# Hardy Spaces and Differentiation of the Integral in the Product Setting

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# ABSTRACT

**Hardy Spaces and Differentiation of the Integral in the Product Setting**

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This work concerns strong differentiation and operators on product Hardy spaces. We show, by counterexample, that strong differentiability of the integral fails even for functions in the intersection of $H^1_{rect}(\mathbb{R} \times \mathbb{R})$ with $L(\log L)^\epsilon(\mathbb{R}^2)$ for all $0 < \epsilon < 1$. Our example is a modification of a function that appears in a work of J. M. Marstrand, where he makes a claim concerning "approximately independent sets". We generalize his claim and, as a corollary, we obtain a version of the second Borel-Cantelli Lemma.

In addition, we prove that a function $f$, created by Papoulis to show that the strong differentiability of $\int f$ does not imply the same behavior for $\int |f|$, belongs to the product Hardy space $H^1(\mathbb{R} \times \mathbb{R})$. The method that we develop to approach this example allows us to relax the sufficient conditions of the Chang-Fefferman atomic decomposition. In analogy with the proof of this result, we demonstrate that a theorem of R. Fefferman, which concludes $H^p \to L^p$, $0 < p \leq 1$, boundedness of two-parameter operators from their behavior on rectangle atoms, can be generalized to settings with more parameters. This generalization enables us to extend a theorem of Pipher concerning boundedness of multiparameter Calderón-Zygmund operators from $H^p$ to $L^p$.

Furthermore, we present variants of Journé's Lemma, two of which hold for the product of $\mathbb{R}$ with a metric measure space satisfying certain conditions.

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# Chapter 1

# Introduction

We describe here the ideas and the main results of this thesis. A more extensive historical background, as well as the definitions that will be used throughout this work, are presented in Chapter 2. The precise statements of our results and their proofs can be found in the subsequent chapters.

The Hardy-Littlewood maximal function maps $L^1$ into weak $L^1$, a property that implies the classical Lebesgue differentiation theorem. Being related to averages of functions on cubes (or convex sets with bounded eccentricity), these results are said to be in the one-parameter setting. By contrast, in the multiparameter (or product) setting, we view $\mathbb{R}^n$ as $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$, $n = n_1 + ... + n_r$, and the cubes are replaced by sets of the form

$$R(x,t) := \left\{ y = (y_1, ..., y_r) \in \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r} : \|x_j - y_j\|_{\mathbb{R}^{n_j}} < t_j \right\},$$

where $x_j \in \mathbb{R}^{n_j}$, $t_j > 0$ and $\|\cdot\|_{\mathbb{R}^{n_j}}$ denotes a norm in $\mathbb{R}^{n_j}$, $1 \leq j \leq r$. We refer to these sets as *rectangles* and we let $\mathcal{R}$ be the collection of all such rectangles.

In the product setting, the analogue of the Hardy-Littlewood maximal function, called the *strong maximal function*, is defined [39] by

$$M_S(f)(x) := \sup \left\{ \frac{1}{|R|} \int_R |f(y)| \, dy : R \in \mathcal{R} \text{ such that } x \in R \right\}, \quad f \in L^1_{loc}(\mathbb{R}^n), \quad (1.1)$$

and the differentiability of the integral is considered with respect to rectangles in $\mathcal{R}$ and known as *strong differentiability* (see Definition 2.1).

The theory of strong differentiation of integrals was originally introduced by Saks in 1933 (see [60] for English translation). Surprisingly, a paper of his [58] and an example by Busemann and Feller [3] showed that, unlike what happens in the one-parameter setting, strong differentiability may fail for the integral of locally integrable functions. Specifically, Saks proved that the set of locally integrable functions $f$ for which strong differentiation of $\int f$ holds a.e. is of first category in $L^1$, while Busemann and Feller exhibited a counterexample which implies, in particular, that $M_S$, viewed as an operator, is not of weak-type (1,1). Later, Papoulis [53] build an integrable function $f$ on $\mathbb{R}^2$ which illustrates the fact that $f$ and $|f|$ may have dramatically distinct behaviors in terms strong of differentiation of the integral. Our Theorem 1.3 is related to Papoulis' example.

As we will see below, there are many results that hold in the one-parameter setting but fail in the product setting. Also, in many cases, a naïve attempt to generalize one-parameter entities to higher parameter settings does not work.

A famous work demonstrating the failure of the naïve approach to the multiparameter theory is Carleson's counterexample [9]. It consists of a measure such that Carleson's condition holds with respect to rectangles, but fails with respect to bounded open sets.

Positive results about the behavior of $M_S$ on the $n$-fold $\mathbb{R} \times \cdots \times \mathbb{R}$ and about strong differentiation in this setting were obtained by Jessen, Marcinkiewicz and Zygmund [39]: $M_S$, viewed as an operator, is bounded on $L^p$, for $p > 1$; $M_S(f)$ is in weak $L^1$ whenever $|f| \log(1 + |f|)^{n-1}$ is integrable; and the integral of a function $f$ is strongly differentiable a.e. whenever $f$ is in $L^p_{loc}$, $p > 1$, or $|f| \log(1 + |f|)^{n-1}$ is in $L^1_{loc}$. The results related to the spaces $L\left(\log^+ L\right)^{n-1}$ were further generalized to products of higher dimensional spaces (see [32]) and, as shown by Saks [59], are as good as possible in the sense that the differentiability of the integral may fail for certain classes of functions satisfying slightly weaker integrability conditions [59]. In particular, the latter showed that for each $0 < \epsilon < 1$, there exists $f$ in $L(\log L)^{\epsilon}(\mathbb{R}^2)$ such that $\int f$ is not strongly differentiable on a set of positive measure. Our result (Theorem 1.1) shows that the strong differentiability of the integral may fail even for function which are simultaneously in all these classes

$L\left(\log^{+} L\right)^{\epsilon}(\mathbb{R}^2)$, $0 < \epsilon < 1$.

After its initial boom, many years elapsed before further successful attempts to extend classical one-parameter results in harmonic analysis to the product setting were made. Some considerable developments took place during the decades of the 1970's and 1980's, with the works of S.Y. Chang [10], R. Fefferman [11], Gundy and Stein [30], Journé [42], [43], and Pipher [54], among others. Other achievements only occurred more recently. They comprise generalizations of results related to Hardy spaces, $BMO$, singular integrals, flag kernels, etc. in product spaces of different levels of generality, varying from the product of unit disks [55] and the product of Euclidean spaces [56], [47], to the product of homogeneous groups [46], [18].

Since many results concerning boundedness of singular operators can be extended from $L^p$, $p > 1$, to one-parameter Hardy spaces $H^1$ [21], [62], the question arose as to whether the strong differentiation of the integral would hold in the one-parameter real Hardy space $H^1$. Compared to the elements of $L^1$, the functions in the Hardy space $H^1$ satisfy stronger integrability conditions and, in addition, have cancellation properties.

While the Hardy spaces $H^p$ originated in the 1920's as spaces of certain complex-valued homomorphic functions on the unit disc, or on the upper half-plane, we are interested in the more recent characterizations of these spaces, specifically in the real-variable ones. The real variable theory of Hardy spaces $H^p$, $0 < p < \infty$, began in 1971 with the nontangential maximal function characterization, by Burkholder, Gundy and Silverstein [2], of the class $H^p$ on the upper half-plane $\mathbb{R}^2_+$. In 1972, their result was extended to $\mathbb{R}^{n+1}_+$ by R. Fefferman and Stein [21], who also established other equivalent definitions for $H^p\left(\mathbb{R}^{n+1}_+\right)$. The innovative aspect of those definitions is that they uncover the real-variable meaning of the classes $H^p$. This freed the study of these classes from the need to deal with holomorphic functions, Poisson integrals, and all the associated entities that were present in previous definitions.

The extension of this real-variable one-parameter theory to the multiparameter setting was first accomplished by Gundy and Stein [30] in 1979. Besides having characterized the $H^p$ spaces on the product of upper half-spaces $\mathbb{R}^{n_1+1}_+ \times ... \times \mathbb{R}^{n_k+1}_+$ in terms of the boundary behavior of the multiharmonic (also called multiply harmonic) functions, they generalized the above mentioned real-variable characterizations to this setting. In particular, they showed that, for $0 < p < \infty$, the *multiparameter*, or *product*, *Hardy space* $H^p(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})$, can be defined by saying that a distribution $f$, in $\mathcal{S}'(\mathbb{R}^n)$, belongs to $H^p(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})$ if and only if its *multiparameter radial maximal function*,

$$\mathfrak{M}_{\varphi^1,\ldots,\varphi^r}(f)(x) := \sup_{t_j>0} \left| \int \varphi^1_{t_1}(y_1) \ldots \varphi^r_{t_r}(y_r) f(x-y)\, dy \right|, \tag{1.2}$$

is in $L^p(\mathbb{R}^{n_1+\ldots+n_r})$, for some fixed Schwartz functions $\varphi^j$ on $\mathbb{R}^{n_j}$ with non-zero integral, where $\varphi^j_{t_j}(x_j) := t_j^{-n_j} \varphi^j\left(t_j^{-1} x_j\right)$, $j = 1, ..., r$.

We consider that the multiparameter radial maximal function (1.2) is the simplest type of multiparameter maximal function after the strong maximal function (1.1). The action of the former depends on the absolute value of smooth averages, while that of the latter is related to averages of the absolute value. This difference yields essentially distinct behaviors.

When working with Hardy spaces $H^p$, $0 < p \leq 1$, it is often convenient to use a result known as the *atomic decomposition*. In the one-parameter setting, this result (Theorem 2.4), proved by Coifman [15] and Latter [45], states that the elements of $H^p$ are infinite linear combinations of special functions, called *atoms* (Definition 2.2), with coefficients in $l^p$.

The intuitive way to extend this result to the multiparameter setting would tell us that the product Hardy space $H^p$, $0 < p \leq 1$, consists of infinite linear combinations of rectangle atoms (Definition 2.4) with coefficients in $l^p$. Surprisingly this is false: the space whose elements are those infinite linear combinations of rectangle atoms is a proper subspace of product $H^p$. This subspace is called *rectangular Hardy space* [13] and is denoted by $H^p_{rect}$. The fact that product $H^p$ is strictly larger than $H^p_{rect}$ came to light due to the above mentioned counterexample of Carleson [9], which implies, by duality, that $H^1_{rect}(\mathbb{R} \times \mathbb{R}) \subsetneq H^1(\mathbb{R} \times \mathbb{R})$. The atomic decomposition for product Hardy spaces was proved by S.-Y. Chang and R. Fefferman [11], [12]. They showed that $H^p(\mathbb{R} \times \mathbb{R})$, $0 < p \leq 1$, consists of infinite linear combinations of product space atoms, called *Chang-*

*Fefferman atoms* (Definition 2.6), with coefficients in $l^p$, but here the definition of the atoms is more sophisticated.

Combining the theory of Hardy spaces and of strong differentiation of the integral, Stokolos [65] gave a negative answer to the question concerning strong differentiability of the integral of functions in the real Hardy space $H^1(\mathbb{R}^2)$. While he considers the one-parameter Hardy space, his example actually belongs to $H^1(\mathbb{R}\times\mathbb{R})$. Due to the multiparameter aspect of the theory of strong differentiation of the integral, the product Hardy spaces seem to be more naturally connected with it than the one-parameter analogues.

The example of Stokolos [65] consists of a modification of a function created by J. M. Marstrand [50]. By making suitable alterations on it, we give a constructive proof to the following:

**Theorem 1.1** ([4]). *There exists a function in the rectangular Hardy space* $H^1_{rect}(\mathbb{R}\times\mathbb{R})$ *which is also in all the Orlicz spaces* $L\left(\log^+ L\right)^\epsilon(\mathbb{R}^2)$ *for* $0<\epsilon<1$*, whose integral is not strongly differentiable almost everywhere on a square of sidelength* 1*.*

This result, which is restated as Theorem 3.1 in Chapter 3, relies on a specific way of evaluating the Orlicz norm of series of functions and it implies, in particular, that $\mathcal{R}$ is not a differentiation basis (see definition in [32], [63], or [64]) for any Orlicz space $L\left(\log^+ L\right)^\epsilon(\mathbb{R}^2)$ with $0<\epsilon<1$.

In Marstrand's work, the approximate independence (in the probabilistic sense) of homothetic copies of certain "hyperbolic-cross" shaped sets:

$$\left\{(x_1,x_2)\in\mathbb{R}^2 : |x_1x_2|\le 1,\ x_1^2+x_2^2\le (n+1)\left(\log(n+1)\right)^2\right\},\quad n\in\mathbb{N}, \tag{1.3}$$

is claimed (see [50], page 210) without a proof. Would a rigorous proof of his claim rely on the particular geometry of the hyperbolic-crosses? Which geometric aspects of the sets are essential? We answer these questions by showing the following generalization of Marstrand's claim:

**Theorem 1.2** ([4])**.** *Let $\{A_k\}_{k\in\mathbb{N}}$ be a family of subsets of $[-2^{-1}, 2^{-1}]^n \subset \mathbb{R}^n$ satisfying $|A_k| > 0$ and $\dim_{upper\, box}\left(\partial\overline{A_k}\right) < n$ for all $k$. There exists $\{m_k\}_{k\in\mathbb{N}}$, a sequence of positive integers, such that if, for each index $k$, we partition $[-2^{-1}, 2^{-1}]^n$ into $m_k^n$ equal sized cubes, and place inside each a homothetic copy of $A_k$, then denoting by $\Lambda_k$ the union of these homothetic copies,*

$$\left|\bigcap_{k\in F}\Lambda_k\right| \sim \prod_{k\in F}|\Lambda_k|\,, \tag{1.4}$$

*for any finite subset $F \subset \mathbb{N}$.*

This result is restated in Theorem 3.2 and has a version of the Second Borel-Cantelli Lemma as a corollary (Corollary 3.1). This illustrates how geometry can lead to results of a probabilistic nature.

The example in [65] is built in the dyadic setting. It has motivated us to show that if we restrict ourselves to sets which are finite unions of dyadic cubes, then we can obtain (1.4) with an equality (Claim 3.2).

Another problem that we investigate in this thesis is the following question, which was raised by Stokolos. Concerning the example created by Papoulis [53] of an integrable function $f$ on $\mathbb{R}^2$ such that the strong derivative of $\int f$ exists a.e., but the upper strong derivative of $\int|f|$ is infinite on a set of positive measure, does such a function belong to the Hardy space $H^1(\mathbb{R}\times\mathbb{R})$? The answer is positive:

**Theorem 1.3** ([5])**.** *The function created by Papoulis belongs to $H^1_{rect}(\mathbb{R}\times\mathbb{R})$, therefore also to $H^1(\mathbb{R}\times\mathbb{R})$.*

This theorem is restated as Theorem 4.1 in Chapter 4. The content of this chapter was submitted to a journal as a manuscript with the same title [5].

The search for the solution to the above mentioned problem raised other questions and lead us to find a variety of other results. Our investigation is outlined below. We present the results in more detail, as well as their proofs, in Chapter 4.

Our first approach to the problem concerning Papoulis' example was to show directly that its two-parameter radial maximal function is integrable. This was accomplished by modifying the standard techniques used in the one-parameter setting. In the proof of

Theorem 1.3, instead exhibiting our initial proof (see proof of Lemma 4.1), we present a more elegant one. Namely, we show that Papoulis' function, $f$, belongs to $H^1_{rect}(\mathbb{R}\times\mathbb{R})$ by exhibiting a decomposition of $f$ in terms of rectangle atoms (Definition 2.4) with coefficients in $l^1$. This means that $f \in H^1_{rect}(\mathbb{R}\times\mathbb{R})$. The method that we developed in our initial approach, enables us to show some crucial estimates (Lemma 4.1) for the action of the radial maximal operator on rectangles atoms on $\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r}$.

The key importance of the estimates of Lemma 4.1 is due to the fact that they allow us to prove the uniform boundedness of the $L^p$-norm of the radial maximal function of Chang-Fefferman atoms [12], without using all the hypotheses on the elementary particles of these atoms. Specifically:

**Theorem 1.4** ([5]). *In to order for a tempered distribution $f$ in $\mathcal{S}'(\mathbb{R}^{n_1+\ldots+n_r})$ to be in $H^p(\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r})$, it suffices that $f$ can be written as $\sum_{k=1}^{\infty}\lambda_k a_k$, with the series converging in $\mathcal{S}'(\mathbb{R}^{n_1+\ldots+n_r})$, where $\{\lambda_k\}_k \in l^p$ and the atoms $a_k$'s are variants of Chang-Fefferman atoms on $\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r}$ without smoothness hypotheses on their elementary particles.*

This result, which is restated in Theorem 4.5, provides new sufficient conditions for tempered distributions to be in $H^p(\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r})$. Since the essential difference between Chang-Fefferman atoms and the atoms of Theorem 1.4 is the that the latter do not need smoothnesses nor continuity on their elementary particles, we denote them as *rough atoms* (Definition 4.3). Note that, without smoothness conditions, an elementary particle is simply a scalar multiple of a rectangle atom. In particular, this result tells us that rectangular atoms are, in a sense, the building blocks of product Hardy spaces $H^p$. For $p = 1$, it was already known [47] that the smoothness of the elementary particles was a superfluous hypothesis for the sufficiency of the atomic decomposition.

Clearly, when one decomposes distributions in $H^p$ as sums of functions, it is better to have as much smoothness as possible. On the other hand, for sufficient conditions, it is more convenient to have less hypotheses on those functions.

Many operators in harmonic analysis belong to the class of singular integral operators: e.g. Riesz transforms, pseudo-differential operators, Cauchy integrals on Lipschitz curves.

The classical theory of singular integral operators began to take shape in the 1950's when Calderón and Zygmund [7] obtained $L^p$ boundedness, $1 < p < \infty$, for certain convolution operators which generalize the Hilbert transform on the real line. In the beginning of the 1980's, their result was extended in two ways: R. Fefferman and Stein [22] extended it to multiparameter convolution operators; David and Journé [17] did it for it one-parameter non-convolution operators (by proving the famous $T(1)$–Theorem). Later, in 1985, Journé [43] obtained this type of boundedness by showing $L^\infty \to$ product $BMO$ boundedness for multiparameter non-convolution type operators and using a generalization of the $T(1)$–Theorem.

Concerning boundedness of singular integral operators on Hardy spaces, in 1978, Miyachi [52] used the Coifman-Latter atomic decomposition to show boundedness from $H^1$ to itself. In 1986, using the Chang-Fefferman atomic decomposition and a lemma proved by Journé [42] in the previous year, R. Fefferman [25] developed a way to conclude $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \to L^p(\mathbb{R}^{n_1+n_2})$ boundedness of an operator from its action on rectangle atoms.

The method that we use to prove Theorem 1.4 enables us to prove a seemingly slightly different (but actually quite relevant) variant of the above mentioned result of R. Fefferman. Combining his proof with the main result in D.-C. Chang et al. [14] (who provided some details which were omitted by him), we prove the result below, which is restated in Theorem 4.2:

**Theorem 1.5** ([5])**.** *Given $0 < p \leq 1$, if $T$ is a bounded operator on $L^2(\mathbb{R}^{n_1+n_2})$ and there exist $\delta_j > 0$, $j = 1, 2$, such that*

$$\int_{(2^k I_1)^c \times \mathbb{R}^{n_2}} |T(a)|^p \leq C\left(2^k\right)^{-\delta_1} \quad \text{and} \quad \int_{\mathbb{R}^{n_1} \times (2^k I_2)^c} |T(a)|^p \leq C\left(2^k\right)^{-\delta_2},$$

*for every $k \in \mathbb{N}$ and every rectangle atom $a$ supported on $I_1 \times I_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then $T$ admits a bounded extension from $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1+n_2})$.*

There are two steps in the proof of theorems of this type:

(i) showing that the behavior of an operator $T$ on rectangle atoms yields uniform boundedness of the $L^p$-norm of the action of $T$ on product $H^p$-atoms;

(ii) proving that this uniform bound implies the existence of a bounded extension of $T$ from product $H^p$ to $L^p$.

The essential aspect of our hypotheses is that, unlike the assumptions in [25], we have distinct dilations on each factor of the product setting. These multiparameter dilations are the essential aspect of our theorem, as they allow a direct extension of step (i) to higher-parameter settings. In particular, they enable us to extend R. Fefferman's result to the three-parameter setting:

**Theorem 1.6** ([5]). *Fix $0 < p \leq 1$. If $T$ is a bounded operator on $L^2\left(\mathbb{R}^{n_1+n_2+n_3}\right)$ and there exist $\delta_j > 0$, $j = 1, 2, 3$, such that*

$$\int_{\left(2^{k_1} I_1\right)^c \times \left(2^{k_2} I_2\right)^c \times \mathbb{R}^{n_3}} |T(a)|^p \leq C \left(2^{k_1}\right)^{-\delta_1} \left(2^{k_2}\right)^{-\delta_2}, \tag{1.5}$$

*for every $k_1, k_2 \in \mathbb{N}$ and every rectangle atom $a$ supported on $I_1 \times I_2 \times I_3 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$, and similar inequalities with $(1,2,3)$ replaced by $(2,3,1)$ and $(3,1,2)$ hold, then $T$ admits a bounded extension from $H^p\left(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}\right)$ to $L^p\left(\mathbb{R}^{n_1+n_2+n_3}\right)$.*

Note that Theorem 1.6 (restated in Theorem 4.3) is not invalidated by Journé's counterexample [43]: our dilations have three parameters instead of one. His example seems to indicate existence of a barrier preventing the extension of R. Fefferman's argument from the two- to the three-parameter setting. Journé [43] himself surpassed this obstacle in the context of convolution operators on product $BMO$. Carbery and Seeger [8] have overcome this difficult in $H^p\left(\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}\right)$, by demonstrating that, under extra assumptions on the operators, R. Fefferman's reasoning is applicable in higher-parameter settings. Working with Hilbert space valued operators and using the Littewood–Paley square function characterization of the multiparameter Hardy spaces, Han et al. [35] exhibited necessary and sufficient conditions for certain Calderón-Zygmund operators to be bounded from $H^p$ into $H^p$, $0 < p \leq 1$.

When a two-parameter operator $T$ is of the type that is commonly known as Calderón-Zygmund operator of Journé type [43] (and referred by Journé as Calderón-Zygmund operator of type $\epsilon$), we can obtain the hypotheses of Theorem 1.6 by imposing certain conditions on the kernel. Our Theorem 4.4 shows that it is sufficient to assume an appropriate Hörmander type condition. This result is similar to one of R. Fefferman [25], which he called the "trivial lemma" and proved for values of $p$ close to 1. We demonstrate that it holds it for all values of $p$ in $(0, 1]$.

The extension of Theorem 4.4 to the three-parameter setting is presented in Theorem 4.6. It exhibits sufficient conditions on the kernel of a Calderón-Zygmund operator for the assumptions of Theorem 4.3 to hold. Our proof of Theorem 4.6 is inspired by an argument of Pipher [54], who showed this result for values of $p$ close to 1 and settings of the form $\mathbb{R} \times ... \times \mathbb{R}$ (she also stated, without a proof, that this result holds for all $p \in (0, 1]$ if more smoothness is assumed on the kernel). Our result is more general: it holds for any $0 < p \leq 1$ and on any multiparameter setting of the form $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$.

In order to prove Theorem 1.6, we adapt the proof of Theorem 1.5 to the three-parameter setting. The original two-parameter discrete Journé's Lemma [42], which is a very useful tool to deal with Chang-Feffeman atoms and its variants, is replaced by a suitable three-parameter variant of it (Lemma 4.4).

In [42], Journé stated that the discrete version of his lemma holds in the setting $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. This result is often used in the context of product Hardy spaces (e.g. [25]), despite the fact that no proof of it can be found in the literature. We state this in Lemma 4.2 and, inspired by proofs found in [54] and [67], we prove it.

There are many variants of Journé's Lemma in the literature: [54], [8], [44], [6] and others. Among them, we find that Pipher's [54] version is the most useful in the context of the Hardy spaces that we deal with. Her result consists of an extension of the discrete Journé's Lemma to the setting $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The variant (Lemma 4.4) that we use in the proof of Theorem 1.6 is an adaptation of her result to the higher-dimensional three-parameter setting $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$. In addition, as she explains in her paper, her proof can be extended, by induction, to the $n$-fold product $\mathbb{R} \times \cdots \times \mathbb{R}$. By the same reasoning, our variant (Lemma 4.4) of Journé's Lemma can be extended to $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$, and therefore Theorem 1.6 can also be extended to this multiparameter setting.

The deep understanding of Journé's Lemma that we obtained through the study of the variants mentioned above empowered us to extend it (Theorem 5.1) to a product setting of the form $X \times \mathbb{R}$, with $X$ being a metric measure space having certain properties. We extend both the non-discrete and the discrete versions of Journé's Lemma. A potential application of our discrete Journé-type lemma is in the context of Hardy spaces on Heisenberg groups.

As we mentioned in the beginning of this chapter, the next one contains a more detailed overview of the theory of strong differentiation of the integral and of Hardy spaces. The content of Chapter 3 is a manuscript that was accepted for publication [4]. It contains the counterexample in $H^1_{rect} \cap \left[\cap_{0<\epsilon<1} L\left(\log^+ L\right)^\epsilon\right]$ and the results related to approximate independence of sets. Chapter 4 consists of a manuscript which was submitted [5]. It has the proof that Papoulis' function is in $H^1_{rect}$; results about $H^p \to L^p$ boundedness of multiparameter singular integral operators; new sufficient conditions for the atomic decomposition on product $H^p$; and variants of Journé's Lemma for the product of higher-dimensional Euclidean spaces. In Chapter 5, we present the variants of Journé's Lemma which we mentioned in the previous paragraph.

# Chapter 2

# Definitions and Background

This chapter is meant to serve as an overview of the background and notation required for the understanding of subsequent chapters. To the interested reader who is unfamiliar with the theory of real Hardy spaces, we would suggest poring over the first four chapters of [62].

## 2.1 Product Spaces and Notation

Our work is on the $n$-dimensional Euclidean space $\mathbb{R}^n$ viewed as a Cartesian product of the form $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$, $n = n_1 + \cdots + n_r$. The points $x$ in $\mathbb{R}^n$ are represented as $(x_1, \ldots, x_r)$, with $x_j \in \mathbb{R}^{n_j}$, $j = 1, ..., r$. We let $\|\cdot\|_{\mathbb{R}^{n_j}}$ denote a norm in $\mathbb{R}^{n_j}$, which will be either the Euclidean or the maximum norm (where no confusion arises, we will denote it by $\|\cdot\|$). We use the term *rectangle* for sets the form

$$R(x,t) := \left\{ y = (y_1, ..., y_r) \in \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r} : \|x_j - y_j\|_{\mathbb{R}^{n_j}} < t_j \right\}, \tag{2.1}$$

where $x_j \in \mathbb{R}^{n_j}$ and $t_j > 0$, $j = 1, ..., r$. We denote by $\mathcal{R}$ the set of rectangles of the form (2.1). Given $R(x,t) \in \mathcal{R}$ and $\epsilon > 0$, we define $\epsilon R := R(x, \epsilon t)$.

The cubes in the collection

$$\mathcal{D}^n := \left\{ 2^{-k}\left(z + [0,1]^n\right) : k \in \mathbb{Z},\ z \in \mathbb{Z}^n \right\}$$

will be called *dyadic cubes*. In the literature, the dyadic cubes are often defined with half-open intervals $[0,1)$ instead of closed $[0,1]$, but the fact that we choose them to be closed will not make any difference in our computations. We say that a set $R \subset \mathbb{R}^{n_1+...+n_r}$ is a *dyadic rectangle* in $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$ if it has the form $R = I_1 \times ... \times I_r$, for some cubes $I_j \in \mathcal{D}^{n_j}$, $j = 1, ..., r$. Thus, when we say that (the interior of) a dyadic rectangle is in $\mathcal{R}$, it means that for each $j = 1, ..., r$, the norm $\|\cdot\|_{\mathbb{R}^{n_j}}$ which we are considering in (2.1) is the maximum norm.

We denote all constants either by $c$ or by $C$, where $c$ (or $C$) may vary from line to line and may depend on the dimension of the space and other fixed entities. Sometimes we will put a subscript on $c$ (or $C$) to indicate on what it depends.

Given a measurable set $A \subset \mathbb{R}^n$, we denote its $n$-dimensional Lebesgue measure by $|A|$ and its characteristic function by $\chi_A$. Given a real number $\alpha$, we use the standard notation and denote its absolute value by $|\alpha|$ (this notation should not be confused with the Lebesgue measure, which is used with sets, not with numbers) and its floor and ceiling (also called roof) by $\lfloor \alpha \rfloor := \max\{n \in \mathbb{Z} : n \leq \alpha\}$ and $\lceil \alpha \rceil := \min\{n \in \mathbb{Z} : n \geq \alpha\}$, respectively. Given $\alpha, \beta$ in $[0, \infty)$, we say that $\alpha$ and $\beta$ are comparable, and write $\alpha \sim \beta$, when there exist positive constants $c, C$, independent of $\alpha$ and $\beta$, such that $c\beta \leq \alpha \leq C\beta$.

Following the usual notation, when $1 \leq p \leq \infty$, we denote the $L^p$ space over $\mathbb{R}^n$ by $L^p(\mathbb{R}^n)$, and we denote the $L^p(\mathbb{R}^n)$ norm by $\|\cdot\|_p$. For $0 < p < 1$, $L^p(\mathbb{R}^n)$ is defined as the space of Lebesgue measurable functions $f$ on $\mathbb{R}^n$ such that $|f|^p \in L^1(\mathbb{R}^n)$. For these values of $p$, the map $f \in L^p(\mathbb{R}^n) \mapsto \|f\|_p := \left(\int |f|^p\right)^{1/p}$ does not define norm: the triangular inequality fails.

For each $0 < \epsilon < \infty$, the *Orlicz space* $L\left(\log^+ L\right)^\epsilon(\mathbb{R}^n)$ [48], can be defined as the set of Lebesgue measurable real-valued functions $f$ on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right| \left(\log\left(1 + \left|\frac{f(x)}{\lambda}\right|\right)\right)^\epsilon dx \leq 1 \quad \text{for some } \lambda > 0. \tag{2.2}$$

The infimum over all $\lambda > 0$ such that (2.2) holds defines a norm in $L\left(\log^+ L\right)^\epsilon$. Endowed with this norm, $L\left(\log^+ L\right)^\epsilon$ is complete.

The *Schwartz space*, consisting of rapidly decreasing smooth functions on $\mathbb{R}^n$, is denoted by $\mathcal{S}(\mathbb{R}^n)$. The dual of $\mathcal{S}(\mathbb{R}^n)$, which is the space of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^n)$. In all other cases, $V^*$ will denote the dual space of $V$.

The Hardy-Littlewood maximal function is defined by

$$M(f)(x) := \sup\left\{\frac{1}{|Q|}\int_Q |f(y)|\,dy : Q \text{ is a cube containing } x\right\}, \qquad f \in L^1_{loc}(\mathbb{R}^n),$$

where the supremum is taken over all cubes $Q$ in $\mathcal{R}$ such that $x \in Q$. Its generalization to the product setting is the so called *strong maximal function*, which is defined [39] by

$$M_S(f)(x) := \sup\left\{\frac{1}{|R|}\int_R |f(y)|\,dy : R \in \mathcal{R} \text{ such that } x \in R\right\}, \quad f \in L^1_{loc}(\mathbb{R}^n). \quad (2.3)$$

Thus the strong maximal function is an operator. In many places in the literature, it is referred to as the *strong maximal operator*.

Recall that in the one-parameter setting, the proof of the Lebesgue differentiation theorem relies on the fact that the Hardy-Littlewood maximal function maps $L^1$ into weak-$L^1$. In the multiparameter setting, the behavior of the strong maximal function is also connected to the strong differentiability of the integral (see [32]). However, unlike in the one-parameter setting, $M_S$, viewed as an operator, is not of weak-type $(1,1)$: there exists a function $f$ in $L^1$ such that $|\{x : M_S(f)(x) = \infty\}| > 0$ [2], [58]. This is one of the many instances where the results of the classical theory do not carry over to the product setting.

## 2.2 Strong Differentiation of the Integral

The strong differentiation of the (indefinite) integral was introduced by Saks [60] (this reference is an English edition of "Théorie de l'intégrale", published in 1933) and consists on a generalization of the classical differentiation of the integral which was developed by Lebesgue in the 1910's. In the strong differentiation, instead of cubes (or balls or convex sets of bounded eccentricity), the infinitesimal averages are taken with respect to rectangles of the form (2.1). A more recent work about this theory a book by Guzmán

[32]. We adopt the notation from it. Specifically:

**Definition 2.1** ([60], [32]). *Given a real-valued function $f \in L^1_{loc}(\mathbb{R}^n)$, $n \geq 2$ and a point $x \in \mathbb{R}^n$, the strong upper derivative and the strong lower derivative of $\int f$ at $x$ are defined by*

$$\overline{D}\left(\int f, x\right) := \sup\left\{\limsup_{k\to\infty}\frac{1}{|R_k|}\int_{R_k} f(y)\,dy : \{R_k\}_{k\in\mathbb{N}} \subset \mathcal{R},\ R_k \to x\right\}$$

*and*

$$\underline{D}\left(\int f, x\right) := \inf\left\{\liminf_{k\to\infty}\frac{1}{|R_k|}\int_{R_k} f(y)\,dy : \{R_k\}_{k\in\mathbb{N}} \subset \mathcal{R},\ R_k \to x\right\},$$

*respectively, where $R_k \to x$ means that $\{R_k\}_{k\in\mathbb{N}}$ satisfies:*

$$x \in \bigcap_{k\in\mathbb{N}} R_k \quad \textit{and} \quad \lim_{k\to\infty} diam(R_k) = 0.$$

*If $\overline{D}\left(\int f, x\right)$ and $\underline{D}\left(\int f, x\right)$ coincide and are finite, then $\lim_{k\to\infty} |R_k|^{-1}\int_{R_k} f(y)\,dy$ exists for any $\{R_k\}_{k\in\mathbb{N}} \subset \mathcal{R}$ with $R_k \to x$, is denoted by $D\left(\int f, x\right)$ and is referred to as the strong derivative of $\int f$ at $x$. In this case we say that $\int f$ is strongly differentiable at $x$.*

Since every cube is a rectangle, if $\int f$ is strongly differentiable at a point $x$, then $D\left(\int f, x\right)$ agrees with the derivative of $\int f$ with respect to cubes at $x$. Thus, the classical differentiation theorem of Lebesgue implies that the equality $D\left(\int f, x\right) = f(x)$ holds for almost every point $x$ in the set where $\int f$ is strongly differentiable.

In a well-known paper of Jessen, Marcinkiewicz and Zygmund, they proved the following: in the $n$-fold product $\mathbb{R} \times \ldots \times \mathbb{R}$,
(i) when $p > 1$, $\|M_S(f)\|_p \leq c\|f\|_p$ for all $f \in L^p$;
(ii) if $f \in L^p_{loc}$, $p > 1$, then $\int f$ is strongly differentiable a.e.;
(iii) $M_S(f)$ is in weak $L^1$ whenever $|f|\log(1+|f|)^{n-1}$ is integrable; and
(iv) $\int f$ is strongly differentiable a.e. whenever $|f|\log(1+|f|)^{n-1}$ is locally integrable.

By contrast, as shown by an example of Busemann and Feller [3], there exists an integrable function whose integral fails to be strongly differentiable on a set of positive measure. Actually, as demonstrated by Saks [58], the set of integrable functions $f$ on $[0,1]^n$, such that $D\left(\int f, x\right) < \infty$ for some point $x \in [0,1]^n$, is of the first category in

$L^1([0,1]^n)$. Furthermore, working in the $n$-fold product $\mathbb{R} \times ... \times \mathbb{R}$, Saks showed that for any measurable function $\sigma : [0,\infty) \to (0,\infty)$ satisfying $\liminf_{t\to\infty} \sigma(t) = 0$, there exists a function $f$ such that $\sigma(|f|)|f|(\log(1+|f|))^{n-1} \in L^1([0,1]^n)$, but $\overline{D}\left(\int f, x\right) = \infty$ for almost every $x \in [0,1]^n$. This suggests that the positive results of Jessen et al. [39] concerning the Orlicz spaces $L\left(\log^+ L\right)^{n-1}(\mathbb{R}^n)$ are sharp.

The results (i) – (iv) of Jessen et al. [39], which we mentioned above, can be generalized to $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$. The weak-type bound for the strong maximal function (item (iii)) is called the strong maximal function theorem. Here we state a version of it which was shown by Zygmund [70] and by Guzmán (with a different proof which works in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$) [31].

**Theorem 2.1** (Strong Maximal Function Theorem [70], [31])**.** *In the multiparameter setting $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$, with $n_1 \in \mathbb{N}$ and $n_2 = ... = n_r = 1$,*

$$|\{x : M_S(f)(x) > \lambda\}| \leq c \int_{\mathbb{R}^{n_1+(r-1)}} \frac{|f(x)|}{\lambda} \left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right)^{r-1} dx,$$

*for any $\lambda > 0$.*

As stated by Guzmán (see [31], Section 3.3), in the product setting $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$, where the number of parameters is $r$, the result of Theorem 2.1 holds with a different constant $c$. There are other versions of the strong maximal function theorem in the literature. Some of them are presented in [32].

The behavior of the strong maximal function is connected with the strong differentiation of the integral (see [32], Chapter III, Section 3).

## 2.3 Background on Hardy Spaces

Originally, the Hardy spaces $H^p$, $0 < p < \infty$, were defined on the upper half of the complex plane $\mathbb{H} := \{z = x + it \in \mathbb{C} : t > 0\}$ (or on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$)

as spaces of homomorphic functions $F : \mathbb{H} \to \mathbb{C}$ satisfying

$$\|F\|_{H^p(\mathbb{H})}^p := \sup_{t>0} \int_{-\infty}^{\infty} |F(x+it)|^p \, dx < \infty \tag{2.4}$$

The map $F \in H^p(\mathbb{H}) \mapsto \|F\|_{H^p(\mathbb{H})}$ defines a norm when $1 \le p < \infty$, but not when $0 < p < 1$. While for $0 < p < 1$, $H^p(\mathbb{H})$ is not a normed space, the map

$$(f,g) \in H^p(\mathbb{H}) \times H^p(\mathbb{H}) \mapsto d_{H^p(\mathbb{H})}(f,g) := \|f-g\|_{H^p(\mathbb{H})}^p .$$

defines a metric and $H^p(\mathbb{F})$ is complete with respect to it (see [62], Chapter 3, Section 5.1). There exists also the space $H^\infty$, which consists of bounded holomorphic functions. Since we are interested in the cases when $0 < p \le 1$, we will not present results for $H^\infty$ here. The reader can find more information about complex Hardy spaces in [28] and in most other books about harmonic analysis.

Recall that if a real-valued harmonic function $u$ on $\mathbb{R}_+^2$ is the real part of a holomorphic function $F$ satisfying (2.4), then the *boundary values* of $u$ are well defined (see [28]). Specifically:

- (Case $0 < p < 1$) If $u = \mathfrak{Re}(F)$ for some holomorphic function $F$ satisfying (2.4), then there exists a bounded distribution (see definition on page 89 of [62]) $f$ in $\mathcal{S}'(\mathbb{R})$ such that $u(\cdot,t) \to f(\cdot)$ as $t \to 0$, and $u(\cdot,t) = (P_t * f)(\cdot)$ for all $t > 0$, where $P_t(x) = \frac{t}{\pi(t^2+x^2)}$ is the Poisson kernel on the line.

- (Case $1 \le p < \infty$) If $u = \mathfrak{Re}(F)$ for some holomorphic function $F$ satisfying (2.4), then there exists $f$ in $L^p(\mathbb{R})$ such that $\|(P_t * f) - f\|_p \to 0$ as $t \to 0$, and $u(\cdot,t) = (P_t * f)(\cdot)$ for all $t > 0$.

We would like to emphasize the main difference among the cases $0 < p < 1$, $p = 1$ and $1 < p < \infty$. For $1 \le p < \infty$, if a real-valued function $f$ is in $L^p(\mathbb{R})$, then $u(x,t) := (P_t * f)(x)$ satisfies

$$\sup_{t>0} \int_{-\infty}^{\infty} |u(x,t)|^p \, dx < \infty, \tag{2.5}$$

and $u(\cdot,t) \to f(\cdot)$ in $L^p$ as $t \to 0$. M. Riesz showed that, for $1 < p < \infty$, if a real-valued harmonic function $u$ on $\mathbb{R}^2_+$ satisfies (2.5), then $u = \mathfrak{Re}\,(F)$ for some $F$ satisfying (2.4), i.e. $F \in H^p\,(\mathbb{H})$. In this case, as $t \to 0$, $u\,(\cdot, t)$ converges in $L^p$ to an $L^p$ function whose Poisson integral is $u$. However, when $p = 1$, there are non-zero real-valued harmonic functions $u$ on $\mathbb{R}^2_+$ which satisfy (2.5) but $u\,(\cdot, t) \to 0$ pointwise as $t \to 0$. In this case $u$ cannot be recovered from its boundary values. For $0 < p \leq 1$, in order for a real-valued harmonic function $u$ on $\mathbb{R}^2_+$ to be the Poisson integral of its boundary values it is necessary and sufficient that $u = \mathfrak{Re}\,(F)$ for some $F \in H^p\,(\mathbb{H})$.

### 2.3.1 One-parameter Real Hardy Spaces

The real variable theory of Hardy spaces $H^p$, $0 < p < \infty$, begun in 1971 with a paper of Burkholder, Gundy and Silverstein [2], where they proved that a real-valued harmonic function $u$ on the upper half-plane $\mathbb{R}^2_+ := \{(x, t) \in \mathbb{R}^2 : t > 0\}$ (which, from a complex variable perspective, is viewed as $\mathbb{H}$) is the real part of some $F$ in $H^p\,(\mathbb{H})$ if and only if the non-tangential maximal function

$$u^*\,(x) := \sup_{(y,t) \in \Gamma(x)} |u\,(y, t)| \tag{2.6}$$

is in $L^p\,(\mathbb{R})$, where $\Gamma\,(x) := \left\{(y, t) \in \mathbb{R}^2_+ : |y - x| < t\right\}$, and in this case, $\|F\|^p_{H^p(\mathbb{H})} \sim \|u^*\|^p_p$. This result allows us say that the real Hardy space $H^p\left(\mathbb{R}^2_+\right)$, $0 < p < \infty$, consists of the harmonic functions $u$ on $\mathbb{R}^2_+$ such that $\int |u^*|^p < \infty$, and we can endow this space with the "norm" $\|u\|_{H^p\left(\mathbb{R}^2_+\right)} := \|u^*\|_p$ (as explained above, this is a norm if and only if $p \geq 1$). Combining this result with the facts about boundary values that we mentioned above, it possible to define $H^p\,(\mathbb{R})$, $0 < p < \infty$, as being the space of tempered distributions $f$ which arise as boundary values of functions $u$ in $H^p\left(\mathbb{R}^2_+\right)$. On this space, we can define $\|f\|_{H^p(\mathbb{R})} := \|u\|_{H^p\left(\mathbb{R}^2_+\right)}$.

It was only in 1972 that the real variable meaning of $H^p$, $0 < p < \infty$, was brought to light. This was achieved by C. Fefferman and Stein [21], who extended the result of Burkholder et al. to the upper half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$ and, in addition, established equivalent characterizations of $H^p$ which do not depend on harmonic functions.

We state some of the equivalent characterizations of $H^p$ in Theorems 2.2 and 2.3 below. In particular, Theorem 2.3 characterizes Hardy spaces using only the behavior of the boundary values.

**Theorem 2.2** ([61] case $1 < p < \infty$; [21] case $0 < p \leq 1$)**.** *Let $0 < p < \infty$ and let $u$ be a harmonic function on $\mathbb{R}^{n+1}_+$. Then the following are equivalent:*

*(i) the non-tangential maximal function $u^*$ is in $L^p(\mathbb{R}^n)$, where $u^*$ is defined by (2.6), except that, in the $n$ dimensional case, $\Gamma(x) := \left\{(y,t) \in \mathbb{R}^{n+1}_+ : \|y - x\| < t\right\}$;*

*(ii) $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in \mathbb{R}^n$, and the Lusin square function $S(u)(x) := \left(\int_{\Gamma(x)} |\nabla u(y,t)|^2 t^{-n+1} dydt\right)^{1/2}$ is in $L^p(\mathbb{R}^n)$.*

*In this case, we say that $u$ is in $H^p\left(\mathbb{R}^{n+1}_+\right)$, we define $\|u\|_{H^p\left(\mathbb{R}^{n+1}_+\right)} := \|u^*\|_p$ and we have $\|u^*\|_p \sim \|S(u)\|_p$.*

While this theorem tells us a way to characterize the real Hardy spaces $H^p\left(\mathbb{R}^{n+1}_+\right)$ which is still dependent on harmonic functions, it enables us to define (as in the case $n = 1$) $H^p(\mathbb{R}^n)$, $0 < p < \infty$, as being the space of boundary values $f$ of functions $u$ in $H^p\left(\mathbb{R}^{n+1}_+\right)$. On this space, $\|f\|_{H^p(\mathbb{R}^n)} := \|u^*\|_p$.

In what follows, for any function $\varphi$ on $\mathbb{R}^n$ and any $t > 0$, we use the standard notation

$$\varphi_t(x) := \frac{1}{t^n}\varphi\left(\frac{x}{t}\right). \tag{2.7}$$

**Theorem 2.3** ([21])**.** *Let $0 < p < \infty$ and let $f \in \mathcal{S}'(\mathbb{R}^n)$. The following are equivalent:*

*(i) for some $\varphi \in \mathcal{S}(\mathbb{R}^n)$, with $\int \varphi = 1$, the radial maximal function*

$$\mathfrak{M}_\varphi(f)(x) := \sup_{t>0} |(\varphi_t * f)(x)|$$

*is in $L^p(\mathbb{R}^n)$;*

*(ii) for some $\varphi$ as in (i), the non-tangential maximal function*

$$\mathfrak{N}_\varphi(f)(x) := \sup_{(y,t)\in\Gamma(x)} |(\varphi_t * f)(y,t)|$$

*is in* $L^p(\mathbb{R}^n)$*;*

*(iii) the grand maximal function*

$$\mathcal{M}_{\mathcal{A}}(f)(x) := \sup_{\varphi \in \mathcal{A}} \sup_{(y,t) \in \Gamma(x)} |(\varphi_t * f)(y,t)|$$

*is in* $L^p(\mathbb{R}^n)$*, where*

$$\mathcal{A} := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + \|x\|)^N \left( \sum_{|\alpha| \leq N} |D^\alpha \varphi(x)|^2 \right) dx \leq 1 \right\}$$

*and* $N = N(p,n)$ *is a fixed positive number;*

*(iv)* $f$ *is a bounded distribution (see definition on page 89 of [62]) and* $\sup_{t>0} |P_t * f(\cdot)|$ *is in* $L^p(\mathbb{R}^n)$*.*

*In this case,* $f \in H^p(\mathbb{R}^n)$ *and*

$$\|f\|_{H^p(\mathbb{R}^n)} \sim \|\mathfrak{M}_\varphi(f)\|_p \sim \|\mathfrak{N}_\varphi(f)\|_p \sim \|\mathcal{M}_{\mathcal{A}}(f)\|_p \sim \left\| \sup_{t>0} |P_t * f(\cdot)| \right\|_p .$$

For $p > 1$, $H^p$ corresponds to $L^p$ (see [62], Chapter 3, Section 1.2.1). To be more precise, the action of each tempered distribution that arises as boundary value of a harmonic function in $H^p(\mathbb{R}^n_+)$ can be represented as integration against a function in $L^p(\mathbb{R}^n)$, and conversely, the Poisson integral of any function in $L^p(\mathbb{R}^n)$ is a harmonic function in $H^p(\mathbb{R}^n_+)$.

By contrast, when $0 < p \leq 1$, $H^p$ is not equivalent to $L^p$. While $H^1$ is a proper subspace of $L^1$, when $0 < p < 1$, the elements of $H^p$ are tempered distributions. The same is true in the product Hardy spaces, which we define soon. Our work is focused on the product Hardy spaces $H^p$, for $0 < p \leq 1$. To deal with these spaces, we decompose their elements into infinite linear combinations of special functions.

The building blocks of the one-parameter real Hardy space $H^p(\mathbb{R}^n)$, $0 < p \leq 1$ are functions known as (one-parameter) *atoms*.

**Definition 2.2** ([15], [45]). *A function $a$ on $\mathbb{R}^n$ is called a (one-parameter) $p$-atom on $\mathbb{R}^n$ if $a$ is supported on a cube $Q$ in $\mathcal{R}$ and satisfies $\|a\|_\infty \leq |Q|^{-1/p}$ and $\int_{\mathbb{R}^n} a(x)\, x^\alpha dx = 0$ for all multi-indices $\alpha \in (\mathbb{Z}_{\geq 0})^n$ of order $|\alpha| \leq \lfloor n(p^{-1}-1) \rfloor$.*

The next result, which is the well-known atomic decomposition, was shown by Coifman [15] and Latter [45]. The former showed the necessity, while the other proved the sufficiency.

**Theorem 2.4** ([15], [45]). *Let $0 < p \leq 1$. A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $H^p(\mathbb{R}^n)$ if and only if*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k, \tag{2.8}$$

*where the series converges in $\mathcal{S}'(\mathbb{R}^n)$, $\{\lambda_k\}_k \in l^p$ and each $a_k$ is a $p$-atom on $\mathbb{R}^n$. In this case*

$$\|f\|_{H^p}^p \sim \sum_{k=1}^{\infty} |\lambda_k|^p .$$

When $p = 1$, the series in (2.8) converges in the $L^1$ norm.

The atomic decomposition is especially useful in the study of the behavior of operators on $H^p$. Still, one must be careful. Even in the case $p = 1$, where the convergence is in $L^1$, there are linear maps $L$ such that the equality $\sum_{k=1}^{\infty} \lambda_k L(a_k) = L\left(\sum_{k=1}^{\infty} \lambda_k (a_k)\right)$ does not hold. This is explained in a paper of Bownik [1], where he gave an example of a linear functional, defined on a dense subspace of $H^1$, which maps all 1-atoms into scalars bounded by 1, but cannot be extended to a bounded linear functional on $H^1$.

The dual space of the one-parameter $H^1(\mathbb{R}^n)$ can be identified with the space of functions of *bounded mean oscillation*, $BMO$. This result is the famous duality theorem of C. Feffermam [20], which was proved using the following definition:

**Definition 2.3** ([40], [20]). *$BMO(\mathbb{R}^n)$ consists, up to additive constants, of locally integrable functions $f$ on $\mathbb{R}^n$ such that*

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_{\mathbb{R}^n} |f(x) - f_Q|\, dx < \infty,$$

*where the supremum is taken over all cubes $Q$ in $\mathcal{R}$ and $f_Q := \frac{1}{|Q|} \int_{\mathbb{R}^n} f(x)\, dx$.*

### 2.3.2 Real Hardy Spaces in the Product Setting

The extension of the one-parameter real-variable theory of Hardy spaces to the product setting was first accomplished by Gundy and Stein [30] in 1979. They characterized $H^p$ spaces, $0 < p < \infty$, on the product of upper half-spaces $\mathbb{R}_+^{n_1+1} \times ... \times \mathbb{R}_+^{n_r+1}$ in terms of the boundary behavior of multiharmonic (also called multiply harmonic: the Laplacian on each parameter is zero) functions. They also extended the equivalences stated in Theorems 2.2 and 2.3 to the product setting. We state some of their results here. We present them in the two-parameter setting, because this way, the notation is simpler. For settings with more than two parameters, the characterizations are similar.

**Theorem 2.5** ([30])**.** *Let $0 < p < \infty$. Then $u \in H^p\left(\mathbb{R}_+^{n_1+1} \times \mathbb{R}_+^{n_2+1}\right)$ if and only if the non-tangential maximal function $u^*(x_1, x_2) := \sup_{(y,t)\in\Gamma(x_1)\times\Gamma(x_2)} |u(y_1, t_1, y_2, t_2)|$ is in $L^p(\mathbb{R}^{n_1+n_2})$. In this case, we define $\|u\|_{H^p\left(\mathbb{R}_+^{n_1+1}\times\mathbb{R}_+^{n_2+1}\right)} := \|u^*\|_p$.*

In Theorem 2.5, we only mentioned one characterization of $H^p\left(\mathbb{R}_+^{n_1+1} \times \mathbb{R}_+^{n_2+1}\right)$. Gundy and Stein [30] extended both equivalences of Theorem 2.3 to the product setting. The characterization of $H^p\left(\mathbb{R}_+^{n_1+1} \times \mathbb{R}_+^{n_1+1}\right)$ in terms of a multiparameter variant of the Lusin area integral was first proved by M. P. Malliavin and P. Malliavin [49], in 1977. They used a complicated algebraic method to show that the area integral is controlled by the non-tangential maximal function.

Since the product space analogues of the facts about boundary values presented in the beginning of Subsection 2.3 hold, Theorem 2.5 enables us to define the *real Hardy space in the product setting*, $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, as the space of tempered distributions $f$ which arise as boundary values of functions $u$ in $H^p\left(\mathbb{R}_+^{n_1+1} \times \mathbb{R}_+^{n_2+1}\right)$, and the "norm" in $H^p\left(\mathbb{R}_+^{n_1+1} \times \mathbb{R}_+^{n_2+1}\right)$ can be set as $\|f\|_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} := \|u^*\|_p$.

The following result, which is an extension of Theorem 2.3, provides a purely real variable characterization of the real Hardy spaces in the product setting. In oder to state it we need to introduce some notion.

For any $\varphi \in \mathcal{S}(\mathbb{R}^{n_1})$, $\psi \in \mathcal{S}(\mathbb{R}^{n_2})$, $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ and any $t_1, t_2 > 0$, we define

$$((\varphi_{t_1} \cdot \psi_{t_2}) * f)(x) := \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \varphi_{t_1}(y_1) \psi_{t_2}(y_2) f(x_1 - y_1, x_2 - y_2) \, dy_1 dy_2,$$

for $x=(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, where $\varphi_{t_1}$ and $\psi_{t_2}$ are defined in (2.7).

**Theorem 2.6** ([30] *(i)*$\Leftrightarrow$*(ii)*$\Leftrightarrow$*(iii)*$\Leftrightarrow f\in H^p$)**.** *Let $0<p<\infty$ and let $f\in\mathcal{S}'(\mathbb{R}^{n_1+n_2})$. The following are equivalent:*

*(i) for some $\varphi^j\in\mathcal{S}(\mathbb{R}^{n_j})$, with $\int\varphi^j=1$, $j=1,2$, the radial maximal function*

$$\mathfrak{M}_{\varphi^1,\varphi^2}(f)(x):=\sup_{t_j>0,\ j=1,2}\left|\left(\left(\varphi^1_{t_1}\cdot\varphi^2_{t_2}\right)*f\right)(x)\right|$$

*is in $L^p(\mathbb{R}^{n_1+n_2})$;*

*(ii) for some $\varphi^1,\varphi^2$ as in (i), the non-tangential maximal function*

$$\mathfrak{N}_{\varphi^1,\varphi^2}(f)(x):=\sup_{(y,t)\in\Gamma(x_1)\times\Gamma(x_2)}\left|\left(\left(\varphi^1_{t_1}\cdot\varphi^2_{t_2}\right)*f\right)(y)\right|$$

*is in $L^p(\mathbb{R}^{n_1+n_2})$;*

*(iii) the grand maximal function*

$$\mathcal{M}_{\mathcal{A}_1,\mathcal{A}_2}(f)(x):=\sup_{\varphi^j\in\mathcal{A}_j,\ j=1,2}\left[\sup_{(y,t)\in\Gamma(x_1)\times\Gamma(x_2)}\left|\left(\left(\varphi^1_{t_1}\cdot\varphi^2_{t_2}\right)*f\right)(y)\right|\right]$$

*is in $L^p(\mathbb{R}^{n_1+n_2})$, where*

$$\mathcal{A}_j:=\left\{\varphi\in\mathcal{S}(\mathbb{R}^{n_j}):\int_{\mathbb{R}^{n_j}}(1+\|x_j\|)^{N_j}\left(\sum_{|\alpha|\le N_j}|D^\alpha\varphi(x_j)|^2\right)dx_j\le 1\right\}$$

*and $N_j=N(p,n_j)$, $j=1,2$, are fixed positive numbers;*

*(iv) $f$ is a bounded distribution and $\sup_{t_j>0}|(P_{t_1}\cdot P_{t_2})*f(\cdot)|$ is in $L^p(\mathbb{R}^{n_1+n_2})$.*

*In this case, $f\in H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$,*

$$\|f\|_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\sim\|\mathfrak{M}_{\varphi^1,\varphi^2}(f)\|_p \tag{2.9}$$

*and*

$$\|\mathfrak{M}_{\varphi^1,\varphi^2}(f)\|_p \sim \|\mathfrak{N}_{\varphi^1,\varphi^2}(f)\|_p \sim \|\mathcal{M}_{\mathcal{A}_1,\mathcal{A}_2}(f)\|_p \sim \left\| \sup_{t_j>0} |(P_{t_1}\cdot P_{t_2}) * f(\cdot)| \right\|_p . \qquad (2.10)$$

In order to have the equivalences (2.9) and (2.10) (which were not included in [30]), we need to add the hypothesis $\int \varphi^j = 1$, $j = 1, 2$, to items (i) and (ii). The proof of (2.9) and (2.10), as well as the equivalence between item (iv) and the previous items, can be done by adapting the argument presented in the proof of Theorem 11 in [21] (stated above as Theorem 2.3) to the product setting. For $p = 1$, Theorem 2.6 holds [47].

Note that Theorem 2.6 implies that $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, $0 < p < \infty$, can be defined by fixing $\varphi^j \in \mathcal{S}(\mathbb{R}^{n_j})$, with non-zero integrals, $j = 1, 2$, and saying that $f$, in $\mathcal{S}'(\mathbb{R}^{n_1+n_2})$, belongs to $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ if and only if $\mathfrak{M}_{\varphi^1,\varphi^2}(f) \in L^p(\mathbb{R}^{n_1+n_2})$. In this case, $c_{\varphi^1,\varphi^2}\|f\|_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \le \|\mathfrak{M}_{\varphi^1,\varphi^2}(f)\|_p \le C_{\varphi^1,\varphi^2}\|f\|_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$, for some constants $c_{\varphi^1,\varphi^2}$, $C_{\varphi^1,\varphi^2}$.

Although $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, $0 < p < 1$, is not a Banach space, it is a complete space with respect to the metric

$$d_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}(f,g) := \|f-g\|^p_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} . \qquad (2.11)$$

We now define the rectangle atoms in the two-parameter setting. The definition in the case of more parameters can be found in Chapter 4 (Definition 4.1).

**Definition 2.4** ([13] and [27] case $q = 2$; [5])**.** *Let $0 < p \le 1 < q \le \infty$. A function $a$ is called a rectangle $(p,q)$-atom on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ if it satisfies supp$(a) \subset R = I_1 \times I_2 \in \mathcal{R}$; $\|a\|_q \le |R|^{1/q-1/p}$; and*

$$\int a(x_1,y_2)\, x_1^{\alpha_1} dx_1 = \int a(y_1,x_2)\, x_2^{\alpha_2} dx_2 = 0 \qquad (2.12)$$

*for all $(y_1,y_2) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ and all $\alpha_j = \left(\alpha_{j,1},...,\alpha_{j,n_j}\right) \in (\mathbb{Z}_{\ge 0})^{n_j}$, $j = 1, 2$, with*

$$|\alpha_j| := \sum_{i=1}^{n_j} \alpha_{j,i} \le \mathcal{N}_j := \left\lfloor n_j\left(\frac{1}{p}-1\right)\right\rfloor .$$

For $0 < p \leq 1$, a subspace of $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is the so called *rectangular Hardy space*. This space is denoted by $H^p_{rect}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. S.-Y. Chang and R. Fefferman [13] dealt with $H^1_{rect}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, which consists of infinite linear combinations of the form $\sum_{k=1}^{\infty} \lambda_k a_k$, where the $a_k$'s are rectangle $(1,2)$-atoms, $\{\lambda_k\} \in l^1$, and the series converges in the $L^1$ norm. This definition can be adapted to the other values of $p$ in $(0,1]$.

**Definition 2.5** ([5])**.** *Let $0 < p \leq 1$. The rectangular Hardy space $H^p_{rect}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ consists of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$ which be written as*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k,$$

*where the series converges in the sense of distributions, $\{\lambda_k\} \in l^p$ and each $a_k$ is a rectangle $(p,2)$–atom on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.*

It is useful to note that the inclusion

$$H^p_{rect}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \subset H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \tag{2.13}$$

is an immediate consequence of Theorem 1.4. This affirmation follows from the fact that any rectangle $(p,2)$-atom is a rough $(p,2)$-atom with one elementary particle. However, the inclusion (2.13) is proper. This fact was first noticed for the case $p = 1$: the counterexample of Carleson [9], the duality [11] $BMO(\mathbb{R} \times \mathbb{R}) \cong (H^1(\mathbb{R} \times \mathbb{R}))^*$ and the characterization [10] of $BMO(\mathbb{R} \times \mathbb{R})$ in terms of Carleson measures imply that $H^1(\mathbb{R} \times \mathbb{R})$ is strictly larger than $H^1_{rect}(\mathbb{R} \times \mathbb{R})$. For $0 < p \leq 1$, since the dual of $H^p(\mathbb{R} \times ... \times \mathbb{R})$ can be characterized in terms of Carleson measures [51], the proper inclusion $H^p_{rect}(\mathbb{R} \times ... \times \mathbb{R}) \subsetneq H^p(\mathbb{R} \times ... \times \mathbb{R})$ also holds.

While not every function in $H^1(\mathbb{R} \times \mathbb{R})$ can be expressed as an infinite linear combination of rectangle atoms (with coefficients in $l^1$), the next theorem, known as the *Chang-Fefferman atomic decomposition*, tells us that we can decompose any element of $H^p(\mathbb{R} \times \mathbb{R})$, $0 < p \leq 1$, as an infinite linear combination of special functions (with coefficients in $l^p$). The statement of this result demands the definition of Chang-Fefferman atoms [12], which requires some notation.

Given an open set $\Omega \subset \mathbb{R}^{n_1+\ldots+n_r}$ with finite measure, denote by $\mathcal{M}(\Omega)$ the set of *maximal dyadic rectangles* [42], [25] of $\Omega$, where a dyadic rectangle $I_1 \times \ldots \times I_r$ is said to be maximal when, for any $j = 1, \ldots r$, if there exists $J_j$ in $\mathcal{D}^{n_j}$ such that $I_j \subset J_j$ and $I_1 \times \ldots \times I_{j-1} \times J_j \times I_{j+1} \ldots \times I_r \subset \Omega$, then $I_j = J_j$.

Now we are able to define Chang-Fefferman atoms.

**Definition 2.6** ([12]). *Let $0 < p \leq 1$. A function $a$ in $L^2(\mathbb{R}^2)$ is a Chang-Fefferman p-atom on $\mathbb{R} \times \mathbb{R}$ if it satisfies:*

*(A) supp($a$) $\subset \Omega$ for some open set $\Omega \subset \mathbb{R}^2$ with $|\Omega| < \infty$;*

*(B) $\|a\|_2 \leq |\Omega|^{1/2-1/p}$,*

*(C) $a$ can be expressed as $a = \sum_{R \in \mathcal{M}(\Omega)} a_R$, where each $a_R$ is a function, called a Chang-Fefferman p-elementary particle, such that*

*(C.1) supp($a_R$) $\subset 3R$ for some distinct maximal dyadic rectangle $R = I_1 \times I_2 \in \mathcal{M}(\Omega)$;*

*(C.2) $a_R$ satisfies the vanishing moment conditions*

$$\int a_R(x_1, y_2)\, x_1^{\alpha_1} dx_1 = \int a_R(y_1, x_2)\, x_2^{\alpha_2} dx_2 = 0,$$

*for all $(y_1, y_2) \in \mathbb{R}^2$ and all $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{Z}_{\geq 0})^2$ with $|\alpha_j| \leq k(p) := \left\lfloor \frac{2}{p} - \frac{3}{2} \right\rfloor$, $j = 1, 2$;*

*(C.3) $a_R$ is of class $C^{k(p)+1}$ and satisfies*

$$\|a_R\|_\infty \leq d_R \text{ and } \left\| \frac{\partial^m a_R}{\partial x_j^m} \right\|_\infty \leq \frac{d_R}{|I_j|^m} \text{ for } 1 \leq m \leq k(p) + 1 \text{ and } j = 1, 2;$$

*(C.4) the constants $d_R$'s of item (iii) satisfy $\sum_{R \in \mathcal{M}(\Omega)} d_R^2 \leq |\Omega|^{1-2/p}$.*

Now we state the original atomic decomposition for product Hardy spaces.

**Theorem 2.7** (Chang-Fefferman Atomic Decomposition [11], [12]). *Let $0 < p \leq 1$. A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to $H^p(\mathbb{R} \times \mathbb{R})$ if and only if*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k,$$

*where the series converges in $\mathcal{S}'(\mathbb{R}^2)$, $\{\lambda_k\}_k \in l^p$ and each $a_k$ is a Chang-Fefferman $p$–atom on $\mathbb{R} \times \mathbb{R}$. In this case,*

$$\|f\|_{H^p} \sim \sum_{k=1}^{\infty} |\lambda_k|^p.$$

Note that the sufficiency in Theorem 2.7 was proved [12] without using Journé's Lemma [42].

The next definition is based on that of Chang-Fefferman atoms (Definition 2.6).

**Definition 2.7** ([5]). *Let $0 < p \leq 1 < q < \infty$. We say that a function $a$ is a rough $(p, q)$-atom on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ if it satisfies*

*(A) supp$(a) \subset \Omega$ for some open set $\Omega \subset \mathbb{R}^{n_1+n_2}$ with $|\Omega| < \infty$;*

*(B) $\|a\|_q \leq |\Omega|^{1/q-1/p}$;*

*(C) $a = \sum_{R \in \mathcal{M}(\Omega)} a_R$, where each $a_R$ is a function supported on $4R$, for some distinct maximal dyadic rectangle $R \in \mathcal{M}(\Omega)$, and satisfies (2.12), and*

$$\sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_q^q \leq |\Omega|^{1-q/p}. \tag{2.14}$$

Many variants of S.-Y. Chang and R. Fefferman's product space atoms, and atomic decomposition, exist in the literature. We list some works where one can find variants of Theorem 2.7. An alternative way of decomposing an element of $H^p(\mathbb{R} \times \mathbb{R})$, $0 < p \leq 1$, into Chang-Fefferman atoms was developed by M. Wilson [68]. R. Fefferman [24] presented atomic decompositions of $H^1(\mathbb{R} \times \mathbb{R})$ with two alternative definitions of product atoms. Li et al. [47] showed that the smoothness and the $L^\infty$-boundedness of the

Chang-Fefferman elementary particles are superfluous for the sufficiency of the atomic decomposition on product $H^1$. Using a discrete Calderón identity, Han et al. [36] showed that every element of $L^p \cap H^p$ can be decomposed into an infinite linear combination, converging in the $L^p$ norm, of variants of Chang-Fefferman $p$-atoms with coefficients in $l^p$.

The dual of $H^1(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})$ is the space known as product $BMO$ [11], which can be defined in terms of Carleson measures, among other characterizations. Below we give a definition of product $BMO$ which will clarify the relationship between Carleson's counterexample [9] and product $H^1$. But first, we will present the counterexample. This requires the following:

**Definition 2.8.** *Given two intervals $I_1, I_2 \subset \mathbb{R}$, the tent over the rectangle $R = I_1 \times I_2$ is defined by $S(R) := T(I_1) \times T(I_2)$, where for any interval $I \subset \mathbb{R}$, with center at $x_0$ and length $2\delta$, $T(I) := \{(x,t) \in \mathbb{R}^2_+ : |x - x_0| < \delta - t\}$ is the tent over $I$. We say that a measure $\mu$ on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ satisfies the Carleson condition with respect to rectangles if*

$$\sup_{R \in \mathcal{R}} \frac{\mu(S(R))}{|R|} < \infty.$$

**Definition 2.9** ([10], [23])**.** *Given a non-empty connected open set $\Omega \subset \mathbb{R}^2$, the Carleson region over $\Omega$ is defined by*

$$S(\Omega) := \bigcup_{\substack{R \subset \Omega \\ R \in \mathcal{R}}} S(R).$$

*We say that a measure $\mu$ on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ satisfies the Carleson condition on the product space $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ if*

$$\sup_{\Omega} \frac{\mu(S(\Omega))}{|\Omega|} < \infty,$$

*where the supremum is taken over all non-empty connected open sets $\Omega \subset \mathbb{R}^2$ with finite measure.*

Carleson's example consists of a measure $\mu$ on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ which satisfies the Carleson condition with respect to rectangles, but does not satisfy it on the product space $\mathbb{R}^2_+ \times \mathbb{R}^2_+$.

From his example, it follows that $BMO(\mathbb{R} \times \mathbb{R})$ is strictly larger than the space *rectangular BMO* on $\mathbb{R}\times\mathbb{R}$, which is denoted by $BMO_{rect}(\mathbb{R}\times\mathbb{R})$ [13]. In order to explain this affirmation, we need to define these spaces.

**Definition 2.10** ([10], [23]). *$BMO_{rect}(\mathbb{R}\times\mathbb{R})$ consists of locally integrable functions $f$ on $\mathbb{R}^2$ such that*

$$\sup_{R=I_1\times I_2\in\mathcal{R}}\frac{1}{|R|}\int_{\mathbb{R}^2}|f(x_1,x_2)-f_{I_1}(x_2)-f_{I_2}(x_1)|^2\,dx<\infty,$$

*where, for each $I_1\times I_2\in\mathcal{R}$,*

$$f_{I_1}(x_2):=\frac{1}{|I_1|}\int_{\mathbb{R}}f(x_1,x_2)\,dx_1 \text{ and } f_{I_2}(x_1):=\frac{1}{|I_2|}\int_{\mathbb{R}}f(x_1,x_2)\,dx_2.$$

This characterization (compare with Definition 2.3) facilitates the proof of the duality $BMO_{rect}(\mathbb{R}\times\mathbb{R})\cong\left(H^1_{rect}(\mathbb{R}\times\mathbb{R})\right)^*$. In fact, this can be shown by adapting the proof of the duality $BMO\cong\left(H^1\right)^*$ in the one-parameter setting, presented in Chapter 4 of [62].

An equivalent way of characterizing $BMO_{rect}(\mathbb{R}\times\mathbb{R})$ is given by the following:

**Theorem 2.8** ([10], [23], [13]). *A function $f$ in $L^1_{loc}(\mathbb{R}^2)$ belongs to $BMO_{rect}(\mathbb{R}\times\mathbb{R})$ if and only if $\int(1+|x_1|)^{-2}(1+|x_2|)^{-2}|f(x)|\,dx<\infty$ and the measure $\mu_f$, defined on $\mathbb{R}^2_+\times\mathbb{R}^2_+$ by*

$$d\mu_f(x_1,t_1,x_2,t_2):=|\nabla_1\nabla_2 u(x_1,t_1,x_2,t_2)|^2\,t_1t_2dx_1dt_1dx_2dt_2, \tag{2.15}$$

*satisfies the Carleson condition with respect to rectangles, where*

$$u(x_1,t_1,x_2,t_2):=\left(\left(P_{t_1}\cdot P_{t_2}\right)*f\right)(x_1,x_2),$$

*and*

$$|\nabla_1\nabla_2 u|^2:=\left|\frac{\partial^2 u}{\partial x_1\partial t_1}\right|^2+\left|\frac{\partial^2 u}{\partial x_1\partial t_2}\right|^2+\left|\frac{\partial^2 u}{\partial x_2\partial t_1}\right|^2+\left|\frac{\partial^2 u}{\partial x_2\partial t_2}\right|^2.$$

**Definition 2.11** ([10], [23]). *A locally integrable function $f$ on $\mathbb{R}^2$ is in $BMO(\mathbb{R}\times\mathbb{R})$ if and only if $\int(1+|x_1|)^{-2}(1+|x_2|)^{-2}|f(x)|\,dx<\infty$ and the measure $\mu_f$, defined in (2.15), satisfies the Carleson condition on $\mathbb{R}^2_+\times\mathbb{R}^2_+$.*

In the light of Carleson's counterexample, the fact that $BMO(\mathbb{R}\times\mathbb{R}) \subsetneq BMO_{rect}(\mathbb{R}\times\mathbb{R})$ becomes clear. On the other hand, $BMO(\mathbb{R}\times\mathbb{R})$ is isomorphic to the dual of $H^1(\mathbb{R}\times\mathbb{R})$ [11]. So

$$\left(H^1(\mathbb{R}\times\mathbb{R})\right)^* \cong BMO(\mathbb{R}\times\mathbb{R}) \subsetneq BMO_{rect}(\mathbb{R}\times\mathbb{R}) \cong \left(H^1_{rect}(\mathbb{R}\times\mathbb{R})\right)^*,$$

and this yields $H^1(\mathbb{R}\times\mathbb{R}) \supsetneq H^1_{rect}(\mathbb{R}\times\mathbb{R})$.

## 2.4 Original Journé's Lemmas

Motivated by the question of how to verify that a measure satisfies the Carleson condition on $\mathbb{R}^2_+\times\mathbb{R}^2_+$, Journé proved a result which is known today as Journé's Lemma (Propositions 1 in [42]). We refer to it as the *discrete Journé's Lemma*. This result is a very useful tool for working in the product setting. It tells us, in particular, that, among the rectangles contained in a bounded open set $\Omega\subset\mathbb{R}^2$, we can select a countable collection with enjoys the two following properties: (i) the union of the rectangles in that collection contains $\Omega$, and (ii) if we multiply the area of each of the rectangles in that collection by a suitable non-zero scalar and sum the results, the sum is controlled by $c|\Omega|$. Instead of proving the discrete Journé's Lemma directly, he showed a non-discrete version of it (Propositions 2 in [42]). We refer to this result as the *non-discrete Journé's Lemma*. The discrete version is a corollary of the non-discrete one.

We present the discrete Journé's Lemma first, in the next lemma. This result is a very useful tool for working in the product setting. We state and prove it in the setting $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ later (Theorem 4.2). While Journé stated that it holds in this setting, he proved it only in the case $n_1=n_2=1$. Moreover, the proofs of it which are available in the literature are also restricted to dimension one.

The statement of the discrete Journé's Lemma requires the establishment of some notation.

Given an open set $\Omega\subset\mathbb{R}^2$ with $|\Omega|<\infty$, we denote by $\mathcal{M}_1(\Omega)$ the set of dyadic rectangles $R:=I_1\times I_2\subset\Omega$ which are maximal with respect to the $x_1$-direction. Specifically, the elements of $\mathcal{M}_1(\Omega)$ are dyadic rectangles $R := I_1 \times I_2 \subset \Omega$ such that if $I_1' \times I_2$ is a dyadic rectangle in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with $R \subset I_1' \times I_2 \subset \Omega$, then $I_1 = I_1'$. The set $\mathcal{M}_2(\Omega)$ is defined analogously. Then $\mathcal{M}(\Omega) = \mathcal{M}_1(\Omega) \cap \mathcal{M}_2(\Omega)$.

For a set $\Omega$, as described in the previous paragraph, let

$$\widetilde{\Omega} := \{x : M_S(\chi_\Omega)(x) > 1/2\}$$

and, to each $R = I_1 \times I_2$ in $\mathcal{M}_2(\Omega)$, let

$$\widehat{I_1} \times I_2 \in \mathcal{M}_1\left(\widetilde{\Omega}\right)$$

be the unique rectangle in $\mathcal{M}_1\left(\widetilde{\Omega}\right)$ such that its first component, $\widehat{I_1}$, contains $I_1$ and its second component is $I_2$. In the literature, the number

$$\gamma_1(I_1 \times I_2, \Omega) := \sup\left\{\frac{|I_1'|}{|I_1|} : I_1' \supset I_1 \text{ and } I_1' \times I_2 \in \mathcal{M}_1\left(\widetilde{\Omega}\right)\right\} = \frac{\left|\widehat{I_1}\right|}{|I_1|} \tag{2.16}$$

is often referred to as the *stretching factor* of $R = I_1 \times I_2$ in the $x_1$-direction.

**Lemma 2.1** (Discrete Journé's Lemma [42]). *Let $\Omega \subset \mathbb{R} \times \mathbb{R}$ be a bounded open set and let $\phi : [0,1] \to [0,\infty)$ be an increasing function satisfying $\int_0^1 \phi(s)\frac{ds}{s} < \infty$. Then*

$$\left|\bigcup_{I \times J \in \mathcal{M}_2(\Omega)} \widehat{I} \times J\right| \leq 8|\Omega| \tag{2.17}$$

*and*

$$\sum_{I \times J \in \mathcal{M}_2(\Omega)} |I \times J| \phi\left(\frac{\left|\widehat{I}\right|}{|I|}\right) \leq 2\left(\sum_{j=0}^{\infty} \phi\left(2^{-j}\right)\right)|\Omega|. \tag{2.18}$$

This lemma is a corollary of the next one. In oder to state the next lemma, we need to introduce some more notation.

For $x \in \mathbb{R}$ and $t > 0$, the open set $E_{x,t} := \left\{y \in \mathbb{R} : \overline{B(x,t)} \times \{y\} \subset \Omega\right\}$ can be written as a countable union of disjoint open intervals, $J_{t,x,k}$, with the indices $k$ in a subset of $\mathbb{N}$, which we call $\Lambda(x,t)$, i.e.

$$E_{x,t} = \underset{k\in\Lambda(x,t)}{\cup} J_{x,t,k}. \tag{2.19}$$

For each $(x,t,k) \in \mathbb{R} \times (0,\infty)$ such that $E_{x,t} \neq \emptyset$ and each $k \in \Lambda(x,t)$, define

$$\tau(x,t,k) := \inf\left\{ s \geq t : \frac{|E_{x,s} \cap J_{x,t,k}|}{|J_{x,t,k}|} \leq \frac{1}{2} \right\}. \tag{2.20}$$

Now we are able to present Journé's Lemma in its non-discrete version.

**Lemma 2.2** (Non-discrete Journé's Lemma [42])**.** *Let $\Omega \subset \mathbb{R} \times \mathbb{R}$ be a bounded open set and let $\phi : [0,1] \to [0,\infty)$ be an increasing function satisfying $\int_0^1 \phi(s)\frac{ds}{s} < \infty$. Then*

$$\left| \bigcup_{\substack{x\in\mathbb{R},\, t>0 \\ k\in\Lambda(x,t)}} (x - \tau(x,t,k), x + \tau(x,t,k)) \times J_{(x,t,k)} \right| \leq 8|\Omega|,$$

*and*

$$\int_0^\infty \int_{-\infty}^\infty \sum_{k\in\Lambda(x,t)} |J_{x,t,k}|\, \phi\left(\frac{t}{\tau(x,t,k)}\right) dx \frac{dt}{t} \leq 2\left(\int_0^1 \phi(s)\frac{ds}{s}\right)|\Omega|, \tag{2.21}$$

*where $J_{x,t,k}$ and $\tau(x,t,k)$ are defined in (2.19) and (2.20), respectively.*

# Chapter 3

# Marstrand's Approximate Independence of Sets

By modifying an example of J. M. Marstrand, we construct a function $f$ belonging to the product Hardy space $H^1(\mathbb{R}\times\mathbb{R})$ and the Orlicz space $L(\log L)^{\epsilon}(\mathbb{R}^2)$ for all $0<\epsilon<1$, such that $\int f$ is not strongly differentiable almost everywhere on a square of sidelength 1.

In addition, we generalize the claim concerning "approximately independent sets" that appears in J. M. Marstrand's work in relation to hyperbolic-cross shaped sets on the plane. Our generalization, which holds for any sets with boundary of sufficiently low complexity in any Euclidean space, has a version of the second Borel-Cantelli Lemma as a corollary.

## 3.1 Introduction

The one-parameter *real Hardy space* $H^1\left(\mathbb{R}^d\right)$ [21] can be defined as the space of distributions $f$ in $\mathcal{S}'\left(\mathbb{R}^d\right)$ such that $\sup_{t>0}\left|t^{-d}\left(f*\varphi\right)\left(t^{-1}x\right)\right|$ is integrable, for some fixed $\varphi\in\mathcal{S}\left(\mathbb{R}^d\right)$ with non-vanishing integral. The *product Hardy space* $H^1\left(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}\right)$ [30] can be defined as the space of distributions $f$ in $\mathcal{S}'\left(\mathbb{R}^{d_1+d_2}\right)$ such that, for some fixed $\varphi\in\mathcal{S}\left(\mathbb{R}^{d_1}\right)$, $\psi\in\mathcal{S}\left(\mathbb{R}^{d_2}\right)$ with non-vanishing integrals,

$$\sup_{t_j>0}\left|t_1^{-d_1}t_2^{-d_2}\int\int\varphi\left(t_1^{-1}y_1\right)\psi\left(t_2^{-1}y_2\right)f\left(x_1-y_1,x_2-y_2\right)dy_1dy_2\right|$$

is in $L^1\left(\mathbb{R}^{d_1+d_2}\right)$, where the points $x$ in $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$ are represented as $x=(x_1,x_2)$, with $x_j\in\mathbb{R}^{d_j}$, $j=1,2$.

For each $0<\epsilon<1$, the *Orlicz space* $L(\log L)^{\epsilon}\left(\mathbb{R}^d\right)$ [48], also denoted $L^{\Phi_\epsilon}\left(\mathbb{R}^d\right)$, can be defined as the set of $f$ real-valued, measurable functions on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d}\Phi_\epsilon\left(\frac{f(x)}{\lambda}\right)dx\le 1,$$

for some $\lambda>0$, where $\Phi_\epsilon(t):=|t|\left(\log(1+|t|)\right)^{\epsilon}$, $t\in\mathbb{R}$. The *Luxemburg norm* on $L^{\Phi_\epsilon}\left(\mathbb{R}^d\right)$ is defined by

$$\|f\|_{\Phi_\epsilon}:=\inf\left\{\lambda>0:\int\Phi_\epsilon\left(\frac{f(x)}{\lambda}\right)dx\le 1\right\}.$$

Endowed with the norm $\|\cdot\|_{\Phi_\epsilon}$, $L^{\Phi_\epsilon}\left(\mathbb{R}^d\right)$ is a complete space.

While the integral of functions in $L^p_{loc}\left(\mathbb{R}^d\right)$, $p>1$, is strongly differentiable a.e. [39] and this property also holds for the integral of functions which are locally in $L\log L\left(\mathbb{R}^2\right)$ [39], it fails for certain classes of functions satisfying slightly weaker integrability conditions [59]. In particular, it fails in $L^1_{loc}\left(\mathbb{R}^d\right)$. Since many results concerning boundedness of singular operators can be extended from $L^p\left(\mathbb{R}^d\right)$, $p>1$, to the Hardy spaces $H^1\left(\mathbb{R}^d\right)$ [62], the question arose as to whether the strong differentiation of the integral would hold in $H^1\left(\mathbb{R}^d\right)$. This was answered negatively by Stokolos [65], who gave an example of a function $f$ in the real Hardy space $H^1\left(\mathbb{R}^2\right)$ such that $\left|\overline{D}\left(\int f,x\right)\right|=\left|\underline{D}\left(\int f,x\right)\right|=\infty$ for almost every $x$ in the unit square. We show that the answer is also negative for the space $H^1(\mathbb{R}\times\mathbb{R})\cap\left(\bigcap_{0<\epsilon<1}L(\log L)^{\epsilon}\left(\mathbb{R}^2\right)\right)$. In particular, $\mathcal{R}$ is not a differentiation basis (see definition in [32], [63], or [64]) for any Orlicz space $L(\log L)^{\epsilon}\left(\mathbb{R}^2\right)$ with $0<\epsilon<1$.

**Theorem 3.1.** *There exists a function $f$ in $H^1(\mathbb{R}\times\mathbb{R})\cap L(\log L)^{\epsilon}\left(\mathbb{R}^2\right)$ for all $0<\epsilon<1$, such that*

$$\left|\overline{D}\left(\int f,x\right)\right|=\left|\underline{D}\left(\int f,x\right)\right|=\infty \tag{3.1}$$

*for almost every $x$ on $\left[-\frac{1}{2},\frac{1}{2}\right]\times\left[-\frac{1}{2},\frac{1}{2}\right]$.*

The proof of this theorem is in Section 3.3. In fact, we will, by modifying the example created by Marstrand [50], construct a function $f$ that belongs to $H^1_{rect}(\mathbb{R}\times\mathbb{R})$ [13], the proper subspace of $H^1(\mathbb{R}\times\mathbb{R})$ which consists of sums of rectangular atoms with coefficients in $l^1$. Then we show that $f$ is in $L(\log L)^\epsilon(\mathbb{R}^2)$ for all $0<\epsilon<1$. The almost everywhere part relies on a variant of the second Borel-Cantelli lemma which extends the version used in [50]. This is a corollary of the theorem below, proved in Section 3.2, which illustrates how geometric properties can yield consequences of a probabilistic nature. In the next result and throughout this text, the notation $\alpha\sim\beta$, for $\alpha,\beta\in[0,\infty)$, means that there exist constants $c, C$ such that $c\alpha\leq\beta\leq C\alpha$.

**Theorem 3.2.** *Let $S_0\subset\mathbb{R}^d$ be the unit cube centered at the origin and let $\{A_n\}_{n\in\mathbb{N}}$ be a family of subsets of $S_0$ satisfying $|A_n|>0$ and $\delta_n := \dim_{upper\,box}\left(\partial\overline{A_n}\right)<d$ for all $n$. There is a sequence $\{m_n\}_{n\in\mathbb{N}}$ of positive integers such that if, for each $n$, we partition $S_0$ into $m_n^d$ cubes of same the size, and place inside each a homothetic copy of $A_n$, then denoting by $\Lambda_n$ the union of these homothetic copies, we have, for any finite subset $F\subset\mathbb{N}$,*

$$\left|\bigcap_{n\in F}\Lambda_n\right|\sim\prod_{n\in F}|\Lambda_n|\,. \tag{3.2}$$

This result generalizes Marstrand's statement ([50], p. 210), where he claims, without proof, the approximately independence (in the probabilistic sense) of homothetic copies of certain "hyperbolic-cross" shaped sets (e.g. 1.3)

Furthermore, we show that if the sets $A_n$ are finite unions of dyadic cubes, then (3.2) holds with an equality.

## 3.2 Approximately Independent Sets

Before we begin, let us fix some notation. By a *cube* we mean a closed cube with sides parallel to the coordinate axes. Given a cube $Q$, we denote its sidelength by $\ell(Q)$ and its interior by $Q^\circ$. Adopting the terminology used in [62], we write that two cubes $P$ and $Q$ *intersect* if $P^\circ\cap Q^\circ\neq\emptyset$ and are *disjoint* if $P^\circ\cap Q^\circ=\emptyset$. For a set $A$ in $\mathbb{R}^d$, we denote its closure by $\overline{A}$ and its the upper box-counting dimension by $\dim_{upper\,box}(A)$, where the latter can be defined [19] as

$$\limsup_{m\to\infty} \frac{\log\left(\#\left\{j\in\mathbb{Z}^d : \left[\frac{j_1-1}{m},\frac{j_1}{m}\right]\times...\times\left[\frac{j_d-1}{m},\frac{j_d}{m}\right]\cap A\neq\emptyset\right\}\right)}{\log(m)}.$$

**Remark 3.1.** *It can be shown that, for any bounded set $A\subset\mathbb{R}^d$, the following are equivalent:*

*(i)* $\dim_{upper\,box}\left(\partial\overline{A}\right)\le\delta_A$*;*

*(ii) for any cube $S$ in $\mathbb{R}^d$ containing $A$, there exist a constant $C_{A,S}>0$ and an integer $\mathcal{N}_{A,S}$ satisfying:*

$$\#\{j\in\{1,...,m^d\}: S^\circ_{m,j}\cap\partial\overline{A}\neq\emptyset\}\le C_{A,S}m^{\delta_A}\qquad\forall m\ge\mathcal{N}_{A,S},\tag{3.3}$$

*where, for each $m>0$, $\{S_{m,j}\}_{j=1}^{m^d}$ is a partition of $S$ into $m^d$ equal sized cubes.*

**Lemma 3.1.** *Consider a cube $S\subset\mathbb{R}^d$, centered at the origin, and a set $A\subset S$ such that $|A|>0$ and $\dim_{upper\ box}\left(\partial\overline{A}\right)\le\delta_A<d$ and let $\epsilon>0$. For any integer $m$ satisfying*

$$m\ge\max\left\{\mathcal{N}_{A,S},\left(\frac{C_{A,S}\,|S|}{\epsilon\,|A|}\right)^{1/(d-\delta_A)}\right\},$$

*where $\mathcal{N}_{A,S}$ and $C_{A,S}$ are as in Remark 3.1, and for any measurable set $E\subset S$, the following holds: if we partition $S$ into $m^d$ equal sized cubes $S_{m,j}$ with center $o_{m,j}$, $j=1,...,m^d$ and denote by $E_{m,j}$ the homothetic copies of $E$, namely*

$$E_{m,j}:=o_{m,j}+\frac{1}{m}E,\qquad j=1,...,m^d,\tag{3.4}$$

*then*

$$(1-\epsilon)\frac{\left|\bigcup_{j=1}^{m^d}E_{m,j}\right|}{|S|}\le\frac{\left|A\cap\left(\bigcup_{j=1}^{m^d}E_{m,j}\right)\right|}{|A|}\le(1+\epsilon)\frac{\left|\bigcup_{j=1}^{m^d}E_{m,j}\right|}{|S|}.\tag{3.5}$$

*Proof.* A counting argument yields

$$\#\{j: S_{m,j}\subset\overline{A}\}\le\frac{|A|}{|S_{m,1}|}=\frac{m^d}{|S|}\,|A|\,,$$

while Remark 3.1 gives us

$$\#\{j : S^\circ_{m,j} \cap \partial\overline{A} \neq \emptyset\} \leq C_{A,S} m^{\delta_A}.$$

If $|S_{m,j} \cap A| > 0$, then either $|S_{m,j} \cap A| = |S_{m,j}|$ or $0 < |S_{m,j} \cap A| < |S_{m,j}|$. Since $|S_{m,j} \cap A| = |S_{m,j}|$ is equivalent to $S_{m,j} \subset \overline{A}$, and since $0 < |S_{m,j} \cap A| < |S_{m,j}|$ implies $S^\circ_{m,j} \cap \partial\overline{A} \neq \emptyset$, it follows that

$$\mathfrak{N}_m = \mathfrak{N}_m(A, S) := \#\left\{j : |S_{m,j} \cap A| > 0\right\} \leq \frac{m^d}{|S|}|A| + C_{A,S} m^{\delta_A}. \tag{3.6}$$

Because the choice of $m$ implies $C_{A,S}\,|S|\,m^{\delta_A - d} \leq \epsilon\,|A|$, we get

$$\mathfrak{N}_m \frac{|S|}{m^d} \leq (1+\epsilon)\,|A|\,. \tag{3.7}$$

As $E_{m_k,j} \subset S_{m,j}$ for each $1 \leq j \leq m^d$, the number of $E_{m,j}$'s satisfying $|A \cap E_{m,j}| > 0$ is at most $\mathfrak{N}_m$. So the proportion of $A$ that lies inside $\bigcup_{j=1}^{m^d} E_{m,j}$ is

$$\frac{\left|A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j}\right)\right|}{|A|} \leq \frac{\mathfrak{N}_m \left|\frac{1}{m}E\right|}{|A|} = \frac{\mathfrak{N}_m\,|E|}{m^d\,|A|} \leq (1+\epsilon)\,\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|},$$

where the last inequality follows by (3.7). Similarly,

$$\frac{\left|A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j}\right)\right|}{|A|} \geq \frac{\left(\#\left\{j : S_{m,j} \subset \overline{A}\right\}\right)\left|\frac{1}{m}E\right|}{|A|} \geq \frac{\left(\frac{m^d}{|S|}\,|A| - \frac{C_{A,S}m^{\delta_A}}{|A|}\right)\left|\frac{1}{m}E\right|}{|A|}$$

$$= \left(1 - \frac{C_{A,S}\,|S|}{m^{d-\delta_A}\,|A|}\right)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|} \geq (1-\epsilon)\,\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|}.$$

□

The example below illustrates a type of set for which the box-counting dimension of the closure is equal to the dimension of the ambient space and (3.5) holds for infinitely many integers $m$.

**Example 3.1.** *Let $\alpha \in (0,1)$ and let $F = F_\alpha$ be the "fat" Cantor set constructed on $[0,1]$ as the Cantor ternary set except that the $2^{k-1}$ intervals removed at step $k$ have length $\alpha/3^k$ instead of $1/3^k$ (see for example [57], p. 64). When $\alpha = p/q \in \mathbb{Q}$, the endpoints of the intervals that remained after the $k$ first steps of the building of $F$ have the form $n/\left(2^k 3^k q\right)$ for some integer $0 \le n \le 2^k 3^k q$. Thus, when we partition $[0,1]$ into $m := 2^k 3^k q$ intervals of the same length, the sum of the lengths of the intervals of that partition which intersect $F$ is exactly the measure of the union of the closed intervals that remained on $[0,1]$ after the $k$-th step of the construction of $F$, i.e.*

$$\frac{1}{m}\#\left\{j \in \mathbb{N} : \left[\frac{j-1}{m}, \frac{j}{m}\right] \cap F \neq \emptyset\right\} = 1 - \left(\frac{\alpha}{3} + 2\frac{\alpha}{3^2} + ... + 2^{k-1}\frac{\alpha}{3^k}\right).$$

*Defining $A := F - 1/2$, it follows that, when we partition $S := [-1/2, 1/2]$ into $m$ intervals $S_{m,j} := [(j-1)/m, j/m] - 1/2$, $j = 1, ..., m$, we obtain*

$$\frac{1}{m}\mathfrak{N}_m = 1 - \frac{\alpha}{3}\sum_{i=1}^{k-1}\left(\frac{2}{3}\right)^i \to 1 - \alpha = |A| \text{ as } k \to \infty,$$

*where $\mathfrak{N}_m$ is as in (3.6). Thus, given $\epsilon > 0$, $\exists k_0 \in \mathbb{N}$ such that (3.7) holds with $m = 2^k 3^k q$ for all $k \ge k_0$. So the argument used to prove Lemma 3.1 yields (3.5).*

In higher dimensions, if a subset $A$ of $S \subset \mathbb{R}^d$ satisfies $|A| > 0$ and

$$\liminf_{m\to\infty}\left(\frac{|S|}{m^d}\mathfrak{N}_m\right) = |A|, \tag{3.8}$$

then (3.5) holds for infinitely many integers $m$. What (3.8) says is that we can approximate the volume of $A$ with a regular grid of boxes. When $\dim_{\textit{upper box}}\left(\partial\overline{A}\right) \le \delta_A < d$, (3.8) holds since (3.6) implies that $|S|\,\mathfrak{N}_m m^{-d}$ converges to $|A|$ as $m \to \infty$.

However, as shown by the example below, the result of Lemma 3.1 fails if $\dim_{\textit{upper box}}\left(\partial\overline{A}\right) = d$.

**Example 3.2.** *Let $G := F_\alpha - 1/2$, where $F_\alpha$ is as in Example (3.1) with $\alpha = 3/4$. We* *define a set A (by filling the gaps in G) as follows*

$$A := G \cup \left\{ \bigcup_{m=1}^{\infty} \left[ \bigcup_{j=1}^{m} \left( -\frac{1}{2} + \frac{j-1/2}{m} + \frac{1}{2^m m} G \right) \right] \right\},$$

*and note that* $A \subset S := [-1/2, 1/2]$ *and* $1/4 \leq |A| \leq 1/2$. *Moreover,* $\mathfrak{N}_m = m$ *for any* $m \in \mathbb{N}$, *since, by construction,*

$$\left| \left[ -\frac{1}{2} + \frac{j-1}{m}, -\frac{1}{2} + \frac{j}{m} \right] \cap A \right| \geq \left| \frac{1}{2^m m} G \right| > 0 \quad \forall 1 \leq j \leq m, \forall m \in \mathbb{N}.$$

*Fix* $m \in \mathbb{N}$ *and let* $E := 2^{-m} G$. *Then (3.5) fails for all* $0 < \epsilon < 1$. *Indeed, using the notation in (3.4),* $E_{m,j} = -\frac{1}{2} + \frac{j-1/2}{m} + \frac{1}{2^m m} G \subset A$, $\forall j$. *So* $|A \cap E_{m,j}| = |E_{m,j}| = 2^{-m} m^{-1} |G|$, $\forall j$, *and it follows that*

$$\left| A \cap \left( \bigcup_{j=1}^{m} E_{m,j} \right) \right| = \sum_{j=1}^{m} |A \cap E_{m,j}| = m \frac{1}{2^m m} |G| = |E| = \left| \bigcup_{j=1}^{m} E_{m,j} \right|. \tag{3.9}$$

*By the choice of A, S and* $\epsilon$, *we have* $\frac{1}{|A|} > \frac{1+\epsilon}{|S|}$, *which, combined with (3.9), implies that (3.5) does not hold.*

Recall that in a probability space $(\Omega, \mathcal{F}, P)$, two events $E_1$, $E_2 \in \mathcal{F}$ are said to be independent if $P(E_1 \cap E_2) = P(E_1) P(E_2)$. Letting $\Omega$ be $S$; $\mathcal{F}$ be the $\sigma$-algebra of Lebesgue measurable subsets of $S$; and $P(E_1) := |E_1| / |S|$ for $E_1 \subset S$ measurable, Lemma 3.1 shows that for certain measurable sets $A \subset S$, there exist arbitrarily large integers $m$ such that, for any measurable set $E \subset S$,

$$P\left( A \cap \left( \bigcup_{j=1}^{m^d} E_{m,j} \right) \right) \sim P(A) P\left( \bigcup_{j=1}^{m^d} E_{m,j} \right),$$

where the $E_{m,j}$'s are as in (3.4). We call this property "approximately independence" and we extend it to infinitely many sets as is (3.2).

*Proof of Theorem 3.2.* We will construct a sequence $\{m_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ with the following property: if we partition $S_0$ into $m_n^d$ cubes $S_{m_n,j}$ of same the size and denote the center of $S_{m_n,j}$ by $o_{m_n,j}$, $j = 1, ..., m_n^d$, then the sets

$$\Lambda_n := \bigcup_{j=1}^{m_n^d} \left( o_{m_n,j} + \frac{1}{m_n} A_n \right), \quad n \in \mathbb{N}, \tag{3.10}$$

satisfy (3.2). It suffices to show that we can choose $\{m_n\}_{n\in\mathbb{N}}$ such that

$$\prod_{i\in F} \left(1 - \frac{1}{4i^2}\right) |\Lambda_i| \le \left| \bigcap_{i\in F} \Lambda_i \right| \le \prod_{i\in F} \left(1 + 2^{-(i-1)}\right) |\Lambda_i| \quad \forall F \subset \{1, ..., n\} \tag{3.11}$$

holds for all $n \in \mathbb{N}$. Indeed, using the representation $\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \prod_{j\in\mathbb{N}} \left(1 - \frac{1}{4j^2}\right)$ and the inequality $1 + t \le e^t \ \ \forall t \in [0, 1]$, we get from (3.11) that, for any finite set $F \subset \mathbb{N}$,

$$\frac{2}{\pi} \prod_{i\in F} |\Lambda_i| = \prod_{j\in\mathbb{N}} \left(1 - \frac{1}{4j^2}\right) \prod_{i\in F} |\Lambda_i| \le \prod_{j\in F} \left(1 - \frac{1}{4j^2}\right) \prod_{i\in F} |\Lambda_i|$$

$$\le \left| \bigcap_{i\in F} \Lambda_i \right| \le \prod_{i\in F} \left(1 + 2^{-(i-1)}\right) |\Lambda_i| \le \prod_{i\in F} e^{2^{-(i-1)}} |\Lambda_i| \le e^2 \prod_{i\in F} |\Lambda_i| .$$

To construct $\{m_n\}_{n\in\mathbb{N}}$, we use induction.

Choose $m_1 = 1$. Then $\Lambda_1 = A_1$ and

$$\left(1 - 4^{-1}\right) |\Lambda_1| \le |\Lambda_1| \le \left(1 + 2^{-(1-1)}\right) |\Lambda_1| .$$

Now, assume that the integers $m_1, ..., m_n$ are chosen such that (3.11) holds. By definition, $\Lambda_k$ is composed of $m_k^d$ homothetic copies of $A_k$. So $\dim_{upper\,box} \left(\partial\overline{\Lambda_k}\right) = \delta_k$, since $\dim_{upper\,box}$ is bi-Lipschitz invariant and finitely stable ([19], p. 48). For any finite subset $F \subset \{1, ..., n\}$, the boundary of the closure of $\Gamma_F := \bigcap_{i\in F} \Lambda_i$ satisfies

$$\dim_{upper\,box} \left(\partial\overline{\Gamma_F}\right) \le \gamma_n := \max\left\{\delta_k : 1 \le k \le n\right\},$$

because $\partial\overline{\Gamma_F} \subset \bigcap_{i\in F} \partial\overline{\Lambda_i}$ and $\dim_{upper\,box}$ is finitely stable [19]. We claim that if

$$C_n := \sum_{k=1}^{n} C_{\Lambda_k,S_0} \quad \text{and} \quad \mathcal{N}_n := \sum_{k=1}^{n} \mathcal{N}_{\Lambda_k,S_0}, \tag{3.12}$$

then it is possible to take $C_{\Gamma_F,S_0} = C_n$ and $\mathcal{N}_{\Gamma_F,S_0} = \mathcal{N}_n$ in (3.3). Indeed, if we take $m \geq \mathcal{N}_n$ and partition $S_0$ into $m^d$ cubes $S_{m,j}$, $j = 1, ..., m^d$, then the number of cubes $S_{m,j}$ which intersect $\partial\overline{\Lambda_k}$ is not greater than $C_{\Lambda_k,S_0} m^{\delta_k}$, $1 \leq k \leq n$. Since $\partial\overline{\Gamma_F} \subset \bigcup_{k=1}^{n} \partial\overline{\Lambda_k}$, the number of cubes $S_{m,j}$ which intersect $\partial\overline{\Gamma_F}$ is not greater than $\sum_{k=1}^{n} C_{\Lambda_k,S_0} m^{\delta_k} \leq C_n m^{\gamma_n}$, and we conclude that our claim holds.

We choose $m_{n+1}$ to be an integer such that

$$m_{n+1} \geq \max\left\{\mathcal{N}_n \max_{I\subset\{1,...,n\}, \left|\bigcap_{i\in I}\Lambda_i\right|>0} \left\{\left(2^n C_n \left|\bigcap_{i\in I}\Lambda_i\right|^{-1}\right)^{1/(d-\gamma_n)}\right\}\right\}, \tag{3.13}$$

and we will show that, for any subset $F \subset \{1, ..., n\}$ such that $\left|\bigcap_{i\in F}\Lambda_i\right| > 0$,

$$\prod_{i\in F\cup\{n+1\}} \left(1 + \frac{1}{4i^2}\right) |\Lambda_i| \leq \left|\bigcap_{i\in F\cup\{n+1\}} \Lambda_i\right| \leq \prod_{i\in F\cup\{n+1\}} \left(1 + 2^{-(i-1)}\right) |\Lambda_i|$$

holds. The case when $\left|\bigcap_{i\in F}\Lambda_i\right| = 0$ is trivial.

Fix $F \subset \{1, ..., n\}$ such that $\Gamma_F := \bigcap_{i\in F}\Lambda_i$ has positive measure. We intend to use Lemma 3.1 with

$$S = S_0, \quad A = \Gamma_F, \quad \epsilon = 2^{-n}, \quad E = A_{n+1}, \quad m = m_{n+1}. \tag{3.14}$$

But first let us verify that the hypotheses are satisfied. We have:

(i) $A \subset S = S_0$ and $S_0$ is a cube centered at the origin;

(ii) $A$ satisfies (3.3) with $C_{A,S} = C_n$ and $\mathcal{N}_{A,S} = \mathcal{N}_n$, since $\Gamma_F$ does;

(iii) $|A| = |\Gamma_F| > 0$, by the choice of $F$;

(iv) $m = m_{n+1} \geq \max\left\{\mathcal{N}_n, \left(\frac{2^n C_n}{|\Gamma_F|}\right)^{1/(d-\gamma_n)}\right\} = \max\left\{\mathcal{N}_{A,S}, \left(\frac{C_{A,S}|S|}{\epsilon|A|}\right)^{1/(d-\gamma_n)}\right\}$.

So we can apply Lemma 3.1 to obtain

$$(1-\epsilon)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|}|A| \leq \left|A \cap \left(\bigcup_{j=1}^{m^d} E_{m,j}\right)\right| \leq (1+\epsilon)\frac{\left|\bigcup_{j=1}^{m^d} E_{m,j}\right|}{|S|}|A|. \tag{3.15}$$

Note that

$$\overset{m^d}{\underset{j=1}{\cup}} E_{m,j} = \overset{m_{n+1}^d}{\underset{j=1}{\cup}} \left( o_{m_{n+1},j} + \frac{1}{m_{n+1}} A_{n+1} \right) = \Lambda_{n+1}.$$

This, combined with (3.14) and (3.15), implies

$$(1-\epsilon) \left| \overset{m^d}{\underset{j=1}{\cup}} E_{m,j} \right| \frac{|A|}{|S|} = \left(1 - 2^{-n}\right) |\Lambda_{n+1}| \, |\Gamma_F| \geq \left[1 - \frac{1}{4\,(n+1)^2}\right] |\Lambda_{n+1}| \, |\Gamma_F| \,,$$

$$\left| A \cap \left( \overset{m^d}{\underset{j=1}{\cup}} E_{m,j} \right) \right| = |\Gamma_F \cap \Lambda_{n+1}| = \left| \left( \underset{i \in F}{\cap} \Lambda_i \right) \cap \Lambda_{n+1} \right| ,$$

and

$$(1+\epsilon) \left| \overset{m^d}{\underset{j=1}{\cup}} E_{m,j} \right| \frac{|A|}{|S|} = \left(1 + 2^{-n}\right) |\Lambda_{n+1}| \, |\Gamma_F| \,.$$

Thus,

$$\prod_{i \in F \cup \{n+1\}} \left(1 - \frac{1}{4i^2}\right) |\Lambda_i| \leq \left[1 - \frac{1}{4\,(n+1)^2}\right] |\Lambda_{n+1}| \left| \underset{i \in F}{\cap} \Lambda_i \right|$$

$$= \left[1 - \frac{1}{4\,(n+1)^2}\right] |\Lambda_{n+1}| \, |\Gamma_F| \leq \left| \left( \underset{i \in F}{\cap} \Lambda_i \right) \cap \Lambda_{n+1} \right| \leq \left(1 + 2^{-n}\right) |\Lambda_{n+1}| \, |\Gamma_F|$$

$$= \left(1 + 2^{-n}\right) |\Lambda_{n+1}| \left| \underset{i \in F}{\cap} \Lambda_i \right| \leq \prod_{i \in F \cup \{n+1\}} \left(1 + 2^{-(i-1)}\right) |\Lambda_i| \,,$$

where the first and last inequalities are due to the induction hypothesis (3.11). We conclude that (3.11) holds for every $n \in \mathbb{N}$. □

**Corollary 3.1.** *Under the hypotheses of Theorem 3.2, if, in addition, the series* $\sum_n |S_0 \cap A_n^c|$ *diverges, then there is a sequence* $\{m_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ *such that when we partition* $S_0$ *into* $m_n^d$ *cubes* $S_{m_n,j}$, $j = 1, ..., m_n^d$, *of the same size and let* $o_{m_n,j}$ *denote the center of* $S_{m_n,j}$ *and*

$$K_n := \overset{m_n^d}{\underset{j=1}{\cup}} \left[ o_{m_n,j} + \frac{1}{m_n} \left(S_0 \cap A_n^c\right) \right], \quad n \in \mathbb{N},$$

*the following holds:*

$$\left| \overset{\infty}{\underset{m=1}{\cap}} \; \overset{\infty}{\underset{n=m}{\cup}} K_n \right| = 1,$$

*i.e. almost every point of* $S_0$ *is contained in infinitely many* $K_n$*'s.*

*Proof.* Indeed, define $\Lambda_n$, $n \in \mathbb{N}$, as in (3.10) and note that

$$S_0 \cap K_n^c = S_0 \cap \left\{ \bigcup_{j=1}^{m_n^d} \left[ o_{m_n,j} + \frac{1}{m_n} \left( S_0 \cap A_n^c \right) \right] \right\}^c$$

$$= S_0 \cap \left\{ \bigcap_{j=1}^{m_n^d} \left[ o_{m_n,j} + \frac{1}{m_n} \left( S_0 \cap A_n^c \right) \right]^c \right\} = \bigcup_{j=1}^{m_n^d} \left( o_{m_n,j} + \frac{1}{m_n} A_n \right) = \Lambda_n.$$

Applying Theorem 3.2 to the family $\{A_n\}_{n\in\mathbb{N}}$, we obtain $\left| \bigcap_{n=k}^{k+l} \Lambda_n \right| \leq e^2 \prod_{n=k}^{k+l} |\Lambda_n|$ for any $k, l \in \mathbb{N}$. Letting $l \to \infty$, we get $\left| \bigcap_{n=k}^{\infty} \Lambda_n \right| \leq e^2 \prod_{n=k}^{\infty} |\Lambda_n|$. We now use this inequality in what is nearly the standard proof on the second Borel-Cantelli lemma:

$$1 - \left| \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} K_n \right| = \left| \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \Lambda_n \right| = \lim_{m\to\infty} \left| \bigcap_{n=m}^{\infty} \Lambda_n \right|$$

$$\leq \lim_{m\to\infty} \left[ e^2 \prod_{n=m}^{\infty} |\Lambda_n| \right] = e^2 \lim_{m\to\infty} \prod_{n=m}^{\infty} (1 - |K_n|)$$

$$\leq e^2 \lim_{m\to\infty} \prod_{n=m}^{\infty} e^{-|K_n|} = e^2 \lim_{m\to\infty} \exp\left( -\sum_{n=m}^{\infty} |K_n| \right) = 0,$$

where the last equality holds because $\sum_n |K_n| = \sum_n |S_0 \cap A_n^c| = \infty$. □

As mentioned above, if we restrict ourselves to sets that are finite unions dyadic cubes, i.e. cubes in the collection

$$\mathcal{D} := \left\{ z + 2^{-k} [0,1]^d : k \in \mathbb{Z},\ z \in 2^{-k}\mathbb{Z}^d \right\},$$

then we have equality in (3.2). The example in [65] is built in the dyadic setting and has motivated us to prove the claims below.

**Claim 3.1.** *Let $S = \left[-2^{k-1}, 2^{k-1}\right]^d$ for some $k \in \mathbb{Z}$ and let $A \subset S$ be a finite union of dyadic cubes. Then, there exists $i_0 \in \mathbb{N}$ such that, for $i \geq k - i_0$, and $m = 2^i$, when we partition $S$ into $m^d$ equal sized cubes $S_{m,j}$ with center $o_{m,j}$, $j = 1, ..., m^d$, the following holds: for any measurable set $E \subset S$, we have (3.5) with $\epsilon = 0$.*

*Proof.* By hypothesis, we can write $A = \bigcup_{i=1}^{n} Q_i$, for some $n \in \mathbb{N}$ and some disjoint cubes

$Q_i \in \mathcal{D}$. Choose

$$i_0 := \min_{1 \le i \le n} \{\log_2 (\ell (Q_i))\} .$$

For any $i \ge k - i_0$, if we set $m := 2^i$ and partition $S$ into $m^d$ cubes $S_{m,j}$, $j = 1, ..., m^d$, of the same size, then $S_{m,j} \in \mathcal{D}$ and $\ell (S_{m,j}) \le 2^{i_0}$. Since each $Q_i$ is a dyadic cube of sidelength $2^j$ for some $j \ge i_0$, it follows that each $Q_i$ is a disjoint union of some of the $S_{m,j}$'s. Therefore so is $A$. Hence

$$\mathfrak{N}_m = \# \left\{ j \in \left\{1, ..., m^d\right\} : |S_{m,j} \cap A| > 0 \right\} = |S_{m,1}|^{-1} |A| = m^d |S|^{-1} |A| .$$

Thus

$$\left| A \cap \left( \bigcup_{j=1}^{m^d} E_{m,j} \right) \right| = \mathfrak{N}_m \left| \frac{1}{m} E \right| = |S|^{-1} |A| |E| = \left| \bigcup_{j=1}^{m^d} E_{m,j} \right| |S|^{-1} |A| . \tag{3.16}$$

Dividing (3.16) by $|A|$, we get (3.5) with $\epsilon = 0$. □

**Claim 3.2.** *Let $S_0 = [-1/2, 1/2]^d$ and let $\{A_n\}_{n \in \mathbb{N}}$ be a family of measurable subsets of $S_0$ such that every $A_n$ is a finite union of dyadic cubes. There is a sequence of integers $\{k_n\}_{n \in \mathbb{N}}$ satisfying: if, for each $n$, we partition $S_0$ into $m_n^d := 2^{k_n d}$ cubes $S_{m_n,j}$, $j = 1, ..., m_n^d$, of the same size and let $o_{m_n,j}$ denote the center of $S_{m_n,j}$ and $\Lambda_n := \bigcup_{j=1}^{m_n^d} \left( o_{m_n,j} + \frac{1}{m_n} A_n \right)$, then for any finite subset $F \subset \mathbb{N}$,*

$$\left| \bigcap_{n \in F} \Lambda_n \right| = \prod_{n \in F} |\Lambda_n| . \tag{3.17}$$

*Proof.* By induction. Choose $k_1 = 0$. Then $m_1 = 1$ and $\Lambda_1 = A_1$.

Now, assume that $k_1, ..., k_n$ are chosen such that, with the above notation,

$$\left| \bigcap_{i \in F} \Lambda_i \right| = \prod_{i \in F} |\Lambda_i| \qquad \forall F \subset \{1, ..., n\} . \tag{3.18}$$

We will choose $k_{n+1}$ such that

$$\left| \bigcap_{i\in F\cup\{n+1\}} \Lambda_i \right| = \prod_{i\in F\cup\{n+1\}} |\Lambda_i| \qquad \forall F \subset \{1, ..., n\}. \tag{3.19}$$

Fix $F \subset \{1, ..., n\}$. By construction, for each $1 \leq i \leq n$, the set $\Lambda_i$ is a finite union of disjoint dyadic cubes. So, for each $1 \leq i \leq n$, we can write $\Lambda_i = \bigcup_{l\in I_i} Q_{i,l}$, for some disjoint dyadic cubes $Q_{i,l}$. We choose

$$m_{n+1} := 2^{-i_n},$$

where $i_n := \min \{\log_2 (\ell (Q_{i,l})) : l \in I_i, 1 \leq i \leq n\}$. When we partition $S$ into $m_{n+1}^d$ cubes $S_{m_{n+1},j}$, $j = 1, ..., m_{n+1}^d$, with $\ell \left(S_{m_{n+1},j}\right) = 2^{i_n}$, each $S^\circ_{m_{n+1},j}$ is either contained in $\bigcap_{i\in F} \Lambda_i$ or in its complement. Thus

$$\# \left\{ j : \left| S_{m_{n+1},j} \cap \left( \bigcap_{i\in F} \Lambda_i \right) \right| > 0 \right\} = \left| S_{m_{n+1},1} \right|^{-1} \left| \bigcap_{i\in F} \Lambda_i \right| = m_{n+1}^d \left| \bigcap_{i\in F} \Lambda_i \right|.$$

So

$$\left| \left( \bigcap_{i\in F} \Lambda_i \right) \cap \Lambda_{n+1} \right| = \left( m_{n+1}^d \left| \bigcap_{i\in F} \Lambda_i \right| \right) \left| \frac{1}{m_{n+1}} A_{n+1} \right| = |\Lambda_{n+1}| \left| \bigcap_{i\in F} \Lambda_i \right|.$$

This and the induction hypothesis (3.18) yield (3.19). Thus (3.17) holds. □

## 3.3 A Counterexample

We divide the proof of Theorem 3.1 into two parts. In the first part we construct a function $f$ in $H^1_{rec} (\mathbb{R} \times \mathbb{R}) \cap L (\log L)^\epsilon (\mathbb{R}^2)$ for all $0 < \epsilon < 1$; in the second, we show that $f$ satisfies (3.1). An analogous reasoning, with $\rho (X_n)$ replacing $X_n$, shows that $\underline{D} \left(\int f, p\right) = -\infty$ for almost every $p$ in $S$.

*Proof of Theorem 3.1 - Part I.* We begin by choosing sequences of positive numbers, $\{\alpha_n\}_n$, $\{\lambda_n\}_n$ and $\{\gamma_n\}_n$, which satisfy the following:

$$\sum_n \frac{\lambda_n}{\alpha_n^4} < \infty, \quad \sum_n \gamma_n < \infty, \tag{3.20}$$

$$\sum_n \frac{\log \alpha_n}{\alpha_n^2} = \infty, \quad \lim_{n\to\infty} \frac{\lambda_n}{\alpha_n^2} = \infty, \tag{3.21}$$

$$\frac{\lambda_n^{-1}\alpha_n^4}{\lambda_{n+1}^{-1}\alpha_{n+1}^4} \leq 1 \tag{3.22}$$

and

$$\frac{\lambda_n}{\kappa_\epsilon \gamma_n \alpha_n^4}\left(\log\left(1+\frac{\lambda_n}{\kappa_\epsilon\gamma_n}\right)\right)^\epsilon \leq 1 \qquad \forall 0<\epsilon<1, \tag{3.23}$$

for some constant $\kappa_\epsilon > 0$, depending on $\epsilon$, but independent of $n$. A suitable choice is described in the end of this section.

We define $S := \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ and we let $\{m_n\}_{n=1}^\infty \subset \mathbb{N}$ be a sequence. The $m_n$'s are required to satisfy certain properties that will be specified later.

We partition $S$ into $m_n^2$ squares $S_{n,j} \in \mathcal{R}$, $j = 1, ..., m_n^2$, of sidelength $1/m_n$. At the center $o_{n,j}$ of each $S_{n,j}$ we place a smaller square

$$Q_{n,j} := \left\{x \in \mathbb{R}^2 : \|o_{n,j} - x\|_\infty \leq \frac{1}{2m_n \lceil \alpha_n \rceil^2}\right\},$$

where here, and in what follows, $\lceil a \rceil := \min\{n \in \mathbb{Z} : n \geq a\}$ for $a \in \mathbb{R}$, and $\|\cdot\|_\infty$ denotes the maximum norm $\|x\|_\infty := \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

For each $j = 1, ..., m_n^2$, we partition $Q_{n,j}$ into 4 squares $Q_{n,j,k} \in \mathcal{R}$, $1 \leq k \leq 4$, of sidelength $1/\left(2m_n \lceil \alpha_n \rceil^2\right)$ and we label the interiors of these 4 squares as black or white in a chessboard pattern with the upper right square being white, as in Figure 3.1. The union of all *white* squares in all squares $Q_{n,j}$'s, $1 \leq j \leq m_n^2$ will be denoted by $\mathcal{W}_n$; that of all *black* squares in all $Q_{n,j}$'s, $1 \leq j \leq m_n^2$, by $\mathcal{B}_n$. Now we define

$$f_n := \lambda_n \chi_{\mathcal{W}_n} - \lambda_n \chi_{\mathcal{B}_n}, \qquad f := \sum_{n=1}^\infty f_n,$$

where $\chi_E$ denotes the characteristic function of a set $E$. Note that $\sum_n |f_n|$ is integrable. Thus the set $W := \{x : \sum_n |f_n(x)| = \infty\}$ has measure zero, a fact the we will use in Part II below.

To see that $f$ is in $H^1(\mathbb{R}\times\mathbb{R})$, we write $f=\sum_{n=1}^{\infty}\sum_{j=1}^{m_n^2}\gamma_n m_n^{-2}a_{n,j}$, where

$$a_{n,j}(x):=m_n^2\gamma_n^{-1}f_n(x)\chi_{Q_{n,j}}(x)\,,\quad 1\le j\le m_n^2,\ n\in\mathbb{N}.$$

The $a_{n,j}$'s are rectangular atoms [13] in $H^1(\mathbb{R}\times\mathbb{R})$ and, by (3.20), the series $\sum_n\left(\sum_{j=1}^{m_n^2}\gamma_n m_n^{-2}\right)$ converges. Hence

$$\sum_{n=1}^{\infty}\sum_{j=1}^{m_n^2}\gamma_n m_n^{-2}a_{n,j}\in H^1_{rect}(\mathbb{R}\times\mathbb{R})\subset H^1(\mathbb{R}\times\mathbb{R})\,.$$

Now, to show that $f$ belongs to $L^{\Phi_\epsilon}(\mathbb{R}^2)$, we write $f=\sum_{n=1}^{\infty}\gamma_n g_n$, where

$$g_n(x):=\gamma_n^{-1}f_n(x)=\sum_{j=1}^{m_n^2}m_n^{-2}a_{n,j},\quad n\in\mathbb{N}.$$

Since $\left(L^{\Phi_\epsilon}(\mathbb{R}^2)\,,\|\cdot\|_{\Phi_\epsilon}\right)$ is complete and the coefficients $\gamma_n$'s satisfy $\sum_n|\gamma_n|<\infty$, to show that $f\in L^{\Phi_\epsilon}(\mathbb{R}^2)$, it suffices to prove that for each $\epsilon\in(0,1)$ we can find a constant $\kappa_\epsilon>0$, independent of $n$, such that

$$\|g_n\|_{\Phi_\epsilon}\le\kappa_\epsilon\quad\text{for all } n\in\mathbb{N}. \tag{3.24}$$

In fact, we claim that (3.24) holds for any $\kappa_\epsilon$ for which (3.23) holds. Indeed, to form each $g_n$, we gathered all the rectangular atoms that compose $f_n$. So

$$|g_n|=\gamma_n^{-1}\lambda_n\chi_{\mathcal{W}_n\cup\mathcal{B}_n},$$

and this yields

$$\int\Phi_\epsilon\left(\frac{g_n(x)}{\kappa_\epsilon}\right)dx=\int\frac{|g_n(x)|}{\kappa_\epsilon}\left[\log\left(1+\frac{|g_n(x)|}{\kappa_\epsilon}\right)\right]^\epsilon dx$$

$$=\frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon}\left[\log\left(1+\frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon}\right)\right]^\epsilon|\mathrm{supp}\,(f_n)|\le\frac{\lambda_n}{\kappa_\epsilon\gamma_n\alpha_n^4}\left[\log\left(1+\frac{\lambda_n}{\kappa_\epsilon\gamma_n}\right)\right]^\epsilon\le 1,$$

for all $n\in\mathbb{N}$, where the last inequality follows from (3.23). This shows that $\kappa_\epsilon$ is an uniform (on $n$) upper bound for the Luxemburg norms $\|g_n\|_{\Phi_\epsilon}$, proving our claim. $\square$

*Proof of Theorem 3.1 - Part II.* The result relies on the construction a sequence $\{K_n\}_{n\in\mathbb{N}}$ of subsets of $S$ such that

$$\left|\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} K_n\right| = 1, \tag{3.25}$$

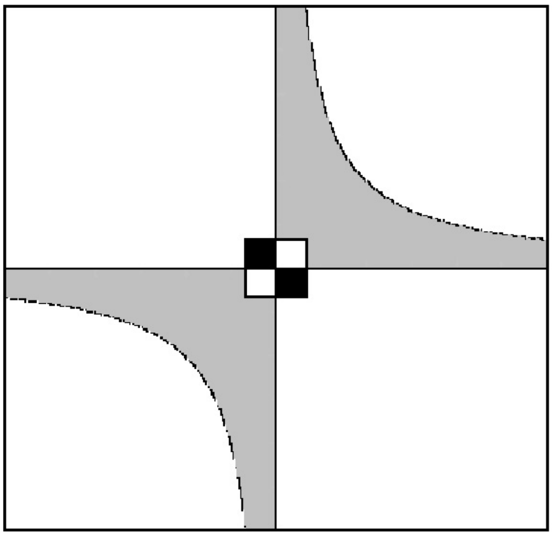
and therefore almost every point in $S$ belongs to $W^c \cap \left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} K_n\right)$.

For each $n \in \mathbb{N}$, we define the set (compare with (1.3))

$$X_n := \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1x_2 \leq \frac{1}{4\lceil\alpha_n\rceil^2},\ \frac{1}{2\lceil\alpha_n\rceil^2} \leq \|(x_1,x_2)\|_\infty \leq \frac{1}{2}\right\}.$$

Since $\partial X_n$ is union of two rectifiable curves, $\dim_{upper\,box}(\partial X_n) = 1$.

Figure 3.1: Homothetic copy of $X_n$



By construction, the dilation of $X_n$ by $1/m_n$ is contained in the square of sidelength $1/m_n$ centered at the origin. In Figure 3.1, we represent a set $o_{n,j} + m_n^{-1}X_n$ in gray and the squares $Q_{n,j,k}$, $1 \leq k \leq 4$, in black and white at the center. So $o_{n,j} + m_n^{-1}X_n \subset S_{n,j}$ for all $1 \leq j \leq m_n^2$. In addition, the area of $X_n$ satisfies (in our proof here, we only need the lower bound for $|X_n|$)

$$\frac{\log\lceil\alpha_n\rceil}{2\lceil\alpha_n\rceil^2} = 2\int_{1/2\lceil\alpha_n\rceil}^{1/2} \frac{1}{4\lceil\alpha_n\rceil^2 t}dt \leq |X_n| \tag{3.26}$$

$$\leq 2\left(\int_0^{1/2\lceil\alpha_n\rceil} tdt + \int_{1/2\lceil\alpha_n\rceil}^{1/2} \frac{1}{4\lceil\alpha_n\rceil^2 t}dt\right) \leq \frac{\log\lceil\alpha_n\rceil}{\lceil\alpha_n\rceil^2}.$$

For a fixed $n \in \mathbb{N}$ and $j \in \{1, ..., m_n^2\}$, every point $p = (p_1, p_2)$ in the set $o_{n,j} + m_n^{-1}X_n$ lies in a rectangle $R_p \in \mathcal{R}$ satisfying $p \in R_p$,

$$|R_p| = \frac{1}{4m_n^2\lceil\alpha_n\rceil^2} \text{ and } |R_p \cap \mathcal{W}_n| - |R_p \cap \mathcal{B}_n| = \frac{1}{4}|Q_{n,j}|. \tag{3.27}$$

Indeed, let $p \in o_{n,j} + m_n^{-1}X_n$. We will construct $R_p$. By symmetry, it suffices to consider $p$ with $0 \leq p_2 - (o_{n,j})_2 \leq p_1 - (o_{n,j})_1$. One of the two cases happens:
(i) If $0 \leq p_2 - (o_{n,j})_2 \leq 1/\left(2m_n\lceil\alpha_n\rceil^2\right)$, then we define

$$R_p := o_{n,j} + \left(\left[0, \frac{1}{2m_n}\right] \times \left[0, \frac{1}{2m_n\lceil\alpha_n\rceil^2}\right]\right)$$

and we observe that (3.27) holds.

(ii) If $p_2 - (o_{n,j})_2 > 1/\left(2m_n\lceil\alpha_n\rceil^2\right)$, then $p_1 - (o_{n,j})_1 > 1/\left(2m_n\lceil\alpha_n\rceil^2\right)$ as well, and we choose

$$R_p := o_{n,j} + \left(\left[0, p_1 - (o_{n,j})_1\right] \times \left[0, \frac{1}{4m_n^2\lceil\alpha_n\rceil^2\left(p_1 - (o_{n,j})_1\right)}\right]\right).$$

With this choice, $p \in R_p$, since $\left(p_2 - (o_{n,j})_2\right)\left(p_1 - (o_{n,j})_1\right) \leq 1/\left(2m_n\lceil\alpha_n\rceil\right)^2$. Also, $R_p$ satisfies (3.27).

Similarly, for every $p \in o_{n,j} + m_n^{-1}\rho(X_n)$, where $\rho$ is the rotation by $\pi/2$ radians, there exists $S_p \in \mathcal{R}$ such that

$$p \in S_p,\ |S_p| = \frac{1}{4m_n^2\lceil\alpha_n\rceil^2} \text{ and } |S_p \cap \mathcal{B}_n| - |S_p \cap \mathcal{W}_n| = \frac{1}{4}|Q_{n,j}|.$$

How does $\lambda_n|Q_{n,1}|$ compare with $\sum_{i=1}^{\infty}\lambda_{n+i}|Q_{n+i,1}|$? The answer given is below and will be used when we deal with the strong upper derivative of the integral of $f$. If

$$m_n \geq 2^4 m_{n-1} \qquad \forall n, \tag{3.28}$$

then $m_{n+i} \geq 2^4 m_{n+i-1} \geq ... \geq 2^{4i} m_n \geq 2^i \left(2^3 m_n\right), \forall n$. This and (3.22) yield

$$\sum_{i=1}^{\infty} \lambda_{n+i} \left|Q_{n+i,1}\right| = \frac{\lambda_n \left|Q_{n,1}\right|}{4} \sum_{i=1}^{\infty} \frac{4\lambda_{n+i} \left|Q_{n+i,1}\right|}{\lambda_n \left|Q_{n,1}\right|}$$

$$= \frac{\lambda_n \left|Q_{n,1}\right|}{4} \sum_{i=1}^{\infty} \frac{4\lambda_{n+i} \left(4m_n^2 \lceil \alpha_n \rceil^4\right)}{\lambda_n \left(4m_{n+i}^2 \lceil \alpha_{n+i} \rceil^4\right)} \leq \frac{\lambda_n \left|Q_{n,1}\right|}{4} \sum_{i=1}^{\infty} \frac{2^2 \lambda_{n+i} m_n^2 \left(2\alpha_n\right)^4}{\lambda_n m_{n+i}^2 \alpha_{n+i}^4}$$

$$= \frac{\lambda_n \left|Q_{n,1}\right|}{4} \sum_{i=1}^{\infty} \left(\frac{\lambda_n^{-1} \alpha_n^4}{\lambda_{n+i}^{-1} \alpha_{n+i}^4}\right) \left(\frac{2^3 m_n}{m_{n+i}}\right)^2$$
$$\leq \frac{\lambda_n \left|Q_{n,1}\right|}{4} \sum_{i=1}^{\infty} \left(2^{-i}\right)^2 = \frac{\lambda_n \left|Q_{n,1}\right|}{12} \qquad \forall n.$$

Thus (3.28) implies

$$\frac{\lambda_n \left|Q_{n,1}\right|}{4} - \sum_{i=1}^{\infty} \frac{\lambda_{n+i} \left|Q_{n+i,1}\right|}{2} \geq \left(\frac{1}{4} - \frac{1}{24}\right) \lambda_n \left|Q_{n,1}\right| = \frac{5}{24} \lambda_n \left|Q_{n,1}\right| \qquad \forall n. \tag{3.29}$$

For each $n$, we define

$$A_n := S \cap X_n^c \text{ and } \Lambda_n := \bigcup_{j=1}^{m_n^2} \left[o_{n,j} + \frac{1}{m_n} A_n\right]. \tag{3.30}$$

Each $A_n$ is contained in $S$ and satisfies $|A_n| > 0$ and $\dim_{upper\, box} \left(\partial \overline{A_n}\right) = 1$. Moreover, since $|S \cap A_n^c| = |X_n|$, estimate (3.26) yields

$$|S \cap A_n^c| \geq \frac{\log \lceil \alpha_n \rceil}{2 \lceil \alpha_n \rceil^2} \geq \frac{\log \alpha_n}{2 \left(2\alpha_n\right)^2}. \tag{3.31}$$

Also, for each $n$, we define

$$K_n := \bigcup_{j=1}^{m_n^2} \left(c_{n,j} + \frac{1}{m_n} X_n\right)$$

and note that $K_n = \bigcup_{j=1}^{m_n^2} \left[c_{n,j} + \frac{1}{m_n} \left(S \cap A_n^c\right)\right]$ and $S \cap K_n^c = \Lambda_n$.

Now we will construct a sequence $\{m_n\}_{n \in \mathbb{N}}$ such that both (3.29) and

$$\left|\bigcap_{i \in F} \Lambda_i\right| \leq \prod_{i \in F} \left(1 + 2^{-(i-1)}\right) |\Lambda_i| \qquad \forall F \subset \{1, ..., n\} \tag{3.32}$$

hold for all $n \in \mathbb{N}$, where the sets $\Lambda_i$ are defined in (3.30). We must choose $\{m_n\}_{n\in\mathbb{N}}$ satisfying (3.28) and (3.13). Condition (3.13) appears in the proof of Theorem 3.2, which we apply to $\{A_n\}_{n\in\mathbb{N}}$. We build $\{m_n\}_{n\in\mathbb{N}}$ by the recurrence relation

$$m_1 = 1, \quad m_n = \left\lceil \max\left\{\mathcal{N}_n, \frac{2^{n-1}C_{n-1}}{\theta_{n-1}}, 2^4\right\}\right\rceil m_{n-1} \lceil \alpha_{n-1} \rceil^2 \quad \text{for } n > 1,$$

where $C_n$ $\mathcal{N}_n$ are as in (3.12), $\theta_n := \min_I \left\{\left|\bigcap_{i\in I} \Lambda_i\right|\right\}$ and the minimum is taken over all finite collections $I \subset \{1, ..., n\}$ satisfying $\left|\bigcap_{i\in I} \Lambda_i\right| > 0$. By construction, with this sequence $\{m_n\}_{n\in\mathbb{N}}$, both (3.28) and (3.13) hold. Hence both (3.29) and (3.32) hold for all $n \in \mathbb{N}$.

From (3.31) and (3.21), we get

$$\sum_{n=1}^{\infty} |S \cap A_n^c| \geq \frac{1}{8} \sum_{n=1}^{\infty} \frac{\log \alpha_n}{\alpha_n^2} = \infty.$$

This, together with (3.32), implies (3.25), as shown in Corollary 3.1.

For each fixed $p \in W^c \cap \left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} K_n\right)$, we will show that $\overline{D}\left(\int f, p\right) = +\infty$. An analogous reasoning, with $\rho(X_n)$ replacing $X_n$, shows that $\underline{D}\left(\int f, p\right) = -\infty$. Indeed, let $\{n_i\}_{i\in\mathbb{N}}$ be such that $p \in K_{n_i}$ $\forall i \in \mathbb{N}$. Then, it suffices to show that

$$\lim_{i\to\infty} \left[\sum_{k=1}^{\infty} \frac{1}{|R_{n_i}(p)|} \int_{R_{n_i}(p)} f_k(x)\, dx\right] = \infty.$$

For each $i \in \mathbb{N}$, $p$ lies in one of the homothetic copies of $X_{n_i}$, say $p \in S_{n_i,j} \cap K_{n_i}$. By (3.27), $p$ lies in a rectangle $R_{n_i}(p) \in \mathcal{R}$ satisfying

$$|R_{n_i}(p)| = \frac{1}{4m_{n_i}^2 \lceil \alpha_{n_i} \rceil^2} \text{ and } |R_{n_i}(p) \cap \mathcal{W}_{n_i}| - |R_{n_i}(p) \cap \mathcal{B}_{n_i}| = \frac{1}{4} |Q_{n_i,1}|. \tag{3.33}$$

Moreover, for any $k \geq 1$, $|R_{n_i}(p) \cap \mathcal{B}_{n_i+k}| - |R_{n_i}(p) \cap \mathcal{W}_{n_i+k}|$ cannot be greater than the area of 2 of the 4 black or white squares that compose each $Q_{n_i+k,j}$, $1 \leq j \leq m_{n_i+k}^2$, i.e.

$$|R_{n_i}(p) \cap \mathcal{B}_{n_i+k}| - |R_{n_i}(p) \cap \mathcal{W}_{n_i+k}| \leq 2\left(\frac{|Q_{n_i+k,1}|}{4}\right) \qquad \forall k \in \mathbb{N}. \tag{3.34}$$

From (3.33), (3.34) and (3.29), we get

$$\int_{R_{n_i}(p)} f_{n_i}(x)\,dx + \sum_{k=1}^{\infty} \int_{R_{n_i}(p)} f_{n_i+k}(x)\,dx$$

$$\begin{aligned} \geq & \lambda_{n_i}\left(\left|R_{n_i}(p)\cap \mathcal{W}_{n_i}\right| - \left|R_{n_i}(p)\cap \mathcal{B}_{n_i}\right|\right) \\ & - \sum_{k=1}^{\infty} \lambda_{n_i+k}\left(\left|R_{n_i}(p)\cap \mathcal{B}_{n_i+k}\right| - \left|R_{n_i}(p)\cap \mathcal{W}_{n_i+k}\right|\right) \end{aligned}$$

$$\geq \frac{\lambda_{n_i}}{4}\left|Q_{n_i,1}\right| - \sum_{k=1}^{\infty} \lambda_{n_i+k}\frac{\left|Q_{n_i+k,1}\right|}{2} \geq \frac{5}{24}\lambda_{n_i}\left|Q_{n_i,1}\right| = \frac{5}{24}\frac{\lambda_{n_i}}{m_{n_i}^2\left\lceil \alpha_{n_i}\right\rceil^4} \quad \forall i \in \mathbb{N}.$$

Then

$$\begin{aligned} & \frac{1}{\left|R_{n_i}(p)\right|}\sum_{k=0}^{\infty}\int_{R_{n_i}(p)} f_{n_i+k}(x)\,dx \\ \geq\, & C\frac{1}{\left(m_{n_i}^2\left\lceil \alpha_{n_i}\right\rceil^2\right)^{-1}}\frac{\lambda_{n_i}}{m_{n_i}^2\left\lceil \alpha_{n_i}\right\rceil^4} \sim \frac{\lambda_{n_i}}{\alpha_{n_i}^2} \to \infty, \end{aligned}$$

as $i \to \infty$, by (3.21). It remains to control $\left|R_{n_i}(p)\right|^{-1}\sum_{k=1}^{n_i-1}\int_{R_{n_i}(p)} f_k(x)\,dx$, $i \in \mathbb{N}$. By construction, for every $i$ and every $k \in \{1,...,n_i-1\}$, $m_{n_i}$ is an integer multiple of $4m_k\left\lceil \alpha_k\right\rceil^2$. This and the fact that the black and white squares $Q_{k,l,v}$, $1 \leq v \leq 4$, that compose each $Q_{k,l}$, $1 \leq l \leq m_k^2$, have sidelength $1/\left(2m_k\left\lceil \alpha_k\right\rceil^2\right)$, yield

$$S_{n_i,j}\cap Q_{m,l,v} \neq \emptyset \Leftrightarrow S_{n_i,j}^{\circ} \subset Q_{m,l,v} \quad \forall 1 \leq k \leq n_i-1,\ 1 \leq l \leq m_k^2,\ 1 \leq v \leq 4.$$

Hence either $R_{n_i}(p)\cap\left(\operatorname{supp}\left(\sum_{k=1}^{n_i-1} f_k\right)\right) = \emptyset$ or $R_{n_i}(p) \subset Q_{k,l,v}$ for some $1 \leq k \leq n_i-1$, $1 \leq l \leq m_k^2$, $1 \leq v \leq 4$. In any of these cases,

$$\frac{1}{\left|R_{n_i}(p)\right|}\int_{R_{n_i}(p)} f_k(x)dx = f_k(p) \qquad \forall 1 \leq k \leq n_i-1,$$

which implies that

$$\left|\sum_{k=1}^{n_i-1}\frac{1}{\left|R_{n_i}(p)\right|}\int_{R_{n_i}(p)} f_m(x)dx\right| \leq \sum_{k=1}^{n_i-1}\left|f_k(p)\right| \leq \sum_{k=1}^{\infty}\left|f_k(p)\right| < \infty \quad \forall i \in \mathbb{N},$$

where the last inequality holds due to the choice of $p$ in $W^c$. Therefore

$$\frac{1}{|R_{n_i}(p)|}\int_{R_{n_i}(p)} f(x)\,dx \geq -\sum_{k=1}^{\infty}|f_k(p)| + \frac{1}{|R_{n_i}(p)|}\sum_{k=n_i}^{\infty}\int_{R_{n_i}(p)} f_k(x)\,dx \to \infty$$

as $i \to \infty$. Thus $\overline{D}\left(\int f, p\right) = +\infty$. $\square$

Here we present a choice of positive numbers satisfying (3.20)–(3.23). For each $n \in \mathbb{N}$, let

$$\alpha_n := 4n^{1/2}\log(4n)\left(\log(\log(4n))\right)^{1/2}, \tag{3.35}$$

$$\lambda_n := n\left(\log(4n)\right)^2\left(\log(\log(4n))\right)^2, \tag{3.36}$$

$$\gamma_n := \frac{1}{4^4 n\log(4n)\left(\log(\log(4n))\right)^2}. \tag{3.37}$$

In addition, let

$$\kappa_\epsilon := \max\left\{2^5, 9^\epsilon \max_{n\in\mathbb{N}}\left\{\frac{\left(\log(\log(4n))\right)^2}{\left(\log(4n)\right)^{1-\epsilon}}\right\}\right\}. \tag{3.38}$$

To see that the sequences $\{\alpha_n\}_n, \{\lambda_n\}_n$ and $\{\gamma_n\}_n$, defined above, satisfy (3.20) and (3.21), it suffices to observe that

$$\frac{\lambda_n}{\alpha_n^4} \sim \frac{1}{n(\log n)^2}, \quad \gamma_n \sim \frac{1}{n(\log n)(\log(\log n))^2},$$

$$\frac{\log\alpha_n}{\alpha_n^2} \sim \frac{1}{n(\log n)(\log(\log n))}, \quad \frac{\lambda_n}{\alpha_n^2} \sim \log(\log n).$$

A direct substitution yields (3.22). The proof of (3.23) requires a bit more work. From (3.35)–(3.38) we obtain

$$1 + \frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon} \leq \frac{2\gamma_n^{-1}\lambda_n}{2^5} = (4n)^2\left(\log(4n)\right)^3\left(\log(\log(4n))\right)^4 \leq (4n)^9. \tag{3.39}$$

Plugging (3.39) into the left-hand-side of (3.23), we get

$$\frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon}\left[\log\left(1+\frac{\gamma_n^{-1}\lambda_n}{\kappa_\epsilon}\right)\right]^\epsilon \frac{1}{\alpha_n^4} \leq \frac{\left(\log(\log(4n))\right)^2}{\kappa_\epsilon\log(4n)}\left[9\log(4n)\right]^\epsilon$$

$$= \frac{9^{\epsilon} \left(\log\left(\log\left(4n\right)\right)\right)^{2}}{\kappa_{\epsilon} \left(\log\left(4n\right)\right)^{1-\epsilon}} \leq 1,$$

where the last inequality follows from the choice of $\kappa_{\epsilon}$.

# Chapter 4

# Differentiation of the Integral, Hardy Spaces and Calderón-Zygmund Operators in the Product Setting

The study that lead to the results of this chapter begun with our investigation on the question as to whether the function created by Papoulis [53] can be in the Hardy space $H^1(\mathbb{R}\times\mathbb{R})$. The search for the answer raised many other questions and lead us to develop a relaxed version of Chang-Fefferman $p$-atoms with a lower number of required vanishing moments and no smoothness needed on the elementary particles. In analogy with the proof of this result, we show a generalization of a theorem of R. Fefferman, which concludes $H^p \to L^p$, $0 < p \leq 1$, boundedness of multiparameter operators from their behavior on rectangle atoms. In addition, we extend a result of Pipher concerning boundedness of multiparameter Calderón-Zygmund operators from $H^p$ to $L^p$.

## 4.1 Introduction and Statement of Results

In the multiparameter setting, we view $\mathbb{R}^n$ as $\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r}$, with $n_1+...+n_r=n$, and the points $x\in\mathbb{R}^n$ are represented as $(x_1,\ldots,x_r)$, with $x_j\in\mathbb{R}^{n_j}$, $j=1,...,r$. The *multi-*

*parameter*, or *product*, *Hardy space* $H^p(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})$, $0 < p \leq \infty$, can be defined [30] by fixing $\varphi^j \in \mathcal{S}(\mathbb{R}^{n_j})$, $j = 1, ..., r$, with non-zero integral, and saying that a distribution $f$, in $\mathcal{S}'(\mathbb{R}^n)$, belongs to $H^p(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})$ if and only if the *multiparameter radial maximal function*

$$\mathfrak{M}_{\varphi^1,...,\varphi^r}(f)(x) := \sup_{t_j>0} \left| \left( \left( \varphi^1_{t_1} ... \varphi^r_{t_r} \right) * f \right)(x) \right| \tag{4.1}$$

is in $L^p(\mathbb{R}^n)$, where

$$\left( \left( \varphi^1_{t_1} ... \varphi^r_{t_r} \right) * f \right)(x) := \int \varphi^1_{t_1}(y_1) ... \varphi^r_{t_r}(y_r) f(x-y)\, dy,$$

$$\varphi^j_{t_j}(x_j) = t_j^{-n_j} \varphi^j \left( t_j^{-1} x_j \right), \quad j = 1, ..., r.$$

In this case, the quasi-norm of $f$ is given by

$$\|f\|_{H^p(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})} := \|\mathfrak{M}_{\varphi^1,...,\varphi^r}(f)\|_p, \tag{4.2}$$

and it is a norm if $p \geq 1$. If the supremum in (4.1) is restricted to $t_1 = ... = t_r$, then we get the *one-parameter Hardy space* $H^p(\mathbb{R}^n)$. Our investigation begins with a question from A. Stokolos. He asked, concerning the example created by Papoulis [53] of an integrable function $f$ on $\mathbb{R}^2$ such that the strong derivative (defined in [32], [60]) of $\int f$ exists a.e., while the upper strong derivative of $\int |f|$ is infinite on a set of positive measure, whether such a function can be in $H^1(\mathbb{R}^2)$. We give a positive answer:

**Theorem 4.1.** *There exists a function* $f$ *in* $H^1_{rect}(\mathbb{R} \times \mathbb{R})$ *such that the strong derivative of* $\int f$ *exists a.e., but the upper strong derivative of* $\int |f|$ *is infinite on a set of positive measure, namely the function created by Papoulis.*

We prove it by exhibiting a decomposition of the function $f$, created by Papoulis (here and below this function will be called *Papoulis' f*, in terms of rectangle $(1, 2)$-atoms (see Definition 4.1) with coefficients in $l^1$. This means that $f \in H^1_{rect}(\mathbb{R} \times \mathbb{R})$ and therefore $f$ is in $H^1(\mathbb{R} \times \mathbb{R})$. In particular, Papoulis' $f$ belongs to $H^1(\mathbb{R}^2)$.

For $p \leq 1$, the space $H^p_{rect}(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})$ (see Remark 4.1) is included in $H^p(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})$ and an example of Carleson [9] implies that $H^1_{rect}(\mathbb{R} \times \mathbb{R})$ is a proper subspace of $H^1(\mathbb{R}\times\mathbb{R})$. It is useful to note that the inclusion $H^p_{rect}(\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r})\subset H^p(\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r})$ is an immediate consequence of Theorem 4.5 below, as any rectangle $(p,2)$-atom on $\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r}$ is an rough $(p,2)$-atom on $\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r}$ (see Definition 4.3) with one elementary particle.

By modifying the techniques used in the one-parameter setting, we are able to show some crucial estimates for the action of the radial maximal function (see Lemma 4.1) on rectangles $(p,q)$-atoms on $\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r}$. The key importance of these estimates is due to the fact that they imply the uniform boundedness of the $L^p(\mathbb{R}^{n_1+\cdots+n_r})$-norm of the radial maximal function of rough $(p,q)$-atoms on $\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r}$, which yields the result of Theorem 4.5, stated below. In the setting $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, this theorem follows from Lemma 4.1 and an argument that R. Fefferman [25] developed to prove that the behavior of an operator on rectangle atoms implies boundedness from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1+n_2})$.

The same techniques we use to prove Theorem 4.5 in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, allow us to enhance R. Fefferman's result [25]. Combining his proof with the main result in D.-C. Chang et al. [14] (who filled some missing steps from [25]), we show the following variant of a theorem of R. Fefferman [25]:

**Theorem 4.2** (Variant of R. Fefferman's Theorem in [25]). *Let $0<p\leq 1$. Given a linear operator $T$ on $L^2(\mathbb{R}^{n_1+n_2})$, if there exist $\delta_j>0$, $j=1,2$, such that*

$$\int_{\left(2^kI_1\right)^c\times\mathbb{R}^{n_2}}|T(a)|^p\leq C\left(2^k\right)^{-\delta_1}\quad and\quad \int_{\mathbb{R}^{n_1}\times\left(2^kI_2\right)^c}|T(a)|^p\leq C\left(2^k\right)^{-\delta_2},\tag{4.3}$$

*for any $k\in\{2,3,4,...\}$ and any rectangle $(p,2)$-atom, $a$, on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, supported on $I_1\times I_2$, then $T$ can be extended to a bounded operator from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1+n_2})$.*

There are two steps in the proof of theorems of this type: (i) showing that the behavior of an operator $T$ on rectangle atoms yields uniform boundedness of the $L^p$-norm of the action of $T$ on product $H^p$-atoms; (ii) proving that this uniform bound implies the existence of a bounded extension of $T$ from product $H^p$ to $L^p$ (see [1] for an example of a linear map which does not admit a bounded extension to $H^1$ despite being uniformly bounded on atoms). Note that the assumptions of Theorem 4.2 are different than those in [25] because, in addition to only requiring dilations by dyadic scalars, we deal with dilations on each factor of the product setting separately. These multiparameter dilations are the essential aspect of our result, as they allow a direct extension of step (i) to higher-parameter settings. In particular, they enable us to prove the following:

**Theorem 4.3** (Three-parameter Variant of Theorem 4.2)**.** *Let* $0 < p \leq 1$*. Given an operator* $T$ *bounded on* $L^2\left(\mathbb{R}^{n_1+n_2+n_3}\right)$*, if there exist* $\delta_j > 0$*,* $j = 1, 2, 3$*, such that*

$$\int_{\left(2^{k_1}I_1\right)^c \times \left(2^{k_2}I_2\right)^c \times \mathbb{R}^{n_3}} |T(a)|^p \leq C \left(2^{k_1}\right)^{-\delta_1} \left(2^{k_2}\right)^{-\delta_2} \quad \text{for all } k_1, k_2 \in \{2, 3, 4, ...\}, \tag{4.4}$$

*for any rectangle* $(p, 2)$*-atom,* $a$*, on* $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$*, supported on* $I_1 \times I_2 \times I_3$*, and similar inequalities with* $(1, 2, 3)$ *replaced by* $(2, 3, 1)$ *and* $(3, 1, 2)$ *hold, then* $T$ *can be extended to a bounded operator from* $H^p\left(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}\right)$ *to* $L^p\left(\mathbb{R}^{n_1+n_2+n_3}\right)$*.*

This result is not in conflict with Journé's counterexample (see [43]) since our dilations have three parameters instead of one. That counterexample seems to suggest the existence of an obstacle to passing R. Fefferman's argument [25] from the two- to the three-parameter setting. Journé [43] overcame this difficulty in the context of convolution operators on product $BMO$. Carbery and Seeger [8] have taken this step in $H^p\left(\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}\right)$ by showing that, with extra hypotheses on the operators, R. Fefferman's reasoning [25] holds in higher-parameter settings. Han et. al. [35], working with Hilbert space valued operators and using the Littewood–Paley square function characterization of $H^p$, established necessary and sufficient conditions for certain classes of singular integral operators to be bounded from multiparameter (three or more parameters) Hardy spaces $H^p$ to $L^p$, $0 < p \leq 1$.

To prove Theorem 4.3, we follow the method used in [25], but instead of the original two-parameter discrete Journé's Lemma [42], we apply a three-parameter variant (Lemma 4.4) proved by Pipher [54]. The discrete Journé's Lemma is a geometric result which can be used to predict the behavior of operators on Chang-Fefferman atoms from their action on rectangle atoms. As shown in Section 4.4, Pipher's variant holds on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$ (not just $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$). In addition, it holds if the hypothesis that "$\Omega$ is bounded" is replaced by "Ω has bounded measure", a fact which is used later in her paper. Since Pipher's variant on settings with more than three parameters follows by induction [54], Theorem 4.3 can be extended to settings of the form $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$.

When we are dealing with Calderón-Zygmund operators, the inequalities (4.3) can be obtained by imposing certain conditions on their kernels. Our next theorem presents some sufficient conditions. The statement is similar to that of the so called "trivial lemma" in [25], but we prove it for all values of $p$ in $(0,1]$; not just for $p$ close to 1 as in [25].

Both the $(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operators and the $p$-$(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operators that appear in Theorem 4.4 are precisely defined in Section 4.2 (see Definitions 4.7 and 4.8). When $n_1 = n_2 = 1$ and $\epsilon_1 = \epsilon_2 = \epsilon$, the $(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operators are the "Calderón-Zygmund operators of type $\epsilon$ on $\mathbb{R} \times \mathbb{R}$" defined by Journé [41].

**Theorem 4.4** (Variant of "trivial lemma" in [25]). *Let $0 < p \leq 1$ and let $T$ be a linear operator on $L^2(\mathbb{R}^{n_1+n_2})$.*
*If $n_1/(n_1-1) < p \leq 1$, assume that $T$ is a $(\beta_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operator for some $\beta_1 > n_1(1/p-1)$ and $\epsilon_2 > n_2(1/p-1)$.*
*If $0 < p \leq n_1/(n_1-1)$, assume that $T$ is a $p$-$(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operator for some $\epsilon_2 > n_2(1/p-1)$ and assume also that the $[CZ(\epsilon_2, n_2)]$-$(\epsilon_1, n_1)$-kernel $K_1$, associated with $T$ as in Definition 4.7, satisfies*

$$\left\| D_{y_1}^{\alpha} K_1(x_1, \xi_1) \right\|_{CZ(\epsilon_2, n_2)} \leq C \left\| x_1 - \xi_1 \right\|^{-\mathcal{N}_1 - 1 - n_1} \tag{4.5}$$

*for each $\alpha \in (\mathbb{Z}{\geq}\, 0)^{n_1}$ with $|\alpha| = \mathcal{N}_1 + 1$ and for almost every $x_1 \neq \xi_1 \in \mathbb{R}^{n_1}$. Under these hypotheses, the first inequality in (4.3) holds for some $\delta_1 > 0$ and for any rectangle $(p,2)$-atom $a$ on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ supported on $I_1 \times I_2$. By symmetry, the second inequality in (4.3) can be obtained by interchanging the variables $x_2$ and $x_1$.*

Observe that, by the definition of $(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operator (Definition 4.7), if $T$ is a such operator, then $T$ is associated with $[CZ(\epsilon_j, n_j)]$-$(\epsilon_i, n_i)$-kernels

(Definition 4.4) $K_i$, $i \in \{1,2\}$, $j \in \{1,2\} \backslash \{i\}$, which satisfy the Hörmander type condition

$$\int_{\|x_i-\xi_i\|>2^s\|y_i-\xi_i\|} \|K_i(x_i,y_i) - K_1(x_1,\xi_i)\|_{CZ(\epsilon_j,n_j)} \, dx_i \leq C\left(2^s\right)^{-\epsilon_i} \text{ for all } s \in \mathbb{N}. \quad (4.6)$$

Theorem 4.6, stated and proved in Section 4.5, gives sufficient conditions on the kernel of a Calderón-Zygmund operator for (4.4) to hold, thus extending Theorem 4.4 to the three-parameter setting. The result of Theorem 4.6 was proved by Pipher [54] for values of $p$ close to 1 and settings of the form $\mathbb{R} \times ... \times \mathbb{R}$. She also stated (without a proof) that this result holds for all $p \in (0,1]$ if more smoothness is assumed on the kernel.

With the crucial estimates of Lemma 4.1, we can relax the sufficient conditions for a distribution to be in $H^p\left(\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}\right)$. Specifically:

**Theorem 4.5.** *Let $0 < p \leq 1 < q < \infty$ and $r \in \{2,3\}$. Suppose that a distribution $f \in \mathcal{S}'\left(\mathbb{R}^{n_1+...+n_r}\right)$ can be written as $f = \sum_{k=1}^{\infty} \lambda_k a_k$ converging in $\mathcal{S}'\left(\mathbb{R}^{n_1+...+n_r}\right)$, where $\{\lambda_k\}_k \in l^p$ and each $a_k$ is a rough $(p,q)$-atom on $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$. Then*

$$f \in H^p\left(\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}\right) \text{ and } \|f\|_{H^p}^p \leq C \sum_{k=1}^{\infty} |\lambda_k|^p ,$$

*for some constant $c$ independent of $f$.*

It is clear that when one decomposes distributions in $H^p$ as sums of functions, it is better to have as much smoothness as possible. However, for the sufficient condition, it is preferable to have fewer requirements on those functions.

Contrasting with Chang-Fefferman atoms [12], our rough atoms do not have any requirement related to smoothness of the elementary particles. Furthermore, the elementary particles of our atoms do not need to be in $L^\infty$ either (this $L^\infty$ boundedness was used in [12] to prove that Chang-Fefferman $p$-atoms are uniformly bounded in $H^p\left(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\right)$). The fact that, for $p = 1$, Chang-Fefferman elementary particles do not need to be smooth nor in $L^\infty$ is known [47]. For $0 < p \leq 1$, it was proved [36] that if a function belongs to $H^p\left(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\right) \cap L^q\left(\mathbb{R}^{n_1+n_2}\right)$, for some $1 < q < \infty$, then it can be written as sum of $(p,q)$-product-atoms with coefficients in $l^p$, where the so called "$(p,q)$-product-atoms"

[36] are rough $(p, q)$-atoms requiring some extra conditions. Unlike the results in [36], which depend on a discrete Calderón's identity, ours rely on variants of Journé's Lemma.

In [12], it was asked whether it is possible to improve the requirement of vanishing moments up to order $\lfloor 2/p - 3/2 \rfloor$ on each coordinate factor of the elementary particles of the atomic decomposition of $H^p(\mathbb{R} \times \mathbb{R})$. By Theorem 4.5, the answer is affirmative: for $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, it is sufficient to have vanishing moments up to order $\lfloor n_j (1/p - 1) \rfloor$ on the $j$-the coordinate, $j = 1, 2$, of the elementary particles. In the case $n_1 = n_2 = 1$, this result was obtained by Han [34]. The reason why we have these lower numbers is that, by working with the radial maximal function instead of the square function (as in [12]), we avoid an early use of Hölder's inequality.

In Section 4.2, we introduce some necessary notation and prove technical results that will be used in the subsequent sections. The proof of Theorem 4.1 is in Section 4.3, where we present, for the sake completeness, Papoulis' construction of a function that proves this result. Section 4.4 is about Journé's Lemma and contains two variants of it. In Section 4.5, we prove Theorems 4.2, 4.3 and 4.4; we present Theorem 4.6 – a three-parameter variant of Theorem 4.4 – and we show Theorem 4.5.

## 4.2 Definitions and Technical Results

We use the standard notation $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$ for $x \in \mathbb{R}$, and by $|A|$ we denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. We say that a set in $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$ is a *rectangle* if it has the form

$$R(x,t) := \left\{(y_1, ..., y_r) \in \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r} : \|x_j - y_j\|_{\mathbb{R}^{n_j}} < t_j, 1 \leq j \leq r\right\}, \tag{4.7}$$

for some $x_j \in \mathbb{R}^{n_j}$, $t_j > 0$, $1 \leq j \leq r$, where $\|\cdot\|_{\mathbb{R}^{n_j}}$ denotes the maximum norm in $\mathbb{R}^{n_j}$ (where no confusion arises, we will denote it by $\|\cdot\|$). We denote by $\mathcal{R}_0$ the set of rectangles of the form (4.7). Given $R(x,t) \in \mathcal{R}_0$ and $\epsilon > 0$, we define $\epsilon R := R(x, \epsilon t)$. Given a cube $I \subset \mathbb{R}^{n_j}$ and a scalar $\epsilon > 0$, we denote by $\epsilon I$ the cube concentric with $I$ and with edge-length $\epsilon$ times the edge-length of $I$.

The multiparameter theory differs from the one-parameter in many ways. For instance, the analogue of Hardy-Littlewood maximal function, which is defined by

$$M_S(f)(x) := \sup\left\{\frac{1}{|R|}\int_R |f(y)|\,dy : R \in \mathcal{R}_0 \text{ such that } x \in R\right\}, \quad f \in L^1_{loc}\left(\mathbb{R}^{n_1+\ldots+n_r}\right), \tag{4.8}$$

and called *strong maximal function* [39], [62], is not [39] of weak-type $(1,1)$. The differentiation of the integrals [60], [32] is considered with respect to rectangles in $\mathcal{R}_0$ and known as *strong differentiability*, but the multiparameter equivalent of the classical differentiation theorem of Lebesgue fails. This is a consequence of the fact that $M_S$ does not map $L^1$ into weak $L^1$ [32].

On the other hand, $M_S$ is bounded on $L^q$, $1 < q \leq \infty$, and satisfies a weak-type $L(\log L)^{r-1}$ inequality [39], [32]. We will refer to this fact as the *strong maximal function theorem*.

The collection of *dyadic cubes* in $\mathbb{R}^{n_j}$ will be denoted by $\mathcal{D}^{n_j}$. We say that a set $R \subset \mathbb{R}^{n_1+\ldots+n_r}$ is a *dyadic rectangle* in $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r}$ if it has the form $R = I_1 \times \ldots \times I_r$, for some cubes $I_j \in \mathcal{D}^{n_j}$, $j = 1, \ldots, r$. Given an open set $\Omega \subset \mathbb{R}^{n_1+\ldots+n_r}$ with finite measure, we let $\mathcal{M}(\Omega)$ be the set of *maximal dyadic rectangles* of $\Omega$ [25], where a dyadic rectangle $I_1 \times \ldots \times I_r$ is said to be maximal when, for any $j = 1, \ldots r$, if there exists $J_j$ in $\mathcal{D}^{n_j}$ such that $I_j \subset J_j$ and $I_1 \times \ldots \times I_{j-1} \times J_j \times I_{j-1} \ldots \times I_r \subset \Omega$, then $I_j = J_j$.

We denote all constants by $c$ or $C$, where $c$ (or $C$) may vary from line to line and may depend on the dimension of the space and other fixed entities. Sometimes we will put a subscript on $c$ (or on $C$) to indicate these dependences.

**Definition 4.1.** *Let $0 < p \leq 1$, $1 < q \leq \infty$. A function $a$ is called a rectangle $(p,q)$-atom on $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r}$ if it satisfies*

$$supp\,(a) \subset R, \text{ where } R = I_1 \times \ldots \times I_r \tag{4.9}$$

*for* $I_j := \left\{y_j \in \mathbb{R}^{n_j} : \|y_j - o_j\|_{\mathbb{R}^{n_j}} < t_j\right\}$, $o_j \in \mathbb{R}^{n_j}$, $t_j > 0$;

$$\|a\|_q \leq |R|^{1/q-1/p}; \tag{4.10}$$

$$\int a(x_1, y_2, \ldots, y_r)\, x_1^{\alpha_1} dx_1 = \ldots = \int a(y_1, \ldots, y_{r-1}, x_r)\, x_r^{\alpha_r} dx_r = 0 \tag{4.11}$$

*for all* $(y_1, \ldots, y_r) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r}$ *and all* $\alpha_j = \left(\alpha_{j1}, ..., \alpha_{jn_j}\right) \in (\mathbb{Z}_{\geq 0})^{n_j}$ *with*

$$|\alpha_j| := \sum_{i=1}^{n_j} \alpha_{ji} \leq \mathcal{N}_j := \left\lfloor n_j \left(\frac{1}{p} - 1\right)\right\rfloor. \tag{4.12}$$

**Remark 4.1.** *The definition above is based on that of rectangle atoms found in [13], [25], [26] and [27]. On these works,* $\|\cdot\|_{\mathbb{R}^{n_j}}$ *is the maximum norm and* $q = 2$*. When* $q = \infty$*, the term* $1/q$ *in (4.10) and in the proof of Lemma 4.1, is defined to be zero.*

**Definition 4.2.** *[13] By* $H^p_{rect}(\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r})$*,* $0 < p \leq 1$*, we denote the space whose elements have the form* $\sum_k \lambda_k a_k$*, where* $\{\lambda_k\}_k \in l^p$*, each* $a_k$ *is a rectangle* $(p, 2)$*-atom on* $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r}$ *and the series converges in* $\mathcal{S}'\left(\mathbb{R}^{n_1+\ldots+n_r}\right)$*.*

The following lemma provides the essential estimates for the behavior of a radial maximal function on rectangle $(p, q)$-atoms.

**Lemma 4.1.** *Let* $0 < p \leq 1 < q \leq \infty$ *and let* $a$ *be a rectangle* $(p, q)$*-atom on* $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_r}$ *supported on* $I_1 \times .... \times I_r$*. For* $\gamma_1, ..., \gamma_r \geq 2$ *and* $1 \leq i \leq r$*, if* $U = (\gamma_1 I_1) \times .... \times (\gamma_{i-1} I_{i-1}) \times \mathbb{R}^{n_i} \times (\gamma_{i+1} I_{i+1}) \times .... \times (\gamma_r I_r)$*, then*

$$\int_{U^c} (\mathfrak{M}(a))^p \leq c_{\varphi^1, .., \varphi^r, p, q} \prod_{\substack{1 \leq j \leq r \\ j \neq i}} \gamma_j^{-(1-\mu_j)p}, \tag{4.13}$$

*where* $\mu_j := n_j (1/p - 1) - \mathcal{N}_j$ *and the multiparameter radial maximal function*

$$\mathfrak{M} := \mathfrak{M}_{\varphi^1, .., \varphi^r} \tag{4.14}$$

*is defined as in (4.1), with each Schwartz function* $\varphi^j$ *being nonnegative, radial (w.r.t. the norm* $\|\cdot\|_{\mathbb{R}^{n_j}}$*), bounded by* 1*, supported on* $\left\{y_j : \|y_j\|_{\mathbb{R}^{n_j}} \leq 1\right\}$ *and having integral* 1*,* $j = 1, ..., r$*.*

*Proof of Lemma 4.1.* In order to simplify the notation, we prove the result in the case $i = r$ and $r = 2$. The modifications in the argument for the case $r > 2$ will be mentioned.

Let $\mathfrak{M}$ be as in (4.14) and let $a$ be a rectangle $(p,q)$-atom on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ supported on $R = I_1 \times I_2$. Since our goal is to show $\int_{(\gamma I_1)^c \times \mathbb{R}^{n_2}} (\mathfrak{M}(a))^p \leq C\gamma^{(\mu_1 - 1)p}$, we can assume w.l.o.g. that $I_j$ is centered at the origin of $\mathbb{R}^{n_j}$, $j = 1, 2$, respectively.

To estimate that integral, we divide $(\gamma I_1)^c \times \mathbb{R}^{n_2}$ into two sets: $(\gamma I_1)^c \times 2I_2$ and $(\gamma I_1)^c \times (2I_2)^c$ (in the case $r > 2$, we divide $(\gamma_1 I_1)^c \times \ldots \times (\gamma_{r-1} I_{r-1})^c \times \mathbb{R}^{n_r}$ into $(\gamma_1 I_1)^c \times \ldots \times (\gamma_{r-1} I_{r-1})^c \times 2I_r$ and $(\gamma_1 I_1)^c \times \ldots \times (\gamma_{r-1} I_{r-1})^c \times (2I_r)^c$; the rest of the argument is analogous to the case $r = 2$). We will integrate $(\mathfrak{M}(a))^p$ on each of these sets separately.

Let $x \in (\gamma I_1)^c \times 2I_2$ and note that

$$\left| \left( \varphi^1_{t_1} . \varphi^2_{t_2} \right) * a(x) \right| \leq \int_{I_2} \left| \int_{I_1} a(y_1, y_2) \varphi^1_{t_1}(x_1 - y_1)\, dy_1 \right| \varphi^2_{t_2}(x_2 - y_2)\, dy_2. \tag{4.15}$$

By (4.11), we can subtract from $\varphi^1\left(t_1^{-1}(x_1 - y_1)\right)$ the Taylor polynomial of order $\mathcal{N}_1$ of $\varphi^1$ at $t_1^{-1} x_1$ evaluated at $t_1^{-1}(x_1 - y_1)$ (in the case $r > 2$, we use the Taylor polynomial of $\varphi^j$ of order $\mathcal{N}_j$ to deal with the integral in each $y_j$, $j = 1, ..., r-1$). This yields

$$\left| \int_{I_1} a(y_1, y_2) \varphi^1_{t_1}(x_1 - y_1)\, dy_1 \right| \leq C \frac{|I_1|^{(\mathcal{N}_1 + 1)/n_1}}{t_1^{n_1 + \mathcal{N}_1 + 1}} \int_{I_1} |a(y_1, y_2)|\, dy_1$$

for any $t_1 > \|x_1\|/2$. So, for $1 < q < \infty$,

$$\left| \int a(y_1, y_2) \varphi^1_{t_1}(x_1 - y_1)\, dy_1 \right| \leq C \frac{|I_1|^{(\mathcal{N}_1 + 1)/n_1 + 1 - 1/q}}{\|x_1\|^{n_1 + \mathcal{N}_1 + 1}} \left( \int_{I_1} |a(y_1, y_2)|^q\, dy_1 \right)^{1/q} =: F(x_1, y_2), \tag{4.16}$$

for any $t_1 > \|x_1\|/2$, and if $q = \infty$, then

$$\left| \int a(y_1, y_2) \varphi^1_{t_1}(x_1 - y_1)\, dy_1 \right| \leq C \frac{|I_1|^{(\mathcal{N}_1 + 1)/n_1 + 1}}{\|x_1\|^{n_1 + \mathcal{N}_1 + 1}} \|a\|_\infty =: F(x_1, y_2), \tag{4.17}$$

for any $t_1 > \|x_1\|/2$. Since $\mathfrak{M}(a)(x) = \sup_{t_1 > \|x_1\|/2;\, t_2 > 0} \left| \left( \left( \varphi^1_{t_1} . \varphi^2_{t_2} \right) * a \right)(x) \right|$ for any $x \in (\gamma I_1)^c \times 2I_2$, plugging (4.16) into (4.15), we get

$$\mathfrak{M}(a)(x) \leq \sup_{t_2 > 0} \int F(x_1, y_2) \varphi^2_{t_2}(x_2 - y_2)\, dy_2 \leq C\, \mathfrak{M}_{\varphi^2}(F(x_1, \cdot))(x_2), \tag{4.18}$$

for any $x \in (\gamma I_1)^c \times 2I_2$, where $\mathfrak{M}_{\varphi^2}(g)(x_2) := \sup_{t_2 > 0} \left| \varphi^2_{t_2} * g(x_2) \right|$ for $g \in \mathcal{S}'(\mathbb{R}^{n_2})$.

Plugging (4.17) into (4.15), we also obtain (4.18). So, in the case $1 < q < \infty$,

$$\int_{(\gamma I_1)^c \times 2I_2} (\mathfrak{M}(a)(x))^p \, dx \leq C \int_{(\gamma I_1)^c} \int_{2I_2} (\mathfrak{M}_{\varphi^2}(F(x_1,\cdot))(x_2))^p \, dx_2 dx_1$$

$$\leq C \int_{(\gamma I_1)^c} |I_2|^{1-p/q} \left[ \int (\mathfrak{M}_{\varphi^2}(F(x_1,\cdot))(x_2))^q \, dx_2 \right]^{p/q} dx_1$$

$$\leq C \, |I_2|^{1-p/q} \int_{(\gamma I_1)^c} \left[ \int (F(x_1,\cdot)(x_2))^q \, dx_2 \right]^{p/q} dx_1$$

$$= C \, |I_2|^{1-p/q} \int_{(\gamma I_1)^c} \frac{|I_1|^{(\mathcal{N}_1+1)p/n_1+p-p/q}}{\|x_1\|^{(n_1+\mathcal{N}_1+1)p}} \left[ \int \int_{I_1} |a(y_1,x_2)|^q \, dy_1 dx_2 \right]^{p/q} dx_1$$

$$= C \, |I_2|^{1-p/q} \, |I_1|^{(\mathcal{N}_1+1)p/n_1+p-p/q} \left( \gamma \, |I_1|^{1/n_1} \right)^{-(n_1+\mathcal{N}_1+1)p+n_1} \|a\|_q^p$$

$$= C \left( \gamma^{\mu_1 - 1} \, |R|^{1/p-1/q} \, \|a\|_q \right)^p, \tag{4.19}$$

while, in the case $q = \infty$,

$$\int_{(\gamma I_1)^c \times 2I_2} (\mathfrak{M}(a)(x))^p \, dx \leq C \, |I_2| \int_{(\gamma I_1)^c} \|F(x_1,\cdot)\|_\infty^p \, dx_1 = C \left( \gamma^{\mu_1 - 1} \, |R|^{1/p} \, \|a\|_\infty \right)^p. \tag{4.20}$$

For $x \in (\gamma I_1)^c \times (2I_2)^c$, we use (4.11) to get

$$\left| \left( \varphi^1_{t_1} . \varphi^2_{t_2} \right) * a(x) \right| \leq C \frac{|I_1|^{(\mathcal{N}_1+1)/n_1} \, |I_2|^{(\mathcal{N}_2+1)/n_2}}{t_1^{n_1+\mathcal{N}_1+1} t_2^{n_2+\mathcal{N}_2+1}} \int_{I_2} \int_{I_1} |a(y_1,y_2)| \, dy_1 dy_2$$

$$\leq C \frac{|I_1|^{(\mathcal{N}_1+1)/n_1} \, |I_2|^{(\mathcal{N}_2+1)/n_2} \, |R|^{1-1/q} \, \|a\|_q}{\|x_1\|^{n_1+\mathcal{N}_1+1} \, \|x_2\|^{n_2+\mathcal{N}_2+1}},$$

for all $t_j > \|x_j\|/2$, $j = 1,2$. Since, when $x$ is in $(\gamma I_1)^c \times (2I_2)^c$, $\mathfrak{M}(a)(x) = \sup_{t_j > \|x_j\|/2} \left| \left( \left( \varphi^1_{t_1} . \varphi^2_{t_2} \right) * a \right)(x) \right|$, we obtain

$$\int_{(\gamma I_1)^c \times (2I_2)^c} (\mathfrak{M}a)^p \leq C \left( \gamma^{\mu_1 - 1} \, |R|^{1/p-1/q} \, \|a\|_q \right)^p. \tag{4.21}$$

To conclude the proof of the case $1 < q < \infty$, we combine the estimates (4.19) and (4.21) and condition (4.10). In the case $q = \infty$, the conclusion follows from (4.20), (4.21) and condition (4.10). □

The definition below is based on that of Chang-Fefferman $p$-atoms on $\mathbb{R}\times\mathbb{R}$ [12], [25].

**Definition 4.3.** *Let $0 < p \leq 1 < q < \infty$. A function $a$ is said to be a rough $(p,q)$-atom on $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$ if it satisfies*

*(A) supp$(a)$ is contained in some open set $\Omega \subset \mathbb{R}^{n_1+...+n_r}$ of finite measure,*

*(B) $\|a\|_q \leq |\Omega|^{1/q-1/p}$,*

*(C) $a = \sum_{R\in\mathcal{M}(\Omega)} a_R$, where each $a_R$ is a function supported on $4R$ for some distinct maximal dyadic rectangle $R \in \mathcal{M}(\Omega)$ and satisfying (4.11), and*

$$\sum_{R\in\mathcal{M}(\Omega)} \|a_R\|_q^q \leq |\Omega|^{1-q/p}. \tag{4.22}$$

Note that, for each function $a_R$ defined above, $|4R|^{1/q-1/p}\|a_R\|_q^{-1} a_R$ is a rectangle $(p,q)$-atom on $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$ as in Definition 4.1.

**Remark 4.2.** *Every $f$ in $H^p(\mathbb{R}\times\mathbb{R})$ can be expressed [12], [25] as a sum $f = \sum_{k=1}^{\infty}\lambda_k a_k$ converging in $\mathcal{S}'(\mathbb{R}^2)$, where $\{\lambda_k\}_k \in l^p$ and each $a_k$ is a Chang-Fefferman p-atom on $\mathbb{R}\times\mathbb{R}$ (in particular, each $a_k$ is a rough $(p,2)$-atom on $\mathbb{R}\times\mathbb{R}$). The argument that was used in [25] to prove this result also works in the setting $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$. Thus every $f$ in $H^p(\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r})$ can be expressed as*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k \text{ in } \mathcal{S}'\left(\mathbb{R}^{n_1+...+n_r}\right), \tag{4.23}$$

*where $\{\lambda_k\}_k \in l^p$ and each $a_k$ is a Chang-Fefferman p-atom on $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$, and the norm $H^p$ norm of $f$ satisfies*

$$\sum_{k=1}^{\infty} |\lambda_k|^p \leq c\,\|f\|_{H^p}^p, \tag{4.24}$$

*where c does not depend of $f$. We will use the decomposition (4.23) in the proofs of Theorems 4.2 and 4.3. In those proofs, we do not need the smoothness of the elementary particles of Chang-Fefferman p-atoms on $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$; all that we need are the fact every Chang-Fefferman p-atom on $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$ is a rough $(p,q)$-atom on $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$ and the bound (4.24).*

As we mentioned in the introduction, we will deal with Calderón-Zygmund operators that satisfy certain conditions which are similar to those which Journé [41] requires on the definition of "Calderón-Zygmund operators of type $\epsilon$". The terminology that we use in our definitions come from [29] and [41].

**Definition 4.4** ([41]). *A locally integrable function $K$, defined on the complement of $\Delta_n := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ and taking values in a normed space $V$, is called an $V$-$(\epsilon, n)$-kernel if there exists $\epsilon > 0$ and a constant $\lambda > 0$ such that*

$$\int_{\|x-\xi\|>2^k\|y-\xi\|} \|K(x,y) - K(x,\xi)\|_V \, dx \leq \lambda \left(2^k\right)^{-\epsilon} \quad \text{for all } k \in \mathbb{N}. \tag{4.25}$$

*We let*

$$|K|_{V,\epsilon} := \inf\{\lambda > 0 : (4.25) \text{ holds}\}. \tag{4.26}$$

**Definition 4.5.** *A locally integrable function $K$, defined on $[(\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}) \backslash \Delta_{n_1}] \times [(\mathbb{R}^{n_2} \times \mathbb{R}^{n_2}) \backslash \Delta_{n_2}]$ and taking values in a normed space $V$, is called an $V$-$(\epsilon_1, \epsilon_2, n_1, n_2)$-kernel if there exist $\epsilon_1, \epsilon_2 > 0$ and a constant $\lambda > 0$ such that*

$$\int_{\substack{\|x_1-\xi_1\|>2^k\|y_1-\xi_1\| \\ \|x_2-\xi_2\|>2^l\|y_2-\xi_2\|}} \|K_{1,2}(x_1,y_1,x_2,y_2) - K_{1,2}(x_1,\xi_1,x_2,\xi_2)\|_V \, dx \leq \lambda \left(2^k\right)^{-\epsilon_1} \left(2^l\right)^{-\epsilon_2} \tag{4.27}$$

*for all $k, l \in \mathbb{N}$. We let*

$$|K|_{V,\epsilon_1,\epsilon_2} := \inf\{\lambda > 0 : (4.27) \text{ holds}\}. \tag{4.28}$$

We now introduce the definitions of the Calderón-Zygmund operators that we will deal with. Our multiparameter operators will be of one-parameter in each variable. So we begin with the definitions in the one-parameter setting.

**Definition 4.6** ([41]). *Let $\epsilon > 0$. A continuous linear map $T_0$ from $C_0^\infty(\mathbb{R}^n)$ into $[C_0^\infty(\mathbb{R}^n)]'$ is a singular integral operator (SIO) of type $(\epsilon, n)$ if there exists a $\mathbb{C}$-$(\epsilon, n)$-kernel, $K$, such that for $f, g$ in $C_0^\infty(\mathbb{R}^n)$ with disjoint supports*

$$\langle T_0(f), g\rangle_{\mathbb{R}^n} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x,y) f(y) g(x) \, dx dy, \tag{4.29}$$

*where* $\langle T_0(f), g\rangle_{\mathbb{R}^n}$ *denotes the action of* $T_0(f)$ *on* $g$.
*Given a bounded linear operator* $T$ *on* $L^2(\mathbb{R}^n)$*, we say that it is an* $(\epsilon, n)$*-Calderón-Zygmund operator if there exists a SIO of type* $(\epsilon, n)$*,* $T_0$*, such that* $T$ *extends* $T_0$*. In this case, we say that* $T$ *is associated to the* $\mathbb{C}$*-*$(\epsilon, n)$*-kernel* $K$*, the kernel corresponding to* $T_0$*, as in (4.29).*

We denote the space of $(\epsilon, n)$-Calderón-Zygmund operators by $CZ(\epsilon, n)$ and we define a norm on it as follows: if an operator $T$, in $CZ(\epsilon, n)$, is associated to a $\mathbb{C}$-$(\epsilon, n)$-kernel $K$, then its $CZ(\epsilon, n)$-norm [41] is

$$\|T\|_{CZ(\epsilon,n)} := \|T\|_{L^2 \to L^2} + |K|_{\mathbb{C},\epsilon}\,. \tag{4.30}$$

Endowed with this norm, the space $CZ(\epsilon, n)$ is a Banach space [41].

**Remark 4.3.** *Note that, if an* $(\epsilon, n)$*-Calderón-Zygmund operator* $T$ *is associated to an* $L^1_{loc}(\mathbb{R}^n \backslash \Delta_n)$ *function* $K$ *as in Definition 4.6, then* $T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy$*, for all* $f \in L^\infty(\mathbb{R}^n)$ *with compact support and almost every* $x \notin supp(f)$*, and this integral representation is absolutely convergent. Note also that if, in addition,* $\int f = 0$ *and* $I$ *is a cube centered at the origin* $0$ *of* $\mathbb{R}^n$ *such that* $supp(f) \subset I$*, then*

$$T(f)(x) = \int_I (K(x, y) - K(x, 0)) f(y)\, dy$$

*for almost every* $x \notin I$*. This and (4.30) yield*

$$\int_{\|x\|>2^k|I|^{1/n}} |T(f)(x)|\, dx \le \int_I \left( \int_{\|x\|>2^k\|y\|} |K(x, y) - K(x, 0)|\, dx \right) |f(y)|\, dy$$

$$\le \left(2^k\right)^{-\epsilon} \|T\|_{CZ(\epsilon,n)} \|f\|_1 \quad \textit{for all } k \in \mathbb{N}.$$

In the two-parameter settings, we have the following.

**Definition 4.7** (Based on Definition 8 in [41])**.** *Let* $\epsilon_j > 0$*,* $j = 1, 2$*. We say that a continuous linear operator* $T_0$ *from* $C_0^\infty(\mathbb{R}^{n_1}) \times C_0^\infty(\mathbb{R}^{n_2})$ *into* $[C_0^\infty(\mathbb{R}^{n_1}) \times C_0^\infty(\mathbb{R}^{n_2})]'$ *is a SIO of type* $(\epsilon_1, \epsilon_2, n_1, n_2)$ *if it satisfies:*

*(1) there exists a* $[CZ(\epsilon_2, n_2)]$*-*$(\epsilon_1, n_1)$*-kernel,* $K_1$ *such that*

$$\langle T_0 (f \otimes \varphi), g \otimes \psi \rangle_{\mathbb{R}^{n_1+n_2}} = \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} \langle [K_1 (x_1, y_1)] (\varphi), \psi \rangle_{L^2(\mathbb{R}^{n_2})} f (y_1) g (x_1) dx_1 dy_1 \tag{4.31}$$

*holds for* $f, g$ *in* $C_0^\infty (\mathbb{R}^{n_1})$ *with disjoint supports and* $\varphi, \psi$ *in* $C_0^\infty (\mathbb{R}^{n_2})$*, where the map*

$$(x_1, y_1) \in (\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}) \setminus \Delta_{n_1} \mapsto K_1 (x_1, y_1)$$

*is a* $[CZ(\epsilon_2, n_2)]$*-valued function,* $[K_1 (x_1, y_1)] (\varphi)$ *denotes the evaluation of* $K_1 (x_1, y_1)$ *at* $\varphi$ *and* $\langle \cdot , \cdot \rangle_{L^2(\mathbb{R}^{n_2})}$ *represents the inner-product on* $L^2 (\mathbb{R}^{n_2})$*;*

*(2) there exists a* $[CZ(\epsilon_1, n_1)]$*-*$(\epsilon_2, n_2)$*-kernel,* $K_2$*, such that*

$$\langle T_0 (\varphi \otimes f), \psi \otimes g \rangle_{\mathbb{R}^{n_1+n_2}} = \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} \langle [K_2 (x_2, y_2)] (\varphi), \psi \rangle_{L^2(\mathbb{R}^{n_1})} f (y_2) g (x_2) dx_2 dy_2 \tag{4.32}$$

*holds for* $f, g$ *in* $C_0^\infty (\mathbb{R}^{n_2})$ *with disjoint supports, and* $\varphi, \psi$ *in* $C_0^\infty (\mathbb{R}^{n_1})$*.*

*A bounded linear operator* $T$ *on* $L^2 (\mathbb{R}^{n_1+n_2})$ *is said to be an* $(\epsilon_1, \epsilon_2, n_1, n_2)$*-Calderón-Zygmund operator if there exists a SIO of type* $(\epsilon_1, \epsilon_2, n_1, n_2)$*,* $T_0$*, such that* $T$ *extends* $T_0$*. In this case, we say that* $T$ *is associated with* $K_1$ *and* $K_2$*.*

We will also deal with operators whose kernels satisfy certain differentiability condition. The order of the derivatives that we require depends on the numbers $\mathcal{N}_1$ and $\mathcal{N}_2$, which are defined in (4.12). Note that $\mathcal{N}_j$ depends on the dimension $n_j$ and on the exponent $p$, $j = 1, 2$.

**Definition 4.8.** *Let* $0 < p \leq 1$ *and* $\epsilon_j > 0$*,* $j = 1, 2$*. A bounded linear operator on* $L^2 (\mathbb{R}^{n_1+n_2})$*,* $T$*, is called a* $p$*-*$(\epsilon_1, \epsilon_2, n_1, n_2)$*-Calderón-Zygmund operator if:*

*(i)* $T$ *is an* $(\epsilon_1, \epsilon_2, n_1, n_2)$*-Calderón-Zygmund operator as in Definition 4.7;*

*(ii-1) the* $[CZ(\epsilon_2, n_2)]$*-valued function*

$$(x_1, y_1) \in (\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}) \setminus \Delta_{n_1} \mapsto K_1 (x_1, y_1)$$

*such that (4.31) holds is of class* $C^{\mathcal{N}_1+1}$ *in the variable* $y_1$ *and for each multi-indice*

*$\alpha \in (\mathbb{Z}_{\geq 0})^{n_1}$ with $|\alpha| \leq \mathcal{N}_1 + 1$, $D^{\alpha}_{y_1} K_1$ is a $CZ(\epsilon_2, n_2)$-$(\epsilon_1, n_1)$-kernel; and*

*(ii–2) item (ii–1) of this definition holds with the roles of 1 and 2 interchanged and the $\alpha$'s replaced by multi-indices $\beta \in (\mathbb{Z}_{\geq 0})^{n_2}$ with $|\beta| = \mathcal{N}_2 + 1$.*

We denote the space of $(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operators by $CZ(\epsilon_1, \epsilon_2, n_1, n_2)$ and we define a norm [41] on it as follows: if an operator $T$ in $CZ(\epsilon_1, \epsilon_2, n_1, n_2)$ corresponds to a pair $(K_1, K_2)$ of a $[CZ(\epsilon_2, n_2)]$-$(\epsilon_1, n_1)$-kernel, $K_1$, and a $[CZ(\epsilon_1, n_1)]$-$(\epsilon_2, n_2)$-kernel, $K_2$, as in (4.31) and (4.32) respectively, then the $CZ(\epsilon_1, \epsilon_2, n_1, n_2)$-norm of $T$ is defined [41] by

$$\|T\|_{CZ(\epsilon_1,\epsilon_2,n_1,n_2)} := \|T\|_{L^2 \to L^2} + |K_1|_{V_2,\epsilon_1} + |K_2|_{V_1,\epsilon_2},$$

where $V_j$ denotes the space $CZ(\epsilon_j, n_j)$, and $|K_i|_{V_j,\epsilon_i}$ is defined in (4.26), $i = 1, 2$, $j \in \{1, 2\}\backslash\{1\}$.

Similarly, we denote the space of $p$-$(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operators by $p$-$CZ(\epsilon_1, \epsilon_2, n_1, n_2)$ and we define the $p$-$CZ(\epsilon_1, \epsilon_2, n_1, n_2)$-norm of an operator $T$ in $p$-$CZ(\epsilon_1, \epsilon_2, n_1, n_2)$ corresponding to a collection of kernels $K_{1,\alpha}$, $K_{2,\beta}$, $\alpha \in (\mathbb{Z}_{\geq 0})^{n_1}$ with $|\alpha| \leq \mathcal{N}_1 + 1$, $\beta \in (\mathbb{Z}_{\geq 0})^{n_2}$ with $|\beta| \leq \mathcal{N}_2 + 1$, as in Definition 4.8, by

$$\|T\|_{p\text{-}CZ(\epsilon_1,\epsilon_2,n_1,n_2)} := \|T\|_{L^2 \to L^2} + \sum_{|\alpha| = \mathcal{N}_1 + 1} \left| D^{\alpha}_{y_1} K_1 \right|_{V_2,\epsilon_1} + \sum_{|\beta| = \mathcal{N}_2 + 1} \left| D^{\beta}_{y_2} K_2 \right|_{V_1,\epsilon_2},$$

where here $V_j$ is the space $CZ(\epsilon_j, n_j)$, and the terms $\left| D^{\alpha}_{y_1} K_1 \right|_{V_2,\epsilon_1}$, $\left| D^{\beta}_{y_2} K_2 \right|_{V_1,\epsilon_2}$ are defined in (4.26).

**Remark 4.4.** *If an $(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operator $T$ is associated with an $[CZ(\epsilon_2, n_2)]$-$(\epsilon_1, n_1)$-kernel $K_1$ as in Definition 4.7, then*

$$\langle T (u \otimes \varphi), v \otimes \psi \rangle_{\mathbb{R}^{n_1+n_2}}$$

$$= \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} \langle [K_1 (x_1, y_1)] (\varphi) - [K_1 (x_1, 0_1)] (\varphi), \psi \rangle_{L^2(\mathbb{R}^{n_2})} u (y_1) v (x_1) \, dx_1 dy_1, \qquad (4.33)$$

*for all $u, v$ in $C^{\infty}_0 (\mathbb{R}^{n_1})$ with disjoint supports, with $u$ satisfying $\int u = 0$, and every $\varphi, \psi$*

*in* $C_0^\infty(\mathbb{R}^{n_2})$. *Since the* $CZ(\epsilon_2, n_2)$*-valued function*

$$(x_1, y_1) \in (\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}) \setminus \Delta_{n_1} \mapsto K_1(x_1, y_1)$$

*is locally integrable on* $(\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}) \setminus \Delta_{n_1}$, *for any* $f$ *in* $L^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *satisfying* $supp(f) \subset I_1 \times I_2$ *for some cubes* $I_1 \subset \mathbb{R}^{n_1}$, $I_2 \subset \mathbb{R}^{n_2}$, *with* $I_j$ *centered at the origin* $0_j$ *of* $\mathbb{R}^{n_2}$, $j = 1, 2$, *and for almost every* $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *with* $x_1 \notin I_1$, *the representation*

$$T(f)(x_1, x_2) = \int_{I_1} [K_1(x_1, y_1)] (f(y_1, \cdot))(x_2)\, dy_1 \tag{4.34}$$

*holds; here* $[K_1(x_1, y_1)](f(y_1, \cdot))(x_2)$ *denotes the evaluation at* $x_2$ *of the operator* $[K_1(x_1, y_1)]$ *on the function* $y_2 \mapsto f(y_1, \cdot)(y_2) := f(y_1, y_2)$. *If, in addition,* $\int f(y_1, y_2)\, dy_1 = 0$ *for all* $y_2 \in \mathbb{R}^{n_2}$, *then (4.33) and (4.34) yield*

$$T(f)(x_1, x_2) = \int_{I_1} ([K_1(x_1, y_1)] - [K_1(x_1, 0_1)]) (f(y_1, \cdot))(x_2)\, dy_1, \tag{4.35}$$

*for almost every* $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *with* $x_1 \notin I_1$.

The analogue of Definition 4.7 for three-parameter settings is:

**Definition 4.9** (Based on Definition 8 in [41]). *Let* $\epsilon_j > 0$, $j = 1, 2, 3$. *We say that a continuous linear map*

$$T_0 : C_0^\infty(\mathbb{R}^{n_1}) \times C_0^\infty(\mathbb{R}^{n_2}) \times C_0^\infty(\mathbb{R}^{n_3}) \to [C_0^\infty(\mathbb{R}^{n_1}) \times C_0^\infty(\mathbb{R}^{n_2}) \times C_0^\infty(\mathbb{R}^{n_3})]'$$

*is a SIO of type* $(\epsilon_1, \epsilon_2, \epsilon_3, n_1, n_2, n_2)$ *if it satisfies:*

*(1) there exists a* $[CZ(\epsilon_3, n_3)]$*-*$(\epsilon_1, \epsilon_2, n_1, n_2)$*-kernel,* $K_{1,2}$,

$$\langle T_0(f_1 \otimes f_2 \otimes \varphi), g_1 \otimes g_2 \otimes \psi \rangle_{\mathbb{R}^{n_1+n_2+n_3}}$$

$$= \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} \begin{pmatrix} \langle [K_{1,2}(x_1, y_1, x_2, y_2)](\varphi), \psi \rangle_{L^2(\mathbb{R}^{n_3})} f_1(y_1) \\ .g_1(x_1) f_2(y_2) g_2(x_2) \end{pmatrix} dx_2 dy_2 dx_1 dy_1 \tag{4.36}$$

*holds for* $f_1, g_1$ *in* $C_0^\infty(\mathbb{R}^{n_1})$ *with disjoint supports,* $f_2, g_2$ *in* $C_0^\infty(\mathbb{R}^{n_2})$ *with disjoint supports and* $\varphi, \psi$ *in* $C_0^\infty(\mathbb{R}^{n_3})$*;*

*(2) the analogue of item (1) of this definition holds with* $(2,3,1)$ *instead of* $(1,2,3)$*; and*

*(3) the analogue of item (1) of this definition holds with* $(3,1,2)$ *instead of* $(1,2,3)$.

*A bounded linear operator* $T$ *on* $L^2(\mathbb{R}^{n_1+n_2})$ *is said to be an* $(\epsilon_1, \epsilon_2, n_1, n_2)$*-Calderón-Zygmund operator if there exists a SIO of type* $(\epsilon_1, \epsilon_2, \epsilon_3, n_1, n_2, n_3)$*,* $T_0$*, such that* $T$ *extends* $T_0$*. In this case, we say that* $T$ *is associated with* $K_{1,2}$*,* $K_{2,3}$ *and* $K_{3,1}$*.*

The analogue of Definition 4.8 for three-parameter settings is:

**Definition 4.10.** *Let* $0 < p \leq 1$ *and* $\epsilon_j > 0$*,* $j = 1, 2, 3$*. A bounded linear operator on* $L^2(\mathbb{R}^{n_1+n_2+n_3})$*,* $T$*, is called a* $p$*-*$(\epsilon_1, \epsilon_2, \epsilon_3, n_1, n_2, n_3)$*-Calderón-Zygmund operator if it satisfies:*

*(i)* $T$ *is an* $(\epsilon_1, \epsilon_2, n_1, n_2)$*-Calderón-Zygmund operator as in Definition 4.9;*

*(ii-1) the* $[CZ(\epsilon_3, n_3)]$*-valued function*

$$(x_1, y_1, x_2, y_2) \in [(\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}) \setminus \Delta_{n_1}] \times [(\mathbb{R}^{n_2} \times \mathbb{R}^{n_2}) \setminus \Delta_{n_2}] \mapsto K_{1,2}(x_1, y_1, x_2, y_2)$$

*such that (4.36) holds is of class* $C^{\mathcal{N}_1+1}$ *in the variable* $y_1$ *and for each multi-indice* $\alpha \in (\mathbb{Z}_{\geq 0})^{n_1}$ *with* $|\alpha| \leq \mathcal{N}_1 + 1$*,* $D^\alpha_{y_1} K_{1,2}$ *is a* $CZ(\epsilon_3, n_3)$*-*$(\epsilon_1, \epsilon_2, n_1, n_2)$*-kernel;*

*(ii–2) the analogue of item (ii–1) of this definition holds with* $(2,3,1)$ *instead of* $(1,2,3)$*, and multi-indices* $\beta \in (\mathbb{Z}_{\geq 0})^{n_2}$ *with* $|\beta| \leq \mathcal{N}_2 + 1$*; and*

*(ii–3) the analogue of item (ii–1) of this definition holds with* $(3,1,2)$ *instead of* $(1,2,3)$*, and multi-indices* $\gamma \in (\mathbb{Z}_{\geq 0})^{n_3}$ *with* $|\gamma| \leq \mathcal{N}_3 + 1$*.*

In Definitions 4.9 and 4.10, we only have three of the six possible permutations of {1,2,3} because those three are all we need.

We denote the space of $(\epsilon_1, \epsilon_2, \epsilon_3, n_1, n_2, n_3)$-Calderón-Zygmund operators by $CZ(\epsilon_1, \epsilon_2, \epsilon_3, n_1, n_2, n_3)$ and we define a norm [41] on it as follows: if an operator $T$ in $CZ(\epsilon_1, \epsilon_2, \epsilon_3, n_1, n_2, n_2)$ corresponding to a triplet $(K_{1,2}, K_{2,3}, K_{3,1})$ of kernels as in

Definition 4.9, then the $CZ(\epsilon_1, \epsilon_2, \epsilon_3, n_1, n_2, n_3)$-norm of $T$ is

$$\|T\|_{CZ(\epsilon_1,\epsilon_2,\epsilon_3,n_1,n_2,n_3)} := \|T\|_{L^2\to L^2} + |K_{1,2}|_{V_3,\epsilon_1,\epsilon_2} + |K_{2,3}|_{V_1,\epsilon_2,\epsilon_3} + |K_{3,1}|_{V_2,\epsilon_3,\epsilon_1},$$

where $V_k$ denotes the space $CZ(\epsilon_k, n_k)$, and the terms $|K_{i,j}|_{V_k,\epsilon_i,\epsilon_j}$'s are defined in (4.28).

Similarly, we denote the space of $p$-$(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operators by $p$-$CZ(\epsilon_1, \epsilon_2, n_1, n_2)$. The $p$-$CZ(\epsilon_1, \epsilon_2, n_1, n_2)$-norm of an operator $T \in p$-$CZ(\epsilon_1, \epsilon_2, n_1, n_2)$ corresponding to kernels $K_{1,2}, K_{2,3}, K_{3,1}$, as in Definition 4.10, is given by

$$\begin{aligned}\|T\|_{p\text{-}CZ(\epsilon_1,\epsilon_2,\epsilon_3,n_1,n_2,n_3)} := \|T\|_{L^2\to L^2} &+ \sum_{|\alpha|=\mathcal{N}_1+1} \left|D^{\alpha}_{y_1} K_{1,2}\right|_{V_3,\epsilon_1,\epsilon_2} \\ &+ \sum_{|\beta|=\mathcal{N}_2+1} \left|D^{\beta}_{y_2} K_{2,3}\right|_{V_1,\epsilon_2,\epsilon_3} + \sum_{|\gamma|=\mathcal{N}_3+1} \left|D^{\gamma}_{y_3} K_{3,1}\right|_{V_2,\epsilon_3,\epsilon_1},\end{aligned}$$

where $V_j = CZ(\epsilon_j, n_j)$, and the terms $\left|D^{\alpha}_{y_1} K_{1,2}\right|_{V_3,\epsilon_1,\epsilon_2}$, $\left|D^{\beta}_{y_2} K_{2,3}\right|_{V_1,\epsilon_2,\epsilon_3}$, $\left|D^{\gamma}_{y_3} K_{3,1}\right|_{V_2,\epsilon_3,\epsilon_1}$ are defined in (4.28).

## 4.3 Papoulis' Function Belongs to $H^1$.

Let $\epsilon > 0$. Papoulis used the construction below to build a function $f$ on $\mathbb{R}^2$ such that the upper strong derivative of $\int|f|$ is equal to $+\infty$ almost everywhere on a set of measure greater than $11/15 - \epsilon$. So any $\epsilon \in (0, 11/15)$ will yield a set of positive measure. Define

$$\sigma(N) := \sum_{j=1}^{N} \frac{1}{j}, \qquad \theta(N) := 1 - \frac{\sigma(N)}{N}, \quad N \in \mathbb{N},$$

and note that $0 < \theta(N) < 1$ and $(\log 2)(\log_2 N) = \log N < \sigma(N)$ $\forall N \in \mathbb{N}$.

Step 1: Let $S_1 := (0,1)\times(0,1)$, $N_1 := 16$ and choose $k_1 \in \mathbb{N}$ such that $(\theta(N_1))^{k_1} < \epsilon/2$. On $S_1$ we consider the following $N_1$ rectangles

$$I_1^{(1,j)} = \left(0, \frac{j}{N_1}\right) \times \left(0, \frac{1}{j}\right), \quad 1 \le j \le N_1.$$

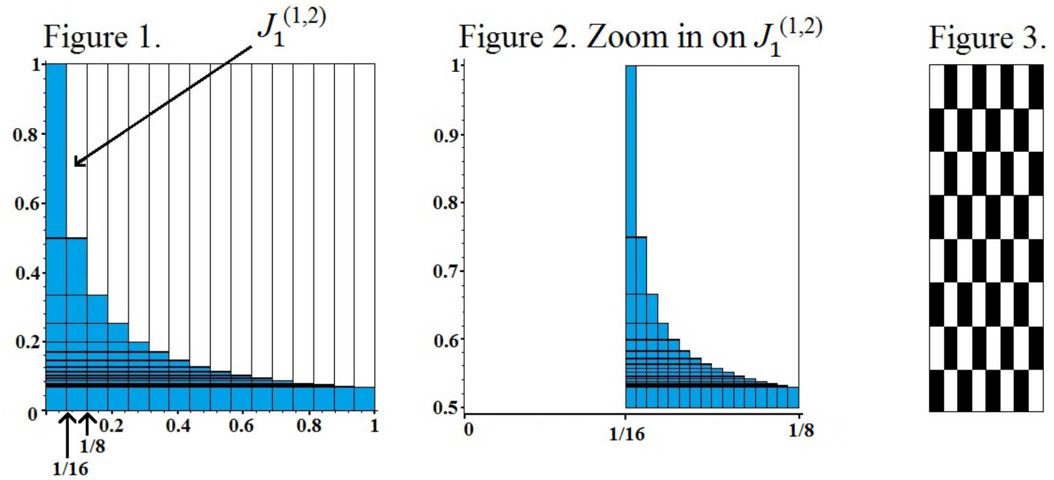
The union of all these $I_1^{(1,j)}$'s is the staircase set denoted by $V_1^{(1)}$ and shown in Figure

1. It satisfies

$$\left|V_1^{(1)}\right| = \left|\bigcup_{1\leq j}^{N}\left(\frac{j-1}{N_1},\frac{j}{N_1}\right)\times\left(0,\frac{1}{j}\right)\right| = \sum_{j=1}^{N_1}\frac{1}{N_1}\frac{1}{j} = |S_1|\,\frac{\sigma\left(N_1\right)}{N_1} = |S_1|\left(1-\theta\left(N_1\right)\right), \tag{4.37}$$

The set $E_1\left(S_1\right) := S_1 - \cup_j I_1^{(1,j)}$ will be called the remainder. Its area is $\left|E_1\left(S_1\right)\right| = \theta\left(N_1\right)|S_1|$.

Figure 4.1: Figures 1, 2 and 3 of Papoulis' example




Through the vertical sides of the rectangles $I_1^{(1,j)}$ we draw lines that are parallel to the $x_2$-axis and that divide $E_1\left(S_1\right)$ into the following $\left(N_1-1\right)$ rectangles

$$J_1^{(1,m)} = \left(\frac{m-1}{N_1},\frac{m}{N_1}\right)\times\left(\frac{1}{m},1\right),\quad 2\leq m\leq N_1.$$

which satisfy $\left|J_1^{(1,m)}\right| = \frac{1}{N_1}\left(1-\frac{1}{m}\right)$, $2\leq m\leq N_1$. They are the non-shaded rectangles in Figure 1 (of Figure 4.1), which represents the case where $N_1=16$.

On each of these $J_1^{(1,m)}$, $2\leq m\leq N_1$, we perform the same process that we did on $S_1$, i.e. we consider the following $N_1$ rectangles

$$I_1^{(2,m,j)} = \left(\frac{m-1}{N_1},\frac{m-1}{N_1}+\frac{j}{N_1^2}\right)\times\left(\frac{1}{m},\frac{1}{m}+\frac{(1-1/m)}{j}\right),\quad 1\leq j\leq N_1,$$

and we form the staircase set $V_1^{(2,m)} = \cup_j I_1^{(2,m,j)}$. By the same argument that yields

(4.37), we obtain $\left|V_1^{(2,m)}\right| = \left|J_1^{(1,m)}\right| (1-\theta(N_1))$ and

$$\left|R_1\left(J_1^{(1,m)}\right)\right| = \left|J_1^{(1,m)} - V_1^{(2,m)}\right| = \theta(N)\left|J_1^{(1,m)}\right|.$$

Through the vertical sides of the rectangles $I_1^{(2,m,j)}$ we draw lines that are parallel to the $x_2$-axis and that divide $E_1\left(J_1^{(1,m)}\right)$ into $(N_1-1)$ rectangles. Figure 2 (of Figure 4.1), which represents the case where $N_1 = 16$ and has different scales on the horizontal and vertical axes, shows $J_1^{(1,m)}$ and the rectangles $I_1^{(2,2,j)}$, $1 \le j \le N_1$, whose union, the staircase set $V_1^{(2,2)}$, is the shaded region.

After repeating the process on all the $J_1^{(1,m)}$'s, $2 \le m \le N_1$, we obtain $(N_1-1)$ staircase sets $V_1^{(2,m)}$ and the new remainder

$$E_2(S_1) := S_1 - V_1^{(1)} - \bigcup_{m=2}^{N_1} V_1^{(2,m)}$$

that has area

$$|E_2(S_1)| = \sum_{m=1}^{N_1}\left|J_1^{(1,m)} - V_1^{(2,m)}\right| = \theta(N_1)\sum_{k=2}^{N_1}\left|J_1^{(1,m)}\right| = (\theta(N_1))^2|S_1|.$$

We proceed recursively, obtaining staircase sets

$$V_1^{(1)},$$

$$V_1^{(2,m)},\ 2 \le m \le N_1,$$

$$V_1^{(3,m_1,m_2)},\ 2 \le m_i \le N_1,\ 1 \le i \le 2,$$

$$\vdots$$

$$V_1^{\left(k_1,m_1,\ldots,m_{k_1-1}\right)},\ 2 \le m_i \le N_1,\ 1 \le i \le k_1-1,$$

and a remainder, $E(S_{k_1}) := S_1 - V_1^{(1)} - \cup_{i=1}^{k_1}\cup_{m_i=2}^{k-1}\cup_{m_1=2}^{N_1} V_1^{(k,m_1,\ldots,m_i)}$, that satisfies

$$|E_{k_1}(S_1)| = (\theta(N_1))^{k_1}, \quad |S_1| = (\theta(N_1))^{k_1} < \frac{\epsilon}{2}.$$

Let

$$\Gamma_1 := \{1\} \cup \{(k, m_1, ..., m_i) : 2 \leq m_i \leq N_1, 1 \leq i \leq k-1, 2 \leq k \leq k_1\}.$$

For each $\gamma \in \Gamma_1$, the staircase set $V_1^{(\gamma)}$ is the union of $N_1$ rectangles $I_1^{(\gamma,j)}$ of equal area, which are defined as in the case $V_1^{(\gamma)} = \cup_j I_1^{(\gamma,j)}$. The intersection of the $I_1^{(\gamma,j)}$'s forms the rectangle

$$A_1^{(\gamma)} := \bigcap_{j=1}^{N_1} I_1^{(\gamma,j)}$$

By construction, $\left|V_1^{(\gamma)}\right| = \left|J_1^{(\gamma)}\right| \sigma(N_1)/N_1$ and $\left|I_1^{(\gamma,j)}\right| = \left|J_1^{(\gamma)}\right|/N_1$. So,

$$\left|A_1^{(\gamma)}\right| = \frac{\left|I_1^{(\gamma,j)}\right|}{N_1} = \frac{\left|V_1^{(\gamma)}\right|}{N_1 \sigma(N_1)} \leq \frac{c\left|V_1^{(\gamma)}\right|}{N_1 \log_2(N_1)}, \quad \forall \gamma \in \Gamma_1. \tag{4.38}$$

We define

$$\mathcal{A}_1 := \bigcup_{\gamma \in \Gamma_1} A_1^{(\gamma)},$$

and we obtain

$$|\mathcal{A}_1| \leq \sum_{\gamma \in \Gamma_1} \left|A_1^{(\gamma)}\right| \leq \sum_{\gamma \in \Gamma_1} \frac{c\left|V_1^{(\gamma)}\right|}{N_1 \log_2(N_1)} \leq \frac{c\,|S_1|}{N_1 \log_2(16)} = \frac{c}{N_1 2^2},$$

using (4.38) and the fact that the staircase sets $V_1^{(\gamma)}$'s are disjoint and are all contained in $S_1$.

<u>Step $n$, for $n > 1$:</u> Let

$$m_{n-1} := \min_{\gamma \in \Gamma_{n-1};\, 1 \leq i \leq p_{n-1}^{-2}} \{\text{length of smallest side of } A_i^{(\gamma)}\}$$

and choose $p_n$ s.t. $0 < p_n < m_{n-1}/N_{n-1}^2$ and $p_n \in \mathbb{N}$. Divide $S_1$ into $p_n^{-2}$ squares $\left\{S_i^{(n)}\right\}_{1 \leq i \leq p_n^{-2}}$, each with side $p_n$. Let $N_n := 2^{2^{2n}}$ and choose $k_n \in \mathbb{N}$ such that $(\theta(N_n))^{k_n} < 2^{-n} p_n^2 \epsilon$.

Perform on each $S_i^{(n)}$, $1 \leq i \leq p_n^{-2}$, the procedure that we did in $S_1$ to obtain a set of disjoint rectangles $\left\{A_i^{(\gamma)}\right\}_{\gamma \in \Lambda_n}$ and a remainder, $E_{k_n}\left(S_i^{(n)}\right)$ satisfying $\left|A_i^{(\gamma)}\right| <$

$\left|S_i^{(n)}\right|/N_n\log_2(N_n)$ and $\left|E_{k_n}\left(S_i^{(n)}\right)\right| = (\theta(N_n))^{k_n}\left|S_i^{(n)}\right|$.

Defining

$$\mathcal{A}_n := \bigcup_{1\le i\le p_n^{-2}}\bigcup_{\gamma\in\Gamma_n} A_i^{(\gamma)},$$

we obtain

$$|\mathcal{A}_n| < \frac{c\,|S_1|}{N_n\log_2(N_n)} = \frac{c}{N_n 2^{2n}},$$

Now, for every $n\in\mathbb{N}$, we proceed as follows: For each $A = I\times J \in \left\{A_i^{(\gamma)}\right\}_{\gamma\in\Lambda_n}$, we divide it into $N_n^4$ congruent rectangles of sides $|I|/N_n^2$ and $|J|/N_n^2$. Then, we make a chessboard pattern by calling half of these rectangles white and the other half black, with the rectangle on the top left corner being white, as in Figure 3 (of Figure 4.1). The set of all white rectangles on $\mathcal{A}_n$ will be denoted by $\mathcal{W}_n$, the set of all black rectangles on $\mathcal{A}_n$, by $\mathcal{B}_n$. Finally, we define

$$f_n := 2^n N_n\left(\chi_{\mathcal{W}_n} - \chi_{\mathcal{B}_n}\right), \qquad f := \sum_{n=1}^{\infty} f_n. \tag{4.39}$$

Since $\int|f_n(x)|\,dx = 2^n N_n|\mathcal{A}_n| \le c2^{-n}$ for all $n$, we conclude that $f\in L^1(\mathbb{R}^2)$.

*Proof of Theorem 4.1.* Let $f$ be the function defined in (4.39). It was shown [53] that the strong derivative of $\int f$ exists a.e. and the upper strong derivative of $\int|f|$ is infinite on a set of positive measure.

To see that $f\in H^1(\mathbb{R}\times\mathbb{R})$, we write

$$f = \sum_{n\in\mathbb{N};\ \gamma\in\Gamma_n;\ 1\le i\le p_n^{-2}} 2^n N_n\left|A_i^{(\gamma)}\right|\frac{f_n\chi_{A_i^{(\gamma)}}}{2^n N_n\left|A_i^{(\gamma)}\right|}, \tag{4.40}$$

and we note that

$$\sum_{n\in\mathbb{N};\ \gamma\in\Gamma_n;\ 1\le i\le p_n^{-2}} 2^n N_n\left|A_i^{(\gamma)}\right| < \infty$$

and that each $f_n\chi_{A_i^{(\gamma)}}/\left(2^n N_n\left|A_i^{(\gamma)}\right|\right)$ is a rectangle $(1,2)$-atom. So, by (4.40), $f\in H^1_{rect}(\mathbb{R}\times\mathbb{R})$. Thus $f\in H^1(\mathbb{R}\times\mathbb{R})$. □

## 4.4 Higher Dimensional Journé's Lemma.

A proof of the discrete version of Journé's Lemma [42] in the setting $\mathbb{R} \times \mathbb{R}$ can be found in [42] and [67]. As mentioned in [42], the discrete version holds in the setting $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and also for sets of finite measure. This known result, which is the content of Lemma 4.2, is often used in the context of product Hardy spaces (e.g. [25]), despite the fact that no proof of it can be found in the literature. Unlike what happens in $\mathbb{R} \times \mathbb{R}$, in the setting $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ the discrete version is not an immediate corollary of the original Journé's Lemma (Proposition 1 in [42]). In higher-dimensional product settings, a different argument is required.

For higher-parameter variants of Journé's Lemma, one can look at [54] and [6]. In Subsection 4.4.2, we show that the proof of Lemma 1.4 in [54] holds in the setting $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$ and for sets of finite measure.

Some notation is necessary. Let $I$ in $\mathcal{D}^{n_j}$ and $i \in \mathbb{N}$. To be consistent with [54] and [67], the unique ($i$-th generation dyadic parent of $I$) $Q$ in $\mathcal{D}^{n_j}$ such that $Q \cap I = Q$ and $|Q| = 2^{in_j} |I|$ will be denoted by $\Delta_i(I)$. The collection of $2^{in_j}$ cubes $Q$ in $\mathcal{D}^{n_j}$ such that $Q \cap I = I$ and $|Q| = 2^{-in_j} |I|$ will be denoted by $\Delta_{-i}(I)$. Note, to avoid confusion, that $\Delta_i(I)$ is cube, while $\Delta_{-i}(I)$ is a set of cubes.

### 4.4.1 Two-parameter Setting

Given an open set $\Omega \subset \mathbb{R}^{n_1+n_2}$ with finite measure, we denote the set of dyadic rectangles $I_1 \times I_2 \subset \Omega$ which are maximal with respect to the $x_1$-direction by $\mathcal{M}_1(\Omega)$. Specifically, the elements of $\mathcal{M}_1(\Omega)$ are dyadic rectangles $I_1 \times I_2 \subset \Omega$ such that if $I_1' \times I_2$ is a dyadic rectangle in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that $I_1' \times I_2 \subset \Omega$, then $I_1 = I_1'$. Define $\mathcal{M}_2(\Omega)$ analogously and note that $\mathcal{M}(\Omega) = \mathcal{M}_1(\Omega) \cap \mathcal{M}_2(\Omega)$.

Define $\Omega_1 := \{M_S(\chi_\Omega) > 1/2\}$ and for each $I_1 \times I_2 \in \mathcal{M}_2(\Omega)$, let $\widehat{I_1} \times I_2$ be the unique rectangle in $\mathcal{M}_1(\Omega_1)$ having the form $I_1' \times I_2$ with $I_1' \supset I_1$.

**Lemma 4.2** (Journé's Lemma [42]). *Let $\Omega \subset \mathbb{R}^{n_1+n_2}$ be an open set with finite measure and let $w$ be a nonnegative, nondecreasing function on $(0, \infty)$ satisfying*

$\sum_{k=1}^{\infty} kw\left(2^{-n_1 k}\right) < \infty$. *Then*

$$\left| \bigcup_{I\times J\in\mathcal{M}_2(\Omega)} \widehat{I}\times J \right| \leq c\,|\Omega| \tag{4.41}$$

*and*

$$\sum_{I\times J\in\mathcal{M}_2(\Omega)} |I\times J|\, w\left(\frac{|I|}{\left|\widehat{I}\right|}\right) \leq c\,|\Omega|\,, \tag{4.42}$$

*where the constant $C$ is independent of $\Omega$.*

The proof of this result follows a reasoning described in [67]. It requires the following lemma:

**Lemma 4.3** (Variant of Lemma 1.40 in [67])**.** *Let $\Omega$ and $w$ be as in Lemma 4.2 and, for each $I \in \mathcal{D}^{n_1}$, consider the collection*

$$E_I(\Omega) := \{J \in \mathcal{D}^{n_2} : I \times J \subset \Omega\}\,.$$

*Then*

$$\sum_{I\in\mathcal{D}^{n_1}} |I| \sum_{k=0}^{\infty} w\left(\frac{|I|}{|\Delta_k(I)|}\right) \left| \bigcup_{J\in E_I(\Omega)\setminus E_{\Delta_{k+1}(I)}(\Omega)} J \right| \leq c\,|\Omega|\,. \tag{4.43}$$

*Proof of Lemma 4.3.* Denote $E_I := E_I(\Omega)$. Since $E_I \backslash E_{\Delta_{k+1}(I)}$ can be written as the disjoint union

$$\left(E_I \backslash E_{\Delta_1(I)}\right) \cup \ldots \cup \left(E_{\Delta_k(I)} \backslash E_{\Delta_{k+1}(I)}\right),$$

the sum in (4.43) equals

$$\sum_{I\in\mathcal{D}^{n_1}} |I| \left[\sum_{k=0}^{\infty} w\left(2^{-n_1 k}\right)\right] \sum_{I_0\in\mathcal{D}^{n_1}: I\subset I_0\subset\Delta_k(I)} \left|\cup_{J\in E_{I_0}\backslash E_{\Delta_1(I_0)}} J\right|$$

$$= \sum_{k=0}^{\infty} w\left(2^{-n_1 k}\right) \sum_{I_0\in\mathcal{D}^{n_1}} \left(\sum_{I\in\mathcal{D}^{n_1}: I\subset I_0\subset\Delta_k(I)} |I| \left|\cup_{J\in E_{I_0}\backslash E_{\Delta_1(I_0)}} J\right|\right). \tag{4.44}$$

For each fixed $k \in \{0, 1, ...\}$ and $I_0 \in \mathcal{D}^{n_1}$,

$$\sum_{I \in \mathcal{D}^{n_1} : I \subset I_0 \subset \Delta_k(I)} |I| \left| \cup_{J \in E_{I_0} \setminus E_{\Delta_1(I_0)}} J \right| = \sum_{I \in \mathcal{D}^{n_1} : I \subset I_0 \subset \Delta_k(I)} |I_0| \left| \cup_{J \in E_{I_0} \setminus E_{\Delta_1(I_0)}} J \right| \frac{|I|}{|I_0|}$$

$$= |I_0| \left| \cup_{J \in E_{I_0} \setminus E_{\Delta_1(I_0)}} J \right| \sum_{j=0}^{k} 2^{-n_1 j} \left( \# \{ I : I \in \Delta_{-j}(I_0) \} \right) = (k+1) |I_0| \left| \cup_{J \in E_{I_0} \setminus E_{\Delta_1(I_0)}} J \right|.$$

Therefore (4.44) is equal to

$$\sum_{k=0}^{\infty} (k+1) w\left(2^{-n_1 k}\right) \left[ \sum_{I_0 \in \mathcal{D}^{n_1}} |I_0| \left| \cup_{J \in E_{I_0} \setminus E_{\Delta_1(I_0)}} J \right| \right]. \tag{4.45}$$

Note that

$$\sum_{I_0 \in \mathcal{D}^{n_1}} |I_0| \left| \cup_{J \in E_{I_0} \setminus E_{\Delta_1(I_0)}} J \right| = |\Omega|. \tag{4.46}$$

This equality holds because $\Omega = \cup_{I \in \mathcal{D}^{n_1}} \left[ I \times \left( \cup_{J \in E_I \setminus E_{\Delta_1(I)}} J \right) \right]$ and the union is disjoint. To obtain (4.43), plug (4.46) into (4.45) and use the assumptions on $w$. □

*Proof of Lemma 4.2.* The estimate (4.41) follows from the definition of $\widehat{I}$ and the strong maximal function theorem.

To prove (4.42), we define

$$A_{I,k+1} := \{ J \in \mathcal{D}^{n_2} : I \times J \in \mathcal{M}_2(\Omega),\ \Delta_k(I) \times J \in \mathcal{M}_1(\Omega_1) \},$$

for each $k \in \mathbb{N}$ and each $I \in \mathcal{D}^{n_1}$. Then we write the sum in (4.42) as

$$\sum_{I \in \mathcal{D}^{n_1}} |I| \sum_{k=0}^{\infty} w\left(2^{-n_1 k}\right) \left( \sum_{J \in \mathcal{D}^{n_2} : I \times J \in \mathcal{M}_2(\Omega),\ \widehat{I} = \Delta_k(I)} |J| \right)$$

$$= \sum_{I \in \mathcal{D}_{n_1}} |I| \sum_{k=0}^{\infty} w\left(2^{-n_1 k}\right) \left[ \sum_{J \in A_{I,k+1}} |J| \right]. \tag{4.47}$$

We claim that for each $k \in \{0, 1, ...\}$ and each $I \in \mathcal{D}^{n_1}$, the following holds:

$$\text{if } J \neq J' \in A_{I,k+1}, \text{ then } J \cap J' = \emptyset.$$

To see this, note that whenever two distinct dyadic cubes have intersecting interiors, one is properly contained in the other. This cannot happen in the case of $J$ and $J'$ because we are assuming that both $I \times J$ and $I \times J'$ belong to $\mathcal{M}_2(\Omega)$. Hence the sum in the brackets in (4.47) is equal to $\left|\cup_{J\in A_{I,k+1}} J\right|$. This yields

$$\sum_{I\times J\in\mathcal{M}_2(\Omega)} |I|\,|J|\,w\left(\frac{|I|}{\left|\widehat{I}\right|}\right) \leq \sum_{I\in\mathcal{D}^{n_1}} |I| \sum_{k=0}^{\infty} w\left(2^{-n_1 k}\right)\left|\cup_{J\in A_{I,k+1}} J\right|.$$

Now, by Lemma 4.3, to finish the proof of (4.42), it suffices to show that

$$\left|\cup_{J\in A_{I,k+1}} J\right| \leq c\left|\cup_{J\in E_I\setminus E_{\Delta_{k+1}(I)}} J\right| \tag{4.48}$$

for all $I \in \mathcal{D}^{n_1}$ and $k \in \{0,1,...\}$. So fix $I$ and $k$, and let $J_0$ be in $A_{I,k+1}$. Then $I \times J_0 \in \mathcal{M}_2(\Omega)$ and $\widehat{I} = \Delta_k(I)$. Since $\widehat{I}$ is the largest dyadic cube satisfying $\widehat{I} \times J_0 \subset \{M_S(\chi_\Omega) > 1/2\}$ and $\widehat{I} \supset I$, its dyadic parent $\Delta_1\left(\widehat{I}\right)$, which is $\Delta_{k+1}(I)$, must satisfy

$$\left|\left(\left(\Delta_1\left(\widehat{I}\right)\right)\times J_0\right)\cap\Omega\right| \leq 2^{-1}\left|\left(\Delta_1\left(\widehat{I}\right)\right)\times J_0\right| = 2^{-1}\left|\Delta_1\left(\widehat{I}\right)\right||J_0| \tag{4.49}$$

By the definition of $E_{\Delta_{k+1}(I)}(\Omega)$, the set $(\Delta_{k+1}(I)) \times \left(\cup_{J\in E_{\Delta_{k+1}(I)}} J\right)$ is contained in $\Omega$. This inclusion and (4.49) yield

$$|\Delta_{k+1}(I)|\left|J_0\cap\left(\cup_{J\in E_{\Delta_{k+1}(I)}} J\right)\right| = \left|(\Delta_{k+1}(I))\times\left(J_0\cap\left(\cup_{J\in E_{\Delta_{k+1}(I)}} J\right)\right)\right|$$

$$\leq |[(\Delta_{k+1}(I))\times J_0]\cap\Omega| \leq 2^{-1}|\Delta_{k+1}(I)|\,|J_0|.$$

This implies

$$\left|J_0\cap\left(\cup_{J\in E_{\Delta_{k+1}(I)}} J\right)^c\right| \geq 2^{-1}|J_0|. \tag{4.50}$$

Since every cube in the collection to $A_{I,k+1}$ is belongs to $E_I$, we have $J_0 \subset \cup_{J\in E_I} J$. So (4.50) yields

$$\left|J_0\cap\left(\cup_{J\in E_I\setminus E_{\Delta_{k+1}(I)}} J\right)\right| = \left|J_0\cap\left(\cup_{J\in E_I} J\right)\cap\left(\cup_{J\in E_{\Delta_{k+1}(I)}} J\right)^c\right| \geq 2^{-1}|J_0|.$$

Letting $V_{I,k} := \cup_{J\in E_I\setminus E_{\Delta_{k+1}(I)}} J$, we obtain $J_0 \subset \{M(\chi_{V_{I,k}}) \geq 1/2\}$, where $M$ is the Hardy-Littlewood maximal function. Since $J_0$ is an arbitrary element of $A_{I,k+1}$, we conclude that

$$\bigcup_{J\in A_{I,k+1}} J \subset \{M(\chi_{V_{I,k}}) > 1/2\}.$$

Then Hardy-Littlewood maximal function theorem yields that $\left|\cup_{J\in A_{I,k+1}} J\right|$ is bounded above by $c\left|\cup_{J\in E_I\setminus E_{\Delta_{k+1}(I)}} J\right|$, which shows that (4.48) holds with a constant $c$ depending only on $n_2$. □

### 4.4.2 Three-parameter Setting

Given an open set $\Omega \subset \mathbb{R}^{n_1+n_2+n_3}$ with finite measure, we denote the set of dyadic rectangles $R = I_1 \times I_2 \times I_3 \subset \Omega$ which are maximal with respect to the $x_1$-direction by $\mathcal{M}_1(\Omega)$. Define $\mathcal{M}_2(\Omega)$ and $\mathcal{M}_3(\Omega)$ analogously and note that $\mathcal{M}(\Omega) = \mathcal{M}_1(\Omega) \cap \mathcal{M}_2(\Omega) \cap \mathcal{M}_3(\Omega)$.

Let

$$\Omega_1 := \{x : M_S(\chi_\Omega)(x) > 1/2\} \text{ and } \Omega_2 := \{x : M_S(\chi_{\Omega_1})(x) > 1/2\}$$

and for each $I_1 \times I_2 \times I_3$ in $\mathcal{M}_3(\Omega)$, let $\widehat{I_1} \times I_2 \times I_3$ be the unique rectangle in $\mathcal{M}_1(\Omega_1)$ having the form $I_1' \times I_2 \times I_3$ with $I_1' \supset I_1$. Similarly, to each $I_1' \times I_2 \times I_3$ in $\mathcal{M}_3(\Omega)$, let $I_1' \times \widehat{I_2} \times I_3$ be the rectangle in $\mathcal{M}_3(\Omega_2)$ having the form $I_1' \times \widehat{I_2} \times I_3$ with $\widehat{I_2} \supset I_2$.

In the statement of Lemma 4.4 below, each rectangle $\widehat{I} \times \widehat{J} \times K$ is constructed from a rectangle $I \times J \times K$ in $\mathcal{M}_3(\Omega)$ according to the following two steps method:

(i) Beginning with $I \times J \times K$ in $\mathcal{M}_3(\Omega)$, we define $\widehat{I} \times J \times K$ to be the unique rectangle in $\mathcal{M}_1(\Omega_1)$ having the form $I_1' \times J \times K$ with $I' \supset I$.

(ii) From $\widehat{I} \times J \times K$, which is a rectangle in $\mathcal{M}_1(\Omega_1)$, we define $\widehat{I} \times \widehat{J} \times K$ to be the unique rectangle in $\mathcal{M}_2(\Omega_2)$ having the form $\widehat{I} \times J' \times K$ with $J' \supset J$.

Note that $\widehat{I}$ depends on $I \times J \times K$, while $\widehat{J}$ depends on $\widehat{I} \times J \times K$.

**Lemma 4.4** (Variant of Pipher's Lemma [54]). *Let $\Omega \subset \mathbb{R}^{n_1+n_2+n_3}$ be an open set with* *finite measure and let $v$ and $w$ be nonnegative nondecreasing functions on $(0,\infty)$ such that $\sum_{k=1}^{\infty} kv\left(2^{-n_1k}\right)<\infty$ and $\sum_{k=1}^{\infty} kw\left(2^{-n_2k}\right)<\infty$. Then*

$$\left|\bigcup_{I\times J\times K\in\mathcal{M}_3(\Omega)} \widehat{I}\times\widehat{J}\times K\right|\leq c\,|\Omega| \tag{4.51}$$

*and*

$$\sum_{I\times J\times K\in\mathcal{M}_3(\Omega)} \left|\widehat{I}\times\widehat{J}\times K\right| v\left(\frac{|I|}{\left|\widehat{I}\right|}\right) w\left(\frac{|J|}{\left|\widehat{J}\right|}\right)\leq c\,|\Omega|\,, \tag{4.52}$$

*where the constant $c$ is independent of $\Omega$.*

As mentioned above, this result can be shown with the argument developed by Pipher [54]. Its proof needs the following three-parameter analogue of Lemma 4.3:

**Lemma 4.5** (Variant of Proposition B in [54])**.** *Let $\Omega$ and $w$ be as in Lemma 4.4 and, for each $I$ in $\mathcal{D}^{n_1}$, consider the collection*

$$E_I(\Omega):=\left\{J\times K\in\mathcal{D}^{n_2}\times\mathcal{D}^{n_3}: I\times J\times K\subset\Omega\right\}.$$

*Then*

$$\sum_{I\in\mathcal{D}^{n_1}} |I|\sum_{k=0}^{\infty} v\left(\frac{|I|}{|\Delta_k(I)|}\right)\left|\bigcup_{J\times K\in E_I(\Omega)\setminus E_{\Delta_{k+1}(I)}(\Omega)} J\times K\right|\leq c\,|\Omega|\,. \tag{4.53}$$

*Proof.* Denoting $E_I:=E_I(\Omega)$, we can express the sum in (4.53) as

$$\sum_{I\in\mathcal{D}^{n_1}} |I|\left[\sum_{k=0}^{\infty} v\left(2^{-n_1k}\right)\right]\sum_{I_0\in\mathcal{D}^{n_1}:I\subset I_0\subset\Delta_k(I)}\left|\cup_{J\times K\in E_{I_0}\setminus E_{\Delta_1(I_0)}} J\times K\right|$$

$$=\sum_{k=0}^{\infty} v\left(2^{-n_1k}\right)\sum_{I_0\in\mathcal{D}^{n_1}}\left[\sum_{I\in\mathcal{D}^{n_1}:I\subset I_0\subset\Delta_k(I)} |I|\left|\cup_{J\times K\in E_{I_0}\setminus E_{\Delta_1(I_0)}} J\times K\right|\right]. \tag{4.54}$$

For each $k\in\{0,1,...\}$ and $I_0\in\mathcal{D}^{n_1}$, the sum in the brackets in (4.54) equals

$$\sum_{I\in\mathcal{D}^{n_1}:I\subset I_0\subset\Delta_k(I)} |I_0|\left|\cup_{J\times K\in E_{I_0}\setminus E_{\Delta_1(I_0)}} J\times K\right|\frac{|I|}{|I_0|}$$

$$= |I_0| \left|\cup_{J\times K\in E_{I_0}\setminus E_{\Delta_1(I_0)}} J\times K\right| \sum_{j=0}^{k} \sum_{I\in\Delta_{-j}(I_0)} 2^{-n_1 j}$$

$$= (k+1)\,|I_0| \left|\cup_{J\times K\in E_{I_0}\setminus E_{\Delta_1(I_0)}} J\times K\right|.$$

Plugging it back in (4.54), we get that (4.54) is equal to

$$\sum_{k=0}^{\infty} (k+1)\, v\left(2^{-n_1 k}\right) \left[\sum_{I_0\in\mathcal{D}^{n_1}} |I_0| \left|\cup_{J\times K\in E_{I_0}\setminus E_{\Delta_1(I_0)}} J\times K\right|\right]. \tag{4.55}$$

Since $\Omega$ can be expressed as $\cup_{I\in\mathcal{D}^{n_1}} I\times\left(\cup_{J\times K\in E_{I_0}\setminus E_{\Delta_1(I_0)}} J\times K\right)$ and this union is disjoint, the sum in the brackets in (4.55) equals $|\Omega|$. This fact and the assumptions on $v$ imply (4.53). □

*Proof of Lemma 4.4.* The estimate (4.51) follows from the strong maximal function theorem.

To prove (4.52), define

$$A_{I,k+1} := \{J\times K\in\mathcal{D}^{n_2}\times\mathcal{D}^{n_3} : I\times J\times K\in\mathcal{M}_3(\Omega),\ \Delta_k(I)\times J\times K\in\mathcal{M}_1(\Omega_1)\},$$

$$\mathcal{A}_{I,k+1} := \bigcup_{J\times K\in A_{I,k+1}} J\times K,$$

for each $k\in\{0,1,...\}$ and each cube $I\in\mathcal{D}^{n_1}$. Note that if $I\times J\times K$ is in $\mathcal{M}_3(\Omega)$ and $\Delta_k(I)\times J\times K$ is in $\mathcal{M}_1(\Omega_1)$, then $J\times K\in\mathcal{M}_2(\mathcal{A}_{I,k+1})$. Recall that $\mathcal{M}_2(\mathcal{A}_{I,k+1})$ is the set of dyadic rectangles in $\mathcal{A}_{I,k+1}$, a subset of $\mathbb{R}^{n_2}\times\mathbb{R}^{n_3}$, which are maximal in the $x_3$-direction, i.e. maximal with respect to the second factor, which is $\mathbb{R}^{n_3}$. Now we apply Lemma 4.2 to the set $\mathcal{A}_{I,k+1}$. For each $J\times K\in\mathcal{M}_2(\mathcal{A}_{I,k+1})$, we let $J'\in\mathcal{D}^{n_2}$ be such that

$$J'\supset J \text{ and } J'\times K\in\mathcal{M}_2\left(\left\{M_S\left(\chi_{\mathcal{A}_{I,k+1}}\right)>1/2\right\}\right).$$

Then

$$\sum_{J\times K\in\mathcal{M}_2\left(\mathcal{A}_{I,k+1}\right)} |J\times K|\, w\left(\frac{|J|}{|J'|}\right) \le c\,|\mathcal{A}_{I,k+1}|. \tag{4.56}$$

For any $I\times J\times K\in\mathcal{M}_3(\Omega)$, if (i) $\Delta_k(I)\times J\times K\in\mathcal{M}_1(\Omega_1)$ and (ii) $\Delta_k(I)\times\widehat{J}\times K\in$

$\mathcal{M}_2(\Omega_2)$, then $J' \subset \widehat{J}$. This follows from the estimate

$$|(\Delta_k(I) \times J' \times K) \cap \Omega_2| > |(\Delta_k(I) \times J' \times K) \cap (\Delta_k(I) \times \mathcal{A}_{I,k+1})|$$

$$= |\Delta_k(I)| \, |(J' \times K) \cap \mathcal{A}_{I,k+1}| > 2^{-1} |\Delta_k(I)| \left| J^{'} \times K \right| = 2^{-1} \left| \Delta_k(I) \times J^{'} \times K \right|.$$

Since $w$ is nondecreasing, the sum in (4.52) can be majorized by

$$\sum_{I \times J \times K \in \mathcal{M}_3(\Omega)} |I \times J \times K| \, v\left(\frac{|I|}{\left|\widehat{I}\right|}\right) w\left(\frac{|J|}{|J'|}\right)$$

$$\leq \sum_{I \in \mathcal{D}^{n_1}} \sum_{k=0}^{\infty} \sum_{I \times J \times K \in \mathcal{M}_3(\Omega):\ \widehat{I} = \Delta_k(I)} |I| \, |J \times K| \, v\left(2^{-n_1 k}\right) w\left(\frac{|J|}{|J'|}\right)$$

$$\leq \sum_{I \in \mathcal{D}^{n_1}} |I| \sum_{k=0}^{\infty} v\left(2^{-n_1 k}\right) \left[ \sum_{J \times K \in \mathcal{M}_2\left(\mathcal{A}_{I,k+1}\right)} |J \times K| \, w\left(\frac{|J|}{|J'|}\right) \right]$$

$$\leq c \sum_{I \in \mathcal{D}^{n_1}} |I| \sum_{k=0}^{\infty} v\left(2^{-n_1 k}\right) |\mathcal{A}_{I,k+1}|, \tag{4.57}$$

where the last inequality is due to (4.56).

Now, to finish this proof, it suffices to show that

$$|\mathcal{A}_{I,k+1}| \leq c \left| \cup_{J \times K \in E_I \backslash E_{\Delta_{k+1}(I)}} J \times K \right|, \tag{4.58}$$

for all $I \in \mathcal{D}^{n_1}$ and $k \in \{0, 1, ...\}$, to plug (4.58) into (4.57) and to apply Lemma 4.5. So fix $I$ and $k$ and define $V_{I,k} := \cup_{J \times K \in E_I \backslash E_{\Delta_{k+1}(I)}} J \times K$. We will show that $\mathcal{A}_{I,k+1} \subset \{M_S(V_{I,k}) > 1/2\}$ and we will conclude, by the strong maximal function theorem, that (4.58) holds. Given $x \in \mathcal{A}_{I,k+1}$, there exists $R_0 \in \mathcal{D}^{n_2} \times \mathcal{D}^{n_3}$ such that $I \times R_0 \in \mathcal{M}_3(\Omega)$ and $\Delta_k(I) \times R_0 \in \mathcal{M}_1(\Omega_1)$. Thus

$$|\Delta_{k+1}(I)| \left| R_0 \cap \left( \cup_{J \times K \in E_{\Delta_{k+1}(I)}} J \times K \right) \right|$$

$$= \left| [\Delta_{k+1}(I) \times R_0] \cap \left[ \Delta_k(I) \times \left( \cup_{J \times K \in E_{\Delta_{k+1}(I)}} J \times K \right) \right] \right|$$

$$\leq \left|\left(\Delta_{k+1}\left(I\right)\times R_0\right)\cap\Omega\right| \leq 2^{-1}\left|\Delta_{k+1}\left(I\right)\times R_0\right| = 2^{-1}\left|\Delta_{k+1}\left(I\right)\right|\left|R_0\right|,$$

where the last inequality holds because $\Delta_k\left(I\right)\times R_0 \in \mathcal{M}_1\left(\Omega_1\right)$. So

$$\left|R_0 \cap \left(\cup_{J\times K\in E_{\Delta_{k+1}(I)}} J\times K\right)^c\right| > 2^{-1}\left|R_0\right|.$$

By the definitions of $\mathcal{A}_{I,k+1}$ and $V_{I,k}$, it follows that $R_0 \subset \mathcal{A}_{I,k+1} \subset V_{I,k}$. Hence

$$\left|R_0\cap V_{I,k}\right| = \left|R_0 \cap \left(\cup_{J\times K\in E_I} J\times K\right)\cap\left(\cup_{J\times K\in E_{\Delta_{k+1}(I)}} J\times K\right)^c\right|$$

$$= \left|R_0 \cap \left(\cup_{J\times K\in E_{\Delta_{k+1}(I)}} \cup J\times K\right)^c\right| > 2^{-1}\left|R_0\right|,$$

and we conclude that $x \in \left\{M_S\left(V_{I,k}\right) > 1/2\right\}$. Therefore (4.58) holds.

To finish the proof of (4.52), plug (4.58) into (4.57) and use the Lemma 4.5. □

## 4.5 Multiparameter Operators on Hardy Spaces

We give a sketch of the proof of the variant of Fefferman's Theorem (Theorem 4.2) just so that it can be compared with the proof of its three-parameter analogue, Theorem 4.3.

*Sketch of the proof of Theorem 4.2.* By Theorem 1.1 in [14] and Remark 4.2 it suffices to show that

$$\left\|T\left(a\right)\right\|_p \leq C \text{ for every rough } \left(p,2\right)\text{-atom on } \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}. \tag{4.59}$$

In fact, the atoms of the statement of Theorem 1.1 in [14] are particular cases of rough $(p,2)$-atoms on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. So the desired conclusion follows from (4.59) and Theorem 1.1 in [14] (with $q = p$ and $\mathcal{B}_q = L^p\left(\mathbb{R}^{n_1+n_2}\right)$, where $q$ and $\mathcal{B}_q$ are defined in [14]).

The proof of this inequality is due to R. Fefferman [25]. Let $a = \sum_{R\in\mathcal{M}(\Omega)} a_R$ be a rough $(p,2)$-atoms on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ associated with a set $\Omega$ as in Definition 4.3. Let

$$\Omega_1 := \left\{M_S\left(\chi_\Omega\right) > 1/2\right\} \quad\text{and}\quad \Omega_2 := \left\{M_S\left(\chi_{\Omega_1}\right) > 1/2\right\},$$

and for each $I\times J$ in $\mathcal{M}\left(\Omega\right)$, let $\widehat{I}\times J$ be the rectangle in $\mathcal{M}_1\left(\Omega_1\right)$ such that $\widehat{I}\supset I$.

Similarly, to each $I' \times J$ in $\mathcal{M}_1(\Omega_1)$, let $I' \times \widehat{J}$ be the rectangle in $\mathcal{M}_2(\Omega_2)$ such that $\widehat{J} \supset J$. Finally, define

$$\widetilde{\Omega} := \bigcup_{R = I \times J \in \mathcal{M}(\Omega)} 16\left(\widehat{I} \times \widehat{J}\right).$$

Here each rectangle $\widehat{I} \times \widehat{J}$ is constructed from a rectangle $I \times J$ in $\mathcal{M}_2(\Omega)$ as follows:

(i) Beginning with $I \times J$ in $\mathcal{M}_2(\Omega)$, we define $\widehat{I} \times J$ to be the unique rectangle in $\mathcal{M}_1(\Omega_1)$ having the form $I' \times J$ with $I' \supset I$.

(ii) From $\widehat{I} \times J$, which is a rectangle in $\mathcal{M}_1(\Omega_1)$, we define $\widehat{I} \times \widehat{J}$ to be the unique rectangle in $\mathcal{M}_2(\Omega_2)$ having the form $\widehat{I} \times J'$ with $J' \supset J$.

Thus $\widehat{I}$ depends on $I \times J$, while $\widehat{J}$ depends on $\widehat{I} \times J$. Note that $\left|\widetilde{\Omega}\right| \leq c\,|\Omega|$.

Using Hölder's inequality, the $L^2$ boundedness of $T$, item (B) of Definition 4.3, and the upper bound for $\left|\widetilde{\Omega}\right|$ we obtain $\int_{\widetilde{\Omega}} |T(a)|^p \leq C$. To show that $\int_{(\widetilde{\Omega})^c} |T(a)|^p \leq C$, it suffices to prove that

$$\sum_{R = I \times J \in \mathcal{M}(\Omega)} \left( \int_{\left[\left(16\widehat{I}\right)^c \times \mathbb{R}^{n_2}\right]} |T(a_R)|^p + \int_{\left[\mathbb{R}^{n_1} \times \left(16\widehat{J}\right)^c\right]} |T(a_R)|^p \right) \leq C.$$

This can be done with the reasoning described in [25].

Note that Lemma (4.2), which is needed to conclude this proof, only requires dilations in one of the factors. This is why, in this setting, it does not matter whether the hypotheses on the behavior of $|T(a_R)|^p$ are stated with respect to integrals over sets of the form $\left[2^k(I_1 \times I_2)\right]^c$ (as in [25]) or of the form $\left(2^k I_1\right)^c \times \mathbb{R}^{n_2}$ and $\mathbb{R}^{n_1} \times \left(2^k I_2\right)^c$. $\square$

To see how Theorem [25] can be extended to higher-parameter settings, one should examine the parallels between the sketch given above and the following:

*Proof of Theorem 4.3.* By Remark 4.2 and a three-parameter analogue of Theorem 1.1 in [14], it suffices to show that $T$ is uniformly bounded on rough $(p,2)$-atoms on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$. So let $a$ be a such atom. By Definition 4.3, to $a$, there corresponds a set $\Omega$ and a decomposition $a = \sum_{R \in \mathcal{M}(\Omega)} a_R$, where each rough elementary particle $a_R$ is supported on $4R$ for some distinct rectangle $R$ in $\mathcal{M}(\Omega)$.

We intend to apply Lemma 4.4 with $v(x) = x^{\delta_i}$ and $w(x) = x^{\delta_j}$ for some fixed $\delta_i, \delta_j > 0$ that will be chosen below. As we will see, the multiparameter dilations are essential to allow the use of this lemma.

We consider

$$\Omega_1 := \{M_S(\chi_\Omega) > 1/2\} \quad \text{and} \quad \Omega_{i+1} := \{M_S(\chi_{\Omega_i}) > 1/2\}, \quad i = 1, 2, 3,$$

and to each $I \times J \times K$ in $\mathcal{M}(\Omega)$, we let $\widehat{I} \times J \times K$ be the unique rectangle in $\mathcal{M}_1(\Omega_1)$ having the form $I' \times J \times K$ with $I' \supset I$. Similarly, given $I' \times J \times K \in \mathcal{M}_1(\Omega_1)$, let $I' \times \widehat{J} \times K \in \mathcal{M}_2(\Omega_2)$ be such that $\widehat{J} \supset J$; given $I' \times J' \times K \in \mathcal{M}_2(\Omega_2)$, let $I' \times J' \times \widehat{K} \in \mathcal{M}_3(\Omega_3)$ be such that $\widehat{K} \supset K$; and given $I' \times J' \times K' \in \mathcal{M}_3(\Omega_3)$, let $\widehat{I} \times J' \times K' \in \mathcal{M}_1(\Omega_4)$ be such that $\widehat{I} \supset I'$. Finally, we define

$$\widetilde{\Omega} := \bigcup_{R = I \times J \times K \in \mathcal{M}(\Omega)} 16\left(\widehat{\widehat{I}} \times \widehat{J} \times \widehat{K}\right).$$

Here each rectangle $\widehat{\widehat{I}} \times \widehat{J} \times \widehat{K}$ is constructed from a rectangle $I \times J \times K$ in $\mathcal{M}_3(\Omega)$ as follows:

(i) Beginning with $I \times J \times K$ in $\mathcal{M}_3(\Omega)$, we define $\widehat{I} \times J \times K$ to be the unique rectangle in $\mathcal{M}_1(\Omega_1)$ having the form $I_1' \times J \times K$ with $I' \supset I$.

(ii) From $\widehat{I} \times J \times K$, which is a rectangle in $\mathcal{M}_1(\Omega_1)$, we define $\widehat{I} \times \widehat{J} \times K$ to be the unique rectangle in $\mathcal{M}_2(\Omega_2)$ having the form $\widehat{I} \times J' \times K$ with $J' \supset J$.

(iii) From $\widehat{I} \times \widehat{J} \times K$, which is a rectangle in $\mathcal{M}_2(\Omega_2)$, we define $\widehat{I} \times \widehat{J} \times \widehat{K}$ to be the unique rectangle in $\mathcal{M}_3(\Omega_3)$ having the form $\widehat{I} \times \widehat{J} \times K'$ with $K' \supset K$.

(iv) From $\widehat{I} \times \widehat{J} \times \widehat{K}$, which is a rectangle in $\mathcal{M}_3(\Omega_3)$, we define $\widehat{\widehat{I}} \times \widehat{J} \times \widehat{K}$ to be the unique rectangle in $\mathcal{M}_1(\Omega_4)$ having the form $I' \times \widehat{J} \times \widehat{K}$ with $I' \supset \widehat{I}$.

Thus $\widehat{I}$ depends on $I \times J \times K$; $\widehat{J}$ depends on $\widehat{I} \times J \times K$; $\widehat{K}$ depends on $\widehat{I} \times \widehat{J} \times K$; and $\widehat{\widehat{I}}$ depends on $\widehat{I} \times \widehat{J} \times \widehat{K}$.

Note that $16S \subset \{M_S(\chi_{\Omega_4}) > 16^{-n_1-n_2-n_3}\}$ for any rectangle $S \in \mathcal{D}^{n_1} \times \mathcal{D}^{n_2} \times \mathcal{D}^{n_3}$ contained in $\Omega$. So $16\left(\widehat{\widehat{I}} \times \widehat{J} \times \widehat{K}\right) \subset \{M_S(\chi_{\Omega_4}) > 16^{-n_1-n_2-n_3}\}$ for all $I \times J \times K \in$

$\mathcal{M}(\Omega)$. This implies: $\left|\widetilde{\Omega}\right| \leq c\,|\Omega|$. As in the sketch of the proof of Theorem 4.2, we use Hölder's inequality, the $L^2$ boundedness of $T$, item (B) of Definition 4.3, and the upper bound on the measure of $\widetilde{\Omega}$, to conclude that $\int_{\widetilde{\Omega}} |T(a)|^p \leq C$.

Since $0 < p \leq 1$ and the inclusion $\left(\widetilde{\Omega}\right)^c \subset \left(16\left(\widehat{\widehat{I}} \times \widehat{J} \times \widehat{K}\right)\right)^c$ holds for any $I \times J \times K \in \mathcal{M}(\Omega)$, the proof will be concluded once we show

$$\sum_{R\in\mathcal{M}(\Omega)} \int_{\left(16\widehat{I}\,\right)^c \times \left(16\widehat{J}\,\right)^c \times \mathbb{R}^{n_3}} |T(a_R)|^p \leq C, \tag{4.60}$$

$$\sum_{R\in\mathcal{M}(\Omega)} \int_{\mathbb{R}^{n_1} \times \left(16\widehat{J}\,\right)^c \times \left(16\widehat{K}\,\right)^c} |T(a_R)|^p \leq C \tag{4.61}$$

and

$$\sum_{R\in\mathcal{M}(\Omega)} \int_{\left(16\widehat{\widehat{I}}\,\right)^c \times \mathbb{R}^{n_2} \times \left(16\widehat{K}\,\right)^c} |T(a_R)|^p \leq C. \tag{4.62}$$

These three inequalities look similar but their proofs are not identical.

To prove (4.60), we first fix $R = I \times J \times K \in \mathcal{M}(\Omega)$. The inclusions

$$4\left(\frac{\left|\widehat{I}\right|}{|I|}\right)^{1/n_1} 4I \subset 16\widehat{I}, \quad 4\left(\frac{\left|\widehat{J}\right|}{|J|}\right)^{1/n_2} 4J \subset 16\widehat{J}, \quad \operatorname{supp}(a_R) \subset 4R,$$

and hypothesis (4.4) yield

$$\int_{\left(16\widehat{I}\right)^c \times \left(16\widehat{J}\right)^c \times \mathbb{R}^{n_3}} |T(a_R)|^p \leq \int_{\left(4\left(\frac{|\widehat{I}|}{|I|}\right)^{1/n_1} 4I\right)^c \times \left(4\left(\frac{|\widehat{J}|}{|J|}\right)^{1/n_2} 4J\right)^c \times \mathbb{R}^{n_3}} |T(a)|^p$$

$$\leq C\,\|a_R\|_2^p\,|R|^{1-p/2} \left(\frac{\left|\widehat{I}\right|}{|I|}\right)^{-\delta_1/n_1} \left(\frac{\left|\widehat{J}\right|}{|J|}\right)^{-\delta_2/n_2}. \tag{4.63}$$

Summing (4.63) over all $R = I \times J \times K$ in $\mathcal{M}(\Omega)$ and using Hölder's inequality, (4.22), and $\mathcal{M}(\Omega) \subset \mathcal{M}_3(\Omega)$, we conclude that the sum in (4.60) is majorized by

$$C\left(|\Omega|^{1-2/p}\right)^{p/2} \left[\sum_{R=I\times J\times K\in\mathcal{M}_3(\Omega)} |R| \left(\frac{\left|\widehat{I}\right|}{|I|}\right)^{-\delta_1'} \left(\frac{\left|\widehat{J}\right|}{|J|}\right)^{-\delta_2'}\right]^{1-p/2},$$

where $\delta_j' := 2\delta_j / [n_j (2-p)]$, $j = 1, 2$. By Lemma 4.4, the term in the brackets is bounded by $c|\Omega|$ (note that this lemma would not be applicable if we had only one-parameter dilations on hypothesis (4.4)). Thus (4.60) holds.

To show (4.61), we apply the reasoning that shows (4.63) to the integral $\int_{\mathbb{R}^{n_1} \times (16\widehat{J})^c \times (16\widehat{K})^c} |T(a_R)|^p$, we sum over all $R = I \times J \times K$ in $\mathcal{M}(\Omega)$ and we conclude that

$$C\left(|\Omega|^{1-2/p}\right)^{p/2} \left[\sum_{R=I\times J\times K\in\mathcal{M}(\Omega)} |R| \left(\frac{\left|\widehat{J}\right|}{|J|}\right)^{-\delta_2'} \left(\frac{\left|\widehat{K}\right|}{|K|}\right)^{-\delta_3'}\right]^{1-p/2} \tag{4.64}$$

is an upper-bound for the left-hand-side of (4.61), where $\delta_3' := 2\delta_3 / [n_3 (2-p)]$. Since to each $I \times J \times K$ in $\mathcal{M}(\Omega)$ there is a unique rectangle $I' \times J \times K$ in $\mathcal{M}_1(\Omega_1)$ having the form $I' \times J \times K$ with $I' \supset I$, the term in the brackets in (4.64) is not greater than

$$\sum_{I'\times J\times K\in\mathcal{M}_1(\Omega_1)} |I' \times J \times K| \left[\left(\frac{|J|}{\left|\widehat{J}\right|}\right)^{\delta_2'} \left(\frac{|K|}{\left|\widehat{K}\right|}\right)^{\delta_3'}\right]^{1-p/2},$$

which, by Lemma 4.4, is majorized by $c|\Omega_1|$. Since $|\Omega_1| \le c|\Omega|$, (4.61) holds.

It remains to show (4.62). Applying to each term $\int_{\left(16\widehat{\widehat{I}}\right)^c \times \mathbb{R}^{n_2} \times \left(16\widehat{K}\right)^c} |T(a_R)|^p$ a reasoning similar to that used in the proof of (4.60), we conclude that the sum in (4.62) is bounded above by

$$C\left(|\Omega|^{1-2/p}\right)^{2/p} \left[\sum_{R=I\times J\times K\in\mathcal{M}(\Omega)} |R| \left(\frac{\left|\widehat{\widehat{I}}\right|}{|I|}\right)^{-\delta_1'} \left(\frac{\left|\widehat{K}\right|}{|K|}\right)^{-\delta_3'}\right]^{1-p/2}. \tag{4.65}$$

Since $|I| \le \left|\widehat{I}\right|$, the term in the brackets in (4.65) is majorized by

$$\sum_{R=I\times J\times K\in\mathcal{M}(\Omega)} |R| \left(\frac{\left|\widehat{\widehat{I}}\right|}{\left|\widehat{I}\right|}\right)^{-\delta_1'} \left(\frac{\left|\widehat{K}\right|}{|K|}\right)^{-\delta_3'}$$

$$\leq \sum_{I'\times J'\times K\in\mathcal{M}_2(\Omega_2)} |I'\times J'\times K| \left(\frac{\left|\widehat{I}\right|}{\left|\widehat{I'}\right|}\right)^{\delta_1'} \left(\frac{|K|}{\left|\widehat{K}\right|}\right)^{\delta_3'}, \tag{4.66}$$

where the inequality in (4.66) holds because to each $I\times J\times K$ in $\mathcal{M}(\Omega)$, there corresponds a unique rectangle of the form $I'\times J'\times K$ in $\mathcal{M}_2(\Omega_2)$ satisfying: $I'\times J\times K\in\mathcal{M}_1(\Omega_1)$, $I'\supset I$, and $J'\supset J$, and in this case $I'=\widehat{I}$ and $J'=\widehat{J}$. So (4.62) follows from Lemma 4.4 applied to $\Omega_2$ and the fact that $|\Omega_2|\leq c\,|\Omega|$. □

*Proof of Theorem 4.4.* Let $\gamma = 2^s$ for some $s\in\mathbb{N}$. W.l.o.g. we can assume that the center of $I_j$ is the origin, $0_j$, of $\mathbb{R}^{n_j}$, $j=1,2$. We will prove the first inequality in (4.3), by following a reasoning described in the beginning of the proof of the "trivial-lemma" in [25]. The idea is to show that there exists $\theta > n_1(1/p-1)$, which does not depend on $s$ nor on the rectangle $(p,2)$-atom $a$, such that

$$\int_{\substack{\|x_1\|\sim 2^k\gamma|I_1|^{1/n_1}\\ \|x_2\|\sim 2^i|I_2|^{1/n_2}}} |T(a)(x)|\,dx \leq c2^{-i\epsilon_2}\left(2^k\gamma\right)^{-\theta}\|a\|_2\,|R|^{1/2} \tag{4.67}$$

and

$$\int_{\substack{\|x_1\|\sim 2^k\gamma|I_1|^{1/n_1}\\ \|x_2\|\leq 4|I_2|^{1/n_2}}} |T(a)(x)|\,dx \leq c\left(2^k\gamma\right)^{-\theta}\|a\|_2\,|R|^{1/2}, \tag{4.68}$$

hold for any integers $i,k\geq 2$, where $r\sim t$ means $r<t\leq 2r$ for $r,t\in(0,\infty)$. Then, by summing (4.67) and (4.68) over the appropriate indexes ($1\leq k<\infty$ and $1\leq l<\infty$ for (4.67), $1\leq k<\infty$ for (4.68)) and using Hölder's inequality and (4.10), we get the first inequality in (4.3) with $\delta_1 = -n_1(1-p)+\theta p$ and the constant $c$ depending on the $CZ(\epsilon_1,\epsilon_2,n_1,n_2)$-norm of $T$ (in the case $n_1/(n_1-1)<p\leq 1$) or on the $p$-$CZ(\epsilon_1,\epsilon_2,n_1,n_2)$-norm of $T$ (in the case $0<p\leq n_1/(n_1-1)$).

We first show (4.67) and (4.68) in the case $n_1/(n_1+1)<p\leq 1$. For $p$ is within this range, the reasoning used in the proof of the "trivial lemma" in [25] works for our purposes.

Let $\{a_s\}_{s\in\mathbb{N}}$ be sequence of functions such that $a_s\in L^\infty(\mathbb{R}^{n_1+n_2})$, $lim_{s\to\infty}\|a_s-a\|_2 = 0$, and each $a_s$ satisfies the cancellation conditions (4.11) and the support condition

$\operatorname{supp}(a_s) \subset 2R$. If we show that

$$\int_{\substack{\|x_1\|\sim 2^k\gamma|2I_1|^{1/n_1} \\ \|x_2\|\sim 2^i|2I_2|^{1/n_2}}} |T(a_s)| \leq c2^{-i\epsilon_2} \left(2^k\gamma\right)^{-\theta} \|a_s\|_2 |2R|^{1/2} \tag{4.69}$$

and

$$\int_{\substack{\|x_1\|\sim 2^k\gamma|2I_1|^{1/n_1} \\ \|x_2\|\leq 2|2I_2|^{1/n_2}}} |T(a_s)| \leq \left(2^k\gamma\right)^{-\theta} \|a_s\|_2 |2R|^{1/2}, \tag{4.70}$$

hold for any integers $i, k \geq 1$, then we can conclude that (4.67) and (4.68) hold for $a$ and for any integers $i, k \geq 2$. Indeed, let $i, k \geq 1$. Then

$$\left| \int_{\substack{\|x_1\|\sim 2^k\gamma|2I_1|^{1/n_1} \\ \|x_2\|\sim 2^i|2I_2|^{1/n_2}}} |T(a_s)| - \int_{\substack{\|x_1\|\sim 2^{k+1}\gamma|I_1|^{1/n_1} \\ \|x_2\|\sim 2^{i+1}|I_2|^{1/n_2}}} |T(a)| \right| \leq \int_{\substack{\|x_1\|\sim 2^k\gamma|2I_1|^{1/n_1} \\ \|x_2\|\sim 2^i|2I_2|^{1/n_2}}} \left| |T(a_s)| - |T(a)| \right|$$

$$\leq \int_{\substack{\|x_1\|\sim 2^k\gamma|2I_1|^{1/n_1} \\ \|x_2\|\sim 2^i|2I_2|^{1/n_2}}} |T(a_s) - T(a)|$$

$$\leq \left| \left\{ (x_1, x_2) : \|x_1\| \sim 2^k\gamma |2I_1|^{1/n_1}, \|x_2\| \sim 2^i |2I_2|^{1/n_2} \right\} \right|^{1/2} \|T(a_s) - T(a)\|_2$$

$$\leq \left| \left\{ (x_1, x_2) : \|x_1\| \sim 2^k\gamma |2I_1|^{1/n_1}, \|x_2\| \sim 2^i |2I_2|^{1/n_2} \right\} \right|^{1/2} \|T\|_{L^2\to L^2} \|a_s - a\|_2 \to 0,$$

as $s \to \infty$, and similarly,

$$\int_{\substack{\|x_1\|\sim 2^k\gamma|2I_1|^{1/n_1} \\ \|x_2\|\leq 2|2I_2|^{1/n_2}}} |T(a_s)| \to \int_{\substack{\|x_1\|\sim 2^{k+1}\gamma|I_1|^{1/n_1} \\ \|x_2\|\leq 4|I_2|^{1/n_2}}} |T(a)|,$$

as $s \to \infty$. Since $lim_{s\to\infty} \|a_s - a\|_2 = 0$, the right-hand sides of the inequalities (4.69) and (4.70) converge to the right-hand sides of (4.67) and (4.68) (with a different constant $c$), respectively, as $s \to \infty$.

To show (4.69), let $x_1 \in \mathbb{R}^{n_1}$ be such that $\|x_1\| \sim 2^k\gamma |2I_1|^{1/n_1}$. By (4.11), we have $\int a_s(y_1, y_2)\, dy_1 = 0$ for all $y_2 \in \mathbb{R}^{n_2}$. This, and the fact that for all $y_2 \in \mathbb{R}^{n_2}$, the support of the function $a_s(\cdot, y_2)$ is contained in $2I_1$, allow us to use the representation (4.35), which yields

$$\int_{\|x_2\|\sim 2^i|2I_2|^{1/n_2}} |T(a_s)(x_1, x_2)|\, dx_2$$

$$= \int_{\|x_2\|\sim 2^i|2I_2|^{1/n_2}} \left| \int_{2I_1} ([K_1(x_1,y_1)] - [K_1(x_1,0_1)])(a_s(y_1,\cdot))(x_2)\, dy_1 \right| dx_2$$

$$\leq \int_{2I_1} \left\{ \int_{\|x_2\|>2^i|2I_2|^{1/n_2}} |([K_1(x_1,y_1)] - [K_1(x_1,0_1)])(a_s(y_1,\cdot))(x_2)|\, dx_2 \right\} dy_1. \qquad (4.71)$$

Since for almost every $y_1 \in \mathbb{R}^{n_1}$, $\int a_s(y_1,y_2)\, dy_2 = 0$ and $([K_1(x_1,y_1)] - [K_1(x_1,0_1)])$ is an $(\epsilon_2, n_2)$-Calderón-Zygmund operator, we can use Remark 4.3 to majorize the term in the curly brackets in (4.71) by

$$\left(2^i\right)^{-\epsilon_2} \|K_1(x_1,y_1) - K_1(x_1,0_1)\|_{CZ(\epsilon_2,n_2)} \int_{2I_2} |a_s(y_1,y_2)|\, dy_2.$$

Therefore

$$\int_{\|x_1\|\sim 2^k\gamma|2I_1|^{1/n_1}} \left( \int_{\|x_2\|\sim 2^i|2I_2|^{1/n_2}} |T(a_s)(x_1,x_2)|\, dx_2 \right) dx_1$$

$$\leq 2^{-i\epsilon_2} \int_{2I_1} \left\{ \int_{\|x_1\|>2^{k-1}\gamma\|y_1\|} \|K_1(x_1,y_1) - K_1(x_1,0_1)\|_{CZ(\epsilon_2,n_2)}\, dx_1 \int_{2I_2} |a_s(y_1,y_2)|\, dy_2 \right\} dy_1$$

$$\leq c 2^{-i\epsilon_2} \left(2^k\gamma\right)^{-\beta_1} \|a_s\|_1 \leq c 2^{-i\epsilon_2} \left(2^k\gamma\right)^{-\beta_1} \|a_s\|_2 |2R|^{1/2}.$$

This proves that (4.69) holds with $\theta = \beta_1$. So (4.67) holds with this value of $\theta$.

Now we will verify that inequality (4.70) also holds with this $\theta$. Let $\|x_1\| \sim 2^k\gamma\, |2I_1|^{1/n_1}$. Since for all $y_2 \in \mathbb{R}^{n_2}$, the function $a_s(\cdot, y_2)$ satisfies $\int a_s(y_1,y_2)\, dy_1 = 0$ and its support is contained in $2I_1$, we can use (4.35) to get

$$\int_{\|x_2\|\leq 2|2I_2|^{1/n_2}} |T(a_s)(x_1,x_2)|\, dx_2$$

$$\leq \int_{2I_1} \left\{ \int_{\|x_2\|\leq 2|2I_2|^{1/n_2}} |([K_1(x_1,y_1)] - [K_1(x_1,0_1)])(a_s(y_1,\cdot))(x_2)|\, dx_2 \right\} dy_1,$$

$$\leq \|K_1(x_1,y_1) - K_1(x_1,0_1)\|_{CZ(\epsilon_2,n_2)} |2I_2|^{1/2} \int_{2I_1} \left( \int_{2I_2} |a_s(y_1,y_2)|^2\, dy_2 \right)^{1/2} dy_1$$

$$\leq \|K_1(x_1,y_1) - K_1(x_1,0_1)\|_{CZ(\epsilon_2,n_2)} \|a_s\|_2 |2R|^{1/2},$$

where the penultimate inequality follows from Cauchy-Schwartz inequality and (4.30).

This and hypothesis (4.6) yield

$$\int_{\|x_1\|\sim 2^k\gamma|2I_1|^{1/n_1}}\left(\int_{\|x_2\|\leq 2|2I_2|^{1/n_2}}|T(a_s)(x_1,x_2)|\,dx_2\right)dx_1$$

$$\leq \|a_s\|_2\,|2R|^{1/2}\int_{\|x_1\|>2^k\gamma\|y_1\|}\|K_1(x_1,y_1)-K_1(x_1,0_1)\|_{CZ(\epsilon_2,n_2)}\,dx_1$$

$$\leq c\,\|a_s\|_2\,|2R|^{1/2}\left(2^k\gamma\right)^{-\beta_1}.$$

Now we let $0<p\leq n_1/(n_1+1)$ and we will show that both (4.69) and (4.70) hold with $\theta=\mathcal{N}_1+1$, and we will concluded that both (4.67) and (4.68) hold with this value of $\theta$. First we prove (4.67). Let $\|x_1\|\sim 2^k\gamma\,|2I_1|^{1/n_1}$. Since for all $y_2\in\mathbb{R}^{n_2}$, the function $a_s(\cdot,y_2)$ satisfies $\int a_s(y_1,y_2)\,y_1^\alpha dy_1=0$ for all $\alpha\in(\mathbb{Z}_{\geq 0})^{n_1}$ of order $|\alpha|\leq\mathcal{N}_1$, and its support is contained in $2I_1$, we can use the argument that proves (4.35) and Taylor's Theorem with Lagrange remainder, to conclude that there exists $\tau\in(0,1)$ such that

$$T(a_s)(x_1,x_2)=\int_{2I_1}[K_1(x_1,y_1)]\,(a_s(y_1,\cdot))(x_2)\,dy_1$$

$$=\int_{2I_1}\left\{\sum_{|\alpha|=\mathcal{N}_1+1}\frac{(-y_1)^\alpha}{\alpha!}\left[D_{y_1}^\alpha K_1(x_1,\tau y_1)\right](a_s(y_1,\cdot))(x_2)\right\}dy_1, \tag{4.72}$$

for almost every $x_2\in\mathbb{R}^{n_2}$. From (4.72), Remark 4.3 and hypothesis (4.5), it follows that

$$\int_{\|x_2\|\sim 2^i|2I_2|^{1/n_2}}|T(a_s)(x_1,x_2)|\,dx_2$$

$$=\int_{\|x_2\|\sim 2^i|2I_2|^{1/n_2}}\left|\sum_{|\alpha|=\mathcal{N}_1+1}\int_{2I_1}\left\{\frac{(-y_1)^\alpha}{\alpha!}\left[D_{y_1}^\alpha K_1(x_1,\tau y_1)\right](a_s(y_1,\cdot))(x_2)\right\}dy_1\right|dx_2$$

$$\leq c\,|2I_1|^{(\mathcal{N}_1+1)/n_1}\sum_{|\alpha|=\mathcal{N}_1+1}\int_{2I_1}\left(\int_{\|x_2\|>2^i\|y_2\|}\left|\left[D_{y_1}^\alpha K(x_1,\tau y_1)\right](a_s(y_1,\cdot))(x_2)\right|dx_2\right)dy_1$$

$$\leq c\,|2I_1|^{(\mathcal{N}_1+1)/n_1}\left(2^i\right)^{-\epsilon_2}\sum_{|\alpha|=\mathcal{N}_1+1}\int_{2I_1}\left(\|K_{1,\alpha}(x_1,\tau y_1)\|_{CZ(\epsilon_2,n_2)}\int_{2I_2}|a_s(y_1,y_2)|\,dy_2\right)dy_1$$

$$\leq c\,|2I_1|^{(\mathcal{N}_1+1)/n_1}\,2^{-i\epsilon_2}\int_{2I_1}\left(\|x_1+\tau y_1\|^{-\mathcal{N}_1-n_1-1}\int_{2I_2}|a_s(y_1,y_2)|\,dy_2\right)dy_1$$

$$\leq c2^{-i\epsilon_2}\left|2I_1\right|^{(\mathcal{N}_1+1)/n_1}\left\|x_1\right\|^{-\mathcal{N}_1-n_1-1}\left\|a_s\right\|_1 .$$

Integrating this over the annulus $\|x_1\| \sim 2^k\gamma\left|2I_1\right|^{1/n_1}$, we conclude that (4.69) holds with $\theta = \mathcal{N}_1 + 1$.

Now we will verify that inequality (4.70) holds with this same value of $\theta$. Again, let $\|x_1\| \sim 2^k\gamma\left|2I_1\right|$. By (4.72) and hypothesis (4.5), we obtain

$$\int_{\|x_2\|\leq 2|2I_2|^{1/n_2}}\left|T\left(a_s\right)\left(x_1,x_2\right)\right| dx_2$$

$$= \int_{\|x_2\|\leq 2|2I_2|^{1/n_2}}\left|\sum_{|\alpha|=\mathcal{N}_1+1}\int_{2I_1}\left\{\frac{\left(-y_1\right)^\alpha}{\alpha!}\left[D^\alpha_{y_1}K_1\left(x_1,\tau y_1\right)\right]\left(a_s\left(y_1,\cdot\right)\right)\left(x_2\right)\right\}dy_1\right|dx_2$$

$$\leq c\left|2I_1\right|^{(\mathcal{N}_1+1)/n_1}\sum_{|\alpha|=\mathcal{N}_1+1}\int_{2I_1}\left(\int_{\|x_2\|\leq 2\|y_2\|}\left|\left[D^\alpha_{y_1}K\left(x_1,\tau y_1\right)\right]\left(a_s\left(y_1,\cdot\right)\right)\left(x_2\right)\right|dx_2\right)dy_1$$

$$\leq c\left|2I_1\right|^{(\mathcal{N}_1+1)/n_1}\sum_{|\alpha|=\mathcal{N}_1+1}\int_{2I_1}\left(\left\|D^\alpha_{y_1}K_1\left(x_1,\tau y_1\right)a_s\left(y_1,\cdot\right)\right\|_2\left|2I_2\right|^{1/2}\right)dy_1$$

$$\leq c\left|2I_1\right|^{(\mathcal{N}_1+1)/n_1}\left|2I_2\right|^{1/2}\sum_{|\alpha|=\mathcal{N}_1+1}\int_{2I_1}\left\|D^\alpha_{y_1}K_1\left(x_1,\tau y_1\right)\right\|_{CZ(\epsilon_2,n_2)}\left\|a_s\left(y_1,\cdot\right)\right\|_2 dy_1$$

$$\leq c\left|2I_1\right|^{(\mathcal{N}_1+1)/n_1}\left|2I_2\right|^{1/2}\int_{2I_1}\left\|a_s\left(y_1,\cdot\right)\right\|_2\left\|x_1-\tau y_1\right\|^{-\mathcal{N}_1-n_1-1}dy_1$$

$$\leq c\left|2I_1\right|^{(\mathcal{N}_1+1)/n_1}\left|2I_2\right|^{1/2}\left\|x_1\right\|^{-\mathcal{N}_1-n_1-1}\left\|a_s\right\|_2 ,$$

where $\tau \in (0,1)$. Then we obtain (4.70) by integrating the above estimate over $\|x_1\| \sim 2^k\gamma\left|2I_1\right|^{1/n_1}$. Thus (4.68) holds with $\theta = \mathcal{N}_1 + 1$. □

Our next result is a three-parameter version of Theorem 4.4. We will deal with $(\epsilon_1,\epsilon_2,\epsilon_3,n_1,n_2,n_2)$-Calderón-Zygmund operators (Definition 4.9) and also with $p$-$(\epsilon_1,\epsilon_2,\epsilon_3,n_1,n_2,n_2)$-Calderón-Zygmund operators (Definition 4.10).

**Theorem 4.6** (Three-parameter variant of Theorem 4.4)**.** *Let $0 < p \leq 1$ and let $T$ be a linear operator on $L^2(\mathbb{R}^{n_1+n_2+n_3})$.*

*If $n_1/\left(n_1-1\right) < p \leq 1$, assume that $T$ is a $(\beta_1,\beta_2,\epsilon_3,n_1,n_2,n_2)$-Calderón-Zygmund operator for some $\beta_j > n_j\left(1/p-1\right)$, $j = 1,2$, and $\epsilon_3 > n_3\left(1/p-1\right)$.*

*If $0 < p \le n_1/(n_1-1)$, assume that $T$ is a $p$-$(\epsilon_1, \epsilon_2, n_1, n_2)$-Calderón-Zygmund operator for some $\epsilon_j > n_j(1/p-1)$, $j = 2, 3$, and assume also that the $[CZ(\epsilon_3, n_3)]$-$(\epsilon_1, \epsilon_2, n_1, n_2)$-kernel $K_{1,2}$, associated with $T$ as in (4.36), satisfies*

$$\int_{\|x_2-\xi_2\|>2^k\gamma_2\|y_2-\xi_2\|} \left\| \begin{array}{c} D^{\alpha}_{y_1} K_{1,2}(x_1, y_1, x_2, y_2) \\ -D^{\alpha}_{y_1} K_{1,2}(x_1, \xi_1, x_2, \xi_2) \end{array} \right\|_{CZ(\epsilon_3, n_3)} dx_2 \le c \|x_1 - \xi_1\|^{-\mathcal{N}_1 - n_1 - 1} \left(2^k \gamma_2\right)^{-\epsilon_2}, \tag{4.73}$$

*$x_1 \ne \xi_1$, for each multi-indice $\alpha \in (\mathbb{Z}_{\ge 0})^{n_1}$ with $|\alpha| = \mathcal{N}_1 + 1$, and for all $k \in \mathbb{N}$ and all $\gamma_2$ of the form $2^s$, $s \in \mathbb{N}$. Under these hypotheses, (4.4) holds for some $\delta_1, \delta_2 > 0$ and for any rectangle $(p,2)$-atom $a$ on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$ supported on $I_1 \times I_2 \times I_3$.*

The inequalities which are similar to (4.4), with $(1,2,3)$ replaced by $(2,3,1)$ and $(3,1,2)$, can be obtained by replacing $(1,2,3)$ by $(2,3,1)$ and $(3,1,2)$, respectively.

Note that, in case $0 < p \le n_1/(n_1-1)$, the hypotheses of Theorem 4.6 imply that $K_{1,2}$, the $[CZ(\epsilon_3, n_3)]$-$(\beta_1, \beta_2)$-kernel associated with $T$ as in (4.36), satisfies

$$\int_{\substack{\|x_1-\xi_1\|>2^k\|y_1-\xi_1\| \\ \|x_2-\xi_2\|>2^l\|y_2-\xi_2\|}} \|K_{1,2}(x_1, y_1, x_2, y_2) - K_{1,2}(x_1, \xi_1, x_2, \xi_2)\|_{CZ(\epsilon_3, n_3)} dx \le c \left(2^k\right)^{-\beta_1} \left(2^l\right)^{-\beta_2} \tag{4.74}$$

for all $k, l \in \mathbb{N}$.

*Proof of Theorem 4.6.* Let $\gamma_1 = 2^{s_1}, \gamma_2 = 2^{s_2}$ for some $s_1, s_2 \in \mathbb{N}$ and assume that $I_j$ is centered at the origin, $0_j$, of $\mathbb{R}^{n_j}$, $j = 1, 2, 3$. In analogy with the proof Theorem 4.4, our goal is to show that there exists $\theta_1, \theta_2 > 0$, which does not depend on $s_1, s_2$ nor on the rectangle $(p,2)$-atom $a$, such that

$$\int_{\substack{\|x_1\|\sim 2^k\gamma_1|I_1|^{1/n_1} \\ \|x_2\|\sim 2^l\gamma_2|I_2|^{1/n_2} \\ \|x_3\|\sim 2^i|I_3|^{1/n_3}}} |T(a)(x)| \, dx \le c 2^{-i\epsilon_3} \left(2^k\gamma_1\right)^{-\theta_1} \left(2^l\gamma_2\right)^{-\theta_2} \|a\|_2 |R|^{1/2} \tag{4.75}$$

and

$$\int_{\substack{\|x_1\|\sim 2^k\gamma_1|I_1|^{1/n_1} \\ \|x_2\|\sim 2^l\gamma_2|I_2|^{1/n_2} \\ \|x_3\|\le 2|I_3|^{1/n_3}}} |T(a)(x)| \, dx \le c \left(2^k\gamma_1\right)^{-\theta_1} \left(2^l\gamma_2\right)^{-\theta_2} \|a\|_2 |R|^{1/2}, \tag{4.76}$$

for any integers $i, k, l \ge 2$. These inequalities imply that (4.4) holds with $\delta_j = -n_j(1-p)$

$+\theta_j p$, $j = 1, 2$.

We begin with the case $p > n_1 / (n_j + 1)$. Again, the argument is analogous to that of the proof of Theorem 4.4. We will omit some of the intermediary steps, that is, we will assume w.l.o.g. that $a$ is in $L^\infty$, just like the $a_s$'s of the proof of Theorem 4.4.

Take $\|x_1\| \sim 2^k \gamma_1 |I_1|^{1/n_1}$ and $\|x_2\| \sim 2^l \gamma_2 |I_2|^{1/n_2}$. The vanishing moments of condition (4.11) and Remark 4.3 yield

$$\int_{\|x_3\| \sim 2^i |I_3|^{1/n_3}} |T(a)(x_1, x_2, x_3)| \, dx_3$$

$$= \int_{\|x_3\| \sim 2^i |I_3|^{1/n_3}} \left| \int_{I_1} \int_{I_2} [K_{1,2}(x_1, y_1, x_2, y_2)] (a(y_1, y_2, \cdot))(x_3) \, dy_2 dy_1 \right| dx_3$$

$$= \int_{\|x_3\| \sim 2^i |I_3|^{1/n_3}} \left| \int_{I_1} \int_{I_2} \begin{pmatrix} [K_{1,2}(x_1, y_1, x_2, y_2)] \\ -[K_{1,2}(x_1, 0_1, x_2, 0_2)] \end{pmatrix} (a(y_1, y_2, \cdot))(x_3) \, dy_2 dy_1 \right| dx_3$$

$$\leq \int_{I_1} \int_{I_2} \left\{ \int_{\|x_3\| > 2^i \|y_3\|} \left| \begin{pmatrix} [K_{1,2}(x_1, y_1, x_2, y_2)] \\ -[K_{1,2}(x_1, 0_1, x_2, 0_2)] \end{pmatrix} (a(y_1, y_2, \cdot))(x_3) \right| dx_3 \right\} dy_2 dy_1$$

$$\leq (2^i)^{-\epsilon_3} \int_{I_1} \int_{I_2} \left\{ \left\| \begin{matrix} K_{1,2}(x_1, y_1, x_2, y_2) \\ -K_{1,2}(x_1, 0_1, x_2, 0_2) \end{matrix} \right\|_{CZ(\epsilon_3, n_3)} \int_{I_3} |a(y_1, y_2, y_3)| \, dy_3 \right\} dx_2 dx_1,$$

where by $([K_{1,2}(x_1, y_1, x_2, y_2)] - [K_{1,2}(x_1, 0_1, x_2, 0_2)])(a(y_1, y_2, \cdot))(x_3)$ we denote the evaluation at the point $x_3$ of the operator $([K_{1,2}(x_1, y_1, x_2, y_2)] - [K_{1,2}(x_1, 0_1, x_2, 0_2)])$ on the function $y_3 \mapsto a(y_1, y_2, \cdot)(y_3) := a(y_1, y_2, y_3)$. Therefore the left-hand side of (4.76) is majorized by

$$2^{-l\epsilon_3} \int_{I_1} \int_{I_2} \int_{\substack{\|x_1\| \sim 2^k \gamma_1 |I_1|^{1/n_1} \\ \|x_2\| \sim 2^l \gamma_2 |I_2|^{1/n_2}}} \left\| \begin{matrix} K_{1,2}(x_1, y_1, x_2, y_2) \\ -K_{1,2}(x_1, 0_1, x_2, 0_2) \end{matrix} \right\|_{CZ(\epsilon_3, n_3)} dx_1 dx_2 \cdot \int_{I_3} |a(y_1, y_2, y_3)| \, dy_3 dy_2 dy_1$$

$$\leq c 2^{-l\epsilon_3} \left(2^k \gamma_1\right)^{-\beta_1} \left(2^l \gamma_2\right)^{-\beta_2} \|a\|_2 |R|^{1/2},$$

where the last inequality follows from (4.74). So (4.75) holds with $\theta_j = \beta_j$, $j = 1, 2$.

Now we will verify that (4.76) holds with these values of $\theta_1$ and $\theta_2$. Let $\|x_1\| \sim 2^k \gamma_1 |I_1|^{1/n_1}$ and $\|x_2\| \sim 2^l \gamma_2 |I_2|^{1/n_2}$. Then

$$\int_{\|x_3\| \leq 2|I_3|^{1/n_3}} |T(a)(x_1, x_2, x_3)| \, dx_3$$

$$= \int_{\|x_3\| \leq 2|I_3|^{1/n_3}} \left| \int_{I_2} \int_{I_1} \begin{pmatrix} [K_{1,2}(x_1, y_1, x_2, y_2)] \\ -[K_{1,2}(x_1, 0_1, x_2, 0_2)] \end{pmatrix} (a(y_1, y_2, \cdot))(x_3) \, dy_1 dy_2 \right| dx_3$$

$$\leq \int_{I_2} \int_{I_1} \left[ \int_{\|x_2\| \leq 2|I_3|^{1/n_3}} \left| \begin{pmatrix} [K_{1,2}(x_1, y_1, x_2, y_2)] \\ -[K_{1,2}(x_1, 0_1, x_2, 0_2)] \end{pmatrix} (a(y_1, y_2, \cdot))(x_3) \right| dx_3 \right] dy_1 dy_2,$$

$$\leq \left\| \begin{matrix} K_{1,2}(x_1, y_1, x_2, y_2) \\ -K_{1,2}(x_1, 0_1, x_2, 0_2) \end{matrix} \right\|_{CZ(\epsilon_3, n_3)} |I_3|^{1/2} \int_{I_2} \int_{I_1} \left( \int_{I_3} |a(y_1, y_2, y_3)|^2 \, dy_3 \right)^{1/2} dy_1 dy_2$$

$$\leq \|K_{1,2}(x_1, y_1, x_2, y_2) - K_{1,2}(x_1, 0_1, x_2, 0_2)\|_{CZ(\epsilon_3, n_3)} \|a\|_2 |R|^{1/2}.$$

So the left-hand side of (4.76) is bounded above by

$$\|a\|_2 |R|^{1/2} \int_{\substack{\|x_1\| > 2^k \gamma_1 \|y_1\| \\ \|x_2\| > 2^l \gamma_2 \|y_2\|}} \|K_{1,2}(x_1, y_1, x_2, y_2) - K_{1,2}(x_1, 0_1, x_2, 0_2)\|_{CZ(\epsilon_3, n_3)} \, dx_1 dx_2$$

$$\leq c \|a\|_2 |R|^{1/2} \left(2^k \gamma_1\right)^{-\beta_1} \left(2^l \gamma_2\right)^{-\beta_2},$$

where the last inequality is due to (4.74).

It remains to prove (4.75) and (4.76) in the case $p \leq n_1/(n_1+1)$.

Let $\|x_1\| \sim 2^k \gamma_1 |I_1|^{1/n_1}$ and $\|x_2\| \sim 2^l \gamma_2 |I_2|^{1/n_2}$. From (4.36), the vanishing moments $\int a(y_1, y_2, y_3) \, y_1^{\alpha} dy_1 = 0$ for all $|\alpha| \leq \mathcal{N}_1$ and Taylor's Theorem with Lagrange remainder, it follows that there exists $\tau \in (0,1)$ such that

$$T(a)(x_1, x_2, x_3) = \int_{I_1} \int_{I_2} [K_{1,2}(x_1, y_1, x_2, y_2)] (a(y_1, y_2, \cdot))(x_3) \, dy_2 dy_1$$

$$= \int_{I_1} \int_{I_2} \left\{ \sum_{|\alpha| = \mathcal{N}_1 + 1} \frac{(-y_1)^{\alpha}}{\alpha!} \left[ D_{y_1}^{\alpha} K_{1,2}(x_1, \tau y_1, x_2, y_2) \right] (a(y_1, y_2, \cdot))(x_3) \right\} dy_2 dy_1$$

for almost every $x_3 \in \mathbb{R}^{n_3}$. It follows that

$$T(a)(x_1, x_2, x_3)$$

$$= \int_{I_1}\int_{I_2} \left\{ \sum_{|\alpha|=\mathcal{N}_1+1} \frac{(-y_1)^\alpha}{\alpha!} \begin{pmatrix} \left[D^\alpha_{y_1} K_{1,2}(x_1, \tau y_1, x_2, y_2)\right] \\ -\left[D^\alpha_{y_1} K_{1,2}(x_1, \tau y_1, x_2, 0_2)\right] \end{pmatrix} (a(y_1, y_2, \cdot))(x_3) \right\} dy_2 dy_1 \tag{4.77}$$

for almost every $x_3 \in \mathbb{R}^{n_3}$. Then

$$\int_{\|x_3\|\sim 2^i|I_3|^{1/n_3}} |T(a)(x_1, x_2, x_3)|\, dx_3$$

$$= \int_{\|x_3\|\sim 2^i|I_3|^{1/n_3}} \left| \int_{I_1}\int_{I_2} \left\{ \begin{array}{c} \sum_{|\alpha|=\mathcal{N}_1+1} \frac{(-y_1)^\alpha}{\alpha!} \\ \cdot \begin{pmatrix} \left[D^\alpha_{y_1} K_{1,2}(x_1, \tau y_1, x_2, y_2)\right] \\ -\left[D^\alpha_{y_1} K_{1,2}(x_1, \tau y_1, x_2, 0_2)\right] \end{pmatrix} \\ \cdot (a(y_1, y_2, \cdot))(x_3) \end{array} \right\} dy_2 dy_1 \right| dx_3$$

$$\leq c\,|I_1|^{(\mathcal{N}_1+1)/n_1} \left(2^i\right)^{-\epsilon_3} \sum_{|\alpha|=\mathcal{N}_1+1} \int_{I_1}\int_{I_2} \begin{pmatrix} \left\| \begin{array}{c} D^\alpha_{y_1} K_{1,2}(x_1, \tau y_1, x_2, y_2) \\ -D^\alpha_{y_1} K_{1,2}(x_1, \tau y_1, x_2, 0_2) \end{array} \right\|_{CZ(\epsilon_3, n_3)} \\ \cdot \int_{I_3} |a(y_1, y_2, y_3)|\, dy_3 \end{pmatrix} dy_2 dy_1,$$

where the last inequality follows from Remark 4.3. So the left-hand side of (4.75) is not greater than

$$\int_{\substack{\|x_1\|\sim 2^k\gamma_1|I_1|^{1/n_1} \\ \|x_2\|\sim 2^l\gamma_2|I_2|^{1/n_2}}} \left[ \begin{array}{c} c\,|I_1|^{(\mathcal{N}_1+1)/n_1}\, 2^{-i\epsilon_3} \\ \cdot \int_{I_1}\int_{I_2} \begin{pmatrix} \left\| \begin{array}{c} D^\alpha_{y_1} K_{1,2}(x_1, \tau y_1, x_2, y_2) \\ -D^\alpha_{y_1} K_{1,2}(x_1, \tau y_1, x_2, 0_2) \end{array} \right\|_{CZ(\epsilon_3, n_3)} \\ \cdot \int_{I_3} |a(y_1, y_2, y_3)|\, dy_3 \end{pmatrix} dy_2 dy_1 \end{array} \right] dx_2 dx_1$$

$$\leq c\left|I_1\right|^{(\mathcal{N}_1+1)/n_1} 2^{-i\epsilon_3}$$

$$\cdot \int_{I_1}\int_{I_2}\left[\begin{array}{c}\left(\int_{\substack{\|x_1\|\sim 2^k\gamma_1|I_1|^{1/n_1}\\ \|x_2\|>2^l\gamma_2\|y_2\|}}\left\|\begin{array}{c} D_{y_1}^{\alpha}K_{1,2}\left(x_1,\tau y_1,x_2,y_2\right)\\ -D_{y_1}^{\alpha}K_{1,2}\left(x_1,\tau y_1,x_2,0_2\right)\end{array}\right\|_{CZ(\epsilon_3,n_3)} dx_2dx_1\right)\\ \cdot\int_{I_3}\left|a\left(y_1,y_2,y_3\right)\right|dy_3\end{array}\right]dy_2dy_1$$

$$\leq c\left|I_1\right|^{(\mathcal{N}_1+1)/n_1} 2^{-i\epsilon_3}\int_{I_1}\int_{I_2}\left[\begin{array}{c}\left(\int_{\|x_1\|\sim 2^k\gamma_1|I_1|^{1/n_1}}\left\|x_1-\tau y_1\right\|^{-\mathcal{N}_1-n_1-1}\left(2^l\gamma_2\right)^{-\epsilon_2}dx_1\right)\\ \cdot\int_{I_3}\left|a\left(y_1,y_2,y_3\right)\right|dy_3\end{array}\right]dy_2dy_1$$

$$\leq c\left(2^i\right)^{-\epsilon_3}\left(2^k\gamma_1\right)^{-\mathcal{N}_1-1}\left(2^l\gamma_2\right)^{-\epsilon_2}\|a\|_2\left|R\right|^{1/2}.$$

This proves that (4.75) holds with $\theta_1=\mathcal{N}_1+1$, $\theta_2=\epsilon_2$.

Now we will verify that (4.76) holds with these values of $\theta_1$ and $\theta_2$. Again, let $\|x_1\|\sim 2^k\gamma_1\left|I_1\right|^{1/n_1}$ and $\|x_2\|\sim 2^l\gamma_2\left|I_2\right|^{1/n_2}$. By (4.77), there exists $\tau\in(0,1)$ such that

$$\int_{\|x_3\|\leq 2|I_3|^{1/n_3}}\left|T\left(a\right)\left(x_1,x_2,x_3\right)\right|dx_3$$

$$=\int_{\|x_3\|\leq 2|I_3|^{1/n_3}}\left|\int_{I_1}\int_{I_2}\left\{\begin{array}{c}\sum_{|\alpha|=\mathcal{N}_1+1}\frac{(-y_1)^{\alpha}}{\alpha!}\left(\begin{array}{c}\left[D_{y_1}^{\alpha}K_{1,2}\left(x_1,\tau y_1,x_2,y_2\right)\right]\\ -\left[D_{y_1}^{\alpha}K_{1,2}\left(x_1,\tau y_1,x_2,0_2\right)\right]\end{array}\right)\\ .\left(a\left(y_1,y_2,\cdot\right)\right)\left(x_3\right)\end{array}\right\}dy_2dy_1\right|dx_3$$

$$\leq c\left|I_1\right|^{(\mathcal{N}_1+1)/n_1}\left|I_3\right|^{1/2}\sum_{|\alpha|=\mathcal{N}_1+1}\int_{I_1}\int_{I_2}\left\|\begin{array}{c}\left(\begin{array}{c}\left[D_{y_1}^{\alpha}K_{1,2}\left(x_1,\tau y_1,x_2,y_2\right)\right]\\ -\left[D_{y_1}^{\alpha}K_{1,2}\left(x_1,\tau y_1,x_2,0_2\right)\right]\end{array}\right)\\ .a\left(y_1,y_2,\cdot\right)\end{array}\right\|_2 dy_2dy_1$$

$$\leq c\left|I_1\right|^{(\mathcal{N}_1+1)/n_1}\left|I_3\right|^{1/2}\sum_{|\alpha|=\mathcal{N}_1+1}\int_{I_1}\int_{I_2}\left\|\begin{array}{c}D_{y_1}^{\alpha}K_{1,2}\left(x_1,\tau y_1,x_2,y_2\right)\\ -D_{y_1}^{\alpha}K_{1,2}\left(x_1,\tau y_1,x_2,0_2\right)\end{array}\right\|_{CZ(\epsilon_3,n_3)}$$

$$\cdot\left\|a\left(y_1,y_2,\cdot\right)\right\|_2 dy_2dy_1$$

where the last inequality follows from (4.30). So the left-hand side of (4.76) can be majorized as follows

$$\int_{\substack{\|x_1\|\sim 2^k\gamma_1|I_1|^{1/n_1}\\ \|x_2\|\sim 2^l\gamma_2|I_2|^{1/n_2}}}\int_{\|x_3\|\leq 2|I_3|^{1/n_3}}|T(a)(x_1,x_2,x_3)|\,dx_3dx_2dx_1$$

$$\leq c\,|I_1|^{(\mathcal{N}_1+1)/n_1}|I_3|^{1/2}\int_{\substack{\|x_1\|\sim 2^k\gamma_1|I_1|^{1/n_1}\\ \|x_2\|\sim 2^l\gamma_2|I_2|^{1/n_2}}}\sum_{|\alpha|=\mathcal{N}_1+1}\int_{I_1}\int_{I_2}\left\|\begin{array}{c} D^{\alpha}_{y_1}K_{1,2}(x_1,\tau y_1,x_2,y_2)\\ -D^{\alpha}_{y_1}K_{1,2}(x_1,\tau y_1,x_2,0_2)\end{array}\right\|_{CZ(\epsilon_3,n_3)}$$

$$.\,\|a(y_1,y_2,\cdot)\|_2dy_2dy_1dx_2dx_1$$

$$\leq c\,|I_1|^{(\mathcal{N}_1+1)/n_1}\left(2^l\gamma_2\right)^{-\epsilon_2}|I_3|^{1/2}\int_{\|x_1\|\sim 2^k\gamma_1|I_1|^{1/n_1}}\|x_1\|^{-\mathcal{N}_1-n_1-1}\,dx_1\int_{I_1}\int_{I_2}\|a(y_1,y_2,\cdot)\|_2\,dy_2dy_1$$

$$\leq c\left(2^k\gamma_1\right)^{-\mathcal{N}_1-1}\left(2^l\gamma_2\right)^{-\epsilon_2}\|a\|_2\,|R|^{1/2},$$

and the proof of (4.76) is concluded. □

*Proof of Theorem 4.5.* Let $\mathfrak{M}=\mathfrak{M}_{\varphi^1,\ldots,\varphi^r}$ be as in (4.14), $r\in\{2,3\}$. By Lemma 4.1, $\mathfrak{M}$ satisfies (4.13).

In the case $r=2$, (4.13) implies (4.3) with $T=\mathfrak{M}$ and $\delta_j=(1-\mu_j)\,p$, $j=1,2$. Similarly, when $r=3$, (4.13) implies that (4.4) (and similar inequalities for the other factors) holds with $T=\mathfrak{M}$ and $\delta_j=(1-\mu_j)\,p$, $j=1,2,3$. So the estimate

$$\|\mathfrak{M}(a)\|\leq c\ \text{ for all rough } (p,2)\text{-atoms on } \mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r} \tag{4.78}$$

follows by the same argument that shows, in the proof Theorems 4.2 (case $r=2$) and 4.3 (case $r=3$), that $\|T(a)\|_p$ is uniformly bounded on these atoms.

Now consider a collection $\{a_k\}_k$ of rough $(p,2)$-atoms $\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r}$ and a sequence of scalars $\{\lambda_k\}_k\in l^p$. Using (4.78) and the reasoning that shows equality (30) on page 107 of [62], we conclude that $\sum_{k=1}^{\infty}\lambda_k a_k$ converges in $H^p(\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r})$ (hence it also converges in $\mathcal{S}'(\mathbb{R}^{n_1+\ldots+n_r})$) and $\|\sum_{k=1}^{\infty}\lambda_k a_k\|_{H^p(\mathbb{R}^{n_1}\times\ldots\times\mathbb{R}^{n_r})}$ is bounded above by $c\sum_{k=1}^{\infty}|\lambda_k|^p$. □

For larger values of $r$, an analogue of Theorem 4.5 holds. To prove it in $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_r}$, $r \geq 4$, one can follow the reasoning described above, but a higher-parameter variant of Lemma 4.4 is required to show (4.78) in this setting.

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# Chapter 5

# Journé-type Lemmas

In this chapter, we adapt the original lemmas of Journé [42] (the non-discrete and the discrete versions) to the product of a metric measure space $X$ and the real line.

## 5.1 Definitions and Notation

Adopting the conventions and the terminology from [37] and the notation from [66], we consider a metric measure space $(X, d_X, \mu_X)$, where

(i) $d_X$ is a metric on $X$;

(ii) the topology on $X$ is the metric topology; and

(iii) $\mu_X$ is a regular, $\sigma$-finite, non-negative, doubling measure on the Borel $\sigma$-algebra (the one generated by the open sets in the metric topology) on $X$, which we denote by $\mathcal{B}_X$.

The doubling property of $\mu_X$ means that for all $x \in X$ and all $r > 0$,

$$0 < \mu_X \left(B\left(x, r\right)\right) < \infty \text{ and } \mu_X \left(B\left(x, 2r\right)\right) \leq c_{\mu_X} \mu_X \left(B\left(x, r\right)\right) \tag{5.1}$$

where $c_{\mu_X}$ is a constant. The second inequality in (5.1) implies that

$$\mu_X \left(B\left(x, R\right)\right) = \mu_X \left(B\left(x, 2^{\log_2(R/r)} r\right)\right) \leq c_{\mu_X}^{\log_2(R/r)} \mu \left(B\left(x, r\right)\right), \text{ for all } 0 < r \leq R. \tag{5.2}$$

Being a metric space, $X$ is a perfectly normal Hausdorff space (i.e. $T_6$). We assume that $X$ is proper and uniformly perfect. The former condition means that all closed balls are compact, and the latter means that there exists a constant $\eta \geq 1$ satisfying

$$\text{for all } r > 0, \text{ if } X \backslash B(x, r) \neq \emptyset, \text{ then } B(x, t) \backslash B\left(x, \eta^{-1} r\right) \neq \emptyset.$$

We take the Cartesian product of $X$ and $\mathbb{R}$, the real line, where in this chapter, we view the line as a metric measure space $(\mathbb{R}, d_Y, \mu_Y)$, where $d_Y(y_1, y_2) := |y_1 - y_2|$ is the distance on $\mathbb{R}$ and $\mathcal{B}_Y$ denotes the Borel $\sigma$-algebra on $\mathbb{R}$.

The product topology on $X \times \mathbb{R}$ can be defined as the coarsest topology for which the projections

$$(x, y) \in X \times \mathbb{R} \mapsto \pi_X(x, y) := x \qquad \text{and} \qquad (x, y) \in X \times \mathbb{R} \mapsto \pi_Y(x, y) := y$$

are continuous. Equivalently, among the topologies on $X \times \mathbb{R}$ which contain all the Cartesian products of an open set in $X$ with an open set in $\mathbb{R}$, the product topology is the one that has the fewest open sets.

Now that we have defined the product topology, we let $\mathcal{B}$ be the Borel $\sigma$-algebra on $X \times \mathbb{R}$, i.e. $\mathcal{B}$ is defined to be the $\sigma$-algebra on $X \times \mathbb{R}$ generated by the open sets in the product topology. Equivalently, $\mathcal{B}$ is the coarsest $\sigma$-algebra on $X \times \mathbb{R}$ with the property that if $E \in \mathcal{B}_X$ and $F \in \mathcal{B}_Y$, then $E \times F \in \mathcal{B}$.

Since both $(X, d_X, \mu_X)$ and $(\mathbb{R}, d_Y, \mu_Y)$ are $\sigma$-finite measure spaces, (by Proposition 1.7.11 in [66]) there exits a unique measure $\mu$ on the product $\sigma$-algebra $\mathcal{B}$ such that $\mu(E \times F) = \mu_X(E) \times \mu_Y(F)$ for every $E \in \mathcal{B}_X$, $F \in \mathcal{B}_Y$. This measure $\mu$ is $\sigma$-finite (by Theorem B in Secion 35 of [33] and the uniqueness of $\mu$), non-negative and monotone. The latter condition means that if $\Omega_1, \Omega_2 \in \mathcal{B}$ and $\Omega_1 \subset \Omega_2$, then $\mu(\Omega_1) \leq \mu(\Omega_1)$.

We say that a set $R \subset X \times \mathbb{R}$ is a *rectangle* if it has the form

$$R = I \times J, \tag{5.3}$$

with

$$I = B\left(x_0, \alpha\right) := \left\{x \in X : d_X\left(x_0, x\right) < \alpha\right\} \text{ for some } x_0 \in X,\ \alpha > 0;$$

$$J = \left\{y \in \mathbb{R} : \left|y - y_0\right| < \beta\right\} \text{ for some } y_0 \in \mathbb{R},\ \beta > 0.$$

In this chapter, we use the symbol $\mathcal{R}$ to denote the collection of all rectangles of the form (5.3).

Let $\Omega \subset X \times \mathbb{R}$ be an open set with $\mu\left(\Omega\right) < \infty$ and define

$$\widetilde{\Omega} := \left\{(x, y) \in X \times \mathbb{R} : M_S\left(\chi_\Omega\right)(x, y) > \frac{1}{2}\right\};$$

here $M_S\left(f\right)$ is defined by

$$(x, y) \in X \times \mathbb{R} \mapsto M_S\left(f\right)(x, y) := \sup_{(x,y)\in R} \frac{1}{\mu\left(R\right)} \int_R \left|f\right| d\mu, \quad f \in L^1_{loc}\left(X \times \mathbb{R}, \mu\right),$$

where the supremum is taken over all rectangles $R$ in $\mathcal{R}$ containing $(x, y)$. Note that $\widetilde{\Omega}$ is open, thus measurable.

For $x \in X$ and $t > 0$, let

$$E_{x,t} := \left\{y \in \mathbb{R} : \overline{B\left(x, t\right)} \times \{y\} \subset \Omega\right\}. \tag{5.4}$$

Since $\overline{B\left(x, t\right)}$ is compact, for every $y$ in $E_{x,t}$, there exists $\delta > 0$ such that $\overline{B\left(x, t\right)} \times (y - \delta, y + \delta)$ is contained in $\Omega$. This shows that $E_{t,x}$ is open.

Since $E_{t,x}$ is an open set of the real line, we can write it as a countable union of disjoint open intervals, $J_{t,x,k}$, with the indices $k$ in a subset of $\mathbb{N}$, which we call $\Lambda\left(x, t\right)$, i.e.

$$E_{x,t} = \bigcup_{k\in\Lambda(x,t)} J_{x,t,k}. \tag{5.5}$$

The possibility that $E_{x,t} = \emptyset$ is been considered. In this degenerate case, the statements above hold trivially: $\Lambda\left(x, t\right)$ is also empty.

Note that, by the definition of $E_{x,t}$ in (5.4), for each triple $(x, t, k) \in X \times (0, \infty) \times \Lambda\left(x, t\right)$, the set $\overline{B\left(x, t\right)} \times J_{x,t,k}$ is contained in $\Omega$, and for any other open interval $J'$, with

$J_{x,t,k} \subsetneqq J'$, the set $\overline{B(x,t)} \times J'$ is not contained in $\Omega$.

Define

$$\tau(x,t,k) := \inf \left\{ s \geq t : \frac{\mu_Y(E_{x,s} \cap J_{x,t,k})}{\mu_Y(J_{x,t,k})} \leq \frac{1}{2} \right\} \tag{5.6}$$

and

$$\widetilde{\tau}(x,t,k) := \sup \left\{ s \geq t : B(x,s) \times J_{x,t,k} \subset \widetilde{\Omega} \right\}, \tag{5.7}$$

for all $(x,t,k) \in X \times (0,\infty)$ such that $E_{x,t} \neq \emptyset$ and all $k \in \Lambda(x,t)$.

## 5.2 Variant of Non-discrete Journé's Lemma

The following result is a variant of Proposition 2 in [42], which we call the non-discrete Journé's Lemma.

**Theorem 5.1** (Variant of Non-discrete Journé's Lemma)**.** *Let $\Omega \subset X \times \mathbb{R}$ be a bounded open set and let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-negative, non-decreasing function of class $C^1$ satisfying $\int_0^1 \phi(s) \frac{ds}{s} < \infty$. Then,*

$$\mu \left( \bigcup_{(x,t) \in X \times \mathbb{R}} \left( \bigcup_{k \in \Lambda(x,t)} B(x, \tau(x,t,k)) \times J_{x,t,k} \right) \right) \leq c\mu(\Omega) \tag{5.8}$$

*and*

$$\int_0^\infty \int_{x \in X} \sum_{k \in \Lambda(x,t)} \mu_Y(J_{x,t,k}) \phi\left( \frac{t}{\tau(x,t,k)} \right) d\mu_X(x) \frac{dt}{t} \leq 2 \left( \int_0^1 \phi(s) \frac{ds}{s} \right) \mu(\Omega), \tag{5.9}$$

*where $J_{x,t,k}$ and $\tau(x,t,k)$ are defined in (5.5) and (5.6), respectively.*

In the proof this theorem we need the following lemma.

**Lemma 5.1.** *For any measurable open set $\Omega \subset X \times \mathbb{R}$, there exists a constant $c$, independent of $\Omega$, such that*

$$\mu\left(\widetilde{\Omega}\right) \leq c\mu(\Omega). \tag{5.10}$$

*Proof of Lemma 5.1.* We will apply Theorem I in Chapter 3 of [69] to the set $\Omega$. First we verify that the hypotheses of this theorem are satisfied.

By Theorem 2.2 in Chapter 2 of [37], there exists a constant $c > 0$ such that, for any measurable set $U \subset X$,

$$\mu_X \left( \{ x \in X : \mathcal{M} \left( \chi_U \right) (x) > \lambda \} \right) \leq \frac{c}{\lambda} \mu_X \left( U \right),$$

for all $\lambda > 0$, where

$$\mathcal{M} \left( f \right) (x) := \sup_{\alpha > 0} \frac{1}{\mu_X \left( B \left( x, \alpha \right) \right)} \int_{B(x,\alpha)} \left| f \left( u \right) \right| d\mu_X(u), \quad f \in L^1_{loc} \left( X, \mu_X \right). \tag{5.11}$$

Inspired by an argument found in Chapter I of [62], we define an "uncentered" version of $\mathcal{M}$ as follows

$$M_X \left( \chi_U \right) (x) := \sup_{x \in B} \frac{1}{\mu_X \left( B \right)} \int_B \left| f \left( u \right) \right| d\mu_X(u), \quad f \in L^1_{loc} \left( X, \mu_X \right),$$

where the supremum is taken over all open balls $B \left( x_0, \alpha \right) \subset X$ containing $x$. We will show that for any measurable set $U \subset X$,

$$M_X \left( \chi_U \right) \geq c \mathcal{M} \left( \chi_U \right), \tag{5.12}$$

where the constant $c$ is independent of $U$. To prove that (5.12) holds, we first note that, since $d_X$ is a metric, for any two points $x_0, u_0 \in X$ and any $r > 0$, if $B \left( x_0, r \right) \cap B \left( u_0, r \right) \neq \emptyset$, then $B \left( x_0, r \right) \subset B \left( u_0, 3r \right)$. Now fix $x \in X$ and take a ball $B \left( x_0, \alpha \right)$ containing $x$. On the one hand, we have $B \left( x_0, \alpha \right) \cap B \left( x, \alpha \right) \neq \emptyset$, which implies $B \left( x_0, \alpha \right) \subset B \left( x, 3\alpha \right)$. So the monotonicity of $\mu_X$ yields

$$\mu_X \left( B \left( x_0, \alpha \right) \cap U \right) \leq \mu_X \left( B \left( x, 3\alpha \right) \cap U \right). \tag{5.13}$$

On the other hand, we also have $B \left( x_0, 3\alpha \right) \cap B \left( x, 3\alpha \right) \neq \emptyset$, and this implies $B \left( x, 3\alpha \right) \subset B \left( x_0, 9\alpha \right)$. So

$$\mu_X \left( B \left( x, 3\alpha \right) \right) \leq \mu_X \left( B \left( x_0, 9\alpha \right) \right) \leq c_{\mu_X}^{\log_2(9)} \mu_X \left( B \left( x_0, \alpha \right) \right). \tag{5.14}$$

Combining (5.13) and (5.14) we obtain

$$\frac{\mu_X\left(B\left(x_0,\alpha\right)\cap U\right)}{\mu_X\left(B\left(x_0,\alpha\right)\right)}\le c_{\mu_X}^{\log_2(9)}\frac{\mu_X\left(B\left(x,3\alpha\right)\cap U\right)}{\mu_X\left(B\left(x,3\alpha\right)\right)}. \tag{5.15}$$

From (5.15) it follows that $M_X\left(\chi_U\right)(x)>c\mathcal{M}\left(\chi_U\right)(x)$, where $c=c_{\mu_X}^{\log_2(9)}$. So (5.12) is proved.

Inequality (5.12) implies that

$$\left\{x\in X:M_X\left(\chi_U\right)(x)>\lambda\right\}\subset\left\{x\in X:\mathcal{M}\left(\chi_U\right)(x)>c_{\mu_X}^{-\log_2(9)}\lambda\right\} \tag{5.16}$$

for any measurable set $U\subset X$ and any $\lambda>0$.

Using (5.16), the monotonicity of $\mu_X$ and (5.11), we get

$$\mu_X\left(\left\{x\in X:M_X\left(\chi_U\right)(x)>\lambda\right\}\right)\le\frac{c_X c_{\mu_X}^{\log_2(9)}}{\lambda}\mu_X\left(U\right) \tag{5.17}$$

for any measurable set $U\subset X$ and any $\lambda>0$.

On the real line, the standard Hardy-Littlewood maximal function theorem guarantees that

$$\mu_Y\left(\left\{y\in\mathbb{R}:M_Y\left(\chi_V\right)(x)>\lambda\right\}\right)\le\frac{c_Y}{\lambda}\mu_Y(V), \tag{5.18}$$

for any measurable set $V\subset\mathbb{R}$ and any $\lambda>0$, where

$$M_Y\left(f\right)(y):=\sup_{y\in J}\frac{1}{\mu_X\left(J\right)}\int_J\left|f\left(v\right)\right|d\mu_Y(v),\quad f\in L_{loc}^1\left(\mathbb{R},\mu_Y\right),$$

with the supremum taken over all open intervals $J\subset\mathbb{R}$ containing $y$.

The hypotheses of Theorem I in Chapter III of [69] are satisfied: (5.17) and (5.18) hold. By this theorem,

$$\mu\left(\left\{(x,y)\in X\times\mathbb{R}:M_S\left(\chi_\Omega\right)(x,y)>\lambda\right\}\right)\le\min_{0<r<\lambda}\left[\left(\frac{c_X c_{\mu_X}^{\log_2(9)}}{r}\right)\left(\frac{c_Y}{\lambda-r}\right)\right]\mu\left(\Omega\right).$$

In particular, when $\lambda = 1/2$, this inequality becomes

$$\mu\left(\widetilde{\Omega}\right) \leq \frac{c_X c_Y c_{\mu_X}^{\log_2(9)}}{16}\mu(\Omega).$$

So (5.10) holds. $\square$

*Proof of Theorem 5.1.* Fix $(x,t) \in X \times (0,\infty)$. Assume, w.l.o.g., that $E_{x,t} \neq \emptyset$ and fix $k \in \Lambda(x,t)$. If $0 < s < \tau(x,t,k)$, then

$$\frac{\mu_Y(E_{x,s} \cap J_{x,t,k})}{\mu_Y(J_{x,t,k})} > \frac{1}{2},$$

and also $B(x,s) \times E_{x,s} \subset \Omega$. Thus, for any $0 < s < \tau(x,t,k)$, we have

$$\begin{aligned}\mu([B(x,s) \times J_{x,t,k}] \cap \Omega) &\geq \mu([B(x,s) \times J_{x,t,k}] \cap [B(x,s) \times E_{x,s}]) \\ &= \mu_X(B(x,s))\,\mu_Y(J_{x,t,k} \cap E_{x,s}) \\ &> \frac{1}{2}\mu_X(B(x,s))\,\mu_Y(J_{x,t,k}).\end{aligned}$$

This shows that $B(x,s) \times J_{x,t,k} \subset \widetilde{\Omega}$ for all $0 < s < \tau(x,t,k)$. From this, it follows that

$$B(x,\tau(x,t,k)) \times J_{x,t,k} \subset \widetilde{\Omega}. \tag{5.19}$$

Since (5.19) holds for all $(x,t) \in X \times (0,\infty)$ such that $E_{x,t} \neq \emptyset$ and all $k \in \Lambda(x,t)$, we conclude that

$$\bigcup_{(x,t)\in X\times\mathbb{R}} \left(\bigcup_{k\in\Lambda(x,t)} B(x,\tau(x,t,k)) \times J_{x,t,k}\right) \subset \widetilde{\Omega}. \tag{5.20}$$

From (5.20) and Lemma 5.1, it follows that (5.8) holds.

Now we will show (5.9).

The Dini condition on $\phi$ implies that $\lim_{s\to\infty}\phi\left(\frac{t}{s}\right) = 0$ for any $t > 0$. So, for any $u, t \in (0,\infty)$,

$$\phi\left(\frac{t}{u}\right) = -\int_u^\infty \left(\frac{d}{ds}\phi\left(\frac{t}{s}\right)\right) ds = \int_u^\infty \phi'\left(\frac{t}{s}\right)\frac{ds}{s^2}. \tag{5.21}$$

Taking $u=\tau(x,t,k)$ in (5.21), we can re-write the left-hand-side of (5.9) as

$$\int_0^\infty \int_{x\in X} \sum_{k\in\Lambda(x,t)} \mu_Y\left(J_{x,t,k}\right)\left[\int_{\tau(x,t,k)}^\infty \phi'\left(\frac{t}{s}\right)\frac{ds}{s^2}\right] d\mu_X(x)\,\frac{dt}{t}$$

$$= \int_0^\infty \int_{x\in X} \sum_{k\in\Lambda(x,t)} \left[\int_{\tau(x,t,k)}^\infty \mu_Y\left(J_{x,t,k}\right)\phi'\left(\frac{t}{s}\right)\frac{ds}{s^2}\right] d\mu_X(x)\,\frac{dt}{t}. \tag{5.22}$$

For a fixed triple $(x,t,k)\in X\times(0,\infty)\times\Lambda(x,t)$, if $s>\tau(x,t,k)$, then $\mu_Y(E_{x,s}\cap J_{x,t,k})\le 2^{-1}\mu_Y\left(J_{x,t,k}\right)$, by the definition of $\tau(x,t,k)$ in (5.6). So, when $s>\tau(x,t,k)$, we have

$$\mu_Y\left(J_{x,t,k}\right)=\mu_Y\left(J_{x,t,k}\cap E_{x,s}\right)+\mu_Y\left(J_{x,t,k}\cap E_{x,s}^c\right)\le 2^{-1}\mu_Y\left(J_{x,t,k}\right)+\mu_Y\left(J_{x,t,k}\backslash E_{x,s}\right),$$

and from this we get $\mu_Y\left(J_{x,t,k}\right)\le 2\mu_Y\left(J_{x,t,k}\backslash E_{x,s}\right)$. Therefore (5.22) is majorized by

$$2\int_0^\infty \int_{x\in X} \sum_{k\in\Lambda(x,t)} \left[\int_{\tau(x,t,k)}^\infty \mu_Y\left(J_{x,t,k}\backslash E_{x,s}\right)\phi'\left(\frac{t}{s}\right)\frac{ds}{s^2}\right] d\mu_X(x)\,\frac{dt}{t}$$

$$= 2\int_{x\in X}\int_0^\infty \sum_{k\in\Lambda(x,t)} \left[\int_{\tau(x,t,k)}^\infty \left(\int_{-\infty}^\infty \chi_{J_{x,t,k}\backslash E_{x,s}}(y)\,d\mu_Y(y)\right)\phi'\left(\frac{t}{s}\right)\frac{ds}{s^2}\right]\frac{dt}{t}d\mu_X(x)$$

$$= 2\int_{x\in X}\int_{-\infty}^\infty\left\{\int_0^\infty \sum_{k\in\Lambda(x,t)} \left[\int_{\tau(x,t,k)}^\infty \chi_{J_{x,t,k}\backslash E_{x,s}}(y)\,\phi'\left(\frac{t}{s}\right)\frac{ds}{s^2}\right]\frac{dt}{t}\right\} d\mu_Y(y)\,d\mu_X(x)$$

$$= 2\int_{x\in X}\int_0^\infty F(x,y)\,d\mu_Y(y)\,d\mu_X(x), \tag{5.23}$$

where

$$F(x,y):=\int_0^\infty \sum_{k\in\Lambda(x,t)} \left[\int_{\tau(x,t,k)}^\infty \chi_{J_{x,t,k}\backslash E_{x,s}}(y)\,\phi'\left(\frac{t}{s}\right)\frac{ds}{s^2}\right]\frac{dt}{t}. \tag{5.24}$$

We claim that $F$ is identically zero outside $\Omega$. To see that this holds, fix $(x,y)\notin\Omega$. Then $y\notin\left\{v\in\mathbb{R}:\overline{B(x,t)}\times\{v\}\subset\Omega\right\}=E_{x,t}$ for all $t>0$. In particular, by (5.5), $y\notin J_{x,t,k}$ for all $t>0$ and all $k\in\Lambda(x,t)$. Therefore $\chi_{J_{x,t,k}\backslash E_{x,s}}(y)=0$ for all $t>0$ and all $k\in\Lambda(x,t)$. Thus, by the expression of $F$ in (5.24), we conclude that $F(x,y)=0$.

Since $F$ vanishes outside $\Omega$, we can rewrite (5.24) as

$$F(x,y) = \chi_{\Omega}(x,y) \int_0^\infty \sum_{k \in \Lambda(x,t)} \left[ \int_{\tau(x,t,k)}^\infty \chi_{J_{x,t,k} \setminus E_{x,s}}(y)\, \phi'\left(\frac{t}{s}\right) \frac{ds}{s^2} \right] \frac{dt}{t}. \tag{5.25}$$

Given $(x,y) \in \Omega$ and $t > 0$ such that $E_{x,t}$ is non-empty, since the intervals $J_{x,t,k}$'s, $k \in \Lambda(x,t)$, are disjoint, the point $y$ cannot be in more than one interval $J_{x,t,k}$, $k \in \Lambda(x,t)$. We denote by $K(x,y,t)$ the index $k$ in $\Lambda(x,t)$ such that $y \in J_{x,t,k}$. With this notation, we can re-write (5.25) as

$$F(x,y) = \chi_{\Omega}(x,y) \int_0^\infty \left[ \int_{\tau(x,t,K(x,y,t))}^\infty \chi_{J_{x,t,K(x,y,t)} \setminus E_{x,s}}(y)\, \phi'\left(\frac{t}{s}\right) \frac{ds}{s^2} \right] \frac{dt}{t}. \tag{5.26}$$

Now we define $T : \Omega \to \mathbb{R}$ by

$$T(x,y) := \sup\left\{ s > 0 : \overline{B(x,s)} \times \{y\} \subset \Omega \right\}.$$

We affirm that if $y \in J_{x,t,K(x,y,t)} \setminus E_{x,s}$ for some $s > \tau(x,t,K(x,y,t))$, then

$$t \le T(x,y) \le s. \tag{5.27}$$

To see that (5.27) holds, note that

(i) if $y \in J_{x,t,K(x,y,t)}$, then $y \in E_{x,t} = \left\{ v \in \mathbb{R} : \overline{B(x,t)} \times \{v\} \subset \Omega \right\}$, which implies that $t \le T(x,y)$; and

(ii) if there exists $s > \tau(x,t,K(x,y,t))$ such that $y \notin E_{x,s}$, then $\overline{B(x,s)} \times \{y\} \not\subset \Omega$, which implies that $T(x,y) \le s$.

Using the bounds in (5.27), we can re-write (5.26) as

$$F(x,y) = \chi_{\Omega}(x,y) \int_0^{T(x,y)} \left[ \int_{T(x,y)}^\infty \chi_{J_{x,t,K(x,y,t)} \setminus E_{x,s}}(y)\, \phi'\left(\frac{t}{s}\right) \frac{ds}{s^2} \right] \frac{dt}{t},$$

which is not greater than

$$\chi_{\Omega}(x,y)\int_0^{T(x,y)}\left[\int_{T(x,y)}^{\infty}\phi'\left(\frac{t}{s}\right)\frac{ds}{s^2}\right]\frac{dt}{t}.$$

So

$$F(x,y)\leq\chi_{\Omega}(x,y)\int_0^{T(x,y)}\left[\int_{T(x,y)}^{\infty}\phi'\left(\frac{t}{s}\right)\frac{ds}{s^2}\right]\frac{dt}{t}=\chi_{\Omega}(x,y)\int_0^{T(x,y)}\phi\left(\frac{t}{T(x,y)}\right)\frac{dt}{t}, \tag{5.28}$$

where the equality follows from (5.21) with $u=T(x,y)$. Now, we plug (5.28) into (5.23) to get (5.9). □

## 5.3 Variant of Discrete Journé's Lemma

The next result is an adaptation of Proposition 1 in [42], which is the discrete version of the lemma of Journé. It requires a generalization of the definition of dyadic rectangles from Euclidean spaces to the product setting $X\times\mathbb{R}$. The non-Euclidean component of $X\times\mathbb{R}$ is the metric measure space $(X,d_X,\mu_X)$. On this kind of metric measure spaces, it is possible to build a collection of sets that imitates the standard system of dyadic cubes of $\mathbb{R}^n$. Some references for this type of construction are [16] and [38]. In the latter, the "dyadic system" is described in Theorem 2.2.

As stated in [37] (Exercise 13.1), there exit constants $C\geq 1$, $\gamma>0$, depending on the doubling constant and on the uniform perfectness constant, such that

$$\frac{\mu_X(B(x,r))}{\mu_X(B(x,R))}\leq C\left(\frac{r}{R}\right)^{\gamma}\quad\text{for all } 0<r<R<diam(X). \tag{5.29}$$

Given open set $\Omega\subset X\times\mathbb{R}$ with $\mu(\Omega)<\infty$, while it makes sense define maximal dyadic rectangles on $\Omega\subset X\times\mathbb{R}$, the variant of Journé's Lemma that we present below does not explicitly need such a definition. Instead we will need a countable collection of rectangles defined as follows:

$$\{R_{i,j,k}\}_{\substack{i\in\mathbb{Z},\\ j\in\Theta_i,\\ k\in\Lambda_{i,j}}}\subset\mathcal{R}$$

satisfies

(i) $R_{i,j,k} = I_{i,j} \times J_{i,j,k} \subset \Omega$ for each $i \in \mathbb{Z},\ j \in \Theta_i,\ k \in \Lambda_{i,j}$;

(ii) $I_{i,j} = B\left(x_j, 2^i\right)$ for each $i \in \mathbb{Z},\ j \in \Theta_i$;

(iii) (bounded overlap) there exists a constant $\theta \geq 1$ such that, for each $i \in \mathbb{Z}$, $\sum_{j\in\Theta_i} \chi_{I_{i,j}} \leq \theta$;

(iv) for fixed $i$ and $j$, each $J_{i,j,k}$ is an open interval of the set

$$E_{i,j} := E_{x_j,2^i} = \left\{y \in \mathbb{R} :\ \overline{B\left(x_j, 2^i\right)} \times \{y\} \subset \Omega\right\},$$

which we express, as in (5.5), as a countable union of disjoint open intervals $J_{i,j,k}$'s, $k \in \Lambda_{i,j} := \Lambda\left(x_j, 2^i\right)$.

**Claim 5.1.** *For each $(x,t) \in X \times (0,\infty)$ such that $E_{x,t} \neq \emptyset$, and each $k \in \Lambda(x,t)$, the definitions of $\tau$ and $\widetilde{\tau}$ ((5.6) and (5.7), respectively) imply*

$$\widetilde{\tau}(x,t,k) \geq \tau(x,t,k)\,. \tag{5.30}$$

*Proof.* Let $s < \tau(x,t,k)$. Then

$$\mu_Y\left(E_{x,s} \cap J_{x,t,k}\right) > \frac{\mu_Y\left(J_{x,t,k}\right)}{2}. \tag{5.31}$$

Using the inclusion $B(x,s) \times E_{x,s} \subset \Omega$ and (5.31), we obtain

$$\mu\left(\left[B(x,s) \times J_{x,t,k}\right] \cap \Omega\right) \geq \mu\left(\left[B(x,s) \times J_{x,t,k}\right] \cap \left[B(x,s) \times E_{x,s}\right]\right)$$

$$= \mu\left(B(x,s) \times \left(J_{x,t,k} \cap E_{x,s}\right)\right) = \mu_X\left(B(x,s)\right)\mu_Y\left(E_{x,s} \cap J_{x,t,k}\right)$$

$$> \mu_X\left(B(x,s)\right)\frac{\mu_Y\left(J_{x,t,k}\right)}{2} = \frac{1}{2}\mu\left(B(x,s) \times J_{x,t,k}\right),$$

and we conclude that

$$B(x,s) \times J_{x,t,k} \subset \widetilde{\Omega}. \tag{5.32}$$

Since (5.32) holds for all $s < \tau(x,t,k)$, it follows that $B(x, \tau(x,t,k)) \times J_{x,t,k} \subset \widetilde{\Omega}$. So

$$\tau(x,t,k) \le \sup\left\{ s \ge t : B(x,s) \times J_{x,t,k} \subset \widetilde{\Omega} \right\} = \widetilde{\tau}(x,t,k).$$

$\square$

**Theorem 5.2** (Variant of Discrete Journé's Lemma). *Let $\Omega \subset X \times \mathbb{R}$ be a bounded open set and consider a countable collection*

$$\{R_{i,j,k}\}_{\substack{i\in\mathbb{Z} \\ j\in\Theta_i, \\ k\in\Lambda_{i,j}}} \subset \mathcal{R}$$

*satisfying the properties (i)–(iv) listed above. If $\phi : \mathbb{R} \to \mathbb{R}$ is a non-negative, non-decreasing function of class $C^1$, such that $\int_0^1 \phi(s) \frac{ds}{s} < \infty$, then*

$$\mu\left( \bigcup_{\substack{i\in\mathbb{Z} \\ j\in\Theta_i \\ k\in\Lambda_{i,j}}} \widehat{I}_{i,j,k} \times J_{i,j,k} \right) \le c\mu(\Omega) \tag{5.33}$$

*and*

$$\sum_{i,j,k} \mu(R_{i,j,k})\, \phi\left( \frac{\mu_X(I_{i,j})}{\mu_X\left(\widehat{I}_{i,j,k}\right)} \right) \le \frac{4\,\theta\, c_{\mu_X}}{\gamma} \left( \int_0^1 \phi(s) \frac{ds}{s} \right) \mu(\Omega), \tag{5.34}$$

*where*

$$\widehat{I}_{i,j,k} := B\left(x_j, \widetilde{\tau}\left(x_j, 2^i, k\right)\right);$$

*$\theta$ is described in (iii); $c_{\mu_X}$ is the doubling constant of $\mu_X$; $\gamma$ is a constant for which (5.29) holds; and the constant $c$ is independent of $\Omega$ and $\{R_{i,j,k}\}_{i\in\mathbb{Z},\, j\in\Theta_i,\, k\in\Lambda_{i,j}}$.*

*Proof.* To show that (5.33) holds, we first observe that, by the definition of $\widetilde{\tau}$,

$$\widehat{I}_{i,j,k} \times J_{i,j,k} \subset \widetilde{\Omega},$$

for all $i \in \mathbb{Z}$ , $j \in \Theta_i$, $k \in \Lambda_{i,j}$. Thus

$$\mu\left(\bigcup_{\substack{i\in\mathbb{Z}\\ j\in\Theta_i\\ k\in\Lambda_{i,j}}} \widehat{I}_{i,j,k} \times J_{i,j,k}\right) \leq \mu\left(\widetilde{\Omega}\right)$$

and, by Lemma 5.1, $\mu\left(\widetilde{\Omega}\right) \leq c\mu\left(\Omega\right)$. So (5.33) is proved.

Now we will show (5.34).

For each $i \in \mathbb{Z},\ j \in \Theta_i$, $t \in (2^{i-2}, 2^{i-1})$, let

$$S_{i,j,t} := \left\{u \in I_{ij} : B\left(u,t\right) \subset I_{ij}\right\},$$

and note that $S_{i,j,t}$ can be represented as $B\left(x_j, (2^i - t)\right)$.

Working with the left-hand-side of (5.34), we find suitable upper bounds for it. The goal is to find an upper bound that allows us to use Theorem (5.1). We preceded as follows:

$$\sum_{\substack{i\in\mathbb{Z}\\ j\in\Theta_i\\ k\in\Lambda_{i,j}}} \mu\left(R_{i,j,k}\right)\phi\left(\frac{\mu_X\left(I_{i,j}\right)}{\mu_X\left(\widehat{I}_{i,j,k}\right)}\right)$$

$$= \sum_{i\in\mathbb{Z}}\sum_{j\in\Theta_i}\mu_X\left(I_{i,j}\right)\sum_{k\in\Lambda_{i,j}}\mu_Y\left(J_{i,j,k}\right)\phi\left(\frac{\mu_X\left(I_{i,j}\right)}{\mu_X\left(\widehat{I}_{i,j,k}\right)}\right)$$

$$= \sum_{i\in\mathbb{Z}}\sum_{j\in\Theta_i}\mu_X\left(I_{i,j}\right)\sum_{k\in\Lambda_{i,j}}\mu_Y\left(J_{i,j,k}\right)\phi\left(\frac{\mu_X\left(I_{i,j}\right)}{\mu_X\left(\widehat{I}_{i,j,k}\right)}\right)\frac{2^{i-1}-2^{i-2}}{2^{i-2}}$$

$$\overset{\text{frist}}{\leq} 2\sum_{i\in\mathbb{Z}}\int_{2^{i-2}}^{2^{i-1}}\sum_{j\in\Theta_i}\mu_X\left(I_{i,j}\right)\sum_{k\in\Lambda_{i,j}}\mu_Y\left(J_{i,j,k}\right)\phi\left(\frac{\mu_X\left(I_{i,j}\right)}{\mu_X\left(\widehat{I}_{i,j,k}\right)}\right)\frac{dt}{t}$$

$$\overset{\text{second}}{\leq} 2\sum_{i\in\mathbb{Z}}\int_{2^{i-2}}^{2^{i-1}}\sum_{j\in\Theta_i}\mu_X\left(2S_{i,j,t}\right)\sum_{k\in\Lambda_{i,j}}\mu_Y\left(J_{i,j,k}\right)\phi\left(\frac{\mu_X\left(I_{i,j}\right)}{\mu_X\left(\widehat{I}_{i,j,k}\right)}\right)\frac{dt}{t}$$

$$\overset{\text{third}}{\leq} 2c_{\mu_X}\sum_{i\in\mathbb{Z}}\int_{2^{i-2}}^{2^{i-1}}\sum_{j\in\Theta_i}\mu_X\left(S_{i,j,t}\right)\sum_{k\in\Lambda_{i,j}}\mu_Y\left(J_{i,j,k}\right)\phi\left(\frac{\mu_X\left(I_{i,j}\right)}{\mu_X\left(\widehat{I}_{i,j,k}\right)}\right)\frac{dt}{t}$$

$$\leq 2c_{\mu_X} \sum_{i\in\mathbb{Z}} \int_{2^{i-2}}^{2^{i-1}} \sum_{j\in\Theta_i} \mu_X (S_{i,j,t}) \sum_{k\in\Lambda(x_j,2^i)} \mu_Y (J_{i,j,k})\, \phi \left( \frac{c_{\mu_X}^2 \mu_X (B(x_j,t))}{\mu_X \left(\widehat{I}_{i,j,k}\right)} \right) \frac{dt}{t}, \tag{5.35}$$

where the first inequality holds because $\frac{1}{2^{i-2}} < \frac{2}{t}$ for any $t \in (2^{i-2}, 2^{i-1})$; the second inequality holds because $S_{i,j,t} = B(x_j, (2^i - t))$ and $t \in (2^{i-2}, 2^{i-1})$ imply

$$2S_{i,j,t} = B\left(x_j, 2\left(2^i - t\right)\right) \supseteq B\left(x_j, 2^i\right) = I_{ij};$$

the third inequality is a direct consequence of the fact that

$$\mu_X (2S_{i,j,t}) \leq c_{\mu_X} \mu_X (S_{i,j,t})$$

which is true because $\mu_X$ is doubling and $S_{i,j,t}$ has the form $B(x_j, (2^i - t))$; and finally the last inequality holds because

$$\mu_X (I_{i,j}) = \mu_X \left(B\left(x_j, 2^2 2^{i-2}\right)\right) \leq \mu_X \left(B\left(x_j, 2^2 t\right)\right) \leq c_{\mu_X}^2 \mu_X (B(x_j,t))$$

for any $t \in (2^{i-2}, 2^{i-1})$ and $\phi$ is non-decreasing.

We claim that for any $i \in \mathbb{Z}$, $j \in \Theta_i$, and $k \in \Lambda(i,j)$,

$$J_{i,j,k} \subset E_{u,t} \text{ for all } t \in \left(2^{i-2}, 2^{i-1}\right) \text{ and all } u \in S_{i,j,t}. \tag{5.36}$$

This holds because if $t \in (2^{i-2}, 2^{i-1})$ and $u \in S_{i,j,t} = B(x_j, (2^i - t))$, then $B(u,t) \subset B(x_j, 2^i) = I_{ij}$, and this inclusion yields

$$J_{i,j,k} = J_{x_j,2^i,k} \subset E_{x_j,2^i} = \left\{ y : \overline{B(x_j,2^i)} \times \{y\} \subset \Omega \right\} \subset \left\{ y : \overline{B(u,t)} \times \{y\} \subset \Omega \right\} = E_{u,t}.$$

Fixed $i \in \mathbb{Z}$, $j \in \Theta_i$, $t \in (2^{i-2}, 2^{i-1})$ and $u \in S_{i,j,t}$, since the interval $J_{x_j,2^i,k}$ is contained in $E_{u,t} = \bigcup_{l\in\Lambda(u,t)} J_{u,t,l}$, there exists one, and only one, index $l$ in $\Lambda(u,t)$, such that $J_{x_j,2^i,k} \subset J_{u,t,l}$. We call that index by $L(i,j,k,u,t)$, i.e.

$$L(i,j,k,u,t) \text{ is the unique element of } \left\{ l \in \Lambda(u,t) : J_{u,t,l} \supset J_{x_j,2^i,k} \right\}. \tag{5.37}$$

For fixed $i \in \mathbb{Z}$, $j \in \Theta_i$, we have

$$\mu_X\left(B\left(x_j, \tau\left(t, u, L\left(i, j, k, u, t\right)\right)\right)\right) \leq c_{\mu_X}\mu_X\left(\widehat{I}_{i,j,k}\right) \tag{5.38}$$

for any $t \in (2^{i-2}, 2^{i-1})$ and $u \in S_{i,j,t}$. To see that (5.38) holds, first fix $t \in (2^{i-2}, 2^{i-1})$ and $u \in S_{i,j,t}$, and note that

$$u \in S_{i,j,t} = B\left(x_j, \left(2^i - t\right)\right) \subset B\left(x_j, \left(2^i - 2^{i-2}\right)\right) \subset B\left(x_j, 2^i\right) = I_{ij}.$$

Note also that the right-hand-side of (5.38) is $c_{\mu_X}\mu_X\left(B\left(x_j, \widetilde{\tau}\left(x_j, 2^i, k\right)\right)\right)$. So (5.38) will be proved once we show that $\tau\left(u, t, L\left(i, j, k, u, t\right)\right) \leq 2\widetilde{\tau}\left(x_j, 2^i, k\right)$. Suppose, to reach a contradiction, that $\tau\left(u, t, L\right) > 2\widetilde{\tau}\left(x_j, 2^i, k\right)$, where here $L := L\left(i, j, k, u, t\right)$. In this case, there exists $\epsilon > 0$, such that $\tau\left(u, t, L\right) = 2\widetilde{\tau}\left(x_j, 2^i, k\right) + \epsilon$, and then

$$B\left(x_j, \widetilde{\tau}\left(x_j, 2^i, k\right) + \epsilon\right) \subset B\left(u, \tau\left(u, t, L\right)\right), \tag{5.39}$$

where the inclusion holds because any $\tilde{x}$ in $B\left(x_j, \widetilde{\tau}\left(x_j, 2^i, k\right) + \epsilon\right)$ satisfies $d_X\left(\tilde{x}, x_j\right) < \widetilde{\tau}\left(x_j, 2^i, k\right) + \epsilon$. This, combined with the fact that $2^i \leq \widetilde{\tau}\left(x_j, 2^i, k\right)$, implies

$$d_X\left(\tilde{x}, u\right) \leq d_X\left(\tilde{x}, x_j\right) + d_X\left(x_j, u\right) < \widetilde{\tau}\left(x_j, 2^i, k\right) + \epsilon + 2^i \leq 2\widetilde{\tau}\left(x_j, 2^i, k\right) + \epsilon = \tau\left(u, t, L\right),$$

which yields $\tilde{x} \in B\left(u, \tau\left(u, t, L\right)\right)$. From (5.39) and the choice of $L$ $(= L\left(i, j, k, u, t\right)$, defined in (5.37)), it follows that

$$\begin{aligned} B\left(x_j, \widetilde{\tau}\left(x_j, 2^i, k\right) + \epsilon\right) \times J_{x_j, 2^i, k} &\subset B\left(u, \tau\left(u, t, L\right)\right) \times J_{x_j, 2^i, k} \\ &\subset B\left(u, \tau\left(u, t, L\right)\right) \times J_{u,t,L} \subset \widetilde{\Omega}. \end{aligned}$$

This contradicts the maximality of $\widetilde{\tau}\left(x_j, 2^i, k\right)$ (see the definition of $\widetilde{\tau}$ in (5.7)). Hence (5.38) is proved.

The expression in (5.35) can be re-written as

$$2c_{\mu_X}\sum_{i\in\mathbb{Z}}\int_{2^{i-2}}^{2^{i-1}}\sum_{j\in\Theta_i}\int_{S_{i,j,t}}\left[\sum_{k\in\Lambda(x_j,2^i)}\mu_Y\left(J_{x_j,2^i,k}\right)\phi\left(\frac{c_{\mu_X}^2\mu_X\left(B\left(x_j,t\right)\right)}{\mu_X\left(\widehat{I}_{i,j,k}\right)}\right)\right]d\mu_X(x)\,\frac{dt}{t}. \tag{5.40}$$

Using (5.38) and the fact that $\phi$ is non-decreasing, we can majorize the term in the brackets by

$$\sum_{k\in\Lambda(x_j,2^i)}\mu_Y\left(J_{x_j,2^i,k}\right)\phi\left(\frac{c_X^3\mu_X\left(B\left(x_j,t\right)\right)}{\mu_X\left(B\left(x_j,\tau\left(x,t,L\left(i,j,k,x,t\right)\right)\right)\right)}\right). \tag{5.41}$$

As shown in (5.36), for each $i\in\mathbb{Z}$, $j\in\Theta_i$, $t\in(2^{i-2},2^{i-1})$ and $x\in S_{i,j,t}$, the inclusion

$$\bigcup_{k\in\Lambda(x_j,2^i)}J_{x_j,2^i,k}\subset E_{x,t}=\bigcup_{l\in\Lambda(x,t)}J_{x,t,l}$$

holds. Since these are disjoint unions of intervals, (5.41) is not grater than

$$\sum_{l\in\Lambda(x,t)}\mu_Y\left(J_{x,t,l}\right)\phi\left(\frac{c_X^3\mu_X\left(B\left(x_j,t\right)\right)}{\mu_X\left(B\left(x_j,\tau\left(x,t,l\right)\right)\right)}\right). \tag{5.42}$$

To assure that this is so, note that the worse that can happen is to have more than one $J_{x_j,2^i,k}$ contained in the same $J_{x,t,l}$. To deal with this scenario, observe that
(i) by the choice of $L\left(i,j,k,x,t\right)$, if $J_{x_j,2^i,k}\subset J_{x,t,l}$, then $L\left(i,j,k,x,t\right)=l$; and
(ii) since the $J_{x_j,2^i,k}$'s are disjoint, $\sum\limits_{\substack{i,j,k\\ J_{x_j,2^i,k}\subset J_{x,t,l}}}\mu_Y\left(J_{x_j,2^i,k}\right)\le\mu_Y\left(J_{x,t,l}\right)$.

Plugging (5.42) into (5.40), we obtain that the latter is bounded above by

$$2c_{\mu_X}\sum_{i\in\mathbb{Z}}\int_{2^{i-2}}^{2^{i-1}}\sum_{j\in\Theta_i}\int_{S_{i,j,t}}\sum_{l\in\Lambda(x,t)}\mu_Y\left(J_{x,t,l}\right)\phi\left(\frac{c_{\mu_X}^3\mu_X\left(B\left(x_j,t\right)\right)}{\mu_X\left(B\left(x_j,\tau\left(x,t,l\right)\right)\right)}\right)d\mu_X(x)\,\frac{dt}{t}$$

$$\le 2c_{\mu_X}\sum_{i\in\mathbb{Z}}\int_{2^{i-2}}^{2^{i-1}}\sum_{j\in\Theta_i}\int_{I_{i,j}}\sum_{l\in\Lambda(x,t)}\mu_Y\left(J_{x,t,l}\right)\phi\left(\frac{c_{\mu_X}^3\mu_X\left(B\left(x_j,t\right)\right)}{\mu_X\left(B\left(x_j,\tau\left(x,t,l\right)\right)\right)}\right)d\mu_X(x)\,\frac{dt}{t}. \tag{5.43}$$

where the inequality follows from the fact that, for each $i\in\mathbb{Z}$ and $t\in(2^{i-2},2^{i-1})$, the inclusion $S_{i,j,t}\subset I_{i,j}$ holds. Now we use will Property (iii) of the collection $\{R_{i,j,k}\}_{i,j,k}$.

This property says that, for each $j \in \Theta_i$, the intervals $I_{i,j}$'s have bounded overlap. So (5.43) is not greater than

$$2\,\theta\, c_{\mu_X} \sum_{i\in\mathbb{Z}} \int_{2^{i-2}}^{2^{i-1}} \sum_{j\in\Theta_i} \int_X \sum_{l\in\Lambda(x,t)} \mu_Y\left(J_{x,t,l}\right) \phi\left(\frac{c_{\mu_X}^3 \mu_X\left(B\left(x_j,t\right)\right)}{\mu_X\left(B\left(x_j,\tau\left(x,t,l\right)\right)\right)}\right) d\mu_X(x)\,\frac{dt}{t},$$

which is equal to

$$2\,\theta\, c_{\mu_X} \int_0^{\infty} \sum_{j\in\Theta_i} \int_X \sum_{l\in\Lambda(x,t)} \mu_Y\left(J_{x,t,l}\right) \phi\left(c_{\mu_X}^3 \frac{\mu_X\left(B\left(x_j,t\right)\right)}{\mu_X\left(B\left(x_j,\tau\left(x,t,l\right)\right)\right)}\right) d\mu_X(x)\,\frac{dt}{t}. \quad (5.44)$$

We intend to apply (5.29) to the balls that appear in the argument of $\phi$ in (5.44). Is $t < \tau\left(x,t,l\right)$? Yes, by the definition of $\tau\left(x,t,l\right)$. Combining (5.29) with the fact that $\phi$ is non-decreasing, we obtain that (5.44) is not greater than

$$2\,\theta\, c_{\mu_X} \int_0^{\infty} \int_{x\in X} \sum_{l\in\Lambda(x,t)} \mu_Y\left(J_{x,t,l}\right) \phi\left(c_{\mu_X}^3 C \left(\frac{t}{\tau\left(x,t,l\right)}\right)^{\gamma}\right) d\mu_X(x)\,\frac{dt}{t}.$$

By theorem 5.1, this can be majorized by

$$4\,\theta\, c_{\mu_X} \int_0^1 \phi\left(c_{\mu_X}^3 C s^{\gamma}\right) \frac{ds}{s} \mu\left(\Omega\right),$$

and a change of variables yields (5.34). □

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