On RG-spaces and the space of prime d-ideals in $C(X)$

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Abstract

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Let A be a commutative semiprime ring with identity. Then A has at least two epimorphic regular extensions namely, the universal epimorphic regular extension $T(A)$, and the epimorphic hull $H(A)$. We are mainly interested in the case of C(X), the ring of real-valued continuous functions defined on a Tychonoff space X. It is a commutative semiprime ring with identity and it has another important epimorphic regular extension namely, the minimal regular extension $G(X)$. In our study we show in chapter 5 that the spectrum of the ring $H(A)$ with the spectral topology is homeomorphic to the space of the prime ζ-ideals in A with the patch topology. In the case of $C(X)$, the spectrum of the epimorphic hull $H(X)$ with the spectral topology is homeomorphic to the space of prime d-ideals in $C(X)$ with the patch topology.

A Tychonoff space X which satisfies the property that $G(X) = C(X_\delta)$ is called an RG-space. We shall introduce a new class of topological spaces namely the class of almost k-Baire spaces, and as a special case of this class we shall have the class of almost Baire spaces. We show that every RG-space is an almost Baire space but it need not be a Baire space. However in the case of RG-spaces of countable pseudocharacter, RG-spaces have to be Baire spaces. Furthermore in this case every dense set in RG-spaces has a dense interior.

The Krull z-dimension and the Krull d-dimension will play an important role to determine which of the extensions $H(X)$ and $G(X)$ has the form of a ring of real-valued continuous functions on some topological space. In [31] the authors gave some techniques to prove that there is no RG-space with infinite Krull z-dimension, but there was an error that we found in the proof of theorem 3.4. In this study we will give an accurate proof which applies to many spaces but the general theorem will remain

open. And we will use the same techniques to prove that if $C(X)$ has an infinite chain of prime d-ideals then $H(X)$ cannot be isomorphic to a ring of real-valued continuous functions.

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This thesis is dedicated to the memory of my loving father, Abohalfya Emhemed, (1924-2003) may Allah (God) have mercy on him.

Farhat M. Abohalfya

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Chapter 1

Introduction

We begin with the assumption that our category is the category of commutative rings and by a ring we mean a commutative ring with identity. By a ring homomorphism we mean a ring homomorphism that preserves the identity. For a ring A, let $Z(A)$ denote the set of zero-divisors of A.

For A, a commutative semiprime ring with identity, there are at least two epimorphic regular extensions namely, the universal epimorphic regular extension $T(A)$ which was defined by [Olivier] in [27], and the epimorphic hull $H(A)$ which was defined and studied by [Storrer] in [33]. Hochster in [10] proved that the spectrum of $T(A)$ as a topological space with the spectral topology can be identified with the spectrum of A with the patch topology. Our first goal in this thesis is to show that the spectrum of the ring $H(A)$ with the spectral topology can be considered as the space of prime ζ-ideals in A with the patch topology. We show that in the case where A satisfies the strongly $a.c.$ condition, then the spectrum of the ring $H(A)$ with the spectral topology can be identified with the space of prime d-ideals of A with the patch topology.

The ring $C(X)$ of real-valued continuous functions defined on a topological space X is a commutative semiprime ring with identity. And it has another epimorphic regular extension namely the minimal regular extension $G(X)$ which was defined and studied by Henriksen, Raphael, and Woods in [20]. Also in the same paper the definition of RG-spaces was given and the properties of such spaces were studied in [20], [31] and [21]. Our second goal in this thesis is to get more results on RG-spaces. This leads us to introduce a new class of topological spaces, the class of almost k-Baire spaces and as a special case of this class we will have the class of almost Baire spaces. In [31] the authors gave some techniques to prove that there is no RG-space with infinite Krull z-dimension. But there was an error that we found in the proof of theorem 3.4. Our third goal in this thesis is to give an accurate proof of this result which applies to many spaces although the general theorem will remain open. We use the same techniques to prove that $H(X)$ cannot be isomorphic to a ring of real-valued continuous functions if $C(X)$ has an infinite chain of prime d-ideals.

This chapter is divided into three sections. The first section is mostly devoted to introducing the basic conceptual machinery in ring theory to be used in this thesis. The second section contains the basic concepts on prime d-ideals and prime ζ-ideals which we need for our research, and the third section will contain a review of the basic concepts on algebraic frames. We have included [23], [24], [32] and [37] as general references.

1.1 Basic concepts

Semiprime and regular rings

Definition 1.1 *An element e of a commutative ring A is said to be idempotent if $e^2 = e$. An element a is said to be nilpotent if there exists a positive integer n with $a^n = 0$. A ring A is called semiprime if A has no nilpotent elements except 0.*

It is well known fact that a ring A is semiprime if and only if the intersection of all prime ideals in A is the zero ideal. And it is clear from the definition that every subring of a semiprime ring is semiprime.

Definition 1.2 *An element a of a ring A is called regular if $\exists\ b \in A$ such that $a = a^2b$. If every element of A is regular then A is said to be a regular ring (in the sense of von Neumann).*

If $a = a^2b$ for some $b \in A$ then ab and $1 - ab$ are idempotent elements.

Remark 1.1 *If A is a regular ring and $a = a^2b$ then $a^* = b^2a$ satisfies the equations $a = a^2a^*$ and $a^* = (a^*)^2a$ and b^2a is called the quasi-inverse of a.*

Lemma 1.1 *Let A be a regular ring. Then the quasi-inverse of a is unique for each $a \in A$.*

Proof.

Let r_1, r_2 be two quasi-inverses of $r \in A$. Then $r = r^2r_1, r_1 = r_1^2r$ and $r = r^2r_2, r_2 = r_2^2r$. Now $r_2 = r_2^2r = r_2^2r^2r_1 = r_2^2r^2r_1^2r = r_2rr_1^2r = rr_1^2 = r_1$. Thus the quasi-inverse of a is unique for each $a \in A$. □

Definition 1.3 *Let A be a ring such that for each $a \in A$ $\exists n \geqslant 1$ and $b \in A$ such that $a^n = (a^n)^2b$. Then A is said to be a Π-regular ring.*

Clearly every regular ring is a Π-regular ring.

Theorem 1.1 *Let A be a ring. Then A is a regular ring if and only if A is a semiprime Π-regular ring.*

Proof.

($\Longrightarrow$) *Let A be a regular ring, and let $x \in A$ such that $x \neq 0$. Then $x = x^2y$ for some $y \in A$. Suppose $x^n = 0$ for some $n \geq 1$, and let n_0 be the smallest integer such that $x^{n_0} = 0$. Then $n_0 \geq 2$, $x^{n_0-1} \neq 0$, and $x^{n_0-1} = (x^{n_0-1})^2z$ for some $z \in A$. Since $2(n_0 - 1) \geq n_0$, it follows that $x^{n_0-1} = 0$, which is a contradiction. Therefore A is semiprime.*

($\Longleftarrow$) *Let A be a semiprime Π-regular ring and let $x \in A$. Then $x^n = (x^n)^2y$ for some $y \in A$, and therefore $x^n - (x^n)^2y = 0$ which means $(x(1 - (x^n)y))^n = 0$. But A is semiprime, so $(x(1 - (x^n)y)) = 0$. If $n = 1$ we are done, if not $x - (x^{n+1})y = 0$ which implies that $x = (x^{n+1})y$ and therefore $x = x^2(x^{n-1})y$. Thus A is regular.* □

Since every finite ring is Π-regular, it follows for finite rings that being regular is the same as being semiprime.

Example 1.1 *$(Z_4, +, .)$ is Π-regular but is neither semiprime nor regular.*

Let A be a ring and let S be a non-empty multiplicative subset of A. Then there is an equivalence relation defined on $A \times S$ by: $(r_1, s_1) \sim (r_2, s_2) \iff s_0(r_1 s_2 - r_2 s_1) = 0$ for some $s_0 \in S$. By denoting the equivalence class $[(r, s)]$ by $\frac{r}{s}$ one can turn the set $S^{-1}A = \{\frac{r}{s} : r \in A, s \in S\}$ into a commutative ring with identity [12]. This ring $S^{-1}A$, denoted A_S, is called the localization of A at S. In particular, if P is a proper prime ideal of A, then $A - P$ is a multiplicative subset and the localization of A at P, denoted A_P, is defined to be A_{S-P}. Moreover, there is a one-to-one correspondence between the set of prime ideals in A_P and the set of prime ideals in A that are contained in P [12, p.147].

For each $s_0 \in S$ we have $\Phi_{s_0} : A \longrightarrow A_S$ defined by $\Phi_{s_0}(a) = \frac{as_0}{s_0}$ is a ring homomorphism. If I, J are ideals of A and A_S respectively then $(\Phi_{s_0})^{-1}(J)$, denoted J^c *or* $J \cap A$, is an ideal of A and $\Phi_{s_0}(I)B$, denoted I^e, is an ideal of A_S. Details appear in [12].

We will use these facts in the next theorem.

Theorem 1.2 *Let A be a semiprime ring. Then the following are equivalent:*

(1) *A is regular.*

(2) *Every prime ideal is maximal.*

(3) *Every principal ideal is generated by an idempotent element.*

(4) *Every finitely generated ideal is generated by an idempotent element.*

Proof.

(1) $\Longrightarrow$ (2) *Let A be a regular ring and $P \neq A$ be a prime ideal. For any element $a \notin P$, we have that $a = a^2b$ for some $b \in A$ which means $(a(1 - ab) = 0)$. Then $1 \in P + aA$ and therefore $P + aA = A$. Hence P is a maximal ideal .*

(2) $\Longrightarrow$ (1) *Suppose that every prime ideal is maximal, and let M be any maximal ideal of A. If I is any prime ideal of A_M, then I^c is a prime ideal of A with $I^c \subseteq M$, which implies that $I = M^e$. Then A_M has a unique prime ideal namely M^e, and therefore M^e is the unique semiprime ideal in A_M. As A is a semiprime ring, then $\{0\}$ is a semiprime ideal in A_M. Then $\{0\} = M^e$ and A_M is a field.*

Clearly every invertible element is regular. Let r be a non-invertible element in A, and let $T = \{t \in A : rt = 0\}$. Then $\langle T, r\rangle$ cannot be contained in any maximal ideal M, as A_M is a field. So $\langle T, r\rangle = A$, which implies that $1 = t + rx$ for some $x \in A$. Then $r = r^2x$ and A is regular.

(1) $\Longrightarrow$ (3) *Let $\langle a\rangle$ be a principal ideal. Since $a = a^2b$ for some $b \in A$. Then $e = ab$ is an idempotent element and $\langle a\rangle = \langle e\rangle$.*

(3) $\Longrightarrow$ (4) *It is enough to prove the statement when I is finitely generated by two idempotent elements. Let $I = \langle e, f\rangle$ where e, f are idempotent elements. Since $(e + f - ef)$ is an idempotent element and $\langle e, f\rangle = \langle (e + f - ef)\rangle$ then I is generated by an idempotent element.*

(4) $\Longrightarrow$ (1) *Let $x \in A$, then $xA = eA$ where e is an idempotent element. Then there are $y, z \in A$ such that $x = ez$, $e = xy = x^2y^2$ which implies that $x = x^2y^2z$. Thus A is regular.* □

Definition 1.4 *If K is any non-empty subset of the commutative ring A, we write $K^* = \{a \in A : aK = \{0\}\}$ and call this the annihilator of K. An ideal I of A is called dense if $I^* = \{0\}$.*

We abbreviate $(K^*)^*$ as K^{**} for each non-empty subset K of A and $\{a\}^*$ as $\langle a\rangle^*$ for each a in A.

Remark 1.2 *(a) K^* is an ideal of A for each non-empty subset K of A.*

(b) If K_1, K_2 are subgroups of A. Then $K_1^* \cap K_2^* = (K_1 + K_2)^*$.

Theorem 1.3 *Let A be a semiprime ring and let K be an ideal of A. Then $K \cap K^* = \{0\}$ and $K + K^*$ is a dense ideal of A.*

Proof.

Let K be an ideal of A. Since $KK^* = 0$ and $(K \cap K^*)^2 \subseteq KK^* = \{0\}$, then $(K \cap K^*)^2 = \{0\}$ and therefore $K \cap K^* = \{0\}$. But $(K + K^*)^* = K^* \cap K^{**}$. Then $(K + K^*)^* = \{0\}$. Thus $K + K^*$ is dense. $\square$

Definition 1.5 *A subring A of a ring S is called large if for every $0 \neq t \in S$ there exists a in A with $0 \neq at \in A$. An ideal I of A is called large if it is a large subring.*

Clearly a subring A of a ring S is large if and only if $A \cap tA \neq \{0\}$ for each nonzero element $t \in S$. And it is clear that every dense ideal is a large ideal.

Theorem 1.4 *Let A be a ring. Then A is semiprime if and only if every large ideal is dense.*

Proof.

($\Longrightarrow$) *Let A be a semiprime, and let L be a large ideal. Let $0 \neq a \in A$. Since $L \cap \langle a \rangle \neq \{0\}$, let $x \in L \cap \langle a \rangle$ such that $x \neq 0$. Then $x \in L$ and $x = ra$ for some $r \in A$. Since $x^2 \neq 0$ and $x^2 \in La$. So $La \neq \{0\}$ for each $a \neq 0$. Thus L is dense.*

($\Longleftarrow$) *Suppose that every large ideal is dense and suppose $\exists\, x \in A$ such that $x \neq 0$ and $x^n = 0$ for some $n \geq 2$. Then there is an element $y \in A$ such that $y \neq 0$ and $y^2 = 0$. Let $I = \langle y \rangle$, then it is clear that $I^2 = \{0\}$. Now suppose $\langle a \rangle \cap I^* = \{0\}$ for some a in A. Since $aI \subseteq \langle a \rangle \cap I^* = \{0\}$ then $aI = \{0\}$ which implies $a = 0$. So I^* is a large ideal and therefore by assumption I^* is dense. But $aI^* = \{0\}$ for each $a \in I$. Therefore $I = \{0\}$, which is a contradiction with $I = \langle y \rangle \neq \{0\}$. Hence A is semiprime.* $\square$

Remark 1.3 *Let D_1, D_2 be two ideals of A. Then:*

(1) *$D_1 \subseteq D_2$ and D_1 dense implies that D_2 is dense ideal.*

(2) *D_1 and D_2 dense implies that D_1D_2 and $D_1 \cap D_2$ are dense ideals too.*

Rings of quotients

The ring B is an extension of A if A is a subring of B and they have the same identity.

If B is an extension of A then the set $c^{-1}A = \{a \in A : ca \in A\}$ is an ideal of A for each $c \in B$.

Definition 1.6 *Let B be an extension of A. Then B is called a ring of quotients of A if $c^{-1}A$ is a dense subring of B for each $c \in B$, in another words, if $\forall\ c \in B$ and $\forall\ 0 \neq d \in B\ \exists\ a \in A$ such that $ac \in A$ and $ad \neq 0$.*

Lemma 1.2 *Let B be a ring extension of A Then:*

(1) *B is a ring of quotients of A if and only if $b^{-1}A$ is dense ideal in A and $b(b^{-1}A) \neq \{0\}\ \forall\ 0 \neq b \in B$.*

(2) *$b(b^{-1}A) \neq \{0\}\ \forall\ 0 \neq b \in B$ implies that $b^{-1}A$ is a large ideal in A for each $b \in B$.*

Proof.

(1) ($\Longrightarrow$) *Obvious*

($\Longleftarrow$) *Suppose $b^{-1}A$ is dense in A and $b(b^{-1}A) \neq \{0\}\ \forall\ 0 \neq b \in B$. Let $b, c \in B$ such that $0 \neq c$. Since $c(c^{-1}A) \neq \{0\}$, then $\exists\ a_1 \in (c^{-1}A)$ such that $a_1c \neq 0$. Since $a_1c \in A$, then $a_1c(b^{-1}A) \neq \{0\}$. Choose $a_2 \in (b^{-1}A)$ such that $a_1a_2c \neq 0$ and let $a = a_1a_2$. Then $a \in A$, $ab \in A$ and $ac \neq 0$. Therefore B is a ring of quotients of A.*

(2) *Suppose $b(b^{-1}A) \neq \{0\}\ \forall\ 0 \neq b \in B$. We need to show that $\langle a \rangle \cap b^{-1}A \neq \{0\}$ for each $0 \neq a \in A$. If $a \in b^{-1}A$ we are done. If not, choose $a_1 \in (ab)((ab)^{-1}A) \neq \{0\}$. Then $aba_1 \in A$ and $aba_1 \neq 0$, which means $aa_1 \in \langle a \rangle$, $aa_1 \in b^{-1}A$, and $aa_1 \neq 0$. Thus $\langle a \rangle \cap b^{-1}A \neq \{0\}$ and $b^{-1}A$ is a large ideal of A.* $\square$

Theorem 1.5 *Let B be a ring extension of a semiprime ring A. Then B is a ring of quotients of A if and only if $b(b^{-1}A) \neq \{0\}\ \forall\ 0 \neq b \in B$.*

Proof.

($\Longrightarrow$) *Obvious.*

($\Longleftarrow$) *Suppose $b(b^{-1}A) \neq \{0\}\ \forall\ 0 \neq b \in B$. Then $b^{-1}A$ is a large ideal [lemma 1.2, (2)], which implies by theorem 1.4, that $b^{-1}A$ is a dense ideal of A. Thus B is a ring of quotients of A [lemma 1.2, (1)].* $\square$

The complete ring of quotients and the classical ring of quotients

The complete ring of quotients of a ring A can be constructed from equivalence classes of module homomorphisms from dense ideals of A into A. Details appear in [14, section 2.3]. The definition of addition and multiplication is natural, and the resulting ring, denoted $Q(A)$, is regular when A is semiprime [14, p.42]. Thus each $a \in A$ has a quasi-inverse a^* in $Q(A)$. Furthermore, if T is any ring of quotients of A then by [14, prop.6] there is a monomorphism of T into $Q(A)$ that induces the canonical morphism of A into $Q(A)$. Therefore $Q(A)$ is the maximal ring of quotients of A. In another words, if T is any ring of quotients of A. Then $A \subseteq T \subseteq Q(A)$.

The classical ring of quotients of a ring A, denoted by $Q_{cl}(A)$, is the subring of $Q(A)$ consisting of all elements of the form ab^{-1} where $a, b \in A$, b is a non zero-divisor in A, and b^{-1} is the inverse of b in $Q(A)$. All non zero-divisors of A are units in $Q_{cl}(A)$, and $A = Q_{cl}(A)$ if and only if each non zero-divisor in A is a unit.

Essential and epimorphic extensions

Definition 1.7 *Let B be an extension of A. Then B is called an essential extension of A if each non-zero ideal of B has non-zero intersection with A.*

Lemma 1.3 *Every ring of quotients B of A is an essential extension of A.*

Proof.

Let I be a non-zero ideal in B, and $0 \neq a \in I$. Since $a^{-1}A$ is dense ideal of A, then $\exists\, b \in A$ such that $ab \in A$, $ab \neq 0$. Thus B is an essential extension of A. $\square$

A morphism $f : A \longrightarrow B$ of a category Ç is called epic (or an epimorphism) if for all objects D and morphisms $g, h \in Hom(B, D)$ we have that $g = h$ whenever $gof = hof$. Clearly the composition of two epimorphisms is an epimorphism and if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are two homomorphisms such that gof is an epimorphism then g is an epimorphism.

Assume that A is a semiprime ring and that our category Ç is the category of commutative rings. It is well-known that any ring epimorphism defined on a regular ring A is surjective [33, 6.1]. A ring extension B of a ring A is called an epimorphic extension of A if the inclusion map from A to B is an epimorphism. So one can conclude that a regular ring has no proper epimorphic extensions.

The epimorphic hull of a ring: Let A be a semiprime ring and S be a regular ring extension of A. Then one can define the smallest regular ring lying between A and S, denoted $G_S(A)$, as follows:

$$G_S(A) = \bigcap\{C : A \subseteq C \subseteq S,\ C\ is\ a\ regular\ ring\}.$$

Theorem 1.6 *Let A be a semiprime ring and S be a regular extension of A. Then $G_S(A)$ is regular and it is the smallest regular ring lying between A and S.*

Proof.

The intersection of subrings is again a subring. Thus $G_S(A)$ is a ring and $A \subseteq G_S(A) \subseteq S$. Let $a \in G_S(A)$, then $\exists!\ a^* \in S$ such that $a = a^2a^*$ and $a^* = {a^*}^2 a$. Since a^* is a unique, then $a^* \in C\ \forall\ C$ a regular ring lying between A and S. Therefore $a^* \in G_S(A)$. Thus $G_S(A)$ is regular. □

Lemma 1.4 *If B is a regular extension of A such that $A \subseteq B \subseteq Q(A)$ then $Q_{cl}(A) \subseteq B$.*

Proof.

Let $ab^{-1} \in Q_{cl}(A)$ where b is a non zero-divisor in A. Then $a, b \in B$ and $b^* = b^{-1}$ is in B and therefore $ab^{-1} \in B$. Hence $Q_{cl}(A) \subseteq B$. □

For each semiprime ring A, the complete ring of quotients $Q(A)$ is a regular ring. Then one can talk about the ring $G_{Q(A)}(A)$ which is called the epimorphic hull of the ring A and denoted $H(A)$. It is clear that $A \subseteq Q_{cl}(A) \subseteq H(A) \subseteq Q(A)$ for each semiprime ring A. Then $H(A)$ can be defined as the smallest regular ring lying between A and $Q(A)$. As $A \subseteq H(A) \subseteq Q(A)$, it follows that $H(A)$ is a ring of quotients of A and therefore an essential extension of A.

Theorem 1.7 *Let A be a semiprime ring. Then $Q_{cl}(A)$ is regular if and only if $\forall\ a \in A\ \exists\ b \in \langle a\rangle^*$ such that $\langle a\rangle^* \cap \langle b\rangle^* = \{0\}$.*

Proof.

($\Longrightarrow$) *Suppose $Q_{cl}(A)$ is regular, and let $a \in A$. Then $\exists z \in Q_{cl}(A)$ such that $a = a^2 z$ where $z = xy^{-1}$. Let $b = y - ax$, then $ab = ay - a^2x = a^2xy^{-1}y - a^2x = 0$ which implies that $b \in \langle a\rangle^*$. To show that $\langle a\rangle^* \cap \langle b\rangle^* = \{0\}$, let $c \in \langle a\rangle^* \cap \langle b\rangle^*$. Then $0 = cb = c(y - ax) = cy - cax = cy$. But y is a non zero-divisor. Then $c = 0$ and therefore $\langle a\rangle^* \cap \langle b\rangle^* = \{0\}$.*

($\Longleftarrow$) *Suppose that $\forall\, a \in A\ \exists\, b \in \langle a\rangle^*$ such that $\langle a\rangle^* \cap \langle b\rangle^* = \{0\}$. Let $ad^{-1} \in Q_{cl}(A)$ where d is a non zero-divisor. Choose $b \in \langle a\rangle^*$ such that $\langle a\rangle^* \cap \langle b\rangle^* = \{0\}$. We claim that $a + bd$ is a non zero-divisor. Suppose $x(a+bd) = 0$. Then $ax(a+bd) = 0$, which implies that $axa = 0$, and therefore $xbd = 0$. Since d is a non zero-divisor, then $xb = 0$ and therefore $x \in \langle a\rangle^* \cap \langle b\rangle^*$. Then $a + bd$ is a non zero-divisor. Therefore $t = (a+bd)^{-1} \in Q_{cl}(A)$, and $[ad^{-1} - (ad^{-1})^2dt] = (ad^{-1})(1 - at) = (ad^{-1})(bdt) = 0$. So $ad^{-1} = (ad^{-1})^2(dt)$. Thus $Q_{cl}(A)$ is regular.*

$\square$

Lemma 1.5 *Let A be a semiprime ring such that $\langle a\rangle^*$ is a principal ideal for each a in A. Then $Q_{cl}(A)$ is regular.*

Proof.

Let $a \in A$, and let $\langle a\rangle^* = bA$. We claim that $\langle a\rangle^* \cap \langle b\rangle^* = \{0\}$. Let $0 \neq y \in \langle a\rangle^* \cap \langle b\rangle^*$. Since $\langle b\rangle^* = cA$ for some $c \in A$ then $y = r_1 b = r_2 c \neq 0$ and $y^2 = 0$, which is a contradiction. Then $\langle a\rangle^* \cap \langle b\rangle^* = \{0\}$, so by theorem 1.7, $Q_{cl}(A)$ is regular. $\square$

The Structure of $H(A)$

Theorem 1.8 *Let T be a regular extension of A, and let S be the subring of T generated by A and B where $B = \{r^*r : r \in A\}$. Then $Q_{cl}(S) = H(S)$.*

Proof.

It is clear that $S = \{\sum_{i=1}^{n} r_i e_i : r_i \in A,\ e_i \in B,\ n \geq 1\}$. Let $x = \sum_{i=1}^{n} r_i e_i$. But $\prod_{i=1}^{n} (e_i + (1 - e_i)) = 1$. Notice that this product can be written as $(\sum_{j=1}^{2^n} f_j) = 1$ where $f_j \in S$, $f_j^2 = f_j\ \forall j = 1, 2, \ldots.., 2^n$ [some of them possibly zeros], and $f_i f_j = 0$ if

$i \neq j$.

Clearly $e_i f_j = 0$ *or* 1 *depending on* i *and* j. *Now* $x = x1 = \sum_{i=1}^{n}(r_i e_i)1 = \sum_{i=1}^{n}(r_i e_i)(\sum_{j=1}^{2^n} f_j) = \sum_{j=1}^{2^n}(\sum_{i=1}^{n}(r_i e_i f_j))$.

Let $a_j = \sum_{i=1}^{n}(r_i e_i f_j)$. *Take* $t_j = r_{m1} + r_{m2} + r_{m3} + \dots\dots + r_{mn}$ *where* $e_{mi} f_j = f_j$. *Then* $a_j = t_j f_j$, *and* $x = \sum_{j=1}^{2^n} a_j$. *Since* $\{f_j : 1 \leq j \leq 2^n\}$ *is an orthogonal set then* $(\sum_{j=1}^{2^n} t_j^* f_j)\,(\sum_{j=1}^{2^n} t_j f_j)^2 = \sum_{j=1}^{2^n}(t_j^* t_j^2 f_j) = \sum_{j=1}^{2^n} t_j f_j$. *It follows that* $x^* = \sum_{j=1}^{2^n} t_j^* f_j$, $xx^* = \sum_{j=1}^{2^n}(t_j^* t_j f_j) \in S$ *and* $1 - xx^* \in S$. *To show that* $\langle x \rangle^*$ *is a principal ideal of* S, *let* $I = \langle x \rangle^* = \{t \in S : tx = 0\}$. *It is clear that* $1 - xx^* \in I$. *Conversely, if* $t \in S$ *such that* $tx = 0$ *then* $t = t(1 - xx^*)$. *Therefore* $\langle x \rangle^* = (1 - xx^*)S$. *So* $\langle x \rangle^*$ *is a principal ideal. Thus by lemma 1.4* $Q_{cl}(S)$ *is regular and* $Q_{cl}(S) = H(S)$. □

Lemma 1.6 *Let* T *be a regular extension of* A. *Then* $G_T(A) = \{\, \sum_{i=1}^{n} r_i s_i^* : r_i, s_i \in A,\ n \geq 1\}$.

Proof.

Let $B = \{r^* r : r \in A\}$, $D = \{r^* : r \in A\}$ *and let* S, K *be the subrings of* T *generated by* A, B *and* A, D *respectively. Then* $K = \{\, \sum_{i=1}^{n} r_i s_i^* : r_i, s_i \in A,\ n \geq 1\}$ *and* $S \subseteq K \subseteq G_T(A)$. *To show that* $Q_{cl}(S) \subseteq K$, *let* x *be any non zero-divisor in* S. *Since* $x = \sum_{i=1}^{n} r_i e_i$ *then as before* $x^* = \sum_{j=1}^{2^n} t_j^* f_j \in K$. *Since* $x(1 - xx^*) = 0$ *and* $(1 - xx^*) \in S$ *then* $1 - xx^* = 0$ *and* x *has* x^{-1} *in* K. *Therefore* $Q_{cl}(S) \subseteq K$. *It follows that* $A \subseteq Q_{cl}(S) = H(S) \subseteq K \subseteq T$. *But* $G_T(A)$ *is the smallest regular ring between* A *and* T. *Then* $G_T(A) = K = H(S)$. □

Corollary 1.1 *Since every finite semiprime ring is regular then* $|A| = |G_T(A)|$ *for any semiprime ring* A *with a regular extension* T.

Let A be a semiprime ring. Since the complete ring of quotients $Q(A)$ is a regular extension of A then $H(A) = \{\, \sum_{i=1}^{n} r_i s_i^* : r_i, s_i \in A,\ n \geq 1\}$, and $|A| = |H(A)|$.

Lemma 1.7 *Let* A *be a semiprime ring and let* S *be a ring extension of* A *such that* $\forall\ s \in S\ \exists\ t \in S,\ \exists\ a, b \in A$ *with* $s = at$ *and* $a = tb$. *Then* S *is an epimorphic* *extension of A.*

Proof.

Let $\imath : A \longrightarrow S$ be the inclusion map and $\alpha, \beta : S \longrightarrow T$ be ring homomorphisms such that $\alpha \circ \imath = \beta \circ \imath$.

Let $s \in S$. Then $\exists\, t \in S$, $\exists\, a, b \in A$ such that $s = at$ and $a = tb$.

Now $\alpha(s) = \alpha(at) = \alpha(a)\alpha(t) = \beta(a)\alpha(t) = \beta(bt)\alpha(t) = \beta(b)\beta(t)\alpha(t) = \alpha(b)\beta(t)\alpha(t) = \beta(t)\alpha(bt) = \beta(t)\alpha(a) = \beta(t)\beta(a) = \beta(ta) = \beta(s)$. Thus $\alpha = \beta$ and S is an epimorphic extension of A. $\square$

Corollary 1.2 *Let A be a semiprime ring. Then $H(A)$ is an epimorphic extension of A.*

Proof.

Let $\alpha, \beta : H(A) \longrightarrow S$ be ring homomorphisms such that $\alpha \circ \imath = \beta \circ \imath$ where $\imath : A \longrightarrow H(A)$ is the inclusion map. For each $r \in A$, let $t = r^{*2}$, $a = r$ and $b = r^3$. Then $s = at$, $a = tb$ and $a, b \in A$. So by the same proof as in lemma 1.7 $\alpha(r^*) = \beta(r^*)$ $\forall r \in A$. Now for any $x \in H(A)$ we have $x = \sum_{i=1}^{n} r_i s_i^*$ and $\alpha(x) = \alpha(\sum_{i=1}^{n} r_i s_i^*) = \sum_{i=1}^{n} \alpha(r_i)\alpha(s_i^*) = \sum_{i=1}^{n} \beta(r_i)\beta(s_i^*) = \beta(\sum_{i=1}^{n} r_i s_i^*) = \beta(x)$. Then $\alpha = \beta$ and $H(A)$ is an epimorphic extension of A. $\square$

The spectrum of a ring

If A is a ring, then the spectrum of A, denoted $spec(A)$, is the set of all prime ideals of A. For any subset E of A and $a \in A$, let $V(E) = \{P \in spec(A) : E \subseteq P\}$, $V(a) = V(\{a\})$, $D(E) = spec(A) - V(E) = \{P \in spec(A) : E \nsubseteq P\}$, $D(a) = D(\{a\})$, $r(E) = \bigcap\{P \in spec(A) : E \subseteq P\}$, and $r_a = r(\{a\})$.

Remark 1.4 *If A is a ring. Then :*

(1) *$V(E) = V(\langle E \rangle) = V(r\langle E \rangle)$ for each E subset of A.*

(2) *$V(\{0\}) = spec(A)$ and $V(A) = \varnothing$.*

(3) $V(\bigcup_{i\in I} E_i) = \bigcap_{i\in I} V(E_i)$ *for each family* $\{E_i : i \in I\}$ *of subsets of* A.

(4) $V(IJ) = V(I \cap J) = V(I) \cup V(J)$ *for all* I, J *ideals in* A.

Lemma 1.8 *Let* A *be a ring. Then* $X = spec(A)$ *with the collection of open sets* $\tau = \{D(E) : E \subseteq A\}$ *forms a topological space which has the collection* $\beta = \{D(a) : a \in A\}$ *as an open base.*

Proof.

Note that $D(\{1\}) = X$, $D(\{0\}) = \varnothing$, $D(E_1) \cap D(E_2) = D(\langle E_1\rangle \cap \langle E_2\rangle)$, *and* $\bigcup_{i\in I} D(E_i) = D(\bigcup_{i\in I} E_i)$. *Thus* (X, τ) *is a topological space. Since* $D(E) = \bigcup_{a\in E} D(a)$ *then* $\beta = \{D(a) : a \in A\}$ *is open base for* (X, τ). $\square$

This topology is called the **spectral topology** on *spec*(A), and it is clear that the closed subsets in this space are those of the form $V(E)$ where E is any subset of A.

Remark 1.5 *Let* A *be a ring,* $X = spec(A)$ *and* $a, b \in A$. *Then :*

(1) $D(a) \cap D(b) = D(ab)$.

(2) $D(a) = \varnothing$ *if and only if* a *is a nilpotent element.*

(3) $D(a) = X$ *if and only if* a *is a unit.*

(4) $D(a) = D(b)$ *if and only if* $r(\langle a\rangle) = r(\langle b\rangle)$.

(5) $(spec(A), \tau)$ *is a* T_0*-space.*

Corollary 1.3 *Let* A *be a ring,* $a \in A$, *and* U *be an open subset of* $spec(A)$. *Then:*

(1) $D(a)$ *is a compact subset.*

(2) U *is a compact subset if and only if* U *is a finite union of sets of the form* $D(a)$.

Proof.

(1) *It is clear that* $D(a) = D(a^n)$. *Suppose* $D(a) \subseteq \bigcup_{\alpha\in\Gamma} D(a_\alpha)$ *then* $V(a) \supseteq \bigcap_{\alpha\in\Gamma} V(a_\alpha) = V(\bigcup_{\alpha\in\Gamma}\langle a_\alpha\rangle)$ *which implies that* $r_a \subseteq r(\oplus_{\alpha\in\Gamma}(\langle a_\alpha\rangle))$. *Since* $a \in r_a$ *then* $\exists\ n_0 \geq 1$ *such that* $a^{n_0} \in \oplus_{\alpha\in\Gamma}(\langle a_\alpha\rangle)$. *Therefore* $\exists\ a_{\alpha_1}, a_{\alpha_2}, \ldots\ldots., a_{\alpha_n}$ *such that* $\langle a^{n_0}\rangle \subseteq \langle a_{\alpha_1}\rangle + \langle a_{\alpha_2}\rangle + \ldots\ldots. + \langle a_{\alpha_n}\rangle$. *It follows that* $\bigcap_{i=1}^{n} V(\langle a_{\alpha_i}\rangle) \subseteq V(\langle a^{n_0}\rangle)$. *Then*

$\bigcup_{i=1}^{n} D(\langle a_{\alpha_i}\rangle) \supseteq D(\langle a^{n_0}\rangle) = D(a)$, *and* $D(a)$ *is a compact subset for each* $a \in A$.

(2) ($\Longrightarrow$) *Suppose* U *is a compact subset. Since* U *is open, then* $U = D(I)$ *for some ideal* I *in* A. *But* $D(I) = \bigcup_{a\in I} D(a)$ *and* U *is a compact subset. Thus* $D(I) = \bigcup_{i=1}^{n} D(a_i)$.

($\Longleftarrow$) *Obvious.* $\square$

Since $spec(A) = D(1)$ then $(spec(A), \tau)$ is a compact space. For each $P \in spec(A)$ we have $cl(\{P\}) = V(P)$. So $\{P\}$ is a closed set if and only if P is a maximal ideal. Therefore the space $spec(A)$ with the spectral topology is a T_1-space if and only if every prime ideal is maximal, or equivalently if and only if A is a regular ring.

Lemma 1.9 *If* $B \subseteq spec(A)$ *then* $cl(B) = V(\bigcap_{Q\in B} Q)$.

Proof.

If $P \notin V(\bigcap_{Q\in B} Q)$, *then* $\bigcap_{Q\in B} Q \nsubseteq P$. *Choose* $a \in (\bigcap_{Q\in B} Q) - P$. *Then* $D(a)\cap B = \varnothing$ *and therefore* $cl(B) \subseteq V(\bigcap_{Q\in B} Q)$. *On the other hand, if* $P \in V(\bigcap_{Q\in B} Q)$ *and* $P \notin cl(B)$ *then* $\exists$ $D(a)$ *such that* $P \in D(a)$, $D(a) \cap B = \varnothing$. *Therefore* $a \in Q$ $\forall$ $Q \in B$, *which implies that* $a \in \bigcap_{Q\in B} Q \subseteq P$ *which contradicts* $P \in D(a)$. *Thus* $P \in cl(B)$ *and* $cl(B) = V(\bigcap_{Q\in B} Q)$. $\square$

Let $M(A)$ be the space of maximal ideals of A as a subspace of $spec(A)$ with the spectral topology. For each $M \in M(A)$, let $\wp_M = \{P : P \subseteq M\}$, and $O^M = \cap_{Q\in\wp_M} Q$.

Corollary 1.4 *Let* A *be a semiprime ring. Then* $O^M = \{a \in A : \exists\, b \notin M,\ D(b) \subseteq V(a)\}$.

Proof.

Let $T = \{a \in A : \exists\, b \notin M,\ D(b) \subseteq V(a)\}$ *and let* $x \in T$. *Then* $\exists$ $y \notin M$ *such that* $V(x) \cup V(y) = spec(A)$. *Since* $xy \in P$ $\forall$ $P \in spec(A)$ *and* $y \notin M$ *then* $x \in P$ $\forall$ $P \in \wp_M$. *Therefore* $x \in O^M$ *and* $T \subseteq O^M$. *On the other hand, suppose* $x \in O^M$ *and let* $S = \{x^n c : n \geq 0,\ c \in A - M\}$. *Then it is clear that* S *is a multiplicative subset. If* $0 \notin S$ *then* $\exists$ P *prime ideal such that* $P \cap S = \varnothing$, *which implies that* $x \notin P$ *and* $A - M \subseteq S$. *Therefore* $P \subseteq M$ *which contradicts the fact that*

$x \in O^M$. Then $0 \in S$. But $x^{n_0}c_0 = 0$ for some $n_0 \geq 1$ implies that $xc_0 = 0$. Thus $V(x) \cup V(c_0) = spec(A)$. Therefore $x \in T$ and $T = O^M$. $\square$

Lemma 1.10 *Let A be a ring and $T \subseteq M(A)$. Then $cl_M(T) = \{N : \bigcap_{M \in T} M \subseteq N\}$*

Proof.

Since $cl_M(T) = cl(T) \cap M(A)$ then $cl_M(T) = \{P : \bigcap_{M \in T} M \subseteq P\} \cap M(A) = \{N : \bigcap_{M \in T} M \subseteq N\}$. $\square$

Definition 1.8 *Let A be a ring. Then A is called a pm-ring if every prime ideal is in a unique maximal ideal.*

If A is a pm-ring, denote the unique maximal ideal containing P by M_P. Then there is a function, $\mu : spec(A) \longrightarrow M(A)$, defined by $\mu(P) = M_P$.

Theorem 1.9 *If A is a pm-ring then μ is a continuous function from $spec(A)$ onto $M(A)$.*

Proof.

Let T be a closed subset of $M(A)$, $J = \bigcap_{M \in T} M$, and $I = \bigcap_{M_P \in T} P$. Then $T = V(J) \cap M(A)$. Let us show that $\mu^{-1}(T) = V(I)$. It is clear that $\mu^{-1}(T) \subseteq V(I)$. Then one just has to show that $I \subseteq Q$ implies $M_Q \in T$. Let $Q \in spec(A)$ such that $Q \subseteq B = \bigcup_{M \in T} M$. Then $Q + J \subseteq B \subsetneq A$ which implies that $Q + J \subseteq M_1$ where M_1 is a maximal ideal containing $Q + J$. Since $J \subseteq M_1$, then $M_1 \in T$ and $\mu(q) = M_1$. Now let $I \subseteq P$ and take $SK = \{sk : s \in S, k \in K\}$ where $S = A - B$ and $K = A - P$. Then SK is a multiplicative subset, and $SK \cap I = \varnothing$. Therefore by Zorn's Lemma $\exists\ Q$ a prime ideal such that $I \subseteq Q$, $Q \cap SK = \varnothing$. Then $Q \subseteq B$, $Q \subseteq P$ which implies $\mu(Q) = M_1 \in T$ and $\mu(Q) = \mu(P)$. So $I \subseteq P$ implies that $\mu(P) \in T$. Thus $\mu^{-1}(T) = V(I)$ and μ is a continuous function. $\square$

A topological space X is called irreducible if each non-empty open set is dense, or equivalently if each pair of non-empty open subsets U, V has a non-empty intersection. It is clear that every T_2-space with more than one point is not an irreducible space. The space $spec(A)$ is irreducible if and only if the ideal of nilpotent elements is prime. It is clear that if A is semiprime then $spec(A)$ is irreducible if and only if A is an integral domain.

Lemma 1.11 *Let F be a closed irreducible subset of $spec(A)$. Then $F = V(P)$ for some $P \in spec(A)$*

Proof.

Let F be a closed irreducible subset of $spec(A)$. Then $F = V(I)$ for some ideal I. Since $V(I) = V(r(I))$, then $F = V(J)$ where $J = r(I)$. So it is enough to show that J is prime.

Let $xy \in J$. Then $V(J) = V(\langle J, x\rangle) \cup V(\langle J, y\rangle)$. Since F is an irreducible subset, then $V(J) = V(\langle J, x\rangle)$ or $V(J) = V(\langle J, y\rangle)$. WLOG, let $V(J) = V(\langle J, x\rangle)$. If $x \notin J = r(I)$ then $\exists$ P a prime ideal such that $J \subseteq P$ and $x \notin P$. Therefore $V(J) \neq V(\langle J, x\rangle)$, which is a contradiction. Thus $x \in J$ and J is prime. □

Definition 1.9 *Let X be a topological space. Then X is called a spectral space if:*

(1) *X is a compact T_0-space.*

(2) *every irreducible closed subset is a closure of one point.*

(3) *X has a base of compact open sets.*

(4) *the intersection of any two compact open sets is compact.*

Clearly for any commutative ring A with identity, the space $spec(A)$ is a spectral space.

There is another topology defined on the space $spec(A)$ called the patch topology. It is stronger than the spectral topology and it turns the space into a Hausdorff space.

Theorem 1.10 *Let A be a ring and let $\beta = \{D(a) \cap V(I) : a \in A, I$ is a finitely generated ideal $\}$. Then β forms an open base for a topology on $spec(A)$.*

Proof.

It is clear that $D(1) \cap V(\{0\}) = spec(A)$. *If* $P_0 \in (D(a_1) \cap V(I_1)) \bigcap (D(a_2) \cap V(I_2))$, *then* $P_0 \in (V(\langle I_1, I_2 \rangle) \cap D(a_1 a_2)) = (D(a_1) \cap V(I_1)) \bigcap (D(a_2) \cap V(I_2))$. *Thus* β *is an open base for a topology on* $spec(A)$. $\square$

This topology is called the **the patch topology**. For any a in A we have $D(a) \cap V(\{0\}) = D(a)$ is an open subset so the spectral topology is weaker than the patch topology. Since $V(a) = V(\langle a \rangle)$ is an open subset then $D(a)$ is clopen subset. i.e. $D(a) \in CO(X)$ $\forall$ $a \in A$. Also $V(a) \in CO(X)$ and $V(\langle a, b \rangle) = V(\langle a \rangle) \cap V(\langle b \rangle)$. Therefore $V(I) \in CO(X)$ for each finitely generated ideal I. Thus $spec(A)$ with the patch topology is a 0-dimensional Hausdorff space. We will denote the spectral topology on $spec(A)$ by τ and the patch topology by $\acute{\tau}$.

Lemma 1.12 *Let* M *be a maximal ideal of* A. *Then* M *is an isolated point in* $(spec(A), \acute{\tau})$ *if and only if* $\exists$ I *a finitely generated ideal such that* $V(I) = \{M\}$.

Proof.

$(\Longrightarrow)$ *Suppose* M *is an isolated point in* X. *Then* $\{M\} = D(a) \cap V(I)$ *where* $I = \langle a_1, a_2, a_3, ..., a_n \rangle$ *is a finitely generated ideal. Since* M *is maximal ideal then* $\exists$ $b \in A$ *such that* $ab - 1 \in M$. *Take* $J = \langle a_1, a_2, a_3, ..., a_n, ab - 1 \rangle$. *Then* J *is a finitely generated ideal and* $J \subseteq M$. *So* $D(a) \cap V(I) = \{M\} \subseteq V(J)$. *On the other hand, if* $P \in V(J)$ *is any prime ideal then* $J \subseteq P$ *which implies that* $ab - 1 \in P$. *So* $P \in D(a) \cap V(I) = \{M\}$. *Then* $V(J) = \{M\}$ *where* J *is a finitely generated ideal.*

$(\Longleftarrow)$ *Obvious.* $\square$

Theorem 1.11 *Let* A *be a ring. Then* A *is regular if and only if* $\tau = \acute{\tau}$.

Proof.

$(\Longrightarrow)$ *Suppose that* A *is a regular ring. Let* $I = \langle a \rangle$ *be a principal ideal of* A. *Since* $a^2 b = a$ *for some* $b \in A$ *then* $V(a) = D(ab - 1)$ *and therefore* $I = V(a)$ *is an open subset. Now if* $I = \langle a_1, a_2, a_3, ..., a_n \rangle$ *is any finitely generated ideal then* $V(I) = \bigcap_{i=1}^{n} V(a_i)$ *is an open subset. Thus* $\tau = \acute{\tau}$.

($\Longleftarrow$) *Suppose $\tau = \acute{\tau}$. Since $(X, \acute{\tau})$ is a Hausdorff space then (X, τ) is a Hausdorff space. Hence A is regular.* $\square$

Using the Alexander subbase lemma one can prove the compactness for $spec(A)$ with the patch topology. Details appear in [10, theorem 1].

Let $f : A \longrightarrow B$ be a ring homomorphism with $f(1_A) = 1_B$ and let I, J be ideals of A, B respectively. Then $f^{-1}(J)$, denoted J^c *or* $J \cap A$, is an ideal of A and $f(I)B$, denoted I^e, is an ideal of B. It is clear that $I \subseteq (I^e)^c$ and $(J^c)^e \subseteq J$. In this case there is another function associated with f, denoted f^a, is defined from $spec(B)$ into $spec(A)$ by $f^a(P) = f^{-1}(P)\ \forall\ P \in spec(B)$. It is a well known fact that if f is an epimorphism then f^a is an injective map [16].

Lemma 1.13 *If $f : A \longrightarrow B$ is a ring homomorphism then $f^a : spec(B) \longrightarrow spec(A)$ is a continuous function with the spectral topology [patch topology] on both spaces.*

Proof.

First, let us start with the spectral topology. Let $E = V(I)$ be a closed subset in $spec(A)$. If $P \in spec(B)$ such that $I^e = f(I)B \subseteq P$ then $I \subseteq f^{-1}(IB) \subseteq f^{-1}(P)$ and therefore $P \in (f^a)^{-1}(V(I))$. On the other hand, let $P \in (f^a)^{-1}(V(I))$. Then $f^a(P) = f^{-1}(P) \in V(I)$ and therefore $f(I)B \subseteq f(f^{-1}(P))B \subseteq P$. Then $(f^a)^{-1}(V(I)) = V(f(I)B)$ and hence f^a is a continuous map.

Secondly, let us consider the patch topology on both spaces. Since $\beta = \{D(a) \cap V(I) : a \in A$ and I is a finitely generated ideal $\}$ is an open base for the patch topology $\acute{\tau}$ on $spec(A)$, then $\Im = \{V(a) \cup D(I) : a \in A,\ I$ is a finitely generated ideal $\}$ is a closed base for $\acute{\tau}$ on $spec(A)$. Then it is enough to show that $(f^a)^{-1}(D(a))$ is a closed set in $(spec(B), \acute{\tau})$ for each $a \in A$. Let $Q \in (f^a)^{-1}(D(a))$. Then $a \notin f^a(Q)$, and therefore $f(a) \notin Q$ i.e. $Q \in D(f(a))$. Hence $(f^a)^{-1}(D(a)) \subseteq D(f(a))$. On the other hand, let $Q \in D(f(a))$. Then $f(a) \notin Q$ which means that $a \notin f^{-1}(Q) = f^a(Q)$. Then $Q \in (f^a)^{-1}(D(a))$, which implies that $D(f(a)) \subseteq (f^a)^{-1}(D(a))$. Hence $(f^a)^{-1}(D(a)) = D(f(a))$ and therefore $(f^a)^{-1}(D(a))$ is a closed subset. Thus f^a is a continuous function under the patch topology. $\square$

1.2 Prime d-ideals and prime ζ-ideals

Let A be a ring and P be a prime ideal. Then P is called a minimal prime ideal if for each $Q \in spec(A)$ such that $Q \subseteq P$ we have $Q = P$. Denote the space of minimal prime ideals by $Min(A)$.

For each $B \subseteq A$ and $a \in A$, let $P_B = \bigcap\{P \in Min(A) : B \subseteq P\}$, $P_a = P_{\{a\}}$, $V_1(B) = V(B) \cap Min(A)$, and $D_1(B) = D(B) \cap Min(A)$.

Theorem 1.12 *Let A be a semiprime ring and $P \in spec(A)$. Then P is minimal prime if and only if $\forall\ a \in P\ \exists\ b \notin P$ such that $ab = 0$.*

Proof.

($\Longrightarrow$) *Let P be a minimal prime ideal, $a \in P$, and let $S = \{sa^k : s \in (A-P)$ and $k \geqslant 1\}$. Then S is a multiplicative subset. If $0 \notin S$, then $\exists\ P_1$ a prime ideal such that $P_1 \cap S = \varnothing$. Then $P_1 \subsetneq P$, which is a contradiction. Therefore $\exists\ b \notin P$ and $k \geq 1$ such that $ba^k = 0$. But A is a semiprime ring, so $ab = 0$.*

($\Longleftarrow$) *If $P_1 \subsetneq P$, then $\exists a \in P$ and $a \notin P_1$. So by hypothesis there is a $b \notin P$ such that $ab = 0$, and therefore $ab \in P_1$ while $a, b \notin P_1$, which is a contradiction. Thus P is a minimal prime ideal.* □

Lemma 1.14 *Let A be a semiprime ring and let B, C be ideals of A such that B is a finitely generated ideal. Then $V_1(B) \subseteq V_1(C)$ if and only if $B^* \subseteq C^*$.*

Proof.

($\Longleftarrow$) *Suppose $B^* \subseteq C^*$ where $B = \langle b_1, b_2, b_3, ..., b_n\rangle$ is a finitely generated ideal. If $B \subseteq P$ where P is a minimal prime ideal, then for each b_i there is $x_i \notin P$ such that $b_i x_i = 0$. Take $x = \prod_{i=1}^{n} x_i$. Then $x \notin P$ and $b_i x = 0$ for each b_i. Therefore $x \in B^* \subseteq C^*$, but $x \notin P$. Then $C \subseteq P$, and therefore $V_1(B) \subseteq V_1(C)$.*

($\Longrightarrow$) *Let $V_1(B) \subseteq V_1(C)$ and suppose $aB = 0$ and $aC \neq 0$. Then $\exists P$ a prime ideal such that $aC \nsubseteq P$. Let P_1 be a minimal prime ideal contained in P. Then $C \nsubseteq P_1$ and $B \subseteq P_1$ which is a contradiction. Thus $B^* \subseteq C^*$.* □

Remark 1.6 *Let A be a semiprime ring and let B, C be finitely generated ideals. Then $V_1(B) = V_1(C)$ if and only if $B^* = C^*$.*

Corollary 1.5 *Let A be a semiprime ring. Then:*

(1) $P_I = I^{**}$ for each finitely generated I of A.

(2) $I^* = \bigcap\{P \in Min(A) : P \in D(I)\}$ for each ideal I of A.

Proof.

(1) Let $I = \langle b_1, b_2, b_3, ..., b_n\rangle$ be a finitely generated ideal. Let $x \in I^{**}$ and $P \in Min(A)$ such that $I \subseteq P$. Then for each $i = 1, 2, ..., n$ $\exists$ $x_i \notin P$ such that $x_i b_i = 0$. Take $b = \prod_{i=1}^{n} x_i$. Then $b \in I^*$ and therefore $xb = 0$. But $b \notin P$. Then $x \in P$, and therefore $I^{**} \subseteq P_I$. On the other hand, if $x \notin I^{**}$ then $xI^* \neq \{0\}$. Therefore $\exists$ P a minimal prime ideal such that $xI^* \nsubseteq P$. Since $II^* = \{0\}$ then $I \subseteq P$, which implies that $x \notin P_I$. Hence $P_I = I^{**}$.

(2) Since $I \nsubseteq P$ implies that $I^* \subseteq P$, then $I^* \subseteq \bigcap\{P \in Min(A) : P \in D(I)\}$. If $x \notin I^*$, then there is a minimal prime ideal P such that $xI \nsubseteq P$. Therefore $P \in D(I)$ and $x \notin P$, which means that $x \notin \bigcap\{P \in Min(A) : P \in D(I)\}$. Thus $I^* = \bigcap\{P \in Min(A) : P \in D(I)\}$. □

Definition 1.10 *Let A be a ring and I be an ideal of A. Then I is called a z-ideal if $\forall$ $a, b \in A$ such that a, b lie in the same maximal ideals, and $a \in I$ implies that $b \in I$.*

Remark 1.7 *Let A be a ring. Then:*

(1) Every z-ideal is a semiprime ideal.

(2) Every maximal ideal is a z-ideal.

(3) An arbitrary intersection of z-ideals is a z-ideal.

Definition 1.11 *Let A be a ring, and I be an ideal in A. Then I is called a d-ideal [ζ-ideal] if $a \in I$ implies $P_a \subseteq I$ [$P_J \subseteq I$ for each finitely generated ideal $J \subseteq I$].*

From the definition we can see that every ζ-ideal is a d-ideal and every minimal prime ideal is a ζ-ideal. It is clear from corollary 1.5 that if A is semiprime, then an ideal

I is a d-ideal [ζ-ideal] if and only if $\langle a\rangle^{**} \subseteq I$ for each $a \in I$ [$J^{**} \subseteq I$ for each finitely generated ideal $J \subseteq I$]. It is obvious that I^* is a ζ-ideal for each ideal I of A.

Remark 1.8 *Let A be a ring. Then:*

(1) *Every d-ideal is a semiprime ideal.*

(2) *An arbitrary intersection of d-ideals [ζ-ideals] is a d-ideal [ζ-ideal].*

(3) *An ideal I is a d-ideal if and only if $I = \sum_{a\in I} P_a = \bigcup_{a\in I} P_a$.*

Proof.

(1) *If $a^n \in I$ for some $n \geq 1$ where I is a d-ideal then $a \in P_a = P_{a^n} \subseteq I$.*

(2) *Obvious.*

(3) *The necessary condition is obvious. In order to prove the sufficient condition, let I be a d-ideal. Since $P_a \subseteq I$ for each $a \in I$ then $\sum_{a\in I} P_a \subseteq I$. Since $a \in P_a$ for each a, then $\sum_{a\in I} P_a = I$. Thus $I \subseteq \bigcup_{a\in I} P_a \subseteq \sum_{a\in I} P_a = I$.* □

Lemma 1.15 *Let A be a semiprime ring. Then TFAE :*

(1) *I is a ζ-ideal.*

(2) *$V_1(J) = V_1(K)$ where K, J are finitely generated ideals and $J \subseteq I$ implies that $K \subseteq I$.*

(3) *$J^* = K^*$ where K, J are finitely generated ideals such that $J \subseteq I$ implies that $K \subseteq I$.*

Proof.

(1) $\Longrightarrow$ (2) *This is clear because $V_1(J) = V_1(K)$ implies that $P_K = P_J$.*

(2) $\Longrightarrow$ (3) *This is clear by remark* 1.6.

(3) $\Longleftarrow$ (2) *This is clear by remark* 1.6.

(3) $\Longrightarrow$ (1) *Let J be a finitely generated ideal such that $J \subseteq I$, and let $b \in J^{**}$. If $y \in \langle b\rangle^*$ then $y(J\langle b\rangle)) = \{0\}$ and therefore $\langle b\rangle^* \subseteq (J\langle b\rangle)^*$. On the other hand, if $y(J\langle b\rangle)) = \{0\}$ then $ybJ = \{0\}$ and therefore $yb^2 = 0$. But A is semiprime, so $yb = 0$ and therefore $\langle b\rangle^* = (J\langle b\rangle)^*$. Since $J\langle b\rangle \subseteq I$ then $\langle b\rangle \subseteq I$. Hence I is a ζ-ideal.* □

Since the d-ideals are a special case of ζ-ideals one can repeat the previous lemma for d-ideals by replacing the finitely generated ideals with principal ideals. Recall that the Jacobson radical, denoted by $J(A)$, is the intersection of all maximal ideals of A.

Theorem 1.13 *Let A be a ring. Then:*

(1) *$J(A) = \{0\}$ implies that every d-ideal is a z-ideal.*

(2) *If A is a semiprime ring such that every d-ideal is a z-ideal. Then $J(A) = \{0\}$.*

Proof.

(1) *Suppose $J(A) = \{0\}$ and let I be a d-ideal such that $a \in I$, $b \notin I$. Since $P_a \subseteq I$ then $b \notin P_a$ which implies that $b\langle a\rangle^* \neq \{0\}$. Therefore by assumption there is a maximal ideal M such that $b\langle a\rangle^* \nsubseteq M$. Then $b \notin M$ and $\langle a\rangle^* \nsubseteq M$. But $a\langle a\rangle^* = \{0\}$, so $a \in M$ and therefore a and b are not in the same maximal ideals. Hence I is a z-ideal.*

(2) *Let A is a semiprime ring and suppose that every d-ideal is a z-ideal. Then $\{0\}$ is a z-ideal, because $\{0\}$ is a d-ideal. If $b \in J(A)$, then 0 and b lie in the same maximal ideals and therefore $b = 0$. Thus $J(A) = \{0\}$.* □

Corollary 1.6 *Let A be a semiprime ring. Then:*

(1) *Every proper d-ideal consists of zero-divisors.*

(2) *Every zero-divisor is in some minimal prime ideal.*

Proof.

(1) *Let $I \subsetneq A$ be a proper d-ideal and let $a \in I$ be a non zero-divisor. Then $\langle a\rangle^{**} = A \nsubseteq I$, which is a contradiction. Therefore I consists of zero-divisors.*

(2) *If $a = 0$ we are done. If not, let $c \neq 0$ such that $ca = 0$. Since $c \neq 0$ then there is a minimal prime ideal P such that $c \notin P$. Hence $a \in P$ for some $P \in Min(A)$.* □

Lemma 1.16 *Let A be a semiprime ring, let I be an ideal of A, and $\varnothing \neq S \subseteq A$. Then:*

(1) *I is a d-ideal implies that $(I : S) = \{a : aS \subseteq I\}$ is a d-ideal.*

(2) *If S is a multiplicative subset, then $O_S = \{a : as_0 = 0$ for some $s_0 \in S\}$ is a*

d-ideal.

(3) *If I is a d-ideal and S is a multiplicative subset, then $I_S = \{a : as_0 \in I$ for some $s_0 \in S\}$ is a d-ideal and $I_S = \bigcup_{s \in S}(I : \{s\}) = \sum_{s \in S}(I : \{s\})$.*

Proof.

(1) *Clearly $(I : S)$ is an ideal of A. Let $a \in (I : S)$, $b \in \langle a \rangle^{**}$, and $s_0 \in S$. Since $as_0 \in I$, then $\langle as_0 \rangle^{**} \subseteq I$. Suppose that $xas_0 = 0$. Then $xbs_0 = 0$ which means that $bs_0 \in \langle as_0 \rangle^{**} \subseteq I$. Thus $b \in (I : S)$ and $(I : S)$ is a d-ideal.*

(2) *Clearly O_S is an ideal in A. Let $a \in O_S$, $b \in \langle a \rangle^{**}$. Then $as_0 = 0$ for some $s_0 \in S$. Since $b \in \langle a \rangle^{**}$ then $xb = 0$, whenever $xa = 0$. Therefore $bs_0 = 0$ and $b \in O_S$. Hence O_S is a d-ideal.*

(3) *Following the same steps as in (1), one can show that I_S is a d-ideal. It is clear that $I_S = \bigcup_{s \in S}(I : \{s\})$ and that $\bigcup_{s \in S}(I : \{s\}) \subseteq \sum_{s \in S}(I : \{s\})$. Let $x = a_1 + a_2 + \ldots\ldots + a_n \in \sum_{s \in S}(I : \{s\})$ where $a_i s_i \in I$ for each $i = 1, 2, \ldots n$. Take $s = \prod_{i=1}^{n} s_i \in S$. Then $xs \in I$ and therefore $x \in I_S$. Thus $I_S = \bigcup_{s \in S}(I : \{s\}) = \sum_{s \in S}(I : \{s\})$.* $\square$

If A is a regular ring then every prime ideal is a minimal prime ideal which means that every prime ideal is a ζ-ideal, and since in regular rings every ideal is semiprime one can conclude that in regular rings all the ideals are ζ-ideals.

Corollary 1.7 *Let A be a semiprime ring. Then A is regular if and only if every principal ideal is a d-ideal.*

Proof.

($\Longrightarrow$) *Obvious.*

($\Longleftarrow$) *Since $\langle a^2 \rangle$ is a d-ideal and $a \in \langle a^2 \rangle^{**} \subseteq \langle a^2 \rangle$ then A is regular.* $\square$

Lemma 1.17 *let A be a semiprime ring. Then:*

(1) *If I is a d-ideal [ζ-ideal] of A, and J is a d-ideal [ζ-ideal] of A/I. Then $\pi^{-1}(J)$ is a d-ideal [ζ- ideal] of A.*

(2) *I is a d-ideal [ζ-ideal] if and only if I is an intersection of prime d-ideals [prime ζ-ideals].*

Proof.

(1) *Let I be a ζ-ideal of A and J be a ζ-ideal of A/I. If $\langle a_1, a_2, ..., a_n\rangle \subseteq \pi^{-1}(J)$, and $b \in \langle a_1, a_2, ..., a_n\rangle^{**}$ then $ca_i = 0\ \forall\ i = 1, 2, ..., n$ implies that $bc = 0$. Since $\pi(a_i) \in J$ for each $i = 1, 2, ..., n$ and A/I is semiprime, then $\langle \pi(a_1), \pi(a_2), ..., \pi(a_n)\rangle^{**} \subseteq J$. So it is enough to show that $b + I \in \langle \pi(a_1), \pi(a_2), ..., \pi(a_n)\rangle^{**}$, which is equivalent to showing that $bx \in I$ whenever $xa_i \in I\ \forall\ i = 1, 2, ..., n$. Suppose that $xa_i \in I$ for each $i = 1, 2, ..., n$. Since I is a ζ-ideal then $\langle xa_1, xa_2, ..., xa_n\rangle^{**} \subseteq I$. Now since $k(xa_i) = 0$ for each $i = 1, 2, ..., n$ implies that $bxk = 0$. Then $bx \in \langle xa_1, xa_2, ..., xa_n\rangle^{**} \subseteq I$. Hence $\pi^{-1}(J)$ is a ζ-ideal.*

The claims for d-ideals follow by the same argument.

(2) ($\Longleftarrow$) *Obvious.*

($\Longrightarrow$) *Since A/I is a semiprime ring, then $\{0 + I\} = \bigcap_{P \in Min(A/I)} P$. Hence $I = \bigcap_{P \in Min(A/I)} \pi^{-1}(P)$ where $\pi^{-1}(P)$ is a prime d-ideal [ζ-ideal] for each $P \in Min(A/I)$.*

$\square$

Let A be a ring and I be a proper d-ideal [proper ζ-ideal] of A. Then I called a maximal d-ideal [maximal ζ-ideal] if it is a maximal element in the set of proper d-ideals [proper ζ-ideals]. By using the Axiom of choice one can prove that every proper d-ideal [proper ζ-ideal] lies in some maximal d-ideal [maximal ζ-ideal]. It is clear that maximal d-ideals and maximal ζ-ideals are not necessarily maximal ideals in general. But they are prime ideals, because by lemma 1.17(2), every maximal d-ideal [maximal ζ-ideal] is an intersection of prime d-ideals [prime ζ-ideals] and therefore it has to be one of them.

Theorem 1.14 *Let A, B be semiprime rings and $f : A \longrightarrow B$ be a ring homomorphism. Then:*

(1) *I^c is a ζ-ideal for each ζ-ideal I of B if and only if P^c is a ζ-ideal for each $P \in Min(B)$.*

(2) *I^c is a d-ideal for each d-ideal I of B if and only if P^c is a d-ideal for each $P \in Min(B)$.*

Proof.

(1) ($\Longrightarrow$) *Obvious.*

($\Longleftarrow$) *Suppose that P^c is a ζ-ideal for each $P \in Min(B)$, and let I be a ζ-ideal of B. If J, K are finitely generated ideals of A such that $J \subseteq I^c$ and $V_1(J) = V_1(K)$ then J^e, K^e are finitely generated ideals of B and $J^e \subseteq I$. We need to show that $V_1(J^e) = V_1(K^e)$. Suppose $J^e \subseteq P$ where P is in $Min(B)$. Then $J \subseteq P^c$. Since P^c is a ζ-ideal, then $K \subseteq P_K = P_J \subseteq P^c$ and therefore $K^e \subseteq P$. On the other hand, if $K^e \subseteq P$ where P is in $Min(B)$, then $K \subseteq P^c$. But P^c is a ζ-ideal. Then $J \subseteq P_J = P_K \subseteq P^c$. Therefore $V_1(J^e) = V_1(K^e)$. Now since I is a ζ-ideal and $J^e \subseteq I$, then by lemma* 1.15 *$K^e \subseteq I$. Hence $K \subseteq I^c$ and I^c is a ζ-ideal.*

(2) *This holds by the argument above with finitely generated ideals replaced by principal ideals.* □

Corollary 1.8 *Let A be a semiprime ring, let S be a multiplicative subset of A, and let I be a d-ideal [ζ-ideal] of A_S. Then ${\Phi_s}^{-1}(I)$ is a d-ideal [ζ-ideal] for each $s \in S$.*

Proof.

Since A and A_S are semiprime rings then it is enough by theorem 1.14, *to show that ${\Phi_s}^{-1}(P)$ is a ζ-ideal for each $P \in Min(A_S)$. Let $P \in Min(A_S)$ and $s \in S$. Then ${\Phi_s}^{-1}(P) \in spec(A)$. If $Q = {\Phi_s}^{-1}(P)$, then $P = Q_S = \{\frac{a}{s} : a \in Q, s \in S\}$ where $Q \cap S = \varnothing$. It suffices to show that Q is a minimal prime ideal. Let $a \in Q$. Then $\frac{a}{s} \in P$ which implies that $\exists \frac{b}{s_1} \notin P$ such that $\frac{a}{s}\frac{b}{s_1} = \frac{0}{s}$. So $\exists s_2 \in S$ such that $s_2abs = 0$. Since $\frac{b}{s_1} \notin P$, then $b \notin Q$ which means that $s_2bs \notin Q$. Hence $Q \in Min(A)$ and Q is a ζ-ideal.* □

Theorem 1.15 *Let B be a regular ring of quotients of A and let J be a finitely generated ideal of A. Then $P_J = J^e \cap A$.*

Proof.

Let $J = \langle a_1, a_2, ..., a_n\rangle$ be a finitely generated ideal of A. Then $J^e = \sum_{i=0}^{n} a_n B$. Since B is a regular ring, then $\sum_{i=1}^{n} a_i B = eB$ for some idempotent element $e \in B$. If *be* $\in J^e \cap A$ *and* $y \in \langle a_1, a_2, ..., a_n\rangle^*$ *then* $ya_i = 0$ *for each* $i = 1, 2, ..., n$ *and therefore* $ybe = 0$. *So* $be \in \langle a_1, a_2, ..., a_n\rangle^{**} = P_J$, *and* $J^e \cap A \subseteq P_J$. *On the other hand, suppose that* $x \in A$, $x \notin eB$. *Then* $x(1-e) \neq 0$. *But* B *is a ring of quotients of* A, *therefore* $\exists\, t \in A$ *such that* $t(1-e) \in A$ *and* $xt(1-e) \neq 0$. *Since* $x^2t(1-e) \neq 0$ *and* $xt(1-e) \in \langle a_1, a_2, ..., a_n\rangle^*$, *then* $x \notin \langle a_1, a_2, ..., a_n\rangle^{**}$. *Hence* $P_J = \langle a_1, a_2, ..., a_n\rangle^{**} \subseteq J^e \cap A$ *and therefore* $P_J = J^e \cap A$. □

Lemma 1.18 *Let* B *be a regular ring of quotients of* A *and let* J *be an ideal of* A. *Then* $J = I \cap A$ *for some ideal* I *of* B *if and only if* J *is a* ζ*-ideal.*

Proof.

($\Longrightarrow$) *Let* $J = I \cap A$ *for some ideal* I *of* B, *and let* $K = \langle a_1, a_2, ..., a_n\rangle \subseteq J$ *be a finitely generated ideal of* A. *Then* $K^e \subseteq J^e \subseteq I$, *and* $P_K = K^e \cap A \subseteq I \cap A = J$. *Thus* J *is a* ζ*-ideal.*

($\Longleftarrow$) *Suppose* J *is a* ζ*-ideal and let* $I = J^e$. *It is clear that* $J \subseteq I \cap A$. *Let* $x \in I \cap A$. *Then* $x = \sum_{i=1}^{n} a_i bi$ *where* $a_i \in J\ \forall\ i = 1, 2, ..., n$. *As* J *is a* ζ*-ideal, then* $\langle a_1, a_2, ..., a_n\rangle^{**} \subseteq J$. *Since* $x \in (\sum_{i=1}^{n} a_i B \cap A) = \langle a_1, a_2, ..., a_n\rangle^{**}$, *then* $x \in J$. *Hence* $J = I \cap A$. □

Theorem 1.16 *Let* A *be a semiprime ring, and let* P *be a prime ideal. Then* $TFAE$:

(1) P *is a* ζ*-ideal.*

(2) $P = N \cap A$ *for some maximal ideal* N *of* $Q(A)$.

(3) $P = M \cap A$ *for some maximal ideal* M *of* $H(A)$.

Proof.

(1) $\Longrightarrow$ (2) *Since* $Q(A)$ *is a regular ring of quotients of* A, *then* $P = J \cap A$ *for some ideal* J *of* $Q(A)$. *Since* $S = A - P$ *is a multiplicative subset of* $Q(A)$, *and* $S \cap J = \varnothing$ *then there is a prime ideal* M *of* $Q(A)$ *and therefore a maximal ideal such that* $S \cap M = \varnothing$ *and* $J \subseteq M$. *Therefore* $P = J \cap A \subseteq M \cap A$. *If* $z \in M \cap A$, *then* $z \notin (A - P)$ *which implies that* $z \in P$. *Thus* $P = M \cap A$ *where* M *is a maximal ideal of* $Q(A)$.

(2) $\Longrightarrow$ (3) *Suppose that* $P = N \cap A$ *for some maximal ideal* N *of* $Q(A)$. *Let* $M = N \cap H(A)$. *Then* M *is a maximal ideal of* $H(A)$ *and* $P = M \cap A$.

(3) $\Longrightarrow$ (1) *Suppose* $P = M \cap A$ *where* M *is a maximal ideal of* $H(A)$. *Thus by lemma* 1.18, P *is a* ζ*-ideal.* □

Remark 1.9 *Let* A *be a semiprime ring. Then:*

(1) *If* $I = J \cap A$ *for some ideal* J *of* $H(A)$ *then* I *is a* ζ*-ideal.*

(2) I *is a proper* ζ*-ideal implies that* $I^e = IH(A) \neq H(A)$.

(3) P *is a prime* ζ*-ideal implies that* $P = M \cap A$ *where* M *is a maximal ideal of* $H(A)$ *such that* $PH(A) \subseteq M$ *and* $M \cap (A - P) = \varnothing$.

(4) P *is a prime* ζ*-ideal ideal does not imply* $PH(A)$ *is a prime ideal.*

(5) *Since* $H(A)$ *is epimorphic extension of* A *then* $i^a : spec(H(A)) \longrightarrow spec(A)$ *is a one-to-one map, that is, for each* P *a prime* ζ*-ideal of* A $\exists!$ M *maximal ideal of* $H(A)$ *such that* $P = M \cap A$.

Corollary 1.9 *Let* A *be a semiprime ring and let* $P \in M(A)$. *Then:*

(1) P *is a* ζ*-ideal if and only if* $PH(A) \neq H(A)$.

(2) P *is a* ζ*-ideal implies that* $M_P = PH(A)$.

(3) $PH(A) \in M(H(A))$ *or* $PH(A) = H(A)$.

Proof.

(1) *If* P *is a* ζ*-ideal then* $P = M_P \cap A$ *for some* $M_P \in M(H(A))$ *such that* $PH(A) \subseteq M_P$. *Therefore* $PH(A) \neq H(A)$.

Conversely, if $PH(A) \neq H(A)$ *then* $P \subseteq (PH(A) \cap A) \neq A$. *Since* P *is a maximal ideal then* $P = (PH(A) \cap A)$. *Hence* P *is a* ζ*-ideal.*

(2) *Let* P *be a* ζ*-ideal. Since* $\imath : A \longrightarrow H(A)$ *and* $\pi : H(A) \longrightarrow H(A)/(PH(A))$ *are ring epimorphisms, then* $\pi \circ \imath$ *is an epimorphism with* $ker(\pi \circ \imath) = P$ *and therefore there is an epimorphic monomorphism* k *from* A/P *into* $H(A)/(PH(A))$. *As* A/P *is a field, then* k *is an onto map and therefore* $H(A)/(PH(A))$ *is a field. Thus* $PH(A)$ *is a maximal ideal and* $M_P = PH(A)$.

(3) *If* $PH(A) \neq H(A)$, *then* P *is a* ζ*-ideal and therefore by* (2), $M_P = PH(A)$. *Hence* $PH(A) \in M(H(A))$. □

Definition 1.12 *Let* A *be a semiprime ring Then:*

(1) *we say that* A *satisfies condition* c *if* $I^* \neq \{0\}$ *for each finitely generated ideal of zero-divisors* I.

(2) *we say that* A *satisfies the condition* a.c. *if for each finitely generated ideal* I, *there is a* b *in* A *such that* $I^* = \langle b \rangle^*$.

(3) *we say that* A *satisfies the strongly* a.c. *condition if for each finitely generated ideal* I *there is a* b *in* I *such that* $I^* = \langle b \rangle^*$.

It is shown in [7] that a semiprime ring satisfies condition c if and only if every ideal consisting entirely of zero-divisors is contained in some proper ζ-ideal. Therefore if A is a semiprime ring which satisfies the condition c, then every maximal ideal of A consisting entirely of zero-divisors is a ζ-ideal.

Lemma 1.19 *Let* A *be a semiprime ring and* $b \in \langle a \rangle^*$. *Then* $\langle b \rangle^{**} = \langle a \rangle^*$ *if and only if* $a + b$ *is a non zero-divisor.*

Proof.

Since $\langle a \rangle^*$ *is a d-ideal then* $\langle b \rangle^{**} \subseteq \langle a \rangle^*$ *for each* $b \in \langle a \rangle^*$.

($\Longrightarrow$) *Suppose that* $\langle b \rangle^{**} = \langle a \rangle^*$. *Then* $xc = 0$, *whenever* $ax = 0$ *and* $cb = 0$. *Suppose* $(a + b)x = 0$. *Then* $a^2x + bax = 0$ *which means* $a^2x = 0$. *Therefore* $ax = 0$ *and hence* $bx = 0$. *But* $ax = bx = 0$ *implies that* $x^2 = 0$. *Thus* $a + b$ *is a non zero-divisor.*

($\Longleftarrow$) *Suppose* $a + b$ *is a non zero-divisor, and let* $x \notin \langle b \rangle^{**}$. *Then* $\exists\ c \in A$ *such that* $cb = 0$, $cx \neq 0$. *If* $xa = 0$, *then* $(a + b)cx = 0$, *which is a contradiction. Then* $x \notin \langle a \rangle^*$. *Hence* $\langle b \rangle^{**} = \langle a \rangle^*$. □

Theorem 1.17 *Let* A *be a semiprime ring. Then TFAE:*

(1) *Minimal prime ideals are the only prime ideals consisting of zero-divisors.*

(2) A *satisfies condition* c *and every prime d-ideal is minimal prime.*

(3) $Q_{cl}(A)$ *is a regular ring.*

(4) *For each* $a \in A$ $\exists$ $b \in A$ *such that* $\langle b\rangle^{**} = \langle a\rangle^{*}$.

Proof.

(1) $\Longrightarrow$ (2) *Let* $I = \langle a_1, a_2, ..., a_n\rangle$ *be a finitely generated ideal consisting of zero-divisors. Then* $I \subseteq \bigcup\{P : P \in Min(A)\}$. *Let* $S = \bigcap\{A - P : P \in Min(A)\}$. *Then* $I \cap S = \varnothing$ *and* S *is a multiplicative subset, which implies that there is a prime ideal* Q *such that* $I \subseteq Q$ *and* $Q \cap S = \varnothing$. *Now since* $x \in Q$ *implies that* $x \notin S$, *therefore* $\exists P_0$ *a minimal prime ideal such that* $x \in P_0$, *i.e.* x *is a zero-divisor. It follows by* (1) *that* Q *is a minimal prime ideal. Since* $I \subseteq Q$, *then* $a_i \in Q$ $\forall i = 1, 2, ..., n$, *and therefore there are* $y_1, y_2, ..., y_n \notin Q$ *such that* $a_i y_i = 0$ $\forall i$. *Take* $y = y_1 y_2 .... y_n$. *Then* $y \notin Q$ *and* $yI = \{0\}$. *Therefore* $I^* \neq \{0\}$ *and hence* A *satisfies condition* c. *It is clear that every prime* d*-ideal is minimal prime.*

(2) $\Longrightarrow$ (3) *Let* M *be a maximal ideal in* $Q_{cl}(A)$. *Since* $M \subseteq Z(Q_{cl}(A))$ *and* $Q_{cl}(A)$ *satisfies condition* c, *then* M *is a* ζ*-ideal, which implies by corollary* 2.8, *that* $M \cap A$ *is a* ζ*-ideal and therefore a prime* d*-ideal. Hence by* (2) $M \cap A$ *is minimal prime. Since* $Q_{cl}(A)$ *is an epimorphic extension of* A, *then* $\imath^a$ *is a one-to-one map. Suppose that* M *is not a minimal prime ideal in* $Q_{cl}(A)$. *Then there is a* $Q \in spec(Q_{cl}(A))$ *such that* $Q \subsetneq M$, *which implies that* $Q \cap A \subsetneq M \cap A$ *and* $Q \cap A$ *is a prime ideal of* A, *which is a contradiction. Thus* M *is a minimal prime ideal of* $Q_{cl}(A)$ *and hence* $Q_{cl}(A)$ *is a regular ring.*

(3) $\Longrightarrow$ (4) *Suppose that* $Q_{cl}(A)$ *is a regular ring, and let* $x \in Q_{cl}(A)$. *Then* $x = x^2 y$ *for some* y *in* $Q_{cl}(A)$. *Take* $e = 1 - xy$. *Then* e *is an idempotent element and* $\langle x\rangle^* = Q_{cl}(A)e$. *For any* a *in* A *we have that* $\langle \frac{a}{1}\rangle^* = Q_{cl}(A)e$ *for some idempotent element* $e = \frac{a_1}{s_1}$. *Take* $b = a_1$. *Then we just have to show that* $\langle b\rangle^{**} = \langle a\rangle^*$. *Let* $xa = 0$ *and* $tb = 0$. *Since* $\frac{x}{1} \in \langle \frac{a}{1}\rangle^* = Q_{cl}(A)e$ *then* $\frac{x}{1} = \frac{rb}{ss_1}$, *and therefore* $\frac{x}{1}\frac{t}{1} = \frac{rbt}{ss_1} = 0$. *Hence* $xt = 0$ *and* $x \in \langle b\rangle^{**}$. *Conversely, let* $x \in \langle b\rangle^{**}$. *Since* $\frac{a}{1}\frac{b}{s_1} = 0$, *then* $ab = 0$. *But* $x \in \langle b\rangle^{**}$, *therefore* $xa = 0$ *and* $x \in \langle a\rangle^*$. *Thus for each* $a \in A$ $\exists$ $b \in A$ *such that* $\langle b\rangle^{**} = \langle a\rangle^*$.

$(4) \Longrightarrow (1)$ *Suppose that for each* $a \in A\ \exists\ b \in A$ *such that* $\langle b\rangle^{**} = \langle a\rangle^{*}$, *and let* P *be a prime ideal in* A *such that* $P \subseteq Z(A)$. *Suppose* $a \in P$ *such that* $\langle a\rangle^{*} \subseteq P$. *Since there is a* b *in* A *such that* $\langle b\rangle^{**} = \langle a\rangle^{*}$ *then* $b \in P$ *and therefore* $a + b \in P$, *which is a contradiction, because* $a + b$ *is not a zero-divisor. Thus* $\langle a\rangle^{*} \nsubseteq P$, *and therefore* P *is a minimal prime ideal.* □

Lemma 1.20 *Let* A *be a semiprime ring that satisfies the strongly* a.c. *condition. Then:*

(1) *A satisfies condition* c.

(2) *Every d-ideal is a* ζ*-ideal.*

(3) *Every maximal ideal is a d-ideal if and only if* $A = Q_{cl}(A)$.

Proof.

(1) *Let* I *be a finitely generated ideal such that* $I \subseteq Z(A)$, *and suppose that* $I^{*} = \{0\}$. *Since* $\exists\ b \in I$ *such that* $I^{*} = \langle b\rangle^{*}$, *therefore* $b \notin Z(A)$, *which is a contradiction. Thus* $I^{*} \neq \{0\}$ *and* A *satisfies condition* c.

(2) *Let* I *be a d-ideal, and* $\{a_1, a_2, ..., a_n\} \subseteq I$. *Then* $\exists\ c \in I$ *such that* $\langle a_1, a_2, ..., a_n\rangle^{*} = \langle c\rangle^{*}$, *which implies that* $\langle a_1, a_2, ..., a_n\rangle^{**} = \langle c\rangle^{**} \subseteq I$. *Hence* I *is a* ζ*-ideal.*

(3) *Suppose that* $A = Q_{cl}(A)$, *and let* M *be a maximal ideal in* A. *Then* $M \subseteq Z(A)$, *and therefore* $M \subseteq N$ *for some d-ideal* N, *which means that* $M = N$. *Hence* M *is a d-ideal. Conversely, suppose that every maximal ideal is a d-ideal, and let* x *be a non-unit element of* A. *Then* $x \in M$ *for some maximal ideal* M, *which implies that* $x \in Z(A)$. *Thus* $A = Q_{cl}(A)$. □

1.3 Algebraic frames

This section is based on the article [23], but it is reworked from the writer's point of view to give a review of the basic concepts on algebraic frames.

A complete lattice L is called Brouwerian if for any two elements a and b in L, the set $\{x \in L : a \wedge x \leq b\}$ has a greatest element. It is well known fact that every Brouwerian lattice is a distributive lattice. A complete lattice is called meet-continuous if for each directed subset D of L, we have $a \wedge (\vee_{x_\alpha \in D}) = \vee_{x_\alpha \in D}(a \wedge x_\alpha)$.

Definition 1.13 *Let L be a complete lattice and $a \in L$. Then:*

(1) a is called a joint-inaccessible element if $a = \bigvee_{x_\alpha \in D} x_\alpha$ where D is a directed set implies that $a = x_\alpha$ for some $x_\alpha \in D$.

(2) a is called a compact element if $a \leq \bigvee_{\alpha \in \Gamma} x_\alpha$ implies that there exist a finite subset β of Γ such that $a \leq \bigvee_{\alpha \in \beta} x_\alpha$.

(3) a is called a meet-irreducible element if $a < 1$ and $a = \bigwedge_{\alpha \in \Gamma} x_\alpha$ implies that $a = x_{\alpha_0}$ for some $\alpha_0 \in \Gamma$.

If L is a complete lattice, then the set of all compact elements in L, denoted by $C(L)$, is closed under finite supremum and need not be closed under finite infimum. If it does, then it is said to have the finite intersection property *F.I.P.*. A complete lattice L is called compact if 1 is a compact element.

Lemma 1.21 *Let L be a complete lattice. Then:*

(1) Every compact element is a joint-inaccessible element.

(2) If L is a meet-continuous lattice then $a \in L$ is compact element if and only if a is a joint-inaccessible element.

Proof.

(1) Let $a = \bigvee_{x_\alpha \in D} x_\alpha$ where D is a directed set. Then $a \leq \bigvee_{x_\alpha \in \Gamma} x_\alpha$ where Γ is a finite subset of D, which means that $a = \bigvee_{x_\alpha \in \Gamma} x_\alpha$. Since there is $x_{\alpha_0} \in D$ such that $x_\alpha \leq x_{\alpha_0}$ for each $\alpha \in \Gamma$ then $a \leq x_{\alpha_0} \leq \bigvee_{x_\alpha \in D} \leq a$. Hence a is a joint-inaccessible element.

(2)($\Longrightarrow$) Obvious.

($\Longleftarrow$) Let a be a joint-inaccessible element and suppose that $a \leq \bigvee_{x_\alpha \in \Gamma} x_\alpha$. Then $D = \{z : z = \vee_{i=1}^{n} x_i, x_i \in \Gamma, n \geq 1\}$ is a directed subset of D, and therefore $a \leq \bigvee_{x_\alpha \in D} x_\alpha$. Then $a = a \wedge (\bigvee_{x_\alpha \in D} x_\alpha) = \bigvee_{x_\alpha \in D}(a \wedge x_\alpha)$. Since the set $\{a \wedge x_\alpha : x_\alpha \in D\}$ is a directed subset and a is a joint-inaccessible element, then $a = a \wedge x_{\alpha_0}$ for some $x_{\alpha_0} \in D$, which implies that $a \leq x_{\alpha_0} = \vee_{i=1}^{n} x_i$. Thus a is a compact element. $\square$

Definition 1.14 *A complete lattice is called an algebraic lattice if every element can be written as a supremum of compact elements.*

Lemma 1.22 *Every algebraic lattice is meet-continuous.*

Proof.

It is clear that $\bigvee_{x_\alpha \in D}(a \wedge x_\alpha) \leq a \wedge (\bigvee_{x_\alpha \in D} x_\alpha)$. *Then, we need only show that* $a \wedge (\bigvee_{x_\alpha \in D} x_\alpha) \leq \bigvee_{x_\alpha \in D}(a \wedge x_\alpha)$. *Write* $a \wedge (\bigvee_{x_\alpha \in D} x_\alpha)$ *as a supremum of compact elements:* $a \wedge (\bigvee_{x_\alpha \in D} x_\alpha) = \bigvee_{\delta \in \Gamma} c_\delta$. *It is clear that* $c_\delta \leq a$ *and* $c_\delta \leq \bigvee_{x_\alpha \in D} x_\alpha$. *Then there is a finite subset F of D such that* $c_\delta \leq \bigvee_{\alpha \in F} x_\alpha$. *But D is a directed subset. Then* $c_\delta \leq x_{\alpha 0}$ *for some* $x_{\alpha 0} \in D$, *and therefore* $c_\delta \leq (a \wedge x_{\alpha 0})$. *So* $c_\delta \leq \bigvee_{x_\alpha \in D}(a \wedge x_\alpha)$ *for each* $\delta \in \Gamma$. *Hence* $a \wedge (\bigvee_{x_\alpha \in D} x_\alpha) \leq \bigvee_{x_\alpha \in D}(a \wedge x_\alpha)$. *Thus L is a meet-continuous lattice.* □

Lemma 1.23 *Let L be an algebraic lattice, and suppose $x < 1$. Then $\exists$ t a meet-irreducible element such that $t \geq x$. Furthermore x is a meet of meet-irreducible elements.*

Proof.

Since $x < 1$, then there exists c a compact element such that $c \nleq x$. Let $S = \{t : t \geq x, t \ngeq c\}$. *Then S is a non-empty partially ordered set. If* $\{x_i : i \in I\}$ *is a chain in S, then* $y = \bigvee_{i \in I} x_i$ *is an element in L and $x_i \leq y$ for each $i \in I$. We need to show that y is in S. It is clear that $y \geq x$, so suppose that $y \geq c$. Since c is a compact element, then* $c \leq (x_1 \vee x_2 \vee .... \vee x_n)$. *But* $\{x_i : i \in I\}$ *is a chain, therefore* $x_1 \vee x_2 \vee .... \vee x_n = x_{i_0}$ *for some* $1 \leq i_0 \leq n$, *so* $x_{i_0} \notin S$, *which is a contradiction. Hence $y \ngeq c$ and therefore $y \in S$. So S has a maximal element t. It is clear that $t < 1$, because $t \ngeq c$. To show that t is a meet-irreducible element, suppose that* $t = \bigwedge_{i \in I} x_i$ *such that $t \neq x_i$ for each $i \in I$. Then $t < x_i$ for each $i \in I$. But $t \geq x$. So $x_i \geq x$ for each i, which implies that $x_i \notin S$, as t is a maximal element in S. Then $x_i \geq c$ for each $i \in I$, and therefore* $t = \bigwedge_{i \in I} x_i \geq c$, *which is a contradiction. Thus t is a meet-irreducible element.*

Finally, to show that x is a meet of meet-irreducible elements, let $T = \{t_i : t_i$ *is a meet*

-irreducible element, $t_i \geq x\}$, and let $y = \bigwedge_{t_i \in T} t_i$. It is clear that $y \geq x$. Suppose $y \nleq x$, since $y = \bigvee_{j \in J} d_j$ where d_j is a compact element for each $j \in J$. Then there exist d_{j0} such that $d_{j0} \nleq x$. Take $S_1 = \{d : d \geq x, d \ngeq d_{j0}\}$. Then S_1 has a maximal element t_1 which will be a meet-irreducible element. Since $t_1 \in T$, then $y \leq t_1$. But $d_{j0} \leq y$ and therefore $d_{j0} \leq t_1$, which is a contradiction. Thus $y \leq x$ and $y = x$. □

Remark 1.10 *If L is a algebraic lattice then $0 = \bigwedge_{t_i \in T} t_i$ where T is the set of all meet-irreducible elements.*

Lemma 1.24 *Every distributive algebraic lattice L is a Brouwerian lattice.*

Proof.

Let $a, b \in L$ and take $S = \{x : a \wedge x \leq b\}$ and $y = \bigvee_{x_i \in S} x_i$. Since L is a distributive lattice, then S is a directed subset. So by lemma 1.22, L is a meet-continuous lattice. Therefore $a \wedge y = a \wedge (\bigvee_{x_i \in S} x_i) = \bigvee_{x_i \in S}(a \wedge x_i) \leq b$. Then $y \in S$ and S has a largest element. Thus L is a Browerian lattice. □

Definition 1.15 *A complete lattice L is called a frame if $a \wedge (\bigvee_{i \in I} x_i) = \bigvee_{i \in I}(a \wedge x_i)$ for each $a \in L$ and for each subset $\{x_i : i \in I\}$ of L.*

Lemma 1.25 *Let L be an algebraic lattice. Then L is a frame if and only if L is a distributive lattice.*

Proof.

($\Longrightarrow$) Suppose L is a frame. Then L is a Browerian lattice and therefore L is a distributive lattice.

($\Longleftarrow$) Suppose L is a distributive lattice, and let $a \in L$ and $\{x_i : i \in I\} \subseteq L$. Take $D = \{\bigvee_{i \in F} x_i :$ where F is a finite subset of $I\}$. Then D is a directed set and $\bigvee_{x_i \in I} x_i = \bigvee_{z_i \in D} z_i$. Since L is a meet-continuous lattice, then $a \wedge (\bigvee_{z_i \in D} z_i) = \bigvee_{z_i \in D}(a \wedge z_i)$. So if $z = x_1 \vee x_2 \vee ... \vee x_n$ is any element in D, then $a \wedge z = a \wedge (x_1 \vee x_2 \vee ... \vee x_n) = \bigvee_{i=1}^{n}(a \wedge x_i)$, which implies that $\bigvee_{z_i \in D}(a \wedge z_i) = \bigvee_{x_i \in I}(a \wedge x_i)$. Then $\bigvee_{x_i \in I}(a \wedge x_i) = a \wedge (\bigvee_{z_i \in D} z_i) = a \wedge (\bigvee_{x_i \in I} x_i)$. Hence L is a frame. □

Definition 1.16 *Let L be a complete lattice and $p \in L$ such that $p < 1$. Then:*

(1) p is called a prime element if $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$ for each $x, y \in L$.

(2) p is called a finite meet-irreducible element if $x \wedge y = p$ implies that $x = p$ or $y = p$ for each $x, y \in L$.

It is clear that for distributive lattices an element p is prime if and only if it is a finite meet-irreducible element.

Lemma 1.26 *Let L be an algebraic lattice and $1 \neq p \in L$. Then p is prime if and only if $a, b \in C(L)$ $a \wedge b \leq p$ implies that $a \leq p$ or $b \leq p$.*

Proof.

Let $x \wedge y \leq p$ and $x = \bigvee_{x_i \in I} x_i$, $y = \bigvee_{y_j \in J} y_j$ where $x_i, y_j \in C(L)$ $\forall$ i, j, and let D_x, D_y be the subsets of finite suprema of the elements of I, J respectively. Then D_x, D_y are directed subsets of $C(L)$ and $\bigvee_{z_j \in D_y}(\bigvee_{z_i \in D_x}(z_j \wedge z_i)) \leq p$. Now if $z_i \leq p$ for each $z_i \in D_x$ we are done. If not, then $\exists z_{i_0} \nleq p$, but $z_{i_0} \wedge z_j \leq p$ for each $z_j \in D_y$. Therefore $z_j \leq p$ for each $z_j \in D_y$. Hence $y \leq p$. The proof of the other implication is obvious. □

Let L be an algebraic frame and $a, b \in L$. Then the element $(a : b)$ is defined to be $\vee\{x : x \wedge a \leq b\}$, and as a special case the element $a^{\perp}$ is defined to be $(a : 0)$. It is clear that $a \wedge (a : b) \leq b$ and, in particular, $a \wedge a^{\perp} = 0$. An element $a \in L$ is called complemented if $a \vee a^{\perp} = 1$. L is said to be a zero-dimensional algebraic frame if every element can be written as a supremum of complemented elements. An element $a \in L$ is called regular if $a = \bigvee\{x : x \preceq a\}$ where $x \preceq a$ means that $x^{\perp} \vee a = 1$. An algebraic lattice L is called regular if every element is regular. Finally, an element $a \in L$ is called polar if there is another element $b \in L$ such that $a = b^{\perp}$. If L is an algebraic frame then the set of all prime elements in L and the set of all polar elements in L are denoted by $spec(L)$ and $p(L)$ respectively.

Remark 1.11 *Let L be an algebraic frame. Then:*

(1) $a \leq a^{\perp\perp}$, $a^{\perp} = a^{\perp\perp\perp}$ *and* $a \leq b$ *implies that* $b^{\perp} \leq a^{\perp}$.

(2) $(a \wedge b)^{\perp} \geq (a^{\perp} \vee b^{\perp})$ *and* $(a \vee b)^{\perp} = (a^{\perp} \wedge b^{\perp})$.

(3) $a \preceq b$ *implies that* $a^{\perp\perp} \leq b$.

(4) *The set of all complemented elements is closed under finite supremum.*

(5) $a \leq b$ *and* $b \preceq c$ *implies that* $a \preceq c$.

(6) $a, b \preceq c$ *implies that* $a \vee b \preceq c$.

(7) $c \preceq a$ *implies that* $c \leq a$.

If (X, τ) is a topological space then the set of all open subsets, denoted $D(X)$, is a partially ordered set. In fact $(D(X), \leq)$ is a complete lattice where the supremum and the infimum are given by $\bigvee_{i \in I} A_i = \bigcup_{i \in I} A_i$ and $\bigwedge_{i \in I} A_i = (\bigcap_{i \in I} A_i)^{\circ}$ respectively. Since $A \wedge (\bigvee_{i \in I} A_i) = A \cap (\cup_{i \in I} A_i) = \cup_{i \in I}(A \cap A_i) = \bigvee_{i \in I}(A \wedge A_i)$, therefore $(D(X), \leq)$ is a frame. However $(D(X), \leq)$ is not an algebraic lattice in general. Clearly it forms one if and only if X has an open base consisting of compact open sets.

Corollary 1.10 *If* (X, τ) *is a topological space then* $(D(X), \leq)$ *is a regular lattice if and only if* X *is regular.*

Proof.

$(\Longrightarrow)$ *Suppose* $(D(X), \leq)$ *is a regular lattice. Then* $U = \bigcup\{V : V \preceq U\}$ *for each* $U \in D(X)$. *Let* F *be any closed subset and* $x \in X$ *such that* $x \notin F$. *Since* $x \in F^c$, *then* $\exists\ V \preceq F^c$ *such that* $x \in V$ *and therefore* $F \subseteq V^{\perp}$. *Since* $(D(X), \leq)$ *is a frame, then* $V \cap V^{\perp} = \varnothing$. *Hence* X *is a regular space.*

$(\Longleftarrow)$ *Suppose* X *is a regular space. If* $V^{\perp} \cup U = X$, *then* $V \subseteq (V^{\perp})^c \subseteq U$. *So* $\bigcup\{V : V \preceq U\} \subseteq U$. *On the other hand, if* $x \in U$, *then* $\exists\ W \in D(X)$ *such that* $x \in W \subseteq cl(W) \subseteq U$. *Since* $U^c \subseteq (cl(W))^c \subseteq W^{\perp}$ *then* $U \cup W^{\perp} = X$ *and therefore* $W \preceq U$. *Thus* $U = \bigcup\{V : V \preceq U\}$ *far each* $U \in D(X)$ *and* $(D(X), \leq)$ *is a regular lattice.* □

Definition 1.17 *Let* L *be an algebraic frame. We say* L *has the compact splitting property CSP if each compact element in* L *is a complemented element.*

Lemma 1.27 *Let L be an algebraic frame. Then L has the CSP if and only if L is a zero-dimensional algebraic frame.*

Proof.

($\Longrightarrow$) *Obvious.*

($\Longleftarrow$) *Suppose L is a zero-dimensional algebraic frame, and let $x \in C(L)$. Then $x = \bigvee_{i \in I} y_i$ where y_i is a complemented element for each $i \in I$, and therefore $x \leq (y_1 \vee y_2 \vee .... \vee y_n) \leq \bigvee_{i \in I} y_i = x$. Hence x is a complemented element.* $\square$

Theorem 1.18 *Let L be an algebraic frame with the F.I.P.. Then there is a one-to-one correspondence between the set of minimal prime elements in L and the set of all ultrafilters on $C(L)$.*

Proof.

Let $p_M = \bigvee\{c^{\perp} : c \in M\}$ for each ultrafilter M on $C(L)$ and $M_P = \{c \in C(L) : c \nleq p\}$ for each minimal prime element p. Suppose that M is an ultrafilter. Firstly, we show that $M = M_{p_M}$. Let $d \in C(L)$ such that $d \leq p_M$. Then there are $c_1, c_2, ..., c_k$ in M such that $d \leq (c_1^{\perp} \vee c_2^{\perp} \vee ... \vee c_k^{\perp})$. But $c = c_1 \wedge c_2 \wedge ... \wedge c_k \in M$ and $d \leq c^{\perp}$. So $d \wedge c = 0$ and therefore $d \notin M$, as $c \in M$. Then $M \subseteq M_{p_M}$. On the other hand, let $d \nleq p_M$ and suppose $c \wedge d = 0$ for some $c \in M$. Then $d \leq c^{\perp}$ and therefore $d \leq p_M$, which is a contradiction. Then $c \wedge d > 0$ for each $c \in M$ which implies that $d \in M$. Then $M = M_{p_M}$. Secondly, we show that p_M is a minimal prime element. Let $x, y \in C(L)$ such that $x, y \nleq p_M$. Then $x, y \in M_{p_M} = M$ which implies that $x \wedge y \in M = M_{p_M}$, therefore $x \wedge y \nleq p_M$. So p_M is a prime element. Suppose there is a prime element q such that $q \leq p_M$. Then $M = M_{p_M} \subseteq M_q$. Since M_q is a filter, then $M = M_{p_M} = M_q$ which implies that $q = p_M$. Hence p_M is a minimal prime element. Finally, if p is any minimal prime element, then M_p is a filter on $C(L)$. So by Zorn's lemma there is an ultrafilter N such that $M_p \subseteq N$, and therefore p_N is a minimal prime element. Suppose $p_N \nleq p$. Then $\exists\ c \in C(L)$ such that $c \leq p_N$ and $c \nleq p$. So $M_p \nsubseteq N_{p_N} = N$, which is a contradiction. Then $p_N \leq p$ and therefore $p_N = p$. So $N = N_{p_N} = M_p$ and M_p is an ultrafilter on $C(L)$. Thus the map $p \longrightarrow M_p$

is a one-to-one correspondence between the set of minimal prime elements in L and the set of all ultrafilters on $C(L)$. □

Lemma 1.28 *Let L be an algebraic frame with the F.I.P. and let p be a prime element. Then p is minimal if and only if $p = \bigvee\{c^{\perp} : c \in C(L), c \nleq p\}$.*

Proof.

($\Longrightarrow$) *Suppose p is a minimal prime element. Then $p = p_{M_p} = \bigvee\{c^{\perp} : c \in C(L), c \nleq p\}$.*

($\Longleftarrow$) *Let p be a prime element such that $p = \bigvee\{c^{\perp} : c \in C(L), c \nleq p\}$. Suppose there is a minimal prime element q such that $q \leq p$ and $p \nleq q$ then $M_p \subseteq M_q$ and M_q is an ultrafilter. Since there is a compact element d such that $d \leq p$ and $d \nleq q$, then $d \leq (c_1^{\perp} \vee c_2^{\perp} \vee ... \vee c_k^{\perp})$ where $c_i \nleq p$ for each $i = 1, 2, ...k$. Let $c = c_1 \wedge c_2 ...... \wedge c_k$. Then $c \in M_p \subseteq M_q$. Since $d \leq c^{\perp}$, then $d \wedge c = 0$ and therefore $d \notin M_q$, which is a contradiction. Thus p is a minimal prime element.* □

Theorem 1.19 *Let L be an algebraic frame. Then L has the CSP if and only if L has the F.I.P. and spec(L) is a trivially ordered set.*

Proof.

($\Longrightarrow$) *Suppose L has the CSP and let $c, d \in C(L)$. Then $c \wedge d = \bigvee_{i \in I} x_i$ where $x_i \in C(L)$ for each $i \in I$. But $d \vee d^{\perp} = 1$ which implies that $c = c \wedge (d \vee d^{\perp}) = (d \wedge c) \vee (c \wedge d^{\perp}) = (\bigvee_{i \in I} x_i) \vee (d^{\perp} \wedge c) = \bigvee_{i \in I}(x_i \vee (d^{\perp} \wedge c))$. Since c is a compact element, there is a finite subset F of I such that $c = \bigvee_{i \in F}(x_i \vee (d^{\perp} \wedge c))$. Therefore $c = x \vee (d^{\perp} \wedge c)$ where $x = \bigvee_{i \in F} x_i$ is a compact element. Then $c \wedge d = (x \vee (d^{\perp} \wedge c)) = (x \wedge d) \vee (c \wedge d \wedge d^{\perp}) = x \wedge d$. But $c \wedge d \geq x$. Thus $c \wedge d = x$ is a compact element and L has has the F.I.P.. Next we show that spec(L) is a trivially ordered set. Let p, q be prime ideals such that $p \leq q$ and $q \nleq p$. Then there is a compact element c such that $c \leq q$ and $c \nleq p$. Since $c \wedge c^{\perp} = 0 \leq p$ then $c^{\perp} \leq p$ and therefore $c \vee c^{\perp} = 1 \leq q$, which is a contradiction. Thus the set spec(L) is a trivially ordered set.*

($\Longleftarrow$) *Suppose L has the F.I.P. and spec(L) is a trivially ordered set. Let $c \in C(L)$*

such that $c \vee c^{\perp} < 1$. *By lemma* 1.23 *there is a meet-irreducible element* b *such that* $c \vee c^{\perp} \leq b$ *and* $b < 1$. *Then* b *is a prime element and therefore a minimal prime element and* $c, c^{\perp} \leq b$, *which is a contradiction. Then* $c \vee c^{\perp} = 1$ *and* L *has the* CSP.

□

Corollary 1.11 *If* L *is an algebraic frame then* a *is a regular element if and only if* $d \leq a$ *implies that* $d \preceq a$ *for each compact element* d.

Proof.

($\Longrightarrow$) *Suppose* a *is a regular element and let* $d \in C(L)$ *such that* $d \leq a$. *Then* $d \leq \bigvee\{x : x \preceq a\}$, *which implies that* $d \leq (x_1 \vee x_2 ...... \vee x_k)$ *where* $x_i \preceq a$ *for each* $i = 1, 2, ...k$. *Let* $x = x_1 \vee x_2 ...... \vee x_k$. *Then* $x \preceq a$ *and* $d \leq x$. *Hence* $d \preceq a$.

($\Longleftarrow$) *Suppose* $d \leq a$ *implies that* $d \preceq a$ *for any compact element* d. *It is clear that* $\bigvee\{x : x \preceq a\} \leq a$. *Since* a *can be written as a supremum of compact elements, then* $a \leq \bigvee\{x : x \preceq a\}$. *Thus* $a = \bigvee\{x : x \preceq a\}$. □

If L is an algebraic frame, and a is a regular element then by corollary 1.11 we can see that $a = \bigvee\{c : c \in C(L),\ c \preceq a\}$.

Lemma 1.29 *Let* L *be an algebraic frame, and* a *be a regular element. Then* $a = \bigvee\{c^{\perp\perp} : c \in C(L),\ c \leq a\}$.

Proof.

Since $a = \bigvee\{c : c \in C(L),\ c \preceq a\} = \bigvee\{c : c \in C(L), c \leqslant a\}$ *and* $c \leqslant c^{\perp\perp}$ *then* $a \leqslant \bigvee\{c^{\perp\perp} : c \in C(L),\ c \leq a\}$. *On the other hand, if* $c \leqslant a$, *then* $c \preceq a$. *Therefore* $(c^{\perp\perp} \wedge c^{\perp}) \vee (c^{\perp\perp} \wedge a) = c^{\perp\perp} \wedge (c^{\perp} \vee a) = c^{\perp\perp}$, *which implies* $c^{\perp\perp} \leqslant a$. *Thus* $\bigvee\{c^{\perp\perp} : c \in C(L),\ c \leq a\} \leqslant a$ *and* $a = \bigvee\{c^{\perp\perp} : c \in C(L),\ c \leq a\}$. □

Definition 1.18 *Let* L *be an algebraic frame and* $a \in L$. *Then* a *is called a* d*-element if* $a = \bigvee\{c^{\perp\perp} : c \in C(L),\ c \leq a\}$.

It is clear from lemma 1.29 that every regular element is a d-element.

Lemma 1.30 *let L be an algebraic frame. Then:*

(1) *Every polar element is a d-element.*

(2) *If L has the F.I.P. then every minimal prime element is a d-element.*

Proof.

(1) *Suppose a is a polar element. Since $a \leq \bigvee\{c^{\perp\perp} : c \in C(L),\ c \leq a\}$ and $a = a^{\perp\perp}$, then $c \leq a$ which implies that $c^{\perp\perp} \leq a^{\perp\perp} = a$. Therefore $\bigvee\{c^{\perp\perp} : c \in C(L),\ c \leq a\} \leq a$ and hence $a = \bigvee\{c^{\perp\perp} : c \in C(L),\ c \leq a\}$.*

(2) *Suppose L has the F.I.P. and let p be a minimal prime element. Then $p = \bigvee\{c^{\perp} : c \in C(L),\ c \not\leq p\}$. Let d be a compact element. If $d \leq p$, then $d^{\perp} \not\leq p$. Therefore $d^{\perp\perp} \leq p$. So $p = \bigvee\{c^{\perp\perp} : c \in C(L),\ c \leq p\}$. Thus p is a d-element.* □

If L is an algebraic frame, and $X = spec(L)$ then X becomes a topological space by taking the subsets of the form $coz(c) = \{p : p \in X, c \not\leq p\}$ as basic open sets for each $c \in C(L)$. This topology is called the hull kernel topology on $spec(L)$.

Theorem 1.20 *Let L be an algebraic frame. Then :*

(1) *Any open subset in $spec(L)$ has the form of $coz(a)$ for some $a \in L$.*

(2) *$coz(a) = coz(b)$ implies that $a = b$.*

(3) *$L \cong D(spec(L))$ as a lattice isomorphism.*

(4) *$spec(L)$ is a T_1-space if and only if $spec(L)$ is a trivially ordered set.*

Proof.

(1) *Suppose U is an open subset of $spec(L)$. Then $U = \bigcup_{i \in I} coz(c_i)$ for some family $\{c_i \in C(L) : i \in I\}$. Let $a = \bigvee_{i \in I} c_i$. If $p \in U$ then $p \in coz(c_{i_0})$ for some $i_0 \in I$ which implies $c_{i_0} \not\leq p$. Then $a \not\leq p$ and therefore $p \in coz(a)$. On the other hand, if $p \in coz(a)$ then $a \not\leq p$ which implies that there exists $i_0 \in I$ such that $c_{i_0} \not\leq p$. So $p \in coz(c_{i_0}) \subseteq U$. Hence $U = coz(a)$.*

(2) *Let $coz(a) = coz(b)$ and suppose that $a \neq b$. Then $a \not\leq b$ or $b \not\leq a$. WLOG, suppose that $a \not\leq b$. Then it is obvious that $b < 1$. Since $a \not\leq b$, then there is a compact element c such that $c \leq a$ and $c \not\leq b$. Take $S = \{t : t \geq b, t \not\geq c\}$. Clearly S is a nonempty subset. By the same argument as that used in lemma 1.23 one can*

prove that there is a meet -irreducible element t such that $t \in S$. Then t is a prime element and $t \in coz(a)$, $t \notin coz(b)$, which is a contradiction. Thus $a = b$.

(3) *Let $f : L \longrightarrow D(spec(L))$ be defined by $f(a) = coz(a)$ for each $a \in L$. Then it is clear from (1) and (2) that f is a one-to-one and onto map. It is clear that $f(a \vee b) = coz(a \vee b) = coz(a) \cup coz(b)$ and $f(a \wedge b) = coz(a \wedge b) = coz(a) \cap coz(b)$ for each $a, b \in L$. Thus f is a lattice isomorphism.*

(4) *First we show that $cl(p) = \{q : q \geq p\}$. Clearly $p \in \{q : q \geq p\}$ and $\{q : q \geq p\} = (coz(p))^c$ is a closed subset. Then $cl(p) \subseteq \{q : q \geq p\}$. If $q \geq p$ and $q \in coz(c)$ for some $c \in C(L)$ then $c \nleq q$ which implies that $c \nleq p$ and $p \in coz(c)$. Therefore $q \in cl(p)$ and hence $cl(p) = \{q : q \geq p\}$. Now if $spec(L)$ is a T_1-space then $cl(p) = \{p\} = \{q : q \geq p\}$ which means that $spec(L)$ is a trivially ordered set. On the other hand, if $spec(L)$ is a trivially ordered set then $cl(p) = \{q : q \geq p\} = \{p\}$ and hence $spec(L)$ is a T_1-space.* □

It is clear from corollary 1.10, that if L is an algebraic frame, then L is a regular if and only if $spec(L)$ is a regular space.

Theorem 1.21 *Let L be an algebraic frame. Then TFAE:*

(1) *L is regular.*

(2) *L has the CSP.*

(3) *$spec(L)$ is a T_1-space and $coz(a) \cap coz(b)$ is a compact subset for each $a, b \in C(L)$.*

(4) *$spec(L)$ is a T_2-space.*

Proof.

(1) $\Longrightarrow$ (2) *Let $c \in C(L)$. Since $c^{\perp\perp}$ is a regular element, and $c \leq c^{\perp\perp}$ then $c \preceq c^{\perp\perp}$ and therefore $c^{\perp\perp}$ is a complemented element for each $c \in C(L)$. Since every regular element is a d-element then $c = \bigvee\{d^{\perp\perp} : d \in C(L), d \leq c\}$ which means that $c = c^{\perp\perp}$. Thus every compact element is a complemented and L has the CSP.*

(2) $\Longrightarrow$ (1) *Let $a \in L$ and $c \in C(L)$ such that $c \leq a$. Since L has the CSP, then $c \vee c^{\perp} = 1$ which implies $a \vee c^{\perp} = 1$, and therefore $c \preceq a$. Thus by corollary 1.11, a is regular.*

$(2) \Longrightarrow (3)$ *Since L has the CSP then by theorem 1.19, L has the F.I.P. and spec(L) is a trivially ordered set. Therefore by theorem 1.20 spec(L) is a T_1-space and coz$(a) \cap$ coz(b) is a compact subset for each $a, b \in C(L)$.*

$(3) \Longrightarrow (2)$ *Since spec(L) is a T_1-space and coz$(a) \cap$ coz(b) is a compact subset for each $a, b \in C(L)$ then L has the F.I.P. and spec(L) is a trivially ordered set. Hence by theorem 1.19 L has the CSP.*

$(3) \Longrightarrow (4)$ *Since spec(L) is a regular T_1-space then spec(L) is a T_2-space.*

$(4) \Longrightarrow (3)$ *Since spec(L) is a T_2-space and coz(a) and coz(b) are compact subsets then coz(a) and coz(b) are closed subsets and therefore coz$(a) \cap$ coz(b) is a closed subset too. Hence coz$(a) \cap$ coz(b) is a compact subset.* □

Let A be a commutative ring with identity and let $I(A)$ be the set of all ideals of A. Then this set can be turned into a complete lattice under the inclusion order where the supremum is given by the sum and the infimum is given by intersection. Since every principal ideal is a compact element, then $I(A)$ is an algebraic lattice which is not a distributive lattice in general unless we take A to be an arithmetical ring. And for this reason since we take A to be a general semiprime ring, we consider only the subset of semiprime ideals, denoted $SP(A)$, which turns out to be a compact algebraic frame where the supremum is given by the square root of the sum and the infimum is given by intersection.

Theorem 1.22 *Let A be a commutative semiprime ring with identity. Then $SP(A)$ is a compact algebraic frame.*

Proof.

Let $T : SP(A) \longrightarrow D(spec(A))$ be defined by $T(I) = coz(I) = \{P : I \nsubseteq P\}$.

It is clear that T is a map. Since $SP(A)$ is the set of all semiprime ideals of A then for any two ideals $I, J \in SP(A)$ we have $I \subseteq J$ if and only if $T(I) \subseteq T(J)$. So T is a one-to-one and order-preserving function. For any open subset coz(K) in $D(spec(A))$, we have $\sqrt{K} \in SP(A)$ and $T(\sqrt{K}) = \{P : \sqrt{K} \nsubseteq P\} = coz(K)$. Therefore T is an onto map.

It is clear that $T(I \wedge J) = coz(I \cap J) = coz(I) \cap coz(J) = T(I) \wedge T(J)$. *If* $\{K_i : i \in \Gamma\}$ *is any collection of semiprime ideals of* $SP(A)$ *and* Q *is a prime ideal then* $\sqrt{\bigoplus_{i\in\Gamma} k_i} \nsubseteq Q$ *if and only if there is an* $i_0 \in \Gamma$ *such that* $K_{i_0} \nsubseteq Q$. *Then* $T(\bigvee_{i\in\Gamma} K_i) = T(\sqrt{\bigoplus_{i\in\Gamma} K_i}) = coz(\sqrt{\bigoplus_{i\in\Gamma} K_i}) = \bigcup_{i\in\Gamma} coz(K_i)) = \bigvee_{i\in\Gamma} T(K_i))$. *Thus* $SP(A)$ *and* $D(spec(A))$ *are isomorphic as compact algebraic frames.* $\square$

Let A be a semiprime ring. Then by corollary 1.3 we can conclude that $C(SP(A)) = \{\sqrt{I} : I$ is a finitely generated ideal $\}$.

Corollary 1.12 *Let* A *be a semiprime ring. Then* $(SP(A), \leq)$ *is a regular algebraic frame if and only if* A *is regular.*

Proof.

($\Longrightarrow$) *Suppose that* $(SP(A), \leq)$ *is a regular algebraic frame. Then* $(D(spec(A)), \leq)$ *is a regular algebraic frame, and therefore* $spec(A)$ *is a regular space. Let* $P \neq Q$ *be any two points in* $spec(A)$ *and assume* $WLOG$ *that* $P \nsubseteq Q$. *Since* $P \notin V(Q)$ *and* $V(Q)$ *is a closed subset, then there are disjoint open subsets* $D(I), D(J)$ *such that* $P \in D(I)$ *and* $V(Q) \subseteq D(J)$. *Hence* $spec(A)$ *is a* T_2*-space and therefore* A *is regular.*

($\Longleftarrow$) *Suppose* A *is regular. Then* $spec(A)$ *is a regular space. Thus* $D(spec(A))$ *is a regular algebraic frame and so is* $SP(A)$. $\square$

Chapter 2

On RG-spaces and Almost Baire Spaces

In the first section of this chapter we shall give a review for the basic concepts on RG-spaces which were originally given in [31]. However we present these results from our point of view. And then we will give new results on RG-spaces which we have obtained in our study. We will introduce a new class of topological spaces which we call almost k-Baire spaces and as special case of this class we will introduce the class of almost Baire spaces.

Throughout this study we start with the assumption that by a topological space X we mean a Tychonoff space, by $C(X)$ we mean the ring of real-valued continuous functions defined on X, and by $F(X)$ we mean the ring of all real-valued functions defined on X. It is clear that both of these rings are commutative semiprime rings with the same identity. Moreover the ring $F(X)$ is a regular ring. The smallest regular ring $G_{F(X)}(C(X))$ lying between $C(X)$ and $F(X)$ is denoted by $G(X)$ [theorem 1.6].

For any function f in $F(X)$, the quasi-inverse of f is given by:

$$f^*(x) = \begin{cases} 0 & \text{if } x \in Z(f) \\ \frac{1}{f(x)} & \text{if } x \in coz(f) \end{cases}$$

where $Z(f) = \{x : f(x) = 0\}$ and $coz(f) = X - Z(f)$. A subset of X is called a zeroset [cozeroset] if it has the form of $Z(f)$ [$coz(f)$] for some function $f \in C(X)$. The set of all zerosets in X is denoted $Z(X)$. An ideal I in the ring $C(X)$ is called a z-ideal if for any two functions f, g in $C(X)$ with the same zerosets, both are in I or both outside of I. Also, an ideal I of $C(X)$ is called an ℓ-ideal if $0 \leq |f| \leq |g|$ and $g \in I$ imply $f \in I$.

2.1 RG-spaces

For a topological space X, a point p in X is called a P-point if p is in the interior of each zeroset containing it. A topological space X is called a P-space if every point in X is a P-point [8, 4L]. A space X is a P-space if and only if $C(X)$ is a regular ring, or equivalently if every G_δ-set is open [8, 4J]. A topological space X is called an almost P-space if every non-empty zeroset has a non-empty interior. Therefore a space X is an almost P-space if and only if $C(X) = Q_{cl}(X)$. Details on almost P-spaces appear in [18]. If (X, τ) is a topological space then the collection $\beta = \{A : A \text{ is a } G_\delta\text{-set}\}$ forms an open base for a potentially stronger topology on X. This topology is called the G_δ topology in the literature. It is denoted τ_δ and the space is denoted X_δ.

Remark 2.1 *Let (X, τ) be a topological space. Then:*

(1) *$\beta = \{A : A \text{ is a } G_\delta \text{ set}\}$ forms an open base for the topology τ_δ on X and $\tau \leq \tau_\delta$.*

(2) *X_δ is a P-space.*

(3) *X is a P-space if and only if $\tau = \tau_\delta$.*

(4) *If Y is any subspace of X then $(\tau_Y)_\delta = (\tau_\delta)_Y$.*

We now discuss the minimal regular extension $G(X)$ of the semiprime ring $C(X)$. By lemma 1.6 and corollary 1.1 we have:

$G(X) = \{ \sum_{i=1}^{n} f_i g_i^* : f_i, g_i \in C(X),\ n \geq 1\}$, and $|C(X)| = |G(X)|$.

Lemma 2.1 *Let X be a topological space. Then:*

(1) $C(X) \subseteq G(X) \subseteq C(X_\delta)$.

(2) X is P-space if and only if $C(X) = G(X)$.

(3) If (X, τ_α) is a Tychonoff space such that $C(X, \tau_\alpha)$ is a regular ring and $\tau \leq \tau_\alpha$ then $\tau_\delta \leq \tau_\alpha$.

(4) $G(X) = C(X, \tau_\alpha)$ for some Tychonoff topology τ_α on X if and only if $G(X) = C(X_\delta)$.

Proof.

(1) Since $C(X) \subseteq C(X_\delta) \subseteq F(X)$ and $C(X_\delta)$ is a regular ring. Then $C(X) \subseteq G(X) \subseteq C(X_\delta)$.

(2) If X is a P-space then $X = X_\delta$ which implies that $C(X) = G(X) = C(X_\delta)$. On the other hand, if $C(X) = G(X)$ then $C(X)$ is a regular ring and therefore X is a P-space.

(3) Since $C(X, \tau_\alpha)$ is a regular ring, then every G_δ set in (X, τ_α) is open. But $\tau \leq \tau_\alpha$. So every G_δ set in (X_δ) is an open subset in (X, τ_α). Hence $\tau_\delta \leq \tau_\alpha$.

(4) ($\Longrightarrow$) Suppose that $G(X) = C(X, \tau_\alpha)$ for some topology τ_α on X. Then (X, τ_α) is a P-space and $C(X) \subseteq C(X, \tau_\alpha)$. Since for any $Z(f) \in Z(X)$, we have $f \in C(X, \tau_\alpha)$ therefore $\tau \leq \tau_\alpha$. Using (3) we can see that $\tau_\delta \leq \tau_\alpha$. Hence $C(X_\delta) \subseteq C(X, \tau_\alpha) = G(X)$ and therefore $G(X) = C(X_\delta)$.

($\Longleftarrow$) Suppose $G(X) = C(X_\delta)$. Since X_δ is a P-space, then X_δ is a 0-dimensional T_1-space. So X_δ is a Tychonoff space and $G(X) = C(X_\delta)$. □

Definition 2.1 *Let X be a topological space and $f \in G(X)$. Then the regularity degree of f, denoted by $rg(f)$, is defined to be:*

$rg(f) = min\{n \in N : f = \sum_{i=1}^{n} g_i h_i^*$ where $g_i, h_i \in C(X),\ n \geq 1\}$.

The regularity degree of the topological space X, denoted by $rg(X)$, is defined to be:

$rg(X) = sup\{rg(f) : f \in G(X)\}$.

Definition 2.2 *Let X be a topological space. Then X is called a regular good space, denoted RG-space, if $G(X) = C(X_\delta)$.*

It is clear from the definition of RG-spaces that every P-space is an RG-space because for P-spaces $G(X) = C(X) = C(X_\delta)$.

Theorem 2.1 *Let X be a topological space. Then $rg(X) = 1$ if and only if X is a P-space .*

Proof.

($\Longrightarrow$) *Suppose $rg(X) = 1$ and let $f \in C(X)$. Then $1 + (-f)f^* \in G(X)$ which implies that there are $g, h \in C(X)$ such that $1 + (-f)f^* = gh^*$. Then $gh^* + ff^* = 1$ and therefore $Z(f) = coz(gh)$. So $Z(f)$ is an open set for each $f \in C(X)$. Thus X is a P-space.*

($\Longleftarrow$) *Suppose X is a P-space. Then $G(X) = C(X)$ which implies that $rg(f) = 1$ for each $f \in G(X)$. Thus $rg(X) = 1$.* □

Definition 2.3 *Let X be a topological space. Then X is called a scattered space if every non empty subspace of X has an isolated point.*

Lemma 2.2 *Let X be a scattered topological space. Then:*

(1) *Every non empty subspace Y of X is scattered.*

(2) *The set $I(X)$ of all isolated points in X is a dense subset.*

Proof.

(1) *Let $\varnothing \neq B \subseteq Y$. So B as a subspace of X has an isolated point $x_0 \in B$. Then there is an open subset U of X such that $U \cap B = \{x_0\}$. Therefore $V = U \cap Y$ is an open subset of Y and $V \cap B = \{x_0\}$. So B has an isolated point as a subspace of Y. Thus Y is a scattered space.*

(2) *It is clear that $I(X) \neq \varnothing$. Since every nonempty open set has an isolated point then $I(X)$ is a dense subset of X.* □

For each ordinal number α define $D_\alpha(X)$ inductively as follows: $D_0(X) = X$, $D_1(X) = X - I(X)$, $D_{\alpha+1}(X) = D_1(D_\alpha(X))$ and if λ is limit ordinal number let $D_\lambda(X) =$

$\bigcap_{\alpha<\lambda} D_\alpha(X)$. Then a space X is scattered if and only if there is ordinal number α_0 such that $D_{\alpha_0}(X) = \varnothing$ [29]. Let X be a scattered space. Then the Cantor-Bendixson number, denoted by $CB(X)$, is defined to be $CB(X) = min\{\alpha : D_\alpha(X) = \varnothing\}$.
It is shown in [13] that if X is a non-empty scattered compact space, then $CB(X)$ is a successor ordinal, $\delta + 1$, and that $D_\delta(X)$ is a finite subset.

Theorem 2.2 *Let X be a topological space and let $f \in G(X)$. Then:*

(1) f is continuous on dense open subset of X.

(2) If T is any non-empty subspace of X then $f|_T \in G(T)$ and $rg(f|_T) \leq rg(f)$.

(3) f can be written as $f = \sum_{i=1}^n f_i.g_i^*$ where $f_i, g_i \in C^*(X)$ for each $i = 1, 2, ...n$ and $rg(f)$ will not change by this new presentation.

(4) If T is C^*-embedded in X then T is G-embedded.

Proof.

(1) Suppose $f = \sum_{i=1}^n f_i g_i^*$ where $f_i, g_i \in C(X)$ for each i. Take $D_i = X - bd(Z(g_i))$ and $D = \bigcap_{i=1}^n D_i$. Since D_i is a dense open subset of X for each i, then D is a dense open subset of X. It is clear that $f_i g_i^*$ is a continuous function on D_i and therefore on D for each $i = 1, 2, ...n$. Since f can be written as a finite sum of real-valued functions each of which is continuous on D, then f is continuous on D.

(2) Suppose $f = \sum_{i=1}^n f_i g_i^*$ where $f_i, g_i \in C(X)$ for each i. Then $f|_T = \sum_{i=1}^n (f_i|_T)(g_i^*|_T)$ where $f_i|_T, g_i|_T \in C(T)$ for each i. Thus $f|_T \in G(T)$. It is clear that $rg(f|_T) \leq rg(f)$.

(3) Let $f = \sum_{i=1}^n f_i.g_i^*$ where $f_i, g_i \in C(X)$ for each $i = 1, 2, ...n$. For each i let $T_i = (1 + f_i^2) \vee (1 + g_i^2)$. Then $T_i \in C(X)$ and $T_i(x) \neq 0$ for each $x \in X$, which implies that $\frac{1}{T_i} \in C(X)$. It is clear that $|\frac{f_i(x)}{T_i(x)}|$, $|\frac{g_i(x)}{T_i(x)}| \leq 1$ for each $x \in X$, therefore $\frac{f_i}{T_i}, \frac{g_i}{T_i} \in C^*(X)$ for each $i = 1, 2, ...n$. Since $f_i g_i^* = (\frac{f_i}{T_i})(\frac{g_i}{T_i})^*$ then $f = \sum_{i=1}^n (\frac{f_i}{T_i})(\frac{g_i}{T_i})^*$. It is clear that this new presentation does not change the regularity degree of the function f.

(4) Suppose T is a C^*-embedded subset of X, and let $h \in G(T)$. Then by (3) there are $n \in N$ and $g_i, k_i \in C^*(T)$ for each $i = 1, 2, ...n$ such that $h = \sum_{i=1}^n g_i k_i^*$. Since T is C^*-embedded, then there are $\tilde{g}_i, \tilde{k}_i \in C^*(X)$ such that $\tilde{g}_i|_T = g_i$ and $\tilde{k}_i|_T = k_i$

for each i. Take $F = \sum_{i=1}^{n}(\tilde{g}_i)(\tilde{k_i}^*)$. Then $F \in G(X)$ and $F|_T = h$. Hence T is a G-embedded subset of X. □

Lemma 2.3 *Let X be an RG-space. Then no dense subset can be written as a countable union of nowhere dense zerosets.*

Proof.

Let X be an RG-space, and suppose $\{Z_i : i \in N\}$ is a collection of nowhere dense zerosets of X such that $S = \bigcup_{i=1}^{\infty} Z_i$ is a dense subset of X. Take $A_1 = Z_1$ and $A_n = Z_n - (\bigcup_{i=1}^{n-1} Z_i)$ for each $n \geq 2$. Then $\{A_i : i \in N\}$ is a collection of disjoint clopen subsets of X_δ and S is an F_σ-subset of X_δ. As X_δ is a P-space, then $\{A_i : i \in N\} \cup \{X - S\}$ is a partition of X_δ. Now define $f : X_\delta \longrightarrow R$ by $f(A_i) = \{i+1\}$ for each $i \in N$ and $f(X - S) = \{1\}$. Then $f \in C(X_\delta) = G(X)$. So by theorem 2.2(1), there is a dense open subset D of X such that $f|D$ is a continuous function. Since $S \cap D \neq \varnothing$ then there exists $p \in S \cap D$, and therefore $p \in A_{n_0}$ for some $n_0 \geq 1$. Since f is continuous at p and $f(p) = n_0 + 1$, then there is a open neighborhood W_p of p such that $f(W_p) \subseteq (n_0 + \frac{2}{3}, n_0 + \frac{4}{3})$. So $W_p \subseteq A_{n_0} \subseteq Z_{n_0}$, which is a contradiction. Thus no dense subset can be written as a countable union of nowhere dense zerosets. □

2.2 Almost Baire Spaces

Recall that a topological space X is called k-Baire where k is a fixed cardinal number if the intersection of fewer than k dense open sets is dense [34]. Thus the usual Baire spaces are $\aleph_1$-Baire spaces. It is clear that the intersection of all dense open subsets of X is a dense subset if and only if X has a dense subset of isolated points, which means that if X has a dense subset of isolated points then X is a k-Baire space for any cardinal number k. So one can conclude that every scattered space is a k-Baire space for any cardinal number k.

For a Tychonoff space X an open subset does not have to be a cozeroset, and the collection of all dense cozerosets may have any cardinal number. For these reasons we will introduce the class of almost k-Baire spaces where k is a fixed cardinal number.

Definition 2.4 *Let X be a topological space and k be a cardinal number. Then we will call X an almost k-Baire space if any collection with cardinal number fewer than k of dense cozerosets intersects in a dense subset, and we will call X almost-Baire if X is an almost $\aleph_1$-Baire space.*

Let X be a topological space and k be a fixed cardinal number. Then X is an almost k-Baire space if and only if the union of any collection with cardinal number fewer than k of nowhere dense zerosets has an empty interior. It is clear that every clopen subspace of almost k-Baire space is an almost k-Baire space and a space X is almost k-Baire if and only if X has a dense subspace which is an almost k-Baire space. Every k-Baire space is an almost k-Baire space, but the converse is not true in general as we will see next.

Theorem 2.3 *Let X be an RG-space and $(Z_n)_{n=1}^{\infty}$ be a sequence of nowhere dense zerosets in X. Then $\bigcup_{n=1}^{\infty} Z_n$ is a nowhere dense subset.*

Proof.

Let $S = \bigcup_{n=1}^{\infty} Z_n$, $A_1 = Z_1$ and $A_m = Z_m - (\bigcup_{i=1}^{m-1} Z_i)$ for each $m \geq 2$. Then $\{A_n : n \in N\}$ is a collection of clopen subsets in X_δ, and therefore $\{A_n : n \in N\} \cup \{X - S\}$ is a clopen partition of X_δ. Let $f : X_\delta \longrightarrow R$ be defined by $f(A_n) = \{n+1\}$ for each $n \in N$ and $f(X - S) = \{1\}$. Then $f \in G(X)$, and since X is an RG-space therefore there is a dense open subset D of X such that $f|D$ is a continuous function. Now suppose $cl(\bigcup_{n=1}^{\infty} Z_n)$ has an interior point p. Then there is an open subset U_p containing p such that $U_p \subseteq cl(\bigcup_{n=1}^{\infty} Z_n)$, which means that for each y in U_p and each neighborhood W_y of y we have $W_y \cap (\bigcup_{n=1}^{\infty} Z_n) \neq \varnothing$. Since D is a dense subset, then $D \cap U_p \neq \varnothing$. Let $y \in D \cap U_p$. There are two cases:

(1) *If $f(y) = 1$, then there is an open neighborhood W_y of y such that $f(W_y) \subseteq (0, \frac{3}{2})$. So $W_y \cap (\bigcup_{n=1}^{\infty} Z_n) = \varnothing$, which is a contradiction.*

(2) If $f(y) = k+1$, then $y \in A_k$, and therefore there is an open neighborhood W_y of y such that $f(W_y) \subseteq (k+\frac{2}{3}, k+\frac{4}{3})$. Then $W_y \subseteq A_k \subseteq Z_k$, which is a contradiction too. Thus $\bigcup_{n=1}^{\infty} Z_n$ is a nowhere dense subset of X. □

If X is an RG-space then it is clear from theorem 2.3 that every countable intersection of dense cozero subsets of X has a dense interior. Recall that a space X is an almost P-space if and only if every non-empty countable intersection of open sets has a non-empty interior. It is clear that every almost P-space is almost k-Baire for each cardinal number k.

Corollary 2.1 *Every RG-space space is an almost-Baire space.*

RG-spaces need not be Baire. In [6] the authors gave two examples. First they gave an example of a regular P-space without any isolated points. Secondly they gave an example of a Tychonoff space X with a dense set of isolated points such that X_δ is not a Baire space. So an RG-space does not have to be a Baire space.
Recall that a topological space X is called separable at a point p if there exists an open set O containing p such that O is separable. A topological space X is called nowhere separable if X is not separable at any of its points. Details appear in [4].

Theorem 2.4 *If X is an RG-space with no almost P-points then X is a nowhere separable space.*

Proof.

Suppose X is an RG-space with no almost P-points and is a somewhere separable space. Then there is a countable subset $\{a_n : n \in N\}$ such that $int(cl(\bigcup_{n=1}^{\infty}\{a_n\})) \neq \varnothing$. For each a_n pick a nowhere dense zeroset Z_n such that $a_n \in Z_n$. Then $int(cl(\bigcup_{n=1}^{\infty} Z_n)) \neq \varnothing$ which is in contradiction to theorem 2.3. Thus X is a nowhere separable space . □

Definition 2.5 *A topological space X is said to be of countable pseudocharacter if every point in X is a G_δ-set.*

Lemma 2.4 *Let X be a topological space. Then:*

(1) *X is of countable pseudocharacter if and only if every point is a zeroset.*

(2) *X is of countable pseudocharacter if and only if X_δ is a discrete space, or equivalently if $C(X_\delta) = F(X)$.*

Proof.

(1) ($\Longrightarrow$) *Suppose X is of countable pseudocharacter. Since for each G_δ-set G and compact set $A \subseteq G$ there is a zeroset Z such that $A \subseteq Z \subseteq G$. Then taking $A = G = \{x_0\}$ implies that $\{x_0\}$ is a zeroset for each $x_0 \in X$.*

($\Longleftarrow$) *Obvious.*

(2) *It is clear that X being of countable pseudocharacter implies that X_δ is a discrete space, and X_δ discrete implies that $C(X_\delta) = F(X)$. Suppose X_δ is a discrete space. Since X_δ has the collection of G_δ-sets as open base. Then every point is a G_δ-set and therefore X is of countable pseudocharacter. On the other hand, suppose $C(X_\delta) = F(X)$. Since the character function $\chi_{\{x_0\}} \in F(X)$. Then $\{x_0\}$ is open subset of X_δ for each $x_0 \in X$. Thus X_δ is a discrete space.* □

Recall that a topological space X is called Blumberg if every real-valued function defined on X can be restricted continuously to a dense subset [36]. J. C. Bradford and C. Goffman in [3] proved that every Blumberg space is Baire and R. Levy in [17] showed that there is consistently a compact Hausdorff space X and therefore Baire space which is not Blumberg.

By theorem 2.2(1) one can conclude that every RG-space of countable pseudocharacter is a Blumberg space.

Theorem 2.5 *Let X be an RG-space of countable pseudocharacter. Then X is a Baire space.*

Proof.

Suppose X is an RG-space of countable pseudocharacter. Since X_δ is a discrete space, then by theorem 2.2 we have that every real-valued function defined on X can be *restricted continuously on dense open subset. So X satisfies the Blumberg's Theorem. Hence X is a Baire space.* $\square$

The following lemma originally appeared in [20].

Lemma 2.5 *Let X be an RG-space and Y be a subspace of X such that Y_δ is C^*-embedded in X_δ. Then Y is an RG-space.*

Proof.

Since every C^*-embedded subset in X_δ is C-embedded, then Y_δ is C-embedded in X_δ. Let $h \in C(Y_\delta)$. Then there exists $H \in C(X_\delta)$ such that $H|_Y = h$. Since X is an RG-space, then $H \in G(X)$ which implies that $H|_Y \in G(Y)$. So $G(Y) = C(Y_\delta)$ and Y is an RG-space. $\square$

It follows that every clopen subset of X_δ of an RG-space X is an RG-space, in particular every cozero and zero subspace of an RG-space is an RG-space.

Corollary 2.2 *The cozerosets and zerosets of RG-spaces are almost-Baire spaces.*

Definition 2.6 *Let X be a topological space. Then the subset gX is defined to be the intersection of all dense cozero subsets of X.*

It is clear that gX is the set of almost P-points in X. If X is an RG-space, then it is clear from theorem 2.4 that every countable subset of $X - gX$ is a nowhere dense subset of X.

Lemma 2.6 *Let X be a topological space. Then:*

(1) X is almost Baire implies that every dense open C^*-embedded subset in X is almost Baire.

(2) X is almost k-Baire for each cardinal number k if and only if gX is dense.

Proof.

(1) Let X be an almost Baire space, let U be a dense open C^*-embedded subset in X and let $V_n, n = 1, 2, 3, ...$ be a collection of dense cozerosets in U. Since U is

C^*-embedded in X then for each n, there is a dense cozeroset W_n in X such that $W_n \cap U = V_n$. But X is almost Baire. Therefore $\bigcap_{n=1}^{\infty} W_n$ is a dense subset of X which implies that $\bigcap_{n=1}^{\infty} W_n \cap U = \bigcap_{n=1}^{\infty} V_n$ is a dense subset of U. Thus U is almost Baire.

(2) ($\Longrightarrow$) *Let X be almost k-Baire for each cardinal number k. Suppose there exists a non-empty open subset U such that $U \cap gX = \varnothing$. For each $x \in U$ choose a nowhere dense zeroset Z_x such that $x \in Z_x$ and let $V_x = X - Z_x$. Then V_x is a dense cozeroset for each $x \in U$ and $U \cap \bigcap_{x \in U} V_x = \varnothing$, which contradicts the fact that X is an almost k-Baire space for each cardinal number k. Thus gX is a dense subset of X.*

($\Longleftarrow$) *This is clear from the fact that gX is contained in every dense cozeroset.* □

The following theorem originally appeared in [20].

Theorem 2.6 *Let X be an RG-space and Y be a subspace of X. Then Y is an RG-space if one of the following hold:*

(1) *Y_δ is a Lindelof space.*

(2) *Y is a scattered Lindelof space.*

(3) *X_δ is a normal space and Y is a realcompact C^*-embedded subset of X.*

(4) *$|X| \leq c$, the continuum hypothesis holds, and Y is a realcompact C^*-embedded subset of X .*

(5) *Y is a countable union of zerosets or cozerosets.*

(6) *X is a paracompact scattered space and Y_δ is a closed subset in X_δ.*

Proof.

(1) *It is clear that Y_δ is completely separated from each zeroset in X_δ disjoint from it. But by [1, 4.2] if a Lindelof subspace of S is completely separated from each zeroset of S disjoint from it, then that subspace is C-embedded. Thus Y_δ is a C-embedded subset of X_δ. Hence by lemma 2.5, Y is an RG-space.*

(2) *If Y is a scattered Lindelof space, then by [29, 5.2] Y_δ is a Lindelof space. Hence by (1) Y is an RG-space.*

(3) *Suppose X_δ is a normal space and Y is a realcompact C^*-embedded subset of X.*

Let $T = cl(Y)$. Then Y is dense and C^*-embedded in T, which implies by [8, 6.7] that $Y \subseteq T \subseteq \beta Y$. Recall that Y is a realcompact space if and only if Y is a G_δ-closed subset of βY. Then $\forall x \in (T - Y)$ there is a G_δ-set D in βY such that $x \in D \subseteq (\beta Y - Y)$. If $E(x) = D \cap T$ then $E(x)$ is a G_δ-set of T. Therefore $E(x) = H(x) \cap T$ for some G_δ-set $H(x)$ of X and $x \in E(x) \subseteq (T - Y)$. Then $X - Y = (X - T) \cup \{H(x) : x \in (T - Y)\}$. So $X - Y$ is an open subset of X_δ, and therefore Y_δ is a closed subset of X_δ. Since every closed subset in a normal space is C^*-embedded then Y_δ is a C^*-embedded subset of X_δ. Hence by lemma 2.5, Y is an RG-space.

(4) *Suppose that $|X| \leq c$ and assume the continuum hypothesis holds. Since every P-space with cardinality no greater than c is paracompact [29] then X_δ is a paracompact space and hence normal. Thus the result follows by (3).*

(5) *If Y is a countable union of zerosets or cozerosets then Y_δ is a clopen subset in X_δ. Hence Y is an RG-space.*

(6) *If X is a paracompact scattered space and Y_δ is a closed subset in X_δ then X_δ is paracompact and hence normal. So by lemma 2.5, Y is an RG-space.* □

The next four results originally appeared in [20]. We include them to give more background on RG-spaces.

Corollary 2.3 *Every countable subspace of RG-space is a scattered subspace.*

Proof.

Suppose S is a countable subspace which is not scattered. Then there is an infinite countable subspace T of S without isolated points. Let $T = \{a_n : n \in N\}$. Then $\{a_n : n \in N\}$ is a countable collection of nowhere dense zerosets in T and $T = \bigcup_{n \in N}\{a_n\}$. But by (1) in theorem 2.6, T is an RG-space, which is a contradiction. Thus S is a scattered subspace. □

Theorem 2.7 *Compact subspaces of RG-space are scattered RG-spaces.*

Proof.

Let X be an RG-space and suppose K is a non-scattered compact subspace of X. Then K has a subspace T without isolated points. Let $L = cl(T)$. Then L is a compact space without isolated points which implies by [29, 3.17] that there is a continuous map from L onto the Cantor set C. So by [28, 6.5] there is a compact subspace M of L such that $f|_M$ is an irreducible continuous function from M onto C. Since C is a compact metric space, then C has a countable dense subset S. For each $s \in S$ pick one point in $f^{-1}\{s\}$ and call it m_s. Let $T = \{m_s : s \in S\} \subseteq M$. Then T is a countable subset of M. Now since $f|_M$ is an irreducible function and C has no isolated points, then T has no isolated points, which is a contradiction with the previous corollary. Thus K must be a scattered space. Since K is scattered and compact then by (2) in theorem 2.6, K is an RG-space. □

Definition 2.7 *Let X be a topological space. Then X is called resolvable if X can be written as a union of two disjoint dense subsets.*

In [2] it is shown that first countable spaces, locally compact Hausdorff spaces, K-spaces, linear topological spaces over a nondiscrete valuated field, and countably compact spaces without isolated points are resolvable spaces. Note that a topological space X is resolvable if and only if it can written as a finite union of sets with void interiors.

Theorem 2.8 *Let X be a topological space. Then:*

(1) *If X is a resolvable space of countable pseudocharacter, then X is not an RG-space.*

(2) *If X is an RG-space of countable pseudocharacter that is countably compact, locally compact, or a K-space, then X is scattered.*

Proof.

(1) *If X is a resolvable space of countable pseudocharacter then $X = A \cup A^c$ where A, A^c are dense subsets. Since the characteristic function χ_A is in $C(X_\delta)$ and it is nowhere continuous then X is not an RG-space.*

(2) *Let X be an RG-space of countable pseudocharacter that is countably compact, locally compact, or a K-space that is not scattered. Then X has a closed subspace T without isolated points. Then T inherits the assumed property from X. But as we mentioned before, T will be a resolvable space of countable pseudocharacter. Therefore T cannot be an RG-space. But by lemma 2.5, T is an RG-space, which is a contradiction. Thus X must be scattered.* □

Corollary 2.4 *Every first countable RG-space is scattered.*

Lemma 2.7 *If X is an RG-space of countable pseudocharacter then every finite intersection of dense subsets is a dense subset.*

Proof.

It is enough to consider the case of two dense subsets. Let A, B be two dense subsets such that $A \cap B$ is not dense. Then there exists a non-empty subset U such that $U \cap A \cap B = \varnothing$. Since the characteristic function $\chi_{(U\cap A)}$ is in $C(X_\delta)$, then there is a dense open set D such that $\chi_{(U\cap A)}|_D$ is a continuous function. Let $y_0 \in U \cap A \cap D$. Then there is an open set V such that $f(V) \subseteq (\frac{1}{2}, \frac{3}{2})$. So $V \cap B = \varnothing$, which is a contradiction. Thus $A \cap B$ is dense. □

Definition 2.8 *Let X be a topological space. Then X is called almost resolvable space if it is a countable union of sets with void interiors.*

Theorem 2.9 *If X is an RG-space of countable pseudocharacter then X is not an almost resolvable space.*

Proof.

Let X be an RG-space of countable pseudocharacter and suppose X is almost resolvable. Then there is a countable collection $\{F_n : n \in N\}$ of sets with void interior such that $X = \bigcup_{n=1}^{\infty} F_n$. Let $A_1 = F_1$ and $A_n = F_n - (\bigcup_{m=1}^{n-1} F_m)$ for each $n \geq 2$. Then it is clear that $\{A_n : n \in N\}$ is a a countable collection of disjoint sets with void interiors and $X = \bigcup_{n=1}^{\infty} A_n$. Define $f : X \longrightarrow R$ by $f(A_n) = n$ for each $n \in N$. Since X_δ

is a discrete space, then $f \in C(X_\delta) = G(X)$, which implies that f is continuous on a dense open subset D, which is a contradiction because f is not continuous at any point. Thus X is not almost resolvable. □

We know from theorem 2.5 that RG-spaces of countable pseudocharacter are Baire. In fact one can do better for RG-spaces of countable pseudocharacter. If X is an RG-space of countable pseudocharacter then every countable union of nowhere dense subsets of X is nowhere dense as we shall see in the next result.

Lemma 2.8 *If X is an RG-space of countable pseudocharacter then every countable union of nowhere dense subsets is nowhere dense.*

Proof.

Let X be an RG-space and $(A_n)_{n=1}^{\infty}$ be a sequence of nowhere subsets in X. Take $S = \bigcup_{n=1}^{\infty} A_n$, $F_1 = A_1$ and $F_m = A_m - (\bigcup_{i=1}^{m-1} A_i)$ for each $m \geq 2$. Then $\{F_n : n \in N\}$ is a collection of disjoint nowhere dense subsets of X, and therefore $\{F_n : n \in N\} \cup \{X - S\}$ is a partition of X. Now define $f : X_\delta \longrightarrow R$ by $f(F_n) = \{n+1\}$ for each $n \in N$ and $f(X - S) = \{1\}$. Then $f \in C(X_\delta) = G(X)$, which implies that there is a dense open subset D of X such that $f|D$ is a continuous function. Suppose $cl(\bigcup_{n=1}^{\infty} F_n)$ has an interior point p. Then there is an open subset U_p containing p such that $U_p \subseteq cl(\bigcup_{n=1}^{\infty} F_n)$, that is $\forall\ y \in U_p$ and for each neighborhood W_y of y we have $W_y \cap (\bigcup_{n=1}^{\infty} F_n) \neq \varnothing$. Since $D \cap U_p \neq \varnothing$, let y be any point in $\in D \cap U_p$. There are two cases:

(1) *If $f(y) = 1$, then there is an open neighborhood W_y of y such that $f(W_y) \subseteq (0, \frac{3}{2})$. Hence $W_y \cap (\bigcup_{n=1}^{\infty} F_n) = \varnothing$, which is a contradiction.*

(2) *If $f(y) = k+1$, then $y \in F_k$. So there is an open neighborhood W_y of y such that $f(W_y) \subseteq (k + \frac{2}{3}, k + \frac{4}{3})$, and therefore $W_y \subseteq F_k \subseteq A_k$, which is a contradiction too. Thus $\bigcup_{n=1}^{\infty} A_n$ is a nowhere dense subset of X.* □

A topological space X can have a dense subset K such that K^c is somewhere dense or even a dense subset. This is will be a very interesting point for RG-spaces.

Lemma 2.9 *Let X be a topological space. Then X is either has the property that every dense subset has a nowhere dense complement or X has a resolvable cozero subspace.*

Proof.

Suppose D is a dense subset such that D^c somewhere dense. Then there is a non-empty cozero subset U such that $U \subseteq cl(D^c)$. Let $A = D \cap U$ and $B = D^c \cap U$. Then A and B are disjoint dense subsets of U. Hence U is a resolvable cozero subspace. □

Theorem 2.10 *Let X be an RG-space of countable pseudocharacter. Then:*

(1) *Every dense subset of X has a nowhere dense complement.*

(2) *Every countable intersection of dense sets has a dense interior.*

(3) *Every dense set has a dense interior.*

Proof.

(1) *Since every cozero subset of X is an RG-space of countable pseudocharacter then it cannot be resolvable. Thus the result follows directly by lemma* 2.9.

(2) *Let $(D_n)_{n=1}^{\infty}$ be a sequence of dense subsets of X. Take $A_n = X - D_n$ for each $n \in N$. Then $(A_n)_{n=1}^{\infty}$ is a sequence of nowhere subsets of X, which implies that $\bigcup_{n=1}^{\infty} A_n$ is a nowhere dense subset of X. Hence the result follows directly from the fact that $cl(T)^c = int(T^c)$ for any subset T of X.*

(3) *This follows directly from* (2). □

Recall that a topological space X is called open hereditarily irresolvable (simply o.h.i) if each open subspace of X is irresolvable [5, def 1.2]. And in [25] Ganster proved that a topological space X is open hereditarily irresolvable if and only if every dense set of X has a dense interior. Since every RG-space of countable pseudocharacter is an open hereditarily irresolvable space, thus one can give a proof of (3) in theorem 2.10 from a somewhat different point of view.

Before finishing this chapter, we would like to use proposition 4.10 which appears in [35] to prove that under the assumption $V = L$, every RG-space of countable pseudocharacter has a dense set of isolated points.

First let us recall proposition 4.10 [Assume $V = L$. Then every space without isolated points is almost resolvable.] [35].

Theorem 2.11 *Assume $V = L$. Then every RG-space of countable pseudocharacter has a dense set of isolated points.*

Proof.

Since every cozero subspace of X is an RG-space of countable pseudocharacter, then by proposition 4.10 *in [35] and theorem* 2.9, *we can see that every cozero set has an isolated point. Thus X has a dense set of isolated points.* $\square$

Chapter 3

The Krull z-dimension for C(X)

The set of prime ideals in the regular ring $G(X)$ is in one-to-one correspondence with the prime z-ideals $PZ(X)$ in the ring $C(X)$. Furthermore $spec(G(X))$ with the spectral topology is homeomorphic to the space $PZ(X)$ as a subspace of $spec(C(X))$ with the patch topology [31]. Therefore the Krull z-dimension will play an important role to determine when the minimal regular ring extension $G(X)$ has the form of a ring of real-valued continuous functions defined on some topological space. In the article (On RG-spaces and regularity degree) by [R.Raphael, R.G.Woods] the authors gave some techniques to prove that there is no RG-space with infinite Krull z-dimension. There was an error that we found in the proof of theorem 3.4. Our goal in this section is to revisit the theorem. We will prove that the theorem is still correct in many cases, but the general theorem will remain open.

3.1 The Krull z-dimension for the ring C(X)

Let $PZ(X)$ be the set of prime z-ideals in the ring $C(X)$. By the Krull z-dimension of a maximal ideal we mean the supremum of the lengths of chains of prime z-ideals lying in it. The Krull z-dimension of $C(X)$ is the supremum of the dimensions of the maximal ideals of $C(X)$. By the fixed Krull z-dimension of $C(X)$ we will mean the supremum of the dimensions of the fixed maximal ideals of $C(X)$. The following lemma originally appeared in [31].

Lemma 3.1 *The collection $\beta = \{D(f) \cap V(g) : f, g \in C(X)\}$ forms an open base for the space $PZ(X)$ with the patch topology.*

Proof.

It is clear from theorem 1.10 *that the collection:*

$\beta = \{D(f) \cap V(I) : f \in C(X),\ I$ *is a finitely generated ideal* $\}$ *is an open base for $PZ(X)$ with the patch topology. Since $V(\langle g_1, g_2 \rangle) = V(g_1) \cap V(g_2)$, then it is enough to show that for any $g_1, g_2 \in C(X)$, we have $V(g_1) \cap V(g_2) = V(h)$ for some $h \in C(X)$. Let $h = {g_1}^2 + {g_2}^2$ and let P be a prime z-ideal. Then $g_1, g_2 \in P$ implies that $h \in P$. On the other hand, if $h \in P$, then $Z(h) = Z(g_1) \cap Z(g_2) \in Z[P]$ where $Z[P]$ is the prime z-filter corresponding to the prime z-ideal P [8, 2.12]. Then $Z(g_1), Z(g_2) \in Z[P]$, which implies that $g_1, g_2 \in P$. Thus $V(g_1) \cap V(g_2) = V(h)$.* $\square$

Recall that a Boolean algebra A is a complemented distributive lattice with 0 and 1. A nonempty subset $\Im$ of A is called an A-filter if $0 \notin \Im$, $a \wedge b \in \Im$ whenever $a, b \in \Im$ and $a \leq b, a \in \Im$ implies that $b \in \Im$. An A-filter $\Im$ is called prime if $a \in \Im$ or $b \in \Im$ whenever $a \vee b \in \Im$, and it is called an A-ultrafilter if it is a maximal element in the set of all A-filters defined on the Boolean algebra A. It is clear that every A-filter is contained in some A-ultrafilter. Details appear in [28].

Remark 3.1 *let A be a Boolean algebra and $\Im$ be an A-filter. Then TFAE:*

(1) *$\Im$ is an A-ultrafilter.*

(2) *If $a \in A$ such that $a \wedge b \neq 0$ for each $b \in \Im$ then $a \in \Im$.*

(3) *$\Im$ is a prime A-filter.*

(4) *For each $a \in A$ we have $a \in \Im$ or $a' \in \Im$, where a' is the complement of a.*

For a Tychonoff space X, the set of clopen subsets $B(X)$, the set of regular closed subsets $R(X)$, and the set of regular open subsets $RO(X)$ are Boolean algebras. On the other hand, if A is any Boolean algebra, then the set $S(A)$ of A-ultrafilters defined

on A can be turned into a topological space by taking the collection $\beta = \{V(a) : a \in A\}$ as an open base where $V(a) = \{U \in S(A) : a \in U\}$ for each $a \in A$. And $S(A)$ under this topology will be a compact 0-dimensional Hausdorff space. Moreover, $B(S(A)) = \{V(a) : a \in A\}$. The map $a \longrightarrow V(a)$ is a Boolean isomorphism from A onto $B(S(A))$. Details appear in [28].
The following theorem originally appeared in [28].

Theorem 3.1 *Let X be a 0-dimensional topological space and A be a subalgebra of $B(X)$ which forms an open base for X. Then:*
(1) The map $\imath : X \longrightarrow S(A)$ defined by $\imath(x) = \{C \in A : x \in C\}$ is a topological homeomorphism from X onto $\imath(X)$.
(2) $\imath(X)$ is a dense subset in $S(A)$.
Proof.
(1) It is clear that $\imath(x)$ is an A-ultrafilter on A. Since for each $C_0 \in A$, we have $\imath^{-1}(V(C_0)) = \{x : \imath(x) \in V(C_0)\} = \{x : C_0 \in \imath(x)\} = C_0$ is an open subset. Then $\imath$ is a continuous map. Suppose $\imath(x_1) = \imath(x_2)$ for some $x_1 \neq x_2 \in X$. Then there are $C_1, C_2 \in A$ such that $x_i \in C_i$ and $C_1 \cap C_2 = \varnothing$, and therefore $\imath(x_1) \neq \imath(x_2)$, which is a contradiction. Thus $\imath$ is a one-to-one map. Finally, we show that $\imath$ is an open map from X onto $\imath(X)$. Let $C_0 \in A$. Then $\imath(C_0) = \imath(X) \cap V(C_0)$ is an open subset of $\imath(X)$. Thus $\imath$ is a homeomorphism from X onto $\imath(X)$.
(2) It is obvious that $\imath(x_0) \in V(C_0) \cap \imath(X)$ for each non-empty subset C_0 in A and for each $x_0 \in C_0$. Therefore $\imath(X)$ is a dense subset of $S(A)$. □

It is clear from theorem 3.1 that if X is a 0-dimensional topological space, then the space $S(A)$ is a 0-dimensional compactification of X where A is a subalgebra of $B(X)$ and an open base of X.
It is clear that the zeroset of any real-valued function in $G(X)$ can be written as a finite union of sets of the form $Z \cap C$ where Z, C^c are both zerosets in X [30, 4.2]. On the other hand, every zeroset and every cozeroset in X is a zeroset for some function in $G(X)$.

The next theorem originally appeared in [31]. It shows that there is a one-to-one correspondence between the set of prime z-ideals of $C(X)$ and the set of maximal ideals of $G(X)$.

Theorem 3.2 *Let X be a Tychonoff space. Then:*

(1) $A = Z(G(X))$ is a subalgebra of $B(X_\delta)$ and it forms an open base for X_δ. Furthermore $S(Z(G(X)))$ is a 0-dimensional compactification of X_δ.

(2) The map $K : S(A) \longrightarrow PF(X)$ defined by $K(\alpha) = \alpha \cap Z(X)$ is a topological homeomorphism from $S(A)$ onto $PF(X)$ with the patch topology, and $K|_{X_\delta}$ is a bijection map onto the fixed z-ultrafilters of $C(X)$.

(3) $S(A)$ can be considered as the space of maximal ideals of $G(X)$ with the spectral topology.

Proof.

(1) It is clear that $\varnothing, X \in Z(G(X))$ and $A_1 \cup A_2$, $A_1 \cap A_2 \in Z(G(X))$ whenever $A_1, A_2 \in Z(G(X))$. Since $Z(G(X)) \subseteq Z(X_\delta) \subseteq B(X_\delta)$, then $Z(G(X))$ is a subalgebra of $B(X_\delta)$ and it forms an open base for X_δ. Hence by theorem 3.1, *$S(Z(G(X)))$ is a 0-dimensional compactification of X_δ.*

(2) It is clear that $\alpha \cap Z(X)$ is a z-filter for any A-ultrafilter α. Suppose Z_1, Z_2 are two zerosets in X such that $Z_1 \cup Z_2 \in K(\alpha)$. Then $Z_1 \in \alpha$ or $Z_2 \in \alpha$ which implies that $K(\alpha) \in PF(X)$ i.e. K is a well-defined function. We first show that K is a one-to-one map. Suppose $\alpha_1 \neq \alpha_2 \in S(A)$. Then there are $Z, C^c \in Z(X)$ such that $Z \cap C \in (\alpha_1 - \alpha_2)$. Therefore $Z \in \alpha_1$ and $Z \cap C \notin \alpha_2$. Now if $Z \notin \alpha_2$ we are done. If not, then $C \notin \alpha_2$ and therefore $C^c \in (\alpha_2 - \alpha_1)$. Then $K(\alpha_1) \neq K(\alpha_2)$ and hence K is a one-to-one map. Secondly, we show that K is an onto map. Let $F \in PF(X)$ be any prime z-filter of $C(X)$, and take $\alpha(F) = \{C \in Z(G(X)) : \exists Z \in F, \exists\, S \in (Z(X) - F), Z \cap S^c \subseteq C\}$. Then $\alpha(F)$ is an A-ultrafilter and $F = K(\alpha(F))$. Hence K is a bijection map from $S(A)$ onto $PF(X)$.

Since $\beta = \{V(Z \cap S) : Z, S^c \in Z(X)\}$ is an open base for $S(A)$, then to prove that

K is an open map, it is enough to show that $K(V(Z \cap S))$ is an open set for each $V(Z \cap S) \in \beta$. Since $K(V(Z \cap S)) = \{\alpha \cap Z(X) : Z \cap S \in \alpha\} = \{\alpha \cap Z(X) : Z \in \alpha, S^c \notin \alpha\}$, then $K(V(Z \cap S)) = \{F \in PF(X) : Z \in F, S^c \notin F\} = V(Z) \cap D(S^c)$. Hence K is an open map. Finally, we show that K is a continuous function. Since $K(V(Z \cap S)) = V(Z) \cap D(S^c)$ and K is a bijection map then $K^{-1}(V(Z) \cap D(S^c)) = V(Z \cap S)$ is an open set. So K is a continuous function. Thus K is a homeomorphism from $S(A)$ onto $PF(X)$ with the patch topology.
Since $X_\delta \cong \imath(X)$ and $K(\imath(x)) = \{Z \in Z(X) : x \in Z\}$ is a fixed z-ultrafilter on X for each $x \in X$, then $K|_{X_\delta}$ is a bijection map onto the fixed z-ultrafilters of $C(X)$.
(3) *Since the map $M \longrightarrow Z(M)$ is a topological homeomorphism from the set of maximal ideals of $G(X)$ with the spectral topology onto $S(A)$, then $S(A)$ can be considered as the space of maximal ideals of $G(X)$ with the spectral topology.* □

Since every prime ideal of $C(X)$ is contained in a unique maximal ideal [8, 2.11], then the ring $C(X)$ is a pm-ring, and therefore the map $\mu : spec(C(X)) \longrightarrow M(X)$ defined by $\mu(P) = M_P$ is a continuous function with the spectral topology on both spaces. The next lemma originally appeared in [31].

Lemma 3.2 *Let X be a Tychonoff space. Then:*
(1) *The identity map $\imath : X_\delta \longrightarrow X$ can be continuously extended to a function: $\hat{\imath} : S(A) \longrightarrow \beta X$.*
(2) *If X is an RG-space then $S(Z(G(X))) = \beta(X_\delta)$.*

Proof.
(1) *We know that $\mu : spec(C(X)) \longrightarrow M(X)$ is a continuous function. Let $\hat{\imath} = \mu|_{PZ(X)}$. As $S(A) \cong PF(X) \cong PZ(X)$, then one can consider $\hat{\imath}$ as a continuous function from $S(A)$ onto the set of maximal ideals of $C(X)$ with the spectral topology on both spaces. This map will be defined by $\hat{\imath}(\alpha) = M_{(\alpha \cap Z(X))}$. Since for any $x_0 \in X$ we have $\hat{\imath}(\imath(x_0)) = M_{(\imath(x_0) \cap Z(X))} = M_{x_0} = \imath(x_0)$, then $\hat{\imath}|_\delta = \imath$.*
(2) *Let X be an RG-space. Then $Z(G(X)) = Z(C(X_\delta))$ and therefore $S(Z(G(X))) = S(Z(C(X_\delta))) = \beta(X_\delta)$.* □

It is clear from the previous discussion that for any Tychonoff space X we have that the space of prime z-ideals $PZ(X)$ of $C(X)$ with the patch topology is homeomorphic to the space of maximal ideals $M(G(X))$ of $G(X)$ with the spectral topology. Furthermore, $PZ(X) = \{M \cap C(X) : M \text{ is a maximal ideal in } G(X)\}$. If P is an arbitrary prime z-ideal of $C(X)$ then there is a unique corresponding maximal ideal in $G(X)$ whose intersection with $C(X)$ equals P. We denote this maximal ideal by M_P.

Lemma 3.3 *Let X be a Tychonoff space. Then:*

(1) *For any $P \in PZ(X)$, we have $Q_{cl}(C(X)/P) \cong G(X)/M_P$ and $Q_{cl}(C(X)/P)$ is an epimorphic image of $G(X)$.*

(2) *If $P_0 \subsetneq P_1 \subsetneq \ldots\ldots \subsetneq P_k$ is a strictly ascending chain of prime z-ideals, then $\prod_{i=0}^{k}(Q_{cl}(C(X)/P_i))$ is an epimorphic image of $G(X)$ and $\prod_{i=0}^{k}(Q_{cl}(C(X)/P_i)) \cong (G(X)/\cap_{i=0}^{k} M_{P_i})$.*

Proof.

(1) *Let P be a prime z-ideal. Since the ring homomorphisms $\imath : C(X) \longrightarrow G(X)$, $\pi : G(X) \longrightarrow G(X)/M_P$ are epimorphisms then $\pi \circ \imath : C(X) \longrightarrow G(X)/M_P$ is an epimorphism with $Ker(\pi \circ \imath) = P$. Therefore the homomorphism $T : C(X)/P \longrightarrow G(X)/M_P$ defined by $T(f + P) = f + M_P$ is a one-to-one epimorphism. Since $G(X)/M_P$ is a field, then there is a unique homomorphism $\widetilde{T} : Q_{cl}(C(X)/P) \longrightarrow G(X)/M_P$ such that $\widetilde{T} \circ \imath = T$. It is clear that $\widetilde{T}$ is a one-to-one epimorphism, and therefore it is an onto map, because it emanates from a field. So $Q_{cl}(C(X)/P) \cong G(X)/M_P$. Since the natural map $\pi_1 : G(X) \longrightarrow G(X)/M_P$ is an epimorphism, then $\widetilde{T}^{-1} \circ \pi_1 : G(X) \longrightarrow Q_{cl}(C(X)/P)$ is an epimorphism. Hence $Q_{cl}(C(X)/P)$ is an epimorphic image of $G(X)$. Let us denote the last map by π_P. Then it is clear that for any $h = \sum_{i=1}^{n} f_i g_i{}^* \in G(X)$, we have $\pi_P(h) = \sum_{i \in L} \frac{f_i + P}{g_i + P}$ where $L = \{i : g_i \notin P\}$.*

(2) *For any $h = gg^* \in G(X)$, we know that $\pi_P(h) = \frac{1+P}{1+P}$ if $g \notin M_P$ and $\pi_P(h) = \frac{0+P}{1+P}$ if $g \in M_P$. Now if $P_0 \subsetneq P_1 \subsetneq \ldots\ldots \subsetneq P_k$ is a strictly ascending chain of prime z-ideals, then the map $\pi : G(X) \longrightarrow \prod_{i=0}^{k}(Q_{cl}(C(X)/P_i))$ defined by*

$\pi(h) = (\pi_{p_0}(h), \pi_{p_1}(h), \ldots\ldots, \pi_{p_k}(h))$ *is a ring homomorphism. To show that* π *is an onto map, fix* i_0 *and for each* $i \neq i_0$*, choose* $g_i \in (M_{P_i} - M_{P_{i_0}})$ *and let* $k_{i_0} = \prod_{i \neq i_0} g_i$ *and* $t_{i_0} = k_{i_0} {k_{i_0}}^*$*. Then* $t_{i_0} \in M_{P_i}$ *for each* $i \neq i_0$ *and* $t_{i_0} \notin M_{P_{i_0}}$*, and therefore* $\pi_{P_i}(t_{i_0}) = \frac{0+P_i}{1+P_i}$ *and* $\pi_{P_{i_0}}(t_{i_0}) = \frac{1+P_{i_0}}{1+P_{i_0}}$*. Let* $(\frac{f_0+P_0}{g_0+P_0}, \frac{f_1+P_1}{g_1+P_1}, \ldots\ldots, \frac{f_k+P_k}{g_k+P_k})$ *be an arbitrary element in* $\prod_{i=0}^{k}(Q_{cl}(C(X)/P_i))$*, and take* $h = \sum_{i=0}^{k}(f_i {g_i}^* t_i)$*. Then* $\pi(h) = (\frac{f_0+P_0}{g_0+P_0}, \frac{f_1+P_1}{g_1+P_1}, \ldots\ldots, \frac{f_k+P_k}{g_k+P_k})$*. Hence* π *is an epimorphic map and therefore* $\prod_{i=0}^{k}(Q_{cl}(C(X)/P_i))$ *is an epimorphic image of* $G(X)$*. Since* $Ker(\pi) = \bigcap_{i=0}^{k} Ker(\pi_{P_i}) = \bigcap_{i=0}^{k} M_{P_i}$*, thus* $\prod_{i=0}^{k}(Q_{cl}(C(X)/P_i)) \cong (G(X)/\cap_{i=0}^{k} M_{P_i})$*.* □

The next theorem originally appeared in [31].

Theorem 3.3 *Let* X *be a Tychonoff space such that* $C(X)$ *has a strictly ascending chain of* $k+1$ *prime z-ideals. Then* $G(X)$ *has a function of regularity degree at least* $k+1$*.*

Proof.

Let $P_0 \subsetneq P_1 \subsetneq \ldots\ldots \subsetneq P_k \subsetneq C(X)$ *be a strictly ascending chain of prime z-ideals. For each* $i = 0, 1, 2, \ldots k-1$*, take* $f_i \in P_{i+1} - P_i$ *and take* $f_k = 1$*. Then the element* $t = (\frac{f_0+P_0}{1+P_0}, \frac{f_1+P_1}{1+P_1}, \ldots\ldots, \frac{1+P_k}{1+P_k}) \in S$ *where* $S = \prod_{i=0}^{k}(Q_{cl}(C(X)/P_i))$*. Now suppose that the* $reg(deg_{C(X)} G(X)) \leq k$*. Since for each* $g \in G(X)$ *we have* $g = \sum_{i=1}^{m} a_i.b_i^*$ *where* $m \leq k$ *and* $a_i, b_i \in C(X)$ *for each* $i = 1, 2, \ldots, m$*, then* $h = \pi(g) = \sum_{i=1}^{m} \pi(a_i).\pi(b_i^*) = \sum_{i=1}^{m} \pi(a_i).\pi(b_i)^*$ *for each* $h \in S$ *and therefore* $reg(deg_{C(X)} S) \leq k$*, which is a contradiction. Thus* $reg(deg_{C(X)} G(X)) \geq k+1$*.* □

We need the next three lemmas before we reach our main results in this section which are to address an error that we found in the proof of [On RG-spaces and regularity degree, Theorem 3.4] and to give an accurate proof which applies to many spaces including those which are cozero complemented.

For any $A \subseteq X_\delta$ we will denote to $cl_{\beta(X_\delta)}(A)$ by $\overline{A}$.

Lemma 3.4 *Let* X *be an RG-space. Then:*

(1) $PZ(X) \cong \beta(X_\delta)$*.*

(2) *There is an algebra isomorphism between* $B(X_\delta)$ *and* $B(\beta(X_\delta))$.

(3) *If* $h \in G(X)$ *and* $A = \overline{coz(h)}$ *then* $A = \overline{A \cap X_\delta}$.

(4) *If* $f \in C(X)$ *then* $\overline{coz(f)} = \{P \in PZ(X) : f \notin P\}$.

Proof.

(1) *Since* $PZ(X) \cong S(Z(G(X)))$ *and* $S(Z(G(X))) = \beta(X_\delta)$, *then* $PZ(X) \cong \beta(X_\delta)$.

(2) *Since* $S(B(X_\delta)) = \beta(X_\delta)$, *then the map* $f : B(X_\delta) \longrightarrow B(\beta(X_\delta))$ *defined by* $f(B) = \overline{B}$ *is a Boolean isomorphism [see the discussion before theorem* 3.1*]. It is clear that* $f^{-1}(A) = A \cap X_\delta = \{U : U$ *is a fixed z-ultafilter of* X_δ, $A \in U\}$ *for each* $A \in B(\beta(X_\delta))$.

(3) *Let* $h \in G(X)$ *and let* $A = \overline{coz(h)}$. *Then* $h \in C(X_\delta)$ *which implies that* $coz(h) \in B(X_\delta)$. *Therefore* $A \in B(\beta(X_\delta))$ *and hence* $A = \overline{A \cap X_\delta}$.

(4) *Let* $f \in C(X)$. *Since* $PZ(X) \cong \beta(X_\delta)$, *then* $coz(f)$ *as a subspace of* $PZ(X)$ *is the set* $\{M_x : x \in coz(f)\}$. *If* $M_{x_0} \in coz(f)$ *then* $x_0 \in coz(f)$ *which means that* $f(x_0) \neq 0$. *Therefore* $f \notin M_{x_0}$ *i.e.* $M_{x_0} \in \{P \in PZ(X) : f \notin P\}$. *So* $coz(f) \subseteq \{P \in PZ(X) : f \notin P\} = D(f)$. *Since* $D(f)$ *is a closed subset of* $PZ(X)$, *then* $\overline{coz(h)} \subseteq \{P \in PZ(X) : f \notin P\}$. *On the other hand, let* $P \in D(f)$ *and let* $V(g) \cap D(k)$ *be any basic open set containing* P *where* $f, k \in C(X)$. *Suppose that* $(Z(g) \cap coz(k) \cap coz(f)) = \varnothing$. *Then* $Z(g) \subseteq (Z(k) \cup Z(f))$. *But* $kf \notin P$. *Then* $Z(k) \cup Z(f) \notin Z[P]$ *and therefore* $Z(g) \notin Z[P]$, *which is a contradiction. So* $(Z(g) \cap coz(k) \cap coz(f)) \neq \varnothing$. *Choose* $x_0 \in (Z(g) \cap coz(k) \cap coz(f))$. *Then* $M_{x_0} \in coz(f)$, $M_{x_0} \in (V(g) \cap D(k))$. *Hence* $coz(f) \cap (V(g) \cap D(k)) \neq \varnothing$. *Thus* $\overline{coz(h)} = D(f) = \{P \in PZ(X) : f \notin P\}$. □

Recall that for a commutative ring with identity A, the set $\Re = \bigcup_{P \in spec(A)} Q_{cl}(A/P)$ can be turned into a topological space by giving $\Re$ the sheaf topology. This topology is defined to be the smallest topology defined on $\Re$ such that $[a, b]$ is a continuous function for each $a, b \in A$. Recall that the function $[a, b] : spec(A) \longrightarrow \Re$ is defined by:

$$[a,b](P) = \begin{cases} \frac{a+P}{b+P} & \text{if } b \notin P \\ \frac{0+P}{1+P} & \text{if } b \in P \end{cases}$$

where the topology which is given on $spec(A)$ is the patch topology. In this case the set $\Re$ will have the collection $\beta = \{[a,b](D(c) \cap V(J)) : a,b,c \in A,\ J$ is a finitely generated ideal$\}$ as an open base [16]. The function $[a,1]$ is often denoted $\hat{a}$.

Corollary 3.1 *Let A be a commutative ring with identity. Then the set $T = \{\frac{0+P}{1+P} : P \in spec(A)\}$ is a closed subset of $\Re = \bigcup_{P \in spec(A)} Q_{cl}(A/P)$.*

Proof.

If $\frac{a+P_1}{b+P_1} \notin T$, then $a, b \notin P_1$. But $[a,b](D\langle a,b\rangle) \cap T = \varnothing$, $\frac{a+P_1}{b+P_1} \in [a,b](D\langle a,b\rangle)$, and $[a,b](D\langle a,b\rangle)$ is open in $\Re$. Thus T is a closed subset of $\Re$. □

Lemma 3.5 *Let X be an RG-space, $P \in B \in B(\beta(X_\delta))$, $A = B \cap X_\delta$, $h = \sum_{i=1}^{m} f_i g_i^* \in G(X)$, and $a \in C(X)$ such that $h|_A = a|_A$. Then $\hat{h}(P) = \hat{a}(P)$ and $\frac{a+P}{1+P} = \sum_{i \in L} \frac{f_i+P}{g_i+P}$ where $L = \{i : g_i \notin P\}$.*

Proof.

Since $h \in G(X) = C(X_\delta)$, then the map $\hat{h} : M(C(X_\delta)) \longrightarrow \bigcup_{M_P \in M(C(X_\delta))}(C(X_\delta)/M_P)$ defined by $\hat{h}(M_P) = h + M_P$ is a continuous function. Since $PZ(X) \cong \beta(X_\delta)$, then $\hat{h}$ can be considered as a continuous function from $PZ(X)$ into $\bigcup_{M_P \in M(C(X_\delta))}(C(X_\delta)/M_P)$ where $\hat{h}(P) = h + M_P$ for each $P \in PZ(X)$.

Since A, as a subset of $PZ(X)$, is the set $\{M_{x_0} : x_0 \in A, M_{x_0} \in M(C(X)\}$ and since $a - h \in C(X_\delta)$, then $(\hat{a} - \hat{h})(M_{x_0}) = (a-h) + \acute{M}_{x_0} = 0 + \acute{M}_{x_0}$ for each $x_0 \in A$. Therefore $(\hat{a} - \hat{h})(P) = 0 + M_P$ and hence $a + M_P = h + M_P$. Now since $a - h \in M_P$, then $\pi_P(a-h) = \frac{0+P}{1+P}$. Hence $\frac{a+P}{1+P} = \sum_{i \in L} \frac{f_i+P}{g_i+P}$ where $L = \{i : g_i \notin P\}$. □

Let us recall that if $\{a_n : n \in N\}$ is a sequence, then the term a_{n_1} is called a peak of the sequence, if $n_2 \geq n_1$ implies $a_{n_1} \geq a_{n_2}$, i.e. if a_{n_1} is greater than or equal to every subsequent term in the sequence.

Lemma 3.6 *If $PZ(X)$ has a chain C with infinite length then C contains an infinite strictly decreasing sequence of prime z-ideals or an infinite strictly increasing sequence of prime z-ideals.*

Proof.

Let $D = \{P_n : n \in N\}$ be any infinite countable subset of C such that $P_{n_1} \neq P_{n_2}$ for each $n_1 \neq n_2$. Then we will have two cases:

(1) *D has an infinite number of peaks. In this case we take P_{n_1} to be the first peak. Then $P_{n_1} \supseteq P_n \ \forall\ n \geq n_1$. Let $n_2 > n_1$ such that P_{n_2} is a peak. Then $P_{n_1} \supsetneq P_{n_2}$. Since D has infinitely many peaks, then this operation will not stop and we get an infinite strictly decreasing sequence of prime z-ideals $P_{n_1} \supsetneq P_{n_2} \supsetneq P_{n_3}, \ldots\ldots\ldots$.*

(2) *If D has only a finite number of peaks namely $P_{m_1}, P_{m_2}, \ldots, P_{m_k}$. In this case we take $L = max\{m_1, m_2, \ldots, m_k\} + 1$ and take $P_{n_1} = P_L$. Since P_{n_1} is not a peak, then there is $n_2 > n_1$ such that $P_{n_1} \subsetneq P_{n_2}$. And since P_{n_2} is not a peak then there is $n_3 > n_2$ such that $P_{n_2} \subsetneq P_{n_3}$. This operation will not stop and we will get an infinite strictly increasing sequence of prime z-ideals $P_{n_1} \subsetneq P_{n_2} \subsetneq P_{n_3}, \ldots\ldots$* □

Our goal is to revisit the theorem 3.4 in [31]. For completeness let us recall the result that was claimed.

[If the Krull z-dimension of $C(X)$ is infinite then X is not an RG-space and $rg(X) = \infty$]. The proof which was given for this theorem is mistaken even though the claim that $rg(X) = \infty$ is still correct [Theorem 3.3]. The assertion that $clB_{k,t}$ contains $Q_{k,t}$ and no other prime from the array is not justified because there is a countably infinite operation used in defining $B_{k,t}$. Our work will be to prove that the theorem is still correct in many cases, but the general theorem will remain open.

Theorem 3.4 *Let X be a Tychonoff space such that $C(X)$ contains an infinite chain of prime z-ideals. Then X is not an RG-space.*

Proof.

Since $C(X)$ contains an infinite chain of prime z-ideals, then by lemma 3.6, we will have two cases:

First case: $C(X)$ contains an infinite strictly increasing sequence of prime z-ideals $P_1 \subsetneq P_2 \subsetneq \ldots\ldots \subsetneq P_n \subsetneq \ldots\ldots$. For each $n \geq 1$, choose $b_n \in P_{n+1} - P_n$ and let $D_n = coz(b_n)$, $B_1 = D_1$, and $B_n = D_n - (\cup_{i=1}^{n-1} D_i)$ for each $n \geq 2$. Then $D_n, B_n \in B(X_\delta)$ for each $n \geq 1$, $\overline{B_n} \cap \overline{B_m} = \varnothing \ \forall \ n \neq m$ and $P_n \in \overline{D_n} \ \forall n \geq 1$. Since $P_n \notin \overline{D_i} \ \forall i = 1, 2, 3, \ldots n-1$ and $\overline{D_i} \cup (\overline{X_\delta - D_i}) = PZ(X)$, then $P_n \in (\overline{X_\delta - D_i}) \ \forall i = 1, 2, 3, \ldots n-1$. Therefore $P_n \in \overline{B_n}$, $P_i \notin \overline{B_n} \ \forall i \neq n$.

Define $h : X_\delta \longrightarrow R$ by $h|_{B_n} = b_n|_{B_n}$ and $h(x) = 0 \ \forall \ x \in (X_\delta - \cup_{n=1}^{\infty} B_n)$. Then $h \in C(X_\delta) = G(X)$ and therefore $h = \sum_{i=1}^{m} c_i . d_i^*$ where $c_i, d_i \in C(X)$. It is clear by lemma 3.5, that $\frac{b_n + P_n}{1 + P_n} = \sum_{i \in L_n} \frac{c_i + P_n}{d_i + P_n}$ where $L_n = \{i : d_i \notin P_n\} \subseteq \{1, 2, \ldots, m\}$. Let $W = \{P_1, P_2, \ldots\ldots, P_{m+2}\}$ and $t = (\frac{b_1 + P_1}{1 + P_1}, \frac{b_2 + P_2}{1 + P_2}, \ldots\ldots, \frac{b_{m+2} + P_{m+2}}{1 + P_{m+2}})$. Then it is clear that $rg(t) \geq m + 2$. At the same time, we have $t = \pi(h)$ and therefore $rg(t) \leq m$, which is a contradiction. Hence X cannot be an RG-space.

Second case: $C(X)$ contains an infinite strictly decreasing sequence of prime z-ideals $P_1 \supsetneq P_2 \supsetneq \ldots\ldots P_n \supsetneq \ldots\ldots$

For each $n \geq 1$, choose $b_n \in P_n - P_{n+1}$ and let $D_n = coz(b_n) - coz(b_{n+1})$. Since $\overline{A - B} = \overline{A} - \overline{B}$ for each $A, B \in B(X_\delta)$, and since $b_n \notin P_m \ \forall \ m \geq n + 1, b_n \in P_m \ \forall \ m \leq n$. Then $P_m \notin \overline{D_n} \ \forall \ m \neq n + 1$, $P_{n+1} \in \overline{D_n}$. Now make the D_n disjoint in the standard way by letting $C_1 = D_1$, $C_2 = D_2 - D_1$, and in general $C_n = D_n - [\bigcup_{i=1}^{n-1} D_i]$.

The $(C_n)_{n=1}^{\infty}$ are non-empty clopen and disjoint subsets of X_δ and each C_n has P_{n+1} but no other P_i in its closure. Define $h : X_\delta \longrightarrow R$ by $h|_{C_n} = b_{n+1}|_{C_n}$ and $h(x) = 0 \ \forall \ x \in (X_\delta - \cup_{n=1}^{\infty} C_n)$. Then $h \in C(X_\delta) = G(X)$, which implies that $h = \sum_{i=1}^{m} c_i . d_i^*$ where $c_i, d_i \in C(X)$. So by lemma 3.5, $\frac{b_n + P_n}{1 + P_n} = \sum_{i \in L_n} \frac{c_i + P_n}{d_i + P_n}$ where $L_n = \{i : d_i \notin P_n\} \subseteq \{1, 2, \ldots, m\}$. Let $W = \{P_{m+3}, P_{m+2}, \ldots\ldots, P_2\}$ and $t = (\frac{b_{m+3} + P_{m+3}}{1 + P_{m+3}}, \ldots\ldots, \frac{b_3 + P_3}{1 + P_3}, \frac{b_2 + P_2}{1 + P_2})$. Then $rg(t) \geq m + 2$. At the same time, $t = \pi(h)$ and therefore $rg(t) \leq m$, which is a contradiction. Hence X cannot be an RG-space. $\square$

The case where the saturated chains in $PZ(X)$ have finite length yet the general dimension is infinite could potentially occur in two ways.

Two cases:

Case A. There exists a countably infinite set of distinct maximal ideals $\{M_n\}$ such that M_n has a chain of length $s_n \geq n$ descending from M_n (these maximal ideals might have finite or infinite dimension).

Case B. With finitely many exceptions all maximal ideals have finite dimension and there is a finite (global) bound for said dimension.

We begin with the case A and show that it does not occur for RG-spaces.

Theorem 3.5 *Suppose that there exists in $C(X)$ a countable infinite set of distinct maximal ideals $M_n, n = 1, 2, 3, ...$ such that for each n, the ideal M_n has a chain of length $s_n \geq n$ descending from it. Then X is not an RG-space.*

Proof.

Let T be the set of the maximal ideals which are at the top of our chains. By [8, 0.13] T as a subspace of βX has an infinite discrete subset T_1. In this case we will be able to choose an infinite sequence of chains Q_k , $k = 1, 2, 3, ...$ such that:

(1) *$k_1 > k_2$ implies that $s_{k_1} > s_{k_2}$ where* s_k *denotes the length of Q_k.*

(2) $Q_{k_1} \cap Q_{k_2} = \varnothing \ \forall \ k_1 \neq \ k_2$.

(3) *Every chain has a maximal ideal M_k from T_1 which is in the top of it.*

Since βX has an open base $\beta_1 = \{D(f) : f \in C(X)\}$, then for each k there exists $f_k \in C(X)$ such that $M_k \in D(f_k)$ and $M_n \notin D(f_k) \ \forall \ n \neq k$. Since the function $\mu : Spec(C(X)) \longrightarrow M(C(X))$ is a continuous function with the spectral topology on both spaces, then $\mu^{-1}(D(f_k))$ is an open set which contains the chain Q_k but no any other chain from our chains. Since $PZ(X)$ is a 0-dimensional space, then for each k there is a clopen subset A_k such that $Q_k \subseteq A_k \subseteq \mu^{-1}(D(f_k))$. Taking $B_1 = A_1$ and $B_n = A_n - (\cup_{i=1}^{n-1} A_i)$ will separate these different chains by disjoint clopen subsets of $\beta(X_\delta)$. For each k choose $g_{k,t} \in P_{k,t+1} - P_{k,t}$ for each $t = 1, 2, ..., S_k - 1$, $g_{k,s_k} = 1$. and let $U_{k,t} = [coz(g_{k,t}) - (\cup_{i=1}^{t-1} coz(g_{k,i}))] \cap X_\delta \cap B_k$. Then $U_{k,t} \in B(X_\delta)$, $P_{k,t} \in \overline{U_{k,t}}$, and $\overline{U_{k_1,t_1}} \cap \overline{U_{k_2,t_2}} = \varnothing$ for each $(k_1, t_1) \neq (k_2, t_2)$. Define $h : X_\delta \longrightarrow R$ by $h|_{U_{k,t}} = g_{k,t}|_{U_{k,t}}$ and $h(x) = 0 \ \forall \ x \in (X_\delta - \cup_{k,t} U_{k,t})$. Then $h \in C(X_\delta)$ which implies *that* $h = \sum_{i=1}^{m} c_i.d_i{}^*$ *where* $c_i, d_i \in C(X)$*, and* $\frac{g_{k,t}+P_{k,t}}{1+P_{k,t}} = \sum_{i\in L_n} \frac{c_i+P_{k,t}}{d_i+P_{k,t}}$ *where* $L_n = \{i : d_i \notin P_{k,t}\} \subseteq \{1, 2, ..., m\}$*. Taking* $t = (\frac{g_{k,1}+P_{k,1}}{1+P_{k,1}}, ..., \frac{1+P_{k,s_k}}{1+P_{k,s_k}})$ *where* $s_k \geq m+2$ *will give us a contradiction as before. Thus* X *cannot be an RG-space.* □

Case B: If case B can also be shown not to hold for RG-spaces, then theorem 3.4 in [On RG-spaces and regularity degree] will be established. To date we are not able to give a general proof that excludes case B for all RG-spaces. We can get the result for a broad class of spaces (the cozero complemented spaces).

We need a wee bit of preparation for the cozero complemented case.

Definition 3.1 *Suppose that* P *is a prime* z*-ideal that lies inside the maximal ideal* M*. We will say that* P *is on level* k *of* M *if there is a saturated chain of prime* z*-ideals of length* $k+1$ *that descends from* M *to* P*.*

Thus, M is on level 0, a prime directly below M is one level 1 and so on. In the case of infinite chains the definition of level may not be pertinent, but in case B we are not concerned with infinite chains. The nature of the inclusions of prime ideals in a ring of the form $C(X)$ is such that the level to which a prime z-ideals belongs is uniquely defined [8].

Proposition 3.1 *Suppose that* M *is a maximal ideal of* $C(X)$ *that contains no infinite chains but does have infinite dimension. Then there is a positive integer* $k > 1$ *for which there are infinitely many prime* z*-ideals on level* k*. Furthermore this set is a* T_2*-space in the relative topology inherited from Spec(*$C(X)$*) with the spectral topology.*

Proof.

The first statement follows immediately from König's infinite lemma [16]. The fact that the prime z*-ideals which all lie on the same level form a Hausdorff space holds by the proof of [22, p.461].* □

Case B does not occur for RG-spaces that are cozero complemented as will be seen in the next theorem.

Theorem 3.6 *Suppose that X is cozero complemented, that $C(X)$ is of infinite Krull z-dimension, and that case B holds. Then X is not an RG-space.*

Proof.

By hypothesis the infinite Krull z-dimension is realized inside a finite number of maximal ideals. Each is obviously itself of infinite dimension. Thus there exists a maximal ideal M that contains for each n a chain C_n of length $s_n \geq n+1$ such that the primes from distinct chains are not comparable.

By proposition 3.1, there is a level, say the k'th, in which there are infinitely many prime z-ideals and these form a Hausdorff subspace of Spec($C(X)$).

Let $T = \{R_n : n \in N\}$ be a countably infinite subset of all the chains which contain primes from the k'th level and below them descend subchains such that chains in T have strictly increasing lengths.

Since the primes on level k form a T-2 space, there is by $[8, 0.13]$ a countably infinite subset $D_1, D_2, ...$ of $R_1, R_2, ...$ which is discrete in the relative topology inherited from Spec($C(X)$) . We can also assume that the D_n have been ordered such that $n_1 > n_2$ implies that $s_{n_1} > s_{n_2}$. Let the top element in the chain D_n be called P_n and the smallest be called Q_n. Since the set $\{P_n\}$ is discrete there exist functions $f_n \in C(X)$ so that $f_n \notin P_n$ but $f_n \in P_m$ $\forall$ $m \neq n$. It follows that $f_n \notin Q_n$ but f_n maybe is not in Q_m for $m \neq n$. We will show that this is not the case if X is a cozero complemented space.

We will need the fact that a cozero subset of a cozero complemented space is also cozero complemented $[9, 1.5(b)]$.

Let $m \neq n$ and let $coz(g)$ be a cozero complement of $coz(f_n)$ inside the space $coz(f_n) \cup coz(f_m)$, i.e. $coz(g) \cup coz(f_n)$ is dense in $coz(f_n) \cup coz(f_m)$ and $coz(g) \cap coz(f_n) = \varnothing$. Therefore $\overline{coz(f_n)} \cup \overline{coz(f_m)} = \overline{coz(g)} \cup \overline{coz(f_n)}$, $\overline{coz(g)} \cap \overline{coz(f_n)} = \varnothing$ and $gf_n = 0$. Since $f_n \in P_m$, then $P_m \in \overline{coz(g)}$ and therefore $g \notin P_m \supseteq Q_m$. But $gf_n = 0$, so $f_n \in Q_m$ for each $n \neq m$.

Thus we have that each chain D_n lies in a different clopen set of $\beta(X_\delta)$ and we can *proceed to build an appropriate function using "the method" of theorem* 3.5. □

Remark 3.2 *The proof above also holds if the functions $\{f_n\}$ are pairwise orthogonal, but actually less is required. It suffices that one be able to appropriately separate the primes in one chain from those in the other ones. The clopen sets used need not be disjoint in $\beta(X_\delta)$. For example, with the sets $\{P_n\}$ and $\{Q_n\}$ chosen as above, suppose that $\bigcap P_n = \bigcap Q_n$. Consider the n'th chain. We have $f_n \notin Q_n$ but $f_n \in P_m$ for all $m \neq n$. Now for a fixed $m \neq n$, the product $f_n f_m \in \bigcap P_n \subseteq Q_m$ a prime ideal, so since $f_m \notin Q_m$, $f_n \in Q_m$. This means that the closures of the different sets $coz(f_n)$ will give clopen sets in $\beta(X_\delta)$ which separate the chains D_n. Then by using "the method" of theorem* 3.5 *one can build the appropriate function. It is easy to see that the condition $\bigcap P_n = \bigcap Q_n$ holds if $\bigcap P_n = O_p$ (see [8, 4I].*

Remark 3.3 *Suppose our chains D_n have been chosen as above. Then one can appropriately separate an infinite sequence of these chains and then prove that Case B does not occur for such RG-space if the following condition holds: for any infinite countable subset T of $\{P_n : n \in N\}$, we have that $\bigcap_{P_n \in T} P_n \subseteq \bigcup_{P_n \in T} Q_n$.*

Proof.

Suppose our chains D_n and our functions $\{f_n\}$ have been chosen as above, and suppose that our condition holds. We will prove our statement by induction as follows: As $f_2 \in P_n$ $\forall$ $n \geq 2$, then there exists Q_{n_1} where $n_1 \geq 2$ such that $f_2 \in Q_{n_1}$. Since our condition also holds for the set $\{P_n : n \geq n_1 + 1\}$, then there exists Q_{n_2} where $n_2 \geq n_1 + 1$ such that $f_2 \in Q_{n_2}$. This operation will never stop and in the end we will be able to replace our chains by a new infinite number of chains which will have the first chain separated from the other ones. Now our condition will alow us to do the same thing for our new chains disregarding the first chain. One can continue this process forever. In the end we will be able to replace our chains by a new infinite number of chains each of which is separated from the set of all the chains which come after it. The latter means that for each chain D_n there exists an open set U of $PZ(X)$

such that $D_n \subseteq U$ and $U \bigcap (\bigcup_{k>n} D_n) = \varnothing$. The prime z-ideals which lie inside the chains which precede D_n may or may not be in U, but in any case we will be able to separate them from D_n because it will be a matter of separating a finite subset inside a Hausdorff space. Thus one can prove that Case B does not occur for such RG-space by following the same steps as in theorem 3.6. □

Recall that the rank of a maximal ideal M of $C(X)$ is the number of minimal prime ideals contained in it if this is a positive integer, and infinity otherwise [19].

Corollary 3.2 *Let X be an RG-space such that every maximal ideal in $C(X)$ has finite rank. Then X has finite Krull z-dimension.*

Proof.

This follows directly from the fact that Case B does not occur for such RG-spaces. □

A topological space X is called an SV-space if $C(X)$ is an SV-ring that is, if $C(X)/P$ is a valuation domain for each minimal prime ideal P of $C(X)$ [15]. It is clear that every F-space in the sense of the Gillman-Jerison text is an SV-space [19]. It is shown in [15], that every SV-space has finite rank.

Remark 3.4 *Let X be an RG-space that is an SV-space. Then $C(X)$ has finite Krull z-dimension.*

Proof.

It follows directly from the fact that every SV-space has finite rank. □

Example 3.1 *Any F-space has rank 1, so corollary 3.2 applies to RG-spaces which are F-spaces. The compact connected space $\beta R^+ - R^+$ of [8, 14.27] is not RG because it has no isolated points. It is clearly not cozeo complemented because in compact cozeo complemented F-spaces, Minspec and Maxspec are homeomorphic as well as basically disconnected.*

Chapter 4

The space of prime d-ideals in C(X) and the Krull d-dimension for C(X)

In the first section of this chapter we will prove that the spectrum of the ring $H(A)$ with the spectral topology is homeomorphic to the space of prime ζ-ideals of A with the patch topology. We will also show that when A satisfies the strongly *a.c.* condition, then the spectrum of $H(A)$ with the spectral topology is homeomorphic to the space of prime d-ideals of A with the patch topology. In the second section we show that the prime ζ-ideals of a semiprime ring A are exactly the prime d-elements inside the algebraic frame $SP(A)$. The third and fourth sections are a review of the prime d-ideals of $C(X)$, and of the structure of $H(X)$ originally given in [24], [30] and [28]. The last section is on the Krull d-dimension for the ring $C(X)$.

4.1 The spectrum of the epimorphic hull $H(A)$

Let A be a semiprime ring and $spec(A)$ be the space of all prime ideals of A. We are interested in three subspaces associated with this space, namely the space of prime z-ideals denoted $PZ(A)$, the space of prime d-ideals denoted $PD(A)$, and the space of prime ζ-ideals denoted $P\zeta(A)$.

Theorem 4.1 *Let A be a semiprime ring. Then $P\zeta(A) \cong spec(H(A))$ as topological spaces with the patch topology on both spaces.*

Proof.

Since the inclusion map $\imath : A \longrightarrow H(A)$ is a ring epimorphism, then by lemma 1.13*, we have that $\imath^a : spec(H(A)) \longrightarrow spec(A)$ is a one-to-one continuous map. Also by remark* 1.9*, we know that $\imath^a : spec(H(A)) \longrightarrow P\zeta(A)$ is an onto map. Then $\imath^a : spec(H(A)) \longrightarrow P\zeta(A)$ is a bijective continuous map. And since $\imath^a$ is a continuous map, $P\zeta(A)$ is a hausdorff space, and $spec(H(A))$ is a compact space, then $\imath^a$ is a closed map. Hence $\imath^a$ is a topological homeomorphism and $P\zeta(A) \cong spec(H(A))$ as topological spaces.* □

Since $H(A)$ is regular, then the patch topology on $spec(H(A))$ coincides with the spectral topology on it.

Corollary 4.1 *Let A be a semiprime ring that satisfies the strongly* a.c. *condition. Then $PD(A) \cong spec(H(A))$ as topological spaces with the patch topology on both spaces.*

Proof.

Since A satisfies the strongly a.c. *condition then $PD(A) = P\zeta(A)$ and therefore $PD(A) \cong spec(H(A))$.* □

Theorem 4.2 *Let A be a semiprime ring and $P \in P\zeta(A)$. Then $Q_{cl}(A/P) \cong H(A)/M$ and $Q_{cl}(A/P)$ is an epimorphic image of $H(A)$ where M is the unique maximal ideal of $H(A)$ such that $M \cap A = P$.*

Proof.

Since $\imath : A \longrightarrow H(A)$ *and* $\pi : H(A) \longrightarrow H(A)/M$ *are ring epimorphisms, then* $\pi \circ \imath :$ $A \longrightarrow H(A)/M$ *is an epimorphism and* $ker(\pi \circ \imath) = \{a : a \in A,\ a+M = 0+M\} = P$. *Then the map* $K : A/P \longrightarrow H(A)/M$ *defined by* $K(a+P) = a+M$ *is an epimorphic monomorphism. But* $H(A)/M$ *is a field which implies that* $\exists!\ \widetilde{K} : Q_{cl}(A/P) \longrightarrow H(A)/M$ *a ring homomorphism such that* $\widetilde{K}(\frac{a+P}{b+P}) = (a+M)(b+M)^{-1}$. *Since* $\widetilde{K}$ *is an epimorphic monomorphism and* $H(A)/M$ *is a field, then* $\widetilde{K}$ *is an onto map. Therefore* $Q_{cl}(A/P) \cong H(A)/M$ *as fields.*

Since $\widetilde{K}^{-1} : H(A)/M \longrightarrow Q_{cl}(A/P)$ *and* $\pi : H(A) \longrightarrow H(A)/M$ *are epimorphisms then* $\widetilde{K}^{-1} \circ \pi : H(A) \longrightarrow Q_{cl}(A/P)$ *is an epimorphism. Hence* $Q_{cl}(A/P)$ *is an epimorphic image of* $H(A)$. □

If this epimorphism from $H(A)$ onto $Q_{cl}(A/p)$ is denoted by π_P, then one can see that $\pi_P(a) = \frac{a+P}{1+P}$, $\pi_P(b^*) = \frac{1+P}{b+P}$ if $b \notin P$, and $\pi_P(b^*) = \frac{0+P}{1+P}$ if $b \in P\ \ \forall\ a, b \in A$. Therefore, if $a_1b_1^* + a_2b_2^* + ... + a_nb_n^* \in H(A)$ then $\pi_P(a_1b_1^* + a_2b_2^* + ... + a_nb_n^*) = \sum_{i\in\Gamma} \pi_P(a_i)\pi_P(b_i^*) = \sum_{i\in\Gamma} \frac{a_i+P}{b_i+P}$ where $\Gamma = \{i : b_i \notin P\}$. Also for each $g \in H(A)$ we have:

$$\pi_P(gg^*) = \begin{cases} \frac{0+P}{1+P} & \text{if } g \in P \\ \frac{1+P}{1+P} & \text{if } g \notin P \end{cases}$$

Lemma 4.1 *Let* $\{P_i\}_{i=0}^n \subseteq P\zeta(A)$ *such that* $P_0 \subsetneq P_1 .... \subsetneq P_n \subsetneq A$. *Then* $\prod_{i=0}^n Q_{cl}(A/P_i)$ *is an epimorphic image of* $H(A)$.

Proof.

Let $\pi : H(A) \longrightarrow \prod_{i=0}^n Q_{cl}(A/P_i)$ *be defined by* $\pi(h) = (\pi_{P_0(h)}, \pi_{P_1(h)}, \pi_{P_2(h)}, ..., \pi_{P_n(h)})$. *It is clear that* π *is a ring homomorphism. We need only show that* π *is an onto map. Since* $\forall\ i\ \exists M_i \in M(H(A))$ *such that* $M_i \cap A = P_i$, *fix* i_0, *let* $g_i \in (M_i - M_{i_0})$ *and let* $k_{i_0} = \prod_{i \neq i_0} g_i$, *and* $T_{i_0} = k_{i_0}k_{i_0}{}^*$. *Then* $T_{i_0} \in M_i\ \forall\ i \neq\ i_0$ *and* $T_{i_0} \notin M_{i_0}$, *which implies that* $\pi_{P_i}(T_{i_0}) = \frac{0+P_i}{1+P_i}$ *and* $\pi_{P_{i_0}}(T_{i_0}) = \frac{1+P_{i_0}}{1+P_{i_0}}$. *Now let* $(\frac{a_0+P_0}{b_0+P_0}, \frac{a_1+P_1}{b_1+P_1}, ..., \frac{a_n+P_n}{b_n+P_n})$ *be an arbitrary element in* $\prod_{i=0}^n Q_{cl}(A/P_i)$. *Then* $h = \sum_{i=0}^n (a_ib_i{}^*T_i) \in H(A)$ *and* $\pi(h) = (\frac{a_0+P_0}{b_0+P_0}, \frac{a_1+P_1}{b_1+P_1}, ..., \frac{a_n+P_n}{b_n+P_n})$. *Hence* π *is an onto map.* □

If π is defined as above, then $ker(\pi) = \bigcap_{i=0}^{n} M_i$ and therefore $ker(\pi) \cap A = P_0$.

Definition 4.1 *Let A be a semiprime ring and $f \in H(A)$. Then the hull regularity degree of f, denoted by $rg_h(f)$, is defined to be:*
$rg_h(f) = min\{n \in N : f = \sum_{i=1}^{n} g_i h_i^* \ , \ g_i, h_i \in A, n \geq 1\}$.
The hull regularity degree of the ring A, denoted by $rg_h(A)$, is defined to be: $rg_h(A) = sup\{rg_h(f) : f \in H(A)\}$.

The lengths of the chains of prime ζ-ideals inside the semiprime ring A will be related to the the hull regularity degree of the ring A as we will see in the next proposition.

Proposition 4.1 *Let A be a semiprime ring with a strictly ascending chain of $k+1$ prime ζ-ideals. Then $H(A)$ has an element of hull regularity degree at least $k+1$.*

Proof.

Suppose $P_0 \subsetneq P_1 \subsetneq \ldots\ldots \subsetneq P_k \subsetneq A$ be a strictly ascending chain of prime ζ-ideals. For each $i = 0, 1, 2, \ldots .k-1$ take $f_i \in P_{i+1} - P_i$ and $f_k = 1$. Then the element $t = (\frac{f_0+P_0}{1+P_0}, \frac{f_1+P_1}{1+P_1}, \ldots\ldots, \frac{1+P_k}{1+P_k}) \in S = \prod_{i=0}^{k}(Q_{cl}(A/P_i))$. We know that $reg(deg_A S) \geq k+1$, and $\pi : H(A) \longrightarrow S$ is an onto map [lemma 4.1]. Now suppose that the $rg_h(A) \leq k$. Since for each $g \in H(A)$ we have $g = \sum_{i=1}^{m} a_i b_i{}^*$ where $m \leq k$ and $a_i, b_i \in A$ for each $i = 1, 2, \ldots, m$. Then $h = \pi(g) = \sum_{i=1}^{m} \pi(a_i)\pi(b_i{}^*) = \sum_{i=1}^{m} \pi(a_i)\pi(b_i)^*$ for each $h \in S$. Therefore $reg(deg_A S) \leq k$, which is a contradiction. Thus $rg_h(A) \geq k+1$. □

The Krull ζ-dimension of the semiprime ring A is defined to be the supremum of of the lengths of chains of prime ζ-ideals lying inside A.

Lemma 4.2 *If the Krull ζ-dimension of the semiprime ring A is infinite then $rg_h(A) = \infty$ and for each $n \in N$ there exists $f \in H(A)$ such that $rg_h(f) \geq n$.*

Proof.

This follows directly from proposition 4.1. □

4.2 ζ-ideals as d-elements in algebraic frames

In this section we show that the definition of a prime ζ-ideal inside the semiprime ring A is exactly the definition of a prime d-element inside the algebraic frame $SP(A)$.

Theorem 4.3 *Let A be a semiprime ring and $J \in SP(A)$. Then:*

(1) $J^{\perp} = J^*$.

(2) J is a prime element in $SP(A)$ if and only if J is a prime ideal.

(3) J is a ζ-ideal if and only if J is a d-element in $SP(A)$.

Proof.

(1) Since A is a semiprime ring, then J^* is a semiprime ideal and $J^* \cap J = \{0\}$. But $J^{\perp} = \bigvee\{I : I \in SP(A), I \cap J = \{0\}\}$. Therefore $J^* \subseteq J^{\perp}$. On the other hand, if $x \in J^{\perp} = \sqrt{\bigoplus_{i\in\Gamma} k_i}$ where $k_i \cap J = \{0\}$ for each i, then there exists $n \geq 1$ such that $x^n \in \bigoplus_{i\in\Gamma} k_i$, and therefore there is a finite subset $\{i_1, i_2, ..., i_m\}$ of Γ such that $x^n = a_{i_1} + a_{i_2} + .. + a_{i_m}$ where $\{a_{i_t} \in K_{i_t}\}$ for each $t = 1, 2, ..., m$. Now if y is any element in J, then $ya_{i_t} \in (K_{i_t} \cap J)$ which implies $ya_{i_t} = 0$ for each $t = 1, 2, ..., m$. So $x^n y = 0$. But A is a semiprime ring so $xy = 0$. This will be true for any element $y \in J$, so $x \in J^*$. Hence $J^{\perp} = J^*$.

(2) ($\Longrightarrow$) Suppose J is a prime element in $SP(A)$, and let I_1, I_2 be ideals of A such that $I_1 \cap I_2 \subseteq J$. Then $\sqrt{I_1} \cap \sqrt{I_2} \subseteq J$, which implies $\sqrt{I_1} \subseteq J$ or $\sqrt{I_2} \subseteq J$. So J is a prime ideal.

($\Longleftarrow$) Obvious.

(3) ($\Longrightarrow$) If J is a ζ-ideal, then J is a semiprime ideal and therefore $J \in SP(A)$. Let $K = \langle a_1, a_2, ..., a_n\rangle$ be a finitely generated ideal such that $\sqrt{K} \subseteq J$. Then $\langle a_1, a_2, ..., a_n\rangle \subseteq J$. But J is a ζ-ideal, therefore $(\sqrt{K})^{**} = K^{**} \subseteq J$. Then $J = \bigvee\{(\sqrt{K})^{**} : \sqrt{K} \subseteq J, K$ is a finitely generated ideal $\}$. Hence J is a d-element in $SP(A)$.

($\Longleftarrow$) Suppose J is a d-element in $SP(A)$. Then $J = \bigvee\{(\sqrt{K})^{**} : \sqrt{K} \subseteq J, K$ is a finitely generated ideal $\}$, which means that $\langle a_1, a_2, ..., a_n\rangle^{**} \subseteq J$ for each $\langle a_1, a_2, ..., a_n\rangle \subseteq$

J. Hence J is a ζ-ideal. □

If A is a semiprime ring, then $SP(A)$ is a compact algebraic frame with the $F.I.P.$, because $D(spec(A))$ has it. And if A satisfies the strongly $a.c.$ condition then an ideal J in $SP(A)$ is a d-element if and only if J is a d-ideal.

4.3 The space of prime d-ideals in $C(X)$

The ring of continuous functions $C(X)$ is a commutative semiprime ring with identity whose complete ring of quotients, $Q(X)$, can be constructed as follows:

Let $D(X)$ be the set of all dense open subsets in X and let $L(X) = \bigcup\{C(U) : U \in D(X)\}$. Then one can define an equivalence relation on $L(X)$ by:

for each $f \in C(V)$, $g \in C(W)$ where $V, W \in D(X)$, let $f \sim g$ if $f|_{V\cap W} = g|_{V\cap W}$.

Then the complete ring of quotients $Q(X)$ will be the equivalence classes of this relation, i.e. $Q(X) = \{[f] : f \in L(X)\}$, and for any two elements $[f], [g] \in L(X)$ where $f \in C(V), g \in C(W)$ the addition and multiplication is defined as follows:

$[f] + [g] = [f|_{V\cap W} + g|_{V\cap W}]$ and $[f][g] = [f|_{V\cap W}g|_{V\cap W}]$.

The classical ring of quotients of $C(X)$, denoted by $Q_{cl}(X)$, is constructed similarly except that $D(X)$ is replaced with the family of all dense cozerosets in X. Details appear in [26].

If X is a topological space, then it follows by lemma 1.4 that $C(X) \subseteq Q_{cl}(X) \subseteq H(X) \subseteq Q(X)$. It is clear that $C(X) = Q_{cl}(X)$ if and only if every non-zero divisor in $C(X)$ is a unit, or equivalently if X is an almost P-space. Taking $C^*(X), Q^*_{cl}(X), Q^*(X)$ to be the bounded functions in $C(X), Q_{cl}(X), Q(X)$ respectively, then one can see that $C^*(X) = Q^*_{cl}(X)$ if and only if every dense cozeroset is C^*-embedded, $C^*(X) = Q^*(X)$ if and only if X is extremally disconnected space, and $C(X) = Q(X)$ if and only if X is extremally disconnected P-space [26, 3.5].

The ring of continuous functions $C(X)$ satisfies the strongly $a.c.$ condition by [24, Ex, p.948], and as a consequence of that $PD(X) = P\zeta(X)$.

Remark 4.1 *If X is a topological space then $spec(H(X)) \cong PD(X)$ with the spectral topology on the first space and the patch topology on the second space.*

It is clear by lemma 1.20(3), that X is an almost P-space if and only if every maximal ideal is a d-ideal. So by corollary 1.9 we can conclude that X is an almost P-space if and only if $MH(X) \in M(H(X))$ for each $M \in H(X)$. Part of this result appears in [30, corollary 3.11].

We need to mention the next definition and two lemmas which appeared in [11]. They give a good description of the d-ideals inside $C(X)$.

Definition 4.2 *Let F be a closed subset of X. Then an ideal I of $C(X)$ is called an o-ideal if $I = \{h \in C(X) : F \subseteq z(h)\}$.*

Lemma 4.3 *Let X be a topological space. Then:*

(1) *Every o-ideal is a z-ideal and an $\ell-$ideal.*

(2) *If I is o-ideal then $I^* = \{h \in C(X) : cl(F^c) \subseteq z(h)\}$.*

Proof.

(1) *It is clear from the definition that every o-ideal is a z-ideal. Let $0 \leq |g| \leq |h|$, $h \in I$. Since $h(x) = 0$ implies that $g(x) = 0$, then $z(h) \subseteq z(g)$ and therefore $g \in I$. Hence I is an $\ell-$ideal.*

(2) *Let $I = \{h \in C(X) : F \subseteq z(h)\}$ be an o-ideal, and suppose $h \in I^*$. Then $hg = 0$ for each $g \in I$. Since $F = \bigcap_{\alpha \in \Gamma} z(g_\alpha)$, then $hg_\alpha = 0$ for each $\alpha \in \Gamma$, which implies that $z(h) \cup F = X$. Then $cl(F^c) \subseteq z(h)$. Conversely, if $cl(F^c) \subseteq z(h)$ and $g \in I$, then for any $x \in X$ we have $x \notin F$ whenever $x \notin z(g)$. Therefore $hg = 0$ and $g \in I^*$. Thus $I^* = \{h \in C(X) : cl(F^c) \subseteq z(h)\}$.* □

Lemma 4.4 *Let X be a topological space. Then:*

(1) *For each $f \in C(X)$, we have $\langle f \rangle^* = \{h \in C(X) : X - int(z(f)) \subseteq z(h)\}$, and $\langle f \rangle^{**} = \{h \in C(X) : int(z(f)) \subseteq int(z(h))\}$.*

(2) *An ideal I is a d-ideal if and only if $int(z(f)) \subseteq int(z(g))$ and $f \in I$ implies that*

$g \in I$.

(3) $O_x = \{f : x \in int(z(f))\}$ *is a d-ideal for each* $x \in X$.

Proof.

(1) *If* $f, h \in C(X)$ *such that* $hf = 0$, *then* $coz(f) \cap coz(h) = \varnothing$. *Suppose* $x \notin z(h)$, *then* $x \notin cl(coz(f))$, *which implies that* $x \in int(z(f))$. *Therefore* $X - int(z(f)) \subseteq z(h)$, *so* $\langle f \rangle^* \subseteq \{h \in C(X) : X - int(z(f)) \subseteq z(h)\}$. *Conversely, if* $X - int(z(f)) \subseteq z(h)$, *then* $x \notin z(f)$ *implies that* $x \in z(h)$, *so* $hf = 0$. *Therefore* $\langle f \rangle^* = \{h \in C(X) : X - int(z(f)) \subseteq z(h)\}$.

Since $\langle f \rangle^*$ *is an o-ideal then* $\langle f \rangle^{**} = \{h \in C(X) : cl(int(z(f))) \subseteq z(h)\}$. *But* $cl(int(z(f))) \subseteq z(h)$ *if and only if* $int(z(f)) \subseteq int(z(h))$. *Thus* $\langle f \rangle^{**} = \{h \in C(X) : int(z(f)) \subseteq int(z(h))\}$.

(2) *This follows directly from* (1).

(3) *Suppose* $int(z(f)) \subseteq int(z(g))$, *and let* $f \in O_x$. *Since* $x \in int(z(f)) \subseteq int(z(g))$, *then* $g \in O_x$. *Hence* O_x *is a d-ideal for each* $x \in X$. □

4.4 The structure of $H(X)$

We present, from our point of view, certain results that appear in [30] and [28]. Our goal is to see the structure of $H(X)$ and to recall the corollary 4.12 in [30] which states that if $H(X)$ is isomorphic to a $C(Y)$ then $Y = (gX)_\delta$. Furthermore, $H(X) = H(gX)$.

Lemma 4.5 *Let* X *be a topological space and* $g \in C(X)$. *Then:*

(1) *The set* $S(g) = coz(g) \cup int(Z(g))$ *is a dense open subset of* X, *and* $[\hat{g}] \in Q(X)$ *where* $\hat{g}$ *is given by:*

$$\hat{g}(x) = \begin{cases} 0 & \text{if } x \in int(Z(g)) \\ \frac{1}{g(x)} & \text{if } x \in coz(g) \end{cases}$$

(2) $[g]^* = [\hat{g}]$ *where* $[g]^*$ *is the quasi-inverse of* $[g]$ *in* $Q(X)$.

(3) *If* $[f] \in H(X)$. *Then* f *can be represented by a function* g *such that* $g \in$

$C(\cap_{i=1}^{n} S(g_i))$ *where* $\{g_i : i = 1, 2, ...n\}$ *is a finite subset of* $C(X)$.

(4) $P(X) = \bigcap\{S(g) : g \in C(X)\}$ *where* $P(X)$ *is the set of all P-points in* X.

Proof.

(1) *It is clear that* $S(g)$ *is a dense open subset and that* $\hat{g} \in C(S(g))$. *Therefore* $[\hat{g}] \in Q(X)$.

(2) *Since* $[g^2][\hat{g}] = [g^2|_{S(g)}\hat{g}] = [g|_{S(g)}] = [g]$, *then* $[g]^* = [\hat{g}]^2[g] = [\hat{g}^2 g|_{S(g)}] = [\hat{g}]$.

(3) *If* $[f] \in H(X)$, *then by lemma 2.6 we know that* $[f] = \sum_{i=1}^{n}[f_i][\hat{g}_i]$ *where* $f_i, g_i \in C(X)$ *for each* $i = 1, 2, ..., n$. *Take* $g = \sum_{i=1}^{n} f_i|_B \hat{g}_i|_B$ *where* $B = \cap_{i=1}^{n} S(g_i)$. *Hence* $g \in C(\cap_{i=1}^{n} S(g_i))$ *and* $[f] = [g]$.

(4) *If* x *is a* P*-point, then* $x \in Z(f)$ *implies that* $x \in int(Z(f))$. *So* $x \in S(g)$ *for all* $g \in C(X)$. *On the other hand, if* $x \in S(g)$ *for all* $g \in C(X)$. *Then* $x \notin coz(f)$ *implies that* $x \in int(Z(f))$. *Hence* x *is a* P*-point.* □

Theorem 4.4 *Let* Y *be a dense* C^**-embedded subset in* X. *Then* $C(Y)$ *is an epimorphic extension of* $C(X)$ *and* $H(X) = H(Y)$, *in particular* $H(X) = H(\beta X)$.

Proof.

We have $\psi : C(X) \longrightarrow C(Y)$ *defined by* $\psi(f) = f|_Y$ *is a ring monomorphism. Thus one can consider* $C(X)$ *to be a subring of* $C(Y)$.

For each $f \in C(Y)$, *let* $g_1 = \frac{f}{1+f^2}$, $g_2 = \frac{f}{(1+f^2)^2}$ *and* $g_3 = f^2 + 1$. *It is clear that* $g_1, g_2 \in C(X)$, $g_3 \in C(Y)$, $f = g_1 g_3$ *and* $g_1 = g_2 g_3$. *Then using lemma 1.7 we can conclude that* $C(Y)$ *is an epimorphic extension of* $C(X)$. *For each* $0 \neq f \in C(Y)$, *we have* $\frac{1}{1+f^2} \in C^*(Y)$ *and* $0 \neq \frac{f}{1+f^2} \in C^*(Y) \subseteq C(X)$. *Therefore* $C(Y)$ *is a ring of quotients of* $C(X)$. *By lemma 1.3 we have* $C(Y)$ *is essential extension of* $C(X)$. *Since* $H(X)$ *is the maximal essential epimorphic extension of* $C(X)$ *then* $C(Y) \subseteq H(X)$. *Since* $U \cap Y$ *is a dense open subset in* Y *for each dense open subset* U *of* X *then* $Q(X) \subseteq Q(Y)$ *and therefore* $C(Y) \subseteq H(X) \subseteq Q(Y)$. *Hence* $H(Y) \subseteq H(X)$. *But* $H(Y)$ *is a ring quotients of* $C(X)$, *so* $C(X) \subseteq H(Y) \subseteq Q(X)$. *Thus* $H(X) \subseteq H(Y)$. *Since* X *is* C^**-embedded in* βX *then* $H(X) = H(\beta X)$. □

Corollary 4.2 *If* X *is an extremally disconnected space then* $H(X) = Q(X)$.

Proof.

Suppose X is an extremally disconnected space and $[f] \in Q(X)$, then $f \in C(V)$ for some dense open set V. Since V is dense and C^*-embedded in X, then by theorem 4.4 $C(V) \subseteq H(X)$. So $Q(X) = H(X)$. $\square$

A topological space X is called realcompact space if every real maximal ideal of $C(X)$ is fixed. For any Tychonoff space X there is a unique (up to homeomorphism) realcompact space υX such that X is dense and C-embedded in υX. Therefore $C(X) \cong C(\upsilon X)$ and $H(X) \cong H(\upsilon X)$. So to study the epimorphic hull $H(X)$ for a topological space X it is enough to study the epimorphic hull for its realcompactification υX. We will assume that all our topological spaces are realcompact spaces. In [28, 5F(7)] the authors proved that X_δ is real-compact space whenever X is. And by [8, 8.9, 8.14] we know that if X is realcompact space then the subspace gX is a realcompact space too.

Lemma 4.6 *Let $\sigma : Y \longrightarrow X$ be a continuous function. Then:*

(1) $\acute{\sigma} : C(X) \longrightarrow C(Y)$ defined by $\acute{\sigma}(g) = g \circ \sigma$ is a ring homomorphism which preserves the identity.

(2) $\acute{\sigma}$ is a monomorphism if and only if $\sigma(Y)$ is a dense subset of X.

(3) $\acute{\sigma}$ is onto if and only if $\sigma : Y \longrightarrow \sigma(Y)$ is a topological homeomorphism and $\sigma(Y)$ is a C-embedded subset in X.

Proof.

(1) Since $((g_1+g_2)\circ\sigma)(x) = ((g_1+g_2)(\sigma(x)) = g_1(\sigma(x))+g_2(\sigma(x))$ and $((g_1g_2)\circ\sigma)(x) = ((g_1g_2)(\sigma(x)) = g_1(\sigma(x))g_2(\sigma(x))$, then $\acute{\sigma}(g_1 + g_2) = \acute{\sigma}(g_1) + \acute{\sigma}(g_2)$ and $\acute{\sigma}(g_1g_2) = \acute{\sigma}(g_1)\acute{\sigma}(g_2)$. Thus $\acute{\sigma}$ is a ring homomorphism. Since $\acute{\sigma}(1)(x) = (1\circ\sigma)(x) = 1(\sigma(x)) = 1$ then $\acute{\sigma}$ preserves the identity.

(2) ($\Longrightarrow$) Suppose $\acute{\sigma}$ is a ring monomorphism. Then $\acute{\sigma}(g) = 0$ implies that $g \equiv 0$ for each $g \in C(X)$, i.e. $g(\sigma(Y)) = \{0\}$ implies that $g \equiv 0$. Suppose that $\sigma(Y)$ is not a dense subset of X. Then there is an open set $U \neq \varnothing$ of X such that $U \cap \sigma(Y) = \varnothing$. Take any point $x_0 \in U$. Then there is a continuous function $f : X \longrightarrow R$ such that

$f(x_0) = 1$ and $f(U^c) = \{0\}$. Since $\sigma(Y) \subseteq U^c$, then $f(\sigma(Y)) = \{0\}$ and $f \not\equiv 0$, which is a contradiction. Thus $\sigma(Y)$ is a dense subset of X.

($\Longleftarrow$) Suppose that $\sigma(Y)$ is a dense subset in X and let $g \in C(X)$ such that $\acute{\sigma}(g) = 0$. Since $g(\sigma(Y)) = \{0\}$ implies that $\sigma(Y) \subseteq Z(g)$, then $Z(g) = X$, because $\sigma(Y)$ is dense in X. Hence $\acute{\sigma}$ is a ring monomorphism.

(3) ($\Longrightarrow$) Let $\acute{\sigma}$ be an onto map. Firstly, we prove that σ is a one-to-one map. Suppose $\sigma(y_1) = \sigma(y_2)$ for some $y_1 \neq y_2 \in Y$. Then $g(\sigma(y_1)) = g(\sigma(y_2))$ for each $g \in C(X)$. But $\acute{\sigma}$ is an onto map, so $\forall f \in C(Y)$ there exists a $g \in C(X)$ such that $\acute{\sigma}(g) = f$. Choose $f \in C(Y)$ such that $f(y_1) \neq f(y_2)$. Then $g(\sigma(y_1)) \neq g(\sigma(y_2))$, which is a contradiction. Thus σ is a one-to-one map. Since σ is a one-to-one map, then one can define the function $\sigma^{-1} : \sigma(Y) \longrightarrow Y$. Now since $\acute{\sigma}$ is an onto map, then $\forall f \in C(Y)$ there exists a $g \in C(X)$ such that $g \circ \sigma = f$, which means that $f \circ \sigma^{-1} = g|_{\sigma(Y)}$ is a continuous function $\forall f \in C(Y)$. Then by [6, 3.8], $\sigma^{-1} : \sigma(Y) \longrightarrow Y$ is a continuous map. So $\sigma : Y \longrightarrow \sigma(Y)$ is a topological homeomorphism. Finally, we show that $\sigma(Y)$ is C-embedded in X. Let $g : \sigma(Y) \longrightarrow R$ be a continuous function. Then $g \circ \sigma \in C(Y)$. Therefore there is an $h \in C(X)$ such that $\acute{\sigma}(h) = g \circ \sigma$, i.e. $g \circ \sigma = h \circ \sigma$. Hence $h|_{\sigma(Y)} = g$ and therefore $\sigma(Y)$ is C-embedded in X.

($\Longleftarrow$) Suppose that $\sigma : Y \longrightarrow \sigma(Y)$ is a topological homeomorphism such that $\sigma(Y)$ is a C-embedded subset in X. Then for any $f \in C(Y)$ we have $f \circ \sigma^{-1} : \sigma(Y) \longrightarrow R$ is a continuous function. Therefore there exists an $h \in C(X)$ such that $h|_{\sigma(Y)} = f \circ \sigma^{-1}$. But $\acute{\sigma}(h)(x) = (h \circ \sigma)(x) = h(\sigma(x)) = (f \circ \sigma^{-1})(\sigma(x)) = f(x)$ for each $x \in X$. Then $\acute{\sigma}(h) = f$. Hence $\acute{\sigma}$ is an onto map. □

Since every non-zero ring homomorphism from the field of the real numbers into itself is the identity homomorphism, then one can conclude that if X is a topological space then every non-zero ring homomorphism φ from $C(X)$ into R is an onto homomorphism and therefore $Ker(\varphi)$ is a real maximal ideal of $C(X)$.

Lemma 4.7 *Let X be a topological space. Then:*

(1) If h_1, h_2 are two homomorphisms from $C(X)$ onto R then $h_1 = h_2$ whenever

$Ker(h_1) = Ker(h_2)$.

(2) *If M is a real maximal ideal of $C(X)$ then there is a unique onto homomorphism $\varphi : C(X) \longrightarrow R$ such that $M = Ker(\varphi)$.*

Proof.

(1) *Let h_1, h_2 be homomorphisms from $C(X)$ onto R such that $Ker(h_1) = Ker(h_2) = K$. Then $\widetilde{h_1}, \widetilde{h_2} : C(X)/K \longrightarrow R$ are isomorphisms, which implies that $\widetilde{h_1} \circ \widetilde{h_2}^{-1}$ is the identity isomorphism on R. So $\widetilde{h_1} = \widetilde{h_2}$. Now for any $f \in C(X)$ we have $h_1(f) = \widetilde{h_1}(f) = \widetilde{h_2}(f) = h_2(f)$. Hence $h_1 = h_2$.*

(2) *Let M be a real maximal ideal of $C(X)$. Then the monomorphism $\pi : R \longrightarrow C(X)/M$ defined by $\pi(r) = r + M$ is an onto map. Since the natural map $\pi_1 : C(X) \longrightarrow C(X)/M$ is an onto map too, then $\pi^{-1} \circ \pi_1 : C(X) \longrightarrow R$ is an onto homomorphism. It is clear that $Ker(\pi^{-1} \circ \pi_1) = \{g : \pi^{-1} \circ \pi_1(g) = 0\} = \{g : \pi^{-1}(g + M) = 0\} = \{g : \pi_1(0) = g + M\} = M$. The uniqueness follows directly from (1).* □

Theorem 4.5 *Let X, Y be topological spaces. Then:*

(1) *X is a realcompact space if and only if for every onto homomorphism $\varphi : C(X) \longrightarrow R$ there is a unique $x_0 \in X$ such that $\varphi(g) = g(x_0)$ for each $g \in C(X)$.*

(2) *If X is a realcompact space and $t : C(X) \longrightarrow C(Y)$ is a ring homomorphism with $t(1) = 1$, then $\exists\ !\ t^* : Y \longrightarrow X$ continuous function such that $\acute{t^*} = t$.*

(3) *Let X be a realcompact space and let Y be a dense C-embedded subset of T. Then for each continuous function $t : Y \longrightarrow X$ there is a continuous function $\widetilde{t} : T \longrightarrow X$ such that $\widetilde{t}|_Y = t$.*

Proof.

(1) ($\Longrightarrow$) *If X is a realcompact space and $\varphi : C(X) \longrightarrow R$ is an onto homomorphism, then $\widetilde{\varphi} : C(X)/K \longrightarrow R$ is an isomorphism where $K = Ker(\varphi)$. Since X is a realcompact space, then there is a unique $x_0 \in X$ such that $K = M_{x_0}$. On the other hand, we have $\phi : C(X) \longrightarrow R$ defined by $\phi(g) = g(x_0)$ is an onto homomorphism with the same kernel, so $\phi = \varphi$. Hence $\varphi(g) = g(x_0)$ for each $g \in C(X)$.*

($\Longleftarrow$) *If* $\varphi : C(X) \longrightarrow R$ *is an onto homomorphism, then there is a unique* $x_0 \in X$ *such that* $\varphi(g) = g(x_0)$ *for each* $g \in C(X)$ *and therefore* $Ker(\varphi) = M_{x_0}$*, i.e. every real maximal ideal of* $C(X)$ *is fixed. Hence* X *is a realcompact space.*

(2) *Let* X *be a realcompact space and let* $t : C(X) \longrightarrow C(Y)$ *be a ring homomorphism with* $t(1) = 1$. *Fix* $y \in Y$ *and let* $H : C(X) \longrightarrow R$ *be defined by* $H(g) = t(g)(y)$. *Then it is clear that* H *is non-zero ring homomorphism and therefore* H *is an onto map. So by* (1) *there is a unique* $t^*(y) \in X$ *such that* $H(g) = g(t^*(y))$ *for each* $g \in C(X)$. *Then one can define a function* $t^* : Y \longrightarrow X$ *by* $y \longrightarrow t^*(y)$. *Since* $H(g) = g(t^*(y)) = (g \circ t^*)(y)$, $H(g) = t(g)(y)$, *then* $t(g)(y) = (g \circ t^*)(y)$, *which implies that* $g \circ t^* = t(g) \in C(Y)$ *for each* $g \in C(X)$. *Thus* $t^* : Y \longrightarrow X$ *is a continuous function. Since* $\acute{t^*} : C(X) \longrightarrow C(Y)$ *is defined by* $\acute{t^*}(g) = g \circ t^* = t(g)$, *hence* $\acute{t^*} = t$. *For the uniqueness, let* $t_1 : Y \longrightarrow X$ *be another continuous function such that* $\acute{t_1} = t$. *Then* $\acute{t_1}(g) = g \circ t_1 = t(g)$ *for each* $g \in C(X)$, *which means that* $(g \circ t_1)(x) = t(g)(x)$, *i.e.* $g(t_1(x)) = g(t^*(x))$ *for each* $g \in C(X)$. *Thus* $t_1(x) = t^*(x)$ *and* t^* *is a unique map.*

(3) *We have* $\acute{t} : C(X) \longrightarrow C(Y)$ *is a homomorphism where* $\acute{t}(g) = g \circ t$ *for each* $g \in C(X)$. *Since* Y *is a* C*-embedded subset in* T, *then there exists* $\widetilde{g \circ t} \in C(T)$ *such that* $\widetilde{g \circ t}|_Y = g \circ t$. *So* $H : C(X) \longrightarrow C(T)$ *defined by* $H(g) = \widetilde{g \circ t}$ *is a ring homomorphism which preserves the identity. Thus by* (2) *there is a unique continuous function* $H^* : T \longrightarrow X$ *such that* $\acute{H^*} = H$. *Take* $\widetilde{t} = H^*$. *Then for any* $y \in Y$ *we have* $g(\widetilde{t}(y)) = H(g)(y) = \widetilde{g \circ t}(y) = (g \circ t)(y) = g(t(y))$ *for each* $g \in C(X)$. *Then* $\widetilde{t}(y) = t(y)$. *Thus* $\widetilde{t}|_Y = t$. □

From theorem 4.5 we see that every ring embedding $t : C(X) \longrightarrow C(Y)$ that preserves the identity can be considered as an embedding which arises from a continuous function $t^* : Y \longrightarrow X$. And the ring $C(X)$ can be replaced by its epimorphic copy $\{f \circ t^* : f \in C(X)\}$, which is a subring of $C(Y)$.

Lemma 4.8 *Let* X, Y *be topological spaces and let* $t : C(X) \longrightarrow C(Y)$ *be a ring embedding that preserves the identity. Then* $C(Y)$ *is a ring of quotients of* $C(X)$

implies that $(t^*)^{-1}(D)$ is a dense subset of Y for each dense subset D of $t^*(Y)$.

Proof.

We have $C(X) \cong \{f \circ t^* : f \in C(X)\}$. Suppose $T_1 = (t^*)^{-1}(D)$ is not a dense subset of Y. Then there is a non-zero function $f \in C(Y)$ such that $f(T_1) = \{0\}$. Since $C(Y)$ is a ring of quotients of $C(X)$, then there is a $g \in C(X)$ such that $0 \neq f(g \circ t^*) \in C(X)$, i.e. there is an $h \in C(X)$ such that $0 \neq f(g \circ t^*) = h \circ t^*$. Since $(t^*)^{-1}\{d\} \neq \varnothing$ for each $d \in D$, then $0 = f(p) = f(p)(g \circ t^*)(p) = (h \circ t^*)(p) = h(d)$ for each $p \in (t^*)^{-1}\{d\}$. Therefore $h = 0$ and so is $f(g \circ t^*)$, which is a contradiction. Thus $(t^*)^{-1}(D)$ is a dense subset of Y. $\square$

Since the identity map j is a continuous map from the space X_δ onto the space X, then the homomorphism $\acute{j} : C(X) \longrightarrow C(X_\delta)$ defined by $\acute{j}(g) = g \circ j$ is an embedding which preserves the identity.

Lemma 4.9 *Let X, Y be topological spaces such that Y is a P-space. Then every continuous function $g : Y \longrightarrow X$ can be considered as a continuous function from Y into X_δ.*

Proof.

We know that the G_δ-sets form an open base for X_δ. Let $G = \bigcap_{i=1}^{\infty} U_i$ be a G_δ-set of X. Then $g^{-1}(\bigcap_{i=1}^{\infty} U_i) = \bigcap_{i=1}^{\infty}(g^{-1}(U_i))$ is a G_δ-set of Y, and hence it is an open subset of Y. Thus $g : Y \longrightarrow X_\delta$ is a continuous function. $\square$

Theorem 4.6 *Let X be a realcompact space. Then $TFAE$:*

(1) *$C(X_\delta)$ is a ring of quotients of $C(X)$.*

(2) *X is an almost P-space.*

(3) *Every dense subset of X is a dense subset of X_δ.*

Proof.

(1) $\Longrightarrow$ (3) *This is clear by lemma 4.8.*

(3) $\Longrightarrow$ (2) *Let $\varnothing \neq Z \in Z(X)$. Since $int(Z) \cup Z^c$ is a dense subset of X, then it is a dense subset of X_δ, which implies that $Z \cap (int(Z) \cup Z^c) \neq \varnothing$, i.e. $int(Z) \neq \varnothing$.*

Hence X is an almost P-space.

(2) $\Longrightarrow$ (1) *Let $0 \neq f \in C(X_\delta)$. Then there exists $r \in R - \{0\}$ such that $f^{-1}(r) \neq \varnothing$. Since $f^{-1}(r) \in Z(X_\delta)$ and $Z(X)$ is an open base for X_δ, then there is a zeroset S of X such that $\varnothing \neq S \subseteq f^{-1}(r)$. Now since X is an almost P-space, then $int_X(S) \neq \varnothing$. Choose $p \in int_X(S)$, and take $g \in C(X)$ such that $g(p) = 1$ and $g(X - int_X(S)) = \{0\}$. Since $f(g \circ j)(p) = r \neq 0$, then $f(g \circ j) \neq 0$. But $rg \in C(X)$, $f(g \circ j)(x) = f(x)g(x)$, and $0 \neq f(g \circ j) = (rg) \circ g \in C(X)$. Thus $C(X_\delta)$ is a ring of quotients of $C(X)$.* □

Corollary 4.3 *If T is a dense almost P-subspace of X then $C(T)$ is a ring of quotients of $C(X)$.*

Proof.

We have $C(X) \cong \{f|_T : f \in C(X)\}$. Let $0 \neq f \in C(T)$. Then there exists $r \in R - \{0\}$ such that $f^{-1}(r) \neq \varnothing$. Since $f^{-1}(r)$ is a zeroset in T and T is an almost P-space, then there is an open subset V of X such that $0 \neq (V \cap T) \subseteq f^{-1}(r)$. Choose $p \in (V \cap T)$ and take $g \in C(X)$ such that $g(p) = 1$ and $g(V^c) = \{0\}$. Then $0 \neq fg|_T = (rg)|_T \in C(X)$. Hence $C(T)$ is a ring of quotients of $C(X)$. □

Lemma 4.10 *Let X be a realcompact space. Then TFAE:*

(1) *gX is a dense subset of X.*

(2) *X has a dense subspace which is an almost P-space.*

(3) *gX is a dense almost P-space and it contains every dense almost P-subspace of X.*

Proof.

(1) $\Longrightarrow$ (2) *Let $p \in Z \in Z(gX)$. Since Z is a G_δ set in gX, then $Z = \bigcap_{i=1}^{\infty}(V_i \cap gX)$ where V_i is an open subset of X for each $i = 1, 2, \ldots\ldots$ Therefore there is a zeroset F of X such that $p \in F \subseteq \bigcap_{i=1}^{\infty} V_i$. Suppose $int(F) = \varnothing$. Then $U = F^c$ is a dense cozeroset and therefore $gX \subseteq U$, which contradicts the fact that $p \in gX$. Then $int(F) \neq \varnothing$. Since gX is a dense subset of X then $int(F) \subseteq int_{gX}(F \cap gX)$, which implies that $int_{gX}(Z) \neq \varnothing$. Thus gX is an almost P-space.*

(2) $\Longrightarrow$ (3) *Let T be a dense almost P-subspace of X. If V is any dense cozeroset of X then $V \cap T$ is a dense cozeroset of T. But T has no proper dense cozerosets. Therefore $V \cap T = T$, i.e. $T \subseteq V$, so $T \subseteq gX$. Hence gX is a dense subset of X. Using the same argument as in the previous proof, we can show that gX is an almost P-space which contains every dense almost P-subspace of X.*

(3) $\Longrightarrow$ (1) *Obvious.* □

Corollary 4.4 *If $t : C(X) \longrightarrow C(Y)$ is a ring embedding such that $C(Y)$ is a regular ring of quotients of $C(X)$, then $C(Y)$ is a regular ring of quotients of $C(T_\delta)$ and $C(T_\delta)$ is a regular ring of quotients of $C(T)$ where T is the image of Y under the map t^*.*

Proof.

Note that T is a dense subset of X, because t is a ring embedding. Since Y is a P-space, then t^* can be considered as a continuous function from Y onto T_δ. Therefore we have the ring embeddings $C(T) \longrightarrow C(T_\delta) \longrightarrow C(Y)$. Since $C(Y)$ is a regular ring of quotients of $C(X)$, then $C(Y)$ is a regular ring of quotients of $C(T)$. Thus $C(Y)$ is a regular ring of quotients of $C(T_\delta)$ and $C(T_\delta)$ is a regular ring of quotients of $C(T)$. □

It is clear by theorem 4.6 that if T is the image of Y under the map t^* as in corollary 4.4, then T has to be an almost P-space whenever T is a realcompact space.

Theorem 4.7 *Let X be a realcompact space. Then TFAE:*

(1) *$C(X)$ has a regular ring of quotients of the form $C(Y)$.*

(2) *gX is a dense subset of X.*

Proof.

(1) $\Longrightarrow$ (2) *Let $t : C(X) \longrightarrow C(Y)$ be the ring embedding such that $C(Y)$ is a regular ring of quotients of $C(X)$, and let $t^* : Y \longrightarrow X$ be the unique continuous function such that $t^{*'} = t$. Then $T = t^*(Y)$ is a dense subset of X and it is an almost P-space. So by lemma 4.10, gX is a dense subset of X.*

(2) $\Longrightarrow$ (1) *If gX is a dense subset of X, then by corollary 4.3 we have that $C(gX)$*

is a ring of quotients of $C(X)$. *But* $C((gX)_\delta)$ *is always a ring of quotients of* $C(gX)$. *Hence* $C((gX)_\delta)$ *is a regular ring of quotients of* $C(X)$. $\square$

Let $C(Y)$ be a regular ring of quotients of $C(X)$ and let T be the image of Y under the map t^*. By lemma 4.10, we know that T is a dense subset of gX where T and gX are both dense almost P-subspaces of X. Then T_δ is a dense subset of $(gX)_\delta$. Hence there is a ring embedding $k : C((gX)_\delta) \longrightarrow C(T_\delta)$. Also we have the ring embeddings $C(X) \longrightarrow C(T) \longrightarrow C(T_\delta) \longrightarrow C(Y)$. So by combining these two embeddings, one has the ring embeddings $C(X) \longrightarrow C(gX) \longrightarrow C((gX)_\delta) \longrightarrow C(Y)$.

Lemma 4.11 *Let* X *be a realcompact space. Then* $H(X)$ *is isomorphic to a ring of real-valued continuous functions if and only if* $H(X) \cong C((gX)_\delta)$ *and in that case* $H(X) \cong H(gX)$.

Proof.

($\Longrightarrow$) *Suppose that* $H(X) \cong C(Y)$ *for some topological space* Y. *Then it follows from theorem* 4.7 *that* gX *is a dense subset of* X, *and therefore there are ring embeddings* $C(X) \longrightarrow C(gX) \longrightarrow C((gX)_\delta) \longrightarrow C(Y)$. *Since* $C(Y)$ *is an epimorphic extension of* $C(X)$, *then the embedding* $C((gX)_\delta) \longrightarrow C(Y)$ *is an epimorphic homomorphism. But* $C((gX)_\delta)$ *is a regular ring, which means that it has no proper epimorphic extension, i.e.* $C(Y) \cong C((gX)_\delta)$. *Thus* $H(X) \cong C((gX)_\delta)$.

($\Longleftarrow$) *Obvious.*

If $H(X) \cong C((gX)_\delta)$, *then we have an epimorphic ring embeddings* $C(X) \longrightarrow C(gX) \longrightarrow C((gX)_\delta)$, *and therefore* $C((gX)_\delta)$ *is a regular epimorphic ring of quotients of* $C(gX)$, *i.e.* $H(gX) = C((gX)_\delta)$. *Thus* $H(X) \cong H(gX)$. $\square$

4.5 The Krull d-dimension for the ring $C(X)$

Let $PD(X)$ be the set of prime d-ideals of the ring $C(X)$. Then by the Krull d-dimension of a maximal ideal we mean the supremum of the lengths of chains of prime d-ideals lying in it. The Krull d-dimension of $C(X)$ is the supremum of the dimensions of the maximal ideals of $C(X)$. Then it is clear by theorem 1.17 that the Krull d-dimension of $C(X)$ is one if and only if $Q_d(X)$ is a regular ring, which means that a necessary and sufficient condition for the Krull d-dimension of $C(X)$ to be one is that the space X be a cozero complemented space.

The space $PD(X)$ as a subspace of $PZ(X)$ with the patch topology has the sets of the form $V(F) \cap D(g)$ as basic open sets where $f, g \in C(X)$. In the next lemma we show that if $H(X) = C((gX)_\delta)$ then the space $PD(X)$ is isomorphic to $\beta((gX)_\delta)$.

Lemma 4.12 *Let X be a topological space such that $H(X) = C((gX)_\delta)$. Then $PD(X) \cong \beta((gX)_\delta)$.*

Proof.

Suppose that $H(X) = C((gX)_\delta)$. We know that $spec(H(X)) \cong PD(X)$ with the patch topology on both of spaces, and that the patch topology on $C((gX)_\delta)$ coincides with the spectral topology on it. Hence $PD(X) \cong \beta((gX)_\delta)$. $\square$

If the epimorphic hull $H(X)$ is isomorphic to a ring of continuous functions $C(Y)$, then by lemma 4.11 we know that $H(X) = C((gX)_\delta)$, where we identify $C(X)$ with the set $\{f|_{gX} : f \in C(X)\}$. Taking $K = gX$, then the topological homeomorphism $\varphi : \beta K \longrightarrow Pd(X)$ is defined by $\varphi(M) = M \cap C(X)$. A function $t \in C(K)$ will be considered as an element in $C(X)$ if there exists a function $g \in C(X)$ such that $g|_{gX} = t$. The subset K as a subset of βK is identified with the set $\{\acute{M}_x : x \in K\}$ where $\acute{M}_x = \{f \in C(K) : f(x) = 0\}$. But K as a subset of $PD(X)$ is identified with the set $\{\acute{M}_x \cap C(X) : x \in K\} = \{M_x : x \in K\}$ where $M_x = \{f \in C(X) : f(x) = 0\}$.

Theorem 4.8 *Let X be a Tychonoff space such that $C(X)$ contains an infinite chain of prime d-ideals. Then $H(X)$ is not isomorphic to $C((gX)_\delta)$.*

Proof.

Since $C(X)$ contains an infinite chain of prime d-ideals, then by lemma 4.6 we will have two cases:

First case: $C(X)$ contains an infinite strictly increasing sequence of prime d-ideals

$P_1 \subsetneq P_2 \subsetneq ..... \subsetneq P_n \subsetneq ....$ For each $n \geq 1$, choose $b_n \in Pn+1 - P_n$ and let $D_n = coz(b_n|_K)$, $B_1 = D_1$, and $B_n = D_n - (\cup_{i=1}^{n-1} D_i)$. Then $D_n, B_n \in B(K_\delta)$ for each $n \geq 1$. We know that $\overline{B_n} \cap \overline{B_m} = \varnothing\ \forall\ n \neq m$ and $P_n \in \overline{D_n}\ \forall\ n \geq 1$. But $P_n \notin \overline{D_i}\ \forall\ i = 1,2,3,...n-1$, and $\overline{D_i} \cup (\overline{X_\delta - D_i}) = Pd(X)$. Then $P_n \in (\overline{X_\delta - D_i})\ \forall\ i = 1,2,3,...n-1$. Therefore $P_n \in \overline{B_n}$ and $P_i \notin \overline{B_n}\ \forall\ i \neq n$.

Define $h : K_\delta \longrightarrow R$ by $h|_{B_n} = b_n|_{B_n}$ and $h(x) = 0\ \forall\ x \in (K_\delta - \cup_{n=1}^{\infty} B_n)$. It is clear that $h \in C(K_\delta) = H(X)$, so $h = \sum_{i=1}^{m} c_i d_i^*$ where $c_i, d_i \in C(X)$. Hence by lemma 3.5 $\frac{b_n+P_n}{1+P_n} = \sum_{i\in L_n} \frac{c_i+P_n}{d_i+P_n}$ where $L_n = \{i : d_i \notin P_n\} \subseteq \{1,2,...,m\}$. Let $W = \{P_1, P_2, ..., P_{m+2}\}$ and $t = (\frac{b_1+P_1}{1+P_1}, \frac{b_2+P_2}{1+P_2}, ..., \frac{b_{m+2}+P_{m+2}}{1+P_{m+2}})$. Then it is clear that $rg(t) \geq m+2$. At the same time, $t = \pi(h)$ and therefore $rg(t) \leq m$, which is a contradiction. Hence X cannot be isomorphic to $C((gX)_\delta)$.

Second case: $C(X)$ contains an infinite strictly decreasing sequence of prime d-ideals $P_1 \supsetneq P_2 \supsetneq .....P_n \supsetneq ....$

For each $n \geq 1$, choose $b_n \in P_n - P_{n+1}$ and let $D_n = coz(b_n|_K) - coz(b_{n+1}|_K)$. Since $\overline{A - B} = \overline{A} - \overline{B}$ for each $A, B \in B(K_\delta)$, $b_n \notin P_m\ \forall\ m \geq n+1$, and $b_n \in P_m\ \forall\ m \leq n$, then $P_m \notin \overline{D_n}$, and $P_{n+1} \in \overline{D_n}\ \forall\ m \neq n+1$. Now make the D_n disjoint in the standard way by letting $C_1 = D_1$, $C_2 = D_2 - D_1$, and in general $C_n = D_n - [\bigcup_{i=1}^{n-1} D_i]$. The $(C_n)_{n=1}^{\infty}$ are non-empty disjoint clopen subsets in K_δ. Each C_n has P_{n+1} but no other P_i in its closure. Define $h : K_\delta \longrightarrow R$ by $h|_{C_n} = b_{n+1}|_{C_n}$ and $h(x) = 0\ \forall\ x \in (K_\delta - \cup_{n=1}^{\infty} C_n)$, so $h \in C(K_\delta) = H(X)$ and therefore $h = \sum_{i=1}^{m} c_i.d_i^*$ where $c_i, d_i \in C(X)$. Then it is clear by lemma 3.5 that $\frac{b_n+P_n}{1+P_n} = \sum_{i\in L_n} \frac{c_i+P_n}{d_i+P_n}$ where $L_n = \{i : d_i \notin P_n\} \subseteq \{1,2,...,m\}$. Let $W = \{P_1, P_2, ..., P_{m+2}\}$ and $t = (\frac{b_1+P_1}{1+P_1}, \frac{b_2+P_2}{1+P_2}, ..., \frac{b_{m+2}+P_{m+2}}{1+P_{m+2}})$. Then it is clear that $rg(t) \geq m+2$. At the same time, $t = \pi(h)$ and therefore $rg(t) \leq m$, which is a contradiction. Hence $H(X)$ cannot be isomorphic to $C((gX)_\delta)$. □

The case where saturated chains in $PD(X)$ have finite length yet the general dimension is infinite could potentially occur in two ways.

Two cases:

Case A. There exists a countably infinite set of distinct maximal ideals $\{M_n\}$ such that M_n has a chain of length $s_n \geq n$ descending from M_n (these maximal ideals might have finite or infinite Krull d-dimension).

Case B. With finitely many exceptions all maximal ideals have finite Krull d-dimension and there is a finite (global) bound for said dimension.

Theorem 4.9 *Suppose that there exists in $C(X)$ a countable infinite set of distinct maximal ideals $M_n, n = 1, 2, 3, ...$ such that for each n, the ideal M_n has a chain of prime d-ideals of length $s_n \geq n$ descending from it. Then $H(X)$ is not isomorphic to $C((gX)_\delta)$.*

Proof.

One follows the same steps as in the proof of the theorem 3.5. □

Corollary 4.5 *If $H(X)$ is isomorphic to $C((gX)_\delta)$ such that every maximal ideal in $C(X)$ has a finite rank then $C(X)$ has finite Krull d-dimension.*

Proof.

It follows directly from the fact that Case B does not occur for such spaces. □

Remark 4.2 *Let X be an RG-space that is an SV-space. Then $C(X)$ has finite Krull d-dimension.*

Proof.

It follows directly from the fact that every SV-space has finite rank. □

Case B remains open. To date we do not know if there is a topological space X such that X has a maximal ideal with infinite Krull d-dimension and still have $H(X)$ isomorphic to $C((gX)_\delta)$.

Chapter 5

Conclusion

5.1 Conclusion

In this thesis we have proved that for a commutative semiprime ring with identity A, the spectrum of the ring $H(A)$ with the spectral topology can be identified with the space of prime ζ-ideals of A under the patch topology. In particular, for a commutative semiprime ring with identity which satisfies the strongly $a.c.$ condition, the spectrum of the ring $H(A)$ with the spectral topology can be identified with the space of prime d-ideals of A under the patch topology.
We have also studied the class of RG-spaces and obtained new results about them. We have introduced the class of almost k-Baire spaces as well as the class of almost Baire spaces. Finally we have studied the Krull z-dimension and the Krull d-dimension for the ring of real-valued continuous functions defined on a Tychonoff space X.

5.2 Our contribution in this thesis

The main new results contributed by this thesis are the following:
(1) For a commutative semiprime ring with identity A, we have $P\zeta(A) \cong spec(H(A))$ as topological spaces with the patch topology on both spaces.

(2) If A is a commutative semiprime ring with identity such that A satisfies the strongly *a.c.* condition then $PD(A) \cong spec(H(A))$ as topological spaces with the patch topology on both spaces.

(3) Every an RG-space is an almost-Baire space .

(4) If X is an RG-space then every countable intersection of dense cozerosets of X has a dense interior.

(5) Every RG-space of countable pseudocharacter X is a Baire space.

(6) If X is an RG-space of countable pseudocharacter then every finite intersection of dense subsets is a dense subset.

(7) If X is an RG-space of countable pseudocharacter then X is not an almost resolvable space.

(8) Assume $V = L$. Then every RG-space of countable pseudocharacter has a dense set of isolated points.

(9) If X is a Tychonoff space such that $C(X)$ contains an infinite chain of prime z-ideals then X is not an RG-space.

(10) If there exists in $C(X)$ a countable infinite set of distinct maximal ideals $M_n, n = 1, 2, 3, ...$ such that for each n, the ideal M_n has a chain of length $S_n > n$ descending from it then X is not an RG-space.

(11) If X is a cozero complemented RG-space then $C(X)$ has finite Krull z-dimension.

(12) If X is an RG-space such that every maximal ideal of $C(X)$ has finite rank then $C(X)$ has finite Krull z-dimension.

(13) Let X be a topological space such that $H(X) = C((gX)_\delta)$. Then $PD(X) \cong \beta((gX)_\delta)$.

(14) Let X be a Tychonoff space such that $C(X)$ contains an infinite chain of prime d-ideals. Then $H(X)$ is not isomorphic to $C((gX)_\delta)$.

(15) If $H(X)$ is isomorphic to $C((gX)_\delta)$ such that every maximal ideal in $C(X)$ has finite rank then $C(X)$ has finite Krull d-dimension.

(16) Every almost P-space is an almost k-Baire space for each cardinal number k.

(18) A topological space X has a dense set of almost P-points if and only if X is an almost k-Baire space for each cardinal number k.

5.3 Open questions and future work

Question (1) Is there an example of RG-space for which all fixed maximal ideals are of finite Krull z-dimension, but there are maximal ideals of infinite Krull z-dimension? If so, is there example where the fixed Krull z-dimension is finite?

Question(2) If X is a topological space with infinite Krull z-dimension such that every chain of prime z-ideals is finite, and for all but finite number of the maximal ideals have a global bound Krull z-dimension. Does this implies that X is not an RG-space? An affirmative answer of this question will establish theorem 3.4.

Question(3) Are all RG-spaces of finite Krull z-dimension of finite regularity degree?

Question(4) If $H(X) = C((gX)_\delta)$ does it follow that $C(X)$ has finite Krull d-dimension?

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