# On the finite-time Gerber-Shiu function

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**Abstract**

We consider a finite-time version of the Gerber-Shiu (G-S) function, defined over a fixed interval $[0, t]$ as follows:

$$m_\delta^{(2)}(u;t) = \mathbb{E}\big[e^{-\delta(t\wedge T)} w(U_{(T\wedge t)-}, U_{T\wedge t}) \mid U_0 = u\big] \,, \qquad u \geq 0,\, t > 0 \,,$$

for general bivariate penalty functions $w$ and spectrally negative Lévy surplus processes $U$.

Our motivation in adapting to a short term the classical G-S function, which was originally defined on an infinite-time ruin horizon, is to allow its use as a risk management tool. Risk management problems are most often set over a short term (a day, a week, a quarter, or a year). Hence the need to redefine the G-S function and analyze it over a finite time.

We first obtain the Laplace transform in $t$ of $m_\delta^{(2)}(u;t)$, for any penalty function, in terms of the scale function of the Lévy process. A closed form inversion of this Laplace transform is given in the case of the classical risk model (compound Poisson with exponential claims + a drift) perturbed by Brownian motion.

Then the numerical evaluation of the finite-time Gerber-Shiu function is illustrated for three special choices of penalty function $w$, that could be applied to the finite-time hedging of insurance portfolios. For these illustrations a compound renewal surplus process is assumed.

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# 1 Introduction

Assume that the insurer's surplus process $\{U_t\}_{t\geq 0}$ is given by

$$U_t = u + ct - S_t, \tag{1.1}$$

where $u = U_0 \geq 0$ is the initial surplus, $c$ is the rate of premium income per unit time, and $S_t$ is the aggregate claims amount up to time $t$ with $S_0 \equiv 0$. Let $T$ denote the time of ruin,

$$T := \inf\{t \geq 0 : U_t < 0 \mid U_0 = u\},$$

with $T = \infty$ if the process $U_t$ never assumes a negative value (no ruin occurs). The infinite time ruin probability is defined as

$$\psi(u) := \mathbb{P}(T < \infty \mid U_0 = u).$$

An additional assumption imposed on the models is that $U_t \to \infty$ almost surely as $t \to \infty$, which is equivalent to saying that ruin is not a certain event. A sufficient condition to guarantee the latter, is the net profit condition $\lim_{t\to\infty} \mathbb{E}[U_t]/t > 0$ [Rolski et al. (1999)].

Similarly, the ruin probability with finite time horizon $t$ is defined as

$$\psi(u,t) := \mathbb{P}(\tau \leq t \mid U_0 = u), \qquad t > 0.$$

Gerber and Shiu (1998) proposed and analyzed the expected discounted penalty function given by

$$m_\delta^{(2)}(u) := \mathbb{E}\big[e^{-\delta T} w(U_{T-}, |U_T|) I(T < \infty) \mid U_0 = u\big], \qquad u \geq 0,\, t > 0, \tag{1.2}$$

where $\delta$ is assumed to be nonnegative, $U_{T-}$ is the surplus immediately before ruin, $|U_T|$ is the deficit at ruin, $S_t$ is a compound Poisson process, $w(x,y)$ is a known function of $x > 0$ and $y > 0$ and $I(A)$ is the indicator function of event $A$. The function $w$ is interpreted as a penalty, when ruin occurs, and $\delta$ as a force of interest. In the case when $w(x,y) = 1$ for all $x$ and $y$, we denote the classical Gerber-Shiu (G-S) function in (1.2) by $m_\delta^{(0)}(u)$, and when $w(x,y) = w(y)$ for all $x$ and $y$, by $m_\delta^{(1)}(u)$.

For a given surplus process $\{U_t\}_{t\geq 0}$ and a penalty function $w$, we define here a finite-time version of the Gerber-Shiu (G-S$_t$) function as follows:

$$m_\delta^{(2)}(u,t) := \mathbb{E}\big[e^{-\delta(T\wedge t)}w(U_{(T\wedge t)-}, U_{T\wedge t}) \mid U_0 = u\big], \qquad u \geq 0,\ t > 0, \tag{1.3}$$

with $U_{t-} = U_t$ a.s., by the quasi-left continuity of $\{U_t\}_{t\geq 0}$.

The renewal techniques commonly used with the classical G-S function do not work for the analysis of $m_\delta^{(2)}(u,t)$, due to the stopping of the process $U$ at time $T\wedge t$. Instead, the next section gives an expression for its Laplace transform derived for spectrally negative Lévy surplus processes $U$ and general penalty functions $w$, in terms of the corresponding scale function. The latter is derived in closed form for some special cases, including a compound Poisson surplus process with exponential claims perturbed by Brownian motion.

Other authors have, simultaneously and independently, investigated alternative definitions of a finite-time G-S function. For instance Kuznetsov and Morales (2014) proposed the following finite-time expected discounted penalty function:

$$\mathbb{E}\big[e^{-\delta T}w(-U_T, U_{T-}, \underline{U}_{T-})I(T < t) \mid U_0 = u\big], \tag{1.4}$$

where $\underline{U}_t = \inf_{0\leq s\leq t} U_s$. Note that the indicator function yields non-zero values only when ruin has occurred in the finite time interval $[0,t]$. Our discounting at the stopped time $T \wedge t$ in (1.3) allows for a more general formulation, still reproducing (1.4) as a special case for bivariate penalty functions equal to 0 when ruin does not occur before time $t$ (that is $U_{(T\wedge t)-} = U_{T\wedge t}$). The definition in (1.4) generalizes that of Kočetova and Šiaulys (2010), which they studied for the special case of a constant penalty function $w \equiv 1$ and compound Poisson $S_t$ with exponential claims.

# 2 The finite-time G-S function for Lévy surplus processes

In this section we consider the G-S$_t$ function in (1.3) for a spectrally negative Lévy surplus process $U$ and general penalty function $w$.

It is known that the Laplace exponent $\Psi(\theta) := \log \mathbb{E}[e^{\theta U_1}]$, for $\theta \geq 0$, allows the Lévy–Khintchine representation

$$\Psi(\theta) = a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} (e^{\theta x} - 1 - x\theta 1_{x>-1})\Pi(dx),$$

where $\Pi$ is a measure on $(-\infty,0)$ satisfying $\int_{(-\infty,0)}(1\wedge x^2)\Pi(dx)<\infty$. Denote by $\Phi:[0,\infty)\mapsto[0,\infty)$ as the right inverse of a convex function $\Psi$ on $(0,\infty)$, i.e.

$$\Phi(q):=\sup\{\theta\geq 0:\Psi(\theta)=q\}\,,\qquad q\geq 0.$$

Then consider the scale functions $\{W^{(q)}:q\geq 0\}$, such that $W^{(q)}(x)=0$ for $x\in(-\infty,0)$, $W^{(q)}$ is right continuous at 0 and continuous on $(0,\infty)$, and

$$\int_0^\infty e^{-\theta x}W^{(q)}(x)dx=\frac{1}{\Psi(\theta)-q}\,,\qquad \text{for }\ \theta>\Phi(q)\,.$$

We write $W$ for $W^{(0)}$.

For a spectrally negative Lévy risk model, consider the classical G-S function

$$m_\delta^{(2)}(u)=\mathbb{E}\big[e^{-\delta T}w(U_{T-},U_T)\,I(T<\infty)\mid U_0=u\big].$$

It is the limit $m_\delta^2(u)=\lim_{t\to\infty}m_\delta^2(u;t)$ of the finite–time Gerber-Shiu G-S$_t$ function in (1.3). Now let us analyze the latter.

The Laplace transform of $m_\delta^{(2)}(u;t)$ in $t$ is given by

$$\begin{aligned}
\widehat{m}_\delta^{(2)}(u,\beta)\ &:=\ \int_0^\infty e^{-\beta t}m_\delta^{(2)}(u;t)\,dt\,,\qquad u\geq 0\,,\\
&=\ \frac{1}{\beta}\,\mathbb{E}\big[m_\delta^{(2)}(u;T_\beta)\big]\\
&=\ \frac{1}{\beta}\,\mathbb{E}\big[e^{-\delta(T_\beta\wedge T)}w(U_{(T_\beta\wedge T)-},U_{T_\beta\wedge T})\mid U_0=u\big]\,,
\end{aligned}$$

where $T_\beta$ is an exponential variate with mean $1/\beta$, independent from $U$.

Equivalently,

$$\begin{aligned}
&\beta\,\widehat{m}_\delta^{(2)}(u,\beta)=\mathbb{E}\big[e^{-\delta T_\beta}w(U_{T_\beta-},U_{T_\beta})\,I(T_\beta\leq T)\mid U_0=u\big]\\
&\qquad+\mathbb{E}\big[e^{-\delta T}w(U_{T-},U_T)\,I(T_\beta>T)\mid U_0=u\big] \qquad (2.1)\\
&=\mathbb{E}\Big[\int_0^T e^{-\delta t}w(U_{t-},U_t)\beta e^{-\beta t}\,dt\mid U_0=u\Big]+\mathbb{E}\big[e^{-(\delta+\beta)T}w(U_{T-},U_T)\mid U_0=u\big]\\
&=\beta\,\mathbb{E}\Big[\int_0^T e^{-(\delta+\beta)t}w(U_t,U_t)\,dt\mid U_0=u\Big]+m_{\delta+\beta}^{(2)}(u)\\
&=\beta\int_0^\infty w(x,x)\Big[e^{-\Phi(\delta+\beta)x}W^{(\delta+\beta)}(u)-W^{(\delta+\beta)}(u-x)\Big]dx\\
&\qquad+\int_{-\infty}^{0-}\int_0^\infty w(x,y)\Big[e^{-\Phi(\delta+\beta)x}W^{(\delta+\beta)}(u)-W^{(\delta+\beta)}(u-x)\Big]dx\Pi(dy-x)\\
&\qquad+\frac{\sigma^2}{2}\big[W^{(\delta+\beta)\prime}(u)-\Phi(\delta+\beta)W^{(\delta+\beta)}(u)\big]\omega(0,0)\,,\qquad u\geq 0\,, \qquad (2.2)
\end{aligned}$$

where, for the first term in the third equality, we need the quasi–left continuity and for the first term in the fourth equality, we need Corollary 8.8 of Kyprianou (2006) on the potential measure. The second term in the last equality of (2.2) is essentially due to Bertoin (1997); also see Zhou (2005). Finally, the last term in the last equality concerns the Laplace transform of the time of ruin caused by Brownian motion. By Corollary 2 of Pistorius (2004) or Kyprianou (2006, p.235),

$$\mathbb{P}\{U_T = 0 | U_0 = u\} = \frac{\sigma^2}{2}\big[W'(u) - \Phi(0)W(u)\big].$$

If we kill the process $U$ at an independent exponential time $\tau$ with rate $\delta + \beta$, then applying the previous result to the killed process $U^K$ we get

$$\begin{aligned}\mathbb{E}\big[e^{-(\delta+\beta)T}; U_T = 0\big] &= \mathbb{P}\{U_T^K = 0\} \\ &= \frac{\sigma^2}{2}\big[W_K'(u) - \Phi_K(0)W_K(u)\big] \\ &= \frac{\sigma^2}{2}\big[W^{(\delta+\beta)\prime}(u) - \Phi(\delta+\beta)W^{(\delta+\beta)}(u)\big],\end{aligned}$$

where $\Phi_K$ and $W_K$ are the root and scale function for $U^K$, respectively.

The evaluation of the Laplace transform in (2.2) requires the numerical integration of the scale function. This is simpler when the latter is known explicitly, as in the following examples.

**Example 2.1** *For Brownian motion with drift the associated Laplace exponent is* $\Psi(\theta) = \mu\theta + \sigma^2\theta^2/2$ *and the scale function (defined below at* (2.10)*)*

$$W^{(q)}(x) = \frac{2}{\sqrt{2q\sigma^2 + \mu}} e^{-\frac{\mu x}{\sigma^2}} \sinh\Big(\frac{x}{\sigma^2}\sqrt{2q\sigma^2 + \mu}\Big), \qquad x > 0,\ W^{(q)}(0) = 0.$$

**Example 2.2** *For a compound Poisson($\lambda$) process with exponential jumps (of mean $\mu$) and drift, the associated Laplace exponent is*

$$\Psi(\theta) = c\theta + \frac{\lambda/\mu}{\theta + 1/\mu} - \lambda, \qquad \theta \geq 0,$$

*and scale function*

$$W^{(q)}(x) = \frac{\big[1/\mu + \rho(q)\big]e^{\rho(q)x} - \big[1/\mu + \bar{\rho}(q)\big]e^{\bar{\rho}(q)x}}{\sqrt{(c/\mu - \lambda - q)^2 + 4cq/\mu}}, \qquad x > 0,\ W^{(q)}(0) = 0,$$

*where*

$$\rho(q) := \frac{1}{2c}\Big[\lambda + q - c/\mu + \sqrt{(c/\mu - \lambda - q)^2 + 4cq/\mu}\Big]$$

*and*

$$\bar{\rho}(q) := \frac{1}{2c}\Big[\lambda + q - c/\mu - \sqrt{(c/\mu - \lambda - q)^2 + 4cq/\mu}\Big].$$

*See Hubalek and Kyprianou (2011) and Zhou (2005).*

**Example 2.3** *The corresponding perturbed compound Poisson surplus process with exponential jumps is then defined as*

$$U_t = u + ct - \sum_{i=1}^{N_t} X_i + \sigma B_t,$$

*where $N_t$ is a Poisson process with rate $\lambda$, the $(X_i)_{i\geq 1}$ are iid exponential random variables with mean $\mu$, $B_t$ is a standard Brownian motion and $\sigma \neq 0$, $c > \lambda\mu$. The moment generating function of $U_t - u$ is given by*

$$E\big[e^{\theta(U_t-u)}\big] = e^{t\Psi(\theta)},$$

*where*

$$\Psi(\theta) = c\theta + \frac{1}{2}\sigma^2\theta^2 - \frac{\theta\lambda}{1/\mu + \theta}, \qquad \theta \geq 0. \tag{2.3}$$

*It follows that here*

$$\frac{1}{\Psi(\theta) - q} = \frac{2(\theta + 1/\mu)}{\sigma^2\theta^3 + (2c + \sigma^2\beta)\theta^2 + 2(c/\mu - q - \lambda)\theta - 2q/\mu}.$$

*Under our assumptions, the denominator in the previous equation*

$$f(\theta) = \sigma^2\theta^3 + (2c + \sigma^2/\mu)\theta^2 + 2(c/\mu - q - \lambda)\theta - 2q/\mu, \qquad \theta \in \mathbb{R},$$

*has three distinct real roots, one positive and two negative, since $f$ is a continuous function with*

$$\lim_{\theta\to\infty} f(\theta) = \infty,\ f(0) < 0,\ f(-1/\mu) > 0 \text{ and } \lim_{\theta\to-\infty} f(\theta) = -\infty.$$

*Hence $f(\theta)$ can be rewritten as*

$$\begin{aligned} f(\theta) &= \sigma^2\Big[\theta^3 + \big(\frac{2c}{\sigma^2} + \frac{1}{\mu}\big)\theta^2 + \frac{2(c/\mu - q - \lambda)}{\sigma^2}\theta - \frac{2q/\mu}{\sigma^2}\Big] \\ &= \sigma^2(\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3). \end{aligned}$$

*Then*

$$\frac{1}{\Psi(\theta) - q} = \frac{2(\theta + 1/\mu)}{\sigma^2(\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3)}, \qquad \theta \geq 0, \tag{2.4}$$

*Using MATHEMATICA we see that one solution to $f(\theta)=0$ is*

$$\theta_1=\frac{\xi^{\frac{1}{3}}}{3\sigma^2}+\frac{6\sigma^2 q+6\sigma^2\lambda+4c^2+\sigma^4/\mu^2-2\sigma^2 c/\mu}{3\sigma^2\xi^{\frac{1}{3}}}-\frac{2c+\sigma^2/\mu}{3\sigma^2},$$

*where*

$$\xi=6\frac{\sigma^2c^2}{\mu}+3\frac{\sigma^4 c}{\mu^2}-18\sigma^2 qc+18\frac{q\sigma^4}{\mu}-18\sigma^2\lambda c-9\frac{\sigma^4\lambda}{\mu}-8c^3-\frac{\sigma^6}{\mu^3}+3\sigma^2\sqrt{\omega},$$

*and*

$$\begin{aligned}\omega \;=\;& -24qc^2\lambda+6\frac{\sigma^4c\lambda}{\mu^3}+24\frac{\sigma^4cq}{\mu^3}+24\frac{c^3\lambda}{\mu}-12\frac{\sigma^2c^2q}{\mu^2}-48\frac{\sigma^2c^2\lambda}{\mu^2}-48\frac{\sigma^2cq^2}{\mu}\\ &+60\frac{\sigma^2c\lambda^2}{\mu}-24\frac{c^3q}{\mu}+12\frac{\sigma^2cq\lambda}{\mu}-24\sigma^2q^3-24\sigma^2\lambda^3+12\frac{\sigma^2c^3}{\mu^3}-12\frac{c^4}{\mu^2}\\ &-3\frac{\sigma^4c^2}{\mu^4}-72\sigma^2q\lambda^2-6\frac{\sigma^6q}{\mu^4}-72\sigma^2q^2\lambda-12q^2c^2+24\frac{\sigma^4q^2}{\mu^2}\\ &-12\lambda^2c^2-3\frac{\sigma^4\lambda^2}{\mu^2}-60\frac{\sigma^4q\lambda}{\mu^2}.\end{aligned}$$

*Since $\theta_1$, $\theta_2$ and $\theta_3$ are roots of $f(\theta)=0$, they must satisfy the following relations*

$$\begin{aligned}\theta_1\theta_2\theta_3 &= \frac{2q/\mu}{\sigma^2},\\ \theta_1+\theta_2+\theta_3 &= -(\frac{2c}{\sigma^2}+\frac{1}{\mu}).\end{aligned}$$

*Solving the previous system of equations leads to the other two real roots:*

$$\begin{aligned}\theta_2 &= -\frac{1}{2}\left(\frac{2c}{\sigma^2}+\frac{1}{\mu}+\theta_1+\sqrt{\left(\frac{2c}{\sigma^2}+\frac{1}{\mu}+\theta_1\right)^2-\frac{8q/\mu}{\sigma^2\theta_1}}\right),\\ \theta_3 &= -\frac{1}{2}\left(\frac{2c}{\sigma^2}+\frac{1}{\mu}+\theta_1-\sqrt{\left(\frac{2c}{\sigma^2}+\frac{1}{\mu}+\theta_1\right)^2-\frac{8q/\mu}{\sigma^2\theta_1}}\right).\end{aligned}$$

*We can now look for the inverse Laplace transform of $\frac{1}{\Psi(\theta)-q}$.*

*The right hand side of (2.4) can be written as a summation of rational fractions. Assume that there exist three unknown $A$, $B$ and $C$ such that*

$$f_1(\theta)=\frac{\frac{2}{\sigma^2}\theta+\frac{2/\mu}{\sigma^2}}{(\theta-\theta_1)(\theta-\theta_2)(\theta-\theta_3)}=\frac{A}{\theta-\theta_1}+\frac{B}{\theta-\theta_2}+\frac{C}{\theta-\theta_3}. \tag{2.5}$$

*Multiplying* (2.5) *by* $(\theta - \theta_1)$ *and letting* $\theta = \theta_1$ *gives*

$$A = \frac{2(\theta_1 + 1/\mu)}{\sigma^2(\theta_1 - \theta_2)(\theta_1 - \theta_3)}. \tag{2.6}$$

*Similarly for* $B$ *and* $C$ *we get:*

$$B = \frac{2(\theta_2 + 1/\mu)}{\sigma^2(\theta_2 - \theta_1)(\theta_2 - \theta_3)}, \tag{2.7}$$

$$C = \frac{2(\theta_3 + 1/\mu)}{\sigma^2(\theta_3 - \theta_1)(\theta_3 - \theta_2)}. \tag{2.8}$$

*Hence, by* (2.4) *and* (2.5) *we have that*

$$\frac{1}{\Psi(\theta) - q} = \frac{A}{\theta - \theta_1} + \frac{B}{\theta - \theta_2} + \frac{C}{\theta - \theta_3}, \qquad \theta \geq 0, \tag{2.9}$$

*where* $A$*,* $B$ *and* $C$ *are given above.*

*Since the Laplace transform of an exponential function is given by*

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \frac{1}{s-a}, \qquad s > a,$$

*where* $a$ *is a constant, then the inverse Laplace transform of* $\frac{1}{\Psi(\theta)-q}$ *is given by*

$$\begin{aligned}
\mathcal{L}^{-1}\Big\{\frac{1}{\Psi(\theta) - q}\Big\} &= \mathcal{L}^{-1}\Big\{\frac{A}{\theta - \theta_1} + \frac{B}{\theta - \theta_2} + \frac{C}{\theta - \theta_3}\Big\}, \qquad \theta > \max(\theta_1, \theta_2, \theta_3), \\
&= \mathcal{L}^{-1}\Big\{\frac{A}{\theta - \theta_1}\Big\} + \mathcal{L}^{-1}\Big\{\frac{B}{\theta - \theta_2}\Big\} + \mathcal{L}^{-1}\Big\{\frac{C}{\theta - \theta_3}\Big\} \\
&= Ae^{\theta_1 x} + Be^{\theta_2 x} + Ce^{\theta_3 x}.
\end{aligned}$$

*Since the scale function* $W^{(q)}(x)$ *is defined as*

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\Psi(\theta) - q}, \qquad \theta \geq 0, \tag{2.10}$$

*from* (2.6)*-*(2.9) *we have*

$$\begin{aligned}
W^{(q)}(x) &= \frac{2(\theta_1 + 1/\mu)}{\sigma^2(\theta_1 - \theta_2)(\theta_1 - \theta_3)} e^{\theta_1 x} + \frac{2(\theta_2 + 1/\mu)}{\sigma^2(\theta_2 - \theta_1)(\theta_2 - \theta_3)} e^{\theta_2 x} \\
&\quad + \frac{2(\theta_3 + 1/\mu)}{\sigma^2(\theta_3 - \theta_1)(\theta_3 - \theta_2)} e^{\theta_3 x}.
\end{aligned}$$

*Note that the above expression is derived in Example 1.1 of Loeffen et al. (2014) for more general Lévy processes.*

*Inserting this closed form expression for* $W^{(q)}$ *back into* (2.2) *allows for the inversion of the Laplace transform* $\widehat{m}_\delta^{(2)}(u, \beta)$ *to obtain numerical values of the G-*$S_t$ *function* $m_\delta^{(2)}(u, t)$*.*

The above example opens the door to the possible use of the G-S$_t$ function as a measure for risk management on the surplus process $U$. The next section considers particular choices of the penalty function $w$, and a compound renewal surplus process, that could be useful in a risk management setting.

# 3 Special penalty functions for risk management

This section investigates further the finite-time Gerber-Shiu (G-S$_t$) function in (1.3), this time for three special cases of penalty function $w$ and for a surplus process $U_t = u + ct - S_t$, where the aggregate claims process is assumed of the form $S_t = \sum_{k=1}^{N_t} X_k$. Here the counting process $\{N_t\}_{t\geq 0}$ is an ordinary renewal process with claim arrival times denoted by $\{T_k\}_{k\geq 1}$ and iid claim inter-arrival times $\{\tau_k = T_k - T_{k-1}\}_{k\geq 1}$, with common probability density function (pdf) $f_\tau$ and cumulative distribution function (cdf) $F_\tau(x) = 1 - \overline{F}_\tau(x)$, where $T_0 = 0$. (By convention, $S_t = 0$ if $N_t = 0$.)

**Case 1**: Consider first the simple case of a constant penalty function $w(x, y) = 1$ for all $x, y > 0$. In this case, the finite-time G-S$_t$ function is denoted by $m_\delta^{(0)}(u, t)$ and defined for $\delta \geq 0$ as

$$m_\delta^{(0)}(u, t) = \mathbb{E}\big[e^{-\delta(t\wedge T)} \mid U_0 = u\big], \qquad u \geq 0,\ t > 0.$$

Clearly, the classical G-S function $m_\delta^{(0)}(u) = \lim_{t\to\infty} m_\delta^{(0)}(u, t)$. The following is an equivalent expression for the G-S$_t$ function $m_\delta^{(0)}(u, t)$, for $\delta \geq 0$, in terms of finite-time ruin probabilities

$$m_\delta^{(0)}(u, t) = \mathbb{E}\big[e^{-\delta T} I(T \leq t) \mid U_0 = u\big] + e^{-\delta t}[1 - \psi(u, t)], \qquad u \geq 0,\ t > 0. \quad (3.1)$$

Using the (defective) density $v(u, t) = \frac{\partial}{\partial t}\psi(u, t)$ of the random variable $T$, relation (3.1) for $\delta \geq 0$ becomes

$$\begin{aligned} m_\delta^{(0)}(u, t) &= \int_0^t e^{-\delta s}\frac{\partial}{\partial s}\psi(u, s)ds + e^{-\delta t}\big[1 - \psi(u, t)\big] \\ &= e^{-\delta s}\psi(u, s)\Big|_0^t + \delta\int_0^t e^{-\delta s}\psi(u, s)ds + e^{-\delta t}\big[1 - \psi(u, t)\big] \\ &= e^{-\delta t} + \delta\int_0^t e^{-\delta s}\psi(u, s)ds, \qquad u \geq 0,\ t > 0, \end{aligned} \quad (3.2)$$

after integration by parts.

Note that letting $t \to \infty$ in relation (3.2) gives the result of Gerber and Shiu (1998) for the G-S function, namely for $\delta > 0$

$$m_\delta^{(0)}(u) = \delta \int_0^\infty e^{-\delta s} \psi(u,s) ds, \qquad u \geq 0. \tag{3.3}$$

Using (3.2) we examine the impact of $u$, $t$ and $\delta$ on the finite-time G-S$_t$ function $m_\delta^{(0)}(u,t)$.

The larger the initial surplus, the less likely it is that ruin will occur in a finite-time period. Hence, by (3.2) it is obvious that the G-S$_t$ function $m_\delta^{(0)}(u,t)$ is decreasing with respect to $u$. So, like finite-time ruin probabilities, it can be used as a solvency measure, with the added advantage that $m_\delta^{(0)}(u)$ is a discounted value of a monetary unit (a sort of *numéraire*).

For given $u$ and $\delta > 0$, the function $m_\delta^{(0)}(u,t)$ is decreasing in $t \in (0,\infty)$ as shown by its partial derivative with respect to $t$:

$$\frac{\partial}{\partial t} m_\delta^{(0)}(u,t) = -\delta e^{-\delta t} \big[1 - \psi(u,t)\big] \leq 0.$$

Finally, to study the impact of $\delta$ on the G-S$_t$ function in (3.2) consider

$$\begin{aligned} \frac{\partial}{\partial \delta} m_\delta^{(0)}(u,t) &= -te^{-\delta t} + \int_0^t e^{-\delta s} \psi(u,s) ds - \delta \int_0^t s e^{-\delta s} \psi(u,s) ds \\ &= -te^{-\delta t} + \int_0^t e^{-\delta s} \psi(u,s) ds + s e^{-\delta s} \psi(u,s)\Big|_0^t \\ &\qquad - \int_0^t e^{-\delta s} \Big[\psi(u,s) + s \frac{\partial}{\partial s} \psi(u,s)\Big] ds \\ &= -te^{-\delta t} \big[1 - \psi(u,t)\big] - \int_0^t e^{-\delta s} s \frac{\partial}{\partial s} \psi(u,s) ds \leq 0, \end{aligned}$$

where the last inequality follows from the fact that the finite-time ruin probability $\psi(u,t)$ is increasing in $t \in (0,\infty)$. Therefore, for given $u$ and $t$, the G-S$_t$ function $m_\delta^{(0)}(u,t)$ is also decreasing in $\delta \in (0,\infty)$, a natural behaviour for a discounted value.

**Remark 3.1** *In the classical risk model the aggregate claims $S_t$ form a compound Poisson process with exponential claim sizes, and an explicit expression for $m_\delta^{(0)}(u,t)$, in* (3.1)*, can easily be derived using both the expression of the finite-time ruin probability [see Seal (1978)] and that of the expectation $\mathbb{E}\big[e^{-\delta T} I(T \leq t) \mid U_0 = u\big]$ established by Kočetova and Šiaulys (2010). Even in this special case numerical integration and numerical summation are still needed to compute these two quantities.*

By finding the Laplace transform of the G-S$_t$ function $m_\delta^{(0)}(u,t)$ and inverting it, values of $m_\delta^{(0)}(u,t)$ can be computed not only for the classical risk model, but also for some compound renewal processes $S_t$. Taking the Laplace transform of both sides of (3.2) in $t$, we obtain for $\delta \geq 0$

$$\begin{aligned} \widehat{m}_\delta^0(u,\beta) &= \int_0^\infty e^{-\beta t} m_\delta^{(0)}(u,t)dt = \frac{1}{\beta+\delta} + \delta \int_0^\infty e^{-\beta t} \int_0^t e^{-\delta s}\psi(u,s)dsdt \\ &= \frac{1}{\beta+\delta} + \frac{\delta}{\beta}\int_0^\infty e^{-(\beta+\delta)t}\psi(u,t)dt, \qquad u \geq 0. \end{aligned} \tag{3.4}$$

By relation (3.3), the Laplace transform given by (3.4) is equivalent for $\delta \geq 0$ to

$$\widehat{m}_\delta^{(0)}(u,\beta) = \frac{1}{\beta+\delta} + \frac{\delta}{\beta(\beta+\delta)} m_{\delta+\beta}^{(0)}(u), \qquad u \geq 0, \tag{3.5}$$

in terms of the G-S function $m_{\delta+\beta}^{(0)}(u)$. The latter is known for some models.

In recent years, the classical Gerber-Shiu function $m_\delta^{(2)}(u)$ has been extensively studied for various risk models: for the compound Poisson process (e.g. Gerber and Shiu (1998), Lin and Willmot (2000)), the renewal Erlang(2) process (e.g. Dickson and Hipp (2000, 2001), Cheng and Tang (2003)), the Erlang(n) renewal risk model considered by Li and Garrido (2004), the generalized Erlang renewal risk model investigated by Gerber and Shiu (2005), the $K_n$ family distribution for the claim inter-arrival times discussed by Li and Garrido (2005).

**Case 2**: Consider the case now where a negative surplus $U_T < 0$ generates a "penalty" of one monetary unit (a loss requiring an injection of capital, so an added surplus), while a positive surplus would trigger a payoff of one unit (a credit or withdrawal from the surplus, a negative penalty). That means a penalty function in relation (1.3) of:

$$w(x,y) = \begin{cases} 1 & \text{for } (x,y) \in \mathbb{R} \times (-\infty, 0) \\ -1 & \text{for } (x,y) \in \mathbb{R} \times [0,\infty) \end{cases}.$$

Therefore, the finite-time Gerber-Shiu function for the univariate penalty above, denoted here by $m_\delta^{(1)}(u,t)$, is for $\delta \geq 0$

$$m_\delta^{(1)}(u,t) = \mathbb{E}\big[e^{-\delta T} I(T \leq t) \mid U_0 = u\big] - e^{-\delta t}\big[1 - \psi(u,t)\big].$$

Note that for $\delta > 0$, $\lim_{t\to\infty} m_\delta^{(1)}(u,t) = m_\delta^{(0)}(u)$, so as in Case 1, the G-S$_t$ function converges to the usual infinite time G-S function.

Arguments similar to those used in Case 1 show that for $\delta \geq 0$, $m_\delta^{(1)}(u,t)$ can be written as:

$$m_\delta^{(1)}(u,t) = -e^{-\delta t} + 2e^{-\delta t}\psi(u,t) + \delta \int_0^t e^{-\delta s}\psi(u,s)ds, \qquad u \geq 0,\ t > 0. \tag{3.6}$$

Since the finite-time ruin probability $\psi(u,t)$ is decreasing in $u \in [0,\infty)$, by (3.6) we have that the G-S$_t$ function $m_\delta^{(1)}(u,t)$ is also a decreasing function of $u$. A desirable property for its use as a solvency measure.

For given $u$ and $\delta \geq 0$, we have that

$$\frac{\partial}{\partial t} m_\delta^{(1)}(u,t) = \delta e^{-\delta t}\big[1-\psi(u,t)\big] + 2e^{-\delta t}\frac{\partial}{\partial t}\psi(u,t) \geq 0,$$

as $\psi(u,t)$ is increasing in $t \in (0,\infty)$. Thus, $m_\delta^{(1)}(u,t)$ is also an increasing function of $t$.

Assume now that, for given $u \geq 0$ and $t > 0$, the function $m_\delta^{(1)}(u,t)$ is non-positive on some interval $(\delta_1,\ \delta_2)$, with $0 < \delta_1 \leq \delta \leq \delta_2 < \infty$. Thus

$$e^{-\delta t} \geq 2e^{-\delta t}\psi(u,t) + \delta \int_0^t e^{-\delta s}\psi(u,s)\,ds.$$

Using this inequality, we further obtain

$$\begin{aligned}
\frac{\partial}{\partial \delta} m_\delta^{(1)}(u,t) &= te^{-\delta t} - 2te^{-\delta t}\psi(u,t) + \int_0^t e^{-\delta s}\psi(u,s)\,ds - \delta\int_0^t se^{-\delta s}\psi(u,s)\,ds \\
&\geq 2te^{-\delta t}\psi(u,t) + \delta t\int_0^t e^{-\delta s}\psi(u,s)\,ds - 2te^{-\delta t}\psi(u,t) \\
&\qquad + \int_0^t e^{-\delta s}\psi(u,s)\,ds - \delta\int_0^t se^{-\delta s}\psi(u,s)\,ds \\
&= \int_0^t e^{-\delta s}\psi(u,s)\,ds + \delta\Big[t\int_0^t e^{-\delta s}\psi(u,s)\,ds - \int_0^t se^{-\delta s}\psi(u,s)\,ds\Big] \\
&\geq 0.
\end{aligned}$$

We conclude that $m_\delta^{(1)}(u,t)$ is an increasing function of $\delta$, under the above assumptions. Again, a natural behaviour for a negative discounted value (when ruin does not occur). Note that the finite-time ruin probabilities in (3.6) are bounded by $\psi(u)$, the ultimate ruin probability. Hence we get that $\psi(u) \leq \big[1-\psi(u)\big]e^{-\delta t}$, or equivalently, $\psi(u) \leq (1+e^{\delta t})^{-1}$ is a sufficient condition for (3.6) to be non-positive. This is not a restrictive condition for small $\delta$, which is the usual case for a force of interest.

Taking a Laplace transform of the G-S$_t$ function in (3.6) with respect to $t$, we obtain for $\delta \geq 0$

$$\begin{aligned}
\widehat{m}^{(1)}_{\delta}(u,\beta) &= -\int_0^\infty e^{-(\beta+\delta)t}dt + 2\int_0^\infty e^{-(\beta+\delta)t}\psi(u,t)dt \\
&\qquad +\delta\int_0^\infty e^{-\beta t}\int_0^t e^{-\delta s}\psi(u,s)dsdt \\
&= -\frac{1}{\beta+\delta} + 2\int_0^\infty e^{-(\beta+\delta)t}\psi(u,t)dt + \frac{\delta}{\beta}\int_0^\infty e^{-(\beta+\delta)t}\psi(u,t)dt \\
&= -\frac{1}{\beta+\delta} + \frac{2\beta+\delta}{\beta(\beta+\delta)}m^{(0)}_{\beta+\delta}(u), \qquad u \geq 0, \qquad (3.7)
\end{aligned}$$

by (3.3). Therefore, inverting numerically the Laplace transform in (3.7), gives values of the G-S$_t$ function $m^{(1)}_{\delta}(u,t)$.

**Case 3**: Finally, consider the more volatile case where a negative surplus $U_T < 0$ triggers a penalty equal to the deficit amount (a ruin requiring an injection of capital to get back to positive assets), while a positive surplus triggers a payoff equal to the full surplus (a credit or withdrawal from the surplus, a negative penalty). That means a penalty function in (1.3) given by:

$$w(x,y) = \left\{ \begin{array}{ll} |y| & \text{for } (x,y) \in \mathbb{R}\times(-\infty,0) \\ -y & \text{for } (x,y) \in \mathbb{R}\times[0,\infty). \end{array} \right\} = -y.$$

Then, the finite-time G-S$_t$ function for this univariate penalty function, also denoted by $m^{(1)}_{\delta}(u,t)$, becomes

$$m^{(1)}_{\delta}(u,t) = \mathbb{E}\big[e^{-\delta T}|U_T|I(T\leq t) \mid U_0 = u\big] - e^{-\delta t}\mathbb{E}\big[U_t I(T>t) \mid U_0 = u\big]. \quad (3.8)$$

In view of the fact that $\Big|e^{-\delta t}\mathbb{E}\big[U_t I(T>t) \mid U_0 = u\big]\Big| \leq te^{-\delta t}\mathbb{E}\big[|U_t|/t \mid U_0 = u\big]$ for $\delta > 0$ and of the net profit condition, letting $t \to \infty$ in (3.8) yields

$$\lim_{t\to\infty} m^{(1)}_{\delta}(u,t) = \mathbb{E}\big[e^{-\delta T}|U_T|I(T<\infty) \mid U_0 = u\big] = m^{(1)}_{\delta}(u), \qquad u \geq 0,$$

which is the classical Gerber-Shiu function in (1.2).

Let us denote by $g^1_{\delta}(u,t) := \mathbb{E}\big[e^{-\delta T}|U_T|I(T\leq t) \mid U_0 = u\big]$ and $g^2_{\delta}(u,t) := e^{-\delta t}\mathbb{E}\big[U_t I(T>t) \mid U_0 = u\big]$. Therefore, relation (3.8) is equivalent to

$$m^{(1)}_{\delta}(u,t) = g^1_{\delta}(u,t) - g^2_{\delta}(u,t).$$

Now, consider the problem of computing both $g^1_{\delta}(u,t)$ and $g^2_{\delta}(u,t)$ numerically.

For the first term $g^1_{\delta}(u,t)$ we can derive an expression of its Laplace transform with respect to $t$. If, for given $U_0 = u$, the joint probability density function of $|U_T|$ and $T$ exists and is denoted by $h(x, s \mid u)$, then for $\delta \geq 0$ this Laplace transform is given by

$$\begin{aligned}
\widehat{g}^1_\delta(u,\beta) &= \int_0^\infty e^{-\beta t}\Big[\int_0^t \int_0^\infty e^{-\delta s} x h(x,s \mid u) dx ds\Big] dt \\
&= \frac{-e^{-\beta t}}{\beta}\int_0^t \int_0^\infty e^{-\delta s} x h(x,s \mid u) dx ds\Big|_{t=0}^{t=\infty} \\
&\quad + \frac{1}{\beta}\int_0^\infty \int_0^\infty e^{-(\beta+\delta)t} x h(x,t \mid u) dx dt \\
&= \frac{1}{\beta}\mathbb{E}\big[e^{-(\beta+\delta)T}|U_T| I(T<\infty) \mid U_0 = u\big] = \frac{1}{\beta} m^{(1)}_{\beta+\delta}(u),
\end{aligned}$$

which is the classical G-S function in (1.2).

For the second term $g^2_\delta(u,t)$ we derive an explicit formula. The claim sizes $\{X_k\}_{k\geq 1}$ are assumed iid, positive, with common cdf $F_X$ and finite mean $\mu$. Define the partial sums $Y_n = X_1 + \ldots + X_n$, for $n \geq 1$, and assume that $h(y_1, \ldots, y_n)$ is the joint density of $(Y_1, ..., Y_n)$ such that $h(y_1, \ldots, y_n) \geq 0$, when $0 \leq y_1 \leq \ldots \leq y_n$, and

$$\int \ldots \int_{0 \leq y_1 \leq \ldots \leq y_n} h(y_1, \ldots, y_n) dy_1 \ldots dy_n = 1.$$

It is further assumed that $\{X_k\}_{k\geq 1}$ and $\{N_t\}_{t\geq 0}$ are mutually independent.

The following proposition gives an explicit expression for $g^2_\delta(u,t)$.

**Proposition 3.1** *Assume that the surplus process $U_t$ is defined by an ordinary renewal claim arrival process. Then, for $\delta \geq 0$*

$$\begin{aligned}
g^2_\delta(u,t) &= e^{-\delta t}\mathbb{E}\big[U_t I(T>t) \mid U_0 = u\big] \\
&= e^{-\delta t}(u+ct)\overline{F}_\tau(t) + e^{-\delta t}\sum_{n=1}^\infty \int_0^{u+ct}\int_{y_1}^{u+ct} \ldots \int_{y_{n-1}}^{u+ct} (u+ct-y_n) \\
&\qquad h(y_1,\ldots,y_n)\int_{\frac{y_1-u}{c}}^t \int_{\max\{\frac{y_2-u}{c},t_1\}}^t \ldots \int_{\max\{\frac{y_n-u}{c},t_{n-1}\}}^t \overline{F}_\tau(t-t_n) \\
&\qquad \prod_{k=1}^n f_\tau(t_k - t_{k-1}) dt_n \ldots dt_2 dt_1 dy_n \ldots dy_2 dy_1,
\end{aligned}$$

*where $y_0 = 0$ and $t_0 = 0$.*

**Proof:** We have that

$$\begin{aligned}
g_\delta^2(u,t) &= e^{-\delta t}\mathbb{E}\big[U_t I(T>t) \mid U_0=u\big] \\
&= e^{-\delta t}\sum_{n=0}^{\infty}\mathbb{P}\{N(t)=n\}\,\mathbb{E}\big[(u+ct-\sum_{k=1}^{n}X_k)I(T>t)\mid N(t)=n\big] \\
&= e^{-\delta t}(u+ct)\mathbb{P}\{N(t)=0\}+e^{-\delta t}\sum_{n=1}^{\infty}\mathbb{P}\{N(t)=n\}\,\mathbb{E}\Big[(u+ct-\sum_{k=1}^{n}X_k) \\
&\qquad I\big\{(T>t)\cap(T_n\le t<T_{n+1})\big\}\mid N(t)=n\Big], \qquad (3.9)
\end{aligned}$$

as $\{N(t)=n\}=\{T_n\le t<T_{n+1}\}$ and $\mathbb{E}\big[XI(A)\mid B\big]=\mathbb{E}\big[XI(A)I(B)\mid B\big]$, for any events $A$, $B$ and random variable $X$. Using the following identity established by Ignatov and Kaishev (2004)

$$(T>t)\cap(T_n\le t<T_{n+1})=\bigcap_{j=1}^{n}\Big(\big(\frac{X_1+\ldots+X_j-u}{c}<T_j\big)\cap(T_n\le t<T_{n+1})\Big),$$

we get that the conditional expectation from (3.9) is then

$$\begin{aligned}
&\mathbb{E}\Big[(u+ct-\sum_{k=1}^{n}X_k)I\big((T>t)\cap(T_n\le t<T_{n+1})\big)\mid N(t)=n\Big] \\
&=\mathbb{E}\Big[(u+ct-\sum_{k=1}^{n}X_k)I\Big(\bigcap_{j=1}^{n}\big(\frac{X_1+\ldots+X_j-u}{c}<T_j\big)\Big)\mid N(t)=n\Big] \\
&=\int_0^{u+ct}\int_{y_1}^{u+ct}\cdots\int_{y_{n-1}}^{u+ct}(u+ct-y_n)h(y_1,\ldots,y_n)\int_{\frac{y_1-u}{c}}^{t}\int_{\max\{\frac{y_2-u}{c},t_1\}}^{t} \\
&\quad\cdots\int_{\max\{\frac{y_n-u}{c},t_{n-1}\}}^{t}\frac{\overline{F}_\tau(t-t_n)f_\tau(t_1)f_\tau(t_2-t_1)\ldots f_\tau(t_n-t_{n-1})}{\mathbb{P}\{N(t)=n\}} \\
&\quad dt_n\ldots dt_2dt_1dy_n\ldots dy_2dy_1. \qquad (3.10)
\end{aligned}$$

The last step of (3.10) uses the fact that, given $N(t)=n$, the conditional joint pdf of $0\le T_1\le\ldots\le T_n\le t$ can be expressed as

$$f_{T_1,T_2,\ldots,T_n|N(t)}(t_1,t_2,\ldots,t_n\mid n)=\frac{\mathbb{P}\{N(t-t_n)=0\}\prod_{k=1}^{n}f_\tau(t_k-t_{k-1})}{\mathbb{P}\{N(t)=n\}},$$

with $t_0=0$ (see Léveillé and Adékambi, 2011). Then (3.9) and (3.10) complete the proof. ■

**Remark 3.2** *In the special case when the claims arrive as a Poisson process with rate $\lambda$, it is well-known that, conditionally on the event $\{N(t)=n\}$, the* *joint distribution of the $n$ arrival times $T_1, \ldots, T_n$ is the same as that of the order statistics of $n$ iid uniformly distributed random variables $U_1, \ldots, U_n$ in the interval $(0,t)$ (see Karlin and Taylor, 1981). More specifically, the conditional joint pdf is given by*

$$f_{T_1,T_2,\ldots,T_n|N(t)}(t_1,t_2,\ldots,t_n \mid n) = \begin{cases} \frac{n!}{t^n} & \text{if } 0 \le t_1 \le \ldots \le t_n \le t \\ 0 & \text{otherwise.} \end{cases}$$

*Therefore, the result in the proposition above reduces to*

$$g_\delta^2(u,t) = e^{-(\lambda+\delta)t}\big[u + ct + \sum_{n=1}^{\infty} \lambda^n \int_0^{u+ct} \int_{y_1}^{u+ct} \cdots \int_{y_{n-1}}^{u+ct} (u + ct - y_n)$$
$$h(y_1,\ldots,y_n) \int_{\frac{y_1-u}{c}}^{t} \int_{\max\{\frac{y_2-u}{c},t_1\}}^{t} \cdots \int_{\max\{\frac{y_n-u}{c},t_{n-1}\}}^{t} dt_n \ldots dt_2 dt_1 dy_n \ldots dy_2 dy_1\big].$$

*Moreover, Ignatov and Kaishev (2004) showed that*

$$\int_{\frac{y_1-u}{c}}^{t} \int_{\max\{\frac{y_2-u}{c},t_1\}}^{t} \cdots \int_{\max\{\frac{y_n-u}{c},t_{n-1}\}}^{t} dt_n \ldots dt_2 dt_1 = A_n(t, \frac{y_1-u}{c}, \ldots, \frac{y_n-u}{c}),$$

*where $A_n(t; w_1, \ldots, w_n)$ are, for $n \ge 1$, the classical Appell polynomials*

$$A_n(t; w_1, \ldots, w_n) = \frac{1}{n!}t^n + a_1 t^{n-1} + \ldots + a_{n-1}t + a_n,$$

*and defined by the following system of $n$ equations containing the first $(n-1)$ derivatives of $A_n(t)$:*

$$A_n(w_n) = 0,\ A_n'(w_{n-1}) = 0, \ldots, A_n^{(n-1)}(w_1) = 0.$$

# 4 Numerical examples and applications

In this section, we provide some numerical values of the finite-time Gerber-Shiu function for each of the first two special cases discussed in Section 3. For this, we adopt Gaver-Stehfest algorithm in order to invert numerically the Laplace transforms. The Gaver-Stehfest algorithm (see Gaver (1966), Stehfest (1970)) is a popular numerical inversion technique for inverting Laplace transforms. More specifically, if $\widehat{f}(\beta) = \int_0^\infty f(t)e^{-\beta t}dt$ is the Laplace transform of $f$, then by the inversion formula we can compute a series approximation of $f$ of the following form:

$$f(t) \approx \frac{\ln 2}{t} \sum_{n=1}^{2M} c_n \widehat{f}(\frac{n \ln 2}{t}), \tag{4.1}$$

where the coefficient $c_n$ is given by

$$c_n = (-1)^{n+M} \sum_{k=[(n+1)/2]}^{\min(n,M)} \frac{k^M (2k)!}{(M-k)!\, k!\, (k-1)!\, (n-k)!\, (2k-n)!},$$

with $[x]$ being the greatest integer less than or equal to $x$.

As pointed out by Stehfest (1970), theoretically the accuracy of formula (4.1) increases with the value of $M$. However, practically $c_n$ takes ever greater absolute values when $M$ becomes too large, and then rounding errors worsen the results. Therefore, the Gaver-Stehfest algorithm requires high system precision for a good accuracy in the computations. In this sense, Abate and Valko (2004), Abate and Whitt (2006) suggested that for $j$ desired significant digits, $M$ should be the positive integer $[1.1j]$, and for given $M$, the system precision should be set at $[2.2M]$. Also, from Abate and Whitt (2006), the weights in (4.1) satisfy the constraint $\sum_{n=1}^{2M} c_n = 0$. For more details on this algorithm, we refer the reader to Abate and Whitt (2006).

In our illustrations, we use $M$ values ranging from 5 to 12, the choice representing the optimum $M$ for all numerical results of the tables given below. All the calculations were carried out with the software MATHEMATICA.

As illustrative examples of the aggregate claim process $S_t$, we consider the compound Poisson process and the compound renewal Erlang(2) process. The claim amounts are assumed to be exponentially distributed with probability density function (pdf) given by $f(x) = \mu e^{-\mu x}$, for $x, \mu > 0$.

Following Gerber and Shiu (1998), in the case of a compound Poisson process with rate $\lambda$, the expected discounted penalty function $m_\delta^{(0)}(u)$ has the following form

$$\begin{aligned} m_\delta^{(0)}(u) &= \mathbb{E}\big[e^{-\delta T} I(T<\infty) \mid U_0 = u\big] \\ &= \frac{\lambda+\delta+\mu c - \sqrt{(\lambda+\delta+\mu c)^2 - 4\lambda\mu c}}{2c\mu} \\ &\qquad \times \exp\Big\{u\Big[\frac{\lambda+\delta-\mu c - \sqrt{(\lambda+\delta+\mu c)^2 - 4\lambda\mu c}}{2c}\Big]\Big\}. \end{aligned} \tag{4.2}$$

For a renewal Erlang(2) process, the claim inter-arrival times, $\{\tau_i\}_{i\geq 1}$, follow an Erlang (2, $\gamma$) distribution with pdf given by $k(x) = \gamma^2 x e^{-\gamma x}$, for $x, \gamma > 0$. Note that, for each $i$, $\mathbb{E}(\tau_i) = 2/\gamma$ and $\mathbb{V}(\tau_i) = 2/\gamma^2$. From Dickson and Hipp (2001), the expected discounted penalty function $m_\delta^{(0)}(u)$ is expressed as

$$m_\delta^{(0)}(u) = \Big(1 - \frac{R}{\mu}\Big) e^{-Ru}, \tag{4.3}$$

where $-R$ is the unique negative root of Lundberg's fundamental equation:

$$\left(\frac{\gamma}{\gamma+\delta-cs}\right)^2\frac{\mu}{\mu+s}=1.$$

By substituting formulas (4.2) and (4.3) into both (3.5) and (3.7), we proceed to invert numerically the Laplace transforms of the finite-time Gerber-Shiu functions, in both Cases 1 and 2. To this purpose, assume a mean claim size of $\mathbb{E}(X)=1/\mu=1$, an arrival rate $\lambda=100$, and a scale parameter $\gamma=200$, so that the mean inter-arrival time is 0.01 in all cases.

For a wide range of different $t$ values, Table 1 reports $m_\delta^{(0)}(u,t)$ values (Case 1) when the force of interest $\delta=0.01, 0.1$ and 0.3, and the premium loading factor $\theta=0$ and 0.5. The initial surplus is assumed to be $u=100$. For these same choices of parameters, Table 2 then reports values of $m_\delta^{(1)}(u,t)$ (Case 2).

The results in Table 1 confirm that increasing either $t$ or $\delta$ leads to decreasing finite-time Gerber-Shiu function values $m_\delta^{(0)}(u,t)$, which is consistent with the results illustrated by Kočetova and Šiaulys (2010) or Kuznetsov and Morales (2014) in this special case of a constant penalty function. Similarly for Table 2, where increasing $t$ yields decreasing $m_\delta^{(1)}(u,t)$ values.

We observe that $m_\delta^{(0)}(u,t)$, for given $u,t,c$, and $\delta$ takes smaller values for the Erlang(2) process than for the compound Poisson process (Case 1), while the opposite is true for $m_\delta^{(1)}(u,t)$ (Case 2). This is due to (3.2) and (3.6), respectively, and the fact that as we move from a Poisson to an Erlang(2) process, keeping the mean inter-arrival constant, the variance of the claim inter-arrival times decreases, and therefore, $\psi(u,t)$ decreases. Additionally, we note that in each table, convergence to the ultimate ruin probability is faster as $\delta$ increases.

Tables 3 and 4 report values of $m_\delta^{(0)}(u,t)$ and $m_\delta^{(1)}(u,t)$, respectively, when $u=25, 100$ and 200. Both properties, $m_\delta^{(0)}(u,t)$ being a decreasing function of $u$ and $m_\delta^{(1)}(u,t)$ an increasing function of $u$, are clearly illustrated by the results in these tables.

# Conclusion

Adapting the definition of the Gerber-Shiu function to a finite time horizon opens the door to its possible use as a risk measure on the losses associated with a surplus process $U$.

In Section 2 we derive an expression for the Laplace transform of the finite-time Gerber-Shiu function (G-S$_t$) for spectrally negative Lévy surplus processes $U$ and general penalty functions $w$, in terms of the corresponding scale function. The latter is derived in closed form for some special cases, including a compound Poisson surplus process with exponential claims perturbed by Brownian motion.

Then Section 3 investigates further the G-S$_t$ function for three special cases of penalty function $w$ and compound renewal surplus processes.

Illustrative examples in Section 4 give values of the G-S$_t$ function for two of the three special cases of penalty function $w$ in Section 3, and surplus processes based on compound Poisson and compound renewal Erlang(2) aggregate claims with exponential severities. These show that the G-S$_t$ function can be computed numerically for different surplus models, and that it produces values in accordance with what is expected of a useful solvency index.

Future research will have to extend the set of surplus processes for which the G-S$_t$ function can be easily computed numerically. More importantly, to use of the G-S$_t$ function as a risk measure for purposes such as establishing capital (surplus) requirements, will necessitate the study of its properties, like homogeneity and sub-additivity, as a function of the penalty $w$.

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Table 1: Values of $m_{\delta}^{(0)}(u,t)$ when $u = 100$ (Case 1)

| | $\delta = 0.01$ | $\delta = 0.1$ | $\delta = 0.3$ | $\delta = 0.01$ | $\delta = 0.1$ | $\delta = 0.3$ |
|---|---|---|---|---|---|---|
| | $(\theta = 0)$ | $(\theta = 0)$ | $(\theta = 0)$ | $(\theta = 0.5)$ | $(\theta = 0.5)$ | $(\theta = 0.5)$ |
| $\tau_i \sim Exp(100)$ | | | | | | |
| $t = 1$ | 0.9900498337 | 0.9048374180 | 0.7408182257 | 0.9900498337 | 0.9048374180 | 0.7408182206 |
| $t = 2$ | 0.9801986841 | 0.8187308438 | 0.5488118246 | 0.9801986733 | 0.8187307531 | 0.5488116361 |
| $t = 5$ | 0.9512472071 | 0.6066510755 | 0.2354885791 | 0.9512294250 | 0.6065306597 | 0.2231301601 |
| $t = 10$ | 0.9054378703 | 0.3707724932 | 0.0516174708 | 0.9048374180 | 0.3678794411 | 0.0497870683 |
| $t = 20$ | 0.8253015571 | 0.1528414483 | 0.0067138541 | 0.8187307531 | 0.1353352832 | 0.0024787521 |
| $t = 100$ | 0.5165855596 | 0.0431489596 | 0.0046118278 | 0.3678794411 | 0.0000453999 | $9.5072 \cdot 10^{-14}$ |
| $t = \infty$ | 0.36604007 | 0.04309732 | 0.00458922 | $2.1955 \cdot 10^{-15}$ | $1.9453 \cdot 10^{-15}$ | $1.4934 \cdot 10^{-15}$ |
| $\tau_i \sim Erlang(2, 200)$ | | | | | | |
| $t = 1$ | 0.9900498334 | 0.9048374178 | 0.7408182238 | 0.9900498336 | 0.9048374108 | 0.7408182206 |
| $t = 2$ | 0.9801986799 | 0.8187308156 | 0.5488117056 | 0.9801986730 | 0.8187307530 | 0.5488116360 |
| $t = 5$ | 0.9512351849 | 0.6065698712 | 0.2234449933 | 0.9512294245 | 0.6065306597 | 0.2231234898 |
| $t = 10$ | 0.9050449704 | 0.3687721188 | 0.0509775111 | 0.9048374180 | 0.3678794410 | 0.0497870683 |
| $t = 20$ | 0.8214209852 | 0.1389334699 | 0.0045473923 | 0.8187307530 | 0.1353352832 | 0.0024787520 |
| $t = 100$ | 0.4885104473 | 0.0357933422 | 0.0028710093 | 0.3678794401 | 0.0000453997 | $9.3422 \cdot 10^{-14}$ |
| $t = \infty$ | 0.31326114 | 0.02643952 | 0.00198283 | $1.9852 \cdot 10^{-19}$ | $1.7698 \cdot 10^{-19}$ | $1.3759 \cdot 10^{-19}$ |

Table 2: Values of $m_\delta^{(1)}(u,t)$ when $u = 100$ (Case 2)

| | $\delta = 0.01$ | $\delta = 0.1$ | $\delta = 0.3$ | $\delta = 0.01$ | $\delta = 0.1$ | $\delta = 0.3$ |
|---|---|---|---|---|---|---|
| | $(\theta = 0)$ | $(\theta = 0)$ | $(\theta = 0)$ | $(\theta = 0.5)$ | $(\theta = 0.5)$ | $(\theta = 0.5)$ |
| $\tau_i \sim Exp(100)$ | | | | | | |
| $t = 1$ | -0.9900498209 | -0.9048374101 | -0.7408182148 | -0.9900498337 | -0.9048374180 | -0.7408182206 |
| $t = 2$ | -0.9801875632 | -0.8187213804 | -0.5488052092 | -0.9801986733 | -0.8187307530 | -0.5488116360 |
| $t = 5$ | -0.9468304904 | -0.6036165854 | -0.1341539861 | -0.9512294245 | -0.6065306597 | -0.2231301601 |
| $t = 10$ | -0.8544889022 | -0.3447597680 | -0.0452193208 | -0.9048374180 | -0.3678794411 | -0.0497870683 |
| $t = 20$ | -0.6243959931 | -0.0868010945 | 0.0023213987 | -0.8187307530 | -0.1353352810 | -0.0024787521 |
| $t = 100$ | 0.1329156173 | 0.0402147621 | 0.0044293042 | -0.3678794411 | -0.0000453999 | $-9.2078 \cdot 10^{-14}$ |
| $t = \infty$ | 0.36604007 | 0.04309732 | 0.00458922 | $2.1955 \cdot 10^{-15}$ | $1.9453 \cdot 10^{-15}$ | $1.4934 \cdot 10^{-15}$ |
| $\tau_i \sim Erlang(2, 200)$ | | | | | | |
| $t = 1$ | -0.9900498214 | -0.9048374145 | -0.7408182162 | -0.9900498337 | -0.9048374180 | -0.7408182207 |
| $t = 2$ | -0.9801940453 | -0.8187279852 | -0.5488106631 | -0.9801986755 | -0.8187307531 | -0.5488116361 |
| $t = 5$ | -0.9507429669 | -0.6054339774 | -0.2206685368 | -0.9512294245 | -0.60653065971 | -0.2231308296 |
| $t = 10$ | -0.8841592379 | -0.3592655502 | -0.0497870683 | -0.9048374180 | -0.3678794412 | -0.0497870684 |
| $t = 20$ | -0.7059748656 | -0.1004232935 | -0.00136864908 | -0.8187307530 | -0.1353352832 | -0.0024787522 |
| $t = 100$ | 0.1049499200 | 0.0134729233 | 0.0003302024 | -0.3678794425 | -0.0000454181 | $-9.3564 \cdot 10^{-14}$ |
| $t = \infty$ | 0.31326114 | 0.02643952 | 0.00198283 | $1.9852 \cdot 10^{-19}$ | $1.7698 \cdot 10^{-19}$ | $1.3759 \cdot 10^{-19}$ |

Table 3: Values of $m_\delta^{(0)}(u,t)$ when $\delta = 0.1$ and $\theta = 0$ (Case 1)

| | $u = 25$ | $u = 100$ | $u = 200$ |
|---|---|---|---|
| $\tau_i \sim Exp(100)$ | | | |
| $t = 1$ | 0.9073683225 | 0.904837418044 | 0.904837418035 |
| $t = 2$ | 0.8338861128 | 0.8187308438 | 0.8187307530 |
| $t = 5$ | 0.6899626771 | 0.6066510755 | 0.6065306603 |
| $t = 10$ | 0.5693628256 | 0.3707724932 | 0.3678801314 |
| $t = 20$ | 0.4813910664 | 0.1528414483 | 0.1354375161 |
| $t = 100$ | 0.4453045653 | 0.0431489596 | 0.0019542421 |
| $t = \infty$ | 0.4449516345 | 0.0430973236 | 0.0019170508 |
| $\tau_i \sim Erlang(2, 200)$ | | | |
| $t = 1$ | 0.9060685584 | 0.9048374178 | 0.9048374175 |
| $t = 2$ | 0.8268284042 | 0.8187308156 | 0.8187307530 |
| $t = 5$ | 0.6499402720 | 0.6065698712 | 0.6065306598 |
| $t = 10$ | 0.5211814554 | 0.3687721188 | 0.3678798939 |
| $t = 20$ | 0.4615819687 | 0.1389334699 | 0.1353243393 |
| $t = 100$ | 0.4236418922 | 0.0357933422 | 0.0009388751 |
| $t = \infty$ | 0.3923139793 | 0.02643952 | 0.0007251258 |

Table 4: Values of $m_\delta^{(1)}(u,t)$ when $\delta = 0.1$ and $\theta = 0$ (Case 2)

| | $u = 25$ | $u = 100$ | $u = 200$ |
|---|---|---|---|
| $\tau_i \sim Exp(100)$ | | | |
| $t = 1$ | -0.7526272826 | -0.9048374101 | -0.9048374180 |
| $t = 2$ | -0.4643389212 | -0.8187213804 | -0.8187307530 |
| $t = 5$ | -0.0176502453 | -0.6036165854 | -0.6065306579 |
| $t = 10$ | 0.2487608556 | -0.3447597680 | -0.3678655765 |
| $t = 20$ | 0.3941865744 | -0.0868010945 | -0.1347129484 |
| $t = 100$ | 0.4432926501 | 0.0402147621 | 0.0018791663 |
| $t = \infty$ | 0.44495163 | 0.04309732 | 0.00191705 |
| $\tau_i \sim Erlang(2, 200)$ | | | |
| $t = 1$ | -0.8759979634 | -0.9048374145 | -0.9048374180 |
| $t = 2$ | -0.6663273556 | -0.8187279852 | -0.8187307532 |
| $t = 5$ | -0.3966999231 | -0.6054339774 | -0.6065308021 |
| $t = 10$ | -0.0305926838 | -0.359265502 | -0.3679384749 |
| $t = 20$ | 0.2369040685 | -0.1004232935 | -0.1353956006 |
| $t = 100$ | 0.3525148814 | 0.0134729233 | -0.0048754247 |
| $t = \infty$ | 0.39231397 | 0.02643952 | 0.00072512 |