ANALYZING EQUITY-INDEXED ANNUITY USING LEE-CARTER MODEL

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A THESIS

IN THE DEPARTMENT
OF
MATHEMATICS AND STATISTICS

PRESENTED IN THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE AT
CONCORDIA UNIVERSITY
MONTREAL, QUEBEC, CANADA
APRIL 2010

Your file Votre référence
ISBN: 978-0-494-67141-2
Our file Notre référence
ISBN: 978-0-494-67141-2

ABSTRACT

Analyzing Equity-Indexed Annuities

Using the Lee-Carter Stochastic Mortality Model

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Equity-indexed annuity (EIA) insurance products have become increasingly sought after since their introduction in 1995. Some of the most important characteristics of these products are that they allow the policyholders to benefit from the equity market's potential growth and ensure that the principals can grow with a minimum guaranteed interest rate.

In this thesis, we show how to derive the closed-form pricing formula of a point-to-point financial guarantee, using the Black-Scholes framework. Moreover, under the complete-market assumption, we construct a replicating portfolio that can hedge a point-to-point financial guarantee.

However, in real financial markets, some of the assumptions required by a complete-market cannot be respected, particularly the continuous-time trading assumption. The replicating portfolio generates hedging errors because companies can only trade discretely. We will show the distribution of the present values of hedging errors for the financial guarantee.

We also introduce the Lee-Carter stochastic mortality model. After presenting how to price a point-to-point equity-indexed annuity with fixed mortality rates, we then take the stochastic mortality rates into consideration to re-evaluate the point-to-point. In both cases, the pricing work for the point-to-point product is done under the assumption of independence between the equity market and the policyholder's time of death.

Furthermore, the replicating portfolio of a point-to-point equity-indexed annuity can be derived based on the replicating portfolio of a point-to-point financial guarantee with the corresponding mortality rates. The distributions of the present values of hedging errors under both fixed and stochastic mortality rates will be presented. Indeed, the replicating portfolio can help companies reduce the risks of issuing EIA products, since it can hedge the EIA very well. The impact of stochastic mortality model is examined at the end of the thesis.

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Introduction

Since their introduction to the U.S market in 1995, the equity-indexed annuities (EIAs) have become increasingly popular. The sales of these products increased over 50% from 2003 to 2004. Moreover, despite the downturn of financial markets in recent years, over 125 billions of EIAs were sold in 2008, see http://www.indexannuity.org.

In addition to the traditional tax deferred advantage offered by most fixed annuities, the EIAs provide the policyholders the opportunity to enjoy the equity market's potential growth while protecting most of their initial premiums.

EIAs have been a popular topic of research since their invention. Brennan and Schwartz (1976), Boyle and Schwartz (1977) were the first to extend the Black-Scholes framework (Black & Scholes, 1973, and Merton, 1973) to equity-linked insurance products. Since then, a lot of research have been done on the subject of EIAs. Hardy (2003) discusses the properties and pricing scheme for EIA life insurance products. Tiong (2000) and Lee (2003) have obtained closed-form formulas for several EIAs. Lin and Tan (2003) consider a more general model for EIAs.

Furthermore, Gaillardetz and Lin (2006) evaluate EIAs in a discrete time framework and relax the mortality diversification assumption under constant interest rates. Gaillardetz (2007) illustrates the EIAs valuation using a stochastic interest rate model.

In this thesis, we focus on the point-to-point (PTP) equity-indexed annuity product. Furthermore, under the complete-market assumption, we can construct a replicating portfolio containing both risky assets and riskless assets to hedge the underlying PTP contracts. Because we can not meet the requirement of trading continuously, we assume that companies will rebalance the replicating portfolio m times per year, and hedging errors will occur at the rebalancing times. We will show the distribution of the present values of hedging errors, obtained through a large number of simulations. Moreover, although most papers assume fixed mortality rates while pricing EIAs, we will explain how a stochastic mortality model effects the hedging errors.

In Chapter 1, we introduce the Black-Scholes financial model and commonly used actuarial notation. We also present how to obtain the Lee-Carter (LC) model parameter estimators, how to forecast and how to generate the stochastic mortality rates with this LC model. We then explain the pricing scheme and the hedging strategy for financial guarantees in Chapter 2, particularly for the point-to-point design. In Chapter 3, we show how to evaluate a PTP equity-indexed annuity product, in the case with both non-stochastic and stochastic mortality rates. The corresponding new hedging strategy and the hedging errors are also discussed. Furthermore, the distributions of the present values of hedging errors for the PTP product are shown in Chapter 4. Finally, we explain the results and make some conclusions.

Chapter 1

Financial Model and Actuarial Notation

1.1 Financial Model

The Black-Scholes (BS) framework has been widely used in mathematical finance since its publication in 1973, Black and Scholes (1973). It provides a fundamental technique to price and hedge financial derivatives. Financial derivatives include financial instruments such as options, stock indices, futures, and swaps. There are numerous references that explain the BS model in depth (Björk, 2004). The BS model contains the following assumptions:

- The borrowing and lending rates are the same, and there is no restriction on short selling;
- There are no transaction or tax costs involved;
- Stocks do not pay dividends and are perfectly divisible;

- Trades can be done continuously;
- The stock prices follow a geometric Brownian motion (GBM) with a constant drift and volatility;
- There are no arbitrage opportunities in the market.

An arbitrage opportunity is a financial design which traders could use to make profits, with probability greater than 0, while investing nothing in the financial market. Note that if the above assumptions are true, the financial market is "complete." A complete market is a market where all the financial instruments can be replicated perfectly by their replicating portfolios.

1.2 Stock Price Dynamic

In the BS model, stock prices are assumed to follow a geometric Brownian motion (GBM) under a continuous-time framework. It is supposed that the stock price process $\{S(t);\, t \geq 0\}$ satisfies the stochastic differential equation given by

$$\begin{cases} dS(t) = \mu S(t)dt + \sigma S(t)dW(t), & t > 0, \\ S(0) = s \quad , \end{cases} \tag{1.1}$$

where μ is a constant drift parameter and can be interpreted as the underlying stock average return; σ is the diffusion coefficient and can be considered as the stock volatility. Suppose we set $S(0) = s$ as the artificial starting value of the stock. Moreover, $dW(t)$ is the derivative of a Wiener process. It represents the randomness of the stock prices in the trading market. A Wiener process $\{W(t);\, t \geq 0\}$ has the following properties:

- $W(0) = 0$;

- $W(t) \sim N(0,t)$, which means that $W(t)$ follows a normal distribution with mean 0 and variance t;

- $W(t)$ has independent increments on non-overlapping time intervals. Suppose $s < t < u < v$, then $W(t) - W(s)$ and $W(v) - W(u)$ are independent random variables. Moreover, $W(t) - W(s) \sim N(0, t-s)$.

Figure 1.1 shows a sample path of a standard Wiener process. The Wiener process is a continuous process tha is not differentiable anywhere. Furthermore, it moves randomly and does not have any particular trend.

So far, everything we have discussed is under the continuous-time frame. In order to simulate the stock prices, we need to relax the assumption of continuous-time. Consider the interval $[t, t + \frac{1}{m}]$, where m is the number of times we want to capture stock prices within a year. Then according to Euler's approximation, we can approximate the stock prices from (1.1) as the following

$$\Delta S(t) = \mu S(t)\Delta t + \sigma S(t)\Delta W(t), \quad t > 0, \tag{1.2}$$

where Δ represents the change . Furthermore, (1.2) is equivalent to

$$S\left(t + \frac{1}{m}\right) - S(t) = S(t)\left[\mu\frac{1}{m} + \sigma\left(W\left(t + \frac{1}{m}\right) - W(t)\right)\right], \tag{1.3}$$

and it leads to

$$S\left(t + \frac{1}{m}\right) = S(t)\left[\mu\frac{1}{m} + \sigma\left(W\left(t + \frac{1}{m}\right) - W(t)\right) + 1\right]. \tag{1.4}$$

Suppose that we start from time 0 and assume $S(0) = 1$, then the following formula is used to simulate stock prices

$$S\left(\frac{i}{m}\right) = S\left(\frac{i-1}{m}\right)\left(1 + \frac{\mu}{m} + \sigma W\left(\frac{1}{m}\right)\right), \tag{1.5}$$

where i takes values of $1, 2, \cdots$.

Figure 1.2 shows four simulated stock price processes with different parameter values over 2 years, where $m = 50$. We assume that the stock process starts from an initial value of 1. In Figure 1.2, the solid line is obtained when setting the stock volatility parameter to $\sigma = 0$. In fact, its shape matches with an exponential function: $e^{\mu t}$. We can also observe that the stock with larger σ value has bigger movements than other stocks. And the stock with higher return rate μ generally leads to higher values of $S(t)$.

We define a new process $\{Z(t); t \geq 0\}$ by $Z(t) = \ln(S(t))$, and according to Itô's formula, we can obtain the following

$$dZ(t) = \frac{1}{S(t)} dS(t) + \frac{1}{2}\left(-\frac{1}{S(t)^2}\right)(dS(t))^2, \quad t > 0, \tag{1.6}$$

with $Z(0) = \ln(s)$. Apply Itô's formula assumptions, we can further derive (1.6) as

$$\begin{aligned} dZ(t) &= \frac{1}{S(t)}\left(\mu S(t)dt + \sigma S(t)dW(t)\right) + \frac{1}{2}\left(-\frac{1}{S(t)^2}\right)\sigma^2 S(t)^2 dt \\ &= \mu dt + \sigma dW(t) - \frac{1}{2}\sigma^2 dt \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t), \quad t > 0. \end{aligned} \tag{1.7}$$

The starting point of the process Z is $Z(0) = \ln(s)$. Integrating (1.7) on both sides leads to

$$Z(t) = \ln(s) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t), \quad t \geq 0. \tag{1.8}$$

Recall that we defined $Z(t) = \ln(S(t))$, so we can derive $S(t)$ by taking the exponential function to (1.8)

$$S(t) = s e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}. \tag{1.9}$$

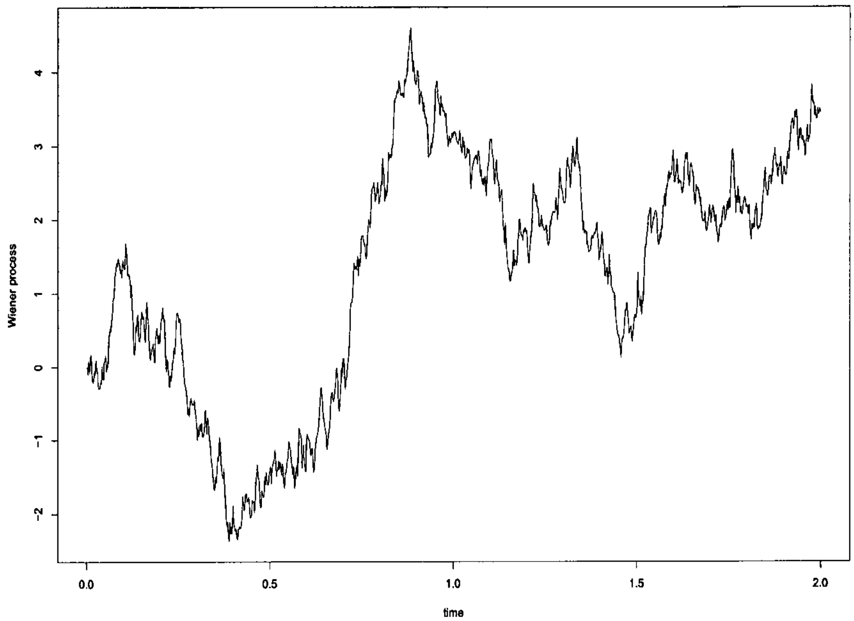

Figure 1.1: Wiener process

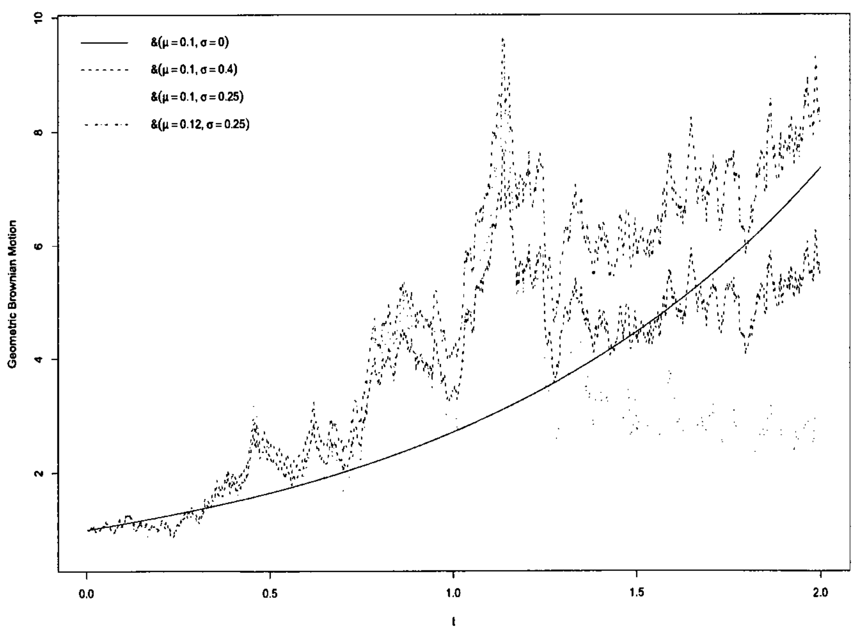

Figure 1.2: Geometric Brownian motion, $\mu = 0.1, 0.12, \sigma = 0, 0.25, 0.4$

Therefore, $S(t)$ has a lognormal distribution with mean $(\mu - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. Moreover, $\frac{S(t+u)}{S(t)}$ itself follows a lognormal distribution: $\frac{S(t+u)}{S(t)} \sim LN(\mu u, \sigma^2 u)$, where LN will be used as the abbreviation for lognormal distribution from hereafter. A lognormal distribution X with mean μt and variance $\sigma^2 t$ has the following density function

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi t}} e^{-\frac{(\ln(x)-\mu t)^2}{2\sigma^2 t}}, \quad x > 0, \tag{1.10}$$

and its cumulative distribution function is

$$F(x) = \Phi\left(\frac{\ln(x) - \mu t}{\sigma\sqrt{t}}\right), \quad x > 0. \tag{1.11}$$

1.3 Arbitrage-Free Valuation

In this section, we briefly discuss the concept of arbitrage-free valuation. Harrison and Pliska (1981) showed that the price of a financial derivative could be calculated as the expected value of a series of future payoffs under a risk-neutral probability measure.

Suppose we have a financial derivative with a maturity of T, and its future payoff at T is $G(T)$, then the price of this financial claim at time $t \leq T$ is

$$P(t,T) = E^Q\left[\frac{G(T)}{e^{r(T-t)}}\middle|\mathcal{F}(t)\right], \tag{1.12}$$

where $E^Q[\ \ ]$ is the expected value under the risk-neutral probability measure Q, $\mathcal{F}(t)$ stands for the information given up to time t, and r is the market risk-free interest rate. The risk-free interest rate is the guaranteed rate of return investors receive when investing in a risk-free asset. Furthermore, under the risk-neutral measure, the stock value dynamic equation is given by

$$\begin{cases} dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t), \quad t > 0, \\ S(0) = s, \end{cases} \tag{1.13}$$

where $\tilde{W}(t)$ is a standard Wiener process under probability measure Q.

Black and Scholes (1973) not only provides the European options pricing formula, but also introduces the replicating portfolio concept, which was later identified by Merton (1973). It says that in a complete market, a financial derivative can be hedged by its corresponding replicating portfolio, whose payoff at T is the same as the financial derivative payoff. Denote $V(T,T)$ as value of the replicating portfolio at T, then

$$V(T,T) = G(T), \tag{1.14}$$

where $G(T)$ is the payoff of the financial derivative at T.

Furthermore, Harrison and Kreps (1979) explain more on the trading strategies in the securities market. They point out that under the arbitrage-free market assumption, a replicating portfolio is self-financing. This means that the hedging portfolio does not need extra investments or withdrawals to adjust itself. Generally, the replicating portfolio is split in two assets: the risky asset, generally is a stock and the risk-free asset, a money-market account $M(t)$. Moreover, Baxter and Rennie (1996) showed that according to the general result of Black and Scholes, the proportion invested in the risky asset, denoted as $b(t,T)$, is calculated through the following formula

$$b(t,T) = S(t)\frac{dP(t,T)}{dS(t)}, \quad 0 < t < T. \tag{1.15}$$

The risk-free asset proportion $a(t,T)$ is obtained as $a(t,T) = P(t,T) - b(t,T)$. In the absence of arbitrage, the replicating portfolio and the financial derivative should have the same price at all times. We denote a hedging portfolio value at t as $V(t,T)$, then

$$P(t,T) = V(t,T) = a(t,T) + b(t,T), \quad 0 < t < T. \tag{1.16}$$

Suppose there are two portfolios with the same payoffs at time T that are sold at different prices, then a trader can buy the cheaper one and short sell the more expensive one. The profit of the trader is the difference between the two prices.

In this thesis, we will need the price formula of a European call option. The European call option gives option holders the right to buy a financial instrument at a predetermined strike price R, at maturity time T. Note that option holders do not have to exercise the option unless the option is "in the money," which means that the underling financial instrument's price at time T is above R. Suppose a European call option is associated with a stock $S(t)$, and $S(t)$ is defined as in (1.1). Let $G(T)$ denote the option payoff at time T, and it is given by

$$\begin{aligned} G(T) &= \max(S(T) - R, 0) \\ &= \begin{cases} 0 & \text{if } S(T) \leq R, \\ S(T) - R & S(T) > R. \end{cases} \end{aligned} \tag{1.17}$$

Using (1.12), the price of the call option at time $t \leq T$ is

$$P_C(t,T) = E^Q \left[\frac{G(T)}{e^{r(T-t)}} \middle| \mathcal{F}(t) \right]. \tag{1.18}$$

Recall that we have assumed that the dynamic stock price $S(t)$ is given by (1.13), then using (1.9), we have

$$S(T) = S(t) e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\tilde{W}(T) - \tilde{W}(t))}, \tag{1.19}$$

where $S(t)$ is given. We can further develop (1.18) as

$$P_C(t,T) = e^{-r(T-t)} \int_0^\infty G(z) f_Z(z) dz, \tag{1.20}$$

where $Z(T-t) \sim N((r - \frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t))$ throughout this thesis. Therefore,

$$\begin{aligned} P_C(t,T) &= e^{-r(T-t)} \left[\int_0^{\ln(\frac{R}{s(t)})} 0 \cdot f_Z(z) dz + \int_{\ln(\frac{R}{s(t)})}^\infty (S(z) - R) \cdot f_Z(z) dz \right] \\ &= e^{-r(T-t)} \int_{\ln(\frac{R}{s(t)})}^\infty S(z) f_Z(z) dz - e^{-r(T-t)} R \int_{\ln(\frac{R}{S(t)})}^\infty f_Z(z) dz. \end{aligned} \tag{1.21}$$

The lower limit $\ln(\frac{R}{S(t)})$ is obtained by solving

$$S(T) - R > 0 \Longrightarrow S(t)e^z > R \Longrightarrow z > \ln(\frac{R}{S(t)}). \tag{1.22}$$

We derive the second integral part in (1.21) to be

$$\int_{\ln(\frac{R}{S(t)})}^\infty f_Z(z) dz = \frac{1}{\hat{\sigma}\sqrt{2\pi}} \int_{\ln(\frac{R}{S(t)})}^\infty e^{-\frac{(z-\hat{\mu})^2}{2\hat{\sigma}^2}} dz, \tag{1.23}$$

where $\hat{\mu} = (r - \frac{1}{2}\sigma^2)(T-t)$ and $\hat{\sigma}^2 = \sigma^2(T-t)$. Let $\frac{z-\hat{\mu}}{\hat{\sigma}} = u$, then $dz = \hat{\sigma} du$. Therefore, (1.23) is further developed as

$$\begin{aligned} \frac{1}{\hat{\sigma}\sqrt{2\pi}} \int_{\frac{\ln(\frac{R}{S(t)})-\hat{\mu}}{\hat{\sigma}}}^\infty e^{-\frac{u^2}{2}} \hat{\sigma} du &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln(\frac{R}{S(t)})-\hat{\mu}}{\hat{\sigma}}}^\infty e^{-\frac{u^2}{2}} du \\ &= Pr^Q \left\{ u \geq \frac{\ln(\frac{R}{S(t)}) - \hat{\mu}}{\hat{\sigma}} \right\} \\ &= \Phi \left(-\frac{\ln(\frac{R}{S(t)}) - \hat{\mu}}{\hat{\sigma}} \right). \end{aligned} \tag{1.24}$$

The first integral in (1.21) is obtained via similar derivations, as is shown in detail in the next chapter. Therefore, using the change of variable method to rearrange (1.21), it can be rewritten in terms of two standard normal distribution functions. That is

$$P_C(t,T) = S(t)\Phi(d_1) - Re^{-r(T-t)}\Phi(d_2), \tag{1.25}$$

where

$$d_1 = \frac{\ln(\frac{S(t)}{R}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{1.26}$$

and

$$\begin{aligned} d_2 &= \frac{\ln(\frac{S(t)}{R}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ &= d_1 - \sigma\sqrt{T-t}. \end{aligned} \tag{1.27}$$

1.4 Actuarial Notation

We use the standard actuarial notation, presented in Bowers (1997) for a life table. Define (x) as a person alive at age of x and define $T(x,s)$ as the continuous future lifetime of (x) who is current living in year s. Moreover, define $K(x,s)$ as the discrete curtate-future-lifetime, that is

$$K(x,s) = \lfloor T(x,s) \rfloor \in \mathbb{N}, \tag{1.28}$$

where $\lfloor \cdot \rfloor$ is a floor function. In fact, $K(x,s)$ is the smallest integer that is close to the actual death time.

Let ${}_hq_{x,s}$ be the probability that (x) living in year s, dies within h years:

$${}_hq_{x,s} = Pr[T(x,s) < h]. \tag{1.29}$$

Denote ${}_hp_{x,s}$ as the probability that (x) in year s survives for h years

$${}_hp_{x,s} = Pr[T(x,s) \geq h] = 1 -_h q_{x,s}. \tag{1.30}$$

Furthermore, the probability that (x) in year s survives exactly h years is defined as

$$\begin{aligned} {}_{h|}q_{x,s} &= Pr[K(x,s) = h] \\ &= Pr[h < T(x,s) < h+1] = Pr[T(x) < h+1] - Pr[T(x) < h]. \quad (1.31) \end{aligned}$$

The probability that (x) survives h years but will die within the following u years is denoted by

$${}_{h|u}q_{x,s} = Pr[h < T(x,s) < h+u]. \tag{1.32}$$

For convenience, we define $q_{x,s}$ as the probability that (x) living in year s dies in the following year.

For a non-deterministic survivorship group (cohort), we denote $\mathcal{L}_{x,s}$ as the random number of population living at the beginning of age interval x in year s, and $\mathcal{D}_{x,s}$ represents the random number of deaths occurred between age x and $x+1$. Furthermore, let $l_{x,s} = E[\mathcal{L}_{x,s}]$ be the expected number of the survivors, and let $d_{x,s} = E[\mathcal{D}_{x,s}]$. Therefore, for a deterministic cohort, we have the following

$$d_{x,s} = l_{x,s} - l_{x+1,s+1}. \tag{1.33}$$

Let $m_{x,s}$ denote the central death rate over the interval from x in year s to $x+1$ in year $s+1$, and it is defined as the following ratio

$$m_{x,s} = \frac{l_{x,s} - l_{x+1,s+1}}{N_{x,s}}, \tag{1.34}$$

where $N_{x,s}$ is the total expected number of years lived by survivors of the initial new-borns between x and $x+1$

$$N_{x,s} = \int_0^1 l_{x+t,s} dt. \tag{1.35}$$

Finally, f_x is defined as the separation factor. It represents the average number of years lived between x and $x+1$ by people who die within the interval of x to $x+1$. It is given by

$$f_x = \frac{\int_0^1 s f_{T(x,s)}(t)dt}{\int_0^1 f_{T(x,s)}(t)dt}, \tag{1.36}$$

where $f_{T(x,s)}(t)$ is the density function of the future lifetime of (x). If we assume that death occurs uniformly on a 1-year interval, then $f_{T(x,s)}(t) = q_{x,s}$, where $0 < t < 1$. Therefore, $f_x = \int_0^1 sds = \frac{1}{2}$.

1.5 Stochastic Mortality Model

The pricing of life insurance products is based on financial models and mortality rates. In this thesis, we use the Lee-Carter (LC) model to forecast future mortality rates. The LC model has been recognized as the leading statistical model, famous for its simplicity and the accuracy of its empirical results. The results generated by the LC model have been used as benchmarks in several countries where historical statistical data are sufficient and complete. The model needs to use data such as the population, the number of deaths per year, the number of new-borns per year, etc., to estimate the parameters. Therefore, the more information we have, the more likely we will make an accurate analysis about mortality rates. Moreover, the LC model allows death rates to decrease without limit and the life expectancy to increase without any additional conditions.

To make our analysis as up-to-date as possible, we have chosen the data for the Canadian population given by the Human Mortality Database from 1921 to 2005. Prior to the year 1921, World War 1 and an epidemic influenza led to high mortality rates. However, those special events only reflect the rare extreme cases, and we do not consider those situation in this thesis. Therefore, our data begins from year 1921. We want to estimate the central death rates $m_{x,s}$ using the following model

$$\ln(m_{x,s}) = \alpha_x + \beta_x k_s + \varepsilon_{x,s}, \tag{1.37}$$

where α_x and β_x are age-parameters that change with age, k_s is the mortality index and changes with time, and $\varepsilon_{x,s}$ represents the error term that is not captured by the model. The error term is assumed to follow a normal distribution with mean 0 and variance ξ.

Because in (1.37) only the $m_{x,s}$'s are given from the population distribution, the usual linear regression will not provide the estimated parameter values. In addition, (1.37) does not have a set of unique estimated parameters values. For example, suppose that $\tilde{\alpha}_x$, $\tilde{\beta}_x$, and $\tilde{k}_s$ are a set of estimated parameters, then for any constant number c, $\tilde{\alpha}_x - c$, $\tilde{\beta}_x$, $\tilde{k}_s + c$ and $\tilde{\alpha}_x$, $\tilde{\beta}_x c$, $\tilde{k}_s / c$ are both possible sets of estimated values. Fortunately, Lee and Carter (1992) pointed out in the paper that the multi-solutions of (1.37) do not affect the uniqueness of the future forecast values. In fact, the likelihood corresponding to the model has a finite number of equivalent maxima, and each set of parameters provides exactly the same forecasts.

Among all the possible estimates, we want to find a set of parameters that is consistent, representative, and easy to calculate. Therefore, the following assumptions were introduced by Lee and Carter (1992):

- The sum of β_x's for all ages x is 1;

- The sum of k_s's is 0.

Furthermore, we only consider the ages up to 99-year old. Mortality rates above 99-year old are obtained through another method. In this thesis, we consider individuals aged between 40 to 50 years old holding life insurance products for 5 to 15 years. Therefore, we focus on estimating mortality rates on the age interval of [0,99]. The matrix containing the logarithm of the central death rates is given by

$$\ln(M)_{(100,85)} = \begin{pmatrix} \ln(m_{0,1921}) & \ln(m_{0,1922}) & \cdots & \ln(m_{0,2005}) \\ \ln(m_{1,1921}) & \ln(m_{1,1922}) & \cdots & \ln(m_{1,2005}) \\ \vdots & \vdots & \ddots & \vdots \\ \ln(m_{99,1921}) & \ln(m_{99,1922}) & \cdots & \ln(m_{99,2005}) \end{pmatrix} \tag{1.38}$$

For each age group (the row in this matrix), we have a total of $H = 85$ equations for s =1921 to 2005, which are in the form of (1.37). If we sum up the 85 equations and ignore the error terms for now, we then get the following equation

$$\sum_{s=1921}^{2005} \ln(m_{x,s}) = H\alpha_x + \beta_x \sum_{s=1921}^{2005} k_s, \quad x = 0, 1, \cdots, 99, \tag{1.39}$$

where H is the year range 85. Using $\sum_{s=1921}^{2005} k_s = 0$, we obtain the estimated $\tilde{\alpha}$ values for each age group

$$\tilde{\alpha}_x = \frac{1}{H} \sum_{s=1921}^{2005} \ln(m_{x,s}). \tag{1.40}$$

The next step is to subtract the obtained $\tilde{\alpha}_x$ from the corresponding age row, and a new matrix is formed as the following

$$\ln(M)_{(100,85)} - \underset{\sim}{\alpha}_x = \begin{pmatrix} \ln(m_{0,1921}) - \tilde{\alpha}_0 & \ln(m_{0,1922}) - \tilde{\alpha}_0 & \cdots & \ln(m_{0,2005}) - \tilde{\alpha}_0 \\ \ln(m_{1,1921}) - \tilde{\alpha}_1 & \ln(m_{1,1922}) - \tilde{\alpha}_1 & \cdots & \ln(m_{1,2005}) - \tilde{\alpha}_1 \\ \vdots & \vdots & \ddots & \vdots \\ \ln(m_{99,1921}) - \tilde{\alpha}_{99} & \ln(m_{99,1922}) - \tilde{\alpha}_{99} & \cdots & \ln(m_{99,2005}) - \tilde{\alpha}_{99} \end{pmatrix} \tag{1.41}$$

In order to solve for β_x and k_s, we use the Singular Value Decomposition (SVD). The SVD provides two sets of left and right eigenvectors for the matrices AA^T and A^TA respectively. This method is pre-programmed in many statistical and computer packages, such as R, matlab, and C++. In this thesis, we use the SVD function in R to decompose (1.41), and we get two vectors containing k_s^*'s and β_x^*'s values. Using the SVD method, the sum of k_s^*'s is 0. However, the sum of β_x^*'s is not 1, and therefore, we normalize the obtained β_x^*'s to get the estimated $\tilde{\beta}_x$ values

$$\tilde{\beta}_x = \frac{\beta_x^*}{\sum_{x=0}^{99} \beta_x^*}. \tag{1.42}$$

So far, the estimated values of $\tilde{\alpha}_x$ and $\tilde{\beta}_x$ have been determined. The original k_s^*'s obtained from the SVD method are not our final estimated values yet. The Lee-Carter model requires that (1.37) matches with the actual historical data, so one more step is needed to find the estimated $\tilde{k}_s$. Keeping $\tilde{\alpha}_x$'s in (1.40), $\tilde{\beta}_x$'s in (1.42), and k_s^*'s obtained with the SVD unchanged, we plug them into (1.37) to get the corresponding forecast central death rates $\tilde{m}_{x,s}$

$$\tilde{m}_{x,s} = e^{\tilde{\alpha}_x + \tilde{\beta}_x k_s^*}, \quad s = 1921, \cdots, 2005. \tag{1.43}$$

Let $D(s)$ denote the total number of death actually observed within year s, then

$$D(s) = \sum_{x=0}^{99} (N_{x,s} \tilde{m}_{x,s}), \tag{1.44}$$

where $N_{x,s}$ is defined in (1.35) and given in the population data. Moreover, from the population distribution, we can calculate the total number of deaths occurred during a year s, for all people from age 0-99. Therefore, using (1.43) to substitute $\tilde{m}_{x,s}$ and forcing (1.44) to equal the observed total number of deaths in year s, we can solve (1.44) iteratively to obtain $\tilde{k}_s$. By applying this procedure to different years from 1921-2005, we obtain all the estimated $\tilde{k}_s$'s. In this thesis, we use the SOLVER function in Excel to solve for $\tilde{k}_s$.

With all the estimated parameter values at hand, we can obtain the forecasted central death rates. Although the obtained values are very close to the actual data, there are still small differences between them, and these differences are the errors. As we mentioned before, we assumed that $\varepsilon_{x,s}$ is a normally distributed variable with mean 0 and variance ξ. The estimated variance is given by

$$\tilde{\xi} = \frac{\sum_{x=0}^{99} \sum_{s=1921}^{2005} (m_{x,s} - \tilde{m}_{x,s})^2}{NH - 1}, \tag{1.45}$$

where N is the age range 100 for x=0 to 99.

In order to forecast future mortality rates, we need to first forecast the future mortality index $\tilde{k}_s$. Lee and Carter (1992) tested several different ARIMA models, and decided to use a simple random walk with drift (RWD) model to fit the $\tilde{k}_s$'s. That is

$$\tilde{k}_s = \tilde{k}_{s-1} + \tilde{\theta} + \epsilon_s, \quad s = 1922, \cdots, 2005, \tag{1.46}$$

where $\tilde{\theta}$ is the average drift parameter on a time interval $[1, SS]$, where SS is an integer number bigger than 1. Moreover, the maximum likelihood estimated $\tilde{\theta}$ value is given by

$$\tilde{\theta} = \frac{\tilde{k}_{SS} - \tilde{k}_1}{SS - 1}. \tag{1.47}$$

Note that the $\tilde{k}_s$'s generally have a decreasing trend, which means people tend to live longer in more recent years s. In addition, the error term ϵ_s in (1.46) is also assumed to follow a normal distribution, with mean 0 and variance υ. The estimated variance of ϵ_s is given by

$$\tilde{\upsilon} = \frac{\sum_{s=1}^{SS} (\tilde{k}_{s+1} - \tilde{k}_s - \tilde{\theta})^2}{SS - 1}. \tag{1.48}$$

Both error terms in (1.37) and (1.46) play an important role when generating random mortality rates.

Table 1.1 shows the forecasted $\tilde{\alpha}, \tilde{\beta}$, and $\tilde{k}$ values of a 50-year old person, from 2005 to 2014, based on the historical data from 1921 to 2005.

Pars.	(50,2005)	(51,2006)	(52,2007)	(53,2008)	(54,2009)	(55,2010)	(56,2011)	(57,2012)	(58,2013)	(59,2014)
$\tilde{\alpha}_x$	-5.14107	-5.12035	-4.96482	-4.90707	-4.81074	-4.73673	-4.63783	-4.55553	-4.44326	-4.37223
$\tilde{\beta}_x$	0.00822	0.00624	0.00787	0.00702	0.00679	0.00678	0.00648	0.00615	0.00683	0.00617
$\tilde{k}_s$	-96.2744	-98.1501	-100.0257	-101.9013	-103.7769	-105.6525	-107.5281	-109.4037	-111.2794	-113.1550
$\tilde{m}_{x,s}$	0.00265	0.00324	0.00318	0.00362	0.00402	0.00428	0.00482	0.00536	0.00550	0.00628

Table 1.1: Estimated parameters values

Recall that $\tilde{\alpha}_x$ is just the average of $\ln(m_{x,s})$ and $\tilde{\beta}_x$ is obtained using the SVD method. Furthermore, $\tilde{k}_{2005}$ is obtained using the SOLVER in Excel, and the rest $\tilde{k}_s$'s are obtained using the RWD method. For instance, we now show how to get k_{2006}.

$$\tilde{k}_{1921} = 61.2772, \text{and} \quad \tilde{k}_{2005} = -96.2744, \quad \text{both obtained using SOLVER},$$
$$\tilde{\theta} = \frac{\tilde{k}_{2005} - \tilde{k}_{1921}}{84} = -1.8756, \quad \text{and therefore},$$
$$\tilde{k}_{2006} = \tilde{k}_{2005} + \tilde{\theta} = -98.1501. \tag{1.49}$$

Furthermore, the $\tilde{m}_{x,s}$'s are obtained using (1.43), and therefore

$$\begin{aligned}
\tilde{m}_{50,2005} &= e^{\tilde{\alpha}_{50}+\tilde{\beta}_{50}\tilde{k}_{2005}} \\
&= e^{-5.14107+0.00822\times(-96.2744)} \\
&= 0.00265.
\end{aligned}$$

From Table 1.1, we can see that the $\tilde{\alpha}_x$'s are increasing with x. This is because central death rates increase as ages increase, and this leads $\tilde{\alpha}_x$, which is the average of the $\ln(m_{x,s})$ to increase as well. Not surprisingly, the $\tilde{k}$'s decrease linearly as we described previously in the RWD model. Indeed, the decreasing trend of the mortality index indicates that mortality rates are decreasing year by year. It seems that mortality rates should decrease as time passes. However, $\tilde{\beta}_x$ which represents the rate that $\ln(m_{x,s})$ declines at, controls how fast $\tilde{k}_s$ can decrease at certain ages. Note that

$$\frac{d\ln(m_{x,s})}{ds} = \beta_x \frac{dk_s}{ds}.$$

Hence, the total impact of α_x, β_x, and k_s determines $m_{x,s}$. Furthermore, the $\tilde{\beta}_x$'s do not follow any particular trend. Note that at age 50, 51, and 52, the $\tilde{\beta}_x$'s are more volatile. It shows that at age 50, mortality declines much more rapidly than at age 51, and that at 51, mortality declines at a much slower rate. It then declines at a relatively stable rate after age 52.

1.5.1 Approximation Method

We need to use an approximation method to convert the forecasted central death rates to annual mortality rates. The following method is generally used by many authors and organizations, such as in Renshaw and Haberman (2003), by the United States Social Security, and Human Mortality Database. This method is recognized for its simplicity and accuracy. Suppose that $x_k \leq 85$, then the following approximation applies

$$q_{x,s} \cong \frac{m_{x,s}}{1 + f_x m_{x,s}}, \tag{1.50}$$

for all x and s. Moreover, recall f_x is the separation factor and is $\frac{1}{2}$ in the case that death occurs uniformly on the time interval [0,1]. Mortality rates of people whose ages are greater than 85 are converted through another approximation method. In this thesis, we consider individuals who were 50 years old in 2005 and held insurance contracts for up to 5-15 years.

With all the estimated values of $\tilde{\alpha}_x$'s, $\tilde{\beta}_x$'s, and $\tilde{k}$'s available, we can first get $\tilde{m}_{x,s}$'s, and $\tilde{q}_{x,s}$'s can be obtained via (1.50). For instance,

$$\tilde{q}_{50,2005} = \frac{\tilde{m}_{50,2005}}{1 + f_x \tilde{m}_{50,2005}} = 0.002646493. \tag{1.51}$$

Table 1.2 shows the future forecasted 10-year mortality rates of a 50-year old living in year 2005. We see that mortality rates increase as age increases, except between ages of 51 and 52, where the mortality rates increase rapidly at age 51 compare with that at age 50. The death rates decrease slightly at age 52 and then keep increasing at a smooth rate. Recall the explanations on this matter in the end of the previous section.

$\tilde{q}_{50,2005}$	$\tilde{q}_{51,2006}$	$\tilde{q}_{52,2007}$	$\tilde{q}_{53,2008}$	$\tilde{q}_{54,2009}$	$\tilde{q}_{55,2010}$	$\tilde{q}_{56,2011}$	$\tilde{q}_{57,2012}$	$\tilde{q}_{58,2013}$	$\tilde{q}_{59,2014}$
0.00265	0.00323	0.00317	0.00361	0.00402	0.00427	0.00481	0.00535	0.00548	0.00626

Table 1.2: Forecasted mortality rates

1.5.2 Generating Stochastic Mortality Rates

We can simulate the stochastic mortalities by first simulating the $\tilde{k}_s$ using the RWD model in (1.46). Recall that $\tilde{k}_{2005} = -96.2744, \tilde{\theta} = -1.87561$, and $\tilde{\upsilon} = 9.94385$ were obtained by using (1.48). Hence,

$$\begin{aligned}\tilde{k}_{2006} &= \tilde{k}_{2005} + \tilde{\theta} + \epsilon_{2005} \\ &= -96.274437 + (-1.875614817) + \epsilon_{2005},\end{aligned}$$

where ϵ_{2005} represents a random variable that follows a normal distribution with mean 0 and variance $\tilde{\upsilon}$. Therefore, $\tilde{k}_{2007}$ is obtained using the same procedures,

$$\begin{aligned}\tilde{k}_{2007} &= \tilde{k}_{2006} + \tilde{\theta} + \epsilon_{2006} \\ &= \tilde{k}_{2005} + \tilde{\theta} + \tilde{\theta} + \epsilon_{2005} + \epsilon_{2006} \\ &= -96.274437 + 2(-1.875614817) + \epsilon_{2005} + \epsilon_{2006},\end{aligned}$$

where ϵ_{2005} and ϵ_{2006} are assumed to be independent and identically distributed.

After generating a series of $\tilde{k}_s$'s, we can then simulate stochastic $\ln(\tilde{m}_{x,s})$'s by using (1.37), where $\tilde{\alpha}_x$, $\tilde{\beta}_x$ were obtained above, and $\varepsilon_{x,s} \sim N(0, \xi)$, where $\tilde{\xi} = 0.01997$. Therefore,

$$\ln(\tilde{m}_{x,s}) = \tilde{\alpha}_x + \tilde{\beta}_x \tilde{k}_s + \varepsilon_{x,s}. \tag{1.52}$$

Taking the exponents of these $\ln(\tilde{m}_{x,s})$'s, we can get the stochastic $\tilde{m}_{x,s}$'s. Using the approximation method, we then obtain the stochastic $\tilde{q}_{x,s}$'s based on the simulated $\tilde{m}_{x,s}$. The forecasted $\tilde{q}_{x,s}$'s are calculated using the procedure in (1.51), where the errors term are ignored.

Figure 1.3 shows 4 simulated future 10 years $\tilde{q}_{x,s}$'s and the forecasted $\tilde{q}_{x,s}$'s, starting from year 2005 and age 50. We can see that the stochastic death rates are around the forecasted death rates. In this figure particularly, we only performed 4 simulations, and the stochastic rates are higher than the forecasted ones most of time. However, if we do enough number of simulations, the averages of the stochastic rates in each year will be very close to the forecasted ones.

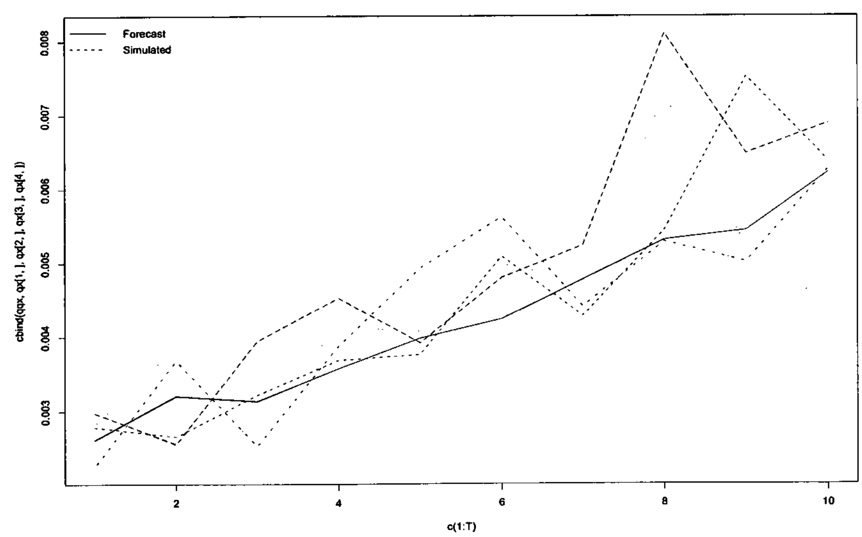

Figure 1.3: Simulated $\tilde{q}_{x,s}$'s and the forecasted $\tilde{q}_{x,s}$'s

Chapter 2

Financial Guarantee

2.1 Introduction

Insurance companies offer various equity-indexed annuities (EIAs) that provide different financial guarantees. Commonly seen financial guarantees are the high-water mark guarantee, the annual reset guarantee, and the point-to-point guarantee. These financial guarantees mainly differ in their payoff designs; however, one thing they have in common is that their payoffs are linked with the performances of certain equity indexes.

These financial guarantee products allow policyholders to enjoy the potential growth from the equity market. Meanwhile, insurance companies offer minimum guaranteed payments to protect policyholders from experiencing a bear market. However, the caps, participation rates, and other limitations prevent policyholders from having full growth from the index performance. In short, at the maturity date, the policyholder can have the higher of the two, the minimum guarantee or the regular payoff which is linked to the index market.

2.2 The Point-to-Point Financial Guarantee

In this thesis, we focus on the point-to-point (PTP) design. The PTP financial guarantee is linked with an underling index stock S whose dynamic equations are given in (1.1). Furthermore, the PTP payoff $G(T)$ is given by

$$G(T) = P(0,T)\max\left(1+\omega\left(\frac{S(T)}{S(0)}-1\right),\rho(1+g)^T\right), \tag{2.1}$$

where $P(0,T)$ is the initial price charged to one policyholder, and ω is denoted as the participation rate, which takes a value between 0 and 1. The participation rate indicates the percentage policyholders expose to the stock index growth. Moreover, $P(0,T)\rho(1+g)^T$ is the minimum guaranteed amount paid at T, where ρ is the percentage that the insurance company guarantees its policyholders they will get from their initial investments, and g is the guaranteed annual rate of return. For example, suppose that $\rho = 80\%$, and $g = 3\%$, then in year T, the minimum guaranteed amount is $0.8(1+3\%)^T$. Furthermore, $\frac{S(T)}{S(0)} - 1$ in (2.1) is the growth rate of the index stock.

From (2.1), we can see that only the starting and ending stock prices are taken into account. In other words, the PTP financial guarantee ignores all the stock index movements throughout the term, except at times 0 and T. In general, investors can enjoy high profit returns if the market experiences an uninterrupted bull market.

Let $Z(T) = (\mu - \frac{1}{2}\sigma^2)T + \sigma W(T)$ be a Brownian motion with drift, then from (1.9), we have

$$\frac{S(T)}{S(0)} = e^{Z(T)}. \tag{2.2}$$

Moreover, recall the geometric Brownian motion in (1.19), given $S(t)$ for $0 < t \leq T$, we can rewrite (2.2) as

$$\frac{S(T)}{S(0)} = \frac{S(t)e^{Z(T-t)}}{S(0)}. \tag{2.3}$$

Therefore, substituting (2.3) into (2.1) for a given $0 < t \leq T$, we can rewrite $G(T)$ as

$$G(T) = P(0,T)\max\left(1 - \omega + \omega\frac{S(t)}{S(0)}e^{Z(T-t)}, \rho(1+g)^T\right). \tag{2.4}$$

For simplicity, we use K to replace $\rho(1+g)^T$ hereafter.

2.3 Pricing

Using (1.12), the price of a PTP financial guarantee at time $t \leq T$ is given by

$$\begin{aligned} P(t,T) &= e^{-r(T-t)}E^Q\left[G(T)\middle|\mathcal{F}(t)\right] \\ &= P(0,T)e^{-r(T-t)}E^Q\left[\max\left(1 - \omega + \omega\frac{S(t)}{S(0)}e^{Z(T-t)}, K\right)\middle|S(t)\right], \end{aligned} \tag{2.5}$$

The second line is true because $\mathcal{F}(t)$ represents the information given up to t, and it is actually just the stock price at time t. Therefore, the price is given by

$$\begin{aligned} P(t,T) &= \frac{P(0,T)}{e^{r(T-t)}}\left[\int_A \left(1 - \omega + \omega\frac{S(t)}{S(0)}e^z\right) f_{Z|S(t)}(z)dz + \int_{A^c} K f_{Z|S(t)}(z)dz\right] \\ &= \frac{P(0,T)}{e^{r(T-t)}}\left[(1-\omega)\int_A f_Z(z)dz + \omega\frac{S(t)}{S(0)}\int_A e^z f_Z(z)dz + K\int_{A^c} f_Z(z)dz\right] \\ &= \frac{P(0,T)}{e^{r(T-t)}}\left[(1-\omega)p_1 + \omega\frac{S(t)}{S(0)}\int_A e^z f_Z(z)dz + K(1-p_1)\right], \end{aligned} \tag{2.6}$$

where A is defined as the event $[1 - \omega + \omega e^{Z(T)} > K]$, and A^c as the event $[1 - \omega + \omega e^{Z(T)} \leq K]$. Furthermore, $p_1 = Pr^Q\{1 - \omega + \omega\frac{S(t)}{S(0)}e^{Z(T-t)} > K\}$ and $p_2 =$

$Pr^Q\{1-\omega+\omega\frac{S(t)}{S(0)}e^{Z(T-t)} \le K\} = 1-p_1$, where Pr^Q indicates the probability under the risk-neutral measure. We can solve p_1 through the following

$$\begin{aligned} p_1 &= Pr^Q\left\{1-\omega+\omega\frac{S(t)}{S(0)}e^{Z(T-t)} > K\right\} \\ &= Pr^Q\left\{Z(T-t) > \ln(\frac{K+\omega-1}{\omega}) - \ln(\frac{S(t)}{S(0)})\right\}. \end{aligned} \tag{2.7}$$

For simplicity, the following notation is introduced

$$\hat{\mu} = (r-\frac{1}{2}\sigma^2)(T-t) \quad \text{and} \quad \hat{\sigma} = \sigma\sqrt{T-t}, \tag{2.8}$$

$$K^* = \ln(\frac{K+\omega-1}{\omega}) - \ln(\frac{S(t)}{S(0)}). \tag{2.9}$$

Therefore, p_1 represents the probability of a standard normal distribution

$$p_1 = \Phi\left(\frac{\hat{\mu}-K^*}{\hat{\sigma}}\right). \tag{2.10}$$

We need to compute the integral $\int_A e^z f_Z(z)dz$ in (2.6). For a normal distribution: $Y \sim N(\mu, \sigma^2)$, we can derive the following

$$\begin{aligned} \int_a^\infty e^y \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy &= \int_a^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{\mu+\frac{\sigma^2}{2}} e^{-\frac{(y-(\mu+\sigma^2))^2}{2\sigma^2}} dy \\ &= e^{\mu+\frac{\sigma^2}{2}} \Phi(\frac{\mu+\sigma^2-a}{\sigma}). \end{aligned} \tag{2.11}$$

Therefore,

$$\int_A e^z f_Z(z)dz = e^{\hat{\mu}+\frac{\hat{\sigma}^2}{2}} \Phi(\frac{\hat{\mu}+\hat{\sigma}^2-K^*}{\hat{\sigma}}). \tag{2.12}$$

In fact, the first integral in (1.21), where we derived Black-Scholes formula, is obtained using the same steps in (2.11):

$$\begin{aligned} e^{-r(T-t)} \int_{\ln(\frac{K}{S(t)})}^\infty S(z) f_Z(z)dz &= e^{-r(T-t)} e^{\hat{\mu}+\frac{\hat{\sigma}^2}{2}} \Phi\left(\frac{\hat{\mu}+\hat{\sigma}^2-\ln(\frac{K}{S(t)})}{\hat{\sigma}}\right) \\ &= S(t)\Phi(d_1), \end{aligned}$$

where d_1 is defined in (1.26).

Therefore, we conclude that the pricing formula at time $t < T$ for a PTP financial guarantee as

$$P(t,T) = P(0,T)\left[\frac{(1-\omega)p_1 + K(1-p_1)}{e^{r(T-t)}} + \omega\frac{S(t)}{S(0)}\Phi\left(\frac{\hat{\mu}+\hat{\sigma}^2-K^*}{\hat{\sigma}}\right)\right], \quad (2.13)$$

where p_1 is given in (2.10), and $t \in [0,T)$. Similar derivations are shown by other authors too, for instance, see Hardy (2003), Chapter 13. Furthermore, note that at time T, $P(T,T) = G(T)$, since $G(T)$ will be the amount insurance company needs to pay out at the maturity date.

The PTP financial guarantee can be viewed as a combination of a fixed amount investment and a call option. We can rearrange $G(T)$ and write it as followings

$$\begin{aligned} G(T) &= P(0,T)\max\left(1+\omega(\frac{S(T)}{S(0)}-1), K\right) \\ &= P(0,T)\left(K+\max\left(1+\omega(\frac{S(T)}{S(0)}-1)-K, 0\right)\right) \\ &= P(0,T)\left[K+\frac{\omega}{S(0)}\max\left(S(T)-S(0)(1+\frac{K-1}{\omega}), 0\right)\right]. \end{aligned} \quad (2.14)$$

We see that the max term in (2.14) matches the European call option's payoff expression in (1.17). Therefore, we can treat the max term as a European call option with strike price $S(0)\left(1+\frac{K-1}{\omega}\right)$. Let $G_C(T)$ denote the payoff of a European call option at maturity T, then (2.14) can be written as

$$G(T) = P(0,T)\left[K+\frac{\omega}{S(0)}G_C(T)\right]. \quad (2.15)$$

Therefore, the PTP financial guarantee's price at time 0 is

$$P(0,T) = P(0,T)\left[Ke^{-rT}+\frac{\omega}{S(0)}P_C(0,T)\right], \quad (2.16)$$

where $P_C(0,T)$ represents the price of a European call option at time 0, and it is given by Black-Scholes formula in (1.25). Note that $P(0,T)$ can be canceled from both sides, and we are left with $1 = Ke^{-rT} + \frac{\omega}{S(0)} P_C(0,T)$.

We can use the price formula in (2.16) to obtain the participation rate ω. Suppose that $T = 10, \sigma = 0.25, r = 0.06, \rho = 0.9, g = 0.03$, and for simplicity, we set $S(0) = 1$ and $P(0,10) = 1$. Using the non-linear minimization function in R to solve ω iteratively, we obtain $\omega = 0.7698524$. In fact, the assumption of $S(0) = 1, P(0,10) = 1$ does not affect the pricing scheme in any way, since ont only $P(0,T)$ can be factored out in (2.16), $S(0)$ can be eliminated by the formula of $P_C(0,T)$. Hence, even if we assign different values for $S(0)$ and $P(0,10)$, we will still obtain the same result for ω. Therefore, for simplicity, we will use the assumption of $P(0,T) = \$1$ and omit $P(0,T)$ throughout this section.

Furthermore, using the same procedures as described for a 10-year term financial design and keeping the same parameter values for σ, r, ρ and g, we can get the participation rates of the 5-year and 15-year contracts, which are 0.7076605 and 0.8117203, respectively. Note that as the contract term extends, the participation rate increases. This is because a contract with a longer term is more likely to be exposed to an up-going market, and hence, a higher participation rate results.

2.4 Hedging Strategy

Using arbitrage-free pricing theory, insurers can reduce risks by creating a hedging portfolio. They need to reinvest the initial premium into other assets, such as stocks and money market accounts. A hedging portfolio can replicate the payoff of a financial derivative with payoff at the end of term T of $G(T)$, which means that

$$V(T, T) = G(T). \tag{2.17}$$

In an arbitrage-free market, the value of the hedging portfolio should be equal to the price of the financial derivative at all times, $V(t, T) = P(t, T)$. The replicating portfolio is called a hedging portfolio for the financial derivative, and the financial derivative is called hedgeable, reachable or replicable.

In a complete market, the hedging portfolio can help the issuer company fulfil its future liabilities without risks. The general way of building a hedging portfolio is to invest in both risky and risk-free assets. Recall (1.15), the amounts invested in risky and risk-free assets are determined as

$$\begin{aligned} b(t, T) &= S(t)\frac{dP(t, T)}{dS(t)}, \quad 0 < t < T, \\ a(t, T) &= P(t, T) - b(t, T). \end{aligned}$$

Furthermore, $\frac{dP(t,T)}{dS(t)}$ is the number of shares of the stock that the issuer should hold in the replicating portfolio at time t. For a PTP financial guarantee, $P(t, T)$ is given in

(2.13), and therefore

$$
\begin{aligned}
b\,(t,T) \;\; &= \;\; \frac{S(t)\left[(1-\omega)\frac{\partial p_1}{\partial S(t)} + K\frac{\partial(1-p_1)}{\partial S(t)}\right]}{e^{r(T-t)}} + \\
&\quad \frac{\omega}{S(0)}\left[S(t)\Phi'\left(\frac{\hat{\mu}+\hat{\sigma}^2-K^*}{\hat{\sigma}}\right) + \Phi\left(\frac{\hat{\mu}+\hat{\sigma}^2-K^*}{\hat{\sigma}}\right)\right],
\end{aligned}
$$

where Φ' represents the derivative of a normal distribution cumulative distribution function, and

$$
\frac{\partial p_1}{\partial S(t)} = \phi\left(\frac{\hat{\mu}-K^*}{\hat{\sigma}}\right)\left(-\frac{\partial(\frac{K^*}{\hat{\sigma}})}{\partial S(t)}\right) = \phi\left(\frac{\hat{\mu}-K^*}{\hat{\sigma}}\right)\frac{1}{\hat{\sigma}S(t)}, \tag{2.18}
$$

$$
\Phi'\left(\frac{\hat{\mu}+\hat{\sigma}^2-K^*}{\hat{\sigma}}\right) = \phi\left(\frac{\hat{\mu}+\hat{\sigma}^2-K^*}{\hat{\sigma}}\right)\frac{1}{\hat{\sigma}S(t)}, \tag{2.19}
$$

where ϕ is the probability density function of a standard normal random variable. Therefore,

$$
b(t,T) = \frac{(1-\omega)\phi(\eta) - K\phi(\eta)}{e^{r(T-t)}\hat{\sigma}} + \frac{S(t)}{S(0)}\omega\left(\Phi(\eta+\hat{\sigma}) + \frac{\phi(\eta+\hat{\sigma})}{\hat{\sigma}}\right), \tag{2.20}
$$

where η is given by

$$
\eta = \frac{\hat{\mu}-K^*}{\hat{\sigma}}. \tag{2.21}
$$

2.4.1 Hedging Errors

Theoretically speaking, a dynamic hedging portfolio is self-financing under the arbitrage-free and continuous-time framework. It means that the hedging portfolio can rebalance itself continuously, and the changes in risky assets should offset the changes in risk-free assets. However, it is impossible to re-adjust the portfolio continuously, since continuous trading cannot be performed on financial markets. Moreover, transaction costs deter investors from trading too often. Therefore, most insurance companies choose a specific time interval, upon which to rebalance the portfolio. The trading periods could be chosen such as to optimize the wealth of the company.

Now assume that the issuers rebalance their portfolios m times in a year. We also suppose that every year is equally divided into m steps of $\frac{1}{m}$, and at $t = \frac{i}{m}$, where $i = 0, 1, \cdots, mT-1$, the portfolio value is given by

$$\begin{aligned} V\left(\frac{i}{m}, T\right) &= P\left(\frac{i}{m}, T\right) \\ &= b\left(\frac{i}{m}, T\right) + a\left(\frac{i}{m}, T\right). \end{aligned} \tag{2.22}$$

Denote the moment immediately before t as t^-, then the accumulated value carried by the replicating portfolio from time $\frac{i}{m}$ to $\frac{i+1}{m}$ is

$$V\left(\left(\frac{i+1}{m}\right)^-, T\right) = \frac{b\left(\frac{i}{m}, T\right)}{S(\frac{i}{m})} S\left(\frac{i+1}{m}\right) + a\left(\frac{i}{m}, T\right) e^{\frac{r}{m}}. \tag{2.23}$$

For instance, suppose $i = 0$, then at $t = \frac{1}{m}$,

$$V\left(\left(\frac{1}{m}\right)^-, T\right) = \frac{b(0,T)}{S(0)} S\left(\frac{1}{m}\right) + a(0,T)e^{\frac{r}{m}}. \tag{2.24}$$

Therefore, the hedging errors occur at times $t = \frac{i+1}{m}$, which are the times when the hedging portfolio is rebalanced. The hedging error is determined as the difference between the contract price and the accumulated value at t

$$HE\left(\frac{i+1}{m}\right) = P\left(\frac{i+1}{m}, T\right) - V\left(\left(\frac{i+1}{m}\right)^-, T\right), \tag{2.25}$$

where $i = 0, 1, \cdots, mT-1$. The random present value of the total hedging errors is

$$PV^f(HE) = \sum_{i=0}^{mT-1} e^{-r\frac{i+1}{m}} HE\left(\frac{i+1}{m}\right), \tag{2.26}$$

where PV^f indicates the present value for a financial guarantee.

In addition, if the hedging error is positive, extra funds need to be invested, since the value of the financial derivative is more than the replicating portfolio accumulated value. In contrast, if the error is negative, the hedging error can be considered as a source of profit, since the actual accumulated value is more than what is needed to rebalance the portfolio. In this case, the company can take away the difference and may use it for other investing purposes. Note that the hedging errors for a financial guarantee are also referred as the financial or tracking errors, since they are caused purely by the random behavior of the risky assets.

Figure 2.1 shows the distribution of the present value of the hedging errors, obtained by performing 50,000 stochastic simulations for one 10-year PTP financial guarantee contract. Figure 2.2 represents the distribution of the hedging errors' present values for 100 contracts. From Figure 2.1, we see that the mean of these present values is 0, and the overall shape is symmetrical, and looks like a normal distribution. The value-at-risk $VaR_{95\%}$ indicates that 95% of the errors are less than 1.02%. The conditional-tail-expectation $CTE_{95\%}$ shows that the expected value of the errors which are bigger than $VaR_{95\%}$ is 1.45%. Although the numerical numbers give us the impression that errors are small, it is important to pay attention to these errors, since if a large amount of contracts are sold, the losses or gains caused by errors are significant. Note that tracking errors are impossible to avoid, and they are also referred as systematic errors. The VaR and CTE in Figure 2.2 are roughly 100 times larger than in Figure 2.1.

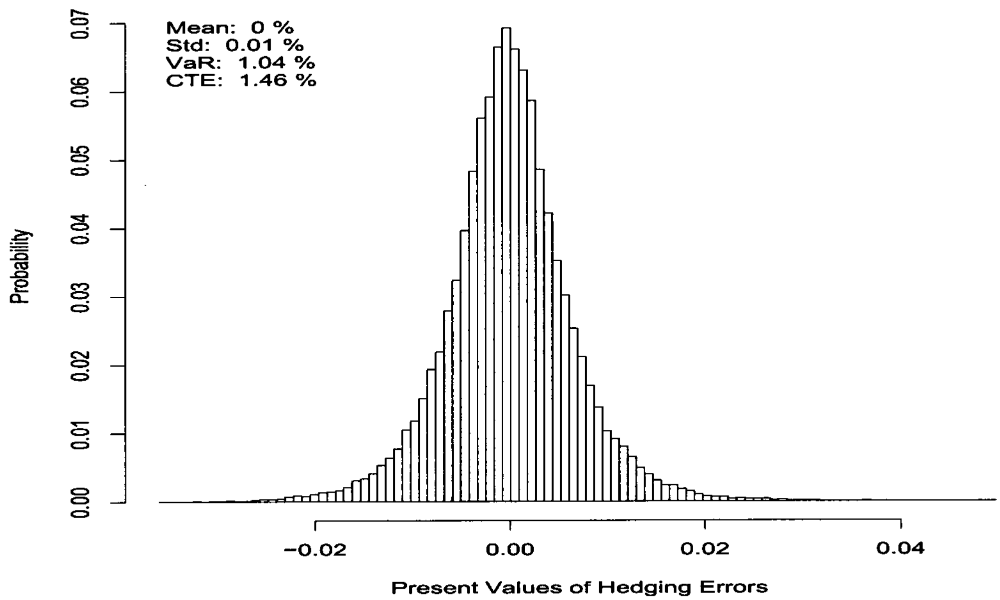

Figure 2.1: Present values of tracking errors for one 10-year financial guarantee.

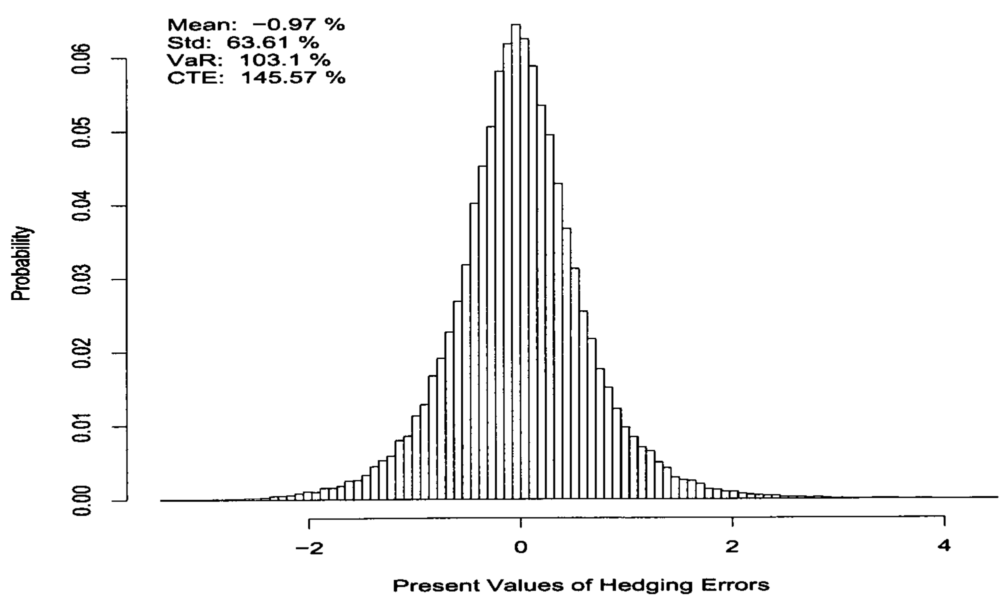

Figure 2.2: Present values of tracking errors for 100 contracts with a 10-year financial guarantee.

Figures 2.3 to 2.6 represent the distributions of the present values of hedging errors for 5-year and 15-year contracts. Their shapes are similar to the 10-year contract. It is important to specify that the hedging portfolio replicates most of the payoffs of the financial derivative, which means that the company is able to reduce the risks using the hedging portfolio. Furthermore, note that as the contract term lengthens, the present values of the hedging errors decrease. This is because that with a longer term, the hedging errors will be discounted back over longer periods. As a result, the present values of hedging errors for longer term contracts are smaller than those with a shorter contract term.

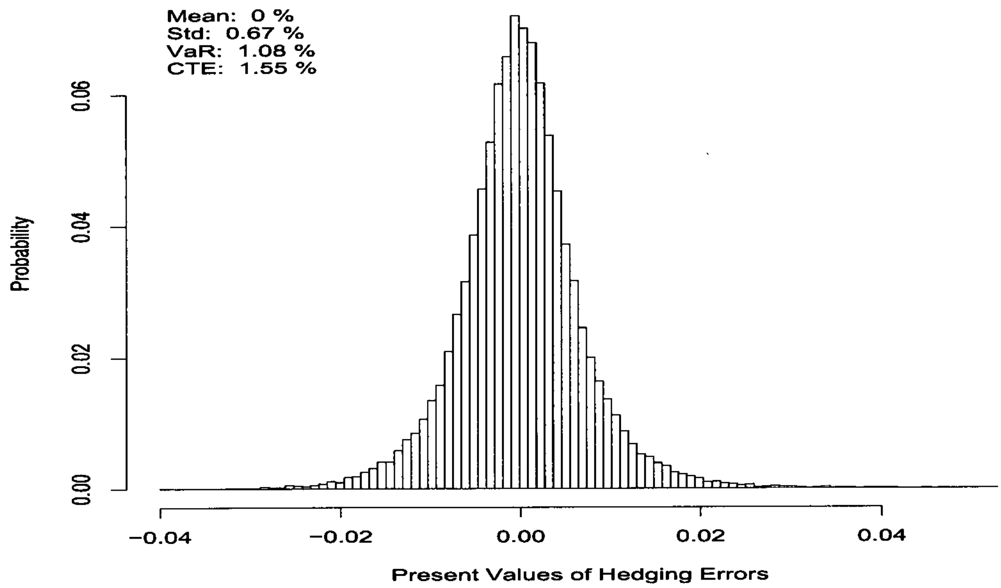

Figure 2.3: Present values of tracking errors for one 5-year financial guarantee.

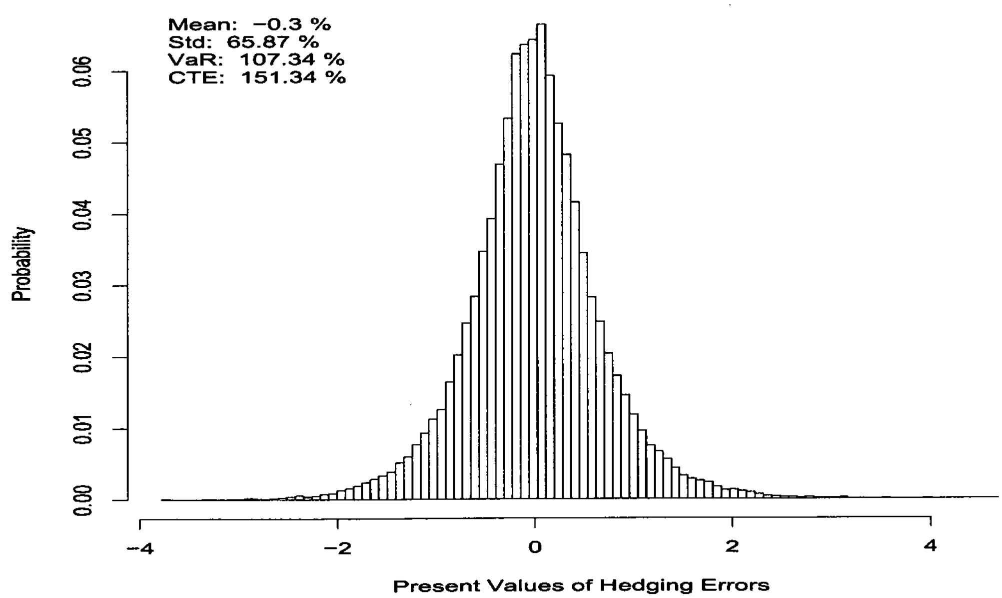

Figure 2.4: Present values of tracking errors for 100 contracts with a 5-year financial guarantee.

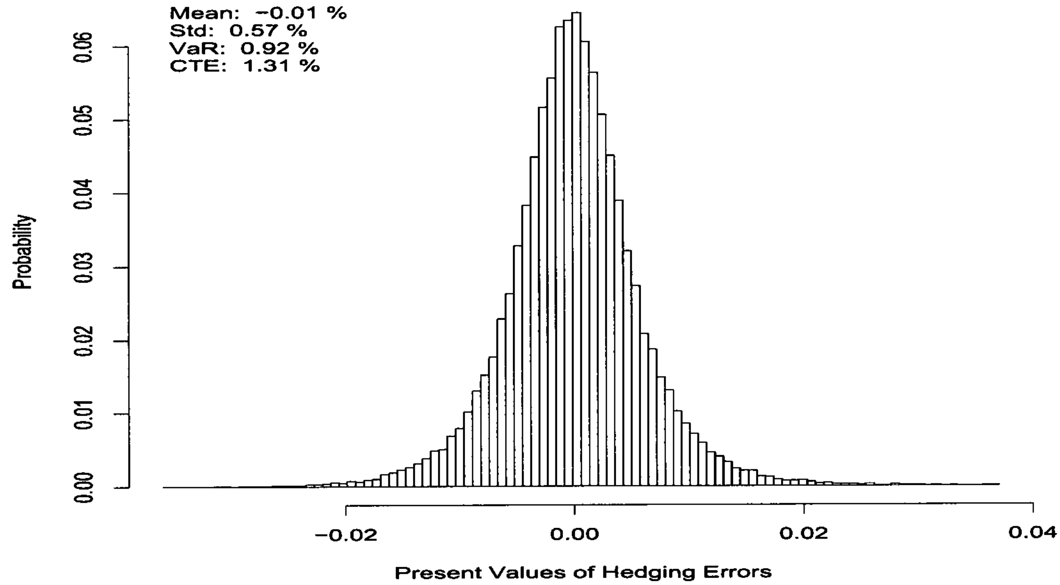

Figure 2.5: Present values of tracking errors for one 15-year financial guarantee.

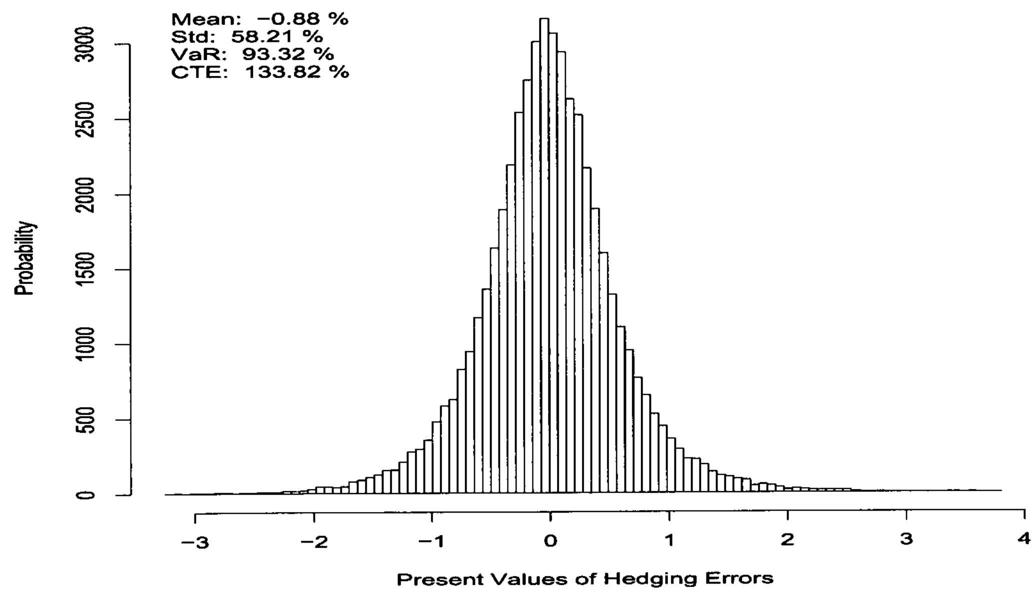

Figure 2.6: Present values of tracking errors for 100 contracts with a 15-year financial guarantee.

Chapter 3

Equity-Indexed Annuity

3.1 Introduction

Since 1995, equity-indexed annuities (EIAs) have been commonly sold by insurance companies, selling over 20 billion annually. The popularity of EIAs has been increasing world-wide. Unlike other traditional annuity products which generally pay fixed amounts to the insureds, the EIAs link the annuity amounts with the performance of the equity market. By offering a limited participation rate, a guaranteed rate of return, and guaranteed minimum payments, an EIA product allows the insured to benefit from the attractive profits occurred in a long-term bull market and protect them from suffering a bear market.

Different EIA products offer different financial guarantees which have different payoff designs. In general, a policyholder can receive an amount of $G(T)$ at the end of the maturity date T, if he survives throughout the contract term. We also consider a mortality option that provides the policyholder the right to receive a certain amount at the end of his year of death. Suppose a policyholder dies at $t < T$, then he is eligible to receive an amount of $G(K(x,s)+1)$ at the end of year $\lfloor t \rfloor + 1$, where we recall that $K(x,s)$ is the curtate lifetime $\lfloor T(x,s) \rfloor$, and $\lfloor \cdot \rfloor$ is the floor function.

3.2 Pricing a Point-to-Point EIA

In this thesis, we focus on the point-to-point equity-indexed annuity (PTP). The PTP product usually charges clients a single premium in year s, at time 0. We denote this premium as $P_{x,s}(0,T)$, where (x,s) indicates that the EIA product is issued to a person age x in year s. Furthermore, the payoff of the PTP is given by

$$\begin{cases} G(K(x,s)+1), & K(x,s) = 0,1,\cdots,T-2 \\ G(T), & K(x,s) = T-1, T, \cdots \end{cases} \tag{3.1}$$

that is

$$\begin{cases} P_{x,s}(0,T)\max\left(1+\omega(\frac{S(K(x,s)+1)}{S(0)}-1), R(K(x,s)+1)\right), K(x,s) = 0,1,\cdots,T-2 \\ P_{x,s}(0,T)\max\left(1+\omega(\frac{S(T)}{S(0)}-1), R(T)\right), K(x,s) = T-1, T, \cdots \end{cases} \tag{3.2}$$

Using (2.5), we can obtain the price at time 0

$$P_{x,s}(0,T) = E^Q\left[I_{K(x,s)<T-1}\frac{G(K(x,s)+1)}{e^{r(K(x,s)+1)}} + I_{K(x,s)\geq T-1}\frac{G(T)}{e^{rT}}\Big|\mathcal{F}(0)\right], \tag{3.3}$$

where I_B equals 1, if B is true, and 0 otherwise. Conditioning on (3.3) at the time of death leads to

$$\begin{aligned} P_{x,s}(0,T) &= \sum_{h=0}^{T-2} Pr^Q[K(x,s)=h]\ E^Q\left[\frac{G(K(x,s)+1)}{e^{r(K(x,s)+1)}}\Big|\mathcal{F}(0), K(x,s)=h\right] + \\ & \quad Pr^Q[K(x,s)\geq T-1]\ E^Q\left[\frac{G(T)}{e^{rT}}\Big|\mathcal{F}(0), K(x,s)=T-1\right]. \end{aligned} \tag{3.4}$$

Moreover, we suppose that the policyholder (x) and the stock index are independent, therefore

$$P_{x,s}(0,T) = \tilde{q}_{x,s}P(0,1) + {}_{1|}\tilde{q}_{x,s}\, P(0,2) + \cdots + {}_{T-2|}\tilde{q}_{x,s}\, P(0,T-1) + {}_{T-1}\tilde{p}_{x,s}\, P(0,T), \tag{3.5}$$

where $P(,)$ is given in (2.16), $Pr^Q[K(x,s) = h]$ and $Pr^Q[K(x,s) \geq T-1]$ are replaced by $\tilde{q}_{x,s}$ and ${}_{T-1}\tilde{p}_{x,s}$ respectively. Note that $\tilde{q}_{x,s}$ is the forecasted mortality rate obtained under the LC model in Chapter 1, Section 1.5.1.

In this thesis, we suppose that mortality probabilities are the same under both the P and Q measures, that is $\tilde{q} = q$ for all x and s. Equation (3.5) shows that $P_{x,s}(0,T)$ is a weighted average price, where the mortality rates are the weights. For simplicity, we assume $P_{x,s}(0,T) = 1$ throughout this thesis. Furthermore, we will use the same parameters as in Chapter 2, Section 2.3, where $\sigma = 0.25$, $\mu = 10\%$, $r = 6\%$, $g = 3\%$, and $\rho = 90\%$, and we will apply the mortality rates obtained in Table 1.2 to our future analysis for the PTP product. Therefore, using the same numerical method–non-linear minimization method as in Chapter 2, we obtain the participation rate ω for a 10-year PTP, which is 0.7687158. Note that this value is very close to the ω obtained for the pure financial guarantee – 0.7698524. This is because most of the weight in (3.5) is from the probability of ${}_{T-1}p_{x,s}$, which means that most people will survive until the end of the contract term. However, the ${}_{k|}q_{x,s}$'s still give some probabilities to shorten the PTP's termination date. Moreover, the participation rates for a 5-year and 15-year PTP are 0.7073852 and 0.8092105. In general, the equity-indexed annuity participation rate increases as the maturity term extends for similar reasons as explained in Chapter 2, Section 2.3.

Denote $P_{x,s}(t,T)$ as the price of a PTP at $t \leq T$, and for $t \in [0, T-1)$, $P_{x,s}(t,T)$ is given by

$$\begin{aligned} P_{x,s}(t,T) &= E^{Q}\left[I_{K(x,s)<T-1}\frac{G(K(x,s)+1)}{e^{r(K(x,s)+1-t)}} + I_{K(x,s)\geq T-1}\frac{G(T)}{e^{r(T-t)}} \Big| \mathcal{F}(t), K(x,s) \geq \llcorner t \lrcorner \right] \\ &= \sum_{h=0}^{T-\llcorner t \lrcorner - 2} {}_{h|}q_{x+\llcorner t \lrcorner, s+\llcorner t \lrcorner}\, P(t, h+\llcorner t \lrcorner + 1) + {}_{T-\llcorner t \lrcorner - 1}p_{x+\llcorner t \lrcorner, s+\llcorner t \lrcorner}\, P(t,T). \end{aligned} \tag{3.6}$$

There is no mortality involved in the last year, and then the PTP price is given by

$$P_{x,s}(t,T) = P(t,T), \tag{3.7}$$

where $T-1 \leq t \leq T$. For instance, if $T = 10$ and $t = 7.7$, then the price is given by

$$\begin{aligned} P_{x,s}(t,10) &= \sum_{h=0}^{10-\llcorner 7.7 \lrcorner - 2} {}_{h|}q_{x+\llcorner 7.7 \lrcorner, s+\llcorner 7.7 \lrcorner} P(t, h+\llcorner 7.7 \lrcorner + 1) + {}_{T-\llcorner 7.7 \lrcorner - 1}p_{x+\llcorner 7.7 \lrcorner, s+\llcorner 7.7 \lrcorner} P(t,T) \\ &= q_{x+7,s+7}P(t,8) + {}_{1|}q_{x+7,s+7}P(t,9) + {}_{2}p_{x+7,s+7}P(t,10). \end{aligned}$$

3.3 Hedging Strategy

In this section, we will present how to extract the dynamic hedging strategy for the PTP product underlying the previous valuation methodology. Recall that $V(t,T)$ – the value of the replicating portfolio for a financial guarantee is given by

$$V(t,T) = b(t,T) + a(t,T) = P(t,T), \tag{3.8}$$

where a and b are the proportions invested in the money market and the equity market. Let $V_{x,s}(t,T)$ denote the value of a PTP hedging portfolio at time $0 \leq t \leq T$, underlying the pricing formula given in (3.6). Recall that, under the complete-market assumption, $V_{x,s}(t,T) = P_{x,s}(t,T)$. Furthermore, the hedging portfolio is a composition of $a_{x,s}(t,T)$

and $b_{x,s}(t,T)$, where $a_{x,s}$ and $b_{x,s}$ are the amounts invested in the risk-free and risky assets. Using (3.6) and (3.8), we can extract the following

$$\begin{aligned}
V_{x,s}(t,T) &= \sum_{h=0}^{T-\llcorner t \lrcorner-2} {}_{h|}q_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner}\ P(t,h+\llcorner t \lrcorner+1) + {}_{T-\llcorner t \lrcorner-1}p_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner}\ P(t,T) \\
&= \sum_{h=0}^{T-\llcorner t \lrcorner-2} {}_{h|}q_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner}\ [b(t,h+\llcorner t \lrcorner+1) + a(t,h+\llcorner t \lrcorner+1)] + \\
&\qquad {}_{T-\llcorner t \lrcorner-1}p_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner}\ [b(t,T) + a(t,T)] \\
&= \sum_{h=0}^{T-\llcorner t \lrcorner-2} {}_{h|}q_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner}\ b(t,h+\llcorner t \lrcorner+1) + {}_{T-\llcorner t \lrcorner-1}p_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner}\ b(t,T) + \\
&\qquad \sum_{h=0}^{T-\llcorner t \lrcorner-2} {}_{h|}q_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner}\ a(t,h+\llcorner t \lrcorner+1) + {}_{T-\llcorner t \lrcorner-1}p_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner}\ a(t,T).
\end{aligned}$$

Therefore, the formulas for $b_{x,s}$ and $a_{x,s}$ are given by

$$\begin{aligned}
b_{x,s}(t,T) &= \sum_{h=0}^{T-\llcorner t \lrcorner-2} {}_{h|}q_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner} b(t,h+\llcorner t \lrcorner+1) + {}_{T-\llcorner t \lrcorner-1}\, p_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner} b(t,T), \\
a_{x,s}(t,T) &= \sum_{h=0}^{T-\llcorner t \lrcorner-2} {}_{h|}q_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner} a(t,h+\llcorner t \lrcorner+1) + {}_{T-\llcorner t \lrcorner-1}\, p_{x+\llcorner t \lrcorner,s+\llcorner t \lrcorner} a(t,T).
\end{aligned} \tag{3.9}$$

From the above equations, we can see that $b_{x,s}$ and $a_{x,s}$ are weighted average of b's and a's. Because there is no mortality involved in the last year of the contract, we have

$$b_{x,s}(t,T) = b(t,T), \quad \text{and} \quad a_{x,s}(t,T) = a(t,T), \tag{3.10}$$

for $t \in [T-1, T)$.

Theoretically, the replicating portfolio can perfectly replicate the financial instrument under a complete market. However, as we have said previously, companies are unable to reset the hedging portfolios continuously in the real financial market. Hence, we suppose that they only rebalance the replicating portfolios m times per year. Furthermore, because the complete-market assumptions are not met, hedging errors will occur.

Therefore, each year is divided into m times of intervals with length of $\frac{1}{m}$, and hedging errors occur at the re-adjustment time $t = \frac{i+1}{m}$, where $i = 0, 1, \cdots, m-1, m, \cdots, mT-1$.

3.3.1 Hedging Errors

Recall that we denote t^- as the moment immediately before reaching time t, then the replicating portfolio accumulated value at t^- is given by

$$V_{x,s}(t^-, T) = \frac{b_{x,s}\left(\frac{i}{m}, T\right)}{S\left(\frac{i}{m}\right)} S\left(\frac{i+1}{m}\right) + a_{x,s}\left(\frac{i}{m}, T\right) e^{\frac{r}{m}}. \tag{3.11}$$

We split the hedging errors into two parts as HE_1 and HE_2, where HE_1 represents the financial tracking errors occurred at rebalancing moments, and HE_2 represents the hedging errors occurring at the end of the year of death, or when the contract is terminated. Here HE_1 and HE_2 are given by

$$HE_1\left(\frac{i+1}{m}\right) = P_{x,s}\left(\frac{i+1}{m}, T\right) - V_{x,s}\left(\left(\frac{i+1}{m}\right)^-, T\right), \tag{3.12}$$

where $i = 0, 1, \cdots, mK(x,s) - 1, \quad K(x,s) = 0, 1, \cdots, T-1$, and

$$HE_2(K(x,s)+1) = G(K(x,s)+1) - V_{x,s}\left((K(x,s)+1)^-, T\right), \tag{3.13}$$

where $K(x,s) = 0, 1, \cdots, T-2$. Suppose the insurance company sells one PTP contract at year s to a policyholder whose age is x, then the replicating portfolio for this contract will provide hedging errors at all the adjusting times $t = \frac{1}{m}, \frac{2}{m}, \cdots, K(x,s)+1$, until the year he dies or the end of the contract term T, whichever happens first. If the policyholder survives throughout the whole contract, then we will only have HE_1 hedging errors, and the random present value of the hedging errors in this case is denoted as PV^s is given by

$$PV^s(HE) = \sum_{i=0}^{mT-1} e^{\frac{-r(i+1)}{m}} HE_1\left(\frac{i+1}{m}\right), \tag{3.14}$$

Moreover, if the policyholder dies within year $u < T$, which implies that $K(x,s) = u-1$, then the random present value of hedging errors is denoted as PV^d, and is given by

$$PV^d(HE) = \sum_{i=0}^{mu-2} e^{\frac{-r(i+1)}{m}} HE_1\left(\frac{i+1}{m}\right) + e^{-ru} HE_2(u). \tag{3.15}$$

We now consider the case that the insurance company issues the PTP equity-indexed annuity to a cohort initially contains $\mathfrak{L}_{x,s}$ policyholders, where $\mathfrak{L}_{x,s} = l_{x,s}$ is a known constant as defined in Chapter 1, Section 1.4. Furthermore, recall that $\mathfrak{D}_{x,s}$ denotes the number of death occurring between age x and $x+1$ in year s. Hence, if $\mathfrak{D}_{x,s}$ policyholders die during the first year, it means that $K(x,s) = 0$, and $\mathfrak{D}_{x,s}$ contracts are terminated at time 1. Therefore, the present value of the hedging errors is denoted as PV^c, and is given by

$$PV^c(HE) = \mathfrak{L}_{x,s} \sum_{i=0}^{m-2} \frac{HE_1(\frac{i+1}{m})}{e^{\frac{r(i+1)}{m}}} + \mathfrak{D}_{x,s} \frac{HE_2(1)}{e^r} + \mathfrak{L}_{x+1,s+1} \sum_{i=m-1}^{mT-1} \frac{HE_1(\frac{i+1}{m})}{e^{\frac{r(i+1)}{m}}}.$$

From the above equation, we see that HE_1 errors occur only to policyholders who are alive, either until one period $\frac{1}{m}$ before they die, or until the contract finishes at T. Moreover, HE_2 occurs to the deceased policyholders at the end of their year of death. Indeed, the more general way of computing the present value of the hedging errors for a pool of policyholders is given by

$$\begin{aligned} PV^c(HE) &= \sum_{h=0}^{T-2} \left[\mathfrak{L}_{x+h,s+h} \sum_{i=mh}^{m(h+1)-1} \frac{HE_1(\frac{i}{m})}{e^{\frac{ri}{m}}} + \mathfrak{D}_{x+h,s+h} \frac{HE_2(h+1)}{e^{r(h+1)}} \right] + \\ & \quad \mathfrak{L}_{x+T-1,s+T-1} \sum_{i=m(T-1)}^{mT} \frac{HE_1(\frac{i}{m})}{e^{\frac{ri}{m}}}, \end{aligned} \tag{3.16}$$

where $HE_1(0)$ is set to be 0, since at time 0, the replicating portfolio value is exactly the same as the initial contact price.

3.3.2 Policyholder's Time of Death

In the previous section, we have obtained formulas to calculate the present values of the hedging errors. In this section we will discuss how to simulate the policyholders time of death, under deterministic mortality rates. We will use the forecasted mortality rates in Table 1.2 to define the probabilities of a 50-year old person living in 2005, who dies in year 2006, 2007,$\cdots$, 2015. Furthermore, recall that ${}_{h|}q_{x,s}$ is the probability that (x) at year s survives exactly h years and then dies in the following year.

$$
\begin{aligned}
{}_{1|}q_{x,s} &= p_{x,s} * q_{x+1,s+1} = (1 - q_{x,s})q_{x+1,s+1} \\
{}_{2|}q_{x,s} &= p_{x,s}p_{x+1,s+1}q_{x+2,s+2} = (1 - q_{x,s})(1 - q_{x+1,s+1})q_{x+2,s+2} \\
&\vdots \\
{}_{h|}q_{x,s} &= \prod_{i=0}^{h-1}(1 - q_{x+i})q_{x+h,s+h} \qquad (3.17)
\end{aligned}
$$

Because a policyholder will either die or survive on the interval of $(0, T-1]$, the sum of the death and survival probabilities is 1.

$$\sum_{h=0}^{T-2} {}_{h|}q_{x,s} + {}_{T-1}p_{x,s} = 1. \qquad (3.18)$$

Therefore, an individual policyholder's year of death can be generated using the inverse transform method to simulate random variables.

Recall that if a discrete variable X takes values $x_1, x_2, \cdots, x_n$ with probabilities $p_1, p_2, \cdots, p_n$, and the sum of p_i's is 1, then $X = x_i$ if the random generated number $v \in (0,1)$, satisfies $F(x_{i-1}) < v \leq F(x_i)$. In fact, the interval (0,1) is divided into n times of small intervals, each with length of $p_1, p_2, \cdots, p_n$, and $X = x_i$ if the generated number falls in the ith interval. Hence, a policyholder's year of death takes values $1, 2, \cdots, T$ and has the following distribution function:

$$\begin{cases} P(K(x,s)=0) = q_{x,s} = p_1, \\ P(K(x,s)=1) = {}_{1|}q_{x,s} = p_2, \\ \vdots \\ P(K(x,s)=T-2) = {}_{T-1|}q_{x,s} = p_{T-1}, \\ P(K(x,s)\geq T-1) = {}_{T-1}p_{x,s} = p_T. \end{cases} \tag{3.19}$$

The years of death for a cohort are simulated differently, but are somehow related. We will still use the forecasted mortality rates in Table 1.2 to simulate the year of death. However, instead of generating one number only, we generate $\mathcal{L}_{x,s}$ random numbers at once. Hence, we get $\mathcal{L}_{x,s}$ numbers u_i, which take values $1, 2, \cdots, T$. These numbers represent the years at which some contracts are terminated, since the deaths of some policyholders take place prior to the contract maturity date. Furthermore, payoffs of $G(u_i)$ must be paid at the end of these years – u_i. Note that, if the simulated numbers take the value T, it means that a policyholder actually has survived until the end of year $T-1$, and the contract will end at its natural maturity date.

After generating all the years of death for the cohort, we then count how many times each number appears, and the total occurrence of that number represents the number of deaths happened in that year. For instance, let $T = 5, \mathcal{L}_{x,s} = 10$. Suppose that one simulation gives us a series numbers (5, 5, 5, 5, 3, 4, 5, 5, 3, 5). Therefore, out of a total of 10 people, two policyholders die in year 3, which implies $\mathcal{D}_{x+2,s+2} = 2$, one in year 4, implying that $\mathcal{D}_{x+3,s+3} = 1$, and the remaining 7 people survive until the end of year 4, $\mathcal{L}_{x+4,s+4} = 7$. In fact, the policyholders will either die or survive in each year, which is equivalent to saying that the death event – I_{Deth} equals 1, if it happens, and equals 0 otherwise. Therefore, the number of deaths that occurred in each year follows a binomial distribution, where the size of the samples is $\mathcal{L}_{x+h,s+h}$, and the probability of one death happening in year h is ${}_{h|}q_{x,s}$, where $h = 0, 1, \cdots, T-2$.

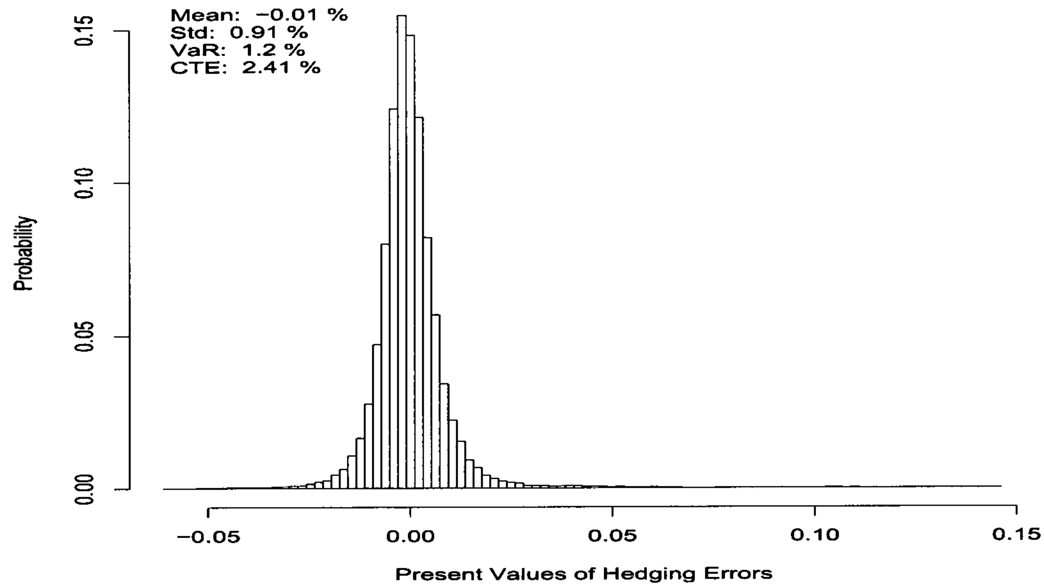

Figure 3.1: Present values of hedging errors for one 10-year PTP contract.

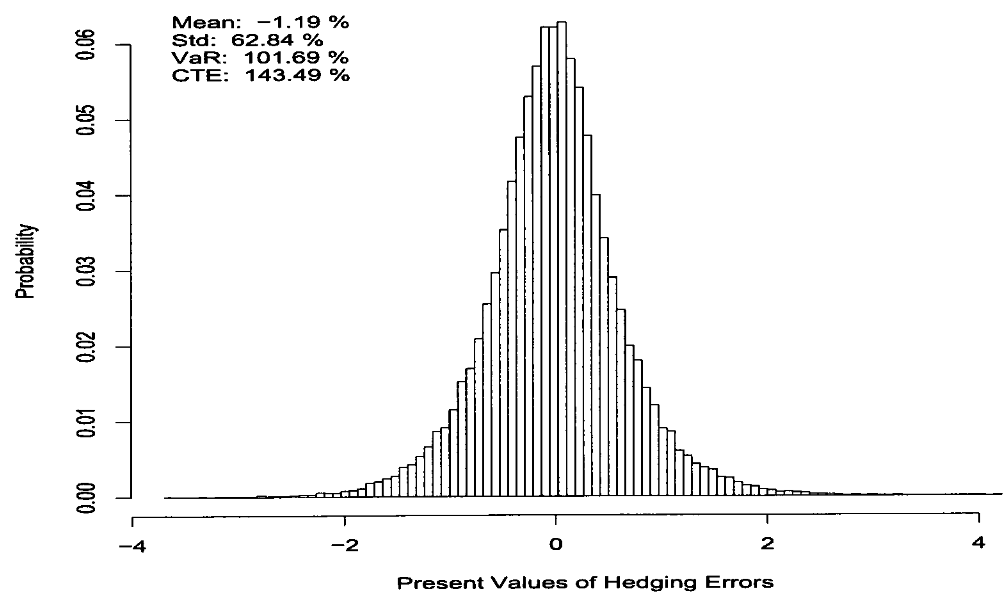

Figure 3.2: Present values of hedging errors for 100 10-year PTP contracts.

Figure 3.1 shows the distribution of the random present value of the hedging errors for a PTP with one policyholder, where the year of death is simulated using the forecasted mortality rates in Table 1.2. From the graph, we see that both the mean and the variance are quite small. The $VaR_{95\%}$ means that 5% of the $PV(HE)$ are greater than 1.21%, and the $CTE_{95\%}$ shows that the expected value of the $PV(HE)$ that are greater than 1.21% is 2.37%.

Figure 3.2 shows the present values of the hedging errors for 100 contracts. Both figures show that the $PV(HE)$ are small, however, Figure 3.1 indicates that the case of one policyholder only is more risky. Even though the bulk of Figure 3.1 is similar to the graphs in Chapter 2, and the range of $PV(HE)$ is wider. By contrast, Figure 3.2 appears to be almost the same as Figure 2.2. The reason for this is that adding more policyholders diversifies the risks caused by mortality, assuming that policyholders are independent. That is why VaR and CTE values in Figure 3.2 are smaller than 100 times the VaR and CTE in Figure 3.1.

Moreover, Figures 3.3 to 3.6 exhibit the distributions of the 5-year and 15-year PTP contracts present values of hedging errors. Note that as the contract term lengthens, VaR values are increasing slightly for only one contract, and CTE values increase a bit faster. However, VaR and CTE values both decrease for 100 contracts.

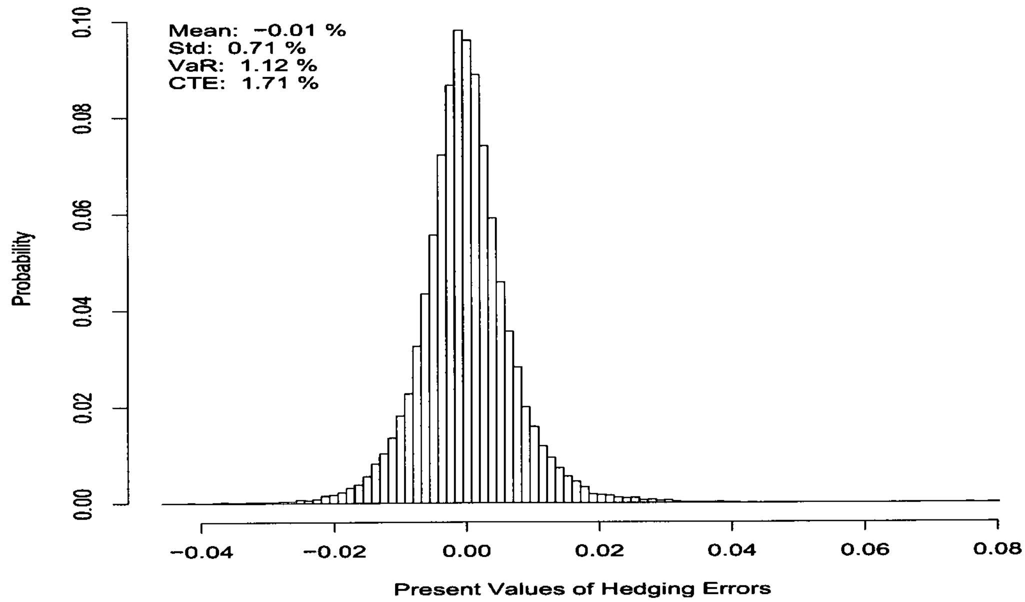

Figure 3.3: Present values of hedging errors for one 5-year PTP contract.

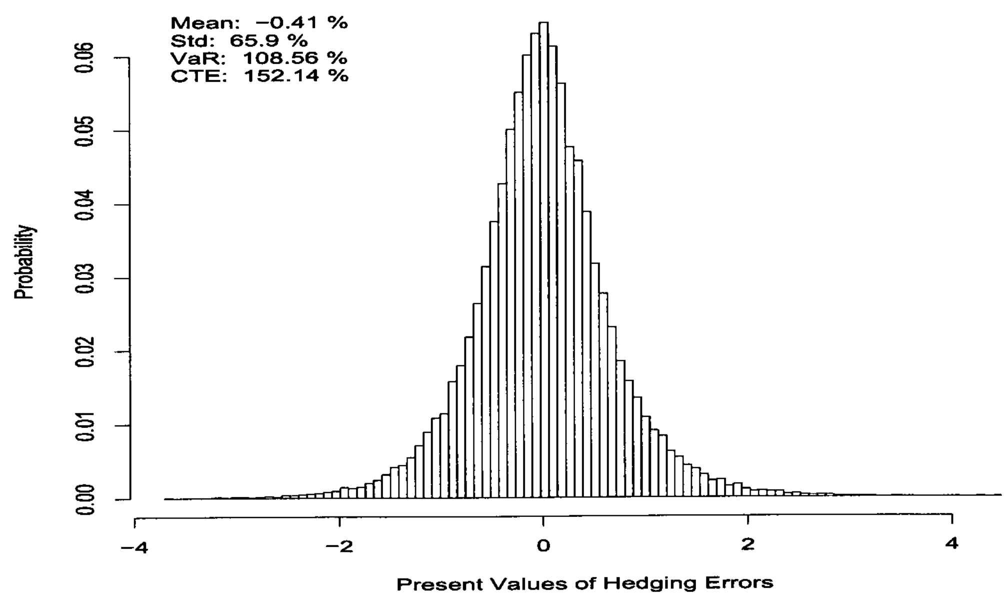

Figure 3.4: Present values of hedging errors for 100 5-year PTP contracts.

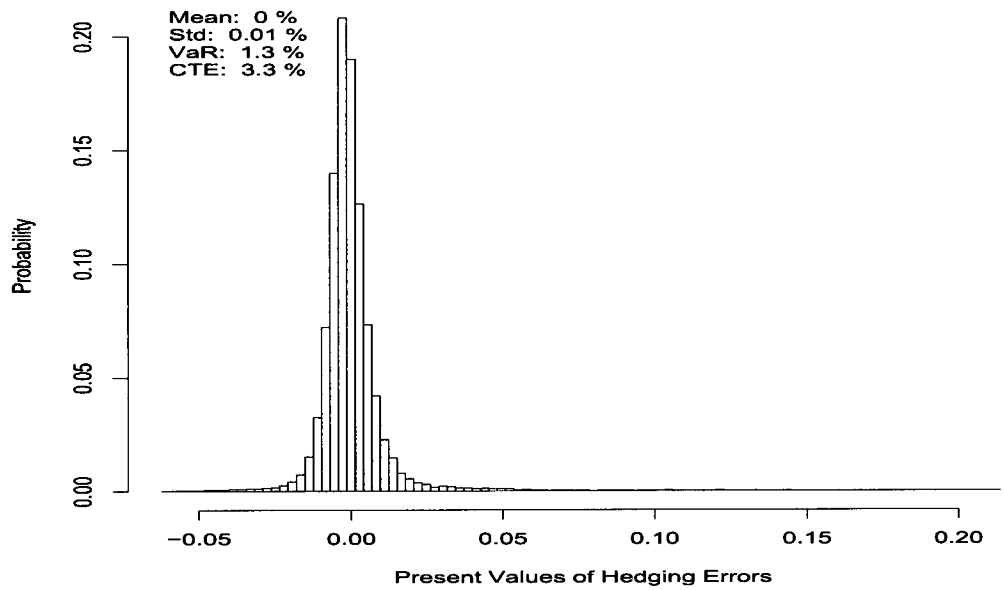

Figure 3.5: Present values of hedging errors for one 15-year PTP contract.

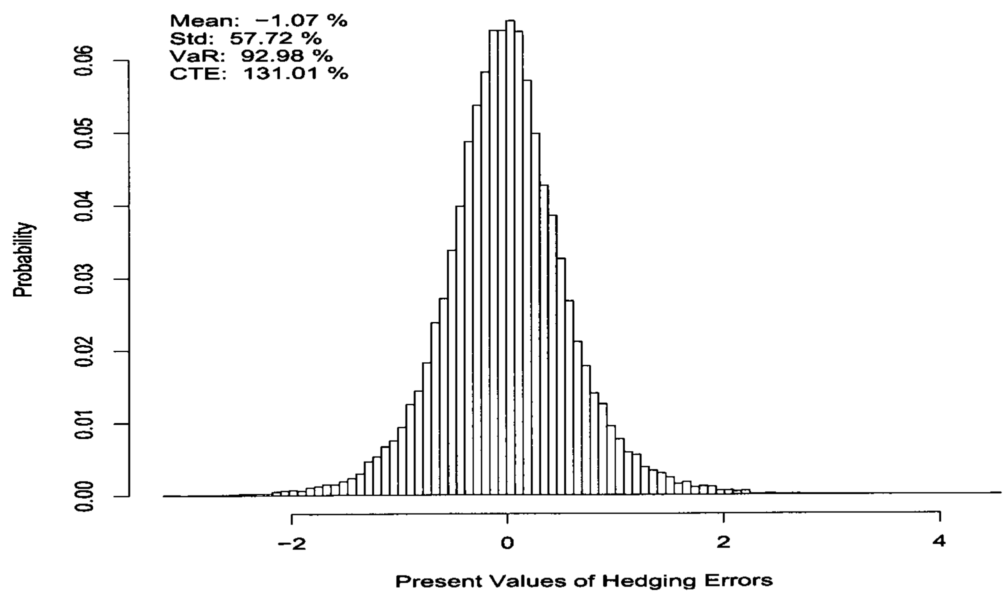

Figure 3.6: Present values of hedging errors for 100 15-year PTP contracts.

3.4 Valuation Under Stochastic Mortality

Previously, we simulated the policyholders' years of death using a fixed mortality Table 1.2. We are now interested in analyzing the effects caused by the stochastic mortality rates. Therefore, we will first simulate a series of stochastic mortality rates, and then use the new rates to generate the years of death, following the same procedure as in the previous section. Recall that the steps of generating stochastic mortality rates were explained in Section 1.6, and in this thesis, we only consider policyholders who are age 50 in year 2005.

Furthermore, recall the pricing formula for $0 < t < T - 1$ in (3.6)

$$P_{x,s}(t,T) \quad = \quad E^Q\left[I_{K(x,s)<T-1}\frac{G(K(x,s)+1)}{e^{r(K(x,s)+1-t)}} + I_{K(x,s)\geq T-1}\frac{G(T)}{e^{r(T-t)}}\middle|\mathcal{F}(t), K(x,s) \geq \llcorner t \lrcorner\right]$$

where $\mathcal{F}(t)$ represents the information from the equity market, as well as the change in mortality up to time t. Moreover, conditioning on the policyholder's time of death, and assuming the policyholder's mortality and equity market are independent, the above equation can be written as

$$\begin{aligned}
&\sum_{h=0}^{T-\llcorner t \lrcorner-2} Pr^Q\left[K(x,s) = \llcorner t \lrcorner + h \middle| \mathcal{F}(t), K(x,s) \geq \llcorner t \lrcorner\right] \times \\
&E^Q\left[\frac{G(K(x,s)+1)}{e^{r(K(x,s)+1-t)}}\middle|\mathcal{F}(t), K(x,s) = h + \llcorner t \lrcorner\right] + \\
&Pr^Q\left[K(x,s) \geq T-1 \middle| \mathcal{F}(t), K(x,s) \geq \llcorner t \lrcorner\right] \; E^Q\left[\frac{G(T)}{e^{r(T-t)}}\middle|\mathcal{F}(t), K(x,s) = T-1\right],
\end{aligned}$$

where $Pr^Q[K(x,s) = \llcorner t \lrcorner + h|\mathcal{F}(t), K(x,s) \geq \llcorner t \lrcorner]$ represents the probability that (x) dies between $x + \llcorner t \lrcorner + h$ and $x + \llcorner t \lrcorner + h + 1$, given that he is alive at age $x + \llcorner t \lrcorner$ and the mortality information up to time t.

Furthermore, we still assume that

$$Pr^{Q}[K(x,s) = \lfloor t \rfloor + h | \mathcal{F}(t), K(x,s) \geq \lfloor t \rfloor] = {}_{h|}\tilde{q}_{x+\lfloor t \rfloor + h, s+\lfloor t \rfloor + h}, \quad h = 0, 1, \cdots, T-1.$$

However, these mortality rates need to be forecasted using the updated information up to time t. Recall that we can use the approach presented in Section 1.5.2 to forecast mortality rates ${}_{h|}\tilde{q}_{x+\lfloor t \rfloor, s+\lfloor t \rfloor}$. More specifically, if (x) has survived through year 2005 and is alive in 2006, then we need to use the mortality index observed in 2006 to generate the following stochastic $\tilde{k}_{h+2006}$'s. In other words, the forecasted mortality rates for $s = 2007, 2008, \cdots$ should be based on the $\tilde{k}_{2006}$ in Table 1.2. Same ideas and procedures should be applied for the consequent years. Furthermore, the age-parameters – α_x, β_x need to be updated as well, since a 50-year old person in 2005 will turn 51-years old in 2006. The remaining calculations for $PV(HE)$ are as in (3.15) and (3.16). Therefore, (3.20) is equivalent to the following

$$P_{x,s}(t,T) = \sum_{h=0}^{T-\lfloor t \rfloor - 2} {}_{h|}\tilde{q}_{x+\lfloor t \rfloor, s+\lfloor t \rfloor}\, P(t, h + \lfloor t \rfloor + 1) + {}_{T-\lfloor t \rfloor - 1}\tilde{p}_{x+\lfloor t \rfloor, s+\lfloor t \rfloor}\, P(t,T),$$

where q is not assumed to be the same as the forecasted $\tilde{q}$ anymore, and it is the simulated mortality rate updated annually.

Figure 3.7 shows the distribution of present values of the hedging errors for one 10-year PTP contract with stochastic mortality rates. The overall shape looks the same as in Figure 3.1, while $VaR_{95\%}$ and $CTE_{95\%}$ are almost the same as in Figure 3.1. This result is somehow surprising, since intuitively, adding extra randomness should make the $PV(HE)$ more volatile. We suppose it is because that the mortality rates are quite small – $0.00265, 0.00323, \cdots$, and therefore, they do not have a big impact on the hedging errors for one contract.

Furthermore, Figure 3.8 represents the distribution of $PV(HE)$ for 100 contracts, obtained under the Lee-Carter model. We can see again that the stochastic mortality rates do not cause significant changes, since Figure 3.8 looks almost the same as Figure 3.2. However, VaR and CTE do increase somewhat compared with Figure 3.2. Note that for a cohort of policyholders, we do not assume they are independent. Indeed, we think this is likely to explain why we have larger values for VaR and CTE in Figure 3.8, since policyholders are likely to die or survive at the same time, and adding more contracts cannot reduce the mortality risks as much as under the fixed mortality rates.

In addition, the differences between simulations may also lead to slight changes in the simulated results. Nonetheless, both two figures indicate that with a large number of policyholders, mortality risks are well diversified. The figures of 5-year and 15-year PTP contracts also follow the same patterns as the 10-year PTP.

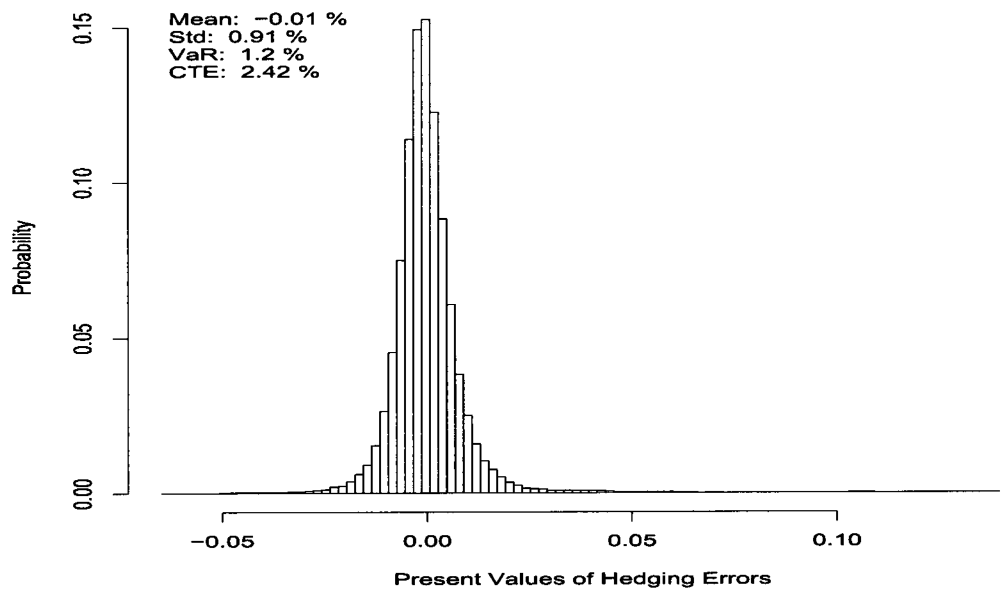

Figure 3.7: Present values of hedging errors for one 10-year PTP contract under Lee-Carter mortality model.

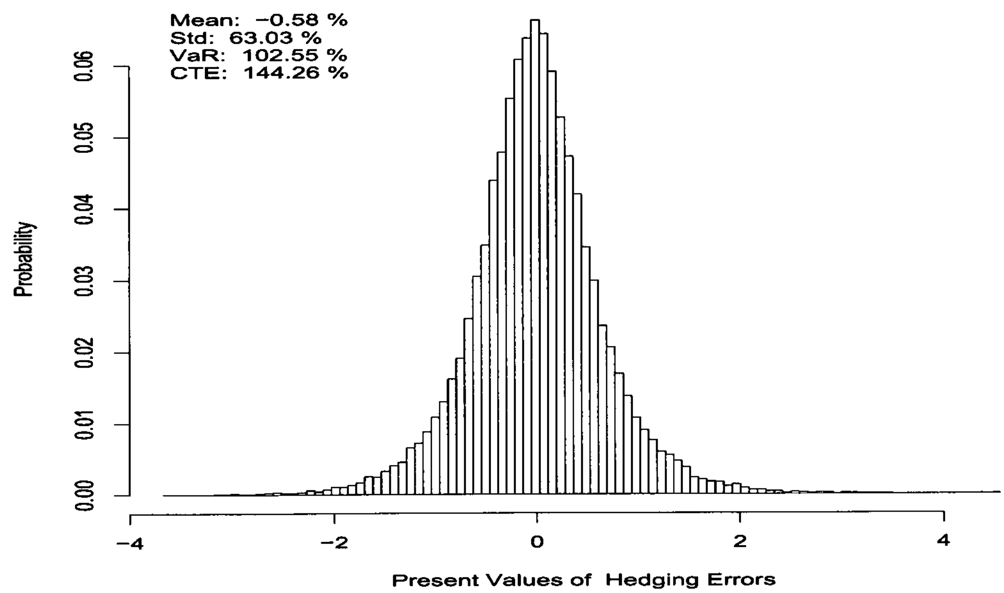

Figure 3.8: Present values of hedging errors for 100 10-year PTP contracts under Lee-Carter mortality model.

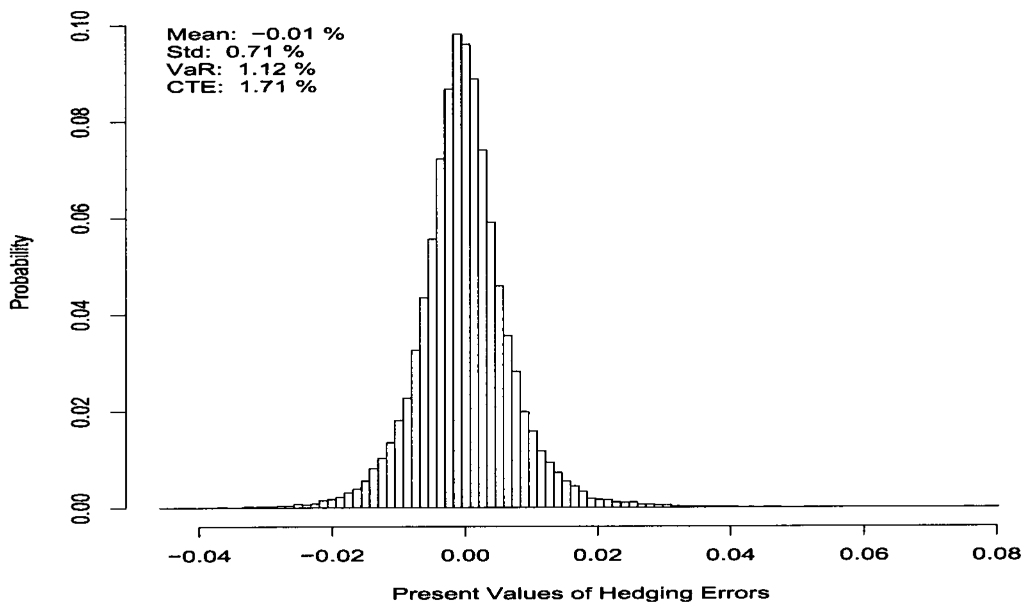

Figure 3.9: Present values of hedging errors for one 5-year PTP contract under Lee-Carter mortality model.

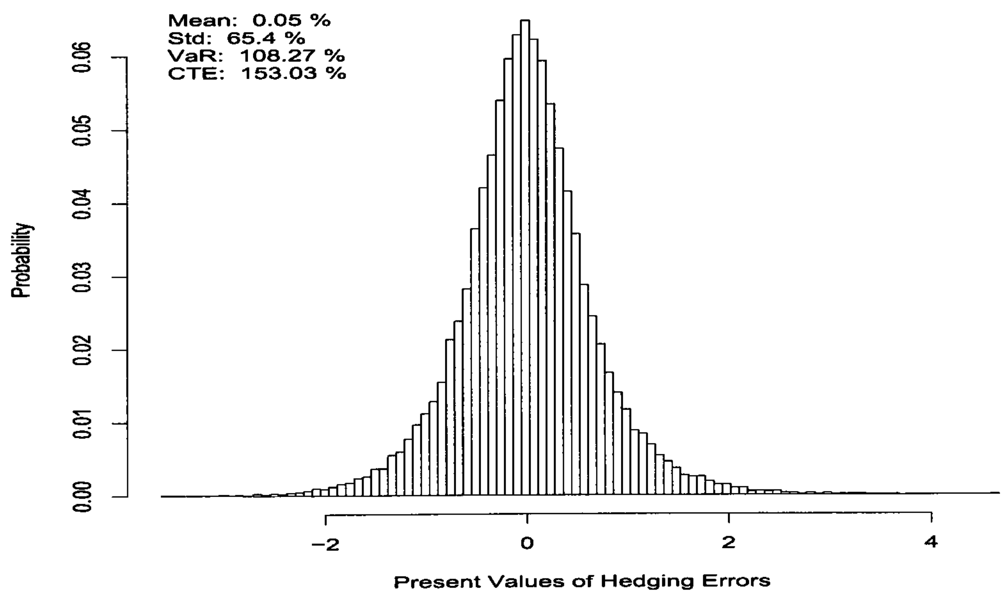

Figure 3.10: Present values of hedging errors for 100 5-year PTP contracts under Lee-Carter mortality model.

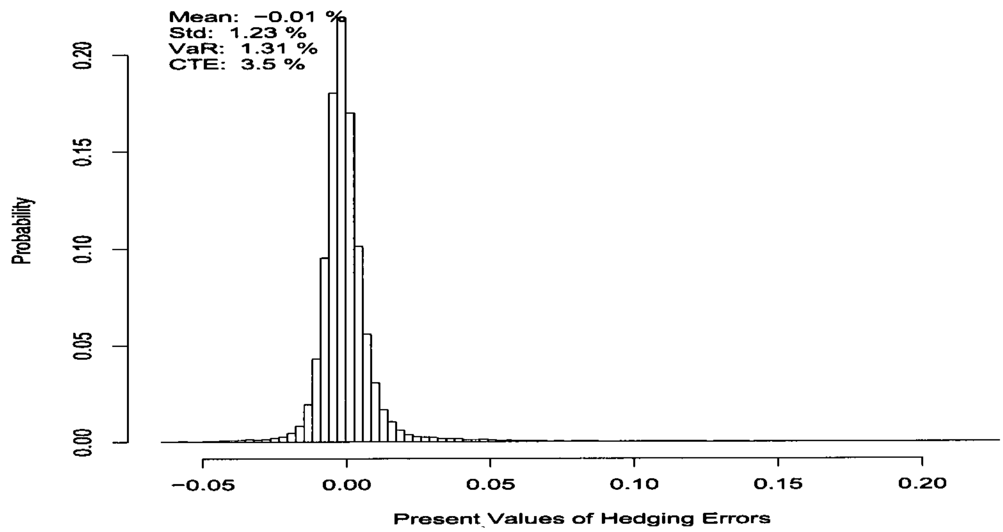

Figure 3.11: Present values of hedging errors for one 15-year PTP contract under Lee-Carter mortality model.

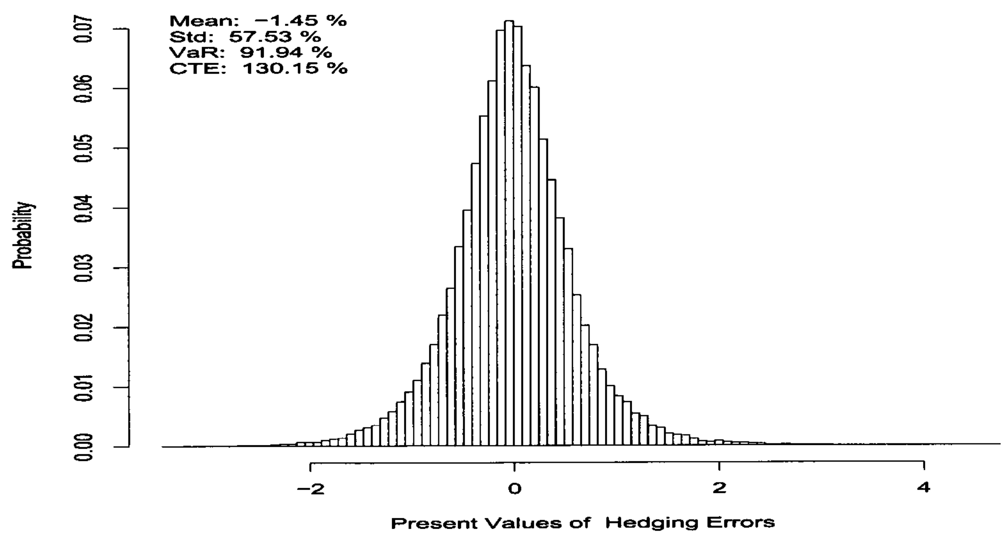

Figure 3.12: Present values of hedging errors for 100 15-year PTP contracts under Lee-Carter mortality model.

Conclusion

From the figures, we can see that for a PTP equity-indexed annuity, the distribution of hedging errors' present values for 100 contracts is more symmetrical than for one contract only. Moreover, because VaR and CTE values for 100 contracts are much less than 100 times the VaR and CTE for one contract only, we can conclude that selling one contract is more risky than selling a number of contracts. By contrast, for a pure financial guarantee, the support of the distribution of the present value of hedging errors for 100 contracts is roughly 100 times larger than the support for one contract only. This is because the pricing for a pure financial guarantee does not take mortality rates into consideration.

The mean and variance of the hedging errors are small in all cases, indicating that the replicating portfolio can effectively reduce the risks of EIA contracts. The figures also show that with the help of a replicating portfolio, insurance companies can be in a balanced-off position most of the time. Moreover, as Gaillardetz and Lakhmiri (2009) points out, insurance companies can charge a security loading to the initial premium, so that the hedging errors tend to be negative. Note that if the distribution of the present values of hedging errors can shift to the left of 0, then companies will be exposed to profits, rather than losses in most of the time. Recall that the hedging errors are defined as the differences between the value of the EIA contracts and the actual value of the portfolio at adjustment time t. Hence, a negative hedging error means that the replicating portfolio is worth more than the actual contract issued by the insurance company. As a result, the company will have larger assets to fulfil the liabilities of the contracts.

In general, a contract with a longer maturity term has a higher participation rate. Furthermore, a pool of contracts with longer terms may experience more risks caused by both the equity market and the changes in mortality rates. However, due to the fact that the replicating portfolio can greatly reduce the financial market risk and mortality rates are quite small, the present values of hedging errors decrease as the contract term increases, particularly because of the larger discount factor (larger T). On the other hand, when there is only one contract, VaR and CTE values increase as the term lengthens, since selling one contract is much more risky. The graphs indicate that if enough contracts can be sold, mortality risks can be diversified greatly, under the assumption of independence between policyholders.

However, there is not a strong evidence that policyholders are independent. In fact, we believe that policyholders are dependent, since for a group of people who are at the same age and exposed to the same environment, they are more likely to either survive or die during the same time, due to the same causes. Therefore, we introduce the stochastic Lee-Carter model to simulate stochastic mortality rates, along with the simulation of stock process. We then compute the price of the EIA and its hedging errors, with the updated stochastic mortality information. Indeed, we expected to have larger VaR and CTE values under the stochastic mortality rates, since theoretically, adding extra risk to the existing process, the overall risks should increase accordingly. However, from the figures obtained under the stochastic mortality model, we do not see significant differences on VaR or CTE values, compared with those obtained with non-stochastic mortality rates.

Even though the graphs do not seem to support our initial hypothesis, we still believe that mortality risks are present, and under the stochastic mortality model, mortality risks can not be diversified as much as under the deterministic mortality rates. Perhaps, Lee-Carter model's specific design is a possible cause for underestimating the mortality risks. Further studies on the effects caused by stochastic mortality rates should be conducted, and one can use another stochastic model to generate the mortality rates to further test the results.

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