HEDGING AND PRICING IN INCOMPLETE MARKETS: THEORY AND APPLICATIONS

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Abstract

Hedging and Pricing in Incomplete Markets: Theory and Applications

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This thesis consists of three essays in financial econometrics. In the first part of the thesis, motivated by different applications of hedging methods in the literature, we propose a general theoretical framework for hedging and pricing. First, we review briefly different strands of literature on hedging which have been developed in various fields such as finance, economics, operations research and mathematics, and then try to come up with a tractable way for hedging and pricing in this paper. By introducing different market principles, we study conditions under which the hedging problem has a solution and pricing is possible. We will conduct an in-depth theoretical analysis of hedging strategies with shortfall risks as well as the spectral risk measures, in particular those associated with Choquet expected utility. We show that asymmetric information results in incorrect risk assessment and pricing. In the second part of the thesis, we will apply our results in the first part to construct an economic risk hedge. We also introduce a general method to estimate the stochastic discount factors associated with different risk measures and different financial models. The third part of the thesis modifies the speculative storage model by embedding staggered price features into the structural model of Deaton and Laroque (1996). In an attempt to replicate the stylized facts of observed commodity price dynamics, we add an additional source of intertemporal linkage to Deaton and Laroque (1996), namely speculation in intermediate-good inventories. The introduction of this type of friction into the model is motivated by its ability to increase price stickiness which gives rise to an increased persistence in the first and higher conditional moments of commodity prices. By incorporating intermediate risk neutral speculators and a final bundler with a staggered pricing rule in the spirit of Calvo (1983) into the storage model, we are able to capture a high degree of serial correlation and conditional heteroskedasticity, which are observed in actual data. The structural parameters of both Deaton and Laroque (1996) and our modified models are estimated using actual prices for 8 agricultural commodities. Simulated data are then employed to assess the effects of our staggered price approach on the time-series properties of commodity prices. Our results lend empirical support to the possibility of staggered prices.

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[1]Centre interuniversitaire de recherche en analyse des organisations

[2]Fonds québécois de la recherche sur la société et la culture

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Introduction

Asset pricing and hedging are two cornerstones of financial economics; seemingly two different areas of research in the literature, in many cases so entangled that it is impossible to distinguish one from the other. While pricing always seems to be a necessary issue, according to the Modigliani and Miller Theorem, there is no need for hedging in financial markets. This theorem holds in an efficient market, with a market price process being the classical random walk, under certain conditions: there is no tax, no bankruptcy costs, no agency costs and no asymmetric information. This suggests that in the absence of each assumption, one has to hedge.

The theory of asset pricing has been developed extensively during past few decades, from different aspects. The most elegant development of the theory of asset pricing in the literature is based on a representative agent's utility maximization problem which gives rise to several interesting discussions in matching the stylized facts with ones suggested by theory. On the other hand, there are theories which find their root directly in finance literature such as factor models (e.g. Arbitrage Price Theory) or the dynamic hedging of Merton, Black and Scholes. The former literature is based on behavioral principles of consumers whereas the latter literature is based on market principles, such as the No Arbitrage assumption. In both cases however, two important facts play a major role: first, both theories try to hedge a financial position against potential risks, either the shocks in asset values, or shocks in news, by manipulating a set of available strategies; second, in both theories the key factor for pricing is the stochastic discount factor, which prices financial positions in a correct way. For a comprehensive discussion on both facts see Cochrane (2009). In this work we develop a methodology which can be properly used for hedging and pricing, including both key facts mentioned earlier. Indeed, motivated by the development of risk measures, their relation to Choquet expected utility, and their application in the literature of finance and financial economics, we model the financial practitioners behavior by risk measures leading to an *aggregate cost* (or *profit*) minimization (maximization) problem in order to hedge and price. We will see how using the aggregate cost (or profit), instead of just using the hedging cost (or portfolio profit), will lead to a more consistent theory which can easily be laid down in the existing literature of finance.

We are mainly motivated by a framework that has been introduced and developed in Assa and Balbás (2011), Balbás, Balbás, and Heras (2009), Balbás, Balbás, and Garrido (2010) and Balbás, Balbás, and Mayoral (2009); we set up our framework by modeling the behavior of a (representative) financial practitioner using risk measures. The key player in this setting is no longer a consumer, but a financial practitioner who minimizes the *aggregate cost* (or maximizes the *profit*). Our aim is to develop a tractable, as well as a general framework, which is also based on strong financial and economic principles. We use different strands of literature developed in the asset-pricing, economic hedging and hedging for pricing, compromising a unique theory of factor hedging.

The last decade has witnessed a surge in commodity prices and a widespread financialization of commodity products. The upward movements and the increased volatility of commodity prices has been largely attributed to strong demand by China and other emerging markets, as well as massive capital flows into the commodity markets by institutional investors, portfolio managers and speculators. While the importance of commodity price movements for economic policy and investors' sentiment has generated a substantial research interest, the behavior and the determination of commodity prices is not yet fully understood. The main objective of this work is to develop a structural model of commodity price determination that reflects the empirical properties (high persistence and conditional heteroskedasticity) of commodity prices. In order to achieve this goal and to gain further understanding of the fundamental factors that drive observed behavior of commodity prices, we modify the structure of the speculative storage model from one where prices adjust almost instantaneously to harvest shocks to a setup in which change slowly and infrequently. More specifically, we depart from the assumption that market prices are determined in a perfectly competitive environment and extend the basic speculative storage model by explicitly introducing intermediate goods speculators with a staggered pricing rule. One appealing aspect of this approach is its ability to mimic some important characteristics of actual commodity prices such as high persistence and conditional heteroskedasticity, which can be generated even in the absence of correlated harvest shocks. Another advantage of our proposed approach is the possibility of conducting policy analysis by tracing the dynamic effects of a harvest shock on commodity prices over time.

Chapter 1

Hedging and Pricing in Incomplete Markets

1.1 Literature on Pricing and Hedging

The theory we develop in this chapter is inspired by various asset pricing models in the literature. First, we use the idea of factor models to the extent that a financial position is approximated by a possible portfolio, namely the mimicking portfolio. Starting with asset pricing, there are numerous theories, including the original capital asset pricing models (CAPM) of Sharpe (1964), Lintner (1965) and Black (1972), which have been used in measuring the relationship between an expected return on a security and its risk. In inter-temporal models the same idea has been developed, for example by Merton (1973), Long (1974), Rubinstein (1976), Breeden (1979), and Cox, Ingersoll, and Ross (1985), and the Arbitrage Pricing Theory (APT) of Ross (1976). These models, in principle, suggest a relation between expected return and one or more measures of exposure to systematic risk. In CAPM, a security's systematic risk is measured by its beta with respect to a diversified stock index, see Blume and Friend (1973), and Fama and MacBeth (1973). However, in the wake of Roll (1977), who criticizes the early studies by notifying that they are tests of that the stock index is mean-variance efficient, many began risk-return analysis. Among these variants of APT, Roll and Ross (1980) use factor-analytic methods to estimate multiple measures of systematic risk.

The other main feature of our theory is to find an appropriate (stochastic) discount factor in order to price correctly a position. Initiated by Black and Scholes (1973) and Merton (1973), the methodology of No Arbitrage (NA) pricing and its relation to martingale measures have been studied in Harrison and Kreps (1979); relating the existence of a correct stochastic discount factor, the No Arbitrage assumption and perfect hedging. This was the beginning of the idea that pricing a derivative of an underlying asset is nothing but a discounted weighted average of the derivative, discounted by a correct stochastic discount factor suggested by the martingale theory. In a different setting though, a stochastic discount factor is assumed to be a probability measure that can correctly price the test assets. The No Arbitrage condition holds in this setting if the probability measure is everywhere positive. Given that typically the stochastic discount factor set is a large set, the No Arbitrage conditions was replaced later by the No Good Deal (NGD) assumption, which was first introduced in Cochrane and Saa-Requejo (2000) in order to incorporate the market efficiency measure. In Černý and Hodges (2002) the authors re-introduce the notion of Good Deals as free and *desirable* financial positions. In Assa and Balbás (2011) one can find a set of equivalent conditions to NGD assumptions when the agents preferences are modeled by coherent risk measures.

In addition to what has been studied in the literature on pricing, we consider the process of pricing as a natural product of a hedging process: meaning, to price the *fully hedged* part and to value the *un-hedged* part. The literature on hedging has been extensively studied in recent years. Apart from corporate hedging[1], there is a large

[1] It is worth mentioning that the corporate hedging is an area has been developed in last few years, however, since this literature has little to do with pricing we will not focus our attention to it, just to mention few work look at Mayers and Smith (1990a), Mayers and Smith (1990b), Tufano (1996), Smith and Stulz (1985) and Jensen and Meckling (1976).

literature on developing methods based on mean-variance utility for economic hedging. Breeden, Gibbons, and Litzenberger (1989) test the consumption-based CAPM by using a portfolio which has maximum correlation with consumption growth. Vassalou (2003) constructs a mimicking portfolio to proxy news related to future GDP growth to explain a cross-section of equity returns. Balduzzi and Kallal (1997) apply smaller variance interval bounds of Hansen and Jagannathan (1991), using hedging portfolios for various economic risk variables. Balduzzi and Robotti (2001) use the minimum-variance kernel of Hansen and Jagannathan to estimate the economic risk premiums. In all of these papers, the mimicking portfolios are constructed by means of an ordinary least squares projection of the risk variables on a set of security returns. In Goorbergh, Roon, and Werker (2003), however, a weighted least squares projection, based on a utility function, yields the hedging on security returns.

Another line of research has been developed in mathematical finance, devoted to hedging financial positions, mainly concerned with the pricing of contingent claims. This literature is developed in different directions. One is based on replicating (or closely replicating) financial positions. For instance, the super-hedging strategy of El Karoui and Quenez (1995), is well studied in the literature; see Karatzas and Shreve (1998). In a different direction the problem of hedging was studied in a mean-variance framework. Mean-variance hedging was first formulated in Duffie and Richardson (1991), while the first ground-breaking result was obtained in Schweizer (1992). For further evidence of this literature see Föllmer and Schweizer (1991), Gourieroux, Laurent, and Pham (1998), Laurent and Pham (1999), Schäl (1994) and Schweizer (1995). Hedging is analyzed by using other decision-making tools, risk measures, in place of mean variance. Föllmer and Leukert (2000) propose replicating contingent claims by minimizing the probability of shortfalls. Similarly Nakano (2004) and Rudloff (2007, 2009) study the problem of minimizing the risk of a shortfall when the risk is measured by a general coherent or convex risk measure.

Another line of research, which is mainly developed in the literature of operation research, is based directly on the concepts of hedging and minimization of risk rather than replication of contingent claims (see Assa and Balbás (2011), Balbás, Balbás, and Heras (2009), Balbás, Balbás, and Garrido (2010) and Balbás, Balbás, and Mayoral (2009)). The main idea is that the financial practitioner minimizes the risk of his/her global position, given the budget constraint on a set of manipulatable positions (a set of accessible portfolios, for example). Assa and Balbás (2011) characterize the existence of a solution to the hedging problem, showing that a solution exists if and only if there is no costless risk-free position (arbitrage opportunity or "Good Deal").

1.2 Preliminaries and Analytical Setup

We start by introducing the main terminology and notation for hedging and pricing financial or economic variables. We assume a finite probability space with a finite[2] event space $\Omega = \{\omega_1, \ldots, \omega_n\}$. We denote the physical measure by $\mathbb{P}$, and the associated expectation by E. To simplify the discussion, we assume that $\mathbb{P}(\omega_i) = 1/n$ for all $i = 1, \ldots, n$. Our theory is developed in a static setting and we only have time 0 and time T. Each random variable represents the random value on a variable at time T. We denote by $\mathbb{R}^n$ the set of all variables. The duality relation is expressed as $(x, y) \mapsto E(xy)\,, \forall x, y \in \mathbb{R}^n$. The risk measure and the pricing rule are expressed in terms of time-zero value and are real numbers.

Let $\mathcal{Y}$ be a subset of $\mathbb{R}^n$. In the subsequent discussion, we will assume that $\mathcal{Y}$ possesses one or several properties from the following list:

S1. Normality if $0 \in \mathcal{Y}$;

S2. Positive homogeneity if $\lambda\mathcal{Y} \subseteq \mathcal{Y}$, for all $\lambda > 0$;

[2] All of the results can be easily extended to a probability space with no atoms in an appropriate space – for instance, $L^2(\Omega)$.

S3. Translation-invariance if $\mathbb{R} + \mathcal{Y} \subseteq \mathcal{Y}$ [3];

S4. Sub-additivity if $\mathcal{Y} + \mathcal{Y} \subseteq \mathcal{Y}$;

S5. Convexity if $\lambda\mathcal{Y} + (1-\lambda)\mathcal{Y} \subseteq \mathcal{Y}$.

1.2.1 Risk Measures

In what follows, we use risk measures to quantify the risk associated with the undiversifiable part of the market exposure.

A *risk measure* ϱ is a mapping from a set $\mathcal{D} \subseteq \mathbb{R}^n$ to the set of real numbers $\mathbb{R}$ which maps each random variable in $\mathcal{D}$ to a real number representing its risk. Each risk measure can have one or more of the following properties:

R1. $\varrho(0) = 0$;

R2. $\varrho(\lambda x) = \lambda\varrho(x)$, for all $\lambda > 0$ and $x \in \mathcal{D}$;

R3. $\varrho(x + c) = \varrho(x) - c$, for all $x \in \mathcal{D}$ and $c \in \mathbb{R}$;

R4. $\varrho(x) \leq \varrho(y)$, for all $x, y \in \mathcal{D}$ and $x \geq y$;

R5. $\varrho(x + y) \leq \varrho(x) + \varrho(y), \forall x, y \in \mathcal{D}$;

R6. $\varrho(\lambda x + (1-\lambda)y) \leq \lambda\varrho(x) + (1-\lambda)\varrho(y)$.

If ϱ satisfies properties R1, R2, R3, R5 or R6, $\mathcal{D}$ has to possess properties S1, S2, S3, S4, or S5, respectively. A risk measure is called an *expectation bounded risk* if it is defined on $\mathbb{R}^n$ and satisfies properties R1, R2, R3 and R5 above. The mean-variance risk measure defined as

$$MV_\delta(x) = \delta\sigma(x) - E(x),$$

where $\sigma(x)$ is the standard deviation of x and δ is a positive number representing the level of risk aversion, is an example of an expectation bounded risk.

[3] in the sequel $\mathbb{R}$ represents the set of all constant random variables $\{(c, ..., c) \in \mathbb{R}^n | c \in \mathbb{R}\}$.

An expectation bounded risk is called a *coherent risk measure* if it also satisfies property R4. Finally, a *convex risk measure* satisfies properties R1, R3, R4 and R6. Coherent and convex risk measures are introduced by Artzner, Delbaen, Eber, and Heath (1999) and Föllmer and Schied (2002), respectively, while expectation bounded risks are first defined in Rockafellar, Uryasev, and Zabarankin (2006).

One popular risk measure is the Value at Risk defined as

$$\mathrm{VaR}_\alpha(x) = -q_\alpha(x)\,,\ \forall x \in \mathbb{R}^n,$$

where $q_\alpha(x) = \inf\{a \in \mathbb{R} | \mathbb{P}[x \leq a] > \alpha\}$ denotes the α-th quantile of the distribution of x. Note that VaR_α is a decreasing risk measure which is neither a coherent risk measure nor an expectation bounded risk. In contrast, the Conditional Value at Risk (CVaR), expressed as the sum over all VaR below α percent

$$\varrho_{\nu_\alpha}(x) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(x)d\beta, \tag{1.1}$$

is a coherent risk measure.

Coherent risk measures are tightly linked to the Choquet expected utility of the form

$$U(x) = \int_0^1 u(F_x^{-1}(t))d\nu(t), \tag{1.2}$$

where u is a utility function and ν is a non-additive probability. The measure ν distorts the probability of different events. The case of a concave ν corresponds to a pessimistic way of weighting events by assigning larger weights to less favorable events and smaller weights to more favorable ones. A convex ν has the opposite effect. In particular, when u is the identity function and $\nu = \nu_\alpha$ such that $d\nu_\alpha = \frac{1}{\alpha}1_{[0,\alpha]}dt$ in equation (1.2), we obtain the coherent risk measure ϱ_{ν_α} defined in (1.1).

We have the following result from Bassett, Koenker, and Kordas (2004) which relates the notion of coherent risk measures to the Choquet expected utility.

Theorem 1.2.1 *Let ϱ be a coherent risk measure. If ϱ is distribution invariant (i.e., $\varrho(x) = \varrho(y)$ for any two random variables $x, y \in \mathbb{R}^n$ such that $F_x = F_y$) and co-monotone additive, then it is pessimistic.*

To further generalize the concept of a risk measure, consider the following family of risk measures.

Definition A risk measure is a *generalized spectral risk* measure if and only if there is a distribution $\varphi : [0,1] \to \mathbb{R}_+$ such that $\int_0^1 \varphi(s)ds = 1$, and

$$\varrho_\varphi(x) = \int_0^1 \varphi(s)\mathrm{VaR}_s(x)ds. \tag{1.3}$$

One can readily see that ϱ_φ is law invariant, i.e., if x and x' are identically distributed, then we have $\varrho_\varphi(x) = \varrho_\varphi(x')$. Indeed, it can be shown that all law-invariant co-monotone additive coherent risk measures can be represented as (1.3); see Kusuoka (2001). Note that, by a change of variables, the spectral risk measure (1.3) coincides with the Choquet utility (1.2) for a risk neutral agent, i.e, when $u(x) = x$. Furthermore, equation (1.3) describes a family of risk measures which are statistically robust. Cont, Deguest, and Scandolo (2010) show that a risk measure $\varrho(x) = \int_0^1 \mathrm{VaR}_\beta(x)\varphi(\beta)d\beta$ is robust if and only if the support of φ is away from zero and one. For example, Value at Risk is a risk measure with this property.

An interesting fact about this type of risk measures is that it can be represented as infimum of a family of coherent risk measures.

Theorem 1.2.2 *If $\varrho_\varphi(x) = \displaystyle\int_0^1 \mathrm{VaR}_\alpha(x)\varphi(\alpha)d\alpha$, for a nonnegative distribution φ with $\displaystyle\int_0^1 \varphi(s)ds = 1$, then we have*

$$\varrho_\varphi(x) = \min\{\varrho(x) | \text{ for all coherent risk measure } \varrho \text{ such that } \varrho \geq \varrho_\varphi\}.$$

Proof See Appendix A.

This theorem provides a motivation for introducing another family of risk measures, called the *infimum risk measures*, which includes all coherent as well as spectral risk measures.

Definition Let $\mathbb{D}$ be a point-wise-closed set of risk measures on $\mathcal{D}$. Then, the infimum risk measure associated with $\mathbb{D}$ is defined as

$$\varrho_{\mathbb{D}}(x) = \min_{\varrho \in \mathbb{D}} \varrho(x). \tag{1.4}$$

1.2.2 Pricing Rules

A *pricing rule* π is a mapping from $\mathcal{X} \subseteq \mathbb{R}^n$ to the set of real numbers $\mathbb{R}$ which maps each random variable in $\mathcal{X}$ to a real number representing its price. The pricing rule can possess one or more of the following properties:

P1. $\pi(0) = 0$;

P2. $\pi(\lambda x) = \lambda \pi(x)$, for all $\lambda > 0$ and $x \in \mathcal{X}$;

P3. $\pi(x + c) = \pi(x) + c$, for all $x \in \mathcal{X}$ and $c \in \mathbb{R}$ (cash-invariance);

P4. $\pi(x) \leq \pi(y)$, for all $x, y \in \mathcal{X}$ and $x \leq y$;

P5. $\pi(x + y) \leq \pi(x) + \pi(y)$, for all $x, y \in \mathcal{X}$;

P6. $\pi(\lambda x + (1 - \lambda y)) \leq \lambda \pi(x) + (1 - \lambda)\pi(y)$.

If π satisfies properties P1, P2, P3, P5 or P6, $\mathcal{X}$ has to satisfy properties S1, S2, S3, S4, or S5, respectively. A pricing rule is *super-additive* if $\pi(x + y) \geq \pi(x) + \pi(y)$, for all $x, y \in \mathcal{X}$.

A pricing rule that satisfies properties P1, P2, P3, P4 and P5 is called a sub-linear pricing rule. Any sub-linear pricing rule can be extended from $\mathcal{X}$ to $\mathbb{R}^n$ as follows

$$\tilde{\pi}(x) = \sup_{\{y \in \mathcal{X} | y \leq x\}} \pi(y). \tag{1.5}$$

Indeed, this supremum exists and is a finite number because (i) $\min(x) \in \{y \in \mathcal{X} | y \leq x\}$ and (ii) for any $x, y \in \mathcal{X}$ such that $y \leq x$, we have $\pi(y) \leq \max(x)$. It can be easily seen that $\tilde{\pi}$ is a sub-linear pricing rule on $\mathbb{R}^n$.

Moreover, any sub-linear pricing rule admits the following representation:

$$\tilde{\pi}(x) = \sup_{z \in \mathcal{R}} E(zx), \tag{1.6}$$

where $\mathcal{R}$ is given by

$$\mathcal{R} := \{z \in \mathbb{R}^n | E(zx) \leq \tilde{\pi}(x), \forall x \in \mathbb{R}^n\}. \tag{1.7}$$

Monotonicity implies that $z \geq 0, \forall z \in \mathcal{R}$ and translation-invariance implies $E(z) = 1, \forall z \in \mathcal{R}$. Therefore, $\mathcal{R}$ is a compact set.

In this paper, the set $\mathcal{R}$ represents the set of nonnegative stochastic discount factors induced by π and

$$\pi(x) = \tilde{\pi}(x) = \sup_{z \in \mathcal{R}} E(zx), \forall x \in \mathcal{X}. \tag{1.8}$$

Also, the condition $z > 0$ is equivalent to the no-arbitrage condition

$$\pi(x) \leq 0 \ \& \ x \geq 0 \Rightarrow x = 0. \tag{1.9}$$

Jouini and Kallal (1995a), Jouini and Kallal (1995b) and Jouini and Kallal (1999) argue that for a wide range of market imperfections such as dynamic market incompleteness, short selling costs and constraints, borrowing costs and constraints, and proportional transaction costs, the pricing rule is sub-linear. Even though the set of sub-linear pricing rules is quite large, it does not cover some practically relevant pricing rules. For example, in a super-hedging context, ask and bid prices defined as

$$\pi^a(x) = \sup_{\mathbb{Q}\in\mathcal{R}} E^{\mathbb{Q}}[x], \tag{1.10}$$

and

$$\pi^b(x) = \inf_{\mathbb{Q}\in\mathcal{R}} E^{\mathbb{Q}}[x], \tag{1.11}$$

where $\mathcal{R}$ is the set of martingale measures of the normalized price processes of traded securities (see Jouini and Kallal (1995a) and Karatzas, Lehoczky, Shreve, and Xu (1991)), are of particular interest (El Karoui and Quenez (1995)). In this case, the bid price is a super-additive pricing rule which does not fulfill the sub-additivity conditions of the sub-linear pricing rule. Furthermore, in insurance applications, the pricing rules are not, in general, sub- or super-additive. As pointed out by Wang, Young, and Panjer (1997), the price of an insurance risk has a Choquet integral representation as in equation (1.2) or (1.3) with respect to a distorted probability. For this reason, we introduce the family of *infimum pricing rules* that subsumes both sub-linear and non-sub-linear pricing rules.

Definition Let $\mathbb{M}$ be a point-wise-closed set of pricing rules on $\mathcal{X}$. Then, the infimum risk measure associated with $\mathbb{M}$ is defined as

$$\pi_{\mathbb{M}}(x) = \min_{\pi\in\mathbb{M}} \pi(x). \tag{1.12}$$

1.2.3 Projection

To put the subsequent discussion in the proper context, assume that we have a set of perfectly-hedged variables denoted by $\mathcal{X}$, where all members of $\mathcal{X}$ are priced according to the pricing rule $\pi : \mathcal{X} \to \mathbb{R}$. As an example, consider the case when $\mathcal{X}$ is equal to the set of all portfolios of given assets $(x_1, ..., x_N)$, i.e., $\mathcal{X} = \text{Span}(x_1, ..., x_N)$ or

$\mathcal{X} = \text{Span}(x_1, ..., x_N)_+$ if the short-selling is forbidden. A variable y is perfectly-hedged if $y \in \mathcal{X}$. In this particular example, y is perfectly-hedged if there is a portfolio whose value is equal to y, i.e., there exist numbers $a_1, ..., a_N$ such that $y = a_1 x_1 + \cdots + a_N x_N$. If any variable y can be perfectly-hedged, we say that the market is complete. Otherwise, if there is at least one variable y whose risk cannot be diversified by the set of perfectly-hedged positions, the market is incomplete. This prompts the need to introduce the mapping (risk measure) ϱ from the set of all variables $\mathcal{D}$ to real numbers which measures the risk generated by the part that cannot be hedged.

We next introduce the idea of projection. Let us consider a financial position y in an incomplete market which has to be hedged or priced. To achieve this, we find a variable, among all perfectly-hedged variables in the set $\mathcal{X}$, that mimics y most closely. In other words, we want to project y on the set $\mathcal{X}$. Assume for a moment that we know this *projection* and denote it by $x \in \mathcal{X}$. Hence, y can be decomposed into two parts: a mimicking strategy (portfolio in our example) x which is *perfectly-hedged*, and an *unhedged* part $y - x$ which generates risk. The cost of the mimicking strategy (or perfectly-hedged) part is given by $\pi(x)$, and the risk generated by the unhedged part, which cannot be diversified by any member of $\mathcal{X}$, is measured by $\varrho(x - y)$. The idea of projection is to minimize the aggregate cost of the hedging given as $\pi(x) + \varrho(y - x)$. Therefore, one can state the problem as follows:

$$\inf_{x \in \mathcal{X}} \left\{ \pi(x) + \varrho(x - y) \right\}. \tag{1.13}$$

In this case, the market imperfections are reflected by the (non-linear) pricing rule π and the risk measure ϱ which capture the market frictions and the market incompleteness, respectively.

Let us now look at this problem from a pricing point of view. Suppose that a financial practitioner wants to price the position (contingent claim, for example)

y. While the pricing of y in complete markets can utilize directly the no-arbitrage approach, the pricing problem in incomplete markets is less straightforward as it needs to incorporate the cost of the unhedged part. As discussed above, the cost of forming the mimicking strategy x is given by $\pi(x)$ and the unhedged risk associated with the unhedged part of y is given by $\varrho(x - y)$. Then, the competitive price for position y can be defined as

$$\pi_\varrho(x) = \inf_{x \in \mathcal{X}} \{\pi(x) + \varrho(x - y)\}. \tag{1.14}$$

As we demonstrate below, if ϱ is a coherent risk measure and π a sub-linear pricing rule, π_ϱ satisfies all of the properties of a sub-linear pricing rule except for the normality condition. As a result, we need to ensure that the normality condition holds for π_ϱ to be a proper pricing rule.

Potential applications of this framework include hedging and pricing contingent claims, insurance underwriting, hedging of economic risk etc. It should be noted that a similar approach to pricing is adopted in Föllmer and Leukert (2000) and Rudloff (2007, 2009) but it is based on minimizing shortfall risk instead of minimizing aggregate cost as we do in this paper. In what follows, we refine the choice of pricing rules and risk measures and analyze their theoretical properties.

1.3 Main Theoretical Results

In this section, we establish some market principles for general risk measures and pricing rules. The results are stated for two different categories: first, for risk measures and pricing rules which satisfy properties R1–R4 and P1–P4 (including non-sub-additive pricing rules and risk measures), and, second, for risk measures and pricing rules that satisfy properties R1, R2, R3, R5 and P1, P2, P3, P5, respectively (including non-monotone ones). Results for the second family make use of the dual representation of pricing rules and risk measures. We then study the conditions under which an arbitrage opportunity is generated.

1.3.1 Market Principles

We start with the following result for π_ϱ defined in (1.14).

Proposition 1.3.1 *Let*

$$\mathcal{X}_\varrho := \{x \in \mathbb{R}^n | \pi_\varrho(x) \in \mathbb{R}\}.$$

Then, the following statements hold:

1. π_ϱ *and* $\mathcal{X}_\varrho$ *are positive homogeneous if* ϱ *and* π *are.*

2. π_ϱ *and* $\mathcal{X}_\varrho$ *are translation-invariant if* ϱ *and* π *are.*

3. π_ϱ *and* $\mathcal{X}_\varrho$ *are sub-additive if* ϱ *and* π *are.*

4. π_ϱ *and* $\mathcal{X}_\varrho$ *are convex if* ϱ *and* π *are.*

   Furthermore,

5. π_ϱ *is monotone if* ϱ *and* π *are.*

Proof See Appendix A.

Note that Proposition 1.3.1 does not say anything about the first property of a pricing rule which warrants some further explanation. It turns out that for the first property of a pricing rule to hold, we need to guarantee that some conditions for $\mathcal{X}$, ϱ and π are satisfied. Below, we explicitly state these conditions as general pricing principles that are valid regardless of the type of pricing or pricing rule.

Normality (N). $\pi_\varrho(0) = 0$.

No Good Deal Assumption (NGD). There is no financial position x such that

$$\varrho(x) < 0\,,\ \pi(x) \leq 0.$$

Consistency Principle (CP). For any member $x \in \mathcal{X}$, π and π_ϱ are consistent, i.e.,

$$\pi(x) = \pi_\varrho(x).$$

Compatibility (C). For a risk measure ϱ and a pricing rule π, (1.13) has a finite infimum.

The first principle simply recognizes that the price of zero is always zero. The second principle states that any risk-free variable has a positive cost (see Cochrane and Saa-Requejo (2000)). The third principle is a consistency condition between a pricing rule π and π_ϱ over $\mathcal{X}$. The last principle points out that the hedging problem always yields a price.

1.3.2 Positive-Homogeneous and Monotone Risk and Pricing Rules

Next, we discuss the equivalence of the market principles for a risk measure ϱ and pricing rule π which satisfy properties R1–R4 and P1–P4.

Theorem 1.3.2 *Let us assume ϱ and π satisfy properties R1–R4 and P1–P4. Then,*

$$(CP) \Rightarrow (N) \Leftrightarrow (NGD) \Leftrightarrow (C).$$

Moreover, if $\mathcal{X}$ is a vector space and π is super-additive, we also have

$$(N) \Rightarrow (CP).$$

Proof See Appendix A.

The following corollary states the conditions under which π_ϱ is a pricing rule.

Corollary 1.3.3 *Given the notation above, $\pi_\varrho : \mathcal{X}_\varrho \to \mathbb{R}$ is a pricing rule if and only if (N) or (NGD) holds.*

1.3.3 Positive-Homogeneous and Sub-Additive Risk and Pricing Rules

In this section, we assume that the risk measure ϱ and the pricing rule π satisfy properties R1, R2, R3, R5 and P1, P2, P3, P5, respectively. In that case, we extend the range of the these mappings to $\mathbb{R} \cup \{+\infty\}$

$$\bar{\varrho}(x) = \begin{cases} \varrho(x) & x \in \mathcal{D} \\ +\infty & \text{otherwise} \end{cases}, \quad \bar{\pi}(x) = \begin{cases} \pi(x) & x \in \mathcal{X} \\ +\infty & \text{otherwise.} \end{cases}$$

This extension allows us to use the dual representation of positive-homogeneous convex functions. Duality theory and sub-gradient analysis prove useful since the risk measures and pricing rules are usually not differentiable. First, we present conditions under which arbitrage opportunities do not exist in terms of the dual sets. Then, we characterize the solution to the hedging problem (1.13) and the pricing rule π_ϱ in (1.14).

We start by introducing some additional notation. From the theory of convex risk measures, any convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ has the following Fenchel-Moreau representation[4]

$$f(x) = \sup_{z \in \mathbb{R}^n} \{E(-zx) - f^*(z)\},$$

[4]For technical reasons, we use $-z$ instead of z.

where $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the dual of f defined as

$$f^*(z) = \sup_{x \in \mathbb{R}^n} \{E(-zx) - f(x)\}.$$

It can be easily seen that for any positive-homogeneous function f, f^* is 0 on a convex closed set, denoted by Δ_f, and infinity otherwise. Therefore, the Fenchel-Moreau representation of a positive homogeneous function f has the form

$$f(x) = \sup_{z \in \Delta_f} E(-zx).$$

As an example, for any coherent risk measure ϱ, Δ_ϱ is a subset of the set of all probability measures, i.e., $\Delta_\varrho \subseteq \{z \in \mathbb{R}^n | z \geq 0, \sum z_i = 1\}$, and, therefore, it is compact (see Artzner, Delbaen, Eber, and Heath (1999)). In contrast, for any expectation bounded risk ϱ, $\Delta_\varrho \subseteq \{z \in \mathbb{R}^n | \sum z_i = 0\}$ (Rockafellar, Uryasev, and Zabarankin (2006)).

Now let us assume that, in general, ϱ and π are positive-homogeneous and sub-additive mappings. Since ϱ and π are positive-homogeneous and sub-additive, and because $\mathcal{X}$ and $\mathcal{D}$ are positive cones, their extensions are also positive-homogeneous and sub-additive. Then, we have the representations

$$\bar{\varrho}(x) = \sup_{z \in \Delta_{\bar{\varrho}}} E(-zx)\,,\ \bar{\pi}(x) = \sup_{z \in \mathcal{R}_{\bar{\pi}}} E(zx) \quad ,\forall x \in \mathbb{R}^n, \tag{1.15}$$

for closed convex sets $\Delta_{\bar{\varrho}}$ and $\mathcal{R}_{\bar{\pi}}$.

In order to obtain the representations for $\bar{\varrho}$ and $\bar{\pi}$, we need to introduce the dual-polar of a scalar-cone of random payoffs. If A is a scalar-cone of a random payoff, the dual-polar of the set A is given by

$$A^\perp := \{z | E(zx) \leq 0 \, \forall x \in A\}.$$

We then have the following proposition.

Proposition 1.3.4 *For any function* $f(x) := \sup_{z \in \Delta_f} E(zx)$*, for some set* Δ_f*, which is defined on a positive cone* A*, we have that*

$$\bar{f}(x) = \sup_{z \in \Delta_f + A^{\perp}} E(zx).$$

Proof See Appendix A.

Proposition 1.3.4 has the important implication that any risk measure $\varrho(x) = \sup_{z \in \Delta_\varrho} E(-zx)$ defined on $\mathcal{D}$, and pricing rule $\pi(x) := \sup_{z \in \mathcal{R}} E(zx)$ defined on $\mathcal{X}$, can be rewritten as

$$\varrho(x) = \sup_{z \in \Delta_{\bar{\varrho}}} E(-zx), \bar{\pi}(x) = \sup_{z \in \mathcal{R}_{\bar{\pi}}} E(zx),$$

where $\Delta_{\bar{\varrho}} = \Delta_\varrho - \mathcal{D}^{\perp}$ and $\mathcal{R}_{\bar{\pi}} = \mathcal{R} + \mathcal{X}^{\perp}$.

The following theorem states the main theoretical result of the paper.

Theorem 1.3.5 *Assume that the risk measure* $\varrho_{\mathbb{D}}$ *is defined as in* (1.4) *and the pricing rule* $\pi_{\mathbb{M}}$ *is defined as in* (1.12)*. Then, the following statements are equivalent:*

1. *The hedging problem* (1.13) *is finite.*

2. $\mathcal{R}_{\varrho,\pi} = (\Delta_\varrho - \mathcal{D}^{\perp}) \cap (\mathcal{R}_\pi + \mathcal{X}^{\perp}) \neq \varnothing$, $\forall \varrho \in \mathbb{D}, \forall \pi \in \mathbb{M}$

   Furthermore, if condition 3 holds for π *and* ϱ*, these statements are equivalent to*

3. *There is no Good Deal in the market.*

In all cases, the price (1.14) *can be represented as*

$$(\pi_{\mathbb{D}})_{\varrho_{\mathbb{M}}}(x) = \inf_{\pi \in \mathbb{M}, \varrho \in \mathbb{D}} \pi_\varrho(x) = \inf_{\pi \in \mathbb{M}, \varrho \in \mathbb{D}} \sup_{z \in \mathcal{R}_{\pi,\varrho}} E(zx).$$

Proof See Appendix A.

In most cases, such as coherent risk measures and deviation measures of risk, the risk measure ϱ is defined on $\mathbb{R}^n$ meaning that $\mathcal{D}^\perp = \{0\}$. We then have the following corollary.

Corollary 1.3.6 *If ϱ is a coherent risk measure and π a sub-linear pricing rule, then there is no Good Deal if and only if $\mathcal{R}_{\pi_\varrho} := \Delta_\varrho \cap (\mathcal{R}_\pi + \mathcal{X}^\perp) \neq \varnothing$, and the pricing rule is given by*

$$\pi_\varrho(y) = \sup_{z \in \mathcal{R}_{\pi_\rho}} E(zx). \tag{1.16}$$

Theorem 1.3.5 and Corollary 1.3.6 illustrate the generality of our approach compared to the existing literature. In the existing literature, the set of stochastic discount factors is constructed either parametrically (using, for example, a semi-martingale process), or empirically and a pricing rule π is then obtained by taking supremum of prices over a closed convex subset $\mathcal{R}$. In order to price all positions in the market, any stochastic discount factor z' is constructed as a positive and linear extension of $z \in \mathcal{R}_\pi$, i.e., $z'|_\mathcal{X} = z$. Therefore, the set of stochastic discount factors is induced by the unique monotonic extension $\tilde{\pi}$ of π (for more details, see Theorem 2.1 in Jouini and Kallal (1995b)). By contrast, in our approach, the extension of the pricing rule is not constructed monotonically but it is obtained within the hedging problem and is affected, in general, by two additional factors: market incompleteness and frictions. In our approach, assuming that ϱ is defined on the whole space so that $\mathcal{D}^\perp = \{0\}$, the set of stochastic discount factors is equal to $\Delta_\varrho \cap (\mathcal{R}_\pi + \mathcal{X}^\perp)$, which is expanded by adding $\mathcal{X}^\perp$ and contracted by intersecting with Δ_ϱ.

Our method can reproduce the existing approach if we assume $\varrho(x) = \tilde{\pi}(-x)$. Indeed, our approach is able to reproduce the pricing rule $\tilde{\pi}$ if and only if the consistency principle holds. If the pricing rule is super-additive, this can be achieved if and only if $\pi(-x) \leq \varrho(x), \forall x \in \mathcal{X}$. This implies that $\mathcal{R} \subseteq \Delta_\varrho$. It can be easily verified that $x \mapsto \tilde{\pi}(-x)$ is the smallest risk measure for which the consistency principle holds. The mapping $x \mapsto \tilde{\pi}(-x)$ is the market measurement of risk and has been proposed by Assa and Balbás (2011). In this case, $\mathcal{D}^{\perp} = \{0\}$ and $\Delta_{\varrho} = \mathcal{R}$, which yields $\pi_{\varrho} = \pi$. Hence, the hedging problem becomes

$$\begin{cases} \min\{\tilde{\pi}(y - x) + \pi(x)\} \\ x \in \mathcal{X}. \end{cases} \tag{1.17}$$

It is clear that since $\tilde{\pi}$ is sub-additive, $x = 0$ is a solution to this hedging problem. Therefore, the pricing rule $\pi_{\tilde{\pi}(-.)}$ equals $\tilde{\pi}$, which reproduces the existing approach in the literature.

1.4 Appendix: Proofs of Propositions and Theorems

1.4.1 Proof of Theorem 1.2.2

From Delbaen (2002), the equality in Theorem 1.2.2 holds for $\varrho_{\alpha} = \mathrm{VaR}_{\alpha}$. Therefore, since the minimum is attained for VaR_{α}, for any α there exists $\varrho^{\alpha} \geq \mathrm{VaR}_{\alpha}$ such that $\varrho^{\alpha}(x_0) = \mathrm{VaR}_{\alpha}(x_0)$. Now introduce $\varrho(x) = \int_0^1 \varrho^{\alpha}(x)\varphi(\alpha)d\alpha$. It is easy to see that ϱ is a coherent risk measure such that $\varrho \geq \varrho_{\varphi}$ and $\varrho(x_0) = \varrho_{\varphi}(x_0)$, which proves the desired result.

1.4.2 Proof of Proposition 1.3.1

We only provide the proof of statement 1 since the proof of statement 2 follows very similar arguments. Let $g \in \mathcal{X}_{\alpha}$ and $t \in \mathbb{R}_+$. Then,

$$\pi_{\varrho}(tg) = \inf_{x \in \mathcal{X}}\{\varrho(x - tg) + \pi(x)\} = \inf_{tx \in \mathcal{X}}\{\varrho(tx - tg) + \pi(tx)\} = t\pi_{\varrho}(x) \in \mathbb{R}.$$

Using the same argument, one can show that for $g \in \mathcal{X}_\varrho$, $\pi_\varrho(x+c) = \pi_\varrho(x) + c$ for all $c \in \mathbb{R}$. Hence, we have that $g + c \in \mathcal{X}_\varrho$.

Now let $g \in \mathcal{X}_\varrho$ and $g \leq h$. Because ϱ is decreasing, we have that

$$\varrho(x-h) + \pi(x) \geq \varrho(x-g) + \pi(x).$$

By taking infimum on $\mathcal{X}$, we obtain that $\pi_\varrho(h) \in \mathbb{R}$.

1.4.3 Proof of Theorem 1.3.2

We begin by showing the equivalence between (N) and (NGD). To this end, we demonstrate that both of them are equivalent to the following inequality:

$$\varrho(x) + \pi(x) \geq 0\,,\ \forall x \in \mathcal{X}. \tag{1.18}$$

First, we show that (N) is equivalent to (1.18). Given (N), we have that $\pi_\varrho(0) = 0$ which, by construction, implies (1.18). On the other hand, given (1.18) it is easy to see that $\pi_\varrho(0) \geq 0$. In addition, by setting $x = 0$ in (1.18), it follows that $\pi_\varrho(0) = 0$.

Second, we show the equivalence between (1.18) and (NGD). Suppose that x is a Good Deal, i.e., $\varrho(x) < 0$ and $\pi(x) \leq 0$, which clearly implies $\varrho(x) + \pi(x) < 0$. On the other hand, if (1.18) does not hold, we have that $\varrho(x) + \pi(x) < 0$ for some position x. By cash-invariance of π and ϱ, it is obvious that $x - \pi(x)$ is a Good Deal.

Next, we demonstrate the equivalence between (NGD) and (C). Assume that (NGD) does not hold. Then, there exists an x such that $\varrho(x) < 0$ and $\pi(x) \leq 0$. Let y be a variable and assume that $c \in \mathbb{R}$ is such that $y \leq c$. Since $tx - y \geq tx - c$ for all $t > 0$,

$$\begin{aligned}\varrho(tx-y) + \pi(tx) &\leq \varrho(tx-c) + \pi(tx)\\ &= \varrho(tx) + c + \pi(tx)\\ &= t(\varrho(x) + \pi(x)) + c \to -\infty,\end{aligned}$$

as t tends to $+\infty$. This shows that (1.13) does not have a finite infimum.

To establish (NGD) $\Rightarrow$ (C), assume that for a variable y, (1.13) does not have a finite infimum. Let $c \in \mathbb{R}$ be such that $c \leq y$. Since $x - c \geq x - y$ for all financial positions $x \in \mathcal{X}$, we have that

$$\begin{aligned} \varrho(x-c) \leq \varrho(x-y) \quad &\Rightarrow \varrho(x) + c \leq \varrho(x-y) \\ &\Rightarrow \varrho(x) + \pi(x) + c \leq \varrho(x-y) + \pi(x). \end{aligned}$$

Since (1.13) is not bounded, then there exists an x such that $\varrho(x-y) + \pi(x) < c$. This yields $\varrho(x) + \pi(x) < 0$. Thus, it is clear that $\tilde{x} = x - \pi(x)$ is a Good Deal.

Finally, we show (N) $\Rightarrow$ (CP) when $\mathcal{X}$ is a vector space and π is super-additive. Let $y \in \mathcal{X}$ and suppose that (N) holds. Since $\mathcal{X}$ is a vector space, we have that, for a given x, $\mathcal{X} - x = \mathcal{X}$. Therefore, by construction,

$$\varrho(x-y) + \pi(x-y) \geq \pi_\varrho(0) = 0$$

and by super-additivity of π,

$$\varrho(x-y) + \pi(x) - \pi(y) \geq \varrho(x-y) + \pi(x-y) \geq 0$$

which implies that $\varrho(x-y) + \pi(x) \geq \pi(y)$. Therefore, $\pi_\varrho(y) = \pi(y)$.

1.4.4 Proof of Proposition 1.3.4

First, note that

$$\begin{aligned} \chi_A^*(z) &= \sup_x \{E(zx) - \chi_A(x)\} \\ &= \sup_{x \in A} E(zx) \qquad , \\ &= \chi_{A^\perp}(z). \end{aligned}$$

Hence, $\chi_A(x) = \sup_{z \in A^\perp} E(zx)$. Then, we have

$$\begin{aligned}\bar{f}(x) &= f(x) + \chi_A(x) = \sup_{z \in \Delta_f} E(zx) + \sup_{z' \in A^\perp} E(z'x) \\ &= \sup_{(z,z') \in \Delta_f \times A^\perp} E((z+z')x) \\ &= \sup_{z \in \Delta_f + A^\perp} E(zx).\end{aligned}$$

1.4.5 Proof of Theorem 1.3.5

First, we prove the result for sub-additive risk measures and pricing rules. The following proposition, which is a standard result in the literature on convex analysis, presents the necessary and sufficient conditions under which solution to the hedging problem exists.

Proposition 1.4.1 *Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be two convex functions. Then, the following equality holds*

$$\inf_{x \in \mathbb{R}^n} \{f_1(y-x) + f_2(x)\} = (f_1^* + f_2^*)^*(x),$$

with the convention that $\sup(\varnothing) = -\infty$.

In the particular case when $f_1 = \bar{\pi}$ and $f_2 = \bar{\varrho}$, we have

$$(f_1^* + f_2^*)(x) = (\chi_{\Delta_\varrho - \mathcal{D}} + \chi_{\mathcal{R} + \mathcal{X}^\perp})(x) = \chi_{(\Delta_\varrho - \mathcal{D}) \cap (\mathcal{R} + \mathcal{X}^\perp)}(x).$$

Therefore,

$$\inf_{x \in \mathcal{X}} \{\varrho(x-y) + \pi(x)\} = \sup_{z \in (\Delta_\varrho - \mathcal{D}) \cap (\mathcal{R} + \mathcal{X}^\perp)} E(zy).$$

This proves the existence of the infimum for the sub-additive case.

In the general case, we have

$$\begin{aligned}\inf_{x\in\mathcal{X}}\{\varrho_{\mathbb{D}}(x-y)+\pi_{\mathbb{M}}(x)\} &= \inf_{x\in\mathcal{X}}\left\{\inf_{\varrho\in\mathbb{D}}\varrho(x-y)+\inf_{\pi\in\mathbb{M}}\pi(x)\right\}\\ &= \inf_{x\in\mathcal{X}}\left\{\inf_{\varrho\in\mathbb{D}\times\pi\in\mathbb{M}}\varrho(x-y)+\pi(x)\right\}\\ &= \inf_{\varrho\in\mathbb{D}\times\pi\in\mathbb{M}}\{\inf_{x\in\mathcal{X}}\varrho(x-y)+\pi(x)\}.\end{aligned}$$

This problem has a finite infimum if for every $\varrho\in\mathbb{D}$ and $\pi\in\mathbb{M}$, the inner problem $\inf_{x\in\mathcal{X}}\varrho(x-y)+\pi(x)$ is finite. Given the discussion above, this proves the statement of the theorem.

Chapter 2

Empirical Assessment and Application

2.1 Estimation Problem

In this section, we illustrate the practical relevance of our theoretical results in the context of hedging economic risk by highlighting the effect of different risk measures on hedging strategies and the role of $\mathcal{X}^{\perp}$. Our analysis of portfolios that track or hedge various economic risk variables follows largely Lamont (2001) and Goorbergh, Roon, and Werker (2003). While these papers employ the mean-variance (MV) framework for constructing the portfolio of assets, we consider the more general and robust CVaR and VaR risk measures. Let y_t denote an economic risk variable to be hedged at time t ($t = 1, 2, ..., T$), $x_t = (x_{t1}, ..., x_{tN})'$ be N securities (traded factors) at time t and $\mathcal{X} = \text{span}\langle x_1, \ldots, x_N \rangle$. The pricing rule is the expected value of the portfolio given by $\pi(x_t'\theta) = E(x_t'\theta)$, where $\theta = (\theta_1, \ldots, \theta_N)'$.

For the mean-variance risk measure, we have that $\varrho(x) = \delta\sigma(x) - E(x)$. To facilitate the comparison with the other risk measures, the risk aversion parameter δ is set equal to 1. By plugging $x = \sum \theta_i x_i - y$, the problem (1.14) reduces to the following OLS problem:

$$\min_{\theta} \frac{1}{T} \sum_{t=1}^{T} \left(\tilde{y}_t - \sum\nolimits_{j=1}^{N} \theta_j \tilde{x}_{tj} \right)^2 , \tag{2.1}$$

where $\tilde{y}_t = y_t - E(y_t)$ and $\tilde{x}_{tj} = x_{tj} - E(x_{tj})$.

For the CVaR risk measure, we rewrite the problem (1.14) with a risk measure $\varrho = \varrho_{\nu_\alpha}$ and a pricing rule $\pi = E$ as

$$\min_{\theta} \left\{ \varrho_{\nu_\alpha} \left(x_t' \theta - y_t \right) + E \left(x_t' \theta \right) \right\} \tag{2.2}$$

or, more conveniently, as

$$\min_{\theta} \left\{ \varrho_{\nu_{1-\alpha}} \left(y_t - x_t' \theta \right) + E \left(x_t' \theta \right) \right\} , \tag{2.3}$$

using that $\varrho_{\nu_\alpha} (x_t'\theta - y_t) = \varrho_{\nu_{1-\alpha}} (y_t - x_t'\theta)$. Then, using translation-invariance and Theorem 2 in Bassett, Koenker, and Kordas (2004), the problem (2.3) can be rewritten equivalently an $(1-\alpha)$-quantile regression problem:

$$\min_{\xi, \theta} \frac{1}{T} \sum_{t=1}^{T} \rho_{1-\alpha} \left(\tilde{y}_t - \xi - \tilde{x}_t' \theta \right) , \tag{2.4}$$

where $\rho_{1-\alpha}(u) = u\left[(1-\alpha)\mathbb{I}\{u > 0\} - \alpha \mathbb{I}\{u \leq 0\}\right]$ and $\mathbb{I}\{\cdot\}$ denotes the indicator function. Note that since 1 is trivially in the intersection of the sub-gradient set of these risk measures and $\mathcal{R}$, then it follows from Theorem 1.3.5 there is no Good Deal and the hedging problem has a solution.

For the VaR hedging problem, we simply minimize the aggregate hedging costs

$$\min_{\theta} \{ \mathrm{VaR}_{1-\alpha}(y_t - x_t'\theta) + E(x_t'\theta) \}.$$

One can easily show that the probability measure $\mathbb{P}$ belongs to the sub-gradient of any law-invariant risk measure which also has properties R2 and R5. Therefore, by using part 2 of Theorem 1.3.5, the risk measures MV and CVaR do not produce any Good Deal with the pricing rule E. For VaR, we use the No-Good-Deal assumption and the theoretical results developed in the previous section. Since $\mathcal{X}$ is a vector space and π is a linear function, then, according to Theorem 1.3.2, the No-Good-Deal assumption holds if and only if π_ϱ (here E_{VaR}) is consistent. Hence,

$$\min_{\theta}\{\text{VaR}_{1-\alpha}(y_t - x_t'\theta) + E(x_t'\theta)\} = E(y_t).$$

2.2 Data Description

Our choice of economic risk variables and security returns is similar to Goorbergh, Roon, and Werker (2003). The data are at monthly frequency for the period February 1952 – December 2012. The traded securities include the risk-free rate, four stock-market factors (Fama and French (1992), Carhart (1997)) and two bond-market factors proxied, respectively, by: (i) the one-month T-bill (from Kenneth French's website), denoted by *RF*, (ii) the excess return (in excess of the one-month T-bill rate) on the value-weighted stock market (NYSE-AMEX-NASDAQ) index (from Kenneth French's website), denoted by *MARKET*, (iii) the return difference between portfolios of stocks with small and large market capitalizations (from Kenneth French's website), denoted by *SMB*, (iv) the return difference between portfolios of stocks with high and low book-to-market ratios (from Kenneth French's website), denoted by *HML*, (v) the momentum factor defined as the average return on the two high prior return portfolios minus the average return on the two low prior return portfolios (from Kenneth French's website), denoted by *MOM*, (vi) *TERM* defined as the difference between the yields of ten-year and one-year government bonds (from the Board of Governors of the Federal Reserve System), and (vii) *DEF* defined the difference between the yields of long-term corporate Baa bonds (from the Board of

Governors of the Federal Reserve System) and long-term government bonds (from Ibbotson Associates).

The macroeconomic risk variables include (i) the inflation rate measured as monthly percentage changes in CPI for all urban consumers (all items, from the Bureau of Labor Statistics), denoted by *INF*, (ii) the real interest rate measured as the monthly real yield on the one-month T-bill (from CRSP, Fama Risk Free Rates), denoted by *RI*, (iii) the term spread measured as the difference between the 10-year Treasury (constant maturity) and 3-month (secondary market) T-bill rate (from the Board of Governors of the Federal Reserve System), denoted by *TS*, (iv) the default spread measured as the difference between corporate Baa and Aaa rated (by Moody's Investor Service) bonds (from the Board of Governors of the Federal Reserve System), denoted by *DS*, (v) the monthly dividend yield on value-weighted stock market portfolio (from the Center for Research in Security Prices, CRSP), denoted by *DIV*, and (vi) the monthly growth rate in real per capita total (seasonally-adjusted) consumption (from the Bureau of Economic Analysis), denoted by *CG*.

2.3 Results

In order to hedge against unexpected economic shocks, we follow Campbell (1996) and replace the variable y_t with the corresponding error term form a six-variable VAR(1) model of y_t ($y = [INF, RI, TS, DS, DIV, CG]$). For VaR and CVaR, we use $\alpha = 0.1$ and 0.05 (i.e., $1 - a = 0.9$ and 0.95). The standard errors for VaR and CVaR are computed by bootstrapping. Statistically significant coefficients at the 5% nominal level are reported in bold font. The results for hedging inflation, real interest rate, term spread, default spread, dividend yield and consumption growth using the three risk measures are presented in tables 2.1 to 2.6, respectively. The last line in each table reports the computed price.

A number of interesting findings emerge from this hedging exercise. First, as it was noted in section 4.1, if the pricing rule E is correctly specified, the price should equal $E(y)$ (in the VaR case we also need to know if y is fully hedged). Tables 2.1 to 2.6 reveal that in all cases, the prices are significantly different from $E(y)$, which is attributed to the unhedged part in pricing y. These results highlight the role of the set $\mathcal{X}^{\perp}$. Indeed, the true stochastic discount factor lies in the larger set $\mathcal{X}^{\perp} \cap \Delta_{\varrho}$ for MV and CVaR, while for VaR we have a family of Δ_{ϱ}'s as in part 2 of Theorem 1.3.5. Our theory suggests that the true SDF has to represented as $P + z$, where z belongs to $\mathcal{X}^{\perp}$.

Second, while there is agreement across the different risk measures in hedging term spread, dividend yield and, to some extent, consumption growth, the hedging of inflation, real interest rate and default spread exhibit substantial heterogeneity both across and within risk measures. For example, CVaR suggests that *RF*, *SMB* and *TERM* prove to be important factors for hedging inflation whereas the other risk measures indicate that these factors are largely insignificant. Furthermore, there are differences across the different quantile regressions for CVaR and in some cases, depending on the level of α, the investor needs to switch from 'long' to 'short' positions in order to hedge the underlying economic risk. This illustrates the potential of alternative risk measures for robustifying the performance of economic portfolios.

2.4 Estimating the Stochastic Discount Factor

In this section we provide estimation methods to estimate Stochastic Discount Factors for two particular cases. Following the literature on pricing we confine ourselves to the set of the admissible stochastic discount factors. This means we are only interested in a set SDF which can correctly price a set of the assets, called test assets. As we have discussed earlier, we denote the set of test assets as $R_1, ..., R_N$ and their correct corresponding price as $p_1, ..., p_N$

$$\text{SDF} = \{z \geq 0 | E(z) = 1, E(zR_i) = p_i, i = 1, ..., N\}.$$

As has been discussed in the same subsection this is equivalent to choosing the following pricing rule

$$\pi(x) = \sup_{\{z \geq 0 | E(z)=1, E(zR_i)=p_i, i=1,...,N\}} E(zx).$$

We have chosen the excess return of Fama and French 25 portfolio assets, denoted by $R_1, ..., R_{25}$, plus the risk free asset, denoted by x_0. In addition we assume the martingale condition for correct prices as follows

$$p_i = 0 \ , \ i = 1, ..., 25 \quad \text{and} \quad p_0 = 1.$$

In order to estimate a SDF we have to make some assumptions. Given discussions in Cochrane (2009) a legitimate economical assumption is to assume that a SDF can be determined by the economy-wide consumer's preferences and consumption. The next legitimate assumption is that the consumer's decision is a function of macroeconomic factors. Therefore, we use a linear model for SDF by using the macro economic factors (as proxy for the systemic factors), as is generally practiced in the literature. Any SDF can model as follows

$$g(\gamma) = f^T \gamma,$$

where f is a $K \times 1$ vector of K systemic factors, and γ is a K vector of SDF parameters.

2.4.1 General Methodology

In this section we develop a general methodology, in order to make use it in the next sections to estimate γ associated with the linear models of the stochastic discount factors. Our strategy is to design a nonnegative function $Q(x)$ such that for a given set Δ

$$x \in \Delta \Leftrightarrow Q(x) = 0.$$

Simply since the function Q attains its minimum at a point x belonging to Δ, by setting the following minimization problem we can estimate the parameter γ

$$\min_{\gamma} Q(g(\gamma)).$$

In the following estimations, we minimize an objective function in the following form

$$Q(\gamma, X) = E(q(\gamma, X)) + \sum_{j=1}^{J} h_j \left(E(r_j(\gamma, X)) \right),$$

where q, r_j, $j = 1, ..., J$ are smooth functions of the parameters and sample data, and h_j are two times differentiable functions. The empirical version can be written as

$$\hat{\gamma} = \text{argmin}_{\gamma} Q_T(\gamma, X) = \text{argmin}_{\gamma} \left\{ \frac{1}{T} \sum_{t=1}^{T} q_t(\gamma, x_t) + \sum_{j=1}^{J} h_j \left(\frac{1}{T} \sum_{t=1}^{T} r_j(\gamma, x_t) \right) \right\}.$$

It is straightforward to see that $\hat{\gamma}$ is consistent. However, the asymptomatic normality of the estimator needs some computational effort which is done in the Appendix. Here we just present the result as follows

$$T^{1/2}(\hat{\gamma} - \gamma_0) \to \mathcal{N}(0, V),$$

where $V = H^{-1} M H^{-1}$,

$$H = E\left(\frac{\partial^2}{\partial\gamma\partial\gamma'} Q(\gamma_0) \right),$$

and

$$M = E\left(\left(\frac{\partial}{\partial\gamma}q + \sum_j h_j'(E(r_j))\frac{\partial}{\partial\gamma}r_j\right)\left(\frac{\partial}{\partial\gamma}q + \sum_j h_j'(E(r_j))\frac{\partial}{\partial\gamma}r_j\right)'\right)\Bigg|_{\gamma=\gamma_0}.$$

Note that h' is the derivative of h (not the transpose).

2.4.2 Set of Admissible SDF

As we mentioned earlier we chose the set of all admissible assets based on true prices of the Fama and French 25 portfolio. Therefore we have to design an appropriate objective function which can capture this set. The objective function we need can be introduced by the following elements

$$\begin{cases} r_i(\gamma, f_t) = \gamma f_t x_{it} & i = 1, ..., 25 \text{ and } t = 1, ..., T \\ h(x) = x^{L_{25}} \end{cases},$$

where L_{25} is a large even number and it is reported in Table 2.7 for each model. In the following we assume different No Good Deal conditions and the associated sets, which will result in different pricing models. In each of them we assume that the objective function is the summation of the one we have introduced above, to keep us in the admissible set, and the one associated with the underling risk measure in use.

2.4.3 No Arbitrage

First we find an stochastic discount factor on which we do not pose any restriction, which is equivalent to No -Arbitrage case. Therefore, we only need $E(g(\gamma)) = 1$ and $g(\gamma) > 0$

$$\begin{cases} q_t(\gamma, f_t) = \mathcal{M}(-\gamma f_t) & t = 1, ..., T \\ r(\gamma, f_t) = \gamma f_t - 1 & t = 1, ..., T \\ h(x) = x^{L_{\text{NA}}} \end{cases},$$

where L_{NA} is a large even number and it is reported in Table 2.7 for each model. In addition, $\mathcal{M}$ is defined as follows

$$\mathcal{M}(x) = \begin{cases} 0 & x < 0 \\ x^{L_{\text{smooth}}} & x \geq 0 \end{cases},$$

where L_{smooth} is a large even number reported on Table 2.7 for each model. The first equation is set to guarantee $g(\gamma) \geq 0$ and the second and third equations are set to ensure $E(g(\gamma)) = 1$. The choice of L is a large number to penalize any deviation from $r = 0$ and are reported in the estimation tables. The choice of L and L' depend on how much we want to penalize deviation from our conditions. These two numbers will play an important role in asymptomatic, implying that we cannot choose them too large (or small), since otherwise the resulting variance-covariance matrix is singular. We will report them once we report the result for the estimation.

2.4.4 Bounded SDF

In this part we assume that the set of SDF consists of all non-negative random variables with mean one whose maximum value is less than or equal to a bound $c > 0$ i.e., $\forall z \in$SDF, $z \leq c$. Since any member of SDF has mean one, it is clear that $c \geq 1$. Taking $\alpha = \frac{1}{c}$, it is clear that this is equivalent to the hedging problem 1.13, when we use $\varrho = \text{CVaR}_\alpha$. According to the parametric linear model we assumed for a SDF this means that we have to estimate γ so that $g(\gamma) \leq c = \frac{1}{\alpha}$. We set for the following functions in the objective

$$\begin{cases} q_t(\gamma, f_t) = \mathcal{M}(\gamma f_t - \frac{1}{\alpha}) + \mathcal{M}(-\gamma f_t) & t = 1, ..., T \\ r(\gamma, f_t) = \gamma f_t - 1 & t = 1, ..., T \\ h(x) = x^{L_{\text{CVaR}}} \end{cases},$$

where L_{CVaR} is a large even number reported on Table 2.7 for each model. The first function is to capture $0 \leq g(\gamma) \leq \frac{1}{\alpha}$ and the second and third one are to capture $E(g(\gamma)) = 1$. Note that M has to be smooth enough, and takes positive values if and only if its argument is positive.

2.4.5 Bounded Variance

In Cochrane and Saa-Requejo (2000), the authors introduced for the first time the concept of Good Deal. They show that in their setting the No Good Deal assumption is equivalent to setting an upper bound on the variance of the set of SDFs. In mathematical terms let c be a positive number, the No Good Deal assumption holds if for any $g \in \text{SDF}$, $\sigma(g) \leq c$. According to the theory we have developed in this chapter, this is equivalent to a hedging problem that uses the CS-risk measure $\varrho = \varrho_c^{CS}$. Given that $E(g) = 1, \forall g \in$SDF, the No Good Deal assumption is equivalent to saying that $E(g^2) \leq c^2 - 1$ and $E(g) = 1$. Therefore we have the following functions

$$\begin{cases} q_t(\gamma, f_t) = \mathcal{M}(-\gamma f_t) & t = 1, ..., T \\ r_1(\gamma, f_t) = \gamma f_t - 1 & t = 1, ..., T \\ r_2(\gamma, f_t) = (\gamma f_t)^2 - c^2 + 1 & \\ h_1(x) = x^{L_{\text{CS}}} & \\ h_2(x) = \mathcal{M}(x) & \end{cases},$$

where L_{CS} is a large even number and it is reported in table 2.7 for each model. The first equation is set to guarantee $g(\lambda) \geq 0$, r_1 and h_1 to ensure $E(g(\gamma)) = 1$ and r_2 and h_2 to set $\sigma(g(\gamma)) \leq c$. The choice of L and L' are reported in the estimation tables.

2.4.6 Mean-Variance

Finally we used the mean-variance risk measure. It is not very difficult to see that the sub-gradient of MV_δ can be represented as:

$$\Delta_{MV_\delta} = \{z \in \mathbb{R}^n | E(z) = 1, \sigma(z) = \delta\}.$$

Therefore,

$$\begin{cases} r_1(\gamma, f_t) = \gamma f_t - 1 & t = 1, ..., T \\ r_2(\gamma, f_t) = (\gamma f_t)^2 - c^2 + 1 & \\ h_1(x) = h_2(x) = x^{L_{\text{MV}}} & \end{cases}.$$

where L_{MV} is a large even number and it is reported in table 2.7 for each model. However, one can see that the members of the set Δ_{MV_δ} are not necessarily positive, which can yield some Arbitrage opportunities.

2.4.7 Models

We use two different models, one model with traded factors and the other non-traded factors.

FF3. The fist model is the most popular factor model in the finance literature, the three factor Fama and French model. In this model f has three factors, Market, Size and Book-to-Market. Therefore,

$$g(\gamma) = \gamma_0 + \gamma_1(RM - RF) + \gamma_2 SMB + \gamma_3 HML.$$

CAY. The second model is the (CC-CAY) which is a conditional version of the CCAPM due to Lettau and Ludvigson (2001). The relation is

$$g(\gamma) = \gamma_0 + \gamma_1 \text{consumption} + \gamma_2 L(\text{cay}) + \gamma_2 \text{consumption} \times L(\text{cay}) \tag{2.5}$$

where *cay*, the conditioning variable, is a consumption-aggregate wealth ratio, and L is the lag operator. This specification is obtained by scaling the constant term and the c factor of a linearized consumption CAPM by a constant and *cay*.

The results are presented in Tables 2.8 to 2.15. There are different observation from the results, but we focus our attention only on the most important ones.

First of all, observe that while the stochastic discount factors resulting from the No Arbitrage restriction and the ones from the No Good Deal assumption using CVaR of 1,5 and 10% are exactly the same. The results for all No Good Deal constraint are similar in direction, which shows the consistency between results; using the FF3 model all go short on RM-RF, SMB and HML, while on the other hand, they go short on *consumption* and $consumption \times L(cay)$, and long on $L(cay)$.

Tantamount results for the NA and CVaR show that all estimation methods find their stochastic discount factors within the smallest set $\Delta_{\mathrm{CVaR}_{0.1}}$.

We have included two different type of factors to compare different risk measures we have used in this chapter, in two different type of models when we have only traded securities or we have only non-traded ones. In all cases we impose the condition that the stochastic discount factor has mean one. Therefore, if we measure the performance of each method by measuring mean of γf, according to its distance from one, then one can see among all different methods by NA and CVaR very well capture mean one and others perform poorly. It is interesting to see that in both models either not imposing any condition but the No Arbitrage or the CVaR (bounded SDF), we get much better results, by our performance measure. It is also interesting the CS and MV conditions (bounded variance), seem less restrictive than CVaR and more restrictive than No Arbitrage, it is impossible to improve upon the performance measure. This might be because the variance controlling conditions along with the methods we have used in this chapter generate some mis-specification problems.

Another interesting point is when we compare the two different models. In the

FF3 model, as one can see, the main load is due to risk free. It can be concluded when we observe how close the mean of the estimated SDF is to the risk free factor, and how small are the other ones. The meaning of this fact is that the prices could be mostly explained by the risk free factor, which does not seem plausible for risky portfolios. This is not the case for the second model, one can see that the variety of the coefficients.

2.5 Appendix on Asymptotic Normality

Let

$$Q_T(\gamma) = \frac{1}{T}\sum_{t=1}^{T} q_t(\gamma, x_t) + \sum_{j=1}^{J} h_j\left(\frac{1}{T}\sum_{t=1}^{T} r_j(\gamma, x_t)\right).$$

By using the first-order Taylor series expansion of $\frac{\partial}{\partial\gamma}Q_T(\hat{\gamma})$ about the true value γ_0 we have

$$0 = \frac{\partial}{\partial\gamma}Q_T(\hat{\gamma}) = \frac{\partial}{\partial\gamma}Q_T(\gamma_0) + \frac{\partial^2}{\partial\gamma\partial\gamma'}Q(\gamma^*)(\hat{\gamma} - \gamma_0), \tag{2.6}$$

where $\gamma*$ is an intermediate point on the line joining $\hat{\gamma}$ and γ_0. This shows that we have to derive the distribution of $\frac{\partial}{\partial\gamma}Q_T(\gamma_0)$. Given the form of Q_T we have

$$T^{1/2}\frac{\partial}{\partial\gamma}Q_T(\gamma_0) = \frac{1}{T^{1/2}}\sum_{t=1}^{T}\frac{\partial q_t}{\partial\gamma}(\gamma_0) + \sum_{j=1}^{J}\left(\frac{1}{T^{1/2}}\sum_{t=1}^{T}\frac{\partial}{\partial\gamma}r_j(\gamma_0, x_t)\right)h_j'\left(\frac{1}{T}\sum_{t=1}^{T} r_j(\gamma_0, x_t)\right). \tag{2.7}$$

As $T \to \infty$, according to Law of Large Numbers, $h_j'\left(\frac{1}{T}\sum\limits_{t=1}^{T} r_j(\gamma_0, x_t)\right) \to h_j'(E(r_j(\gamma_0)))$. Combining this with (2.7) yields

$$T^{1/2}\frac{\partial}{\partial\gamma}Q_T(\gamma_0) \to \mathcal{N}(0, M), \tag{2.8}$$

where

$$M = E\left(\left(\frac{\partial}{\partial\gamma}q + \sum_j h_j'(E(r_j))\frac{\partial}{\partial\gamma}r_j\right)\left(\frac{\partial}{\partial\gamma}q + \sum_j h_j'(E(r_j))\frac{\partial}{\partial\gamma}r_j\right)'\right)\Bigg|_{\gamma=\gamma_0}.$$

This relation with (2.6), and by using the consistency as $T \to \infty$, yield the desired relation. ∎

2.6 Tables

	MV	$CVaR_{0.9}$	$CVaR_{0.95}$	$VaR_{0.9}$	$VaR_{0.95}$
Intercept		**0.0026**	**0.0022**		
		(0.0001)	(0.0002)		
RF	0.0072	**-0.6737**	**-0.7844**	0.0058	0.0027
	(0.0558)	(0.0705)	(0.1106)	(0.0041)	(0.0019)
MARKET	**-0.0048**	**-0.0072**	**-0.0123**	-0.0038	**-0.0066**
	(0.0029)	(0.0030)	(0.0043)	(0.0013)	(0.0023)
SMB	-0.0008	**0.0131**	**0.0383**	-0.0008	-0.0003
	(0.0030)	(0.0048)	(0.0065)	(0.0005)	(0.0002)
HML	0.0022	0.0013	-0.0009	**0.0038**	0.0031
	(0.0042)	(0.0013)	(0.0006)	(0.0016)	(0.0023)
UMD	0.0015	0.0006	0.0002	0.0019	0.0023
	(0.0027)	(0.0006)	(0.0001)	(0.0012)	(0.0015)
TERM	0.0084	**-0.1427**	**-0.1440**	0.0025	0.0104
	(0.0109)	(0.0162)	(0.0263)	(0.0018)	(0.0070)
DEF	**-0.0265**	**0.1063**	**0.1370**	**-0.0246**	**-0.0227**
	(0.0242)	(0.0209)	(0.0369)	(0.0080)	(0.0117)
Price	**0.0023**	**0.0077**	**0.0093**	**0.0025**	**0.0037**
	(0.0000)	(0.0000)	(0.0001)	(0.0000)	(0.0001)

Table 2.1: Hedging Inflation.

The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

	MV	$CVaR_{0.9}$	$CVaR_{0.95}$	$VaR_{0.9}$	$VaR_{0.95}$
Intercept		**0.0021**	**0.0035**		
		(0.0001)	(0.0002)		
RF	-0.0304	**-0.1473**	**-0.6805**	-0.0147	-0.0296
	(0.0563)	(0.0621)	(0.1000)	(0.0063)	(0.0380)
MARKET	**0.0049**	-0.0020	0.0094	**0.0048**	0.0038
	(0.0028)	(0.0017)	(0.0045)	(0.0015)	(0.0032)
SMB	0.0013	-0.0009	0.0042	**0.0009**	0.0013
	(0.0029)	(0.0007)	(0.0033)	(0.0003)	(0.0020)
HML	-0.0029	**-0.0142**	**0.0188**	**-0.0031**	-0.0031
	(0.0042)	(0.0042)	(0.0072)	(0.0012)	(0.0037)
UMD	-0.0008	**-0.0346**	-0.0011	-0.0007	-0.0007
	(0.0027)	(0.0031)	(0.0008)	(0.0003)	(0.0013)
TERM	-0.0167	**-0.0664**	**-0.1187**	**-0.0284**	-0.0166
	(0.0109)	(0.0123)	(0.0239)	(0.0065)	(0.0133)
DEF	0.0205	**0.1095**	**0.2810**	**0.0222**	0.0226
	(0.0244)	(0.0194)	(0.0305)	(0.0073)	(0.0186)
Price	**0.0023**	**0.0075**	**0.0110**	**0.0027**	**0.0035**
	(0.0000)	(0.0000)	(0.0001)	(0.0000)	(0.0001)

Table 2.2: Hedging Real Interest Rate.

The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

	MV	$CVaR_{0.9}$	$CVaR_{0.95}$	$VaR_{0.9}$	$VaR_{0.95}$
Intercept		-0.0000	-0.0000		
		(0.0012)	(0.0018)		
RF	**0.3958**	**0.3782**	**0.3696**	**0.3886**	**0.3883**
	(0.0922)	(0.0877)	(0.1094)	(0.0526)	(0.0898)
MARKET	0.0011	-0.0023	-0.0016	**0.0018**	0.0006
	(0.0043)	(0.0061)	(0.0095)	(0.0005)	(0.0006)
SMB	0.0034	0.0047	0.0033	**0.0026**	0.0028
	(0.0048)	(0.0092)	(0.0138)	(0.0008)	(0.0016)
HML	0.0071	0.0010	0.0013	**0.0091**	0.0078
	(0.0053)	(0.0227)	(0.0230)	(0.0021)	(0.0042)
UMD	-0.0038	-0.0017	-0.0015	**-0.0065**	-0.0051
	(0.0042)	(0.0069)	(0.0099)	(0.0018)	(0.0026)
TERM	**0.1346**	**0.1055**	**0.1024**	**0.1277**	**0.1532**
	(0.0163)	(0.0194)	(0.0242)	(0.0126)	(0.0251)
DEF	**-0.0691**	**-0.0789**	-0.0756	**-0.0926**	**-0.0571**
	(0.0296)	(0.0307)	(0.0341)	(0.0138)	(0.0268)
Price	**0.0033**	**0.0057**	**0.0082**	**0.0029**	**0.0044**
	(0.0000)	(0.0001)	(0.0002)	(0.0001)	(0.0002)

Table 2.3: Hedging Term Spread.

The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

	MV	$CVaR_{0.9}$	$CVaR_{0.95}$	$VaR_{0.9}$	$VaR_{0.95}$
Intercept		**0.0010**	**0.0011**		
		(0.0000)	(0.0001)		
RF	**-0.0616**	0.0358	-0.0017	**-0.0572**	**-0.0359**
	(0.0335)	(0.0214)	(0.0010)	(0.0119)	(0.0133)
MARKET	-0.0008	0.0002	**0.0115**	**-0.0011**	-0.0005
	(0.0018)	(0.0001)	(0.0026)	(0.0003)	(0.0002)
SMB	-0.0015	0.0030	**-0.0090**	**-0.0016**	**-0.0027**
	(0.0014)	(0.0015)	(0.0036)	(0.0005)	(0.0011)
HML	-0.0005	**0.0091**	**0.0177**	-0.0004	**-0.0001**
	(0.0023)	(0.0015)	(0.0040)	(0.0001)	(0.0001)
UMD	-0.0003	**0.0026**	**0.0097**	**-0.0002**	**-0.0000**
	(0.0012)	(0.0010)	(0.0026)	(0.0001)	(0.0000)
TERM	**-0.0147**	**-0.0153**	-0.0169	**-0.0135**	-0.0005
	(0.0049)	(0.0051)	(0.0082)	(0.0025)	(0.0003)
DEF	**0.0503**	**0.1078**	**0.0965**	**0.0498**	**0.0882**
	(0.0125)	(0.0063)	(0.0158)	(0.0052)	(0.0079)
Price	**0.0011**	**0.0036**	**0.0048**	**0.0009**	**0.0016**
	(0.0000)	(0.0000)	(0.0001)	(0.0000)	(0.0001)

Table 2.4: Hedging Default Spread.

The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

	MV	$CVaR_{0.9}$	$CVaR_{0.95}$	$VaR_{0.9}$	$VaR_{0.95}$
Intercept		**0.0007**	**0.0011**		
		(0.0001)	(0.0001)		
RF	**-0.0735**	-0.0323	**-0.0794**	**-0.0774**	**-0.0736**
	(0.0148)	(0.0291)	(0.0352)	(0.0074)	(0.0104)
MARKET	**-0.0309**	**-0.0313**	**-0.0343**	**-0.0293**	**-0.0313**
	(0.0010)	(0.0015)	(0.0017)	(0.0012)	(0.0016)
SMB	-0.0000	0.0004	0.0000	**-0.0000**	**-0.0000**
	(0.0012)	(0.0007)	(0.0000)	(0.0000)	(0.0000)
HML	**-0.0028**	-0.0026	-0.0020	**-0.0029**	**-0.0032**
	(0.0014)	(0.0021)	(0.0019)	(0.0004)	(0.0005)
UMD	**-0.0012**	0.0016	-0.0028	**-0.0013**	**-0.0015**
	(0.0008)	(0.0012)	(0.0015)	(0.0002)	(0.0002)
TERM	-0.0036	**0.0146**	**-0.0163**	**-0.0035**	**-0.0040**
	(0.0029)	(0.0069)	(0.0067)	(0.0006)	(0.0008)
DEF	-0.0038	-0.0152	**0.0276**	**-0.0037**	**0.0010**
	(0.0044)	(0.0083)	(0.0105)	(0.0008)	(0.0002)
Price	**0.0007**	**0.0018**	**0.0028**	**0.0007**	**0.0010**
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Table 2.5: Hedging Dividend Yield.

The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

	MV	$CVaR_{0.9}$	$CVaR_{0.95}$	$VaR_{0.9}$	$VaR_{0.95}$
Intercept		**0.0059**	**0.0080**		
		(0.0003)	(0.0005)		
RF	**-0.2533**	-0.0510	0.0696	**-0.2882**	-0.1159
	(0.1112)	(0.0718)	(0.1184)	(0.0672)	(0.0788)
MARKET	0.0082	0.0079	-0.0067	**0.0074**	0.0048
	(0.0056)	(0.0062)	(0.0102)	(0.0023)	(0.0037)
SMB	**0.0256**	**0.0315**	**0.0467**	**0.0288**	**0.0237**
	(0.0080)	(0.0097)	(0.0169)	(0.0051)	(0.0094)
HML	0.0132	0.0079	-0.0096	**0.0034**	0.0108
	(0.0080)	(0.0089)	(0.0141)	(0.0020)	(0.0075)
UMD	-0.0034	**0.0334**	**0.0546**	**-0.0032**	-0.0032
	(0.0052)	(0.0073)	(0.0120)	(0.0015)	(0.0028)
TERM	**-0.0509**	**-0.0581**	-0.0070	**-0.0835**	**-0.0819**
	(0.0241)	(0.0223)	(0.0160)	(0.0165)	(0.0263)
DEF	-0.0568	0.0005	**0.1904**	**-0.0562**	**-0.0802**
	(0.0312)	(0.0007)	(0.0648)	(0.0178)	(0.0332)
Price	**0.0054**	**0.0159**	**0.0223**	**0.0064**	**0.0083**
	(0.0000)	(0.0001)	(0.0002)	(0.0001)	(0.0002)

Table 2.6: Hedging Consumption Growth.

The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

Model	L_{smooth}	L_{25}	L_{NA}	L_{CVaR}	L_{CS}	L_{MV}
FF3	6	6	2	2	2	2
CAY	6	6	2	2	2	2

Table 2.7: Values for $L_{\text{smooth}}, L_{25}, L_{\text{NA}}, L_{\text{CVaR}}, L_{\text{CS}}, L_{\text{MV}}, L_{\text{NAMV}}$.

γ_0	γ_1	γ_2	γ_3	$mean(\gamma f)$
0.9975	-0.0407	-0.0146	-0.1140	0.9964
(0.0019)	(0.0028)	(0.0061)	(0.0047)	(0.0010)

Table 2.8: FF3 model, No Arbitrage

α	γ_0	γ_1	γ_2	γ_3	$mean(\gamma f)$
1,5,10%	0.9975	-0.0407	-0.0146	-0.1140	0.9964
	(0.0019)	(0.0028)	(0.0061)	(0.0047)	(0.0010)

Table 2.9: FF3 model, CVaR

c	γ_0	γ_1	γ_2	γ_3	$mean(\gamma f)$
0.1	0.2311	-0.0026	-0.0023	-0.0016	0.2316
	(0.0001)	(0.0001)	(0.0003)	(0.0002)	(0.0001)
0.25	0.2467	-0.0031	-0.0027	-0.0019	0.2473
	(0.0001)	(0.0001)	(0.0003)	(0.0002)	(0.0001)
1	0.6281	-0.0161	-0.0196	-0.0388	0.6321
	(0.0008)	(0.0011)	(0.0020)	(0.0010)	(0.0005)

Table 2.10: FF3 model, CS

δ	γ_0	γ_1	γ_2	γ_3	$mean(\gamma f)$
0.1	0.3132	-0.0015	-0.0013	-0.0010	0.3135
	(0.0001)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
0.25	0.3223	-0.0017	-0.0016	-0.0011	0.3227
	(0.0001)	(0.0001)	(0.0002)	(0.0001)	(0.0001)
1	0.5814	-0.0120	-0.0112	-0.0085	0.5840
	(0.0003)	(0.0006)	(0.0011)	(0.0008)	(0.0002)

Table 2.11: FF3 model, MV

γ_0	γ_1	γ_2	γ_3	$mean(\gamma f)$
1.2822	-0.6862	0.5187	-0.3580	0.9444
(0.0492)	(0.0792)	(0.1171)	(0.2623)	(0.0134)

Table 2.12: CAY, No Arbitrage

α	γ_0	γ_1	γ_2	γ_3	$mean(\gamma f)$
1,5,10%	1.2822	-0.6862	0.5187	-0.3580	0.9444
	(0.0492)	(0.0792)	(0.1171)	(0.2623)	(0.0134)

Table 2.13: CAY, CVaR

c	γ_0	γ_1	γ_2	γ_3	$mean(\gamma f)$
0.1	0.2338	-0.0046	0.0009	-0.0005	0.2313
	(0.0013)	(0.0019)	(0.0008)	(0.0014)	(0.0006)
0.25	0.2499	-0.0068	0.0014	-0.0008	0.2462
	(0.0019)	(0.0028)	(0.0012)	(0.0020)	(0.0008)
1	0.7772	-0.3847	0.0880	-0.0484	0.5657
	(0.0586)	(0.0874)	(0.0726)	(0.1219)	(0.0101)

Table 2.14: CAY, CS

δ	γ_0	γ_1	γ_2	γ_3	$mean(\gamma f)$
0.1	0.3147	-0.0005	0.0001	-0.0001	0.3144
	(0.0001)	(0.0002)	(0.0001)	(0.0001)	(0.0001)
0.25	0.3241	-0.0006	0.0001	-0.0001	0.3238
	(0.0002)	(0.0003)	(0.0001)	(0.0002)	(0.0001)
1	0.6287	-0.1168	0.0258	-0.0156	0.5645
	(0.0219)	(0.0321)	(0.0199)	(0.0334)	(0.0049)

Table 2.15: CAY, MV

Chapter 3

A Staggered Pricing Approach to Modeling Speculative Storage: Implications For Commodity Price Dynamics

The last decade has witnessed a surge in commodity prices and a widespread financialization of commodity products. The upward movements and the increased volatility of the commodity prices have been largely attributed to strong demand by China and other emerging markets as well as massive capital flows into the commodity markets by institutional investors, portfolio managers and speculators. While the importance of commodity price movements for the economic policy and investors' sentiment has generated a substantial research interest, the behavior and the determination of commodity prices is not yet fully understood. The main objective of this paper is to develop a structural model of commodity price determination that reflects the empirical properties (high persistence and conditional heteroskedasticity) of commodity prices. In order to achieve this goal and to gain further understanding into the fundamental factors that drive the observed behavior of commodity prices, we modify the structure of the speculative storage model from one where the prices adjust almost instantaneously to harvest shocks to a setup where they change slowly and infrequently. More specifically, we depart from the assumption that market prices are determined in a perfectly competitive environment and extend the basic speculative storage model by explicitly introducing intermediate goods speculators with a staggered pricing rule. One appealing aspect of this approach is its ability to mimic some important characteristics of the actual commodity prices such as high persistence and conditional heteroskedasticity, which can be generated even in the absence of correlated harvest shocks.

The speculative storage model for commodity prices can be dated back to Gustafson (1958) who defines a set of optimal storage rules that state how much grain should be carried over into the next period given the current year supply. Moreover, by introducing intertemporal storage arbitrage and supply shocks, Gustafson (1958) incorporates rational expectations. This line of research is further elaborated in Muth (1961). Samuelson (1971) develops a model for commodities which determines the behavior of the prices as the solution to a stochastic dynamic programming problem. Furthermore, Beck (1993) builds upon the work by Muth (1961) and provides a theoretical basis for treating the variance of storable commodities as serially correlated which suggests that commodity prices may exhibit conditional heteroskedasticity. The presence of storage is instrumental in ensuring that the price variance in one period directly affects inventory variance which in turn is transmitted to next period's price variation. Williams and Wright (1991) provide a comprehensive discussion of the basic storage model and its extensions, and summarize the time series properties of storable commodities. Williams and Wright (1991) put an emphasis on the complex non-linear storage behavior resulting from the fact that aggregate storage cannot be negative.

Deaton and Laroque (1992, 1995, 1996) develop a partial equilibrium structural model of commodity price determination and apply numerical methods to test and estimate the model parameters, confronting for the first time the storage model with the documented behavior of actual prices. Their analysis suggests that the introduction of speculative inventories and serially correlated supply shocks do not appear to generate sufficient persistence in commodity prices although they prove to be successful in replicating the substantial volatility observed in the actual data.

More recently, numerous studies have focused on modifying the storage model in order to accommodate the persistence of commodity prices. Chambers and Bailey (1996) relax the *iid* assumption on harvest shocks, and study the price fluctuations of storable commodities, assuming that shocks are either time dependent or that the model exhibits periodic disturbances. Ng and Ruge-Murcia (2000) incorporate additional features into the storage model in an attempt to generate a higher degree of persistence in commodity prices. In particular, Ng and Ruge-Murcia (2000) allow for serially correlated shocks assuming that harvest follows a first-order moving average (MA(1)) process. They also examine the ability of production lags and heteroskedastic supply shocks, multi-period forward contracts and convenience yields to generate an increased persistence in commodity prices. Cafiero, Bobenrieth, Bobenrieth, and Wright (2011) demonstrate that the competitive storage model can give rise to high levels of serial correlation observed in commodity prices if more precise numerical methods are employed. Moreover, estimates for seven commodities supported the specification of the speculative storage model with positive constant marginal costs and no deterioration, which is in line with Gustafson (1958).

Furthermore, Cafiero, Bobenrieth, Bobenrieth, and Wright (2011) use a maximum likelihood framework to estimate the storage model with stock-outs, which is extended to include unbounded harvests and free disposal. Their results produce more accurate small sample estimates of the structural parameters of the model compared to the previous studies based on the pseudo-maximum likelihood procedure. Miao and Funke (2011) add shocks to the trends of output and demand. Evans and Guthrie (2007) include transaction cost frictions into the speculative storage model. One important finding that emerges from their analysis is that these frictions tend to have explanatory power for the dynamic behavior of spot and futures commodity prices. In a competitive equilibrium framework, the model of Evans and Guthrie (2007) is able to capture the serial correlation and GARCH characteristics of commodity prices. Finally, Arseneau and Leduc (2012) embed the speculative storage model into a general equilibrium framework. Their main result is that the interaction between storage and interest rates in general equilibrium increases the impact of competitive storage on commodity prices and leads to higher persistence than the one observed in the storage model with fixed interest rate.

In spite of this extensive literature for understanding the determinants and the dynamic patterns of commodity prices, reproducing the documented high persistence and conditional heteroskedasticity of actual prices within a well-articulated structural model proved to be a challenging task. In this paper, we address the issues regarding the commodity price dynamics in a unified fashion by embedding a staggered pricing mechanism into the speculative storage model. While Arseneau and Leduc (2012) also suggest to "introduce staggered price setting on the part of the final goods producing firm" in a general equilibrium framework as a possible extension for future research, our paper is the first to implement this approach and assess the properties of the model-generated commodity prices against the observed data.

In an attempt to depart from the assumption of perfect competition at both the production and storage activity, Newbery (1984), Williams and Wright (1991), and McLaren (1999) investigate the effects of market power on the storage behavior. Our model differs from their work along the dimension that the final bundler does not store the good and the storage is only done by intermediate risk neutral speculators. The final bundler only bundles intermediate prices in order to set the final price. Finally, Mitraille and Thille (2009) examine the market power in production with competitive storage by analyzing the effects that competitive storage has on the behavior of a monopolist. Using his market power, the monopolist can influence speculative activity by manipulating prices and consequently affect the distribution of prices. One of the findings of Mitraille and Thille (2009) is that stockouts occur less frequently under monopoly.

The focus of this paper is on the improved ability of the storage model with staggered prices to account for the empirical features of commodity prices. The main impact of staggered prices in our model is to dampen the movements in prices as well as the market power of intermediate speculators to affect prices. This leads to gradual adjustments and persistent responses of prices following a harvest shock. In addition to generating sufficient persistence in commodity prices, the staggered pricing approach allows us to match other important moments in the unconditional and conditional distributions of the commodity prices.

Nominal price rigidity is often incorporated in dynamic general equilibrium models with two widely used nominal price rigidity specifications in the literature. On one hand, the partial adjustment model developed by Calvo (1983), Rotemberg (1987), and Rotemberg (1996) allows for only a randomly chosen fraction of firms to adjust their prices according to some constant hazard rate in any given period. On the other hand, the staggered price setting rule adopted by Taylor (1980) and Blanchard and Fisher (1989) permits all firms to optimize their prices after a fixed number of periods.

In this paper, we assume that the pricing decisions are staggered as in Calvo (1983) and use a similar modeling framework as the one developed in McCandless (2008). Even though the staggered pricing is not generated endogenously within the model, it serves as a useful device to impart the inefficiencies in the agricultural commodity markets such as price floors, subsidies, import/export quotas and controls, government strategic stock reserves, collusion etc. that prevent prices to adjust instantaneously to changes in economic conditions. Note that these types of market inefficiencies induce some product differentiation and allow us to depart from the typical assumption in the literature that commodities tend to be homogeneous products whose prices are fully flexible and equal to their marginal costs. Our results confirm the importance of staggered prices for commodity price dynamics and suggest that the staggered pricing mechanism appears to be consistent with the behavior of the actual data. Moreover, we show how our model can be used to analyze the response of commodity prices to harvest shocks which provides a framework for economic and policy evaluation.

The remainder of the paper is organized as follows. The competitive storage model with staggered prices as well as the statistical characterizations of this model are presented in Section 2. Section 3 studies the practical implications of our staggered price speculative storage model using simulated data. Section 4 contains a brief description of the data and the estimation method used in the paper, and presents the main empirical results. Section 5 concludes.

3.1 Competitive Storage Model with Staggered Prices

This section introduces the model setup and characterizes the equilibrium and statistical behavior of the model-generated commodity prices.

3.1.1 Model and Equilibrium Price Behavior

The rational expectations model determines the optimal inventory decisions by risk-neutral speculators. The basic version of the model developed by Deaton and Laroque

(1992, 1995, 1996)[1] incorporates competitive storage into the consumer demand and supply dynamics and establishes the concept of stationary rational expectations equilibrium (SREE). The model with serial correlation in harvest shocks is tested by Ng and Ruge-Murcia (2000). In their paper, Ng and Ruge-Murcia (2000) consider an MA(1) specification for the model harvest shocks. Our model complements and extends the original DL model by embedding a staggered price setting into the speculative storage model. Regarding the harvest shock specification, we consider both (i) *iid* harvest shocks and (ii) MA(1) harvests shocks.

Our modified model has three types of commodity market participants: final consumers, intermediate risk neutral speculators and a bundler[2] who bundles the commodities in order to set the final price. In the absence of storage, the behavior of final consumers is characterized by a linear inverse demand function

$$p_t = P(z_t) = a + bz_t,$$

where a and $b < 0$ are parameters to be estimated and z_t denotes the harvest in period t.

Let the harvest z_t be given by

$$z_t = \bar{z} + u_t,$$

where $\bar{z}$ is constant (perfectly inelastic) and u_t is a random disturbance term which

[1]For brevity, we denote hereafter the basic speculative storage model of Deaton and Laroque by DL.

[2]In the literature, it is common to use the term "monopolist" instead of the term "bundler" that we employ in this paper. The reason that we prefer the latter is the following: in the staggered pricing literature, the final goods producer maximizes profits by setting the price. In this paper, we do not consider any profit maximization and any type of price setting for the final goods producer. Instead, we use directly the final goods prices as set in (3.6).

is assumed either to be *iid* or to follow an MA(1) process

$$u_t = e_t + \rho e_{t-1},$$

where e_t is $iid(0, \sigma^2)$. If $\rho = 0$, we have the case of *iid* shocks as in DL, and when $\rho > 0$, we have MA(1) shocks as in Ng and Ruge-Murcia (2000). In this paper, we investigate both cases and show that when we add staggered prices, the case for $\rho = 0$ gives better results compared to the case of non-staggered prices and $\rho > 0$.

Intermediate risk neutral speculators or inventory holders know the current year harvest and demand the commodity to transfer to the next period. They will do so whenever they expect to make a profit above the storage and interest cost. The depreciation rate of storage is denoted by δ. A simple form of proportional deterioration is considered which means that if in period t the speculators store I units of the commodity, they have at their disposal $(1-\delta)I$ units at the beginning of the next period. Moreover, speculators have to pay the real interest rate on the value of their storage. Let r denotes the constant exogenous real interest rate. The sum of harvest and inherited inventories, denoted by x_t, is referred to as the amount on hand and is given by

$$x_t = (1-\delta)I_{t-1} + z_t.$$

The relationship between the amount of storage and its net profit can be summarized as

$$\begin{cases} I_t > 0 \text{ if } (1-\delta)/(1+r)\mathbb{E}_t[p_{t+1}] = p_t, \\ I_t = 0 \text{ otherwise}, \end{cases}$$

where $\mathbb{E}_t$ denotes the expectation given the information at time t.

The condition for non-negative inventories is the crucial source of non-linearity in the model. This specification does not allow the market participants to borrow commodities that have not yet been grown. In addition, intermediate speculators benefit from market power that reflects their ability to affect the price. In this framework, we assume that there is a continuum of intermediate speculators (of unit mass indexed by $k \in [0, 1]$) and final big players in the market. Final players collect all the commodities from intermediate speculators and bundle intermediate speculators' prices into the final price in order to sell the commodity to consumers. In reality, the price level of many commodities is influenced either through the formation of cartels by producers or through government intervention by imposing export control agreements or keeping strategic stock reserves. Although some of those cartels brake up in the long run, as discussed in Gilbert (1987), all of them have a strong influence on commodity prices, at least in the short-run. Hence, the introduction of these final big players who bundle prices tends to generate persistence in commodity prices over consecutive periods.

For simplicity, we assume that there exists a bundler who bundles all intermediate speculators' prices into a single one. Each period t, a fraction $1-\gamma$ $(0 < 1-\gamma < 1)$ of the speculators is able to exploit their market power and to reset the prices of their commodities $P_t^*(k)$. In contrast, those who did not benefit from their market power to affect prices, retain their last period prices: $P_t^*(k) = P_{t-1}^*(k)$. Given this staggered pricing rule, along with the assumptions that speculators are risk neutral and have rational expectations, intermediate speculators' current and expected future prices must satisfy

$$P_t^*(k) = \max\left\{p(x_t), (1-\gamma)\frac{1-\delta}{1+r}\mathbb{E}_t[P_{t+1}^*(k)] + \gamma P_t^*(k)\right\}. \tag{3.1}$$

The first term in the brackets represents the price if the harvest is sold to consumers in period t and no inventories are carried over to the next period. The second term is known as the intertemporal Euler equation. This is the value of one unit stored if $1-\gamma$ of the speculators benefit form their market power to affect the price. This, in turn, occurs if the speculators expect to cover their costs (after depreciation) from buying the commodity at time t. Since the current period bundler prices are not yet determined, it is important to stress that speculators, who do not reset their prices, use their own current prices and not the market ones in order to determine $P_t^*(k)$ in (3.1).

Finally, the bundler will bundle all intermediate prices together according to the following pricing rule (see McCandless (2008))

$$P_t^{1-\psi} = \gamma P_{t-1}^{1-\psi} + (1-\gamma) P_t^*(k)^{1-\psi},$$

where P_t denotes the bundler final price of the good, the parameter ψ is the gross markup of the intermediate goods speculators and $P_t^*(k)$ represents the price for intermediate goods speculators who can set their prices. Since all intermediate goods speculators who can fix their prices are assumed to have the same markup over the same marginal costs, $P_t^*(k)$ is the same for all intermediate risk neutral speculators who adjust their prices. Prices for intermediate speculators who cannot set their prices are the same as the previous period prices denoted by P_{t-1}.

In order to simplify the bundler's pricing rule, we use the log-linearized version of this equation so that the final price becomes

$$\tilde{p}_t = \gamma \tilde{p}_{t-1} + (1-\gamma) \tilde{p}_t^*(k), \tag{3.2}$$

where $\tilde{p}_t$ and $\tilde{p}_t^*$ denote the logarithm of P_t and P_t^*, respectively.

After completing the description of our model, we elaborate on some important implications of equation (3.1). As implied by this equation, the intermediate risk neutral speculators' price follows a non-linear first-order Markov process with a kink at the price above which we do not have inventories. In the case of *iid* shocks, the kink is determined by

$$\hat{p} = (1-\gamma)\frac{1-\delta}{1+r}\mathbb{E}p(z) + \gamma \hat{p}.$$

This implies that

$$\hat{p} = \frac{1-\delta}{1+r}\mathbb{E}p(z) \tag{3.3}$$

which coincides with the kink given in DL.

However, as in Chambers and Bailey (1996), the price kink $\hat{p}$ in the case of correlated harvests shocks is no longer constant and varies with the current harvest. This is due to the fact that with serially correlated harvest shocks, speculators form their price forecasts using all the information contained in the current shock.

Under some regularity conditions, most notably $r + \delta > 0$ and that z has a compact support, DL establish the existence of a solution to (3.1) when $\gamma = 0$ and shocks are independent. Indeed, to show the existence of the demand function for non-independent shocks, it is enough to prove the independent case conditioning on time t. In our case, we proceed by following a similar approach to proving that such an equilibrium exists. Assume that the demand x_t always lies in a subset $\mathbb{X} = [\underline{z}, +\infty)$ of the real numbers and that the harvest shock z_t belongs to a compact set $\mathbb{Z} = [\underline{z}, \bar{z}]$.

Definition Assume that $\gamma \in [0, 1)$. A staggered stationary rational expectation equilibrium (SSREE) is a price function $f : \mathbb{X} \times \mathbb{Z} \to \mathbb{R}$ which satisfies the following equation

$$p_t = f(x_t, z_t) = \max\left\{p(x_t), (1-\gamma)\frac{1-\delta}{1+r}\mathbb{E}_t f(z_{t+1} + (1-\delta)I_t, z_{t+1}) + \gamma f(x_t, z_t)\right\}$$

where

$$I_t = x_t - p^{-1}(p_t) = x_t - p^{-1}(f(x_t, z_t)). \tag{3.4}$$

This defines the price function

$$P_t^*(k) = f(x_t, z_t),$$

where $f(x_t, z_t)$ is the unique, monotone decreasing in its first argument, solution to the functional equation. Since this price function is non-linear, numerical techniques similar to the ones adopted by DL and Michaelides and Ng (2000) are used to solve for $f(x_t, z_t)$

$$f(x_t, z_t) = \max\left\{p(x_t), (1-\gamma)\frac{1-\delta}{1+r}\mathbb{E}_t f((z_{t+1} + (1-\delta)I_t), z_{t+1}) + \gamma f(x_t, z_t)\right\}.$$

In the case of independent shocks, we can remove the time subscript and the shocks in f.

When $\gamma = 0$ and the shocks are *iid*, we have the same model as the one considered by DL. Hence, the equilibrium is simply called SREE. In the following theorem we show that the staggered stationary rational expectation equilibrium (SSREE) coincides with the stationary rational expectation equilibrium (SREE) derived from the basic DL speculative storage model.

Theorem 3.1.1 *If shocks are iid, then SSREE=SREE.*

Proof See Appendix A. ■

Remark Theorem 3.1.1 shows that $p_t = P_t^*$. This allows us to use all of the results for the process p_t, that are available in the literature, for the process P_t^*.

We next show that the final demand for the bundler in our staggered speculative model is different form the one in DL. It proves useful to compare the price processes in the speculative storage model with and without staggered prices for the market participants who can reset their prices. In the basic speculative storage model of DL, the market participants cannot hold negative inventories. If prices are expected to increase or decrease by less than the cost of carrying the commodity from one period to another, inventories are zero. If inventories are positive, the expected price next period is equal to the current price plus the storage costs. The final price of the commodity in the basic speculative storage model satisfies

$$p_t = \max\left\{p(x_t), \frac{1-\delta}{1+r}\mathbb{E}_t p_{t+1}\right\}.$$

Hence,

$$\begin{cases} p_t = \frac{1-\delta}{1+r}\mathbb{E}_t p_{t+1} & \text{if } I_t > 0; \\ p_t = p(x_t) & \text{if } I_t = 0. \end{cases}$$

However, as stated in the description of our speculative storage model with staggered prices, the intermediate risk neutral speculators price function satisfies

$$P_t^* = \max\left\{p(x_t), (1-\gamma)\frac{1-\delta}{1+r}\mathbb{E}_t P_{t+1}^* + \gamma P_t^*\right\}.$$

In this case,

$$\begin{cases} P_t^* = (1-\gamma)\frac{1-\delta}{1+r}\mathbb{E}_t P_{t+1}^* + \gamma P_t^*, & \text{if } I_t > 0; \\ P_t^* = p(x_t) & \text{if } I_t = 0. \end{cases} \tag{3.5}$$

It can be easily seen from (3.5) that the prices for intermediate risk neutral speculators who can adjust them satisfy the same equation as the one that speculators face in the basic storage model of DL.

Since the final price process in the speculative storage model with staggered prices is given by

$$\tilde{p}_t = \gamma \tilde{p}_{t-1} + (1-\gamma)\tilde{p}_t^*(k), \tag{3.6}$$

one can infer that the demand of the bundler (the final demand) will be different from the demand presented by DL in the basic speculative storage model. We expect the final demand for speculative storage model with staggered prices to be in between the DL demand and the regular market demand. Moreover, we expect this demand to be more inelastic than the one derived from the basic speculative storage model. This is more consistent with the commodity elasticities estimated from actual data.

3.1.2 Statistical Characterization

Under the assumption of *iid* harvests shocks, the final log-price process satisfies equation (3.6). The bundler price can then be written as

$$P_t = {P_{t-1}}^{\gamma}\ {P_t^{*}}^{1-\gamma}. \tag{3.7}$$

The persistence of commodity prices is then simply an outcome of the staggered prices which is extensively discussed in the literature on staggered pricing. Here, we provide an alternative explanation. From the logarithmic form of the relation (3.7), we have by induction that

$$\tilde{p}_{t+1} = (1-\gamma)\sum_{i=0}^{t} \gamma^i \tilde{p}^{*}_{t+1-i}$$

which in turn yields

$$P_{t+1} = \left(\prod_{i=0}^{t} {P^{*}_{t+1-i}}^{\gamma^i}\right)^{1-\gamma}.$$

This shows that P_{t+1} shares overlapping terms prices in previous periods which gives rise to high persistence.

Next, we show that the final prices of the bundler exhibit conditional heteroskedasticity which is another salient characteristic of the observed commodity prices. Note that from (3.7), we have

$$\mathbb{E}_{t-1}(P_t^2) = {P_{t-1}}^{2\gamma}\mathbb{E}_{t-1}({P_t^{*}}^{2(1-\gamma)}) \tag{3.8}$$

and

$$(\mathbb{E}_{t-1}P_t)^2 = {P_{t-1}}^{2\gamma}(\mathbb{E}_{t-1}({P_t^{*}}^{1-\gamma}))^2. \tag{3.9}$$

Combining (3.8) and (3.9) and assuming that the shocks are *iid*, the conditional variance of the final prices is given by

$$\mathrm{Var}_{t-1}(P_t) = {P_{t-1}}^{2\gamma} \left[\mathbb{E}\left(f(z+(1-\delta)I_{t-1})^{2(1-\gamma)}\right) - (\mathbb{E}(f(z+(1-\delta)I_{t-1})^{1-\gamma})^2\right]. \tag{3.10}$$

In the absence of inventories in the previous period, $I_{t-1} = 0$, the variance reduces to

$$\mathrm{Var}_{t-1}(P_t) = {P_{t-1}}^{2\gamma}\mathrm{Var}\left(f(z)^{1-\gamma}\right). \tag{3.11}$$

From (3.10) and (3.11), we can see that the variance is time-varying and, as a result, the final commodity prices derived from our model exhibit conditional heteroskedasticity. In addition, it is worth noting that the variance also depends on the value of γ.

It is interesting to point out that the form of the conditional variance in (3.11) bears strong resemblance to modeling the conditional heteroskedasticity in interest rate models (see, for instance, Brenner, Harjes, and Kroner (1996)). In these models, there is a parameter that allows the volatility of interest rates to depend on the level of the process. Similarly, higher values of the parameter γ in equation (3.11) indicate that the volatility of commodity prices is more sensitive to their past level which generates volatility clustering.

3.2 Model Comparisons Using Simulated Data

In this section we examine the statistical properties of the simulated data from our commodity price model with staggered pricing. In order to assess the qualitative and quantitative implications of our model, we compare it to the basic speculative storage model of DL and the modified version of the speculative model of Ng and Ruge-Murcia (2000). The model of Ng and Ruge-Murcia (2000) extends the DL model by adding serially correlated harvest shocks that follow an MA(1) process, as well as gestation lags, heteroskedastic supply shocks, multi-period forward contracts and convenience yields.

In our simulations, we calibrate the models using the parameter values estimated by Deaton and Laroque (1996) for a set of 12 commodities. These parameters (a, b, δ), presented in Table 3.1, are the same as the parameters used by Ng and Ruge-Murcia (2000). The data are simulated using *iid* harvest shocks or MA(1) harvest shocks with an MA parameter $\rho = 0.8$. We denote our speculative storage model with staggered prices by ADG.

Table 3.2 presents the results for the first-order autocorrelation of the simulated prices from the different models. The first column of Table 3.2 reports the autocorrelations from the actual data used in Deaton and Laroque (1996), the second column shows the results from the basic DL model ($\rho = 0$) and the third column contains the results obtained using DL model with MA(1) shocks ($\rho = 0.8$). The highest autocorrelation for the simulated prices from the DL model is for Maize (0.413 for the basic DL model and 0.644 for the specification with MA(1) harvest shocks). For all other commodities, the serial correlation in the simulated prices is well below the persistence in the actual prices.

The last two columns of Table 3.2 report the results from our model. For all commodities, the autocorrelation coefficients of the simulated prices based on the ADG model are much higher than those of the DL model specifications and are very close to the autocorrelations obtained from actual data. Once we account for staggered pricing, the additional effect of serially correlated harvest shocks is minimal.

Furthermore, Table 3.3 lends additional support to our ADG model with staggered prices. In this table, we compare the autocorrelation coefficients for the model by Ng and Ruge-Murcia (2000) with gestation lags, overlapping contracts and convenience yields to those computed from our ADG model in columns 4 and 5 of Table 3.2.

Ng and Ruge-Murcia (2000) add gestations lags to the DL basic specification in an attempt to reduce the number of periods where the intertemporal price link between periods with and without production is severed. Consequently, this increases the serial correlation in prices. For this purpose, Ng and Ruge-Murcia (2000) assume that there are odd and even periods and that harvest takes place in the even periods. Hence, the random disturbance term of the harvest process has a variance that could differ if the period is odd (σ_1) or even (σ_2). The highest autocorrelations are reached for a value of $\frac{\sigma_2}{\sigma_1} = 1.8$. This model is denoted by GS. The results from the GS specification are reported in column 2 of Table 3.3.

Ng and Ruge-Murcia (2000) also show, in contrast to the earlier literature on storage where contracts are absent and stockholders are free to roll-over their inventories, that a model with overlapping contracts can partially explain the high serial correlation in prices. Column 3, denoted by OV in Table 3.3 reports the corresponding autocorrelation coefficients.

Finally, Ng and Ruge-Murcia (2000) add a convenience yield to the DL model. Since inventory holders might derive convenience from holding inventories, Ng and Ruge-Murcia (2000) introduce both a speculative and a convenience motive for inventory holding. Hence, since the convenience yield partially compensates inventory holders for the expected loss when the basis is below carrying charges, their model with convenience yield generates a smaller number of stock-outs and, as a result, the demand for inventories for convenience purposes strengthens the intertemporal link resulting in a higher persistence of prices. Results for $c = 50$ are reported in column 4 of Table 3.3. The model is denoted by CY.

Overall, the results in Table 3.3 suggest that the different specifications of Ng and Ruge-Murcia (2000) cannot generate autocorrelation coefficients greater than 0.640 and they are below the autocorrelation coefficients from our ADG model and the actual data across all commodities.

3.3 Empirical Application

This section presents new empirical results from estimating the structural parameters of our proposed model using monthly data for four agricultural commodities..

3.3.1 Data

The data set employed in this empirical application consists of prices for four agricultural commodities: sugar, soybeans, soybean oil, and wheat. The commodity prices are obtained from the Commodity Research Bureau and are available at daily frequency for the period March 1983 – July 2008. The trading characteristics of these commodities are summarized in Table 3.4.

The spot price is approximated by the price of the nearest futures contract. Monthly commodity price series are constructed from daily data by averaging the daily prices in the corresponding month. The monthly frequency is convenient for studying the persistence and conditional heteroskedasticity in commodity prices. The real commodity prices are obtained by deflating the nominal spot prices by the CPI (seasonally adjusted) index obtained from the Bureau of Labor Statistics (BLS). Each deflated price series is then further normalized by dividing by the sample average. By performing this additional normalization, each series has a historical mean of one which allows us to conduct easier comparisons of the estimated parameters across various price series.

3.3.2 Estimation Method: Simulated Method of Moments

This section provides a brief description of the simulated method of moments (SMM) which is used for estimating the model parameters. The main advantage of SMM lies in its flexibility of the choice of moment conditions that allow us to identify the staggered pricing parameter γ. See Pakes and Pollard (1989), Lee and Ingram

(1991) and Duffie and Singleton (1993) for a detailed description of the method and its asymptotic properties, and Michaelides and Ng (2000) for an investigation of its finite-sample properties in the context of the speculative storage model.

The SMM estimator requires repeatedly solving the model for given values of the structural parameters. For this reason, we present some computational details regarding the solution of the model. The function $f(x)$ is approximated using cubic splines and 100 grid of points for x. This function is calculated using an iterative procedure, starting with an initial value $f_0(x) = \max[p(x_t), 0]$. As in DL, the interest rate r is not estimated but it is fixed at 5 percent per annum or 0.41 percent ($r = 1.05^{\frac{1}{12}} - 1 = 0.0041$) per month. In addition, we calibrate the depreciation rate δ and set it equal to 0.04 per month. One reason to calibrate δ is that the SMM estimator tends to over-estimate δ as indicated by Michaelides and Ng (2000). Finally, the harvest shocks z are discretized using a discrete approximation of a standard normal random variable with z taking one of the following 10 values: $(\pm 1.755, \pm 1.045, \pm 0.677, \pm 0.386, \pm 0.126)$, with equal probability of 0.1.

It is worth noting that the prices used for estimation of ADG model parameters represent the prices of intermediate risk neutral speculators, not the final prices that are given by the data set described above. Hence, we first retrieve the prices of intermediate risk neutral speculators from the final prices given by the time series of commodity prices using the equation

$$P_t^* = \left(\frac{P_t}{{P_{t-1}}^{\gamma}} \right)^{\frac{1}{1-\gamma}}. \tag{3.12}$$

Let $\theta = (a, b, \gamma)'$ denote the vector of structural parameters of the model. Sample paths of commodity prices can be simulated from the assumed structural model for a candidate value of θ. In what follows, we simulate one sample path of prices $\tilde{P}_t(\theta)$ of length TH, where $H = 20$ and T is the sample size of the observed prices P_t. The SMM estimator of θ is then obtained by minimizing the weighted distance (using an optimal weighting matrix) between the moments of the observed data P_t (empirical moments) and simulated data $\tilde{P}_t(\theta)$ (theoretical moments). Let $m(P_t)$ and $m(\tilde{P}_t(\theta))$ denote the set of moments from the observed and simulated data. Then, the SMM estimator $\hat{\theta}$ is defined as

$$\hat{\theta} = \text{Argmin}_\theta D_T(\theta) V_T^{-1} D_T(\theta), \tag{3.13}$$

where

$$D_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} m(P_t) - \frac{1}{TH}\sum_{t=1}^{TH} m(\tilde{P}_t(\theta)),$$

and V_T denotes a consistent estimator of

$$V = \lim_{T\to\infty} \text{Var}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} m(P_t)\right).$$

The vector of moments

$$m(P_t) = [P_t, (P_t - \bar{P})^i, (P_t - \bar{P})(P_{t-1} - \bar{P})]', \text{ for } i = 2, 3, 4, \tag{3.14}$$

is chosen to capture the dynamics and the higher-order unconditional moments of actual commodity prices. The long-run variance V is estimated using the Parzen window

$$w(x) = \begin{cases} 1 - 6x^2 + 6|x|^3 & \text{if } |x| \leq 1/2, \\ 2(1 - |x|^3) & \text{if } 1/2 \leq |x| \leq 1 \end{cases} \tag{3.15}$$

with four lags.

Under some regularity conditions, Lee and Ingram (1991) and Duffie and Singleton (1993) show that the SMM estimator is asymptotically normally distributed

$$\sqrt{T}(\hat{\theta} - \theta_0) \to N(0, \Omega_H), \tag{3.16}$$

where $\Omega_H = \left(1 + \frac{1}{H}\right) \left(\mathbb{E}\left[\frac{\partial m(\tilde{P}_t(\theta_0))}{\partial \theta}\right]' V^{-1} \mathbb{E}\left[\frac{\partial m(\tilde{P}_t(\theta_0))}{\partial \theta}\right] \right)^{-1}$. The derivatives $\partial m / \partial \theta$ are computed numerically and Ω_H is replaced by a consistent estimator in constructing the standard errors of the parameter estimates.

3.3.3 Empirical Results

The estimation results for the ADG model parameters are presented in Table 3.5. The standard errors of the estimated parameters, based on the asymptotic approximation described above, are reported in parentheses below the parameter estimates. The standard errors for the staggered price parameter γ are low for all of the four commodities indicating that γ is well identified and significantly different from zero. The mean of γ for the four commodities is equal to 0.85. The parameter estimates for b satisfy the constraint $b < 0$. For most of the cases, the standard errors of the estimated parameters a and b are relatively low.

In this paper, we argue that the high persistence and the conditional heteroskedasticity in commodity prices appear to be primarily driven by the staggered price parameter γ. To illustrate this, we simulate 200 price series, each of length of 300 observations. The set of parameters used to conduct the simulations is $(a, b, \delta) = (.7, -3, .04)$ and $r = .0041$. We compute the first-order autocorrelation for each series and then calculate the average over the Monte Carlo replications. We repeat the same exercise for four different values of γ, $\gamma = (0, 0.3, 0.6, 0.9)$. In the first three columns of Table 3.7 we report the first-order autocorrelation for the actual data, ADG and DL models, respectively. Table 3.6 shows that incorporating staggered prices into the speculative storage model does increase the first-order autocorrelation of the prices and makes it comparable to the sample autocorrelation of the actual data. More specifically, as γ increases from $\gamma = 0$ (which represents the case for the DL model) to $\gamma = 0.9$, the first-order autocorrelation increases from 0.6 to 0.9.

To visualize the differences between the two models, Figure 3.1 plots the actual price of soybean, the simulated prices generated by our ADG model with *iid* harvest shocks and estimated parameters $(a, b, \gamma) = (0.352, -4.787, 0.909)$, and the simulated prices generated by DL model with estimated parameters $(a, b, \delta) = (0.723, -0.394, 0.130)$. It is clear from the graph that our staggered price model generates more persistent data with volatility clustering which is closer to the actual price dynamics of soybean prices presented in Figure 3.1. Also, in Figure 3.2 we trace the dynamic responses of the simulated commodity prices following a negative harvest shock. The gradual adjustment of the commodity prices from the ADG model stands in sharp contrast with the stronger but short-lived impact of the harvest shock on commodity prices in the DL model.

Next, in order to reveal the advantages of our ADG model in matching the dynamics in the first two conditional moments of the data, we simulate 200 series of prices, each of length of 300 observations, using the parameters estimated from ADG model (reported in Table 3.5). We repeat the same exercise, using the same values for the parameters a and b but setting $\gamma = 0$, which represents the case for the DL model. We filter the simulated prices from both the DL and ADG models using an AR(1) model and then fit a GARCH(1,1) model to each of the pre-filtered series using the following equations:

$$\begin{aligned} P_t &= a_0 + a_1 P_{t-1} + \epsilon_t \\ \epsilon_t &= \sigma_t z_t \\ \sigma_t^2 &= \kappa + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2. \end{aligned}$$

Figures 3.3 and 3.4 plot the distribution of the parameter estimates $\hat{\beta}$ and $\hat{\alpha}$ for the ADG and DL models. The figures clearly suggest that the ADG model provides an improvement over DL model by better capturing the conditional heteroskedasticity. In fact, the medians for $\hat{\beta}$ and $\hat{\alpha}$, generated by ADG model, are much closer to the parameters (denoted by bullets) estimated from actual data. Table 3.7 summarizes the results by reporting the means of the autocorrelations and the GARCH parameters for the ADG and DL models against the statistics from the actual data. Overall, the results lend strong support to the staggered pricing feature of the modified speculative storage model of commodity price determination.

3.4 Conclusion

The main objective of this paper is to propose a model which is able to reproduce the statistical characteristics of the actual commodity prices. Our modified speculative storage model embeds a staggered price feature into the DL storage model. The staggered pricing rule is incorporated by introducing intermediate good speculators and a final goods bundler. We examine the empirical relevance of the structural modification by comparing our model performance with several models in the literature, namely DL and the extended DL version of Ng and Ruge-Murcia (2000). Our analysis suggests that the proposed model outperforms the existing models along several dimensions such as matching the serial correlation and GARCH dynamics of the observed commodity prices. We also estimate the vector of structural parameters for the ADG model with uncorrelated harvest shocks using monthly data for four agricultural commodity prices. The results tend to suggest that the staggered price parameter is large and it proves to be instrumental in generating the documented persistence and conditional heteroskedasticity of commodity prices.

While our paper provides convincing evidence for the importance of integrating staggered pricing features in modeling the dynamics of commodity prices, it only serves as an initial step towards better understanding of the source of the gradual adjustment of commodity prices and the role of market power and government intervention in commodity price determination. Explicitly incorporating institutional arrangements, different risk preferences as well as possible existence of financial hedges in some commodity markets might offer a more solid justification for the sluggishness of commodity prices adopted in this paper. Finally, developing a full structural model in which staggered pricing is generated endogenously within the model appears to be a promising direction for future research.

3.5 Appendix: Proof of Theorem 3.1.1

First, we state the assumptions for the theorem.

Assumptions: Assume that

A.1 $r + \delta > 0$.

A.2 The harvest shocks z belong to a compact set $\mathbb{Z} = [\underline{z}, \bar{z}]$;

A.3 The function $p^{-1} : (q_0, q_1) \to \mathbb{R}$ is continuous and strictly decreasing such that

$$\lim_{q \to q_0} p^{-1}(q) = +\infty.$$

Furthermore, we have that $\underline{z} \in p^{-1}(p_0, p_1)$ and $p(\underline{z}) \in \mathbb{R}_+ \setminus \{0\}$.

Following Deaton and Laroque (1992), for any function g on the set $\mathbb{X} = [\underline{z}, +\infty)$ we introduce a function G on $\mathbb{Y} = \{(q, x) | x \in \mathbb{X}\,,\, p(x) \leq q < q_1\}$ which has the form

$$G(q, x) = (1 - \gamma)\frac{1 - \delta}{1 + r}\mathbb{E}g(z + (1 - \delta)(x - p^{-1}(q))) + \gamma q. \tag{3.17}$$

If $\gamma = 0$, then G is the same as in Deaton and Laroque (1992). Let G^{DL} denote the function when $\gamma = 0$:

$$G^{DL}(q, x) = \frac{1 - \delta}{1 + r}\mathbb{E}g(z + (1 - \delta)(x - p^{-1}(q))).$$

It can be seen that $G = (1 - \gamma)G^{DL} + \gamma p$.

Theorem 3.1.1 aims to find a function f such that

$$f(x) = \max\{G(f(x), x), p(x)\}, \ \forall x \in \mathbb{X}, \tag{3.18}$$

where we also have $f = g$. To prove the theorem, we use the following lemma.

Lemma 3.5.1 *For a given g, the unique solution $f : \mathbb{X} \to \mathbb{R}$ to* (3.18) *equals f^{DL}, where f^{DL} is the unique solution to the same problem when $\gamma = 0$.*

Proof For each x, $f(x)$ is the solution to the following equation for q

$$\max\{G(q, x) - q, p(x) - q\} = 0. \tag{3.19}$$

It can be seen that

$$G(q, x) - q = (1 - \gamma)G^{DL}(q, x) + \gamma q - q = (1 - \gamma)(G^{DL}(q, x) - q).$$

Thus, the solution q is a solution to

$$\max\{(1 - \gamma)(G^{DL}(q, x) - q), p(x) - q\} = 0. \tag{3.20}$$

But this is equivalent to solving[3]

$$\max\{G^{DL}(q, x) - q, p(x) - q\} = 0, \tag{3.21}$$

which gives the desired result. ■

This lemma shows that for any g, there is a unique f which is the solution to (3.18). Therefore, we can introduce an operator $\mathbb{T}$ and denote f with $\mathbb{T}g$.

Proof of Theorem 3.1.1 From Lemma A.1 it follows that $\mathbb{T}$ is the same as the operator introduced in Deaton and Laroque (1992). It is shown in Deaton and Laroque (1992) that $\mathbb{T}$ is an operator from the set of non-increasing and continuous functions on $\mathbb{X}$ to itself and has a unique fixed point f, i.e., $f = \mathbb{T}f$. It then follows that this unique fixed point is the unique SSREE or SREE. This completes the proof of Theorem 3.1.1. ■

[3] For a positive number θ and two real numbers a, b, we have that $\max\{a, b\} = 0 \Leftrightarrow \max\{\theta a, b\} = 0$.

3.5.1 Tables

Table 3.1: Parameter estimates from the DL (1996) model.

Commodity	a	b	δ
Cocoa	0.162	-0.221	0.116
Coffee	0.263	-0.158	0.139
Copper	0.545	-0.326	0.069
Cotton	0.642	-0.312	0.169
Jute	0.572	-0.356	0.096
Maize	0.635	-0.636	0.059
Palm oil	0.461	-0.429	0.058
Rice	0.598	-0.336	0.147
Sugar	0.643	-0.626	0.177
Tea	0.479	-0.211	0.123
Tin	0.256	-0.170	0.148
Wheat	0.723	-0.394	0.130

Table 3.2: Comparing autocorrelations for DL and ADG models based on 5000 observations.

Commodity	Actual	DL	DL	ADG	ADG
		$\rho = 0$	$\rho = 0.8$	$\rho = 0$	$\rho = 0.8$
		$\gamma = 0$	$\gamma = 0$	$\gamma = 0.8$	$\gamma = 0.8$
Cocoa	0.834	0.352	0.609	0.7715	0.8446
Coffee	0.804	0.219	0.576	0.7811	0.8501
Copper	0.838	0.335	0.619	0.8918	0.9074
Cotton	0.884	0.173	0.564	0.8626	0.9053
Jute	0.713	0.289	0.589	0.8817	0.9072
Maize	0.756	0.413	0.644	0.9246	0.9180
Palm oil	0.730	0.397	0.637	0.9079	0.9050
Rice	0.829	0.237	0.579	0.8700	0.9078
Sugar	0.621	0.266	0.583	0.8860	0.9184
Tea	0.778	0.213	0.571	0.8332	0.8893
Tin	0.895	0.238	0.567	0.7547	0.8462
Wheat	0.863	0.250	0.602	0.8834	0.9198

Table 3.3: Comparing autocorrelations for Ng and Ruge-Murcia and ADG models based on 5000 observations.

Commodity	Actual	GL	OV	CY	ADG	ADG
					$\rho = 0$	$\rho = 0.8$
					$\gamma = 0.8$	$\gamma = 0.8$
Cocoa	0.834	0.511	0.462	0.522	0.7715	0.8446
Coffee	0.804	0.433	0.385	0.530	0.7811	0.8501
Copper	0.838	0.526	0.394	0.608	0.8918	0.9074
Cotton	0.884	0.365	0.337	0.473	0.8626	0.9053
Jute	0.713	0.486	0.365	0.545	0.8817	0.9072
Maize	0.756	0.620	0.418	0.623	0.9246	0.9180
Palm oil	0.730	0.640	0.438	0.625	0.9079	0.9050
Rice	0.829	0.398	0.334	0.475	0.8700	0.9078
Sugar	0.621	0.427	0.370	0.424	0.8860	0.9184
Tea	0.778	0.428	0.302	0.509	0.8332	0.8893
Tin	0.895	0.428	0.355	0.472	0.7547	0.8462
Wheat	0.863	0.411	0.368	0.505	0.8834	0.9198

Table 3.4: Description of commodity prices data.

Description	Exchange	Contract size	Contract month
Foodstuffs			
SB : Sugar No.11/World raw	NYBOT	112,000 lbs.	H,K,N,V
Grains and Oilseeds			
S : Soybean/No.1 Yellow	CBOT	5,000 bu.	F,H,K,N,Q,U,X
BO : Soybean Oil/Crude	CBOT	60,000 lb.	F,H,K,N,Q,U,V,Z
W : Wheat/No.2 Soft red	CBOT	5,000 bu.	H,K,N,U,Z

Notes: This table provides a brief description about each commodity. The first column presents the symbol description and the second one lists the futures exchange where the commodity is traded. In this table, CBOT refers to Chicago Board of Trade, NYBOT: New York Board of Trade. The third column states the contract size and the last column provides the contract months denoted by: F = January, G = February, H = March, J = April, K = May, M = June, N= July, Q = August, U = September, V = October, X = November and Z = December.

Table 3.5: Parameters estimation for ADG model using SMM, with $\delta = 0.04$ and $r = 0.004$.

Commodity	a	b	γ
W	0.4227	-4.6606	0.9476
	(0.0102)	(0.2929)	(0.0086)
BO	0.7860	-2.1265	0.7621
	(0.0177)	(0.1354)	(0.0237)
S	0.7209	-2.7562	0.8524
	(0.0454)	(0.3256)	(0.0343)
SB	0.2264	-5.6592	0.9474
	(0.0195)	(0.4351)	(0.0099)

Table 3.6: First order autocorrelations for simulated price series.

	$\gamma = 0$	$\gamma = 0.3$	$\gamma = 0.6$	$\gamma = 0.9$
Auto. corr.	0.6122	0.7899	0.9172	0.9838

Table 3.7: First order autocorrelation, β and α parameter for GARCH(1,1) of actual prices.

	Auto. corr.			β			α		
Com.	Actual	ADG	DL	Actual	ADG	DL	Actual	ADG	DL
W	0.9648	0.9899	0.6387	0.6977	0.6719	0.4834	0.2283	0.3006	0.5138
BO	0.9679	0.9550	0.5989	0.7903	0.5160	0.4089	0.1473	0.4709	0.5804
S	0.9697	0.9765	0.6180	0.3413	0.5674	0.4476	0.3410	0.4194	0.5483
SB	0.9620	0.9902	0.6680	0.9018	0.6781	0.4852	0.0798	0.2977	0.5126

3.5.2 Figures

Figure 3.1: Actual data of soybean and the Simulated data

From models with staggered pricing and without staggered pricing.

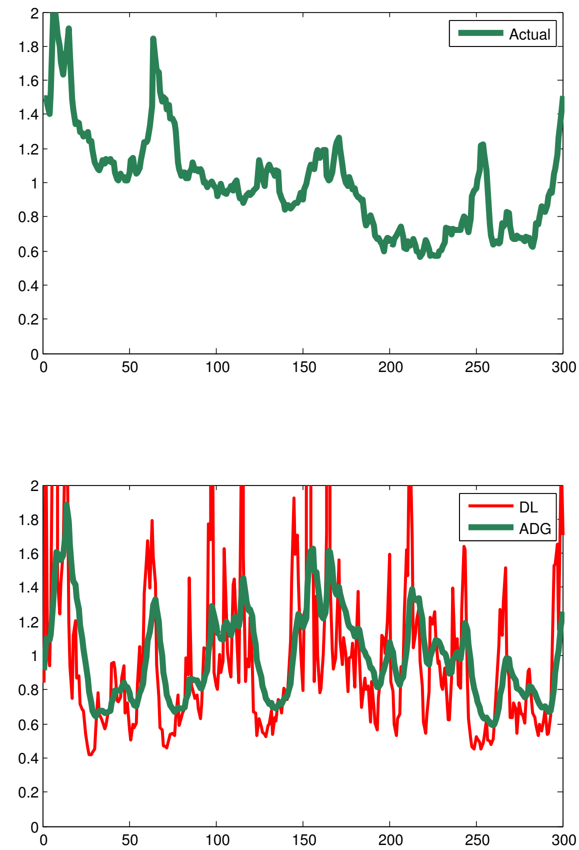


Figure 3.2: Impulse response function based on simulated data.

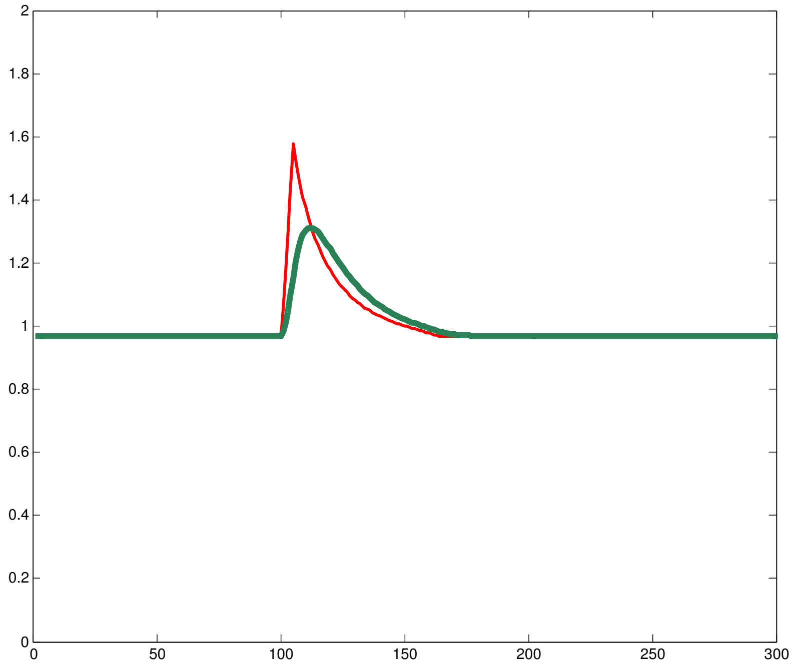


Figure 3.3: Distribution of β.

For simulated data from models with and without staggered pricing. The dash-point indicates the ADG model and the other one is the DL model. Bullet indicates β for the actual data. Simulation is conducted based on a sample of 300 periods, repeated 200 times.

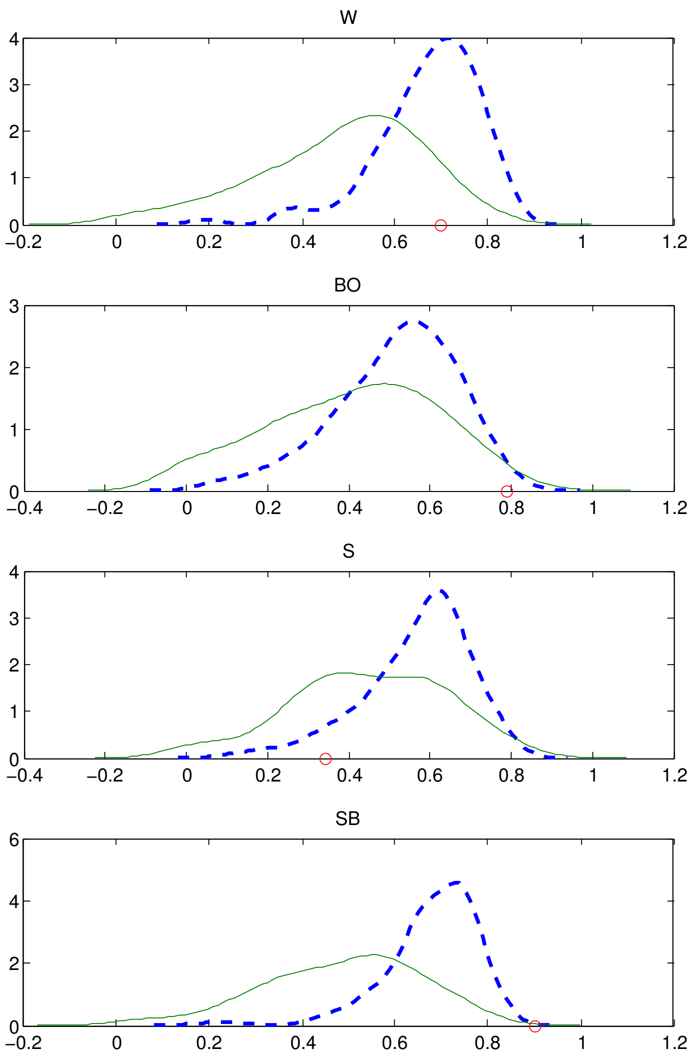
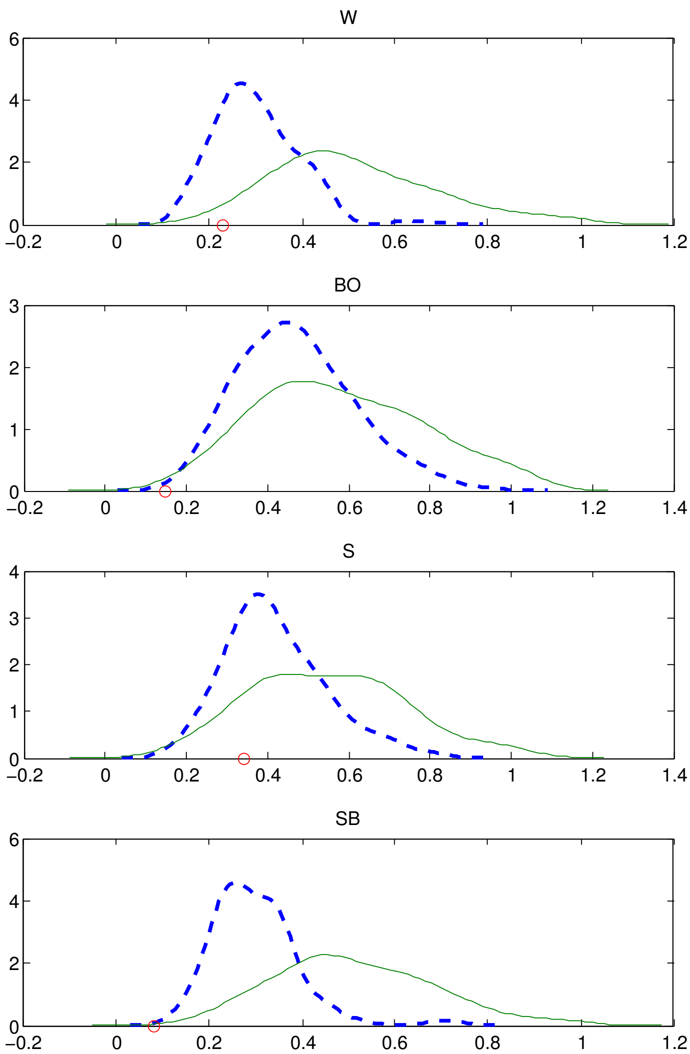


Figure 3.4: Distribution of α

For simulated data from models with and without staggered pricing. The dash-point indicates the ADG model and the other one is the DL model. Bullet indicates α for the actual data. Simulation is conducted based on a sample of 300 periods, repeated 200 times.



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