On variational formulas on spaces of quadratic differentials

Shahab Azarfar

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By: Shahab Azarfar

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______Marco Bertola______ Chair

______S. Twareque Ali______ Examiner

______Marco Bertola______ Examiner

______Dmitry Korotkin______ Supervisor

Approved by ______Jose Garrido______
Chair of Department or Graduate Program Director

______Andre Roy______
Dean of Faculty

Date ______August 29, 2014______

ABSTRACT

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We study the variational formulas for the normalized Abelian differentials and matrix of b-periods on Hurwitz spaces, the moduli spaces of holomorphic Abelian differentials and quadratic differentials over compact Riemann surfaces. As the main result of the thesis, we find a complete set of local vector fields on the non-hyperelliptic connected component of the principal stratum of the moduli space of holomorphic quadratic differentials preserving the moduli of the base Riemann surface.

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DEDICATION

To my family

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$\mathbb{C}$	the field of complex numbers
$\mathbb{Z}$	the ring of integers
I_n	the $n \times n$ identity matrix
$\mathrm{M}(n, \mathbb{C})$	the set of complex $n \times n$ matrices
$\mathcal{C}^\infty(U)$	the ring of C^∞ complex-valued functions defined on an open set U
$\mathcal{O}(U)$	the ring of holomorphic functions defined on an open set U
$\mathcal{O}^*(U)$	the multiplicative group of nonzero holomorphic functions defined on an open set U
Ω^1_C	the sheaf of holomorphic 1-forms on a compact Riemann surface C
$\mathcal{K}_C$	the canonical line bundle of a compact Riemann surface C

Chapter 1
Introduction and Background Material

1.1 Introduction

The goal of this thesis is to study the variational formulas of Ahlfors-Rauch type for the normalized Abelian differentials and the matrix of b-periods on Hurwitz spaces, the moduli spaces of holomorphic 1-forms and holomorphic quadratic differentials over compact Riemann surfaces.

Consider a compact Riemann surface C of genus g. Denote by $\mathbb{B}$ the matrix of b-periods of C (see Sect. 1.2 below). Consider a *Beltrami differential* $\mu \in \Gamma\left(\overline{\mathcal{K}}_C \otimes \mathcal{K}_C^{-1}\right)$ on C, where $\mathcal{K}_C$ is the canonical line bundle of C and μ is a smooth section of the line bundle $\overline{\mathcal{K}}_C \otimes \mathcal{K}_C^{-1}$. The Beltrami differential μ represents a tangent vector to the Teichmuller space $\mathcal{T}_g$ of marked compact Riemann surfaces of genus g, at the point C. The classical Ahlfors-Rauch variational formula gives the directional derivative of $\mathbb{B}$ at C in direction μ.

Let $\{(U_i, z_i)\}$ be a holomorphic atlas on C, where $z_i : U_i \to \mathbb{C}$ is the local coordinate on the open neighbourhood $U_i \subset C$. The local expression of the Beltrami differential μ on U_i is given by

$$\mu|_{U_i} = \mu_i(z_i)\frac{d\overline{z}_i}{dz_i}\,, \tag{1.1}$$

where $\{\mu_i \in \mathcal{C}^\infty(U_i)\}$ is a collection of local smooth complex-valued functions satisfying

$$\mu_i(z_i) = \mu_j(z_j)\,\overline{\left(\frac{dz_j}{dz_i}\right)}\Big/\left(\frac{dz_j}{dz_i}\right) \qquad \text{on } U_i \cap U_j\,. \tag{1.2}$$

Let $\{\zeta_i \in \mathcal{C}^\infty(U_i)\}$ be a collection of local diffeomorphisms defined as solutions of the following *Beltrami equation*:

$$\mu_i = \left(\frac{\partial \zeta_i}{\partial \overline{z}_i}\right)\Big/\left(\frac{\partial \zeta_i}{\partial z_i}\right) \qquad \text{on } U_i\,. \tag{1.3}$$

By substituting (1.3) into (1.2), we get

$$\left(\frac{\partial \zeta_i}{\partial \overline{z}_i}\right)\Big/\left(\frac{\partial \zeta_i}{\partial z_i}\right) = \left(\frac{\partial \zeta_j}{\partial \overline{z}_i}\right)\Big/\left(\frac{\partial \zeta_j}{\partial z_i}\right) \qquad \text{on } U_i \cap U_j\,. \tag{1.4}$$

According to (1.4), the complex dilatation of the transition map $\zeta_j \circ \zeta_i^{-1}$ is equal to one, i.e. the mapping $\zeta_j \circ \zeta_i^{-1}$ is holomorphic on $\zeta_i(U_i \cap U_j) \subset \mathbb{C}$. Therefore, each Beltrami differential μ on C (modulo *the infinitesimally trivial Beltrami differentials*) corresponds to a Riemann surface C^μ given by the holomorphic atlas $\{(U_i, \zeta_i)\}$.

Let $\{v_\alpha\}_{\alpha=1,\cdots,g}$ and $\{\tilde{v}_\alpha\}_{\alpha=1,\cdots,g}$ be the normalized basis of the space of holomorphic 1-forms on C and C^μ, respectively. Notice that each v_α and $\tilde{v}_\alpha$ is a smooth, complex-valued, closed 1-form on the underlying smooth surface X of C, since they are each holomorphic with respect to *some* complex structure on X (i.e. the holomorphic atlases $\{(U_i, z_i)\}$ and $\{(U_i, \zeta_i)\}$, respectively).

Denote by $\mathbb{B} = [\mathbb{B}_{\alpha\beta}]$ and $\widetilde{\mathbb{B}} = [\widetilde{\mathbb{B}}_{\alpha\beta}]$ the matrices of b-periods of C and C^μ, respectively. To compare $\mathbb{B}_{\alpha\beta}$ with $\widetilde{\mathbb{B}}_{\alpha\beta}$, we apply the Riemann's bilinear relation to the closed 1-forms v_β and $(\tilde{v}_\alpha - v_\alpha)$. So we get

$$\begin{aligned} \widetilde{\mathbb{B}}_{\alpha\beta} - \mathbb{B}_{\alpha\beta} = \int_{b_\beta} \tilde{v}_\alpha - \int_{b_\beta} v_\alpha &= \iint_C v_\beta \wedge (\tilde{v}_\alpha - v_\alpha) \\ &= \iint_C v_\beta \wedge \tilde{v}_\alpha \,, \end{aligned} \tag{1.5}$$

since $v_\beta \wedge v_\alpha \equiv 0$.

Let $z : U \to \mathbb{C}$ and $\zeta : U \to \mathbb{C}$ be the local coordinates in an open neighbourhood $U \subset X$ induced by the complex structure of C and C^μ, respectively. The local expression of the 2-form $v_\beta \wedge \tilde{v}_\alpha$ on U is given by

$$\begin{aligned} v_\beta \wedge \tilde{v}_\alpha &= [v_\beta(z) dz] \wedge [\tilde{v}_\alpha(\zeta) d\zeta] \\ &= [v_\beta(z) dz] \wedge \left[\tilde{v}_\alpha(\zeta) \left(\frac{\partial \zeta}{\partial z} dz + \frac{\partial \zeta}{\partial \overline{z}} d\overline{z} \right) \right] \\ &= \left[v_\beta(z) \tilde{v}_\alpha(\zeta) \mu \frac{\partial \zeta}{\partial z} \right] dz \wedge d\overline{z} \\ &= [v_\beta(z) v_\alpha(z) \mu] \, dz \wedge d\overline{z} \\ &\quad + \left[v_\beta(z) \left(\tilde{v}_\alpha(\zeta) \mu \frac{\partial \zeta}{\partial z} - v_\alpha(z) \mu \right) \right] dz \wedge d\overline{z} \,. \end{aligned} \tag{1.6}$$

Thus (1.5) can be written in the following form:

$$\widetilde{\mathbb{B}}_{\alpha\beta} - \mathbb{B}_{\alpha\beta} = \iint_C (v_\alpha \otimes v_\beta) \, \mu + E_{\alpha\beta} \,, \tag{1.7}$$

where the “error term" $E_{\alpha\beta}$ is given by

$$E_{\alpha\beta} = \iint_C v_\beta \otimes \left(\tilde{v}_\alpha \frac{\partial \zeta}{\partial z} - v_\alpha \right) \mu \,. \tag{1.8}$$

It can be shown that $E_{\alpha\beta} = O\left((||\mu||_\infty)^2 \right)$ as $||\mu||_\infty$ tends to zero [11]. So we get the following classical Ahlfors-Rauch variational formula:

$$d\mathbb{B}_{\alpha\beta}\big|_C ([\mu]) = \iint_C (v_\alpha \otimes v_\beta) \, \mu \,. \tag{1.9}$$

The variation of matrix of b-periods under variation of the local homological coordinates on the strata of the moduli space $\mathcal{H}_g$ of holomorphic Abelian differentials over Riemann surfaces of genus g is investigated in [5]. Our primary aim is to study the variational formula for matrix of b-periods, similar to (1.9), on the principal stratum $\mathcal{Q}_g\left([1^{4g-4}]\right)$ of the moduli space of holomorphic quadratic differentials, i.e. the space of pairs (C, q), where C is a compact Riemann surface of genus $g \geq 2$ and q is a holomorphic quadratic differential on C with simple zeros.

Corresponding to each pair $(C, q) \in \mathcal{Q}_g\left([1^{4g-4}]\right)$, there exists a canonical 2-sheeted branched covering $\pi : \widehat{C} \to C$ such that the pullback quadratic differential $\pi^* q$ has $4g-4$ zeros of multiplicity 4 on the Riemann surface $\widehat{C}$ of genus $4g-3$. Therefore, the holomorphic Abelian differential $\sqrt{\pi^* q}$ on $\widehat{C}$ can be constructed such that

$$\sqrt{\pi^* q} \otimes \sqrt{\pi^* q} = \pi^* q \,. \tag{1.10}$$

The 1-form $\sqrt{\pi^* q}$ has $4g-4$ zeros of multiplicity 2 on $\widehat{C}$ so, the pair $(\widehat{C}, \sqrt{\pi^* q})$ is in the stratum $\mathcal{H}_{4g-3}\left([2^{4g-4}]\right)$. Considering the above-mentioned correspondence between $(C, q) \in \mathcal{Q}_g\left([1^{4g-4}]\right)$ and $(\widehat{C}, \sqrt{\pi^* q}) \in \mathcal{H}_{4g-3}\left([2^{4g-4}]\right)$, we have the following local embedding:

$$\begin{gathered} \mathcal{S} : \mathcal{Q}_g\left(\left[1^{4g-4}\right]\right) \to \mathcal{H}_{4g-3}\left(\left[2^{4g-4}\right]\right) . \\ (C, q) \mapsto (\widehat{C}, \sqrt{\pi^* q}) \end{gathered} \tag{1.11}$$

The spaces $\mathcal{Q}_g\left([1^{4g-4}]\right)$ and $\mathcal{H}_{4g-3}\left([2^{4g-4}]\right)$ are of dimension $6g-6$ and $12g-11$, respectively.

There exists a biholomorphic involution $\mu : \widehat{C} \to \widehat{C}$ on $\widehat{C}$ which interchange the points in each fiber of $\pi : \widehat{C} \to C$. The pullback μ^* of the involution $\mu : \widehat{C} \to \widehat{C}$ is an isomorphism on the space

$\Lambda^1_{\widehat{C}} = H^0(\widehat{C}, \Omega^1_{\widehat{C}})$ of holomorphic Abelian differentials on $\widehat{C}$. The $(4g-3)$-dimensional vector space $\Lambda^1_{\widehat{C}}$ splits into two eigenspaces Λ_+ and Λ_- corresponding to the eigenvalues ± 1 of μ^*.

The involution $\mu : \widehat{C} \to \widehat{C}$ induces an isomorphism

$$\mu_* : H_1(\widehat{C}; \mathbb{C}) \to H_1(\widehat{C}; \mathbb{C}) \tag{1.12}$$

on the complex homology group $H_1(\widehat{C}; \mathbb{C})$ of $\widehat{C}$. The $(8g-6)$-dimensional complex vector space $H_1(\widehat{C}; \mathbb{C})$ splits into two eigenspaces H_+ and H_- corresponding to the eigenvalues ± 1 of μ_*.

Let

$$\{a_j, a_j^*, \tilde{a}_k, b_j, b_j^*, \tilde{b}_k\}, \quad j = 1, \cdots, g\,;\; k = 1, \cdots, 2g-3 \tag{1.13}$$

be a symplectic basis of $H_1(\widehat{C}; \mathbb{C})$ such that

$$\mu_* a_j = a_j^*, \quad \mu_* b_j = b_j^*, \quad \mu_* \tilde{a}_k + \tilde{a}_k = \mu_* \tilde{b}_k + \tilde{b}_k = 0\,.$$

We construct the symplectic bases $\{\alpha_j^+, \beta_j^+\}_{j=1,\cdots,g}$ and $\{\alpha_l^-, \beta_l^-\}_{1=1,\cdots,3g-3}$ for the eigenspaces H_+ and H_-, from the basis (1.13), respectively.

Let

$$\{u_j, u_j^*, \tilde{u}_k\}\,, \quad j = 1, \cdots, g\,,\; k = 1, \cdots, 2g-3 \tag{1.14}$$

be the normalized basis of $\Lambda^1_{\widehat{C}}$ associated with the canonical basis (1.13). Consider the following vectors

$$\mathrm{U} = \begin{bmatrix} u \\ u^* \\ \tilde{u} \end{bmatrix} \in \left(H^1(\widehat{C}; \mathbb{C})\right)^{4g-3}; \quad \mathrm{A} = \begin{bmatrix} a \\ a^* \\ \tilde{a} \end{bmatrix}, \; \mathrm{B} = \begin{bmatrix} b \\ b^* \\ \tilde{b} \end{bmatrix} \in \left(H_1(\widehat{C}; \mathbb{C})\right)^{4g-3}. \tag{1.15}$$

The basis (1.14) is normalized by the following condition on its periods over the 1-cycles $\{a_j, a_j^*, \tilde{a}_k\}$ (see Sect. 1.2 below):

$$\Pi\,(\mathrm{U}, \mathrm{A}) = I_{4g-3}\,. \tag{1.16}$$

The matrix of b-periods $\widehat{\mathbb{B}}$ of $\{u_j, u_j^*, \tilde{u}_k\}$ with respect to the homology basis (1.13) is given by

$$\widehat{\mathbb{B}} = \Pi\,(\mathrm{U}, \mathrm{B})\;. \tag{1.17}$$

We construct the normalized bases $\{u_j^+\}_{j=1,\cdots,g}$ and $\{u_l^-\}_{l=1,\cdots,3g-3}$ for the eigenspaces Λ_+ and Λ_-, from the basis (1.14), respectively, such that

$$\Pi\left(\begin{bmatrix} u^+ \\ u^- \end{bmatrix}, \begin{bmatrix} \alpha^+ \\ \alpha^- \end{bmatrix}\right) = \begin{bmatrix} I_g & 0 \\ 0 & I_{3g-3} \end{bmatrix} . \tag{1.18}$$

Since

$$\Pi\left(u_j^+, \beta_l^-\right) = \Pi\left(\mu^* u_j^+, \beta_l^-\right) = \Pi\left(u_j^+, \mu_* \beta_l^-\right) = -\Pi\left(u_j^+, \beta_l^-\right) , \tag{1.19}$$

we have

$$\Pi\left(\begin{bmatrix} u^+ \\ u^- \end{bmatrix}, \begin{bmatrix} \beta^+ \\ \beta^- \end{bmatrix}\right) = \begin{bmatrix} \mathbb{B}^+ & 0 \\ 0 & \mathbb{B}^- \end{bmatrix} , \tag{1.20}$$

where $\mathbb{B}^+ \in \mathrm{M}(g, \mathbb{C})$ and $\mathbb{B}^- \in \mathrm{M}(3g-3, \mathbb{C})$ are called the matrices of β-periods.

To shorten the notation, we denote the basis $\{\alpha_l^-, \beta_l^-\}$ for the eigenspace H_- by $\{r_i\}_{i=1,\cdots,6g-6}$, where

$$r_l = \alpha_l^- , \quad r_{3g-3+l} = \beta_l^- , \quad l = 1, \cdots, 3g-3 . \tag{1.21}$$

The periods of $\sqrt{\pi^* q}$ on the cycles $\{r_i\}_{i=1,\cdots,6g-6}$, i.e.

$$\eta_i := \Pi\left(\sqrt{\pi^* q}, r_i\right) , \quad i = 1, \cdots, 6g-6 , \tag{1.22}$$

provide a system of local homological coordinates on $\mathcal{S}\left(\mathcal{Q}_g\left([1^{4g-4}]\right)\right)$. Let $\{\zeta_n\}_{n=1,\cdots,12g-11}$ be the set of natural homological coordinates on the stratum $\mathcal{H}_{4g-3}\left([2^{4g-4}]\right)$. Let $\mathbb{B}$ be the matrix of b-periods of the base Riemann surface C. Using the chain rule and the formula for variation of $\widehat{\mathbb{B}}$ with respect to $\{\zeta_n\}_{n=1,\cdots,12g-11}$, we derive the formula for variation of $\mathbb{B}$ under variation of $\{\eta_i\}_{i=1,\cdots,6g-6}$.

The difference between the dimensions of $\mathcal{Q}_g\left([1^{4g-4}]\right)$ and the moduli space of compact Riemann surfaces of genus $g \geq 2$ is equal to $3g-3$. Therefore, there must exist $3g-3$ local independent vector fields $\{W_l\}_{l=1,\cdots,3g-3}$, defined on a neighbourhood of the point $(C, q) \in \mathcal{Q}_g\left([1^{4g-4}]\right)$, which preserve the complex structure of the Riemann surface C, i.e.

$$W_l(\mathbb{B}) = 0_{g\times g} , \qquad l = 1, \cdots, 3g-3 . \tag{1.23}$$

As the main result of the thesis, we find the following expression for W_l , $l = 1, \cdots, 3g-3$, as a linear combination of the vector fields $\{\partial/\partial\eta_i\}_{i=1,\cdots,6g-6}$, on the non-hyperelliptic connected component of $\mathcal{Q}_g\left([1^{4g-4}]\right)$:

$$W_l = \frac{\partial}{\partial \eta_l} + \sum_{k=1}^{3g-3} \mathbb{B}^-_{lk} \frac{\partial}{\partial \eta_{3g-3+k}}, \quad l = 1, \cdots, 3g-3, \tag{1.24}$$

where $\mathbb{B}^-$ is the matrix of β^--periods given by (1.20).

The thesis is organized as follows. In Section 1.2 we review the basic concepts from the theory of compact Riemann surfaces which are necessary in the sequel. In Section 2.1 we introduce the Hurwitz space $H(d; r_1, \cdots, r_n)$ as the space of branched coverings $f : C \to \mathbb{CP}^1$ of a fixed combinatorial type $(d; r_1, \cdots, r_n)$ of the Riemann sphere or, equivalently, the space of meromorphic functions on Riemann surfaces C of fixed genus. In Section 2.2 we study the variation of the normalized Abelian differentials and matrix of b-periods of the Riemann surface C under the variation of one of the branch points of the meromorphic function $f : C \to \mathbb{CP}^1$ [4].

In Section 3.1 we introduce the set of homological coordinates on the strata of the moduli space of holomorphic Abelian differentials over Riemann surfaces of fixed genus. In Section 3.2 we study the variational formulas, given in [5], for the normalized Abelian differentials and matrix of b-periods with respect to these homological coordinates.

In Section 4.1 we introduce the canonical 2-sheeted branched covering $\pi : \widehat{C} \to C$ corresponding to a pair $(C, q) \in \mathcal{Q}_g\left([1^{4g-4}]\right)$. The main objective of this section is to discuss the decomposition of the (co)homology group of $\widehat{C}$ into invariant and anti-invariant subspaces under the action of the involution $\mu : \widehat{C} \to \widehat{C}$. In Section 4.2 we introduce the induced homological coordinates on $\mathcal{Q}_g\left([1^{4g-4}]\right)$ and derive the variational formulas for matrix of b-periods of C and $\widehat{C}$ under variation of these coordinates. In addition, we find a complete set of local vector fields in a neighbourhood of the point $(C, q) \in \mathcal{Q}_g\left([1^{4g-4}]\right)$ which preserve the complex structure of C.

1.2 Basic Objects on Compact Riemann Surfaces

Let C be a compact Riemann surface of genus g. Let $\{a_l, b_l\}_{l=1,\cdots,g}$ be a canonical basis of the integral homology group $H_1(C;\mathbb{Z})$ of C. The intersection numbers of these 1-cycles are given by

$$a_i \circ b_j = \delta_{ij}, \qquad a_i \circ a_j = b_i \circ b_j = 0\,, \quad i,j = 1,\cdots,g\,. \tag{1.25}$$

The relations (1.25) can be represented in the following matrix form:

$$\begin{bmatrix} a \\ b \end{bmatrix} \circ \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}\,,$$

(we denote the $n \times n$ identity matrix by I_n).

Let Ω^1_C be the sheaf of holomorphic 1-forms on C. Denote by $H^0(C,\Omega^1_C)$ the g-dimensional vector space of holomorphic 1-forms on C. Let $\{v_k\}_{k=1,\cdots,g}$ be the basis of $H^0(C,\Omega^1_C)$ normalized by $\oint_{a_l} v_k = \delta_{kl}$. The matrix of b-periods $\mathbb{B} = [\mathbb{B}_{kl}]$ of C is defined by

$$\mathbb{B}_{kl} := \oint_{b_l} v_k\,, \quad k,l = 1,\cdots,g\,. \tag{1.26}$$

Let $\mathcal{A}^k(C,\mathbb{C})\,,\ k = 0,1,2$ be the space of smooth complex-valued k-forms on C. Consider the following two subspaces of $\mathcal{A}^1(C,\mathbb{C})$:

$$\mathcal{Z}^1(C;\mathbb{C}) = \left\{\alpha \in \mathcal{A}^1(C,\mathbb{C}) \middle|\, d\alpha = 0\right\} = \{\text{smooth closed 1-forms on } C\}\,,$$
$$\mathcal{B}^1(C;\mathbb{C}) = \left\{df \in \mathcal{A}^1(C,\mathbb{C}) \middle|\, f \in \mathcal{A}^0(C,\mathbb{C})\right\} = \{\text{smooth exact 1-forms on } C\}\,.$$

The De Rham cohomology group $H^1(C;\mathbb{C})$ of C is a $(2g)$-dimensional complex vector space defined in the following way:

$$H^1(C;\mathbb{C}) = \frac{\mathcal{Z}^1(C;\mathbb{C})}{\mathcal{B}^1(C;\mathbb{C})}\,. \tag{1.27}$$

Let α be a closed 1-form on C and γ be a closed path in C. The *period* $\Pi(\alpha,\gamma)$ of α on γ is defined [8] by

$$\Pi(\alpha,\gamma) := \oint_\gamma \alpha\,. \tag{1.28}$$

The integral $\oint_\gamma \alpha$ depends only on the cohomology class $[\alpha] \in H^1(C;\mathbb{C})$ of α and the homology class $[\gamma] \in H_1(C;\mathbb{C})$ of γ. Therefore, there exists a bilinear *period* mapping

$$\Pi : H^1(C;\mathbb{C}) \times H_1(C;\mathbb{C}) \to \mathbb{C}\,, \tag{1.29}$$

defined by $\Pi([\alpha],[\gamma]) := \int_\gamma \alpha$.

Let $(H^1(C;\mathbb{C}))^n$ and $(H_1(C;\mathbb{C}))^n$ be the Cartesian product of n copies of $H^1(C;\mathbb{C})$ and $H_1(C;\mathbb{C})$, respectively. Consider the vectors

$$\mathrm{X} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in \left(H^1(C;\mathbb{C})\right)^n \quad \text{and} \quad \mathrm{Y} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} \in \left(H_1(C;\mathbb{C})\right)^n. \tag{1.30}$$

We define the bilinear period mapping

$$\Pi : \left(H^1(C;\mathbb{C})\right)^n \times \left(H_1(C;\mathbb{C})\right)^n \to \mathrm{M}(n,\mathbb{C}) \tag{1.31}$$

in the following way:

$$\Pi(\mathrm{X},\mathrm{Y}) := \begin{bmatrix} \Pi(\alpha_1,\gamma_1) & \dots & \Pi(\alpha_1,\gamma_n) \\ \vdots & & \vdots \\ \Pi(\alpha_n,\gamma_1) & \dots & \Pi(\alpha_n,\gamma_n) \end{bmatrix}, \tag{1.32}$$

where $\mathrm{M}(n,\mathbb{C})$ denotes the vector space of $n \times n$ complex matrices.

Definition 1. *A meromorphic differential on C is called an Abelian differential of the second kind if the residues at all of its poles are equal to zero.*

The normalized Abelian differential of second kind, denoted by $\omega_p^{(n)}$, has only one singularity at $p \in C$ that is of the form

$$\omega_p^{(n)}\big|_U = \left(\frac{1}{z^n} + O(1)\right) dz, \tag{1.33}$$

where $z : U \to \mathbb{C}$ is the local coordinate on $U \subset C$ with $z(p) = 0$. It satisfies the normalization condition $\oint_{a_l} \omega_p^{(n)} = 0 \quad l = 1, \cdots, g$.

Denote by $E(x,y)$ the prime form on C. The Bergman bidifferential $B(x,y)$ is defined by

$$B(x,y) = d_x\, d_y\, log E(x,y). \tag{1.34}$$

The bidifferential $B(x,y)$ is symmetric, i.e. $B(x,y) = B(y,x)$, and $B(x,y_0)$ is a differential on C with a single pole of order 2 at y_0 for each fixed $y_0 \in C$.

Let U be an open subset of C with local coordinate $z : U \to \mathbb{C}$. The Bergman bidifferential restricted to the open subset $C \times U \subset C \times C$ is given by

$$B(x,y)|_{C\times U} = \pi_1^*(\omega_p^{(2)}) \otimes \pi_2^*(dz), \tag{1.35}$$

where $\omega_p^{(2)}$ is the normalized Abelian differential of second kind on C with a double pole at $p \in U$. The b-period of $\omega_p^{(2)}$ is given by

$$\oint_{b_k} \omega_p^{(2)} = 2\pi i \left(\frac{v_k}{dz}\right)(p), \tag{1.36}$$

where $\{v_l\}_{l=1,\cdots,g}$ is the normalized basis of the space of holomorphic 1-forms on C. So we have

$$\oint_{b_k} B(x,p) := \left(\oint_{b_k} \omega_p^{(2)}(x)\right) dz(p) = 2\pi i v_k(p)\,, \tag{1.37}$$

$$\oint_{a_k} B(x,p) := \left(\oint_{a_k} \omega_p^{(2)}(x)\right) dz(p) = 0\,. \tag{1.38}$$

The differential $\omega_p^{(n)}$ can be expressed in terms of Bergman bidifferential as follows. Let $\iota_p^* B$ be the pullback of the Bergman bidifferential B by the map $\iota_p : C \hookrightarrow C \times C$ defined by $\iota_p(q) = (q,p)$. Let V be an arbitrary open subset of C with local coordinate $y : V \to \mathbb{C}$. The 1-form $\omega_p^{(n)}$ restricted to V satisfies the following equality:

$$\omega_p^{(n)}\big|_V = \frac{1}{(n-1)!}\left[\left(\frac{\partial}{\partial x}\right)^{n-2}\left(\frac{\iota_x^* B}{dy}\right)\Big|_{x=p}\right] dy. \tag{1.39}$$

We can rewrite (1.39) in a more concise way as follows:

$$\omega_p^{(n)} = \frac{1}{(n-1)!}\left[\left(\frac{\partial}{\partial x}\right)^{n-2}(\iota_x^* B)\,\Big|_{x=p}\right]. \tag{1.40}$$

The Bergman bidifferential has the following asymptotics near the diagonal divisor on $C \times C$:

$$B(p,q) = \left(\frac{1}{(\zeta(p)-\zeta(q))^2} + O(1)\right) d\zeta(p) d\zeta(q) \tag{1.41}$$

as $p \to q$, where $\zeta : W \to \mathbb{C}$ is a local coordinate in the neighbourhood $W \subset C$ containing both p and q.

Chapter 2
Variational Formulas on Hurwitz Spaces

In this chapter, we introduce the Hurwitz space $H(d; r_1, \cdots, r_n)$ as the space of d-sheeted branched coverings $f : C \to \mathbb{CP}^1$ of a fixed combinatorial type $(d; r_1, \cdots, r_n)$ of the Riemann sphere or, equivalently, the space of meromorphic functions on Riemann surfaces C of fixed genus. In addition, we study the variational formulas for the normalized Abelian differentials and matrix of b-periods of the Riemann surface C, given in [4], under the variation of one of the branch points of the meromorphic function $f : C \to \mathbb{CP}^1$.

2.1 Hurwitz Spaces

Let (C, f) be a d-sheeted branched covering of the Riemann sphere $\mathbb{CP}^1$, where C is a compact Riemann surface and $f : C \to \mathbb{CP}^1$ is a non-constant holomorphic mapping. The critical points $\{p_i\}_{i=1,\cdots,n}$ of f are the points where $df(p_i) = 0$. These points are called the *ramification points* of f and we call their images $\lambda_i = f(p_i)$ the *branch points* of f. We assume that none of the branch points coincide with $\{\infty\} \in \mathbb{CP}^1$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. The natural local coordinate $x_i(q)$ of a point $q \in C$ in a neighbourhood V_i of the ramification point $p_i \in V_i$ is given by:

$$x_i(q) = (f(q) - f(p_i))^{1/r_i} = (\lambda - \lambda_i)^{1/r_i}, \quad q \in V_i, \tag{2.1}$$

where r_i is the *ramification index* of p_i.

Let $\widetilde{\mathbb{CP}^1} = \mathbb{CP}^1 \setminus \{\lambda_i\}_{i=1,\cdots,n}$ and $\widetilde{C} = C \setminus \{f^{-1}(\lambda_i)\}_{i=1,\cdots,n}$. Consider a fixed point $\lambda_0 \in \widetilde{\mathbb{CP}^1}$. The mapping f induces a topological covering map $\widetilde{f} : \widetilde{C} \to \widetilde{\mathbb{CP}^1}$ of degree d which is a local homeomorphism. In addition, the d-sheeted branched covering (C, f) induces a representation of the fundamental group of $\widetilde{\mathbb{CP}^1}$, called the monodromy map, $\mu : \pi_1(\widetilde{\mathbb{CP}^1}, \lambda_0) \to \mathfrak{S}_d$ where $\mathfrak{S}_d$ is the symmetric group on d letters.

Theorem 2.1.1. (Riemann's Existence Theorem [3]) *Let $B = \{p_i\}_{i=1,\cdots,n}$ be a finite subset of $\mathbb{CP}^1$ and $\lambda_0 \in \widetilde{\mathbb{CP}^1} = \mathbb{CP}^1 \setminus B$. Then for each positive integer d the following sets are in natural one-to-one correspondence:*

(i) the set of equivalence classes of d-sheeted branched coverings of $\mathbb{CP}^1$ with branch points located at B;

(ii) the set of equivalence classes of topological coverings of degree d of $\widetilde{\mathbb{CP}^1}$;

(iii) the subset of $Hom(\pi_1(\widetilde{\mathbb{CP}^1}, \lambda_0), \mathfrak{S}_d)$ represented by homomorphisms whose images are transitive subgroups of $\mathfrak{S}_d$.

Let $\{\lambda_i\}_{i=1,\cdots,n}$ be the set of branch points of the d-sheeted branched covering (C, f). There exist loops $\gamma_1, \cdots, \gamma_n \subset \widetilde{\mathbb{CP}^1}$ based at $\lambda_0 \in \widetilde{\mathbb{CP}^1}$, satisfying the following conditions:

(i) The homotopy classes $[\gamma_i]$ of γ_i's generate $\pi_1(\widetilde{\mathbb{CP}^1}, \lambda_0)$;

(ii) Their homotopy classes satisfy the relation $[\gamma_1][\gamma_2]\cdots[\gamma_n] = 1$;

(iii) Each γ_i is homotopic to a small loop around λ_i .

The corresponding monodromy map $\mu : \pi_1(\widetilde{\mathbb{CP}^1}, \lambda_0) \to \mathfrak{S}_d$ is defined by the permutations

$$\mu([\gamma_i]) = \sigma_i \in \mathfrak{S}_d\,, \quad i = 1, \cdots, n\,, \tag{2.2}$$

where $\sigma_1\sigma_2\cdots\sigma_n = 1$. The cycle type of the permutation σ_i is $(1^{d-r_i}, r_i)$.

Let $\mathcal{U} = (U_1, \cdots, U_n)$ be a tuple of small open disks $U_i \subset \mathbb{CP}^1 \setminus \{\infty\}$ centred at $\lambda_i,\ i = 1, \cdots, n$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Fix a representation $\mu : \pi_1(\widetilde{\mathbb{CP}^1}, \lambda_0) \to \mathfrak{S}_d$ of the fundamental group of $\widetilde{\mathbb{CP}^1} = \mathbb{CP}^1 \setminus \{\lambda_i\}_{i=1,\cdots,n}$, corresponding to the monodromy map of the d-sheeted branched covering (C, f). If we allow the λ_i's to vary in U_i's, then we get the set $\Delta_{\mathcal{U},\mu}(C, f)$ of d-sheeted branched coverings $(\widetilde{C}, \tilde{f})$ with the branch points $\tilde{\lambda}_i \in U_i\,,\ i = 1, \cdots, n$ and monodromy map μ. The set $\Delta_{\mathcal{U},\mu}(C, f)$ is an open neighbourhood of (C, f) in the *Hurwitz space* $H(d; r_1, \cdots, r_n)$ [12].

2.2 Variational Formulas

Let (C^0, f^0) be a d-sheeted branched covering of $\mathbb{CP}^1$ with branch points $\{\lambda_i^0\}_{i=1,\cdots,n}$ and monodromy map μ. Consider a small open disk $U_m \subset \mathbb{CP}^1 \setminus \{\infty\}$ of radius R centered at λ_m^0

such that

$$\lambda_i^0 \in \mathbb{C} \setminus \overline{U_m}\,, \quad i \neq m\,,\, 1 \leq i \leq n\,. \tag{2.3}$$

Consider a parameter $\epsilon \in \mathbb{C}$ such that $|\epsilon| < R$. Using the fixed monodromy map μ, we construct a set of d-sheeted branched coverings $\left\{(C^\epsilon, f^\epsilon)\right\} \subset \Delta_{\mathcal{U},\mu}(C^0, f^0)$, where the branch points of (C^ϵ, f^ϵ) are $\{\lambda_1^0, \cdots, \lambda_m^\epsilon, \cdots, \lambda_n^0\}$ and $\lambda_m^\epsilon = \lambda_m^0 + \epsilon$. The complex structure of the Riemann surface C^ϵ depends on the parameter ϵ.

According to the Riemann-Hurwitz formula, the genus g of the underlying smooth surface of C^ϵ's is given by

$$g = 1 - d + \sum_{i=1}^{n} \frac{r_i - 1}{2}\,. \tag{2.4}$$

Denote by $\{v_k^\epsilon\}_{k=1,\cdots,g}$ the normalized basis of the space of holomorphic 1-forms on the Riemann surface C^ϵ. Let $\mathbb{B}^\epsilon = [\mathbb{B}_{kl}^\epsilon]$ be the matrix of b-periods of C^ϵ, where

$$\mathbb{B}_{kl}^\epsilon = \oint_{b_l} v_k^\epsilon\,, \quad k, l = 1, \cdots, g\ . \tag{2.5}$$

We want to investigate the variation of v_k^ϵ and $\mathbb{B}_{kl}^\epsilon$ under variation of ϵ.

Let $\mathcal{K}_{C^\epsilon} := T^{*\,(1,0)}C^\epsilon$ be the canonical line bundle over C^ϵ. The exact form df^ϵ is a meromorphic section of $\mathcal{K}_{C^\epsilon}$, while v_k^ϵ are holomorphic sections of $\mathcal{K}_{C^\epsilon}$. Therefore, $v_k^\epsilon = \phi_k^\epsilon\,.\,df^\epsilon$, where ϕ_k^ϵ is a meromorphic function on C^ϵ. The derivative of the basic holomorphic differential v_k with respect to the position of the branch point λ_m is defined as follows [5]:

$$\left(\frac{\partial}{\partial \lambda_m} v_k\right)(q) := \left(\frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \left(\phi_k^\epsilon(q)\Big|_{f^\epsilon(q)=const}\right)\right) df^0. \tag{2.6}$$

Theorem 2.2.1. *Let $\{s_m\}_{m=1,\cdots,n}$ be small loops with positive orientation around the ramification points $\{p_m\}_{m=1,\cdots,n}$ of the branched covering (C^0, f^0). The following variational formulas hold:*

$$\left(\frac{\partial}{\partial \lambda_m} v_k\right)(p) = \frac{1}{2\pi i} \oint_{s_m} \frac{v_k(q) B(p,q)}{df(q)} \tag{2.7}$$

$$\frac{\partial}{\partial \lambda_m} \mathbb{B}_{kl} = \oint_{s_m} \frac{v_k\, v_l}{df}, \tag{2.8}$$

where $m = 1, \cdots, n$.

Proof. Let $x_i : V_i \to \mathbb{C}$ be the natural coordinate, defined in (2.1), on a neighbourhood V_i of the ramification point p_i with ramification index r_i. We find the local expression for $\phi_k^\epsilon = v_k^\epsilon / df^\epsilon$

restricted to V_i for each $i \in \{1, \cdots, n\}$. Let $\lambda := f^\epsilon(q)$ for $q \in V_i$. The 1-form v_k^ϵ is a holomorphic differential on C^ϵ. So we have

$$v_k^\epsilon(x_i) = \left(\sum_{j=0}^{\infty} c_j^\epsilon (x_i)^j\right) dx_i = \left(\sum_{j=0}^{\infty} c_j^\epsilon (\lambda - \lambda_i^\epsilon)^{j/r_i}\right) dx_i, \tag{2.9}$$

where the coefficients c_j^ϵ are holomorphic functions of the parameter ϵ. Using

$$dx_i = d\left((f^\epsilon(q) - f^\epsilon(p_i))^{1/r_i}\right) = d\left((\lambda - \lambda_i^\epsilon)^{1/r_i}\right) = \frac{1}{r_i}(\lambda - \lambda_i^\epsilon)^{\frac{1-r_i}{r_i}} d\lambda, \tag{2.10}$$

we get

$$\phi_k^\epsilon|_{V_i} = \frac{1}{r_i}\left(\sum_{j=0}^{\infty} c_j^\epsilon (\lambda - \lambda_i^\epsilon)^{\frac{j+1-r_i}{r_i}}\right). \tag{2.11}$$

Now we calculate the derivative of $\phi_k^\epsilon|_{V_m}$ with respect to ϵ, while λ is kept constant:

$$\begin{aligned} \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \left(\phi_k^\epsilon(x_m)\big|_{\lambda=const}\right) &= \frac{1}{r_m}\left(\sum_{j=0}^{\infty} c_j^0 \left(\frac{r_m - 1 - j}{r_m}\right)(\lambda - \lambda_m^0)^{\frac{j+1-2r_m}{r_m}}\right) \\ &+ \frac{1}{r_m}\left(\sum_{j=0}^{\infty}\left(\frac{dc_j^\epsilon}{d\epsilon}\Big|_{\epsilon=0}\right)(\lambda - \lambda_m^0)^{\frac{j+1-r_m}{r_m}}\right). \end{aligned} \tag{2.12}$$

Therefore, according to the definition (2.6), the 1-form $\frac{\partial}{\partial\lambda_m} v_k$ restricted to the open neighbourhood V_m of the ramification point p_m has the following form:

$$\left(\frac{\partial}{\partial\lambda_m} v_k\right)(x_m) = \left[\left(\sum_{j=0}^{r_m-2} c_j^0\left(1 - \frac{j+1}{r_m}\right)\frac{1}{x_m^{r_m-j}}\right) + O(1)\right] dx_m \tag{2.13}$$

If $i \neq m$, then

$$\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} \left(\phi_k^\epsilon(x_i)\big|_{\lambda=const}\right) = \frac{1}{r_i}\left(\sum_{j=0}^{\infty}\left(\frac{dc_j^\epsilon}{d\epsilon}\Big|_{\epsilon=0}\right)(\lambda - \lambda_i^0)^{\frac{j+1-r_i}{r_i}}\right). \tag{2.14}$$

Therefore, with similar calculations, it can be shown that $\left(\frac{\partial}{\partial\lambda_m} v_k\right)\big|_{V_i}$ is holomorphic on the neighbourhood V_i of the ramification point p_i.

Hence, the 1-form $\frac{\partial}{\partial\lambda_m} v_k$ is a meromorphic differential on C^0 with only one pole of order r_m at the point p_m. Its principal part at p_m is given by (2.13). Since $\oint_{a_l} v_k^\epsilon = \delta_{kl}$, we have

$$\oint_{a_l} \frac{\partial}{\partial\lambda_m} v_k = 0\,, \qquad l = 1, \cdots, g\,. \tag{2.15}$$

Thus, $\frac{\partial}{\partial \lambda_m} v_k$ can be expressed in terms of normalized Abelian differentials of second kind $\omega_{p_m}^{(j)}$ as follows:

$$\frac{\partial}{\partial \lambda_m} v_k = \sum_{j=0}^{r_m-2} c_j^0 \left(1 - \frac{j+1}{r_m}\right) \omega_{p_m}^{(r_m-j)}. \tag{2.16}$$

Note that

$$c_j^0 = \frac{1}{j!} \left(\frac{d}{dx_m}\right)^j \left(\frac{v_k^0}{dx_m}\right) \Big|_{x_m=0}. \tag{2.17}$$

According to (1.40), we have

$$\omega_{p_m}^{(r_m-j)} = \frac{1}{(r_m-1-j)!} \left[\left(\frac{\partial}{\partial x_m}\right)^{r_m-2-j} \left(\iota_{x_m}^* B\right) \Big|_{x_m=0} \right]. \tag{2.18}$$

So, using the general Leibniz rule [1] , we can rewrite (2.16) in the following form:

$$\frac{\partial}{\partial \lambda_m} v_k = \frac{1}{r_m(r_m-2)!} \left\{ \left(\frac{\partial}{\partial x_m}\right)^{r_m-2} \left[\left(\frac{v_k^0}{dx_m}\right) \left(\iota_{x_m}^* B\right) \right] \Big|_{x_m=0} \right\}. \tag{2.19}$$

There is an alternative way to rewrite (2.16) in terms of Bergman bidifferential. After substituting (2.18) into (2.16), we have

$$\frac{\partial}{\partial \lambda_m} v_k = \frac{1}{2\pi i} \sum_{j=0}^{r_m-2} \oint_{s_m} \frac{1}{r_m(r_m-2-j)!} \left(\frac{c_j^0}{x_m}\right) \left[\left(\frac{\partial}{\partial x_m}\right)^{r_m-2-j} \left(\iota_{x_m}^* B\right) \Big|_{x_m=0} \right] dx_m . \tag{2.20}$$

Now, we can use "integration by parts" to get the following relation

$$\begin{aligned} \frac{\partial}{\partial \lambda_m} v_k &= \frac{1}{2\pi i} \sum_{j=0}^{r_m-2} \oint_{s_m} \frac{1}{r_m} \left(\frac{c_j^0}{{x_m}^{r_m-1-j}}\right) \left(\iota_{x_m}^* B\right) dx_m \\ &= \frac{1}{2\pi i} \oint_{s_m} \frac{1}{r_m} \left(\frac{\sum_{j=0}^{r_m-2} c_j^0 x_m^j}{{x_m}^{r_m-1}}\right) \left(\iota_{x_m}^* B\right) dx_m \\ &= \frac{1}{2\pi i} \oint_{s_m} \left(\frac{v_k^0}{df^0}\right) \Big|_{V_m} \left(\iota_{x_m}^* B\right) dx_m, \end{aligned} \tag{2.21}$$

[1] Let f and g be two smooth functions. The n-th derivative of their product is given by

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (D^k f)(D^{n-k} g),$$

where $\binom{n}{k}$ are the binomial coefficients.

which, after omitting the upper index 0, can be written in a coordinate-independent form as follows:

$$\left(\frac{\partial}{\partial\lambda_m}v_k\right)(p) = \frac{1}{2\pi i}\oint_{s_m}\frac{v_k(q)B(p,q)}{df(q)}, \tag{2.22}$$

where $q \in V_m$.

To get (2.8) it suffices to integrate (2.21) over the b-cycle b_l and use the following relation

$$\oint_{b_l}\iota_q^* B = \oint_{b_l}\omega_q^{(2)} = 2\pi i\left(\frac{v_l}{dx_m}\right)(q), \quad q \in V_m. \tag{2.23}$$

$\square$

Chapter 3
Variational Formulas on Spaces of Abelian Differentials over Compact Riemann Surfaces

3.1 Space of Holomorphic Abelian Differentials

For integer $g \geq 2$ the space $\mathcal{H}_g$ of holomorphic Abelian differentials over compact Riemann surfaces of genus g is the moduli space of pairs (C, ω), where C is a compact Riemann surface of genus g and ω is a holomorphic 1-form on C which is not identically equal to zero. The space $\mathcal{H}_g$ is a fiber bundle over the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g. Its fiber over $[C] \in \mathcal{M}_g$ is the punctured vector space of holomorphic 1-forms on C, denoted by $H^0(C, \Omega^1_C) \setminus \{0\}$ [6]. It is well known that the dimension of $\mathcal{M}_g$ and $H^0(C, \Omega^1_C)$ is equal to $3g-3$ and g, respectively. So $\mathcal{H}_g$ is a complex orbifold of dimension $4g-3$.

Let (ω) be the divisor of a holomorphic 1-form ω on C. As a known corollary of the Riemann-Roch theorem, we have

$$deg(\omega) = 2g - 2\,. \tag{3.1}$$

The space $\mathcal{H}_g$ is stratified according to multiplicities of zeroes of ω. Let $k_1, \cdots, k_n$ be a sequence of positive integers with $\sum_{i=1}^n k_i = 2g-2$. We denote by $\mathcal{H}_g(k_1, \cdots, k_n)$ the stratum of $\mathcal{H}_g$ consisting of pairs (C, ω) where ω has exactly n zeroes $\{p_1, \cdots, p_n\}$ such that their multiplicities are equal to $k_1, \cdots, k_n$. The principle stratum $\mathcal{H}_g(1, 1, \cdots, 1)$ is of dimension $4g-3$. For an arbitrary stratum $\mathcal{H}_g(k_1, \cdots, k_n)$ we have

$$\dim_{\mathbb{C}} \mathcal{H}_g(k_1, \cdots, k_n) = (4g-3) - \sum_{i=1}^{n} (k_i - 1) = 2g + n - 1\,. \tag{3.2}$$

Let $\mathcal{Z} = \{p_i\}_{i=1,\cdots,n}$ be a finite set of points $p_i \in C$. Let $\{a_l, b_l\}_{l=1,\cdots,g}$ be a canonical basis of the absolute homology group $H_1(C; \mathbb{Z})$. We assume that all the simple closed curves $\{a_l, b_l\}_{l=1,\cdots,g}$ pass through one point $p_0 \in C$. Let the "fundamental polygon" $\widetilde{C}$ be the simply connected open subset of the compact Riemann surface C that we get after cutting C along the curves $\{a_l, b_l\}_{l=1,\cdots,g}$. We choose $n-1$ paths $\gamma_m \subset \widetilde{C}$ which connect the point $p_1 \in \mathcal{Z}$ to the other points $p_m \in \mathcal{Z} \quad m = 2, \cdots, n$. The following set of paths gives a basis of the relative homology group $H_1(C, \mathcal{Z}; \mathbb{Z})$:

$$\{a_l, b_l, \gamma_m\} \qquad l = 1, \cdots, g;\ m = 2, \cdots, n. \tag{3.3}$$

We can use the basis $\{a_l, b_l, \gamma_m\}$ to get a set of local coordinates $\{A_l, B_l, z_m\}$ on $\mathcal{H}_g(k_1, \cdots, k_n)$ defined as follows (see [6], [5]):

$$A_l := \oint_{a_l} \omega, \quad B_l := \oint_{b_l} \omega, \quad z_m := \int_{\gamma_m} \omega, \tag{3.4}$$

$l = 1, \cdots, g;\ m = 2, \cdots, n.$

To shorten the notations, we denote the basis $\{a_l, b_l, \gamma_m\}$ by $\{s_j\}_{j=1,\cdots,2g+n-1}$, where

$$\begin{aligned} s_l := a_l\,, \quad s_{g+l} :&= b_l\,, \quad l = 1, \cdots, g\,, \\ s_{2g+m-1} :&= \gamma_m\,, \quad m = 2, \cdots, n. \end{aligned} \tag{3.5}$$

Also the local coordinates $\{A_l, B_l, z_m\}$ are denoted by $\{\zeta_j\}_{j=1,\cdots,2g+n-1}$, where

$$\zeta_j := \int_{s_j} \omega. \tag{3.6}$$

Let $\{\widetilde{\gamma}_m\}_{m=2,\cdots,n}$ be a set of small loops $\widetilde{\gamma}_m$ with positive orientation around the point $p_m \in \mathcal{Z}$, $m = 2, \cdots, n$. The following set $\{s_j^*\}_{j=1,\cdots,2g+n-1}$ gives a basis, dual to (3.5), of the homology group $H_1(C \setminus \mathcal{Z}; \mathbb{Z})$:

$$\begin{aligned} s_l^* := -b_l\,, \quad s_{g+l}^* :&= a_l\,, \quad l = 1, \cdots, g\,, \\ s_{2g+m-1}^* :&= \widetilde{\gamma}_m\,, \quad m = 2, \cdots, n\,. \end{aligned} \tag{3.7}$$

The intersection numbers of the bases (3.5) and (3.7) are given by

$$s_i^* \circ s_j = \delta_{ij}, \quad i, j = 1, \cdots, 2g + n - 1\,. \tag{3.8}$$

3.2 Variational Formulas

Let $\{\zeta_j\}_{j=1,\cdots,2g+n-1}$ be the set of local coordinates in a neighbourhood of the point $(C,\omega)\in\mathcal{H}_g(k_1,\cdots,k_n)$. The complex structure of C generically changes under the variation of ζ_j. We want to investigate the variation of the normalized basis $\{v_\alpha\}_{\alpha=1,\cdots,g}$ of the space of holomorphic 1-forms on C and the matrix of b-periods $\mathbb{B}=[\mathbb{B}_{\alpha\beta}]$ under the variation of the coordinate ζ_j.

Let $\mathcal{Z}=\{p_i\}_{i=1,\cdots,n}$ be the set of zeroes of the holomorphic Abelian differential ω, where p_m is a zero of multiplicity $k_m\,,\quad m=1,\cdots,n$. The Abelian integral $z(p)=\int_{p_1}^{p}\omega$ is a well-defined function on the fundamental polygon $\widetilde{C}$. For each point $q\in\widetilde{C}\setminus\mathcal{Z}$, there exists an open neighbourhood $\widetilde{V}\subset\widetilde{C}$ such that the function $z(p)=\int_q^p\omega,\quad p\in\widetilde{V}$ is univalent and provides a local coordinate on $\widetilde{V}$. The local coordinate $x_i:V_i\to\mathbb{C}$ on a neighbourhood $V_i\subset\widetilde{C}$ of $p_i\in\mathcal{Z}\,,\quad i=1,\cdots,n$ is given by

$$x_i(p)=\left(\int_{p_1}^{p}\omega-\int_{p_1}^{p_i}\omega\right)^{1/(k_i+1)}=(z(p)-z_i)^{1/(k_i+1)},\quad p\in V_i. \tag{3.9}$$

Consider the local universal family $\pi:\mathcal{X}\to\mathcal{H}_g(k_1,\cdots,k_n)$. Its fiber over the point $(C,\omega)\in\mathcal{H}_g(k_1,\cdots,k_n)$ is the Riemann surface C. The local coordinates on $\mathcal{X}\setminus(\omega)$ are given by the set $\{z(p):=\int^p\omega,\zeta_1,\cdots,\zeta_{2g+n-1}\}$. A vicinity of a point $((C,\omega),p)$ in the level set

$$H_{z(p)}:=\{x\in\mathcal{X}|z(x)=z(p)\} \tag{3.10}$$

is biholomorphically mapped onto a vicinity of the point $(C,\omega)\in\mathcal{H}_g(k_1,\cdots,k_n)$ via the projection $\pi:\mathcal{X}\to\mathcal{H}_g(k_1,\cdots,k_n)$.

The 1-forms v_α and ω are two global sections of the canonical line bundle $\mathcal{K}_C$. So there exists a function

$$\phi_\alpha:=\frac{v_\alpha}{\omega}:\mathcal{X}\to\mathbb{CP}^1 \tag{3.11}$$

such that $\left((\pi|_{H_{z(p)}})^{-1}\right)^*\left(\phi_\alpha|_{H_{z(p)}}\right)$ is a local holomorphic function on the stratum $\mathcal{H}_g(k_1,\cdots,k_n)$. The derivative of the normalized Abelian differential v_α with respect to ζ_j is defined as follows [5]:

$$\begin{aligned}\frac{\partial v_\alpha(p)}{\partial\zeta_j}\Big|_{z(p)}:&=\left\{\frac{\partial}{\partial\zeta_j}\left[\left((\pi|_{H_{z(p)}})^{-1}\right)^*\left(\phi_\alpha|_{H_{z(p)}}\right)\right]\right\}\omega(p)\\ &=\left[\frac{\partial}{\partial\zeta_j}\Big|_{z(p)=const}\phi_\alpha\right]\omega(p)\end{aligned} \tag{3.12}$$

Note that the map $p \mapsto z(p)$ is not globally defined on C. Therefore, the 1-forms $\partial v_\alpha(p)/\partial\zeta_j$ are local meromorphic differentials defined within $\widetilde{C}$. They do not necessarily correspond to global 1-forms on C itself.

Theorem 3.2.1. *The following variational formulas hold:*

$$\frac{\partial v_\alpha(p)}{\partial \zeta_j}\Big|_{z(p)} = \frac{1}{2\pi i}\oint_{s_j^*} \frac{v_\alpha(q) B(q,p)}{\omega(q)}\,; \tag{3.13}$$

$$\frac{\partial \mathbb{B}_{\alpha\beta}}{\partial \zeta_j} = \oint_{s_j^*} \frac{v_\alpha v_\beta}{\omega}\,, \tag{3.14}$$

where $j = 1, \cdots, 2g+n-1$. *We assume that the local coordinate* $z(p) = \int^p \omega$ *is kept constant under differentiation.*

Proof. We start with the proof of formulas (3.13) for $j = 1, \cdots, 2g$. For example, consider the derivative of v_α with respect to the coordinate B_l (defined in (3.4)). First, we analyse the 1-form $\partial v_\alpha/\partial B_l$ on the fundamental polygon $\widetilde{C}$. Let $\mathcal{Z} = \{p_i\}_{i=1,\cdots,n}$ be the set of zeroes of ω, where p_i is a zero of multiplicity $k_i\,,\quad i = 1, \cdots, n$. Let $x_i(p) = (z(p) - z_i)^{1/(k_i+1)}$ be the local coordinate, defined in (3.9), on a neighbourhood $V_i \subset \widetilde{C}$ of $p_i \in \mathcal{Z}\,,\quad i = 1, \cdots, n$. The holomorphic 1-forms ω and v_α restricted to V_i have the following local expressions:

$$\omega(x_i) = (k_i+1)x_i^{k_i}\,dx_i\,, \tag{3.15}$$

$$v_\alpha(x_i) = \left(\sum_{r=0}^{\infty} c_r (x_i)^r\right) dx_i\,, \tag{3.16}$$

where the coefficients c_r are holomorphic functions of the moduli parameters $\{\zeta_j\}$. So we get

$$\phi_\alpha|_{V_i} := \frac{v_\alpha(x_i)}{\omega(x_i)} = \frac{1}{k_i+1}\left(\sum_{r=0}^{\infty} c_r (x_i)^{r-k_i}\right). \tag{3.17}$$

Thus, according to (3.12), the local expression of the 1-form $\partial v_\alpha/\partial B_l$ restricted to $V_i \subset \widetilde{C}$ is given by

$$\begin{aligned}\left(\frac{\partial v_\alpha}{\partial B_l}\right)(x_i) &= \left[\frac{\partial}{\partial B_l}\Big|_{z(p)=const}\left(\sum_{r=0}^{\infty} c_r (x_i)^{r-k_i}\right)\right]\left(x_i^{k_i}\,dx_i\right)\\ &= \left(\sum_{r=0}^{\infty}\left(\frac{dc_r}{dB_l}\right)x_i^r\right)dx_i,\end{aligned} \tag{3.18}$$

which is a local holomorphic 1-form on V_i. Furthermore, similar calculations show that $\partial v_\alpha/\partial B_l$ restricted to an open neighbourhood $\widetilde{V} \subset \widetilde{C}$ of an arbitrary point $q \in \widetilde{C} \setminus \mathcal{Z}$ is holomorphic on $\widetilde{V}$. Therefore, the 1-form $\partial v_\alpha/\partial B_l$ is holomorphic on $\widetilde{C}$.

To understand global properties of $\partial v_\alpha/\partial B_l$ on C, we should analyse how it behaves near the boundary $\partial\widetilde{C}$. Let $\widehat{\pi} : U \to C$ be the universal covering of C. Let G be the group of deck transformations of U. Let $T : \pi_1(C, p_0) \to G$ be the group isomorphism of the fundamental group $\pi_1(C, p_0)$ onto G. The fundamental group $\pi_1(C, p_0)$ is generated by simple closed curves $\{a_l, b_l\}_{l=1,\cdots,g}$ based at $p_0 \in C$. Denote by T_{a_l} and T_{b_l} the deck transformations which correspond to a_l and b_l, respectively. The sides a_l^+ and a_l^- of the fundamental cell $\widetilde{C} \subset U$ are “glued" together by the deck transformation T_{b_l}.

The mapping $z(p) := \int^p \omega$ is single-valued on U. Consider an open neighbourhood $D \subset \widetilde{C}$ of an arbitrary point $q \in \widetilde{C} \setminus \mathcal{Z}$ such that the function $z(p) = \int^p \omega, \quad p \in D$ is univalent on D. Denote by $\widetilde{D}$ the image of D under the mapping $p \mapsto z(p)$. Consider the domain $T_{b_l}[D]$ lying in the fundamental cell $T_{b_l}[\widetilde{C}]$ as well as its image in the z-plane $\widetilde{D}_{b_l} = \{z + B_l | z \in \widetilde{D}\}$. We can always take sufficiently small domain D such that $\widetilde{D} \cap \widetilde{D}_{b_l} = \emptyset$.

Let $\widehat{v_\alpha} := \widehat{\pi}^*(v_\alpha)$ and $\widehat{\omega} := \widehat{\pi}^*(\omega)$ be the pull-back of v_α and ω under the covering map $\widehat{\pi} : U \to C$, respectively. Consider the meromorphic function $\widehat{\phi_\alpha} := \widehat{v_\alpha}/\widehat{\omega}$. Since the deck transformation T_{b_l} is a fiber-preserving biholomorphic map of U to itself, i.e. $\widehat{\pi} \circ T_{b_l} = \widehat{\pi}$, we have

$$T_{b_l}^* \widehat{\phi_\alpha} = \frac{T_{b_l}^* \widehat{v_\alpha}}{T_{b_l}^* \widehat{\omega}} = \frac{\widehat{v_\alpha}}{\widehat{\omega}} = \widehat{\phi_\alpha} \,. \tag{3.19}$$

The local expression of (3.19), restricted to the open neighbourhood $D \subset \widetilde{C}$, in terms of coordinate z is as follows::

$$\widehat{\phi_\alpha}(z + B_l) = \left(T_{b_l}^* \widehat{\phi_\alpha}\right)(z) = \widehat{\phi_\alpha}(z) \,, \quad z \in \widetilde{D} \,. \tag{3.20}$$

Note that in the right hand side of (3.20), the function $\widehat{\phi_\alpha}$ and its argument $z + B_l$ both depend on the moduli parameter B_l. Thus, if we differentiate (3.20) with respect to B_l while z is kept constant, then we get:

$$\frac{\partial \widehat{\phi_\alpha}}{\partial B_l}(z + B_l) = \frac{\partial \widehat{\phi_\alpha}}{\partial B_l}(z) - \frac{\partial \widehat{\phi_\alpha}}{\partial z}(z) \,, \quad z \in \widetilde{D} \,, \tag{3.21}$$

where we use the equality $\left(\partial\widehat{\phi_\alpha}/\partial z\right)(z+B_l)=\left(\partial\widehat{\phi_\alpha}/\partial z\right)(z)$ as a corollary of (3.20). The local expression of $\widehat{\omega}$ restricted to D is given by $\widehat{\omega}|_D = dz$. So we can rewrite (3.21) in the following form:

$$T_{b_l}^*\left(\frac{\partial\widehat{\phi_\alpha}}{\partial B_l}(z)dz\right)=\left(\frac{\partial\widehat{\phi_\alpha}}{\partial B_l}(z)dz\right)-\left(\frac{\partial\widehat{\phi_\alpha}}{\partial z}(z)dz\right)\,,\quad z\in\widetilde{D}\,. \tag{3.22}$$

Denote by Φ_α the derivative of $\widehat{v_\alpha}$ with respect to B_l. The holomorphic 1-form Φ_α on U is given by

$$\Phi_\alpha(p):=\frac{\partial\widehat{v_\alpha}(p)}{\partial B_l}\Big|_{z(p)}=\left[\frac{\partial}{\partial B_l}\Big|_{z(p)=const}\widehat{\phi_\alpha}\right]\widehat{\omega}(p)\,,\quad p\in U\,. \tag{3.23}$$

Using (3.23), the equality (3.22) can be written in the following coordinate-independent form:

$$\left(T_{b_l}^*\,\Phi_\alpha\right)(p)=\Phi_\alpha(p)-d\widehat{\phi_\alpha}(p)\,,\quad p\in D\,. \tag{3.24}$$

Let $s_j\in\{a_i,b_i\}_{i=1,\cdots,g},\quad j\neq g+l$ be one of the generators of the fundamental group $\pi_1(C,p_0)$ which is not homotopic to b_l. Since $\widehat{\phi_\alpha}$ is invariant under the deck transformation T_{s_j}, we have

$$\widehat{\phi_\alpha}(z+\zeta_j)=\left(T_{s_j}^*\,\widehat{\phi_\alpha}\right)(z)=\widehat{\phi_\alpha}(z)\,,\quad z\in\widetilde{D}\,, \tag{3.25}$$

where $\zeta_j:=\int_{s_j}\omega$. Differentiating (3.25) with respect to B_l assuming z to be constant, we get:

$$\frac{\partial\widehat{\phi_\alpha}}{\partial B_l}(z+\zeta_j)=\frac{\partial\widehat{\phi_\alpha}}{\partial B_l}(z)\,,\quad z\in\widetilde{D}\,. \tag{3.26}$$

Therefore, we have

$$\left(T_{b_i}^*\,\Phi_\alpha\right)(p)=\Phi_\alpha(p)\,,\quad i\in\{1,\cdots,g\}\text{ and }i\neq l\,,\quad p\in D \tag{3.27}$$

and

$$\left(T_{a_i}^*\,\Phi_\alpha\right)(p)=\Phi_\alpha(p)\,,\quad i\in\{1,\cdots,g\}\,,\quad p\in D\,. \tag{3.28}$$

Since the formulas (3.24), (3.27) and (3.28) are valid on a neighbourhood D of any point of $\widetilde{C}\setminus\mathcal{Z}$, and the differential Φ_α is holomorphic on $\widetilde{C}$, we conclude that these formulas are valid on $\widetilde{C}$. Since Φ_α is not invariant under the group of deck transformations of U, the 1-form $\partial v_\alpha/\partial B_l$ is not globally defined on C. Indeed, we can consider $\partial v_\alpha/\partial B_l$ as a 1-form which is holomorphic everywhere except the cycle a_l, where it has the additive jump given by the exact form $-d\left(v_\alpha/\omega\right)$.

Consider the cycle $a_i,\ i \neq l$. Consider a finite number of open sets $\widetilde{V}_r \subset C$, $r = 1, \cdots, k$ such that $a_i \subset \bigcup_{r=1}^{k} \widetilde{V}_r$, and $z_r(p) = \int^p \omega, \quad p \in \widetilde{V}_r$ provides local coordinate on $\widetilde{V}_r$. There exists a finite set of smooth functions $\{\psi_r\}_{r=1,\cdots,k}$, called *partition of unity*, with the following properties (see [10], p.54):

(i) $0 \leq \psi_r(p) \leq 1$ for all $r \in \{1, \cdots, k\}$ and all $p \in a_i$,

(ii) supp$(\psi_r) \subset \widetilde{V}_r$,

(iii) $\sum_{r=1}^{k} \psi_r(p) = 1$ for all $p \in a_i$.

On each $\widetilde{V}_r$, the local expression of ω and v_α is given by

$$\omega|_{\widetilde{V}_r} = dz_r \quad \text{and} \quad v_\alpha|_{\widetilde{V}_r} = \phi_{\alpha r}\, dz_r\, . \tag{3.29}$$

Now we can compute the a_i-period of $\partial v_\alpha / \partial B_l$ as follows:

$$\begin{aligned}
\oint_{a_i} \frac{\partial v_\alpha}{\partial B_l} &= \oint_{a_i} \left(\frac{\partial}{\partial B_l}\Big|_{z(p)=const} \left(\frac{v_\alpha}{\omega} \right) \right) \omega(p) \\
&= \sum_{r=1}^{k} \int_{a_i \cap \widetilde{V}_r} \left(\frac{\partial}{\partial B_l}\Big|_{z(p)=const} (\psi_r \phi_{\alpha r}) \right) dz_r \\
&= \frac{\partial}{\partial B_l} \left(\sum_{r=1}^{k} \int_{a_i \cap \widetilde{V}_r} (\psi_r \phi_{\alpha r})\, dz_r \right) \\
&= \frac{\partial}{\partial B_l} \left(\oint_{a_i} v_\alpha \right) = \frac{\partial}{\partial B_l} (\delta_{i\alpha}) = 0.
\end{aligned} \tag{3.30}$$

The integral of the “jump differential" $-d\,(v_\alpha/\omega)$ over any closed loop is equal to zero. Therefore, all the a_i-periods $i = 1, \cdots, g$ of $\partial v_\alpha / \partial B_l$ are well-defined and equal to zero.

To write down an explicit formula for $\partial v_\alpha / \partial B_l$, we recall the Plemelj formula on the complex plane $\mathbb{C}$. Let γ be a positively oriented simple closed curve in the complex plane. Let $f(x)$ be a holomorphic function defined on a tubular neighbourhood of γ. Define the function $F(y),\ y \in \mathbb{C} \setminus \gamma$ by the following contour integral taken in positive direction of γ:

$$F(y) := \oint_\gamma \frac{f(x)}{(x-y)^2} dx\, . \tag{3.31}$$

Denote by $F^{(R)}(y)$ and $F^{(L)}(y)$ the function $F(y)$ restricted to the exterior and the interior of γ, respectively. The boundary values of holomorphic functions $F^{(R)}(y)$ and $F^{(L)}(y)$ are related by the following Plemelj formula [5]:

$$\lim_{y\to x}\left(F^{(R)}(y)-F^{(L)}(y)\right)=-\frac{df}{dy}(x)\,. \tag{3.32}$$

Consider the 1-form Ψ defined as follows:

$$\Psi(p):=\frac{1}{2\pi i}\oint_{a_l}\phi_\alpha(q)B(q,p)\,, \tag{3.33}$$

where $B(q,p)$ is the Bergman bidifferential, and $\phi_\alpha:=v_\alpha/\omega$. Consider a small strip $\mathcal{A}\subset C$ around a_l in the shape of an annulus. Let $a_l^{(R)}$ and $a_l^{(L)}$ be the right and left parts of $\mathcal{A}\setminus a_l$. Denote by $\widetilde{\mathcal{A}}$ the biholomorphic image of $\mathcal{A}$ in the complex plane. The local expression of Ψ restricted to $\mathcal{A}$ is given by

$$\Psi(y)=\left(\oint_{a_l}\frac{\phi_\alpha(x)}{(x-y)^2}dx\right)dy\,,\quad x,y\in\widetilde{\mathcal{A}}\,. \tag{3.34}$$

Thus, according to Plemelj formula (3.32), the 1-form $\Psi(p)$ has a “jump" equal to $-d\phi_\alpha$ on a_l as p moves from $a_l^{(L)}$ to $a_l^{(R)}$. In addition, according to (1.38), all the a_i-periods $i=1,\cdots,g$ of Ψ are equal to zero.

The 1-forms $\partial v_\alpha/\partial B_l$ and Ψ are holomorphic on $\widetilde{C}$ and have the same discontinuity on a_l. Also, all of their a-periods are equal to zero. So their difference $\Upsilon:=\Psi-(\partial v_\alpha/\partial B_l)$ is a global holomorphic 1-form on C. Since all the a-periods of Υ are equal to zero, we have $\Upsilon=0$. Therefore, the 1-form $\partial v_\alpha/\partial B_l$ is given by

$$\frac{\partial v_\alpha(p)}{\partial B_l}\Big|_{z(p)}=\frac{1}{2\pi i}\oint_{a_l}\frac{v_\alpha(q)B(q,p)}{\omega(q)}\,. \tag{3.35}$$

Formula (3.35) implies (3.13) for $j=g+1,\cdots,2g$.

With similar calculations, we find that the 1-form $\partial v_\alpha(p)/\partial A_l$ has a “jump" equal to $-d(v_\alpha/\omega)$ on the cycle b_l as p moves from $b_l^{(R)}$ to $b_l^{(L)}$. Also, all of its a-periods are equal to zero. So, we have

$$\frac{\partial v_\alpha(p)}{\partial A_l}\Big|_{z(p)}=\frac{1}{2\pi i}\oint_{-b_l}\frac{v_\alpha(q)B(q,p)}{\omega(q)}\,. \tag{3.36}$$

Note that, according to (3.32), we should take the corresponding integral along $-b_l$. Indeed, the interchange of “right" and “left" in this case is due to the asymmetry of the intersection number of a_l and b_l, i.e.

$$a_l\circ b_l=-(b_l\circ a_l)=1\,. \tag{3.37}$$

Let us now prove the formulas (3.13) for $j = 2g+1, \cdots, 2g+n-1$. The proof is parallel to the proof of (2.7). For example, consider the derivative of v_α with respect to the coordinate z_m (defined in (3.4)). First, we analyse the 1-form $\partial v_\alpha / \partial z_m$ on the fundamental polygon $\widetilde{C}$. Using (3.17), the function $\phi_\alpha := v_\alpha/\omega$ restricted to the open neighbourhood $V_m \subset \widetilde{C}$ of the point $p_m \in \mathcal{Z}$ is given by

$$\phi_\alpha|_{V_m} = \frac{1}{k_m+1}\left(\sum_{r=0}^{\infty} c_r (x_m)^{r-k_m}\right) = \frac{1}{k_m+1}\left(\sum_{r=0}^{\infty} c_r (z-z_m)^{\frac{r-k_m}{k_m+1}}\right) \tag{3.38}$$

If we differentiate (3.38) with respect to z_m for fixed $z(p)$, then we get:

$$\begin{aligned}\frac{\partial \phi_\alpha(x_m)}{\partial z_m}\Big|_{z(p)=const} &= \frac{1}{k_m+1}\left(\sum_{r=0}^{\infty} c_r \left(\frac{k_m - r}{k_m+1}\right)(z-z_m)^{\frac{r-1-2k_m}{k_m+1}}\right) \\ &+ \frac{1}{k_m+1}\left(\sum_{r=0}^{\infty}\left(\frac{dc_r}{dz_m}\right)(z-z_m)^{\frac{r-k_m}{k_m+1}}\right).\end{aligned} \tag{3.39}$$

Therefore, according to (3.12), the 1-form $\partial v_\alpha/\partial z_m$ restricted to V_m has the following form:

$$\left(\frac{\partial v_\alpha}{\partial z_m}\right)(x_m) = \left[\left(\sum_{r=0}^{k_m-1} c_r \left(1 - \frac{r+1}{k_m+1}\right)\frac{1}{x_m^{k_m+1-r}}\right) + O(1)\right] dx_m\,. \tag{3.40}$$

With similar calculations, we find that $\partial v_\alpha/\partial z_m$ is holomorphic on a neighbourhood $\widetilde{V} \subset \widetilde{C}$ of each point $q \in \widetilde{C} \setminus \{p_m\}$.

Now we analyse the boundary behaviour of $\partial v_\alpha/\partial z_m$ near $\partial\widetilde{C}$. Let $s_j \in \{a_i, b_i\}_{i=1,\cdots,g}$ be one of the generators of the fundamental group $\pi_1(C, p_0)$. Since the function $\widehat{\phi_\alpha} := (\widehat{\pi}^* v_\alpha)/(\widehat{\pi}^*\omega)$ is invariant under all of the deck transformations T_{s_j}, we have

$$\widehat{\phi_\alpha}(z+\zeta_j) = \left(T_{s_j}^*\,\widehat{\phi_\alpha}\right)(z) = \widehat{\phi_\alpha}(z)\,, \quad z \in \widetilde{D} \subset \widetilde{C}\,, \tag{3.41}$$

where $\zeta_j := \int_{s_j}\omega\,, \quad j = 1, \cdots, 2g$. By differentiating (3.41) with respect to z_m while $z(p)$ is kept constant, we get:

$$\frac{\partial\widehat{\phi_\alpha}}{\partial z_m}(z+\zeta_j) = \frac{\partial\widehat{\phi_\alpha}}{\partial z_m}(z)\,, \quad z \in \widetilde{D}\,. \tag{3.42}$$

Therefore, $\partial v_\alpha/\partial z_m$ is a meromorphic global 1-form on C with only one pole of order k_m+1 at p_m. Similarly to (3.30), we find that all the a-periods of $\partial v_\alpha/\partial z_m$ are equal to zero. Thus, the 1-form $\partial v_\alpha/\partial z_m$ can be expressed in terms of normalized Abelian differentials of second kind $\omega_{p_m}^{(r)}$

as follows :

$$\frac{\partial v_\alpha}{\partial z_m} = \sum_{r=0}^{k_m-1} c_r \left(1 - \frac{r+1}{k_m+1}\right) \omega_{p_m}^{(k_m+1-r)}. \tag{3.43}$$

Using (1.40), we can rewrite (3.43) in the following way:

$$\frac{\partial v_\alpha}{\partial z_m} = \tag{3.44}$$
$$\frac{1}{2\pi i} \sum_{r=0}^{k_m-1} \oint_{\widetilde{\gamma}_m} \frac{1}{(k_m+1)(k_m-1-r)!} \left(\frac{c_r}{x_m}\right) \left[\left(\frac{\partial}{\partial x_m}\right)^{k_m-1-r} \left(\iota_{x_m}^* B\right) \big|_{x_m=0} \right] dx_m \,,$$

where $\widetilde{\gamma}_m$ is a small loop with positive orientation around the point p_m. Now using “integration by parts" in the same way as (2.21), we get:

$$\begin{aligned} \frac{\partial v_\alpha}{\partial z_m}(p) &= \frac{1}{2\pi i} \oint_{\widetilde{\gamma}_m} \left(\frac{v_\alpha}{\omega}\right) \Big|_{V_m} \left(\iota_{x_m}^* B\right)(p)\, dx_m \\ &= \frac{1}{2\pi i} \oint_{\widetilde{\gamma}_m} \frac{v_\alpha(q) B(q,p)}{\omega(q)} \,, \end{aligned} \tag{3.45}$$

which implies (3.13) for $j = 2g+1, \cdots, 2g+n-1$.

According to (1.37), if we integrate (3.13) over the cycle b_β and change the order of integration, we get (3.14).

□

Chapter 4
Variational Formulas on the Space of Quadratic Differentials over Compact Riemann Surfaces

Let C be a compact Riemann surface of genus g. Let $\mathcal{K}_C$ be the canonical line bundle of C. A global holomorphic section of the line bundle $\mathcal{K}_C \otimes \mathcal{K}_C$ is called a *holomorphic quadratic differential* on C. The set of all holomorphic quadratic differentials on C forms a complex vector space denoted by $\mathrm{QD}(C)$. As a corollary of the Riemann-Roch theorem, we have

$$\dim_{\mathbb{C}} \mathrm{QD}(C) = 3g - 3, \quad g \geq 2\,. \tag{4.1}$$

Let $\mathcal{Z} = \{p_i\}_{i=1,\cdots,n}$ be the set of zeroes of a holomorphic quadratic differential $q \in \mathrm{QD}(C) \setminus \{0\}$, where $p_i \in C$ is a zero of multiplicity k_i , $i = 1, \cdots, n$. According to (3.1), the degree of the canonical divisor on C is equal to $2g - 2$; thus $\sum_{i=1}^n k_i = 4g - 4$.

For integer $g \geq 2$ the space $\mathcal{Q}_g$ of holomorphic quadratic differentials over compact Riemann surfaces of genus g is the moduli space of pairs (C, q), where C is a compact Riemann surface of genus g and $q \in \mathrm{QD}(C)$ is a holomorphic quadratic differentials on C. The space $\mathcal{Q}_g$ is the cotangent bundle of the Teichmuller space $\mathcal{T}_g$ of marked compact Riemann surfaces of genus g.

The space $\mathcal{Q}_g$ is stratified according to the multiplicities of zeroes of holomorphic quadratic differentials. Let $k_1, \cdots, k_n$ be a sequence of positive integers such that $\sum_{i=1}^n k_i = 4g - 4$. We denote by $\mathcal{Q}_g(k_1, \cdots, k_n)$ the stratum of $\mathcal{Q}_g$ consisting of pairs (C, q) where q has exactly n zeroes $\{p_1, \cdots, p_n\}$ with multiplicities $k_1, \cdots, k_n$.

4.1 Geometry of the canonical double covering

Consider an arbitrary holomorphic quadratic differential $q \in \mathrm{QD}(C) \setminus \{0\}$ on C. In general, there is no global holomorphic Abelian differential $v \in H^0(C, \Omega_C^1)$ such that $q = v \otimes v$.

To represent q as a square of an Abelian differential, we construct a canonical double covering $\pi : \widehat{C} \to C$ of C provided with a holomorphic Abelian differential $\omega \in H^0(\widehat{C}, \Omega^1_{\widehat{C}})$ on $\widehat{C}$ such that $\pi^*(q) = \omega \otimes \omega$ [1].

Let $\mathcal{Z} = \{p_i\}_{i=1,\cdots,n}$ be the set of zeroes of q, where $p_i \in C$ is a zero of multiplicity k_i, $i = 1, \cdots, n$. Consider an atlas $\{(U_j, z_j)\}_{j\in J}$ on the punctured Riemann surface $C \setminus \mathcal{Z}$, where $z_j : U_j \to \mathbb{C}$ is a local coordinate on the connected and simply-connected open neighbourhood $U_j \subset C \setminus \mathcal{Z}$. The quadratic differential q restricted to U_j is given by $q|_{U_j} = f_j(z_j)(dz_j)^2$, where the nonzero holomorphic functions $f_j \in \mathcal{O}^*(U_j)$ satisfy the following relation:

$$f_j(z_j(z_l))\left(\frac{dz_j}{dz_l}\right)^2 = f_l(z_l)\,, \quad z_l \in U_j \cap U_l\,. \tag{4.2}$$

Denote by $h_j^{(k)}(z_j),\ k = 1, 2$ the two branches of $\sqrt{f_j(z_j)}$. Consider two copies $U_j^{(k)},\ k = 1, 2$ of each open neighbourhood U_j. For each j, consider the local holomorphic 1-forms $h_j^{(k)} dz_j,\ k = 1, 2$ defined on the open neighbourhoods $U_j^{(k)},\ k = 1, 2$. Now identify the part of $U_j^{(1)}$ corresponding to $U_j \cap U_l$ with the part of one of $U_l^{(k)},\ k = 1, 2$ corresponding to $U_j \cap U_l$ in such a way that

$$h_j^{(1)}(z_j(z_l))\left(\frac{dz_j}{dz_l}\right) = h_l^{(k)}(z_l)\,, \quad z_l \in U_j^{(1)} \cap U_l^{(k)}\,, \tag{4.3}$$

for either $k = 1$ or $k = 2$. Apply the similar identification to all the open neighbourhoods $\{U_j^{(1)}, U_j^{(2)}\}_{j\in J}$. We get a Riemann surface $\widehat{C}_0$ with punctures provided with a holomorphic Abelian differential ω_0 on it, where the local expression of ω_0 on $U_j^{(k)}$ is given by $h_j^{(k)} dz_j$. By construction we have a double covering $\pi_0 : \widehat{C}_0 \to C \setminus \mathcal{Z}$.

After "filling" the punctures lying over $\mathcal{Z} \subset C$, we obtain a compact Riemann surface $\widehat{C}$ and a (possibly ramified) double covering $\pi : \widehat{C} \to C$. The covering map π has a ramification point over each zero $p_m \in \mathcal{Z}$ of odd multiplicity $k_m = 2r + 1,\ r \in \mathbb{N} \cup \{0\}$ of the quadratic differential q. In addition, the 1-form ω_0 extends to a holomorphic Abelian differential ω on $\widehat{C}$ such that $\pi^*(q) = \omega \otimes \omega$. The 1-form ω is commonly denoted by $\sqrt{\pi^* q}$.

Let $\mu : \widehat{C} \to \widehat{C}$ be the biholomorphic involution of $\widehat{C}$ which interchanges the points in each fiber of $\pi : \widehat{C} \to C$. The fixed points of μ are the preimage of the zeroes of odd multiplicity of q under π. According to the above-mentioned construction, the holomorphic 1-form $\sqrt{\pi^* q}$ is anti-invariant under μ, i.e.

$$\mu^*\left(\sqrt{\pi^* q}\right) = -\sqrt{\pi^* q}\,. \tag{4.4}$$

Henceforth, we only consider the pairs (C, q) in the principal stratum $\mathcal{Q}_g\left([1^{4g-4}]\right)$ of the moduli space of holomorphic quadratic differentials. In other words, we shall study the space of holomorphic quadratic differentials with simple zeroes on compact Riemann surfaces of genus g.

Let $\mathcal{Z} = \{p_i\}_{i=1,\cdots,4g-4}$ be the set of zeroes of the holomorphic quadratic differential $q \in \mathrm{QD}(C)$. The canonical double covering $\pi : \widehat{C} \to C$ defined by q has exactly $4g-4$ ramification points $\{\widehat{p_i}\}_{i=1,\cdots,4g-4}$, where $\widehat{p_i} = \pi^{-1}(p_i)$. Using the Riemann-Hurwitz formula, we find that the compact Riemann surface $\widehat{C}$ is of genus $\widehat{g} = 4g-3$.

Let $\xi_i : D_i \to \mathbb{C}$ be the local coordinate on an open neighbourhood $D_i \subset C$ of $p_i \in \mathcal{Z}$ such that $\xi_i(p_i) = 0$ and the quadratic differential q restricted to D_i has the following local expression:

$$q|_{D_i} = \xi_i (d\xi_i)^2\,, \tag{4.5}$$

for $i = 1, \cdots, 4g-4$. The natural local coordinate $x_i : \widehat{D_i} \to \mathbb{C}$ on the open neighbourhood $\widehat{D_i} = \pi^{-1}(D_i) \subset \widehat{C}$ of $\widehat{p_i} \in \widehat{C}$ is given by

$$x_i = \sqrt{\xi_i}\,. \tag{4.6}$$

The local expression of the double covering map $\pi : \widehat{C} \to C$ restricted to $\widehat{D_i}$ has the following form:

$$\left(\xi_i \circ \pi \circ x_i^{-1}\right)(y) = y^2\,, \quad y \in x_i\left(\widehat{D_i}\right) \subset \mathbb{C}\,. \tag{4.7}$$

So we have

$$\begin{aligned} \pi^*\left(q|_{D_i}\right) &= \pi^*\left(\xi_i (d\xi_i)^2\right) \\ &= (x_i)^2 \left(d({x_i}^2)\right)^2 \\ &= 4(x_i)^4 (dx_i)^2\,. \end{aligned} \tag{4.8}$$

Therefore, the holomorphic 1-form $\sqrt{\pi^* q}$ has a zero of multiplicity two at each $\widehat{p_i} \in \widehat{C}$. Since the degree of the canonical divisor on $\widehat{C}$ is equal to $2\widehat{g} - 2 = 8g - 8$, the set $\widehat{\mathcal{Z}}$ of all the zeroes of $\sqrt{\pi^* q}$ is equal to $\{\widehat{p_i}\}_{i=1,\cdots,4g-4}$.

Thus, the correspondence between a pair $(C, q) \in \mathcal{Q}_g\left([1^{4g-4}]\right)$ and the canonical double covering $\pi : \widehat{C} \to C$ defined by q induce a mapping

$$\mathcal{S} : \mathcal{Q}_g\left(\left[1^{4g-4}\right]\right) \to \mathcal{H}_{4g-3}\left(\left[2^{4g-4}\right]\right) \tag{4.9}$$
$$(C, q) \mapsto (\widehat{C}, \sqrt{\pi^* q})$$

which is a local embedding. According to (3.2), the stratum $\mathcal{H}_{4g-3}\left([2^{4g-4}]\right)$ is of dimension $12g - 11$. Locally, the space $\mathcal{S}\left(\mathcal{Q}_g\left([1^{4g-4}]\right)\right)$ forms a subspace of codimension $6g - 5$ in $\mathcal{H}_{4g-3}\left([2^{4g-4}]\right)$.

Denote by $\Lambda^1_{\widehat{C}}$ the space $H^0(\widehat{C}, \Omega^1_{\widehat{C}})$ of holomorphic Abelian differentials on $\widehat{C}$. The pullback μ^* of the involution $\mu : \widehat{C} \to \widehat{C}$ is an isomorphism on $\Lambda^1_{\widehat{C}}$. The $(4g-3)$-dimensional vector space $\Lambda^1_{\widehat{C}}$ splits into two eigenspaces Λ_+ and Λ_- corresponding to the eigenvalues ± 1 of μ^*.

Let $z : U \to \mathbb{C}$ be the local coordinate on an arbitrary connected and simply-connected open subset $U \subset C \setminus \mathcal{Z}$. Let $\hat{z} := z \circ \pi : U^{(k)} \to \mathbb{C}$ be the induced local coordinate on the k-th connected component $U^{(k)} \subset \widehat{C}$ of $\pi^{-1}(U)$ for $k = 1, 2$. Consider an open neighbourhood $\widehat{D_i} \subset \widehat{C}$ of the ramification point $\widehat{p_i}$ of $\pi : \widehat{C} \to C$. Let $x_i : \widehat{D_i} \to \mathbb{C}$ be the natural local coordinate on $\widehat{D_i}$ defined in (4.6). Consider an arbitrary holomorphic 1-form $\widehat{\alpha}$ on $\widehat{C}$ which is invariant under μ, i.e. $\widehat{\alpha} \in \Lambda_+ \subset \Lambda^1_{\widehat{C}}$. Consider the following local expressions of $\widehat{\alpha}$ on $U^{(k)}$ and $\widehat{D_i}$:

$$\widehat{\alpha}|_{U^{(k)}} = g^{(k)} d\hat{z} , \quad g^{(k)} \in \mathcal{O}(U^{(k)}) , \quad k = 1, 2 , \tag{4.10}$$

$$\widehat{\alpha}|_{\widehat{D_i}} = \left(\sum_{r=0}^{\infty} c_r (x_i)^r \right) dx_i , \quad c_r \in \mathbb{C} . \tag{4.11}$$

Since $\mu(\widehat{D_i}) \subset \widehat{D_i}$ and $x_i \circ \mu = -x_i$, we have

$$\mu^* \left(\widehat{\alpha}|_{\widehat{D_i}} \right) = - \left(\sum_{r=0}^{\infty} c_r (-1)^r (x_i)^r \right) dx_i . \tag{4.12}$$

From the relation $\mu^* \widehat{\alpha} = \widehat{\alpha}$, we deduce that $g^{(1)} = g^{(2)} \circ \mu$ and

$$c_{2m} = 0, \quad m \in \mathbb{N} \cup \{0\} . \tag{4.13}$$

Thus, the local expressions (4.10) and (4.11) are simplified in the following way:

$$\widehat{\alpha}|_{U^{(k)}} = \pi^* \left(\phi(z) dz \right) , \quad k = 1, 2 , \quad \phi \in \mathcal{O}(U) , \tag{4.14}$$

$$
\begin{aligned}
\widehat{\alpha}|_{\widehat{D_i}} &= \left(\sum_{s=0}^{\infty} c_{2s+1}(x_i)^{2s}\right)(x_i dx_i) \\
&= \pi^*\left(\left(\sum_{s=0}^{\infty} \frac{c_{2s+1}}{2}(\xi_i)^s\right) d\xi_i\right) .
\end{aligned} \tag{4.15}
$$

According to (4.14) and (4.15), each 1-form $\widehat{\alpha} \in \Lambda_+ \subset \Lambda^1_{\widehat{C}}$ is the pullback of a holomorphic 1-form $\alpha \in H^0(C, \Omega^1_C)$ under the canonical double covering map $\pi : \widehat{C} \to C$, i.e. $\widehat{\alpha} = \pi^*\alpha$. On the other hand, since $\pi \circ \mu = \pi$, for each $\tilde{\alpha} \in H^0(C, \Omega^1_C)$ we have

$$
\mu^*\left(\pi^*\tilde{\alpha}\right) = (\pi \circ \mu)^*\tilde{\alpha} = \pi^*\tilde{\alpha} . \tag{4.16}
$$

Therefore, the canonical double covering map $\pi : \widehat{C} \to C$ induce the following linear isomorphism:

$$
\begin{aligned}
\pi^* : H^0(C, \Omega^1_C) &\to \Lambda_+ \subset \Lambda^1_{\widehat{C}} \\
\tilde{\alpha} &\mapsto \pi^*\tilde{\alpha}
\end{aligned} \tag{4.17}
$$

Considering the isomorphism (4.17), since

$$
\dim_{\mathbb{C}} H^0(C, \Omega^1_C) = g \quad \text{and} \quad \dim_{\mathbb{C}} H^0(\widehat{C}, \Omega^1_{\widehat{C}}) = 4g - 3 , \tag{4.18}
$$

we have

$$
\dim_{\mathbb{C}} \Lambda_- = 3g - 3 = \dim_{\mathbb{C}} \mathrm{QD}(C) . \tag{4.19}
$$

Consider an arbitrary holomorphic 1-form $\beta \in \Lambda_- \subset \Lambda^1_{\widehat{C}}$. Consider the following local expressions of β on $\widehat{D_i}$ and $U^{(k)}$:

$$
\beta|_{\widehat{D_i}} = \left(\sum_{r=0}^{\infty} \tilde{c}_r (x_i)^r\right) dx_i , \quad \tilde{c}_r \in \mathbb{C} , \tag{4.20}
$$

$$
\beta|_{U^{(k)}} = t^{(k)} d\hat{z} , \quad t^{(k)} \in \mathcal{O}(U^{(k)}) , \quad k = 1, 2 . \tag{4.21}
$$

From the relation $\mu^*\beta = -\beta$, we deduce that $t^{(1)} = -\left(t^{(2)} \circ \mu\right)$ and

$$
\tilde{c}_{2m+1} = 0, \quad m \in \mathbb{N} \cup \{0\} . \tag{4.22}
$$

Let $q|_U = f(z)(dz)^2,\ f \in \mathcal{O}^*(U)$ be the local expression of the quadratic differential q on $U \subset C$. Let $h^{(k)} \in \mathcal{O}^*(U^{(k)}),\ k = 1, 2$ be the two branches of $\sqrt{f}$. Using (4.8), we get the following local expressions of the 1-form $\sqrt{\pi^* q}$ on $\widehat{D_i}$ and $U^{(k)}$:

$$\sqrt{\pi^* q}|_{\widehat{D_i}} = 2(x_i)^2 dx_i\,, \tag{4.23}$$

$$\sqrt{\pi^* q}|_{U^{(k)}} = h^{(k)} d\hat{z}\,, \quad k = 1, 2\,, \tag{4.24}$$

where $h^{(1)} = -\left(h^{(2)} \circ \mu\right)$.

So the local expression of the quadratic differential $(\beta \otimes \sqrt{\pi^* q}) \in \mathrm{QD}(\widehat{C})$ on $\widehat{D_i}$ and $U^{(k)}$ are given by:

$$\begin{aligned}\left(\beta \otimes \sqrt{\pi^* q}\right)|_{\widehat{D_i}} &= 2\left(\sum_{s=0}^{\infty} \tilde{c}_{2s}(x_i)^{2s}\right)(x_i dx_i)^2 \\ &= \pi^*\left(\left(\sum_{s=0}^{\infty} \frac{\tilde{c}_{2s}}{2}(\xi_i)^s\right)(d\xi_i)^2\right)\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}\left(\beta \otimes \sqrt{\pi^* q}\right)|_{U^{(k)}} &= \left(t^{(k)} h^{(k)}\right)(d\hat{z})^2 \\ &= \pi^*\left(\psi(z)(dz)^2\right)\,, \quad k = 1, 2\,, \quad \psi \in \mathcal{O}(U)\,,\end{aligned} \tag{4.26}$$

respectively. According to (4.25) and (4.26), for each 1-form $\beta \in \Lambda_- \subset \Lambda^1_{\widehat{C}}$, there exists a unique quadratic differential $\Psi \in \mathrm{QD}(C)$ such that

$$\beta \otimes \sqrt{\pi^* q} = \pi^* \Psi\,. \tag{4.27}$$

Therefore, considering (4.19), there is an isomorphism between $\mathrm{QD}(C)$ and $\Lambda_- \subset \Lambda^1_{\widehat{C}}$ given by:

$$\begin{aligned} L : \mathrm{QD}(C) &\to \Lambda_- \\ \Psi &\mapsto \frac{\pi^* \Psi}{\sqrt{\pi^* q}}\,. \end{aligned} \tag{4.28}$$

The involution $\mu : \widehat{C} \to \widehat{C}$ induces an isomorphism

$$\mu_* : H_1(\widehat{C}; \mathbb{C}) \to H_1(\widehat{C}; \mathbb{C}) \tag{4.29}$$

on the complex homology group $H_1(\widehat{C};\mathbb{C})$ of $\widehat{C}$. The $(8g-6)$-dimensional complex vector space $H_1(\widehat{C};\mathbb{C})$ splits into two eigenspaces H_+ and H_- corresponding to the eigenvalues ± 1 of μ_*.

Let

$$\{a_j, a_j^*, \tilde{a}_k, b_j, b_j^*, \tilde{b}_k\}, \quad j = 1, \cdots, g\,;\ k = 1, \cdots, 2g-3 \tag{4.30}$$

be a set of $8g-6$ cycles on $\widehat{C}$ providing a canonical basis of $H_1(\widehat{C};\mathbb{C})$ such that

$$\mu_* a_j = a_j^*, \quad \mu_* b_j = b_j^*, \quad \mu_* \tilde{a}_k + \tilde{a}_k = \mu_* \tilde{b}_k + \tilde{b}_k = 0\,,$$

and their intersection matrix is given by

$$\begin{bmatrix} a \\ a^* \\ \tilde{a} \\ b \\ b^* \\ \tilde{b} \end{bmatrix} \circ \begin{bmatrix} a & a^* & \tilde{a} & b & b^* & \tilde{b} \end{bmatrix} = \begin{bmatrix} 0 & I_{4g-3} \\ -I_{4g-3} & 0 \end{bmatrix}.$$

Using the symplectic basis (4.30) of $H_1(\widehat{C};\mathbb{C})$, we construct a symplectic basis of the $(2g)$-dimensional eigenspace $H_+ \subset H_1(\widehat{C};\mathbb{C})$ in the following way:

$$\alpha_j^+ = \frac{1}{2}(a_j + a_j^*)\,, \quad \beta_j^+ = b_j + b_j^*\,, \quad j = 1, \cdots, g\,, \tag{4.31}$$

where

$$\begin{bmatrix} \alpha^+ \\ \beta^+ \end{bmatrix} \circ \begin{bmatrix} \alpha^+ & \beta^+ \end{bmatrix} = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}.$$

In addition, the following 1-cycles form a symplectic basis of the $(6g-6)$-dimensional eigenspace $H_- \subset H_1(\widehat{C};\mathbb{C})$:

$$\begin{aligned} \alpha_l^- &= \frac{1}{2}(a_l - a_l^*)\,, & \beta_l^- &= b_l - b_l^*\,, & l &= 1, \cdots, g\,, \\ \alpha_l^- &= \tilde{a}_{l-g}\,, & \beta_l^- &= \tilde{b}_{l-g}\,, & l &= g+1, \cdots, 3g-3\,. \end{aligned} \tag{4.32}$$

Let $\{u_j, u_j^*, \tilde{u}_k\}\,,\ j = 1, \cdots, g\,,\ k = 1, \cdots, 2g-3$ be the normalized basis of $\Lambda^1_{\widehat{C}}$ associated with the canonical basis (4.30). To shorten the notations, we consider the following vectors:

$$\mathrm{U} = \begin{bmatrix} u \\ u^* \\ \tilde{u} \end{bmatrix} \in \left(H^1(\widehat{C};\mathbb{C})\right)^{4g-3}; \quad \mathrm{A} = \begin{bmatrix} a \\ a^* \\ \tilde{a} \end{bmatrix}\,,\ \mathrm{B} = \begin{bmatrix} b \\ b^* \\ \tilde{b} \end{bmatrix} \in \left(H_1(\widehat{C};\mathbb{C})\right)^{4g-3}. \tag{4.33}$$

According to the definition of the bilinear period mapping

$$\Pi : \left(H^1(C;\mathbb{C})\right)^n \times \left(H_1(C;\mathbb{C})\right)^n \to \mathrm{M}(n,\mathbb{C})\,, \tag{4.34}$$

given by (1.32), we have

$$\Pi\left(\mathrm{U},\mathrm{A}\right) = I_{4g-3}\,. \tag{4.35}$$

The action of μ_* on the vector $\mathrm{A} \in \left(H_1(C;\mathbb{C})\right)^{4g-3}$ is given by $\mu_*\mathrm{A} = \mathrm{SA}$, where

$$\mathrm{S} = \begin{bmatrix} 0 & I_g & 0 \\ I_g & 0 & 0 \\ 0 & 0 & -I_{2g-3} \end{bmatrix}. \tag{4.36}$$

Suppose the action of μ^* on the vector $\mathrm{U} \in \left(H^1(C;\mathbb{C})\right)^{4g-3}$ is given by

$$\mu^*\mathrm{U} = \widetilde{\mathrm{S}}\mathrm{U}\,. \tag{4.37}$$

Since

$$\Pi\left(\mu^*\mathrm{U},\mathrm{A}\right) = \Pi\left(\mathrm{U},\mu_*\mathrm{A}\right)\,, \tag{4.38}$$

we have

$$\widetilde{\mathrm{S}}\,\Pi\left(\mathrm{U},\mathrm{A}\right) = \Pi\left(\widetilde{\mathrm{S}}\mathrm{U},\mathrm{A}\right) = \Pi\left(\mathrm{U},\mathrm{SA}\right) = \Pi\left(\mathrm{U},\mathrm{A}\right)\mathrm{S}^t\,. \tag{4.39}$$

Therefore, using (4.35), we get

$$\widetilde{\mathrm{S}} = \mathrm{S}^t = \mathrm{S}\,. \tag{4.40}$$

So the holomorphic 1-forms $u_j^+ = u_j + u_j^*\,,\ j = 1, \cdots, g$ provide a basis of the eigenspace Λ_+, whereas a basis of Λ_- is given by $\{u_l^-\}_{l=1,\cdots,3g-3}$, where

$$u_l^- = \begin{cases} u_l - u_l^*\,, & l = 1, \cdots, g\,, \\ \tilde{u}_{l-g}\,, & l = g+1, \cdots, 3g-3\,. \end{cases} \tag{4.41}$$

Considering (4.35), the 1-forms $\{u_j^+, u_l^-\}$ are normalized with respect to the 1-cycles $\{\alpha_j^+, \alpha_l^-\}$, i.e.

$$\Pi\left(\begin{bmatrix} u^+ \\ u^- \end{bmatrix}, \begin{bmatrix} \alpha^+ \\ \alpha^- \end{bmatrix}\right) = \begin{bmatrix} I_g & 0 \\ 0 & I_{3g-3} \end{bmatrix}. \tag{4.42}$$

Since

$$\Pi\left(u_j^+, \beta_l^-\right) = \Pi\left(\mu^* u_j^+, \beta_l^-\right) = \Pi\left(u_j^+, \mu_* \beta_l^-\right) = -\Pi\left(u_j^+, \beta_l^-\right)\,, \tag{4.43}$$

we have

$$\Pi\left(\begin{bmatrix} u^+ \\ u^- \end{bmatrix}, \begin{bmatrix} \beta^+ \\ \beta^- \end{bmatrix}\right) = \begin{bmatrix} \mathbb{B}^+ & 0 \\ 0 & \mathbb{B}^- \end{bmatrix}, \tag{4.44}$$

where $\mathbb{B}^+ \in \mathrm{M}(g, \mathbb{C})$ and $\mathbb{B}^- \in \mathrm{M}(3g-3, \mathbb{C})$ are called the matrices of β-periods [7]. Let

$$\widehat{\mathbb{B}} = \Pi\left(\mathrm{U}, \mathrm{B}\right) \tag{4.45}$$

be the matrix of b-periods of $\{u_j, u_j^*, \tilde{u}_k\}$ with respect to the canonical homology basis (4.30). We have

$$\begin{bmatrix} u^+ \\ u^- \end{bmatrix} = \mathrm{TU} \quad \text{and} \quad \begin{bmatrix} \beta^+ \\ \beta^- \end{bmatrix} = \mathrm{TB}\,, \tag{4.46}$$

where the matrix T is given by

$$\mathrm{T} = \begin{bmatrix} I_g & I_g & 0 \\ I_g & -I_g & 0 \\ 0 & 0 & I_{2g-3} \end{bmatrix}. \tag{4.47}$$

So the matrix $\widehat{\mathbb{B}}$ is related to $\mathbb{B}^+$ and $\mathbb{B}^-$ in the following way:

$$\widehat{\mathbb{B}} = \Pi\left(\mathrm{U}, \mathrm{B}\right) = \Pi\left(\mathrm{T}^{-1}\begin{bmatrix} u^+ \\ u^- \end{bmatrix}, \mathrm{T}^{-1}\begin{bmatrix} \beta^+ \\ \beta^- \end{bmatrix}\right) \tag{4.48}$$

$$= \mathrm{T}^{-1}\,\Pi\left(\begin{bmatrix} u^+ \\ u^- \end{bmatrix}, \begin{bmatrix} \beta^+ \\ \beta^- \end{bmatrix}\right)\left(\mathrm{T}^{-1}\right)^t$$
$$= \mathrm{T}^{-1}\begin{bmatrix} \mathbb{B}^+ & 0 \\ 0 & \mathbb{B}^- \end{bmatrix}\left(\mathrm{T}^{-1}\right)^t .$$

Consider the mapping $\pi_* : H_1(\widehat{C};\mathbb{C}) \to H_1(C;\mathbb{C})$ induced by the canonical double covering map $\pi : \widehat{C} \to C$. Let $\{\mathfrak{a}_m, \mathfrak{b}_m\}_{m=1,\cdots,g}$ be the canonical basis of the homology group $H_1(C;\mathbb{C})$ of C given by

$$\mathfrak{a}_m = \pi_*(a_m) = \pi_*(a_m^*) \quad \text{and} \quad \mathfrak{b}_m = \pi_*(b_m) = \pi_*(b_m^*)\,, \quad m = 1,\cdots,g\,. \tag{4.49}$$

Let the holomorphic 1-forms $\{v_n\}_{n=1,\cdots,g}$ be the basis of the g-dimensional vector space $H^0(C,\Omega^1_C)$ normalized by $\oint_{\mathfrak{a}_m} v_n = \delta_{nm}$.

According to (4.16), the 1-forms $\pi^* v_n\,,\ n = 1,\cdots,g$ are invariant under μ, i.e. $\pi^* v_n \in \Lambda_+ \subset \Lambda^1_{\widehat{C}}$. So each $\pi^* v_n$ has a unique expansion in the following way:

$$\pi^* v_n = \sum_{j=1}^{g} c_{nj}\, u_j^+\,, \quad c_{nj} \in \mathbb{C}\,. \tag{4.50}$$

Using (4.42), the period of r.h.s. and l.h.s. of (4.50) over the 1-cycle α_m^+, $m = 1,\cdots,g$ are given by

$$\Pi\left(\sum_{j=1}^{g} c_{nj}\, u_j^+, \alpha_m^+\right) = \sum_{j=1}^{g} c_{nj}\Pi\left(u_j^+, \alpha_m^+\right) = c_{nm} \tag{4.51}$$

and

$$\Pi\left(\pi^* v_n, \alpha_m^+\right) = \Pi\left(v_n, \pi_* \alpha_m^+\right) = \Pi\left(v_n, \mathfrak{a}_m\right) = \delta_{nm}\,, \tag{4.52}$$

respectively. Thus, we have

$$\pi^* v_n = u_n^+\,, \quad n = 1,\cdots,g\,. \tag{4.53}$$

Let $\mathbb{B} = [\mathbb{B}_{nm}]$ be the matrix of $\mathfrak{b}$-periods of $\{v_n\}$ with respect to the homology basis $\{\mathfrak{a}_m, \mathfrak{b}_m\}$, where $\mathbb{B}_{nm} = \Pi(v_n, \mathfrak{b}_m)\,,\ n, m = 1,\cdots,g\,.$ The matrix of β^+-periods $\mathbb{B}^+$ is related to $\mathbb{B}$ in the following way:

$$\begin{aligned}\mathbb{B}^+_{nm} = \Pi\left(u_n^+, \beta_m^+\right) = \Pi\left(\pi^* v_n, \beta_m^+\right) &= \Pi\left(v_n, \pi_* \beta_m^+\right) \\ &= 2\,\Pi\left(v_n, \mathfrak{b}_m\right) = 2\,\mathbb{B}_{nm}\,.\end{aligned} \tag{4.54}$$

4.2 Variational Formulas

For integer $g \geq 2$, consider the stratum $\mathcal{H}_{4g-3}\left([2^{4g-4}]\right)$ consisting of the pairs (X, ω), where X is a compact Riemann surface of genus $4g-3$ and ω is a holomorphic 1-form on X with $4g-4$ zeroes $\{\tilde{p}_i\}_{i=1,\cdots,4g-4}$ of multiplicity 2. Denote the set $\{\tilde{p}_i\}_{i=1,\cdots,4g-4}$ of zeros by $\widetilde{\mathcal{Z}}$. Let

$$\{a_j, a_j^*, \tilde{a}_k, b_j, b_j^*, \tilde{b}_k\}, \quad j = 1, \cdots, g\,;\; k = 1, \cdots, 2g-3 \tag{4.55}$$

be a canonical basis, i.e.

$$\begin{bmatrix} a \\ a^* \\ \tilde{a} \\ b \\ b^* \\ \tilde{b} \end{bmatrix} \circ \begin{bmatrix} a & a^* & \tilde{a} & b & b^* & \tilde{b} \end{bmatrix} = \begin{bmatrix} 0 & I_{4g-3} \\ -I_{4g-3} & 0 \end{bmatrix},$$

of $H_1(X;\mathbb{C})$ which, in the case of the pairs $(X,\omega) = (\widehat{C}, \sqrt{\pi^* q}) \in \mathcal{S}\left(\mathcal{Q}_g\left([1^{4g-4}]\right)\right)$ in the image of the local embedding (4.9), coincide with the basis (4.30). Consider the paths $\{\gamma_m\}_{m=2,\cdots,4g-4}$ connecting $\tilde{p}_1$ to $\tilde{p}_m$, $m = 2, \cdots, 4g-4$, and not intersecting the cycles (4.55). The paths

$$\{a_j, a_j^*, \tilde{a}_k, b_j, b_j^*, \tilde{b}_k, \gamma_m\}, \quad j = 1, \cdots, g\,;\; k = 1, \cdots, 2g-3\,;\; m = 2, \cdots, 4g-4 \tag{4.56}$$

form a basis of the relative homology group $H_1(X, \widetilde{\mathcal{Z}}; \mathbb{C})$.

To shorten the notations, we denote the basis (4.56) by $\{s_n\}_{n=1,\cdots,12g-11}$, where

$$\begin{aligned} s_j &:= a_j\,, & s_{4g-3+j} &:= b_j\,, & & \\ s_{g+j} &:= a_j^*\,, & s_{5g-3+j} &:= b_j^*\,, & & j = 1, \cdots, g\,, \\ s_{2g+k} &:= \tilde{a}_k\,, & s_{6g-3+k} &:= \tilde{b}_k\,, & & k = 1, \cdots, 2g-3\,, \\ & & s_{8g-7+m} &:= \gamma_m\,, & & m = 2, \cdots, 4g-4\,. \end{aligned} \tag{4.57}$$

The set of local homological coordinates $\{\zeta_n\}_{n=1,\cdots,12g-11}$ on the stratum $\mathcal{H}_{4g-3}\left([2^{4g-4}]\right)$ is given by

$$\zeta_n := \int_{s_n} \omega\,, \quad n = 1, \cdots, 12g-11\,. \tag{4.58}$$

Now consider the pairs $(\widehat{C}, \sqrt{\pi^* q})$ in the image of the local embedding

$$\begin{aligned} \mathcal{S} : \mathcal{Q}_g\left(\left[1^{4g-4}\right]\right) &\to \mathcal{H}_{4g-3}\left(\left[2^{4g-4}\right]\right) , \\ (C, q) &\mapsto (\widehat{C}, \sqrt{\pi^* q}) \end{aligned} \tag{4.59}$$

where $\pi : \widehat{C} \to C$ is the canonical double covering defined by the quadratic differential $q \in \mathrm{QD}(C)$. The canonical basis of $H_1(\widehat{C}; \mathbb{C})$ is given by (4.30). Let $\widehat{\mathcal{Z}} = \{\hat{p}_i\}_{i=1,\cdots,4g-4}$ be the set of zeros of the holomorphic 1-form $\sqrt{\pi^* q}$. Consider $4g-5$ curves

$$\gamma_m : [0,1] \to \widehat{C} , \quad m = 2, \cdots , 4g-4 \tag{4.60}$$

such that $\gamma_m(0) = \hat{p}_1$, $\gamma_m(1) = \hat{p}_m$ and they do not intersect the loops (4.30). The curve

$$\varphi_m(t) = \begin{cases} \gamma_m(t) & t \in [0,1] \\ \mu\left(\gamma_m(2-t)\right) & t \in [1,2] \end{cases} \tag{4.61}$$

represents a 1-cycle in the absolute homology group $H_1(\widehat{C}; \mathbb{C})$ of $\widehat{C}$ for each $m = 2, \cdots , 4g-4$. Indeed, each 1-cycle φ_m can be decomposed into a linear combination of $\{\tilde{a}_k, \tilde{b}_k\}_{k=1,\cdots,2g-3}$.

Using (4.4), we get

$$\begin{aligned} \oint_{\varphi_m} \sqrt{\pi^* q} &= \int_{\gamma_m} \sqrt{\pi^* q} - \int_{\mu(\gamma_m)} \sqrt{\pi^* q} \\ &= \int_{\gamma_m} \sqrt{\pi^* q} + \int_{\mu(\gamma_m)} \mu^*\left(\sqrt{\pi^* q}\right) \\ &= 2 \int_{\gamma_m} \sqrt{\pi^* q} \, . \end{aligned} \tag{4.62}$$

Thus, for pairs $(\widehat{C}, \sqrt{\pi^* q}) \in \mathcal{S}\left(\mathcal{Q}_g\left([1^{4g-4}]\right)\right)$, the local homological coordinates $\{\zeta_{8g-7+m}\}_{m=2,\cdots,4g-4}$ are linear combinations of $\{\zeta_{2g+k}, \zeta_{6g-3+k}\}_{k=1,\cdots,2g-3}$ with half-integer coefficients.

Furthermore, we have

$$\begin{aligned} \Pi\left(\sqrt{\pi^* q}, a_j^*\right) = \Pi\left(\sqrt{\pi^* q}, \mu_* a_j\right) = \Pi\left(\mu^* \sqrt{\pi^* q}, a_j\right) = -\Pi\left(\sqrt{\pi^* q}, a_j\right) , \\ \Pi\left(\sqrt{\pi^* q}, b_j^*\right) = \Pi\left(\sqrt{\pi^* q}, \mu_* b_j\right) = \Pi\left(\mu^* \sqrt{\pi^* q}, b_j\right) = -\Pi\left(\sqrt{\pi^* q}, b_j\right) . \end{aligned} \tag{4.63}$$

So the homological coordinates $\{\zeta_j, \zeta_{g+j}, \zeta_{4g-3+j}, \zeta_{5g-3+j}\}_{j=1,\cdots,g}$ of each point in $\mathcal{S}\left(\mathcal{Q}_g\left([1^{4g-4}]\right)\right)$ satisfy the following relations :

$$\zeta_{g+j} = -\zeta_j\,, \quad \zeta_{5g-3+j} = -\zeta_{4g-3+j}\,, \quad j = 1, \cdots, g\,. \tag{4.64}$$

According to (4.64), the period $\Pi\left(\sqrt{\pi^* q}, \varrho\right)$ of $\sqrt{\pi^* q}$ on any invariant 1-cycle $\varrho \in H_+ \subset H_1(\widehat{C}, \mathbb{C})$ is equal to zero.

Let $\{\alpha_l^-, \beta_l^-\}$ be the symplectic basis of the $(6g-6)$-dimensional eigenspace $H_- \subset H_1(\widehat{C}; \mathbb{C})$ given by (4.32). Put for brevity

$$r_l = \alpha_l^-\,, \quad r_{3g-3+l} = \beta_l^-\,, \quad l = 1, \cdots, 3g-3\,. \tag{4.65}$$

The set of complex parameters

$$\eta_i := \Pi\left(\sqrt{\pi^* q}, r_i\right)\,, \quad i = 1, \cdots, 6g-6 \tag{4.66}$$

provides a system of local coordinates on $\mathcal{S}\left(\mathcal{Q}_g\left([1^{4g-4}]\right)\right)$ [7]. Our next goal is to study the variation of the matrices of β-periods $\mathbb{B}^+$ and $\mathbb{B}^-$ under the variation of the coordinate η_i.

Let $\widetilde{\gamma}_m$ be a small loop with positive orientation around the point $\hat{p}_m \in \widehat{\mathcal{Z}}$ for each $m = 2, \cdots, 4g-4$. The following set $\{s_n^*\}_{n=1,\cdots,12g-11}$ gives a basis, dual to (4.57), of the homology group $H_1(\widehat{C} \setminus \widehat{\mathcal{Z}}; \mathbb{C})$:

$$\begin{aligned} s_j^* &:= -b_j\,, \quad s_{4g-3+j}^* := a_j\,, \\ s_{g+j}^* &:= -b_j^*\,, \quad s_{5g-3+j}^* := a_j^*\,, \quad j = 1, \cdots, g\,, \\ s_{2g+k}^* &:= -\tilde{b}_k\,, \quad s_{6g-3+k}^* := \tilde{a}_k\,, \quad k = 1, \cdots, 2g-3\,, \\ & \qquad s_{8g-7+m}^* := \widetilde{\gamma}_m\,, \quad m = 2, \cdots, 4g-4\,. \end{aligned} \tag{4.67}$$

The intersection numbers of the bases (4.57) and (4.67) are given by

$$s_n^* \circ s_l = \delta_{nl}\,, \quad n, l = 1, \cdots, 12g-11\,. \tag{4.68}$$

Furthermore, the following set $\{\tilde{r}_i\}_{i=1,\cdots,6g-6}$ gives a basis, dual to (4.65), of the eigenspace $H_- \subset H_1(\widehat{C}; \mathbb{C})$:

$$\tilde{r}_l = -\beta_l^-\,, \quad \tilde{r}_{3g-3+l} = \alpha_l^-\,, \quad l = 1, \cdots, 3g-3\,, \tag{4.69}$$

where

$$\tilde{r}_i \circ r_j = \delta_{ij}\,, \quad i,j = 1,\cdots,6g-6\,. \tag{4.70}$$

Lemma 4.2.1. *The following variational formula holds:*

$$\frac{\partial}{\partial \eta_i}\begin{bmatrix} \mathbb{B}^+ & 0 \\ 0 & \mathbb{B}^- \end{bmatrix} = \int_{\tilde{r}_i} \frac{1}{\sqrt{\pi^* q}} \begin{bmatrix} u^+ (u^+)^t & 0 \\ 0 & u^-(u^-)^t \end{bmatrix}\,, \tag{4.71}$$

where $i = 1,\cdots,6g-6$.

Proof. According to (3.14), the derivative of the matrix of b-periods $\widehat{\mathbb{B}}$ with respect to ζ_n is given by

$$\frac{\partial}{\partial \zeta_n}\widehat{\mathbb{B}} = \oint_{s_n^*} \frac{1}{\sqrt{\pi^* q}} \mathrm{U}\mathrm{U}^t\,, \tag{4.72}$$

where $\mathrm{U} = \begin{bmatrix} u \\ u^* \\ \tilde{u} \end{bmatrix} \in \left(H^1(\widehat{C};\mathbb{C})\right)^{4g-3}$.

Using (4.48), (4.72) and (4.46), we get

$$\begin{aligned} \frac{\partial}{\partial \eta_i}\begin{bmatrix} \mathbb{B}^+ & 0 \\ 0 & \mathbb{B}^- \end{bmatrix} &= \mathrm{T}\left(\frac{\partial}{\partial \eta_i}\widehat{\mathbb{B}}\right)\mathrm{T}^t = \mathrm{T}\left(\sum_{n=1}^{12g-11}\frac{\partial \zeta_n}{\partial \eta_i}\left(\frac{\partial}{\partial \zeta_n}\widehat{\mathbb{B}}\right)\right)\mathrm{T}^t \\ &= \mathrm{T}\left(\sum_{n=1}^{12g-11}\frac{\partial \zeta_n}{\partial \eta_i}\oint_{s_n^*}\frac{1}{\sqrt{\pi^* q}}\mathrm{U}\mathrm{U}^t\right)\mathrm{T}^t \\ &= \sum_{n=1}^{12g-11}\frac{\partial \zeta_n}{\partial \eta_i}\oint_{s_n^*}\frac{1}{\sqrt{\pi^* q}}\left(\mathrm{T}\mathrm{U}\right)\left(\mathrm{T}\mathrm{U}\right)^t \\ &= \int_{\nu_i}\frac{1}{\sqrt{\pi^* q}}\begin{bmatrix} u^+ (u^+)^t & 0 \\ 0 & u^-(u^-)^t \end{bmatrix}\,, \end{aligned} \tag{4.73}$$

where

$$\nu_i = \sum_{n=1}^{12g-11}\frac{\partial \zeta_n}{\partial \eta_i}\, s_n^* \tag{4.74}$$

is a 1-cycle in $H_1\left(\widehat{C}\setminus\widehat{\mathcal{Z}};\mathbb{C}\right)$.

Each entry of the matrices $\frac{u^+(u^+)^t}{\sqrt{\pi^* q}}$ and $\frac{u^-(u^-)^t}{\sqrt{\pi^* q}}$ is an anti-invariant 1-form under the action of μ, whereas the small loop $\widetilde{\gamma}_m$ can be chosen invariant under μ. Therefore, we have

$$\oint_{\widetilde{\gamma}_m} \frac{1}{\sqrt{\pi^* q}} \begin{bmatrix} u^+(u^+)^t & 0 \\ 0 & u^-(u^-)^t \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{4.75}$$

for each $m = 2, \cdots, 4g-4$.

Furthermore, the coordinates $\{\zeta_n\}_{n=1,\cdots,8g-6}$ depend to $\{\eta_i\}_{i=1,\cdots,6g-6}$ in the following way:

$$\begin{aligned} &\zeta_j = -\zeta_{g+j} = \eta_j\,, \\ &\zeta_{4g-3+j} = -\zeta_{5g-3+j} = \tfrac{1}{2}\eta_{3g-3+j}\,, \qquad j = 1, \cdots, g\,; \end{aligned} \tag{4.76}$$

$$\begin{aligned} &\zeta_{2g+k} = \eta_{g+k}\,, \\ &\zeta_{6g-3+k} = \eta_{4g-3+k}\,, \qquad k = 1, \cdots, 2g-3\,. \end{aligned} \tag{4.77}$$

Thus, the 1-cycles $\{\nu_i\}_{i=1,\cdots,6g-6}$ can be simplified to the following form:

$$\begin{aligned} &\nu_j = b_j^* - b_j\,, \\ &\nu_{3g-3+j} = \tfrac{1}{2}(a_j - a_j^*)\,, \qquad j = 1, \cdots, g\,; \end{aligned} \tag{4.78}$$

$$\begin{aligned} &\nu_{g+k} = -\tilde{b}_k\,, \\ &\nu_{4g-3+k} = \tilde{a}_k\,, \qquad k = 1, \cdots, 2g-3\,. \end{aligned} \tag{4.79}$$

The relations (4.73), (4.78) and (4.79) leads to (4.71) for $i = 1, \cdots, 6g-6$. $\square$

The dimensions of the space $\mathcal{Q}_g\left([1^{4g-4}]\right)$ and the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geq 2$ are equal to $6g-6$ and $3g-3$, respectively. Therefore, there must exist $3g-3$ local independent vector fields $\{W_l\}_{l=1,\cdots,3g-3}$, defined on a neighbourhood of the point $(C, q) \in \mathcal{Q}_g\left([1^{4g-4}]\right)$, which preserve the complex structure of the Riemann surface C. We use the same notation $\{W_l\}_{l=1,\cdots,3g-3}$ for their image under the local embedding (4.9).

According to Torelli's theorem [11], the vector field W_l preserve the complex structure of C if and only if $W_l(\mathbb{B}) = 0_{g\times g}$, where $\mathbb{B}$ is the matrix of $\mathfrak{b}$-periods of C. Using (4.54), we get the following condition for W_l which is equivalent to $W_l(\mathbb{B}) = 0_{g\times g}$:

$$W_l\left(\mathbb{B}^+_{nk}\right) = 0\,, \quad n, k = 1, \cdots, g\,;\ l = 1, \cdots, 3g-3\,. \tag{4.80}$$

According to [9], the stratum $\mathcal{Q}_g\left([1^{4g-4}]\right)$ may have a hyperelliptic connected component. In what follows, we only consider the non-hyperelliptic connected component of $\mathcal{Q}_g\left([1^{4g-4}]\right)$. The following theorem, as the main result of the thesis, provide an expression for $W_l\,,\ l=1,\cdots,3g-3\,,$ as a linear combination of the local vector fields $\{\partial/\partial\eta_i\}_{i=1,\cdots,6g-6}$, on the non-hyperelliptic connected component of $\mathcal{Q}_g\left([1^{4g-4}]\right)$.

Theorem 4.2.2. *Let $(C,q)\in\mathcal{Q}_g\left([1^{4g-4}]\right)$ be a point in the non-hyperelliptic connected component of the stratum $\mathcal{Q}_g\left([1^{4g-4}]\right)$, where C is a non-hyperelliptic compact Riemann surface of genus $g\geq 2$ and q is a holomorphic quadratic differential with simple zeroes on C. Let $\{\eta_i\}_{i=1,\cdots,6g-6}$ be the local homological coordinates, given by* (4.66)*, in a neighbourhood of the point (C,q). The local vector fields*

$$W_l=\frac{\partial}{\partial\eta_l}+\sum_{k=1}^{3g-3}\mathbb{B}^-_{lk}\,\frac{\partial}{\partial\eta_{3g-3+k}}\,,\quad l=1,\cdots,3g-3 \tag{4.81}$$

preserve the complex structure of C, where $\mathbb{B}^-$ is the matrix of β^--periods given by (4.44) *.*

Proof. Let $W_l\,,\ l=1,\cdots,3g-3$ be the local vector fields on $\mathcal{S}\left(\mathcal{Q}_g\left([1^{4g-4}]\right)\right)$ which preserve $\mathbb{B}^+$. Consider the following expression of W_l in terms of the local vector fields $\{\partial/\partial\eta_i\}_{i=1,\cdots,6g-6}$:

$$W_l=\sum_{i=1}^{6g-6}\mathrm{F}_{li}\frac{\partial}{\partial\eta_i}\,,\qquad l=1,\cdots,3g-3\,, \tag{4.82}$$

where $\mathrm{F}=[\mathrm{F}_{li}]$ is a $(3g-3)\times(6g-6)$ matrix of rank $3g-3$. Using (4.71), we can rewrite the condition (4.80) in the following way:

$$0=W_l\left(\mathbb{B}^+_{nk}\right)=\sum_{i=1}^{6g-6}\mathrm{F}_{li}\frac{\partial\mathbb{B}^+_{nk}}{\partial\eta_i}=\sum_{i=1}^{6g-6}\mathrm{F}_{li}\int_{\tilde{r}_i}\frac{u_n^+u_k^+}{\sqrt{\pi^*q}}\,, \tag{4.83}$$

where $n,k=1,\cdots,g$ and $l=1,\cdots,3g-3$.

Let $\{v_j\}_{j=1,\cdots,g}$ be the normalized basis of the space of holomorphic 1-forms on C. Since $u_j^+=\pi^*v_j\,,\ j=1,\cdots,g$, considering the isomorphism (4.28), we have

$$\frac{u_n^+u_k^+}{\sqrt{\pi^*q}}=\frac{\pi^*\left(v_nv_k\right)}{\sqrt{\pi^*q}}\in\Lambda_-\subset\Lambda^1_{\widehat{C}}\,. \tag{4.84}$$

Since the Riemann surface C is non-hyperelliptic, the quadratic differentials

$$v_n v_k\,,\quad n,k=1,\cdots,g \tag{4.85}$$

span the whole space of holomorphic quadratic differentials on C [2]. So we deduce from equalities (4.83) and (4.84) that each holomorphic 1-form $w\in\Lambda_-$ satisfy the following relation:

$$\Pi\left(w,\sum_{i=1}^{6g-6}\mathrm{F}_{li}\,\tilde{r}_i\right)=\sum_{i=1}^{6g-6}\mathrm{F}_{li}\int_{\tilde{r}_i}w=0\,,\quad l=1,\cdots,3g-3\,. \tag{4.86}$$

Therefore, we have

$$\Pi\left(u_m^-,\sum_{i=1}^{6g-6}\mathrm{F}_{li}\,\tilde{r}_i\right)=0\,,\quad l,m=1,\cdots,3g-3\,, \tag{4.87}$$

where $\{u_m^-\}_{m=1,\cdots,3g-3}$ is the basis of Λ_- given by (4.41).

Since F is a full-row-rank matrix, we may assume, without loss of generality, that

$$\mathrm{F}=\begin{bmatrix}I_{3g-3} & \mathrm{G}\end{bmatrix}\,, \tag{4.88}$$

where I_{3g-3} is the identity matrix and $\mathrm{G}\in\mathrm{M}(3g-3,\mathbb{C})$. Thus, using (4.42) and (4.44), we can rewrite (4.87) in the following form:

$$\begin{aligned}\Pi\left(u_m^-,\sum_{i=1}^{6g-6}\mathrm{F}_{li}\,\tilde{r}_i\right)&=\Pi\left(u_m^-,-\beta_l^-\right)+\Pi\left(u_m^-,\sum_{k=1}^{3g-3}\mathrm{G}_{lk}\,\alpha_k^-\right)\\&=-\mathbb{B}_{lm}^-+\sum_{k=1}^{3g-3}\mathrm{G}_{lk}\,\delta_{mk}=0\,,\end{aligned} \tag{4.89}$$

where $l,m=1,\cdots,3g-3$. So we have $\mathrm{G}=\mathbb{B}^-$ and, by (4.88), we get (4.81). □

Chapter 5
Summary and Outlook

The main results of this thesis are related to the variational formulas for the matrix of b-periods on the moduli space of holomorphic quadratic differentials over compact Riemann surfaces.

We study the variational formulas of Ahlfors-Rauch type for the normalized Abelian differentials and the matrix of b-periods on Hurwitz spaces and the moduli space of holomorphic Abelian differentials.

We introduced the canonical 2-sheeted branched covering $\pi : \widehat{C} \to C$ corresponding to a pair (C, q) in the principal stratum $\mathcal{Q}_g\left([1^{4g-4}]\right)$ of moduli space of holomorphic quadratic differentials. The decomposition of the homology and cohomology groups of $\widehat{C}$ into invariant and anti-invariant subspaces under the action of the natural involution $\mu : \widehat{C} \to \widehat{C}$ was discussed.

We derived the variational formulas for matrix of b-periods of C and $\widehat{C}$ under variation of the induced homological coordinates $\{\eta_i\}_{i=1,\cdots,6g-6}$ on $\mathcal{Q}_g\left([1^{4g-4}]\right)$. Also, we found a complete set of local vector fields $\{W_l\}_{l=1,\cdots,3g-3}$, in terms of vector fields $\{\partial/\partial\eta_i\}_{i=1,\cdots,6g-6}$, on the non-hyperelliptic connected component of $\mathcal{Q}_g\left([1^{4g-4}]\right)$ preserving the moduli of the base Riemann surface C.

The first open problem is to find the set of $3g-3$ vector fields on the hyperelliptic connected component of $\mathcal{Q}_g\left([1^{4g-4}]\right)$ which preserve the complex structure of the base Riemann surface C.

The second question is whether there exists a set of local functions $\{\xi_l\}_{l=1,\cdots,3g-3}$ on $\mathcal{Q}_g\left([1^{4g-4}]\right)$ such that the vector fields $\{W_l\}_{l=1,\cdots,3g-3}$ have the following local expression:

$$W_l = \frac{\partial}{\partial \xi_l}\,, \quad l = 1, \cdots, 3g-3\,. \tag{5.1}$$

In other words, the problem is whether the local coordinates on $\mathcal{Q}_g\left([1^{4g-4}]\right)$ can split into natural $3g-3$ coordinates on moduli space $\mathcal{M}_g$ of Riemann surfaces and the coordinates $\{\xi_1, \cdots, \xi_{3g-3}\}$ in the fiber of $\mathcal{Q}_g\left([1^{4g-4}]\right)$ over $[C] \in \mathcal{M}_g$.

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