Krasovskii-based conditions are presented in terms of LMIs to address stability and stabilization of linear multi-rate sampled-data systems with incommensurate sampling rates.

In contrast to [77–83, 85, 88, 89] and Chapter 3, this chapter addresses sensor allocation with guaranteed exponential stability for linear multi-rate sampled-data systems. Suppose there exists a continuous-time linear controller that stabilizes a linear system. Given sensors with incommensurate sampling rates, the objective is to allocate each state to a sensor such that the resulting multi-rate sampled-data system is exponentially stable. In other words, the goal is to partition the state vector among sensors such that exponential stability of the closed-loop system is guaranteed. Different ways of partitioning the state vector among sensors may result in stable or unstable systems as shown in Example 4.1.

The main contribution of this chapter is to propose sufficient Krasovskii-based conditions that partition the state vector among sensors such that the resulting linear multi-rate sampled-data system is exponentially stable. The problem of finding a partition that guarantees exponential stability is cast as a mixed integer program subject to LMIs. Mixed integer programs are NP-hard. However, mixed integer problems of small size can be solved using free optimization software such as the BNB solver which is shipped with YALMIP [17].

Example 4.1. *Consider a path following problem where the objective is to control a*

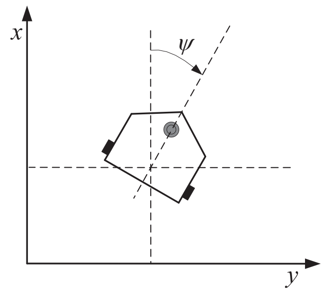


Figure 4.1: Unicycle path following problem

unicycle to follow the line $y = 0$ *in the* $x - y$ *plane (see Fig. 4.1). The dynamics of the system are represented by*

$$\begin{bmatrix} \dot{y} \\ \dot{\psi} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -k/I \end{bmatrix} \begin{bmatrix} y \\ \psi \\ r \end{bmatrix} + \begin{bmatrix} v\sin(\psi) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/I \end{bmatrix} u,$$

where y *represents the distance from the line* $y = 0$, ψ *and* r *are the heading angle and its time derivative, respectively,* $v = 0.25$ *(m/s) is the unicycle's velocity,* u *denotes the torque input about the* z *axis,* $I = 1$ *(kgm*^{2}*) is the unicycle's moment of inertia with respect to its center of mass, and* $k = 0.15$ *(Nms) is the damping coefficient. Linearizing the system about the origin leads to the linear system* $\dot{x}(t) = Ax(t) + Bu(t)$*, where*

$$x = \begin{bmatrix} y \\ \psi \\ r \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0.25 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -0.15 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Assume that a continuous-time linear controller with gain $K = -\begin{bmatrix} 0.8 & 1.4 & 1.6 \end{bmatrix}$ *is designed for the linearized system. Consider a single rate scenario where all the states are sampled at the same sampling instants. In other words, the information about the states arrives at the controller at the same time. According to Theorem 2.2, the single rate sampled-data system is guaranteed to remain exponentially stable with a decay rate smaller than* $0.0001/2$ *for nonuniform sampling periods of up to* 1.19 *(s).*

In reality, however, not all the states are sampled at the same instant. Assuming a dual rate scenario, let the first state y *be acquired via a camera (at sampling intervals of up to* 0.3 *(s)) and the rate of the heading angle* r *be acquired by an inertial measurement unit (at sampling periods of up to* 1.1 *(s)). The heading angle* ψ *can be measured either via the camera or via the inertial measurement unit (by integrating*

Table 4.1: Simulation result for Example 4.1

State vector partition and sampling interval	Simulation result
$\begin{bmatrix} \underbrace{y}_{0.3\ (s)} & \underbrace{\psi \quad r}_{1.1\ (s)} \end{bmatrix}$	Stable
$\begin{bmatrix} \underbrace{y \quad \psi}_{0.3\ (s)} & \underbrace{r}_{1.1\ (s)} \end{bmatrix}$	Unstable

the rate of the heading angle). The question is whether to measure ψ via the camera or via the inertial measurement unit in order to ensure that the system is exponentially stable. Notice that both sampling intervals 0.3 *(s) and* 1.1 *(s) are smaller than the guaranteed stability limit for the single rate case which is* 1.19 *(s). Intuitively, grouping ψ with y that has the faster sampling rate seems to have more chance of stabilizing the system than grouping ψ with r. Simulation results, however, show the opposite (see Table 4.1). The system is stable if ψ is sampled at the slower rate along with r (i.e. sampling intervals of up to* 1.1 *(s)). The system becomes unstable if ψ is sampled along with y at the faster rate (i.e. sampling periods of up to* 0.3 *(s)).* □

The results of this chapter also find application in the field of sensor networks. In these networks, m sensors are deployed over a region to acquire data from $n > m$ points of interest. The sensors can possibly have different sampling rates. The problem of assigning each point of interest to a sensor such that a global objective (stability) is achieved is an application of the theorems in this chapter.

The rest of the chapter is organized as follows. Section 4.2 is dedicated to problem formulation and preliminary notions. The main result of the chapter is presented in Section 4.3. A numerical example is provided in Section 4.4, followed by conclusion in Section 4.5. In terms of the notation used in this chapter,

Notation. The matrix entry that is located on the i^{th} row and j^{th} column of a matrix Z is represented by $Z_{(i,j)}$.

4.2 Problem Formulation

Consider a stabilizable linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{4.1}$$

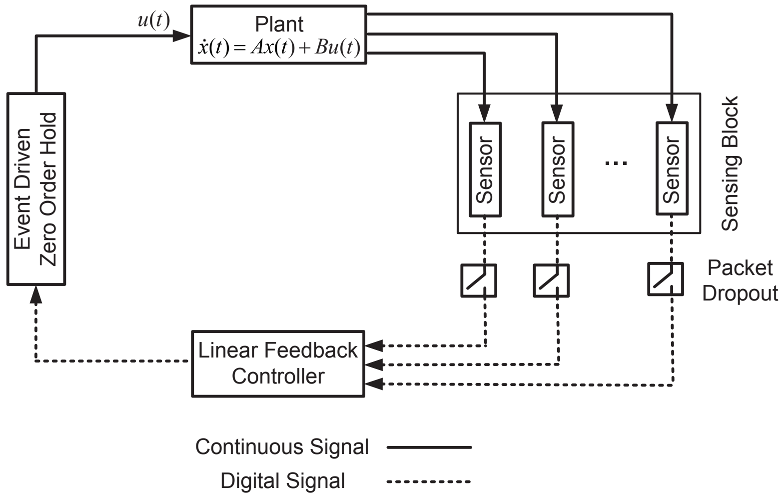


Figure 4.2: The schematic diagram of a linear multi-rate sampled-data system.

where $x \in \mathbb{R}^{n_x}$ denotes the state vector, $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, and $u \in \mathbb{R}^{n_u}$ is the control input. Let a continuous-time stabilizing linear controller for (4.1) be defined by

$$u(t) = Kx(t), \tag{4.2}$$

where $K \in \mathbb{R}^{n_u \times n_x}$. In practice, the controller is located in the feedback loop between a sensing block and an event driven zero order hold. Furthermore, it is assumed that the non-ideal communication links experience data packet dropouts (see Fig. 4.2). Packet dropouts are modeled via a switch. When the switch is closed, data is transmitted through the communication link. When the switch is open, however, data is assumed to be dropped. The sensing block comprises n_s sensors represented by S^i, $i \in \mathbb{I}$, where

$$\mathbb{I} = \{1, \ldots, n_s\}.$$

Definition 4.1. *The set $\mathcal{P} = \{\mathbb{I}_i | i \in \mathbb{I}\}$ is called a partition of $\{1, \ldots, n_x\}$ if it satisfies the following properties*

$$\mathbb{I}_i \neq \emptyset, \forall\, i \in \mathbb{I}, \tag{4.3a}$$

$$\mathbb{I}_i \bigcap \mathbb{I}_{i'} = \emptyset, \forall\, i \neq i', \tag{4.3b}$$

$$\bigcup_{i \in \mathbb{I}} \mathbb{I}_i = \{1, \ldots, n_x\}. \tag{4.3c}$$

Let $\mathcal{P} = \{\mathbb{I}_i | i \in \mathbb{I}\}$ represent a partition of $\{1, \ldots, n_x\}$. The multi-rate structure of the problem discussed in this chapter is presented in the following assumption.

Assumption 4.1. *Each sensor* S^i*,* $i \in \mathbb{I}$*, is dedicated to sampling the states* x_j*,* $j \in \mathbb{I}_i$*.*

Next, we parametrize the partition $\mathcal{P}$ by a matrix $\mathcal{L} \in \{0,1\}^{n_x \times n_s}$. Each row of the matrix $\mathcal{L}$ corresponds to one of the states. The columns of $\mathcal{L}$ correspond to the n_s sensors and represent the sets $\mathbb{I}_i$, $i \in \mathbb{I}$. The entries of $\mathcal{L}$ are defined in the Boolean domain $\{0,1\}$. If $\mathcal{L}_{(j,i)} = 1$ then the state x_j is sampled by the sensor S^i or equivalently $j \in \mathbb{I}_i$. In a similar way, $\mathcal{L}_{(j,i)} = 0$ implies $j \notin \mathbb{I}_i$.

Lemma 4.1. *Partition* $\mathcal{P}$ *can be parametrized by a matrix* $\mathcal{L} \in \{0,1\}^{n_x \times n_s}$ *under the constraints*

$$\mathcal{L}\boldsymbol{1} = \boldsymbol{1}, \tag{4.4}$$

$$\boldsymbol{1}^T\mathcal{L} \succeq \boldsymbol{1}^T, \tag{4.5}$$

where $\succeq$ *represents an elementwise inequality and* $\boldsymbol{1}$ *denotes the column vector with all elements equal to* 1*.*

Proof. Equation (4.4) guarantees that each row of $\mathcal{L}$ has one and only one element equal to one, which imply (4.3c) and (4.3b), respectively. In addition, inequality (4.5) guarantees that each column of $\mathcal{L}$ has at least one element equal to one, i.e. (4.3a). This concludes the proof. □

Remark 4.1. *Ignoring condition (4.5), we can address the scenario where one or more sensors do not measure any of the states. While this scenario violates (4.3a) in Definition 4.1, it is compatible with the main results of the chapter.*

The sensors are assumed to have uncertain and non-uniform but bounded sampling intervals as formulated in the next assumption.

Assumption 4.2. *The sensor* S^i*,* $i \in \mathbb{I}$*, performs measurement at instants* s_k^i*, where* $0 < \epsilon < s_{k+1}^i - s_k^i < \tau^i$*,* $\forall k \in \mathbb{N}$*.*

The positive constant ϵ models the fact that a sensor cannot measure a particular phenomenon twice at the same instant. The number τ^i denotes the longest interval between two consecutive samplings by sensor S^i. Without loss of generality, by the index $k \in \mathbb{N}$, we denote only the instants s_k^i for which a data packet is not lost. For each sensor, the time elapsed since the sensor's last sampling instant is denoted by a sawtooth function $\rho^i(t)$ (see Fig. 4.3) defined as

$$\rho^i(t) = t - s_k^i, \text{ for } t \in [s_k^i, s_{k+1}^i). \tag{4.6}$$

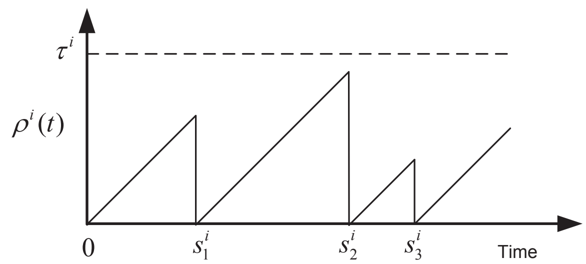


Figure 4.3: The sawtooth function $\rho^i(t)$

Equation (4.6) and Assumption 4.2 yield

$$0 \leq \rho^i < \tau^i. \tag{4.7}$$

Next, we formulate the control signal equation in a multi-rate sampled-data structure. To this end, let diagonal matrices $D_i \in \{0,1\}^{n_x \times n_x}$, $i \in \mathbb{I}$, be defined as

$$D_{i(j,j)} = \begin{cases} 1 & \text{if } j \in \mathbb{I}_i, \\ 0 & \text{otherwise,} \end{cases} \tag{4.8}$$

where $\mathbb{I}_i \in \mathcal{P}$. It is assumed that the sensors are time driven and the controller and the zero order hold are event driven. In other words, the controller and the zero order hold are updated as soon as new data becomes available. Therefore, the control signal (4.2) can be rewritten as

$$u(t) = K\overline{x}(t), \tag{4.9}$$

where

$$\overline{x}(t) = \sum_{i \in \mathbb{I}} D_i x(t - \rho^i(t)). \tag{4.10}$$

Lemma 4.2. *Let v_i and w_i, $i \in \mathbb{I}$, be arbitrary vectors in $\mathbb{R}^{n_x}$, Z be an arbitrary matrix in $\mathbb{R}^{n_x \times n_x}$, and D_i, $i \in \mathbb{I}$, be diagonal matrices defined in (4.8). Then,*

(a) $\sum_{i \in \mathbb{I}} D_i = I$,

(b) $D_i D_{i'} = \begin{cases} 0 & \text{if } i \neq i', \\ D_i & \text{if } i = i', \end{cases}$

(c) $\sum_{i \in \mathbb{I}} \left[(D_i v_i)^T (D_i w_i)\right] = \left(\sum_{i \in \mathbb{I}} D_i v_i\right)^T \left(\sum_{i \in \mathbb{I}} D_i w_i\right)$,

(d) $\sum_{i\in\mathbb{I}}\big[(D_iv_i)^TZ(D_iw_i)\big] = \big(\sum_{i\in\mathbb{I}} D_iv_i\big)^T\big(\sum_{i\in\mathbb{I}} D_iZD_i\big)\big(\sum_{i\in\mathbb{I}} D_iw_i\big)$.

Proof. The proof of parts (a) and (b) are straightforward and follow from the definition of matrix D_i in (4.8). Using part (b), the proof of part (c) is as follows

$$\begin{aligned}\Big(\sum_{i\in\mathbb{I}} D_iv_i\Big)^T\Big(\sum_{i\in\mathbb{I}} D_iw_i\Big) &= \sum_{i\in\mathbb{I}}\Big[v_i^TD_i\Big(\sum_{i'\in\mathbb{I}} D_{i'}w_{i'}\Big)\Big] = \sum_{i\in\mathbb{I}} v_i^TD_iw_i\\ &= \sum_{i\in\mathbb{I}}\big[(D_iv_i)^T(D_iw_i)\big].\end{aligned}$$

The proof of part (d) is similar to the proof of part (c) and is therefore omitted. □

The instants at which (at least) one of the n_s sensors performs a sampling action constitute an increasing sequence in time, represented by $\{t_n\}$, $n\in\mathbb{N}$ (see Fig. 4.4). Therefore, each time instant t_n, $n\in\mathbb{N}$, is associated with (i.e. is equal to) at least one and at most n_s instants s_k^i, $k\in\mathbb{N}$, $i\in I$. The time elapsed since the last sampling instant by any of the n_s sensors is denoted by $\rho(t)$, i.e.

$$\begin{aligned}\rho(t) &= t-t_n,\ \ t\in[t_n,t_{n+1})\\ &= \min_i \rho^i(t),\ \ i\in\mathbb{I}.\end{aligned}\tag{4.11}$$

Therefore, based on (4.7)

$$0\le\rho(t)<\tau,\tag{4.12}$$

where

$$\tau=\min_i\tau^i,\ \ i\in\mathbb{I}.\tag{4.13}$$

Figure 4.4 illustrates $\rho(t)$ for a system with two sensors. Based on Assumption 4.2, there exists a lower bound on the length of the interval (s_k^i, s_{k+1}^i), $k\in\mathbb{N}$, $i\in\mathbb{I}$. The length of the interval (t_n,t_{n+1}), $n\in\mathbb{N}$, however, can approach zero because a sensor might possibly sample right after another sensor. Nonetheless, the following lemma holds for a scalar T defined as

$$T=\max_i\tau^i,\ \ i\in\mathbb{I}.\tag{4.14}$$

Lemma 4.3. *For any interval spanning a time period longer than* $(n_s+1)T$*, there exists at least one interval* $(t_{n'},t_{n'+1})$*,* $n'\in\mathbb{N}$*, with a length greater than* ϵ/n_s*.*

Proof. The proof is similar to the proof of Lemma 3.1 and is therefore omitted. □

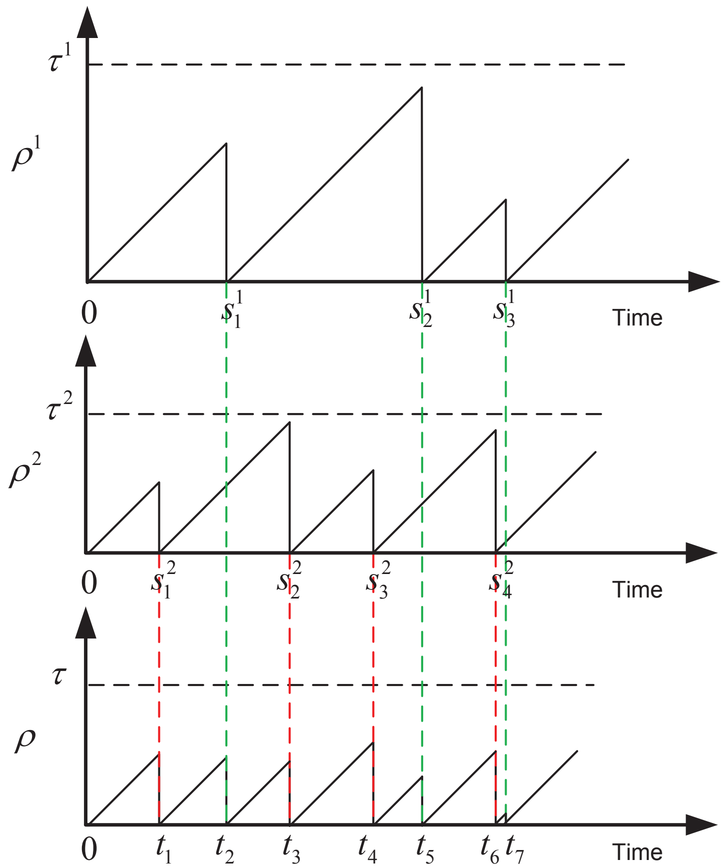


Figure 4.4: The sawtooth functions $\rho^1(t)$, $\rho^2(t)$, and $\rho(t)$ in a multi-rate sampled-data structure with two sensors.

The following lemma exploits the special structure of matrices $\mathcal{L}$ and D_i, $i \in \mathbb{I}$, and will be used in the proof of the main result.

Lemma 4.4. *Let D_i, $i \in \mathbb{I}$, be diagonal matrices defined in (4.8) and assume that the matrix $\mathcal{L}$ satisfies the conditions in Lemma 4.1. Consider symmetric matrices $R_i \in \mathbb{R}^{n_x \times n_x}$, $i \in \mathbb{I}$, satisfying*

$$|R_{i(j,k)}| \leq \beta \min\{\mathcal{L}_{(j,i)}, \mathcal{L}_{(k,i)}\}, \ \forall\, i \in \mathbb{I} \text{ and } j, k \in \{1, \ldots, n_x\}, \tag{4.15}$$

where β is an arbitrary positive scalar. Then,

(a) $D_i R_{i'} = R_{i'} D_i = 0$, $\forall\, i, i' \in \mathbb{I}$, $i \neq i'$,

(b) $D_i R_i = R_i D_i = R_i$, $\forall\, i \in \mathbb{I}$,

(c) $D_i(\sum_{i' \in \mathbb{I}} R_{i'}) = (\sum_{i' \in \mathbb{I}} R_{i'}) D_i = R_i$, $\forall\, i \in \mathbb{I}$,

(d) $D_i(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'}) = (\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})D_i = R_i/\tau^i$, $\forall\, i \in \mathbb{I}$,

(e) $\sum_{i\in\mathbb{I}} \big[\gamma_i D_i(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1} D_i\big] = \big[\sum_{i\in\mathbb{I}} R_i/(\tau^i\gamma_i)\big]^{-1}$, *where* $\gamma_i \neq 0$ *are real scalars.*

Proof. The inequalities in (4.15) yield

$$\begin{cases} -\beta \le R_{i(j,k)} \le \beta & \text{if } j \in \mathbb{I}_i \text{ and } k \in \mathbb{I}_i, \\ R_{i(j,k)} = 0 & \text{otherwise.} \end{cases}$$

Considering the special structure of matrices D_i and R_i, $i \in \mathbb{I}$, the proofs of parts (a) and (b) are straightforward and follow the definition of matrix multiplication. Parts (c) and (d) are easily derived based on (a) and (b). The proof of part (e) is as follows

$$\begin{aligned} &\sum_{i\in\mathbb{I}} \big[\gamma_i D_i(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1} D_i\big] \\ &= \sum_{i\in\mathbb{I}} \big[\gamma_i D_i(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1} D_i\big] \big[(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1}\big] \\ &= \sum_{i\in\mathbb{I}} \big[\gamma_i D_i(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1} D_i(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})\big](\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1}. \end{aligned}$$

Based on the first equality in part (d) we can write

$$\begin{aligned} \sum_{i\in\mathbb{I}} \big[\gamma_i D_i(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1} D_i\big] &= \sum_{i\in\mathbb{I}} \big[\gamma_i D_i(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1}(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'}) D_i\big](\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1} \\ &= \sum_{i\in\mathbb{I}} \big[\gamma_i D_i\big](\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1}, \end{aligned}$$

where we used Lemma 4.2(b) in the last equality. Lemma 4.2(b) also yields

$$(\sum_{i\in\mathbb{I}} \gamma_i D_i)(\sum_{i\in\mathbb{I}} D_i/\gamma_i) = I.$$

Therefore,

$$\begin{aligned} \sum_{i\in\mathbb{I}} \big[\gamma_i D_i(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})^{-1} D_i\big] &= \big[(\sum_{i'\in\mathbb{I}} R_{i'}/\tau^{i'})(\sum_{i\in\mathbb{I}} D_i/\gamma_i)\big]^{-1} \\ &= \big[\sum_{i\in\mathbb{I}} R_i/(\tau^i\gamma_i)\big]^{-1}, \end{aligned}$$

where we used Lemma 4.4(d) in the last equality. □

It is easy to verify (see e.g. [92]) that the norm function is convex, the pointwise minimum of a set of linear functions is concave, and the subtraction of a concave function from a convex function is convex. Therefore, the inequalities in (4.15) are convex. The main result of this chapter is presented in the next section.

4.3 Main Results

This section addresses stability of linear multi-rate sampled-data systems. It is assumed that a stabilizing controller is already designed in continuous-time and n_s sensors with different sampling rates are available. The upper bounds τ^i, $i \in \mathbb{I}$, on the nonuniform and uncertain sampling intervals are assumed to be known. Our objective is to partition the state vector and assign each part to a sensor such that exponential stability of the closed-loop sampled-data system is guaranteed.

Theorem 4.1. *Consider the closed-loop linear multi-rate sampled-data system defined in (4.1) and (4.9) under Assumptions 4.1 and 4.2. Given positive scalars α and β, there exists a partition of the state vector among sensors, parametrized by a matrix $\mathcal{L}$, that guarantees global uniform exponential stability with a decay rate greater than $\alpha/2$, if there exist positive definite matrices P, R_0, and X, symmetric matrices R_i, $i \in \mathbb{I}$, and matrices N and $\overline{N}$, with appropriate dimensions, satisfying conditions (4.4), (4.5), (4.15), and*

$$\begin{bmatrix} \Psi + \tau M_1 & \star & \star \\ \overline{R}\begin{bmatrix} A & BK & 0 \end{bmatrix} & -\sum_{i\in\mathbb{I}} R_i/\tau^i & \star \\ \overline{N}^T & 0 & -\sum_{i\in\mathbb{I}} R_i/(\tau^i e^{\alpha\tau^i}) \end{bmatrix} < 0 \tag{4.16}$$

$$\begin{bmatrix} \Psi + \tau M_2 & \star & \star & \star \\ \overline{R}\begin{bmatrix} A & BK & 0 \end{bmatrix} & -\sum_{i\in\mathbb{I}} R_i/\tau^i & \star & \star \\ \overline{N}^T & 0 & -\sum_{i\in\mathbb{I}} R_i/(\tau^i e^{\alpha\tau^i}) & \star \\ \tau N^T & 0 & 0 & -\tau e^{-\alpha\tau} R_0 \end{bmatrix} < 0 \tag{4.17}$$

$$\overline{R} = \sum_{i\in\mathbb{I}} R_i > 0 \tag{4.18}$$

where τ is defined in (4.13) and

$$\Psi = \begin{bmatrix} A & BK & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 \end{bmatrix} + \begin{bmatrix} P & 0 & 0 \end{bmatrix}^T \begin{bmatrix} A & BK & 0 \end{bmatrix} + \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
&- \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T N^T - N \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} I & -I & 0 \end{bmatrix}^T \overline{N}^T - \overline{N} \begin{bmatrix} I & -I & 0 \end{bmatrix} \\
&- \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
M_1 = &\begin{bmatrix} A & BK & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T R_0 \begin{bmatrix} A & BK & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} A & BK & 0 \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix} \\
&+ \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} A & BK & 0 \end{bmatrix} + \alpha \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix},
\end{aligned}
$$

$$
M_2 = - \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T N^T - N \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.
$$

Proof. Here we show that the LMIs in Theorem 4.1 are sufficient conditions for the LKF (4.19) to satisfy the conditions of Theorem 2.1. To this end, let $\mathcal{W}([-T,0],\mathbb{R}^{n_x})$ be the space of absolutely continuous functions with square integrable first-order derivatives, mapping the interval $[-T,0]$ to $\mathbb{R}^{n_x}$. We define the function $x_t \in \mathcal{W}$ as $x_t(r) = x(t+r), \ -T \le r \le 0$, and denote its norm by

$$
||x_t||_{\mathcal{W}} = \sum_{i=\mathbb{I}} \max_{r\in[-T,0]} |D_i x_t(r)| + \left[\int_{-T}^{0} |\dot{x}_t(r)|^2 \,\mathrm{d}r \right]^{\frac{1}{2}}.
$$

Consider the Krasovskii functional candidate

$$
V(t,x_t) = V_1 + V_2 + V_3 + V_4, \ t \in [t_n, t_{n+1}), \tag{4.19}
$$

where

$$
\begin{aligned}
V_1 =& x^T(t) P x(t), \\
V_2 =& (\tau - \rho) \int_{t-\rho}^{t} e^{\alpha(s-t)} \begin{bmatrix} \dot{x}^T(s) & \overline{x}^T(s) & x^T(t_n) \end{bmatrix} R_0 \begin{bmatrix} \dot{x}^T(s) & \overline{x}^T(s) & x^T(t_n) \end{bmatrix}^T \mathrm{d}s, \\
V_3 =& \sum_{i=\mathbb{I}} (\tau^i - \rho^i) \int_{t-\rho^i}^{t} e^{\alpha(s-t)} (D_i \dot{x}(s))^T \overline{R} (D_i \dot{x}(s)) \,\mathrm{d}s, \\
V_4 =& (\tau - \rho)(x(t) - x(t_n))^T X (x(t) - x(t_n)),
\end{aligned}
$$

where $\overline{R}$ is defined in (4.18), P, R_0, and X are symmetric matrices, and α is a given positive scalar which represents the decay rate. Similar to Chapter 3, it can be shown that $P > 0$, $R_0 > 0$, $\overline{R} > 0$, and $X > 0$ are sufficient conditions for the Krasovskii functional (4.19) to satisfy inequalities (2.10) and (2.11) in Theorem 2.1. In what follows, we prove that LMIs (4.16) and (4.17) are sufficient conditions for $\dot{V} + \alpha V \leq 0$ in the interval between two sampling instants. The time derivative of V_1 is

$$\dot{V}_1 = \dot{x}^T P x + x^T P \dot{x}. \tag{4.20}$$

From (4.11) we have $\dot{\rho} = 1$. Hence, applying the Leibniz integral rule to V_2 yields

$$\begin{aligned}\dot{V}_2 = &- \int_{t-\rho}^{t} e^{\alpha(s-t)} \left[\dot{x}^T(s) \quad \overline{x}^T(s) \quad x^T(t_n)\right] R_0 \left[\dot{x}^T(s) \quad \overline{x}^T(s) \quad x^T(t_n)\right]^T \mathrm{d}s \\ &+ (\tau - \rho) \left[\dot{x}^T \quad \overline{x}^T \quad x^T(t_n)\right] R_0 \left[\dot{x}^T \quad \overline{x}^T \quad x^T(t_n)\right]^T - \alpha V_2. \end{aligned} \tag{4.21}$$

Since R_0 is positive definite, $\alpha > 0$, and $\rho < \tau$ (see (4.12)), for any $s \in [t-\rho, t]$ and any arbitrary time varying vector $h(t) \in \mathbb{R}^{3n_x}$ we can write

$$\begin{bmatrix} \begin{bmatrix} \dot{x}(s) \\ \overline{x}(s) \\ x(t_n) \end{bmatrix} \\ h \end{bmatrix}^T \begin{bmatrix} e^{\alpha(s-t)} R_0 & -I \\ -I & e^{\alpha\tau} R_0^{-1} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \dot{x}(s) \\ \overline{x}(s) \\ x(t_n) \end{bmatrix} \\ h \end{bmatrix} \geq 0. \tag{4.22}$$

This inequality can be verified using Schur complement. Hence, for all $s \in [t-\rho, t]$,

$$\begin{aligned} &-e^{\alpha(s-t)} \left[\dot{x}^T(s) \quad \overline{x}^T(s) \quad x^T(t_n)\right] R_0 \left[\dot{x}^T(s) \quad \overline{x}^T(s) \quad x^T(t_n)\right]^T \\ &\leq h^T e^{\alpha\tau} R_0^{-1} h - \left[\dot{x}^T(s) \quad \overline{x}^T(s) \quad x^T(t_n)\right] h - h^T \left[\dot{x}^T(s) \quad \overline{x}^T(s) \quad x^T(t_n)\right]^T. \end{aligned}$$

For s varying between $t - \rho$ and t, the vectors $\overline{x}(s)$ and $x(t_n)$ are constant, and $x(s) = x_s(0) \in \mathcal{W}$ is absolutely continuous. Therefore, integrating both sides with respect to s, yields

$$\begin{aligned} &- \int_{t-\rho}^{t} e^{\alpha(s-t)} \left[\dot{x}^T(s) \quad \overline{x}^T(s) \quad x^T(t_n)\right] R_0 \left[\dot{x}^T(s) \quad \overline{x}^T(s) \quad x^T(t_n)\right]^T \mathrm{d}s \\ &\leq \rho h^T e^{\alpha\tau} R_0^{-1} h - \left[x^T - x^T(t-\rho) \quad \rho\overline{x}^T \quad \rho x^T(t_n)\right] h \\ &\quad - h^T \left[x^T - x^T(t-\rho) \quad \rho\overline{x}^T \quad \rho x^T(t_n)\right]^T. \end{aligned} \tag{4.23}$$

Based on (4.11), $t-\rho = t_n$. Hence, replacing (4.23) in (4.21), we get

$$\begin{aligned}\dot{V}_2 \leq & \rho h^T e^{\alpha\tau} R_0^{-1} h - \left[x^T - x^T(t_n) \quad \rho\overline{x}^T \quad \rho x^T(t_n)\right] h \\ & - h^T \left[x^T - x^T(t_n) \quad \rho\overline{x}^T \quad \rho x^T(t_n)\right]^T \\ & + (\tau-\rho)\left[\dot{x}^T \quad \overline{x}^T \quad x^T(t_n)\right] R_0 \left[\dot{x}^T \quad \overline{x}^T \quad x^T(t_n)\right]^T - \alpha V_2. \end{aligned} \tag{4.24}$$

According to (4.18), $\overline{R}$ is positive definite. Therefore, a similar inequality can be written for the time derivative of V_3 as follows

$$\begin{aligned}\dot{V}_3 \leq & \sum_{i=\mathbb{I}} \Big(\rho^i (D_i h_i)^T e^{\alpha\tau^i} \overline{R}^{-1} (D_i h_i) - \left(D_i(x - x(t-\rho^i))\right)^T (D_i h_i) \\ & \qquad - (D_i h_i)^T \left(D_i(x - x(t-\rho^i))\right) + (\tau^i - \rho^i)(D_i\dot{x})^T \overline{R}(D_i\dot{x})\Big) - \alpha V_3 \\ \leq & \sum_{i=\mathbb{I}} \Big(\tau^i (D_i h_i)^T e^{\alpha\tau^i} \overline{R}^{-1} (D_i h_i) - \left(D_i(x - x(t-\rho^i))\right)^T (D_i h_i) \\ & \qquad - (D_i h_i)^T \left(D_i(x - x(t-\rho^i))\right) + \tau^i (D_i\dot{x})^T \overline{R}(D_i\dot{x})\Big) - \alpha V_3, \end{aligned} \tag{4.25}$$

where $h_i(t) \in \mathbb{R}^{n_x}$, $i \in \mathbb{I}$, are arbitrary time-varying vectors and we used (4.7) to make the second inequality independent of the coefficient ρ^i. Leaving ρ^i in inequality (4.25) decreases conservatism of the sufficient conditions in Theorem 4.1 but increases the number of LMIs to 2^{n_s} and causes scalability issues. Let $\overline{h} = \sum_{i=\mathbb{I}} D_i h_i$. Based on Lemma 4.2(a) and 4.2(c) inequality (4.25) yields

$$\begin{aligned}\dot{V}_3 \leq & \sum_{i=\mathbb{I}} \Big((D_i h_i)^T e^{\alpha\tau^i} \overline{R}^{-1} \tau^i (D_i h_i) + (D_i\dot{x})^T \overline{R}(\overline{R}^{-1}\tau^i \overline{R})(D_i\dot{x})\Big) \\ & - (x-\overline{x})^T \overline{h} - \overline{h}^T (x - \overline{x}) - \alpha V_3, \end{aligned} \tag{4.26}$$

where $\overline{x}$ is defined in (4.10). Let $\overline{\tau}$ be a diagonal matrix defined as $\overline{\tau} = \sum_{i\in\mathbb{I}} \tau^i D_i$. Based on Lemma 4.2(b), we can write $\tau^i D_i = \overline{\tau} D_i$. Therefore, using the definition of $\overline{R}$ in (4.18) and the first equality in Lemma 4.4(c), inequality (4.26) yields

$$\begin{aligned}\dot{V}_3 \leq & \sum_{i=\mathbb{I}} \Big((D_i h_i)^T e^{\alpha\tau^i} \overline{R}^{-1} \overline{\tau} (D_i h_i) + (D_i\overline{R}\dot{x})^T \overline{R}^{-1} \overline{\tau}(D_i\overline{R}\dot{x})\Big) \\ & - (x-\overline{x})^T \overline{h} - \overline{h}^T (x - \overline{x}) - \alpha V_3. \end{aligned}$$

Note that $\overline{\tau}$ is invertible and $\overline{\tau}^{-1} = \sum_{i\in\mathbb{I}} D_i/\tau^i$. Therefore, Lemma 4.2(d) yields

$$\begin{aligned}\dot{V}_3 \leq & \overline{h}^T[\sum_{i=\mathbb{I}} e^{\alpha\tau^i} D_i(\overline{\tau}^{-1}\overline{R})^{-1}D_i]\overline{h} + \dot{x}^T\overline{R}[\sum_{i=\mathbb{I}} D_i(\overline{\tau}^{-1}\overline{R})^{-1}D_i]\overline{R}\dot{x} \\ & - (x-\overline{x})^T\,\overline{h} - \overline{h}^T\,(x-\overline{x}) - \alpha V_3 \\ = & \overline{h}^T[\sum_{i=\mathbb{I}} e^{\alpha\tau^i} D_i(\sum_{i'\in\mathbb{I}} \frac{D_{i'}}{\tau^{i'}} \sum_{i''\in\mathbb{I}} R_{i''})^{-1}D_i]\overline{h} + \dot{x}^T\overline{R}[\sum_{i=\mathbb{I}} D_i(\sum_{i'\in\mathbb{I}} \frac{D_{i'}}{\tau^{i'}} \sum_{i''\in\mathbb{I}} R_{i''})^{-1}D_i]\overline{R}\dot{x} \\ & - (x-\overline{x})^T\,\overline{h} - \overline{h}^T\,(x-\overline{x}) - \alpha V_3.\end{aligned}$$

Using Lemma 4.4(c) we can write

$$\begin{aligned}\dot{V}_3 \leq & \overline{h}^T[\sum_{i=\mathbb{I}} e^{\alpha\tau^i} D_i(\sum_{i'\in\mathbb{I}} \frac{R_{i'}}{\tau^{i'}})^{-1}D_i]\overline{h} + \dot{x}^T\overline{R}[\sum_{i=\mathbb{I}} D_i(\sum_{i'\in\mathbb{I}} \frac{R_{i'}}{\tau^{i'}})^{-1}D_i]\overline{R}\dot{x} \\ & - (x-\overline{x})^T\,\overline{h} - \overline{h}^T\,(x-\overline{x}) - \alpha V_3.\end{aligned}$$

Next, Lemma 4.4(e) and Lemma 4.2(a) yield

$$\dot{V}_3 \leq \overline{h}^T\big(\sum_{i\in\mathbb{I}} \frac{R_i}{\tau^i e^{\alpha\tau^i}}\big)^{-1}\overline{h} + \dot{x}^T\overline{R}\big(\sum_{i\in\mathbb{I}} \frac{R_i}{\tau^i}\big)^{-1}\overline{R}\dot{x} - (x-\overline{x})^T\,\overline{h} - \overline{h}^T\,(x-\overline{x}) - \alpha V_3. \quad (4.27)$$

The time derivative of V_4 is computed as

$$\dot{V}_4 = -\,(x - x(t_n))^T X(x - x(t_n)) + (\tau - \rho)\left(\dot{x}^T X(x - x(t_n)) + (x - x(t_n))^T X\dot{x}\right). \quad (4.28)$$

We now define an augmented vector $\zeta(t)$ as

$$\zeta(t) = \begin{bmatrix} x^T(t) & \overline{x}^T(t) & x^T(t_n)\end{bmatrix}^T,\ t \in [t_n, t_{n+1}).$$

Recalling (4.1) and (4.9), the closed-loop vector field can be written as

$$\dot{x}(t) = \begin{bmatrix} A & BK & 0\end{bmatrix}\zeta(t). \quad (4.29)$$

Replacing (4.29) in (4.20), (4.24), (4.27), and (4.28), and setting $h(t) = N^T\zeta(t)$ and $\overline{h}(t) = \overline{N}^T\zeta(t)$, where $N \in \mathbb{R}^{3n_x\times 3n_x}$ and $\overline{N} \in \mathbb{R}^{3n_x\times n_x}$, yields

$$\dot{V} + \alpha V = \ \textstyle\sum_{l=1}^{4}(\dot{V}_l + \alpha V_l)$$

$$
\begin{aligned}
\leq \zeta^T \Bigg(& \begin{bmatrix} A & BK & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} A & BK & 0 \end{bmatrix} \\
& + \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} + \rho N e^{\alpha\tau} R_0^{-1} N^T - \begin{bmatrix} I & 0 & -I \\ 0 & \rho I & 0 \\ 0 & 0 & \rho I \end{bmatrix}^T N^T \\
& - N \begin{bmatrix} I & 0 & -I \\ 0 & \rho I & 0 \\ 0 & 0 & \rho I \end{bmatrix} + (\tau - \rho) \begin{bmatrix} A & BK & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T R_0 \begin{bmatrix} A & BK & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\
& + \overline{N} \Bigg(\sum_{i \in \mathbb{I}} \frac{R_i}{\tau^i e^{\alpha \tau^i}} \Bigg)^{-1} \overline{N}^T - \begin{bmatrix} I & -I & 0 \end{bmatrix}^T \overline{N}^T - \overline{N} \begin{bmatrix} I & -I & 0 \end{bmatrix} \\
& + \begin{bmatrix} A & BK & 0 \end{bmatrix}^T \overline{R} \Bigg(\sum_{i \in \mathbb{I}} \frac{R_i}{\tau^i} \Bigg)^{-1} \overline{R} \begin{bmatrix} A & BK & 0 \end{bmatrix} \\
& + (\alpha(\tau - \rho) - 1) \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix} \\
& + (\tau - \rho) \begin{bmatrix} A & BK & 0 \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix} \\
& + (\tau - \rho) \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} A & BK & 0 \end{bmatrix} \Bigg) \zeta. \qquad (4.30)
\end{aligned}
$$

Using Schur complement, for $\rho = 0$, LMI (4.16) implies $\dot{V} + \alpha V < 0$. Similarly, LMI (4.17) implies $\dot{V} + \alpha V < 0$ for $\rho = \tau$. Since (4.30) is affine in ρ, LMIs (4.16) and (4.17) are sufficient conditions for $\dot{V} + \alpha V < 0$ to hold for any $\rho \in (0, \tau)$, i.e. inequality (2.12) in Theorem 2.1 is satisfied. According to Assumption 4.2, for any time interval with a length smaller than ϵ, there exists a finite number of (at most n_s) sampling instants t_n, $n \in \mathbb{N}$. Therefore, inequality (2.13) is satisfied with $q = n_s$. This finishes the proof. □

Based on Theorem 4.1, the problem of finding a partition (parametrized by the matrix $\mathcal{L}$) that guarantees exponential stability is formulated as

Problem 4.1.

$$
\begin{aligned}
&\textit{find} && \mathcal{L} \in \{0,1\}^{n_x \times n_s} \\
&\textit{subject to} && P > 0,\ R_0 > 0,\ X > 0,\ R_i = R_i^T,\ i \in \mathbb{I},\ \alpha > 0,\ \beta > 0, \\
& && (4.4),\ (4.5),\ (4.15) - (4.18).
\end{aligned}
$$

According to (4.15), the entries of matrices R_i, $i \in \mathbb{I}$, are bounded by an arbitrary positive scalar β. When solving Problem 4.1 in an optimization software, we assign the biggest acceptable number by the software to β. Mixed integer programs are NP-hard. However, mixed integer problems of small size can be solved using free optimization software such as the BNB solver which is shipped with YALMIP [17].

4.4 Numerical Example

Example 4.2. *Consider the unicycle path following problem with parameters defined in Example 4.1. Let the first state y be acquired via a camera (at sampling intervals of up to* 0.3 *(s)) and the rate of the heading angle r be acquired by an inertial measurement unit (at sampling periods of up to* 0.6 *(s)). As explained in Example 4.1, the heading angle ψ can be measured either via the camera or via the inertial measurement unit. With states y and r already assigned to sensors (camera and inertial measurement unit, respectively), we use the algorithm in Problem 4.1 to find which sensor should be used to sample ψ such that exponential stability is guaranteed. In Example 4.1, it was shown via simulations that measuring the heading angle ψ with the inertial measurement unit stabilizes the system while sampling ψ with the camera leads to instability. Solving Problem 4.1, the heading angle ψ is assigned to the inertial measurement unit. This sensor allocation strategy is compatible with simulation results in Example 4.1.*

4.5 Conclusion

Sensor allocation with guaranteed exponential stability for linear multi-rate sampled-data systems was addressed. Sufficient Krasovskii-based conditions that yield a partition of the state vector were proposed such that the resulting closed-loop multi-rate sampled-data system is exponentially stable. The problem of finding a partition that guarantees exponential stability was cast as a mixed integer program subject to LMIs. In the next chapter, we address the observer design problem for linear multi-rate sampled-data systems.

Chapter 5

Observer Design for Linear Multi-rate Sampled-data Systems

This chapter addresses observer design for linear systems with multi-rate sampled output measurements. The sensors are assumed to be asynchronous and to have uncertain nonuniform sampling intervals. The contributions of this chapter are twofold. Given the MASP for each sensor, the main contribution of the chapter is to propose sufficient Krasovskii-based conditions for design of linear observers. The designed observers guarantee exponential convergence of the estimation error to the origin. Most importantly, the sufficient conditions are cast as a set of LMIs that can be solved efficiently. As a second contribution, given an observer gain, the problem of finding MASPs that guarantee exponential stability of the estimation error is also formulated as a convex optimization program in terms of LMIs. The theorems are applied to a unicycle path following example.

5.1 Introduction

In multi-rate sampled-data systems, data is gathered through several sensors that work at different sampling rates. One reason is that different phenomena (e.g. temperature, pressure, or voltage) are measured with different sensors that work at different sampling rates. As a second reason, different methods of sensing the same phenomenon can lead to different sampling frequencies (e.g. measuring an angle with a potentiometer, an encoder, or a camera through image processing). Finally, even if the sensors are synchronized, the inevitable delays and packet losses in non-ideal communication links result in the data arriving at the controller at different rates. The reader is referred to Chapter 3 and the references therein for stability analysis and controller synthesis of multi-rate sampled-data systems.

Design of single-rate sampled-data observers and single-rate sampled-data output feedback controllers have been the subject of numerous research (see [93–95] and the references therein). In [96], dual-rate output feedback control of a class of nonlinear systems is addressed using a high-gain observer. Observer-based robust fault detection of linear multi-rate sampled-data systems is addressed in [97–100]. A common drawback of [96–100] is the assumption that the sampling rate of the sensors are uniform and their ratios are rational numbers (commensurate samplings). However, the uniform and commensurate sampling assumptions do not hold in practice. For instance, in the servo control of brushless DC motors via Hall-effect sensors, the sampling intervals depend on the motor speed and are not uniform [88]. Reference [101] addresses observer design for a class of nonlinear single-rate sampled-data systems with nonuniform samplings using a Krasovskii-based theorem and LMIs. In contrast to [93–101], we address observer design for linear systems with multi-rate sampled output measurements, where the sensors are assumed to be asynchronous and to have uncertain nonuniform sampling intervals. To the best of the authors' knowledge, the continuous-time state estimation problem using asynchronous multi-rate discrete-time output measurements was not studied before.

The contributions of this chapter are twofold. Given the MASP for each sensor, the main contribution of the chapter is to propose sufficient Krasovskii-based conditions for design of linear multi-rate sampled-data observers. The designed observers guarantee exponential convergence of the estimation error to the origin. The sufficient conditions are cast as a set of LMIs that can be solved efficiently using available optimization software [16, 17]. As a second contribution, given an observer gain, the problem of finding MASPs that guarantee exponential stability of the estimation error is formulated as a convex optimization program in terms of LMIs. The importance of choosing the right sensors with adequate sampling frequencies is shown through a path following example.

The rest of the chapter is organized as follows. Section 5.2 is dedicated to preliminary notions and problem statement. The main results of the chapter are presented in Section 5.3. A path following example is provided in Section 5.4, followed by the concluding remarks in Section 5.5.

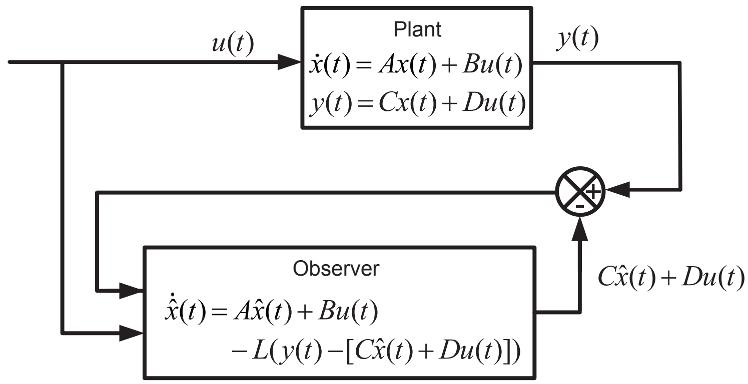


Figure 5.1: The schematic diagram of a linear plant and a continuous-time observer.

5.2 Problem Statement

Let an observable continuous-time linear system be defined as

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.1a}$$

$$y(t) = Cx(t) + Du(t), \tag{5.1b}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector, $y \in \mathbb{R}^{n_y}$ represents the output vector, $u \in \mathbb{R}^{n_u}$ is the control input, and A, B, C, and D are matrices of the appropriate dimension. Consider a continuous-time Luenberger observer with gain L (see Fig. 5.1). Let $\hat{x}(t)$ denote the observer state vector and $e(t) = x(t) - \hat{x}(t)$ be the estimation error. Therefore, the rate of change of $\hat{x}$ and e can be written as

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) - L\Big(y(t) - [C\hat{x}(t) + Du(t)]\Big) + Bu(t)\\ &= A\hat{x}(t) - LCe(t) + Bu(t),\\ \dot{e}(t) &= (A + LC)e(t),\end{aligned}$$

In practice, the output vector is measured via different sensors (see Fig. 5.2). In this chapter, the sensors are modeled as asynchronous sample and hold devices. Assume that the output vector comprises m components, i.e. $y^T = \begin{bmatrix} y_1^T & y_2^T & \dots & y_m^T \end{bmatrix}^T$. Each of these components is measured by a dedicated sensor S^i, $i \in \{1, \dots, m\}$, with an uncertain nonuniform sampling interval.

Assumption 5.1. *The sensor S^i, $i \in \{1, \dots, m\}$, samples the i^{th} component of the output vector (i.e. y_i) at sampling instants s_k^i, where $0 < \epsilon < s_{k+1}^i - s_k^i < \tau^i$, $\forall k \in \mathbb{N}$.*

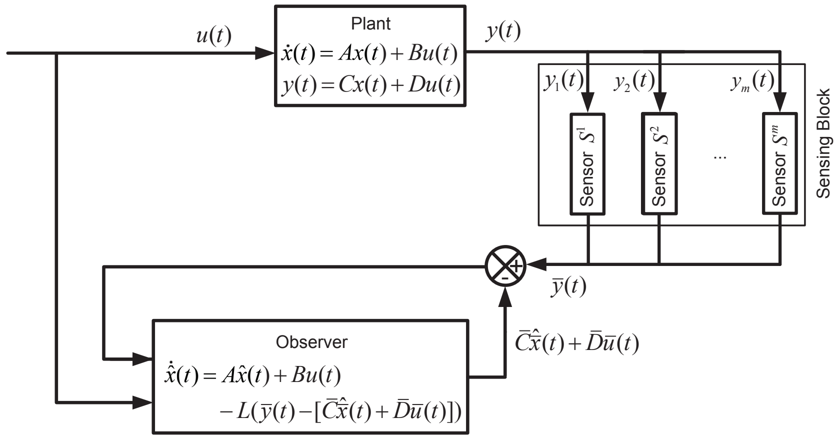


Figure 5.2: The schematic diagram of a linear multi-rate sampled-data observer.

The positive constant ϵ models the fact that a sensor cannot measure an output twice at the same instant. It is used later in the proof of the main result to rule out the occurrence of the Zeno phenomenon. The number τ^i denotes the longest interval between two consecutive samplings by sensor S^i, $i \in \{1, \ldots, m\}$. For each sensor, the time elapsed since the sensor's last sampling instant is denoted by a sawtooth function $\rho^i(t)$ (see Fig. 4.3) defined as

$$\rho^i(t) = t - s^i_k, \text{ for } t \in [s^i_k, s^i_{k+1}). \tag{5.2}$$

Based on Assumption 5.1, equation (4.6) yields

$$0 \leq \rho^i < \tau^i. \tag{5.3}$$

The sensors are assumed to be asynchronous. Hence, in the multi-rate sampled-data structure, the output of the sample and hold devices (the sensors) is written as

$$\overline{y}(t) = \begin{bmatrix} y_1^T(t - \rho^1(t)) & \ldots & y_m^T(t - \rho^m(t)) \end{bmatrix}^T.$$

Let C_i represent the row of the matrix C corresponding to the output y_i, i.e. $C = \begin{bmatrix} C_1^T & \ldots & C_m^T \end{bmatrix}^T$. Similarly, let D_i, represent the rows of the matrix D corresponding to the output y_i. Considering (5.1b), the vector $\overline{y}(t)$ can be rewritten as

$$\overline{y}(t) = \overline{C}\overline{x}(t) + \overline{D}\overline{u}(t),$$

where

$$\overline{C} = \begin{bmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & C_m \end{bmatrix}, \ \overline{D} = \begin{bmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & D_m \end{bmatrix},$$

$$\overline{x}(t) = \Big[x^T(t-\rho^1(t)) \quad \dots \quad x^T(t-\rho^m(t))\Big]^T,$$

$$\overline{u}(t) = \Big[u^T(t-\rho^1(t)) \quad \dots \quad u^T(t-\rho^m(t))\Big]^T.$$

In the multi-rate sampled-data structure, the rate of change of $\hat{x}$ and e can be written as

$$\begin{aligned} \dot{\hat{x}}(t) =& A\hat{x}(t) - L\Big(\overline{y}(t) - [\overline{C}\hat{\overline{x}}(t) + \overline{D}\overline{u}(t)]\Big) + Bu(t) \\ =& A\hat{x}(t) - L\overline{C}\overline{e}(t) + Bu(t), \\ \dot{e}(t) =& Ae(t) + L\overline{C}\overline{e}(t), \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} \hat{\overline{x}}(t) =& \Big[\hat{x}^T(t-\rho^1(t)) \quad \dots \quad \hat{x}^T(t-\rho^m(t))\Big]^T, \\ \overline{e}(t) =& \overline{x} - \hat{\overline{x}} = \Big[e^T(t-\rho^1(t)) \quad \dots \quad e^T(t-\rho^m(t))\Big]^T. \end{aligned} \tag{5.5}$$

The instants at which (at least) one of the m sensors performs a sampling action constitute an increasing sequence in time, represented by $\{t_n\}$, $n \in \mathbb{N}$ (see Fig. 4.4 for a system with two sensors). Therefore, each time instant t_n, $n \in \mathbb{N}$, is associated with (i.e. is equal to) at least one and at most m instants s_k^i, $k \in \mathbb{N}$, with different $i \in \{1, \dots, m\}$. Based on Assumption 5.1, there exists a lower bound on the length of the interval (s_k^i, s_{k+1}^i), $\forall k \in \mathbb{N}$, $i \in \{1, \dots, m\}$. The length of the interval (t_n, t_{n+1}), $\forall n \in \mathbb{N}$, however, can approach zero because two sensors might possibly sample right after each other. Nonetheless, the following lemma holds for a scalar T defined as

$$T = \max_i \tau^i, \ i \in \{1, \dots, m\}. \tag{5.6}$$

Lemma 5.1. *For any interval spanning a time period longer than $(m+1)T$, there exists at least one interval $(t_{n'}, t_{n'+1})$, $n' \in \mathbb{N}$, with a length larger than ϵ/m.*

Proof. The proof is similar to the proof of Lemma 3.1 and is therefore omitted. □

The time elapsed since the last sampling instant by any of the m sensors is denoted by $\rho(t)$ defined as

$$\begin{aligned}\rho(t) &= t - t_n,\ t \in [t_n, t_{n+1}) \\ &= \min_i \rho^i(t),\ i \in \{1, \dots, m\}.\end{aligned} \tag{5.7}$$

Therefore, based on (5.3),

$$0 \leq \rho(t) < \tau, \tag{5.8}$$

where

$$\tau = \min_i \tau^i,\ i \in \{1, \dots, m\}. \tag{5.9}$$

The function $\rho(t)$ is illustrated in Fig. 4.4 for a system with two sensors. Let $\mathcal{W}([-T, 0], \mathbb{R}^{n_x})$ be the space of absolutely continuous functions, with square integrable first-order derivatives, mapping the interval $[-T, 0]$ to $\mathbb{R}^{n_x}$. We define the function $e_t \in \mathcal{W}$ as

$$e_t(r) = e(t + r), \quad -T \leq r \leq 0,$$

and denote its norm by

$$||e_t||_{\mathcal{W}} = \max_{r \in [-T,0]} |e_t(r)| + \left[\int_{-T}^{0} |\dot{e}_t(r)|^2 \, \mathrm{d}r\right]^{\frac{1}{2}}. \tag{5.10}$$

Sufficient Krasovskii-based conditions for exponential stability of the estimation error are presented in the next section.

5.3 Main results

The following theorem provides a set of sufficient conditions for which the estimation error of a linear multi-rate sampled-data observer exponentially converge to the origin.

Theorem 5.1. *Consider the linear system defined in (5.1) and a multi-rate sampled-data observer with estimation error (5.4). Under Assumption 5.1, the estimation error is globally uniformly exponentially stable with a decay rate greater than $\alpha/2$ if there exist symmetric positive definite matrices P, R_0, R_i, $i \in \{1, \dots, m\}$, and X,* *and matrices N and $\overline{N}$, with appropriate dimensions, satisfying*

$$\begin{bmatrix} \Psi+\tau M_1 & \star & \star & \star \\ \tau R_0 \begin{bmatrix} A & L\overline{C} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} & -\tau R_0 & \star & \star \\ \overline{E}\overline{\tau}\overline{N}^T & 0 & -\overline{E}\overline{\tau}\overline{R} & \star \\ \overline{R}\overline{\tau}(\mathbf{1}\otimes \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}) & 0 & 0 & -\overline{R}\overline{\tau} \end{bmatrix} < 0 \tag{5.11}$$

$$\begin{bmatrix} \Psi+\tau M_2 & \star & \star & \star \\ \overline{E}\overline{\tau}\overline{N}^T & -\overline{E}\overline{\tau}\overline{R} & \star & \star \\ \overline{R}\overline{\tau}(\mathbf{1}\otimes \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}) & 0 & -\overline{R}\overline{\tau} & \star \\ \tau N^T & 0 & 0 & \frac{-\tau R_0}{\exp(\alpha\tau)} \end{bmatrix} < 0 \tag{5.12}$$

where τ is defined in (5.9) and

$$\overline{\tau} = \mathrm{diag}(\tau^1 I, \tau^2 I, \ldots, \tau^m I), \tag{5.13}$$

$$\overline{E} = \mathrm{diag}(\exp(\alpha\tau^1)I, \exp(\alpha\tau^2)I, \ldots, \exp(\alpha\tau^m)I), \tag{5.14}$$

$$\overline{R} = \mathrm{diag}(R_1, R_2, \ldots, R_m), \tag{5.15}$$

$$\begin{aligned} \Psi = {} & \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 \end{bmatrix} + \begin{bmatrix} P & 0 & 0 \end{bmatrix}^T \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix} + \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} \\ & - \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T N^T - N \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathbf{1}\otimes I & -I & 0 \end{bmatrix}^T \overline{N}^T \\ & - \overline{N} \begin{bmatrix} \mathbf{1}\otimes I & -I & 0 \end{bmatrix} - \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix}, \\ M_1 = {} & \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix} + \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix} \\ & + \alpha \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix}, \\ M_2 = {} & - \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T N^T - N \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \end{aligned}$$

Proof. Consider the candidate LKF

$$V(t, e_t) = V_1 + V_2 + V_3 + V_4, \; t \in [t_n, t_{n+1}), \tag{5.16}$$

where

$$
\begin{aligned}
V_1 =& e^T(t)Pe(t),\\
V_2 =& (\tau-\rho)\int_{t-\rho}^{t}\exp(\alpha(s-t))\left[\dot{e}^T(s)\quad \overline{e}^T(s)\quad e^T(t_n)\right]R_0\left[\dot{e}^T(s)\quad \overline{e}^T(s)\quad e^T(t_n)\right]^T\mathrm{d}s,\\
V_3 =& \sum_{i=1}^{m}(\tau^i-\rho^i)\int_{t-\rho^i}^{t}\exp(\alpha(s-t))\dot{e}^T(s)R_i\dot{e}(s)\,\mathrm{d}s,\\
V_4 =& (\tau-\rho)(e(t)-e(t_n))^TX(e(t)-e(t_n)),
\end{aligned}
$$

where P, R_0, R_i, $i\in\{1,\dots,m\}$, and X are symmetric positive definite matrices, α is a positive scalar representing the decay rate, and functions ρ^i and ρ are defined in (5.2) and (5.7), respectively. Here we show that the LMIs in Theorem 5.1 are sufficient conditions for the LKF (5.16) to satisfy the conditions of Theorem 2.1. To this end, similar to Chapter 3, it is easy to show that the LKF (5.16) satisfies inequalities (2.10) and (2.11) in Theorem 2.1, i.e.

$$c_1|e_t(0)|^2\le V(t,e_t)\le c_2||e_t||^2_{\mathcal{W}}, \tag{5.17}$$

$$V(t_n,e_{t_n})\le V(t_n^-,e_{t_n^-}),\ \forall\, n\in\mathbb{N}, \tag{5.18}$$

where c_1 and c_2 are positive scalars, and $V(t_n^-,e_{t_n^-})=\lim_{t\nearrow t_n}V(t,e_t)$. In order to prove exponential stability, we next prove that the LKF (5.16) satisfies inequality (2.12), i.e.

$$\dot{V}(t,e_t)+\alpha V(t,e_t)<0,\ \forall\, t\neq t_n,\ n\in\mathbb{N}. \tag{5.19}$$

The time derivative of V in the interval between two consecutive sampling instants $t\in(t_n,t_{n+1})$ is composed of four terms computed as follows. The time derivative of V_1 is

$$\dot{V}_1=\dot{e}^TPe+e^TP\dot{e}. \tag{5.20}$$

From (5.7) we have $\dot{\rho}=1$. Hence, applying the Leibniz integral rule to V_2 yields

$$
\begin{aligned}
\dot{V}_2=&-\int_{t-\rho}^{t}\exp(\alpha(s-t))\left[\dot{e}^T(s)\quad \overline{e}^T(s)\quad e^T(t_n)\right]R_0\left[\dot{e}^T(s)\quad \overline{e}^T(s)\quad e^T(t_n)\right]^T\mathrm{d}s\\
&+(\tau-\rho)\left[\dot{e}^T\quad \overline{e}^T\quad e^T(t_n)\right]R_0\left[\dot{e}^T\quad \overline{e}^T\quad e^T(t_n)\right]^T-\alpha V_2.
\end{aligned}\tag{5.21}
$$

Since R_0 is positive definite, $\alpha>0$, and $\rho<\tau$ (see (5.8)), for any $s\in[t-\rho,t]$ and any arbitrary time varying vector $h(t) \in \mathbb{R}^{(2+m)n_x}$ we can write

$$\begin{aligned}&\left[\begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} \quad h^T\right] \times \\ &\begin{bmatrix}\exp(\alpha(s-t))R_0 & -I \\ -I & \exp(\alpha\tau)R_0^{-1}\end{bmatrix} \times \left[\begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} \quad h^T\right]^T \geq 0. \qquad (5.22)\end{aligned}$$

This inequality can be verified using Schur complement. Hence, for all $s \in [t-\rho, t]$,

$$\begin{aligned}&-\exp(\alpha(s-t))\begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} R_0 \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix}^T \\ &\leq h^T \exp(\alpha\tau)R_0^{-1}h - \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} h - h^T \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix}^T.\end{aligned}$$

For s varying between $t-\rho$ and t, the vectors $\overline{e}(s)$ and $e(t_n)$ are constant, and $e(s) = e_s(0) \in \mathcal{W}$ is absolutely continuous. Therefore, integrating both sides with respect to s, yields

$$\begin{aligned}&-\int_{t-\rho}^{t} \exp(\alpha(s-t))\begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} R_0 \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix}^T \mathrm{d}s \\ &\quad\leq \rho h^T \exp(\alpha\tau)R_0^{-1}h - \begin{bmatrix}e^T - e^T(t-\rho) & \rho\overline{e}^T & \rho e^T(t_n)\end{bmatrix} h \\ &\qquad - h^T \begin{bmatrix}e^T - e^T(t-\rho) & \rho\overline{e}^T & \rho e^T(t_n)\end{bmatrix}^T. \qquad (5.23)\end{aligned}$$

Based on (5.7), $t-\rho = t_n$. Hence, replacing (6.27) in (5.21), we get

$$\begin{aligned}\dot{V}_2 \leq& \rho h^T \exp(\alpha\tau)R_0^{-1}h - \begin{bmatrix}e^T - e^T(t_n) & \rho\overline{e}^T & \rho e^T(t_n)\end{bmatrix} h \\ &- h^T \begin{bmatrix}e^T - e^T(t_n) & \rho\overline{e}^T & \rho e^T(t_n)\end{bmatrix}^T \\ &+ (\tau-\rho)\begin{bmatrix}\dot{e}^T & \overline{e}^T & e^T(t_n)\end{bmatrix} R_0 \begin{bmatrix}\dot{e}^T & \overline{e}^T & e^T(t_n)\end{bmatrix}^T - \alpha V_2. \qquad (5.24)\end{aligned}$$

Similarly, we can write the following inequality

$$\begin{aligned}\dot{V}_3 \leq \sum_{i=1}^{m}\Big(&\rho^i h_i^T \exp(\alpha\tau^i)R_i^{-1}h_i - \left[e - e(t-\rho^i)\right]^T h_i \\ &- h_i^T \left[e - e(t-\rho^i)\right] + (\tau^i - \rho^i)\dot{e}^T R_i \dot{e}\Big) - \alpha V_3, \qquad (5.25)\end{aligned}$$

where $h_i(t) \in \mathbb{R}^{n_x}$, $i \in \{1, \ldots, m\}$, are arbitrary time-varying vectors. Based on (5.3), $\tau^i h_i^T \exp(\alpha\tau^i)R_i^{-1}h_i$ and $\tau^i \dot{e}^T R_i \dot{e}$ are upper bounds for $\rho^i h_i^T \exp(\alpha\tau^i)R_i^{-1}h_i$ and $(\tau^i - \rho^i)\dot{e}^T R_i \dot{e}$, respectively. Hence, inequality (5.25) can be rewritten in a more compact form as

$$\dot{V}_3 \leq \overline{h}^T \overline{\tau} \overline{ER}^{-1} \overline{h} - [\mathbf{1} \otimes e - \overline{e}]^T \overline{h} - \overline{h}^T [\mathbf{1} \otimes e - \overline{e}] + (\mathbf{1} \otimes \dot{e})^T \overline{\tau} \overline{R} (\mathbf{1} \otimes \dot{e}) - \alpha V_3, \tag{5.26}$$

where $\overline{e}$, $\overline{\tau}$, $\overline{E}$, and $\overline{R}$ are defined in (5.5), (5.13)-(5.15), and $\overline{h} = \begin{bmatrix} h_1^T & h_2^T & \dots & h_m^T \end{bmatrix}^T$. The time derivative of V_4 is computed as

$$\dot{V}_4 = -(e - e(t_n))^T X (e - e(t_n)) + (\tau - \rho) \dot{e}^T X (e - e(t_n)) + (\tau - \rho)(e - e(t_n))^T X \dot{e}. \tag{5.27}$$

We now define an augmented vector $\zeta(t)$ as

$$\zeta(t) = \begin{bmatrix} e^T(t) & \overline{e}^T(t) & e^T(t_n) \end{bmatrix}^T, \ t \in [t_n, t_{n+1}). \tag{5.28}$$

Therefore, recalling (5.4),

$$\dot{e}(t) = \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix} \zeta(t). \tag{5.29}$$

Replacing (4.29) in (5.20), (9.42), (9.43), and (9.44), setting $h(t) = N^T \zeta(t)$ and $\overline{h}(t) = \overline{N}^T \zeta(t)$, where $N \in \mathbb{R}^{(2+m)n_x \times (2+m)n_x}$ and $\overline{N} \in \mathbb{R}^{(2+m)n_x \times mn_x}$, and using Lemma 3.2, yields

$$\begin{aligned}
\dot{V} + \alpha V = & \sum_{j=1}^{4} (\dot{V}_j + \alpha V_j) \\
\leq \zeta^T \Big(& \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix} \\
& + \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} + \rho N \exp(\alpha \tau) R_0^{-1} N^T \\
& - \begin{bmatrix} I & 0 & -I \\ 0 & \rho I & 0 \\ 0 & 0 & \rho I \end{bmatrix}^T N^T - N \begin{bmatrix} I & 0 & -I \\ 0 & \rho I & 0 \\ 0 & 0 & \rho I \end{bmatrix} \\
& + (\tau - \rho) \begin{bmatrix} A & L\overline{C} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T R_0 \begin{bmatrix} A & L\overline{C} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\
& + \overline{N} \overline{\tau} \overline{ER}^{-1} \overline{N}^T - \begin{bmatrix} \mathbf{1} \otimes I & -I & 0 \end{bmatrix}^T \overline{N}^T - \overline{N} \begin{bmatrix} \mathbf{1} \otimes I & -I & 0 \end{bmatrix} \\
& + (\mathbf{1} \otimes \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix})^T \overline{\tau} \overline{R} (\mathbf{1} \otimes \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}) \\
& + (\alpha(\tau - \rho) - 1) \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix}
\end{aligned}$$

$$+ (\tau - \rho) \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix}$$
$$+ (\tau - \rho) \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix} \Big) \zeta. \tag{5.30}$$

For $\rho = 0$, using Schur complement, inequality (5.11) implies $\dot{V} + \alpha V < 0$. Similarly, inequality (5.12) implies $\dot{V} + \alpha V < 0$ for $\rho = \tau$. Since (9.51) is affine in ρ, inequalities (5.11) and (5.12) are sufficient conditions for $\dot{V} + \alpha V < 0$ to hold for any $\rho \in (0, \tau)$. Therefore, based on (5.7), inequalities (5.11) and (5.12) are sufficient conditions for inequality (5.19) to be satisfied. Note that inequality (5.19) is the same as inequality (2.12) with the estimation error e replace with x. By Assumption 5.1, for any time interval with a length smaller than ϵ, there exists a finite number of (at most m) sampling instants t_n, $n \in \mathbb{N}$. Therefore, inequality (2.13) is satisfied with $q = m$. Hence, all the conditions of Theorem 2.1 are satisfied and the estimation error is exponentially stable with a decay rate greater than $\alpha/2$. □

Next, the sufficient conditions in Theorem 5.1 are used to address two problems in sampled-data observers. In the first problem, it is assumed that an observer gain L is available which exponentially stabilizes the estimation error in continuous-time. The objective is to find the MASP for the sensor S^j, $j \in \{1, ..., m\}$, such that exponential stability of the estimation error is preserved. Given L, α, and τ^i, $i \in \{1, ..., m\}$, the sufficient conditions in Theorem 5.1 become LMIs. These LMIs can be solved efficiently using available optimization software [16, 17]. Following the line search strategy, the problem of finding a lower bound[1] on the MASP for the sensor S^j, $j \in \{1, ..., m\}$, that guarantees exponential stability of the estimation error is formulated as

Problem 5.1.

maximize τ^j

subject to $P > 0,\ R_0 > 0,\ R_i > 0,\ i \in \{1, ..., m\},\ X > 0,$ (5.11) *and* (5.12).

We denote the computed lower bound on the MASP that guarantees exponential stability of the estimation error by $\tau^j_{\max}$. In the second problem, it is assumed that the upper bound on the sampling intervals for each sensor (i.e. τ^i, $i \in \{1, ..., m\}$) is known and the decay rate α is given. The objective is to design an observer gain L such that exponential stability of the estimation error is guaranteed. With L as an

[1] The computed value is *a lower bound* on the MASP because the LMIs in Theorem 5.1 are sufficient conditions.

optimization variable, the conditions in Theorem 5.1 are bilinear matrix inequalities. Using a change of variables, however, these conditions can be written in the form of LMIs as shown in the following corollary.

Corollary 5.1. *Given τ^i, $i \in \{1, ..., m\}$, and $\alpha > 0$, suppose there exist positive definite matrices P and $\mathcal{R}_0$, matrices Y, N, and $\overline{N}$, with appropriate dimensions, and a positive scalar ϵ_X, satisfying*

$$\begin{bmatrix} \Psi_s + \tau M_{1s} & \star & \star & \star \\ \begin{bmatrix} PA & \begin{bmatrix} Y\overline{C} & 0 \end{bmatrix} \\ 0 & \mathcal{R}_0 \end{bmatrix} & -\tau R_{0s} & \star & \star \\ \overline{E}\overline{\tau}\overline{N}^T & 0 & -\overline{E}\overline{\tau}\overline{R} & \star \\ \overline{\tau}(\mathbf{1} \otimes \begin{bmatrix} PA & Y\overline{C} & 0 \end{bmatrix}) & 0 & 0 & -\overline{R}\overline{\tau} \end{bmatrix} < 0 \tag{5.31}$$

$$\begin{bmatrix} \Psi_s + \tau M_{2s} & \star & \star & \star \\ \overline{E}\overline{\tau}\overline{N}^T & -\overline{E}\overline{\tau}\overline{R} & \star & \star \\ \overline{\tau}(\mathbf{1} \otimes \begin{bmatrix} PA & Y\overline{C} & 0 \end{bmatrix}) & 0 & -\overline{R}\overline{\tau} & \star \\ \tau N^T & 0 & 0 & \frac{-\tau R_{0s}}{\exp(\alpha\tau)} \end{bmatrix} < 0 \tag{5.32}$$

where τ, $\overline{\tau}$, $\overline{E}$, and $\overline{R}$ are defined in (5.9), (5.13)-(5.15), and

$$R_{0s} = \mathrm{diag}(P, \mathcal{R}_0), \tag{5.33}$$

$$\begin{aligned} \Psi_s = & \begin{bmatrix} PA & Y\overline{C} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T + \begin{bmatrix} PA & Y\overline{C} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} \\ & - \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T N^T - N \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathbf{1} \otimes I & -I & 0 \end{bmatrix}^T \overline{N}^T \\ & - \overline{N} \begin{bmatrix} \mathbf{1} \otimes I & -I & 0 \end{bmatrix} - \epsilon_X \begin{bmatrix} I & 0 & -I \end{bmatrix}^T P \begin{bmatrix} I & 0 & -I \end{bmatrix}, \\ M_{1s} = & \epsilon_X \begin{bmatrix} PA & Y\overline{C} & 0 \\ 0 & 0 & 0 \\ -PA & -Y\overline{C} & 0 \end{bmatrix}^T + \epsilon_X \begin{bmatrix} PA & Y\overline{C} & 0 \\ 0 & 0 & 0 \\ -PA & -Y\overline{C} & 0 \end{bmatrix} \\ & + \alpha\epsilon_X \begin{bmatrix} I & 0 & -I \end{bmatrix}^T P \begin{bmatrix} I & 0 & -I \end{bmatrix}, \end{aligned}$$

$$M_{2s} = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T N^T - N \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then there exists an observer gain $L = P^{-1}Y$ and a set of matrix variables R_0, R_i, $i \in \{1, ..., m\}$, and X that satisfy the conditions in Theorem 5.1, for the same values of τ^i, $i \in \{1, ..., m\}$, α, P, N, and $\overline{N}$.

Proof. The proof is straightforward and consists of verifying that inequalities (5.31) and (5.32) are equivalent to (5.11) and (5.12) with the change of variables $L = P^{-1}Y$, $R_0 = R_{0s}$ (see (5.33)), $R_i = P$, $i = \{1, \ldots, m\}$, and $X = \epsilon_X P$. □

Therefore, given τ^i, $i \in \{1, \ldots, m\}$, and $\alpha > 0$ the problem of designing an observer gain L that guarantees exponential stability of the estimation error is formulated as

Problem 5.2.

$$\begin{aligned} &\textit{find} \quad Y \\ &\textit{subject to } P > 0,\ \mathcal{R}_0 > 0,\ \epsilon_X > 0,\ (5.31),\ \textit{and } (5.32). \end{aligned}$$

The observer gain is then computed as $L = P^{-1}Y$. The conditions in Corollary 5.1 are sufficient conditions for the inequalities in Theorem 5.1 and therefore are more conservative. However, they can be used to design linear observers by solving a convex optimization program that can be solved efficiently using available software packages.

5.4 Numerical Example

Consider the path following problem in Chapter 4 where the objective is to control a unicycle to follow the line $y = 0$ in the $x - y$ plane (see Fig. 4.1). The dynamics of the system are represented by

$$\begin{bmatrix} \dot{\psi} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -k/I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ r \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v\sin(\psi) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I \\ 0 \end{bmatrix} u, \tag{5.34}$$

where y represents the distance from the line $y = 0$, ψ and r are the heading angle and its time derivative, respectively, $v = 1$ (m/s) is the unicycle's velocity, u denotes

Table 5.1: The computed MASP ($\tau^2_{\max}$) for sensor S^2 that guarantees exponential stability of the estimation error with $\alpha = 0.1$

	$\tau^2_{\max}$
For $\tau^1 = 0.1$ (s)	0.24 (s)
For $\tau^1 = 0.15$ (s)	0.18 (s)
For $\tau^1 = 0.17$ (s)	0.05 (s)

the torque input about the z axis, $I = 1$ (kgm^2) is the unicycle's moment of inertia with respect to its center of mass, and $k = 0.01$ (Nms) is the damping coefficient. Linearizing the system about the origin leads to the linear system (5.1), with

$$x = \begin{bmatrix} y \\ \psi \\ r \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.01 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Assume that the states y and r are measured by asynchronous dedicated sensors S^1 and S^2, respectively. Let L be an observer gain that places the poles of the continuous-time estimation error vector field at $-0.25 \pm 1.5j$ and -3.5. Therefore,

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; L = \begin{bmatrix} 0 & -2 \\ -2 & 0 \\ 0 & -4 \end{bmatrix}. \tag{5.35}$$

Assume that the sampling intervals of sensor S^1 have a known upper bound, i.e. τ^1 is fixed. Solving Problem 5.1 for different values of τ^1, the lower bound on the MASP (that guarantees exponential stability of the estimation error) for sensor S^2 is presented in Table 5.1. As expected, when the allowable length of sampling intervals for sensor S^1 increases, the computed value for $\tau^2_{\max}$ decreases. In other words, as the sampling frequency of the first sensor decreases, the second sensor must sample faster to guarantee convergence of the estimation error to the origin.

Now suppose that sensors S^1 and S^2 perform measurements at unknown nonuniform sampling intervals smaller than $\tau^1 = 0.5$ (s) and $\tau^2 = 0.3$ (s), respectively. Simulation results in Fig. 5.3 show that the estimation error does not converge to the origin in this case with the choice of observer gain (5.35). Solving Problem 5.2 with

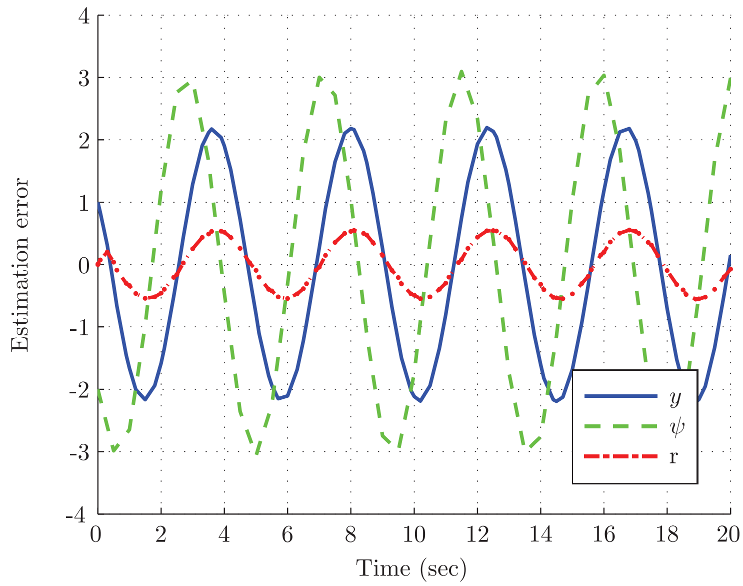


Figure 5.3: State estimation error for $\tau^1 = 0.5$ (s) and $\tau^2 = 0.3$ (s) and observer gain L defined in (5.35).

$\alpha = 0.1$ and $\epsilon_X = 1$, we find a new observer gain

$$L' = \begin{bmatrix} -0.8079 & -0.2555 \\ -0.2071 & -0.0550 \\ -0.7609 & -0.7714 \end{bmatrix}. \tag{5.36}$$

The new observer gain L' guarantees exponential stability of the estimation error as shown in Fig. 5.4.

5.5 Conclusion

The observer design problem for linear systems with multi-rate sampled output measurements was addressed. Given the MASP for each sensor, sufficient Krasovskii-based conditions for design of linear observers were proposed in terms of LMIs. Furthermore, given an observer gain, the problem of finding MASPs that guarantee exponential stability of the estimation error was formulated as a convex optimization program subject to LMIs.

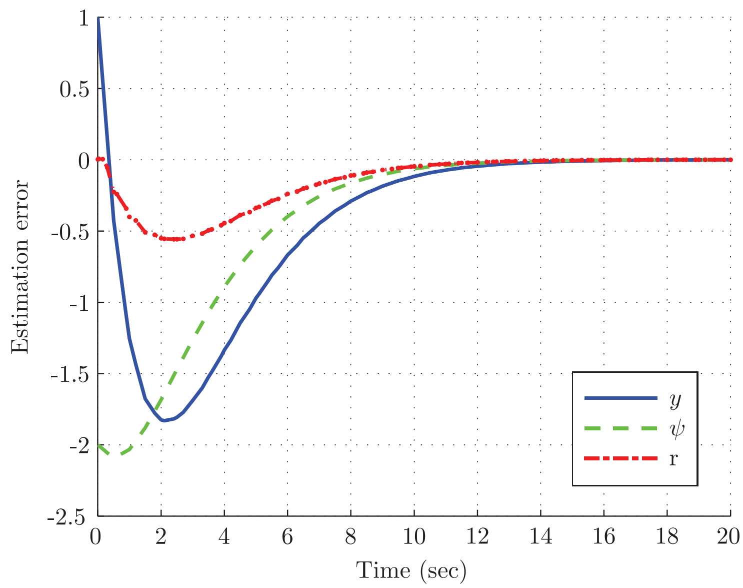


Figure 5.4: State estimation error for $\tau^1 = 0.5$ (s) and $\tau^2 = 0.3$ (s) and observer gain L' defined in (5.36).

Chapter 6

Stability Analysis of Piecewise Affine Sampled-data Systems

This chapter addresses stability analysis of sampled-data PWA slab systems. In particular, we study the case in which a PWA plant is in feedback with a discrete-time emulation of a PWA controller. The contributions of this chapter are threefold. First, a modified LKF is presented for studying PWA sampled-data systems that is less conservative when compared to previously suggested alternatives. Second, based on the new LKF, sufficient conditions are provided for asymptotic stability of sampled-data PWA slab systems to the origin. These conditions become LMIs in the case of a PWL controller. Finally, we present an algorithm for finding a lower bound on the MASP that preserves asymptotic stability. Therefore, the output of the algorithm provides an upper bound on the minimum sampling frequency that guarantees asymptotic stability of the sampled data system.

6.1 Introduction

PWA systems form a special class of hybrid systems that is often considered as a framework for modeling and approximating nonlinearities that arise in physical systems [23]. Stability analysis of continuous-time PWA systems was addressed using Lyapunov-based methods in [25–27, 29]. Lyapunov-based synthesis methods for PWA systems were presented in [26, 29, 30, 32]. However, to be implementable in microprocessors, the resulting continuous-time controllers must be emulated as a discrete-time controller.

While sampled-data control of linear systems is a well-studied subject (e.g. see [37]), its extension to PWA systems has not received many research contributions. The term "sampled-data PWA system" was probably used for the first time in [102, 103], although the systems described there do not possess the typical structure of a continuous-time plant being controlled by a discrete-time controller. By contrast, reference [53] addresses the classical structure of a sampled-data system in which a continuous-time system is controlled in discrete-time inside a computer. Assuming constant sampling rate, the author studies the stability of sampled-data PWA systems using a quadratic Lyapunov function. The paper provides a set of LMIs as sufficient conditions for exponential convergence of the sampled-data system to an invariant set containing the origin.

In sampled-data systems, the discrete-time controller can also be modeled as a continuous-time controller with time varying delay. The time-delay representation has been implemented in nonlinear and linear sampled-data systems using Razumikhin-type theorems [41], and LKFs [48]. Following the time-delay approach, reference [54] studies the stability of sampled-data PWA systems with variable sampling rate. The paper uses an LKF to prove that if a set of LMIs are satisfied, the trajectories of the sampled-data system converge to an invariant set containing the origin.

In contrast to previous work, we address asymptotic stability to the origin rather than stability to an invariant set for sampled-data PWA slab systems. To the best of our knowledge, asymptotic stability of sampled-data PWA systems to the origin was not proved before. We study a continuous-time PWA slab plant in feedback with a PWA slab controller that appears between a sampler, with variable sampling rate, and a zero-order-hold. The contributions of this chapter are threefold. First, a modified LKF is presented for studying PWA sampled-data systems that is less conservative when compared to previously suggested alternatives. Second, based on the new LKF, sufficient conditions are provided for asymptotic stability of sampled-data PWA slab systems to the origin. Finally, following the time-delay approach, we present an algorithm for finding a lower bound on the MASP that preserves asymptotic stability. This result provides an upper bound on the minimum sampling frequency that guarantees asymptotic stability of the sampled data system.

The chapter is organized as follows. Section 6.2 presents basic information about sampled-data PWA slab systems. In Section 6.3, a modified LKF is introduced first. Next, we present a theorem that provides sufficient conditions for asymptotic stability of sampled-data PWA slab systems. Furthermore, we present an algorithm for finding a lower bound on the MASP that preserves asymptotic stability. Finally, the new results are applied to a unicycle example in Section 6.4, followed by conclusions.

6.2 Preliminaries

Consider the PWA slab system

$$\dot{x}(t) = A_i x(t) + a_i + Bu(t), \text{ for } x(t) \in \mathcal{R}_i \text{ and } i \in \mathcal{I}, \tag{6.1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^{n_x}$ denotes the state vector, $A_i \in \mathbb{R}^{n_x \times n_x}$, $a_i \in \mathbb{R}^{n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $u \in \mathbb{R}^{n_u}$ is the control input, and $\mathcal{I} = \{1, ..., M\}$ is the set of indices of the slab regions $\mathcal{R}_i$ that partition the state space $\mathcal{X}$. The slab regions are defined as

$$\mathcal{R}_i = \{x \in \mathbb{R}^{n_x} | \sigma_i < c^T x < \sigma_{i+1}\}, \ i \in \mathcal{I}, \tag{6.2}$$

where $c \in \mathbb{R}^{n_x}$, $c \neq 0$, and $\sigma_1 < \sigma_2 < ... < \sigma_{M+1}$ are finite scalars. We denote the closure of $\mathcal{R}_i$ by $\overline{\mathcal{R}_i}$. The state space is represented by the union of the closure of all regions, i.e.

$$\mathcal{X} = \bigcup_{i \in \mathcal{I}} \overline{\mathcal{R}_i} = \{x \in \mathbb{R}^{n_x} | \sigma_1 \leq c^T x \leq \sigma_{M+1}\}. \tag{6.3}$$

Based on (6.3) and (6.2), the state space $\mathcal{X}$ and the regions $\mathcal{R}_i$ are only bounded in the direction of vector c. Each slab region $\mathcal{R}_i$ can also be described by a degenerate ellipsoid as

$$\mathcal{R}_i = \{x | ||L_i x + l_i|| < 1\}, \tag{6.4}$$

where $L_i = 2c^T/(\sigma_{i+1} - \sigma_i)$ and $l_i = -(\sigma_{i+1} + \sigma_i)/(\sigma_{i+1} - \sigma_i)$ (see [30] for more details).

Lemma 6.1. *For the slab regions defined in (6.4), $x \in \mathcal{R}_i$ if and only if*

$$\begin{bmatrix} x^T & 1 \end{bmatrix} \begin{bmatrix} L_i^T L_i & L_i^T l_i \\ l_i L_i & l_i^2 - 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0.$$

Proof. According to (6.4), $x \in \mathcal{R}_i$ if and only if $||L_i x + l_i|| < 1$. Therefore,

$$\begin{aligned} x \in \mathcal{R}_i &\iff (L_i x + l_i)^2 < 1 \\ &\iff \left(\begin{bmatrix} x^T & 1 \end{bmatrix} \begin{bmatrix} L_i^T \\ l_i \end{bmatrix} \right) \left(\begin{bmatrix} L_i & l_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \right) < 1 \\ &\iff \begin{bmatrix} x^T & 1 \end{bmatrix} \begin{bmatrix} L_i^T L_i & L_i^T l_i \\ l_i L_i & l_i^2 - 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < 0. \end{aligned}$$

□

Let a PWA controller for (6.1) be defined by

$$u(t) = K_i x(t) + k_i, \text{ for } x(t) \in \mathcal{R}_i,$$

where $K_i \in \mathbb{R}^{n_u \times n_x}$ and $k_i \in \mathbb{R}^{n_u}$. We now present the assumptions used in this work.

Assumption 6.1. *The vector field of the open-loop system (6.1) for $u(t) = 0$ is continuous across the boundaries of any neighboring regions.*

We denote the region containing the origin by $\mathcal{R}^*$.

Assumption 6.2. *The open-loop and closed-loop systems are linear in $\mathcal{R}^*$, i.e. $a_i = 0$ and $k_i = 0$ for $\mathcal{R}_i = \mathcal{R}^*$.*

Assumption 6.3. *The state vector is measured at the sampling instants t_n, $n \in \mathbb{N}$, where $0 < t_\epsilon \leq t_{n+1} - t_n \leq \tau$ for all $n \in \mathbb{N}$.*

The positive constant t_ϵ is an arbitrary small number that models the fact that two transmissions cannot occur simultaneously in practice. It is also used later to rule out the existence of the Zeno phenomenon. The number τ denotes the longest interval between two consecutive sampling times. According to Assumption 6.3, the control input is rewritten as

$$u(t) = K_j x(t_n) + k_j, \text{ for } t \in [t_n, t_{n+1}), \ x(t_n) \in \mathcal{R}_j, \text{ and } j \in \mathcal{I}.$$

We denote the time elapsed since the last sampling instant by

$$\rho(t) = t - t_n, \text{ for } t \in [t_n, t_{n+1}). \tag{6.5}$$

Assuming $x(t) \in \mathcal{R}_i$ and $x(t_n) \in \mathcal{R}_j$, for $t \in [t_n, t_{n+1})$, we can rewrite (6.1) as

$$\dot{x}(t) = A_i x(t) + a_i + B(K_j x(t_n) + k_j) \tag{6.6a}$$
$$= A_i x(t) + a_i + B(K_i x(t_n) + k_i) + Bw(t), \tag{6.6b}$$

where $w \in \mathbb{R}^{n_u}$ is a piecewise constant vector defined by

$$w(t) = (K_j - K_i) x(t_n) + (k_j - k_i). \tag{6.7}$$

The vector w is associated with the fact that the state and its most recent sample can possibly be in different regions. To be of later use in the proofs we must define bounds on the mismatch between controller gain matrices K_i and affine vectors k_i, $i \in \mathcal{I}$. To that end, let $B_\mu(0)$ be the ball with radius $\mu > 0$ centered at the origin and $\mathcal{I}_\mu = \{p \in \mathcal{I} | \overline{\mathcal{R}_p} \bigcap B_\mu(0) \neq \emptyset\}$. We define non-negative scalars ΔK_μ and δk_μ as

$$\Delta K_\mu = \max_{i\in\mathcal{I},\ j\in\mathcal{I}_\mu} ||K_j - K_i||,\ \delta k_\mu = \max_{i\in\mathcal{I},\ j\in\mathcal{I}_\mu} ||k_j - k_i||. \tag{6.8}$$

Furthermore, let non-negative scalars ΔK and δk be defined as

$$\Delta K = \max_{i,j\in\mathcal{I}} ||K_j - K_i||,\ \delta k = \max_{i,j\in\mathcal{I}} ||k_j - k_i||. \tag{6.9}$$

The following lemma presents a bound on the vector w which is used in the proof of the main result.

Lemma 6.2. *For $t \in [t_n, t_{n+1})$, if $||x(t_n)|| < \mu$, then*

$$\begin{bmatrix} -I & (\Delta K_\mu \mu + \delta k_\mu)\mathbf{1} \\ I & (\Delta K_\mu \mu + \delta k_\mu)\mathbf{1} \end{bmatrix} \begin{bmatrix} w(t) \\ 1 \end{bmatrix} \succ 0, \tag{6.10}$$

where $\mathbf{1}^T = \begin{bmatrix} 1 \cdots 1 \end{bmatrix}_{1\times n_u}$ and $\succ$ represents an elementwise inequality.

Proof. If $||x(t_n)|| < \mu$, according to (6.7) and (6.8) we can write

$$||w(t)|| \leq ||K_j - K_i|| ||x(t_n)|| + ||k_j - k_i|| < \Delta K_\mu \mu + \delta k_\mu. \tag{6.11}$$

For single input systems, inequality (6.11) can be written as

$$\begin{bmatrix} -1 & \Delta K_\mu \mu + \delta k_\mu \\ 1 & \Delta K_\mu \mu + \delta k_\mu \end{bmatrix} \begin{bmatrix} w(t) \\ 1 \end{bmatrix} \succ 0.$$

For the case of multi-input systems, a more conservative inequality can be written as (6.10), i.e. the absolute value of each element of w is less than $\Delta K_\mu \mu + \delta k_\mu$. □

6.3 Main Results

In this section, we first present a modified LKF for studying PWA sampled-data systems. Let $V(t, x_t)$ be an LKF defined as

$$V(t, x_t) = V_1 + V_2 + V_3, \tag{6.12}$$

where

$$V_1 = x^T(t)Px(t),$$
$$V_2 = \int_{-\tau}^{0}\int_{t+r}^{t}\Big[\dot{x}(s) - B(K_jx(t_n)+k_j)\Big]^T R\Big[\dot{x}(s) - B(K_jx(t_n)+k_j)\Big]\,\mathrm{d}s\,\mathrm{d}r,$$
$$V_3 = (\tau-\rho)(x(t)-x(t_n))^T X(x(t)-x(t_n)),$$

where P, R, and X are symmetric positive definite matrices in $\mathbb{R}^{n_x\times n_x}$, $t_n \le t$ is the most recent sampling instant, and j is the index of the region containing the most recent sampled state, i.e. $x(t_n) \in \mathcal{R}_j$.

Note that the second component of the LKF introduced in (6.12) is different from its corresponding term in previously studied LKFs such as [48, 54]. By subtracting $B(K_jx(t_n)+k_j)$ from $\dot{x}$ in the definition of V_2, we omit an unfavorable positive definite term involving w^Tw from $\dot{V}$. This modification considerably improves the stability results as shown in Section 6.4. We now define stability in the context of retarded functional differential equations.

Definition 6.1. *[20] The solution of (6.6a) is said to be uniformly stable if for any $\epsilon > 0$, there is a $\delta = \delta(\epsilon)$ such that $||x_{t_0}||_{\mathcal{W}} < \delta$ implies $||x(t)|| < \epsilon$ for $t \ge t_0$. The solution of (6.6a) is uniformly asymptotically stable if it is uniformly stable and there is $\delta_a > 0$ such that for any $\eta > 0$, there is a $T = T(\delta_a, \eta)$, such that $||x_{t_0}||_{\mathcal{W}} < \delta_a$ implies $||x(t)|| < \eta$ for $t \ge t_0 + T$.*

The following theorem provides a set of sufficient conditions for which the trajectories of a sampled-data PWA slab system asymptotically converge to the origin.

Theorem 6.1. *Consider the sampled-data PWA slab system defined in (6.6b) and (6.7) subject to Assumptions 6.1-6.3. The system is uniformly asymptotically stable to the origin if there exist symmetric positive definite matrices P, R, and X, symmetric matrices Λ_i with non-negative entries, matrices $\widetilde{N}$ and N_i, with appropriate dimensions, and positive scalars γ, $\theta < 1$, η, λ_i, σ, and ϵ, with $i \in \mathcal{I}$, satisfying*

$$\Delta K^2\gamma < \theta \tag{6.13}$$

- *for all i such that $\mathcal{R}_i \neq \mathcal{R}^*$*

$$\Omega_i + \overline{\Omega}_i + \overline{\Omega}_i^T + \tau(\overline{M}_{1i} + \overline{M}_{1i}^T + \overline{M}_{2i}) + S_{1i} + D < 0 \tag{6.14}$$

$$\begin{bmatrix} \Omega_i + \overline{\Omega}_i + \overline{\Omega}_i^T + \tau(\overline{M}_{2i} + \overline{M}_{3i} + \overline{M}_{3i}^T) + S_{1i} + D & \tau N_i \\ \tau N_i^T & -\tau R \end{bmatrix} < 0 \tag{6.15}$$

$$\Omega_i + \overline{\Omega}_i + \overline{\Omega}_i^T + \tau(\overline{M}_{1i} + \overline{M}_{1i}^T + \overline{M}_{2i}) + S_{1i} + S_{3i} + \epsilon I < 0 \tag{6.16}$$

$$\begin{bmatrix} \Omega_i + \overline{\Omega}_i + \overline{\Omega}_i^T + \tau(\overline{M}_{2i} + \overline{M}_{3i} + \overline{M}_{3i}^T) + S_{1i} + S_{3i} + \epsilon I & \tau N_i \\ \tau N_i^T & -\tau R \end{bmatrix} < 0 \tag{6.17}$$

- *for i such that $\mathcal{R}_i = \mathcal{R}^*$*

$$\Omega_i + \tau(M_{1i} + M_{1i}^T + M_{2i}) + S_{1i} - S_{2i} + D < 0 \tag{6.18}$$

$$\begin{bmatrix} \Omega_i + \tau(M_{2i} + M_{3i} + M_{3i}^T) + S_{1i} - S_{2i} + D & \tau N_i \\ \tau N_i^T & -\tau R \end{bmatrix} < 0 \tag{6.19}$$

$$\Omega_i + \tau(M_{1i} + M_{1i}^T + M_{2i}) + S_{1i} - S_{2i} + S_{3i} + \epsilon I < 0 \tag{6.20}$$

$$\begin{bmatrix} \Omega_i + \tau(M_{2i} + M_{3i} + M_{3i}^T) + S_{1i} - S_{2i} + S_{3i} + \epsilon I & \tau N_i \\ \tau N_i^T & -\tau R \end{bmatrix} < 0 \tag{6.21}$$

$$\Psi_i + \tau \widetilde{M}_{1i} + \widetilde{M}_{3i} + \epsilon I < 0 \tag{6.22}$$

$$\begin{bmatrix} \Psi_i + \tau \widetilde{M}_{2i} + \widetilde{M}_{3i} + \epsilon I & \tau \widetilde{N}_i \\ \tau \widetilde{N}_i^T & -\tau R \end{bmatrix} < 0 \tag{6.23}$$

where

$$\begin{aligned}
\Omega_i &= \begin{bmatrix} \Psi_i & \begin{bmatrix} PB & 0 \\ 0 & 0 \end{bmatrix} \\ \star & 0 \end{bmatrix} - \begin{bmatrix} N_i & -N_i & 0 & 0 \end{bmatrix}^T - \begin{bmatrix} N_i & -N_i & 0 & 0 \end{bmatrix}, \\
\Psi_i &= \begin{bmatrix} A_i^T P + PA_i - X & PBK_i + X \\ \star & -X \end{bmatrix}, \\
\overline{\Omega}_i &= \begin{bmatrix} P & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & a_i + Bk_i \end{bmatrix}, \\
\overline{M}_{1i} &= \begin{bmatrix} X & -X & 0 & 0 \end{bmatrix}^T \begin{bmatrix} A_i & BK_i & B & a_i + Bk_i \end{bmatrix}, \\
\overline{M}_{2i} &= \begin{bmatrix} A_i & 0 & 0 & a_i \end{bmatrix}^T R \begin{bmatrix} A_i & 0 & 0 & a_i \end{bmatrix}, \\
\overline{M}_{3i} &= N_i \begin{bmatrix} 0 & BK_i & B & Bk_i \end{bmatrix}, \\
S_{1i} &= -\lambda_i \left(\begin{bmatrix} L_i & 0 & 0 & l_i \end{bmatrix}^T \begin{bmatrix} L_i & 0 & 0 & l_i \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right), \\
S_{2i} &= -\sigma \left(\begin{bmatrix} 0 & L_i & 0 & l_i \end{bmatrix}^T \begin{bmatrix} 0 & L_i & 0 & l_i \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right),
\end{aligned}$$

$$
\begin{aligned}
S_{3i} =& \begin{bmatrix} 0 & 0 & -I & (\Delta K_\mu \mu_\tau + \delta k_\mu)\boldsymbol{1} \\ 0 & 0 & I & (\Delta K_\mu \mu_\tau + \delta k_\mu)\boldsymbol{1} \end{bmatrix}^T \Lambda_i \begin{bmatrix} 0 & 0 & -I & (\Delta K_\mu \mu_\tau + \delta k_\mu)\boldsymbol{1} \\ 0 & 0 & I & (\Delta K_\mu \mu_\tau + \delta k_\mu)\boldsymbol{1} \end{bmatrix}, \\
\mu_\tau =& \frac{\delta k}{\sqrt{\theta/\gamma} - \Delta K}, \\
D =& \operatorname{diag}(\eta I, \ I, \ -\gamma I, \ \eta), \\
M_{1i} =& \begin{bmatrix} X & -X & 0 & 0 \end{bmatrix}^T \begin{bmatrix} A_i & BK_i & B & 0 \end{bmatrix}, \\
M_{2i} =& \begin{bmatrix} A_i & 0 & 0 & 0 \end{bmatrix}^T R \begin{bmatrix} A_i & 0 & 0 & 0 \end{bmatrix}, \\
M_{3i} =& N_i \begin{bmatrix} 0 & BK_i & B & 0 \end{bmatrix}, \\
\widetilde{M}_{1i} =& \begin{bmatrix} A_i^T X + X A_i + A_i^T R A_i & -A_i^T X + XBK_i \\ \star & -K_i^T B^T X - XBK_i \end{bmatrix}, \\
\widetilde{M}_{2i} =& \begin{bmatrix} 0 \\ K_i^T B^T \end{bmatrix} \widetilde{N}_i^T + \widetilde{N}_i \begin{bmatrix} 0 & BK_i \end{bmatrix} + \begin{bmatrix} A_i^T R A_i & 0 \\ \star & 0 \end{bmatrix}, \\
\widetilde{M}_{3i} =& - \begin{bmatrix} \widetilde{N}_i & -\widetilde{N}_i \end{bmatrix}^T - \begin{bmatrix} \widetilde{N}_i & -\widetilde{N}_i \end{bmatrix}.
\end{aligned}
\tag{6.24}
$$

Proof. Similar to the approach in Chapter 2, it can be shown that the LKF (6.12) is positive definite, radially unbounded, and decrescent. The first two components, V_1 and V_2, are continuous functions. The last component, V_3, is equal to zero at the sampling instants ($x(t)|_{t=t_n} = x(t_n)$) and greater than zero at other times. Therefore, the LKF is non-increasing at the sampling times. To prove uniform asymptotic stability of the closed-loop trajectories to the origin, it suffices to show that inequalities (6.13)-(6.23) are sufficient conditions for V to be strictly decreasing between any two consecutive sampling times. The time derivative of V for $t \in (t_n, t_{n+1})$ is computed as follows. First, the time derivative of V_1 is

$$
\dot{V}_1 = \dot{x}^T P x + x^T P \dot{x}. \tag{6.25}
$$

Second, applying the Leibniz integral rule to V_2 yields

$$
\begin{aligned}
\dot{V}_2 =& \int_{-\tau}^{0} \left[\dot{x} - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x} - B(K_j x(t_n) + k_j)\right] \mathrm{d}r \\
& - \int_{-\tau}^{0} \left[\dot{x}(t+r) - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x}(t+r) - B(K_j x(t_n) + k_j)\right] \mathrm{d}r.
\end{aligned}
$$

According to (6.5), we have $\rho < \tau$. Therefore,

$$\begin{aligned}\dot{V}_2 \leq & \tau \left[\dot{x} - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x} - B(K_j x(t_n) + k_j)\right] \\ & - \int_{-\rho}^{0} \left[\dot{x}(t+r) - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x}(t+r) - B(K_j x(t_n) + k_j)\right] \mathrm{d}r \\ = & \tau \left[\dot{x} - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x} - B(K_j x(t_n) + k_j)\right] \\ & - \int_{t-\rho}^{t} \left[\dot{x}(v) - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x}(v) - B(K_j x(t_n) + k_j)\right] \mathrm{d}v. \end{aligned} \tag{6.26}$$

Since R is positive definite, for any arbitrary time varying vector $h_i(t) \in \mathbb{R}^{n_x}$ we can write

$$\begin{bmatrix} (\dot{x}(v) - B(K_j x(t_n) + k_j))^T & h_i^T \end{bmatrix} \begin{bmatrix} R & -I \\ -I & R^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}(v) - B(K_j x(t_n) + k_j) \\ h_i \end{bmatrix} \geq 0.$$

Therefore,

$$\begin{aligned} & - \left[\dot{x}(v) - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x}(v) - B(K_j x(t_n) + k_j)\right] \\ & \quad \leq h_i^T R^{-1} h_i - \left[\dot{x}(v) - B(K_j x(t_n) + k_j)\right]^T h_i - h_i^T \left[\dot{x}(v) - B(K_j x(t_n) + k_j)\right]. \end{aligned}$$

Integrating both sides from $t - \rho$ to t, we have

$$\begin{aligned} & - \int_{t-\rho}^{t} \left[\dot{x}(v) - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x}(v) - B(K_j x(t_n) + k_j)\right] \mathrm{d}v \\ & \quad \leq \rho h_i^T R^{-1} h_i - \left[x - x(t_n) - \rho B(K_j x(t_n) + k_j)\right]^T h_i \\ & \quad\quad - h_i^T \left[x - x(t_n) - \rho B(K_j x(t_n) + k_j)\right]. \end{aligned} \tag{6.27}$$

Here, we used the facts that for $v \in [t - \rho, t]$, $u = K_j x(t_n) + k_j$ is constant and therefore $\dot{x}(v)$ is continuous by Assumption 6.1, and $t - \rho = t_n$. Replacing (6.27) in (6.26), we have

$$\begin{aligned} \dot{V}_2 \leq & \tau \left[\dot{x} - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x} - B(K_j x(t_n) + k_j)\right] + \rho h_i^T R^{-1} h_i \\ & - \left[x - x(t_n) - \rho B(K_j x(t_n) + k_j)\right]^T h_i - h_i^T \left[x - x(t_n) - \rho B(K_j x(t_n) + k_j)\right]. \end{aligned} \tag{6.28}$$

Using (6.7) to replace $K_j x(t_n) + k_j$ by $(K_i x(t_n) + k_i) + w$ in the last two components of (6.28) yields

$$\begin{aligned}\dot{V}_2 \leq & \tau \left[\dot{x} - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x} - B(K_j x(t_n) + k_j)\right] + \rho h_i^T R^{-1} h_i \\ & - \left[x - x(t_n) - \rho B((K_i x(t_n) + k_i) + w)\right]^T h_i \\ & - h_i^T \left[x - x(t_n) - \rho B((K_i x(t_n) + k_i) + w)\right]. \end{aligned} \tag{6.29}$$

From (6.5) we have $\dot{\rho} = 1$. Hence, the time derivative of V_3 is computed as

$$\dot{V}_3 = (\tau - \rho)\left[\dot{x}^T X (x - x(t_n))\right] + (\tau - \rho)\left[(x - x(t_n))^T X \dot{x}\right] - (x - x(t_n))^T X (x - x(t_n)). \tag{6.30}$$

Since $\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3$, adding (6.25), (6.29), and (6.30) yields

$$\begin{aligned}\dot{V} \leq & \dot{x}^T P x + x^T P \dot{x} + \rho h_i^T R^{-1} h_i + \tau \left[\dot{x} - B(K_j x(t_n) + k_j)\right]^T R \left[\dot{x} - B(K_j x(t_n) + k_j)\right] \\ & - \left[x - x(t_n) - \rho B((K_i x(t_n) + k_i) + w)\right]^T h_i \\ & - h_i^T \left[x - x(t_n) - \rho B((K_i x(t_n) + k_i) + w)\right] + (\tau - \rho)\left[\dot{x}^T X (x - x(t_n))\right] \\ & + (\tau - \rho)\left[(x - x(t_n))^T X \dot{x}\right] - (x - x(t_n))^T X (x - x(t_n)). \end{aligned} \tag{6.31}$$

For $t \in (t_n, t_{n+1})$ and $x(t) \in \mathcal{X}$ we consider the following three possibilities;

1. $x(t) \notin \mathcal{R}^*$,
2. $x(t) \in \mathcal{R}^*$ and $x(t_n) \notin \mathcal{R}^*$,
3. $x(t) \in \mathcal{R}^*$ and $x(t_n) \in \mathcal{R}^*$.

The rest of the proof is divided into three parts corresponding to the above possibilities.

- *Part 1:* For $x(t) \in \mathcal{R}_i \neq \mathcal{R}^*$, based on (6.6), we have

$$\dot{x}(t) = \begin{bmatrix} A_i & BK_i & B & a_i + Bk_i \end{bmatrix} \zeta(t), \tag{6.32}$$

and

$$\dot{x}(t) - B(K_j x(t_n) + k_j) = \begin{bmatrix} A_i & 0 & 0 & a_i \end{bmatrix} \zeta(t), \tag{6.33}$$

with $\zeta(t) = \begin{bmatrix} x^T(t) & x_{t_n}^T & w^T(t) & 1 \end{bmatrix}^T \in \mathbb{R}^{2n_x + n_u + 1}$. Replacing (6.32) and (6.33) in (6.31) and setting $h_i(t) = N_i^T \zeta(t)$ with $N_i \in \mathbb{R}^{(2n_x + n_u + 1) \times n_x}$, we can write

$$\dot{V} \leq \zeta^T \Bigg(\begin{bmatrix} A_i & BK_i & B & a_i + Bk_i \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
&+\begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} A_i & BK_i & B & a_i + Bk_i \end{bmatrix} + \rho N_i R^{-1} N_i^T \\
&+\tau \begin{bmatrix} A_i & 0 & 0 & a_i \end{bmatrix}^T R \begin{bmatrix} A_i & 0 & 0 & a_i \end{bmatrix} - \begin{bmatrix} I & -I-\rho BK_i & -\rho B & -\rho Bk_i \end{bmatrix}^T N_i^T \\
&-N_i \begin{bmatrix} I & -I-\rho BK_i & -\rho B & -\rho Bk_i \end{bmatrix} \\
&+(\tau-\rho) \begin{bmatrix} A_i & BK_i & B & a_i + Bk_i \end{bmatrix}^T X \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix} \\
&+(\tau-\rho) \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} A_i & BK_i & B & a_i + Bk_i \end{bmatrix} \\
&- \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix} \Bigg) \zeta.
\end{aligned} \tag{6.34}
$$

Hence, for $\rho = 0$, LMI (6.14) implies

$$
\dot{V} < -\eta x^T x - x(t_n)^T x(t_n) + \gamma w^T w - \eta - \zeta^T S_{1i} \zeta. \tag{6.35}
$$

Using Schur complement, LMI (6.15) implies that (6.35) holds for $\rho = \tau$. Since (6.34) is affine in ρ, LMIs (6.14)-(6.15) are sufficient conditions for (6.35) to hold for any $\rho \in (0, \tau)$. Recalling (6.7) and (6.9), we can write

$$
||w|| \leq \Delta K ||x(t_n)|| + \delta k. \tag{6.36}
$$

Considering (6.24) and (6.13), for $||x(t_n)|| \geq \mu_\tau$ we have

$$
\sqrt{\theta/\gamma} ||x(t_n)|| - \Delta K ||x(t_n)|| \geq \delta k.
$$

Therefore, based on (6.36), for $||x(t_n)|| \geq \mu_\tau$ we can write

$$
\sqrt{\theta/\gamma} ||x(t_n)|| \geq ||w||. \tag{6.37}
$$

Adding and subtracting $\theta x(t_n)^T x(t_n)$, $0 < \theta < 1$, in inequality (6.35) and using (6.37), for $||x(t_n)|| \geq \mu_\tau$, we get

$$
\dot{V} < -\eta x^T x - (1-\theta) x(t_n)^T x(t_n) - \eta - \zeta^T S_{1i} \zeta. \tag{6.38}
$$

Furthermore, considering (6.34) for $\rho = 0$, inequality (6.16) implies

$$
\dot{V} < \zeta^T (-\epsilon I - S_{1i} - S_{3i}) \zeta. \tag{6.39}
$$

Using Schur complement, inequality (6.17) implies that (6.39) holds at $\rho = \tau$. Since (6.34) is affine in ρ, inequalities (6.16)-(6.17) are sufficient conditions for (6.39) to hold for any $\rho \in (0, \tau)$.

According to Lemma 6.1, $\zeta^T S_{1i} \zeta > 0$ if $x(t) \in \mathcal{R}_i$. Furthermore, using Lemma 6.2, $\zeta^T S_{3i} \zeta > 0$ if $||x(t_n)|| < \mu_\tau$. Hence considering (6.38), LMIs (6.13)-(6.15) are sufficient conditions for V to be strictly decreasing between two consecutive sampling times for $||x(t_n)|| \geq \mu_\tau$. Moreover, considering (6.39), inequalities (6.16)-(6.17) are sufficient conditions for V to be strictly decreasing between two consecutive sampling times for $||x(t_n)|| < \mu_\tau$.

Therefore, inequalities (6.13)-(6.17) are sufficient conditions for V to be strictly decreasing for any $t \in (t_n, t_{n+1})$ and $x(t) \notin \mathcal{R}^*$, regardless of the magnitude of $x(t_n)$.

• *Part 2:* For $x(t) \in \mathcal{R}_i = \mathcal{R}^*$ and $x(t_n) \notin \mathcal{R}^*$, based on Assumption 6.2, we have $a_i = 0$ and $k_i = 0$. Setting $a_i = 0$ and $k_i = 0$ in (6.34), for $\rho = 0$, LMI (6.18) implies

$$\dot{V} < -\eta x^T x - x(t_n)^T x(t_n) + \gamma w^T w - \eta + \zeta^T(-S_{1i} + S_{2i})\zeta. \tag{6.40}$$

Using Schur complement, LMI (6.19) implies that (6.40) holds for $\rho = \tau$. Since (6.34) is affine in ρ, LMIs (6.18)-(6.19) are sufficient conditions for (6.40) to hold for any $\rho \in (0, \tau)$.

Adding and subtracting $\theta x(t_n)^T x(t_n)$ with $0 < \theta < 1$ in (6.40) and recalling (6.37) for $||x(t_n)|| \geq \mu_\tau$, we get

$$\dot{V} < -\eta x^T x - (1-\theta) x(t_n)^T x(t_n) - \eta + \zeta^T(-S_{1i} + S_{2i})\zeta. \tag{6.41}$$

Furthermore, considering (6.34) with $a_i = 0$, $k_i = 0$, and for $\rho = 0$, inequality (6.20) implies

$$\dot{V} < \zeta^T(-\epsilon I - S_{1i} + S_{2i} - S_{3i})\zeta. \tag{6.42}$$

Using Schur complement, inequality (6.21) implies that (6.42) holds for $\rho = \tau$. Since (6.34) is affine in ρ, inequalities (6.20)-(6.21) are sufficient conditions for (6.42) to hold for any $\rho \in (0, \tau)$.

Based on Lemma 6.1, $\zeta^T S_{1i} \zeta > 0$ if $x(t) \in \mathcal{R}_i$. Furthermore, $\zeta^T S_{2i} \zeta < 0$ if $x(t_n) \notin \mathcal{R}_i$. Finally, using Lemma 6.2, $\zeta^T S_{3i} \zeta > 0$ if $||x(t_n)|| < \mu_\tau$. Hence considering (6.41), LMIs (6.13) and (6.18)-(6.19) are sufficient conditions for V to be strictly decreasing between two consecutive sampling times for $||x(t_n)|| \geq \mu_\tau$. Moreover, considering (6.42), inequalities (6.20)-(6.21) are sufficient conditions for V to be strictly decreasing between two consecutive sampling times for $||x(t_n)|| < \mu_\tau$.

Therefore, inequalities (6.13) and (6.18)-(6.21) are sufficient conditions for V to be strictly decreasing for any $t \in (t_n, t_{n+1})$, $x(t) \in \mathcal{R}^*$, and $x(t_n) \notin \mathcal{R}^*$, regardless of the magnitude of $x(t_n)$.

• *Part 3:* For $x(t) \in \mathcal{R}_i = \mathcal{R}^*$ and $x(t_n) \in \mathcal{R}_i = \mathcal{R}^*$, According to (6.7) and Assumption 6.2, we have $a_i = 0$, $k_i = 0$, and $w = 0$. Replacing N_i by $\begin{bmatrix} \widetilde{N}_i^T & 0_{n_x \times (n_u+1)} \end{bmatrix}^T$, $\widetilde{N}_i \in \mathbb{R}^{2n_x \times n_x}$, and setting $a_i = 0$ and $k_i = 0$ in (6.34), LMI (6.22) implies

$$\dot{V} < -\epsilon \widetilde{\zeta}^T \widetilde{\zeta} \tag{6.43}$$

for $\rho = 0$, where $\widetilde{\zeta} = \begin{bmatrix} x^T(t) & x(t_n)^T \end{bmatrix}^T$. Using Schur complement, LMI (6.23) implies that (6.43) holds for $\rho = \tau$. Since (6.34) is affine in ρ, LMIs (6.22)-(6.23) are sufficient conditions for V to be strictly decreasing for any $\rho \in (0, \tau)$, $x(t) \in \mathcal{R}^*$, and $x(t_n) \in \mathcal{R}^*$.

Therefore, inequalities (6.13)-(6.23) are sufficient conditions for V to be strictly decreasing between any two consecutive sampling times over the state space. According to Assumption 6.3, any sampling interval (t_n, t_{n+1}), $n \in \mathbb{N}$, has a length greater than or equal to $t_\epsilon > 0$. Hence $V|_{t_{n+1}^-} < V|_{t_n}$, where $V|_{t_{n+1}^-} = \lim_{t \nearrow t_{n+1}} V$.

Note that we computed $\dot{V}$ for the three possibilities in which the state vector $x(t)$ belongs in the state space $\mathcal{X}$. Therefore, we must ensure that $x(t)$ remains in $\mathcal{X}$ during the evolution of the sampled-data system. To this end, consider the following bounds on V over the boundaries of the state space,

$$C_1 = \min_{c^T x = \sigma_1} V(t, x_t), \ \forall \ x_t \in \mathcal{W}, \ \rho \in [0, \tau), \tag{6.44a}$$

$$C_{M+1} = \min_{c^T x = \sigma_{M+1}} V(t, x_t), \ \forall \ x_t \in \mathcal{W}, \ \rho \in [0, \tau), \tag{6.44b}$$

$$C = \min \{C_1, \ C_{M+1}\}. \tag{6.44c}$$

Note that the minima in (6.44) exist since V_1 is positive definite and radially unbounded, and V_2 and V_3 are non-negative. Let $\widetilde{C} \in (0, C)$ and define the set Ω as

$$\Omega = \{(t, x_t) | V(t, x_t) \leq \widetilde{C}\}. \tag{6.45}$$

Since V is strictly decreasing in the sampling intervals and non-increasing at the sampling instants, the set Ω is forward invariant. Considering (6.44), it can be shown by contradiction that the projection of the set Ω onto $\mathcal{X}$ lies in the interior of $\mathcal{X}$. Therefore, for any trajectory starting in Ω, the state vector remains in $\mathcal{X}$. Assuming that the system's trajectories start in Ω, based on Lyapunov-Krasovskii theorem [20], the closed-loop sampled data PWA slab system is uniformly asymptotically stable to the origin. Note that the Zeno phenomenon does not occur since, by Assumption 6.3, there exists $t_\epsilon > 0$ such that $t_{n+1} - t_n \geq t_\epsilon$ for all $n \in \mathbb{N}$. □

In the proof of Theorem 6.1, we showed that inequality (6.13) and inequalities (6.14)-(6.23) are sufficient conditions for the LKF to be decreasing, between two consecutive sampling times. Table 6.1 summarizes the correspondence between inequalities (6.13)-(6.23) and the portion of the state space that they refer to.

Remark 6.1. *In intuitive terms, relaxing Assumption 6.3 by letting the sampling intervals approach zero, yields $\tau \to 0$ and $x(t) = x(t_n)$ for $t_n \leq t < t_{n+1}$. Therefore, V_2 and V_3 vanish and the inequalities in Theorem 6.1 reduce to the LMI conditions for stability of continuous-time PWA slab systems (e.g. see [30]).*

We now present the result for a PWA slab system in feedback with a sampled-data PWL controller as a corollary.

Corollary 6.1. *Consider the sampled-data PWA slab system defined in (6.6) and (6.7) subject to Assumptions 6.1-6.3. Assume that the controller is piecewise linear (PWL), i.e. $k_i = 0$, $\forall i \in \mathcal{I}$. The system is uniformly asymptotically stable to the origin if there exist symmetric positive definite matrices P, R, and X, matrices $\widetilde{N}$ and N_i, with appropriate dimensions, and positive scalars γ, $\theta < 1$, η, λ_i, σ, and ϵ, with $i \in \mathcal{I}$, satisfying (6.13)-(6.15), (6.18)-(6.19), and (6.22)-(6.23).*

Proof. Since $k_i = 0$ for all $i \in \mathcal{I}$, we get $\delta k = 0$. Hence, equation (6.24) yields $\mu_\tau = 0$. According to the proof of Theorem 6.1, LMIs (6.13)-(6.15), (6.18)-(6.19), and (6.22)-(6.23) are sufficient conditions for the LKF (6.12) to be strictly decreasing for any $t \in (t_n, t_{n+1})$ and $||x(t_n)|| \geq 0$ (i.e. the whole state space). Since the LKF is non-increasing at the sampling instants, similar to the proof of Theorem 6.1, a forward invariant set can be found. Assuming that the trajectories start in the invariant set, the closed-loop sampled data PWA slab system is uniformly asymptotically stable to the origin. □

Remark 6.2. *For PWL controllers we have $\mu_\tau = 0$. Therefore according to Table 6.1, Corollary 6.1 contains only those inequalities of Theorem 6.1 that correspond to $||x(t_n)|| \geq \mu_\tau$. Consequently, the inequalities in Corollary 6.1 can be solved efficiently as a set of LMIs. For PWA controllers, however, the inequalities in Theorem 6.1 do not constitute a set of LMIs.*

Note that the matrix S_{3i} is a nonlinear function of the variables γ and θ. Hence, inequalities (6.13)-(6.23) cannot be solved simultaneously using LMI solvers. However, inequalities (6.13)-(6.15), (6.18)-(6.19), and (6.22)-(6.23) are linear in γ and θ

Table 6.1: The correspondence between inequalities of Theorem 6.1 and the state space.

	$\|\|x(t_n)\|\| \geq \mu_\tau$	$\|\|x(t_n)\|\| < \mu_\tau$
$x(t) \notin \mathcal{R}^*$	(6.13) and (6.14)-(6.15)	(6.16)-(6.17)
$x(t) \in \mathcal{R}^*$ and $x(t_n) \notin \mathcal{R}^*$	(6.13) and (6.18)-(6.19)	(6.20)-(6.21)
$x(t) \in \mathcal{R}^*$ and $x(t_n) \in \mathcal{R}^*$	(6.22)-(6.23)	

and constitute a set of LMIs. Moreover, treating γ and θ as constant parameters, inequalities (6.16)-(6.17) and (6.20)-(6.21) become a set of LMIs. Based on the above observations, we propose a two-phase algorithm for solving inequalities (6.13)-(6.23). To this end, consider the following remark.

Remark 6.3. *The variable $0 < \theta < 1$ appears only in inequality (6.13) and the matrix S_{3i}. Without loss of generality, we assume θ=1-**eps**, where **eps** is the machine epsilon. To justify this assumption, note that if (6.13) is satisfied for a θ, it is also satisfied for any larger θ. Moreover, based on (6.24), a larger θ yields a smaller μ_τ, which in turn provides a tighter bound on the mismatch vector w (see Lemma 6.2). A tighter bound on w makes LMIs (6.16)-(6.17) and (6.20)-(6.21) less conservative through the $\mathcal{S}$-procedure term S_{3i}. This in turn allows the algorithm to yield a larger lower bound on the longest sampling interval that preserves asymptotic stability.*

Algorithm 1 finds a lower bound on the longest interval between two consecutive sampling times $\tau_{\max}$ which preserves asymptotic stability. In the first phase of the algorithm, given τ, we solve the following optimization problem.

Problem 6.1.

$$\begin{aligned} &\textbf{\textit{minimize}}\ \gamma \\ &\textbf{\textit{subject to}}\ P > 0,\ R > 0,\ X > 0,\ \gamma > 0,\ \eta > 0,\ \sigma > 0,\ \lambda_i > 0,\ i \in \mathcal{I}, \\ &\qquad\qquad (6.13) - (6.15),\ (6.18),\ (6.19),\ (6.22),\ \textit{and}\ (6.23). \end{aligned}$$

If Problem 6.1 is feasible, according to Table 6.1, the LKF is decreasing for any $t \in (t_n, t_{n+1})$ and $||x(t_n)|| \geq \mu_\tau$. Note that minimizing γ leads to a smaller μ_τ which relaxes the inequalities that will be solved in the next phase (see Remark 6.3). Treating γ, P, R, and X as constant parameters computed in Problem 6.1, we solve the following feasibility problem in the second phase.

Name	Algorithm 1
Goal	Find a lower bound on the longest interval between two consecutive sampling times ($\tau_{\max}$) that preserves asymptotic stability
Inputs	A PWA slab system and a PWA slab continuous-time controller
Outputs	A lower bound on the longest interval between two consecutive sampling times ($\tau_{\max}$) and an LKF which proves asymptotic stability

Initialization: set $\tau_{\max} := 0$, $\theta := 1 - \mathbf{eps}$, $\tau_l := 0$, $\tau_u := \mathcal{M}$, where $\mathcal{M}$ is a large number, and choose a finite threshold > 0
while $\tau_u - \tau_l >$ threshold:
  set $\tau := (\tau_l + \tau_u)/2$
  if Problem 6.1 is infeasible:
    set $\tau_u := \tau$
  elseif the controller is PWL:
    set $\tau_{\max} := \tau$ and $\tau_l := \tau$
  else:
    (Using γ, P, R, and X from solution of Problem 6.1)
    if Problem 6.2 is infeasible:
      set $\tau_u := \tau$
    else:
      set $\tau_{\max} := \tau$ and $\tau_l := \tau$

Problem 6.2.

$$\begin{aligned} &\textbf{\textit{find}} && \epsilon > 0,\ \sigma > 0,\ \lambda_i > 0,\ \Lambda_i \succeq 0,\ i \in \mathcal{I} \\ &\textbf{\textit{subject to}} && (6.16),\ (6.17),\ (6.20),\ \textit{and}\ (6.21). \end{aligned}$$

If Problem 6.2 is feasible, based on Table 6.1, the LKF is decreasing for any $t \in (t_n, t_{n+1})$ and $||x(t_n)|| < \mu_\tau$.

Remark 6.4. *In Problem 6.2, matrices P, R, and X are treated as constant parameters and replaced with the numerical values computed in Problem 6.1, so that the same LKF is used both outside and inside the ball of radius μ_τ.*

In the next section, we use Algorithm 1 to compute $\tau_{\max}$ in a unicycle example.

6.4 Numerical Example

Consider the line following example of Chapter 4, whose objective is to control a unicycle to follow the line $y = 0$ in the $x - y$ plane (see Fig. 4.1). The dynamics of the system are represented by

$$\begin{bmatrix}\dot{\psi}\\ \dot{r}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -k/I & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\psi\\ r\\ y\end{bmatrix} + \begin{bmatrix}0\\ 0\\ v\sin(\psi)\end{bmatrix} + \begin{bmatrix}0\\ 1/I\\ 0\end{bmatrix}u, \tag{6.46}$$

where ψ and r are the heading angle and its time derivative, respectively, y is the distance from the line $y = 0$, v represents the unicycle's velocity, u is the torque input about the z axis, $I = 1$ (kgm^2) is the unicycle's moment of inertia with respect to its center of mass, and k =0.01 (Nms) is the damping coefficient. The state vector of the system is represented by $z = \begin{bmatrix}\psi & r & y\end{bmatrix}^T$. We assume that the unicycle has a constant velocity v =1 (m/s) and the heading angle ψ is restricted to the interval $[-3\pi/5, 3\pi/5]$, i.e. the state space is defined as $\mathcal{Z} = [-3\pi/5, 3\pi/5] \times \mathbb{R}^2$.

The system's nonlinearity, $\sin(\psi)$, is approximated by a PWA function. The PWA approximation is defined over the following five regions:

$$\begin{aligned}
&\mathcal{R}_1 = \left\{z \in \mathbb{R}^3 | \psi \in (-3\pi/5, -\pi/5)\right\},\ \mathcal{R}_5 = \left\{z \in \mathbb{R}^3 | \psi \in (\pi/5, 3\pi/5)\right\},\\
&\mathcal{R}_2 = \left\{z \in \mathbb{R}^3 | \psi \in (-\pi/5, -\pi/15)\right\},\ \mathcal{R}_4 = \left\{z \in \mathbb{R}^3 | \psi \in (\pi/15, \pi/5)\right\},\\
&\mathcal{R}_3 = \left\{z \in \mathbb{R}^3 | \psi \in (-\pi/15, \pi/15)\right\}.
\end{aligned}$$

Consider the PWA controller

$$u = K_i z + k_i, \text{ for } z \in \mathcal{R}_i,\ i \in \{1, ..., 5\}, \tag{6.47}$$

with

$$\begin{aligned}
K_1 &= [-49.907\ \ -9.468\ \ -13.925], \quad k_1 = -0.617,\\
K_2 &= [-48.315\ \ -9.330\ \ -13.812], \quad k_2 = 0.384,\\
K_3 &= [-50.147\ \ -9.468\ \ -13.742], \quad k_3 = 0,\\
K_4 &= [-48.316\ \ -9.330\ \ -13.812], \quad k_4 = -0.384,\\
K_5 &= [-49.907\ \ -9.468\ \ -13.925], \quad k_5 = 0.617.
\end{aligned}$$

The vector gains K_i, $i \in \{1, ..., 5\}$, are taken from the PWL controller proposed in [30]. The affine gains k_i, $i \in \{1, ..., 5\}$, are added to the controller such that the continuous-time PWA controller becomes continuous at the boundaries of the regions. Our goal is to find a lower bound on the longest interval between two consecutive sampling times such that asymptotic stability is guaranteed. Using Algorithm 1, with

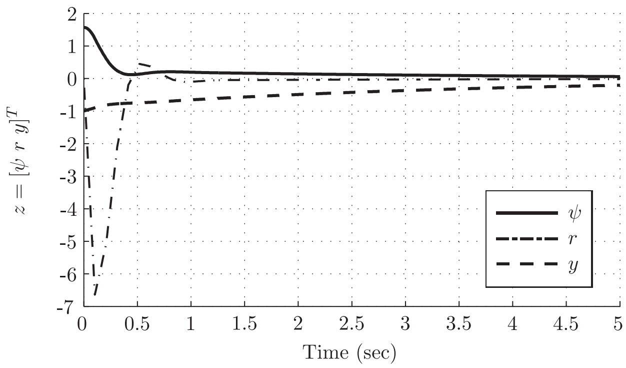


Figure 6.1: Unicycle's states for $T_s = \tau_{\max}$.

$\tau_u = 0.2$ and threshold=0.001, after eight iterations, we get

$$\tau_{\max} = 0.104\,(\text{sec})$$

and

$$P = \begin{bmatrix} 14.75 & 0.45 & 4.20 \\ 0.45 & 0.19 & 0.13 \\ 4.20 & 0.13 & 5.80 \end{bmatrix},\ X = \begin{bmatrix} 98.74 & 9.75 & 30.12 \\ 9.75 & 1.37 & 6.73 \\ 30.12 & 6.73 & 790.61 \end{bmatrix},$$
$$R = \begin{bmatrix} 8.29 & 72.57 & 1.18 \\ 72.57 & 7112.51 & -17.32 \\ 1.18 & -17.32 & 5.00 \end{bmatrix}. \tag{6.48}$$

Similar to (6.45), an invariant set Ω' can be computed by considering the quadratic term V_1 in the LKF, i.e. $\Omega' = \{(t, z_t) | V(t, z_t) \leq \widetilde{C'}\}$, where $\widetilde{C'} \in (0, C')$ and $C' = \min_{|\psi|=3\pi/5} V_1(t, z_t) \leq \min_{|\psi|=3\pi/5} V(t, z_t)$. Since $V_1 = z^T P z$, with P computed in (6.48), we find $C' = 39.245$. Let $\widetilde{C'} = 39.24 < C'$ and choose the system's trajectories to start in Ω'. Theorem 6.1 guarantees that if controller (6.47) is implemented in the unicycle via sample-and-hold, with variable sampling rates greater than $1/\tau_{\max} = 9.62$ (Hz), the PWA closed-loop system asymptotically converges to the origin.

Figures 6.1- 6.2, illustrate the simulation results for the unicycle system (6.46) with PWA feedback (6.47). The initial condition is $z_0(\alpha) = [\pi/2, 0, -1]^T$, $-0.104 \leq \alpha \leq 0$, and $\rho(0) = 0$. The simulation is performed for sampling time $T_s = \tau_{\max} = 0.104$ (sec). According to Fig. 6.1 the state vector asymptotically converges to the origin. The solid line in Fig. 6.2 shows the torque input for the sampled-data PWA controller. The dashed curve in Fig. 6.2 illustrates the torque input if the PWA controller was

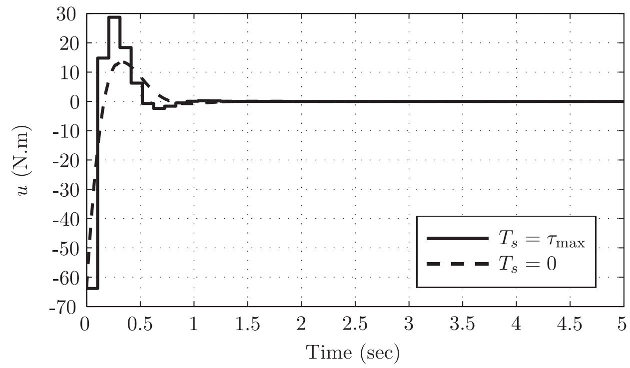


Figure 6.2: Control input for $T_s = \tau_{\max}$ and $T_s = 0$.

Table 6.2: Comparison of two stability theorems applied to the unicycle problem

Method	Stability Result	$\tau_{\max}$ (sec)
Theorem 1 in [54]	Convergence to the invariant set $\{V \leq 4.296 \times 10^6\}$	0.098
Algorithm 1	Asymptotic stability to the origin	0.104

implemented in continuous-time. As expected, more control energy is required to stabilize the system with the sample-and-hold controller.

Simulating the system with the same initial condition z_0 for $T_s = 0.213$ (sec), the closed-loop sampled-data trajectories do not converge to the origin. Therefore, in this example, the error in the computed lower bound on the longest sampling interval that preserves asymptotic stability is at most 51%. Still, as shown in Table 6.2, the $\tau_{\max}$ provided by Algorithm 1 is less conservative than the previous results in the literature. Moreover, Algorithm 1 provides a stronger stability result (asymptotic stability to the origin) than Theorem 1 in [54].

6.5 Conclusion

In this chapter, based on a modified LKF, sufficient conditions were provided for asymptotic stability of sampled-data PWA slab systems to the origin. It was shown that these conditions become LMIs in the case of a PWL controller. An algorithm was presented for finding a lower bound on the MASP that preserves asymptotic stability. The output of the algorithm provides an upper bound on the minimum sampling frequency that guarantees asymptotic stability of the sampled data system.

It was shown that our results compare favorably with the results available in the literature. Controller synthesis for PWA sampled-data systems will be addressed in the next chapter.

Chapter 7

Controller Synthesis for Piecewise Affine Sampled-data Systems

This chapter addresses exponential stability and stabilization of PWA slab systems with PWL sampled-data feedback. The PWL controller is assumed to be located in the feedback loop between a sampler with an unknown nonuniform sampling rate and a zero-order-hold. Convex Krasovskii-based sufficient conditions are proposed for exponential stability and stabilization of the sampled-data PWA slab system. The main contributions of this chapter are twofold. First, the direct sampled-data controller synthesis problem for PWA slab systems is formulated as a convex optimization program with the maximum allowable sampling period as a parameter. Second, sufficient conditions for exponential stability of PWA sampled-data systems are presented. The stability analysis and controller synthesis conditions are cast as LMIs. The results are successfully applied to a unicycle path following problem.

7.1 Introduction

PWA systems are a class of state-based switched systems where the vector field is affine in each mode or region. PWA systems arise in many engineering problems (e.g. systems with saturation, deadband, and hysteresis). They are also used as a tool for approximating nonlinear systems (see [23, 104, 105] and the references therein). Stability analysis and controller synthesis of PWA systems has received an increasing number of contributions since the late nineties. The reader is referred to [25–29] for stability analysis and to [26, 29, 30, 32] for controller synthesis in continuous-time. To be implementable in a microprocessor (or any sample and hold device), however, the designed continuous-time controllers must be emulated as a discrete-time controller.

In a PWA sampled-data system, the continuous-time PWA plant is controlled in discrete-time by a controller which is located in the feedback loop between a sampler and a zero-order-hold. This chapter is focused on stability and stabilization of PWA slab systems with PWL sampled-data feedback.

According to [37], there are three main approaches to sampled-data controller synthesis. In the emulation approach [40, 41], a continuous-time controller is designed based on the continuous-time plant, then approximated in discrete-time, and finally implemented via a sample and hold device. In the second approach, the discrete-time controller is designed based on an approximate discretized model of the plant [39, 44]. A common drawback of the first two approaches is that the "exact discrete-time models of continuous-time processes are typically impossible to compute" [39, 40]. Finally, the *direct sampled-data design approach* is more mathematically involved because it addresses the continuous-time plant and the discrete-time control signal simultaneously. Its advantage, however, is that the approximation step in the other two approaches is obviated.

The direct sampled-data design approach has recently gained an increasing interest in the literature of linear sampled-data systems. In this approach, the sampled-data system is usually modelled as either a continuous-time system with a time-varying *input delay* [15, 47] or a *hybrid* (impulsive) system with jumps at the sampling instants [48, 62]. Razumikhin or Krasovskii-type theorems [20] are then exploited to develop sufficient stability and stabilization conditions for the sampled-data system. While the Razumikhin-type theorems are based on classical Lyapunov *functions*, Krasovskii-type theorems use Lyapunov *functionals* and are known to be less conservative [15, 20].

Stability and stabilization of PWA sampled-data systems are challenging problems since the resulting hybrid systems simultaneously involve state-based (due to the PWA vector field) and event-based switching (due to the sampling). Given a PWA plant and a stabilizing continuous-time controller, references [53, 54] study the stability of the closed-loop PWA system in a sampled-data framework. Assuming uniform sampling intervals, reference [53] uses a quadratic Lyapunov function to provide sufficient conditions for convergence of the PWA sampled-data system to an invariant set containing the origin. Following the time-delay approach and using Krasovskii functionals, reference [54] addresses the same stability problem for the case of samplers with unknown nonuniform sampling intervals. The PWA sampled-data structure discussed in [55, 56] is different from the one in this chapter. In [55, 56], the switching is only event-based (i.e. occurs at the sampling instants), whereas in this chapter the switching is both state-based and event-based.

The main contributions of this chapter are twofold. First, the direct sampled-data controller synthesis problem for PWA slab systems is formulated as a convex optimization program with the maximum allowable sampling period (MASP) as a parameter. From an engineering perspective, without this formulation, there is no guarantee that a designed controller satisfies the MASP dictated by the sensing equipment. To the best of the authors' knowledge, the convex formulation for the controller synthesis problem of sampled-data PWA slab systems is presented here for the first time. Second, sufficient stability conditions for exponential stability of PWA sampled-data systems (as opposed to asymptotic stability in [53, 54]) are provided.

In this chapter, we follow the direct sampled-data design approach and input delay modeling to address stability and stabilization of PWA slab systems with PWL sampled-data feedback. For stability analysis, a PWL controller is assumed to be available which stabilizes the PWA system in continuous-time. The objective is to find a lower bound on the MASP that preserves exponential stability of the closed-loop sampled-data system. For controller synthesis, the desired MASP is assumed to be known. In this case, the objective is to design a PWL sampled-data controller that exponentially stabilizes the PWA slab system for the desired MASP. The stabilization results are successfully applied to a PWA model of a unicycle path following example. For the same example, it is shown that our sufficient stability conditions are less conservative when compared to other work in the literature [54].

The chapter is organized as follows. Section 7.2 provides preliminary information on PWA sampled-data systems. The stability and stabilization results are presented in Section 7.3 and Section 7.4, respectively. Finally, the new approach is applied to a unicycle path following example in Section 7.5.

Notation. The $n \times n$ identity matrix and the $n \times n$ zero matrix are denoted by I_n and 0_n, respectively. Non-square zero matrices and vectors of the appropriate size are simply represented by 0.

7.2 Preliminaries

Consider the PWA system

$$\dot{x}(t) = A_i x(t) + a_i + Bu(t),\ \forall\, x(t) \in \mathcal{R}_i,\ i \in \mathcal{I}, \tag{7.1}$$

where x denotes the state vector, $A_i \in \mathbb{R}^{n_x \times n_x}$, $a_i \in \mathbb{R}^{n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, and $u \in \mathbb{R}^{n_u}$ is the control input. The set $\mathcal{I} = \{1, \ldots, M\}$ contains the indices of the regions $\mathcal{R}_i$ that partition the state space $\mathcal{X} \subseteq \mathbb{R}^{n_x}$. The state space is represented by the union of the closure of all regions, i.e. $\mathcal{X} = \bigcup_{i \in \mathcal{I}} \overline{\mathcal{R}}_i$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. The regions are defined to be non-overlapping except on their closures. Two regions with overlapping closures are called neighbors. PWA slab systems constitute a special class of PWA systems where the state space is partitioned along a linear combination of the states, i.e.

$$\mathcal{R}_i = \{x | \sigma_i < \mathfrak{c}^T x < \sigma_{i+1}\}, \tag{7.2}$$

where $\mathfrak{c} \in \mathbb{R}^{n_x}$ and $\sigma_1 < \ldots < \sigma_{M+1}$ are scalars. The vector $\mathfrak{c}$ is usually a vector of zeros except for one element corresponding to the state that represents the state based switching (the nonlinearity of the system). Each slab region $\mathcal{R}_i$ can be represented [30] by a degenerate ellipsoid

$$\overline{\mathcal{R}}_i = \epsilon_i = \{x | \, |E_i x + e_i| \leq 1\}, \tag{7.3}$$

where

$$E_i = 2\mathfrak{c}^T / (\sigma_{i+1} - \sigma_i), \; e_i = -(\sigma_{i+1} + \sigma_i)/(\sigma_{i+1} - \sigma_i). \tag{7.4}$$

Let $\mathcal{I}(x) = \{i | x(t) \in \overline{\mathcal{R}}_i\}$. In the PWA literature, one often imposes a continuity assumption on the vector field across the boundaries of neighboring regions to avoid the occurrence of sliding modes (see [28] for more details).

Assumption 7.1. *For $u(t) = 0$, the open-loop vector field of system (7.1) is continuous across the boundaries of any neighboring regions.*

Remark 7.1. *In the case where the PWA system comes from an approximation of a continuous nonlinear function, the condition in Assumption 7.1 can be imposed using the algorithms in [23, 104, 105] and the references therein.*

Assumption 7.2. *The open-loop system is linear in the regions that contain the origin in their closure, i.e. $a_i = 0, \; \forall i \in \mathcal{I}(0)$. In other words, the origin is assumed to be an equilibrium point of the open-loop system.*

Let a continuous-time PWL controller for (7.1) be defined by

$$u(t) = K_i x(t), \; \forall \, x(t) \in \mathcal{R}_i, \; i \in \mathcal{I}, \tag{7.5}$$

where $K_i \in \mathbb{R}^{n_u \times n_x}$. In a sampled-data system, the state vector is measured at sampling intervals that might be uncertain and nonuniform. The following assumption imposes lower and upper bounds on the sampling interval.

Assumption 7.3. *The state vector is measured at sampling instants* t_n*,* $n \in \mathbb{N}$*, where* $0 < t_\epsilon \leq t_{n+1} - t_n \leq \tau$ *for all* $n \in \mathbb{N}$*.*

The positive constant t_ϵ is an arbitrary small number that models the fact that two sampling instants cannot occur simultaneously in practice. For sampled-data systems, the control input (7.5) can be rewritten as

$$u(t) = K_j x(t_n), \text{ for } t \in [t_n, t_{n+1}), \; x(t_n) \in \mathcal{R}_j, \text{ and } j \in \mathcal{I}. \tag{7.6}$$

For $x(t) \in \mathcal{R}_i$, $x(t_n) \in \mathcal{R}_j$, and $t \in [t_n, t_{n+1})$, equations (7.1) and (7.6) yield

$$\begin{aligned} \dot{x}(t) =& A_i x(t) + a_i + BK_j x(t_n) \\ =& A_i x(t) + a_i + BK_i x(t_n) + Bw(t), \end{aligned} \tag{7.7}$$

where $w \in \mathbb{R}^{n_u}$ is a piecewise constant vector defined by

$$w(t) = (K_j - K_i)x(t_n). \tag{7.8}$$

The vector w is associated with the fact that the current state vector and its most recent sample can possibly be in different regions. In order to address the nonuniform and unknown nature of the sampling intervals, the sampled-data system is modeled as a time-delay system with time-varying delays. To this end, the delay induced by the sampler is defined by

$$\rho(t) = t - t_n, \text{ for } t \in [t_n, t_{n+1}), \; n \in \mathbb{N}. \tag{7.9}$$

The function $\rho(t)$ denotes the time elapsed since the last sampling instant. Let $\mathcal{W}([-\tau, 0], \mathcal{X})$ be the space of absolutely continuous functions mapping the interval $[-\tau, 0]$ to $\mathcal{X}$. Consider the function $x_t \in \mathcal{W}$ defined as

$$x_t(r) = x(t + r), \quad -\tau \leq r \leq 0.$$

Similar to [47], we denote the norm of x_t by

$$||x_t||_{\mathcal{W}} = \max_{r \in [-\tau, 0]} |x_t(r)| + \left[\int_{-\tau}^{0} |\dot{x}_t(r)|^2 \, \mathrm{d}r \right]^{\frac{1}{2}}.$$

For $x(t) \in \mathcal{R}_i$ and $x(t_n) = x_t(-\rho(t)) \in \mathcal{R}_j$, the PWA sampled-data system (7.7) can now be rewritten as

$$\begin{aligned}\dot{x}(t) =& A_i x(t) + a_i + BK_i x_t(-\rho(t)) + Bw(t),\\ x_0(r) =& \phi(r), \ r \in [-\tau, 0],\end{aligned} \tag{7.10}$$

where ϕ is a vector-valued function specifying the initial condition in the interval $[-\tau, 0]$.

Definition 7.1. *The solution of system (7.10) is said to be locally uniformly exponentially stable with decay rate λ if there exist $\Omega \subseteq \mathcal{W}([-\tau, 0], \mathcal{X})$, $\delta > 0$, and $\lambda > 0$, such that for any initial condition $x_0 \in \Omega$, the solution $x(t) \in \mathcal{X}$ is defined for all $t \geq 0$ and satisfies*

$$|x(t)| \leq \delta e^{-\lambda t} ||x_0||_{\mathcal{W}}. \tag{7.11}$$

Moreover, if (7.11) is verified, the state space $\mathcal{X}$ is equal to $\mathbb{R}^{n_x}$, and $\Omega = \mathcal{W}([-\tau, 0], \mathbb{R}^{n_x})$, then the solution is globally uniformly exponentially stable.

7.3 Stability Analysis

Assume that a PWL controller is designed to stabilize the PWA system (7.1) in continuous-time. In practice, however, the controller will be located between a sampler and a zero-order-hold in the feedback loop. In this section, our objective is to find a lower bound on the MASP that preserves exponential stability of the closed-loop PWA system. To this end, we first present a Krasovskii functional. The functional is then used to propose sufficient stability and stabilization conditions in the form of LMIs. The LMIs can be solved efficiently using available software packages such as SeDuMi [16] and YALMIP [17]. For $t \in [t_n, t_{n+1})$, let $V(t, x_t)$ be a Krasovskii functional defined as

$$V(t, x_t) = V^{(1)}(x) + V^{(2)}(t, x_t) + V^{(3)}(t, x_t), \tag{7.12}$$

where

$$\begin{aligned}V^{(1)} =& x^T(t) P x(t),\\ V^{(2)} =& (\tau - \rho) \int_{t-\rho}^{t} e^{\alpha(s-t)} \begin{bmatrix} \dot{x}^T(s) & x^T(t_n) \end{bmatrix} R \begin{bmatrix} \dot{x}^T(s) & x^T(t_n) \end{bmatrix}^T \mathrm{d}s,\\ V^{(3)} =& (\tau - \rho) \begin{bmatrix} x^T(t) & x^T(t_n) \end{bmatrix} X \begin{bmatrix} x^T(t) & x^T(t_n) \end{bmatrix}^T,\end{aligned}$$

with ρ defined in (7.9), and

$$X = \begin{bmatrix} I_{n_x} & -I_{n_x} \end{bmatrix}^T X_1 \begin{bmatrix} I_{n_x} & -I_{n_x} \end{bmatrix}, \tag{7.13}$$

where $P > 0$, $R > 0$, and $X_1 > 0$, are matrices of appropriate dimensions and α is a positive scalar. One of the contributions of this work is the introduction of the functional $V^{(2)}$ in a new format (compare [47, 48, 54]). The functional $V^{(2)}$ penalizes the derivative of the state as well as the sampled state error in the interval $[t-\rho, t]$. The new functional allows us to prove exponential stability of the PWA system and to provide less conservative sufficient conditions (as will be shown in Section 7.5). Theorem 7.1 provides a set of sufficient conditions for which the trajectories of a PWA system in feedback with a PWL sampled-data controller, with sampling intervals smaller than τ, exponentially converge to the origin.

Theorem 7.1. *Consider the sampled-data PWA slab system defined in (7.1) with a given PWL controller subject to Assumptions 7.1-7.3. The system is locally uniformly exponentially stable with a decay rate larger than $\alpha/2$ if there exist symmetric matrices $P > 0$, $R > 0$, and $X_1 > 0$, matrices $\overline{N}_i$, $i \in \mathcal{I}$, with appropriate dimensions, and positive scalars c_{1i}, $i \in \mathcal{I}$, λ_i, $i \in \mathcal{I}\backslash\mathcal{I}(0)$, η, and γ satisfying*

$$\Delta K^2 \gamma < 1 \tag{7.14}$$

- *for all $i \in \mathcal{I}\backslash\mathcal{I}(0)$*

$$\overline{\Omega}_i + \tau \overline{M}_{1i} + \overline{S}_i < 0 \tag{7.15}$$

$$\begin{bmatrix} \overline{\Omega}_i + \tau \overline{M}_{2i} + \overline{S}_i & \tau \overline{N}_i \\ \tau \overline{N}_i^T & -\tau e^{-\alpha\tau} R \end{bmatrix} < 0 \tag{7.16}$$

- *for all $i \in \mathcal{I}(0)$*

$$\Omega_i + \tau M_{1i} < 0 \tag{7.17}$$

$$\begin{bmatrix} \Omega_i + \tau M_{2i} & \tau N_i \\ \tau N_i^T & -\tau e^{-\alpha\tau} R \end{bmatrix} < 0 \tag{7.18}$$

where X is defined in (7.13), ΔK is a positive parameter defined as

$$\Delta K = \max_{i,j \in \mathcal{I}} ||K_j - K_i||, \tag{7.19}$$

and

$$
\begin{aligned}
\overline{\Omega}_i =& \begin{bmatrix} A_i & BK_i & B & a_i \end{bmatrix}^T P \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} + \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} A_i & BK_i & B & a_i \end{bmatrix} \\
&+ \alpha \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} \\
&- \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T \overline{N}_i^T - \overline{N}_i \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} \\
&- \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix} + \mathrm{diag}(\eta I_{n_x}, I_{n_x}, -\gamma I_{n_u}, 0), \\
\overline{M}_{1i} =& \begin{bmatrix} A_i & BK_i & B & a_i \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}^T R \begin{bmatrix} A_i & BK_i & B & a_i \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix} + \alpha \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} A_i & BK_i & B & a_i \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix} + \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} A_i & BK_i & B & a_i \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} \\
\overline{M}_{2i} =& - \begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0 \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}^T \overline{N}_i^T - \overline{N}_i \begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0 \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}, \\
\overline{S}_i =& -\lambda_i \left(\begin{bmatrix} E_i & 0 & 0 & e_i \end{bmatrix}^T \begin{bmatrix} E_i & 0 & 0 & e_i \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right), \\
\Omega_i =& \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{\Omega}_i \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix}^T, \\
M_{1i} =& \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{M}_{1i} \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix}^T, \\
M_{2i} =& \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{M}_{2i} \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix}^T, \\
N_i =& \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{N}_i.
\end{aligned}
$$

Moreover, if the state space is equal to $\mathbb{R}^{n_x}$ *then the system is globally uniformly exponentially stable.*

Proof. Consider the Krasovskii functional (7.12). The proof consists of showing that LMIs (7.14)-(7.18) are sufficient conditions for the Krasovskii functional to satisfy $\dot{V}(t, x_t) + \alpha V(t, x_t) < 0$, $t \neq t_n$, $n \in \mathbb{N}$. The main steps of the proof are similar to the proof of Theorem 6.1. Therefore, the rest of the proof is omitted. □

Based on Theorem 7.1, the problem of finding a lower bound on the largest sampling interval that preserves exponential stability is formulated as

Problem 7.1.

maximize τ

subject to $P > 0,\ R > 0,\ X_1 > 0,\ \eta > 0,\ \gamma > 0,\ \lambda_i > 0,\ i \in \mathcal{I}\backslash\mathcal{I}(0),\ (7.14) - (7.18).$

The controller synthesis problem is addressed in the next section.

7.4 Controller Synthesis

In the controller synthesis problem the controller gains K_i are unknown. Therefore, the LMIs in Theorem 7.1 turn into non-convex matrix inequalities that cannot be solved efficiently. Theorem 7.2 addresses this issue and provides sufficient conditions for controller synthesis that can be cast as a convex optimization program.

Theorem 7.2. *Consider the sampled-data PWA slab system defined in (7.1) subject to Assumptions 7.1-7.3. There exists an exponentially stabilizing PWL controller with gains* $K_i = Y_i Q^{-1}$ *if there exist a symmetric matrix* Q*, matrices* Y_i*,* $\overline{\mathcal{N}}_i$*,* $i \in \mathcal{I}$*, with appropriate dimensions, and positive scalars* λ_i*,* $i \in \mathcal{I}\backslash\mathcal{I}(0)$*,* γ*,* μ*, and* ϵ_1*, satisfying*

$$Q > \gamma I_{n_x} \tag{7.20}$$

$$\begin{bmatrix} -\gamma & ||Y_i - Y_j|| \\ ||Y_i - Y_j|| & -1 \end{bmatrix} < 0,\ \forall i, j \in \mathcal{I} \tag{7.21}$$

- *for all* $i \in \mathcal{I}\backslash\mathcal{I}(0)$

$$\begin{bmatrix} \overline{\Gamma}_i + \tau\overline{\mathcal{M}}_{1i} + \mathcal{S}_i & \star & \star & \star \\ \tau\begin{bmatrix} A_iQ & BY_i & B & a_i \\ 0_{n_x} & Q & 0 & 0 \end{bmatrix} & -\tau\overline{Q} & \star & \star \\ \begin{bmatrix} \overline{Q} & 0 & 0 \end{bmatrix} & 0_{2n_x} & -\operatorname{diag}(\mu I_{n_x}, I_{n_x}) & \star \\ \lambda_i\begin{bmatrix} e_ia_i^T + E_iQ & 0 & 0 & 0 \end{bmatrix} & 0 & 0 & -\lambda_i(e_i^2 - 1) \end{bmatrix} < 0 \tag{7.22}$$

$$\begin{bmatrix} \overline{\Gamma}_i + \tau\overline{\mathcal{M}}_{2i} + \mathcal{S}_i & \star & \star & \star \\ \tau\overline{\mathcal{N}}_i^T & -\tau e^{-\alpha\tau}\overline{Q} & \star & \star \\ \begin{bmatrix} \overline{Q} & 0 & 0 \end{bmatrix} & 0_{2n_x} & -\operatorname{diag}(\mu I_{n_x}, I_{n_x}) & \star \\ \lambda_i\begin{bmatrix} e_ia_i^T + E_iQ & 0 & 0 & 0 \end{bmatrix} & 0 & 0 & -\lambda_i(e_i^2 - 1) \end{bmatrix} < 0 \tag{7.23}$$

• *for all* $i \in \mathcal{I}(0)$

$$\begin{bmatrix} \Gamma_i + \tau \mathcal{M}_{1i} & \tau \begin{bmatrix} QA_i^T & 0_{n_x} \\ Y_i^T B^T & Q \\ B^T & 0 \end{bmatrix} & \begin{bmatrix} \overline{Q} \\ 0 \end{bmatrix} \\ \star & -\tau \overline{Q} & 0_{2n_x} \\ \star & \star & -\operatorname{diag}(\mu I_{n_x}, I_{n_x}) \end{bmatrix} < 0 \tag{7.24}$$

$$\begin{bmatrix} \Gamma_i + \tau \mathcal{M}_{2i} & \tau \mathcal{N}_i & \begin{bmatrix} \overline{Q} \\ 0 \end{bmatrix} \\ \star & -\tau e^{-\alpha\tau} \overline{Q} & 0_{2n_x} \\ \star & \star & -\operatorname{diag}(\mu I_{n_x}, I_{n_x}) \end{bmatrix} < 0 \tag{7.25}$$

where

$$\begin{aligned} \overline{Q} =& \operatorname{diag}(Q, Q), \qquad (7.26) \\ \mathfrak{C} =& \epsilon_1 \begin{bmatrix} I_{n_x} & -I_{n_x} \end{bmatrix}^T \begin{bmatrix} I_{n_x} & -I_{n_x} \end{bmatrix}, \\ \overline{\Gamma}_i =& \begin{bmatrix} A_i Q & BY_i & B & a_i \end{bmatrix}^T \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} + \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T \begin{bmatrix} A_i Q & BY_i & B & a_i \end{bmatrix} \\ & + \alpha \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T Q \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} - \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T \overline{\mathcal{N}}_i^T \\ & - \overline{\mathcal{N}}_i \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} - \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix}^T \mathfrak{C} \overline{Q} \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix} \\ & + \operatorname{diag}(0_{n_x}, 0_{n_x}, -\gamma I_{n_u}, 0), \\ \overline{\mathcal{M}}_{1i} =& \alpha \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix}^T \mathfrak{C} \overline{Q} \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix} + \begin{bmatrix} A_i Q & BY_i & B & a_i \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \mathfrak{C} & 0 & 0 \end{bmatrix} \\ & + \begin{bmatrix} \mathfrak{C} & 0 & 0 \end{bmatrix}^T \begin{bmatrix} A_i Q & BY_i & B & a_i \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}, \\ \overline{\mathcal{M}}_{2i} =& - \begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0 \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}^T \overline{\mathcal{N}}_i^T - \overline{\mathcal{N}}_i \begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0 \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\mathcal{S}_i = -\lambda_i \begin{bmatrix} a_i a_i^T & 0_{n_x} & 0 & QE_i^T e_i \\ \star & 0_{n_x} & 0 & 0 \\ \star & \star & 0_{n_u} & 0 \\ \star & \star & \star & e_i^2 - 1 \end{bmatrix}, \tag{7.27}$$

$$\begin{aligned} \Gamma_i &= \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{\Gamma}_i \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix}^T, \\ \mathcal{M}_{1i} &= \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{\mathcal{M}}_{1i} \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix}^T, \\ \mathcal{M}_{2i} &= \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{\mathcal{M}}_{2i} \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix}^T, \\ \mathcal{N}_i &= \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{\mathcal{N}}_i. \end{aligned}$$

Moreover, if the state space is equal to $\mathbb{R}^{n_x}$ *then the system is globally uniformly exponentially stable.*

Proof. Here, we prove that inequalities (7.20)-(7.25) are sufficient conditions for the LMIs in Theorem 7.1 to be verified. Suppose there exist a symmetric matrix Q, matrices Y_i, and $\overline{\mathcal{N}}_i$, $i \in \mathcal{I}$, with appropriate dimensions, and positive scalars λ_i, $i \in \mathcal{I} \backslash \mathcal{I}(0)$, γ, μ, and ϵ_1, satisfying (7.20)-(7.25). Let

$$\begin{aligned} &P = Q^{-1},\ X_1 = \epsilon_1 Q^{-1},\ R = \overline{Q}^{-1},\ \eta = \mu^{-1}, \\ &K_i = Y_i Q^{-1},\ i \in \mathcal{I},\ \overline{N}_i = \widetilde{Q}^{-1} \overline{\mathcal{N}}_i \overline{Q}^{-1},\ i \in \mathcal{I}, \end{aligned} \tag{7.28}$$

where $\overline{Q}$ is defined in (7.26) and

$$\widetilde{Q} = \text{diag}(Q, Q, I_{n_u}, 1). \tag{7.29}$$

The rest of the proof is divided into four parts where we prove

1. (7.20) and (7.21) $\Rightarrow$ (7.14),
2. (7.22) and (7.23) $\Rightarrow$ (7.15) and (7.16),
3. (7.24) and (7.25) $\Rightarrow$ (7.17) and (7.18).

• *Part 1:* According to (7.20), the matrix Q is invertible. Therefore, LMI (7.20) yields

$$\gamma < \lambda_{\min}(Q) \Leftrightarrow \gamma < 1/\lambda_{\max}(Q^{-1}) \Leftrightarrow \gamma < 1/||Q^{-1}|| \tag{7.30}$$

where the right most inequality holds because Q is symmetric. Using Schur complement, LMI (7.21) implies $||Y_i - Y_j||^2 < \gamma,\ \forall i, j \in \mathcal{I}$. Multiplying both sides by $\gamma > 0$

and considering inequality (7.30), we can write

$$
\begin{aligned}
&||Y_i - Y_j||^2\gamma < ||Q^{-1}||^{-2},\ \forall\, i,j \in \mathcal{I} \\
&\Leftrightarrow (||Y_i - Y_j||\, ||Q^{-1}||)^2\gamma < 1,\ \forall\, i,j \in \mathcal{I}.
\end{aligned} \tag{7.31}
$$

Note that $||\Psi\Upsilon|| \leq ||\Psi||\, ||\Upsilon||$, where Ψ and Υ are matrices of appropriate dimensions (see [35], Appendix A). Therefore, inequality (7.31) yields $(||(Y_i - Y_j)Q^{-1}||)^2\gamma < 1,\ \forall\, i,j \in \mathcal{I}$. Hence, using the change of variables in (7.28), we can write

$$
||K_i - K_j||^2\gamma < 1,\ \forall\, i,j \in \mathcal{I} \Rightarrow (\max_{i,j\in\mathcal{I}} ||K_i - K_j||)^2\gamma < 1,
$$

which based on (7.19) is equivalent to LMI (7.14). This concludes the first part of the proof.

• *Part 2:* For $i \in \mathcal{I}\backslash\mathcal{I}(0)$, we first multiply inequality (7.22) from left and right by $\mathrm{diag}(\widetilde{Q}^{-1}, I_{2n_x}, I_{2n_x}, 1)$, where $\widetilde{Q}$ is defined in (7.29). Using Schur complement, we can write

$$
\begin{aligned}
&\mathrm{diag}(\widetilde{Q}^{-1}, I_{2n_x}, I_{2n_x})\times \\
&\begin{bmatrix} \overline{\Gamma}_i + \tau\overline{\mathcal{M}}_{1i} + \mathfrak{S}_i & \star & \star \\ \tau\begin{bmatrix} A_iQ & BY_i & B & a_i \\ 0_{n_x} & Q & 0 & 0 \end{bmatrix} & -\tau\overline{Q} & \star \\ \begin{bmatrix} \overline{Q} & 0 & 0 \end{bmatrix} & 0 & -\mathrm{diag}(\mu I_{n_x}, I_{n_x}) \end{bmatrix} \\
&\times \mathrm{diag}(\widetilde{Q}^{-1}, I_{2n_x}, I_{2n_x}) < 0,\ i \in \mathcal{I}\backslash\mathcal{I}(0),
\end{aligned} \tag{7.32}
$$

where

$$
\mathfrak{S}_i = \mathcal{S}_i + \lambda_i \begin{bmatrix} e_ia_i^T + E_iQ & 0 & 0 & 0 \end{bmatrix}^T (e_i^2 - 1)^{-1} \begin{bmatrix} e_ia_i^T + E_iQ & 0 & 0 & 0 \end{bmatrix}. \tag{7.33}
$$

Consider the term $\widetilde{Q}^{-1}\mathfrak{S}_i\widetilde{Q}^{-1}$ which appears in the first diagonal entry of the matrix inequality (7.32). Equations (7.27) and (7.33) yield

$$
\widetilde{Q}^{-1}\mathfrak{S}_i\widetilde{Q}^{-1} = -\lambda_i \begin{bmatrix} \theta_i & 0_{n_x} & 0 & E_i^Te_i \\ \star & 0_{n_x} & 0 & 0 \\ \star & \star & 0_{n_u} & 0 \\ \star & \star & \star & e_i^2 - 1 \end{bmatrix}, \tag{7.34}
$$

where

$$\theta_i = (e_i a_i^T Q^{-1} + E_i)^T (1 - e_i^2)^{-1} (e_i a_i^T Q^{-1} + E_i) + Q^{-1} a_i a_i^T Q^{-1}. \tag{7.35}$$

Equivalently, adding and subtracting the same terms, equation (7.35) can be written as

$$\begin{aligned}
\theta_i =& (e_i a_i^T Q^{-1} + E_i)^T (1 - e_i^2)^{-1} (e_i a_i^T Q^{-1} + E_i) + Q^{-1} a_i a_i^T Q^{-1} \\
&+ \Big(E_i^T (1 + e_i^2) E_i + (e_i a_i^T Q^{-1})^T E_i + E_i^T (e_i a_i^T Q^{-1}) + E_i^T (1 - e_i^2) E_i \\
&\quad - (e_i a_i^T Q^{-1} + E_i)^T E_i - E_i^T (e_i a_i^T Q^{-1} + E_i) \Big) \\
=& (e_i a_i^T Q^{-1} + E_i - (1 - e_i^2) E_i)^T (1 - e_i^2)^{-1} (e_i a_i^T Q^{-1} + E_i - (1 - e_i^2) E_i) \\
&+ Q^{-1} a_i a_i^T Q^{-1} + E_i^T (1 + e_i^2) E_i + (e_i a_i^T Q^{-1})^T E_i + E_i^T (e_i a_i^T Q^{-1}) \\
=& (e_i a_i^T Q^{-1} + e_i^2 E_i)^T (1 - e_i^2)^{-1} (e_i a_i^T Q^{-1} + e_i^2 E_i) + Q^{-1} a_i a_i^T Q^{-1} \\
&+ E_i^T E_i + E_i^T e_i^2 E_i + (e_i a_i^T Q^{-1})^T E_i + E_i^T (e_i a_i^T Q^{-1}) \\
=& E_i^T E_i - (a_i^T Q^{-1} + e_i E_i)^T (-1 - e_i (1 - e_i^2)^{-1} e_i) (a_i^T Q^{-1} + e_i E_i) \\
=& E_i^T E_i - (a_i^T Q^{-1} + e_i E_i)^T (e_i^2 - 1)^{-1} (a_i^T Q^{-1} + e_i E_i).
\end{aligned} \tag{7.36}$$

Next, we replace (7.36) and (7.34) in (7.32). Using Schur complement twice it can be verified that (7.32) is a sufficient condition for

$$\overline{\Omega}_i + \tau \overline{M}_{1i} + \overline{S}_i + \mathcal{L}_i < 0, \; i \in \mathcal{I} \backslash \mathcal{I}(0), \tag{7.37}$$

where $\overline{\Omega}_i$, $\overline{M}_{1i}$, and $\overline{S}_i$ are defined in Theorem 7.1 with the change of variables (7.28), and matrices $\mathcal{L}_i$, $i \in \mathcal{I} \backslash \mathcal{I}(0)$, are defined as

$$\mathcal{L}_i = \lambda_i \Big[(a_i^T P + e_i E_i) \quad 0 \quad 0 \quad 0 \Big]^T (e_i^2 - 1)^{-1} \Big[(a_i^T P + e_i E_i) \quad 0 \quad 0 \quad 0 \Big], \tag{7.38}$$

with change of variables (7.28). So far, we have shown that (7.22) is a sufficient condition for (7.32) which in turn is a sufficient condition for (7.37). Therefore, inequality (7.22) implies (7.37). Similarly, it can be shown that (7.23) is a sufficient condition for

$$\begin{bmatrix} \overline{\Omega}_i + \tau \overline{M}_{2i} + \overline{S}_i + \mathcal{L}_i & \tau \overline{N}_i \\ \tau \overline{N}_i^T & -\tau e^{-\alpha \tau} R \end{bmatrix} < 0, \; i \in \mathcal{I} \backslash \mathcal{I}(0), \tag{7.39}$$

where $\overline{\Omega}_i$, $\overline{M}_{2i}$, and $\overline{S}_i$ are defined in Theorem 7.1 with the change of variables (7.28), and matrices $\mathcal{L}_i$, $i \in \mathcal{I}\backslash\mathcal{I}(0)$, are defined in (7.38). Now, our goal is to show that inequalities (7.37) and (7.39) are sufficient conditions for LMIs (7.15) and (7.16). Comparing (7.37) and (7.39) with LMIs (7.15) and (7.16), the goal is achieved if we prove that the matrices $\mathcal{L}_i$ are positive semi-definite. Considering (7.2), for $i \in \mathcal{I}\backslash\mathcal{I}(0)$, the bounds σ_i and σ_{i+1} have the same sign. Therefore, based on (7.4), $|e_i| > 1$, $i \in \mathcal{I}\backslash\mathcal{I}(0)$. Hence, $e_i^2 - 1 > 0$ and, according to (7.38), the matrices $\mathcal{L}_i$, $i \in \mathcal{I}\backslash\mathcal{I}(0)$, are positive semi-definite. Therefore, inequalities (7.37) and (7.39) imply LMIs (7.15) and (7.16). This concludes the proof of Part 2, since we have shown that (7.22) and (7.23) are sufficient conditions for LMIs (7.15) and (7.16) to be verified.

• *Part 3:* For $i \in \mathcal{I}(0)$, multiplying (7.24) and (7.25) from left and right by $\text{diag}(\text{diag}(Q^{-1}, Q^{-1}, I_{n_u}), I_{2n_x}, I_{2n_x})$ and using Schur complement yields LMIs (7.17) and (7.18) with the change of variables (7.28). Note that in this case, the vector field of the PWA system is linear (Assumption 7.2) and the need for defining the auxiliary matrices $\mathcal{L}_i$ is eliminated.

The proof is complete since for any set of matrix variables satisfying inequalities (7.20)-(7.25), there exists a set of matrix variables (7.28) satisfying the stability criteria in Theorem 7.1. □

Remark 7.2. *The stabilization criteria in Theorem 7.2 are sufficient conditions for the stability criteria in Theorem 7.1 and therefore are more conservative. However, they can be used to design PWL controllers by solving a convex optimization program that can be solved efficiently using available software packages [16, 17].*

Based on Theorem 7.2 and using the line search strategy, the problem of designing an exponentially stabilizing PWL controller that maximizes the lower bound on the longest sampling interval is formulated as

Problem 7.2.

$$\begin{aligned}&\textit{maximize} \;\; \tau \\ &\textit{subject to} \;\; \lambda_i > 0,\; i \in \mathcal{I}\backslash\mathcal{I}(0),\; \gamma > 0,\; \mu > 0,\; \epsilon_1 > 0,\; (7.20) - (7.25).\end{aligned}$$

The controller gain is then computed as $K_i = Y_i Q^{-1}$, $i \in \mathcal{I}$.

7.5 Numerical Examples

In the literature of sampled-data systems, the lower bound on the MASP that preserves exponential stability is usually used as a criteria for comparing the conservativeness of stability theorems. The greater is the computed lower bound, the less conservative is the stability theorem. In the following examples, we use the same approach to demonstrate the effectiveness of the proposed sufficient stability and stabilization conditions.

Example 7.1. *Consider the path following problem in Chapter 4, whose objective is to control a unicycle to follow the line $y = 0$ in the $x - y$ plane (see Fig. 4.1). The dynamics of the system are represented by*

$$\begin{bmatrix} \dot{\psi} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -k/I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ r \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v\sin(\psi) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I \\ 0 \end{bmatrix} u, \tag{7.40}$$

where ψ and r are the heading angle and its time derivative, respectively, y is the distance from the line $y = 0$, v represents the unicycle's velocity, u is the torque input about the z axis, $I = 1$ (kgm^2) is the unicycle's moment of inertia with respect to its center of mass, and k =0.01 (Nms) is the damping coefficient. The state vector of the system is represented by $z^T = \begin{bmatrix} \psi & r & y \end{bmatrix}$. We assume that the unicycle has a constant velocity v =1 (m/s) and the heading angle ψ is restricted to the interval $[-\pi/2, \pi/2]$, i.e. the state space is defined as $\mathcal{Z} = [-\pi/2, \pi/2] \times \mathbb{R}^2$.

The system's nonlinearity, $\sin(\psi)$, is approximated by a PWA function. The PWA approximation is defined over the following five regions:

$$\begin{aligned}
\mathcal{R}_1 &= \left\{ z \in \mathbb{R}^3 | \psi \in (-\pi/2, -\pi/5) \right\}, \\
\mathcal{R}_2 &= \left\{ z \in \mathbb{R}^3 | \psi \in (-\pi/5, -\pi/15) \right\}, \\
\mathcal{R}_3 &= \left\{ z \in \mathbb{R}^3 | \psi \in (-\pi/15, \pi/15) \right\}, \\
\mathcal{R}_4 &= \left\{ z \in \mathbb{R}^3 | \psi \in (\pi/15, \pi/5) \right\}, \\
\mathcal{R}_5 &= \left\{ z \in \mathbb{R}^3 | \psi \in (\pi/5, \pi/2) \right\}.
\end{aligned}$$

Consider the PWL controller

$$u = K_i z, \ \textit{for } z \in \mathcal{R}_i, \ i \in \{1, ..., 5\}, \tag{7.41}$$

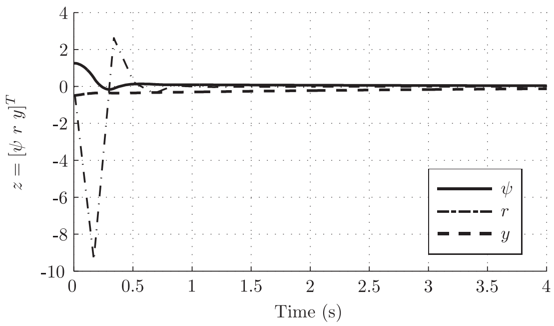


Figure 7.1: Unicycle's states for $T_s = \text{MASP} = 0.166$ (s) in Example 7.1.

with

$$K_1 = K_5 = \begin{bmatrix} -49.907 & -9.468 & -13.925 \end{bmatrix},$$
$$K_2 = K_4 = \begin{bmatrix} -48.315 & -9.330 & -13.812 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} -50.147 & -9.468 & -13.742 \end{bmatrix}.$$

The vector gains K_i, $i \in \{1, ..., 5\}$, are taken from the PWL controller proposed in [30]. Our goal is to find a lower bound on the longest interval between two consecutive sampling times such that exponential stability is guaranteed. Solving problem 7.1, for $\alpha = 0.0001$, *yields*

$$\text{MASP} = 0.166\,(\text{s}).$$

The decay rate ($\alpha/2$) was chosen to be small to make the results comparable with the existing methods in the literature that can only prove asymptotic stability. Theorem 7.1 guarantees that if controller (7.41) is implemented in the unicycle via sample-and-hold, with variable sampling rates greater than $1/\text{MASP} = 5.92$ *(Hz), the closed-loop PWA system exponentially converges to the origin.*

Now, consider a scenario in which the unicycle system (7.40) with PWL feedback (7.41) starts from the initial condition $z_0^T(r) = \begin{bmatrix} 2\pi/5 & 0 & -0.5 \end{bmatrix}$, $-0.166 \leq r \leq 0$, *and* $\rho(0) = 0$. *The simulation is performed with sampling intervals equal to* $T_s = \text{MASP} = 0.166$ *(s). According to Fig. 7.1 the state vector exponentially converges to the origin. The solid line in Fig. 7.2 shows the torque input for the PWL sampled-data controller. The dashed curve in Fig. 7.2 illustrates the torque input if the PWL controller was implemented in continuous-time. As expected, more control energy is required to stabilize the system with the sample-and-hold controller.*

Simulating the system with the same initial condition z_0 for $T_s = 0.213$ *(s), the*

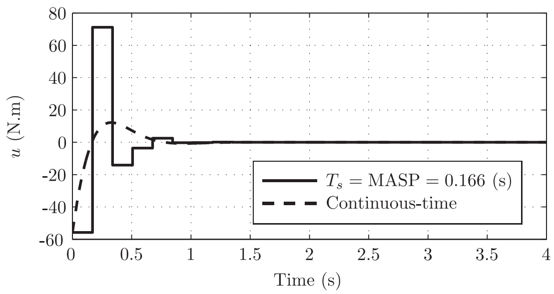


Figure 7.2: Control input for T_s = MASP = 0.166 (s) and the continuous-time case in Example 7.1.

Table 7.1: Comparison of different stability theorems applied to Example 7.1

Method	Stability result	MASP (s)
[54]	Convergence to an invariant set containing the origin	0.098
Theorem 7.1	Exponential stability to the origin	0.166

closed-loop sampled-data trajectories do not converge to the origin. Therefore, in this example, the error in the computed lower bound on the MASP that preserves exponential stability is at most 21%. As shown in Table 7.1, the value of the MASP provided by Theorem 7.1 is less conservative than the previous results in the literature. Moreover, Theorem 7.1 provides a stronger stability result (exponential stability to the origin).

Example 7.2. *Consider again the unicycle path following problem in Example 7.1. It was shown by simulation that the system is unstable for sampling intervals greater than* 0.213 *(s). In this example, our goal is to design a PWL controller that exponentially stabilizes the closed-loop sampled-data system for sampling intervals as large as* 0.213 *(s). Solving Problem 7.2 to design a PWL controller that provides the largest lower bound on the longest sampling interval that preserves exponential stability yields*

$$\mathrm{MASP} = 0.133\,(\mathrm{s}),$$

and

$$K_1 = K_5 = \begin{bmatrix} -21.235 & -8.100 & -17.980 \end{bmatrix},$$

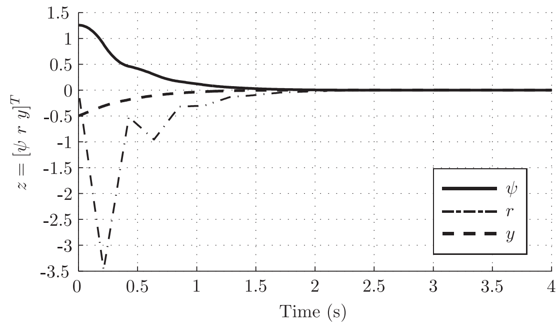


Figure 7.3: Unicycle's states for $T_s = 0.213$ (s) in Example 7.2.

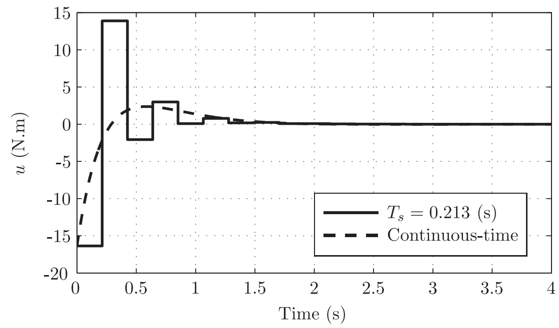


Figure 7.4: Control input for $T_s = 0.213$ (s) and the continuous-time case in Example 7.2.

$$\begin{aligned} K_2 = K_4 &= \begin{bmatrix} -21.071 & -8.081 & -18.814 \end{bmatrix}, \\ K_3 &= \begin{bmatrix} -21.153 & -8.090 & -18.928 \end{bmatrix}. \end{aligned} \tag{7.42}$$

The value of MASP provided by Problem 7.2 is not as large as we desired ($0.133 < 0.213$*). However, we have already shown that the convex formulation of the controller synthesis problem in Theorem 7.2 leads to extra conservatism in the sufficient conditions when compared to Theorem 7.1 (see Remark 7.2). Hence, in order to find a less conservative estimation of the MASP that preserves exponential stability, we solve Problem 7.1 with the new controller gains defined in (7.42). This yields*

$$\text{MASP} = 0.217\,(\text{s}).$$

Therefore, the designed PWL controller (7.42) is guaranteed to stabilize the sampled-data PWA model if the nonuniform sampling intervals are smaller than 0.217 *(s). Since* $0.213 < 0.217$*, the objective of this example is accomplished. Fig. 7.3 and*

Fig. 7.4 demonstrate the state vector and the control input, respectively, in a simulation with the new controller gains (7.42), initial condition z_0 *(same as Example 7.1), and sampling interval* $T_s = 0.213$ *(s).*

7.6 Conclusion

Exponential stability and stabilization of PWA slab systems with PWL sampled-data feedback was addressed. Convex Krasovskii-based sufficient conditions were proposed for exponential stability and stabilization of the sampled-data PWA slab system. The direct sampled-data controller synthesis problem for PWA slab systems was formulated as a convex optimization program with the MASP as a parameter.

Chapter 8

Stability and Stabilization of a Class of Nonlinear Sampled-data Systems

This chapter addresses exponential stability and stabilization of a class of uncertain nonlinear systems with PWL sampled-data feedback. The PWL controller is assumed to be located in the feedback loop between a sampler with an unknown nonuniform sampling rate and a zero-order-hold. First, the open-loop nonlinear system is bounded by a PWA differential inclusion. Next, convex Krasovskii-based sufficient conditions are proposed for exponential stability and stabilization of the closed-loop PWA sampled-data differential inclusion. The contributions of this chapter are twofold. The main contribution is the formulation of the direct sampled-data controller synthesis problem for a class of uncertain nonlinear systems as a convex optimization program with the maximum allowable sampling period as a parameter. Additionally, as the second contribution, sufficient conditions for exponential stability of a class of nonlinear sampled-data systems are presented using a piecewise smooth Krasovskii functional. This decreases the conservativeness of the proposed sufficient conditions when compared with the use of smooth Krasovskii functionals. The stability analysis and controller synthesis conditions are cast as LMIs. It is shown through an example that the proposed method can perform favorably when compared to other methods in the literature of nonlinear systems.

8.1 Introduction

In a nonlinear sampled-data system, the continuous-time nonlinear plant is controlled in discrete-time by a controller which is located in the feedback loop between a sampler and a zero-order-hold. This chapter is focused on stability and stabilization of a class of uncertain nonlinear systems with PWL sampled-data state feedback. First, the open-loop nonlinear system is bounded by a PWA differential inclusion. Next, sufficient conditions are proposed for exponential stability and stabilization of the closed-loop PWA sampled-data differential inclusion.

According to [37, 38], there are three main approaches to sampled-data controller synthesis. In the emulation approach, a continuous-time controller is designed based on the continuous-time plant, then approximated in discrete-time, and finally implemented via a sample and hold device. In this method, the controller can easily be designed based on performance specifications. The performance, however, is only guaranteed for sufficiently high sampling frequencies. In other words, the MASP should be sufficiently small. This results in a trade-off between performance and the cost of sensing equipment. In the second approach, the discrete-time controller is designed based on an approximate discretized model of the plant. The advantage of this approach is its simplicity at the cost of ignoring the inter-sample behaviour of the system. A common drawback of the first two approaches is that the "exact discrete-time models of continuous-time nonlinear processes are typically impossible to compute" [39, 40]. Finally, the direct sampled-data design approach is more mathematically involved because it addresses the continuous-time plant and the discrete-time control signal simultaneously. Its advantage, however, is that the approximation step in the other two approaches is obviated. In this chapter, we focus on the direct sampled-data design approach.

A general framework for the design of nonlinear controllers using the emulation approach is presented in [40]. First, a dissipation property is used to design a continuous-time controller. Next, the authors propose conditions that should be satisfied by the approximate discretized controller in order to preserve the dissipation property. Following the emulation approach, reference [41] addresses input-to-state stability of nonlinear systems with dynamic sampled-data controllers. A controller redesign scheme can later be used to improve the performance of the designed controller [42, 43].

For a discrete-time controller design based on an approximate discrete-time model of the plant, we refer the reader to [38, 39, 44] and the references therein. First, a parametrized family of approximate discrete-time models of the plant is developed. Next, a corresponding family of discrete-time controllers is designed for the approximate models. Reference [39] provides conditions to guarantee that the exact nonlinear sampled-data system is stable for sufficiently small modeling parameters and uniform samplings. As mentioned earlier, ignoring the inter-sample behaviour is a drawback of this approach. One way to address this issue is the lifting technique [37], where the closed-loop sampled-data system is modeled as a finite dimensional discrete-time system. The reader is referred to [45] for a study of sampled-data tracking problems and to [46] for H_∞ sampled-data control using the lifting technique.

The direct sampled-data design approach has recently gained an increasing interest in the literature of linear sampled-data systems (see [15, 47–49] and the references therein). In this approach, the sampled-data system is usually modelled as either a continuous-time system with a time-varying *input delay* [15, 47] or a *hybrid* (impulsive) system with jumps at the sampling instants [48]. Razumikhin or Krasovskii-type theorems [20] are then used to develop sufficient stability and stabilization conditions for the sampled-data system. These conditions are usually cast in terms of linear matrix inequalities (LMIs) which can be efficiently solved using software packages such as SeDuMi [16] and YALMIP [17]. While the Razumikhin-type theorems are based on classical Lyapunov *functions*, Krasovskii-type theorems use Lyapunov *functionals* and are known to be less conservative [9, 15, 20]. For direct sampled-data design of linear systems using the lifting technique the reader is referred to [37].

There are scarce references in the literature of nonlinear sampled-data systems where the *input delay* model (for static controllers) [41] or the *hybrid* model [50–52] of the system is studied. In all these references, however, a continuous-time controller is assumed to be available. In other words, the controller synthesis is performed while ignoring the sample and hold structure of the feedback. Therefore, similar to the emulation approach, references [41, 50–52] cannot be used to design controllers that provide a desired MASP. In contrast, one of the main contributions of this chapter is to propose a controller synthesis technique based on the direct sampled-data design approach where the MASP is considered as a parameter in the controller design problem. The proposed technique uses PWA differential inclusions and PWA systems which are discussed in the next subsection.

The main contributions of this chapter are twofold. First, the direct sampled-data controller synthesis problem for a class of uncertain nonlinear systems is formulated as a convex optimization program with the MASP as a parameter. From an engineering perspective, without this formulation, there is no guarantee that a designed controller satisfies the MASP dictated by the sensing equipment. To the best of the authors' knowledge, there is no other *direct sampled-data design* based approach in the literature of nonlinear systems that can be used to design sampled-data controllers for a desired MASP. Second, sufficient conditions for exponential stability of a class of nonlinear sampled-data systems are presented using a Krasovskii functional that is piecewise smooth in the state vector. This decreases the conservativeness of the proposed sufficient conditions when compared with the use of smooth Krasovskii functionals. Note that the piecewise smooth Krasovskii functional in our work is different from [106] and [107] where the *complete* Krasovskii functional is approximated by functionals that are piecewise linear in time and piecewise polynomial in time, respectively. This chapter also makes contributions in the field of PWA sampled-data systems. In particular, sufficient conditions for exponential stability of PWA sampled-data systems (as opposed to asymptotic stability in 6) are provided using piecewise smooth Krasovskii functionals.

In this chapter, we follow the direct sampled-data design approach and the input delay modeling to address stability and stabilization of a class of nonlinear systems with PWL sampled-data feedback. For stability analysis (see Section 8.3), a PWL controller is assumed to be available which stabilizes the nonlinear system in continuous-time. The objective is to find a lower bound on the MASP that preserves exponential stability of the closed-loop sampled-data system. For controller synthesis (see Section 8.4), the desired MASP is assumed to be known. In this case, the objective is to design a PWL sampled-data controller that exponentially stabilizes the nonlinear system for the desired MASP. Note that, in contrast to previous work in the literature, no pre-designed continuous-time controller is required in the controller synthesis problem. We show through examples that the proposed methods can perform favorably when compared to other methods in the literature of nonlinear systems.

The chapter is organized as follows. Section 8.2 provides preliminary information on differential inclusions, PWA sampled-data systems, and nonsmooth analysis. The stability and stabilization results are presented in Section 8.3 and Section 8.4, respectively. Finally, the new approach is applied to two examples in Section 8.5.

Notation. The Euclidean norms of a vector and a matrix are represented by $|.|$ and $||.||$, respectively. The $n \times n$ identity matrix and the $n \times n$ zero matrix are denoted by I_n and 0_n, respectively. Non-square zero matrices and vectors of the appropriate size are represented by 0. The symbol 0^+ denotes the $\lim_{\epsilon \searrow 0} \epsilon$.

8.2 Preliminaries

In this section, we present preliminary notions on PWA differential inclusions, PWA sampled-data systems, and nonsmooth analysis.

8.2.1 Nonlinear systems and piecewise affine differential inclusions

Consider the class of uncertain nonlinear systems

$$\dot{x} = f(x) + Bu,\ \forall\, x \in \mathcal{X} \subseteq \mathbb{R}^{n_x}, \tag{8.1}$$

where x denotes the absolutely continuous state vector, $\mathcal{X}$ represents the state space, $f : \mathcal{X} \to \mathbb{R}^{n_x}$ is an uncertain continuous nonlinear function, $f(0) = 0$, $B \in \mathbb{R}^{n_x \times n_u}$, and $u \in \mathbb{R}^{n_u}$ is the control input. The dynamic equations of many mechanical systems fall into the class of nonlinear systems (8.1) because the input (usually a force or a torque) appears linearly in Newton's second law of motion. The continuity condition on f is to rule out the possibility of sliding modes.

Assumption 8.1. *The open-loop vector field is bounded by a PWA differential inclusion defined as* $f(x(t)) \in \operatorname{conv}\{A_{i\kappa}x(t) + a_{i\kappa},\ \kappa = 1, 2\},\ \forall\, x(t) \in \mathcal{R}_i,\ i \in \mathcal{I}$, *where* conv *represents the convex hull of a set,* $A_{i\kappa} \in \mathbb{R}^{n_x \times n_x}$, $a_{i\kappa} \in \mathbb{R}^{n_x}$, *and the set* $\mathcal{I} = \{1, \ldots, M\}$ *contains the indices of the regions* $\mathcal{R}_i$ *that partition the state space* $\mathcal{X}$.

Remark 8.1. *Clearly, studying a family of functions as opposed to one particular function adds to the conservatism of the results. However, the family of functions described by a PWA differential inclusion are more tractable (due to the affine dynamics in each region) than the original nonlinear function. Furthermore, in general, the PWA differential inclusion in Assumption 8.1 can be arbitrarily tight at the cost of increasing the number of regions. In [108] (see Section* 4.3.1*), a convex optimization algorithm is provided to find the tightest possible PWA differential inclusion that bounds a given nonlinear function.*

According to Assumption 8.1, equation (8.1) yields

$$\dot{x}(t) \in \operatorname{conv}\{A_{i\kappa}x(t) + a_{i\kappa} + Bu(t),\ \kappa = 1, 2\},\ \forall\, x(t) \in \mathcal{R}_i,\ i \in \mathcal{I}. \tag{8.2}$$

Definition 8.1. *Given a Lebesgue integrable input $u(t)$ and an initial condition vector x_{IC}, an absolutely continuous function $x(t)$ is a solution of (8.2), if $x(t) \in \mathcal{X}$, $\forall\, t \geq 0$, $\dot{x}(t)$ is defined for almost all $t \geq 0$ and satisfies (8.2), and $x(0) = x_{IC}$.*

The two extreme dynamics corresponding to $\kappa = 1$ and $\kappa = 2$ in the PWA differential inclusion (8.2) represent the equations of two PWA systems

$$\dot{x}(t) = A_{i\kappa}x(t) + a_{i\kappa} + Bu(t),\ \forall\, x(t) \in \mathcal{R}_i,\ i \in \mathcal{I},\ \kappa = 1, 2. \tag{8.3}$$

The main idea behind using PWA differential inclusions to prove Lyapunov stability is summarized in the following lemma. Similar arguments can be found in the literature of linear differential inclusions [74] and PWA differential inclusions [26].

Lemma 8.1. *Let there exist a positive definite candidate Lyapunov function that is decreasing along the solution of each of the two extreme dynamics (8.3) corresponding to the PWA differential inclusion (8.2). Then the candidate Lyapunov function is also decreasing along every trajectory of the nonlinear system (8.1).*

Proof. Suppose that a positive definite candidate Lyapunov function V is decreasing along the extreme dynamics (8.3) of the PWA differential inclusion (8.2), i.e.

$$\dot{V} = \frac{\partial V}{\partial x}(A_{i1}x(t) + a_{i1} + Bu(t)) < 0,\ \forall\, x(t) \in \mathcal{R}_i,\ i \in \mathcal{I},$$

and

$$\dot{V} = \frac{\partial V}{\partial x}(A_{i2}x(t) + a_{i2} + Bu(t)) < 0,\ \forall\, x(t) \in \mathcal{R}_i,\ i \in \mathcal{I}.$$

Let $0 \leq \beta \leq 1$. Therefore, $\forall\, x(t) \in \mathcal{R}_i,\ i \in \mathcal{I}$,

$$\beta\frac{\partial V}{\partial x}(A_{i1}x(t) + a_{i1} + Bu(t)) + (1-\beta)\frac{\partial V}{\partial x}(A_{i2}x(t) + a_{i2} + Bu(t)) < 0.$$

According to (8.2), any trajectory of the nonlinear system (8.1) lies in the convex hull of the two extreme dynamics of the PWA differential inclusion. Therefore, $\dot{V} = \frac{\partial V}{\partial x}\dot{x} < 0$. This concludes the proof. □

In the rest of this subsection, we provide more details on PWA differential inclusions and PWA systems. The regions $\mathcal{R}_i$ are defined to be non-overlapping except on their closures. Two regions with overlapping closures are called neighbors. The state space is represented by the union of the closure of all regions, i.e. $\mathcal{X} = \bigcup_{i\in\mathcal{I}} \overline{\mathcal{R}}_i$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. In polytopic partitioning, each region $\mathcal{R}_i$ is defined as the intersection of p_i open half spaces in $\mathbb{R}^{n_x}$, i.e. $\mathcal{R}_i = \{x | G_i x + g_i \succ 0\}$, where

$G_i \in \mathbb{R}^{p_i \times n_x}$, $g_i \in \mathbb{R}^{p_i}$, and $\succ$ represents an elementwise inequality (see [23] for an algorithm to generate polytopic regions). Slab regions constitute a special class of polytopic regions where the state space is partitioned along a linear combination of the states. Each slab region is defined as

$$\mathcal{R}_i = \{x | \sigma_i < \mathfrak{c}_{\mathfrak{s}}\mathfrak{d}_{\mathfrak{n}}\mathfrak{l}^T x < \sigma_{i+1}\},$$

where $\mathfrak{c}_{\mathfrak{s}}\mathfrak{d}_{\mathfrak{n}}\mathfrak{l} \neq 0 \in \mathbb{R}^{n_x}$ and $\sigma_1 < \ldots < \sigma_{M+1}$ are scalars. Every polytopic region $\mathcal{R}_i$ can be outer approximated by a (possibly degenerate) ellipsoid as

$$\overline{\mathcal{R}}_i \subseteq \epsilon_i = \{x | \, |E_i x + e_i| \leq 1\}, \tag{8.4}$$

where $E_i \in \mathbb{R}^{n_e \times n_x}$, $e_i \in \mathbb{R}^{n_e}$, and $n_e \leq n_x$ (this inequality is strict for degenerate ellipsoids). A more detailed discussion on ellipsoidal approximations can be found in [29] and the references therein. In the case of slab regions [30], we have $n_e = 1$ and

$$E_i = \frac{2\mathfrak{c}_{\mathfrak{s}}\mathfrak{d}_{\mathfrak{n}}\mathfrak{l}^T}{\sigma_{i+1} - \sigma_i}, \;\; e_i = -\frac{\sigma_{i+1} + \sigma_i}{\sigma_{i+1} - \sigma_i}. \tag{8.5}$$

Moreover, for slab regions, $\overline{\mathcal{R}}_i = \epsilon_i$ (i.e. the ellipsoidal approximation is exact) if σ_i and σ_{i+1} are finite. A parametric description of the boundary between two polytopic regions $\mathcal{R}_i$ and $\mathcal{R}_j$ where $\overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset$ can be obtained as (see [29, 109] for more details)

$$\overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \subseteq \{x | x = F_{ij} s + f_{ij}, \;\; s \in \mathbb{R}^{n_x - 1}\}, \tag{8.6}$$

where $F_{ij} \in \mathbb{R}^{n_x \times n_x - 1}$ and $f_{ij} \in \mathbb{R}^{n_x}$. Finally, we define the set $\mathcal{I}(x)$ as

$$\mathcal{I}(x) = \{i | x(t) \in \overline{\mathcal{R}}_i\}. \tag{8.7}$$

The sampled-data structure of the system is addressed in the next subsection.

8.2.2 Piecewise affine sampled-data systems

In contrast to previous work [37–41, 44–46], samplers with unknown nonuniform sampling intervals will be considered in this work. The following assumption imposes lower and upper bounds on the sampling interval.

Assumption 8.2. *The state vector is measured at sampling instants t_n, $n \in \mathbb{N}$, where $0 < t_\epsilon \leq t_{n+1} - t_n \leq \tau, \;\forall n \in \mathbb{N}$.*

The positive constant t_ϵ is an arbitrary small number that models the fact that two sampling instants cannot occur simultaneously in practice. It is used in the proof of the main results to rule out the occurrence of the Zeno phenomenon. In this chapter, we are particularly interested in PWL sampled-data feedback, i.e.

$$u(t) = K_j x(t_n), \text{ for } t \in [t_n, t_{n+1}), \ x(t_n) \in \mathcal{R}_j, \text{ and } j \in \mathcal{I}. \tag{8.8}$$

Note that subscript j is used in (8.8) (as opposed to subscript i in (8.2)) to illustrate the fact that the current state vector $x(t)$ and its most recent sample $x(t_n)$ might be in different regions. For $x(t) \in \mathcal{R}_i$, $x(t_n) \in \mathcal{R}_j$, and $t \in [t_n, t_{n+1})$, equations (8.2) and (8.8) yield

$$\begin{aligned} &\dot{x}(t) \in \mathrm{conv}\{A_{i\kappa}x(t) + a_{i\kappa} + BK_j x(t_n), \ \kappa = 1, 2\} \\ \Rightarrow\ &\dot{x}(t) \in \mathrm{conv}\{A_{i\kappa}x(t) + a_{i\kappa} + BK_i x(t_n) + Bw(t), \ \kappa = 1, 2\}, \end{aligned} \tag{8.9}$$

where $w \in \mathbb{R}^{n_u}$ is a piecewise constant vector defined by

$$w(t) = (K_j - K_i)x(t_n). \tag{8.10}$$

The vector w is associated with the fact that the state vector and its most recent sample can possibly be in different regions. Following the input delay modelling technique [47], the sampled-data system is modeled as a time-delay system with a time-varying delay. To this end, the delay induced by the sampler is defined by

$$\rho(t) = t - t_n, \text{ for } t \in [t_n, t_{n+1}), \ n \in \mathbb{N}. \tag{8.11}$$

The function $\rho(t)$ denotes the time elapsed since the last sampling instant. Based on (8.11) and Assumption 8.2, the induced delay $\rho(t)$ is a saw-tooth function, bounded in the interval $[0, \tau)$, and with derivative $\dot{\rho}(t) = 1$. Let $\mathcal{W}([-\tau, 0], \mathcal{X})$ be the space of absolutely continuous functions mapping the interval $[-\tau, 0]$ to $\mathcal{X}$. Consider the function $x_t \in \mathcal{W}$ defined as $x_t(r) = x(t + r)$, $-\tau \le r \le 0$. For $x(t) \in \mathcal{R}_i$ and $x(t_n) = x_t(-\rho(t)) \in \mathcal{R}_j$, the PWA sampled-data system (8.9) can now be rewritten as

$$\begin{aligned} &\dot{x}(t) \in \mathrm{conv}\{A_{i\kappa}x(t) + a_{i\kappa} + BK_i x_t(-\rho(t)) + Bw(t), \ \kappa = 1, 2\}, \\ &x_0(r) = \phi(r), \ r \in [-\tau, 0], \end{aligned} \tag{8.12}$$

where $w(t)$ and $\rho(t)$ are defined in (8.10) and (8.11), respectively, and ϕ is a vector-valued function specifying the initial condition in the interval $[-\tau, 0]$. In the PWA literature, one often imposes a continuity assumption on the vector field across the boundaries of neighboring regions to avoid the occurrence of sliding modes (see [28] for more details).

Assumption 8.3. *For $u(t) = 0$, the open-loop vector fields of the PWA systems in (8.3) are continuous across the boundaries of any neighboring regions.*

The next assumption, combined with (8.2), guarantees that the origin is an equilibrium point of the open-loop nonlinear system.

Assumption 8.4. *For $u(t) = 0$, the open-loop vector fields of the PWA systems in (8.3) are linear in the regions that contain the origin in their closure, i.e. $a_{i\kappa} = 0,\ \forall\, i \in \mathcal{I}(0),\ \kappa \in \{1, 2\}$.*

Next, two important properties of the PWA systems defined in (8.3) are proved.

Lemma 8.2. *The solution $x(t)$ of each of the PWA systems described in (8.3) is absolutely continuous.*

Proof. Let $\kappa = 1$ (or $\kappa = 2$). Integrating (8.3), the function $x(t)$ is an indefinite integral and therefore absolutely continuous (see [73] Chapter 5, Theorem 13). $\square$

Lemma 8.3. *In the interval between two consecutive sampling instants, i.e. $\forall\, t \in (t_n, t_{n+1})$, $n \in \mathbb{N}$, the closed-loop vector field $\dot{x}(t)$ of each of the PWA systems in (8.3) is continuous everywhere (including at the boundaries of neighboring regions).*

Proof. Let $\kappa = 1$ (or $\kappa = 2$). The open-loop vector field is continuous in the interior of any region because it is affine according to equation (8.3). Moreover, the open-loop vector field is continuous across the boundaries of neighboring regions (as stated in Assumption 8.3). For $t \in (t_n, t_{n+1})$, $n \in \mathbb{N}$, the control signal u defined in (8.8) is constant. Therefore, according to (8.3), the closed-loop vector field $\dot{x}(t)$ is continuous everywhere in the interval between two consecutive sampling instants. $\square$

In Section 8.3, we use a piecewise smooth Krasovskii functional to prove stability of PWA sampled-data differential inclusions. The next subsection presents preliminary notions for nonsmooth functions.

8.2.3 Nonsmooth analysis

For functions defined in a finite dimensional space, the concept of gradient is generalized in the nonsmooth analysis literature through the following definition.

Definition 8.2. ***(Clarke's generalized gradient)****[110] Let the function $\overline{W} : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and let $\Omega_{\overline{W}}$ denote the set of measure zero where the function $\overline{W}$ fails to be differentiable. The generalized gradient of $\overline{W}$ at x is defined by*

$$\partial\overline{W}(x) = \mathrm{conv}\{\lim \nabla\overline{W}(x^p) : x^p \to x,\ x^p \notin \Omega_{\overline{W}}\}, \tag{8.13}$$

where conv *is the convex hull of a set and $x^p \to x$ represents any sequence converging to x.*

The following lemma presents the chain rule in nonsmooth analysis.

Lemma 8.4. *[111] If $\overline{W} : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz and $x : \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous, then for almost all t there exists $p \in \partial\overline{W}(x(t))$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\overline{W}(x(t)) = p\dot{x}(t)$.*

8.3 Stability Analysis

In this section, we address stability analysis of a class of nonlinear sampled-data systems. It is assumed that a stabilizing PWL controller is already designed in continuous-time. The objective in this section is to find a lower bound on the MASP that preserves exponential stability of the closed-loop sampled-data system. The controller synthesis problem for nonlinear sampled-data systems is addressed in Section 8.4. The main results of this section are provided in two theorems. Theorem 8.1 is a Krasovskii-type theorem which uses a piecewise smooth functional to propose sufficient conditions for exponential stability of a class of nonlinear sampled-data systems. In Theorem 8.2, a piecewise smooth Krasovskii functional is presented which enables one to formulate the sufficient stability conditions of Theorem 8.1 as an optimization program in terms of LMIs. We start by a preliminary result that will be used in the proofs of Theorem 8.1 and Theorem 8.2.

Since Clarke's generalized gradient (Definition 8.2) is only valid in finite dimensional spaces, a special structure is assumed for the piecewise smooth Krasovskii functional.

Lemma 8.5. *Let $\overline{W}(x) : \mathbb{R}^{n_x} \to \mathbb{R}^+$ be a locally Lipschitz piecewise smooth function defined as*

$$\overline{W}(x) = \overline{W}_i(x),\ \forall\, x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I}, \tag{8.14}$$

and let $\widetilde{W}(t, x_t) : \mathbb{R}^+ \times \mathcal{W} \to \mathbb{R}^+$ be a functional that is continuously differentiable with respect to time except possibly at instants $t = t_n$, $n \in \mathbb{N}$. Then the piecewise smooth Krasovskii functional defined as

$$W(t, x_t) = \overline{W}(x) + \widetilde{W}(t, x_t) \tag{8.15a}$$

$$= \overline{W}_i(x) + \widetilde{W}(t, x_t),\ \forall x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I} \tag{8.15b}$$

$$= W_i(t, x_t),\ \forall\, x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I}. \tag{8.15c}$$

is continuous for all $t \in (t_n, t_{n+1})$.

Proof. For $t \in (t_n, t_{n+1})$, the functional $W(t, x_t)$ is continuous because it is the sum of a locally Lipschitz continuous function $\overline{W}(x)$ and a functional $\widetilde{W}(t, x_t)$ that is differentiable and therefore continuous (see (8.15a)). □

Let $W_i(t_n^-, x_{t_n^-})$ denote the $\lim_{t \nearrow t_n} W_i(t, x_t)$. The following theorem provides sufficient conditions for exponential stability of a class of nonlinear sampled-data systems.

Theorem 8.1. *Consider the nonlinear system (8.1) with a sampled-data feedback subject to Assumptions 8.1-8.4. The closed-loop system is locally uniformly exponentially stable if there exists a piecewise smooth functional $W(t, x_t)$, with the structure defined in (8.15), such that*

$$c_{1i}|x_t(0)|^2 \le W_i(t, x_t) \le c_{2i}||x_t||^2_{\mathcal{W}},\ \forall\, x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I} \tag{8.16}$$

$$W_i(t_n, x_{t_n}) \le W_i(t_n^-, x_{t_n^-}),\ \forall\, x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I},\ \forall\, n \in \mathbb{N} \tag{8.17}$$

and the solution of each of the two extreme dynamics (8.3) satisfies

$$\nabla \overline{W}_i(x)\dot{x}(t) + \dot{\widetilde{W}}(t, x_t) + \alpha_i W_i(t, x_t) < 0,\ \forall\, x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I},\ \forall\, t \ne t_n,\ n \in \mathbb{N}, \tag{8.18}$$

where c_{1i}, c_{2i}, and α_i are positive scalars. If the state space is equal to $\mathbb{R}^{n_x}$ then the system is globally uniformly exponentially stable.

Proof. The main steps of the proof are similar to the proof of Lemma 8.1. First, it is proved that the conditions stated in the theorem are sufficient conditions for the Krasovskii functional W to be decreasing along the solution of each of the two extreme dynamics (8.3). Next, we use Lemma 8.1 to conclude that the Krasovskii functional is also decreasing along every trajectory of the nonlinear system (8.1). To this end, consider the extreme vector field corresponding to $\kappa = 1$ (or $\kappa = 2$). Note that according to (8.17), $W_i(t, x_t)$ is non-increasing at the sampling instants. Next, we analyze the interval between two sampling instants. Based on Lemma 8.5, in the interval (t_n, t_{n+1}), $n \in \mathbb{N}$, the function $\overline{W}(x)$ is locally Lipschitz and the functional $\widetilde{W}(t, x_t)$ is continuously differentiable. Furthermore, the solution $x(t)$ of the extreme dynamics with $\kappa = 1$ (or $\kappa = 2$) is absolutely continuous according to Lemma 8.2. Therefore, Lemma 8.4 yields

$$\dot{W}(t, x_t) = p\dot{x}(t) + \dot{\widetilde{W}}(t, x_t),\ \forall\, t \in (t_n, t_{n+1}), \tag{8.19}$$

where

$$p \in \partial\overline{W}(x) = \operatorname{conv}\{\nabla\overline{W}_i(x)|i \in \mathcal{I}(x)\}, \tag{8.20}$$

and $\mathcal{I}(x)$ is defined in (8.7). Consider the following two cases

1. the state vector is in the interior of a region,
2. the state vector is at the boundary of two or more regions.

• *Case 1.* According to (8.20), if $x(t) \in \mathcal{R}_i$, $i \in \mathcal{I}$, then $p = \nabla\overline{W}_i(x)$. Therefore, replacing (8.19) in (8.18) and considering (8.15c) yields

$$\dot{W}(t, x_t) + \alpha_i W(t, x_t) < 0,\ \forall\, x(t) \in \mathcal{R}_i,\ \forall\, t \in (t_n, t_{n+1}).$$

Let $\alpha = \min_{i\in\mathcal{I}} \alpha_i$. Since the functional $W(t, x_t)$ is non-negative according to (8.16), we can write

$$\dot{W}(t, x_t) + \alpha W(t, x_t) < 0,\ \forall\, x(t) \in \mathcal{R}_i,\ \forall\, t \in (t_n, t_{n+1}). \tag{8.21}$$

• *Case 2.* Assume that the state vector $x(t)$ is at the boundary of two or more regions $\mathcal{R}_i$, $i \in \mathcal{I}(x)$. Let β_i, $i \in \mathcal{I}(x)$, be positive scalars satisfying $\sum_{i\in\mathcal{I}(x)} \beta_i = 1$. For $x(t) \in \bigcap_{i\in\mathcal{I}(x)} \overline{\mathcal{R}}_i$ and $t \in (t_n, t_{n+1})$, inequality (8.18) yields

$$\sum_{i\in\mathcal{I}(x)} \beta_i \left(\nabla\overline{W}_i(x)\dot{x}(t) + \dot{\widetilde{W}}(t, x_t) + \alpha_i W_i(t, x_t)\right) < 0.$$

For $t \in (t_n, t_{n+1})$, according to Lemma 8.3 and Lemma 8.5, $\dot{x}(t)$, $\dot{\widetilde{W}}(t, x_t)$, and $W(t, x_t)$ are continuous at the boundaries of neighboring regions. Therefore at the boundary, the values of $\dot{x}(t)$, $\dot{\widetilde{W}}(t,x_t)$, and $W(t,x_t)$ are independent of the region they are defined in. Hence, for $x(t) \in \bigcap_{i\in\mathcal{I}(x)} \overline{\mathcal{R}}_i$ and $t \in (t_n, t_{n+1})$, we can write

$$\Big(\sum_{i\in\mathcal{I}(x)} \beta_i \nabla \overline{W}_i(x)\Big)\dot{x}(t) + \Big(\sum_{i\in\mathcal{I}(x)} \beta_i\Big)\dot{\widetilde{W}}(t,x_t) + \Big(\sum_{i\in\mathcal{I}(x)} \beta_i\alpha_i\Big) W(t,x_t) < 0,$$

where we used (8.15c) in the last summand. According to inequality (8.16), the functional $W(t,x_t)$ is non-negative. Therefore, for $x(t) \in \bigcap_{i\in\mathcal{I}(x)} \overline{\mathcal{R}}_i$ and $t \in (t_n, t_{n+1})$,

$$\Big(\sum_{i\in\mathcal{I}(x)} \beta_i \nabla \overline{W}_i(x)\Big)\dot{x}(t) + \dot{\widetilde{W}}(t,x_t) + \alpha W(t,x_t) < 0, \tag{8.22}$$

where we used the fact that $\sum_{i\in\mathcal{I}(x)} \beta_i = 1$. Replacing (8.19) and (8.20) in (8.22) yields

$$\dot{W}(t,x_t) + \alpha W(t,x_t) < 0,\ \forall\, x(t) \in \bigcap_{i\in\mathcal{I}(x)} \overline{\mathcal{R}}_i,\ \forall\, t \in (t_n, t_{n+1}). \tag{8.23}$$

Considering (8.21) and (8.23) the following inequality holds everywhere on $\mathcal{X}$

$$\dot{W}(t,x_t) + \alpha W(t,x_t) < 0,\ \forall\, t \in (t_n, t_{n+1}). \tag{8.24}$$

Therefore, solving (8.24) and using (8.17) yields

$$W(t,x_t) \le e^{-\alpha(t-t_n)} W(t_n, x_{t_n}) \le e^{-\alpha(t-t_n)} W(t_n^-, x_{t_n^-}) \le \ldots < e^{-\alpha t} W(0, x_0).$$

The last inequality is strict because it corresponds to the solution of (8.24) in at least one sampling interval with a nonzero length (note that, according to Assumption 8.2, any interval (t_n, t_{n+1}), $n \in \mathbb{N}$, has a length of at least $t_\epsilon > 0$). Based on Lemma 8.1, equation (8.24) is also valid for every trajectory of the nonlinear system (8.1). Let $c_1 = \min_{i\in\mathcal{I}} c_{1i}$ and $c_2 = \max_{i\in\mathcal{I}} c_{2i}$. Inequality (8.16) yields

$$|x(t)| = |x_t(0)| \le \left(\frac{W(t,x_t)}{c_1}\right)^{\frac{1}{2}} \le \left(\frac{e^{-\alpha t} W(0,x_0)}{c_1}\right)^{\frac{1}{2}} \le \left(\frac{c_2}{c_1}\right)^{\frac{1}{2}} e^{-\frac{\alpha}{2}t} ||x_0||_{\mathcal{W}}. \tag{8.25}$$

Note that the functional W is defined over $\mathbb{R} \times \mathcal{W}$. Therefore, equation (8.25) is only valid for $(t, x_t) \in \mathbb{R} \times \mathcal{W}$, i.e. $x(t+r) \in \mathcal{X}$, for all $-\tau \le r \le 0$. If the state space $\mathcal{X}$ is equal to $\mathbb{R}^{n_x}$, then (8.25) holds globally. In this case, the closed-loop nonlinear sampled-data system (8.1) is globally uniformly exponentially stable with a decay rate larger than $\alpha/2$ and an overshoot smaller than $\sqrt{c_2/c_1}$. On the other hand, if the state space $\mathcal{X}$ is a subset of $\mathbb{R}^{n_x}$ we must find a forward invariant set inside $\mathbb{R} \times \mathcal{W}$ to ensure the validity of (8.25). Note that if the state space $\mathcal{X}$ is a subset of $\mathbb{R}^{n_x}$, then there exists at least one $x(t) \in \partial\mathcal{X}$ with a finite norm. Consider the following bound on $W(t, x_t)$ over the boundary of the state space $\partial\mathcal{X}$

$$c = \inf_{x(t) \in \partial\mathcal{X}} W(t, x_t), \ \forall\, t \in \mathbb{R}^+, \ \forall\, x_t \in \mathcal{W}. \tag{8.26}$$

The existence of c follows from (8.16) and the fact that there exists at least one $x(t) \in \partial\mathcal{X}$ that has a finite norm. Consider the set $\Omega_{c_s d_n l} \subset \mathbb{R} \times \mathcal{W}$ defined as

$$\Omega_{c_s d_n l} = \{(t, x_t) | W(t, x_t) < c\}. \tag{8.27}$$

Since $W(t, x_t)$ is strictly decreasing in the sampling intervals (equation (8.24)) and non-increasing at the sampling instants (equation (8.17)), the set $\Omega_{c_s d_n l}$ is forward invariant. Therefore, for any initial condition $(0, x_0) \in \Omega_c$, the pair (t, x_t), $t \in \mathbb{R}^+$, remains in Ω_c. Next, we show that the state vector $x(t)$ remains in the interior of the state space for all $t \in \mathbb{R}^+$. To this end, let the projection of a point $(t, x_t) \in \Omega_{c_s d_n l}$ onto $\mathbb{R}^{n_x}$ be defined as $\text{Proj}(t, x_t) = x(t)$. We define the projection of the set $\Omega_{c_s d_n l}$ onto $\mathbb{R}^{n_x}$ as the union of the projections of all its members, i.e. $\text{Proj}(\Omega_c) = \{\text{Proj}(t, x_t) | (t, x_t) \in \Omega_c\}$. It is now shown by contradiction that $\text{Proj}(\Omega_c)$ lies in the interior of $\mathcal{X}$. Assume that this is not true. Then, there exists a point $x^*(t^*) \in \text{Proj}(\Omega_c) \bigcap \partial\mathcal{X}$ corresponding to a point $(t^*, x^*_{t^*}) \in \Omega_c$ for which $W(t^*, x^*_{t^*}) < c$. This contradicts (8.26). Therefore, based on (8.25), assuming that the system's trajectories start in $\Omega_{c_s d_n l}$, the closed-loop nonlinear sampled-data system (8.1) is locally uniformly exponentially stable with a decay rate larger than $\alpha/2$ and an overshoot smaller than $\sqrt{c_2/c_1}$. The possibility of sliding modes is avoided since the function $\dot{x}(t)$ is continuous everywhere in the state space according to Lemma 8.3. Note that the Zeno phenomenon does not occur either since, by Assumption 8.2, in any time interval with a length smaller than t_ϵ, there exists at most one sampling instant t_n, $n \in \mathbb{N}$. □

Remark 8.2. *The results of Theorem 8.1 are valid for any nonlinear system (8.1) in feedback with a sampled-data controller that verifies (8.18), regardless of the structure (i.e. linear, PWL, PWA, etc.) of the feedback signal u.*

In the rest of the chapter, we focus on nonlinear systems that are controlled by PWL sampled-data controllers. The inequality conditions in Theorem 8.1 cannot be directly coded in optimization software. In fact, condition (8.17) corresponds to an infinite number of inequality conditions parametrized by t_n. In the following, we will present a piecewise smooth Krasovskii functional that is formed by quadratic terms. Using this functional, the conditions in Theorem 8.1 are formulated as LMIs that can be solved efficiently using available software packages such as SeDuMi [16] and YALMIP [17]. For $t \in [t_n, t_{n+1})$, $i \in \mathcal{I}$, let $V(t, x_t)$ be a piecewise smooth Krasovskii functional defined as

$$
\begin{aligned}
V(t,x_t) =& V^{(1)}(x) + V^{(2)}(t,x_t) + V^{(3)}(t,x_t) \\
=& V_i^{(1)}(x) + V^{(2)}(t,x_t) + V^{(3)}(t,x_t),\ \forall x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I} \\
=& V_i(t,x_t),\ \forall\, x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I}
\end{aligned} \tag{8.28}
$$

where

$$
V^{(1)} = V_i^{(1)} = \overline{x}^T(t)\overline{P}_i\overline{x}(t),\ \forall x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I}, \tag{8.29}
$$

$$
V^{(2)} = (\tau - \rho)\int_{t-\rho}^{t} e^{\alpha(s-t)} \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix} R \begin{bmatrix}\dot{x}^T(s) & x^T(t_n)\end{bmatrix}^T \mathrm{d}s, \tag{8.30}
$$

$$
V^{(3)} = (\tau - \rho)\begin{bmatrix}x^T(t) & x^T(t_n)\end{bmatrix} X \begin{bmatrix}x^T(t) & x^T(t_n)\end{bmatrix}^T, \tag{8.31}
$$

with $\overline{x}(t) = \begin{bmatrix}x^T(t) & 1\end{bmatrix}^T$, ρ defined in (8.11), and

$$
\overline{P}_i = \begin{bmatrix}P_i & 0 \\ 0 & 0\end{bmatrix},\ \forall\, i \in \mathcal{I}(0), \tag{8.32}
$$

$$
X = \begin{bmatrix}I_{n_x} & -I_{n_x}\end{bmatrix}^T X_1 \begin{bmatrix}I_{n_x} & -I_{n_x}\end{bmatrix}, \tag{8.33}
$$

where $\overline{P}_i$, $i \in \mathcal{I}$, R, $X_1 = X_1^T$, and X_2 are matrices of appropriate dimensions and α is a positive scalar. It is easy to verify that the Krasovskii functional (8.28) falls into the structure of the functional (8.15), with $\overline{W} = V^{(1)}$ and $\widetilde{W} = V^{(2)} + V^{(3)}$.

Lemma 8.6. *The Krasovskii functional (8.28) satisfies conditions (8.16) and (8.17) in Theorem 8.1 if there exist symmetric matrices P_i, $i \in \mathcal{I}(0)$, $\overline{P}_i$, $i \in \mathcal{I}\backslash\mathcal{I}(0)$, $R > 0$, and $X_1 > 0$, with appropriate dimensions, positive scalars c_{1i}, $i \in \mathcal{I}$, and non-negative scalars λ_i', $i \in \mathcal{I}\backslash\mathcal{I}(0)$, satisfying*

$$
\overline{F}_{ij}^T(\overline{P}_i - \overline{P}_j)\overline{F}_{ij} = 0,\ \forall i,j \in \mathcal{I} : \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset \tag{8.34}
$$

$$
\overline{P}_i - S_i \geq \mathrm{diag}(c_{1i}I_{n_x}, 0),\ \forall\, i \in \mathcal{I}\backslash\mathcal{I}(0) \tag{8.35}
$$

$$
P_i \geq c_{1i}I_{n_x},\ \forall\, i \in \mathcal{I}(0) \tag{8.36}
$$

where F_{ij} and f_{ij}, $i, j \in \mathcal{I}$, are defined in (8.6), and

$$\overline{F}_{ij} = \begin{bmatrix} F_{ij} & f_{ij} \\ 0 & 1 \end{bmatrix}, \ \forall i, j \in \mathcal{I} : \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset,$$
$$S_i = -\lambda_i' \left(\begin{bmatrix} E_i & e_i \end{bmatrix}^T \begin{bmatrix} E_i & e_i \end{bmatrix} - \begin{bmatrix} 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \end{bmatrix} \right).$$

Proof. Equation (8.34) guarantees that the piecewise quadratic function $V^{(1)}(x)$ is continuous at the boundary of neighboring regions (see [109] for more details). The LKF (8.28) is similar to the LKF used in Chapter 2. Following the techniques presented in Chapter 2 it is easy to prove that the Krasovskii functional (8.28) is non-increasing at the sampling instants (i.e. inequality (8.17) is verified).

In order to prove that the Krasovskii functional (8.28) satisfies condition (8.16), we divide the state space into two parts

1. $x(t) \in \overline{\mathcal{R}}_i, \ i \in \mathcal{I} \backslash \mathcal{I}(0)$,

2. $x(t) \in \overline{\mathcal{R}}_i, \ i \in \mathcal{I}(0)$.

• *Part 1:* For $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I} \backslash \mathcal{I}(0)$, and non-negative λ_i', it follows from (8.4) that $\overline{x}^T S_i \overline{x} \geq 0$. Therefore, for $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I} \backslash \mathcal{I}(0)$, LMI (8.35) is a sufficient condition for the following inequality to hold

$$c_{1i}|x(t)|^2 = c_{1i}|x_t(0)|^2 \leq V_i^{(1)}. \tag{8.37}$$

Moreover, $R > 0$ and $X_1 > 0$ are sufficient conditions for $V^{(2)}$ and $V^{(3)}$ to be non-negative for $t \in [t_n, t_{n+1})$. Therefore, equation (8.28) and inequality (8.37) yield $c_{1i}|x_t(0)|^2 \leq V_i^{(1)} \leq V_i$. This proves that the Krasovskii functional V_i verifies the left hand side inequality in (8.16) for regions $\overline{\mathcal{R}}_i$, $i \in \mathcal{I} \backslash \mathcal{I}(0)$. Similarly, one can prove that the right hand side inequality in (8.16) is satisfied for regions $\overline{\mathcal{R}}_i$, $i \in \mathcal{I} \backslash \mathcal{I}(0)$.

• *Part 2:* In a similar way, it is easy to show that LMI (8.36) is a sufficient condition for inequality (8.16) to hold for all $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I}(0)$. □

Assume that a PWL controller is designed to stabilize the nonlinear system (8.1) in continuous-time. In practice, however, the controller will be located between a sampler and a zero-order-hold in the feedback loop. The objective is to find a lower bound on the MASP that preserves exponential stability of the closed-loop sampled-data system. To this end, Theorem 8.2 provides a set of sufficient conditions for which the trajectories of a class of nonlinear systems in feedback with a PWL sampled-data controller, with sampling intervals smaller than τ, exponentially converge to the origin. Later, we use Theorem 8.2 to cast the problem of finding a lower bound on the MASP as an optimization program in term of LMIs.

Theorem 8.2. *Consider the nonlinear system (8.1) and a given PWL sampled-data controller (8.8) subject to Assumptions 8.1-8.4. The closed-loop system is locally uniformly exponentially stable with a decay rate larger than $\alpha/2$ if there exist symmetric matrices P_i, $i \in \mathcal{I}(0)$, $\overline{P}_i$, $i \in \mathcal{I}\backslash\mathcal{I}(0)$, $R > 0$, and $X_1 > 0$, matrices $\overline{N}_{i\kappa}$, $i \in \mathcal{I}$, $\kappa \in \{1,2\}$, with appropriate dimensions, non-negative scalars λ'_i, $i \in \mathcal{I}\backslash\mathcal{I}(0)$, and positive scalars c_{1i}, $i \in \mathcal{I}$, $\lambda_{i\kappa}$, $i \in \mathcal{I}\backslash\mathcal{I}(0)$, $\kappa \in \{1,2\}$, η, and γ satisfying the LMIs in Lemma 8.6 and*

$$\Delta K^2\gamma < 1 \tag{8.38}$$

- *for all $i \in \mathcal{I}\backslash\mathcal{I}(0)$ and $\kappa \in \{1,2\}$*

$$\overline{\Omega}_{i\kappa} + \tau\overline{M}_{1i\kappa} + \overline{S}_{i\kappa} < 0 \tag{8.39}$$

$$\begin{bmatrix} \overline{\Omega}_{i\kappa} + \tau\overline{M}_{2i\kappa} + \overline{S}_{i\kappa} & \tau\overline{N}_{i\kappa} \\ \tau\overline{N}_{i\kappa}^T & -\tau e^{-\alpha\tau}R \end{bmatrix} < 0 \tag{8.40}$$

- *for all $i \in \mathcal{I}(0)$ and $\kappa \in \{1,2\}$*

$$\Omega_{i\kappa} + \tau M_{1i\kappa} < 0 \tag{8.41}$$

$$\begin{bmatrix} \Omega_{i\kappa} + \tau M_{2i\kappa} & \tau N_{i\kappa} \\ \tau N_{i\kappa}^T & -\tau e^{-\alpha\tau}R \end{bmatrix} < 0 \tag{8.42}$$

where $\overline{P}_i$, $i \in \mathcal{I}(0)$, and X are defined in (8.32) and (8.33), respectively, ΔK is a positive parameter defined as

$$\Delta K = \max_{i,j\in\mathcal{I}} ||K_j - K_i||, \tag{8.43}$$

and

$$\overline{\Omega}_{i\kappa} = \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa} \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \overline{P}_i \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$+ \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \overline{P}_i \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
&+\alpha\begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}^T \overline{P}_i \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} - \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0\\ 0_{n_x} & 0_{n_x} & 0 & 0\end{bmatrix}^T \overline{N}_{i\kappa}^T\\
&-\overline{N}_{i\kappa}\begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0\\ 0_{n_x} & 0_{n_x} & 0 & 0\end{bmatrix} - \begin{bmatrix} I_{2n_x} & 0 & 0\end{bmatrix}^T X \begin{bmatrix} I_{2n_x} & 0 & 0\end{bmatrix}\\
&+\operatorname{diag}(\eta I_{n_x}, I_{n_x}, -\gamma I_{n_u}, 0),
\end{aligned}
$$

$$
\begin{aligned}
\overline{M}_{1i\kappa} =& \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa}\\ 0_{n_x} & I_{n_x} & 0 & 0\end{bmatrix}^T R \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa}\\ 0_{n_x} & I_{n_x} & 0 & 0\end{bmatrix} + \alpha\begin{bmatrix} I_{2n_x} & 0 & 0\end{bmatrix}^T X \begin{bmatrix} I_{2n_x} & 0 & 0\end{bmatrix}\\
&+\begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa}\\ 0_{n_x} & 0_{n_x} & 0 & 0\end{bmatrix}^T X \begin{bmatrix} I_{2n_x} & 0 & 0\end{bmatrix} + \begin{bmatrix} I_{2n_x} & 0 & 0\end{bmatrix}^T X \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa}\\ 0_{n_x} & 0_{n_x} & 0 & 0\end{bmatrix}
\end{aligned}
$$

$$
\overline{M}_{2i\kappa} = -\begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0\\ 0_{n_x} & I_{n_x} & 0 & 0\end{bmatrix}^T \overline{N}_{i\kappa}^T - \overline{N}_{i\kappa}\begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0\\ 0_{n_x} & I_{n_x} & 0 & 0\end{bmatrix},
$$

$$
\overline{S}_{i\kappa} = -\lambda_{i\kappa}\left(\begin{bmatrix} E_i & 0 & 0 & e_i\end{bmatrix}^T \begin{bmatrix} E_i & 0 & 0 & e_i\end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1\end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 1\end{bmatrix}\right),
$$

$$
\Omega_{i\kappa} = \begin{bmatrix} I_{2n_x+n_u} & 0\end{bmatrix} \overline{\Omega}_{i\kappa} \begin{bmatrix} I_{2n_x+n_u} & 0\end{bmatrix}^T,
$$

$$
M_{1i\kappa} = \begin{bmatrix} I_{2n_x+n_u} & 0\end{bmatrix} \overline{M}_{1i\kappa} \begin{bmatrix} I_{2n_x+n_u} & 0\end{bmatrix}^T,
$$

$$
M_{2i\kappa} = \begin{bmatrix} I_{2n_x+n_u} & 0\end{bmatrix} \overline{M}_{2i\kappa} \begin{bmatrix} I_{2n_x+n_u} & 0\end{bmatrix}^T,
$$

$$
N_{i\kappa} = \begin{bmatrix} I_{2n_x+n_u} & 0\end{bmatrix} \overline{N}_{i\kappa}.
$$

Moreover, if the state space is equal to $\mathbb{R}^{n_x}$ then the system is globally uniformly exponentially stable.

Proof. Consider the Krasovskii functional (8.28). Based on Lemma 8.6, LMIs (8.34)-(8.36), $R > 0$, $X_1 > 0$, and non-negative λ_i', $i \in \mathcal{I}\backslash\mathcal{I}(0)$, are sufficient conditions for the Krasovskii functional (8.28) to satisfy conditions (8.16) and (8.17) in Theorem 8.1. Hence, it suffices to prove that the remaining LMIs in Theorem 8.2 (LMIs (8.38)-(8.42)) are sufficient conditions for the solution of each of the two extreme dynamics (8.3) to satisfy inequality (8.18), i.e.

$$
\begin{aligned}
&\nabla V_i^{(1)}(x)\dot{x}(t) + \dot{V}^{(2)}(t,x_t) + \dot{V}^{(3)}(t,x_t) + \alpha_i V_i(t,x_t) < 0,\\
&\forall\, x(t) \in \overline{\mathcal{R}}_i,\ i \in \mathcal{I},\ \forall\, t \neq t_n,\ n \in \mathbb{N},
\end{aligned}
\tag{8.44}
$$

for arbitrary $\alpha_i \geq \alpha > 0$. Without loss of generality, we assume

$$\alpha_i = \alpha, \ \forall i \in \mathcal{I}. \tag{8.45}$$

For the first summand in inequality (8.44), equation (8.29) yields

$$\nabla V_i^{(1)}(x)\dot{x}(t) = \dot{\overline{x}}^T \overline{P}_i \overline{x} + \overline{x}^T \overline{P}_i \dot{\overline{x}}, \ \forall\, x(t) \in \overline{\mathcal{R}}_i, \ i \in \mathcal{I}. \tag{8.46}$$

For $t \in (t_n, t_{n+1})$, following the technique presented in Chapter 2 yields

$$\begin{aligned}\dot{V}^{(2)} \leq & \rho h_i^T e^{\alpha\tau} R^{-1} h_i - \begin{bmatrix} x^T(t) - x^T(t_n) & \rho x^T(t_n) \end{bmatrix} h_i - h_i^T \begin{bmatrix} x^T(t) - x^T(t_n) & \rho x^T(t_n) \end{bmatrix}^T \\ & + (\tau - \rho) \begin{bmatrix} \dot{x}^T(t) & x^T(t_n) \end{bmatrix} R \begin{bmatrix} \dot{x}^T(t) & x^T(t_n) \end{bmatrix}^T - \alpha V^{(2)}. \end{aligned} \tag{8.47}$$

For $t \in (t_n, t_{n+1})$, the time derivative of $V^{(3)}$, defined in (8.31), is computed as

$$\begin{aligned}\dot{V}^{(3)} = & - \begin{bmatrix} x^T(t) & x^T(t_n) \end{bmatrix} X \begin{bmatrix} x^T(t) & x^T(t_n) \end{bmatrix}^T + (\tau - \rho) \begin{bmatrix} \dot{x}^T(t) & 0 \end{bmatrix} X \begin{bmatrix} x^T & x^T(t_n) \end{bmatrix}^T \\ & + (\tau - \rho) \begin{bmatrix} x^T(t) & x^T(t_n) \end{bmatrix} X \begin{bmatrix} \dot{x}^T & 0 \end{bmatrix}^T. \end{aligned} \tag{8.48}$$

We now divide the state space into two parts

1. $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I} \backslash \mathcal{I}(0)$,
2. $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I}(0)$.

In the rest of the proof, we check the requirements for inequality (8.44) to hold in each part of the state space.

• *Part 1:* For $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I} \backslash \mathcal{I}(0)$, based on (8.9), the two extreme dynamics are defined as

$$\dot{x}(t) = \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa} \end{bmatrix} \zeta(t), \ \kappa = 1 \text{ or } \kappa = 2, \tag{8.49}$$

where $\zeta(t) = \begin{bmatrix} x^T(t) & x^T(t_n) & w^T(t) & 1 \end{bmatrix}^T \in \mathbb{R}^{2n_x + n_u + 1}$. In what follows, we represent the left hand side of inequality (8.44) by LHS. Let $\kappa = 1$ (or $\kappa = 2$), replace (8.49) in (8.46), (8.47), and (8.48) and set $h_i(t) = \overline{N}_{i\kappa}^T \zeta(t)$, where $\overline{N}_{i\kappa}$ is a matrix in $\mathbb{R}^{(2n_x + n_u + 1) \times 2n_x}$. Considering (8.45) we can write

$$\text{LHS} = \nabla V_i^{(1)}(x)\dot{x}(t) + \dot{V}^{(2)}(t, x_t) + \dot{V}^{(3)}(t, x_t) + \alpha_i V_i(t, x_t)$$

$$
\begin{aligned}
\leq \zeta^T \Bigg(& \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa} \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \overline{P}_i \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
& + \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \overline{P}_i \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
& + \alpha \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \overline{P}_i \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \rho \overline{N}_{i\kappa} e^{\alpha\tau} R^{-1} \overline{N}_{i\kappa}^T \\
& - \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & \rho I_{n_x} & 0 & 0 \end{bmatrix}^T \overline{N}_{i\kappa}^T - \overline{N}_{i\kappa} \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & \rho I_{n_x} & 0 & 0 \end{bmatrix} \\
& + (\tau - \rho) \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa} \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}^T R \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa} \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix} \\
& + (\alpha(\tau - \rho) - 1) \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix} \\
& + (\tau - \rho) \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa} \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix} \\
& + (\tau - \rho) \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix}^T X \begin{bmatrix} A_{i\kappa} & BK_i & B & a_{i\kappa} \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} \Bigg) \zeta. \qquad (8.50)
\end{aligned}
$$

For $\rho = 0$, LMI (8.39) implies

$$
\text{LHS} < -\eta x^T x - x^T(t_n) x(t_n) + \gamma w^T w - \zeta^T \overline{S}_{i\kappa} \zeta. \qquad (8.51)
$$

Using Schur complement, LMI (8.40) implies that (8.51) holds for $\rho = \tau$. Since (8.50) is affine in ρ, LMIs (8.39) and (8.40) are sufficient conditions for (8.51) to hold for any $\rho \in (0, \tau)$. Recalling (8.10) and (8.43), we can write

$$
||w|| \leq \Delta K ||x(t_n)||, \qquad (8.52)
$$

which considering (8.38), yields $||w||^2 < \frac{1}{\gamma} ||x(t_n)||^2$, or equivalently

$$
0 < x^T(t_n) x(t_n) - \gamma w^T w. \qquad (8.53)
$$

Adding inequality (8.53) to inequality (8.51) yields LHS $< -\eta x^T x - \zeta^T \overline{S}_{i\kappa} \zeta$. For $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I} \backslash \mathcal{I}(0)$, and positive $\lambda_{i\kappa}$, it follows from (8.4) that $\zeta^T \overline{S}_{i\kappa} \zeta \geq 0$. Hence, LMIs (8.38), (8.39), and (8.40) are sufficient conditions for inequality (8.44) and therefore (8.18) to hold for any $t \in (t_n, t_{n+1})$, $n \in \mathbb{N}$, and $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I}\backslash\mathcal{I}(0)$.

• *Part 2:* For $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I}(0)$, based on Assumption 8.4, we have $a_{i\kappa} = 0$, $\kappa \in \{1, 2\}$. Setting $a_{i\kappa} = 0$ and $N_{i\kappa} = \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{N}_{i\kappa}$ in (8.50), LMI (8.41) implies

$$\text{LHS} < -\eta x^T x - x^T(t_n)x(t_n) + \gamma w^T w \tag{8.54}$$

for $\rho = 0$. Using Schur complement, LMI (8.42) implies that (8.54) holds for $\rho = \tau$. Since (8.50) is affine in ρ, LMIs (8.41) and (8.42) are sufficient conditions for (8.54) to hold for any $\rho \in (0, \tau)$. Adding inequality (8.53) to inequality (8.54) yields LHS $< -\eta x^T x$. Hence, LMIs (8.38), (8.41), and (8.42) are sufficient conditions for inequality (8.44) and therefore (8.18) to hold for any $t \in (t_n, t_{n+1})$, $n \in \mathbb{N}$, and $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I}(0)$.

Based on the conclusions in the two parts of the state space, inequality (8.18) holds for all $x(t) \in \overline{\mathcal{R}}_i$, $i \in \mathcal{I}$, and $t \in (t_n, t_{n+1})$. Hence, all the conditions of Theorem 8.1 are satisfied and the closed-loop nonlinear sampled-data system is uniformly exponentially stable with decay rate larger than $\alpha/2$. Based on Theorem 8.1, if the state space is equal to $\mathbb{R}^{n_x}$ then the conditions of Theorem 8.2 are sufficient conditions for the nonlinear sampled-data system to be globally uniformly exponentially stable. If the state space is a subset of $\mathbb{R}^{n_x}$, however, the conditions of Theorem 8.2 are sufficient conditions for the nonlinear sampled-data system to be locally uniformly exponentially stable. □

Based on Theorem 8.2, the problem of finding a lower bound on the MASP that preserves exponential stability is formulated as

Problem 8.1.

$$\begin{aligned} &\textit{maximize} \quad \tau \\ &\textit{subject to} \quad R > 0,\ X_1 > 0,\ \lambda'_i \geq 0,\ i \in \mathcal{I}\backslash\mathcal{I}(0),\ c_{1i} > 0,\ i \in \mathcal{I},\ \eta > 0,\ \gamma > 0, \\ &\qquad\qquad\quad \lambda_{i\kappa} > 0,\ i \in \mathcal{I}\backslash\mathcal{I}(0),\ \kappa \in \{1, 2\},\ (8.34) - (8.36),\ (8.38) - (8.42). \end{aligned}$$

8.4 Controller Synthesis

In this section, we address controller synthesis for a class of nonlinear sampled-data systems where the MASP is considered as a parameter. In the controller synthesis problem, the controller gains K_i, $i \in \mathcal{I}$, are unknown. Therefore, the LMIs in Theorem 8.2 turn into non-convex matrix inequalities that cannot be solved efficiently.

Theorem 8.3 addresses this issue and provides sufficient conditions for controller synthesis that can be cast as a convex optimization program. The price of the convex formulation of the controller synthesis problem is an extra condition on the structure of the PWA differential inclusion, which is formulated in Assumption 8.5.

Assumption 8.5. *The regions in the PWA differential inclusion (8.2) are slabs.*

Considering Assumption 8.5, in this section, e_i, $i \in \mathcal{I}$, are scalars and E_i, $i \in \mathcal{I}$, are vectors (see (8.5)).

Theorem 8.3. *Consider the nonlinear system (8.1) subject to Assumptions 8.1-8.5. Given τ as the desired MASP, there exists a PWL controller with gains $K_i = Y_i Q^{-1}$ that locally uniformly exponentially stabilizes the closed-loop sampled-data system, if there exist a symmetric matrix Q, matrices Y_i, $\overline{\mathcal{N}}_{i\kappa}$, $i \in \mathcal{I}$, $\kappa \in \{1, 2\}$, with appropriate dimensions, positive scalars $\lambda_{i\kappa}$, $i \in \mathcal{I}\backslash\mathcal{I}(0)$, $\kappa \in \{1, 2\}$, γ, μ, and ϵ_X, satisfying*

$$Q > \gamma I_{n_x} \tag{8.55}$$

$$\begin{bmatrix} -\gamma & ||Y_i - Y_j|| \\ ||Y_i - Y_j|| & -1 \end{bmatrix} < 0, \ \forall i, j \in \mathcal{I} \tag{8.56}$$

- *for all $i \in \mathcal{I}\backslash\mathcal{I}(0)$ and $\kappa \in \{1, 2\}$*

$$\begin{bmatrix} \overline{\Gamma}_{i\kappa} + \tau\overline{\mathcal{M}}_{1i\kappa} + \mathcal{S}_{i\kappa} & \star & \star & \star \\ \tau \begin{bmatrix} A_{i\kappa}Q & BY_i & B & a_{i\kappa} \\ 0_{n_x} & Q & 0 & 0 \end{bmatrix} & -\tau\overline{Q} & \star & \star \\ \begin{bmatrix} \overline{Q} & 0 & 0 \end{bmatrix} & 0_{2n_x} & -\operatorname{diag}(\mu I_{n_x}, I_{n_x}) & \star \\ \lambda_{i\kappa} \begin{bmatrix} e_i a_{i\kappa}^T + E_i Q & 0 & 0 & 0 \end{bmatrix} & 0 & 0 & -\lambda_{i\kappa}(e_i^2 - 1) \end{bmatrix} < 0 \tag{8.57}$$

$$\begin{bmatrix} \overline{\Gamma}_{i\kappa} + \tau\overline{\mathcal{M}}_{2i\kappa} + \mathcal{S}_{i\kappa} & \star & \star & \star \\ \tau\overline{\mathcal{N}}_{i\kappa}^T & -\tau e^{-\alpha\tau}\overline{Q} & \star & \star \\ \begin{bmatrix} \overline{Q} & 0 & 0 \end{bmatrix} & 0_{2n_x} & -\operatorname{diag}(\mu I_{n_x}, I_{n_x}) & \star \\ \lambda_{i\kappa} \begin{bmatrix} e_i a_{i\kappa}^T + E_i Q & 0 & 0 & 0 \end{bmatrix} & 0 & 0 & -\lambda_{i\kappa}(e_i^2 - 1) \end{bmatrix} < 0 \tag{8.58}$$

- *for all $i \in \mathcal{I}(0)$ and $\kappa \in \{1, 2\}$*

$$\begin{bmatrix} \Gamma_{i\kappa} + \tau \mathcal{M}_{1i\kappa} & \tau \begin{bmatrix} QA_{i\kappa}^T & 0_{n_x} \\ Y_i^T B^T & Q \\ B^T & 0 \end{bmatrix} & \begin{bmatrix} \overline{Q} \\ 0 \end{bmatrix} \\ \star & -\tau \overline{Q} & 0_{2n_x} \\ \star & \star & -\operatorname{diag}(\mu I_{n_x}, I_{n_x}) \end{bmatrix} < 0 \tag{8.59}$$

$$\begin{bmatrix} \Gamma_{i\kappa} + \tau \mathcal{M}_{2i\kappa} & \tau \mathcal{N}_{i\kappa} & \begin{bmatrix} \overline{Q} \\ 0 \end{bmatrix} \\ \star & -\tau e^{-\alpha\tau} \overline{Q} & 0_{2n_x} \\ \star & \star & -\operatorname{diag}(\mu I_{n_x}, I_{n_x}) \end{bmatrix} < 0 \tag{8.60}$$

where

$$\begin{aligned} \overline{Q} =& \operatorname{diag}(Q, Q), \qquad (8.61) \\ \mathfrak{C} =& \epsilon_X \begin{bmatrix} I_{n_x} & -I_{n_x} \end{bmatrix}^T \begin{bmatrix} I_{n_x} & -I_{n_x} \end{bmatrix}, \\ \overline{\Gamma}_{i\kappa} =& \begin{bmatrix} A_{i\kappa}Q & BY_i & B & a_{i\kappa} \end{bmatrix}^T \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T \begin{bmatrix} A_{i\kappa}Q & BY_i & B & a_{i\kappa} \end{bmatrix} \\ &+ \alpha \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T Q \begin{bmatrix} I_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} - \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T \overline{\mathcal{N}}_{i\kappa}^T \\ &- \overline{\mathcal{N}}_{i\kappa} \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} - \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix}^T \mathfrak{C}\overline{Q} \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix} \\ &+ \operatorname{diag}(0_{n_x}, 0_{n_x}, -\gamma I_{n_u}, 0), \\ \overline{\mathcal{M}}_{1i\kappa} =& \alpha \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix}^T \mathfrak{C}\overline{Q} \begin{bmatrix} I_{2n_x} & 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{i\kappa}Q & BY_i & B & a_{i\kappa} \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \mathfrak{C} & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \mathfrak{C} & 0 & 0 \end{bmatrix}^T \begin{bmatrix} A_{i\kappa}Q & BY_i & B & a_{i\kappa} \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}, \\ \overline{\mathcal{M}}_{2i\kappa} =& - \begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0 \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}^T \overline{\mathcal{N}}_{i\kappa}^T - \overline{\mathcal{N}}_{i\kappa} \begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0 \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\mathcal{S}_{i\kappa} = -\lambda_{i\kappa} \begin{bmatrix} a_{i\kappa}a_{i\kappa}^T & 0_{n_x} & 0 & QE_i^T e_i \\ \star & 0_{n_x} & 0 & 0 \\ \star & \star & 0_{n_u} & 0 \\ \star & \star & \star & e_i^2 - 1 \end{bmatrix},$$

$$\Gamma_{i\kappa} = \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{\Gamma}_{i\kappa} \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix}^T,$$

$$\mathcal{M}_{1i\kappa} = \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{\mathcal{M}}_{1i\kappa} \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix}^T,$$

$$\mathcal{M}_{2i\kappa} = \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{\mathcal{M}}_{2i\kappa} \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix}^T,$$

$$\mathcal{N}_{i\kappa} = \begin{bmatrix} I_{2n_x+n_u} & 0 \end{bmatrix} \overline{\mathcal{N}}_{i\kappa}.$$

Moreover, if the state space is equal to $\mathbb{R}^{n_x}$ then the system is globally uniformly exponentially stable.

Proof. Here, we prove that inequalities (8.55)-(8.60) are sufficient conditions for the LMIs in Theorem 8.2 to be verified. Suppose there exist a symmetric matrix Q, matrices Y_i, and $\overline{\mathcal{N}}_{i\kappa}$, $i \in \mathcal{I}$, $\kappa \in \{1,2\}$, with appropriate dimensions, positive scalars $\lambda_{i\kappa}$, $i \in \mathcal{I}\backslash\mathcal{I}(0)$, $\kappa \in \{1,2\}$, γ, μ, and ϵ_X, satisfying (8.55)-(8.60). Let

$$P_i = Q^{-1},\ i \in \mathcal{I}(0),\ \overline{P}_i = \text{diag}(Q^{-1}, 0),\ i \in \mathcal{I}\backslash\mathcal{I}(0),\ X_1 = \epsilon_X Q^{-1},\ R = \overline{Q}^{-1},$$
$$\eta = \mu^{-1},\ K_i = Y_i Q^{-1},\ i \in \mathcal{I},\ \overline{N}_{i\kappa} = \widetilde{Q}^{-1}\overline{\mathcal{N}}_{i\kappa}\overline{Q}^{-1},\ i \in \mathcal{I},\ \kappa \in \{1,2\},$$

where $\overline{Q}$ is defined in (8.61) and $\widetilde{Q} = \text{diag}(Q, Q, I_{n_u}, 1)$. The rest of the proof is similar to the proof of Theorem 7.2 and therefore omitted. □

Remark 8.3. *The stabilization criteria in Theorem 8.3 are sufficient conditions for the stability criteria in Theorem 8.2 and therefore are more conservative. However, they can be used to design PWL controllers by solving a convex optimization program that can be solved efficiently using available software packages. Numerical examples will show the effectiveness of this approach (see Section 8.5).*

Based on Theorem 8.3, the problem of designing an exponentially stabilizing PWL controller that maximizes the lower bound on the MASP is formulated as

Problem 8.2.

$$\begin{aligned} &\textit{maximize} \ \ \tau \\ &\textit{subject to} \ \ \lambda_{i\kappa} > 0,\ i \in \mathcal{I}\backslash\mathcal{I}(0),\ \kappa \in \{1,2\},\ \gamma > 0,\ \mu > 0,\ \epsilon_X > 0,\ (8.55) - (8.60). \end{aligned}$$

8.5 Numerical Examples

In this section, the theorems of Sections 8.3 and 8.4 are applied to two examples of linear and nonlinear sampled-data systems. In the literature of sampled-data systems, the lower bound on the MASP is usually used as a criterion for comparing the conservativeness of stability theorems. The greater is the computed lower bound, the less conservative is the stability theorem. In the following example, we use the same criterion to demonstrate the effectiveness of the proposed sufficient stability and stabilization conditions.

Example 8.1. *Consider the nonlinear system* $\dot{x} = f(x) + Bu$ *with*

$$f(x) = \begin{bmatrix} -x_1 + |x_2| + 0.5\cos(x_1)\frac{x_2}{1+x_2^2} + 0.1x_2\sin(x_2) \\ 0 \end{bmatrix}, \tag{8.62}$$

where $B^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in \mathbb{R}^2$ *is the state vector, and* u *is the control input. The open-loop system is bounded by the PWA differential inclusion* $f(x(t)) \in \mathrm{conv}\{A_{i\kappa}x(t) + a_{i\kappa},\ \kappa = 1,2\},\ \forall\, x(t) \in \mathcal{R}_i,\ i \in \mathcal{I}$, *where* $\mathcal{I} = \{1, \ldots, 6\}$. *The slab regions are defined as*

$$\mathcal{R}_1 = \{x | x_2 \in (-\infty, -5)\},\ \mathcal{R}_2 = \{x | x_2 \in (-5, -1.2)\},\ \mathcal{R}_3 = \{x | x_2 \in (-1.2, 0)\},$$
$$\mathcal{R}_4 = \{x | x_2 \in (0, 1.2)\},\ \mathcal{R}_5 = \{x | x_2 \in (1.2, 5)\},\ \mathcal{R}_6 = \{x | x_2 \in (5, \infty)\},$$

and

$$A_{11} = -\begin{bmatrix} 1 & 0.88 \\ 0 & 0 \end{bmatrix},\ A_{21} = -\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ A_{31} = -\begin{bmatrix} 1 & 0.5 \\ 0 & 0 \end{bmatrix},\ A_{41} = \begin{bmatrix} -1 & 0.5 \\ 0 & 0 \end{bmatrix},$$
$$A_{51} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},\ A_{61} = \begin{bmatrix} -1 & 0.88 \\ 0 & 0 \end{bmatrix},\ a_{11} = a_{31} = a_{41} = a_{61} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ a_{21} = a_{51} = \begin{bmatrix} -0.6 \\ 0 \end{bmatrix},$$
$$A_{12} = -\begin{bmatrix} 1 & 1.12 \\ 0 & 0 \end{bmatrix},\ A_{22} = -\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ A_{32} = -\begin{bmatrix} 1 & 1.5 \\ 0 & 0 \end{bmatrix},\ A_{42} = \begin{bmatrix} -1 & 1.5 \\ 0 & 0 \end{bmatrix},$$
$$A_{52} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},\ A_{62} = \begin{bmatrix} -1 & 1.12 \\ 0 & 0 \end{bmatrix},\ a_{12} = a_{32} = a_{42} = a_{62} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ a_{22} = a_{52} = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}.$$

Figure 8.1(a) illustrates the nonlinear function and the corresponding PWA differential inclusion. Consider the linear feedback controller

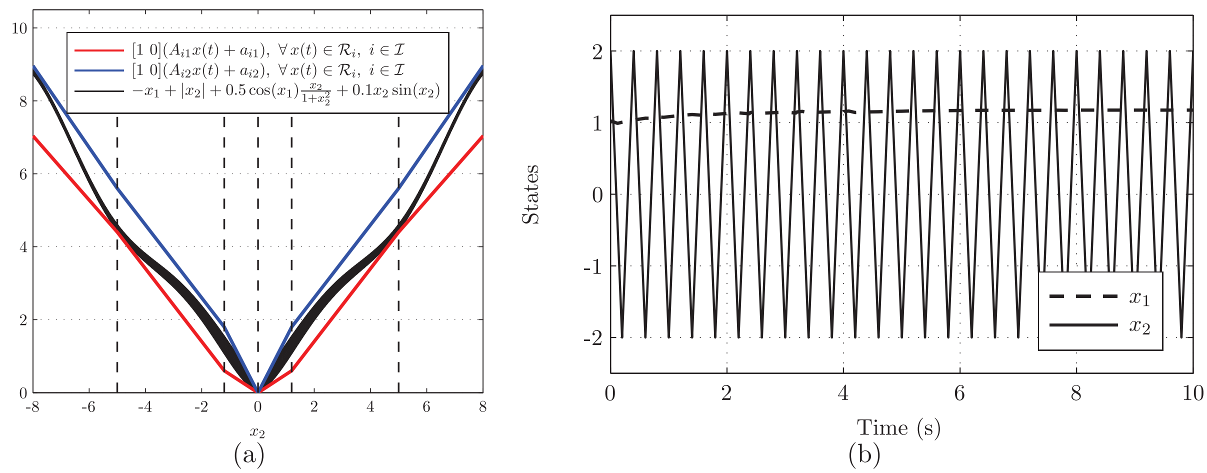


Figure 8.1: The left figure illustrates the nonlinear function and the corresponding PWA differential inclusion (as seen from the angle generated by the MATLAB® command `view(90,-33.6)`). The vertical dashed lines represent the boundaries of the slab regions. The right figure shows the response of the system for sampling intervals equal to 0.2 (s).

$$u = Kx, \ K = \begin{bmatrix} 0 & -10 \end{bmatrix}. \tag{8.63}$$

The controller can be considered as a PWL controller where the controller gain is equal in all regions. It is easy to see that (8.63) is a stabilizing controller for x_2. Since $f([x_1 \ 0]) = [-x_1 \ 0]^T$, one can conclude that the continuous-time controller (8.63) asymptotically stabilizes the nonlinear system. Now assume that the controller is implemented via a sample and hold device. Simulation results (with initial condition $x_0^T(r) = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $-0.2 \le r \le 0$, and $\rho(0) = 0$) show that the system becomes unstable for sampling intervals greater than 0.2 *(s) (see Fig. 8.1(b)). Our first goal is to find a lower bound on the MASP such that the closed-loop nonlinear system remains stable. Table 8.1 compares the values provided by Problem 8.1 for the lower bound on the MASP that guarantees global uniform exponential stability with other methods in the literature.*[1] *Note that the lower bound on the MASP computed using the approach in [50] decreases drastically as the decay rate increases. In this example, comparing the data for the case where $\alpha = 0^+$, the lower bound on the MASP given by the approach proposed in this chapter is twice as large as the lower bound on the MASP provided*

[1] *When solving Example 8.1 based on the approach of [50], the following Lyapunov function candidates were used: $V(x) = 3 \times 10^{-5}x_1^2 + 10x_2^2$ and $W(e) = |e|$ (please see [50] for notation). The coefficients of the quadratic Lyapunov function were optimized to provide the largest lower bound on the MASP.*

Table 8.1: Comparison of the computed lower bound on the MASP that guarantees global uniform exponential stability for different decay rates ($\alpha/2$)

[50]	MASP $= 0.100$ (s) for $\alpha = 0^+$
	MASP $= 0.033$ (s) for $\alpha = 2.4 \times 10^{-6}$
	MASP $= 0^+$ (s) for $\alpha = 4.8 \times 10^{-6}$
Problem 8.1	MASP $= 0.199$ (s) for $\alpha = 0^+$
	MASP $= 0.195$ (s) for $\alpha = 0.5$
	MASP $= 0.190$ (s) for $\alpha = 1$

by [50]. Moreover, in this example, the calculated lower bound on the MASP is more than 99% *accurate (recall that the system becomes unstable for sampling intervals greater than* 0.2 *(s)).*

Now consider a scenario in which the sampling intervals of the available sensors are as large as 0.35 *(s). Clearly, controller (8.63) cannot stabilize the sampled-data system because the sampling intervals might be longer than* 0.2 *(s). Here, our goal is to design a new controller that guarantees global uniform exponential stability of the nonlinear sampled-data system for the MASP* $= 0.35$ *(s). To the best of the authors' knowledge, there is no other direct sampled-data design based approach in the literature of nonlinear systems that can be used to design sampled-data controllers for a desired MASP. Solving Problem 8.2 for the MASP* $= 0.35$ *(s) and* $\alpha = 0.05$ *yields the following PWL controller*

$$K_1 = \begin{bmatrix} 0.2033 & -2.4987 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.2431 & -2.5960 \end{bmatrix},\ K_3 = \begin{bmatrix} 0.2738 & -2.6302 \end{bmatrix},$$
$$K_4 = \begin{bmatrix} -0.2738 & -2.6302 \end{bmatrix},\ K_5 = \begin{bmatrix} -0.2431 & -2.5960 \end{bmatrix},\ K_6 = \begin{bmatrix} -0.2033 & -2.4987 \end{bmatrix}. \tag{8.64}$$

As mentioned earlier, the convex formulation of the controller synthesis problem in Theorem 8.3 leads to extra conservatism in the sufficient conditions when compared to Theorem 8.2 (see Remark 8.3). Hence, in order to find a less conservative estimation of the MASP, we solve Problem 8.1 with the new controller gains defined in (8.64). This yields the MASP $= 0.57$ *(s). Therefore, the designed PWL controller (8.64) is guaranteed to stabilize the closed-loop nonlinear sampled-data system if the nonuniform sampling intervals are smaller than* 0.57 *(s).*

8.6 Conclusion

Exponential stability and stabilization of a class of uncertain nonlinear systems with PWL sampled-data feedback was addressed. The direct sampled-data controller synthesis problem for a class of uncertain nonlinear systems was formulated as a convex optimization program with the MASP as a parameter. Sufficient conditions for exponential stability of a class of nonlinear sampled-data systems were presented using a piecewise smooth Krasovskii functional. The stability analysis and controller synthesis conditions were cast as LMIs. It was shown that the proposed methods can perform favorably when compared to other methods in the literature of nonlinear systems.

Chapter 9

Linear Networked Control Systems

9.1 Introduction

In networked control systems, sensory information and feedback signals are exchanged among different components of the system (i.e. sensors, actuators, and controllers) through a communication network. In a modern long-range aircraft for instance, there exist about 170 (Km) of signal wiring which account for almost 700 (Kg) of the weight of the aircraft [4]. Other than weight, the main drawbacks of wired communication links include connector/pin failures, cracked insulation issues, arc faults, and maintenance/upgrade difficulties [5]. The inherent benefits of wireless communication systems and the recent advancements in this field have led to a growing interest in wireless flight control systems (i.e. fly-by-wireless) [6]. However, the effects of non-ideal communication networks on stability and performance of the system become more prominent in the case of wireless communication networks [7] and motivate a thorough study of networked control systems. We refer the reader to [1–3] for applications of networked control systems in document printing control, air vehicle systems and satellites, and an inverted pendulum, respectively.

In a networked control system (as well as a sampled-data system and a time-delay system, as special cases of networked control systems), the vector field is defined as a function of the current and the past values of the state vector. *Retarded functional differential equations* [19] are widely used as a framework for modeling, stability analysis, and controller synthesis of deterministic and stochastic networked control systems (see [19–21] and the references therein). The main approaches for studying networked control systems include the lifting approach [37, 45, 60, 61], the impulsive model approach [1, 11, 48, 62], and the input delay approach [12, 15, 47, 63, 64].

In the lifting approach, the retarded system is modeled as a finite dimensional discrete-time system. Lifting is used in studying sampled-data systems with constant or uncertain sampling rates [49]. However, the lifting approach is not applicable to systems with uncertain parameters. In the impulsive model approach, the retarded system is modeled as an impulsive system which exhibits continuous state evolutions (described by ordinary differential equations) and instantaneous state jumps. In the input delay approach, the retarded system is modeled as a continuous-time system with a delayed control input. Both the impulsive model and input delay approaches use Razumikhin-type [20] or Krasovskii-type [112] theorems to prove stability of the retarded system. While the Razumikhin-type theorems are based on classical Lyapunov *functions*, Krasovskii-type theorems use Lyapunov *functionals* and are known to be less conservative [9, 15, 20]. The evolution of LKFs over the past decade has yielded less conservative stability conditions. These conditions are usually cast in terms of LMIs which can efficiently be solved using software packages such as SeDuMi [16] and YALMIP [17].

In a networked control system, a continuous-time plant is in feedback with a discrete-time emulation of a controller. The control signal is computed using state measurements that are sampled in intervals that are not necessarily uniform [3, 47, 48]. These signals go through a quantization process [57], and experience uncertain and time varying delays [58, 59], data packet dropouts, and congestion over the communication network. Most of the works in the literature focus on only one aspect of networked control systems. There are papers, however, that study two or more features of a networked control system at the same time. Reference [2] studies H_∞ control of a class of uncertain stochastic networked control systems with both network-induced delays and packet dropouts. Sufficient conditions are proposed to ensure exponential stability in mean square of the closed-loop system subject to a performance measure. The robust filtering problem is addressed in [65] for a class of discrete-time uncertain nonlinear networked systems with both multiple stochastic time-varying communication delays and multiple packet dropouts. A method for designing linear full-order filter is proposed such that the estimation error converges to zero exponentially in the mean square while the disturbance rejection attenuation is constrained to a given level. Reference [66] studies the distributed finite- horizon filtering problem for a class of time-varying systems over lossy sensor networks with quantization errors and successive packet dropouts. Through available output measurements from a sensor and its neighbors (according to a given topology), a sufficient condition is established for the desired distributed finite-horizon filter to ensure that the prescribed average filtering performance constraint is satisfied.

The networked control system considered in [67] comprises a linear sampled-data controller and an uncertain, time varying delay. Two drawbacks of that model are that the sampling intervals are assumed to be constant and the delay is assumed to be upper bounded by the sampling period. A more general model of networked control systems is studied in [11, 12], where a linear sampled-data controller with uncertain sampling rates, the possibility of data packet dropouts, and an unknown, time varying delay are considered. While the stability theorems in [12] are less conservative than the corresponding theorems in [11], they are more computationally expensive as they involve solving two times as many LMIs. Moreover, due to the complexity of the LKF in [12], the number of LMIs increases even more if additional information on the time varying delay (e.g. a lower bound) is available.

Similar to [11, 12], in this chapter we focus on linear networked control systems. In particular, we study a continuous-time linear plant in feedback with a linear sampled-data controller with an unknown, time varying sampling rate, the possibility of data packet dropout, and an uncertain, time varying delay. In contrast to [12], this chapter improves the stability conditions of [11] without increasing the computational cost of the resulting optimization program. We first consider the general case where information on the lower and upper bounds of the time-delay are available, and then study the case with limited information on the time-delay. In all those scenarios, our goal is to find a lower bound on the maximum network-induced delay that preserves exponential stability of the system.

The main contribution of this chapter is the derivation of new sufficient stability conditions for linear networked control system taking into account all of the factors mentioned before. The stability conditions are based on a modified LKF. The stability results are also applied to the case where limited information on the delay bounds is available. Furthermore, this chapter also formulates the problem of finding a lower bound on the maximum network-induced delay that preserves exponential stability as a convex optimization program in terms of LMIs. This problem can be solved efficiently from both a practical and theoretical point of view. Finally, as a comparison, we show that the stability conditions proposed in this chapter compare favorably with the ones available in the open literature for different benchmark problems.

The chapter is organized as follows. Section 9.2 presents the linear networked control system model. Section 9.3 starts by introducing a modified LKF. Next, we present theorems that provide sufficient conditions for exponential stability of linear networked control systems. Furthermore, the problem of finding a lower bound on the

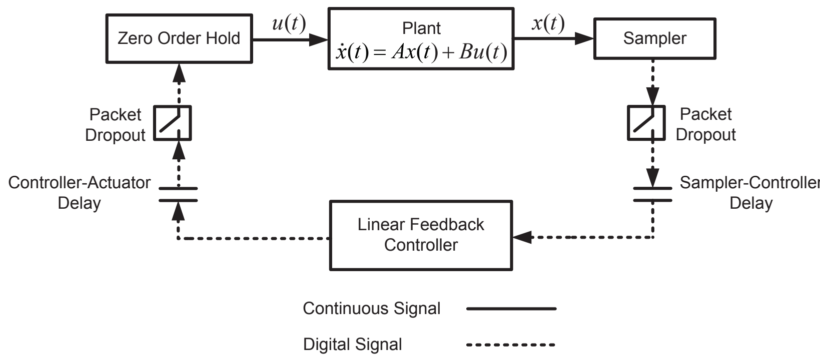


Figure 9.1: A linear networked control system

maximum network-induced delay that preserves exponential stability as an optimization program is formulated in terms of LMIs. Finally, the new approach is applied to different examples in Section 9.4.

9.2 Preliminaries

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{9.1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector, $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, and $u \in \mathbb{R}^{n_u}$ is the control input. Let a continuous-time linear controller for (9.1) be defined by

$$u(t) = Kx(t), \tag{9.2}$$

where $K \in \mathbb{R}^{n_u \times n_x}$. In this chapter, we study the stability of system (9.1) where controller (9.2) is implemented through a network. The network comprises a time driven sampler and an event driven zero order hold (see Figure 9.1). The possibility of data packet dropout and communication delays are also considered in the network's model. The networked controller is characterized through Assumptions 9.1-9.4.

Assumption 9.1. *The state vector is measured at the sampling instants s_k, $k \in \mathbb{N}$. Each sampled state vector is sent over the network in one data packet.*

Since the controller is static and time-invariant, without loss of generality [1, 8, 113], the delay between the sensor (sampler) and the controller, the delay between the controller and the actuator, and the computation delay in the controller are modeled as one single delay.

Assumption 9.2. *The state vector sampled at s_k, $k \in \mathbb{N}$, experiences an uncertain, time varying delay η_k as it is transmitted through the network. The delay η_k is bounded, i.e., $0 \le \eta_{\min} \le \eta_k \le \eta_{\max}$.*

Note that our model allows the delay η_k to grow larger than the sampling interval $[s_k, s_{k+1}]$ as opposed to the model in [67]. The possibility of data packet dropout is modeled via a switch in Figure 9.1. When the switch is closed, the data is transmitted through the network. When the switch is open, however, the data is assumed to be dropped. The actuator is updated with new control signals at the instants t_k,

$$t_k = s_k + \eta_k,\ k \in \mathbb{N}. \tag{9.3}$$

An event driven zero order hold keeps the control signal constant through the interval $[t_k, t_{k+1})$, i.e. until the arrival of new data at t_{k+1}.

Assumption 9.3. *The control signals arrive at the actuator in the same order that their corresponding state vectors are sampled, i.e. $s_i < s_j \Longrightarrow t_i < t_j$, $\forall\ i, j \in \mathbb{N}$. If a sampled state vector arrives after a more recent sampled vector has arrived, the older sampled vector is dropped (cf. s_d and s_2 in Figure 9.2).*

Without loss of generality, by the index $k \in \mathbb{N}$, we denote only the instants s_k and t_k for which a data packet is not dropped. In the interval between two actuator update instants t_k and t_{k+1}, the network-induced delay represented by ρ_s is defined as the time elapsed since the last available sampling instant s_k (see Figure 9.2), i.e.

$$\rho_s(t) = t - s_k = t - t_k + \eta_k,\ t \in [t_k, t_{k+1}), \tag{9.4}$$

where equation (9.3) is used in the second equality. Based on Assumption 9.2, the network-induced delay is greater than or equal to $\eta_{\min}$. We denote the largest network-induced delay by τ, i.e.

$$\tau = \sup\left(\rho_s(t)\right) = \sup_{k \in \mathbb{N}} \left(t_{k+1} - s_k\right).$$

Therefore,

$$\eta_{\min} \le \rho_s(t) \le \tau. \tag{9.5}$$

Furthermore, the time elapsed since the last actuator update instant t_k is denoted by ρ_t, i.e.

$$\rho_t(t) = t - t_k = t - s_k - \eta_k = \rho_s(t) - \eta_k,\ t \in [t_k, t_{k+1}). \tag{9.6}$$

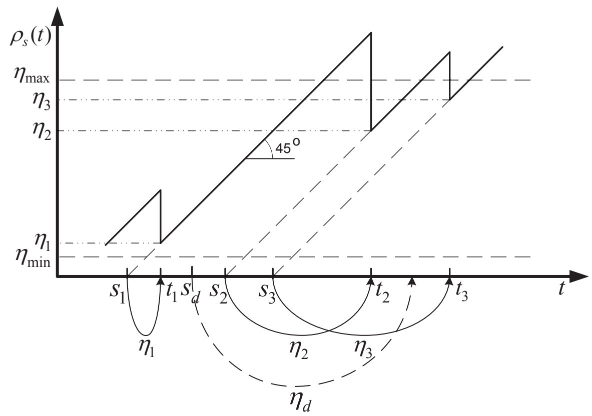


Figure 9.2: Network-induced delay

Equation (9.5), equation (9.6), and Assumption 9.2 yield

$$0 \leq \rho_t(t) \leq \tau - \eta_{\min}. \tag{9.7}$$

The following assumption models the fact that two actuator updates cannot occur simultaneously in practice. It is used in Section 9.3 to rule out the occurrence of the Zeno phenomenon and also plays an essential role in proving the convergence of the closed-loop vector field to the origin.

Assumption 9.4. *There exists $\epsilon > 0$ such that $t_{k+1} - t_k > \epsilon$ for any $k \in \mathbb{N}$.*

The control signal (9.2) is now redefined in the networked control system framework as the piecewise constant function

$$u(t) = Kx(s_k), \ t \in [t_k, t_{k+1}), \tag{9.8}$$

with jumps at the actuator update instants t_k, $k \in \mathbb{N}$. Given a controller gain K that exponentially stabilizes the continuous-time system (9.1)-(9.2), our objective is to find a lower bound on the maximum network-induced delay that preserves exponential stability for the networked control system defined by (9.1) and (9.8). To this end, we use the input delay approach to draw an analogy between networked control systems and time-delay systems. Considering (9.4), we can rewrite (9.8) as

$$u(t) = Kx(t - \rho_s), \ t \in [t_k, t_{k+1}). \tag{9.9}$$

The linear networked control system (9.1) with control input (9.9) can be viewed as a linear system with a discontinuous time varying input delay $d(t) = \rho_s$. In the literature of time-delay systems, LKFs are widely used to devise stability conditions (see [21, 58, 59, 114] and the references therein). Different LKFs are used for networked control systems in [11, 12, 67] and sampled-data systems in [15, 47, 48, 64]. The subject of LKFs and stability of linear networked control systems will be addressed in the next section where we present the main results of the chapter.

9.3 Main Results

First, a modified LKF is presented. Next, the LKF is used to provide new sufficient conditions for stability of linear networked control systems. The problem of finding a lower bound on the maximum network-induced delay that preserves exponential stability is cast as an optimization program in terms of LMIs. Let $V(t, x_t)$ be an LKF defined as

$$V(t, x_t) = \sum_{j=0}^{8} V_j,\ t \in [t_k, t_{k+1}), \tag{9.10}$$

where

$$V_0 = x^T(t)Px(t), \tag{9.11}$$

$$V_1 = (\tau - \rho_s)\int_{t-\rho_t}^{t} [\dot{x}(r) - Bu(r)]^T R_1 [\dot{x}(r) - Bu(r)]\,\mathrm{d}r, \tag{9.12}$$

$$V_2 = (\tau - \rho_s)\int_{t-\rho_t}^{t} \dot{x}^T(r)R_2\dot{x}(r)\,\mathrm{d}r, \tag{9.13}$$

$$V_3 = \int_{t-\eta_{\min}}^{t} (\eta_{\min} - t + r)\dot{x}^T(r)R_3\dot{x}(r)\,\mathrm{d}r, \tag{9.14}$$

$$V_4 = \int_{t-\rho_s}^{t-\eta_{\min}} (\tau - t + r)\dot{x}^T(r)R_4\dot{x}(r)\,\mathrm{d}r + (\tau - \eta_{\min})\int_{t-\eta_{\min}}^{t} \dot{x}^T(r)R_4\dot{x}(r)\,\mathrm{d}r, \tag{9.15}$$

$$V_5 = \int_{t-\rho_s}^{t} (\tau - t + r)\dot{x}^T(r)R_5\dot{x}(r)\,\mathrm{d}r, \tag{9.16}$$

$$V_6 = \int_{t-\rho_s}^{t} (\tau - t + r)\dot{x}^T(r)R_6\dot{x}(r)\,\mathrm{d}r, \tag{9.17}$$

$$V_7 = \int_{t-\eta_{\min}}^{t} x^T(r)Zx(r)\,\mathrm{d}r, \tag{9.18}$$

$$V_8 = (\tau - \rho_s)\begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix} X \begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix}^T, \tag{9.19}$$

Table 9.1: Comparison of the LKF in (9.10) with the LKFs proposed in the literature

LKF in (9.10)	V_0	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
LKF in [11]	✓	✗	✗	✓	✓	✗	✓	✓	✗
LKF in [47]	✓	✗	✓	✗	✗	✗	✗	✗	✓

ρ_s and ρ_t are defined in (9.4) and (9.6), respectively, and

$$X = \begin{bmatrix} X_1 & -X_2 \\ -X_2^T & X_2 + X_2^T - X_1 \end{bmatrix}, \tag{9.20}$$

where $P > 0$, $R_i > 0$, $i \in \{1, \dots, 6\}$, $Z > 0$, $X_1 = X_1^T$, and X_2 are matrices in $\mathbb{R}^{n_x \times n_x}$. The reason for defining two similar functionals V_5 and V_6 becomes clear in the next subsection where we use V_5 to provide stability conditions that are independent of $\eta_{\max}$ and use V_6 to devise stability conditions for the case when $\eta_{\max}$ is known (see equations (9.45) and (9.46)). Table 9.1 compares the LKF in equation (9.10) with the LKFs in [11, 47]. The sign ✓ (respectively, ✗) denotes that a functional exists (respectively, does not exist) in the corresponding LKF. Using the new functional V_1 and the proper use of the functional V_5, enables one to achieve less conservative stability criteria.

The following theorem provides a set of sufficient conditions for which the trajectories of the linear networked control system are globally uniformly exponentially stable to the origin.

Theorem 9.1. *Consider the linear networked control system defined in (9.1) and (9.8) with Assumptions 9.1-9.4. Given the controller gain K and the scalars τ, $\eta_{\min}$, and $\eta_{\max}$, the networked control system is globally uniformly exponentially stable if there exist symmetric positive definite matrices P, R_i, $i \in \{1, \dots, 6\}$, and Z, a symmetric matrix X_1, and matrices X_2, N_j, $j \in \{1, \dots, 5\}$, N_{6_a}, and N_{6_b}, with appropriate dimensions, satisfying*

$$\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + (\tau - \eta_{\min})X > 0 \tag{9.21}$$

$$\begin{bmatrix} \Psi + \tau M_1 + (\tau - \eta_{\min})(M_2 + M_4) + \eta_{\min} M_3 & \eta_{\min} N_3 & \eta_{\min} N_5 & \eta_{\max} N_{6_a} \\ \eta_{\min} N_3^T & -\eta_{\min} R_3 & 0 & 0 \\ \eta_{\min} N_5^T & 0 & -\eta_{\min} R_5 & 0 \\ \eta_{\max} N_{6_a}^T & 0 & 0 & -\eta_{\max} R_6 \end{bmatrix} < 0 \tag{9.22}$$

$$\begin{bmatrix} \Psi + \tau M_1 + (\tau - \eta_{\min})(M_4 + M_5) + \eta_{\min} M_3 & \overline{N} \\ \overline{N}^T & D \end{bmatrix} < 0 \tag{9.23}$$

where X is defined in (9.20) and

$$\begin{aligned}
\Psi = & \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} P & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix} \\
& - \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}^T (N_1^T + N_2^T + N_{6_b}^T) - (N_1 + N_2 + N_{6_b}) \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix} \\
& - \begin{bmatrix} I & 0 & 0 & -I \end{bmatrix}^T N_3^T - N_3 \begin{bmatrix} I & 0 & 0 & -I \end{bmatrix} - \begin{bmatrix} 0 & 0 & -I & I \end{bmatrix}^T N_4^T \\
& - N_4 \begin{bmatrix} 0 & 0 & -I & I \end{bmatrix} - \begin{bmatrix} I & 0 & -I & 0 \end{bmatrix}^T N_5^T - N_5 \begin{bmatrix} I & 0 & -I & 0 \end{bmatrix} \\
& - \begin{bmatrix} 0 & I & -I & 0 \end{bmatrix}^T N_{6_a}^T - N_{6_a} \begin{bmatrix} 0 & I & -I & 0 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}^T Z \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} \\
& - \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}^T Z \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

$$M_1 = \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T (R_5 + R_6) \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix},$$

$$\begin{aligned}
M_2 = & \begin{bmatrix} A & 0 & 0 & 0 \end{bmatrix}^T R_1 \begin{bmatrix} A & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T R_2 \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix} \\
& + \begin{bmatrix} A & 0 & BK & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} X & 0 \end{bmatrix} + \begin{bmatrix} X \\ 0 \end{bmatrix} \begin{bmatrix} A & 0 & BK & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

$$M_3 = \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T R_3 \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T R_4 \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 0 & 0 & BK & 0 \end{bmatrix}^T N_1^T + N_1 \begin{bmatrix} 0 & 0 & BK & 0 \end{bmatrix},$$

$$\overline{N} = \begin{bmatrix} (\tau - \eta_{\min}) N_1 & (\tau - \eta_{\min}) N_2 & \eta_{\min} N_3 & (\tau - \eta_{\min}) N_4 & \tau N_5 & \eta_{\max} N_{6_a} & (\tau - \eta_{\min}) N_{6_b} \end{bmatrix},$$

$$\begin{aligned}
D = \mathrm{diag}\Big(& (\eta_{\min} - \tau) R_1, \; (\eta_{\min} - \tau) R_2, \; -\eta_{\min} R_3, \; (\eta_{\min} - \tau) R_4, \; -\tau R_5, \; -\eta_{\max} R_6, \\
& (\eta_{\min} - \tau) R_6 \Big).
\end{aligned}$$

Proof. First we show that $P > 0$, $R_i > 0$, $i \in \{1, \ldots, 6\}$, $Z > 0$, and LMI (9.21) are sufficient conditions for the LKF (9.10) to satisfy

$$c_1|x_t(0)|^2 \leq V(t, x_t) \leq c_2||x_t||^2_{\mathcal{W}}, \tag{9.24}$$

for some $c_1 > 0$ and $c_2 > 0$. Adding V_0 and V_8 yields

$$V_0 + V_8 = \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}^T \left(\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + (\tau - \rho_s)X \right) \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}, \tag{9.25}$$

for $t \in [t_k, t_{k+1})$. Based on (9.5), ρ_s varies between $\eta_{\min}$ and τ. Since (9.25) is affine in ρ_s, LMI (9.21) and $P > 0$ are sufficient conditions for the existence of a sufficiently small $c_1 > 0$ such that

$$\begin{bmatrix} c_1 I & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + (\tau - \rho_s)X,$$

for any $\rho_s \in [\eta_{\min}, \tau)$. Therefore, based on (2.7) and (9.25) we can write

$$c_1|x(t)|^2 = c_1|x_t(0)|^2 \leq V_0 + V_8.$$

Moreover, note that the constraints $R_i > 0$, $i \in \{1, \ldots, 6\}$, and $Z > 0$ are sufficient conditions for V_j, $j \in \{1, \ldots, 7\}$, to be non-negative at any time. Therefore, the lower bound on V in inequality (9.24) is computed as

$$c_1|x_t(0)|^2 \leq V_0 + V_8 \leq V.$$

Considering (9.5) and (2.8), observe that at any time t and for all $\alpha \in [-\rho_s, 0]$, $|x_t(\alpha)| \leq ||x_t||_{\mathcal{W}}$. Equivalently,

$$|x(r)| \leq ||x_t||_{\mathcal{W}}, \ \forall \ r \in [t - \rho_s, t]. \tag{9.26}$$

Therefore, $\left| \begin{bmatrix} x^T(t) & x^T(t_k) \end{bmatrix}^T \right| < \sqrt{2}||x_t||_{\mathcal{W}}$. Based on (9.25),

$$V_0 + V_8 \leq 2 \max_{\rho_s \in [\eta_{\min}, \tau]} \left\{ \lambda_{\max} \left(\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + (\tau - \rho_s)X \right) \right\} ||x_t||^2_{\mathcal{W}}. \tag{9.27}$$

Note that

$$[\dot{x}(r) - Bu(r)]^T R_1 [\dot{x}(r) - Bu(r)] \leq \lambda_{\max}(R_1)|\dot{x}(r) - Bu(r)|^2.$$

Moreover, according to the parallelogram law [115], $|v_1 - v_2|^2 + |v_1 + v_2|^2 = 2|v_1|^2 + 2|v_2|^2$, where v_1 and v_2 are vectors in $\mathbb{R}^m$. Therefore, $|v_1 - v_2|^2 \leq 2|v_1|^2 + 2|v_2|^2$. Thus, using (9.5),

$$V_1 \leq (\tau - \eta_{\min})\lambda_{\max}(R_1)\left(\int_{t-\rho_t}^{t} 2|\dot{x}(r)|^2\,\mathrm{d}r + \int_{t-\rho_t}^{t} 2|Bu(r)|^2\,\mathrm{d}r\right). \tag{9.28}$$

With a change of variables, considering (9.7), and using the definition of norm in (2.8), we can write

$$\int_{t-\rho_t}^{t} 2|\dot{x}(r)|^2\,\mathrm{d}r = 2\int_{-\rho_t}^{0} |\dot{x}(t+\alpha)|^2\,\mathrm{d}\alpha = 2\int_{-\rho_t}^{0} |\dot{x}_t(\alpha)|^2\,\mathrm{d}\alpha \leq 2||x_t||_{\mathcal{W}}^2. \tag{9.29}$$

Based on (9.8), note that $u(r) = Kx(s_k)$ is constant for $r \in [t - \rho_t, t] = [t_k, t]$, $t \in [t_k, t_{k+1})$. According to (9.26), $|x(s_k)| \leq ||x_t||_{\mathcal{W}}$. Therefore, considering (9.7),

$$\int_{t-\rho_t}^{t} 2|Bu(r)|^2\,\mathrm{d}r = 2\int_{t-\rho_t}^{t} |BKx(s_k)|^2\,\mathrm{d}r \leq 2(\tau - \eta_{\min})\lambda_{\max}(K^T B^T BK)||x_t||_{\mathcal{W}}^2. \tag{9.30}$$

From (9.28)-(9.30),

$$V_1 \leq 2(\tau - \eta_{\min})\lambda_{\max}(R_1)\left(1 + (\tau - \eta_{\min})\lambda_{\max}(K^T B^T BK)\right)||x_t||_{\mathcal{W}}^2. \tag{9.31}$$

Similarly, it can be shown that

$$V_2 \leq (\tau - \eta_{\min})\lambda_{\max}(R_2)||x_t||_{\mathcal{W}}^2, \tag{9.32}$$
$$V_3 \leq \eta_{\min}\lambda_{\max}(R_3)||x_t||_{\mathcal{W}}^2, \tag{9.33}$$
$$V_4 \leq 2(\tau - \eta_{\min})\lambda_{\max}(R_4)||x_t||_{\mathcal{W}}^2, \tag{9.34}$$
$$V_5 \leq \tau\lambda_{\max}(R_5)||x_t||_{\mathcal{W}}^2, \tag{9.35}$$
$$V_6 \leq \tau\lambda_{\max}(R_6)||x_t||_{\mathcal{W}}^2. \tag{9.36}$$

Based on (9.5), $[-\eta_{\min}, 0] \subset [-\rho_s, 0]$, i.e. $[t - \eta_{\min}, t] \subset [t - \rho_s, t]$. Therefore, using (9.26),

$$V_7 \leq \eta_{\min}\lambda_{\max}(Z)||x_t||_{\mathcal{W}}^2. \tag{9.37}$$

Adding inequalities (9.27) and (9.31)-(9.37) leads to the upper bound on V in (9.24), i.e.

$$\begin{aligned}c_2 =& 2 \max_{\rho_s \in [\eta_{\min}, \tau]} \left\{ \lambda_{\max} \left(\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + (\tau - \rho_s) X \right) \right\} + (\tau - \eta_{\min})(\lambda_{\max}(R_2) + 2\lambda_{\max}(R_4)) \\ &+ 2(\tau - \eta_{\min})\lambda_{\max}(R_1) \left(1 + (\tau - \eta_{\min})\lambda_{\max}(K^T B^T B K)\right) \\ &+ \tau(\lambda_{\max}(R_5) + \lambda_{\max}(R_6)) + \eta_{\min}(\lambda_{\max}(R_3) + \lambda_{\max}(Z)).\end{aligned}$$

So far, it was shown that the LKF is positive definite and decrescent. Following Lyapunov theorem, to prove stability, it suffices to show that the LKF is decreasing. Since the LKF is discontinuous at actuator update instants t_k, we first show that the LKF is non-increasing at $t = t_k$, $k \in \mathbb{N}$. Next, computing the time derivative of V for $t \in (t_k, t_{k+1})$, $k \in \mathbb{N}$, it is proved that LMIs (9.22) and (9.23) are sufficient conditions for the LKF to be decreasing in the interval between two actuator update instants. To this end, note that V_j, $j \in \{1, \ldots, 7\}$, and $V_0 + V_8$ are always non-negative. Also observe that V_0, V_3, and V_7 are continuous functions. The functionals V_1 and V_2 vanish at the actuator update instants since $\rho_t = 0$ at $t = t_k$. The first integral in the functional V_4 is non-increasing at the actuator update instants $t = t_k$ because the integrand is non-negative and based on Assumption 9.3 the lower limit of the integral changes from s_{k-1} to s_k (see Figure 9.2). Note that the second part of V_4 is a continuous function. Using the same reasoning, the functionals V_5 and V_6 are non-increasing at the actuator update instants $t = t_k$ because the integrands are non-negative and the lower limit of the integrals change from s_{k-1} to s_k. The last component of the LKF, i.e. V_8, vanishes at the actuator update instants because $x(t) = x(t_k)$ at $t = t_k$ and the sum of the entries of X is equal to zero. Therefore, the LKF is non-increasing at instants t_k, $k \in \mathbb{N}$. The LKF is differentiable in the interval between two actuator update instants. For $t \in (t_k, t_{k+1})$, $\dot{V}$ is composed of nine terms computed as follows. The time derivative of V_0 is

$$\dot{V}_0 = \dot{x}^T P x + x^T P \dot{x}. \tag{9.38}$$

From (9.4) and (9.6), we have $\dot{\rho}_s = \dot{\rho}_t = 1$. Hence, applying the Leibniz integral rule to V_1 yields

$$\dot{V}_1 = -\int_{t-\rho_t}^{t} [\dot{x}(r) - Bu(r)]^T R_1 [\dot{x}(r) - Bu(r)] \, \mathrm{d}r + (\tau - \rho_s) \left[\dot{x} - Bu\right]^T R_1 \left[\dot{x} - Bu\right]. \tag{9.39}$$

Since R_1 is positive definite, for any arbitrary time varying vector $h_1(t) \in \mathbb{R}^{n_x}$ we can write

$$\begin{bmatrix} \dot{x}(r) - Bu(r) \\ h_1 \end{bmatrix}^T \begin{bmatrix} R_1 & -I \\ -I & R_1^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}(r) - Bu(r) \\ h_1 \end{bmatrix} \geq 0.$$

Therefore,

$$-[\dot{x}(r) - Bu(r)]^T R_1 [\dot{x}(r) - Bu(r)] \leq h_1^T R_1^{-1} h_1 - [\dot{x}(r) - Bu(r)]^T h_1 - h_1^T [\dot{x}(r) - Bu(r)].$$

Note that $u(r) = Kx(s_k)$ is constant for $r \in (t_k, t_{k+1})$, and $x(r) = x_r(0) \in \mathcal{W}$ is absolutely continuous. Therefore, integrating both sides from $t - \rho_t$ to t, with respect to r, we have

$$\begin{aligned} -\int_{t-\rho_t}^{t} [\dot{x}(r) - Bu(r)]^T R_1 [\dot{x}(r) - Bu(r)] \, \mathrm{d}r \leq & \rho_t h_1^T R_1^{-1} h_1 - [x - x(t_k) - \rho_t Bu]^T h_1 \\ & - h_1^T [x - x(t_k) - \rho_t Bu]. \end{aligned} \tag{9.40}$$

Replacing (9.40) in (9.39), yields

$$\begin{aligned} \dot{V}_1 \leq & \rho_t h_1^T R_1^{-1} h_1 - \left[x - x(t_k) - \rho_t Bu\right]^T h_1 - h_1^T \left[x - x(t_k) - \rho_t Bu\right] \\ & + (\tau - \rho_s) \left[\dot{x} - Bu\right]^T R_1 \left[\dot{x} - Bu\right]. \end{aligned} \tag{9.41}$$

Similarly, we can write the following equations

$$\dot{V}_2 \leq \rho_t h_2^T R_2^{-1} h_2 - \left[x - x(t_k)\right]^T h_2 - h_2^T \left[x - x(t_k)\right] + (\tau - \rho_s) \dot{x}^T R_2 \dot{x}, \tag{9.42}$$

$$\begin{aligned} \dot{V}_3 = & -\int_{t-\eta_{\min}}^{t} \dot{x}^T(r) R_3 \dot{x}(r) \, \mathrm{d}r + \eta_{\min} \dot{x}^T R_3 \dot{x} \\ \leq & \eta_{\min} h_3^T R_3^{-1} h_3 - \left[x - x(t - \eta_{\min})\right]^T h_3 - h_3^T \left[x - x(t - \eta_{\min})\right] + \eta_{\min} \dot{x}^T R_3 \dot{x}, \end{aligned} \tag{9.43}$$

$$\dot{V}_4 = (\tau - \eta_{\min}) \dot{x}^T(t - \eta_{\min}) R_4 \dot{x}(t - \eta_{\min}) - \int_{t-\rho_s}^{t-\eta_{\min}} \dot{x}^T(r) R_4 \dot{x}(r) \, \mathrm{d}r + (\tau - \eta_{\min}) \dot{x}^T R_4 \dot{x}$$

$$
\begin{aligned}
&\quad - (\tau - \eta_{\min})\dot{x}^T(t-\eta_{\min})R_4\dot{x}(t-\eta_{\min}) \\
&\le (\rho_s - \eta_{\min})h_4^T R_4^{-1} h_4 - \left[x(t-\eta_{\min}) - x(s_k)\right]^T h_4 - h_4^T\left[x(t-\eta_{\min}) - x(s_k)\right] \\
&\quad + (\tau - \eta_{\min})\dot{x}^T R_4 \dot{x}, && (9.44)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_5 &= -\int_{t-\rho_s}^{t} \dot{x}^T(r)R_5\dot{x}(r)\,\mathrm{d}r + \tau\dot{x}^T R_5\dot{x} \\
&\le \rho_s h_5^T R_5^{-1} h_5 - \left[x - x(s_k)\right]^T h_5 - h_5^T\left[x - x(s_k)\right] + \tau\dot{x}^T R_5\dot{x}, && (9.45)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_6 &= -\int_{t-\rho_s}^{t-\rho_t} \dot{x}^T(r)R_6\dot{x}(r)\,\mathrm{d}r - \int_{t-\rho_t}^{t} \dot{x}^T(r)R_6\dot{x}(r)\,\mathrm{d}r + \tau\dot{x}^T R_6\dot{x} \\
&\le \eta_k h_{6_a}^T R_6^{-1} h_{6_a} - \left[x(t_k) - x(s_k)\right]^T h_{6_a} - h_{6_a}^T\left[x(t_k) - x(s_k)\right] + \rho_t h_{6_b}^T R_6^{-1} h_{6_b} \\
&\quad - \left[x - x(t_k)\right]^T h_{6_b} - h_{6_b}^T\left[x - x(t_k)\right] + \tau\dot{x}^T R_6\dot{x}, && (9.46)
\end{aligned}
$$

$$
\dot{V}_7 = x^T Z x - x^T(t-\eta_{\min})Zx(t-\eta_{\min}), \qquad (9.47)
$$

$$
\begin{aligned}
\dot{V}_8 &= -\begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix} X \begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix}^T + (\tau - \rho_s)\begin{bmatrix} \dot{x}^T(t) & 0\end{bmatrix} X \begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix}^T \\
&\quad + (\tau - \rho_s)\begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix} X \begin{bmatrix} \dot{x}^T(t) & 0\end{bmatrix}^T, && (9.48)
\end{aligned}
$$

where $h_j(t)$, $j \in \{2, \ldots, 5\}$, $h_{6_a}(t)$, and $h_{6_b}(t)$ are arbitrary time varying vectors in $\mathbb{R}^{n_x}$. Although the functionals V_5 and V_6 were defined similarly in equations (9.16) and (9.17), their time derivatives were approximated differently in equations (9.45) and (9.46). $\dot{V}_6$ is approximated by a delay dependant functional and is used to devise stability conditions for the case when $\eta_{\max}$ is known. Since $\dot{V} = \sum_{i=0}^{8} \dot{V}_i$, adding (9.38) and (9.41)-(9.48) yields

$$
\begin{aligned}
\dot{V} &\le \dot{x}^T P x + x^T P \dot{x} + \rho_t h_1^T R_1^{-1} h_1 - \left[x - x(t_k) - \rho_t B u\right]^T h_1 - h_1^T\left[x - x(t_k) - \rho_t B u\right] \\
&\quad + (\tau - \rho_s)\left[\dot{x} - Bu\right]^T R_1 \left[\dot{x} - Bu\right] + \rho_t h_2^T R_2^{-1} h_2 - \left[x - x(t_k)\right]^T h_2 - h_2^T\left[x - x(t_k)\right] \\
&\quad + (\tau - \rho_s)\dot{x}^T R_2 \dot{x} + \eta_{\min} h_3^T R_3^{-1} h_3 - \left[x - x(t-\eta_{\min})\right]^T h_3 - h_3^T\left[x - x(t-\eta_{\min})\right] \\
&\quad + \eta_{\min}\dot{x}^T R_3 \dot{x} + (\rho_s - \eta_{\min})h_4^T R_4^{-1} h_4 - \left[x(t-\eta_{\min}) - x(s_k)\right]^T h_4 \\
&\quad - h_4^T\left[x(t-\eta_{\min}) - x(s_k)\right] + (\tau - \eta_{\min})\dot{x}^T R_4 \dot{x} + \rho_s h_5^T R_5^{-1} h_5 - \left[x - x(s_k)\right]^T h_5 \\
&\quad - h_5^T\left[x - x(s_k)\right] + \tau\dot{x}^T R_5 \dot{x} + \eta_k h_{6_a}^T R_6^{-1} h_{6_a} - \left[x(t_k) - x(s_k)\right]^T h_{6_a} \\
&\quad - h_{6_a}^T\left[x(t_k) - x(s_k)\right] + \rho_t h_{6_b}^T R_6^{-1} h_{6_b} - \left[x - x(t_k)\right]^T h_{6_b} - h_{6_b}^T\left[x - x(t_k)\right] \\
&\quad + \tau\dot{x}^T R_6 \dot{x} + x^T Z x - x^T(t-\eta_{\min})Zx(t-\eta_{\min}) \\
&\quad - \begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix} X \begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix}^T + (\tau - \rho_s)\begin{bmatrix} \dot{x}^T(t) & 0\end{bmatrix} X \begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix}^T \\
&\quad + (\tau - \rho_s)\begin{bmatrix} x^T(t) & x^T(t_k)\end{bmatrix} X \begin{bmatrix} \dot{x}^T(t) & 0\end{bmatrix}^T. && (9.49)
\end{aligned}
$$

Recalling (9.1) and (9.8), we can write

$$\dot{x}(t) = \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix} \zeta(t), \text{ and } \dot{x}(t) - Bu(t) = \begin{bmatrix} A & 0 & 0 & 0 \end{bmatrix} \zeta(t), \tag{9.50}$$

where $\zeta(t) = \begin{bmatrix} x^T(t) & x^T(t_k) & x^T(s_k) & x^T(t-\eta_{\min}) \end{bmatrix}^T$, $t \in (t_k, t_{k+1})$. Replacing (9.8) and (9.50) in (9.49), setting $h_j(t) = N_j^T \zeta(t)$, $j \in \{1, \ldots, 5\}$, $h_{6_a}(t) = N_{6_a}^T \zeta(t)$, and $h_{6_b}(t) = N_{6_b}^T \zeta(t)$, where N_j, $j \in \{1, \ldots, 5\}$, N_{6_a}, and N_{6_b} are matrices in $\mathbb{R}^{4n_x \times n_x}$, and replacing ρ_t and η_k with $\rho_s - \eta_{\min}$ and $\eta_{\max}$, respectively, yields

$$\begin{aligned}
\dot{V} \leq \zeta^T \Bigg(& \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix} \\
& + (\rho_s - \eta_{\min}) N_1 R_1^{-1} N_1^T - \begin{bmatrix} I & -I & -(\rho_s - \eta_{\min})BK & 0 \end{bmatrix}^T N_1^T \\
& - N_1 \begin{bmatrix} I & -I & -(\rho_s - \eta_{\min})BK & 0 \end{bmatrix} + (\tau - \rho_s) \begin{bmatrix} A & 0 & 0 & 0 \end{bmatrix}^T R_1 \begin{bmatrix} A & 0 & 0 & 0 \end{bmatrix} \\
& + (\rho_s - \eta_{\min}) N_2 R_2^{-1} N_2^T - \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}^T N_2^T - N_2 \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix} \\
& + (\tau - \rho_s) \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T R_2 \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix} + \eta_{\min} N_3 R_3^{-1} N_3^T \\
& - \begin{bmatrix} I & 0 & 0 & -I \end{bmatrix}^T N_3^T - N_3 \begin{bmatrix} I & 0 & 0 & -I \end{bmatrix} \\
& + \eta_{\min} \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T R_3 \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix} + (\rho_s - \eta_{\min}) N_4 R_4^{-1} N_4^T \\
& - \begin{bmatrix} 0 & 0 & -I & I \end{bmatrix}^T N_4^T - N_4 \begin{bmatrix} 0 & 0 & -I & I \end{bmatrix} \\
& + (\tau - \eta_{\min}) \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T R_4 \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix} + \rho_s N_5 R_5^{-1} N_5^T \\
& - \begin{bmatrix} I & 0 & -I & 0 \end{bmatrix}^T N_5^T - N_5 \begin{bmatrix} I & 0 & -I & 0 \end{bmatrix} \\
& + \tau \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T R_5 \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix} + \eta_{\max} N_{6_a} R_6^{-1} N_{6_a}^T \\
& - \begin{bmatrix} 0 & I & -I & 0 \end{bmatrix}^T N_{6_a}^T - N_{6_a} \begin{bmatrix} 0 & I & -I & 0 \end{bmatrix} + (\rho_s - \eta_{\min}) N_{6_b} R_6^{-1} N_{6_b}^T \\
& - \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}^T N_{6_b}^T - N_{6_b} \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix} \\
& + \tau \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix}^T R_6 \begin{bmatrix} A & 0 & BK & 0 \end{bmatrix} + \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}^T Z \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} \\
& - \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix}^T Z \begin{bmatrix} 0 & 0 & 0 & I \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

$$+ (\tau - \rho_s) \begin{bmatrix} A & 0 & BK & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} X & 0 \end{bmatrix} + (\tau - \rho_s) \begin{bmatrix} X \\ 0 \end{bmatrix} \begin{bmatrix} A & 0 & BK & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Bigg) \zeta. \tag{9.51}$$

Based on (9.5), ρ_s varies between $\eta_{\min}$ and τ. Considering (9.51) and using Schur complement [74], for $\rho_s = \eta_{\min}$, LMI (9.22) implies $\dot{V} < 0$. Similarly, LMI (9.23) implies $\dot{V} < 0$ for $\rho_s = \tau$. Since (9.51) is affine in ρ_s, LMIs (9.22) and (9.23) are sufficient conditions for $\dot{V} < 0$ to hold for any $\rho_s \in [\eta_{\min}, \tau]$, i.e. $\forall (t_k, t_{k+1})$, $k \in \mathbb{N}$. Note that there exists a sufficiently small scalar $c_3 > 0$ such that $\dot{V}(t, x_t) < -c_3 ||x_t||^2_{\mathcal{W}}$, for all $t \neq t_k$, $k \in \mathbb{N}$. Hence, inequality (9.24) yields

$$\dot{V}(t, x_t) < -\frac{c_3}{c_2} V(t, x_t), \; \forall t \neq t_k, \; k \in \mathbb{N}. \tag{9.52}$$

Therefore, for any $k \in \mathbb{N}$,

$$V(t_k^-, x_{t_k^-}) \leq e^{-\frac{c_3}{c_2}(t_k - t_{k-1})} V(t_{k-1}, x_{t_{k-1}}) \leq V(t_{k-1}, x_{t_{k-1}}),$$

where $V(t_k^-, x_{t_k^-}) = \lim_{t \nearrow t_k} V(t, x_t)$. The second inequality is strict when the length of the interval (t_{k-1}, t_k) is nonzero. Note that according to Assumption 9.4, any interval (t_{k-1}, t_k), $k \in \mathbb{N}$, has a length greater than or equal to $\epsilon > 0$. Furthermore, it was shown at the beginning of the proof that V is non-increasing at the actuator update instants, i.e.

$$V(t_k, x_{t_k}) \leq V(t_k^-, x_{t_k^-}), \; k \in \mathbb{N}.$$

Therefore, for any $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$,

$$\begin{aligned} V(t, x_t) &\leq e^{-\frac{c_3}{c_2}(t - t_k)} V(t_k, x_{t_k}) \leq e^{-\frac{c_3}{c_2}(t - t_k)} V(t_k^-, x_{t_k^-}) \\ &\leq e^{-\frac{c_3}{c_2}(t - t_{k-1})} V(t_{k-1}, x_{t_{k-1}}) \leq e^{-\frac{c_3}{c_2}(t - t_{k-1})} V(t_{k-1}^-, x_{t_{k-1}^-}) \\ &\;\;\vdots \\ &\leq e^{-\frac{c_3}{c_2} t} V(0, x_0). \end{aligned} \tag{9.53}$$

A similar conclusion could be drawn from Comparison Lemma [35]. Now, inequalities (9.24) and (9.53) yield

$$|x(t)| \leq \left(\frac{V(t, x_t)}{c_1} \right)^{\frac{1}{2}} \leq \left(\frac{e^{-\frac{c_3}{c_2} t} V(0, x_0)}{c_1} \right)^{\frac{1}{2}} \leq \left(\frac{c_2}{c_1} \right)^{\frac{1}{2}} e^{-\frac{c_3}{2c_2} t} ||x_0||_{\mathcal{W}}.$$

Hence, the networked control system is globally uniformly exponentially stable. Note that the Zeno phenomenon does not occur since, by Assumption 9.4, there exists $\epsilon > 0$ such that $t_{k+1} - t_k > \epsilon$. This finishes the proof. $\square$

In Theorem 9.1, given the value of the network-induced delay τ and the lower and upper bounds on the delay, i.e. $\eta_{\min}$ and $\eta_{\max}$, we presented sufficient conditions for exponential stability of linear networked control systems. In some practical problems, however, such information about the delay might not be available. Here, we present sufficient conditions for exponential stability of linear networked control systems under limited information about the delay. The following corollary addresses the case where the upper bound on the delay $\eta_{\max}$ is unknown. To the best of our knowledge, this scenario was not studied in the literature before.

Corollary 9.1. *Consider the linear networked control system defined in (9.1) and (9.8) with Assumptions 9.1-9.4. Given the controller gain K and the scalars τ and $\eta_{\min}$, the networked control system is globally uniformly exponentially stable if there exist symmetric positive definite matrices P, R_i, $i \in \{1, \ldots, 5\}$, and Z, a symmetric matrix X_1, and matrices X_2, N_j, $j \in \{1, \ldots, 5\}$, with appropriate dimensions, satisfying*

$$\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + (\tau - \eta_{\min})X > 0$$

$$\begin{bmatrix} \Psi + \tau M_1 + (\tau - \eta_{\min})(M_2 + M_4) + \eta_{\min} M_3 & \eta_{\min} N_3 & \eta_{\min} N_5 \\ \eta_{\min} N_3^T & -\eta_{\min} R_3 & 0 \\ \eta_{\min} N_5^T & 0 & -\eta_{\min} R_5 \end{bmatrix} < 0$$

$$\begin{bmatrix} \begin{pmatrix} \Psi + \tau M_1 \\ +\eta_{\min} M_3 + \\ (\tau - \eta_{\min})\times \\ (M_4 + M_5) \end{pmatrix} & (\tau - \eta_{\min})N_1 & (\tau - \eta_{\min})N_2 & \eta_{\min} N_3 & (\tau - \eta_{\min})N_4 & \tau N_5 \\ (\tau - \eta_{\min})N_1^T & (\eta_{\min} - \tau)R_1 & 0 & 0 & 0 & 0 \\ (\tau - \eta_{\min})N_2^T & 0 & (\eta_{\min} - \tau)R_2 & 0 & 0 & 0 \\ \eta_{\min} N_3^T & 0 & 0 & -\eta_{\min} R_3 & 0 & 0 \\ (\tau - \eta_{\min})N_4^T & 0 & 0 & 0 & (\eta_{\min} - \tau)R_4 & 0 \\ \tau N_5^T & 0 & 0 & 0 & 0 & -\tau R_5 \end{bmatrix} < 0$$

where Ψ, M_j, $j \in \{1, \ldots, 5\}$, are defined in Theorem 9.1 with $R_6 = 0$ and $N_{6_a} = N_{6_b} = 0$.

Proof. Let an LKF be defined as $\sum_m V_m$, $m \in \{0, \ldots, 5, 7, 8\}$. Here, we omit the functional V_6 because its derivative is approximated by a functional that depends on η_k (see inequality (9.46)). In turn, η_k is replaced in (9.51) by the upper bound $\eta_{\max}$. In this corollary, however, $\eta_{\max}$ is assumed to be unknown. Using the modified LKF, the rest of the proof is similar to the proof of Theorem 9.1. □

If the lower bound on the delay $\eta_{\min}$ is unknown, based on Assumption 9.2, we set $\eta_{\min} = 0$. The next corollary provides sufficient conditions for exponential stability of linear networked control systems where $\eta_{\min}$ is unknown or similarly where $\eta_{\min} = 0$.

Corollary 9.2. *Consider the linear networked control system defined in (9.1) and (9.8) with Assumptions 9.1-9.4. Given the controller gain K and the scalars τ and $\eta_{\max}$, the networked control system is globally uniformly exponentially stable if there exist symmetric positive definite matrices P, R_1, R_2, R_5, and R_6, a symmetric matrix X_1, and matrices X_2, N_1, N_2, N_5, N_{6_a}, and N_{6_b}, with appropriate dimensions, satisfying*

$$\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + \tau X > 0$$

$$\begin{bmatrix} \Psi + \tau(M_1 + M_2) & \eta_{\max} N_{6_a} \\ \eta_{\max} N_{6_a}^T & -\eta_{\max} R_6 \end{bmatrix} < 0$$

$$\begin{bmatrix} \Psi + \tau(M_1 + M_5) & \tau N_1 & \tau N_2 & \tau N_5 & \eta_{\max} N_{6_a} & \tau N_{6_b} \\ \tau N_1^T & -\tau R_1 & 0 & 0 & 0 & 0 \\ \tau N_2^T & 0 & -\tau R_2 & 0 & 0 & 0 \\ \tau N_5^T & 0 & 0 & -\tau R_5 & 0 & \\ \eta_{\max} N_{6_a}^T & 0 & 0 & 0 & -\eta_{\max} R_6 & 0 \\ \tau N_{6_b}^T & 0 & 0 & 0 & 0 & -\tau R_6 \end{bmatrix} < 0$$

where Ψ, M_1, M_2, and M_5 are defined in Theorem 9.1 with $R_3 = R_4 = Z = 0$ and $N_3 = N_4 = 0$, and all the zero rows and columns (corresponding to $x(t - \eta_{min})$) are omitted.

Proof. Let an LKF be defined as $\sum_m V_m$, $m \in \{0, 1, 2, 5, 6, 8\}$. Here, the functionals V_3, V_7, and the second term in V_4 are omitted because they vanish when $\eta_{\min} = 0$. Also note that when $\eta_{\min} = 0$, the first part of V_4 becomes identical to the functionals V_5 and V_6. Therefore, the first part of V_4 is dispensable in this case. Using the modified LKF, the rest of the proof is similar to the proof of Theorem 9.1. □

The following proposition presents sufficient conditions for exponential stability of linear networked control systems with uncertain parameters.

Proposition 9.1. *Suppose that the pair of system matrices* $\Omega = \begin{bmatrix} A & B \end{bmatrix}$ *in (9.1) is unknown but satisfies the following condition*

$$\Omega \in \left\{ \sum_{j=1}^{p} \alpha_j \Omega_j,\ 0 \leq \alpha_j \leq 1,\ \sum_{j=1}^{p} \alpha_j = 1 \right\},$$

where $\Omega_j = \begin{bmatrix} A_j & B_j \end{bmatrix}$, $j \in \{1, ..., p\}$, *denote the vertices of a convex polytope. If the LMIs in Theorem 9.1 (or Corollaries 9.1 and 9.2) hold for each* Ω_j, $j \in \{1, ..., p\}$, *with the same matrix variables* P, R_i, $i \in \{1, \ldots, 6\}$, Z, X_1, *and* X_2, *then the uncertain linear networked control system is globally uniformly exponentially stable.*

Proof. The proof is similar to the proof of Proposition 2.1 and is hence omitted. □

The LMIs in Theorem 9.1 are affine in τ, $\eta_{\min}$, and $\eta_{\max}$. Therefore, keeping two of these variables constant, we can use a line search approach to optimize for the other variable. For instance, given the lower and upper bounds on the delay, the problem of finding a lower bound on the maximum network-induced delay that preserves exponential stability is formulated as

Problem 9.1.

maximize τ

subject to $P > 0,\ R_i > 0,\ i \in \{1, \ldots, 6\},\ Z > 0,\ X_1 = X_1^T,\ (9.21) - (9.23).$

We denote the computed lower bound on the maximum network-induced delay that preserves exponential stability by $\tau_{\max}$. Similarly, the LMIs in Corollaries 9.1 and 9.2 can be used to write suitable optimization programs.

9.4 Numerical Examples

In this section, we apply our stability theorems to a benchmark problem in the literature.

Example 9.1. *[11, 12, 47, 48] Consider the linear networked control system defined in (9.1) and (9.8) with the following parameters*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},\ K = -\begin{bmatrix} 3.75 & 11.5 \end{bmatrix}.$$

Table 9.2: Comparison of the computed lower bound on the maximum network-induced delay $\tau_{\max}$ (s) for $\eta_{\max} = 0.8$ (s) and different values of $\eta_{\min}$ in Example 9.1.

$\eta_{\min}$ (s)	0	0.2	0.4	0.6	0.75
[116]	1.04	-	-	-	-
[11]	0.87	0.89	0.92	0.97	1.02
([11] plus V_5) $\equiv$ (Theorem 9.1 with $V_1 = 0$)	0.87	0.89	0.93	0.98	1.03
([11] plus V_1) $\equiv$ (Theorem 9.1 with $V_5 = 0$)	1.06	1.02	1.00	1.01	1.03
[12]	1.10	-	-	-	-
Theorem 9.1	1.14	1.09	1.06	1.05	1.07

Here, we assume that $\eta_{\max} = 0.8$ (s) and solve Problem 9.1 to find a lower bound on the maximum network-induced delay that preserves exponential stability for different values of $\eta_{\min}$. Table 9.2 shows the computed τ_{max} by Theorem 9.1 and the Theorems in [11, 12, 116]. According to Table 9.2, the stability criteria of Theorem 9.1 are less conservative (i.e. provide larger lower bounds on the maximum network-induced delay) for this benchmark problem than the previously existing results.

9.5 Conclusion

In this chapter, we addressed exponential stability of linear networked control systems. We introduced a modified LKF that contains a functional in terms of the open-loop vector field of the linear system. Next, based on the modified LKF, new sufficient stability conditions were derived for linear networked control systems. Furthermore, the problem of finding a lower bound on the maximum network-induced delay that preserves exponential stability was formulated as a convex optimization program in terms of LMIs. The stability conditions of this chapter were shown to be less conservative than previously existing results when applied to a benchmark problem.

Chapter 10

Conclusions

In this thesis, we developed computationally efficient methods for stability analysis, controller synthesis, and observer design for sampled-data networked control systems. A diverse range of systems were studied in this thesis. These systems can be categorized by their vector fields as linear systems (Chapters 2-5 and 9), PWA systems (Chapters 6 and 7), and nonlinear systems (Chapter 8). The network structures addressed in this can be divided into three main categories; single-rate sampled-data networked control systems (Chapters 2, 6-8), multi-rate sampled-data networked control systems (Chapters 3-5), and single-rate sampled-data networked control systems with time-varying delays (Chapter 9).

We proposed Krasovskii-based sufficient conditions to address stability, stabilization, and estimation problems. The controller design problem usually leads to non-convex optimization problems. Therefore, convex relaxation techniques were used to formulate the sufficient stabilization criteria as convex optimization programs. In particular, the sufficient conditions were formulated in terms of LMIs that can be solved efficiently using available optimization software. For the first time, sufficient conditions for exponential stability of PWA and nonlinear sampled-data systems were presented using a piecewise smooth Krasovskii functional. This decreases the conservativeness of the proposed sufficient conditions when compared with the use of smooth Krasovskii functionals. The proposed stability and stabilization conditions are applicable to systems with polytopic uncertainty in the model parameters. In practice, the results of this thesis improve performance and reliability of networked control systems by allowing engineers

1. To estimate the MASP that guarantees exponential stability. Depending on the vector field of the model selected for the system and the network structure, the

sufficient conditions that are formulated as LMIs in Chapters 2, 3, and 6-8 may be used.

2. To estimate the maximum allowable delay in communication links that guarantee exponential stability. The theorems in Chapter 9 provide a set sufficient conditions in terms of LMIs to estimate the maximum allowable delay.

3. To design controllers that guarantee exponential stability for the MASP dictated by the sensing equipment. Depending on the vector field of the model selected for the system and the network structure, the sufficient conditions that are formulated as LMIs in Chapters 2, 3, and 6-8 may be used.

4. To design observers that guarantee exponential convergence of the estimation error for the MASP dictated by the sensing equipment. The theorems in Chapter 5 provide a set sufficient conditions in terms of LMIs to design sampled-data observers.

5. To allocate sensors to states such that exponential stability is guaranteed for a desired set of MASPs. In other words, to determine which states should be sampled at a higher rate and which states should be sampled at a lower rate. The theorem in Chapter 4 provides a set sufficient conditions in terms of LMIs to address the sensor allocation problem.

The methodology developed in this thesis serves as a guideline to address several new problems such as

1. PWA and nonlinear multi-rate sampled-data networked control systems. This line of research is an extension of Chapter 3 to PWA and nonlinear systems.

2. Output feedback control of multi-rate sampled-data networked control systems. This can be achieved by extending the observer design technique in Chapter 5 to systems with control feedback.

3. Fault detection in multi-rate sampled-data networked control systems. This can be seen as an extension of the observer design technique in Chapter 5.

4. PWA and nonlinear sampled-data networked control systems with time-varying delays. This line of research is an extension of Chapter 9 to PWA and nonlinear systems described in Chapters 6-8.

5. Stability and stabilization of systems that are described by neutral functional differential equations. In neutral (as opposed to retarded) functional differential equations, the highest order derivative contains delayed variables. Population dynamics models are examples of systems that are defined by functional differential equations of neutral type.

6. As a measure of performance, H_∞ control of sampled-data networked control systems can be addressed using Krasovskii-based approaches similar to the ones developed in this thesis. Minimum attention control and event-triggered control are also interesting approaches that can be used to guarantee certain performance requirements for networked control systems.

Finally, two open problems in the field of networked control systems are

1. to present necessary and sufficient conditions for stability of linear networked control systems with multi-rate samplers, time-varying delays, and data packet losses.

2. to address stability analysis and controller synthesis problems for sampled-data networked control systems with general nonlinear vector fields.

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