LAGRANGIAN RELAXATION FOR q-HUB ARC LOCATION PROBLEMS

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ABSTRACT

Lagrangian Relaxation for q-Hub Arc Location Problems

Majid Bazrafshan

The topic of this Master thesis is an in-depth research study on a specific type of network systems known as hub-and-spoke networks. In particular, we study q-Hub Arc Location Problems that consist, at a strategical level, of selecting q hub arcs and at most p hub nodes, and of the routing of commodities through the so called hub level network. We propose strong formulations to two variants of the problem, namely the q-hub arc location problem and the q-hub arc location problem with isolated hub nodes. We present a Lagrangian relaxation that exploits the structure of these problems by decomposing them into $|K| + 2$ independent easy-to-solve subproblems and develop Lagrangian heuristics that yield high quality feasible solutions to both models. We, further, provide some insights on the structure of the optimal solutions to both models and investigate the cost benefit of incomplete hub networks with and without isolated hub nodes. Finally, computational results on a set of benchmark instances with up to 100 nodes are reported to assess the performance of the proposed MIP formulations and of our algorithmic approach.

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Dedicated to my Parents

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Chapter 1

Introduction

Hub Location has become an important research area of location theory over the past two decades. This is due, in large part, to the wide variety of applications of hub and spoke networks in modern transportation systems namely air passenger travel, air freight travel, express shipments, large trucking systems, postal operations, rapid transit systems, and telecommunication networks. Hub networks, rather than serving every origin-destination (O/D) pair with a direct link, provide service via a smaller set of links between O/D nodes and hubs, and between pairs of hubs where economies of scale exists in the cost for such travel or communication. The location of the hubs as well as the selection of paths for sending flows between O/D pairs are the most challenging decisions with the objective of minimizing the total flow cost (Campbell and O'Kelly, 2012).

Several variants of HLPs, arising from different objective functions in the design of hub and spoke networks have been studied in literature such as fixed-cost hub location, p-hub location, p-hub center, and hub covering problem (Alumur and Kara, 2008). There are two basic assumptions underlying most HLPs. The first is that commodities have to be routed via a set of hubs, and thus paths between O/D pairs include at least one hub facility. The second assumption is that hubs are fully interconnected with more effective, higher volume, pathways that enable a discount factor α, $(0 < \alpha < 1)$, to be applied to all transportation costs associated to the commodities routed through a hub arc (Campbell et al., 2005a,b). Literature surveys on research on HLPs are numerous (Zanjirani-Farahani et al., 2013; Campbell and O'Kelly, 2012; Alumur and Kara, 2008; Campbell et al., 2002).

Among all classes of HLPs, the p-hub median problem and its variants have been comprehensively studied and addressed in recent research on HLPs (Marın et al., 2006). The solution of a p-hub median problem is a (connected) network in which $p(p-1)/2$ (undirected) hub arcs connect all hub pairs, and the remaining access arcs connect nonhubs to hubs. The assumption in hub median models that hub arcs form a complete graph on hub nodes simplifies the network design and the routing of flows; however, it imposes a topology and cost structure that may not be desired or realistic in many settings (Campbell et al., 2005a). For instance, Less-than-Truckload (LTL) might have many hubs (break-bulk terminals), each connecting several of its neighboring hubs, but not all hubs in the network. The other assumption in hub location problems is to encourage concentration of flows between all hubs by providing a discounted flow independent cost on hub arcs. However, optimal solutions to hub median and related models tend to result in some hub arcs carrying considerably less flow than some access arcs in the network. This is because the degree of concentration of flows depends on the demand pattern and spatial distribution of the hub nodes (Campbell et al., 2005a). Thus, the basic assumption in hub median models that flow costs are discounted on hub arcs to reflect high volumes may lead to a possible mismatch between the abstracted model and the underlying motivations of the model as reported by Campbell et al. (2005a). That is, the efficient structure of hub-and-spoke networks suggests less than full connectivity at hub-level network.

Campbell et al. (2005a) recommend the relaxation of the restriction that hubs should be fully interconnected but retain the other assumptions in the hub median problem. This results in a more general class of hub location models, namely Hub Arc Location Problems (HALPs), in which hub-arcs may (or may not) require a particular topological structure and may form either a connected or disconnected hub level network. Rather than viewing a hub-and-spoke network design problem from a hub-node location perspective, HALPs view it from a hub-arc location perspective where, instead of deciding the location of fully connected hubs, decide the location of hub arcs, each of which connects two terminal nodes that are hubs by definition. Note that, the hub median model could then be viewed as a special case of the hub arc location models, where hub arcs form a complete graph on the hub nodes (Campbell et al., 2005a). Later, Alumur et al. (2009) and Calık et al. (2009) study the incomplete hub network design where there is no assumption on the topological structure but that the hub level networks form a single connected component. There are also several studies considering a particular topological structure such as single assignment tree-star hub network in which hub level network forms a tree (Contreras et al., 2010; Kim and Tcha, 1992; Martins de Sá et al., 2013a). Some other works study single assignment hub-line networks in which hubs are connected via a single or multiple-disconnected lines (Martins de Sá et al., 2014), and single assignment cycle star network where hubs are connected by means of a cycle (Contreras et al., 2013).

Analogous to the research on incomplete hub-and-spoke networks, some authors evaluate the economical aspect of introducing isolated hubs. Models that allow isolated hub nodes have attracted a growing attention due to their ever increasing importance in today's express delivery systems. Hall (1989) describes how large US overnight delivery express companies established isolated hub facilities at disperse areas of their service (e.g. east and wets coasts) through time. O'Kelly and Lao (1991) devise optimization models for the routing operations in hub and spoke networks that consider isolated mini-hubs where act as isolated hubs. O'Kelly (1998) extends this work by allowing multiple isolated hubs on network and solves the problem to optimality for some small instances. More recently, Campbell (2010) studies the q-Hub Arc Location Problems (q-HALPs) and highlights the necessity of opening isolated hubs or hub arcs when the intensity of demand pattern in a specific region increases, and discusses the importance of isolated hubs to companies with the interest of expanding their geographic service region.

Hub arc location models and q-HALPs, in particular, constitute a challenging class of NP-hard combinatorial optimization problems combining hub-arc location and network design decisions. One of the main difficulties in solving HALPs is the huge number of variables and constraints needed to model them. Several exact and approximate solution methodologies have been developed for classical hub location problems some of which are Benders decomposition (Contreras et al., 2011a), Branch and price (Contreras et al., 2011b), Lagrangian relaxation (Contreras et al., 2009a; Aykin, 1994), and many metaheuristic algorithms. There is, however, only one solution methodology developed for hub arc location problems in literature. Campbell et al. (2005b) introduce several classes of HALPs, provide flow based formulations and develop an Enumeration-Based algorithm to solve some small instances to optimality. And more recently, Campbell (2010) provides a path bathed formulation for a time definite hub and spoke network that considers both isolated and non isolated hubs in the network, and solves some small instances to optimality using a general commercial solver. There is, to the best of our knowledge, no work studying possible exact or approximate algorithms for solving large scale hub arc location problems. This calls for an in-depth investigation of possible solution methodologies for large scale HALPs.

In this thesis, we study two classes of HALPs namely q-Hub Arc Location Problem (q-HALP) and q-Hub Arc Location Problem with Isolated Hubs (q-HALPIH). We provide a path-based formulation for the q-HALPs that yields tight Linear Programming (LP) bounds. We develop a Lagrangian relaxation that exploits the structure of the problem by decomposing it into $|K| + 2$ independent subproblems that can be solved very efficiently. We propose primal heuristics that extract primal information from Lagrangian function to obtain feasible solutions for both variants of the problem. We run a set of computational tests to assess the efficiency of the proposed MIP formulations and of the solution algorithms and, further, compare results and provide insights on the structure solutions to both models and investigate the cost benefit of incomplete hub networks with and without isolated hub nodes.

This thesis is organized as follows. Chapter 2 provides a comprehensive review of current research on the design of hub-and-spoke networks together with solutions methodologies employed. To make this document self-contained, a brief description Lagrangian relaxation and subgradient optimization are also presented in this chapter. In chapter 3, we formally define the studied problems and present Mixed Integer Programming (MIP) formulations as well as the proposed Lagrangian relaxation algorithms. Chapter 4 provides some insights on the structure of optimal solutions to both models and reports the results of computational experiments. Finally, Chapter 5 details some conclusions and future research avenues.

Chapter 2

Literature Review

Transportation systems such as mail, freight, passenger and even telecommunication systems frequently employ a hub and spoke networks. A well-designed network guarantees a strong balance between high service quality and low costs, and results in an economically competitive operation. That is, finding an efficient design of a hub and spoke network is very critical to the success of any competing transportation company. However, real life situations are more complicated, dynamic and, therefore, modeling the question of what is an optimal hub and spoke network structure and finding an optimal solution is challenging. Many researchers and practitioners have tried to address this issue by making several assumptions and simplifications on the behavior of such systems to allow mathematical models to be formulated and solved optimally or near optimally within a practical timeframe. In this chapter, we review the related literature and provide insights on some simplifying assumptions that have been usually considered in well-known HLPs.

2.1 Hub Location Problems

Facility location research has attracted a growing attention over the past few decades due, in large measure, to their vast applications in real life problems. We start with the definition of facility location and conclude this section with a review of recent hub location research.

Facility location problems investigate where to physically, or even virtually, locate a set of facilities (resources) so as to minimize the cost of satisfying some set of demands (customers) subject to some set of constraints. Location decisions are integral to a particular system's ability to satisfy its demands in an efficient manner, and these decisions correspond, most often, to long term strategical decisions that usually have lasting impacts on system performance and will also affect the system's flexibility to meet these demands as they evolve over time.

Several variants of facility location models have been developed in literature that are used in a wide variety of applications. These include, but are not limited to, locating organ transplant centers in a healthcare setup, locating warehouses within a supply chain to minimize the average delivery time to market, locating hazardous material sites to minimize exposure to the public, locating railroad stations to minimize the variability of delivery schedules, locating automatic teller machines to best serve the bank's customers, and locating a coastal search and rescue station to minimize the maximum response time to maritime accidents. These problems fall under the realm of facility location research, yet they all have different objective functions and some specific set of constraints. Facility location models can differ in several decision indices such as their objective function, the distance metric, the number and size of the facilities to locate, etc. (Owen and Daskin, 1998). That is, the specific application, comprehension, inclusion and consideration of respective indices will lead to very different location models.

Hub location research is a branch of location theory that further combines network design decisions and facility location decisions. The blend of location and network design gives rise to special challenges in the formulation and solution of this type of problems. The fundamental hub location models have been extended, in many ways, analogous to the extensions in facility location research (e.g., with capacities, competition, reliabilities, stochasticity, etc.) with features from network design problems (e.g., allocation constraints and restricted network topologies). We next define the terminology of hub location problems and discuss the relevant literature.

Hub Location Problems (HLPs) have been subject of intensive studies over the last 25 years given that these problems frequently arise in modern transportation and telecommunication systems such as air passenger travel, air freight travel, express shipments, large trucking systems, postal operations, rapid transit systems and telecommunication networks, etc. (Campbell et al., 2002). The term hub corresponds to the functionality of some points on the transportation and the telecommunication systems that collect and distribute flows originated from some starting points and must be transferred to some destination points. Rather than serving every demand with a direct link from its origin to its destination, hub and spoke networks route demand via smaller subset of links. The use of fewer links concentrates flows and allows economies of scale to be exploited in transportation costs for that commodity or demand.

The classical HLPs consist of selecting hubs and also the assignment of origin and destination points to the established hubs. HLPs are classified, based on their objective function, into four main groups namely p-hub location, fixed cost hub location, p-hub center, and hub covering problem. For each of these classes of problems, there exists several variants arising from various assumptions, such as hub capacities or single vs multiple assignments (Alumur and Kara, 2008).

There are several assumptions in these models that have become classical assumptions in the hub location theory: (i) there is a constant discount factor $0 < \alpha < 1$ that applies to all transportation costs between hub nodes to concentrate flow between hubs to exploit economies of scale in the cost for transportation, (ii) the hub level network forms a complete graph, meaning all hub nodes are connected via a hub-arc, (iii) finally, the model assumes that there is no direct connection between non-hub nodes.

HLPs can also be classified into two categories according to how the non-hub nodes (demand nodes) are connected to hub nodes, namely single and multiple allocation (see Figure 2.1.). If a non-hub node is restricted to send its flow via a single hub, the allocation type is called single allocation. On the other hand, in multiple allocation

HLPs a non-hub node is allowed to use different hubs for sending or receiving flows.

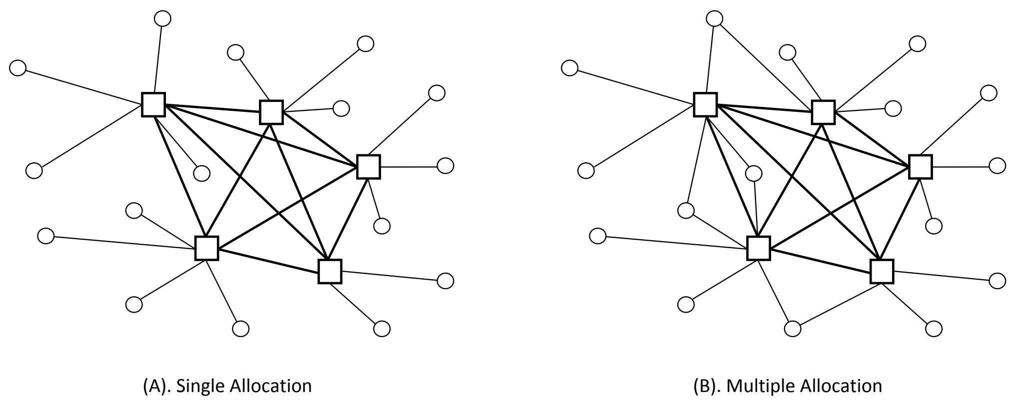

Figure 2.1: Single vs. multiple allocation hub location problems.

In what follows, the hub location literature is analyzed in detail together with specific applications and solution methodologies.

2.1.1 p-Hub Median Problems

This first class of HLPs to be analyzed is the p-Hub Location Problem (p-HLP). The p-HLP is one class of classical HLPs that has received an increasing attention in literature over the past two decades because of its frequent use in telecommunication and modern transportation systems. The objective to p-hub median problems is to minimize the total transportation cost of routing commodities through the network. The hub level network given to the p-hub median problems includes p hubs which are fully interconnected. Different versions of the problem arise from single and multiple allocations and capacity constraints. The p-hub median problem belongs to the class of NP-hard problems. For case of single allocation, even when the set of hubs is given, the subproblem of optimal allocation of non-hub nodes to hubs is still NP-hard (Sohn and Park, 2000).

O'Kelly (1986b) provides the first quadratic integer programming formulation for the p-hub median problems. In a subsequent study, Campbell (1994) proposes the first linear integer programming formulation to p-HALPs. This formulation includes $(n^4 + n^2 + n)$ variables out of which $(n^2 + n)$ are of binary nature and also has $(n^4 + 2n^2 + n + 1)$ linear constraints. Later Skorin-Kapov et al. (1996) describes that, in terms of LP bounds, the formulation provided by Campbell (1994) is poor and proposes a new MIP formulation for the single allocation p-hub median problem. The resulting formulation has $(n^4 + n^2)$ variables in which n^2 are binary and also includes $(2n^3 + n^2 + n + 1)$ linear constraints. Ernst and Krishnamoorthy (1996), in another attempt to solve larger instances, propose a flow based formulation to p-hub median problem and solves some larger instances to optimality. The proposed formulation has (n^3) continuous variables, (n^2) binary variables and also $(2n^2 + n + 1)$ constraints. Note that comparing to the previous path based formulations that yield tight bound in terms of their linear relaxations, this flow based formulation is weaker and includes fewer variables and constraints. Ebery (2001) later proposes another formulation to single allocation p-hub location problem which uses fewer number of constraints and variables to formulate the problem than the previous models. Hamacher et al. (2004) also propose a path based formulation to multiple allocation p-HLP that yields tighter LP bounds compared to formulations proposed earlier in literature. Marın et al. (2006) introduces a new formulation to multiple allocation p-HLP that provides even tighter bounds than that of Hamacher et al. (2004).

Several efficient solution methodologies, including approximate and exact methods, have been developed for single and multiple allocation p-hub median problems. The first approximate algorithm for solving p-hub median problem was developed by O'Kelly (1986b). He proposes an enumerative based heuristic that searches all possibilities of p hub selection and uses nearest hub for assignment of nonhubs to hub nodes. Klincewicz (1991) could be considered as second in proposing heuristics for both single and mutiple allocation p-hub median problems.

There are also several other heuristics such as tabu search and GRASP (Greedy Randomized Adaptive Search Procedure) for single and multiple allocation p-HLPs that outperform the earlier heuristics Klincewicz (1992). Skorin-Kapov and Skorin-Kapov (1994) propose another tabu search that further outperforms the previous heuristics but weaker in terms of computational time. There are several other heuristics developed to obtain good quality solutions to larger instances of p-hub median problems. Ernst and Krishnamoorthy (1996) propose a simulated annealing to p-HLPs that outperforms the tabu search presented in Skorin-Kapov and Skorin-Kapov (1994). This work is one of the earliest successful attempts in obtaining optimal solutions to p-hub median problems. However, they cannot solve instances with more than 50 nodes in the network. In a subsequent study, Ernst and Krishnamoorthy (1998b) develop a branch-and-bound algorithm that incorporates solving a shortest path problem for obtaining better lower bounds and also starts with a set of root nodes. The proposed algorithm tends to outperform the one proposed in Ernst and Krishnamoorthy (1996) in terms of computational time and memory requirement but only for small values of p. Later, Pirkul and Schilling (1998) use Lagrangian relaxation for obtaining better lower bounds to p-hub median problems. They present a subgradient algorithm to obtain better lower bounds and good quality solutions.

When it comes to multiple allocation p-HLPs, the allocation decisions are trivial once the location of hubs are fixed: each pair of nodes sends flows via the shortest path in the given hub network. This idea was first presented and employed in Ernst and Krishnamoorthy (1998a). Later Boland et al. (2004) study the structure of the problem and identify some characteristics to optimal solutions to the problem and perform a preprocessing phase to decrease the size of the formulation. The preprocessing is known to be very effective in solving larger instances since the size of such models grows rapidly as the number of nodes increases. Milanović (2010) develop a new evolutionary based algorithm for uncapacitated multiple allocation p-HLPs. In another study, García et al. (2012) propose new formulations and a branch-and-cut algorithm for this problem. Kratica (2013) develops an electromagnetism-like metaheuristic for the uncapacitated multiple allocation p-HLP. Some other authors also study the allocation strategies in HLPs (Yaman, 2011). Peiró et al. (2014) study r-allocation p-HLPs where the number of hub nodes to be assigned to each nonhub does not exceed r in routing of commodities.

In p-hub location models, the number of hubs is exogenously given by the decision maker (who can solve problems with different values of p which is generally easy to solve as the number of hubs is typically very small). On the other hand, instead of prespecifying the number of hubs one could allow the model to decide the optimal number of hubs by including a term in the objective function that express the fixed set-up costs that are incurred when hubs are established. The objective will then be to minimize the costs of routing flows through the network plus the costs of establishing hubs. This would be the hub-equivalent of the simple facility location problem. This problem is studied under "Fixed Cost Hub Location Problems" title in next section.

2.1.2 Fixed Cost Hub Location Problems

Analogous to p-HLPs, several authors have studied HLPs with fixed costs. This class of HLPs is very simillar to p-hub median problems but differs in relaxation of the cardinality constraint on the number of hubs to be established and an extra term in their objectives. Several variants of this problem arise from capacity constraints and single and multiple assignment at spoke network. The complexity of this problems increases as the number of hubs is not fixed and is also a decision to model.

O'Kelly (1992) introduces the first single allocation uncapacitated hub location problem to literature by considering the set-up cost for opening hubs and formulates the problem as a quadratic integer program. Campbell (1994) later presents the first mixed integer linear programming formulation for both single and multiple allocation hub location problems with capacity constraints without testing them computationally. The capacity constraints in this model are introduced by putting some restriction on maximum flow allowed on links. Abdinnour-Helm and Venkataramanan (1998) propose, based on the idea of multi-commodity flows in network, another quadratic formulation to single allocation uncapacitated hub location problem and develop a Branch-and-bound that uses a combinatorial relaxation for obtaining better lower bounds and a genetic algorithm to get fast upperbounds. In a subsequent study, Abdinnour-Helm (1998) present a hybrid heuristic composed of a genetic algorithm and a tabu search that attempt to identify the number and location of hubs and the allocation decisions, respectively. There are several other studies developing heuristic algorithms for obtaining good quality solutions to both single and multiple allocation problems among which are genetic algorithm by Topcuoglu et al. (2005) that outperforms previous heuristics, hybrid heuristic combining simulated annealing with genetic algorithm (Cunha and Silva, 2007) and with tabu search (Chen, 2007). Each of the presented algorithms outperforms its preceding and the later two hybrids are considered to be the best heuristics proposed for the single allocation hub location problem to date.

The multiple allocation hub location problem (MAHLP) has been comprehensively studied in literature and several solution methodologies have been proposed to solve large scale MAHLP. The first Branch-and-bound algorithm to MAHLP was proposed by Mayer and Wagner (2002) in which they compare their results with that of Klincewicz (1996) and show that their algorithm performs faster and obtains better lower bounds. However, their algorithm was not able to outperform IBM Ilog CPLEX for some instances. In a subsequent study, Cánovas et al. (2007) propose another Branch-and-bound based on a dual-ascend technique. The proposed algorithm again outperforms all previous works and is able to solve some instances with up to 120 nodes in the network. Note that the solution to more realistic size instances still remains challenging.

The other exact algorithm that have been applied to multiple allocation uncapacitated hub location problem (MAULHLP) is Benders decomposition. de Camargo et al. (2008) propose the first Benders decomposition algorithm to MAUHLP and solve some instances with up to 200 nodes in the network. The second algorithm that further takes advantage of several features of model such as the use of multi-cut reformulation, the generation of strong optimality cuts and some reductions tests to a reduced sized strong path based formulation is an enhanced Benders decomposition presented in Contreras et al. (2011a). This proposed algorithm outperforms any other algorithm when comparable and enables optimal solution to some large instances with up to 500 nodes. These instances are, to the best of our knowledge, the largest instances ever solved in literature.

Analogous to the uncapacitated hub location problems (UHLP), the capacitated version of this problem has considered, to some extend, to be more realistic (Ebery et al., 2000; Sasaki and Fukushima, 2003; Boland et al., 2004; Contreras et al., 2011b) and as a result frequently addressed in literature. Several exact and approximate algoirthms have also been developed for CHLPs among which are, Lagrangian relaxation (Contreras et al., 2009a), Branch-and-cut (Labbé et al., 2005), Branch-and-price (Contreras et al., 2011b).

2.2 Hub Arc Location Problems

In what presented above, we observe that the basic assumptions proposed by O'Kelly (1986a) in the design of hub-and-spoke networks are later considered to be underlying assumptions to all HLPs. We next discuss the limitations brought by each of these assumptions and how they can lead to a hub-and-spoke network that under-performs for some settings and, then, review recent research on Hub Arc Location Problems (HALPs).

The first assumption in hub location models that the transportation costs are discounted between all hubs simplifies the network design while it imposes a network structure that may not be desired or realistic. This is because the location of hubs and as a result the hub-level network is highly sensitive to the discount factor $0 < \alpha < 1$. O'Kelly and Bryan (1998) discuss how flow independent discount factors may lead to the design of a sub-optimal hub-and-spoke network and propose a non-linear cost function that allows transportation costs to increase with a decreasing rate as flows increase. The authors approximate this non-linear cost function by a piecewise-linear concave function and employ it in the multiple allocation uncapacitated hub location problem. They show several illustrative examples demonstrating that the optimal solution to most instances changes using their cost function. In a subsequent work, Bryan (1998) extends this work by presenting several variations of the formulation presented in O'Kelly and Bryan (1998) and by considering capacities and minimum flow on hub arcs and also flow dependent cost in all links. In another attempt Horner and O'Kelly (2001) introduce a new cost function, which is also non-linear, that characterizes a flow discount factor that could be applied along any portion of a route that has a sufficient volume. Several other studies employ the idea of flow-based discount factor in the design of hub-and-spoke networks, see for example (Cunha and

Silva, 2007; Racunica and Wynter, 2005; Wagner, 2008; Kimms, 2006).

The other research stream in the current research on HLPs deals with the design of incomplete hub-and-spoke networks namely HALPs. The assumption in classical hub location models that hub arcs form a complete graph on hub nodes also simplifies the network design and routing decisions. However, it imposes a topology and cost structure that might not be desired or realistic in many settings (Nickel et al., 2001; Campbell et al., 2005a). Nickel et al. (2001) introduce new hub location models that could be applied to design public transportation networks. They, to the best of our knowledge, were the first in literature to consider fixed set-up costs for establishing hub-arcs as opposed to hub nodes. Later Podnar et al. (2002) present a model that instead of locating hubs, locates hub arcs each of which connects two hubs. In this model hub arcs carrying a flow that is larger than a threshold value will benefit from a discount factor α. Recall that the assumption that all hub arcs have a unit discount was applied in order to consolidate flow over hub-level network. Campbell et al. (2005a), however, provide several illustrative examples showing that, at optimal solution to HLPs, some hub-arcs carry considerably less flow than some other access arcs and still benefit from the discount factor α. This means the assumption that hub-level network forms a complete graph leads to a possible mismatch between abstracted model and its outcome. Campbell et al. (2005a) suggest relaxation of second assumption that hub-arcs form a complete graph on hub nodes but retain the other assumptions in classical HLPs. They propose new models called q-Hub Arc Location Problems (q-HALP) where instead of locating hub facilities, locate hub arcs, which have a reduced unit flow cost. They examine four classes of q-HALPs which differ from each other in the way the origin-destination paths are formed such as the length of the paths, the number of hub arcs allowed and some topological structures in some models. These models also employ a new type of arcs, namely bridge arcs, that connect two hubs but without any discount factor in the cost for transportation. They provide several flow based formulations, and in a companion paper, use an enumeration based algorithm to solve some instances with up to 25 nodes (Campbell et al., 2005b).

Several studies address incomplete hub networks and hub networks with a particular topological structure. Alumur et al. (2009) relax fully interconnectivity assumption but assume that the hub level network forms a single connected component. The study several variants of the problem arose from different objective functions and solve some instances to optimality with a commercial optimization solver. There are also some other works studying the hub network design problem with a particular topological structure (see Figure 2.2.). Lee et al. (1993b) and Contreras et al. (2009b) study tree-star topology where hub-level network forms a tree. These studies consider the single allocation of nonhubs to hub nodes. Lee et al. (1993a) develop a heuristic for obtaining feasible solutions for some medium size instances. Contreras et al. (2009b) propose a Lagrangian heuristic for obtaining upper and lower bounds on the optimal solution of this problem and report computational experiments on instances with up to 100 nodes. In another recent work, Martins de Sá et al. (2013b) develop a refined benders decomposition to solve the tree of hubs location problem. Several other topologies have been suggested in this area. Contreras et al. (2013) study ring-star (cycle-star) topologies. In this model, hubs are connected via a ring and non-hub nodes are connected to a single hub. They further develop a branch and cut method coupled with a mixed-dicut inequalities that improve the LP bounds of their formulation. The cycle hub location models are known to be very challenging and therefore they further propose an Greedy Randomized Adaptive Search Procedure to obtain high quality feasible solutions to some instances with up to 100 nodes. The single assignment hub-line networks have also been studied in recent hub location research.

Martins de Sá et al. (2014) study the hub line location model in which hubs are connected via a single or multiple lines. They propose a refined Benders decomposition algorithm and further enhance the efficiency of their algorithm by a local branching method that enables their methodology to efficiently solve some instances with up to 100 nodes. The other topological structure suggested in hub location research is star-star networks where all hubs are connected to a central hub and nonhub nodes again are connected to only one hub namely single allocation (Labbé and Yaman, 2008; Yaman, 2008). Labbé and Yaman (2008) provide some analyses on their MIP formulation and propose an Lagrangian heuristic to obtain upper and lower bounds on some benchmark instances with up to 100 nodes. Yaman (2008) also studies the star p-hub median problem with modular arc capacities and develops a Lagrangian heuristic to obtain promising feasible solutions.

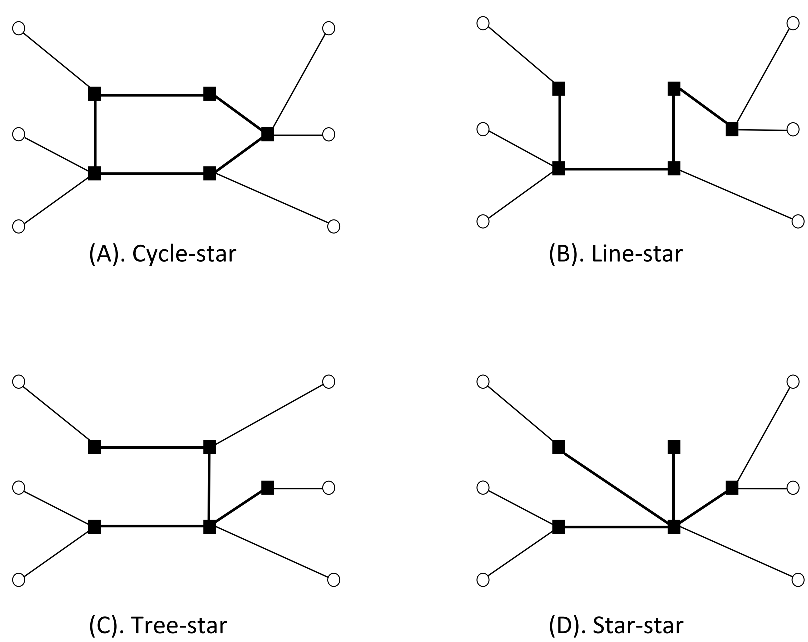

Figure 2.2: Incomplete hub-and-spoke networks with particular topological structures

2.3 Solution Methodologies

There is often a large collection of integer linear programming optimization problems that commonly arise in different real-life situations such as production, finance, enterprise networks, telecommunications, logistics and transportation. From a theoretical point of view, one would say, it is possible to solve many of these problems to optimality by enumerating all feasible solutions and evaluating each one of them with respect to the objective function. It is not, however, an applicable strategy given the huge number of solution alternatives that might exist, even for medium-size instances. Besides, there are usually time restrictions to obtain solutions in order to make opportune decisions. After the introduction of mathematical linear programming by Dantzig in 1947, a great effort has been made to develop methods that do not require to explicitly enumerate all solutions. After several decades, there are still a significant number of applications to which the employment of these methods yet fails to solve real-size problems or ends up being intractable within a limited time frame. This has motivated the formation and development of a research stream, namely Approximate methods.

The problem that managers need to decide within a very limited time frame has strained decision makers to design algorithms, namely heuristics, that provide most often not an optimal but a good quality solution. In this case, good solutions sought-after but with no guarantee on the solution quality. These algorithms are classified as heuristics, and as approximate solution techniques when they have a solution quality guarantee, i.e., lower bounds for minimization problems. Analogous to the development of approximate methods several exact methodologies have also been proposed and developed by scientists that are also competing with approximate algorithms. Exact algorithms are exact in the sense that the solution found is proven to be optimal. That is, there does not exist any other solution with a better objective function value. Exact algorithms for Mixed Integer Programing (MIP) problems are often based on enumeration schemes and may be impractical in practice due to computational burden and long running times.

The effectiveness of both methods, and in particular exact methods, depends on the way they are employed to specific problems and whether or not that algorithm can exploit some structure of the problem. In this chapter we will provide the reader with a brief review of Lagrangian relaxation and subgradient optimization. Our aim is not an in-depth study of this solution methodology but rather a short discussion of this technique, necessary to make this thesis self-contained.

2.3.1 Lagrangian Relaxation

Lagrangian relaxation (LR) was one of the first methods proposed for solving linear programs (Kuhn and Tucker, 1958). This idea was first applied to the traveling salesman problem by Held and Karp (1971) and, later, Geoffrion (1974) extended it to Integer programs. The reader is referred to Geoffrion (1974), Guignard (2003) and Fisher (2004) for a comprehensive review of theory and applications of Lagrangian relaxation.

Consider the following integer program:

$$\begin{aligned} \text{(IP)} \quad Z^* = \text{minimize} \quad & cx \\ \text{subject to} \quad & Ax \geq b \\ & x \in X = \{x \in \mathbb{Z}^n_+ | Dx \geq d\}. \end{aligned}$$

Where (A,b) and (D,d) are $m \times (n+1)$ and $m' \times (n+1)$ matrices, respectively, and variable x is an n-vector of non-negative integers. Without loss of generality we assume X to be the set of integral points satisfying the constraint $Dx \geq d$. The problem (IP) is called primal problem, and any solution to so called primal problem is a primal solution. Now assume that the constraints $Ax \geq b$ are complicating constraints. That is, the problem (IP) is interpreted as an easy to solve problem without these constraints. The Lagrangian relaxation problem is derived by removing the complicating constraint set from the set of constraints with an associated Lagrangian multiplier and adding them to the objective function of the original problem by penalizing the violation of each of the complicating constraints. The lagrangian dual problem is then the problem of finding the best lagrangian multipliers enabling the highest value for the Lagrangian function. Note that, the Lagrangian dual problem of the problem (IP) is an alternative approach to approximate the optimal integer solution to the primal problem. Relaxing the first set of constraints by introducing the Lagrangian multiplier $u \geq 0$, which is an m-vector, the Lagrangian function to this problem can be described as following:

$$\ell(x, u) = cx + u(b - Ax)$$

The subproblem obtained is called LR problem, and the problem of finding the best Lagrangian multipliers in order to obtain the sharpest lower bound is called Lagrangian dual problem. Given this, the LR problem can be stated as:

$$\text{(SP(u))} \qquad \begin{aligned} \phi(u) = \text{minimize} \quad & \ell(x, u) \\ \text{subject to} \quad & x \in X \end{aligned}$$

Note that, it can be shown that for any $u \in \mathbb{R}^m_+$, the Lagrangian function value will provide a lower bound on the optimal solution to (IP). Note also that for any

$u \in \mathbb{R}^m_+$, $x \in X \cup \{x|Ax \geq b\}$, the Lagrangian function value is a lower bound for the primal problem and we have:

$$\phi(u) \leq \ell(x, u) \leq cx$$

And at any optimal solution x^* to the original problem (IP) it holds that

$$\phi(u) \leq \ell(x^*, u) \leq cx^* = z^*$$

Now the problem is to find the best (highest) lower bound. In order to find the sharpest lower bound, we need to solve the problem of finding the best Lagrangian multipliers by which the LR problem obtains its highest value. We now present the Lagrangian dual problem to (IP) as following:

$$\text{(LD)} \quad \phi^* = \max_{u \geq 0} \ \phi(u)$$

Note that the problem (SP(u)) is a problem in the space of x, whereas (LD) is in the dual space of the Lagrangian multipliers u.

One question that arises here is how one can solve the Lagrangian dual problem. The objective of the Lagrangian dual is a non-differentiable concave function where standard ascend methods relying on gradients cannot be applied. There are however several methods for solving nonsmooth optimization problems, among which are the subgradient optimization algorithm (Camerini et al., 1975), the volume algorithm (Barahona and Anbil, 2000) and the Bundle method (Helmberg and Kiwiel, 2002). The subgradient method is an iterative procedure that is designed to solve the problem of maximizing a non-differentiable concave function. This procedure initially presented in Held and Karp (1970) and has been successfully applied to many minimization (maximization) combinatorial optimization problems using the following procedure:

- Choose and initial point $u^0 \in \Omega$.
- Construct a sequence of points $\{u^k\} \subseteq \Omega$ that ultimately converge to an optimal solution employing the rule

$$u^{k+1} = P_{\Omega}(u^k + \Theta^k s^k)$$

  where $P_{\Omega}(\cdot) \subseteq \Omega$ is a projection on the set Ω, $\Theta_k > 0$ is a proper step size and s^k is step direction (usually the subgradient vector).

- Continue until some stopping criteria is satisfied.

One of the difficulties in subgradient algorithm is the calculation of step direction and step size.

The step direction and step size have to be determined at each iteration and play a key role in order for algorithm to be able to converge to an optimal solution. Subgradient optimization methods can be categorized mainly into pure subgradient, deflected subgradient and conditional subgradient methods, depending on particular strategies applied in finding the step direction. The pure subgradient algorithm uses the subgradient of the Lagrangian function in the space of Lagrangian multipliers in Lagrangian function–or simply the coefficients of the Lagrangian multipliers–to calculate direction of motion.

Chapter 3

Problem Statement and Mathematical Formulation

In this chapter we formally define q-Hub Arc Location Problems. We next provide Mixed Integer Linear Programming (MILP) formulations and propose a LR scheme for obtaining lower and upper bounds on the optimal solution to both versions of the problem.

3.1 Problem Definition

Let $G = (V, A)$ be a complete digraph, where V is the set of nodes, and A is the set of edges. Let $N \subseteq V$ be the set of potential hubs, and K represent the set of commodities which origin and destination points correspond to V. We define a hub edge set E, where E is the set of subsets of N containing one or two hubs. We denote E^2 as the set of arcs $e = \{e_1, e_2\}$, $|e| = 2$ and E^1 as the set of loops (hubs) $e = \{e_1\}$ i.e. $|e| = 1$ where $E = E^1 \cup E^2$. For each commodity $k \in K$, W_k is the amount of flow to be routed from origin $o(k) \in V$ to destination $d(k) \in V$. The transportation costs, or distances $d_{ij} \geq 0$, between nodes i and j, are assumed to be symmetric and to satisfy the triangle inequality.

3.1.1 q-Hub Arc Location Problem

The q-HALP consists of locating q hub arcs and at most p hubs and of determining the routing of commodities, with the objective of minimizing the total transportation cost. Note that unlike the HALPs presented in Campbell et al. (2005a), we limit the number of hubs in the hub-and-spoke network of q-HALP to at most p. This model also restricts the establishment of hubs by not allowing the opening of isolated hub nodes. Similar to the majority of hub location models, q-hub arc location problems assume that every path between an origin and a destination will contain at least one and at most two hubs meaning there is no direct connection between origin-destination pairs. That is, a path between two nodes are of the form $(o(k), i, j, d(k))$, where $i, j \in N$ correspond to the potential hub nodes to which $o(k)$ and $d(k)$ are assigned. The transportation cost is, therefore, obtained by

$$F_{ek} = W_k(\beta d_{o(k)i} + \alpha d_{ij} + \gamma d_{jd(k)})$$

where i and j are the end points of hub arc or loop $e \in E$, and β, α and γ are collection, transfer and distribution discount factors through the path. To reflect economies of scale in cost for transportation on hub arcs, we assume $\alpha \leq \beta$ and $\alpha \leq \gamma$ where $0 \leq \alpha \leq 1$ is the discount factor associated to the hub-arc transportation.

Recall that in q-HALPs no longer require hubs to be connected only via hub arcs. This creates the possibility of employing a new type of arcs namely bridge arcs that connect two hubs but without a discounted unit flow cost. Bridge arcs may coincide with hub arcs and can occur as the first (or the last) arc in any path when origin (or destination) node is a hub. They may also appear when the origin and destination are hub nodes and they are not connected via a hub arc. Note also that, because of triangle inequality, no origin destination path will include a bridge arc adjacent to an access arc.

With the above assumptions, we introduce our first set of binary routing variables x_{ek} to be 1 if and only if commodity $k \in K$ is routed via hubs $\{i, j\} \in e$ and the second set of binary binary location variables z_i to be 1 if only if hub node $i \in N$ is established. And finally we define our last set of binary variables y_e to be 1 if and only if hub arc $e \in E^2$ is open. Given this, the q-HALP can be formulated as follows:

$$
\begin{aligned}
\text{(M1)} \qquad & \text{minimize} \sum_{k \in K} \sum_{e \in E} F_{ek} x_{ek} \\
& \text{subject to} \sum_{e \in E} x_{ek} = 1 && \forall k \in K && (3.1) \\
& \sum_{e \in E^2} y_e = q && && (3.2) \\
& \sum_{i \in N} z_i \leq p && && (3.3) \\
& x_{ek} \leq y_e && \forall k \in K, e \in E^2 && (3.4) \\
& x_{ek} \leq z_{e_1} && \forall k \in K, e \in E^1 && (3.5) \\
& y_e \leq z_{e_1} && \forall e \in E^2 && (3.6) \\
& y_e \leq z_{e_2} && \forall e \in E^2 && (3.7) \\
& z_i \leq \sum_{\{e \in E^2, \{i\} \in e\}} y_e && \forall i \in N && (3.8) \\
& x_{ek} \geq 0 && \forall e \in E, k \in K && (3.9) \\
& z_i \in \{0, 1\} && \forall i \in N && (3.10) \\
& y_e \in \{0, 1\} && \forall e \in E^2. && (3.11)
\end{aligned}
$$

The objective is to minimize the total cost of routing flows between origins and destinations via hub nodes and hub arcs. Constraints (3.1) guarantee that there is a unique path connecting the origin and destination of every commodity. Constraints (3.2) ensure that there are exactly q hub-arcs open in the hub and spoke network. The cardinality constraint (3.3) ensures that at most p hub nodes are established. Constraints (3.4) and (3.5) guarantee that no commodity is routed via non hub-arcs and non hub-nodes, respectively. Constraints (3.6) and (3.7) ensure that no hub arc is established unless its end points are hub nodes. Constraints (3.8) guarantee that no hub node is established unless there is at least one established hub arc associated with it. Finally, constraints (3.9), (3.10) and (3.11) are the standard integrability and non-negativity constraints. Observe that even though x_{ek} variables have a binary interpretation, we can define them as nonnegative continuous variables as the formulation will enforce them to take binary values. That is, when the set of open hubs and hubs arcs are known, the problem is decomposable into $|K|$ subproblems and for each commodity k, the optimal route will be the one with minimum transportation cost and because the problem is uncapacitated, only one edge will be selected for the routing of that commodity.

3.1.2 q-Hub Arc Location Problem with Isolated Hubs

The q-HALPIH is composed of locating q hub arcs and at most p hubs and of the routing of commodities, with the objective of minimizing the total transportation cost. Similar to q-HALP and unlike the HALPs presented in Campbell et al. (2005a), we restrict the number of hub nodes allowed in the hub and spoke network to be at most p. The definitions and assumptions described in q-HALP presented in section 3.1.1 are valid in q-HALPIH except that we allow isolated hubs to be established. That is, hubs without adjoining hub arcs could exist in the optimal solution to q-HALPIH.

An interesting aspect of the path based formulation to q-HALPs is the possibility of adapting it for the case in which it is possible to have isolated hubs by simply removing one set of constraints. The economical evaluation of introducing isolated hubs has attracted a growing attention due, in large measure, to the vast applications in carrier and express delivery systems. Hall (1989) discuss how gradually large US overnight delivery express companies established isolated hub facilities at disperse areas of their service. The establishment of isolated hubs is known to be a potent strategy for companies with the interest of expanding their geographic service region and also when the intensity of demand pattern in a specific region increases. We now present our mathematical formulation for the q-HALPIH, where isolated hubs are allowed on the hub-and-spoke network. The q-HALPIH could be derived from model (M1) as follows.

$$\text{(M2)} \qquad \text{minimize} \quad \sum_{k \in K} \sum_{e \in E} F_{ek} x_{ek}$$

$$\text{subject to} \quad (3.1) - (3.7), (3.9) - (3.11).$$

Note that the only difference in modeling of q-HALPIH is that constraints (3.8) are not longer required.

3.2 Properties of Optimal Solutions and Preprocessing

One of the main drawbacks of path based formulations to HALPs is the huge number of variables needed to model them. There are, however, several properties of optimal solutions to hub location models that can be used to reduce the size of the formulations. In this section, we summarize some of the relevant results and observations on optimal solutions to HLPs that can be directly extended to q-HALPs. As described in Hamacher et al. (2004), we eliminate approximately half of the x_{ek} variables by considering only optimal direction of an undirected hub edge namely *undirected* transportation cost. That is, for every commodity $k \in K$ and every edge $e \in E^2$,

$$F_{ek} = min\{\hat{F}_{\{e_1,e_2\}k}, \hat{F}_{\{e_2,e_1\}k}\}$$

It can also be shown that in any optimal HLP solution, no commodity will be routed using an edge unless the cost of transfer through that edge is cheaper than routing it via the single hub, i.e., end points of that edge (Boland et al., 2004; Marın et al., 2006).

Proposition 3.1. *For every $k \in K$ and $e \in E^2$ such that $F_{ek} \geq min\{F_{\{e_1\}k}, F_{\{e_2\}k}\}$ then $x_{ek} = 0$ in any optimal solution to q-HALPs.*

Now consider the particular case of commodities having the same origin and destination points. Boland et al. (2004) show that such commodities will always be routed via the closest sorting single hub facility at symmetric distances, i.e. $d_{ij} = d_{ji}$.

Proposition 3.2. *For every $k \in K$ such that $o(k) = d(k)$ and every $e \in E^2$, $x_{ek} = 0$ in any optimal solution to q-HALPs.*

Using the above mentioned properties, the q-HALPs can be modeled via a very compact formulation and with fewer number of constraints. We now define a set of potential hub edges for each commodity $k \in K$ as

$$E_k = \begin{cases} \{e \in E^1\} \cup \{e \in E^2 : (F_{ek} < \min\{F_{\{e_1\}k}, F_{\{e_2\}k})\} & o(k) \neq d(k) \\ \\ \{e \in E^1\}, & \text{otherwise} \end{cases}$$

Similarly we partition E_k into two subsets $E_k^1 = \{e \in E_k : |e| = 1\}$ and $E_k^2 = \{e \in E_k : |e| = 2\}$. Given the above redefinition of candidate hub edges, the reduced formulation of the q-HALP can be stated as

$$
\begin{aligned}
\text{(P1)} \quad \text{minimize} \quad & \sum_{k\in K}\sum_{e\in E_k} F_{ek}x_{ek} \\
\text{subject to} \quad & (3.2)-(3.3),(3.6)-(3.8),(3.10) \\
& \sum_{e\in E_k} x_{ek} = 1 && \forall k \in K && (3.12) \\
& x_{ek} \le y_e && \forall k \in K, e \in E_k^2 && (3.13) \\
& x_{ek} \le z_{e_1} && \forall k \in K, e \in E_k^1 && (3.14) \\
& x_{ek} \ge 0 && \forall k \in K, e \in E_k. && (3.15)
\end{aligned}
$$

Constraints (3.12), (3.13), (3.14) and (3.15), have the same interpretation as (3.1), (3.4), (3.5) and (3.9), respectively. The reduced formulation (P2) to q-HALPIH, again, can be derived by removing constraints (3.8) from model (P1).

3.3 Lagrangian Relaxation to q-HALPs

LR is a well-known method commonly applied to combinatorial optimization problems (Guignard, 2003). It exploits the inherent structure of the problem to obtain lower bounds on the value of the optimal solution where the resulting bound is at least as good as the LP relaxation bound. The LR uses the idea of relaxing the complicating constraints by bringing them into the objective function with associated Lagrange multipliers. We refer to the corresponding problem as the Lagrangian relaxation or Lagrangian sub-problem.

If we relax, in a Lagrangian fashion, constraints (3.6), (3.7),(3.8), (3.13) and (3.14) from constraint sets of model (P1) and weight their violations with associated multiplier vectors α, β, γ, μ and ν, we obtain the following Lagrangian function:

$$
\begin{aligned}
\mathrm{L}(\alpha,\beta,\gamma,\mu,\nu) = \text{minimize} \quad & \sum_{k\in K}\sum_{e\in E_k} F_{ek}x_{ek} + \sum_{k\in K}\sum_{e\in E_k^2} \alpha_{ek}(x_{ek}-y_e) \\
& + \sum_{k\in K}\sum_{e\in E_k^1:e=\{i\}} \beta_{ek}(x_{ek}-z_i) \\
& + \sum_{e\in E^2} \gamma_e \left(y_e - z_{e_1}\right) \\
& + \sum_{e\in E^2} \mu_e \left(y_e - z_{e_2}\right), \\
& + \sum_{i\in N} \nu_i \left(z_i - \sum_{e\in E^2:\{i\}\in e} y_e\right) \\
\text{subject to} \quad & (3.2)-(3.3), (3.10)-(3.11), (3.12) \textit{ and } (3.15).
\end{aligned}
$$

The Lagrangian function to formulation (P2), $\mathrm{L}(\alpha,\beta,\gamma,\mu,\nu)$, can also be devised by removing $\sum_{i\in N}\nu_i\left(z_i - \sum_{e\in E^2:\{i\}\in e} y_e\right)$ from the Lagrangian function of the Lagrangian subproblem or simply setting them to zero. Observe that $\mathrm{L}(\alpha,\beta,\gamma,\mu,\nu)$ can be decomposed into three subproblems. These problems will be in the space of x, y and z, respectively. After some algebra, the first subproblem can be devised as

$$
\begin{aligned}
L_x(\alpha,\beta) = \text{minimize} \quad & \sum_{k\in K}\left(\sum_{e\in E_k^2}\left(F_{ek}+\alpha_{ek}\right)x_{ek} + \sum_{e\in E_k^1}\left(F_{ek}+\beta_{ek}\right)x_{ek}\right), \\
\text{subject to} \quad & (3.12), \textit{and } (3.15),
\end{aligned}
$$

The second subproblem in the space of arcs can be expressed as

$$L_y(\alpha, \gamma, \mu, \nu) = \text{minimize} \sum_{e \in E^2} \left(\gamma_e + \mu_e - \nu_{e_1} - \nu_{e_2} - \sum_{k \in K} \alpha_{ek} \right) y_e,$$

$$\text{subject to} \quad (3.2)$$

$$y_e \in \{0, 1\}, \qquad \forall e \in E^2 \quad (3.16)$$

and finally, the third subproblem can be sketched in the space of loops as

$$L_z(\beta, \gamma, \mu, \nu) = \text{minimize} \sum_{i \in N} \left(\nu_i + \sum_{\substack{e \in E^2 \\ e=\{i,e_2\}}} \gamma_e + \sum_{\substack{e \in E^2 \\ e=\{e_1,i\}}} \mu_e - \sum_{k \in K} \sum_{\substack{e \in E_k^1 \\ e=\{i\}}} \beta_{ek} \right) z_i$$

$$\text{subject to} \quad (3.3), (3.10).$$

with the above notations, we drive the following result.

Proposition 3.3. $L(\alpha, \beta, \gamma, \mu, \nu) = L_x(\alpha, \beta) + L_y(\alpha, \gamma, \mu, \nu) + L_z($
$beta, \gamma, \mu, \nu)$.

We define Lagrangian relaxation to q-HALPIH, once again, by removing constraints (3.8) and the associated Lagrangian multipliers from the Lagrangian function $L(\alpha, \beta, \gamma, \mu, \nu)$. That is, we simply set the Lagrangian multipliers associated to constraints (3.8) to zero. The resulting Lagrangian relaxation has the same properties as isolated version and is decomposable into $|k| + 2$ subproblems.

3.3.1 Solution to Subproblem $L_x(\alpha, \beta)$

Note that $L_x(\alpha, \beta)$ can be further decomposed into $|K|$ independent subproblems corresponding to commodity $k \in K$, each of the form

$$L_x(\alpha,\beta)_k = \text{minimize} \quad \sum_{\substack{e\in E_k\\|e|=2}} (F_{ek}+\alpha_{ek})\,x_{ek} + \sum_{\substack{e\in E_k\\|e|=1}} (F_{ek}+\beta_{ek})\,x_{ek},$$

$$\text{subject to} \quad \sum_{e\in E_k} x_{ek} = 1 \qquad (3.17)$$

$$x_{ek} \geq 0. \qquad \forall e \in E_k \quad (3.18)$$

The solution to $L_x(\alpha,\beta)_k$ corresponds to the routing of commodity k through hub edges or hub nodes, e. Each subproblem k is an easy $0-1$ problem which has the integrality property and can be solved very efficiently. That is, for every commodity k the path with the smalled coefficient in the objective function of $L_x(\alpha,\beta)_k$ is optimal.

3.3.2 Solution to Subproblem $L_y(\alpha,\gamma,\mu,\nu)$

Subproblem $L_y(\alpha,\gamma,\mu,\nu)$ corresponds to the establishment of q hub edges and has integrality property which makes this problem an easily solvable subproblem. In other words, the q hub arcs with the smalled coefficients in the objective function of $L_y(\alpha,\gamma,\mu,\nu)$ will be established as hub arcs.

3.3.3 Solution to Subproblem $L_z(\beta,\gamma,\mu)$

The optimal solution of $L_z(\beta,\gamma,\mu)$ define the opening of, at most, p hub nodes $i \in N$ in the hub and spoke network. This subproblem also satisfies the integrality property and as a result is an easy subproblem that can be solved very efficiently. That is, if negative, at most p hub nodes with smalled coefficients will be selected as hubs. Observe that each subproblem captures one inherent structure of the original problem.

3.3.4 The Solution of the Lagrangian Dual

To obtain the sharpest possible lower bound, we need to solve the optimization problem of finding the best Lagrangian multipliers, namely the Lagrangian Dual problem.

$$\text{(LD)} \quad Z_D = \max_{\lambda \geq 0} \; L(\alpha, \beta, \gamma, \mu, \nu) \tag{3.19}$$

The objective of LD is a non-differentiable concave function where standard ascend methods relying on gradients cannot be applied. We use the classical subgradient optimization method to solve LD. Bellow follows an implementation of the subgradient algorithm as depicted in Algorithm 1. The output of the algorithm is a lower bound Z_D and an upper bound $\bar{\eta}$ on the optimal value of primal problem (P1).

Algorithm 1: Subgradient Algorithm

Iteration 0

Initialize $Z_D = -\infty$, $\lambda^0 = 0$, $\xi^t = 2$.

Let $\bar{\eta}$ to be known upper bound on the optimal solution value.

Iteration t

Solve the Lagrangian function $L(\lambda^t)$

if $L(\lambda^t) > Z_D$ **then**

$Z_D \leftarrow L(\lambda^t)$

end if

Evaluate subgradient $\gamma(\lambda^t)$

Calculate Step length $\theta^t = \frac{\xi^k(\bar{\eta} - Z_D)}{||\gamma(\lambda^t)||^2}$

Set $(\lambda^{t+1}) = (\lambda^t + \theta^t \gamma(\lambda^t))^+$

Set $t \leftarrow t + 1$

3.3.5 Upper Bound from Lagrangian Heuristics

An important aspect of subgradient optimization is the richness of some primal information it provides. This information can be used to construct good quality feasible solutions via heuristics embedded in subgradient optimization search procedure, see for example, (Gavish and Pirkul, 1986; Mazzola and Neebe, 1986). Let $S = (E_s^1, E_s^2)$ a feasible hub-level network solution, where E_s^1 represents the set of open hub nodes and E_s^2 represents the set of open hub-arcs. The solution to the routing subproblem of commodities is trivial once the established hub nodes and hub arcs are known. That is, for each commodity $k \in K$, the optimal route is the one with the minimum transportation cost as follows.

$$e^*(k) = \arg\min\{F_{ek} | e \in E_s^1 \cup E_s^2\}.$$

We next present the Lagrangian heuristics developed for obtaining upperbounds on the optimal solution of problems q-HALP and q-HALPIH.

Lagrangian heuristic to q-HALP

Our heuristic to q-HALP consists of a construction phase and a local improvement phase containing two neighborhoods. Let $\hat{x}^t$, $\hat{y}^t$ and $\hat{z}^t$ be the optimal solution to the Lagrangian subproblems at iteration t of the subgradient algorithm. At construction phase, we only need to ensure that there are exactly q arcs open while the number of corresponding end-point hubs does not exceed p. In order to construct promising feasible solutions, we extract some primal information of the Lagrangian dual. We firstly identify hub arcs/nodes carrying considerably higher flow volumes given the current solution to subproblems $L_y(\cdot)$. We define $\psi_e^1 = \sum_{k \in K: e^*(k)=e} W_k$ as the amount of total flow carried by open hub edges (the solution to $L_y(\cdot)$). We next extract some primal information from the first subproblem $L_x(\cdot)$ and calculate $\psi_e^2 = \sum_{k \in K} W_k \hat{x}_{ek}^t$, the total flow carried by arc e based on the solution to routing subproblem.

To construct a good quality feasible solution we use a simple procedure (as described in Algorithm 2) that is composed of the following steps. At every iteration of the procedure, a new edge is added along with its associated end point hub nodes, based on the amount of flow it carries (ψ_e^1). This procedure continuous until we assess all arcs in the solution of $L_y(\cdot)$. If this procedure, because of the cardinality constraint on the number of hubs, fails to construct a feasible solution, meaning q hub arcs are not established yet ($|E_s^2| \neq q$), we, based on the function ψ_e^2, establish, one at a time, more hub arcs while taking into account that the number of hubs should not exceed p ($|E_s^1| \leq p$). The described methodology continuous until exactly q hub arcs are selected and always generates a feasible solution that can be the starting solution for the neighborhood search procedure.

We define a local search for improving the initial solutions. Our first neighborhood, namely node-shift, contains a set of solutions corresponding to the set of nodes that are not in E_s^1. This neighborhood closes an open hub node $i \in E_s^1$ and its joint hub arcs, $B_i^r = \{e | e \in E_s^2, i \in e\}$, to open a non-hub node $j \in N \backslash E_s^1$ and a new set of hub arcs $B_j^a = \{e^{'} | e^{'} \in E^2 \backslash E_s^2, j \in e^{'}, e_1^{'} \in N_{B_i^r} \ or \ e_2^{'} \in N_{B_i^r}\}$ where $|B_j^a| = |B_i^r|$ and $N_{B_i^r} = \{l \in N | l \in e, e \in B_i^r\} \backslash \{i\}$, represents the end points of removing arcs in B_i^r excluding i.

By this definition, the node-shift neighborhood can be stated as:

$$N_{ns}(S) = \{S' = (E_s^{1'}, E_s^{2'}) | \exists! i \in E_s^1, \exists! j \in N \backslash E_s^1, E_s^{1'} = E_s^1 \cup \{j\} \backslash \{i\}, E_s^{2'} = E_s^2 \cup B_j^a \backslash B_i^r\}.$$

The second neighborhood search corresponds to the set of open hub arcs and the potential neighboring arcs, namely arc-shift method. This neighborhood search scheme closes one open hub arc at a time in search of a better replacing hub arc while retaining the feasibility of the solution. The arc-shift procedure can be described as

$$\begin{aligned} N_{as}(S) = \{S' = (E_s^{1'}, E_s^{2'}) | E_s^{2'} = E_s^2 \cup e^{'} \backslash e, E_s^{1'} = E_s^1 \cup \{e_1^{'}\}, \{e_2^{'}\} \backslash \{e_1\}, \{e_2\}, \\ \exists! e \in E_s^2, \exists! e^{'} \in E \backslash E_s^2, e_1^{'}, e_2^{'} \in E_s^1\}. \end{aligned} \tag{3.20}$$

Observe that $N_{as}(S)$ may contain infeasible solutions. We thus restrict the search procedure to feasible solutions as follows. Let d_i denote the degree of hub node $i \in E_s^1$. Note that three scenarios may arise:

1. If the degree of both end nodes of a closing hub arc greater than one (i.e. $d_{e_1} > 1$ and $d_{e_2} > 1$), then the end points of the replacing hub arc should belong to hub set E_s^1, i.e. $e_1^{'} \in E_s^1$ and $e_2^{'} \in E_s^1$.

2. If the degree of one end node of a closing hub arc is equal to one, the replacing arc must have at least one end point in E_s^1, i.e. $\{e_1^{'}, e_2^{'}\} \cap E_s^1 \neq \phi$.

3. There is, however, no restriction on opening a hub arc when the degree associated with the end-nodes of the closing arc are equal to *one* (i.e. $d_{e_1} = 1 and d_{e_2} = 1$).

The local search procedure of our Lagrangian heuristic iteratively explores $N_{ns}(S)$ and then $N_{as}(S)$ with a best improvement strategy until there are no better solutions in their neighborhoods. Note that the computational demand of heuristic should be kept minimum while making sure the Lagrangian heuristic is searching every good solution to Lagrangian dual. The proposed Lagrangian heuristic to p-HALPs attempts to build feasible solution and a local optima at iteration t of the subgradient algorithm if and only if $t = 2$, $t \geq 500$ and $L(\lambda^t) > Z_D$. The outline of the overall primal heuristic is depicted in Algorithm 2.

Algorithm 2: Lagrangian Heuristic to q-HALP

Step 0: Initialization

Let $\hat{x}^t$, $\hat{y}^t$ and $\hat{z}^t$, $E_s^1 = \phi$, $E_s^2 = \phi$, Ψ^1 and Ψ^2

Step 1: Construction Phase

```
while (Ψ ≠ φ) do
    e* = arg max{ψ_e | e ∈ Ψ¹}
    Ψ¹ = Ψ¹ \ {e*}
    if (e*_1 ≠ e*_2 and |E¹_s ∪ e*_1 ∪ e*_2| ≤ p) do
        E²_s = E²_s ∪ e*
        E¹_s = E¹_s ∪ e*_1 ∪ e*_2
    end-if
end while
while (|E_s| ≠ q) do
    e* = arg max{ψ²_e | e ∈ Ψ²}
    Ψ² = Ψ² \ {e*}
    if (e*_1 ≠ e*_2 and |E¹_s ∪ e*_1 ∪ e*_2| ≤ p) do
        E²_s = E²_s ∪ e*
        E¹_s = E¹_s ∪ e*_1 ∪ e*_2
    end-if
end while
```

Step 2: Local Search Phase

```
stoppingcriteria ← false
while ( stoppingcriteria = false) do
    explore N_ns(S)
    explore N_as(S)
    if(Solution has not been updated in N_ns(S) and N_as(S) ) then
        stoppingcriteria ← true
    end-if
end while
```

where Ψ^1 represents the solution to subproblem $L_y(\cdot)$ and Ψ^2 is the set of edges employed in the solution to subproblem $L_x(\cdot)$

Lagrangian Heuristic to q-HALPIH

Similar to the heuristic developed for q-HALPNH, the heuristic to q-HALPIH consists of a construction phase and a local search phase containing two neighborhoods. We use the same notation in the description of our construction phase and the two neighborhoods. In order to construct high quality feasible solutions, we again extract some primal information of the Lagrangian dual following the same procedure described in previous section. We also revisit the definition of $\psi_e^1 = \sum_{k \in K: e^*(k)=e} W_k$ as the amount of total flow carried by open hub edges and nodes(loops) (the solution to $L_y(\cdot)$ and $L_z(\cdot)$).

The Lagrangian heuristic to q-HALPIH that is composed of the following steps. The construction phase to q-HALPIH is the same as the construction phase developed for q-HALP, but differs in allowing isolated hubs to be established solely. That is, if in the solution to subproblems of the Lagrangian dual, a hub node carries more flow than some other hub arcs then it will be established before hub arcs in the construction phase.

We next define our local search procedure for obtaining local optimal solutions. The first local search, node-shift, developed for q-HALP can be applied to q-HALPIH without any modifications. However, we need to adapt the second neighborhood search procedure to in searching neighboring solutions to the q-HALPIH where can be described as:

$$
\begin{aligned}
N_{as}(S) = \{S' = (E_s^{1'}, E_s^{2'}) | E_s^{2'} = E_s^2 \cup e' \backslash e, E_s^{1'} = E_s^1, \\
\exists! e \in E_s^2, \exists! e' \in E^2 \backslash E_s^2, e_1' \in E_s^1 \text{ and } e_2' \in E_s^1\}
\end{aligned}
\tag{3.21}
$$

Similar to the Lagrangian heuristic to q-HALP, the local search procedure defined for the q-HALPIH firstly explores $N_{as}(S)$ and then $N_{as}(S)$ with a best improvement strategy. The computational demand of heuristic should be kept at its minimum yet allowing the Lagrangian heuristic to search neighborhood solutions to every good solution to Lagrangian dual. The proposed Lagrangian heuristic to q-HALPIH attempts to build feasible solution and a local optima at iteration t of the subgradient algorithm if and only if $t = 2$, $t \geq 500$ and $L(\cdot^t) > Z_D$. The construction phase and the two neighborhood search procedures to q-HALPIH are described in Algorithm 3.

Algorithm 3: Lagrangian Heuristic to q-HALPIH

Step 0: Initialization

Let $\hat{x}^t$, $\hat{y}^t$ and $\hat{z}^t$, $E_s^1 = \phi$, $E_s^2 = \phi$, Ψ^1 and Ψ^2

Step 1: Construction Phase

while $(\Psi^1 \neq \phi)$ **do**
  $e^* = \arg\max\{\psi_e^1 | e \in \Psi^1\}$, $\Psi^1 = \Psi^1 \backslash \{e^*\}$
  if $(e_1^* \neq e_2^*$ *and* $|E_s^1 \cup \{e_1^*\} \cup \{e_2^*\}| \leq p)$ **do**
    $E_s^2 = E_s^2 \cup e^*$, $E_s^1 = E_s^1 \cup \{e_1^*\} \cup \{e_2^*\}$
  end-if
  if $(e_1^* = e_2^*$ and $|E_s^1 \cup \{e_1^*\}| \leq p)$ **do**
    $E_s^1 = E_s^1 \cup \{e_1^*\}$
  end if
end while
while $(|E_s| \neq q)$ **do**
  $e^* = \arg\max\{\psi_e^2 | e \in \Psi^2\}$, $\Psi^2 = \Psi^2 \backslash \{e^*\}$
  if $(e_1^* \neq e_2^*$ *and* $|E_s^1 \cup \{e_1^*\} \cup \{e_2^*\}| \leq p)$ **do**
    $E_s^2 = E_s^2 \cup e^*$, $E_s^1 = E_s^1 \cup \{e_1^*\} \cup \{e_2^*\}$
  end-if
  if $(e_1^* = e_2^*$ *and* $|E_s^1 \cup \{e_1^*\}| \leq p)$ **do**
    $E_s^1 = E_s^1 \cup \{e_1^*\}$
  end if
end while

Step 2: Local Search Phase

stoppingcriteria $\leftarrow$ *false*
while (*stoppingcriteria* = *false*) **do**
  explore $N_{ns}(S)$
  explore $N_{as}(S)$
  if(Solution has not been updated in $N_{ns}(S)$ and $N_{as}(S)$) **then**
    stoppingcriteria $\leftarrow$ *true*
  end-if
end while

Chapter 4

Computational Experiments

In this chapter we present the results of computational experiments to analyze and compare the performance of our proposed formulations and solution algorithms. The algorithms were coded in $C++$ language and run on a Lenovo ThinkStation with an Intel Xeon CPU E31230 processor at 3.20 GHz and 16 GB of RAM under a Windows 7 environment. To conduct the computational experiments, we have used the well-known Australian Postal data set that can be downloaded at *mscmga.ms.ic.ac.uk/jeb/orlib/phubinfo.html*. This data set is commonly used in the hub location literature. It consists of the Euclidean distances d_{ij} between each pair of nodes, and of the values of W_k representing flows between pairs of nodes and also each instance has a strictly positive flow between every pair of nodes. For comparison purposes, we also solved the *q-HALPs* using Concert Technology of IBM ILOG CPLEX 12.5.

4.1 Network Structure of q-HALPs

In this section, we provide some insights on the cost and the solution network structure of q-HALPs to measure the impact of isolated hubs on optimal solutions. The cardinality constraints (3.2) and (3.3) play an important role in the structure of optimal solutions to q-HALPs. Particular configurations of q and p can lead to formation of connected, disconnected, complete or incomplete hub level networks, as depicted in Figure 4.1.

Allowing isolated hub nodes in the network has also a significant impact on the optimal transportation cost. Figure 4.1 displays several important mechanisms in the optimal solution to q-HALP and q-HALPIH on an instance of size $N = 20$ with different configurations of q and p. The cost savings (in percentage) gained by q-HALPIH, for the same q and p configurations, as compared to the optimal value of q-HALP

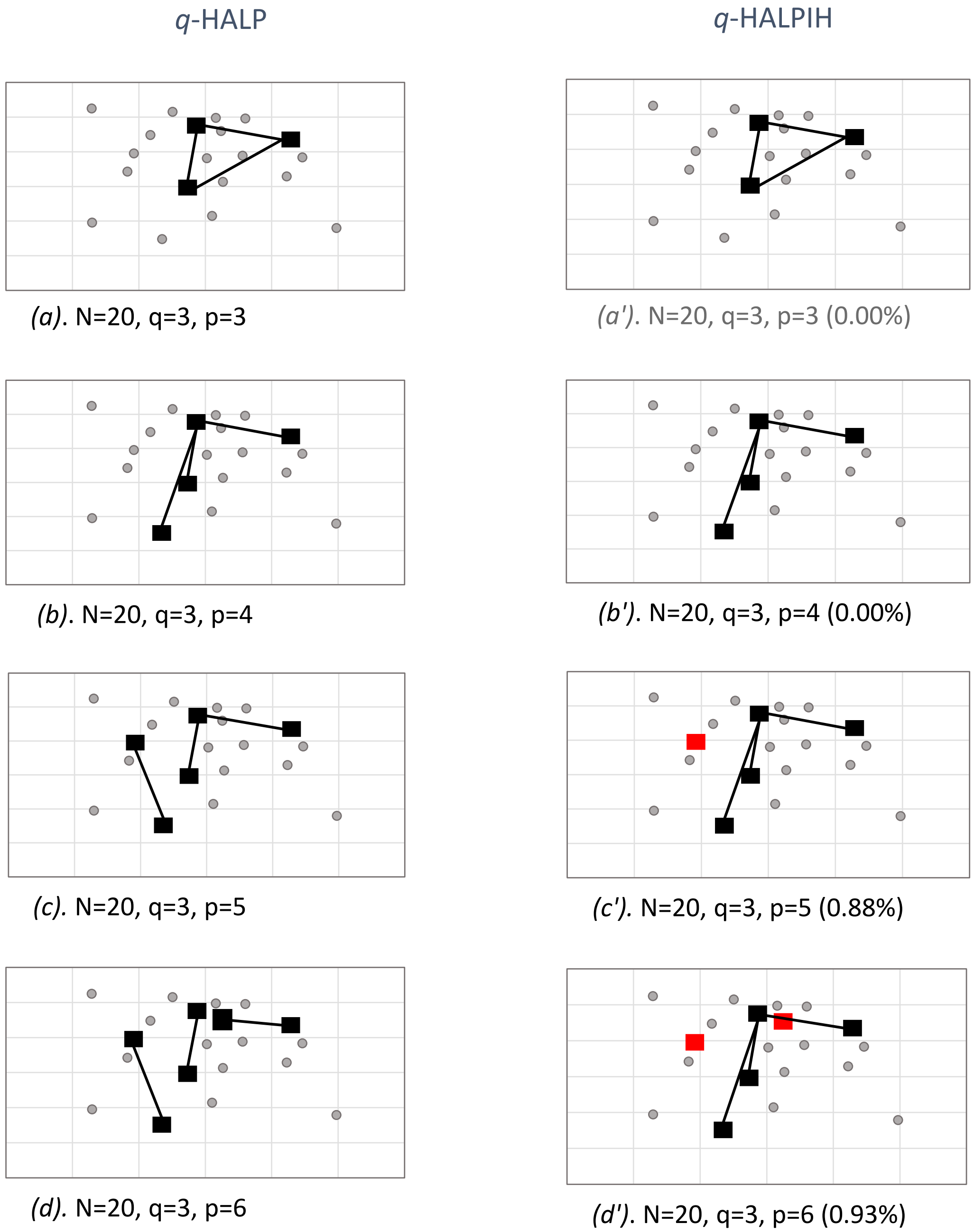

Figure 4.1: Network structure of q-HALP with and without isolated hubs

is also reported. Figure 4.1(a) and 4.1(a′) present the optimal solution to q-HALP and q-HALPIH for $q = 3$ and $p = 3$, respectively. Note that by this configuration, $q = p(p-1)/2$, the resulting network is equivalent to the p-hub median problem for both versions. That is, the optimal network forms a complete graph at hub level network for both versions of the problem. Figure 4.1(b) and 4.1(b′) show the optimal hub level network structure for the same number of hub arcs ($q = 3$) but an extra number of hub nodes, $p = 4$. The resulting optimal hub level network to q-HALP (Figure 4.1(b)) has a connected incomplete structure where its transportation cost decreases by 5.9% as compared to 4.1(a). Observe that the optimal solution to q-HALP and q-HALPIH are equivalent when there are no isolated hubs at optimal solution to p-HALPIH and when the configuration of p and q forms a complete network.

Now consider case (b) and (c) in Figure 4.1. Both networks represent optimal solution to q-HALP for $q = 3$, and $p = 4$ and $p = 5$, respectively. Note that the configuration of p and q in the former case forces the hub level network to be connected. However, the optimal network might (or might not) have connected structure in the latter one. Note that optimal solution to q-HALPIH for $q = 3$, $p = 5$ includes one isolated hub. This is, however, the optimal solution to q-HALP, case (c), includes the same set of hub nodes but a different set of hub arcs. Note that due to the establishment of one isolated hub node in the solution, the transportation cost of isolated model (c′) decreases by 0.88% as compared to (c) where isolated hub nodes are not allowed. By further increasing the number of hub nodes, ($p = 6$), the optimal solution to q-HALPs tend to open a new hub node with different sets of hub arcs as shown in Figure 4.1 (d)-(d′) where their transportation costs decreases by 2.82% and 2.78% compared to transportation cost of similar models with $p = 5$, respectively. The total cost of q-HALPIH also decreases by 0.93% as compared to the transportation cost of q-HALP for $q = 3$ and $p = 6$ (see Figure 4.1(d) and Figure 4.1(d′)).

We observe that when the optimal solution to q-HALPIH contains isolated hubs has a lower transportation cost compared to that of p-HALP with the same configuration of q and p. That is, for each q and p, the transportation cost to the isolated version is always smaller than or equal to that of non-isolated version. This highlights the significance and benefit of allowing isolated hubs in the hub level network. Another observation could be ability of q-HALP to control the hub level network dispersion by particular configuration of number of hub arcs (q) and hub nodes (p) in the network.

4.2 Algorithm Performance

We now present the results of computational experiments performed to assess the behavior of LR and the subgradient algorithm. We set the discount factor as $\alpha = 0.5$ at all instances. Our set of instances contains 50 small to large size instances with up to 100 nodes. This set consists of ten instances of each of the size $|V| = 10, 20, 25, 40, 50, 60, 70, 75, 90$ and 100. The computational experiments focus on evaluating the performance of Lagrangian heuristic over q-$HALP$ and q-HALPIH for the described instances.

We consider several stopping criteria over subgradient algorithm including: (1) the maximum number of iterations(Itr_{max}), (2) the maximum time limit ($Time_{max}$), (3) duality gap is bellow a threshold value $\left(\frac{\bar{\eta}-Z_D}{\bar{\eta}} < \varepsilon\right)$, and (4) the percentage improvement on the lower bound is below a threshold value δ after l consecutive iterations. After some tunning, we set the following parameters to: $Itr_{max} = 8000$, $Time_{max} = 25000(sec)$, $l = 1500$, $\epsilon = 10^{-6}$, $\delta = 0.002\%$. Furthermore, the parameter ξ^k is halved after 25 consecutive iterations without improvement on the lower bound and is reset to 2 every 200 iterations.

Our first computational results analyze the performance of the subgradient algorithm using the mentioned set of instances for q-HALP. We compare the Lagrangian relaxation with the LP relaxation of the formulation to q-HALP obtained by CPLEX. The detailed results of this comparison are presented in Table 4.1. The first three columns give the number of nodes $|V|$, number of hub arcs q, and number of hubs p. The second three columns under the heading *Deviation(%)* depict the LP gap with respect to optimal solution, $LP = 100\left(Opt - LP\right)/Opt$, the Lagrangian relaxation gap with respect to heuristic upper bound, $LR = 100\left(UB - Z_D\right)/UB$, and the percentage deviation of the upper bound (UB) obtained by the Lagrangian heuristic with respect to optimal vale (Opt) obtained by CPLEX. That is, $Heu = 100(UB - Opt)/UB$. The branching time for CPLEX to obtain the optimal solution and the time of the Lagrangian heuristic for obtaining upper and lower bounds, in seconds, are presented in the columns under heading *Time*. In the presentation of the results, the letter *'time'* and *'memory'* refer to the failing of CPLEX or subgradient algorithm to solve an instance due to the time limit or to the lack of memory, respectively.

Instance			Deviation (%)			Time (*sec*)	
$\|V\|$	q	p	LP	LR	Heu	$CPLEX$	SG
10	2	4	0.95	1.79	0.00	0.48	0.39
10	3	5	0.04	1.14	0.00	0.40	0.38
10	4	5	0.22	0.30	0.00	0.40	0.36
10	5	5	0.40	0.54	0.00	0.55	0.41
10	5	6	0.05	1.11	0.00	0.48	0.42
20	3	6	0.41	1.82	0.00	4.98	1.76
20	4	6	0.47	1.06	0.00	3.76	1.73
20	4	7	0.37	1.56	0.35	10.28	1.36
20	5	6	1.02	1.22	0.00	5.71	1.33
20	5	7	0.09	0.34	0.00	2.59	1.42
25	3	6	0.79	3.83	1.90	24.85	6.47
25	4	6	0.67	1.50	0.00	23.48	3.99
25	5	6	0.89	1.49	0.00	22.79	3.38
25	5	7	0.86	1.57	0.00	26.32	3.60
25	6	7	1.27	1.69	0.00	24.93	3.59
40	4	6	0.25	1.48	0.00	602.30	54.94
40	4	7	0.39	1.81	0.00	631.40	59.92
40	4	8	0.09	1.80	0.00	414.81	57.88
40	5	7	0.38	1.41	0.00	746.41	58.73
40	5	8	0.32	1.53	0.00	543.15	39.50
50	4	6	0.43	1.97	0.00	5416.73	193.61
50	4	7	0.57	2.16	0.00	5216.71	166.23
50	4	8	0.10	2.32	0.00	2555.28	145.55
50	5	7	0.61	2.14	0.00	5693.48	171.46
50	5	8	0.19	1.93	0.00	2975.92	147.83
60	4	6	Time	2.61	n.a.	n.a.	514.97
60	5	8	Time	2.69	n.a.	n.a.	385.68
60	6	9	Time	2.61	n.a.	n.a.	382.25
60	6	10	Time	2.79	n.a.	n.a.	370.36
60	8	10	Time	2.40	n.a.	n.a.	303.92
70	4	6	Memory	2.61	n.a.	n.a.	1088.97
70	5	7	Memory	2.41	n.a.	n.a.	888.39
70	6	8	Memory	2.32	n.a.	n.a.	995.76
70	6	9	Memory	2.65	n.a.	n.a.	1050.69
70	7	10	Memory	2.80	n.a.	n.a.	1038.93
75	4	6	Memory	2.81	n.a.	n.a.	1595.35
75	6	8	Memory	3.03	n.a.	n.a.	1296.47
75	6	10	Memory	3.14	n.a.	n.a.	1673.03
75	7	10	Memory	3.08	n.a.	n.a.	1906.62
75	7	12	Memory	3.07	n.a.	n.a.	1646.71
90	4	6	Memory	3.45	n.a.	n.a.	4143.49
90	6	8	Memory	3.92	n.a.	n.a.	4188.62
90	6	10	Memory	3.94	n.a.	n.a.	4470.55
90	7	10	Memory	3.84	n.a.	n.a.	4019.88
90	7	12	Memory	3.75	n.a.	n.a.	4902.56
100	4	6	Memory	3.72	n.a.	n.a.	11512.81
100	6	8	Memory	3.81	n.a.	n.a.	8248.22
100	6	10	Memory	4.19	n.a.	n.a.	7865.55
100	7	10	Memory	3.88	n.a.	n.a.	7607.84
100	7	12	Memory	3.94	n.a.	n.a.	12040.49

Table 4.1: Performances of the Lagrangian heuristic and CPLEX on p-HALP.

The results presented in Table 4.1 are satisfactory. Because of the the path based formulation, the LR bound obtained by CPLEX for instances that are known does not exceed 1.27% in our set of instances showing that the formulation proposed in this study is strong. Recall that because of the integrality property of our Lagrangian relaxation, the best lower bound that subgradient can obtain in not greater than the LP value. The LR value obtained reaches a very close but not exact LP due to some convergence issues of the subgradient algorithm. The marginal gap between LR and LP, however, never exceeds 1.50% which in our opinion is good.

As can be seen, the duality gap is always bellow 4.19% where the average is 2.38%. Observe that the Lagrangian heuristic is able to obtain the optimal solution in 23 out of 25 instance that CIPLEX could prove optimality of the obtained solutions. Note also that CPLEX cannot solve larger instances because of the CPU time limits and/or lack of memory.

We now present the computational results for the LR developed for the q-HALPIH in Table 4.2. The columns in Table 4.2. have the same interpretation as in Table 4.1.

Instance			Deviation (%)			Time (*sec*)	
$\|V\|$	q	p	LP	LR	Heu	$CPLEX$	SG
10	2	4	1.07	1.11	0.00	0.58	0.52
10	3	5	0.13	0.30	0.00	0.49	0.45
10	4	5	0.24	0.48	0.00	0.51	0.47
10	5	5	0.40	0.69	0.00	0.53	0.33
10	5	6	0.12	0.34	0.00	0.48	0.23
20	3	6	0.83	1.12	0.00	4.09	1.72
20	4	6	0.91	1.36	0.00	4.26	1.69
20	4	7	0.20	0.74	0.00	2.57	1.78
20	5	6	1.03	1.35	0.00	4.24	1.66
20	5	7	0.25	0.54	0.00	2.04	1.90
25	3	6	0.43	1.17	0.00	11.920	3.86
25	4	6	0.76	1.25	0.00	14.49	3.95
25	5	6	1.02	1.51	0.00	17.60	4.12
25	5	7	1.02	1.67	0.00	22.79	3.94
25	6	7	1.24	1.55	0.00	17.50	3.89
40	4	6	0.51	1.52	0.00	735.67	53.05
40	4	7	0.50	1.65	0.00	645.20	59.23
40	4	8	0.38	1.60	0.00	540.87	62.02
40	5	7	0.60	1.47	0.00	614.83	76.20
40	5	8	0.44	1.50	0.00	556.56	65.20
50	4	6	0.61	1.98	0.00	6209.51	170.79
50	4	7	0.53	1.87	0.00	6979.98	195.17
50	4	8	0.23	1.90	0.00	2867.72	165.63
50	5	7	0.75	2.21	0.00	8984.35	160.37
50	5	8	0.46	2.01	0.00	5882.63	165.66
60	4	6	Time	2.58	n.a.	n.a.	547.45
60	5	8	Time	2.54	n.a.	n.a.	507.91
60	6	9	Time	2.49	n.a.	n.a.	508.40
60	6	10	Time	2.57	n.a.	n.a.	590.59
60	8	10	Time	2.39	n.a.	n.a.	609.45
70	4	6	Memory	2.47	n.a.	n.a.	1451.22
70	5	7	Memory	2.19	n.a.	n.a.	1589.19
70	6	8	Memory	2.43	n.a.	n.a.	1277.05
70	6	9	Memory	2.57	n.a.	n.a.	1116.45
70	7	10	Memory	2.66	n.a.	n.a.	1244.18
75	4	6	Memory	2.89	n.a.	n.a.	1673.77
75	6	8	Memory	2.89	n.a.	n.a.	1643.68
75	6	10	Memory	2.95	n.a.	n.a.	1803.93
75	7	10	Memory	2.98	n.a.	n.a.	2192.85
75	7	12	Memory	2.94	n.a.	n.a.	2030.44
90	4	6	Memory	3.49	n.a.	n.a.	5672.25
90	6	8	Memory	3.73	n.a.	n.a.	5392.63
90	6	10	Memory	3.97	n.a.	n.a.	4251.82
90	7	10	Memory	3.96	n.a.	n.a.	4522.32
90	7	12	Memory	3.80	n.a.	n.a.	4884.93
100	4	6	Memory	3.81	n.a.	n.a.	8578.94
100	6	8	Memory	3.82	n.a.	n.a.	9670.71
100	6	10	Memory	4.22	n.a.	n.a.	6997.31
100	7	10	Memory	3.88	n.a.	n.a.	7888.70
100	7	12	Memory	3.55	n.a.	n.a.	11481.40

Table 4.2: Performances of the Lagrangian heuristic and CPLEX on p-HALPIH.

The results presented in Table 4.2 further confirm the efficiency of our Lagrangian heuristic. We firstly show that the duality gap obtained by our Lagrangian heuristic is bellow 4.22% where the average is 2.22%. Secondly, we note that the performance of our heuristic is very good as it is able to obtain the optimal solutions for all instances that CPLEX could solve the problem to optimality. Note that once more for instances of size $N = 60$ or more CPLEX could not solve the problem within the time limit and was not able to solve the MIP problem because of memory issues for all larger instances. This provides a clear indication of the complexity of q-Hub Arc Location Problems.

Chapter 5

Conclusion and Future Research Avenues

In this thesis, we studied q-HALPs. We presented a strong MIP formulation to the q-HALPs and examined several properties of optimal solutions to q-HALPs to reduce the size of this formulation. We then introduced a Lagrangian relaxation that exploits the inherent structure of the problem by decomposing it into $|K| + 2$ independent subproblems. We developed two Lagrangian heuristics to q-HALP and q-HALPIH that yield very close to optimal solutions by employing Lagrangian relaxation solutions. We further analyzed economical benefits in allowing isolated hub nodes in hub and spoke networks and studied several interesting structures of optimal solutions to q-HALPs. Computational results on benchmark instances with up to 100 nodes were also reported to confirm the efficiency and robustness of the proposed approaches. To the best of authors' knowledge, this is the first attempt at solving and/or providing very close to optimal solutions for large-size q-HALPs.

Future studies might consider developing exact algorithms for solving larger instances to optimality. Because of the structure of the path based formulation provided in this thesis, Benders decomposition could be a very suitable approach to be developed for solving large-scale hub arc location problems. The dynamic and stochastic version of HALPs in uncertain environments are also subject to further investigations. Hub and spoke networks are frequently employed in fast delivery carrier systems where the service quality is measured by the access and the speed of the delivery. Given this, another interesting research avenue could be studying hub arc location models from a set covering perspective.

Chapter 6

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