Determinants of Pseudo-Laplacians on compact Riemannian manifolds and uniform bounds of eigenfunctions on tori

Tayeb Aissiou

A Thesis
In the Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy (Mathematics and Statistics) at
Concordia University
Montreal, Quebec, Canada

December 2013

© Tayeb Aissiou, 2013

CONCORDIA UNIVERSITY

SCHOOL OF GRADUATE STUDIES

This is to certify that the thesis prepared

By: **Tayeb Aissiou**

Entitled: **Determinants of Pseudo-Laplacians on compact Riemannian manifolds and uniform bounds of eigenfunctions on tori**

and submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY (Mathematics)

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final examining committee:

______________________________ Chair
Dr. S. Misra

______________________________ External Examiner
Dr. I. Polterovich

______________________________ External to Program
Dr. M. Frank

______________________________ Examiner
Dr. M. Bertola

______________________________ Examiner
Dr. S.T. Ali

______________________________ Thesis Co-Supervisor
Dr. A. Kokotov

______________________________ Thesis Co-Supervisor
Dr. D. Korotkin

Approved by ______________________________
Dr. J. Garrido, Graduate Program Director

December 12, 2013

____
Professor J. Locke, Interim Dean
Faculty of Arts and Science

ABSTRACT

In the f rst part of this thesis, we derive comparison formulas relating the zeta-regularized determinant of an arbitrary self-adjoint extension of the Laplace operator with domain $C_c^\infty(X \setminus \{P\}) \subset L_2(X)$ to the zeta-regularized determinant of the Laplace operator on X. Here X is a compact Riemannian manifold of dimension 2 or 3; $P \in X$. In the second part, we provide a proof of a conjecture by Jakobson, Nadirashvili, and Toth stating that on an n-dimensional f at torus $\mathbb{T}^n$, and the Fourier transform of squares of the eigenfunctions $|\varphi_\lambda|^2$ of the Laplacian have uniform l^n bounds that do not depend on the eigenvalue λ. The thesis is based on the papers [2] and [1].

ACKNOWLEDGEMENTS

I would like to thank my supervisors, Professor Alexey Kokotov and Professor Dmitri Korotkin for their support and their endless source of inspiration. Working under their supervision taught me so much about mathematics and for this, I am eternally grateful. Thank you so much!

I would also like to thank my supervisor Dmitry Jakobson for all his support, for believing in me when I started my graduate studies in mathematics, for accompanying me during my learning, his availability and his patience. A sentence can def nitely not do justice to how much I am thankful to him.

I would also like to thank my family for everything they have given me. As a father to be, I just realized that I can never value enough their support. Thank you!

The author of this thesis was partially funded by the FQRNT

DEDICATION

To my family.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1
Introduction

In this thesis, we consider two different topics from the spectral theory of the Laplace operator on compact Riemannian manifolds. First, we study the determinants of self-adjoint extensions of the Laplacian on compact Riemannian manifolds X, more precisely, we give a formula that compares the determinants of two distinct self-adjoint extensions of the Laplacian with domain $\mathcal{D}$ consisting of smooth functions supported on the complement of a point $P \in X$, i.e.,

$$\mathcal{D} = C_0^\infty(X \setminus \{P\}) \subset L^2(X).$$

In the second part of the thesis we provide a proof of the conjecture formulated in [18, 19] which states that on a n-dimensional f at torus $\mathbb{T}^n$, the Fourier transform of squares of the eigenfunctions $|\varphi_\lambda|^2$ of the Laplacian have uniform l^n bounds that do not depend on the eigenvalue λ. To this end we prove a geometric lemma that bounds the number of codimension-one simplices which satisfy a certain restriction on an n-dimensional sphere $S^n(\lambda)$ of radius $\sqrt{\lambda}$, we think that this lemma might be of its own interest.

The thesis is based on the papers [2] and [1].

The study of self-adjoint extensions started in the last century together with development of rigorous mathematical foundations of quantum mechanics. It is well known from quantum physics, that all measurable quantities in nature (i. e., observables) , correspond to linear operators that have real eigenvalues. The latter condition is guaranteed by self-adjointness of the operators.

In the 1930's and 1940's, physicists were interested in Hamiltonians given by the heuristic form:

$$H = -\Delta + \mathcal{P}, \tag{1.1}$$

where the potential $\mathcal{P}$ is supported on a discrete set, say, $\mathcal{P} = c_P \delta_P(\cdot)$, where δ_P is the Dirac delta function supported at a point P and c_P is a coupling constant and Δ is the Laplace operator in $\mathbb{R}^3$.

Of course, the heuristic "Hamiltonian" H does not define any operator in $L_2(\mathbb{R}^3)$ and its mathematical meaning remains unclear.

To the best of our knowledge, the first paper that clarified the meaning of the expression 1.1 and proposed its mathematical interpretation goes back to 1961 and is due to Berezin and Faddeev (see [6]).

The main idea of Berezin and Faddeev is to consider the operator (1.1) as a self-adjoint extension of the Laplacian operator with domain $C_0^\infty(\mathbb{R}^3 \setminus \{P\})$ which is smaller than

$C_0^\infty(\mathbb{R}^3)$. In contrast to the Laplacian with domain $C_0^\infty(\mathbb{R}^3)$ the latter symmetric operator is no longer essentially self-adjoint, its def ciency indices are $(1,1)$. It is the self-adjoint extension (actually there is a one-parametric family of such extensions) of this operator that should be identif ed with the Hamiltonian (1.1). (Actually, Berezin and Faddeev worked in the impulse representation: their operator is the multiplication by p^2 and not the Laplacian - both pictures are, of course, unitary equivalent via Fourier transform.)

It should be noted that the properties of the Laplacian with domain $C_0^\infty(\mathbb{R}^d \setminus \{P\})$ depend on the dimension d; the corresponding symmetric operator is essentially self-adjoint and its closure coincides with the standard self-adjoint Laplace operator in $L_2(\mathbb{R}^d)$ if and only if $d \geq 4$, this fact for a long time remained in folklore (to the best of our knowledge, it was for the f rst time distinctly formulated in one of the exercises in the second volume of the Reed-Simon textbook in Functional Analysis [31]), the recent paper [26] gives its accurate and detailed proof.

For $d = 1, 2, 3$ the Hamiltonian (1.1) can be considered as a perturbation of the free Hamiltonian Δ and the corresponding scattering theory appeared to be extremely interesting (due to simplicity of the perturbation most of the formulas of the scattering theory in this situation could be made pretty explicit) and became a subject of numerous papers and the well known monograph of Albeverio, Gesztesy, Hoegh-Krohn and Holden ”Solvable Models in Quantum Mechanics” ([3]).

In contrast to the just described subject, we wish to consider here the Laplacians on *compact* manifolds and not on the space $\mathbb{R}^d$. On the one hand, this situation is very similar to the case of $\mathbb{R}^d$: if X is a d-dimensional compact Riemannian manifold, Δ is the Laplacian on it and a point P belongs to X, then the operator Δ with domain $C_0^\infty(X \setminus \{P\})$ is essentially self-adjoint if and only if $d \geq 4$; in case $d = 1, 2, 3$ it has equal non-zero def ciency indices (in fact, $(1, 1)$) and inf nitely many self-adjoint extension (the proofs of these fact are almost the same as in the case of $\mathbb{R}^d$). On the other hand, this new situation is *dramatically* different from the case of noncompact space $\mathbb{R}^d$: the spectra of all the self-adjoint extensions of the Laplacian are now *discrete* and all the questions of the scattering theory which were natural for noncompact situation become completely irrelevant.

We also note that one motivation for the study of the pseudo-Laplacians came from the works of P. Cartier and D. Hejhal [8] on the study of the zeros of the Riemann zeta function. The description of the spectra of the self-adjoint extensions of the operator Δ with domain $C_0^\infty(X \setminus \{P\})$ was given in the seminal paper of Y. Colin de Verdière [11] (we devote the next chapter of the thesis to a brief survey of these results). It should be also mentioned that the name ”PSEUDO-LAPLACIANS” was given to these extensions of the Laplacian by Colin de Verdière and we choose to follow this terminology throughout our thesis.

The main goal of the present thesis is to study *the ζ-regularized determinants* of the pseudo-Laplacians. The latter determinant is an extremely important spectral characteristic of operators with discrete spectrum being (in case of the Laplacian on a Riemann surface or even in the case of the general elliptic self-adjoint pseudo-differential operator acting in the space of sections of vector bundles) the subject of immense number of works in spectral geometry and the string theory.

Let us say here a few words about this spectral characteristic.

For a finite dimensional operator, the determinant is defined as the product of its eigenvalues. For infinite dimensional operators (such as the Laplacian) the product of the eigenvalues diverges. In order to make sense of such a quantity (i. e. to define the determinant), Ray and Singer proposed to use the so-called zeta-regularization [30].

Denote by $\zeta(s, A)$ the Minakshisundaram-Pleijel zeta function associated to the operator A. For $\Re(s)$ large enough, it is given by

$$\zeta(s, A) = \sum_{\mu_k \in \sigma^*(A)} \mu_k^{-s}, \tag{1.2}$$

where we assume that A has a discrete spectrum $\sigma(A)$ and $\sigma^*(A)$ the spectrum of A without the zero eigenvalue, i.e.,

$$\sigma^*(A) := \sigma(A) - \{0\}.$$

Formally differentiating the function $\zeta(s, A)$ term-wise with respect to s, we get

$$\zeta'(s, A) = -\sum_{\mu_k \in \sigma^*(A)} \mu_k^{-s} \log \mu_k. \tag{1.3}$$

Evaluating the derivative (1.3) at $s = 0$, we get

$$\zeta'(0, A) = -\sum_{\mu_k \in \sigma^*(A)} \log \mu_k. \tag{1.4}$$

The determinant of the operator A is then (formally) def ned by

$$\begin{aligned} \det(A) &= \exp\left(-\zeta'(0, A)\right) \\ \text{“}={}\text{''} &\ \exp\left(\sum_{\mu_k \in \sigma^*(A)} \log \mu_k\right) \\ \text{“}={}\text{''} &\ \prod_{\mu_k \in \sigma^*(A)} \mu_k \end{aligned} \tag{1.5}$$

In order to validate the evaluation of $\zeta'(s, A)$ at $s = 0$ in (1.4), one needs to prove that the Minakshisundaram-Pleijel zeta function of the operator A (the Laplacian in our case) is regular at $s = 0$. Doing so, the zeta regularized determinant becomes well def ned.

As it was shown by Colin de Verdière, the operator Δ on a compact Riemannian manifold X of dimension $d = 2, 3$ with domain $C_0^\infty(X \setminus \{P\})$ admits a one-parameter family of self-adjoint extensions Δ_α (pseudo-Laplacians). We are to study the zeta-regularized determinants of these operators.

It should be noted that the case $d = 1$ lies beyond the scope of the present thesis - in this case it is natural to consider not a relatively simple case of one-dimensional compact smooth manifold (i. e. the disjoint union of circles) but a more complicated case of a graph with several edges adjoint to the vertex P. The arising problem belongs to the theory of quantum graphs (a very popular topic at the moment) and is considered in the recent papers Kirsten, Loya & Park [21, 20, 23].

The main result of the f rst part of our thesis is a comparison formula relating $\det(\Delta_{\alpha,P} - \lambda)$ to $\det(\Delta - \lambda)$, for $\lambda \in \mathbb{C} \setminus (\sigma(\Delta) \cup \sigma(\Delta_{\alpha,P}))$, here $\Delta =: \Delta_\infty$ denotes the standard self-adjoint Laplacian on X. Passing to the limit $\lambda \to 0$ one gets the formulas relating the determinants of the pseudo-Laplacians to the determinants of standard Laplace operators on Riemannian manifolds. The main results are formulated in Proposition 3 and in Corollary 2.

We also consider separately the cases of three-dimensional f at tori and the sphere S^3 with standard round metric, in these cases all our formulas could be made pretty explicit (the explicit expressions for the determinants of Laplacians on these manifolds can be found in [14] and [24]).

It should be mentioned that in case of two-dimensional manifolds, the zeta regularization of $\det(\Delta_{\alpha,P} - \lambda)$ is not that standard, since the corresponding operator zeta-function has logarithmic singularity at 0. The main result in this case is formulated in Proposition 4 and in Corollary 3.

It should be also mentioned that in the case when the manifold X_d is flat in a vicinity of the point P we deal with a very special case of the situation (Laplacian on a manifold with conical singularity) considered in [27], [23], [22] and, via other method, in [15]. Notice also a recent paper [42] that is devoted to the case where X_d is a compact Riemann surface equipped with Poincaré metric. In a sequence of papers, Ueberschär & Rudnick studied some properties of the eigenvalue and the eigenfunctions of the pseudo-Laplacians on flat tori (Cf. [32, 33]).

The general scheme of the present work is close to that of [15], although some calculations from [22] also appear very useful for us.

The results of the first part of the thesis were published in [2].

In the second part of the thesis we consider the Laplacian Δ on the n-dimensional flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The eigenvalues of $-\Delta$ are denoted by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, and the corresponding eigenfunctions are denoted by φ_j. We normalize $||\varphi_j||_2 = 1$.

The following Proposition was proved in [43] for $n = 2$, in [18] for $n = 3$, and in [19] for $n = 4$.

Theorem 1. *Let $2 \leq n \leq 4$, then the Fourier series of $|\varphi_j|^2$ have uniformly bounded l^n norms, where the bound is independent of λ_j.*

We remark that it is well-known that the *multiplicity* of λ_j becomes unbounded for $n \geq 2$, and therefore so does $||\varphi_j||_\infty$.

It was conjectured in [18] that the conclusion of Theorem 1 holds for arbitrary n. The main result of this paper is the proof of that conjecture:

Proposition 1. *For any $n \geq 5$, there exists $C = C_n < \infty$, such that for every eigenfunction $\Delta\varphi_j + \lambda_j\varphi_j = 0$, $||\varphi_j||_2 = 1$, the Fourier series of $g := |\varphi_j|^2$ satisf es*

$$||\widehat{g}||_{l^n} \leq C(n)||\varphi_j||_2^2 \tag{1.6}$$

We stress that the bound C does not depend on the eigenvalue λ_j. The bound $C(n)$ is computed at the end of the proof and tends to 2 as $n \to \infty$.

Proposition 1 implies (by an argument in [18]) a statement about limits of eigenfunctions on $\mathbb{T}^{n+2}$. Consider weak limits of the probability measures $d\mu_j = |\varphi_j|^2 dx$, and denote the limit measure as $\lambda_j \to \infty$ by $d\nu$, one can prove that all such limit measures $d\nu$ are absolutely continuous in any dimension with respect to the Lebesgue measure on $\mathbb{T}^n$ (Cf.

[18]). Accordingly, by Radon-Nikodym theorem, one can conclude that $d\nu$ has a density $h(x) \in L^1(\mathbb{T}^n)$ such that $d\nu = h(x)\, dx$. Then, we consider the Fourier expansion of $h(x)$:

$$h(x) \sim \sum_{\tau \in \mathbb{Z}^n} c_\tau \, e^{i(\tau, x)} \tag{1.7}$$

In dimension $n = 2$, it was shown in [18] that the density of every such limit is a trigonometric polynomial with at most *two* different magnitudes for the frequency. It was also shown in[18, 19] that on $\mathbb{T}^n$ for $3 \leq n \leq 6$, the Fourier expansion of the limit measure $d\nu$ is in l^{n-2}, that is,

$$\sum_{\tau \in \mathbb{Z}^n} |b_\tau|^{n-2} < \infty. \tag{1.8}$$

The proofs in dimensions $4 \leq n \leq 6$ used Proposition 1 and results in [18] that reduced estimates for limits on $\mathbb{T}^{n+2}$ to estimates for eigenfunctions on $\mathbb{T}^n$. The estimate (1.8) implies that on $\mathbb{T}^3$, the density of any limit $d\nu$ has an absolutely convergent Fourier series, whereas on $\mathbb{T}^4$, we conclude that $h(x) \in L^2(\mathbb{T}^4)$.

Combining Proposition 1 with the results in [18], we immediately obtain

Proposition 2. *Given the Fourier expansion* (1.7) *of the limit measure $d\nu$ on $\mathbb{T}^{n+2}$, we have*

$$\left(\sum_{\tau \in \mathbb{Z}^{n+2}} |b_\tau|^n \right)^{1/n} \leq C(n) < \infty \tag{1.9}$$

A generalization of B. Connes' result [10] proved in [18] shows that the constant $C(n)$ appearing in Proposition 2 on $\mathbb{T}^{n+2}$ coincides with the constant in Proposition 1 on $\mathbb{T}^n$. The bound $C(n)$ will be computed at the end of the proof and we will f nd that it tends to 2 as $n \to \infty$.

An important question about eigenfunctions of the Laplacian is the following: given $\varphi(x)$ satisfying $\Delta\varphi_j + \lambda_j\varphi_j = 0$ and $||\varphi||_2 = 1$ on a general n-dimensional smooth Riemannian manifold $\mathcal{M}$, what is the asymptotic growth rate of the L^p norms of the eigenfunction? That is, how fast does $||\varphi_j||_{L^p}$ grow as the eigenvalue $\lambda_j \to \infty$.

On a two dimensional compact boundaryless Riemannian manifold, Sogge showed in [37] that for $2 \leq p \leq \infty$:

$$||\varphi_j||_p \leq C\lambda_j^{\delta(p)}||\varphi_j||_2 \tag{1.10}$$

where

$$\delta(p) = \begin{cases} \frac{1}{4}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \leq p \leq 6 \\ \\ \frac{1}{2}\left(\frac{1}{2} - \frac{2}{p}\right), & 6 \leq p \leq \infty \end{cases} \tag{1.11}$$

This bound turned out to be sharp on the round sphere S^2.

In a remarkable result, Zygmund [43] provides a uniform bound for the L^4-norm of the eigenfunctions of the Laplacian on $\mathbb{T}^2$. That is,

$$\frac{||\varphi||_4}{||\varphi||_2} \leq 5^{1/4} \tag{1.12}$$

The bound (1.12) provided in [43] is independent of the eigenvalue.

Before we mention the next result, we give the following definition:

$$M_{n,p}(\lambda) := \sup_{\substack{(\Delta+\lambda)\varphi=0 \\ \varphi \text{ on } \mathbb{T}^n}} \frac{||\varphi||_p}{||\varphi||_2} \tag{1.13}$$

The question of the growth rate mentioned earlier can be translated into, what is the asymptotic behavior of $M_{n,p}(\lambda)$. It is sometimes possible to obtain uniform bounds (independent of λ) for $M_{n,p}(\lambda)$ for a restricted set of eigenvalues.

In particular, Mockenhaupt proved in [29] the following: given a finite subset $D = \{q_1, q_2, \ldots, q_k\}$ of prime integers with $q_j \equiv 1 \pmod 4$, we consider the set Λ_D consisting of all eigenvalues $\lambda \in \mathbb{N}$ such that all prime divisors q of λ with the property $q \equiv 1 \pmod 4$, belong to D. Then, for all $\lambda \in \Lambda_D$ and for all $p < \infty$, we have $M_{2,p} \leq C(p,k) < \infty$, where $C(p,k)$ is a constant.

A legitimate question to ask is whether or not there exists a uniform bound for $M_{n,p}$ for general n and p. The question is still open, although there exist results about the rate of growth of $M_{n,p}(\lambda)$ as $\lambda \to \infty$. Bourgain showed in [7] that on $\mathbb{T}^n$ with $n \geq 4$, we have $M_{n,p} \ll \lambda^{(n-2)/4-n/2p}$ for $p \geq 2(n+1)/(n-3)$.

We notice that Proposition 1 does not imply a bound on eigenfunctions since there is no converse to Hausdorff-Young inequality. For $1 < p \leq 2 \leq q < \infty$ with $p^{-1} + q^{-1} = 1$, we have:

$$||b_\tau||_{l^q} \ll ||\varphi||^2_{L^{2p}}. \tag{1.14}$$

Although the bound $C(n)$ from Proposition 1 does not depend on the eigenvalue λ, it does not give us information about the bound $M_{n,p}$ in (1.13).

There exist bounds for the L^∞ norm of the eigenfunctions as well. Hörmander showed (cf. [17, 16]) that on any compact Riemannian manifold M, we have

$$||\varphi_\lambda||_\infty \leq C\,\lambda^{\frac{n-1}{4}},$$

where n is the dimension of the manifold M. This bound is attained for some manifolds such as S^n, but not for others such as $\mathbb{T}^n$. Manifolds for which this bound is sharp are called manifolds with *maximal eigenfunction growth*.

Y. Safarov studied the asymptotic behavior of the spectral function, the remainder in Weyl's law, and of eigenfunctions in many papers including [34, 35].

C. Sogge, J. Toth and S. Zelditch studied, in a series of papers (Cf. [38, 39, 41]) the following question: what characterizes the manifolds with maximal eigenfunction growth?

They established that the manifolds with maximal eigenfunction growth must have a point x where the set of geodesic loops at that point has a positive measure in $S^*_x M$. The converse turned out to be false as they constructed a counterexample in [39].

An older question of the same type is: how fast does the spectral function and the remainder term in Weyl's formula grow as $\lambda \to \infty$? The spectral function is given by:

$$N_{x,y}(\lambda) = \sum_{0<\sqrt{\lambda_j}<\sqrt{\lambda}} \varphi_j(x)\overline{\varphi_j(y)} \tag{1.15}$$

If we consider the diagonal when $x = y$, we obtain $N_{x,x}(\lambda)$. If we integrate the latter over the volume of the manifold M (assumed to be compact), we obtain the eigenvalue counting function $N(\lambda)$ def ned by:

$$N(\lambda) = \#\{\lambda_i < \lambda\}. \tag{1.16}$$

The remainder term in Weyl's formula is given by:

$$R(\lambda) = N(\lambda) - c_n \operatorname{vol}(M)\, \lambda^{n/2}, \tag{1.17}$$

where c_n is a constant that depends on the dimension n.

The asymptotic behavior of the spectral function and the remainder term were studied by many people, cf. [4, 12, 16, 25, 36] and the references therein for a detailed exposition of the subject.

The results of this paper appear in [1].

Chapter 2
Preliminaries: The Spectral Theory of Pseudo-Laplacians and the Krein Formula

In this chapter we, first, (very briefly) remind the reader the basic statements of the Von Neumann theory of self-adjoint extensions of a symmetric operator in a Hilbert space, and then describe the content of the work of Y. Colin de Verdière on the spectral theory of the pseudo-Laplacians. We also remind the reader the classical Krein formula for the trace of the difference of resolvents of two self-adjoint extensions of a symmetric operator (in the simplest case of deficiency indices $(1, 1)$ - this is the only case we need in what follows).

2.1 Short reminder: Self-adjoint extensions of symmetric operators in Hilbert space

All the statements of this section are pretty basic and (in this or in an equivalent form) can be found in any textbook devoted to the spectral theory of unbounded operators in Hilbert space (see e. g. [31]). We include this small section only for convenience of a reader.

Let A be a symmetric operator in a Hilbert space H with dense domain $\mathcal{D}(A)$ and let A^* be its adjoint.

Def ne the scalar product on $\mathcal{D}(A^*)$ via

$$\langle\langle u, v\rangle\rangle = \langle u, v\rangle_H + \langle A^*u, A^*v\rangle_H.$$

One has the following orthogonal decomposition of the domain of the adjoint with respect to this scalar product:

$$\mathcal{D}(A^*) = \mathcal{D}(\bar{A}) \oplus_{\langle\langle\, ,\, \rangle\rangle} K_+ \oplus_{\langle\langle\, ,\, \rangle\rangle} K_-, \tag{2.1}$$

where $\bar{A}$ is the closure of A and $K_\pm$ are the def ciency subspaces.

The dimensions of the spaces $K_\pm$

$$n_\pm := \dim K_\pm,$$

are called the *def ciency indices* of the operator A.

The operator A is self-adjoint if and only if $n_+ = n_- = 0$. On the other hand, if A is not self-adjoint then its self-adjoint extension exists (in the *same* Hilbert space) if and only if $n_+ = n_-$. In case $n_- = n_+$ there is a bijection between the set of unitary isomorphisms $U : K_+ \longrightarrow K_-$ and the set of self-adjoint extensions $(A_U, \mathcal{D}_U)$ of A given by $\mathcal{D}_U = \mathcal{D}(\bar{A}) \oplus_{\langle\langle\, ,\, \rangle\rangle}$ (the graph of U) and $A_U = A^*_{\restriction \mathcal{D}_U}$, the restriction of A^* to the domain $\mathcal{D}_U$.

2.2 Results of Colin de Verdière

Let Δ be the Laplace-Beltrami operator on a compact $d-$dimensional Riemannian manifold X with domain $\mathcal{D}(X) = C_0^\infty(X)$. In this section we denote by Δ_∞ the unique self-adjoint extension on the Hilbert space $L^2(X, dx)$ where dx is the Riemannian volume element. The domain of the extension Δ_∞ is the Sobolev space $\mathcal{H}^2(X)$.

For any point $P \in X$, we define the operator Δ_P with domain $C_0^\infty(X \setminus \{P\})$ such that $\Delta_P f = \Delta f$ for every function in the domain. In [11], Yves Colin de Verdière obtained detailed results concerning the spectra of the self-adjoint extensions of the operator Δ_P (pseudo-Laplacians). We are presenting his results here.

Theorem 2 (Colin de Verdière)**.** *If $d \geq 4$, the only self-adjoint extension of Δ_P is Δ_∞. If $d = 2$ or 3, the deficiency indices $n_+ = n_- = 1$, and there is a continuum of self-adjoint extensions of Δ_P parametrized by $\alpha \in [0, \pi)$ and denoted by $\Delta_{\alpha,P}$. Their domain is given by:*

$$\begin{aligned} \mathcal{D}(\Delta_{\alpha,P}) &= \{f \in \mathcal{D}(\Delta_P^*) \mid \exists \lambda \in \mathbb{C}, \text{ where in the vicinity of } P \text{, we have} \\ &\quad f(x) = \lambda(\sin\alpha \cdot G_d(r) + \cos\alpha) + o(1) \text{ as } r \to 0\}, \end{aligned} \tag{2.2}$$

where,

$$\mathcal{D}(\Delta_P^*) = \mathcal{H}^2(X - P) = \{f \in L^2(X) \mid \exists C \in \mathbb{C}, \Delta f - C\delta(P) \in L^2(X)\},$$

r is the geodesic distance between x and P,

$$G_d(r) = \begin{cases} \frac{1}{2\pi}\log(r), & d = 2 \\ \\ \frac{-1}{4\pi r}, & d = 3 \end{cases}$$

and hence $\Delta_{\alpha,P} f = \Delta f - C\delta(P)$.

Theorem 2 characterizes all the self-adjoint extensions of Δ_P and their domains on complete smooth Riemannian manifolds.

Remark 1. *For $\alpha = 0$, the extension $\Delta_{0,P}$ coincides with the Friedrich's extension Δ_∞. For convenience, we will refer to it by Δ_∞ instead of $\Delta_{0,P}$. As for $\alpha \neq 0$, we will refer to $\Delta_{\alpha,P}$ as non-Friedrich's extensions*

In the same paper [11], Colin de Verdière studied the spectra of such extensions but restricted to the case where X is compact of dimension 2 or 3. We state his results.

For any $\alpha \in [0, \pi)$, the operators $\Delta_{\alpha,P}$ have compact resolvent operators. Hence $\Delta_{\alpha,P}$ have discrete spectra accumulating to ∞.

Let the sequence $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the spectrum of Δ_∞ on X and $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions for $L^2(X)$ satisfying

$$\Delta_\infty \varphi_n = \lambda_n \varphi_n. \tag{2.3}$$

We denote by E_λ the eigenspace corresponding to the eigenvalue λ. For any $\lambda \notin \sigma(\Delta_\infty)$, the resolvent kernel of $(\Delta_\infty - \lambda)^{-1}$ is given by

$$R(\lambda; x, y) = \sum_{n=0}^{\infty} \frac{\varphi_n(x)\,\overline{\varphi_n(y)}}{\lambda_n - \lambda}. \tag{2.4}$$

We translate Lemma 2 in [11] that gives a description of $R(\lambda; x, y)$.

Lemma 1 (Colin de Verdière). *The resolvent kernel $R(\lambda;\ \cdot\ , P) \in \mathcal{D}(\Delta_P^*)$. In the vicinity of P, it has the following asymptotics*

$$-R(\lambda; x, P) = G_d(r) + F(\lambda; P) + o(1), \tag{2.5}$$

where $F(\lambda; P)$ is meromorphic in λ, have (simple) poles at $\lambda \in \sigma(\Delta_\infty)$ for which the eigenspace E_λ contains a function that is nonzero at P. Moreover, the restriction $F_{|\mathbb{R}}$ is strictly decreasing in between the poles.

Remark 2. *Note that we have sign difference in* (2.5) *from the original result because we consider the resolvent $(\Delta_\infty - \lambda)^{-1}$ as opposed to $(\lambda - \Delta_\infty)^{-1}$ used by Colin de Verdière. Hence the negative sign (-) in front of $R(\lambda; x, y)$.*

The function $F(\lambda; P)$ is a global invariant on X usually very diff cult to compute. In the case where X is homogeneous (in what follows we will be particularly interested in the cases $X = S^3$ and $X = \mathbb{T}^3$), the function $F(\lambda; P)$ is independent of P and in that case, the following equality holds:

$$F'(\lambda; P) = -\frac{1}{\text{Vol(X)}} \sum_{n=0}^{\infty} \frac{1}{(\lambda - \lambda_n)^2}.$$

Colin de Verdière used (2.2) and (2.5) to describe the spectrum of $\Delta_{\alpha,P}$. He proved the following

Theorem 3 (Colin de Verdière)**.** *The spectrum of $\Delta_{\alpha,P}$ is composed of*

1. *The set $\lambda \in \sigma(\Delta_\infty)$ such that the eigenspace E_λ contains at least one function that vanishes at P. In this case the multiplicity $n(\lambda;\alpha)$ is*

$$n(\lambda;\alpha) = \begin{cases} \dim E_\lambda - 1, & \textit{if } E_\lambda \textit{ contains at least one} \\ & \textit{function not vanishing at } P, \\ \dim E_\lambda + 0, & \textit{if } F(\lambda; P) \neq \cot\alpha, \\ \dim E_\lambda + 1, & \textit{if } F(\lambda; P) = \cot\alpha, \end{cases}$$

2. *The set of $\lambda \notin \sigma(\Delta_\infty)$ such that $F(\lambda; P) = \cot\alpha$ with multiplicity 1,*

$$\mu_0(\alpha) < \mu_1(\alpha) < \mu_2(\alpha) < \dots$$

2.3 Krein Formula

In this section, we remind the reader the classical formula of M. G. Krein (See appendix A in [3]) for the difference of resolvent operators of two self-adjoint extensions of the same symmetric operator with equal f nite def ciency indices. We restrict ourselves to the case $n_{\pm} = 1$, since it is the only only case we need in what follows.

Theorem 4 (Krein's Formula for the case $n_{\pm} = 1$)**.** *Let A_1 and A_2 be any self-adjoint extensions of a densely def ned closed symmetric operator A on a Hilbert space H. Then the following identity holds:*

$$(A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1} = k(\lambda)\,(\varphi_\lambda,\ \cdot\ )\,\varphi_\lambda \tag{2.6}$$

where $k(\lambda) \neq 0$, $\lambda \notin \sigma(A_1) \cup \sigma(A_2)$ and k and φ may be chosen to be analytic in $\lambda \notin \sigma(A_1) \cup \sigma(A_2)$. In fact, φ_λ may be def ned as

$$\varphi_\lambda = \varphi_{\lambda_0} + (\lambda - \lambda_0)\,(A_2 - \lambda)^{-1}\,\varphi_{\lambda_0} \tag{2.7}$$

where φ_{λ_0} satisf es

$$A^* \varphi_{\lambda_0} = \lambda_0 \varphi_{\lambda_0}, \tag{2.8}$$

for $\lambda_0 \in \mathbb{C} - \mathbb{R}$.

Notice that (2.8) implies that $\varphi_{\lambda_0} \in K_{\lambda_0}$ where K_{λ_0} is the def ciency subspace.

If we apply Krein's formula (2.6) to compare the resolvent kernels of the two different self-adjoint extensions $\Delta_{\alpha,P}$ and Δ_∞ keeping in mind that the def ciency indices $n_{\pm} = 1$, we obtain:

$$R_\alpha(\lambda; x, y) - R(\lambda; x, y) = k(\lambda; P) R(\lambda; x, P) R(\lambda; P, y) \tag{2.9}$$

where $R_\alpha(\lambda; x, y)$ is the resolvent kernel of $\Delta_{\alpha,P}$ and $R(\lambda; x, y)$ is as in (2.4).

Chapter 3
Comparison formula for determinants of Pseudo-Laplacian

In this chapter, we will present our main result: the comparison formula for determinants of Pseudo-Laplacian. In the f rst place, we prove a lemma that determines the relation between the coeff cient $k(\lambda)$ from Krein's formula (cf. (2.9)) and the scattering coeff cient $F(\lambda; P)$ that appears in (2.5). Then, we state a lemma about the analytic properties of the zeta function. Its proof coincides verbatim with that of Proposition 5.9 from [15], but for the sake of completeness, we will provide a complete version of its proof. After that, we derive the comparison formulas in dimensions 3 and 2 and f nally conclude the chapter with the examples of scattering coeff cients for S^3 and $\mathbb{T}^3$.

3.1 The Scattering Coeff cient

Krein's formula (2.9) compares the resolvent kernel $R_\alpha(\lambda; x, y)$ and $R(\lambda; x, y)$. From [11], we know almost everything about $R(\lambda; x, y)$, but we don't know much about $R_\alpha(\lambda; x, y)$ except the fact that it lies in $\mathcal{D}(\Delta_{\alpha,P})$.

We want to understand the behavior of the resolvent kernel $R_\alpha(\lambda; x, y)$ of the pseudo-Laplacian near the point P. We will compare the asymptotic behavior from both sides of the Krein formula (2.9) as $x \to P$ to obtain a closed form for $k(\lambda)$. We summarize the result in the following

Lemma 2. *In the limit as $x \to P$, the Krein coeff cient $k(\lambda)$ has the following closed form*

$$k(\lambda; P) = \frac{\sin\alpha}{F(\lambda; P)\sin\alpha - \cos\alpha}, \tag{3.1}$$

where $F(\lambda; P)$ is the scattering coeff cient introduced in lemma 1.

Proof. The idea of the proof is to compare the behavior of both the right hand side and the left hand side of (2.9) in the limit as $x \to P$. On one hand, the resolvent kernel $R_\alpha(\lambda\ ; \cdot\ , y) \in \mathcal{D}(\Delta_{\alpha,P})$, so the following expression from (2.2) in the limit as $x \to P$ holds,

$$R_\alpha(\lambda; x, y) = C(y) \cdot (\sin\alpha \cdot G_d(x, P) + \cos\alpha) + o(1),$$

where $G_d(x, P) = G_d(r)$ with r being the geodesic distance between x and P. On the other hand, the asymptotic behavior of $R(\lambda; x, y)$ is given by (2.5), that is:

$$-R(\lambda; x, y) = G_d(x, y) + F(\lambda) + o(1).$$

In the limit as $x \to P$, the left hand side of (2.9) can be simplif ed in the following way:

$$\begin{aligned} R_\alpha(\lambda; x, y) - R(\lambda; x, y) &= C(y) \cdot (\sin\alpha \cdot G_d(x, P) + \cos\alpha) \\ &\quad + G_d(x, y) + F(\lambda) + o(1) \\ &= C(y)\sin\alpha \cdot G_d(x, P) \\ &\quad + C(y)\cos\alpha + G_d(x, y) + F(\lambda) + o(1), \end{aligned} \tag{3.2}$$

whereas the right hand side becomes:

$$\begin{aligned} k(\lambda)R(\lambda; x, P)R(\lambda; P, y) &= k(\lambda)\left(G_d(x, P) + F(\lambda) + o(1)\right) \\ &\quad \times \left(G_d(P, y) + F(\lambda) + o(1)\right) \\ &= G_d(x, P)\left[k(\lambda)G_d(P, y) + k(\lambda)F(\lambda) + o(1)\right] \\ &\quad + k(\lambda)F(\lambda)G_d(P, y) + k(\lambda)F^2(\lambda) + o(1) \end{aligned} \tag{3.3}$$

In both equations (3.2) and (3.3), the term with $G_d(x, P)$ will dominate in the limit as $x \to P$. If we compare the coeff cients of the dominating terms, we get:

$$C(y)\sin\alpha = k(\lambda)G_d(y, P) + F(\lambda)k(\lambda) \tag{3.4}$$

Then, if we compare the remaining terms, we obtain

$$C(y)\cos\alpha + G_d(P, y) + F(\lambda) = k(\lambda)F(\lambda)G_d(P, y) + k(\lambda)F^2(\lambda). \tag{3.5}$$

What remains to do now is to isolate $G_d(y, P)$ from (3.4) and substitute it into (3.5). This yields the following relation:

$$\cos\alpha + \frac{\sin\alpha}{k(\lambda)} = F(\lambda)\sin\alpha, \tag{3.6}$$

from which we isolate $k(\lambda)$ and that finishes the proof of the lemma. □

From Krein's formula (2.9), it follows that the difference of the resolvent operators $(\Delta_{\alpha,P} - \lambda)^{-1}$ and $(\Delta - \lambda)^{-1}$ is a rank one operator. That is, the range of $(\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1}$ is a one dimensional subspace. We will prove a key result in the following

Lemma 3. *The difference of the resolvent operators $(\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1}$ is a trace class operator. The expression for its trace is given by :*

$$\mathrm{Tr}\left((\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta - \lambda)^{-1}\right) = \frac{F'_\lambda(\lambda; P)\sin\alpha}{\cos\alpha - F(\lambda; P)\sin\alpha}, \tag{3.7}$$

where $F(\lambda; P)$ is the scattering coefficient introduced in lemma 1.

Proof. We first recall the resolvent operator identity that holds for the Laplacian Δ_∞ with λ and μ regular points:

$$(\Delta_\infty - \lambda)^{-1} - (\Delta_\infty - \mu)^{-1} = (\lambda - \mu)(\Delta_\infty - \lambda)^{-1}(\Delta_\infty - \mu)^{-1}. \tag{3.8}$$

If we apply it to the resolvent kernel of $(\Delta_\infty - \lambda)^{-1}$, we obtain (using the "convolution" of two integral operators) the following equation:

$$\frac{R(\lambda; x, y) - R(\mu; x, y)}{(\lambda - \mu)} = \int R(\lambda; x, z) R(\mu; z, y) dz. \tag{3.9}$$

Taking the limit as $\mu \to \lambda$ in both sides of (3.9), we obtain

$$\left.\frac{dR(\mu; x, y)}{d\mu}\right|_{\mu=\lambda} = \int R(\lambda; x, z) R(\lambda; z, y)\, dz. \tag{3.10}$$

On the other hand, differentiating the expansion (2.5) with respect to λ yields

$$\frac{dR(\lambda; x, y)}{d\lambda} = -F'_\lambda(\lambda). \tag{3.11}$$

Equating both (3.10) and (3.11) gives us the following equation

$$\int R(\lambda; x, z) R(\lambda; z, y)\, dz = -F'_\lambda(\lambda). \tag{3.12}$$

Then, if we consider Krein's formula (2.9), and take the trace by integrating the resolvent kernels along the diagonal, we get

$$\begin{aligned} \operatorname{Tr}\left((\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta_\infty - \lambda)^{-1}\right) &:= \int R_\alpha(\lambda, x, x) - R(\lambda, x, x)\, dx \\ &= k(\lambda; P) \int R(\lambda; z, P) R(\lambda; P, z)\, dz \\ &= -k(\lambda; P)\, F'_\lambda(\lambda), \end{aligned} \tag{3.13}$$

where in the last equality of (3.13), we used (3.12). Finally, using the expression of $k(\lambda; P)$ from lemma 2, we obtain

$$\mathrm{Tr}\left((\Delta_{\alpha,P}-\lambda)^{-1}-(\Delta_{\infty}-\lambda)^{-1}\right)=\frac{F'_{\lambda}(\lambda;P)\sin\alpha}{\cos\alpha-F(\lambda;P)\sin\alpha} \tag{3.14}$$

and that finishes the proof. □

We introduce the domain $\Omega_{\alpha,P}\subset\mathbb{C}$ to be the set where we remove a downward vertical cut starting at each eigenvalue of Δ_{∞} or $\Delta_{\alpha,P}$,

$$\Omega_{\alpha,P}=\mathbb{C}\backslash\{\lambda-it,\lambda\in\sigma(\Delta_{\infty})\cup\sigma(\Delta_{\alpha,P});t\in[0,\infty)\}. \tag{3.15}$$

On this domain $\Omega_{\alpha,P}$, we can write the expression for the trace of the difference of resolvents (3.7) as the derivative (with respect to λ) of a logarithmic function.

That is,

$$\begin{aligned}\mathrm{Tr}\left((\Delta_{\alpha,P}-\lambda)^{-1}-(\Delta_{\infty}-\lambda)^{-1}\right) &= \frac{F'_{\lambda}(\lambda,P)\sin\alpha}{\cos\alpha-F(\lambda;P)\sin\alpha}\\ &=: 2\pi i\,\tilde{\xi}'(\lambda),\end{aligned} \tag{3.16}$$

where

$$\tilde{\xi}(\lambda):=\frac{-1}{2\pi i}\log\left(\cos\alpha-F(\lambda;P)\sin\alpha\right). \tag{3.17}$$

3.2 Contour Integration

We recall our goal to prove a comparison formula for the determinants of different self-adjoint extensions of the pseudo-Laplacian. In order to do that, we need to study the following expression

$$\zeta(s, \Delta_\infty - \tilde{\lambda}) - \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) \tag{3.18}$$

where $\tilde{\lambda}$ is a parameter which will be sent to 0 eventually.

We will consider the contour γ that encloses all the eigenvalues of $\Delta_{\alpha,P}$ and Δ_∞. The contour may be tailored to avoid any inconvenient intersections with the branch cut $c_{\tilde{\lambda}}$.

We have for $\Re(s) > 1$,

$$\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) = \frac{1}{2\pi i} \mathrm{Tr} \left(\int_{\tilde{\gamma}} \lambda^{-s} \ \left((\Delta_{\alpha,P} - \tilde{\lambda}) - \lambda \right)^{-1} d\lambda \right) \tag{3.19}$$

where $\tilde{\gamma}$ is the shifted contour that encloses the eigenvalues of $(\Delta_{\alpha,P} - \tilde{\lambda})$ and the eigenvalues of $(\Delta_\infty - \tilde{\lambda})$.

With a simple change of variable : $\mu = \tilde{\lambda} + \lambda$, then calling μ once again λ, the equation (3.19) can be written as

$$\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) = \frac{1}{2\pi i} \mathrm{Tr} \left(\int_{\gamma} (\lambda - \tilde{\lambda})^{-s} \ (\Delta_{\alpha,P} - \lambda)^{-1} \, d\lambda \right). \tag{3.20}$$

Similarly one gets an expression for the Laplacian:

$$\zeta(s, \Delta_\infty - \tilde{\lambda}) = \frac{1}{2\pi i} \mathrm{Tr} \left(\int_\gamma (\lambda - \tilde{\lambda})^{-s} \, (\Delta_\infty - \lambda)^{-1} d\lambda \right). \tag{3.21}$$

For the rest of this section, we will establish the groundwork, similar to the one adopted in [15] and in [2] to prove a lemma (C.f. Lemma 6) used in the proof of the main results.

Let C be a large enough positive real number and consider a branch $c_{\tilde{\lambda}}$ that lies in $\Omega_{\alpha,P}$ and that goes from $(-\infty + 0\,i)$ to $(-C + 0\,i)$ along the real axis, then from $(-C + 0\,i)$ to the point $\tilde{\lambda}$; we will denote the two pieces of this cut by $c_{\tilde{\lambda};1}$ and $c_{\tilde{\lambda};2}$ respectively. We then consider the contour $c_{\tilde{\lambda},\varepsilon}$ that follows the branch $c_{\tilde{\lambda}}$ at a distance ε. See f gure 3–1.

We note that the integrand that is common to (3.20) and (3.21) is holomorphic in the exterior of $c_{\tilde{\lambda};\varepsilon}$ union the exterior of γ. This is because the resolvent operators $(\Delta_{\alpha,P} - \lambda)^{-1}$ are compact away from the points enclosed by γ for any $\alpha \in [0, \pi)$ and the fact that $(\lambda - \tilde{\lambda})^{-s}$ can be extended to a holomorphic function anywhere away from the branch cut $c_{\tilde{\lambda}}$.

In addition to that, when the contour $c_{\tilde{\lambda};\varepsilon}$ shrinks around the cut $c_{\tilde{\lambda}}$, i.e., in the limit as $\varepsilon \to 0$, we obtain the following jump condition:

$$\lim_{\lambda \downarrow c_{\tilde{\lambda}}} e^{(-i\pi s)} (\lambda - \tilde{\lambda})^{-s} = \lim_{\lambda \uparrow c_{\tilde{\lambda}}} e^{(i\pi s)} (\lambda - \tilde{\lambda})^{-s} =: (\lambda - \tilde{\lambda})_0^{-s}. \tag{3.22}$$

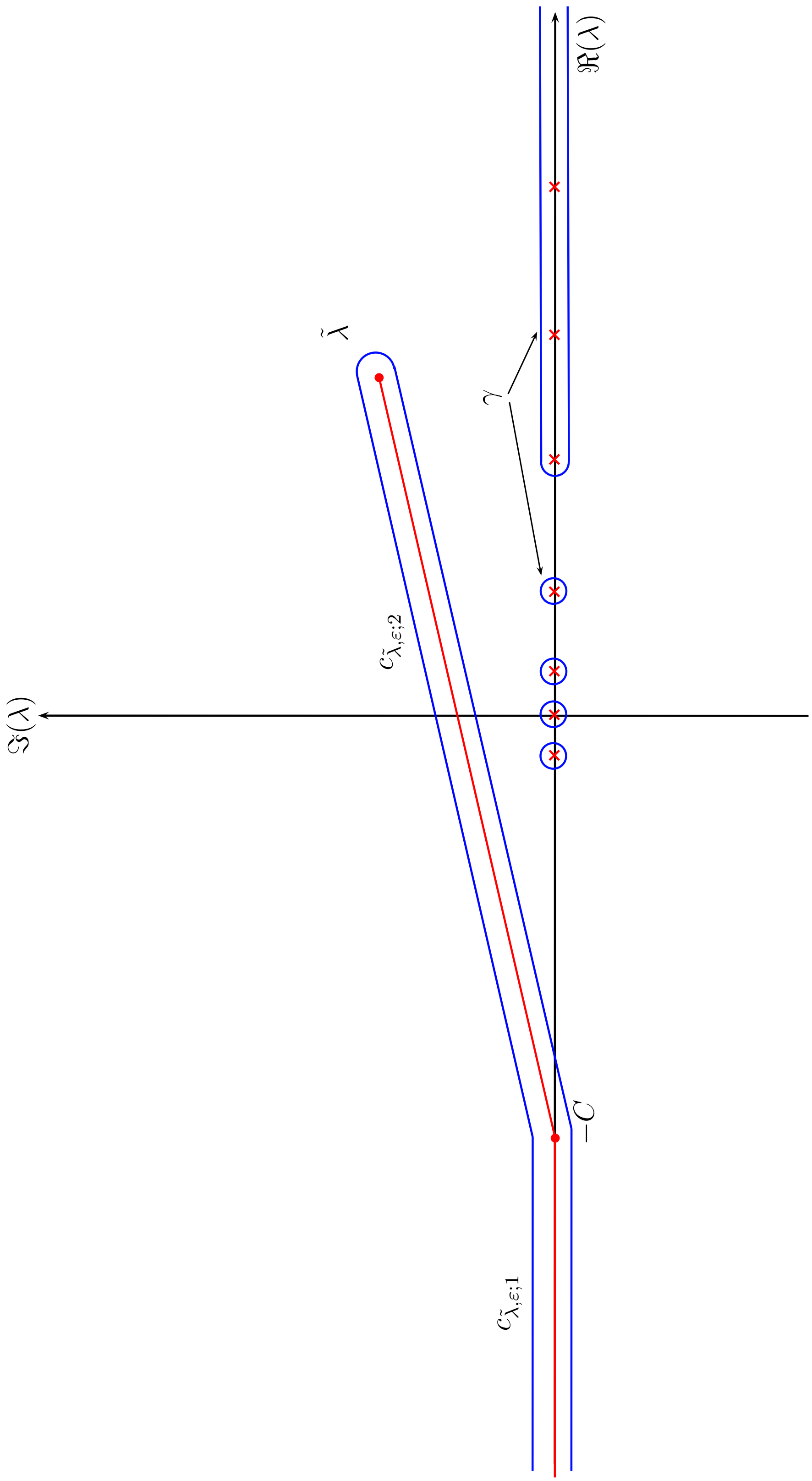

Figure 3–1: Contours for integration

Moreover, when $\Re(s) > 1$, the contribution from large circles centered at $\tilde{\lambda}$ tends to 0 as the radius of the circle tends to infinity. Therefore, by Cauchy integral formula, we can replace the contour γ by $c_{\tilde{\lambda};\varepsilon}$ in (3.20) and in (3.21) to obtain:

$$\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) - \zeta(s, \Delta_\infty - \tilde{\lambda}) = \frac{1}{2\pi i} \mathrm{Tr} \left(\int_{c_{\tilde{\lambda};\varepsilon}} (\lambda - \tilde{\lambda})^{-s} \left((\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta_\infty - \lambda)^{-1} \right) d\lambda \right) \tag{3.23}$$

Since the difference $(\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta_\infty - \lambda)^{-1}$ is a trace class by lemma 3, we can interchange the contour integral and the trace operation, which yields :

$$\zeta(s, \Delta_\infty - \tilde{\lambda}) - \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) = \frac{1}{2\pi i} \int_{c_{\tilde{\lambda},\varepsilon}} (\lambda - \tilde{\lambda})^{-s} \mathrm{Tr} \left((\Delta_{\alpha,P} - \lambda)^{-1} - (\Delta_\infty - \lambda)^{-1} \right) d\lambda \tag{3.24}$$

Then, we substitute eq (3.16) into (3.24) to obtain

$$\begin{aligned} \zeta(s, \Delta_\infty - \tilde{\lambda}) - \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) &= \frac{1}{2\pi i} \int_{c_{\tilde{\lambda},\varepsilon}} (\lambda - \tilde{\lambda})^{-s} \tilde{\xi}'(\lambda) d\lambda \\ &= \frac{1}{2\pi i} \int_{c_{\tilde{\lambda},\varepsilon;1}} (\lambda - \tilde{\lambda})^{-s} \tilde{\xi}'(\lambda) d\lambda \\ &+ \frac{1}{2\pi i} \int_{c_{\tilde{\lambda},\varepsilon;2}} (\lambda - \tilde{\lambda})^{-s} \tilde{\xi}'(\lambda) d\lambda \\ &=: \zeta_1(s) + \zeta_2(s) \end{aligned}$$

where $c_{\tilde{\lambda},\varepsilon;1}$ and $c_{\tilde{\lambda},\varepsilon;2}$ are the contours at a distance ε from the pieces $c_{\tilde{\lambda};1}$ and $c_{\tilde{\lambda};2}$ respectively.

The functions $\zeta_2(s)$ extends to an entire function of s. For all $s \in \mathbb{C}$ with $\Re(s) < 1$, we take the limit as $\varepsilon \to 0$, then use the jump condition expressed in (3.22) to obtain the following limit :

$$\hat{\zeta}_2(s) = \lim_{\varepsilon \to 0} \zeta_2(s) = 2i \sin(\pi s) \int_{-C}^{\tilde{\lambda}} (\lambda - \tilde{\lambda})_0^{-s} \tilde{\xi}'(\lambda) d\lambda. \tag{3.25}$$

where $(\lambda - \tilde{\lambda})_0^{-s}$ is the common limit def ned in (3.22).

Using the same jump condition, we f nd that in the limit as $\varepsilon \to 0$, the function $\zeta_1(s)$ becomes :

$$\hat{\zeta}_1(s) = \lim_{\varepsilon \to 0} \zeta_1(s) = 2i \sin(\pi s) \int_{-\infty}^{-C} (\lambda - \tilde{\lambda})_0^{-s} \tilde{\xi}'(\lambda) d\lambda. \tag{3.26}$$

But instead of using (3.26), we will express it in a form that we can study.

We f rst introduce the function

$$\rho(s, z) := (1 - z)^{-s} - 1 \tag{3.27}$$

def ned on $\mathbb{C} \times \{|z| < 1\}$ with properties given in the

Lemma 4 (Technical Lemma)**.** *For any $r < 1$ and any $R > 0$, the following upper bound for $\rho(s, z)$ hold for any $|z| < r$ and $|s| > R$:*

$$|\rho(s, z)| \leq \frac{e^{\frac{-Rr}{1-r}}}{1 - r} \cdot |s| \cdot |z| \tag{3.28}$$

Proof. Consider the power series of $(1-z)^{-s}$ that is given by

$$(1-z)^{-s} = e^{-s\log(1-z)} = \sum_{n=0}^{\infty} \frac{(-s)^n\left[\log(1-z)\right]^n}{n!}. \tag{3.29}$$

We can bound each $\log(1-z)$, in the case where $|z| \le r < 1$, as follows:

$$|\log(1-z)| = \left|\sum_{n=1}^{\infty} \frac{z^n}{n}\right| \le |z|\left|\sum_{n=0}^{\infty} z^n\right| \le \frac{|z|}{1-r}. \tag{3.30}$$

Substituting the bound (3.30) in the equation (3.29) yields the bound on $|\rho(s,z)|$. That is,

$$\left|(1-z)^{-s} - 1\right| \le \sum_{n=1}^{\infty} \frac{|s|^n\left(\frac{|z|}{1-r}\right)^n}{n!} = e^{\frac{|s||z|}{1-r}} - 1. \tag{3.31}$$

What remains now is to notice that the right hand side of previous expression (3.31) is nothing else but the following integral

$$e^{\frac{|s||z|}{1-r}} - 1 = \int_0^{\frac{|s||z|}{1-r}} e^u du \le e^{\frac{|s||z|}{1-r}} \cdot \frac{|s||z|}{1-r} = \frac{e^{\frac{Rr}{1-r}}}{1-r} \cdot |s| \cdot |z|, \tag{3.32}$$

and this finishes the proof of lemma 4. ☐

In the case where $z = \frac{\tilde{\lambda}}{\lambda}$, we have this useful

Corollary 1. *For any fixed* $\tilde{\lambda}$*, the function* $\rho(s, \tilde{\lambda}/\lambda) = O\left(|\lambda|^{-1}\right)$ *in the limit as* $\lambda \to -\infty$.

We will now turn back to the expression $\zeta_1(s)$. We start with

Lemma 5. *For* C *a large enough positive real number, there exist* $r < 1$*, such that* $\left|\frac{\tilde{\lambda}}{\lambda}\right| < r$ *for any* λ *with* $\Re(\lambda) \le -C$*. Moreover, the limit as* $\varepsilon \to 0$ *yields the following expression* *for the function:*

$$\zeta_1(s) = 2i\sin(\pi s)\int_{-\infty}^{-C} |\lambda|^{-s}\tilde{\xi}'(\lambda)\, d\lambda + 2i\sin(\pi s)\int_{-\infty}^{-C} |\lambda|^{-s}\tilde{\xi}'(\lambda)\rho(s,\tilde{\lambda}/\lambda)d\lambda\,. \quad (3.33)$$

Proof. We note that for $\lambda \notin (-\infty, -C)$, we can write

$$(\lambda - \tilde{\lambda})^{-s} = \lambda^{-s}\left(1 + \rho(s, \tilde{\lambda}/\lambda)\right). \quad (3.34)$$

The function $\zeta_1(s)$ is the contour integral

$$\zeta_1(s) = \left\{\int_{-\infty+i\varepsilon}^{-C+i\varepsilon} - \int_{-\infty-i\varepsilon}^{-C-i\varepsilon}\right\}(\lambda - \tilde{\lambda})^{-s}\tilde{\xi}'(\lambda)d\lambda\,. \quad (3.35)$$

Substituting (3.34) in one of the previous integrals (the f rst integral say) then taking the limit as $\varepsilon \to 0$ yields:

$$\begin{aligned} \int_{-\infty+i\varepsilon}^{-C+i\varepsilon}(\lambda - \tilde{\lambda})^{-s}\tilde{\xi}'(\lambda)d\lambda &= \int_{-\infty+i\varepsilon}^{-C+i\varepsilon}\lambda^{-s}\left(1 + \rho(s, \tilde{\lambda}/\lambda)\right)\tilde{\xi}'(\lambda)d\lambda \\ &= \int_{-\infty}^{-C}(\lambda - i\varepsilon)^{-s}\left(1 + \rho(s, \tilde{\lambda}/(\lambda - i\varepsilon)\right)\tilde{\xi}'(\lambda - i\varepsilon)d\lambda \\ &\xrightarrow{\varepsilon\to 0} e^{i\pi s}\int_{-\infty}^{-C}|\lambda|^{-s}\left(1 + \rho(s, \tilde{\lambda}/\lambda\right)\tilde{\xi}'(\lambda)d\lambda, \end{aligned} \quad (3.36)$$

where we used the fact that $\rho(s, \lambda)$ and $\tilde{\xi}'(\lambda)$ are continuous functions of λ.

Similarly, we obtain the following limit for the second integral in (3.35)

$$\int_{-\infty-i\varepsilon}^{-C-i\varepsilon}(\lambda - \tilde{\lambda})^{-s}\tilde{\xi}'(\lambda)d\lambda \xrightarrow{\varepsilon\to 0} e^{-i\pi s}\int_{-\infty}^{-C}|\lambda|^{-s}\left(1 + \rho(s, \tilde{\lambda}/\lambda\right)\tilde{\xi}'(\lambda)d\lambda \quad (3.37)$$

All that remains is to subtract the two integrals, (3.37) from (3.36) to obtain

$$\begin{aligned}
\zeta_1(s) &= e^{i\pi s}\int_{-\infty}^{-C}|\lambda|^{-s}\left(1+\rho(s,\tilde{\lambda}/\lambda)\right)\tilde{\xi}'(\lambda)d\lambda - e^{-i\pi s}\int_{-\infty}^{-C}|\lambda|^{-s}\left(1+\rho(s,\tilde{\lambda}/\lambda)\right)\tilde{\xi}'(\lambda)d\lambda \\
&= \left(e^{i\pi s}-e^{-i\pi s}\right)\int_{-\infty}^{-C}|\lambda|^{-s}\left(1+\rho(s,\tilde{\lambda}/\lambda)\right)\tilde{\xi}'(\lambda)d\lambda \\
&= 2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\left(1+\rho(s,\tilde{\lambda}/\lambda)\right)\tilde{\xi}'(\lambda)d\lambda \\
&= 2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\,d\lambda + 2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\rho(s,\tilde{\lambda}/\lambda)d\lambda\,,
\end{aligned}$$

and this f nishes the proof. □

The second integral in the right hand side of (3.33) is of a special importance and will be studied independently in Lemma 6 in the next section. For this purpose, it is convenient to introduce the function:

$$R_C(s,\tilde{\lambda}) := 2\,i\,\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\rho(s,\tilde{\lambda}/\lambda)d\lambda\,, \tag{3.38}$$

so that we can write in the limit as $\varepsilon \to 0$

$$\begin{aligned}
\zeta(s,\Delta_{\alpha,P}-\tilde{\lambda})) - \zeta(s,\Delta-\tilde{\lambda}) &= \hat{\zeta}_1(s)+\hat{\zeta}_2(s) \\
&= \Bigg[2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\,d\lambda \\
&+ 2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\rho(s,\tilde{\lambda}/\lambda)d\lambda\Bigg] + \hat{\zeta}_2(s) \\
&= 2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)\,d\lambda + R_C(s,\tilde{\lambda}) + \hat{\zeta}_2(s).
\end{aligned}$$

3.3 The Auxiliary Lemma

The lemmas proved in the previous section are used to study the regularity of the function $R_C(s, \tilde{\lambda})$ at the point $s = 0$. To alleviate the notation, we will drop the subscript C in $R_C(s, \tilde{\lambda})$ and simply write $R(s, \tilde{\lambda})$ instead.

Let us prove the auxiliary Lemma which is due to Hillairet and Kokotov in [15]

Lemma 6 (Auxiliary Lemma)**.** *Suppose that the function* $\tilde{\xi}'(\lambda)$ *from* (3.16) *is* $O\left(|\lambda|^{-1}\right)$ *as* $\lambda \to -\infty$*. Then the function*

$$R(s, \tilde{\lambda}) = \zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})) - \zeta(s, \Delta - \tilde{\lambda}) - 2i\sin(\pi s)\int_{-\infty}^{-C} |\lambda|^{-s}\tilde{\xi}'(\lambda)d\lambda - \hat{\zeta}_2(s) \quad (3.39)$$

can be analytically continued to $\Re s > -1$ *with* $R(0, \tilde{\lambda}) = \frac{d}{ds} R(s, \tilde{\lambda})\Big|_{s=0} = 0.$

Proof. Given the assumptions of the lemma, we note that the asymptotic behavior of the integrand of (3.38) is $O\left(|\lambda|^{-\Re(s)-2}\right)$ since $\rho(s, \tilde{\lambda}/\lambda) = O\left(|\lambda|^{-1}\right)$. Hence the integral will converge for any s with $\Re(s) > -1$. Moreover, since the integral is multiplied by $\sin(\pi s)$, we obtain $R(0, \tilde{\lambda}) = 0$.

For $\Re(s) > -1$, we can differentiate the expression (3.38) of $R(s, \tilde{\lambda})$ with respect to s to obtain

$$\begin{aligned} R_s'(s,\tilde{\lambda}) &= 2\pi\, i\, \cos(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) \rho(s, \tilde{\lambda}/\lambda) d\lambda \\ &+ 2\pi\, i\, \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \left(-\log|\lambda|\right) \tilde{\xi}'(\lambda) \rho(s, \tilde{\lambda}/\lambda) d\lambda \\ &- 2\pi\, i\, \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) \rho(s, \tilde{\lambda}/\lambda) \log(1 - \tilde{\lambda}/\lambda) d\lambda \qquad (3.40) \end{aligned}$$

The three integrands in the right hand side of (3.40) have, respectively, the following asymptotic behavior in the $\lambda \to -\infty$ regime: $O\left(|\lambda|^{-\Re(s)-2}\right)$, $O\left(|\lambda|^{-\Re(s)-2}\log|\lambda|\right)$ and $O\left(|\lambda|^{-\Re(s)-3}\right)$. All the integrals converge when $\Re(s) > -1$. Moreover, when $s = 0$, the second and the third integrals of the right hand side of (3.40) vanish because of the $\sin(\pi s)$ factor, while the f rst integral vanishes because $\rho(0, \tilde{\lambda}/\lambda) = 0$. Thus, we conclude that $R_s'(0, \tilde{\lambda}) = 0$ and that f nishes the proof of Lemma 6. □

The zeta function $\zeta(s, \Delta_\infty - \tilde{\lambda})$ is regular at $s = 0$ (in fact, it is true for any arbitrary elliptic differential operator on any compact manifold) and the function $\hat{\zeta}_2(s)$ is entire. Hence, by Lemma 6, the behavior of the function $\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})$ at $s = 0$ is fully determined by the properties of the analytic continuation of the term

$$2i \sin(\pi s) \int_{-\infty}^{-C} |\lambda|^{-s} \tilde{\xi}'(\lambda) d\lambda \qquad (3.41)$$

that appears in (3.39). These properties in their turn are determined by the asymptotic behavior of the function $\tilde{\xi}'(\lambda)$ as $\lambda \to -\infty$.

But, it turned out that the behavior of $\tilde{\xi}'(\lambda)$ depends on the dimension d of the manifold. In the next two sections, we will study both cases where $d = 3$ and $d = 2$ case.

3.4 Determinants of pseudo-Laplacians on three-dimensional manifolds

We will describe the asymptotic behavior as $\lambda \to -\infty$ of the scattering coeff cient $F(\lambda; P)$. We start with

Lemma 7. *Let X be a three-dimensional compact Riemannian manifold. The scattering coeff cient $F(\lambda; P)$ has the following asymptotic behavior:*

$$F(\lambda; P) = \frac{1}{4\pi}\sqrt{-\lambda} + c_1(P)\frac{1}{\sqrt{-\lambda}} + O\left(|\lambda|^{-3/2}\right) \tag{3.42}$$

as $\lambda \to -\infty$.

Proof. Consider Minakshisundaram-Pleijel asymptotic expansion ([28])

$$H(x, P; t) = (4\pi t)^{-3/2} e^{-r^2/(4t)} \sum_{k=0}^{\infty} u_k(x, P) t^k \tag{3.43}$$

for the heat kernel in a small vicinity of P. Where $r = \text{dist}(x, P)$ is the geodesic distance from x to P and the functions $u_k(\cdot, P)$ are smooth in a vicinity of P. The equality in (3.43) is understood in the sense of asymptotic expansions. We will make use of the standard relation

$$R(x, P; \lambda) = \int_0^{+\infty} H(x, P; t) e^{\lambda t} \, dt \, . \tag{3.44}$$

We f rst truncate the sum in (3.43) at some f xed $k = N - 1$ so that the remainder $r_n(t; x, P) = O\left(t^N\right)$ for small values of t, that is,

$$\sum_{k=0}^{\infty} u_k(x, P) t^k = \sum_{k=0}^{N-1} u_k(x, P) t^k + \underbrace{u_N(x, P) t^N + u_{N+1}(x, P) t^{N+1} + \dots}_{r_n(t; x, P)} \tag{3.45}$$

We def ne

$$\tilde{R}_N(x, P; -\lambda) := \int_0^{\infty} r_n(t; x, P) e^{t\lambda} dt \, . \tag{3.46}$$

Then, in the limit as $\lambda \to -\infty$, we see that

$$\tilde{R}_N(x, P; \lambda) = O\left(|\lambda|^{-(N+1)}\right) \tag{3.47}$$

uniformly with respect to x belonging to a small vicinity of P.

Now, for each $0 \leq k \leq N - 1$, we have to address the following quantity

$$R_k(x, P; \lambda) := \frac{u_k(x, P)}{(4\pi)^{3/2}} \int_0^{\infty} t^{k - \frac{3}{2}} e^{-\frac{r^2}{4t}} e^{\lambda t} dt. \tag{3.48}$$

Using well known explicit expression for the latter integral in the case $k = 0$ (see, e. g., [40]) we obtain the following asymptotic for $\lambda \ll 0$

$$R_0(x, P; \lambda) = \frac{u_0(x, P)}{4\pi r} e^{-r\sqrt{-\lambda}} = \frac{1}{4\pi r} - \frac{1}{4\pi}\sqrt{-\lambda} + o(1), \qquad (3.49)$$

as $r \to 0$. We also used that $u_0(P, P) \equiv 1$ (see construction of the parametrix in [28]).

On the other hand, for $k \geq 1$ one has the following asymptotic for $\lambda \ll 0$

$$\begin{aligned} R_k(x, P; \lambda) &= \frac{u_k(x, P)}{(4\pi)^{3/2}} 2^{3/2-k} \left(\frac{r}{\sqrt{-\lambda}}\right)^{k-1/2} K_{k-\frac{1}{2}}(r\sqrt{-\lambda}) \\ &= u_k(x, P) \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2k-3)}{\pi \cdot 2^{k+2}} \cdot \frac{1}{(\sqrt{-\lambda})^{2k-1}} + o(1) \\ &=: -c_k(P)\frac{1}{(\sqrt{-\lambda})^{2k-1}} + o(1) \end{aligned} \qquad (3.50)$$

as $r \to 0$ (see [5], p. 146, f-la 29). In the previous equation (3.50), the function $K_m(x)$ is the modif ed Bessel function of the second type (Bessel K function) of order m. Now (3.42) follows from (3.43), (3.44), (3.49) and (3.50). That is,

$$\begin{aligned} R(x, P; \lambda) &= R_0(x, P; \lambda) + R_1(x, P; \lambda) + R_2(x, P; \lambda) + \cdots \\ &= \frac{1}{4\pi r} - \frac{1}{4\pi}\sqrt{-\lambda} + \frac{u_1(x, P)}{8\pi}\frac{1}{\sqrt{-\lambda}} + \frac{3u_2(x, P)}{16\pi}\frac{1}{\left(\sqrt{-\lambda}\right)^3} + \cdots \\ &= \frac{1}{4\pi r} - \frac{1}{4\pi}\sqrt{-\lambda} - c_1(P)\frac{1}{\sqrt{-\lambda}} - c_2(P)\frac{1}{\left(\sqrt{-\lambda}\right)^3} + \cdots \\ &= -G_3(r) - F(\lambda; P) + o(1) \end{aligned}$$

where in the last step, we compared the previous calculations with the expression provided by Colin de Verdière in (2.5) (recall that $G_3(r)$ is the Green function for the Laplacian in $\mathbb{R}^3$) and this proves the expression for the scattering coeff cient $F(\lambda; P)$ stated in the lemma. □

In section 3.1, we obtained the following expression for

$$2\pi i\tilde{\xi}'(\lambda) = \frac{F'_\lambda(\lambda; P)\sin\alpha}{\cos\alpha - F(\lambda; P)\sin\alpha}. \tag{3.51}$$

If we substitute the expansion of $F(\lambda; P)$ obtained in Lemma 7 in (3.51), and expand in the the variable λ in the limit as $\lambda \to -\infty$, we obtain:

$$2\pi i\tilde{\xi}'(\lambda) = -\frac{1}{2\lambda} + O\left(|\lambda|^{-3/2}\right). \tag{3.52}$$

Therefore, we can rewrite (3.41) as

$$\begin{aligned} 2i\sin(\pi s)\int_{-\infty}^{-C} |\lambda|^{-s}\,\tilde{\xi}'(\lambda)\,d\lambda &= \frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C} |\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda} - \frac{1}{2\lambda}\right)d\lambda \\ &= \frac{\sin(\pi s)}{\pi}\left\{\int_{-\infty}^{-C} |\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda}\right)d\lambda + \frac{C^{-s}}{2s}\right\} \end{aligned}$$

where in the last equality, we evaluated one piece of the integral. We also know from (3.52) that $2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda} = O\left(|\lambda|^{-3/2}\right)$. Hence, the integral

$$\frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C} |\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda}\right)d\lambda \tag{3.53}$$

is analytic for $\Re(s) > -\frac{1}{2}$. Therefore, from the discussion at the end of section 3.3 and equation (3.39) in Lemma 6, we conclude that $\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})$ is regular at $s = 0$.

Now, we can write the usual zeta-regularized expression

$$\det(\Delta_{\alpha,P} - \tilde{\lambda}) = \exp\left\{-\zeta'(0, \Delta_{\alpha,P} - \tilde{\lambda})\right\} \tag{3.54}$$

for the determinant of the shifted pseudo-Laplacian $(\Delta_{\alpha,P} - \tilde{\lambda})$.

From Lemma 6, we know that $R'(0, \tilde{\lambda}) = 0$, thus,

$$\frac{d}{ds}\left\{\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})) - \zeta(s, \Delta_{\infty} - \tilde{\lambda})\right\}\Big|_{s=0} = \frac{d}{ds}\left\{2i\sin(\pi s)\int_{-\infty}^{-C} |\lambda|^{-s}\tilde{\xi}'(\lambda)d\lambda - \hat{\zeta}_2(s)\right\}\Big|_{s=0} \tag{3.55}$$

We will rewrite the right hand side of the previous equation as a sum of three terms that we will treat individually:

$$\frac{d}{ds}\left\{\frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C} |\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda}\right)d\lambda + \frac{\sin(\pi s)\; C^{-s}}{2\pi s} - \hat{\zeta}_2(s)\right\}\Big|_{s=0}. \tag{3.56}$$

We start by differentiating the f rst term using the product rule and obtain:

$$\cos(\pi s)\int_{-\infty}^{-C} |\lambda|^{-s}\left(2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda}\right)d\lambda - \frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C} |\lambda|^{-s}\log(|\lambda|)\left(2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda}\right)d\lambda. \tag{3.57}$$

Since both integrals in (3.57) are convergent for $s = 0$, the limit as $s \to 0$ makes the second term vanish because $\sin(\pi s) \to 0$; whereas the first term becomes

$$\int_{-\infty}^{-C} \left(2\pi i \tilde{\xi}'(\lambda) + \frac{1}{2\lambda}\right) d\lambda. \tag{3.58}$$

The derivative of the second term is calculated as follows:

$$\frac{d}{ds}\frac{\sin(\pi s)\ C^{-s}}{2\pi s} = \frac{\cos(\pi\, s)\, C^{-s}}{2s} - \frac{\sin(\pi\, s)\, C^{-s}\log(C)}{2\pi\ s} - \frac{\sin(\pi\, s)\, C^{-s}}{2\pi\ s^2} \tag{3.59}$$

and in the limit as $s \to 0$, we obtain

$$\lim_{s\to 0}\frac{d}{ds}\frac{\sin(\pi s)\ C^{-s}}{2\pi s} = -\frac{1}{2}\log(C). \tag{3.60}$$

For the last term in (3.56), we recall that $\hat{\zeta}_2(s)$ is entire, hence, we will differentiate its expression from (3.25) using the product rule and obtain

$$\begin{aligned}
\frac{d}{ds}\hat{\zeta}_2(s) &= \frac{d}{ds}\ 2i\sin(\pi s)\int_{-C}^{\tilde{\lambda}} (\lambda - \tilde{\lambda})_0^{-s}\tilde{\xi}'(\lambda)\ d\lambda \\
&= 2\pi i\cos(\pi s)\int_{-C}^{\tilde{\lambda}} (\lambda - \tilde{\lambda})_0^{-s}\tilde{\xi}'(\lambda)\ d\lambda + 2i\sin(\pi s)\frac{d}{ds}\int_{-C}^{\tilde{\lambda}} (\lambda - \tilde{\lambda})_0^{-s}\tilde{\xi}'(\lambda)\ d\lambda
\end{aligned} \tag{3.61}$$

and in the limit as $s \to 0$, the first term in (3.61) becomes

$$2\pi i\int_{-C}^{\tilde{\lambda}} \tilde{\xi}'(\lambda)\ d\lambda = 2\pi i\left(\tilde{\xi}(\tilde{\lambda}) - \tilde{\xi}(-C)\right). \tag{3.62}$$

The second term in (3.61) needs more attention because $(\lambda - \tilde{\lambda})_0^{-s}$ is given by a limit in (3.22). To deal with it, we need to differentiate first with respect to s, take the limit as $s \to 0$, then take the limit as $\varepsilon \to 0$ (that is, shrink the contour $c_{\tilde{\lambda};\varepsilon}$ around the cut $c_{\tilde{\lambda}}$). In that case, the term vanishes and hence the only contribution to the third term in (3.56) comes from (3.62).

Now that we obtained the derivative of every term in (3.56), we can rewrite (3.55) using the contributions from (3.58), (3.60) and (3.62) as

$$\frac{d}{ds}\left\{\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})) - \zeta(s, \Delta_\infty - \tilde{\lambda})\right\}\Big|_{s=0} =$$
$$2\pi i\left(\tilde{\xi}(\tilde{\lambda}) - \tilde{\xi}(-C)\right) + \int_{-\infty}^{-C}\left(2\pi i\tilde{\xi}'(\lambda) + \frac{1}{2\lambda}\right)\, d\lambda - \frac{1}{2}\log(C). \tag{3.63}$$

We note that the choice of the real value C in section 3.2 was arbitrary, therefore the expression (3.63) should not depend on C and hence we can send $C \to \infty$. In that case, the integral of the middle term in (3.63) vanishes. For the term $2\pi i\tilde{\xi}(-C)$, we will use (3.17) and the asymptotics of $F(\lambda; P)$ from Lemma 7 to obtain

$$\begin{aligned} -2\pi i\tilde{\xi}(-C) &= \log\left(\cos\alpha - F(-C;P)\sin\alpha\right) \\ &= \log\left(\cos\alpha - \left(\frac{\sqrt{C}}{4\pi} + O\left(\frac{1}{\sqrt{C}}\right)\right)\sin\alpha\right). \end{aligned} \tag{3.64}$$

Now, for a f xed α and when $C \to \infty$, the term $\left(\frac{\sqrt{C}}{4\pi}\sin\alpha\right)$ dominates in the logarithm. Hence, (3.64) has the asymptotics

$$-2\pi i\tilde{\xi}(-C) = \log\left(-\frac{\sqrt{C}}{4\pi}\sin\alpha\right) + o(1), \tag{3.65}$$

for large values of λ and which reduces to

$$-2\pi i\tilde{\xi}(-C) = i\pi + \frac{1}{2}\log(C) - \log(4\pi) + \log(\sin\alpha) + o(1). \tag{3.66}$$

When we substitute (3.66) in (3.63) and take the limit as $C \to \infty$, the equation (3.63) becomes

$$\begin{aligned}
\frac{d}{ds}\left\{\zeta(s,\Delta_{\alpha,P}-\tilde{\lambda})) - \zeta(s,\Delta_\infty-\tilde{\lambda})\right\}\Big|_{s=0} &= 2\pi i\tilde{\xi}(\tilde{\lambda}) + i\pi - \log(4\pi) + \log(\sin\alpha) \\
&= -\log\left(\cos\alpha - F(\tilde{\lambda};P)\sin\alpha\right) + \log(\sin\alpha) \\
&\quad - \log(4\pi) + i\pi \\
&= -\log\left(\cot\alpha - F(\tilde{\lambda};P)\right) - \log(4\pi) + i\pi \\
&= -\log\left(4\pi\left(F(\tilde{\lambda};P) - \cot\alpha\right)\right). \qquad (3.67)
\end{aligned}$$

Now we are equipped to prove the f rst main

Proposition 3. *For $d = 3$, let $\Delta_{\alpha,P}$ be a non-Friedrich's extension of the pseudo-Laplacian Δ_P on X ($\alpha \neq 0$) and Δ_∞ the usual (Friedrich's) Laplacian and let $\tilde{\lambda} \in \mathbb{C} \setminus \{\sigma(\Delta) \cup \sigma(\Delta_\infty)\}$.*

Then the following relation holds

$$\det\left(\Delta_{\alpha,P}-\tilde{\lambda}\right) = -4\pi\left(\cot\alpha - F(\tilde{\lambda};P)\right)\det\left(\Delta_{\infty}-\tilde{\lambda}\right). \tag{3.68}$$

Proof. We starts by using the def nition of the zeta regularized determinant given in (3.54).

We can write

$$\begin{aligned}
\frac{\det\left(\Delta_{\infty}-\tilde{\lambda}\right)}{\det\left(\Delta_{\alpha,P}-\tilde{\lambda}\right)} &= \exp\left(\frac{d}{ds}\left\{\zeta(s,\Delta_{\alpha,P}-\tilde{\lambda})) - \zeta(s,\Delta_{\infty}-\tilde{\lambda})\right\}\Big|_{s=0}\right) \\
&= \exp\left(-\log\left(4\pi\left(F(\tilde{\lambda};P)-\cot\alpha\right)\right)\right) \\
&= \frac{1}{-4\pi\left(\cot\alpha - F(\tilde{\lambda};P)\right)}
\end{aligned}$$

□

For the rest of this section, we will investigate the behavior of equation (3.68) when $\tilde{\lambda} \to 0$. Since 0 is a simple eigenvalue of the Laplacian Δ_{∞}, and $\varphi \equiv 1$ is an eigenfunction corresponding to the eigenvalue $\lambda = 0$, it follows from Theorem 2 in [11] that $\lambda = 0 \notin \sigma(\Delta_{\alpha,P})$ for any $\alpha \in (0,\pi)$. Thus, every $\Delta_{\alpha,P}$ has one strictly negative eigenvalue. Hence, the determinant in the left hand side of 3.68 is well def ned for $\tilde{\lambda} = 0$, whereas the determinant of the right hand side has the following asymptotics as $\tilde{\lambda} \to 0^{-}$

$$\det\left(\Delta_{\infty}-\tilde{\lambda}\right) \sim (-\tilde{\lambda})\det{}^{*}(\Delta_{\infty}), \tag{3.69}$$

where $\det^{*}$ is the modif ed determinant of an operator with zero mode.

The resolvent kernel $R(x, y; \lambda)$ has the following asymptotics when $\lambda \to 0$ and dist$(x, y) \to 0$,

$$-R(x, y; \lambda) = \frac{1}{\text{Vol}(X)} \frac{1}{\lambda} + G_3(r) + O(1). \tag{3.70}$$

Thus, one gets the following expression for

$$F(\lambda; P) = \frac{1}{\text{Vol}(X)} \frac{1}{\lambda} + O(1) \tag{3.71}$$

as $\lambda \to 0$.

Let us take the limit in (3.68) as $\tilde{\lambda} \to 0$ and use the asymptotics for the determinants that we just derived in (3.69) and (3.71). We state the result in the following corollary of Proposition 3

Corollary 2. *For $\alpha \in (0, \pi)$, the following relation holds true*

$$\det(\Delta_{\alpha,P}) = -\frac{4\pi}{\text{Vol(X)}} \det{}^*(\Delta_\infty). \tag{3.72}$$

Remark 3. *We note the minus* $(-)$ *sign in* (3.72) *that "confrms" the existence of a negative eigenvalue for every pseudo-Laplacian.*

Remark 4. *Note also that equation* (3.72) *doesn't depend on the parameter α which implies that all the self-adjoint extensions (except the one of Friedrich's) have the same determinant.*

Remark 5. *It is also important to notice that* (3.72) *is independent of the chosen point* P *since the leading coeff cients in* (3.71) *and in the scattering coeff cient are both independent of the point* P*.*

3.5 Determinants of pseudo-Laplacians on two-dimensional manifolds

As we mentioned in the introduction, the zeta regularized determinant is not that standard in the case of a two-dimensional Riemannian surface X. We will show that it has a logarithmic singularity at $s = 0$.

We start by studying the scattering coeff cient $F(\lambda; P)$ from the resolvent kernel $R(\lambda; z, w)$ of the Laplacian on X. We introduce the isothermal local coordinates (x, y) and set $z := x + iy$. Then we can write the area element on X as

$$\rho^{-2}(z)\ |dz|^2. \tag{3.73}$$

The following estimate of the resolvent kernel, $R(\lambda; z, w)$, of the Laplacian on X was found by J. Fay (see [13]; Theorem 2.7 on page 38 and the formula preceding Corollary 2.8 on page 39). Notice that Fay works with negative Laplacian, so we had to change the signs when using his formulas).

Lemma 8 (J. Fay)**.** *The resolvent kernel of the Laplacian on a a two-dimensional manifold has the following asymptotic expression*

$$\begin{aligned} -R(\lambda; z, w) &= G_2(r) + O(r) + \frac{1}{2\pi}\left[\gamma + \log\frac{\sqrt{|\lambda|+1}}{2} + \hat{R}(\lambda; z, w)\right. \\ &\left. -\frac{1}{2\left(|\lambda|+1\right)}\left(1 + \frac{4}{3}\rho^2(z)\partial^2_{z\bar{z}}\log\rho(z)\right)\right], \end{aligned} \tag{3.74}$$

where $O(r)$ is λ-independent, $\hat{R}(\lambda; z, w)$ is continuous for w near z, and

$$\hat{R}(\lambda; z, z) = O\left(|\lambda|^{-2}\right)$$

uniformly with respect to $z \in X$ as $\lambda \to -\infty$; $r = \operatorname{dist}(z, w)$, $\gamma = 0.57721566$ is the Euler constant.

After comparing (3.74) with the expression (2.5) for the resolvent introduced by Colin de Verdière, we obtained the following expression for the scattering coeff cient

$$\begin{aligned} F(\lambda; P) &= \frac{1}{4\pi}\log\left(|\lambda|+1\right) + \frac{\gamma - \log 2}{2\pi} \\ &- \frac{1}{4\pi\left(|\lambda+1|\right)}\left[1 + \frac{4}{3}\rho^2(z)\partial^2_{z\bar{z}}\log\rho\left(z\right)\right] + O\left(|\lambda|^{-2}\right). \end{aligned} \tag{3.75}$$

Therefore, from the def nition (3.16), we can write

$$\begin{aligned} 2\pi i\tilde{\xi}'(\lambda) &= \frac{F'_\lambda(\lambda; P)\sin\alpha}{\cos\alpha - F(\lambda; P)\sin\alpha} \\ &= -\frac{\frac{1}{4\pi(|\lambda|+1)} - \frac{B}{(|\lambda|+1)^2} + O\left(|\lambda|^{-3}\right)}{\cot\alpha - A - \frac{1}{4\pi}\log(|\lambda|+1) + \frac{B}{|\lambda|+1} + O\left(|\lambda|^{-2}\right)}, \end{aligned} \tag{3.76}$$

where $A = \frac{1}{2\pi}(\gamma - \log 2)$ and $B = \frac{1}{4\pi}\left(1 + \frac{3}{4}\rho^2(z)\,\partial^2_{z\bar{z}}\log\rho(z)\right)$. We note that both expressions A and B are λ-independent. When considering $-\infty < \lambda \leq -C$, the asymptotics of $2\pi i\tilde{\xi}'(\lambda)$ will be given by:

$$2\pi i\tilde{\xi}'(\lambda) = \frac{1}{|\lambda|\,(\log|\lambda| - 4\pi\cot\alpha + 4\pi A)} + f(\lambda) \tag{3.77}$$

with $f(\lambda) = O\left(|\lambda|^{-2}\right)$ as $\lambda \to -\infty$.

All that remains now is to study the behavior of (3.41) knowing (3.77). That is,

$$\begin{aligned} 2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)d\lambda \quad &= \quad \frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}\frac{|\lambda|^{-s-1}}{(\log|\lambda| - 4\pi\cot\alpha + 4\pi A)}\,d\lambda \\ &+ \quad \frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}|\lambda|^{-s}f(\lambda)\,d\lambda \end{aligned} \tag{3.78}$$

The second integral in the right hand side of (3.78) is regular for $s = 0$, whereas the f rst one is a special integral that was studied by Kirsten-Loya-Park in [22]. We will present their result in the following

Lemma 9 (Kirsten-Loya-Park)**.** *Let $C > e^{\kappa} > 0$, then the following relation holds*

$$\int_C^{\infty}\frac{\lambda^{-s-1}}{\log(\lambda) - \kappa}d\lambda = -e^{-s\kappa}\log\left(\frac{s}{2}\right) - e^{-s\kappa}\left(\gamma + \log\left(2\log C - \kappa\right) + e(s)\right) \tag{3.79}$$

where $e(s)$ is an entire function of s that is $O(s)$ at $s = 0$.

In their proof, Kristen, Loya and Park started by a series of change of variables, $u := \log(\lambda) - \kappa$, then $y = su$ to write the original integral in terms of the exponential integral

$\mathrm{Ei}(z)$, then use its power series solution:

$$\mathrm{Ei}(z) := -\int_{-z}^{\infty} e^{-y}\frac{dy}{y} = \gamma + \log(-z) + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!}. \tag{3.80}$$

Note that the series in the right hand side of (3.80) has better convergence properties than the exponential function.

So, we can expand the f rst integral of the right hand side of (3.78) as follows:

$$\begin{aligned} \frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C} \frac{|\lambda|^{-s-1}}{(\log|\lambda| - \kappa)}\, d\lambda &= -\frac{\sin(\pi s)}{\pi} e^{-s\kappa}\mathrm{Ei}\left(-s\left(\log C - \kappa\right)\right) \\ &= -\frac{\sin(\pi s)}{\pi} e^{-s\kappa}\left(\gamma + \log\left(s(\log C - \kappa)\right) + e(s)\right) \\ &= -s\log s + g(s) \end{aligned} \tag{3.81}$$

where $g(s)$ is differentiable at $s = 0$, $e(s)$ is an entire function such that $e(0) = 0$ and $\kappa = 4\pi\cot\alpha - 2\gamma + \log 4$.

Thus, we conclude that the zeta regularized determinant as def ned in (1.5) has a logarithmic singularity, but we can still associate a natural def nition to the determinant by subtracting the singularity which motivates

Def nition 1. *Let $\Delta_{\alpha,P}$ be the pseudo-Laplacian on a two-dimensional compact Riemannian manifold. Then, the zeta-regularized determinant of the operator $\left(\Delta_{\alpha,P} - \tilde{\lambda}\right)$ with*

$\tilde{\lambda} \in \mathbb{C} \setminus \sigma\left(\Delta_{\alpha,P}\right)$ *is defined as*

$$\det\left(\Delta_{\alpha,P} - \tilde{\lambda}\right) = \exp\left\{-\frac{d}{ds}\left[\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda}) + s\log s\right]\Big|_{s=0}\right\}. \tag{3.82}$$

With this definition, we are ready to obtain a formula relating $\det(\Delta_{\alpha,P} - \tilde{\lambda})$ and $\det\left(\Delta_\infty - \tilde{\lambda}\right)$.

In Lemma 6, we showed that $R'(0, \tilde{\lambda}) = 0$. This means that

$$\frac{d}{ds}\left\{\zeta(s, \Delta_{\alpha,P} - \tilde{\lambda})) + s\log s - \zeta(s, \Delta_\infty - \tilde{\lambda})\right\}\Big|_{s=0} =$$

$$\frac{d}{ds}\left\{2i\sin(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}\tilde{\xi}'(\lambda)d\lambda + s\log s + \hat{\zeta}_2(s)\right\}\Bigg|_{s=0} \tag{3.83}$$

Thus, substituting (3.78) in (3.83) and using (3.81) yields the equation

$$\frac{d}{ds}\hat{\zeta}_2(s)\Bigg|_{s=0} + \frac{d}{ds}\left(\frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}|\lambda|^{-s}f(\lambda)d\lambda\right)\Bigg|_{s=0} -$$

$$\frac{d}{ds}\left(\frac{\sin(\pi s)}{\pi}e^{-s\kappa}\left[\gamma + \log\left(s(\log C - \kappa)\right) + e(s)\right] - s\log s\right)\Bigg|_{s=0} \tag{3.84}$$

that we will study term by term.

The first term is treated the same way we did in the dimension 3 case. It will become

$$\frac{d}{ds}\hat{\zeta}_2(s)\Bigg|_{s=0} = 2\pi i\left(\tilde{\xi}(\tilde{\lambda}) - \tilde{\xi}(-C)\right) \tag{3.85}$$

In the second term, we differentiate using the product rule to obtain

$$\frac{d}{ds}\left(\frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}|\lambda|^{-s}f(\lambda)d\lambda\right) = -\frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}|\lambda|^{-s}\log|\lambda|f(\lambda)d\lambda + \cos(\pi s)\int_{-\infty}^{-C}|\lambda|^{-s}f(\lambda)d\lambda, \tag{3.86}$$

then consider the limit as $s \to 0$. The f rst term in (3.86) goes to 0 because the function $f(\lambda) = O\left(|\lambda|^{-2}\right)$ which makes the integral converge but the $\sin(\pi s)$ will make the whole term vanish. So the only contribution that comes from (3.86) is the limit as $s \to 0$ of the second integral in the right hand side and which yields:

$$\frac{d}{ds}\left(\frac{\sin(\pi s)}{\pi}\int_{-\infty}^{-C}|\lambda|^{-s}f(\lambda)d\lambda\right) = \int_{-\infty}^{-C}f(\lambda)d\lambda. \tag{3.87}$$

After differentiating the last term of (3.86) with respect to s using the product rule, we obtain:

$$\cos(\pi s)e^{-s\kappa}\left(\gamma + \log(s) + \log\left(\log(C) - \kappa\right) + e(s)\right) + \frac{\sin\left(\pi s\right)}{\pi}e^{-s\kappa}\left(s^{-1} + \frac{d}{ds}e(s)\right)$$
$$-\frac{\sin(\pi s)}{\pi}\kappa e^{-s\kappa}\left(\gamma + \log(s) + \log\left(\log(C) - \kappa\right) + e(s)\right) - \log(s) - 1,$$

and when taking the limit as $s \to 0$, the only terms that remain are

$$-\gamma - \log\left(\log(C) - \kappa\right). \tag{3.88}$$

Now that we investigated all the terms in (3.84), we can put together their contributions and obtain

$$\frac{d}{ds}\left[\zeta(s,\Delta_{\alpha,P}-\tilde{\lambda})+s\log s-\zeta(s,\Delta_{\infty}-\tilde{\lambda})\right]\Big|_{s=0} = 2\pi i\tilde{\xi}(\tilde{\lambda})-\gamma+\int_{-\infty}^{-C}f(\lambda)\,d\lambda-2\pi i\tilde{\xi}(-C)$$

$$-\log\left[\log C-4\pi\cot\alpha+2\gamma-\log 4\right].\quad(3.89)$$

But the choice of C large enough was arbitrary, so (3.89) should not depend on C. Hence we can take the limit as $-C\to-\infty$ and make use of (3.75) to obtain

$$\frac{d}{ds}\left[\zeta(s,\Delta_{\alpha,P}-\tilde{\lambda})+s\log s-\zeta(s,\Delta-\tilde{\lambda})\right]\Big|_{s=0} = 2\pi i\tilde{\xi}(\tilde{\lambda})-\gamma+\log\left(\sin\frac{\alpha}{4\pi}\right)-i\pi$$

Now, we are equipped to prove the second main proposition in the 2 dimensional case.

Proposition 4. *For $d=2$, let $\Delta_{\alpha,P}$ be a non-Friedrich's extension of the pseudo-Laplacian Δ_P on X ($\alpha\neq 0$) and Δ_{∞} the usual (Friedrich's) Laplacian and let $\tilde{\lambda}\in\mathbb{C}\setminus\{\sigma(\Delta)\cup\sigma(\Delta_{\infty})\}$. Let also the zeta-regularized determinant of $\Delta_{\alpha,P}$ be defined as in Definition 1. Then the following relation holds*

$$\det(\Delta_{\alpha,P}-\tilde{\lambda})=-4\pi e^{\gamma}(\cot\alpha-F(\tilde{\lambda},P))\det(\Delta-\tilde{\lambda})\,.\quad(3.90)$$

The proof is done the same way as the proof in dimension 3 case so will omit the details. As for the rest of this section, we will also consider the limit as $\tilde{\lambda}\to 0$ which will yield the next

Corollary 3. *For $\alpha \in (0,\pi)$ the following relation holds true*

$$\det\Delta_{\alpha,P} = -\frac{4\pi e^{\gamma}}{\mathrm{Vol}(X)}\det{}^{*}\Delta\,. \tag{3.91}$$

Similar remarks as in the 3 dimensional case hold after analyzing the previous results.

Remark 6. *The minus $(-)$ sign in* (3.91) *also "conf rms" the existence of a negative eigenvalue for every pseudo-Laplacian.*

Remark 7. *Note also that equation* (3.91) *doesn't depend on the parameter α which implies that all the self-adjoint extensions (except the one of Friedrich's) have the same determinant in 2 dimensions also.*

Remark 8. *It is also important to notice that* (3.91) *is also independent of the chosen point P.*

3.6 Scattering coeff cients for S^3 and $\mathbb{T}^3$

In the following section, we will provide an explicit formula for the scattering coeff - cient $F(\lambda; P)$ when $X = S^3$ and when $X = \mathbb{T}^3$. The case $X = \mathbb{T}^3$ is trivial, whereas the case $X = S^3$ is somewhat more complicated; Colin de Verdière mentions the possibility to f nd this scattering coeff cient explicitly but we failed to f nd such an expression in the literature.

Lemma 10. *The scattering coefficient $F(\lambda; P)$ on S^3 with the usual round metric is given by*

$$F(\lambda) = \frac{1}{4\pi}\coth\left(\pi\sqrt{-\lambda-1}\right)\cdot\sqrt{-\lambda-1} \tag{3.92}$$

Moreover, its asymptotic behavior as $\lambda \to -\infty$ is given by

$$F(\lambda) = \frac{1}{4\pi}\sqrt{|\lambda|-1} + O(|\lambda|^{-\infty}) \tag{3.93}$$

Proof. We will make use the well-known identity (see, e. g., [5], p. 146, f-la 28):

$$\int_0^{+\infty} e^{\lambda t} t^{-3/2} e^{-\frac{d^2}{4t}}\,dt = 2\frac{\sqrt{\pi}}{|d|}e^{-|d|\sqrt{-\lambda}}; \tag{3.94}$$

for $\lambda < 0$ and $d \in \mathbb{R}$ and the following explicit formula for the operator kernel $e^{-t}H(x,y;t)$ of the operator $e^{-t(\Delta+1)}$, where Δ is the (positive) Laplacian on S^3 (see [9], (2.29)):

$$e^{-t}H(x,y;t) = -\frac{1}{2\pi}\frac{1}{\sin \mathrm{dist}(x,y)}\frac{\partial}{\partial z}\bigg|_{z=d(x,y)}\Theta(z,t)\,. \tag{3.95}$$

Here $\mathrm{dist}(x,y)$ is the geodesic distance between $x, y \in S^3$ and

$$\Theta(z,t) = \frac{1}{\sqrt{4\pi t}}\sum_{k=-\infty}^{+\infty} e^{-(z+2k\pi)^2/4t}$$

is the theta-function.

Denoting $\mathrm{dist}(x,y)$ by θ and using (3.95) and (3.94), one gets

$$R(x,y;\lambda-1) = \int_0^{+\infty} e^{\lambda t}e^{-t}H(x,y;t)\,dt =$$

$$\frac{1}{4\pi}\frac{1}{\sin\theta}\left(-\sum_{k<0} e^{(\theta+2k\pi)\sqrt{-\lambda}}+\sum_{k\geq 0} e^{-(\theta+2k\pi)\sqrt{-\lambda}}\right)=$$

$$\frac{1}{4\pi}\frac{1}{\sin\theta}\frac{1}{1-e^{-2\pi\sqrt{-\lambda}}}\left[-e^{-2\pi\sqrt{-\lambda}}e^{\theta\sqrt{-\lambda}}+e^{-\theta\sqrt{-\lambda}}\right]=$$

$$\frac{1}{4\pi\theta}-\frac{1}{4\pi}\frac{1+e^{-2\pi\sqrt{-\lambda}}}{1-e^{-2\pi\sqrt{-\lambda}}}\sqrt{-\lambda}+o(1) \tag{3.96}$$

as $\theta \to 0$, which implies the Lemma. □

For the f at $3d$-torus, we let $\{\mathbf{A},\mathbf{B},\mathbf{C}\}$ be a basis of $\mathbb{R}^3$ and let $\mathbb{T}^3$ be the quotient of $\mathbb{R}^3$ by the lattice $\{m\mathbf{A}+n\mathbf{B}+l\mathbf{C} : (m,n,l)\in\mathbb{Z}^3\}$ provided with the usual f at metric.

The free resolvent kernel in R^3 is given by:

$$\frac{e^{-\sqrt{-\lambda}||x-y||}}{4\pi||x-y||}.$$

Therefore,

$$R(x,y;\lambda)=\frac{e^{-\sqrt{-\lambda}||x-y||}}{4\pi||x-y||}+\frac{1}{4\pi}\sum_{(m,n,l)\in\mathbb{Z}^3\setminus(0,0,0)}\frac{e^{-\sqrt{-\lambda}||x-y+m\mathbf{A}+n\mathbf{B}+l\mathbf{C}||}}{||x-y+m\mathbf{A}+n\mathbf{B}+l\mathbf{C}||}. \tag{3.97}$$

From (3.97) it follows that

$$F(\lambda)=\frac{1}{4\pi}\sqrt{-\lambda}-\frac{1}{4\pi}\sum_{(m,n,l)\in\mathbb{Z}^3\setminus(0,0,0)}\frac{e^{-\sqrt{-\lambda}||m\mathbf{A}+n\mathbf{B}+l\mathbf{C}||}}{||m\mathbf{A}+n\mathbf{B}+l\mathbf{C}||}=$$

$$\frac{1}{4\pi}\sqrt{-\lambda}+O\left(|\lambda|^{-\infty}\right)$$

as $\lambda\to-\infty$.

Remark 9. *It should be noted that explicit expressions for* $\det^*\Delta$ *in case* $X = S^n$ *are given in [24]. In particular, results:*

1. *For* S^2*, the determinant is expressed in terms of the Glaisher-Kinkelin constant* $A \approx 1.282427$ *and the result is:* $A^4 e^{1/6} \approx 3.19531.$

2. *For* S^3*, the determinant is expressed in terms of the zeta function* $\zeta(s)$ *and the result is:* $\pi e^{\zeta(3)/\pi^2} \approx 3.548496.$

For the case of $\mathbb{T}^2$, $\mathbb{T}^3$ *there are explicit formulas for the determinant can be found in in the paper of Furutani and Gosson [14]. These expressions are a bit cumbersome and we do not copy them here.*

Chapter 4
The l^n norm of the eigenfunctions of the Laplacian $|\phi_j|^2$

4.1 Notation and Main results

Let us define the notation that will be used throughout the argument. For $\varphi_j(x)$, an L^2-normalized eigenfunction of the Laplacian on an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with eigenvalue λ_j, we let its Fourier expansion be:

$$\varphi_j(x) \sim \sum_{\substack{\eta\in\mathbb{Z}^n \\ |\eta|^2=\lambda_j}} a_\eta\, e^{i(x,\eta)}$$

The Fourier series of $g(x) = |\varphi_j(x)|^2$ (recall the definition from the introduction) is as follows:

$$|\varphi_j(x)|^2 \sim \sum_{\substack{\tau=\xi-\eta \\ |\xi|^2=|\eta|^2=\lambda_j}} b_\tau\, e^{i(x,\tau)} \tag{4.1}$$

$$b_\tau = \sum_{\substack{\xi-\eta=\tau \\ |\xi|^2=|\eta|^2=\lambda_j}} a_\xi \bar{a}_\eta \tag{4.2}$$

$$\sum_{\substack{\eta\in\mathbb{Z}^n \\ |\eta|^2=\lambda_j}} |a_\eta|^2 \equiv 1 \tag{4.3}$$

We will write $\mathbf{S}^{n-1}(\lambda_j)$ for the $(n-1)$-sphere of radius $\sqrt{\lambda_j}$ and S_{n-1,λ_j} for the set of lattice points on $\mathbf{S}^{n-1}(\lambda_j)$.

In the spirit of this new notation, the last three equations may be written as follows:

$$|\varphi_j(x)|^2 \sim \sum_{\substack{\tau=\xi-\eta \\ \xi,\eta\,\in S_{n-1,\lambda_j}}} b_\tau\, e^{i(x,\tau)} \tag{4.4}$$

$$b_\tau = \sum_{\substack{\xi,\eta\,\in S_{n-1,\lambda_j} \\ \xi-\eta=\tau}} a_\xi \bar{a}_\eta \tag{4.5}$$

$$\sum_{\eta\,\in S_{n-1,\lambda_j}} |a_\eta|^2 \equiv 1 \tag{4.6}$$

We can assume, without loss of generality, the coeff cients a_ξ to be real and then we have $|a_\xi| = |\bar{a}_\xi| = |a_{-\xi}|$. For the case where $\tau = \mathbf{0}$, we have:

$$b_{\mathbf{0}} = \sum_{0=\tau=\xi-\eta} a_\xi \bar{a}_\eta = \sum_{\xi\,\in S_{n-1,\lambda_j}} |a_\xi|^2 = 1. \tag{4.7}$$

The proof of Proposition 1 requires a lemma that will be proved at the end of this section.

Lemma 11. *Given n points $\{\xi_i\}_{i=1}^n$ on $\mathbf{S}^{n-1}(\lambda_j) \cap \mathbb{Z}^n$, no two of which are diametrically opposite, that form codimension-one simplex, assume that there exists $\tau \in \mathbb{Z}^n$ and another n points $\{\eta_i\}_{i=1}^n$ on $\mathbf{S}^{n-1}(\lambda_j) \cap \mathbb{Z}^n$ such that*

$$\xi_i - \eta_i = \pm\tau, \qquad \forall 1 \leq i \leq n. \tag{4.8}$$

Then, there can be at most 2^{n-1} such different vectors τ satisfying (4.8).

Remark 10. *Given $m > n$ points on $\mathbf{S}^{n-1}(\lambda_j) \cap \mathbb{Z}^n$, we will still have the same bound, 2^{n-1} on the number of possible τ's. In other words, adding more points augments the number of* *restrictions, which, in principle, might reduce the number of possibilities for the different τ's.*

Remark 11. *We also notice that the bound we obtained is independent of the eigenvalue λ_j. This fact is crucial in the proof of Proposition 1.*

The proof of Proposition 1 is done by strong induction, the base case being done in [18] for the case of $n = 3$ and in [19] for the case of $n = 4$. We will provide a proof for the case of $n = 5$ f rst. This will give a feeling of how the proof of the general case goes.

4.2 Proof of proposition 1 for the case $n = 5$

The aim of the following calculations is to bound the sum $\sum_\tau |b_\tau|^5$. Given (4.7), we will consider the sum with nonzero τ:

$$\sum_{\tau \neq \mathbf{0}} |b_\tau|^5 \quad \leq \quad \sum_{\tau \neq \mathbf{0}} \left(\sum_{\xi_j - \eta_j = \tau} \prod_{j=1}^{5} |a_{\xi_j}||a_{\eta_j}| \right) \tag{4.9}$$

The trick that we shall use is to bound the right-hand side of (4.9) by:

$$\sum_{\tau \neq 0} \sum_{\xi_i - \eta_i = \tau} \frac{1}{2} \left(\prod_{i=1}^{5} |a_{\xi_i}|^2 + \prod_{i=1}^{5} |a_{\eta_i}| \right) \tag{4.10}$$

then, we interchange the order of summation in (4.10) and f nally we use lemma 11 to obtain a f nite upper bound.

In doing so, we will encounter several conf gurations of the points ξ_i's on $\mathbf{S}^4(\lambda_j) \cap \mathbb{Z}^5$. Each conf gurations needs to be studied separately. An obvious case is when two or more points ξ_i coincide, equation (4.9) reduces to,

$$\sum_{\tau \neq 0} \sum_{\xi_0 - \eta_0 = \tau} |a_{\xi_0}|^2 |a_{\eta_0}|^2 \left(\sum_{\xi_i - \eta_i = \xi_0 - \eta_0} \left(\prod_{i=3}^{5} |a_{\xi_i}||a_{\eta_i}| \right) \right) \tag{4.11}$$

and one can bound the terms $|a_{\xi_i}||a_{\eta_i}|$ inside the product of (4.11) by $\frac{1}{2}(|a_{\xi_i}|^2 + |a_{\eta_i}|^2)$. Then, we can bound this case by,

$$\frac{1}{2^3} \sum_{\tau \neq 0} \sum_{\xi_0 - \eta_0 = \tau} |a_{\xi_0}|^2 |a_{\eta_0}|^2 \left(\sum_{\xi, \eta \in S_{4,\lambda_j}} |a_\xi|^2 |a_\eta|^2 \right) \tag{4.12}$$

where the former is bounded by $\frac{1}{2^3}$.

Now, we may suppose that no two points coincide. We end up with f ve points in $\mathbb{R}^5$. These points will either lie in a 4 dimensional aff ne subspace (where they will form a 4-simplex), a 3 dimensional aff ne subspace or a 2 dimensional aff ne subspace.

In the case where the points form a 4-simplex, we can use lemma 11 and interchange the order of summation in (4.10) as follows,

$$\frac{1}{2} \sum_{\xi_i \in S_{4,\lambda_j}} \sum_{\tau \neq 0} \sum_{\xi_i - \eta_i = \pm\tau} \left(\prod_{i=1}^{5} |a_{\xi_i}|^2 + \prod_{i=1}^{5} |a_{\eta_i}|^2 \right). \tag{4.13}$$

The former will be bounded by

$$\frac{1}{2}\sum_{\xi_i\in S_{4,\lambda_j}} 2^4\cdot 2\prod_{i=1}^{5}|a_{\xi_i}|^2 \tag{4.14}$$

which by the L^2 normalization will not exceed 2^4.

In the case where the points ξ_i lie in a 3 dimensional affine subspace namely α, they will form a codimension 2 simplex. There will be 3 different configurations that need to be considered.

The first case is when $\{\xi_i\}_{i=1\ldots5} \in \alpha$ and at least one of the $-\eta_i \notin \alpha$. Without loss of generality, we may suppose that $-\eta_5 \notin \alpha$. Then, the simplex formed by $(\xi_1, \xi_2, \xi_3, \xi_4, -\eta_5)$ is a parallel translate of the simplex formed by $(\eta_1, \eta_2, \eta_3, \eta_4, -\xi_5)$ and these simplices do *not* lie in a 3-dimensional subspace. They form a non-degenerate 4-simplex. Hence, we are reduced to the case just studied above and we obtain the same bound, that is, 2^4.

In the next case, we suppose that the points $\{\xi_i\} \in \alpha$, $\{-\eta_i\} \in \alpha$ but $\{\eta_i\} \notin \alpha$ for all $i = 1\ldots5$. The trick we will be using is to consider the subspace that contains both α and η_1 say, namely γ. The subspace γ is a 4 dimensional subspace that contains $\mathbf{0}$ since both η_1 and $-\eta_1$ lie in γ. Thus, $\gamma \cap \mathbf{S}^4(\lambda_j)$ is the great 3-sphere, where the great k-sphere is defined to be the intersection of $\mathbf{S}^n(\lambda_j)$ with a k dimensional hyperplane passing through the origin. Hence, by lemma 11 and remark 10, we have the same bound on the number of τ's as to have 4 points on S_{3,λ_j}, and this will lead to a bound of 2^3.

The last scenario that needs to be considered in the case where $\{\xi_i\}_{i=1\ldots5} \in \alpha$ is when $\{-\eta_i\}_{i=1\ldots5} \in \alpha$ and at least one of the $\eta_i \in \alpha$, say η_1. Since both η_1 and $-\eta_1$ are in α, $\mathbf{0} \in \alpha$ and all of $\pm\eta_i$, $\pm\xi_i \in \alpha$. Hence, $\alpha \cap \mathbf{S}^4(\lambda_j)$ is the great 2-sphere. Once again, lemma 11 and remark 10 will lead us to a bound that is equal to 2^2.

It may happen that the points lie in a 2-dimensional aff ne subspace say, β. We will study the possible cases in the same manner we did previously. In the f rst case, we suppose that $\{\xi_i\}_{i=1\ldots5} \in \beta$ with $\{-\eta_i\} \in \beta$ for all i. We consider the 3-dimensional subspace γ_1 that contains both β and η_1 say. Then, $\mathbf{0} \in \gamma_1$, which implies that $\pm\eta_i, \pm\xi_i$ all lie in $\gamma_1 \cap \mathbf{S}^4(\lambda_j)$, which is the great 2-sphere. We are back in one of the cases studied previously and once again, lemma 11 and remark 10 will guarantee us a bound of 2^2.

In the very last case, we lose a bit of control on where the η_i might be. We let $\xi_i \in \beta$, but at least one of the $-\eta_i \notin \beta$, $-\eta_5$ say Then, the points $\{\xi_1, \xi_2, \xi_3, \xi_4, \eta_5\}$ lie in a 3-dimensional aff ne subspace and we are back to case where the $\xi_i \in \alpha$. Hence, we have a total bound equal to $2^2 + 2^3 + 2^4 = 28$.

Summing all the bounds, we obtain $C(n) \approx 2.384729...$

4.3 Proof of the general case

We shall now turn into the proof of the general case, that is, the sum (4.15) given below is convergent for any n. The proof is done by strong induction. That is, we suppose that the sum (4.15) is bounded in any dimension $k < n$.

$$\sum_{\tau\in\mathbb{Z}^n\cap\mathbf{S}^{n-1}(\lambda_j)} |b_\tau|^n = 1 + \sum_{0\neq\tau\in\mathbb{Z}^n\cap\mathbf{S}^{n-1}(\lambda_j)} |b_\tau|^n \tag{4.15}$$

As in the proof of the $n = 5$ case, we have,

$$\sum_{\tau\neq\mathbf{0}} |b_\tau|^n \leq \sum_{\tau\neq\mathbf{0}} \sum_{\xi_i-\eta_i=\tau} \prod_{i=1}^{n} |a_{\xi_i}||a_{\eta_i}| \tag{4.16}$$

The same trick is used as before, that is, we will bound the right-hand side of (4.16) by (4.17), then interchange the order of summation in the latter, and f nally use Lemma 11 to obtain a f nite upper bound.

$$\sum_{\tau\neq\mathbf{0}} \sum_{\xi_i-\eta_i=\tau} \frac{1}{2}\left(\prod_{i=1}^{n} |a_{\xi_i}|^2 + \prod_{i=1}^{n} |a_{\eta_i}|^2\right) \tag{4.17}$$

Once again, several cases need to be studied. We will do so in the same manner as for the $n = 5$ case. Instead of 5 points, we now have n points $\{\xi_i\}_{i=1}^n$ on the surface of the sphere $\mathbf{S^{n-1}}(\lambda_j)$

The trivial case where two or more points coincide gives a bounded contribution to the sum (4.15) that is equal to $\frac{1}{2^{n-2}}$ by the same computations done in the $n = 5$ case. In the subsequent cases, we may assume that no two points ξ_i coincide.

The second trivial case is when the points $\{\xi_i\}$ form a non-degenerate codimension-1 simplex. A change of order of summation in (4.17) and Lemma 11 yield a bound equal to 2^{n-1}.

The non trivial cases are when the points $\{\xi_i\}$ lie in smaller subspaces. Providing an upper bound to each of these cases finishes the proof.

The first of such non trivial cases is when the points $\{\xi_i\}$ lie in a $(n-2)$ dimensional affine subspace, namely α_{n-2}. Let us suppose $\{\xi_i\}_{i=1}^{n} \in \alpha_{n-2}$ with all the $\{-\eta_i\} \in \alpha_{n-2}$ as well. If either one of the η_i's or $-\xi_i$'s is an element of α_{n-2}, then the origin $\mathbf{0} \in \alpha_{n-2}$, which implies that $\alpha_{n-2} \cap \mathbf{S}^{n-1}(\lambda_j)$ is the great $(n-2)$-sphere. Hence, we have n points on S_{n-2,λ_j} and by the induction hypothesis, this gives us a bounded contribution to the sum (4.15). Suppose now that none of the η_i's or $-\xi_i$'s is an element of α_{n-2}. Then, we consider the subspace β_{n-2} containing both α_{n-2} and η_1 say. We get an $(n-1)$-dimensional subspace including $\mathbf{0}$, and $\beta_{n-2} \cap \mathbf{S}^{n-1}(\lambda_j)$ is the great $(n-2)$-sphere. Remark 10 implies that the resulting case is one of the cases in our induction hypothesis and this gives a bounded contribution to the sum (4.15).

In order to prove it for the rest of the cases; i.e., when the points $\{\xi_i\}$ lie in a $(n - k) < (n-2)$ dimensional affine subspace, namely α_{n-k}, we will use a *second* (reversed) induction on the dimension of the affine subspace α_{n-k} where the points $\{\xi_i\}$ might lie. That is, assuming we have a bounded contribution from all the α_{n-k+1} for some k with $3 < k < (n-1)$, we will prove that we have a bounded contribution from the case where the $\{\xi_i\} \in \alpha_{n-k}$. Once again, we have the two subcases depending on whether or not $-\eta_j$ belong to α_{n-k}.

For the first subcase, we may assume, without loss of generality, that $-\eta_1 \notin \alpha_{n-k}$. Then, the simplex $(-\eta_1, \xi_2, \ldots, \xi_n)$ is a parallel translate of $(-\xi_1, \eta_2, \ldots, \eta_n)$ and the last two simplices lie in a $(n-k+1)$-dimensional subspace. Hence, we are reduced to the *second* induction hypothesis which yields a bounded contribution to the sum (4.15).

Let us now turn our attention to the second subcase: if all the $\{\xi_i\}_{i=1}^n$ and $\{-\eta_i\}_{i=1}^n$ lie in α_{n-k} with *none* of the η_i's in α_{n-k}, we consider the subspace β_{n-k} containing both α_{n-k} and η_1 say. This is a $(n-k+1)$-dimensional subspace that includes $\mathbf{0}$. We can see that $\beta_{n-k} \cap \mathbf{S}^{n-k+1}(\lambda_j)$ is the great $(n-k)$-sphere. Hence, we have n points on S_{n-k,λ_j} and by the strong *first* induction hypothesis, we obtain a finite contribution from this subcase to the sum (4.15).

We note that if all the $\{\xi_i\}_{i=1}^n$ and $\{-\eta_i\}_{i=1}^n$ lie in α_{n-k} with at least one of the η_i's in α_{n-k}, then $\mathbf{0} \in \alpha_{n-k}$ and $\alpha_{n-k} \cap \mathbf{S}^{n-1}(\lambda_j)$ is the great $(n-k-1)$-sphere and this case gives a bounded contribution to the sum (4.15) by once again the strong *f rst* induction hypothesis.

We have exhausted all the possible cases, each giving a bounded contribution to the sum (4.15). Therefore, the sum is bounded and this f nishes the proof of the conjecture in [18].

4.4 Proof of the Geometric Lemma 11

Suppose we are given $\{\xi_i\}_{i=1}^n$, n points on S_{n-1,λ_j}, no two of which are diametrically opposite, and such that the simplex with vertices $\{\xi_i\}_{i=1}^n$ is non-degenerate. That is, the points $\{\xi_i\}_{i=1}^n$ cannot be in any (aff ne) subspace of dimension *strictly* less than $n-1$. Then, given n equal parallel "chords" $\{\mathbf{v_i}\}_{i=1}^n$ of S_{n-1,λ_j} (not equal to $\overline{\xi_i\xi_j}, \forall i, j$) such that ξ_i is an endpoint of $\mathbf{v_i}$, we denote the other endpoint of $\mathbf{v_i}$ by η_i and the diametrically opposite points of ξ_i (respectively η_i) by ξ_i' (respectively η_i'). The question we would like to pose is: *where on S_{n-1,λ_j} can $\{\eta_i\}_{i=1}^n$ lie?* We will see that there are *f nitely* many places where the $\{\eta_i\}_{i=1}^n$ can be. In fact, there are $\left[\frac{n}{2}\right]$ different scenarios, and we will study each of them.

If $\overline{\xi_i\eta_i}$ are equal $\forall i$, then $\eta_1 = \eta_i + \overline{\xi_i\xi_1}$ for all $i = 1 \ldots n$. Hence, the points $\eta_1 + \overline{\xi_1\xi_i}$ lie on S_{n-1,λ_j} $\forall i$. Since $\mathbf{S}^{n-1}(\lambda_j)$ is strictly convex, there is at most *one* point (other than ξ_1), namely η_1, for which the points $\eta_1 + \overline{\xi_1\xi_i}$ for all $i = 1 \ldots n$ lie on S_{n-1,λ_j}.

In the next scenario, we suppose $\overline{\xi_i\eta_i}$ are equal for all i, except at one point k, where $\overline{\xi_i\eta_i} = \overline{\eta_k\xi_k}$. Then, the points $\eta_1 + \overline{\xi_1\xi_i}$ for all $i \neq k$ and $\eta_1 + \overline{\xi_1\xi_k'}$ lie on S_{n-1,λ_j}. Again, by the convexity of $\mathbf{S}^{n-1}(\lambda_j)$ and the fact that $\{\xi_i\}$ form a codimension-1 simplex, there is at most one point (other than ξ_1), namely η_1, for which the points $\eta_1 + \overline{\xi_1\xi_i}$ for $i \neq k$ and $\eta_1 + \overline{\xi_1\xi_k'}$ lie on S_{n-1,λ_j}. However, the last equation gives us at most one possibility for η_1 for every $k = 1 \ldots n$. Hence, we have a total of $n = \binom{n}{1}$ possibilities for η_1.

In the next case, we assume $\overline{\xi_i\eta_i}$ are equal for all $i \neq k, l$, where $\overline{\xi_i\eta_i} = \overline{\eta_k\xi_k} = \overline{\eta_l\xi_l}$. Here again, $\eta_1 = \eta_i + \overline{\xi_i\xi_1}$ for all $i \neq k, l$ and $\eta_1 = \eta_k' + \overline{\xi_k'\xi_1} = \eta_l' + \overline{\xi_l'\xi_1}$, making the points $\eta_1 + \overline{\xi_1\xi_i}$ for $i \neq k, l$, $\eta_1 + \overline{\xi_1\xi_k'}$ and $\eta_1 + \overline{\xi_1\xi_l'}$ lie on S_{n-1,λ_j}. The convexity of $\mathbf{S}^{n-1}(\lambda_j)$ implies the uniqueness of such $\eta_1 \neq \xi_1$ for every pair k, l. Hence, we have $\binom{n}{2}$ possibilities for η_1 in this scenario.

Similarly, we will get $\binom{n}{3}$ for the next and so on, until $\binom{n}{n}$. However, the $\binom{n}{n}$ case is the same as the very f rst case $\binom{n}{0}$ in which we will simply change the sign of all the vectors $\overline{\xi_i\eta_i}$. The $(n-1)^{\text{th}}$ scenario is similar to the second scenario, and so on; hence, counting twice every case. The total number of possibilities will be the sum of the possibilities in every scenario and is:

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} = 2^{n-1} \tag{4.18}$$

4.5 Estimating the Bound $C(n)$

The bound follows from the proof of Proposition 1, and use the bounds given by Lemma 11. We do not claim that $C(n)$ is a sharp bound. In fact we suspect that one can improve the bound obtained in Lemma 11 and get a better f nal bound that would approach 1 in the limit. In our setting, and for $n \geq 5$ the result will be:

$$C(n) = \left(2^{2-n} + \left(\frac{5n}{4} - 4\right)2^n + 5\right)^{1/n} \tag{4.19}$$

It is clear that $C(n) \to 2$ as $n \to \infty$.

References

[1] Tayeb Aissiou. Semiclassical limits of eigenfunctions on f at n-dimensional tori. *Canadian Mathematical Bulletin*, 56(1):3–12, 2013.

[2] Tayeb Aissiou, Luc Hillairet, and Alexey Kokotov. Determinant of pseudo-laplacians. *Mathematical Research Letters*, 19(3):3–12, 2013.

[3] S Albeverio, F Gesztesy, R Høegh-Krohn, and H Holden. Solvable models in quantum mechanics, with an appendix by p. *Exner, AMS Chelsea*, 2005.

[4] Vojislav G Avakumović. Über die eigenfunktionen auf geschlossenen riemannschen mannigfaltigkeiten. *Mathematische Zeitschrift*, 65(1):327–344, 1956.

[5] Harry Bateman. *Tables of Integral Transforms: Vol. 1*. MacGraw-Hill Book, 1954.

[6] FA Berezin and LD Faddeev. Remark on the schrödinger equation with singular potential. In *Dokl. Akad. Nauk SSSR*, volume 137, pages 1011–1014, 1961.

[7] Jean Bourgain. Eigenfunction bounds for the laplacian on the n-torus. *International Mathematics Research Notices*, 1993(3):61–66, 1993.

[8] Pierre Cartier and Dennis A Hejhal. *Sur les zéros de la fonction zêta de Selberg*. Institut des hautes études scientif ques, 1979.

[9] Jeff Cheeger and Michael Taylor. On the diffraction of waves by conical singularities. i. *Communications on Pure and Applied Mathematics*, 35(3):275–331, 1982.

[10] Bernard Connes. Sur les coeff cients des séries trigonométriques convergentes sphériquement. *CR Acad. Sci. Paris Sér. AB*, 283(4), 1976.

[11] Yves Colin De Verdière. Pseudo-laplaciens. i. In *Annales de l'institut Fourier*, volume 32, pages 275–286. Institut Fourier, 1982.

[12] Johannes J Duistermaat. On the spectrum of positive elliptic operators and periodic bicharacteristics. In *Fourier Integral Operators and Partial Differential Equations*, pages 15–22. Springer, 1975.

[13] John Fay. *Kernel functions, analytical torsion, and moduli spaces*. Number 464. AMS Bookstore, 1992.

[14] Kenro Furutani and Serge de Gosson. Determinant of laplacians on heisenberg manifolds. *Journal of Geometry and Physics*, 48(2):438–479, 2003.

[15] Luc Hillairet and Alexey Kokotov. Krein formula and s-matrix for euclidean surfaces with conical singularities. *Journal of Geometric Analysis*, pages 1–32, 2010.

[16] Lars Hörmander. The spectral function of an elliptic operator. *Acta mathematica*, 121(1):193–218, 1968.

[17] Lars Hörmander. *The analysis of linear partial differential operators IV: Fourier Integral Operators*. Springer, 2009.

[18] Dmitry Jakobson. Quantum limits on f at tori. *Annals of mathematics*, pages 235–266, 1997.

[19] Dmitry Jakobson, Nikolai Nadirashvili, and John Toth. Geometric properties of eigenfunctions. *Russian Mathematical Surveys*, 56(6):1085, 2001.

[20] Klaus Kirsten and Paul Loya. Calculation of determinants using contour integrals. *American Journal of Physics*, 76:60, 2008.

[21] Klaus Kirsten, Paul Loya, and Jinsung Park. The very unusual properties of the resolvent, heat kernel, and zeta function for the operator- d2/ dr2- 1/(4r2). *Journal of mathematical physics*, 47:043506, 2006.

[22] Klaus Kirsten, Paul Loya, and Jinsung Park. Exotic expansions and pathological properties of ζ-functions on conic manifolds. *Journal of Geometric Analysis*, 18(3):835–888, 2008.

[23] Klaus Kirsten, Paul Loya, and Jinsung Park. Functional determinants for general self-adjoint extensions of laplace-type operators resulting from the generalized cone. *manuscripta mathematica*, 125(1):95–126, 2008.

[24] H Kumagai. The determinant of the laplacian on the n-sphere. *ACTA ARITHMETICA-WARSZAWA-*, 91(3):199–208, 1999.

[25] Boris Moiseevich Levitan. On the asymptotic behavior of the spectral function of a self-adjoint differential equation of the second order. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 16(4):325–352, 1952.

[26] Paul Lin. Non-friedrichs self-adjoint extensions of the laplacian in rˆ d. *arXiv preprint arXiv:1103.6089*, 2011.

[27] Paul Loya, Patrick McDonald, and Jinsung Park. Zeta regularized determinants for conic manifolds. *Journal of Functional Analysis*, 242(1):195–229, 2007.

[28] S Minakshisundaram and A Pleijel. Some properties of the eigenfunctions of the laplace-operator on riemannian manifolds. *Can. J. Math*, 1:242–256, 1949.

[29] Gerd Mockenhaupt. *Bounds in Lebesgue spaces of oscillatory integral operators*. PhD thesis, Universität-Gesamthochschule-Siegen, 1996.

[30] DB Ray and IM Singer. Analytic torsion for complex manifolds. *The Annals of Mathematics*, 98(1):154–177, 1973.

[31] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics: Vol.: 2.: Fourier Analysis, Self-Adjointness*. Academic Press, 1975.

[32] Zeev Rudnick and Henrik Ueberschär. On the eigenvalue spacing distribution for a point scatterer on the f at torus. *arXiv preprint arXiv:1208.5920*, 2012.

[33] Zeév Rudnick and Henrik Ueberschär. Statistics of wave functions for a point scatterer on the torus. *Communications in Mathematical Physics*, 316(3):763–782, 2012.

[34] Yu G Safarov. Asymptotic behavior of the spectrum of a pseudodifferential operator with periodic bicharacteristics. *Journal of Soviet Mathematics*, 40(5):645–652, 1988.

[35] Yu G Safarov. Asymptotic of the spectral function of a positive elliptic operator without the nontrap condition. *Functional Analysis and Its Applications*, 22(3):213–223, 1988.

[36] Yuri Safarov and D Vassiliev. *The asymptotic distribution of eigenvalues of partial differential operators*, volume 155. American Mathematical Soc., 1997.

[37] Christopher D Sogge. Concerning the l^p norm of spectral clusters for second-order elliptic operators on compact manifolds. *Journal of functional analysis*, 77(1):123–138, 1988.

[38] Christopher D Sogge, John A Toth, and Steve Zelditch. About the blowup of quasimodes on riemannian manifolds. *Journal of Geometric Analysis*, 21(1):150–173, 2011.

[39] Christopher D Sogge and Steve Zelditch. Riemannian manifolds with maximal eigenfunction growth. *Duke Mathematical Journal*, 114(3):387–437, 2002.

[40] M.E. Taylor. *Partial Differential Equations II: Qualitative Studies of Linear Equations*. Applied Mathematical Sciences. Springer London, Limited, 2012.

[41] John A Toth and Steve Zelditch. Riemannian manifolds with uniformly bounded eigenfunctions. *Duke Mathematical Journal*, 111(1):97–132, 2002.

[42] Henrik Ueberschär. The trace formula for a point scatterer on a compact hyperbolic surface. *Journal of Mathematical Physics*, 53:012108, 2012.

[43] Antoni Zygmund. On fourier coeff cients and transforms of functions of two variables. *Studia Mathematica*, 50(2):189–201, 1974.