A Crystalline Criterion for Good Reduction on Semi-stable

$K3$-Surfaces over a p-Adic Field

Jesús Rogelio Pérez Buendía

Mathematics & Statistics

Concordia University

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By: Jesús Rogelio Pérez Buendía

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Dr. Adrian Iovita — Thesis Supervisor

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____
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Abstract

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In this thesis we prove a p-adic analogous of the Kulikov-Persson-Pinkham classification theorem for the central fibre of a degeneration of $K3$-surfaces in terms of the nilpotency degree of the monodromy of the family [Persson & Pinkham, 1981].

Namely, let X_K be a be a smooth, projective $K3$-surface which has a minimal semistable model X over $\mathcal{O}_K$. If we let N_{st} be the monodromy operator on $D_{st}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$, then we prove that the degree of nilpotency of N_{st} determines the type of the special fibre of X. As a consequence we give a criterion for the good reduction of the semi-stable $K3$-surface X_K over the p-adic field K in terms of its p-adic representation $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, which is similar to the criterion of good reduction for p-adic abelian varieties and curves given by [Coleman & Iovita, 1997] and [Iovita *et al.*, 2013].

A mi compañera de aventuras
a mi soporte y mi aliento
a la mujer que amo
a la madre de mis hijos:
A Yuriria

A Canek y Alina que son el mejor proyecto de mi vida
con todo mi amor para ellos.

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Contents

Chapter 1

Introduction

The main object of study of this thesis is the interplay between the geometry of algebraic varieties and their cohomology. In general it is known that the geometry of an algebraic variety over a field determines the various cohomology groups with their extra structure. For example if X is a smooth, proper algebraic variety over the complex numbers $\mathbb{C}$, then the Hodge structure on its Betti cohomology is pure with determined weights.

Similarly, if X is a smooth, proper algebraic variety over a p-adic field K, then its p-adic étale cohomology groups $V_i := H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ are p-adic $G_K := \text{Gal}(\bar{K}/K)$-representations whose type is determined by the geometry of various integral models of X. For instance if X has good reduction then the V_i's are crystalline G_K representations, if X has semi-stable reduction, then the V_i's are semi-stable representations, etc.

In general it is not true that the cohomology groups of an algebraic variety determine their geometric properties, however, for certain very special classes of varieties it has been known for some time that this might happen.

Here are some examples:

For Abelian varieties over $\mathbb{C}$ we have the Torelli theorem:

Theorem 1.0.1. *An abelian variety over $\mathbb{C}$ is determined by its periods. More precisely, if A, A' are complex polarised abelian varieties, and we have an isomorphism of Hodge structures*

$$\phi : H^1(A, \mathbb{Z}) \to H^1(A', \mathbb{Z})$$

then the abelian varieties A and A' are isomorphic.

Moreover, if A is an Abelian variety over a p-adic field K we have:

Theorem 1.0.2.

- *A has good reduction if and only if $H^1_{ét}(A_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline G_K-representation.*
- *A has semi-stable reduction if and only if $H^1_{ét}(A_{\overline{K}}, \mathbb{Q}_p)$ is a semi-stable G_K-representation.*

The class of $K3$-surfaces is another very interesting class of algebraic varieties which resembles the class of Abelian varieties. More precisely, they satisfy a Torelli theorem Looijenga & Peters [1980]:

Theorem 1.0.3 (Weak Torelli Theorem)**.** *Two complex $K3$-surface s X, X' are isomorphic if and only if there exists an isometry from $H^2(X, \mathbb{Z})$ to $H^2(X', \mathbb{Z})$ which sends $H^{2,0}(X, \mathbb{C})$ to $H^{2,0}(X', \mathbb{C})$ (see page 9 and theorem (2.1.6)).*

Also, if $\mathfrak{X} \to \Delta$ is a degeneration of $K3$-surfaces over the open unit complex disk Δ, we have the important theorem of Kulikov and Persson & Pinkham.

Theorem 1.0.4. *(see theorem (3.1.9)) Let $\pi : \mathfrak{X} \to \Delta$ be a semi-stable degeneration of $K3$-surfaces with all components of the central fibre $\mathfrak{X}_0 = \pi^{-1}(0) = \bigcup V_i$ algebraic.*

Let $N = \log T : H^2(\mathcal{X}_t, \mathbb{Z}) \to H^2(\mathcal{X}_t, \mathbb{Z})$ be the monodromy operator. After birational modification we may assume that $\pi : \mathcal{X} \to \Delta$ is a Kulikov model (definition (3.1.4)). Then the central fibre $\mathcal{X}_0$ is one of the following:

1. *(Type I) $\mathcal{X}_0$ is a $K3$-surface and $N = 0$.*

2. *(type II) $\mathcal{X}_0 = V_0 \cup V_1 \cdots V_r$, where V_0, V_r are smooth rational, and $V_1, \ldots, V_{r-1}$ are smooth elliptic ruled and $V_i \cap V_j \neq \emptyset$ if and only if $j = i \pm 1$. $V_i \cap V_j$ is then a smooth elliptic curve and a section of the ruling on V_i, if V_i is elliptic ruled. $N \neq 0$ but $N^2 = 0$.*

3. *(Type III) $\mathcal{X}_0 = \cup V_i$, with each $\overline{V}_i$ smooth rational and all double curves are cycles of rational curves. The dual graph Γ is a triangulation of S^2. In this case $N^2 \neq 0$, but $N^3 = 0$.*

Remark 1.0.5. Note that in particular $\mathcal{X}_0$ is smooth if and only if $N = 0$.

Let now K be a p-adic field for a prime number $p > 3$, and let X_K be a smooth, projective $K3$-surface over K, having a minimal semi-stable model X over the ring of integers of K. The main result of this thesis is the following theorem theorem (5.1.2):

Theorem 1.0.6. *X_K has good reduction if and only if $H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline G_K-representation.*

Remark 1.0.7. Matsumoto proves a version of theorem (1.0.6) in [Matsumoto, 2012] for the case of $K3$-surfaces coming from Abelian varieties, more precisely for $K3$-surfaces with Shioda-Inose structure.

Remark 1.0.8. In fact we prove more, namely let N_{st} be the monodromy operator on $D_{st}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$. Then the degree of nilpotency of N_{st} determines the type of the special fibre of the minimal integral model X of X_K, as follows: If $N_{st} = 0$

then the special fibre $\mathcal{X}_0$ is a smooth $K3$-surface. If $N_{st} \neq 0$ but $N_{st}^2 = 0$ then $\mathcal{X}_0 = V_0 \cup V_1 \cdots V_r$, where V_0, V_r are smooth rational, and $V_1, \ldots, V_{r-1}$ are smooth elliptic ruled and $V_i \cap V_j \neq \emptyset$ if and only if $j = i \pm 1$. If $N_{st}^2 \neq 0$ but $N_{st}^3 = 0$ then $\mathcal{X}_0 = \cup V_i$, with each $\overline{V}_i$ smooth rational and all double curves are cycles of rational curves.

In fact our method is more general: let us suppose that $\mathcal{A}$ is a class of varieties over various fields satisfying the following two properties:

1. if X is a scheme over $\mathcal{O}_K$ such that its generic fibre X_K is a smooth, proper variety in $\mathcal{A}$ and its special fibre $\bar{X}$ is a semi-stable variety over k, then $\bar{X}$, a log scheme (with the natural log structure), has global deformations over $W(k)[[t]]$ of the type described in proposition (3.3.16).

2. if Y is a family of varieties in $\mathcal{A}$ over the complex open unit disk Δ, degenerating exactly at 0, then there is a Kulikov-type theorem saying that: the family is smooth if and only if the monodromy operator of the log cohomology of its special fiber vanishes.

   Then, following the same steps as in chapter (5) one would be able to prove a theorem of type theorem (1.0.6) for a variety X_K in $\mathcal{A}$ over a p-adic field K.

As we have mentioned before, the cohomology dose not always determine the geometry of the algebraic varieties. For example, it is known that the geometry of curves is not determined by the structure of their cohomology groups. Nevertheless, their geometry is determined by the quotients of their unipotent fundamental groups as follows [Iovita *et al.*, 2013]:

Let K be a finite extension of $\mathbb{Q}_p$ and suppose that X_K is a curve with semi-stable reduction. Assume also that the genus of X_K is larger or equal to 2. For a fix geometric point b of X_K let, for every prime l, $\pi^{(l)}$ be the maximal pro-l quotient of the geometric étale fundamental group $\pi_1(X_{\bar{K}}, b)$ of X_K and let $\left\{\pi_1^{(l)}[n]\right\}_{n\geq 1}$ be the lower central series of $\pi_1^{(l)}$.

Theorem 1.0.9 (Oda)**.** *X_K has good reduction if and only if for some prime integer $l \neq p$ the outer representations $\pi_1^{(l)}/\pi_1^{(l)}[n]$ are unramified for all $n > 1$.*

The theorem of Iovita *et al.* [2013] is a p-adic analogue of theorem (1.0.9).

Theorem 1.0.10. *If $G^{ét}$ denotes the unipotent p-adic étale fundamental group of $X_{\bar{K}}$ for the base point b, then X_K has good reduction if and only if $G^{ét}$ is a crystalline G_K-representation.*

This raises the very interesting question: given a class of algebraic varieties, are there combinatorial (linear algebra type) objects attached to them which determine their geometry? If yes, what are they?

Chapter 2

Introduction to $K3$-Surfaces

2.1 $K3$-Surfaces

Definition 2.1.1. A compact smooth complex manifold X of dimension 2 is a $K3$-surface if:

1. The canonical bundle ω_X is trivial.

2. The first Betti number $b_1(X) := \operatorname{rank} H_1(X, \mathbb{Z}) = 0$.

Since we are working with algebraic $K3$-surfaces, we have that the irregularity $q := \dim H^1(X, \mathcal{O}_X) = 2b_1$ [Beauville, 2011], so we can define also a $K3$-surface by asking to have $q = 0$. Since the irregularity and the canonical divisors are defined for any algebraic surface (over any field), then we can give a more ad hoc definition of $K3$-surface.

Note that since the canonical bundle $\omega_X := \wedge^2 \Omega^1_X$ is trivial, there exists a nowhere vanishing 2-form on X.

Definition 2.1.2. Let K be any field and let X be a non singular proper algebraic variety over K of dimension 2. X is a $K3$ surface if

1. The canonical sheaf ω_X is trivial, and

2. The irregularity $q = \dim_K H^1(X, \mathcal{O}_X) = 0$.

Proposition 2.1.3. *The Betti numbers of a complex projective $K3$-surface are $b_0 = b_4 = 1$, $b_1 = b_3 = 0$ and $b_2 = 22$; Moreover $H^2(X, \mathbb{Z})$ is torsion free.*

Proposition 2.1.4. *The Hodge diamond of a $K3$-surface is given by the following diagram [Barth et al., 1984]:*

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0,} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

From the Hodge diamond above we can also read the Betti numbers as the sum of the numbers of every row by Hodge theory. In particular we see that $\dim H^2(X, \mathbb{Z}) = 22$.

Proposition 2.1.5. *For a $K3$-surface X the second cohomolology group $H^2(X, \mathbb{Z})$ is a non-degenerated lattice. That is $H^2(X, \mathbb{Z})$ is torsion free and together with the quadratic form induced by the cup product it is a non-degenerated quadratic space. Moreover we can choose a basis so as to have an isomorphism of lattices:*

$$H^2(X, \mathbb{Z}) \simeq U^3 \oplus (-E_8)^2$$

where U is the standard hyperbolic lattice, whose matrix associated to its quadratic form is given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and E_8 is the root lattice determined by the Dynkin diagram E_8 (Figure 2.1), that is E_8 has 8 generators $e_1, e_2, \ldots, e_8$ in bijection with the points

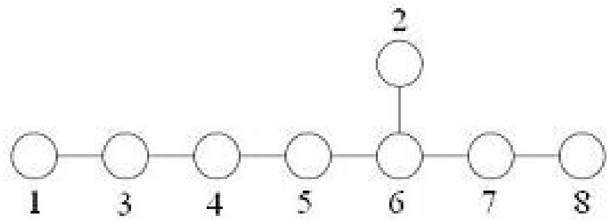


Figure 2.1: Dynkin diagram E_8.

of the diagram in figure 2.1 and $B_Q(e_i, e_i) = 2$ and $B_Q(e_i, e_j) = 0$ or -1 according as the corresponding vertex are unjoined or joined. $-E_8$ is the same but with the opposite signs. That is $\pm E_8 = \mathbb{Z}^8$ with the quadratic form Figure 2.2.

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Figure 2.2: Matrix of the quadratic space E_8.

Now we know that $H^2(X, \mathbb{Z})$ is a lattice of rank 22 that with the quadratic form induced by the cup product and Poincaré duality it is unimodular quadratic space.

Finally I want to remark that De Rham's theorem tells us that

$$H^2(X, \mathbb{C}) \simeq H_{dR}(X/\mathbb{C})$$

and that the cup product corresponds under this isomorphism to

$$\langle \overline{\omega}, \overline{\nu} \rangle \longrightarrow \int_X \omega \wedge \nu.$$

A very important result on complex $K3$-surfaces is the Torelli Theorem that we mention now.

Let X be a complex $K3$-surface. From the previous sections we know that the second cohomology group $H^2(X, \mathbb{Z})$ together with the quadratic form induced by the intersection paring (the cup product) $\cup$ is a quadratic lattice isomorphic to $U^3 \oplus (-E_8)^2$.

Moreover by Hodge theory we have that $H^2(X, \mathbb{Z})$ is a Hodge structure of weight 2. This is to say that we have a Hodge decomposition:

$$H^2(X, \mathbb{C}) \simeq H^{0,2} \oplus H^{1,1} \oplus H^{0,2}$$

is such that $\overline{H}^{2,0} = H^{0,2}$ and $H^{1,1}$ is orthogonal to $H^{0,2} \oplus H^{2,0}$. Also we know that $H^{p,q} = H^q(X, \Omega^p)$ and that $h^{0,2} = h^{2,0} = 1$ and $h^{1,1} = 20$.

Torelli's theorem tells us basically that if the Hodge structures of a $K3$ surface are isomorphic then the $K3$-surfaces are isomorphic. More precisely:

Theorem 2.1.6. *Suppose X, X' are $K3$-surfaces and*

$$\phi : (H^2(X, \mathbb{Z}), \cup) \longrightarrow (H^2(X', \mathbb{Z}), \cup)$$

is a Hodge isometry, that is an isomorphism of lattices such that

$$\phi(H^{2,0}(X)) = H^{2,0}(X').$$

Then X is isomorphic to X'.

The previous theorem is known as the weak Torelli's theorem. We have a strong Torelli's theorem and it is related to the surjectivity of the period map:

Theorem 2.1.7. *Let L be the lattice $H^3 \oplus (-E_8)^2$. Consider $v \in L \otimes \mathbb{C}$ a vector such that $\langle v,v \rangle = 0$ and $\langle v,\overline{v} \rangle > 0$. Then there exists a $K3$-surface X and a marking of the $K3$-surface (that is an isomorphism $\phi : (H^2(X,\mathbb{Z}), \cup) \to (L, \langle\ ,\ \rangle)$ such that $\phi_{\mathbb{C}}(H^{2,0}(X)) = span_{\mathbb{C}}\{v\}$.*

For a reference about Torelli's theorem ([Looijenga & Peters, 1980]) and for the general theory of algebraic complex surfaces and in particular for $K3$-surfaces we follow ([Barth *et al.*, 1984]).

First we want to study $K3$-surfaces defined over a p-adic field K or over the algebraic closure of a finite field.

Let K be a field of characteristic $p \geq 0$. By a surface over K we understand a separated geometrically integral scheme of finite type $X \to \mathrm{Spec}(K)$ of relative dimension 2.

Definition 2.1.8. A smooth proper surface X over K is a $K3$-surface if it has trivial canonical bundle and its irregularity is zero. In other words a $K3$ surface over K is a surface such that

- $q = \dim_K H^1(X, \mathcal{O}_X) = 0$
- $\omega_X = \Omega^2_X \simeq \mathcal{O}_X$.

As before ω_X denotes the canonical sheaf of X and $\mathcal{O}_X$ its structural sheaf. The canonical divisor K_X is just the class of divisors associated to the line bundle ω_X. Therefore for a $K3$-surface we have $K_X = 0$.

For a $K3$ surface over K we have also the same Hodge diamond (2.1.4) as in the complex case.

Indeed for a $K3$-surface the Hodge to de Rham spectral sequence

$$E_1^{pq} = H^q(X, \Omega_X^p) \Longrightarrow H_{\mathrm{dR}}^{p+q}(X/K)$$

degenerates at E_1 because any $K3$-surface over a field of characteristic p lifts to a $K3$-surface of characteristic zero [Deligne & Illusie, 1981]. Then is true for any field (also see Rizov thesis [Rizov, 2005]). This, together with Poincaré duality, implies that if as before $h^{q,p} = \dim_K H^q(X, \Omega_X^p)$, then we have the usual Hodge diamond for a $K3$-surface.

2.2 Some Examples of $K3$-Surfaces

Example 2.2.1 (Complete intersections)**.** In this example we consider complete intersections on a projective space. Let X be a smooth projective surface which is a complete intersection of n hypersurfaces of degree $d_1, \ldots, d_n$ in $\mathbb{P}^{n+2}$ over $\mathbb{C}$.

The adjunction formula tells us that $\Omega^2_{X/k} \cong \mathcal{O}_X(d_1, \ldots, d_n - n - 3)$. We want X to be a $K3$ surface, therefore we need that $d_1 + \ldots + d_n = n + 3$ in order to have trivial canonical bundle.

We have the following options for small n:

$$\begin{aligned} n &= 1 & d_1 &= 4 \\ n &= 2 & d_1 &= 2, d_2 = 3 \\ n &= 3 & d_1 &= d_2 = d_3 = 2. \end{aligned}$$

On the other hand, we have that for a general complete intersection Y of dimension n, the cohomology groups $H^i(Y, \mathcal{O}_M(m))$ are equal to 0 for all $m \in \mathbb{Z}$ and $1 \leq i \leq n-1$ [Hartshorne, 1977]. Therefore on the cases above we have $H^1(X, \mathcal{O}_X) = 0$ and X is a $K3$ surface. That is to say, a quartic in $\mathbb{P}^3$, the complete intersection of a cubic and a quadric on $\mathbb{P}^4$ and the complete intersection of three quadrics in $\mathbb{P}^5$ are examples of $K3$-surfaces.

Example 2.2.2 (Kummer surfaces)**.** Let A be an abelian surface. Let $\tau : Y \to Y$ be an involution (for example the inverse $x \to x^{-1}$ using the group law on A). Consider the quotient of A under the action of τ: $A/ < \tau >$ which is a normal surface with $2^4 = 16$ singularities (corresponding to the fixed points of τ). Let $\tilde{A} \to A/ < \tau >$ be the blow up along the singularities. Then $X = \tilde{A}$ is a $K3$-surface called the *Kummer surface* associated to A.

Chapter 3

Semi-stable $K3$-Surfaces

3.1 Kulikov Degeneration's Theorem

We will briefly describe the Kulikov-Persson-Pinkham's classification theorem of the central fibre of a semi-stable family of complex $K3$-surfaces over the open disk. The main references are [Morrison, 1984; Nishiguchi, 1983; Persson & Pinkham, 1981].

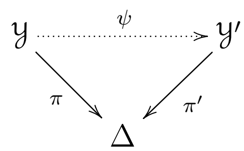
Denote by $\Delta := \{z \in \mathbb{C} : |z| < \varepsilon\}$ the open small disk and by Δ^* the punctured disk, that is $\Delta^* = \Delta \setminus \{0\}$.

Definition 3.1.1. A degeneration of $K3$-surfaces is a flat proper holomorphic map $\pi : \mathcal{Y} \to \Delta$ of relative dimension 2 such that $\mathcal{Y}_t := \pi^{-1}(t)$ is a smooth $K3$-surface for $t \neq 0$. We call $Y_0 := \pi^{-1}(0)$ the degenerated fibre or central fibre. We assume that $\mathcal{Y}$ is Kähler.

If we have a fixed $K3$-surface Y, then a degeneration of Y is a degeneration of $K3$-surfaces such that for some $t \neq 0$, $\mathcal{Y}_t = Y$.

Definition 3.1.2. A degeneration $\pi : \mathcal{Y} \to \Delta$ is semi-stable if the central fibre Y_0 is a reduced divisor with normal crossings, that is the union $Y_0 = \cup Y_i$ of irreducible components with each Y_i smooth and the Y_i's meeting transversally so that locally π is defined by an equation of the form $0 = x_1 x_2 \dots x_k$ for some k.

Definition 3.1.3. A degeneration $\pi' : \mathcal{Y}' \to \Delta$ is called a **modification** of a degeneration $\pi : \mathcal{Y} \to \Delta$; if there exists a bimeromorphic map $\psi : \mathcal{Y} \to \mathcal{Y}'$ such that the diagram commutes:



and the restriction of ϕ to $\pi^{-1}(\Delta^*)$ gives a biholomorphic map

$$\pi^{-1}(\Delta^*) \xrightarrow{\phi} \pi'^{-1}(\Delta^*)$$

over Δ^*.

Thanks to Mumford's semi-stable reduction theorem, every degeneration can be made semi-stable after base change and modifications.

Definition 3.1.4. A semi-stable degeneration $\pi : \mathcal{Y} \to \Delta$ of $K3$-surfaces with trivial canonical bundle $K_Y \equiv 0$ is called a *Kulikov model* or a *good model*.

We have the following theorem of Kulikov and Persson-Pinkham [Persson & Pinkham, 1981] and [Kulikov, 1977]:

Theorem 3.1.5. *Let $\pi : \mathcal{Y} \to \Delta$ be a semi-stable degeneration of $K3$-surfaces such that all components of the special fibre are algebraic. Then there exists a modification $\pi' : \mathcal{Y} \to \Delta$ of $\pi : \mathcal{Y} \to \Delta$ which is a Kulikov model.*

Given a Kulikov model, Kulikov [Kulikov, 1977] and Persson-Pinkham [Persson & Pinkham, 1981] give a description of the cohomology of its special fibre in terms of the monodromy operator acting on cohomology.

Let $\pi : \mathcal{Y} \to \Delta$ be a degeneration of $K3$-surfaces, and let $\pi^* : \mathcal{Y}^* \to \Delta^*$ be the restriction to the punctured disk. Fix a smooth fibre $Y := \mathcal{Y}_t$, which is a $K3$-surface. Since π^* is a fibration, the fundamental group of Δ^* acts on the cohomology $H^2(Y, \mathbb{Z})$.

Definition 3.1.6. The map

$$T : H^2(Y, \mathbb{Z}) \longrightarrow H^2(Y, \mathbb{Z})$$

induced by the action of $\pi_1(\Delta^*)$ is called the *Picard-Lefschetz transformation*.

We have the following theorem of Landman [Landman, 2010]

Theorem 3.1.7.

- *T is quasi-unipotent, with index of unipotency at most 2. In other words, there is some k such that*

$$(T^k - I)^3 = 0.$$

- *If $\pi : \mathcal{Y} \to \Delta$ is semi-stable, then T is unipotent, that is $k = 1$.*

Therefore for a Kulikov model of a $K3$-surface we have that the Picard-Lefschetz transformation is unipotent. Moreover we can define the logarithm of T (in the semi-stable case):

Definition 3.1.8. The *Monodromy operator* N on $H^2(Y, \mathbb{Z})$ is defined as:

$$N := \log T = (T - I) - \frac{1}{2}(T - I)^2.$$

N is nilpotent, and the index of unipotency of T coincides with the index of nilpotency of N; in particular, $T = I$ if and only if $N = 0$.

The main theorem of this section is the classification theorem of Kulikov [Kulikov, 1977] and Persson-Pinkham [Persson & Pinkham, 1981] of the central fibre of a Kulikov model:

Theorem 3.1.9. *Let $\pi : \mathcal{Y} \to \Delta$ be a semi-stable degeneration of $K3$-surfaces with all components of the central fibre $\mathcal{Y}_0 = \pi^{-1}(0) = \bigcup V_i$ algebraic.*

Let $N = \log T : H^2(\mathcal{Y}_t, \mathbb{Z}) \to H^2(\mathcal{Y}_t, \mathbb{Z})$ be the monodromy operator. After birational modifications we may assume that $\pi : \mathcal{Y} \to \Delta$ is a Kulikov model. Then the central fibre $\mathcal{Y}_0$ is one of the following:

1. *(Type I) $\mathcal{Y}_0$ is a $K3$-surface and $N = 0$.*

2. *(type II) $\mathcal{Y}_0 = V_0 \cup V_1 \cdots V_r$, where V_0, V_r are smooth rational, and $V_1, \ldots, V_{r-1}$ are smooth elliptic ruled and $V_i \cap V_j \neq \emptyset$ if and only if $j = i \pm 1$. $V_i \cap V_j$ is then a smooth elliptic curve and a section of the ruling on V_i, if V_i is elliptic ruled. $N \neq 0$ but $N^2 = 0$.*

3. *(Type III) $\mathcal{Y}_0 = \cup V_i$, with each $\overline{V}_i$ smooth rational and all double curves are cycles of rational curves. The dual graph Γ is a triangulation of S^2. In this case $N^2 \neq 0$, but $N^3 = 0$.*

The proof of these results uses the Clemens-Schmid exact sequence. An account of this sequence is the paper [Morrison, 1984] in which as application we have the proof of the previous theorem.

3.2 Logarithmic Structures

Definition 3.2.1. A *monoid* is a commutative semi-group with a unit. A morphism of monoids is required to preserve the unit element. Denote by **Mon** the category of monoids.

Definition 3.2.2. Let X be a scheme. A *pre-log structure* on X is a sheaf of monoids M_X (on the étale or Zariski site of X) together with a morphism of sheaves of monoids: $\alpha : M_X \longrightarrow \mathcal{O}_X$, called the *structure morphism*, where we consider $\mathcal{O}_X$ a monoid with respect to the multiplication.

A pre-log structure is called a *log structure* if $\alpha^{-1}(\mathcal{O}_X^*) \simeq \mathcal{O}_X^*$ via α.

The pair (X, M_X) is called a *log scheme* and it will be denoted by $X^{\log}$.

We have a functor i from the category of log structure of X to the category of pre-log structure of X by sending a log structure M in X to itself considered as a pre-log structure $i(M)$. Vice-versa given a pre-log structure we can construct a log structure M^{ls} out of it in such a way that $(\ )^{ls}$ is left adjoint of i, so $j(M)$ is *universal.* (see [Kato, 1989]).

Remark 3.2.3. The category of schemes is a full subcategory of the category of log.schemes. Indeed, given a scheme X the trivial inclusion $\mathcal{O}_X^* \longrightarrow \mathcal{O}_X$ gives the trivial log structure on X, which is, in fact, an initial object in he category of log structure over X. Also we have the identity map $\mathcal{O}_X \longrightarrow \mathcal{O}_X$ which gives a different log structure on X, which is in fact a final object.

To clarify the action of this inclusion on morphisms we need the following definitions.

Definition 3.2.4. Let $f : X \to Y$ be a morphism of schemes. Given a log structure M_Y on Y we can define a log structure on X, called *the inverse image* of M_Y, to be the log structure associated to the pre-log structure

$$f^{-1}(M_Y) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X,$$

it is denoted by $f^*(M_Y)$.

Definition 3.2.5. A morphism of log.schemes $X^* \longrightarrow Y^*$ consists of a morphism of underlying schemes $f : X \to Y$ and a morphism $f^b : f^*M_Y \to M_X$ of log structure on X.

One of the main examples of interest for us is the following:

Example 3.2.6. Let X be a regular scheme (we can take for example a $K3$-surface over K or a proper model of it). Let D be a divisor of X. We can define a log structure M on X associated to the divisor D as

$$M(U) := \left\{ g \in \mathcal{O}_X(U) : g|_{U\setminus D} \in \mathcal{O}_X^*(U \setminus D) \right\}.$$

Let P be a monoid and R a ring. We denote by $R[P]$ the monoid algebra. The natural inclusion $P \longrightarrow R[P]$ induces a pre-log structure on $\mathrm{Spec}(R[P])$. The associated log structure is called the *canonical log structure* on $\mathrm{Spec}(R[P])$. The log structure on $\mathrm{Spec}(R[P])$ is the inverse image of the log structure on $\mathrm{Spec}(\mathbb{Z}[P])$ under the natural map $\mathrm{Spec}(R[P]) \longrightarrow \mathrm{Spec}(\mathbb{Z}[P])$.

Definition 3.2.7. Let (X, M_X) be a log scheme and P be a monoid. Denote by P_X the constant sheaf associated to P. A *chart* for M_X is a morphism $P_X \to M_X$ such that we have an isomorphism between the log structures

$$P^a \to M_X$$

where P^a is the log structure associated to the pre-log structure given by the map $P_X \to M_X \to \mathcal{O}_X$. Equivalently a chart of M_X is a morphism $X \to \mathrm{Spec}(\mathbb{Z}[P])$

of log structures, such that its morphism of log structures on X, $P_X \to M_X$, is an isomorphism.

Definition 3.2.8. Let $f : X \longrightarrow Y$ be a morphism of log schemes. Consider the constant sheaves P_X and Q_Y on X and Y associated to the monoids P and Q respectively. A *chart* for the morphism f is the data $(P_X \to M_X, Q_Y \to M_Y, Q \to P)$ such that:

- The maps $P_X \to M_X$ and $Q_Y \longrightarrow M_Y$ are charts of M_X and M_Y.
- We have a commutative diagram:

$$\begin{array}{ccc} Q_X & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ f^*M_Y & \longrightarrow & M_X \end{array}$$

where the top arrow is induced by the map $Q \to P$.

Remember that given a monoid P we can associate to P an abelian group (the Grothendieck group) denoted by P^{gp}. Explicitly we have that

$$P^{gp} = \{(p, q) \in P \times P : (p, q) \sim (r, s)\}$$

where we say that $(p, q) \sim (r, s)$ if there exists $t \in P$ such that $p + s + t = q + r + t$. It is a group with addition coordinate-wise and zero the class of (p, p).

We have a canonical map $P \to P^{pg}$ sending $q \to (p, e)$ where e is the neutral element of P. This group satisfies the universal property that any morphism of monoids from P to an abelian group G factors trough P^{gp} in a unique way.

Definition 3.2.9. A monoid P is called *integral* if the canonical map $P \to P^{gp}$ is injective. It is called *saturated* if it is integral and for any $p \in P^{gp}$, if $np \in P$ for some positive integer n, then $p \in P$.

Definition 3.2.10. A log scheme (X, M_X) is said to be *fine* if (étale) locally there is a chart $P \to M_X$ with P a finitely generated integral monoid.

The scheme (X, M_X) is *fine and saturated* (fs) if P is also saturated. Equivalently a log scheme is fs if for any geometric point $\tilde{x} \to X$ the monoid $M_{\tilde{x},X}$ is finitely generated and saturated.

If moreover $P \simeq N^r$ for some r, then we say that the log structure is *locally free*.

Definition 3.2.11. A morphism of log schemes $f : (X, M_X) \to (Y, M_Y)$ is called *strict*, if the morphism on log structures $f^* M_Y \to M_X$ is an isomorphism.

Definition 3.2.12. A morphism of log schemes $\imath : (X, M_X) \to (Y, M_Y)$ is called a *closed immersion* (resp. an exact closed immersion) if the underlying morphism of schemes $X \to Y$ is a closed immersion and $i^* M_Y \to M_X$ is surjective (resp. an isomorphism).

Definition 3.2.13. A morphism $f : X \to Y$ of fine log schemes is *log smooth* (respectively log étale) if étale locally (on X and Y) f admits a chart

$$(P_X \to M_X, Q_Y \to M_Y, Q \to P),$$

such that:

- The kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $Q^{gp} \to P^{gp}$ are finite groups of order invertible on X.

- The induced morphism of log schemes

$$(X, M_X) \longrightarrow (Y, M_Y) \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$$

is étale in the classical sense.

Proposition 3.2.14. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a morphism of log schemes. Consider the sheaves of log differentials $\Lambda^1_{Y/Z}$, $\Lambda^1_{X/Y}$ and $\Lambda^1_{X/Z}$. Then we have the following:*

1. *The sequence $f^*\Lambda^1_{Y/Z} \longrightarrow \Lambda^1_{X/Z} \longrightarrow \Lambda^1_{X/Y} \longrightarrow 0$ is exact.*
2. *If f is log smooth, then $\Lambda^1_{X/Y}$ is a locally free $\mathcal{O}_X$-module. Moreover the sequence*

$$0 \longrightarrow f^*\Lambda^1_{Y/Z} \longrightarrow \Lambda^1_{X/Y} \longrightarrow \Lambda^1_{X/Z} \longrightarrow 0$$

   is exact.

3. *If $g \circ f$ is log smooth and the sequence in part (2) is exact and splits locally, then f is log smooth.*

Proof. [Ogus, 2006, sec 2.3]. □

3.3 Simple Normal Crossing Log $K3$-Surfaces

We are assuming that all schemes are noetherian and that all morphisms are of finite type.

Definition 3.3.1. Let k be a field[1]. A *normal crossing variety* Y/k over k is a geometrically connected scheme Y over k, whose irreducible components are geo-

[1] We are interested in the case k the residue field of a p-adic field K. So in particular we can consider k perfect (or algebraically closed) of characteristic p.

metrically irreducible and of the same dimension d, and such that Y is isomorphic to $\operatorname{Spec} k[x_0, \dots, x_d]/(x_0 \cdots x_r))$ étale locally over Y, where $0 \le r \le d$ is a natural number that depends on étale neighborhoods.

We denote by Y_{sing} the *singular locus* of Y. So $Y_{\text{sing}} := D_1 \cup D_2 \cup \cdots \cup D_m$ is a disjoint union of the m connected components of Y_{sing}. We assume that each D_i is geometrically connected.

Definition 3.3.2. A scheme Z over k is d-semistable if there is an isomorphism $\operatorname{Ext}^1_{\mathcal{O}_Z}(\Omega_{Z/k}, \mathcal{O}_Z) \simeq \mathcal{O}_{\text{sing}}$

Definition 3.3.3. By a *log point* we mean the scheme $\operatorname{Spec} k$ with the log structure induced by the morphism

$$\mathbb{N}^m \longrightarrow k; \quad e_i \mapsto 0; \tag{3.1}$$

where e_i stands for the canonical generator of $\mathbb{N}^m$. Here m is the number of geometrically connected components of Y_{sing}.

Note that for every $1 \le i \le m$ we have a log structure on

$$\operatorname{Spec} k[x_1, \dots, x_d]/(x_1 \cdots x_r),$$

which is the one associated to the pre-log structure given by the map

$$\mathbb{N}^{i-1} \oplus \mathbb{N}^{r+1} \oplus \mathbb{N}^{m-i} \longrightarrow \operatorname{Spec} k[x_1, \dots, x_d]/(x_1 \cdots x_r)$$

such that for the basic elements $e_j \in \mathbb{N}^{m+r}$

$$e_j \mapsto \begin{cases} 0 & e_j \in \mathbb{N}^{i-1} \\ x_j & e_j \in \mathbb{N}^{r+1} \\ 0 & e_j \in \mathbb{N}^{m-i}. \end{cases} \tag{3.2}$$

Note that this log structure commutes with the log structure over the $\operatorname{Spec} k^{\log}$ since we have a commutative diagram

$$\begin{array}{ccc} \mathbb{N}^{i-1} \oplus \mathbb{N} \oplus \mathbb{N}^{m-i} & \longrightarrow & k \\ \downarrow & & \downarrow \\ \mathbb{N}^{i-1} \oplus \mathbb{N}^{r+1} \oplus \mathbb{N}^{m-i} & \longrightarrow & \operatorname{Spec} k[x_1, \ldots, x_d]/(x_1 \cdots x_r) \end{array}$$

where the upper horizontal morphism sends $e_j \mapsto 0$ for $e_j \in \mathbb{N}^m$ and the left vertical map is $id \oplus \text{diagonal} \oplus id$. Let Y be a proper d-semistable normal crossing variety over k. We endow Y with the log structure given by:

1. étale locally on the neighborhood of a smooth point of Y, the log structure is given by the pull back of the log structure of $\operatorname{Spec} k^{\log}$;

2. étale locally on the neighborhood of a point of D_i, the log structure is the pull back of the log structure of $\operatorname{Spec} k[x_1, \ldots, x_d]/(x_1 \cdots x_r)$ described above.

Definition 3.3.4. We denote $Y^{log}/\operatorname{Spec} k^{log}$ the log scheme described above and we call it a *normal crossing log variety* (NCL).

We say that the NCL variety $Y^{log}/\operatorname{Spec} k^{log}$ is *simple* if the underlying scheme Y is a simple normal crossing variety, where simple means that all its irreducible components are smooth and geometrically irreducible (SNCL).

Now we follow the paper of Kato, F. 1996. Log Smooth Deformation Theory. Tohoku Mathematical Journal [Kato, 1996].

Let R be a fixed complete noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. We are mainly interested in the case $R = W(k)$. Let Q be a fine and saturated (fs) non torsion monoid. Let $R[[Q]]$ be the completion of the monoid ring $R[Q]$ with respect to the maximal ideal $\mathfrak{m}+R[Q\backslash 1]$. If the monoid is $\mathbb{N}$, then $R[[Q]]$ is isomorphic to $R[[t]]$ as a local R-algebra.

Let $C_{R[[Q]]}$ be the category Artinian local $R[[Q]]$-algebras with residue field k, and $\widehat{C}_{R[[Q]]}$ be the category of pro-objets of $C_{R[[Q]]}$.

Definition 3.3.5. For an object A of $C_{R[[Q]]}$, we endow $\operatorname{Spec} A$ with a log structure whose chart is $Q \to A$. We denote this log scheme by $\operatorname{Spec} A^{log}$. This data is equivalent to the following: A is a R-algebra and there is a global chart

$$\operatorname{Spec} A^{log} \longrightarrow (\operatorname{Spec} \mathbb{Z}[[Q]], Q).$$

Let $\beta : \operatorname{Spec} k^{log} \to (\operatorname{Spec} \mathbb{Z}[[Q]], Q)$ be a morphism of log.sch induced by a morphism

$$Q \setminus \{0\} \to k; \quad q \mapsto 0.$$

Let Y^{log} be a fs log scheme that is log smooth and integral over $\operatorname{Spec} k^{log}$.

Definition 3.3.6. An fs log.sch Y_A^{log} over $\operatorname{Spec} A^{log}$ is called a a charted deformation of Y^{log} over $\operatorname{Spec} K^{log}$, if Y_A^{log} is a log smooth scheme over $\operatorname{Spec} A^{log}$ and

$$Y^{log} \simeq Y_A^{log} \times_{\operatorname{Spec} A^{log}} \operatorname{Spec} k^{log}$$

in the category of the fs log schemes.

We have that Y_A^{log} is automatically integral over $\operatorname{Spec} A^{log}$.

The charted deformations of $Y^{log}/\operatorname{Spec} A^{log}$ define a functor

$$D_{(Y^{log},\beta)} \longrightarrow (Sets).$$

Then we have the following:

Proposition 3.3.7. *If Y is proper, then the functor $D_{(Y^{log},\beta)}$ has a hull.*

The proof is in [Kato, 1996].

In our situation of interest, that is when we have a semi-stable model of a $K3$ surface X_K over a local field K (with residue field k), that is a diagram:

$$\begin{array}{ccc} X_K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} \mathcal{O}_K \end{array}$$

with special fibre $\bar{X} = X \otimes k$. We set $Y = \bar{X}$. Since Y has a smoothing, that is, Y lifts to a smooth $K3$-surface, then it is d-semi-stable [Friedman, 1983] and [Olsson, 2004] then we can endow it with the log structure studied in this chapter. We can take $R = W := W(k)$ as the ring of Witt vectors with coefficients in k. Then proposition 3.3.7 is telling us that the deformation functor of the special fibre has a hull.

Definition 3.3.8. Let X^{log}/k^{log} be a NCL variety of pure dimension 2. We say that X^{log}/k^{log} is a *normal crossing log $K3$-surface* if the underlying scheme X is a proper scheme over $\operatorname{Spec} k$ and

1. $H^1(X, \mathcal{O}_X) = 0$

2. $\Lambda^2_{X/k} \simeq \mathcal{O}_X$.

Here Λ^1_X is the sheaf of logarithmic differentials as in [Kato, 1994a, sec. 5].

Definition 3.3.9. Let X be a proper surface over a field k. Let $\overline{k}$ be an algebraic closure of k. X is a combinatorial $K3$ surface if it satisfies one of the following conditions:

Type I X is a smooth $K3$ surface over k.

Type II $X \otimes_k \overline{k} = X_1 \cup X_2 \cup \cdots \cup X_N$ is a chain of smooth surfaces with X_1 and X_N rational and the other elliptic ruled and two double curves on each of them are rulings. The dual graph of $X \otimes_K \overline{k}$ is a segment with end points X_1 and X_2.

Type III $X \otimes_K \overline{k} = X_1 \cup X_2 \cup \cdots \cup X_N$ is a chain of smooth surfaces and every X_i is rational, and the double curves on X_i are rational and form a cycle on X_i.

Under this conditions Nakkajima proves that $H^1(X, \mathcal{O}_X) = 0$ [Nakkajima, 2000].

By the previous section, X has a log structure whose charts are given by its local normal crossing components and $\Lambda^2_{X/k} \simeq \mathcal{O}_X$.

Theorem 3.3.10. *Let X be a combinatorial Type II or Type III $K3$ surface over a field k. Then $\Gamma(X, \Lambda^1_{X/k}) = 0$.*

Proposition 3.3.11. *Let $X^{log}/\operatorname{Spec} k^{log}$ be SNCL $K3$ surface. Then $X \otimes_k \overline{k}$ is a combinatorial $K3$ surface.*

Definition 3.3.12. We say that a SNCL $K3$ surface is of type (I, II or III) if X is of the respective type.

Theorem 3.3.13. *Let k be an algebraically closed field of characteristic $p > 0$. Let X^{log} be a* ***projective*** *SNCL $K3$ surface over $\operatorname{Spec} k^{log}$. Then there exists a log smooth* *family* $\mathfrak{X}^{log}$ *over* $\operatorname{Spec} W[[u_1, \ldots, u_m]]^{log}$ *which is a charter deformation of* X^{log} *(Automatically* $\Lambda^2_{\mathfrak{X}/W[[u_1,\ldots,u_m]]}$ *is trivial). Where* m *is the number of geometrically connected components of* X_{sing}*. Moreover, the deformation functor has the information of the deformations of the log structure associated to the irreducible components* $X_1, \ldots, X_N$ *of* $\bar{X}$*, in such a way that there exist closed subschemes* $\mathfrak{X}_1, \ldots, \mathfrak{X}_N$*, deformations of* $X_1, \ldots X_N$ *respectively, and the log structure on* $\mathfrak{X}$ *is the one associated with* $\mathfrak{X}_1, \ldots, \mathfrak{X}_N$ *as on page 23.*

If $\bar{X}$ is smooth, that is of type I, then this is the result of Deligne [Deligne & Illusie, 1981]. If $\bar{X}^{\log}$ is of type I type III, then it is [Nakkajima, 2000, prop. 5.9 and prop. 6.8].

Nakkajima also give the following corollaries:

Corollary 3.3.14. *Let* X *be a projective SNCL* $K3$ *surface over* k*. The following holds:*

- *There exists a projective semi-stable family* $\mathfrak{Y}$ *over* $\operatorname{Spec} W$ *whose special fibre is* X*.*
- *There exists a projective semi-stable family* $\mathfrak{Y}$ *over* $\operatorname{Spec} k[[t]]$ *whose special fibre is* X*.*

Corollary 3.3.15.

- *Let* K_0 *be the fraction field of* W *(resp.* $k[[t]]$*). The generic fibre* X_{K_0} *of* $\mathfrak{Y}$ *is a smooth* $K3$ *surface.*
- *Let* k *be a field of characteristic* $p > 0$ *and let* X^{log} *be a projective SNCL* $K3$ *surface over* $\operatorname{Spec} k^{log}$*. Then* $\dim_k H^1(X, \Lambda^1_{X/k}) = 20$*.*

In the argument for the proof, Nakkajima considers the family

$$\mathfrak{X} \to \operatorname{Spec} W[[u_1, \ldots, u_m]]$$

and specializes $W[[u_1, \dots, u_m]] \to W$ by sending $u_i \to p$ getting the desired $\mathcal{Y} \to \mathrm{Spec}(W)$. Similarly he considers the map

$$W[[u_1, \dots, u_m]] \to W[[t]] \to W[[t]]/p = k[[t]]$$

and sends $u_i \to t$ and then reduces modulo p to get the $\mathcal{Y} \to \mathrm{Spec}(k[[t]])$.

These results are Nakkajima's generalization, for the logarithmic case, of the results of Deligne [Deligne & Illusie, 1981] and Friedman [Friedman, 1983].

Now we have the following proposition, which is the main result of this chapter:

Proposition 3.3.16. *Let p be a prime number and consider K be a finite extension of $K_0 = W(k)[1/p]$ with k algebraically closed. Let $X_K \to \mathrm{Spec}(K)$ be a semi-stable $K3$ surface with semi-stable model $X \to \mathrm{Spec}(\mathcal{O}_K)$ and projective SNCL (therefore combinatorial) special fibre $\bar{X} = X \otimes k \to \mathrm{Spec}(k)$.*

Then there exists a deformation $\mathcal{X} \to S := \mathrm{Spec}(W[[t]])$ of $\bar{X}$ satisfying the following:

- *We denote by 0 the point of $S \otimes_W K_0$ corresponding to the maximal ideal $t(W[[t]] \otimes_W K_0)$, then $(\mathcal{X} \otimes_W K_0)_0$ is a combinatorial $K3$-surface over K_0 of the same type of $\bar{X}$.*
- *For every point $x \in S \otimes_W K_0$, with $x \neq 0$, then $(\mathcal{X} \otimes_W K_0)_x$ is a smooth $K3$-surface over $k(x)$.*

Proof. By theorem (3.3.13) there exists a deformation of $\bar{X}$:

$$\mathfrak{X}^{\log} \longrightarrow \mathfrak{S} := \mathrm{Spec}(W[[u_1, u_2, \dots, u_m]])^{\log}.$$

Let $\mathfrak{S} \otimes_W K_0$ be the scheme $\mathrm{Spec}(W[[u_1, u_2, \dots, u_m]] \otimes_W K_0)^{\log}$ and let us denote by:

$$(\mathfrak{S} \otimes_W K_0)^{\text{sing}} := \{x \in (\mathfrak{S} \otimes_W K_0) | (\mathfrak{X} \otimes_W K_0)_x \text{ is singular}\}.$$

Denote by $\mathfrak{S}^{\text{sing}}$ the Zariski closure of $(\mathfrak{S} \otimes_W K_0)^{\text{sing}}$ in $\mathfrak{S}$.

Since being singular is a closed condition and $(\mathfrak{S} \otimes_W K_0)^{\text{sing}} \subset \mathfrak{S} \otimes_W K_0$ is a proper contention, we have that $\mathfrak{S}^{\text{sing}} \subset \mathfrak{S}$ is a proper closed immersion and therefore

$$0 \leq \dim \mathfrak{S}^{\text{sing}} \leq \dim \mathfrak{S} - 1.$$

Let x_0 a closed point of $(\mathfrak{S} \otimes_W K_0)^{\text{sing}}$ such that $(\mathcal{X} \otimes_W K_0)_{x_0}$ is a $K3$-surface over K_0 of the same type of $\bar{X}$, and let y_0 be a closed point of $\mathfrak{S}^{\text{sing}}$ extending x_0.

Let C be a smooth curve in $\mathfrak{S}$ containing y_0 and normal to $\mathfrak{S}^{\text{sing}}$. Let

$$\widehat{\mathcal{O}}_{C,y_0} \simeq W[[t]]$$

denote the completion of the local ring of C at y_0 with respect to the maximal ideal of y_0. So we have a natural morphism $S := \text{Spec}(W[[t]]) \to \mathfrak{S}$. Let $\mathcal{X} \to S$ be the pull back of $\mathfrak{X} \to \mathfrak{S}$ with respect to $S \to \mathfrak{S}$. Then $\mathcal{X} \to S$ satisfies the desired properties. □

Remark 3.3.17. If X is the minimal semi-stable model for X_K (which there exists for $p > 3$ [Kawamata, 1993, 1998]), then its special fibre $\bar{X}$ is automatically a SNCL $K3$-surface [Maulik, 2012, sec 4] and [Nakkajima, 2000]. Therefore, if X is the minimal semi-stable model of X_K the previews proposition follows without assuming that its special fibre $\bar{X}$ is a SNCL $K3$-surface.

Chapter 4

Comparison Isomorphisms for Logarithmic K3-Surfaces

4.1 p-Adic Hodge Theory

4.1.1 Witt Vectors

Even if we have been using Witt vectors before, I would like to give a fast review of them, that will be useful to recall the construction of the rings of periods.

Let R be a perfect ring of characteristic p (for example our residue field k).

Definition 4.1.1. A *strict* p*-ring with respect to* R is a ring A (as always commutative and with one) such that p is not nilpotent and A is complete and separated with respect to the p-adic topology with residue ring $A/pA = R$.

The ring of Witt vectors with coefficients in R is a strict p-ring with respect to R, and since R is perfect, it is possible to construct at least one strict p-ring that in fact is unique, up to unique isomorphism. This ring is the ring of Witt vectors $W(R)$

over R. Moreover by uniqueness W is functorial in R, that is, if we have a morphism $\phi : R \to S$ then it lifts to a map $W(\phi) : W(R) \longrightarrow W(S)$. In particular we have a lift of the Frobenius automorphism of R, also called the Frobenius automorphism.

For example, if $R = \mathbb{F}_p$, then $W(\mathbb{F}_p) = \mathbb{Z}_p$. In general if $\mathbb{F}$ is a finite field, then $W(R)$ is the ring of integers of the unique unramified extension of $\mathbb{Q}_p$ whose residue field is R. As a particular case, we have that if K is a finite extension of $\mathbb{Q}_p$ (as in our case of study) and $k = \mathcal{O}_K/\pi\mathcal{O}_K$ is its residue field, then $\mathrm{Frac}(W(k)) = K_0$ is the maximal unramified extension of $\mathbb{Q}_p$ in K. Another important example is when $R = \overline{\mathbb{F}}_p$ the algebraic closure of a finite field; in this case $W(R) = \mathcal{O}_{\widehat{\mathbb{Q}_p^{unr}}}$. Now we want to understand the ring structure of $W(R)$.

For $x = x_0 \in R$ and for every n, choose a lifting $\widetilde{x}_n \in W(R)$ whose image in R is $x^{p^{-n}}$. The sequence $\{\widetilde{x}_n\}$ converges in $W(R)$.

Definition 4.1.2. We define the *Teichmüler map*

$$[\ ] : R \longrightarrow W(R); \quad x \mapsto [x] := \lim_n \widetilde{x}_n.$$

The elements on the image of this map are called the *Teichmüler elements.*

In fact the Teichmüler map is multiplicative and it is a section of the natural projection. It turns out that the Teichmüler elements allow us to write any element $x \in W(R)$ in a unique way as:

$$x = \sum_{n=0}^{\infty} p^n [x_n], \quad x_n \in R.$$

Moreover, given two elements $x, y \in A$ we have that

$$x + y = \sum_{n=0}^{\infty} p^n [S_n(x_n, y_n)] \quad \text{and} \quad xy = \sum_{n=0}^{\infty} p^n [P_n(x_n, y_n)]$$

where S_n, P_n are polynomials in $\mathbb{Z}[X_i^{p^{-n}}, Y_i^{p^{-n}}]$.

Remark 4.1.3. If R is not perfect, it is still possible to have strict p-rings over R, however we do not have uniqueness.

The standard reference for the proofs, properties and construction of Witt vectors is the book of Serre [Serre, 1979].

Definition 4.1.4. A p-adic field is a field K of characteristic 0 which is complete with respect to a fixed discrete valuation that has a prefect residue field k of characteristic p.

Given a p-adic field K, we denote by $\mathcal{O}_K$ its valuation ring, and we fix once and for all a uniformizer $\pi \in \mathcal{O}_K$. Finally denote by $\mathbb{C}_K = \widehat{\overline{K}}$. In case $k \subset \overline{\mathbb{F}}_p$ we have that $\mathbb{C}_K = \mathbb{C}_p$, the field of complex p-adic numbers, that is $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$.

Later we will need to assume that our p-adic field K has algebraically closed residue field. That is $k = \overline{k}$. We fix once and for all an algebraic closure $\overline{K}$ of K and we denote by $G_K := \mathrm{Gal}(\overline{K}/K)$ its absolute Galois group.

Let $\mu_{p^\infty} = \varprojlim_n \mu_{p^n}$ where $\mu_{p^n} := \left\{x \in \overline{K} : x^{p^n} = 1\right\}$ with morphisms for every n, m such that $n > m$:

$$\phi_{m,n} : \mu_{p^m} \longrightarrow \mu_{p^n}; \quad x \to x^{p^{n-m}}.$$

Fix a primitive element $\xi \in \mu_{p^\infty}$ that is a sequence of primitive elements

$$\xi = (1, \xi^{(1)}, \ldots, \xi^{(n)}, \ldots)$$

such that $(\xi^{(n+1)})^p = \xi^{(n)}$.

We have the following chain of fields:

$$K_0 \subset K \subset K_n := K(\mu_{p^n}) \subset K_\infty := K(\mu_{p^\infty}) \subset \overline{K} = \overline{K}_0 \subset \mathbb{C}_K.$$

If we denote by $\chi : G_K \longrightarrow \mathbb{Z}_p^*$ the cyclotomic character of G_K, that is the homomorphism of groups defined by $\chi(\sigma) = \xi^{\chi(\sigma)}$ for every $\sigma \in G_K$, we have that the kernel of χ is exactly $H_K := \mathrm{Gal}(\overline{K}/K_\infty)$ and therefore χ identifies $\Gamma_K := \mathrm{Gal}(K_\infty/K) = G_K/H_K$ with the image of χ which is an open subgroup of $\mathbb{Z}_p^*$.

Denote by $\mathcal{O} := W[[Z]]$. Consider the W-algebra homomorphism

$$\mathcal{O} \longrightarrow \mathcal{O}_K; \quad Z \to \pi.$$

Finally denote by $P_\pi(Z)$ the minimal polynomial of π.

4.1.2 p-Adic Representations

The main examples of p-adic representations (for us) are the p-adic étale cohomology groups of a $K3$-surface. Indeed, if X is a scheme of finite type over a field K, we know that the étale cohomlogy groups $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ are finitely generated $\mathbb{Q}_p$-vector spaces. Moreover they admit a natural action of G_K because we have a natural action of G_K on $X_{\overline{K}} := X \times_K \mathrm{Spec}(\overline{K})$ and then by functoriality it extends to an action on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Remark that since G_K is a profinite group, it is in particular a topological group and this action is continuous.

Definition 4.1.5. A p-adic representation of G_K of dimension d is a continuous group homomorphism $\rho : G_K \longrightarrow GL(V)$ for a finite dimensional (of dimension d) $\mathbb{Q}_p$-vector space V.

The collection of p-adic representations form a category whose morphisms are given by $\mathbb{Q}_p$-linear and equivariant G_K-maps. We denote by $\mathbf{Rep}_{\mathbb{Q}_p}(G_K)$ the category of p-adic representations. This category is an abelian category with tensor products.

Example 4.1.6. An important family of p-adic representations of dimension one are the so called Tate twists of $\mathbb{Q}_p$. Precisely, let $r \in \mathbb{Z}$ and define $\mathbb{Q}_p(r)$ to be the one dimensional $\mathbb{Q}_p$-vector space $\mathbb{Q}_p e_r$ with action of G_K given by twisting by the r-power of the cyclotomic character, that is $\sigma(e_r) = \chi(\sigma)^r e_r$ for every $\sigma \in G_K$. This is called the r-th Tate twist of $\mathbb{Q}_p$. Moreover if V is another p-adic representation, we can construct a new p-adic representation by twisting V:

$$V(r) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r).$$

This is again a p-adic representation of dimension $\dim V$.

Example 4.1.7. If A_K is an abelian variety over K, then the Tate module

$$V_p := T_p A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a p-adic representation of dimension $d = 2 \dim A$. For example, if A_K is an elliptic curve, then V_p is a p-adic representation of dimension 2.

Example 4.1.8. If X_K is a $K3$-surface over K, then $V = H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a p-adic representation of dimension 22.

4.1.3 Rings of Periods

In order to study p-adic representations, Fontaine et al. constructed certain rings that are known as *rings of periods*. They are topological $\mathbb{Q}_p$-algebras B, together with an action of G_K and depending on B, some additional structures like filtrations, Frobenius, monodromy operator, etc. He also observed that the B^{G_K}-modules $D_B(B)$ defined as

$$D_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_K}$$

reveal important properties of the p-adic representation V.

The $\mathbb{Q}_p$-algebra B is G_K-regular if for any $b \in B$ such that the line $\mathbb{Q}_p b$ is G_K-stable, we have that $b \in B^*$. Note that if B is G_K regular, then for every $b \neq 0$ in G^{G_K} the line $\mathbb{Q}_p b$ is G_K-stable, therefore for every $b \in B^{G_K}$, $b \in B^*$ and since b^{-1} is also in B^{G_K} we have that B^{G_K} is a field.

If B is G_K regular, then we have that $\dim_{B^{G_K}} D_B(V) \leq \dim_{\mathbb{Q}_p} V$ [Brinon & Conrad, 2008].

Definition 4.1.9. A p-adic representation V is B-admissible if

$$\dim_{B^{G_K}} D_B(V) = \dim_{\mathbb{Q}_p} V.$$

Our final objective will be to construct B_{cris} and $B_{\log}$ in this section. In order to define B_{cris} we first have to talk about another ring of periods: B_{dR}.

4.1.3.1 The Ring of Periods B_{dR}

Let R be the set of sequences $x = (x^{(0)}, x^{(1)}, \dots, x^{(n)}, \dots)$ of elements in $\mathcal{O}_{\mathbb{C}_K}$ such that $(x^{(n+1)})^p = x^{(n)}$. We endow R with a structure of a ring with product $*$ and sum

+ laws defined as:

$$x * y = (x^{(n)}y^{(n)})_{n\in\mathbb{N}} \quad \text{and} \quad x + y = (s^{(n)})_{n\in\mathbb{N}}$$

where

$$s^{(n)} = \lim_{m\to\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$$

which converge in $\mathcal{O}_{\mathbb{C}_K}$. With these operations R is a commutative domain whose unit element is $1 = (1, 1, \ldots, 1)$. This ring is usually denoted by $\widetilde{\mathbf{E}}^+ = \widetilde{\mathbf{E}}^+_{\mathbb{C}_\mathbf{K}}$. Also note that

$$p * 1 = \lim_{m\to+\infty} \underbrace{(1 + \ldots + 1)}_{p-\text{times}}^{p^m} = 0,$$

thus, R is of characteristic p. The Frobenius $x = (x^{(n)}) \mapsto x^p = ((x^{(n)})^p)$ on R is an isomorphism, and so, R is perfect ring.

Even more, we have a natural action of $\mathrm{Gal}(\overline{K}/K)$ on R trough its action on $\mathcal{O}_{\mathbb{C}_K}$ and a valuation defined as: $\mathrm{val}(x) = \mathrm{val}(x^{(0)})$. With the topology induced by the valuation, R is separated and complete with a residue field $R/\{x|\mathrm{val}(x) > 0\} \cong \overline{k}$.

Since $R = \widetilde{\mathbf{E}}^+$ is a perfect ring we can consider the Witt vectors $A_{\text{inf}} := W(R)$ with coefficients in R. Every element of A_{inf} can be written in a unique way as:

$$\sum_{n=0}^{+\infty} p^n[x_n]$$

where $x_n \in R$ and $[x_n]$ is its multiplicative representative or Teichmüler representative in $W(R) = A_{\text{inf}}$. We have a surjection

$$\theta : A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_K}; \qquad \sum_{n=0}^{+\infty} p^n[x_n] \mapsto \sum_{n=0}^{+\infty} p^n x_n^{(0)}.$$

Remark that $\theta([\overline{\pi}]) = \pi$ and $\theta([\overline{p}]) = p$ and that $\ker\theta$ is a principal ideal generated by $p - [\overline{p}]$, where $\overline{p} := (p^{(n)}) \in R$ is such that $p^{(0)} = p$, also $\overline{\pi} \in R$ such that if $\overline{\pi} = (\pi^{(n)})$ then $\pi^{(0)} = \pi$. Finally note that also the element ξ is in R and that also $\theta(1 - [\xi]) = 0$.

The ring A_{inf} is complete for the topology defined by the ideal $(p, \ker(\theta)) = (p, [\underline{p}])$.

Definition 4.1.10. The ring B_{dR}^{+} is the completion of $A_{\text{inf}}[1/p]$ with respect to the ideal $\ker(\theta) = (p - [\overline{p}])$.

We extend the surjection $\theta : A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_K}$ to $\theta : B_{\text{dR}}^{+} \to \mathbb{C}_K$. We have that B_{dR}^{+} is a complete ring with a discrete valuation and maximal ideal $\ker\theta = (p - [\overline{p}])B_{\text{dR}}^{+}$ and residue field

$$B_{\text{dR}}^{+}/\ker(\theta) \cong \mathbb{C}_K.$$

We can consider several topologies in B_{dR}^{+}. We endow B_{dR}^{+} with the topology so that $p^m W(R) + (\ker\theta)^k$ forms a base of neighbourhoods of 0, where $(m,k) \in \mathbb{N}^2$. B_{dR}^{+} is complete and separated for this topology.

There exists a natural and continuous action of G_K in B_{dR}^{+} through the action on R and this action commutes with θ.

$\overline{\mathbb{Q}}_p$ is identified canonically with the algebraic closure of $\mathbb{Q}_p$ in B_{dR}^{+} and the following diagram commutes:

$$\begin{array}{ccc} \overline{\mathbb{Q}}_p & \longrightarrow & B_{\text{dR}}^{+} \\ \| & & \downarrow{\scriptstyle\theta} \\ \overline{\mathbb{Q}}_p & \longrightarrow & \mathbb{C}_K. \end{array}$$

In fact, in the case where we give $\overline{\mathbb{Q}}_p$ the topology induced by B_{dR}^{+} (which is not p-adic), Colmez proved that B_{dR}^{+} is the completion of $\overline{\mathbb{Q}}_p$ for this topology, and thus $\overline{\mathbb{Q}}_p$ is dense in B_{dR}^{+}.

Definition 4.1.11. We define B_{dR} as the fraction field of B_{dR}^+, that is

$$B_{\mathrm{dR}} := \mathrm{Frac}(B_{\mathrm{dR}}^+).$$

We extend naturally θ to B_{dR} and we give to it a filtration defined as $\mathrm{Fil}^i B_{\mathrm{dR}} := (\ker(\theta))^i$.

Remark that if $x \in \mathrm{Fil}^1(B_{\mathrm{dR}}) = \ker(\theta)$, is non zero, then $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[x^{-1}]$.

Since $\theta(1-[\xi]) = 0$ the element $1-[\xi]$ is small with respect to the topology on B_{dR}^+ and the logarithm of this element converges in B_{dR}^+, that is, there exists an element $t \in B_{\mathrm{dR}}^+$ such that

$$t = \log([\xi]) := -\sum_{n=1}^{\infty} \frac{(1-[\xi])^n}{n}.$$

If $\sigma \in G_K$ then

$$\sigma * t = \sigma(\log([\xi])) = \log([\xi^{\chi(\sigma)}]) = \chi(\sigma)t.$$

Moreover since $t \in Fil^1(B_{\mathrm{dR}})$ we also have that $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$ and the filtration is such that $\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B_{\mathrm{dR}}^+$.

The field B_{dR} satisfies that $B_{\mathrm{dR}}^{G_K} = K$.

Definition 4.1.12. We say that a p-adic representation V is de Rham if V is B_{dR}-admissible, that is if $\dim_K D_{\mathrm{dR}}(V) = \dim V$ where $D_{\mathrm{dR}}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$.

4.1.3.2 The Ring of Periods $B_{\mathbf{cris}}$

We recall the definition of B_{cris}.

Remember that $\pi \in \mathcal{O}_K$ is our fixed uniformizer for K.

A_{cris} is the p-adic completion of the divided power envelope of A_{inf} with respect to the ideal generated by p and $\ker(\theta)$. We endow A_{cris} with the p-adic topology and the divided power filtration.

Remember that $t := \log([\xi])$.

Definition 4.1.13. We define B_{cris} as the ring:

$$B_{\text{cris}} := A_{\text{cris}}[1/t]$$

with the inductive limit topology and filtration given by:

$$\operatorname{Fil}^i B_{\text{cris}} := \sum_{m \in \mathbb{N}} t^{-m} \operatorname{Fil}^{m+i} A_{\text{cris}}.$$

We have that $B_{\text{cris}}^{G_K} = K_0$. B_{cris} has a Frobenius ϕ compatible with the Frobenius of W and such that $\phi(t) = pt$.

Definition 4.1.14. A p-adic representation V is *crystalline*, if it is B_{cris}-admissible, that is if $\dim_{K_0} D_{\text{cris}}(V) = \dim_{\mathbb{Q}_p} V$, where

$$D_{\text{cris}}(V) := (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

So we have that B_{cris} is an algebra over K_0 which is a subring of B_{dR}, G_K-stable.

4.1.3.3 The Ring of Periods B_{st}

Definition 4.1.15. We define B_{st} as the ring of polynomials $B_{\text{cris}}[Y]$ on the variable Y such that:

- We extend the Frobenius ϕ of B_{cris} to B_{st} by letting $\phi(Y) = Y^p$.

- We extend the action of G_K on B_{cris} by

$$\sigma * Y = Y + c(\sigma)t : \quad \text{for } \sigma \in G_K$$

  where $c(\sigma)$ is defined by the formula $\sigma(p^{1/p^n}) = p^{1/p^n}(\xi^{(n)})^{c(\sigma)}$.

- We define a *Monodromy* operator on it as $N_{st} := -d/dY$.

Definition 4.1.16. A p-adic representation V is *semistable* if it is B_{st}-admissible.

We have then that B_{st} is a K_0-algebra with an action of G_K and containing B_{cris}. Moreover we have that $B_{\text{st}}^{G_K} = K_0$ and that $B_{\text{st}}^{N_{st}=0} = B_{\text{cris}}$.

4.1.3.4 The Ring of Periods $B_{\log}$

We denoted by $\mathcal{O} = W[[Z]]$. We denote by $\mathcal{O}_{\text{cris}}$ the p-adic completion of the divided power envelope of $\mathcal{O}$ with respect to the ideal $(p, P_\pi(Z))$, where $P_\pi(Z)$ is the minimal polynomial of π with coefficients on W. We extend the Frobenius of W to $\mathcal{O}$ by letting it act on Z as $Z \mapsto Z^p$ and the usual Frobenius on W. Finally let $\omega^1_{\text{cris}/W} \simeq \mathcal{O}_{\text{cris}} \frac{dZ}{Z}$ be the continuous log 1-differential forms of $\mathcal{O}_{\text{cris}}$ relative to W.

Definition 4.1.17. Define $A_{\log}$ as the p-adic completion of the log divided power envelope of the morphism ring $A_{\text{inf}} \otimes_W \mathcal{O}$ with respect to the kernel of the morphism

$$\theta \otimes \theta_{\mathcal{O}} : A_{\text{inf}} \otimes_W \mathcal{O} \longrightarrow \mathcal{O}_{\mathbb{C}_K}.$$

Consider the element $u := \frac{[\pi]}{Z}$. Then we have that $A_{\log}$ is isomorphic to the p-adic completion $A_{\text{cris}}\{\langle V\rangle\}$ of the divided power polynomial ring over A_{cris} in the variable

V by a morphism:

$$A_{\text{cris}}\{\langle V\rangle\} \longrightarrow A_{\log}; \quad V \mapsto \frac{[\overline{\pi}]}{Z} - 1 = u - 1.$$

Then $A_{\log} \simeq A_{\text{cris}}\{\langle u-1\rangle\}$.

We endow $A_{\log}$ with the p-adic topology and the divided power filtration.

Definition 4.1.18. Remember we used t to denote $\log([\xi])$. We define the ring $B_{\log}$ as the ring

$$B_{\log} := A_{\log}[t^{-1}]$$

with the inductive limit topology and filtration defined by

$$\operatorname{Fil}^n B_{\log} := \sum_{m\in\mathbb{N}} \operatorname{Fil}^{n+m} A_{\log} t^{-m}.$$

We have a Frobenius on $A_{\log}$ that extends the Frobenius on A_{cris} by letting $u \mapsto u^p$ and we extend it to $B_{\log}$ by letting $t \mapsto pt$.

We have a continuous action on $B_{\log}$ of the group G_K acting trivialy on W and on $\mathcal{O}$, and acting on A_{inf} through the action on $\mathcal{O}_{\mathbb{C}_K}$. Moreover we have a derivation on $B_{\log}$

$$d : B_{\log} \longrightarrow B_{\log}\frac{dZ}{Z}$$

which is B_{cris} linear and satisfies $d((u-1)^{[n]}) = (u-1)^{[n-1]}u\frac{dZ}{Z}$ [Kato, 1994b].

Definition 4.1.19. The Monodromy operator on $B_{\log}$ is the operator

$$N_{\log} : B_{\log} \longrightarrow B_{\log}; \qquad \text{such that} \quad d(f) = N_{\log}(f)\frac{dZ}{Z}.$$

We can recover the ring B_{st} from $B_{\log}$ by considering the largest subring of $B_{\log}$

in which $N_{\log}$ acts as a nilpotent operator [Fontaine, 1982].

4.2 p-Adic Comparison Isomorphisms for $K3$-Surfaces

In this section I will recall the comparison isomorphism of Andreatta & Iovita for the special case in which X_K is a smooth proper $K3$-surface over a p-adic field K.

As before we let $\mathcal{O}_K$ be the ring on integers of K and we fix a uniformizer $\pi \in \mathcal{O}_K$. We also denote by $k = \mathcal{O}_K/\pi\mathcal{O}_K$ the residue field, which we assume to be algebraically closed.

4.2.0.5 An Admissibility Criterion

We recall the admissibility criterion of [Andreatta & Iovita, 2012, 2.1.1] which is very similar to the admissibility criteria described above defined by Colmez & Fontaine.

Let M be a finite free $B_{\log}^{G_K}$-module, which is a finite (ϕ, N)-module. The map

$$B_{\log} \to B_{dR}; \quad Z \to \pi$$

has image $\bar{B}_{\log}$. We define

$$V_{\log}^0(M) := (B_{\log} \otimes_{B_{\log}} M)^{N=0,\, \phi=1}$$

and

$$V_{\log}^1(M) := (B_{\log} \otimes_{\bar{B}_{\log}} M)/\operatorname{Fil}^0(\bar{B}_{log} \otimes_{B_{\log}^{G_K}} M).$$

Let $\delta(M) : V_{\log}^0 \longrightarrow V_{\log}^1(M)$ be the map given by the composite of the inclusion and projection

$$V^0_{\log}(M) \subset B_{\log} \otimes_{B^{G_K}_{\log}} M \longrightarrow \bar{B}_{\log} \otimes_{B^{G_K}_{\log}} M.$$

We define $V_{\log}(M) := \ker(\delta(M))$. Then

Proposition 4.2.1.
- *A filtered (ϕ, N)-module M over $B^{G_K}_{\log}$ is* admissible *if and only if $V_{\log}(M)$ is a finite dimensional $\mathbb{Q}_p$-vector space and $\delta(M)$ is surjective.*
- *Moreover, if M is admissible then $V := V_{\log}(M)$ is a finite dimensional, semi-stable G_K-representation and $D_{\log}(V) = M$.*

4.2.0.6 The Comparison Isomorphisms

Let us recall that we fixed a smooth, projective $K3$-surface X_K over a p-adic field K which has a minimal semi-stable model X over $\mathcal{O}_K$. We also suppose that the residue field $k := \mathcal{O}_K/\pi\mathcal{O}_K$ is algebraically closed of characteristic $p > 3$.

We consider on X the induced log structure given by its special fibre $\bar{X}$, which is a normal crossing divisor and we give to $\bar{X}$ the pull back log structure as in section (3.2) denoted by $X^{\log}$ and $\bar{X}^{\log}$ as usual.

Let $S^{\log} := \mathrm{Spec}(W[[t]])^{\log}$ where $W = W(k)$ and the log structure on S is the induced by the pre log structure $\mathbb{N} \to W[[t]];\ n \to t^n$. We have seen on (3.3.16) that the deformation $\mathcal{X}^{\log} \to S^{\log}$ of the special fibre $\bar{X}$ may be chosen such that it has the properties (3.3.16):

- $(\mathcal{X} \otimes_W K_0)_0$ is a combinatorial $K3$-surface over K_0 of the same type of $\bar{X}$.
- For every point $x \in S \otimes_W K_0$, with $x \neq 0$, then $(\mathcal{X} \otimes_W K_0)_x$ is a smooth $K3$-surface over $k(x)$.

Remember that $\mathbb{Y} := (\mathcal{X} \otimes_W K_0)_0$ is of the same type of $\bar{X}$, in particular $\bar{X}$ is smooth if and only if $\mathbb{Y}$ is smooth.

We consider on $\mathcal{X}_{K_0} = \mathcal{X} \otimes_W K_0$ the log structure defined by the divisor with normal crossings $\mathbb{Y} \to \mathcal{X}_{K_0}$ and on $\mathbb{Y}$ the inverse image log structure.

Denote by $D := H^2_{dR}(\mathbb{Y})$ the log de Rham cohomology of $\mathbb{Y}/K_0$. Then D has a natural structure of filtered, (ϕ, N)-module over K_0 obtained by its identification with $H^2_{\text{cris}}(\bar{X}/W)[1/p]$ with the log crystalline cohomology of $\bar{X}$ over W. More precisely the structure of filtered (ϕ, N)-module of D can be explicitly described as follows:

Let $\mathcal{H} = H^2_{\text{dR}}(\mathcal{X}/S)$ denote the locally free $\mathcal{O}_S$-module of relative log de Rham cohomology of $\mathcal{X}$ over S. It is endowed with a log integrable connection ∇, the Gauss-Manin connection, and a Frobenius ϕ. Moreover, $\mathcal{H}$ can be naturally identified with $H^2_{\text{cris}}(\bar{X}/W[[t]])[1/p]$ therefore we have the identifications:

- $\mathcal{H}_0 := \mathcal{H}/t\mathcal{H} \simeq D$;
- $\mathcal{H}_0 \otimes_{K_0} K \simeq H^2_{\text{dR}}(X_K)$.

Hence, we have natural identifications $D_K := D \otimes_{K_0} K \simeq H^2_{dR}(X_K)$ and so we define the filtration on D_K to be the inverse image of the Hodge filtration on $H^2_{dR}(X_K)$.

Moreover we define the p-adic monodromy operator N_p on D to be the residue of ∇ modulo $t\mathcal{H}$ and the Frobenius ϕ_0 on D to be the reduction modulo $t\mathcal{H}$.

We also denote by $V := H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$; it is a p-adic G_K-representation.

In [Andreatta & Iovita, 2012], the following theorem is proved:

Theorem 4.2.2 (Comparison Isomorphisms). *[Andreatta & Iovita, 2012, sec. 2.3.9]. V is a semi-stable G_K-representation and we have a natural isomorphism of filtered, (ϕ, N)-modules: $D_{st}(V) \simeq D$.*

For the proof, one considers $M := D \otimes_{K_0} (B_{\log}^{G_K})$ with its induced filtered (ϕ, N)-module structure and one proves that:

- M is an admissible filtered (ϕ, N)-module and
- V and $V_{\log}(M)$ are isomorphic as G_K-representations.

Proposition (4.2.1) now implies that:

Proposition 4.2.3.

$$D_{\log}(V) = M = D \otimes_{K_0} B_{\log}^{G_K}$$

and so $D \simeq D_{st}(V)$, as filtered, Frobenius, monodromy modules. In particular there is an identification $N_p = N_{st}$.

Chapter 5

The Main Theorem

Assume $p > 3$ is a fixed prime number. Let K be a p-adic field with algebraically closed residue field k. That is K is a totally ramified finite extension of the field $K_0 = \mathrm{Frac}(W(k))$. Let $\mathcal{O}_K$ be the ring of integers of K.

5.1 The Main Theorem

The following theorem is an analogue of the Kulikov; Persson & Pinkham-classification theorem of the central fibre of a semi-stable degeneration of complex $K3$-surfaces in terms of the monodromy, but now over a p-adic field.

The new part of the theorem is that we can distinguish the three possible types of the special fibre of a semi-stable $K3$-surface over a p-adic field K, in terms of the (p-adic) monodromy operator N_{st} on $D_{st}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$. As a consequence of this result, we get a criterion for the good reduction of the semi-stable $K3$-surface in terms of the p-adic representation $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ analogous to the Coleman & Iovita-theorem for abelian varieties and Iovita *et al.*-theorem for curves.

Theorem 5.1.1. *Let $X_K \to \operatorname{Spec}(K)$ be a smooth projective $K3$-surface and let $X \to \operatorname{Spec}(\mathcal{O}_K)$ be a semi-stable minimal model of X_K. Let $\bar{X}$ be the special fibre of X. We denote $D_{st} = D_{st}(H^2_{ét}(X_{\overline{K}}, \mathbb{Q}_p))$ and let $N_{st} : D_{st} \to D_{st}$ be the monodromy operator on D_{st}. Then we have 3 possibilities for the special fibre $\bar{X}$, distinguished in terms of the nilpotency degree of the monodromy operator N_{st}, as follows:*

I. $N_{st} = 0$ if and only if $\bar{X}$ is a nonsingular $K3$ surface.

II. $N_{st} \neq 0$ but $N_{st}^2 = 0$ if and only if $\bar{X} = \cup_{i=1}^n V_i$ where the V_i are rational surfaces and $V_2, \ldots, V_{n-1}$ are elliptic ruled surfaces.

III. $N_{st}^2 \neq 0$ but $N_{st}^3 = 0$ if and only if $\bar{X} = \cup_{i=1}^n V_i$ where all the V_i are rational surfaces.

Proof. As $X \to \operatorname{Spec}(\mathcal{O}_K)$ is a minimal semi-stable model of $X_K \to \operatorname{Spec}(K)$, the special fibre is a SNCL $K3$-surface [Maulik, 2012; Nakkajima, 2000], and therefore it is a combinatorial $K3$-surface by proposition (3.3.11), i.e. it is of type I, II or III. So the remaining thing to prove is that we can distinguish these 3 cases in terms of the nilpotency degree of the monodromy operator N_{st}.

Step 1. We consider on X the induced log structure given by its special fibre $\bar{X}$, which is a normal crossing divisor and we give to $\bar{X}$ the pull back log structure as in section (3.2) denoted by $X^{\log}$ and $\bar{X}^{\log}$ as usual.

Remember that by proposition (3.3.16) there exists a deformation

$$\mathcal{X} \to S := \operatorname{Spec}(W[[t]])$$

of $\bar{X}$ such that:

- If we let 0 denote the point of $S \otimes_W K_0$ corresponding to the maximal ideal

$t(W[[t]] \otimes_W K_0)$. Then $\mathbb{Y} := (\mathcal{X} \otimes_W K_0)_0$ is a combinatorial $K3$-surface over K_0 of the same type of $\bar{X}$.

- For every point $x \in S \otimes_W K_0$, with $x \neq 0$, then $(\mathcal{X} \otimes_W K_0)_x$ is a smooth $K3$-surface over $k(x)$.

We considered on $\mathcal{X}_{K_0} := \mathcal{X} \otimes_W K_0$ the log structure defined by the divisor with normal crossings $\mathbb{Y} \to \mathcal{X} \otimes_W K_0$ and on $\mathbb{Y}$ the inverse image log structure.

Denote by $D := H^2_{dR}(\mathbb{Y})$ the log de Rham cohomology of $\mathbb{Y}/K_0$ where $K_0 = \mathrm{Frac}(W(k))$. Then D has a natural structure of filtered, (ϕ, N)-module over K_0 obtained by its identification of $H^2_{\mathrm{cris}}(\bar{X}/W)[1/p]$ with the log crystalline cohomology of $\bar{X}$ over W.

By proposition (4.2.3), the monodromy operator N_{st} on $D_{st}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ can be identified with the residue N_p of the Gauss-Manin connection ∇ modulo $t\mathcal{H}$, that is N_p is an endomorphism of $H^2_{\mathrm{dR}}(\mathbb{Y}/S[1/p]))$.

Step 2. Now fix once and for all an embedding of $K_0 \to \mathbb{C}$. Consider the base change of $\mathcal{X}_{K_0} := \mathcal{X} \otimes_W K_0$ with respect to the induced embedding $W[[t]] \otimes_W K_0 \to \mathbb{C}[[t]]$. We have a complex family $\mathcal{X}_{\mathbb{C}} := \mathcal{X}_{K_0} \otimes \mathbb{C} \longrightarrow \mathrm{Spec}(\mathbb{C}[[t]])$ with special fibre $Y_{\mathbb{C}} = \mathbb{Y} \otimes \mathbb{C} \to \mathrm{Spec}(\mathbb{C})$ a combinatorial $K3$-surface and generic fiber a smooth $K3$-surface $X_{\mathbb{C}((t))}$. Let $\mathcal{S} = S[1/p] \otimes \mathbb{C} = \mathrm{Spec}(\mathbb{C}[[t]])$.

We endow $\mathcal{X}_{\mathbb{C}}, \mathcal{S}, Y_{\mathbb{C}}$ with the usual log structures and, we denote them as $\mathcal{X}^{\log}_{\mathbb{C}}$, $S^{\log}$, $Y^{\log}_{\mathbb{C}}$ respectively.

Consider now the log de Rham cohomology $H^2_{\mathrm{dR}}(\mathcal{X}^{\log}_{\mathbb{C}}/\mathcal{S}^{\log})$, it is a free $\mathcal{O}_{\mathcal{S}}$-module or rank 22 with an integrable logarithmic connection (The log Gauss-Manin connection):

$$\nabla : H^2_{\mathrm{dR}}(\mathcal{X}^{\log}_{\mathbb{C}}/\mathcal{S}^{\log})) \longrightarrow H^2_{\mathrm{dR}}(\mathcal{X}^{\log}_{\mathbb{C}}/\mathcal{S}^{\log})) \otimes_{\mathcal{O}_{\mathcal{S}}} \Lambda^1_{\mathcal{S}/\mathbb{C}}.$$

The fibre of $H^2_{\mathrm{dR}}(\mathcal{X}_{\mathbb{C}}^{\log}/\mathcal{S}^{\log}))$ at the special point is $H^2_{\mathrm{dR}}(Y_{\mathbb{C}}^{\log})$ that is

$$H^2_{\mathrm{dR}}(Y_{\mathbb{C}}^{\log}) \simeq H^2_{\mathrm{dR}}(\mathcal{X}_{\mathbb{C}}^{\log}/\mathcal{S}^{\log}))/tH^2_{\mathrm{dR}}(\mathcal{X}_{\mathbb{C}}^{\log}/\mathcal{S}^{\log})).$$

We also have the operator $N_{\mathbb{C}} := \mathrm{Res}_{t=0}\nabla$ which is a $\mathbb{C}$-linear, nilpotent operator on $H^2_{\mathrm{dR}}(Y_{\mathbb{C}}^{\log})$.

Let us notice that the pair $\left(H^2_{\mathrm{dR}}(Y_{\mathbb{C}}^{\log}/\mathbb{C}^{\log}), N_{\mathbb{C}}\right)$ is the base change to $\mathbb{C}$, via the embedding $K_0 \subset \mathbb{C}$, of the pair $\left(H^2_{\mathrm{dR}}(\mathbb{Y}^{\log}/K_0^{\log}), N_p\right)$.

Step 3. Now we use [Artin, 1969]:

We associate to the family $\mathcal{X}_{\mathbb{C}} \to \mathcal{S} = \mathrm{Spec}(\mathbb{C}[[t]])$ above a family of $K3$-surfaces $\mathcal{Y} \to \Delta$, over the complex open unit disk Δ. This family has the property that if we base change it over $\mathcal{S}$, we obtain a family $\mathcal{X}'_{\mathbb{C}}$ which is congruent to $\mathcal{X}_{\mathbb{C}}$ modulo $t^m\mathbb{C}[[t]]$ for some large $m > 1$. It follows that:

- $\mathcal{Y}|_{\Delta-\{0\}}$ is a smooth projective family of $K3$-surfaces.
- The central fibre $\mathcal{Y}_0 \simeq \mathcal{X}'_0 \simeq Y_{\mathbb{C}}^{an}$.

Here $Y_{\mathbb{C}}^{an}$ denotes the complex analytic variety associated to the complex points of $Y_{\mathbb{C}}$ (the usual GaGa functor).

Now we use the Monodromy criterion [Morrison, 1984, pag. 112] given by the Clemens-Schmidt exact sequence to the family $\mathcal{Y} \to \Delta$ (this criterion leads to the proof of the classical Kulikov; Persson & Pinkham-classification theorem, as we can see in [Morrison, 1984, pag. 113]).

Consider the GaGa functor an sending a complex algebraic Z variety to its associated complex analytic variety Z^{an}. We let N_{an} be the monodromy operator on $H^2_{\mathrm{dR}}(Y_{\mathbb{C}}^{\log,an})$. By [Deligne, 1970] it can be seen, up to non-zero constant, as the residue at zero of the Gauss-Manin connection:

$$\nabla_{an} : H^2_{\mathrm{dR}}(\mathcal{Y}/\Delta^{\log}) \longrightarrow H^2_{\mathrm{dR}}(\mathcal{Y}/\Delta^{\log}) \otimes \Lambda^1_{\mathcal{Y}/\Delta}.$$

Therefore, N_{an} can be seen (by the previous analysis) as the residue of the Gauss-Manin connection ∇' on $H^2_{\mathrm{dR}}((\mathcal{X}'_{\mathbb{C}})^{\log}/\mathcal{S}^{\log})$. But we have:

$$H^2_{\mathrm{dR}}((\mathcal{X}'_{\mathbb{C}})^{\log}/\mathcal{S}^{\log})/t^m H^2_{\mathrm{dR}}((\mathcal{X}'_{\mathbb{C}})^{\log}/\mathcal{S}^{\log}) \simeq H^2_{\mathrm{dR}}(\mathcal{X}^{\log}_{\mathbb{C}}/\mathcal{S}^{\log}))/t^m H^2_{\mathrm{dR}}(\mathcal{X}^{\log}_{\mathbb{C}}/\mathcal{S}^{\log})) \tag{5.1}$$

and $\nabla' \equiv \nabla \pmod{t^m \mathbb{C}[[t]]}$.

This implies that the residue of ∇_{an} ∇ and ∇' are the same under the identification

$$H^2_{\mathrm{dR}}(\mathcal{Y}_0) \simeq H^2_{\mathrm{dR}}(Y^{\log,an}_{\mathbb{C}}) \simeq H^2_{\mathrm{dR}}((\mathcal{X}'_0)^{\log}).$$

In other words $N_{an} = N_{\mathbb{C}}$, which is the base change to $\mathbb{C}$ of N_p.

Now we apply the description of $\mathcal{Y}_0$ in therms of N_{an} for the family $\mathcal{Y}$ given by the Clemens-Schmidt exact sequence [Morrison, 1984, pag. 113]. So if $N_{an} = 0$ then $\mathcal{Y}_0$ is of type I. If $N_{an} \neq 0$ but $N^2_{an} = 0$ then $\mathcal{Y}_0$ is of type II and if $N^2_{an} \neq 0$ but $N^3_{an} = 0$ then $\mathcal{Y}_0$ is of type III. Where the type I, type II and type III are as in the theorem (5.1.1).

Since $Y^{an}_{\mathbb{C}} = \mathcal{Y}_0$, then also $Y_{\mathbb{C}}$ is of the same type, and since $\mathbb{Y} \otimes \mathbb{C} = Y_{\mathbb{C}}$ we have that also $\mathbb{Y}$ is of the same type, hence $\bar{X}$ is of the same type.

Moreover we have seen that $N_{\mathbb{C}} = N_{an} = N_p = N_{st}$ up to constants. Which implies that we can distinguish the three possible types of $\bar{X}$ in terms of the nilpotency degree on N_{st} as stated in the theorem. □

The following theorem, which is in fact a corollary of theorem (5.1.1), is the main objective of this thesis.

Theorem 5.1.2. *Let $X_K \to \operatorname{Spec}(K)$ be a semi-stable $K3$-surface over the p-adic field K with minimal semi-stable integral model $X \to \operatorname{Spec} \mathcal{O}_K$ and with projective special fiber $\bar{X}$ over the algebraic closed field $k = \mathcal{O}_K/\pi\mathcal{O}_K$. Let $V := H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)$. Then X_K has good reduction (i.e $\bar{X}$ is smooth), if and only if V is a crystalline representation of $G_K := \operatorname{Gal}(\bar{K}, K)$.*

Proof. If $\bar{X}$ is smooth, that is if X_K has good reduction, then this is the theorem of Faltings et al [Faltings, 1988, 1992].

Now assume that V is crystalline representation of G_K. Then V is B_{cris}-admissible, but the B_{cris}-admissible representations are those semi-stable representations for which $N_{st} = 0$ [Breuil, 1997] or [Conrad, 2010]. So by theorem (5.1.1) $\bar{X}$ is of type I, that is, $\bar{X}$ is smooth and so X_K has good reduction. $\square$

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