OPTIMAL SURRENDER AND ASSET ALLOCATION STRATEGIES FOR EQUITY-INDEXED INSURANCE INVESTORS

WENXIA LI

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ABSTRACT

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Wenxia Li

Equity-indexed annuity (EIA) products are becoming more and more popular since they were first introduced in 1995. The growing popularity stems from the fact that the EIAs allow investors to earn part or all the equity accumulated and enjoy a minimum guaranteed growth rate on the principal. An EIA investor may consider surrendering the contract before maturity and invest in the stock index in order to earn the full stock growth. He may also invest in a risk-free asset for protection from downside risk.

We consider an EIA policyholder who seeks the optimal surrender strategy, and asset allocation strategy after surrender, in order to maximize his expected discounted utility at expiration of the contract that is either the maturity or his time of death, whichever comes first. We derive the Hamilton-Jacobi-Bellman equations, satisfied by the optimal value function, and derive the optimal strategies. We find that the optimal surrender strategies appear in the form of a continuation region or two surrender boundaries. That is, the investor stays in the contract when the contract value is within the region. We also study the impact of product features, market assumptions and individual behavior patterns on the surrender boundaries.

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Introduction

Since first introduced in 1995, equity-indexed annuities (EIAs) have enjoyed significant growth. In 2009, more than $30 billion of EIA products were sold in the USA (AnnuitySpecs.com, Indexed Sales & Market Report, 4Q2009), a 470% increase over the 2000 level of around $5.25 billion. EIAs are regulated and distributed in the same way as fixed annuities. An EIA is a type of fixed annuity and thus provides the advantages that most fixed annuities offer. Moreover, EIAs always protect policyholders from downturns in the market while offering upside interest crediting potential based on a stock index, like the S&P 500. Because of this feature, EIAs are a conservative product suitable for managing retirement money.

EIAs have received significant attention in the actuarial literature ever since they were introduced. Brennan and Schwartz (1976), and Boyle and Schwartz (1977) are the first to extend the Black- Scholes framework to equity-linked insurance products. The pricing and reserving of EIAs are examined by Tiong (2000), Gerber and Pafumi (2000), Imai and Boyle (2001), Lee (2003), Gerber and Shiu (2003), Fung and Li (2003), and Lin and Tan (2003). Chen and Huang (2007) explore the relationship between wealth share dynamics and relative risk aversions. In Young and Zariphopoulou (2002), the dynamic method is applied to study the equity-indexed life insurance assuming the insurance risks are independent of the financial risks. Moore and Young (2003) price equity-indexed pure endowments using utility theory, as introduced by Gerber (1976), in a dynamic setting. Young (2003) applies the principle of equivalent utility to price and reserve equity-indexed life insurance. Cheung and Yang (2005) study the optimal EIA surrender strategy in a discrete-time model with regime-switching. Moore and Young (2005) study the optimal design of a perpetual equity-indexed annuity based on the buyer's utility. Moore (2009) determines the optimal time for an individual to surrender a given EIA.

This thesis investigates the optimal investment strategy of EIA investors using utility theory in a dynamic setting. The optimal investment strategy is in the form of surrender boundaries within which the investor should keep the contract. The optimal asset allocation strategy is also taken into account. The criteria in determining the optimal strategy is to maximize the expectation of the present value of the policyholder's terminal utility.

In Chapter 1, we introduce the stock price dynamics and commonly used actuarial notation. Utility theory and utility functions used are also presented. We define the optimal value function and then derive the Hamilton-Jacobi-Bellman equation it satisfies. The optimal investment dynamics are then investigated to see how various features impact investors' strategies without insurance risk. Chapter 2 incorporates insurance risk with financial risk by considering a perpetual equity-linked life insurance. We study the value function with insurance risk to find the optimal surrender thresholds for policyholders. In Chapter 3, we focus on an equity-indexed annuity with fixed maturity and investigate the impact of the maturity on the investors' surrender boundaries.

Chapter 1

Models and Notation

In this chapter, we derive the optimal asset allocation strategy for an investor. Using utility theory, we maximize the investor's expected discounted utility at the investor's time of death. We first present the wealth dynamics of the investor, which are based on risky and risk-free assets. Then, fundamental actuarial notation are introduced afterwards. Next, we introduce utility theory in Section 1.3. In Section 1.4, we observe that the optimal value function, which is the goal of the investment, satisfies a Hamilton-Jacobi-Bellman equation from which the optimal dynamic portfolio strategy is extracted.

1.1 Wealth Dynamics

Geometric Brownian motion (GBM) has widely been used to model stocks and indices. In this thesis, we assume that the underlying stock's stochastic process follows a continuous GBM. A stochastic process $\{S(t) : t \geq 0\}$ is a GBM if it satisfies the following stochastic differential equation:

$$\begin{cases} dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \\ S(0) = s \geq 0. \end{cases} \tag{1.1}$$

Equation(1.1) means that the stock price follows a GBM with a drift μ and diffusion σ under the initial condition $S(0) = s \geq 0$. In finance, the coefficients μ and σ are interpreted as the mean of return and the volatility of the underlying stock price, and they are both supposed to be given as positive constants. The process B is a standard Brownian motion $\{B(t) : t \geq 0\}$ on a probability space $(\Omega, \mathfrak{F}, P)$. The standard Brownian motion, also called Wiener process, is characterized by:

(1) $B(0) = 0$,

(2) B is almost surely continuous,

(3) for any $t > s$, the increments $B(t) - B(s)$ are independent and follow a normal distribution with mean 0 and variance $t - s$, denoted by $N(0, t - s)$.

A standard Brownian motion is a continuous-time stochastic process which is not differentiable anywhere. According to Oksendal (2003), the quadratic variation of standard Brownian motion is given by $\langle B(t), B(t) \rangle = t$.

The stochastic differential equation for GBM can be viewed as a linear ordinary differential equation $dS(t) = S(t)(\mu dt + \sigma dB(t))$ which would have an exponential solution if it were deterministic, according to Björk (2004). We follow a similar approach to solve the stochastic differential equation. First define a new process $\{Z(t) : t \geq 0\}$ by

$Z(t) = ln(S(t))$ assuming $S(t)$ is a solution to (1.1) that is strictly positive. Applying $It\hat{o}$'s formula (Oksendal, 2003) to $Z(t) = ln(S(t))$, we have

$$\begin{aligned} dZ(t) &= \frac{1}{S(t)} dS(t) + \frac{1}{2}(-\frac{1}{S(t)^2})(dS(t))^2 \\ &= \frac{1}{S(t)}(\mu dt + \sigma dB(t))S(t) + \frac{1}{2}(-\frac{1}{S(t)^2})\sigma^2 S(t)^2 (dB(t))^2 \\ &= (\mu dt + \sigma dB(t)) - \frac{1}{2}\sigma^2 dt \\ &= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB(t). \end{aligned} \tag{1.2}$$

Thus the new process $Z(t)$ follows the stochastic differential equation

$$\begin{cases} dZ(t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB(t), \\ Z(0) = ln(S(0)) = lns. \end{cases} \tag{1.3}$$

Integrating (1.3) gives

$$Z(t) = Z(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma B(t).$$

Since $Z(t) = ln(S(t))$, we have

$$S(t) = S(0) \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B(t)\}.$$

Recall that $B(t) = B(t) - B(0)$ follows a normal distribution with mean 0 and variance t. Therefore, $S(t)$ follows a lognormal (LN) distribution and $E[S(t)] = S(0)e^{\mu t}$. The return factor $\frac{S(t+s)}{S(t)} \sim LN(\mu s, \sigma^2 s)$. This result can be used to simulate the stock prices.

Now consider an investor with total wealth $W(t)$ at time t. His investment consists of two parts: a risk free asset (for example, a treasury bill) and a risky asset (for example, a stock index). Let $\pi(t)$ denote the dollar value invested in the risky asset at time t. Thus the dollar value invested in the risk free account is $\pi^0(t) = W(t) - \pi(t)$. Assume that the portfolio is self-financing, *i.e.*, after the initial investment the investor does not make additional deposits or withdrawals from the portfolio. In this way, the investor's only strategy is to dynamically reallocate the shares in both assets.

Assume that the constant r is the rate of return of the risk-free asset, compounded continuously. The rate r is also referred to as the force of interest. Given the rate of return, the growth in the risk-free asset over a time interval dt is measured by rdt. Further assume that $\mu > r > 0$. The investor's overall rate of return is the weighted average of the rates of return in the stock and the risk-free asset:

$$\begin{aligned}
dW^\pi(t) &= \pi^0(t)rdt + \pi(t)\frac{dS(t)}{S(t)} \\
&= [W^\pi(t) - \pi(t)]rdt + \pi(t)(\mu dt + \sigma dB(t)) \\
&= [rW^\pi(t) + (\mu - r)\pi(t)]dt + \sigma\pi(t)dB(t).
\end{aligned}$$

Here $W^\pi(t)$ represents the wealth at time t from using strategy $\pi(t)$. When there is no confusion about the strategy being used, we drop the superscript π.

Given the initial investment w, the total wealth of the investor follows the stochastic differential equation:

$$\begin{cases}
dW^\pi(t) = [rW^\pi(t) + (\mu - r)\pi(t)]dt + \sigma\pi(t)dB(t), \\
W^\pi(0) = w.
\end{cases} \tag{1.4}$$

In the next section we introduce some actuarial notation that shall be needed in the remainder of the thesis.

1.2 Actuarial Notation

Let X denote the lifetime of an individual. Then X is a continuous random variable because of the uncertainty of the individual's time of death. Its cumulative probability function and probability density function are defined as

$$F_X(t) = Pr[X \leq t] \text{ and } f_X(t) = dF_X(t).$$

Define the survival function as

$$s(t) = Pr[X > t] = 1 - F_X(t).$$

The force of mortality $\lambda(t)$, also called hazard rate or morality rate, represents the annualized instantaneous rate of mortality at a certain age. It can be interpreted as the probability the individual dies at the instant t given that he is alive at this time. That is

$$\lambda(t) = \frac{f_X(t)}{1 - F_X(t)}.$$

Then we can express $s(t)$ and $f_X(t)$ in terms of the force of mortality:

$$s(t) = e^{-\int_0^t \lambda(s)ds} \text{ and } f_X(t) = \lambda(t)e^{-\int_0^t \lambda(s)ds}.$$

Table (1.1) shows several frequently used mortality models. In this thesis, the constant mortality and Gompertz model will be used.

Model	$\lambda(x)$	$s(x)$	Parameters
constant	a	e^{-ax}	$a > 0$
de Moivre (1729)	$\frac{1}{w-x}$	$1 - \frac{x}{w}, 0 < x < w$	$w > 0$
Gompertz (1825)	ae^{bx}	$\exp(-\frac{a}{b}e^{bx}), x \geq 0$	$a > 0, b > 0$
Makeham (1860)	$c + ae^{bx}$	$\exp(-cx - \frac{a}{b}e^{bx})$	$a > 0, b > 0, c \geq -a$
Weibull (1939)	ax^b	$\exp(-\frac{a}{b+a}x^{b+1})$	$a > 0, b > 0$

Table 1.1: Mortality models

Let (x) denote an individual at the age of x and τ_d be his future lifetime, given survival at age x. The cumulative distribution function of τ_d is given by

$$F_{\tau_d}(t) = Pr[X - x \leq t \mid X > x] = 1 - \frac{s(x+t)}{s(x)} = {}_tq_x,$$

which is the probability that (x) dies within t years. Let ${}_tp_x$ denote the probability that (x) survives t years:

$${}_tp_x = \frac{s(x+t)}{s(x)} = e^{-\int_x^{x+t}\lambda(s)ds} = 1 - {}_tq_x.$$

Therefore, the probability density function of τ_d is

$$f_{\tau_d}(t) = F'_{\tau_d}(t) = \frac{dF_X(x+t)}{s(x)} = \frac{f(x+t)}{s(x)}.$$

The force of mortality is

$$\lambda_x(t) = \lim_{s\to 0}\frac{{}_sq_{x+t}}{s} = \frac{f_{\tau_d}(t)}{1 - F_{\tau_d}(t)} = \frac{f(x+t)/s(x)}{s(x+t)/s(x)} = \lambda(x+t).$$

The probability that (x) dies between time u and $u+t$ is

$${}_{u|t}q_x = Pr[u < \tau_d < u+t] = {}_up_x\ {}_tq_{x+u} = {}_up_x - {}_{u+t}p_x = {}_{u+t}p_x - {}_uq_x.$$

1.3 Utility Function

In utility theory, investors seek to gain profits in order to satisfy their preferences. The satisfaction, though abstract, can be measured quantitatively by the concept of utility. Utility function describes one individual's relative satisfaction from wealth, various products or even services. Higher utility values imply higher satisfaction levels. Utility can be evaluated with different functions which are dependent on variables like wealth or price of goods. Different types of utility functions describe different human behavior patterns. In this thesis, we only consider utility as a function of the wealth an investor possesses. In general the more wealthy a person is, the more satisfied. Therefore, utility is an increasing function of wealth. In addition, the utility function is assumed to be differentiable.

There are three behavior pattens in financial markets: risk neutral, risk averse and risk seeker.

• A risk neutral investor has constant marginal utility. Same increases in wealth always bring the same increases in satisfaction for any initial wealth. Risk neutral investors have linear utility functions. In real markets, most people are not risk neutral.

• A risk averse investor has decreasing marginal utility. Same increases in wealth bring smaller increases in satisfaction when the investor is more wealthy. This shows that risk averse investors have concave utility functions. Most people are risk averse in financial markets. They like profits but want to avoid risks.

• A risk seeker has increasing marginal utility. Same increases in wealth bring large increases in satisfaction when the investor is more wealthy. They have convex utility functions. Unlike risk averse investors, they take on risks because of the potential high profits they bring.

In this thesis, we consider that all investors are risk averse. For these investors, the utility function $u : \mathbb{R} \to \mathbb{R}$ is increasing, concave, and differentiable. The concept of risk aversion is used to measure the reluctance of an investor to accept a bargain with an uncertain payoff rather than another bargain with a more certain, but possibly lower, expected payoff. We also assume that the utility function has constant relative risk aversion (CRRA). In other words, the percentage of wealth the investor is willing to expose to risk remains unchanged as wealth changes. Denote the level of relative risk aversion by γ. Then by definition, $\gamma = -\dfrac{w\ u''(w)}{u'(w)}$ and hence the utility function is given by $u(w) = \dfrac{w^{1-\gamma}}{1-\gamma}$, $\gamma > 0$ and $\gamma \neq 1$, according to Moore (2009).

1.4 Optimal Investment Strategy

The goal of the investor is to optimize his expected discounted utility at the time of death given his current wealth by taking a dynamic investment strategy within a family $\mathcal{A} = \{\pi(t)\}$. The reason why the discounted utility is preferred to the utility is that the utility function does not stay the same over time for each individual.

Humans usually prefer a reward which arrives sooner than later. An investor would rather receive one dollar today than tomorrow. Thus utility is discounted at the investor's subjective discount rate which is denoted by ρ. Every person has his own subjective discount rate. A higher subjective discount rate implies a more impatient individual.

1.4.1 Hamilton-Jacobi-Bellman Equation

Given that the investor's total wealth is w at time t, his goal is to maximize

$$E[e^{-\rho(\tau_d-\tau)}u(W^{\pi}(\tau_d)) \mid W(\tau)=w],$$

where τ_d is the time of death.

Define the value function as

$$V^{\pi}(w,\tau)=E[e^{-\rho(\tau_d-\tau)}u(W^{\pi}(\tau_d)) \mid W(\tau)=w]. \tag{1.5}$$

Our main interest is to maximize this value function with respect to investment strategies $\pi(t)$ to get the optimal value function:

$$\begin{aligned} V^{\hat{\pi}}(w,\tau) &= E[e^{-\rho(\tau_d-t)}u(W^{\hat{\pi}}(\tau_d)) \mid W(\tau)=w] \\ &= \sup_{\pi(t)\in\mathcal{A}} V^{\pi}(w,\tau), \end{aligned} \tag{1.6}$$

where $\hat{\pi}$ is the optimal investment strategy. The family $\mathcal{A}$ is the set of all admissible strategies $\pi(t)$ which are $\mathfrak{F}_t$-measurable. $\mathfrak{F}_t$ is a filtration The admissible policies are the ones satisfying the following conditions:

(1) $W^{\pi}(t)=\pi(t)+\pi^0(t)$, for $\tau<t<\tau_d$;

(2) The portfolio is self-financing, which means there is no exogenous infusion or withdrawal of money;

(3) $E[\int_{\tau}^{\tau_d} \pi^2(t)dt] < +\infty$.

The value function $V^{\pi}(w,\tau)$ is the expected discounted utility using the investment strategy $\pi(t) \in \mathcal{A}$ over the time interval $[\tau, \tau_d]$ given the wealth is w at time τ. The optimal value function gives the optimal expected discounted utility over the same time period under the same initial conditions. Therefore $V^{\hat{\pi}}(w,\tau)$ is the optimal among all the value functions $V^{\pi}(w,\tau)$.

Proposition 1.4.1. *Assume that there exists an optimal strategy and the optimal value function $V^{\hat{\pi}}$ is smooth, then the function $V^{\hat{\pi}}$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation*

$$\begin{cases} V_\tau + rwV_w + \max_{\pi(\tau)\in\mathcal{A}}[(\mu - r)\pi(\tau)V_w + \frac{1}{2}\sigma^2\pi(\tau)^2 V_{ww}] + \lambda_x(\tau)u(w) = V[\rho + \lambda_x(\tau)] \\ V(w,\tau_d) = u(w). \end{cases} \tag{1.7}$$

In order to simplify the notations, V_τ and V_w are used to represent partial derivatives $\frac{\partial V}{\partial \tau}$ and $\frac{\partial V}{\partial w}$. Similarly, V_{ww} is used to represent second derivative $\frac{\partial^2 V}{\partial w^2}$.

Proof. The optimal value function $V^{\hat{\pi}}(w,\tau)$ is the solution of the optimal control problem

$$V^{\hat{\pi}}(w,\tau) = \sup V^{\pi}(w,\tau) \ \text{ subject to } \ \pi(t) \in \mathcal{A}.$$

The value of $V^{\pi}(w,\tau)$ is controlled by the dynamic strategy $\pi(t)$ over the time period $[\tau, \tau_d]$. Consider two strategies over this time interval. One is optimal, called Strategy I and denoted by π_I. The other one is not optimal on time interval $[\tau, \tau+h]$ and optimal afterwards. It is called Strategy II and denoted by

$$\pi_{II}(t) = \begin{cases} \tilde{\pi}(t) & t \in [\tau,\ \tau+h] \\ \hat{\pi}(t) & t \in (\tau+h,\ \tau_d] \end{cases}$$

- Under Strategy I, $\pi_I(t) = \hat{\pi}(t)$, and $V^{\pi_I}(w,\tau) = V^{\hat{\pi}}(w,\tau)$. It is trivial by the definition of the value function. Under the optimal strategy, the value function is optimized.

- Under Strategy II, there are two outcomes. If the individual dies during $[\tau, \tau+h]$, $V^{\pi_{II}}$ is given by $E[e^{-\rho(\tau_d-\tau)}u(W^{\tilde{\pi}}(\tau_d)) \mid W(\tau)=w]$. Otherwise he dies after $\tau+h$, $V^{\pi_{II}}$ is equal to the value function discounted to time τ since strategy II is optimal after $\tau+h$. So over the entire time period, $V^{\pi_{II}}$ is given by

$$\begin{aligned} V^{\pi_{II}}(w,\tau) = E[\ & I_{\{\tau_d>\tau+h\}}e^{-\rho h}V^{\hat{\pi}}(W^{\tilde{\pi}}(\tau+h),\tau+h) \\ & +I_{\{\tau_d\le\tau+h\}}e^{-\rho(\tau_d-\tau)}u(W^{\tilde{\pi}}(\tau_d)) \mid W(\tau)=w], \end{aligned} \tag{1.8}$$

where $I_{\{A\}} = \begin{cases} 1 & \text{if A is true} \\ 0 & \text{otherwise} \end{cases}$.

Since Strategy I is optimal, the inequality $V^{\pi_I}(w,\tau) \ge V^{\pi_{II}}(w,\tau)$ must hold. From

(1.8), we have

$$\begin{aligned} V^{\hat{\pi}}(w,\tau) \geq\ & E[I_{\{\tau_d>\tau+h\}}e^{-\rho h}V^{\hat{\pi}}(W^{\tilde{\pi}}(\tau+h),\tau+h) \\ & +I_{\{\tau_d\leq\tau+h\}}e^{-\rho(\tau_d-\tau)}u(W^{\tilde{\pi}}(\tau_d)) \mid W(\tau)=w] \\ \geq\ & E[I_{\{\tau_d>\tau+h\}}e^{-\rho h}V^{\hat{\pi}}(W^{\tilde{\pi}}(\tau+h),\tau+h) \\ & +I_{\{\tau_d\leq\tau+h\}}e^{-\rho h}u(W^{\tilde{\pi}}(\tau_d)) \mid W(\tau)=w] \\ =\ & e^{-\int_\tau^{\tau+h}\lambda_x(t)dt}e^{-\rho h}E[V^{\hat{\pi}}(W^{\tilde{\pi}}(\tau+h),\tau+h) \mid W(\tau)=w,\tau_d>\tau+h] \\ & +(1-e^{-\int_\tau^{\tau+h}\lambda_x(t)dt})e^{-\rho h}E[u(W^{\tilde{\pi}}(\tau_d)) \mid W(\tau)=w,\tau_d\leq\tau+h]. \end{aligned}$$

Multiplying $e^{\int_\tau^{\tau+h}\lambda_x(t)dt}e^{\rho h}$ on both sides,

$$\begin{aligned} e^{\int_\tau^{\tau+h}\lambda_x(t)dt}e^{\rho h}V^{\hat{\pi}}(w,\tau) \geq\ & E[V^{\hat{\pi}}(W^{\tilde{\pi}}(\tau+h),\tau+h) \mid W(\tau)=w,\tau_d>\tau+h] \\ & +(e^{\int_\tau^{\tau+h}\lambda_x(t)dt}-1)E[u(W^{\tilde{\pi}}(\tau_d)) \mid W(\tau)=w,\tau_d\leq\tau+h]. \end{aligned} \tag{1.9}$$

Since $V^{\hat{\pi}}(w,\tau)$ is assumed to have continuous partial derivatives, $It\hat{o}'s$ formula can be applied using (1.4). It leads to

$$\begin{aligned} V^{\hat{\pi}}(W^{\tilde{\pi}}(\tau+h),\tau+h) =\ & V^{\hat{\pi}}(w,\tau)+\int_\tau^{\tau+h}V_\tau^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)dt+\int_\tau^{\tau+h}V_w^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)dW^{\tilde{\pi}}(t) \\ & +\frac{1}{2}\int_\tau^{\tau+h}V_{ww}^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)d\langle W^{\tilde{\pi}},W^{\tilde{\pi}}\rangle_t \\ =\ & V^{\hat{\pi}}(w,\tau)+\int_\tau^{\tau+h}V_\tau^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)dt+\int_\tau^{\tau+h}[rW^{\tilde{\pi}}(t) \\ & +(\mu-r)\tilde{\pi}(t)]V_w^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)dt+\int_\tau^{\tau+h}\sigma\tilde{\pi}(t)V_w^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)dB(t) \\ & +\frac{1}{2}\int_\tau^{\tau+h}\sigma^2\tilde{\pi}(t)^2V_{ww}^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)dt \\ =\ & V^{\hat{\pi}}(w,\tau)+\int_\tau^{\tau+h}\{V_\tau^{\hat{\pi}}(W^{\tilde{\pi}}(t),t) \\ & +[rW^{\tilde{\pi}}(t)+(\mu-r)\tilde{\pi}(t)]V_w^{\hat{\pi}}(W^{\tilde{\pi}}(t),t) \\ & +\frac{1}{2}\sigma^2\tilde{\pi}(t)^2V_{ww}^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)\}dt+\int_\tau^{\tau+h}\sigma\tilde{\pi}(t)V_w^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)dB(t). \end{aligned} \tag{1.10}$$

Substitute (1.10) into (1.9)

$$
\begin{aligned}
e^{\int_\tau^{\tau+h}\lambda_x(t)dt+\rho h} \quad & V^{\bar{\pi}}(w,\tau) \geq E[V^{\bar{\pi}}(w,\tau) + \textstyle\int_\tau^{\tau+h}\{V_\tau^{\bar{\pi}}(W^{\bar{\pi}}(t),t) \\
& +[rW^{\bar{\pi}}(t) + (\mu - r)\bar{\pi}(t)]V_w^{\bar{\pi}}(W^{\bar{\pi}}(t),t) \\
& +\frac{1}{2}\sigma^2\bar{\pi}(t)^2 V_{ww}^{\bar{\pi}}(W^{\bar{\pi}}(t),t)\}dt \\
& +\textstyle\int_\tau^{\tau+h}\sigma\bar{\pi}(t)V_w^{\bar{\pi}}(W^{\bar{\pi}}(t),t)dB(t) \mid W(\tau) = w, \tau_d > \tau + h] \\
& +(e^{\int_\tau^{\tau+h}\lambda_x(t)dt} - 1)E[u(W^{\bar{\pi}}(\tau_d)) \mid W(\tau) = w, \tau_d \leq \tau + h].
\end{aligned}
\tag{1.11}
$$

Since $\sigma\bar{\pi}(t)V_w^{\bar{\pi}}(W^{\bar{\pi}}(t),t)$ is $\mathfrak{F}_t$-measurable, we further assume that

$$
\int_\tau^{\tau+h}\sigma\bar{\pi}(t)V_w^{\bar{\pi}}(W^{\bar{\pi}}(t),t)dt < +\infty.
$$

Then the stochastic integral $\int_\tau^{\tau+h}\sigma\bar{\pi}(t)V_w^{\bar{\pi}}(W^{\bar{\pi}}(t),t)dB(t)$ is a martingale and its expectation is equal to $\int_\tau^{\tau}\sigma\bar{\pi}(t)V_w^{\bar{\pi}}(W^{\bar{\pi}}(t),t)dB(t) = 0$. The stochastic integral term in (1.11) vanishes, leaving us with the following inequality:

$$
\begin{aligned}
(e^{\int_\tau^{\tau+h}\lambda_x(t)dt+\rho h} - 1) \quad & V^{\bar{\pi}}(w,\tau) \geq E[\textstyle\int_\tau^{\tau+h}\{V_\tau^{\bar{\pi}}(W^{\bar{\pi}}(t),t) \\
& +[rW^{\bar{\pi}}(t) + (\mu - r)\bar{\pi}(t)]V_w^{\bar{\pi}}(W^{\bar{\pi}}(t),t) \\
& +\frac{1}{2}\sigma^2\bar{\pi}(t)^2 V_{ww}^{\bar{\pi}}(W^{\bar{\pi}}(t),t)\}dt \mid W(\tau) = w, \tau_d > \tau + h] \\
& +(e^{\int_\tau^{\tau+h}\lambda_x(t)dt} - 1)E[u(W^{\bar{\pi}}(\tau_d)) \mid W(\tau) = w, \tau_d \leq \tau + h].
\end{aligned}
\tag{1.12}
$$

Next divide both sides by h, move h within the expectation and let h tend to zero. The left hand side of (1.12) gives

$$
\begin{aligned}
& \lim_{h\to 0}\frac{(e^{\int_\tau^{\tau+h}\lambda_x(t)dt+\rho h} - 1)V^{\bar{\pi}}(w,\tau)}{h} \\
= \; & [\lambda_x(\tau) + \rho]V^{\bar{\pi}}(w,\tau)\lim_{h\to 0} e^{\int_\tau^{\tau+h}\lambda_x(t)dt+\rho h} \\
= \; & [\lambda_x(\tau) + \rho]V^{\bar{\pi}}(w,\tau)
\end{aligned}
\tag{1.13}
$$

On the right hand side, we have $\tau_d \to \tau$ as $h \to 0$ implying that $W^{\bar{\pi}}(\tau_d) \to W(\tau) = w$.

So the last term of (1.12) has the following limit

$$\lim_{h\to 0}\frac{(e^{\int_\tau^{\tau+h}\lambda_x(t)dt}-1)}{h}E[u(W^{\tilde{\pi}}(\tau_d))\mid W(\tau)=w,\tau_d\le\tau+h]=\lambda_x(\tau)u(w). \quad (1.14)$$

Assume that

$$\frac{\int_\tau^{\tau+h}\{V_\tau^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)+[rW^{\tilde{\pi}}(t)+(\mu-r)\tilde{\pi}(t)]V_w^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)+\frac{1}{2}\sigma^2\tilde{\pi}(t)^2V_{ww}^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)\}dt}{h}$$

is dominated by some integrable function. By the dominated convergence theorem,

$$\begin{aligned}
&\lim\nolimits_{h\to 0}E[\frac{1}{h}\int_\tau^{\tau+h}\{V_\tau^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)+[rW^{\tilde{\pi}}(t)+(\mu-r)\tilde{\pi}(t)]V_w^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)\\
&+\tfrac{1}{2}\sigma^2\tilde{\pi}(t)^2V_{ww}^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)\}dt]\\
=\;&E[\lim\nolimits_{h\to 0}\frac{1}{h}\int_\tau^{\tau+h}\{V_\tau^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)+[rW^{\tilde{\pi}}(t)+(\mu-r)\tilde{\pi}(t)]V_w^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)\\
&+\tfrac{1}{2}\sigma^2\tilde{\pi}(t)^2V_{ww}^{\hat{\pi}}(W^{\tilde{\pi}}(t),t)\}dt] \qquad (1.15)\\
=\;&E[V_\tau^{\hat{\pi}}(W^{\tilde{\pi}}(\tau),\tau)+[rW^{\tilde{\pi}}(\tau)+(\mu-r)\tilde{\pi}(\tau)]V_w^{\hat{\pi}}(W^{\hat{\pi}}(\tau),\tau)\\
&+\frac{1}{2}\sigma^2\tilde{\pi}(\tau)^2V_{ww}^{\hat{\pi}}(W^{\hat{\pi}}(\tau),\tau)]\\
=\;&V_\tau^{\hat{\pi}}(w,\tau)+[rw+(\mu-r)\tilde{\pi}(\tau)]V_w^{\hat{\pi}}(w,\tau)+\frac{1}{2}\sigma^2\tilde{\pi}(\tau)^2V_{ww}^{\hat{\pi}}(w,\tau)
\end{aligned}$$

Plugging (1.13), (1.14) and (1.15) into (1.11), we have

$$[\lambda_x(\tau)+\rho]V^{\hat{\pi}}\ge V_\tau^{\hat{\pi}}+[rw+(\mu-r)\tilde{\pi}(\tau)]V_w^{\hat{\pi}}+\frac{1}{2}\sigma^2\tilde{\pi}(\tau)^2V_{ww}^{\hat{\pi}}+\lambda_x(\tau)u(w). \quad (1.16)$$

Since the controlling strategy $\tilde{\pi}(\tau)\in\mathcal{A}$ is arbitrary, (1.16) holds for all strategies $\tilde{\pi}(\tau)$. If $\tilde{\pi}(\tau)$ is equal to the optimal one $\hat{\pi}(\tau)$, Strategy II is optimal over the entire time period thus it is identical to Strategy I and the equality in (1.16) is attained. Thus the optimal value function $V^{\hat{\pi}}(w,\tau)$ satisfies the following equation:

$$V_\tau+rwV_w+\max_{\pi(\tau)}[(\mu-r)\pi(\tau)V_w+\frac{1}{2}\sigma^2\pi(\tau)^2V_{ww}]+\lambda_x(\tau)u(w)=[\lambda_x(\tau)+\rho]V(1.17)$$

If the initial time τ is equal to the time of death τ_d, by definition the value function is given by $V^{\tilde{\pi}}(w,\tau) = V^{\hat{\pi}}(w,\tau) = u(w)$. It is independent of the investment strategy. Therefore the optimal value function is also equal to the utility of the initial wealth. This gives the boundary condition $V(w,\tau_d) = u(w)$ in the HJB equation.

□

Björk (2004) gives a theorem stating that the if the HJB equation (1.7) has a unique and smooth solution which follows the strategy that obtain the supremum involved in the equation, the optimal value function exists and it equals to the unique solution. The strategy at which the supremum involved is obtained is the optimal strategy.

By the verification theorem in Clarke, Ledyaev, Stern and Wolenski (1998), the above Hamilton-Jacobi-Bellman equation has a unique viscosity solution which is equal to the optimal value function defined in (1.6). In the rest part of this thesis, we assume that the optimal value function V is smooth. Otherwise we can always work with sub-differentials ∂_D instead of classic differentials ∂.

In order to find the optimal control strategy which we denote by $\hat{\pi}$, consider the following problem

$$\max_{\pi(\tau)}[(\mu - r)\pi(\tau)V_w + \frac{1}{2}\sigma^2\pi(\tau)^2V_{ww}]. \tag{1.18}$$

Notice that this is an optimization problem of a quadratic function of $\pi(\tau)$. If we are given $V_{ww} < 0$, the optimal $\pi(\tau)$ could be found using the first-order necessary condition. In fact, the linearity implied by the wealth process together with the concavity of the utility function $u(w)$ indicate that the optimal value function $V(w,\tau)$ is concave with respect to w, *i.e.* $V_{ww} < 0$ under the smoothness assumption of V. Therefore, the maximum in (1.18) is attained at

$$\hat{\pi}(w,\tau) = -\frac{\mu - r}{\sigma^2}\frac{V_w}{V_{ww}}. \tag{1.19}$$

The optimal investment strategy in the risky asset is

$$\hat{\pi}(\tau) = \hat{\pi}(W^{\hat{\pi}}(\tau),\tau) = -\frac{\mu - r}{\sigma^2}\frac{V_w(W^{\hat{\pi}}(\tau),\tau)}{V_{ww}(W^{\hat{\pi}}(\tau),\tau)}, \tag{1.20}$$

in which $W^{\hat{\pi}}(\tau)$ is the optimal wealth process based on the optimal strategy process $\hat{\pi}(\tau)$ instead of $\pi(t)$.

Using this optimal strategy, the Hamilton-Jacobi-Bellman equation can be rewritten as

$$V_\tau + rwV_w - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{V_w^2}{V_{ww}} + \lambda_x(\tau)u(w) = (\lambda_x(\tau) + \rho)V. \tag{1.21}$$

This is a regular partial differential equation. In order to solve (1.21), Moore (2009) proposed that the solution could be in the form of

$$V(w,\tau) = \xi(\tau)u(w). \tag{1.22}$$

Recall that the utility function is given by $u(w) = \dfrac{w^{1-\gamma}}{1-\gamma}$. Plugging (1.22) into (1.21):

$$\begin{aligned}
\xi_\tau\frac{w^{1-\gamma}}{1-\gamma} + rw\xi w^{-\gamma} - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{\xi^2 w^{-2\gamma}}{\xi(-\gamma w^{-\gamma-1})} + \lambda_x(\tau)\frac{w^{1-\gamma}}{1-\gamma} &= (\lambda_x(\tau) + \rho)\xi\frac{w^{1-\gamma}}{1-\gamma} \\
\xi_\tau\frac{w^{1-\gamma}}{1-\gamma} + r\xi w^{1-\gamma} + \frac{1}{2}\frac{(\mu - r)^2}{\gamma\sigma^2}\xi w^{1-\gamma} + \lambda_x(\tau)\frac{w^{1-\gamma}}{1-\gamma} &= (\lambda_x(\tau) + \rho)\xi\frac{w^{1-\gamma}}{1-\gamma} \\
\xi_\tau + [r + \frac{1}{2}\frac{(\mu - r)^2}{\gamma\sigma^2}](1-\gamma)\xi + \lambda_x(\tau) &= (\lambda_x(\tau) + \rho)\xi.
\end{aligned}$$

Therefore ξ satisfies the ordinary differential equation

$$\xi_\tau = [\lambda_x(\tau) + \rho - (r + \frac{1}{2}\frac{(\mu - r)^2}{\gamma\sigma^2})(1-\gamma)]\xi - \lambda_x(\tau). \tag{1.23}$$

with the boundary condition $\xi(\tau_d) = 1$.

Since V is given by (1.22) and $\hat{\pi}(t)$ is given by (1.20), we have

$$\begin{aligned} \hat{\pi}(\tau) &= -\frac{\mu - r}{\sigma^2}\frac{V_w(W^{\hat{\pi}}(\tau),\tau)}{V_{ww}^{\hat{\pi}}(W^{\hat{\pi}}(\tau),\tau)} \\ &= -\frac{\mu - r}{\sigma^2}\frac{\xi(\tau)u'(W^{\hat{\pi}}(\tau))}{\xi(\tau)u''(W^{\hat{\pi}}(\tau))} \\ &= -\frac{\mu - r}{\sigma^2}\frac{W^{\hat{\pi}}(\tau)^{-\gamma}}{-\gamma W^{\hat{\pi}}(\tau)^{-\gamma-1}} \\ &= \frac{\mu - r}{\gamma\sigma^2}W^{\hat{\pi}}(\tau), \end{aligned} \tag{1.24}$$

where $W^{\hat{\pi}}(\tau)$ is the optimally controlled wealth process. Here the optimal strategy is to keep $\frac{\mu - r}{\gamma\sigma^2}$ of the total wealth in the risky asset. This result comes from the assumption of constant relative risk aversion. If an investor has a constant relative risk aversion, the proportion of total wealth he is willing to expose to risk is independent of total wealth. Also note that, as the risk aversion increases, as measured by γ, the ratio invested in the risky asset decreases. An investor with higher relative risk aversion is more reluctant to expose his investment to risk.

Now that the optimal strategy is a constant ratio of total wealth, we can write the optimal wealth dynamic:

$$\begin{aligned} dW^{\hat{\pi}}(t) &= [rW^{\hat{\pi}}(t) + (\mu - r)\hat{\pi}(t)]dt + \sigma\hat{\pi}(t)dB(t) \\ &= [r + \frac{(\mu - r)^2}{\gamma\sigma^2}]W^{\hat{\pi}}(t)dt + \frac{\mu - r}{\gamma\sigma}W^{\hat{\pi}}(t)dB(t). \end{aligned} \tag{1.25}$$

(1.25) could be rewritten into

$$\frac{dW^{\hat{\pi}}(t)}{W^{\hat{\pi}}(t)} = [r + \frac{(\mu - r)^2}{\gamma\sigma^2}]dt + \frac{\mu - r}{\gamma\sigma}dB(t). \tag{1.26}$$

The above stochastic differential equation shows that $W^{\hat{\pi}}(t)$ is a geometric Brownian motion. Using the same approach as in Section 1.1, we have

$$dln(W^{\hat{\pi}}(t)) = [r + \frac{(\mu-r)^2}{\gamma\sigma^2} - \frac{1}{2}\frac{(\mu-r)^2}{\gamma^2\sigma^2}]dt + \frac{\mu-r}{\gamma\sigma}dB(t). \tag{1.27}$$

The solution of the above stochastic differential equation with the initial condition $W^{\hat{\pi}}(\tau) = w$ is given by

$$W^{\hat{\pi}}(t) = w\exp\{[r + \frac{(\mu-r)^2}{\gamma\sigma^2} - \frac{1}{2}\frac{(\mu-r)^2}{\gamma^2\sigma^2}](t-\tau) + \frac{\mu-r}{\gamma\sigma}B(t-\tau)\}. \tag{1.28}$$

From the definition in (1.6) of V, we have that

$$\begin{aligned} V^{\hat{\pi}}(w,\tau) &= E[e^{-\rho(\tau_d-\tau)}u(W^{\hat{\pi}}_{\tau_d}) \mid W(\tau) = w] \\ &= E\left[\frac{e^{-\rho(\tau_d-\tau)}}{1-\gamma}[w\exp\{(r + \tfrac{(\mu-r)^2}{\gamma\sigma^2} - \tfrac{1}{2}\tfrac{(\mu-r)^2}{\gamma^2\sigma^2})(\tau_d-\tau) + \tfrac{\mu-r}{\gamma\sigma}B(\tau_d-\tau)\}]^{1-\gamma}\right] \\ &= \frac{w^{1-\gamma}}{1-\gamma}E[e^{-\rho(\tau_d-\tau)}e^{[(r+\frac{(\mu-r)^2}{\gamma\sigma^2}-\frac{1}{2}\frac{(\mu-r)^2}{\gamma^2\sigma^2})(\tau_d-\tau)+\frac{\mu-r}{\gamma\sigma}B(\tau_d-\tau)](1-\gamma)}] \\ &= u(w)E[e^{-\rho(\tau_d-\tau)}e^{[(r+\frac{(\mu-r)^2}{\gamma\sigma^2}-\frac{1}{2}\frac{(\mu-r)^2}{\gamma^2\sigma^2})(\tau_d-\tau)+\frac{\mu-r}{\gamma\sigma}B(\tau_d-\tau)](1-\gamma)}] \\ &= u(w)\xi(\tau), \end{aligned}$$

where

$$\xi(\tau) = E[e^{-\rho(\tau_d-\tau)}e^{[(r+\frac{(\mu-r)^2}{\gamma\sigma^2}-\frac{1}{2}\frac{(\mu-r)^2}{\gamma^2\sigma^2})(\tau_d-\tau)+\frac{\mu-r}{\gamma\sigma}B(\tau_d-\tau)](1-\gamma)}]. \tag{1.29}$$

In order to find $V^{\hat{\pi}}(w,\tau)$ we only need to obtain $\xi(\tau)$ numerically. There are two ways to evaluate $\xi(\tau)$. One is to solve the ODE (1.23) with the boundary condition and the other is to simulate the expectation (1.29).

1.4.2 Numerical Illustration

In this section, we illustrate the result of the previous section with numerical examples. We compare the optimal strategies to see the impact of the parameters to the portfolio. Since the optimal investment in the risky asset is a constant ratio of the total wealth, they follow the same stochastic pattern but with different initial value. Because of this linear relationship, their expectation and variance are related:

$$E[\hat{\pi}(t)] = \frac{\mu - r}{\gamma\sigma^2} E[W^{\hat{\pi}}(t)]; \qquad Var[\hat{\pi}(t)] = \frac{(\mu - r)^2}{\gamma^2\sigma^4} Var[W^{\hat{\pi}}(t)].$$

Since the optimal investment in the risky asset is a fixed portion of the total wealth, its expected value and volatility are fixed portions of those of the total wealth.

We choose the following parameters as the base scenario:

- The rate of return on the risk free asset $r = 0.04$;
- The rate of return on the risky asset $\mu = 0.08$;
- The volatility on the risky asset $\sigma = 0.2$;
- The relative risk aversion $\gamma = 2$;
- The initial total wealth $w = 1$.

The optimal wealth dynamic (1.25) becomes

$$\begin{cases} dW^{\hat{\pi}}(t) = 0.06W^{\hat{\pi}}(t)dt + 0.1W^{\hat{\pi}}(t)dB(t) \\ W^{\hat{\pi}}(0) = 1 \end{cases}$$

and the optimal investment process (1.24) becomes

$$\hat{\pi}(t) = 0.5W_t^{\hat{\pi}}.$$

In the base scenario, the optimal investment strategy is to keep 50% of the total wealth in the risky asset. That is, the risky asset and the risk-free asset have the same value. The total wealth follows a GBM with drift 0.06 and diffusion 0.1. All the dynamics are simulated using an 1-year horizon on a daily basis (365 days).

Example 1.4.1. *μ=0.08 versus μ=0.1 (Fig. 1.1).*

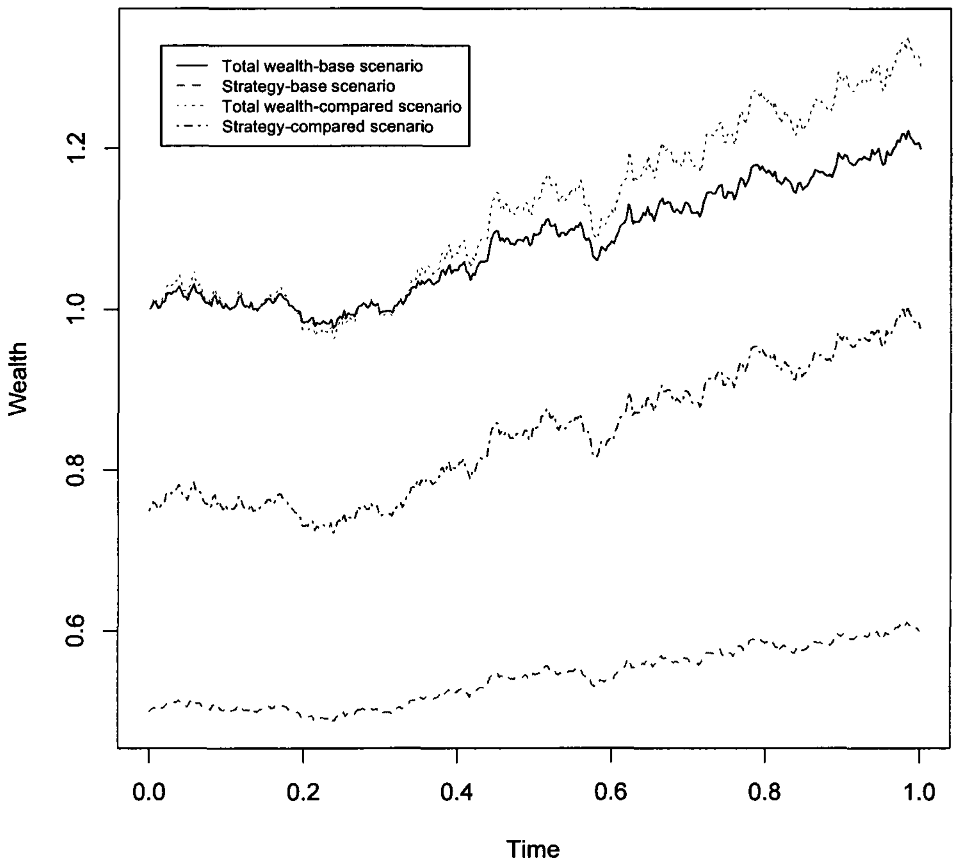

Figure 1.1: Base Scenario: μ=0.08 versus Compared Scenario: μ=0.1

In this example, we increase μ from 0.08 to 0.1. The drift and volatility of the wealth dynamic both increase:

$$
\begin{aligned}
&dW^{\hat{\pi}}(t) = 0.085W^{\hat{\pi}}(t)dt + 0.15W^{\hat{\pi}}(t)dB(t);\\
&\hat{\pi}(t) = 0.75W^{\hat{\pi}}(t).
\end{aligned}
$$

Under higher returns of the risky asset, the exposure to risk not only takes a larger portion of total wealth but also has greater volatility, as shown in Fig. 1.1. Particularly, when the return of the risky asset gets as high as 0.12, the optimal investment strategy is to put all the money into the risky asset.

Example 1.4.2. *σ=0.2 versus σ=0.15 (Fig. 1.2).*

In this example, the volatility of the risky asset σ drops from 0.2 to 0.15. However, in spite of this decrease, the volatility of total wealth increases due to the greater growth in the share of the risky asset.

$$
\begin{aligned}
&dW^{\hat{\pi}}(t) = 0.0756W^{\hat{\pi}}(t)dt + 0.1333W^{\hat{\pi}}(t)dB(t);\\
&\hat{\pi}(t) = 0.8889W^{\hat{\pi}}(t).
\end{aligned}
$$

One special case is that when $\dfrac{\mu - r}{\gamma\sigma^2} = 1$ the optimal strategy is to keep all the money in the risky asset. In this example, it occurs when σ reduces to $\sqrt{0.02} \approx 0.1414$.

Example 1.4.3. *γ=2 versus γ=4 (Fig. 1.3).*

In this example, we compare two investors with different relative risk aversion coefficients (Fig. 1.3).

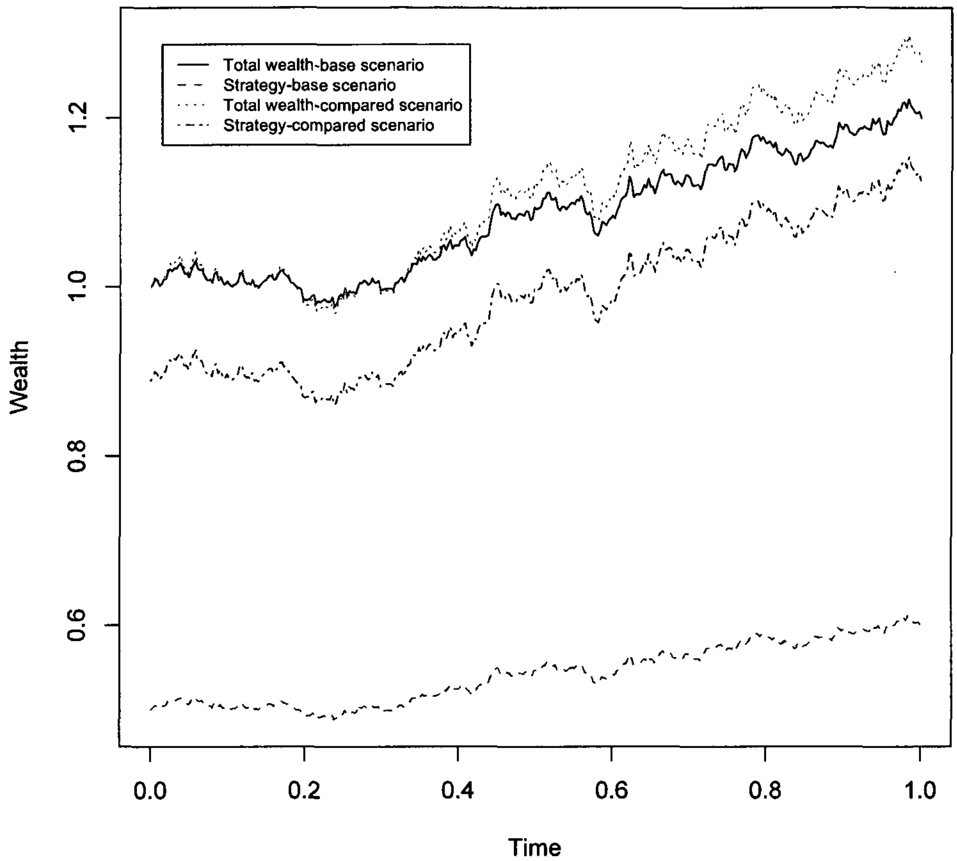

Figure 1.2: Base Scenario: σ=0.2 versus Compared Scenario: σ=0.15

$$dW^{\hat{\pi}}(t) = 0.05W^{\hat{\pi}}(t)dt + 0.05W^{\hat{\pi}}(t)dB(t);$$

$$\hat{\pi}(t) = 0.25W^{\hat{\pi}}(t).$$

In the same financial market with everything being equal, an investor with higher relative risk aversion (RRA) has lower portfolio volatility since the percentage he is willing to expose to risk is lower. This is because a higher RRA implies a greater

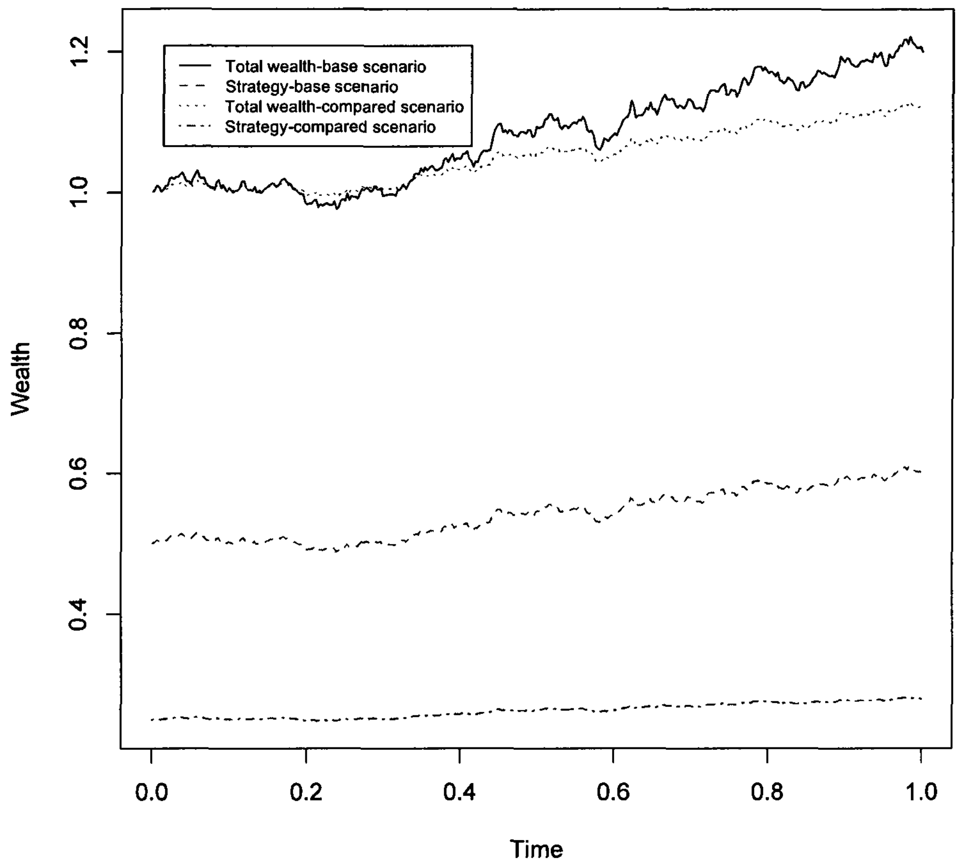

Figure 1.3: Base Scenario: γ=2 versus Compared Scenario: γ=4

reluctance to risk exposure. The optimal strategy for an investor with a unit RRA is to put all the money in the risky asset.

Example 1.4.4. . *r=0.04 versus r=0.06 (Fig. 1.4)*

In this example, the expected return of the risk-free asset increases from 0.04 to

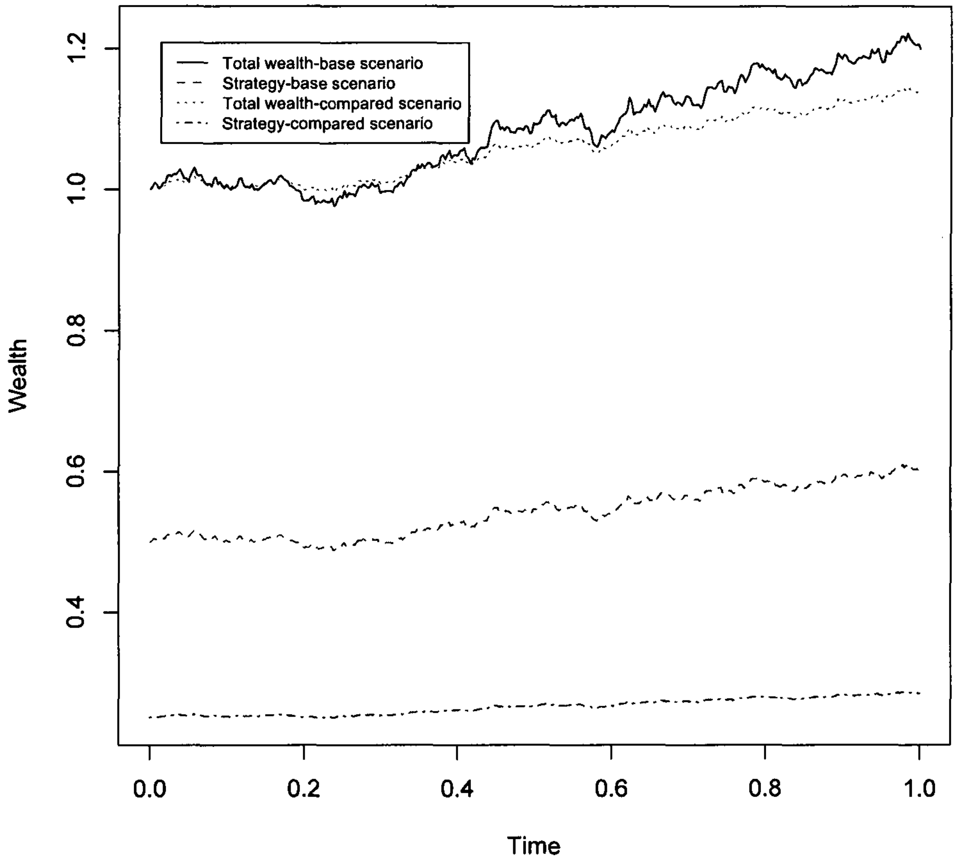

Figure 1.4: Base Scenario: r=0.04 versus Compared Scenario: r=0.06

0.06.

$$dW^{\hat{\pi}}(t) = 0.065W^{\hat{\pi}}(t)dt + 0.05W^{\hat{\pi}}(t)dB(t);$$

$$\hat{\pi}(t) = 0.25W^{\hat{\pi}}(t).$$

Fig. 1.4 shows that when the return of the risk-free asset increases, it is more attractive to investors, thus the share in the risky asset decreases. Since the risk-free asset, which is less volatile dominates the portfolio, the volatility of total wealth drops as the return of the risk-free asset increases.

Chapter 2

Equity-Linked Life Insurance

2.1 Introduction

Equity-linked products are insurance policies with payoffs linked to the performance of an underlying financial instrument. This kind of insurance contract contains not only mortality risk but also financial risk. Equity-linked products are popular among investors since they provide returns linked to the financial market.

In addition to the above common features, equity-linked insurance contracts may include some other options such as the option to surrender the contract, a guaranteed minimum maturity or a death benefit which provides higher benefits than a given level. In this thesis, we investigate the surrender option which gives policyholders the right to terminate the contract before the maturity date set in the contract. This is the possibility of withdrawing the value of the contract with certain penalties.

In this chapter, we will apply the results of Chapter 1 to study the surrender strategies of the policyholders.

2.2 Optimal Value Function

We consider a perpetual equity-linked life insurance policy. The perpetual product enables the policyholder to surrender the contract before his time of death. If he chooses to surrender, the contract terminates and the investor receives a surrender benefit. A death benefit is received upon death if the contract is never surrendered. The benefits are related to the stock index.

Suppose that the investor is aged x when he enters the contract at time 0. He receives an equity-linked death benefit $D(S(\tau_d), \tau_d)$ at the time of death $\tau = \tau_d$. The total value of the contract $S(\tau)$ is related to the underlying risky asset. At the time of surrender τ_s, the investor receives a surrender benefit $B(S(\tau_s), \tau_s)$ which is also linked to the risky asset. This benefit will then be moved to financial markets such that the discounted utility is maximized using the optimal investment strategy described in (1.24). After the policyholder surrenders the insurance contract, the only risk that remains is the financial risk, since the insurance contract is terminated.

2.2.1 Optimal Value Function After Surrender

In order to determine the optimal time to surrender the contract, we need to investigate the optimal value function when the investor is in the contract. He should consider what strategies he will take after surrender to determine an optimal one over the entire horizon. To do that, we first need the optimal value function after surrender.

Since the only risk that remains after the policyholder surrenders the contract is the financial risk, we are in the same setting as in Chapter 1. Since there is no confusion, we simply use the notation V to represent the optimal value function $V^{\hat{\pi}}$ which is obtained by using the optimal investment strategy $\hat{\pi}$. Then at any time $\tau < \tau_s$ the investor's optimal value function is defined as in (1.6):

$$V(w,\tau) = \sup_{\pi(t)\in\mathcal{A}} E[e^{-\rho(\tau_d-\tau)}u(W^{\pi}(\tau_d)) \mid W(\tau) = w].$$

It satisfies the HJB equation (1.21)

$$V_\tau + rwV_w - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\frac{V_w^2}{V_{ww}} + \lambda_x(\tau)u(w) = [\lambda_x(\tau)+\rho]V.$$

Therefore the optimal investment strategy (1.24) is

$$\hat{\pi}(\tau) = \frac{\mu-r}{\gamma\sigma^2}W^{\hat{\pi}}(\tau).$$

2.2.2 Optimal Value Function Before Surrender

Before surrendering the contract, the investor could decide at any time $\tau < \tau_d$ whether to surrender the contract or continue his position. If he chooses to surrender the contract, he receives the payoff $B(S(\tau_s), \tau_s)$ and invests the money optimally in the financial market. His wealth at the time of surrender is $B(S(\tau_s), \tau_s)$, so his optimal value function is $V(B(S(\tau_s), \tau_s), \tau_s)$, which is given in Section 2.2.1. If the investor dies prior to surrender, *i.e.* $\tau_d \leq \tau_s$, he receives the death benefit $D(S(\tau_d), \tau_d)$ and the utility is $u(D(S(\tau_d)), \tau_d)$. Thus the optimal value function prior to surrender is given by

$$\begin{aligned} U(S,\tau) = \sup\nolimits_{\tau_s} E[\quad & I_{\{\tau_s<\tau_d\}} e^{-\rho(\tau_s-\tau)} V(B(S(\tau_s),\tau_s),\tau_s) \\ & + I_{\{\tau_d\leq\tau_s\}} e^{-\rho(\tau_d-\tau)} u(D(S(\tau_d),\tau_d)) \mid S(\tau)=s]. \end{aligned} \tag{2.1}$$

If $\tau_s = \tau < \tau_d$, $I_{\{\tau<\tau_d\}} = 1$, $I_{\{\tau_d\leq\tau\}} = 0$, and (2.1) is reduced to

$$\begin{aligned} U(S,\tau) \quad & = \sup\nolimits_{\tau_s} E[V(B(S(\tau),\tau),\tau) \mid S(\tau)=s] \\ & = V(B(S,\tau),\tau). \end{aligned} \tag{2.2}$$

Surrendering at the initial time τ for $B(S,\tau)$ and then following the optimal strategy to obtain $V(B(S,\tau),\tau)$ is one trajectory among all possible strategies. It is not necessarily the optimal one ,allowing surrender option afterwards. Therefore the optimal value function before surrender $U(S,\tau)$ must be higher or equal to $V(B(S,\tau),\tau)$.

Using the approach in Section 1.4.1, we subdivide the time interval into two parts: $[\tau, \tau+h]$ and $(\tau+h, \infty)$, for h close to zero, and compare the strategy which is sub-optimal over $(\tau + h, \infty)$ to the optimal strategy. We get the following inequality:

$$\begin{aligned} U(S,\tau) \geq \;& E[I_{\{\tau_d > \tau+h\}} e^{-\rho h} U(S(\tau+h), \tau+h) \\ & + I_{\{\tau_d \leq \tau+h\}} e^{-\rho(\tau_d-\tau)} u(D(S(\tau_d), \tau_d)) \mid S(\tau) = s] \\ \geq \;& e^{-\int_\tau^{\tau+h}} e^{-\rho h} E[U(S(\tau+h), \tau+h) \mid S(\tau) = s, \tau_d > \tau+h] \\ & +(1 - e^{-\int_\tau^{\tau+h} \lambda_x(t)dt}) e^{-\rho h} E[u(D(S(\tau_d), \tau_d)) \mid S(\tau) = s, \tau_d \leq \tau+h]. \end{aligned} \tag{2.3}$$

Reorganizing (2.3), we have the inequality

$$\begin{aligned} e^{\int_\tau^{\tau+h} \lambda_x(t)dt} e^{\rho h} U(S,\tau) \;& \geq E[U(S(\tau+h), \tau+h) \mid S(\tau) = s, \tau_d > \tau+h] \\ & +(e^{\int_\tau^{\tau+h} \lambda_x(t)dt} - 1) E[u(D(S(\tau_d), \tau_d)) \\ & \mid S(\tau) = s, \tau_d \leq \tau+h]. \end{aligned} \tag{2.4}$$

In order to transform the first term in the above inequality, we apply *Itô's* formula to $U(W, \tau)$ and have

$$\begin{aligned} U(S(\tau+h), \tau+h) = \;& U(S,\tau) + \int_\tau^{\tau+h} U_\tau(S(t), t)dt + \int_\tau^{\tau+h} U_S(S(t), t)dS(t) \\ & + \frac{1}{2} \int_\tau^{\tau+h} U_{SS}(S(t), t) d\langle S, S\rangle_t. \end{aligned} \tag{2.5}$$

Recall that the stock price $S(t)$ follows a geometric Brownian motion in (1.1). Substitute it into (2.5):

$$\begin{aligned} U(S(\tau+h), \tau+h) = \;& U(S,\tau) + \int_\tau^{\tau+h} [U_\tau(S(t), t) + \mu S(t) U_S(S(t), t) \\ & + \frac{1}{2}\sigma^2 S^2(t) U_{SS}(S(t), t)]dt + \int_\tau^{\tau+h} \sigma S(t) U_S(S(t), t) dB(t). \end{aligned} \tag{2.6}$$

With this equation, inequality (2.4) is transformed into a partial differential inequality for $U(S, \tau)$:

$$\begin{aligned} & (e^{\int_\tau^{\tau+h} \lambda_x(t)dt} e^{\rho h} - 1) U(S,\tau) \\ \geq \;& E[\int_\tau^{\tau+h} \{U_\tau(S(t), t) + \mu S(t) U_S(S(t), t) + \frac{1}{2}\sigma^2 S^2(t) U_{SS}(S(t), t)\}dt \\ & + \int_\tau^{\tau+h} \sigma S(t) U_S(S(t), t) dB(t) \mid S(\tau) = s, \tau_d > \tau+h] \\ & +(e^{\int_\tau^{\tau+h} \lambda_x(t)dt} - 1) E[u(D(S(\tau_d), \tau_d)) \mid S(\tau) = s, \tau_d \leq \tau+h]. \end{aligned}$$

Divide both sides by h, move it within the expectation and let it go to zero. By $l'H\hat{o}pital$'s rule, the left hand side has the limit

$$\begin{aligned}\lim_{h\to 0}\frac{(e^{\int_\tau^{\tau+h}\lambda_x(t)dt}e^{\rho h}-1)U(S,\tau)}{h} &= \lim_{h\to 0}\frac{d(e^{\int_\tau^{\tau+h}\lambda_x(t)dt}e^{\rho h}-1)}{dh}U(S,\tau)\\ &= [\lambda_x(\tau)+\rho]U(S,\tau).\end{aligned}$$

Similarly to Section 1.4.1, the right hand side of the inequality becomes

$$U_\tau(S,\tau)+\mu SU_S(S,\tau)+\frac{1}{2}\sigma^2S^2U_{SS}(S,\tau)+\lambda_x(\tau)u(D(S,\tau)).$$

Based on the above transformations, the inequality (2.4) is simplified to the partial differential inequality

$$[\lambda_x(\tau)+\rho]U(S,\tau)\geq U_\tau(S,\tau)+\mu SU_S(S,\tau)+\frac{1}{2}\sigma^2S^2U_{SS}(S,\tau)+\lambda_x(\tau)u(D(S,\tau))\tag{2.7}$$

Similarly to the optimal value function after surrender V, the optimal value function before surrender U satisfies the inequalities

$$[\lambda_x(\tau)+\rho]U \geq U_\tau+\mu SU_S+\frac{1}{2}\sigma^2S^2U_{SS}+\lambda_x(\tau)u(D(S,\tau)),\tag{2.8}$$

$$U(S,\tau) \geq V(B(S,\tau),\tau).\tag{2.9}$$

In (2.9), if $U(S,\tau)>V(B(S,\tau),\tau)$, the expected utility from keeping the equity-linked contract is strictly greater than the expected utility from surrendering the contract. Thus the investor should not surrender the contract. However as the value of (S,τ) varies, the inequality in (2.9) switches to equality at some points. The set of such paired values are defined as the surrender boundary. In the no-surrender region, $U(S,\tau)$ is free which means its value is not restricted by $V(B(S,\tau),\tau)$. Moreover, since $U(S,\tau)$ is optimal by definition and the inequality (2.8) holds arbitrarily, the equality of (2.8) holds when U is free. Therefore the equality holds in at least one of (2.8) and (2.9).

Because we are interested in the surrender boundaries at which investors can surrender at no loss in terms of utility, we want to investigate the non-surrender region before it hits the boundaries. Then instead of studying two inequalities at the same time, we only need to solve

$$\begin{cases} [\lambda_x(\tau)+\rho]U \geq U_\tau + \mu S U_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \lambda_x(\tau)u(D(S,\tau)) \\ U(S,\tau) \geq V(B(S,\tau),\tau) \\ \{U(S,\tau) - V(B(S,\tau),\tau)\} \\ \times \left\{[\lambda_x(\tau)+\rho]U - U_\tau - \mu S U_S - \frac{1}{2}\sigma^2 S^2 U_{SS} - \lambda_x(\tau)u(D(S,\tau))\right\} = 0 \end{cases} \tag{2.10}$$

The third equation in (2.10) shows that at least one of the two inequalities must hold.

2.2.3 Solving For Boundaries

In order to solve (2.10), we implement the Projected SOR algorithm suggested by Wilmott et al. (2000). The Projected SOR algorithm is an iterative method for solving partial differential equations subject to an inequality constraint. We first solve (2.10) on a domain containing the boundary and then find the boundary from the inequality. The boundary points are the points where the inequality switches to equality. The details of the solution are given in the Appendix.

Before applying the algorithm explained in the appendix, we first transform the problem (2.10) into the form of (.21).

Let $t = T - \tau$, $v = \ln S$ where T is a fixed large number. Let $h(v,t) = U(S,\tau)$. Thus we have the differentials

$$
\begin{aligned}
U_\tau &= \frac{\partial h}{\partial t}\frac{\partial t}{\partial \tau} = -h_t, \\
U_S &= \frac{\partial h}{\partial v}\frac{\partial v}{\partial S} = \frac{1}{S}h_v, \\
U_{SS} &= -\frac{1}{S^2}h_v + \frac{1}{S}h_{vv}\frac{1}{S} = \frac{1}{S^2}(h_{vv} - h_v).
\end{aligned}
$$

Plugging the differentials into (2.10), the problem is transformed into

$$
\begin{cases}
[\lambda_x(T-t)+\rho]h \geq -h_t + \mu h_v + \frac{1}{2}\sigma^2(h_{vv}-h_v) + \lambda_x(T-t)u(D(e^v, T-t)) \\
h \geq V(B, T-t) \\
\{h - V(B,T-t)\}\times \\
\Big\{[\lambda_x(T-t)+\rho]h + h_t - \mu h_v - \frac{1}{2}\sigma^2(h_{vv}-h_v) - \lambda_x(T-t)u(D)\Big\} = 0
\end{cases}
\tag{2.11}
$$

Next we approximate the terms of partial differentials by finite differences on a regular mesh with step sizes Δt and Δv. We solve the problem within $v \in [v_1,\ v_2]$ and $t \in [0,T]$ where $v_1 \approx \ln(0)$ and $v_2 \approx \ln(\infty)$. Then intervals are truncated such that

$$v_1 = N_1\Delta v \leq v = n\Delta v \leq N_2\Delta v = v_2,$$

$$0 = M_1\Delta t \leq t = m\Delta t \leq M_2\Delta t = T.$$

We use the general finite-difference approximation

$$
\begin{aligned}
h_t &= \frac{h_n^{m+1} - h_n^m}{\Delta t} + O(\Delta t), \\
h_v &= \frac{h_{n+1}^{m+1} - h_n^{m+1}}{\Delta v} + O(\Delta v), \\
h_{vv} &= \frac{h_{n+1}^{m+1} - 2h_n^{m+1} + h_{n-1}^{m+1}}{(\Delta v)^2} + O((\Delta v)^2)
\end{aligned}
$$

where $h_n^m = h(n\Delta v, m\Delta t)$.

Let l_n^m denote the solution of the finite-difference approximation to the exact solution to h_n^m. The condition $h(v,t) \geq V(B, T-t)$ implies that

$$l_n^m \geq V_n^m \quad \text{for } m \geq 1 \tag{2.12}$$

where $V_n^m = V(B(e^{n\Delta v}, T - m\Delta t), T - m\Delta t)$.

We want that $U(S,\tau) \geq V(B(S,\tau),\tau)$. By definition of these two optimal value functions, they must satisfy three conditions

$$\begin{aligned} U(S,T) &= V(B(S,T),T) = h(v,0), \\ U(0,\tau) &= V(B(0,\tau),\tau) = h(-\infty,t), \\ U(\infty,\tau) &= V(B(\infty,\tau),\tau) = h(\infty,t). \end{aligned} \tag{2.13}$$

Thus the boundary and initial conditions for l are

$$l_n^0 = V_n^0, \quad l_{N_1}^m = V_{N_1}^m, \quad l_{N_2}^m = V_{N_2}^m. \tag{2.14}$$

Using the finite-difference approximation, the other condition $[\lambda_x(T-t)+\rho]h \geq -h_t + \mu h_v + \frac{1}{2}\sigma^2(h_{vv} - h_v) + \lambda_x(T-t)u(D(e^v, T-t))$ is approximated by

$$\begin{aligned} &-l_n^m + l_n^{m+1}[\Delta t(\lambda_x(T-(m+1)\Delta t)+\rho) + 1 + \frac{\Delta t}{\Delta v}(\mu - \frac{1}{2}\sigma^2) + \alpha\sigma^2] \\ &-l_{n+1}^{m+1}[\frac{\Delta t}{\Delta v}(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\alpha\sigma^2] - l_{n-1}^{m+1}[\frac{1}{2}\alpha\sigma^2] \\ &-\Delta t\lambda_x(T-(m+1)\Delta t)u(D(e^{n\Delta v}, T-(m+1)\Delta t) \geq 0, \end{aligned} \tag{2.15}$$

where $\alpha = \dfrac{\Delta t}{(\Delta v)^2}$.

Let $\mathbf{l}^m$ denote the vector of approximate values at time-step $m\Delta t$ and $\mathbf{V}^m$ be the vector representing the constraint at the same time:

$$\mathbf{l}^m = \begin{pmatrix} l^m_{N_1+1} \\ \vdots \\ l^m_{N_2-1} \end{pmatrix}, \quad \mathbf{V}^m = \begin{pmatrix} V^m_{N_1+1} \\ \vdots \\ V^m_{N_2-1} \end{pmatrix}.$$

The terms $l^m_{N_1}$ and $l^m_{N_2}$ are not included because they are determined explicitly by conditions (2.14). We further define

$$\mathbf{b}^m = \begin{pmatrix} b^m_{N_1+1} \\ \vdots \\ b^m_{N_2-1} \end{pmatrix},$$

where b^m_n is defined as

$$b^m_n = l^m_n + \Delta t \lambda_x (T - (m+1)\Delta t) u(D(e^{n\Delta v}, T - (m+1)\Delta t). \tag{2.16}$$

Let d^0, d^{+1} and d^{-1} represent the coefficients of l^m_n, l^m_{n+1} and l^m_{n-1} correspondingly:

$$\begin{aligned} d^0 &= \Delta t(\lambda_x(T-(m+1)\Delta t) + \rho) + 1 + \frac{\Delta t}{\Delta v}(\mu - \frac{1}{2}\sigma^2) + \alpha\sigma^2, \\ d^{+1} &= -[\frac{\Delta t}{\Delta v}(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}\alpha\sigma^2], \\ d^{-1} &= -\frac{1}{2}\alpha\sigma^2, \end{aligned}$$

we then have the coefficient matrix for $\mathbf{l}^m$

$$\mathbf{C} = \begin{pmatrix} d^0 & d^{+1} & 0 & \cdots & 0 \\ d^{-1} & d^0 & d^{+1} & \cdots & 0 \\ 0 & d^{-1} & d^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & d^{-1} & d^0 \end{pmatrix}, \tag{2.17}$$

which is a $(N_2 - N_1 - 2)$-square tridiagonal matrix. The Problem (2.11) can be rewritten into matrix form

$$\mathbf{C}\mathbf{l}^{m+1} - \mathbf{b}^m \geq 0, \quad \mathbf{l}^{m+1} - \mathbf{V}^{m+1} \geq 0, \quad (\mathbf{l}^{m+1} - \mathbf{V}^{m+1})(\mathbf{C}\mathbf{l}^{m+1} - \mathbf{b}^m) = 0. \quad (2.18)$$

This is the general form of the constrained matrix problem. To solve it, we apply the Projected SOR algorithm in the Appendix. More specifically, we choose the parameters as $\Delta v = 0.01$, $\Delta t = 0.1$, $\omega = 1.8$ and the tolerance level $\epsilon = 0.00001$.

2.3 Numerical Illustration

In this section, we consider a policyholder that invests in a perpetual equity-linked life insurance. We assume that he is aged 50 and his force of mortality follows the Gompetz (exponential) model:

$$\lambda_x(t) = \frac{1}{b} \exp\left(\frac{x+t-m}{b}\right), \quad (2.19)$$

with $b = 9$ and $m = 90$ as in Moore (2009). His expected future lifetime is 35.3 years. The equity-linked insurance products specify the surrender benefit and death benefit in the contract. They often include guaranteed minimum benefits. In this section, we follow Bernard and Lemieux (2008) and assume the following death benefit function:

$$D(S(\tau_d), \tau_d) = dw_0 \max\left((1+\tilde{g})^{\tau_d}, \frac{S(\tau_d)}{S(0)}\right), \quad (2.20)$$

assuming the participation rate is 1. Here $\tilde{g}$ is the guaranteed minimum growth rate. When the policyholder dies he receives the higher of the guaranteed rate and stock price growth rate on a proportion d of the premium w_0. This could be used to model the death benefit the investor receives at the time of death while the contract is not terminated.

Also assume that the surrender benefit function is given by

$$B(S(\tau_s), \tau_s) = sw_0 \max\left((1+g)^{\tau_s}, (1-f(\tau_s))\frac{S(\tau_s)}{S(0)},\right) \tag{2.21}$$

where g is the guaranteed minimum growth rate for surrender benefits which is usually equal to $\tilde{g}$. The surrender benefit that the investor receives at the time of surrender is the higher of the guaranteed rate and the return of the stock, less a penalty on a proportion s of the premium w_0. The function $f(\tau_s)$ is a time dependent penalty, or the surrender charge. When death occurs before surrender, the penalty is waived. The parameters s, d, g and $\tilde{g}$ are specified in the contract.

We choose the parameters r, μ, σ, and γ as in the base scenario of Section 1.4.2. In addition, the base scenario has the following parameters, which are similar to the base scenario studied in Moore (2009).

- The proportion of the initial premium returned upon death $d = 1$;
- The proportion of the initial premium returned upon surrender $s = 1$;
- The guaranteed minimum return rate upon surrender and death $g = \tilde{g} = 0.03$;
- The subjective discount rate $\rho = 0.04$;
- The initial premium invested $w_0 = 1$;
- The initial stock price $S(0) = 1$;
- The surrender charge $f(\tau_s) = 0$.

Under the above assumptions, the contract payoff at the time of surrender is

$$B(S(\tau_s), \tau_s) = \max\left(1.03^{\tau_s}, S(\tau_s)\right) \tag{2.22}$$

and the death benefit is

$$D(S(\tau_d), \tau_d) = \max\left(1.03^{\tau_d}, S(\tau_d)\right). \tag{2.23}$$

As explained at the end of Section 2.2.2, in order to study the surrender boundaries we need to first find the optimal value function after surrender $V(S, \tau)$, and further solve (2.10) for the optimal value function before surrender $U(S, \tau)$. Since by our assumption $V(S, \tau) = \xi(\tau)u(S)$, the first step is to calculate $\xi(\tau)$ numerically by solving (1.23) or calculating the expectation in (1.29).

Fig. 2.1 shows the optimal value function U (solid) with contract and the constraint function $V \circ B$ (dashed) without contract for $t = 5, 10, 15$ and 20. The value of U must be no less than the value of $V \circ B$, which is the optimal discounted terminal utility given the surrender benefit as initial wealth. The surrender boundaries S^l and S^u are the points at which the inequality in $U \geq V \circ B$ switches to equality .For example, when $t = 5$, the lower boundary $S^l \approx 1.1388$ and the upper boundary $S^u \approx 1.6000$. For $S \in (S^l, S^u)$, the expected discounted utility from holding the equity-linked insurance exceeds the expected discounted utility from surrendering the contract. The investor should not surrender the contract when the stock price is between S^l and S^u. When out of the non-surrender region, the investor should surrender the contract.

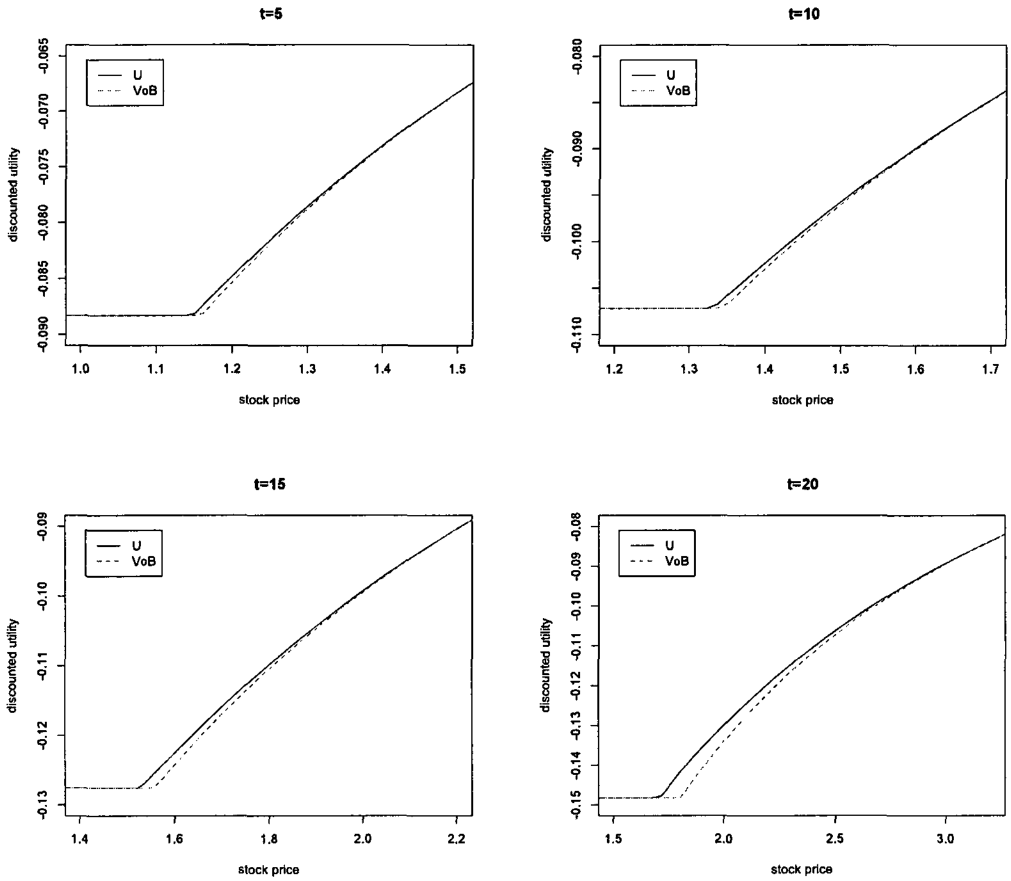

Figure 2.1: The value function U and the constraint function $V \circ B$

Fig. 2.2 shows the evolution of the surrender boundaries for $t \in (0, 30)$. The upper surrender boundary increases from $S^u \approx 1.39$ at time $t = 0$ to $S^u \approx 4.18$ at time $t = 30$. This means that the investor is less likely to surrender at high stock prices in the long run. The lower surrender boundary increases from $S^l \approx 0.99$ at time $t = 0$ to $S^l \approx 2.16$ at $t = 30$. The investor is less likely to surrender at low stock prices in the short term. However, note that the lower surrender boundary starts from 0.99 which is very close to the initial stock price $S(0) = 1$, there's a great chance that the

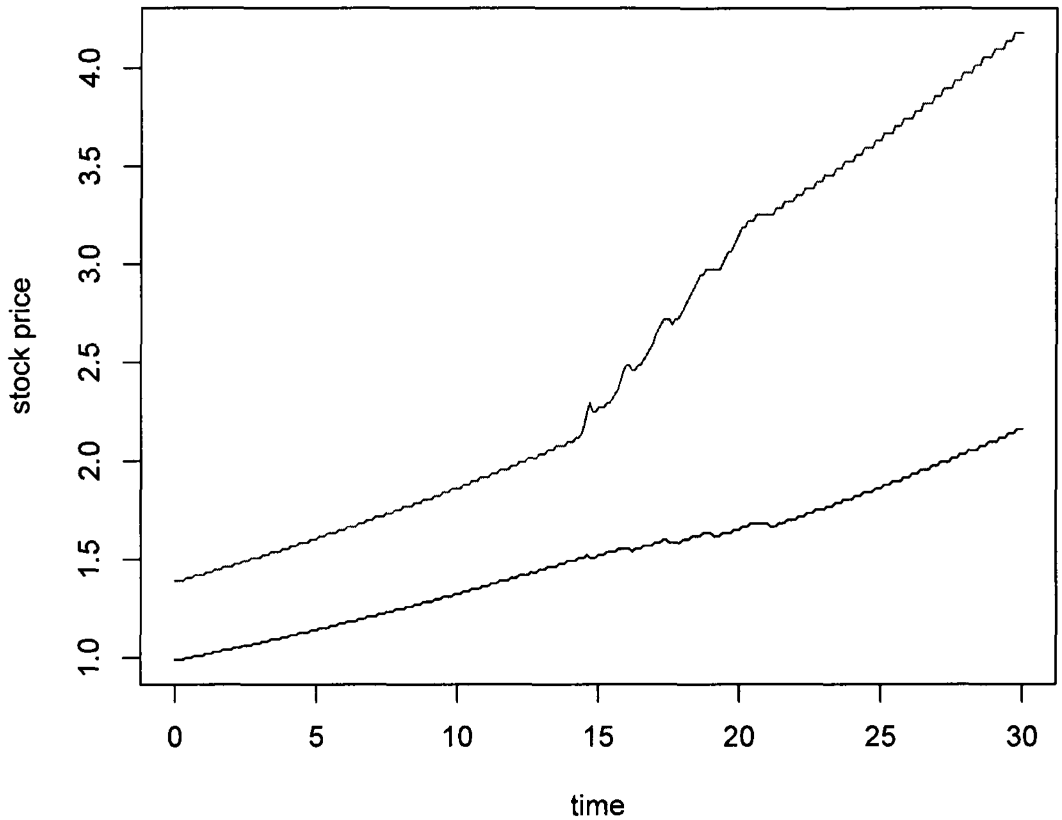

Figure 2.2: The evolution of the free boundary for the base scenario

surrender occurs very soon.

In the examples that follow, we investigate the impact of changing product features. We vary the parameters and then compare them with the above base scenario.

Example 2.3.1. *Impact of changing the relative risk aversion,* $\gamma = 2$ *vs.* $\gamma = 1.2$ *vs.* $\gamma = 4$ *(Fig. 2.3).*

The change in the relative risk aversion γ influences the surrender boundaries by

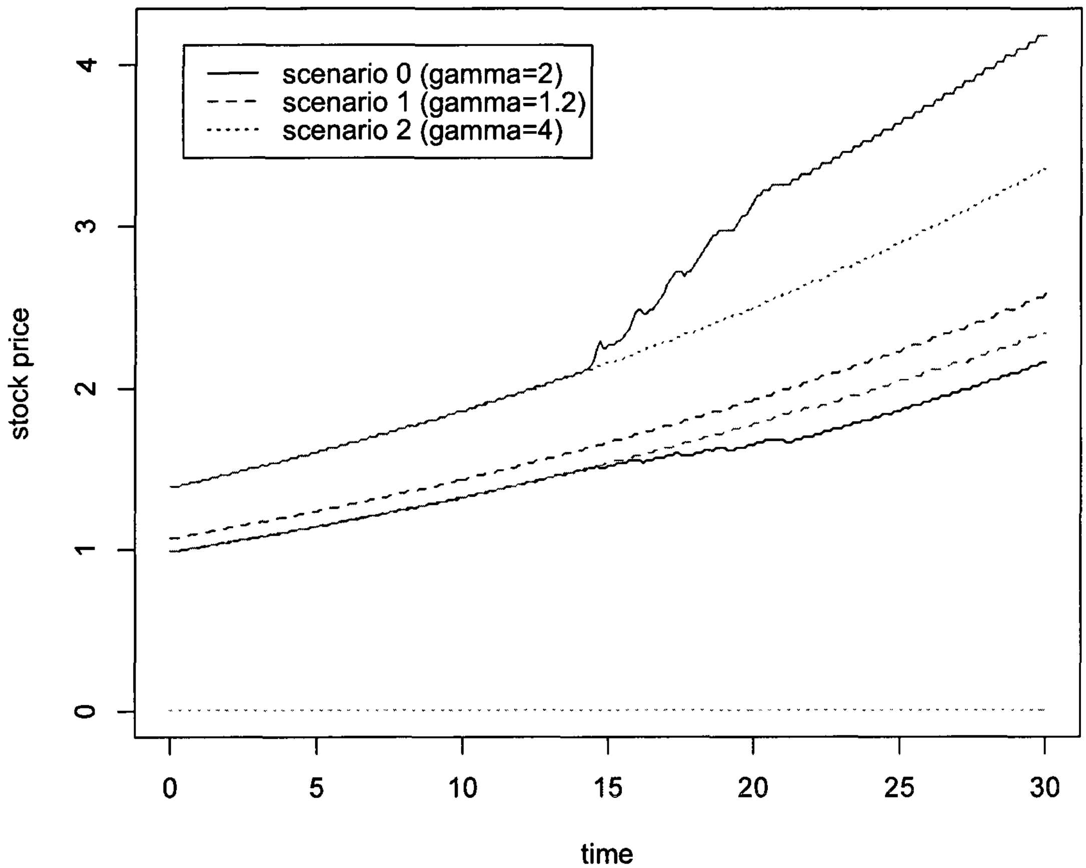

Figure 2.3: The impact of γ on surrender boundaries

the way of $\xi(\tau)$. We consider the following three scenarios:

- Scenario 0: Base scenario: $\gamma = 2$;
- Scenario 1: $\gamma = 1.2$;
- Scenario 2: $\gamma = 4$.

In Fig. 2.3, we see that in a common financial market, everything else being equal, the investor with higher RRA is more likely to surrender at lower stock prices, especially on the long run. For an investor with $\gamma = 4$, there is not even a lower threshold.

Since he is very reluctant to expose to risk, when the market goes down he can always enjoy the guaranteed minimum return provided by the equity-linked product. In this way the investor is protected from downward markets.

For the investor with the lowest RRA, the no-surrender boundary is the narrowest at all times, although the region tends to widen over time. This result again reflects the fact that the investors with lower RRA are more willing to be exposed to risk.

Example 2.3.2. *Impact of changing the guaranteed minimum growth rates (Fig. 2.4).*

In this example, we investigate the impact of changing the guaranteed minimum growth rate g and $\tilde{g}$. Fig. 2.4 contrasts the surrender boundaries for the following three scenarios:

- Scenario 0: Base scenario: $g = \tilde{g} = 3\%$;
- Scenario 1: $g = \tilde{g} = 0\%$;
- Scenario 2: $g = \tilde{g} = 4\%$.

In Fig. 2.4, we can see that although the boundaries have different trends for three scenarios, the boundaries at $t = 0$ are the same. Both the upper and lower boundary decrease in this scenario, though there is a small increase in the upper threshold between the eighth and tenth year. When the minimum guaranteed growth rate is non-zero, both the upper and lower boundaries increase over time. The higher the guaranteed rate, the faster the boundaries increase. High stock prices are required to give up the high minimum guaranteed rate.

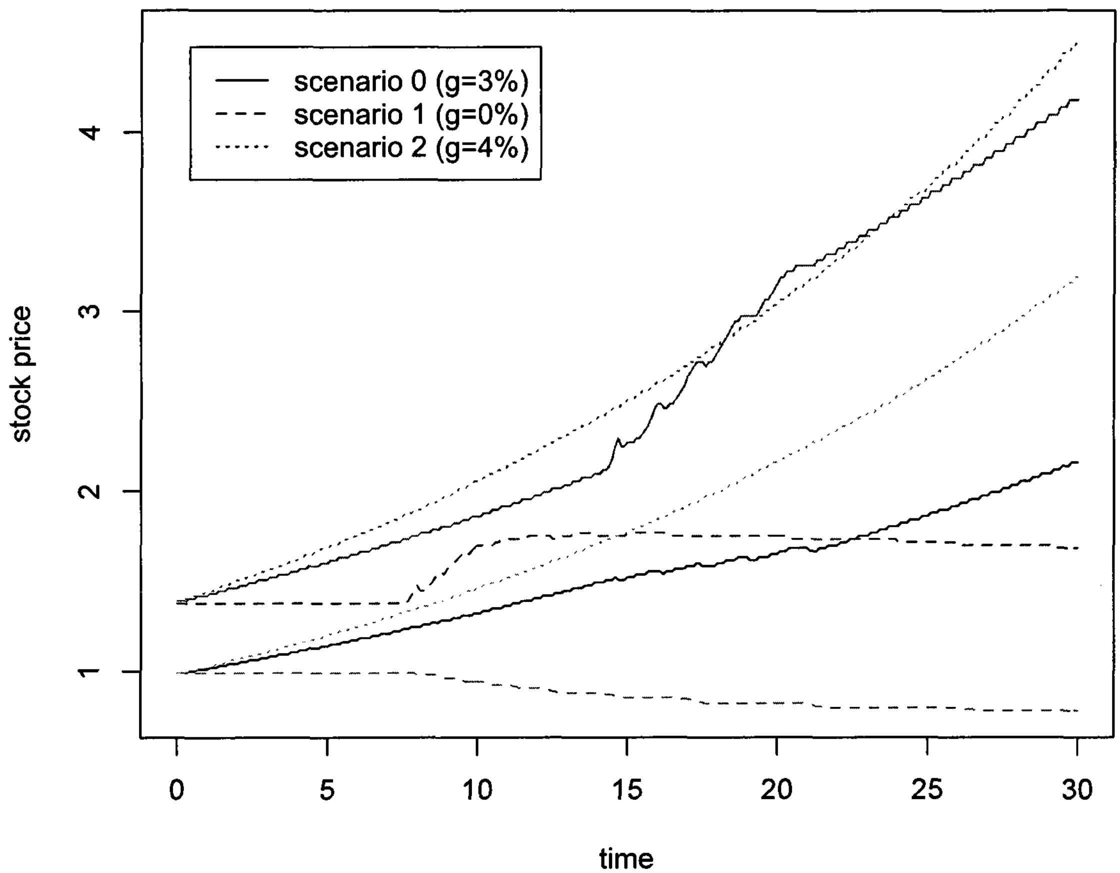

Figure 2.4: The impact of guaranteed minimum growth rate on surrender boundaries

Example 2.3.3. *Impact of changing the surrender charge (Fig. 2.5).*

In the base scenario, we did not consider any penalty upon surrender. However, there is always a surrender charge for real equity-linked products with surrender options. In this example we examine the impact of the surrender charge on the free

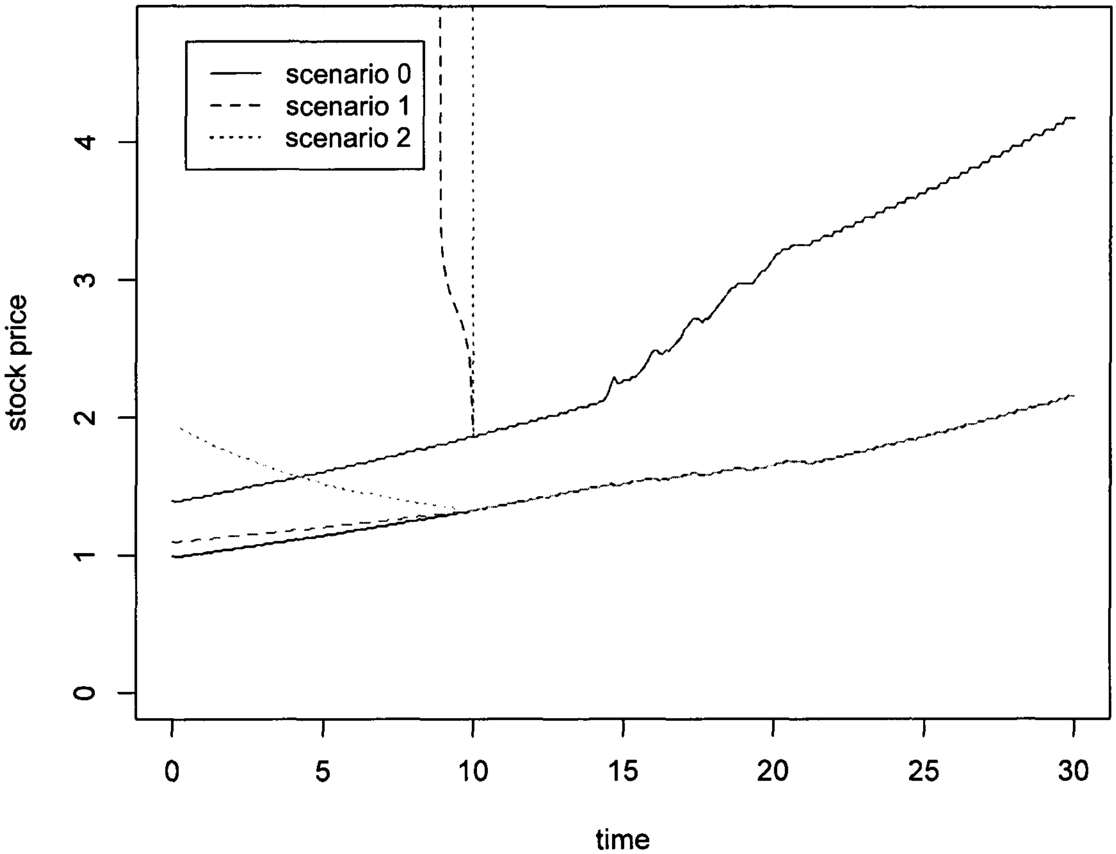

Figure 2.5: The impact of surrender charge on surrender boundaries

boundaries under two scenarios:

- Scenario 0: Base scenario: $f(\tau_s) = 0$;
- Scenario 1: $f(\tau_s) = \begin{cases} 0.005(10-\tau_s) & 0 \le \tau_s \le 10; \\ 0 & \tau_s > 10. \end{cases}$
- Scenario 2: $f(\tau_s) = \begin{cases} 0.01(10-\tau_s) & 0 \le \tau_s \le 10; \\ 0 & \tau_s > 10. \end{cases}$

In Scenario 1, the surrender charge starts at 1% and reduces linearly to 0 in ten years. In Scenario 2 the initial surrender charge is 5% and also reduces to 0 in ten years. The incorporation of the surrender charge influences the surrender boundaries by the way of the surrender benefit.

Fig. 2.5 shows the lower and upper surrender boundaries for these scenarios. We see that they have the same boundaries after the tenth year because the surrender charge only applies for the first ten years. During the first ten years when the penalty is applicable, the surrender boundaries are higher in order to compensate the surrender charge. The higher the penalty is the higher the boundaries are. Moreover, there is no upper surrender threshold in Scenarios 1 and 2, during the first eight years. The investor will not surrender the contract regardless how high the stock price goes, while the surrender charge is also high. Finite upper surrender boundaries appear between the eighth and ninth year in Scenario 1 and close to the end of the tenth year in Scenario 2.

Chapter 3

Equity-Indexed Annuity

3.1 Introduction

An equity-indexed annuity (EIA) is an investment product that is linked to financial markets. Its return is based on the changes of an underlying equity index (*e.g.* S&P 500). Moreover, an equity-indexed annuity provides a guaranteed rate such that the investor receives a minimum guaranteed payments, even if the financial market is not performing well. By offering a minimum guaranteed return, an equity-linked product provides investors with downside protection as well as potential upside return. Because of the secured returns and their strong ties with financial markets, equity linked insurance products are becoming more and more popular with investors.

The key features of EIA contracts usually includes an indexing method. The indexing method is how the return is measured. There are two commonly used indexing classes: annual reset and point-to-point. In this thesis, we use the point-to-point method. With this method, the interest rate used is based on the difference between the equity index value at the end and the beginning of the term and it is credited at maturity.

The rate of return used to credit interest is proportional to the equity index return μ. The ratio is called the participation rate, usually denoted by p. Although the participation rate can be any number between 0 and 100%, set by the issuing company, in most cases it is set to 100% making the crediting rate of return equal to the return of the linked equity index.

In addition to the indexing method and the participation rate, EIA contracts can also specify features like a minimum guaranteed rate of return, a maximum rate of return, usually called a cap, and a spread which is a maintenance fee.

Upon entering an EIA contract, the investor is offered with the option to surrender the contract anytime before maturity. If he chooses to surrender the contract, he will receive a surrender benefit linked to the value of the EIA contract, less a surrender charge.

In Chapter 2 we considered the perpetual equity-linked product which the investor may surrender at any time or continue with the contract indefinitely. Although it provides qualitative behavior of the surrender boundaries, it is not a realistic product. In reality, most products have a fixed maturity date. In this section, we will consider an EIA product with fixed maturity date. The analysis can be adapted from the derivation in Chapter 2.

3.2 Value Functions and Surrender Boundaries

Throughout the investment horizon, the investor seeks to maximize the expectation of the present value of his utility at the time of death or at maturity. After surrender, the investor starts a new investment with the surrender benefits, in this case his optimal value function is given as the function V defined in the Chapter 1. However prior to surrender, the investor must decide at every instant $t < \tau_d$ whether to surrender the contract or keep his position. If he surrenders, say at time τ_s, he receives the surrender benefit and then continues with the optimal investment strategy $\hat{\pi}_t$ to obtain the optimal value function V. Otherwise, if he dies before surrender he receives the death benefit.

Consider an investor aged at x with an initial wealth $\tilde{w}_0$. We assume that his utility function follows the constant relative risk aversion model

$$u(w) = \frac{w^{1-\gamma}}{1-\gamma}, \tag{3.1}$$

where γ is the coefficient of the constant relative risk aversion.

Suppose that the investor pays $\tilde{w}_0$ for the EIA contract which earns participation rate p of the stock return rate. The net premium is

$$w_0 = (1 - f_0)\tilde{w}_0, \tag{3.2}$$

where f_0 is the initial spread fee charged at the beginning of the contract. Most of the time, the initial fee is waived, *i.e.* f_0=0. In addition to f_0, the investor is also required to pay the maintenance fee f_a each year. This spread is assumed to be collected continuously. Let A_t denote the value in the EIA account which is also the total wealth of the investor at time t. Then the value in the EIA follows a stochastic process $\{A(t) : t \geqslant 0\}$

$$\begin{cases} \dfrac{dA(t)}{A(t)} = p\dfrac{dS(t)}{S(t)} - f_a dt \\ A(0) = w_0. \end{cases} \tag{3.3}$$

Equation (3.3)shows that the rate of return of the EIA contract is p times the rate of return of the stock less the annual maintenance fee, assuming that the annual maintenance fee is paid continuously.

The stock price is assumed to follow a geometric Brownian motion as described in equation (1.1) where the constants μ and σ are the expected return and volatility of the stock. Thus, reorganizing (3.3), we have the wealth process

$$\begin{cases} dA(t) = (p\mu - f_a)A(t)dt + p\sigma A(t)dB(t) \\ A(0) = w_0. \end{cases} \tag{3.4}$$

Note that the wealth in the EIA contract is also a geometric Brownian motion with drift $p\mu - f_a$ and volatility $p\sigma$.

We consider a fixed maturity EIA instead of perpetual products. Suppose that the investor enters the contract at time 0 and the maturity date of the contract is T. Before maturity, the investor can choose to surrender the contract and receives the surrender benefit

$$B(A(\tau_s), \tau_s) = \max\{s(1+g)^{\tau_s} w_0, (1 - f(\tau_s))A(\tau_s)\} \tag{3.5}$$

at the time of surrender. The investor is forced to surrender at maturity.

If death occurs before the expiration, the contract pays the death benefit

$$D(A(\tau_d), \tau_d) = \max\{d(1+\tilde{g})^{\tau_d} w_0, A(\tau_d)\}. \tag{3.6}$$

The parameters s, d, g, and $\tilde{g}$ are the same as in Chapter 2. The function f is the surrender penalty charged at the time of surrender.

With the maturity fixed at T, the investor seeks to maximize the expected present value of his utility at maturity. Then the optimal value function without EIA is

$$\bar{V}(w, \tau) = \sup_{\pi(t) \in \mathcal{A}} E[e^{-\rho((\tau_d \wedge T) - \tau)} u(W^{\pi}(\tau_d \wedge T)) \mid W(\tau) = w], \quad \tau \leq T, \tag{3.7}$$

which is the same with (1.6) except that the termination time is the maturity T or the time of death τ_d, whichever comes first. The time interval is subdivided into two intervals: $[\tau, \tau + h]$ and $(\tau + h, T]$. Then $\bar{V}(w, \tau)$ satisfies

$$\begin{aligned} \bar{V}(w, \tau) \geq \ & E[I_{\{\tau_d > \tau + h\}} e^{-\rho h} \bar{V}(W(\tau + h), \tau + h) \\ & + I_{\{\tau_d \leq \tau + h\}} e^{-\rho(\tau_d - \tau)} u(W(\tau_d)) \mid W(\tau) = w], \end{aligned} \tag{3.8}$$

which is exactly the same inequality for $V(w,\tau)$ on perpetual horizon. Therefore the function $\bar{V}(w,\tau)$ solves the Hamilton-Jacobi-Bellman equation (1.7) and has the optimal strategy (1.24).

Substituting the optimal wealth process (1.28) into the definition of $\bar{V}(w,\tau)$, we have

$$
\begin{aligned}
\bar{V}(w,\tau) =\ & E[e^{-\rho((\tau_d\wedge T)-\tau)}u(W^{\hat{\pi}}(\tau_d\wedge T)) \mid W(\tau)=w] \\
=\ & E\{\frac{e^{-\rho((\tau_d\wedge T)-\tau)}}{1-\gamma}[w\exp\{(r+\tfrac{(\mu-r)^2}{\gamma\sigma^2}-\tfrac{1}{2}\tfrac{(\mu-r)^2}{\gamma^2\sigma^2})((\tau_d\wedge T)-\tau) \\
& +\tfrac{\mu-r}{\gamma\sigma}B((\tau_d\wedge T)-\tau)\}]^{1-\gamma}\} \\
=\ & \frac{w^{1-\gamma}}{1-\gamma}E[e^{-\rho((\tau_d\wedge T)-\tau)}e^{[(r+\frac{(\mu-r)^2}{\gamma\sigma^2}-\frac{1}{2}\frac{(\mu-r)^2}{\gamma^2\sigma^2})((\tau_d\wedge T)-\tau)+\frac{\mu-r}{\gamma\sigma}B((\tau_d\wedge T)-\tau)](1-\gamma)}] \\
=\ & u(w)E[e^{-\rho((\tau_d\wedge T)-\tau)}e^{[(r+\frac{(\mu-r)^2}{\gamma\sigma^2}-\frac{1}{2}\frac{(\mu-r)^2}{\gamma^2\sigma^2})((\tau_d\wedge T)-\tau)+\frac{\mu-r}{\gamma\sigma}B((\tau_d\wedge T)-\tau)](1-\gamma)}] \\
=\ & u(w)\bar{\xi}(\tau).
\end{aligned}
\tag{3.9}
$$

Then the function $\bar{\xi}$ can be found by calculating the expectation

$$
\bar{\xi}(\tau) = E[e^{-\rho((\tau_d\wedge T)-\tau)}e^{[(r+\frac{(\mu-r)^2}{\gamma\sigma^2}-\frac{1}{2}\frac{(\mu-r)^2}{\gamma^2\sigma^2})((\tau_d\wedge T)-\tau)+\frac{\mu-r}{\gamma\sigma}B((\tau_d\wedge T)-\tau)](1-\gamma)}], \tag{3.10}
$$

the same as (1.29).Moreover, as discussed in Chapter 1 we can also solve the function $\bar{\xi}$ from the equation (1.23).

Now that we know how the policyholder would invest and his optimal value function after surrender, we can study his value function before surrendering the contract and see the optimal surrender time.

The optimal value function with insurance risk is defined as

$$\begin{aligned}\bar{U}(w,\tau) = \sup\nolimits_{0<\tau_s<\tau_d} E[\ & I_{\{\tau_s<\tau_d\wedge T\}} e^{-\rho(\tau_s-\tau)} \bar{V}(B(A(\tau_s),\tau_s),\tau_s) \\ & + I_{\{\tau_s>\tau_d,\tau_d<T\}} e^{-\rho(\tau_d-\tau)} u(D(A(\tau_d),\tau_d)) \\ & + I_{\{\tau_s>T,\tau_d\leq T\}} e^{-\rho(T-\tau)} u(B(A(T),T)) \mid A(\tau)=w].\end{aligned} \tag{3.11}$$

Also consider two intervals $[\tau,\tau+h]$ and $(\tau+h,T]$ and let h be close to zero. Then we can find the inequality for $\bar{U}(w,\tau)$:

$$\begin{aligned}\bar{U}(w,\tau) \geq\ & E[I_{\{\tau_d>\tau+h\}} e^{-\rho h} \bar{U}(A(\tau+h),\tau+h) \\ & + I_{\{\tau_d\leq\tau+h\}} e^{-\rho(\tau_d-\tau)} u(D(A(\tau_d),\tau_d)) \mid A(\tau)=w]\end{aligned} \tag{3.12}$$

which is the same as with (2.4) except that $\bar{U}$ is a function of value in EIA product while U is a function of stock price.

Following the same approach as in Section 2.2.2 and employing the wealth process (3.4), the function $\bar{U}(w,\tau)$ satisfies the inequalities:

$$\begin{cases} [\lambda_x(\tau)+\rho]\bar{U} \geq \bar{U}_\tau + (p\mu - f_a)w\bar{U}_w + \frac{1}{2}p^2\sigma^2 w^2 \bar{U}_{ww} + \lambda_x(\tau)u(D(w,\tau)), \\ \bar{U}(w,\tau) \geq \bar{V}(B(w,\tau),\tau), \\ \{\bar{U} - \bar{V}(B,\tau)\} \\ \times \left\{[\lambda_x(\tau)+\rho]\bar{U} - \bar{U}_\tau - (p\mu - f_a)w\bar{U}_w - \frac{1}{2}p^2\sigma^2 w^2 \bar{U}_{ww} - \lambda_x(\tau)u(D)\right\} = 0 \end{cases} \tag{3.13}$$

If $\bar{U}(w,\tau) > \bar{V}(B(w,\tau),\tau)$, the expected utility from keeping the EIA contract exceeds the expected utility from surrendering. Thus the investor should not surrender. The surrender boundary is defined as the values of (w,τ), where the inequality in $\bar{U}(w,\tau) \geq \bar{V}(B(w,\tau),\tau)$ switches to equality.

The projected SOR algorithm presented in the Appendix will be used to solve the problem (3.13). First (3.13) needs to be rewritten in terms of matrices. This is done by discretization, using the finite-difference approximation.

Let $t = T - \tau$ and $v = \ln w$, where T is a fixed large number. Using the same approximations as in Section 2.2.3, the conditions of the problem are discretized into

$$\begin{aligned}
&-\bar{l}_n^m + \bar{l}_n^{m+1}[\Delta t(\lambda_x(T-t) + \rho + 1 + \frac{\Delta t}{\Delta v}(p\mu - f_a - \frac{1}{2}p^2\sigma^2) + p^2\sigma^2\alpha] \\
&-\bar{l}_{n+1}^{m+1}[\frac{\Delta t}{\Delta v}(p\mu - f_a + \frac{1}{2}p^2\sigma^2) - \frac{1}{2}p^2\sigma^2\alpha] - \bar{l}_{n-1}^{m+1}[\frac{1}{2}p^2\sigma^2\alpha] \\
&-\Delta t\lambda_x(T-(m+1)\Delta t)u(D(e^{n\Delta v},\ T-(m+1)\Delta t)) \geq 0; \qquad (3.14) \\
&\bar{l}_n^m \geq \bar{V}_n^m \quad \text{for } m \geq 1; \\
&\bar{l}_n^0 = \bar{V}_n^0, \quad \bar{l}_{N_1}^m = \bar{V}_{N_1}^m, \quad \bar{l}_{N_2}^m = \bar{V}_{N_2}^m.
\end{aligned}$$

The vectors are defined as

$$\bar{\mathbf{l}}^m = \begin{pmatrix} \bar{l}_{N_1+1}^m \\ \vdots \\ \bar{l}_{N_2-1}^m \end{pmatrix}, \quad \bar{\mathbf{V}}^m = \begin{pmatrix} \bar{V}_{N_1+1}^m \\ \vdots \\ \bar{V}_{N_2-1}^m \end{pmatrix}, \quad \bar{\mathbf{b}}^m = \begin{pmatrix} \bar{b}_{N_1+1}^m \\ \vdots \\ \bar{b}_{N_2-1}^m \end{pmatrix},$$

where b_n^m is defined as

$$\bar{b}_n^m = \bar{l}_n^m + \Delta t\lambda_x(T-(m+1)\Delta t)u(D(e^{n\Delta v},\ T-(m+1)\Delta t)). \qquad (3.15)$$

The coefficient of $\bar{\mathbf{l}}^m$ is a $(N_2 - N_1 - 2)$-square, tridiagonal matrix

$$\bar{\mathbf{C}} = \begin{pmatrix} \bar{d}^0 & \bar{d}^{+1} & 0 & \cdots & 0 \\ d^{-1} & \bar{d}^0 & d^{+1} & \cdots & 0 \\ 0 & \bar{d}^{-1} & \bar{d}^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{d}^{-1} & d^0 \end{pmatrix}, \qquad (3.16)$$

where

$$\bar{d}^0 = \Delta t(\lambda_x(T-t) + \rho + 1 + \frac{\Delta t}{\Delta v}(p\mu - f_a - \frac{1}{2}p^2\sigma^2) + p^2\sigma^2\alpha,$$
$$\bar{d}^{+1} = -[\frac{\Delta t}{\Delta v}(p\mu - f_a + \frac{1}{2}p^2\sigma^2) - \frac{1}{2}p^2\sigma^2\alpha],$$

and

$$\bar{d}^{-1} = -\frac{1}{2}p^2\sigma^2\alpha.$$

Now that problem (3.13) has a general matrix form:

$$\bar{\mathbf{C}}\bar{\mathbf{I}}^{m+1} - \bar{\mathbf{b}}^m \geq 0, \quad \bar{\mathbf{I}}^{m+1} - \bar{\mathbf{V}}^{m+1} \geq 0, \quad (\bar{\mathbf{I}}^{m+1} - \bar{\mathbf{V}}^{m+1})(\bar{\mathbf{C}}\mathbf{I}^{m+1} - \bar{\mathbf{b}}^m) = 0, \quad (3.17)$$

we can use the Projected SOR method in the Appendix to solve the problem. The parameters are chosen to be the same as in Section 2.2.3.

3.3 Numerical Illustration

In this section, we investigate the impact of the participation rate on surrender boundaries and compare the surrender boundaries of fixed maturity EIA products with those of the corresponding perpetual products. We choose $r, \mu, \sigma, \gamma, s, d, g, \tilde{g}$ and ρ, $\tilde{w}_0$ to take the same values as in the base scenario of Section 2.3 and the other parameters of the base scenario are as follows.

- The participation rate $p = 0.9$;
- The time of maturity $T = 10$;
- The initial spread is $f_0 = 0$;

- The annual maintenance fee is $f_a = 0$;
- The time dependent surrender charge is $f(\tau_s) = \begin{cases} 0.01(10-\tau_s) & 0 \leq \tau_s \leq 10; \\ 0 & \tau_s > 10. \end{cases}$

Under the above assumptions, the surrender benefit is given as

$$B(A(\tau_s), \tau_s) = \max(1.03^{\tau_s}, (1 - f(\tau_s))A(\tau_s)) \tag{3.18}$$

and the death benefit is

$$D(A(\tau_d), \tau_d) = \max(1.03^{\tau_d}, A(\tau_d)). \tag{3.19}$$

The value in the EIA contract follows the following dynamics

$$\begin{cases} dA(t) = 0.072W(t)dt + 0.36A(t)dB(t) \\ W(0) = 1. \end{cases} \tag{3.20}$$

Example 3.3.1. *Fixed maturity T=10 vs. perpetual EIA (Fig. 3.1).*

Fig. 3.1 shows the surrender boundaries for a 10-year product and the perpetual product. We see that the lower boundaries are close in early years. Near the maturity time $T = 10$, there is less uncertainty about the fund value and the mortality risk over the remaining horizon for the fixed maturity product. So the minimum guarantee and death benefit are less valuable for the fixed maturity EIA than for the perpetual product. Therefore the investor is more likely to surrender the fixed maturity product near the maturity date.

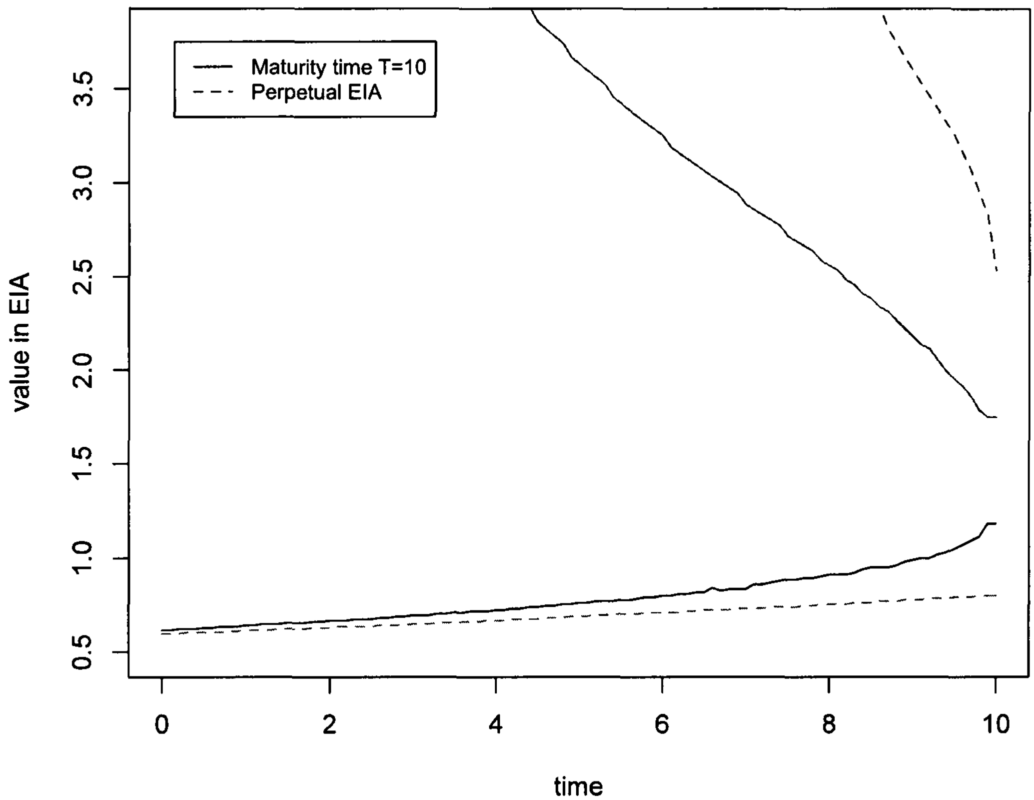

Figure 3.1: The surrender boundaries of a fixed maturity EIA with $T = 10$ and perpetual EIA

Example 3.3.2. *Impact of changing the participation rate (Fig. 3.2).*

In this example, we investigate the impact of changing the participation rate p. Fig 3.2 contrasts the boundaries for the following scenarios:

- Scenario 0: Base scenario: $p = 0.9$;
- Scenario 1: $p = 1$;
- Scenario 2: $p = 0.5$.

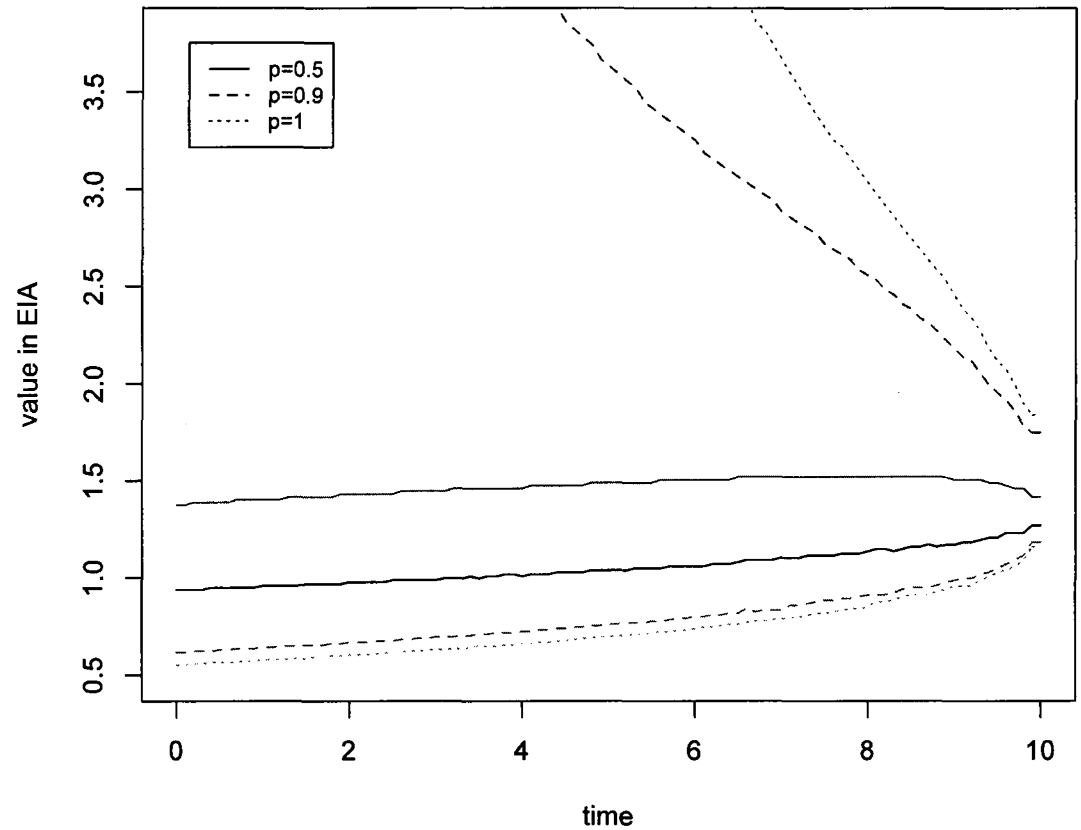

Figure 3.2: The impact of changing the participation rate

Fig. 3.2 shows the impact of different participation rates on the surrender boundary of a 10-year fixed maturity EIA product. It shows that the no-surrender region is wider when the participation rate is higher. With higher participation rates, the index growth contributes more to the EIA product, so the investor is more inclined to hold the contract. The no-surrender region narrows over time as the surrender charge decreases such that the investor becomes more likely to surrender the contract.

Example 3.3.3. *Impact of changing the annual fee (Fig. 3.3).*

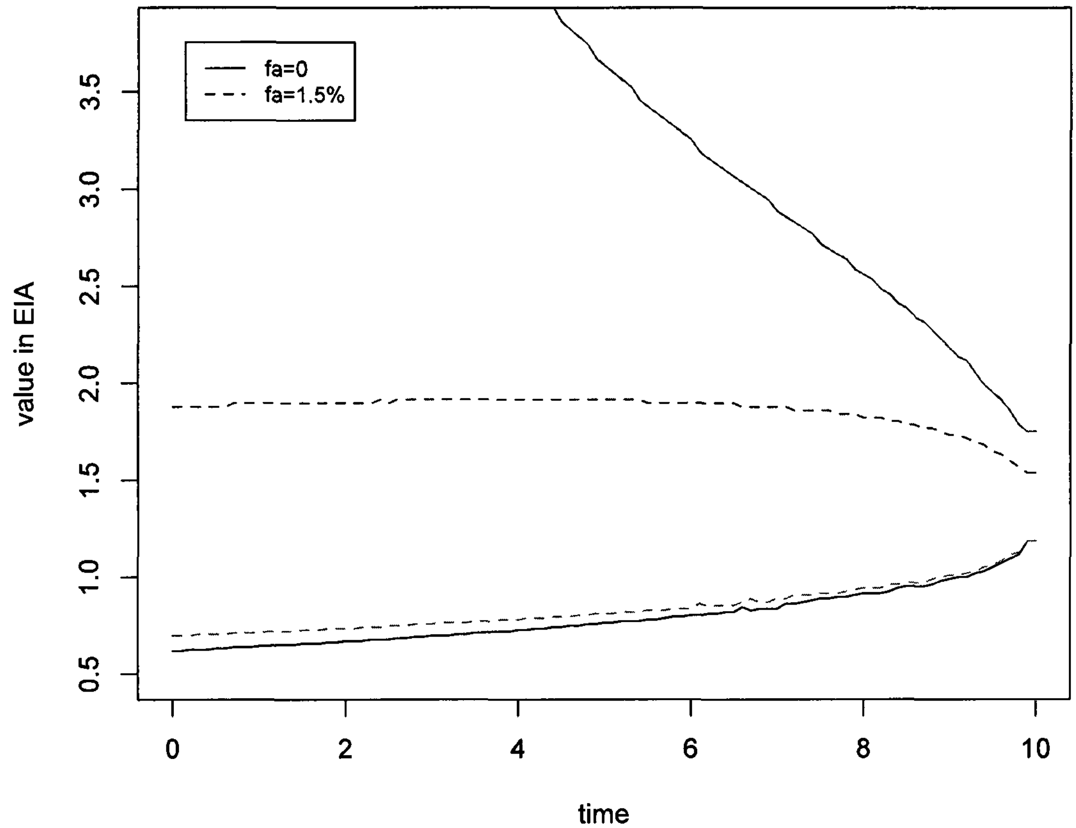

Figure 3.3: The impact of changing the annual fee

In this example, we examine the impact of changing the annual maintenance fee f_a. We consider the following two scenarios:

- Scenario 0: Base scenario: $f_a = 0$;
- Scenario 1: $f_a = 1.5\%$.

Fig. 3.3 contrasts the surrender boundaries with different annual maintenance fees. We see that the no-surrender region is wider when the annual fee is smaller. Higher annual fees imply higher costs of holding the EIA contract, thus the investor is more likely to surrender.

Example 3.3.4. *Impact of changing the initial spread (Fig. 3.4).*

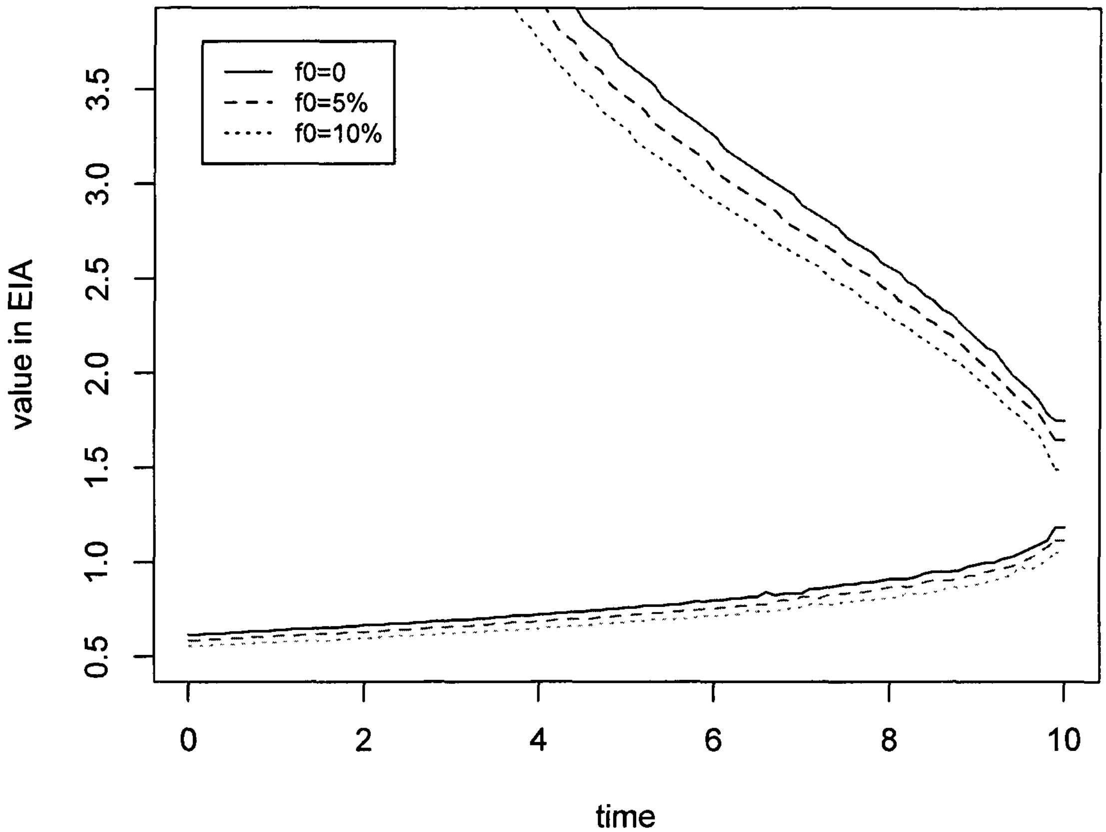

Figure 3.4: The impact of changing the initial spread

In this example, we examine the impact of changing the initial spread f_0. We consider the following three scenarios:

- Scenario 0: Base scenario: $f_0 = 0$;

- Scenario 1: $f_0 = 5\%$.
- Scenario 1: $f_0 = 10\%$.

The charge of the initial spread changes the effective initial investment only. Denote the initial investment by $\tilde{w}_0$. Then the effective initial investment is given by $w_0 = \tilde{w}_0(1 - f_0)$. The corresponding effective initial investments in the three scenarios are

- Scenario 0: $w_0 = 1$;
- Scenario 1: $w_0 = 0.95$;
- Scenario 2: $w_0 = 0.9$.

Fig. 3.4 contrasts the surrender boundaries with different initial spreads. Since the change in the effective initial investment does not affect the product itself, the surrender boundaries change with the effective initial investment. When the initial spread increases, the effective initial investment shrinks thus the surrender boundaries shrink respectively. In Fig. 3.4 we see that both the upper and lower thresholds shrink and the shrinkage reflects the rate of the initial spread.

Conclusion

From the figures presented in the thesis, we can see that for an investment in both risky and risk-free assets, the optimal investment strategy is to keep a fixed percentage of the total wealth in the risky asset. This maximizes his expected discounted utility at the time of death, which is called the value function. The ratio of the amount invested in the risky asset to the total wealth is a constant which depends on the financial market performance. The ratio is related to the return of the risk-free asset, the expected return and the volatility of the stock index, as well as the investor's coefficient of relative risk aversion.

When the expected return of the stock index rises or the volatility drops, the risky asset is more attractive to the investors. Thus the share invested in the risky asset increases and the total wealth has greater volatility. Similarly, when the return of the risk-free asset increases, the risky asset becomes less attractive to the investors so the share invested in the risky asset decreases as does the volatility of the total wealth. On the other hand, even if in the same market, different investors may have different optimal investment strategies. The optimal share invested in the risky asset of an investor with a higher coefficient of relative risk aversion is smaller than that of an investor with a lower coefficient of relative risk aversion. This is because higher coefficients of RRA imply more reluctance to risk exposure.

The idea of maximizing the value function can also be applied to determine the optimal surrender strategies for policyholders of equity-indexed insurance products. Given the optimal strategy the policyholder would take, he can find the optimal surrender boundary such that his expected discounted utility at his time of death is maximized. Between the upper and lower boundaries is the non-surrender region. When the stock price falls in the non-surrender region, the policyholder should not surrender the contract and stays invested in the contract. Otherwise, the investor should surrender the contract when the value of contract hits the boundaries. The non-surrender region changes over time depending on the product features and the policyholder's behavior pattern. The coefficient of RRA affects the surrender boundaries more on the long run. The increase in the surrender charge raises the surrender boundaries since the cost of surrendering increases.

Extending the above analysis to a more realistic product, the equity-indexed annuity (EIA), we obtain similar results. Instead of expiring at the time of death, the EIAs usually have fixed maturities. Thus the expiration time of EIAs is either the policyholder's time of death or the maturity set in the EIA contract, whichever comes first. The fixed maturity shrinks the non-surrender region such that the policyholder is more willing to surrender both at upper and lower thresholds. The participation rate represents how much of the stock index is to be credited to the investor. Higher participation rates imply higher rates of return when the market performs well, thus is more attractive to investors. In this case policyholders are less willing to surrender the contract and the non-surrender region for higher participation rates are wider than others. In addition to the value of the contract, the policyholders also need to pay annual fees and an initial spread. The charge of these fees reduces the attractiveness of the EIAs. Therefore the higher the fees are, the narrower the non-surrender region is. However, the initial and annual fees affect the boundaries in different ways. The initial fee shrinks the boundaries linearly since the initial fee only changes the effective initial investment. The shrinkage reflects the rate of the initial spread. The influence of the annul fees is greater at the beginning of the contract and decreases over the investment horizon.

For insurance companies, knowing when the policyholders would surrender the contract may help evaluate the present value of the EIAs and therefore effectively price and reserve for EIAs. However, since people have different coefficients of RRA and subjective discount factors, companies may need to identify customers' behavior patterns.

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Appendix A

The acronym SOR is short for Successive Over Relaxation which is a method used to solve certain classes of matrix equations. The projected SOR is a modified SOR method to solve constrained matrix problems.

Consider a general problem:

$$\mathbf{Ax} \geq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{c}, \quad (\mathbf{x} - \mathbf{c}) \cdot (\mathbf{Ax} - \mathbf{b}) = 0, \tag{.21}$$

where $\mathbf{A}$ is a given matrix and $\mathbf{b}$ and $\mathbf{c}$ are given vectors. Assuming only that the matrix $\mathbf{A}$ is invertible and that it is positive definite (*i.e.* $\mathbf{x} \cdot (\mathbf{Ax})$ for any $\mathbf{x} \neq 0$), then there is one and only one solution vector $\mathbf{x}$ for this problem.

The algorithm for finding the solution is iterative. Start with an initial guess $\mathbf{x}^0 \geq \mathbf{c}$ (the algorithm may not converge if $\mathbf{x}^0 < \mathbf{c}$). During each iteration we form a new vector

$$\mathbf{x}^{k+1} = (x_1^{k+1},\ x_2^{k+1},\ \ldots,\ x_n^{k+1}), \tag{.22}$$

from the current vector

$$\mathbf{x}^k = (x_1^k,\ x_2^k,\ ...,\ x_n^k), \tag{.23}$$

by the following two-step process. For each $i = 1, 2, ..., n$ we *sequentially* form the intermediate quantity y_i^{k+1}, given by

$$y_i^{k+1} = \frac{1}{A_{ii}}\left(b_i - \Sigma_{j=1}^{i-1} A_{ij} x_j^{k+1} - \Sigma_{j=i+1}^{n} A_{ij} x_j^k\right) \tag{.24}$$

and then define the new x_i^{k+1} to be

$$x_i^{k+1} = \max(c_i,\ x_i^k + \omega(y_i^{k+1} - x_i^k)). \tag{.25}$$

Note that it is important to perform these two steps in sequence; we need the new value of x_{i-1}^{k+1} in order to find x_i^{k+1}. The only difference between this method and the classical SOR method is the test to make sure that $x_i^{k+1} \geq c_i$. The constant ω is called a relaxation parameter, and provided that $\mathbf{x}^0 < \mathbf{c}$ and $0 < \omega < 2$, the method converges. (It can be shown that the convergence be optimized by choosing a particular value of $\omega \in (1,2)$ which depends on the matrix $\mathbf{A}$.)

At each iteration this defines a new vector $\mathbf{x}^{k+1} \geq \mathbf{c}$; as $k \to \infty$ $\mathbf{x}^k \to \mathbf{x}$, the solution of the problem. In practice, naturally enough, we do not iterate forever. We stop once we have satisfied a condition of the form

$$|\mathbf{x}^{k+1} - \mathbf{x}^k| < \epsilon \tag{.26}$$

where $\epsilon > 0$ is some pre-chosen small tolerance. We then take $\mathbf{x}^{k+1}$ as the solution.