Solutions of the Inverse Frobenius-Perron Problem

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Abstract

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The Frobenius-Perron operator describes the evolution of density functions in a dynamical system. Finding the fixed points of this operator is referred to as the Frobenius-Perron problem. This thesis discusses the inverse Frobenius-Perron problem (IFPP), which seeks the transformation that generates a prescribed invariant probability density. In particular, we present in detail five different ways of solving the IFPP, including approaches using conjugation and differential equation, and two matrix solutions. We also generalize Pingels method [27] to the case of two-pieces maps.

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Contents

Chapter 1

Introduction

For many dynamical systems, iterating a discrete-time map, for a given initial point x_0, results in a random-like trajectory. However, if we consider the density function which describes the probability of landing anywhere in the space, one can get a clear statistical understanding of the long term dynamics of the map. To do this, an operator that reflects how density functions evolve on iteration, is studied. When an invariant density is eventually reached, one knows where iterates end up on average. More specifically, if the initial density function on the space I is $f(x)$, for a map $\tau : I \to I$, the density ϕ under the action of τ is $\phi = P_\tau(f)$, where the operator P_τ is called the Frobenius-Perron operator (FPO), or transfer operator, corresponding to τ. Representing the statistical distribution of iterates, the invariant density is a fixed point of P_τ: $P_\tau(f) = f$. The existence of invariant densities for a class of chaotic point transformations has been proved by Lasota and Yorke [20]. FPO is a special class of Markov operators [21], and possesses nice properties such as linearity, positivity, preservation of integrals, etc [4]. The introduction of FPO opened a new area of study in dynamical systems and chaos research.

The FPO has a nice representation for piecewise monotonic and expanding τ, when its derivatives are not singular. In 1960, Ulam [36] conjectured that the FPO could be approximated by a Markov map defined on a partition of the given interval. To do this, the interval I is partitioned into m equal subintervals, $I_1, I_2, ..., I_m$. At stage n, we define a constant ρ_i on each subinterval such that the density function is a step function $f_n = \sum_{i=1}^{m} \rho_i \cdot \chi_{I_i}(x)$. Ulam hypothesized that the FPO of τ can be approximated by $f_{n+1} = Af_n$, where $A = \{a_{ij}\}$ is a $m \times m$ transition matrix with each element a_{ij} representing the probability of interval I_j being mapped into interval I_i:

$$a_{ij} = \frac{m[I_i \cap \tau^{-1}(I_j)]}{m(I_i)},$$

where $m(\cdot)$ denotes Lebesgue measure and τ^{-1} may have multiple branches. A is also called the Ulam matrix. This conjecture was proved by Li [22] in 1976, using bounded variation tools under the conditions that τ is piecewise in C^2 and $|\tau^{'}| > 2$. Li showed that as $n \to \infty$, f_{n+1} converges to the fixed point of P_τ in one-dimension. In 1991, Boyarsky and Lou [5] generalized the result to Jablonski transformations in n-dimensions. After the methods of Li, the result was also extended to higher-dimensional cases by Froyland [15], Ding and Zhou [11], Proppe *et al.* [28]. These results are seminal since it allows us to study a chaotic system by investigating the corresponding linear equation and associated non-negative matrix.

However, in many practical situations only stochastic data is observed, while the underlying dynamical system remains unknown. Thus, solving the Frobenius-Perron problem inversely plays an important role in real life. The inverse Frobenius-Perron problem (IFPP) concerns generating chaotic maps which give rise to a prescribed invariant density. By solving the IFPP, the interaction between the statistical behaviours and the dynamics becomes obvious. The IFPP is an active area of research and was considered by numerous groups.

An early solution was given by Grossmann and Thomae [17] in 1977. This method is known as the conjugation transformation approach and was used to construct symmetric maps with not only observed invariant densities but also their stationary correlation functions. It revealed the relationship between conjugate maps and their corresponding invariant densities. Later on, in 1982, Friedman and Boyarsky [14] offered a different solution with different conditions. They dealt with the situation that seeks ergodic transformations for classes of densities which are piecewise constant functions. Graph-theoretic methods were used and the class of density functions were required to take on the value zero at all relative minima points. This is restrictive and not easy to generalize. In 1990, a numerical algorithm was developed by Ershov and Malinetskii [13]. They provided a way to construct a unimodal transformation τ whose unique invariant density function is the targeted f. It requires us to compute the integral $\phi(x) = \int_x^1 f(y)dy$. The key to this method is the converting function $R(x)$, which maps the left preimage to the right one and vice versa. In 1991, Koga [19] proposed a differential equation approach for two special types of transformations on the unit interval: $\tau(1-x) = \tau(x)$ and $\tau(x+\frac{1}{2}) = \tau(x)$. Assuming the slope of τ is positive on $[0, \frac{1}{2}]$ and $\tau(0) = 0$, Koga derived an ordinary differential equation associated with τ and f for each type, so that τ could be found by solving the differential equation.

In order to completely generalize the result of [13], Gora and Boyarsky [16] introduced a special transformation named 3-band transformation in 1993. The 3-band transformation is a class of $\mathscr{P}$-semi-Markov piecewise linear transformation, where $\mathscr{P}$ denotes a partition of the interval considered. Using matrix analysis, they revealed the relation between the density function and the 3-band transformation τ. Thirteen years later, Gora generalized their 3-band matrix approach to the N-band case [1], and presented an application in finance. In 1999, Pingel, Schmelcher and Diakonos [27] established a general solution of IFPP for the class of maps that are unimodal, symmetric and each branch covers the whole interval. Quite similarly to [13], they approached the problem by considering the converting function h_τ (in [13] it is $R(x)$) and computing the integral $\mu(x) = \int_0^x f(x)dx$. Furthermore, they solved the inverse problem for maps with beta distributions as their invariant densities. The same year, this group used the parametrization of τ by h_τ as a starting point, to develop a Monte-Carlo optimization method based on the Metropolis algorithm [9], and used a stochastic method to construct a dynamical system with given time correlation. In 2004, Lozowski, Lysetskiy, and Zurada [23] used an optimization algorithm to determine each element of the transition matrix. In the same year, Rogers, Shorten and Heffernan [29] addressed IFPP based on the previously-mentioned Ulam's conjecture. They started with a matrix used in synchronised communication networks, which is also a column stochastic matrix and can be treated as Ulam's transition matrix. The formula for its principle eigenvector is known. Thus, by expressing the given density in the form of the leading eigenvector, one can determine the Ulam matix and hence the chaotic map. Later in 2008, based on this method, they studied [30] properties and the Lyapunov exponent and switching between chaotic maps. A relatively recent solution was given by Nie and Coca [26], assuming τ is an unknown semi-Markov transformation. By observing the experimental data from some stage, they first determined the partition $\mathscr{R}$, and assembled two consecutive densities F_0 and F_1 under one iteration. Then the Frobenius-Perron matrix M of τ satisfies the equation $F_1 = F_0 \cdot M$. After M is solved, one can translate it back to the piecewise linear map τ.

Besides solving IFPP directly, a control problem is also considered. In 2000, Bollt [2, 3] studied dynamical systems with some perturbations. More specifically, to produce a new dynamical system near a given one (in the sup-norm sense), but with remarkably different prescribed invariant density. Bollt proceeded by using the Penrose pseudoinverse and an open-loop perturbation approach. As discussed in [3],the existence of l^2 solutions cannot be determined, while in the space l^∞, Bollt proved a sharp theorem on the non-existence.

By the nature of the inverse Frobenius-Perron problem, IFP models can be applied in various fields. In biological systems [23], the authors constructed models of olfactory bulbs' temporal sequences with stationary interspike interval distribution. This gives a way to generate realizations of neural signals. It is also applied to the field of signal processing [34] and computer network [25].

The main goal of this thesis is to study the Inverse Frobenius-Perron Problem. The thesis is organized as follows. In Chapter 2, we introduce some relevant concepts of dynamical system including necessary theorems from measure theory and ergodic theory. In Chapter 3 we discuss the inverse Frobenius-Perron problem — its definition, the existence of the invariant density and some representations. In Chapter 4 some theoretical solutions in [17], [19] and [27] are presented on the inverse problem. Chapter 5 provides 3 different matrix approaches based on the work of [16], [29], and [1]. In Chapter 6 we draw our conclusions.

Chapter 2

Preliminaries

The results presented in this chapter are derived from the books of Lasota and Mackey (1994), Gora and Boyarsky (1997), Royden and Fitzpatrick (2010), Brucks and Bruin(2004).

2.1 Measure space

Let us consider a set X with a metric. It is usually assumed to be a compact metric space.

Definition 2.1.1. *A family $\mathscr{B}$ of subsets of a set X is a σ-algebra if:*

1. $X \in \mathscr{B}$;
2. *When* $B \in \mathscr{B}$ *then* $X/B \in \mathscr{B}$;
3. *Given a finite or infinite sequence* $\{B_k\}$ *of subsets of* X, $B_k \in \mathscr{B}$, *then* $\bigcup_k B_k \in \mathscr{B}$.

A σ-algebra of X is usually denoted as $\sigma(X)$.

Definition 2.1.2. *A function $\lambda : \mathscr{B} \to \mathbb{R}^+$ is called a measure on $\mathscr{B}$ if*

1. $\lambda(\emptyset) = 0$;

2. If $\{B_k\}$ is a finite or infinite sequence of pairwise disjoint sets from $\mathscr{B}$, then

$$\lambda\Big(\bigcup_k B_k\Big) = \sum_k \lambda(B_k).$$

Definition 2.1.3. *The triplet $(X, \mathscr{B}, \lambda)$ is called a measure space, where $\mathscr{B}$ and μ are defined above. The set B is said to be measurable if $B \in \mathscr{B}$. In particular, if $\lambda(X) = 1$, we say it is a normalized measure space, or probability space.*

Definition 2.1.4. *If there is a sequence $\{B_k\}, B_k \in \mathscr{B}$ satisfying*

$$X = \bigcup_{k=1}^{\infty} B_k \quad \text{and} \quad \lambda(B_k) < \infty \quad \text{for all } k,$$

then the measure space $(X, \mathscr{B}, \lambda)$ is called σ-finite.

Definition 2.1.5. *Let $\mathfrak{O}$ denote a family of open sets of X. Then the σ-algebra $\mathfrak{B} = \sigma(\mathfrak{O})$ is called the Borel σ-algebra of X. Its elements are called Borel subsets of X.*

If a property is true except for a subset having measure zero, then we say this property is true almost everywhere (abbreviated as a.e.). Next we define a relation between two measures.

Definition 2.1.6. *Let μ and λ be two measures on the same measurable space $(X, \mathscr{B})$. Then μ is said to be absolutely continuous with respect to λ provided the following holds:*

$$\text{if } E \in \mathscr{B} \text{ and } \lambda(E) = 0, \text{ then } \mu(E) = 0.$$

We write $\mu << \lambda$.

In this thesis, we focus our attention on Lebesgue measure, which we define as follows.

Definition 2.1.7. *Let I be a nonempty interval on the real line. Denote its length by $l(I)$. For a set of $A \subset \mathbb{R}$, consider a family $\mathscr{F}$ of countable coverings of A by open bounded intervals, i.e., a family of coverages $\{I_k\}$ for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$. Then the Lebesgue outer measure is defined by*

$$\lambda^*(A) = \inf_{\mathscr{F}} \Big\{ \sum_{k=1}^{\infty} l(I_k) \Big\}.$$

Definition 2.1.8. *A set E is said to be measurable if, for any set A,*

$$\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^C).$$

If E is measurable, then the Lebesgue measure of E is defined by its outer measure $m(E) = \lambda^*(E)$.

2.2 Lebesgue integration

Based on Lebesgue measure, we introduce a type of integration — the Lebesgue integral, which is more general than the commonly used Riemann integral. Lebesgue integral is an essential tool for Frobenius-Perron operator. So now we consider the real-valued function on the measure space $(X, \mathfrak{B}, \lambda)$.

Definition 2.2.1. *A function $f : X \to \mathbb{R}$ is measurable if $f^{-1}(\Delta)$ is measurable for any Borel set $\Delta \subset \mathbb{R}$, that is, $f^{-1}(\Delta) \in \mathfrak{B}$.*

In developing the concept of the Lebesgue integral, we will use characteristic functions and simple functions.

Definition 2.2.2. *For any set A, the characteristic function χ_A is defined by*

$$\chi_A(x) = \begin{cases} 1, & \text{if} \quad x \in A \\ 0, & \text{if} \quad x \notin A. \end{cases}$$

Definition 2.2.3. *A real-valued function ψ defined on a measurable set E is called simple if it is measurable and takes only a finite number of values. That is, there exists constants a_i, $i = 1, ..., n$ such that*

$$\psi(x) = \sum_{i=1}^{n} a_i \cdot \chi_{E_i}(x),$$

where E_i's are measurable sets.

We will first define the Lebesgue integral for simple functions, for bounded functions, non-negative functions, and finally define it in general.

Definition 2.2.4. *For a simple function ψ defined on a set E of finite measure, we define the integral of ψ as*

$$\int_E \psi d\lambda = \sum_{i=1}^{n} a_i \cdot \lambda(E_i).$$

Definition 2.2.5. *For a bounded real-valued f on a set E of finite measure, the lower and upper Lebesgue integral is defined respectively as*

$$\sup \left\{ \int_E \varphi d\lambda \middle| \varphi \text{ simple and } \varphi \leqslant f \text{ on } E \right\}$$

and

$$\inf \left\{ \int_E \psi d\lambda \middle| \psi \text{ simple and } \psi \geqslant f \text{ on } E \right\}$$

A function is said to be Lebesgue integrable if the lower and upper Lebesgue integrals have the same value, and this common value is called its Lebesgue integral denoted by $\int_E f d\lambda$.

Note that for a bounded function f defined on a closed interval $[a, b]$, if f is Riemann integrable over $[a, b]$, then it is Lebesgue integrable over $[a, b]$, and the two integrals share the same value.

Definition 2.2.6. *For a nonnegative measurable function f on E, we define the integral of f over E by*

$$\int_E f d\lambda = \sup \left\{ \int_E h d\lambda \middle| h \text{ bounded, measurable, of finite support and } 0 \leqslant h \leqslant f \right\}$$

Definition 2.2.7. *A nonnegative measurable function over a measurable set E is integrable if*

$$\int_E f d\lambda < \infty.$$

Definition 2.2.8. *For an extended real-valued function f, define*

$$f^+(x) = \max\{f(x), 0\} \quad \text{and} \quad f^-(x) = \max\{-f(x), 0\}$$

f is integrable if $|f| = f^+ + f^-$ is integrable, and

$$\int_E f d\lambda = \int_E f^+ d\lambda - \int_E f^- d\lambda.$$

For convenience, we introduce the following notation.

Definition 2.2.9. *Let $1 \leqslant p < \infty$. Then the collection of real-valued measurable functions (or rather a.e. -equivalence classes of them) $f : X \to \mathbb{R}$ satisfying*

$$\int_X |f|^p d\lambda < \infty$$

is called the $\mathscr{L}^p(X, \mathscr{B}, \lambda)$ space.

Then for Lebesgue measure λ, the space $\mathscr{L}^1(X, \mathscr{B}, \lambda)$ contains all Lebesgue integrable functions. It will be denoted by $\mathscr{L}^1(\lambda)$. The $\mathscr{L}^p$ norm of f is defined by

$$\| f \|_p = (\int_X |f(x)|^p d\mu)^{\frac{1}{p}}.$$

Definition 2.2.10. *Let $(X, \mathscr{B}, \mu)$ be a normalized measure space. Let*

$$\mathfrak{D} = \mathfrak{D}(X, \mathscr{B}, \mu) = \{f \in \mathscr{L}^1(X, \mathscr{B}, \mu) : f \geq 0 \text{ and } \| f \|_1 = 1\}$$

denote the space of probability density functions. A function $f \in \mathfrak{D}$ is called a density function or simply a density.

The Lebesgue integral has some nice properties such as linearity, monotonicity, additivity, etc. We state the following theorems.

Theorem 2.2.1. *[32] If $f, g \in \mathscr{L}^1(X, \mathscr{B}, \lambda)$, then for any α and β, $\int(\alpha f + \beta g)d\lambda$ is integrable and*

$$\int(\alpha f + \beta g)d\lambda = \alpha \int f d\lambda + \beta \int g d\lambda.$$

Furthermore, if $f \leqslant g$ then

$$\int f d\lambda \leqslant \int g d\lambda.$$

Theorem 2.2.2. *[32] Let f be a integrable function on E and $\{E_i\}_{i=1}^{\infty}$ be a countable disjoint measurable subsets of E and $E = \bigcup_{i=1}^{\infty} E_i$, then*

$$\int_E f d\lambda = \sum_{i=1}^{\infty} \int_{E_i} f d\lambda = \sum_{i=1}^{\infty} \int_E f \cdot \chi_{E_i} d\lambda.$$

Like the Riemann integral, the Lebesgue integral connects to differentiation, and the variable can be changed. These facilitate greatly our further analysis.

Theorem 2.2.3. *[32] If f is Lebesgue integrable on the closed interval $[a,b]$, then*

$$\frac{d}{dx}\int_a^x f d\lambda = f(x)$$

for almost all $x \in (a,b)$.

If $\mu << \lambda$, the Radon-Nikodym Theorem allow us to represent μ in terms of λ.

Theorem 2.2.4. *[32] Let $(X, \mathscr{B})$ be a measurable space and let μ and λ be two normalized measures on $(X, \mathscr{B})$. If $\mu << \lambda$, then there exists a unique element $f \in \mathscr{L}^1(X, \mathscr{B}, \lambda)$ such that for every $A \in \mathscr{B}$*

$$\mu(A) = \int_A f d\lambda.$$

f is called the Radon-Nikodym derivative and is denoted by $\frac{d\mu}{d\lambda}$.

Definition 2.2.11. *A measurable transformation $S : X \to X$ on a measure space $(X, \mathscr{B}, \mu)$ is nonsingular if $\mu(S^{-1}(B)) = 0$ for all $B \in \mathscr{B}$ such that $\mu(B) = 0$.*

By virtue of the Radon-Nikodym theorem, we can state a change of variables theorem.

Theorem 2.2.5. *[21] Let $(X, \mathscr{B}, \lambda)$ be a measure space, $S : X \to X$ a non-singular transformation, and $f : X \to X$ a measurable function such that $f \circ S \in \mathscr{L}^1(X, \mathscr{B}, \lambda)$. Then for every $B \in \mathscr{B}$,*

$$\int_{S^{-1}(B)} (f \circ S) d\lambda = \int_B f d(\lambda S^{-1}) = \int_B f \cdot J^{-1} d\lambda$$

where λS^{-1} denotes the measure

$$\lambda S^{-1}(B) = \lambda(S^{-1}(B)), \qquad \text{for } B \in \mathscr{B},$$

and J^{-1} is the density of λS^{-1} with respect to λ, that is

$$\lambda(S^{-1}(B)) = \int_B J^{-1} d\lambda \qquad \text{for } B \in \mathscr{B}.$$

For differentiable invertible transformations on $\mathbb{R}^d$, $J(x)$ is the determinant of the Jacobian matrix:

$$J(x) = \left|\frac{dS(x)}{dx}\right|.$$

Note $J^{-1}(x) = \left|\frac{dS^{-1}(x)}{dx}\right|$.

2.3 Ergodic Theory

Let $(X, \mathscr{B}, \mu)$ be a normalized measure space.

Definition 2.3.1. *A measurable function $\tau : X \to X$ is called measure-preserving, or we say μ is τ-invariant if $\mu(\tau^{-1}(B)) = \mu(B)$ for all $B \in \mathscr{B}$.*

Now we can present the definition of a dynamical system.

Definition 2.3.2. *Let τ preserve μ. The quadruple $(X, \mathscr{B}, \mu, \tau)$ is called a dynamical system.*

Let $\tau : X \to X$ be a transformation. We denote the nth iterate of τ by τ^n: $\tau^n(x) = \tau \circ \tau \circ ... \circ \tau(x)$. In a dynamical system, we are interested in the properties of the orbit $\{\tau^n(x)\}_{n\geq 0}$. For example, the property that starting in a specific set, the orbit returns to the set infinitely many times; or that every orbit is eventually "attracted" by some set. The Poincare Recurrence Theorem gives us a powerful result on the recurrence:

Theorem 2.3.1. *[4] Let τ be a measure-preserving transformation on $(X, \mathscr{B}, \mu)$. Let $E \in \mathscr{B}$ such that $\mu(E) > 0$. Then almost all points in E return infinitely often to E under iterations of τ, i.e.,*

$$\mu(\{x \in E| \text{ there exists } N \text{ such that } \tau^n(x) \notin E \text{ for all } n > N\}) = 0.$$

Proof: Let $B \subset E$ be the set of points that *never* return to E. Since

$$B = E \cap \tau^{-1}(X \setminus E) \cap \tau^{-2}(X \setminus E) \cap ...,$$

B is measurable. Suppose $\mu(B) > 0$. If $x \in B$, then $\tau(x)$, $\tau^2(x)$, ..., $\tau^n(x)$ do not inside B. Therefore B is disjoint from $\tau^{-n}(B)$ for all positive n. Moreover, they are all pairwise disjoint since

$$\tau^{-i}(B) \cap \tau^{-(i+j)}(B) = \tau^{-i}(B \cap \tau^{-j}(B)),$$

Since τ is measure preserving, $\mu(B) = \mu(\tau^{-1}(B)) = ... = \mu(\tau^{-n}(B)) > 0$, and thus $\mu(\bigcup_{n=0}^{\infty} \tau^{-n}(B)) = \infty$. This contradicts the fact that $\mu(X) = 1$. Therefore $\mu(B) = 0$. One can obtain the same result for τ^i, because τ is measure preserving.

Let $F_1 \subset E$ denote the set of points that eventually return to E under some iteration of τ. Clearly $F_1 = E \setminus B$ and $\mu(F_1) = \mu(E)$. Similarly, let $F_n \subset E$ denote the set of points that eventually return to E under some iteration of τ^n, then $\mu(F_n) = \mu(E)$ for all positive n. Denote $F = \bigcap_{n=1}^{\infty} F_n$. F consists of all points of E that return infinitely often to E. Since $F_1 \supset F_2 \supset F_3 \supset ...$ we have $\mu(F) = \mu(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n) = \mu(E)$. $\square$

Definition 2.3.3. *A measure-preserving transformation τ is ergodic if for any $B \in \mathscr{B}$, such that $\tau^{-1}(B) = B$, $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.*

Next we state the Birkhoff Ergodic Theorem, a fundamental theorem in ergodic theory.

Definition 2.3.4. *Suppose $\tau : (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ is measure-preserving, where $(X, \mathscr{B}, \mu)$ is σ-finite, and $f \in \mathscr{L}^1(\mu)$. Then there exists a function $f^* \in \mathscr{L}^1(\mu)$ such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k(x)) \to f^*, \mu - a.e.$$

Furthermore, $f^* \circ \tau = f^*$, $\mu - a.e.$ and if $\mu(X) < \infty$, then $\int_X f^* d\mu = \int_X f d\mu$.

Chapter 3

The Inverse Frobenius-Perron Problem

Studying a dynamical system by analysing time-dependent orbits is usually difficult, since the tiny difference of the initial status can result in dramatically different trajectory. Hence, instead of focusing on each single orbit, we see the system from a macroscopic perspective — consider the probability of points locating in a interval after nth iterates. This give rise to the study of Frobenius-Perron operator.

3.1 Definition

The inverse Frobenius-Perron Problem is associated with the Frobenius-Perron operator.

Assume the space under consideration is the interval $I = [a, b]$ and points are distributed by a probability density function $f \in \mathscr{L}^1$. That is, the probability of the initial point being in any measurable set $A \subset I$ is

$$Prob\{x \in A\} = \int_A f d\lambda,$$

where λ is the normalized Lebesgue measure on I. Let points being transformed by a map τ. After the transformation, the distribution over I would be different. Assume the new density is ϕ, then the probability function becomes

$$\int_A \phi d\lambda = Prob\{\tau(x) \in A\} = Prob\{x \in \tau^{-1}(A)\} = \int_{\tau^{-1}(A)} f d\lambda.$$

The existence of ϕ is given by the Radon-Nikodym Theorem (see Theorem 2.2.4). It is easy to see that ϕ is determined by τ and f. We let $P_\tau f$ denote ϕ. Then

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}(A)} f d\lambda. \tag{3.1}$$

Definition 3.1.1. *Let $(X, \mathscr{B}, \lambda)$ be a measure space. If $\tau : X \to X$ is a non-singular transformation, the unique operator $P_\tau : \mathscr{L}^1 \to \mathscr{L}^1$ defined by equation (3.1) is called the Frobenius-Perron operator corresponding to τ.*

The Frobenius-Perron operator describes the evolution of density functions.

Let $A = [a, x]$, then

$$\int_a^x P_\tau f d\lambda = \int_{\tau^{-1}([a,x])} f d\lambda.$$

Differentiating both sides, we obtain, by Theorem 2.2.3,

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}([a,x])} f d\lambda \qquad a.e.$$

Next we will state some useful properties for Frobenius-Perron operator in general.

Proposition 3.1.1. *The Frobenius-Perron operator has the following properties:*

1. *P_τ is a linear operator:*

$$P_\tau(\alpha f_1 + \beta f_2) = \alpha \cdot P_\tau f_1 + \beta \cdot P_\tau f_2 \tag{3.2}$$

for $\alpha, \beta \in \mathbb{R}$ and $f_1, f_2 \in \mathscr{L}^1$;

2. *Let $f \in \mathscr{L}^1$ and assume $f \geq 0$. Then $P_\tau f \geq 0$;*

3. *Suppose I is the space. P_τ preserves the integral:*

$$\int_I P_\tau f d\lambda = \int_I f d\lambda$$

4. *For non-singular transformations $\tau : I \to I$ and $\sigma : I \to I$, $P_{\tau\circ\sigma} f = P_\tau \circ P_\sigma f$. In particular, $P_{\tau^n} f = P_\tau^n f$.*

Proof:

1)Linearity:

$$\begin{aligned}
\int_A P_\tau(\alpha f_1 + \beta f_2)d\lambda &= \int_{\tau^{-1}A} (\alpha f_1 + \beta f_2)d\lambda \\
&= \alpha \int_{\tau^{-1}A} f_1 d\lambda + \beta \int_{\tau^{-1}A} f_2 d\lambda \\
&= \alpha \int_A P_\tau f_1 d\lambda + \beta \int_A P_\tau f_2 d\lambda \\
&= \int_A (\alpha \cdot P_\tau f_1 + \beta \cdot P_\tau f_2)d\lambda
\end{aligned}$$

Since A is arbitrary, Equation (3.2) is true.

2)Positivity: If $f \geq 0$, then for any measurable set the integral is nonnegative. Therefore,

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}A} f d\lambda \geq 0$$

for arbitrary mearable set A. So $P_\tau f \geq 0$.

3)Preservation of integrals: Note that $\tau^{-1}I = I$. So

$$\int_I P_\tau f d\lambda = \int_{\tau^{-1}I} f d\lambda = \int_I f d\lambda$$

4)Composition property: For any measurable set A, since

$$\begin{aligned}
\int_A P_{\tau\circ\sigma} f d\lambda &= \int_{\sigma^{-1}\circ\tau^{-1}A} f d\lambda \\
&= \int_{\tau^{-1}A} P_\sigma f d\lambda \\
&= \int_A P_\tau \circ P_\sigma f d\lambda
\end{aligned}$$

the result follows. If $\sigma = \tau$, then $P_{\tau^2} f = P_\tau^2 f$. By mathematical induction, $P_{\tau^n} f = P_\tau^n f$. $\square$

The Frobenius-Perron problem is to find a fixed point of P_τ, i.e., $P_\tau f = f$. Such f is also called the invarant density. If we are given a density function f, the inverse Frobenius-Perron problem involves in determining a point transformation τ such that the dynamical system $x_{i+1} = \tau(x_i)$ has f as its unique invariant probability density function.

3.2 The existence of the invariant density

In this section we show the existence of the invariant density.

Definition 3.2.1. *The function τ is said to be of class C^r if the derivatives $\tau', \tau'', ..., \tau^{(r)}$ exist and are continuous.*

Definition 3.2.2. *Let $f : [a,b] \to \mathbb{R}$ and let $\mathscr{P} = \mathscr{P}\{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$. If there exist a positive number M such that*

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M$$

for all partitions $\mathscr{P}$, then f is said to be of bounded variation on $[a,b]$.

The symbol $\bigvee_a^b f$ denote the variation of f over the closed interval $[a,b]$.

The first existence theorem is given by Lasota and Yorke in 1973.

Theorem 3.2.1. *Let $\tau : [0,1] \to [0,1]$ be a piecewise C^2 function such that $\inf|\tau'| > 1$. Then for any $f \in \mathscr{L}$ the sequence*

$$\frac{1}{n}\sum_{k=0}^{n-1} P_\tau^k f$$

is convergent in norm to a function $f^* \in \mathscr{L}^1$. The limit function has the following properties:

1. *$f \geq 0 \Rightarrow f^* \geq 0$.*

2. *$\int_0^1 f^* dm = \int_0^1 f dm$.*

3. *$P_\tau f^* = f^*$ and consequently the measure $d\mu^* = f^* dm$ is invariant under τ.*

4. *Then function f^* is of bounded variation; moreover, there exists a constant c independent of the choice of initial f such that the variation of the limiting f^* satisfies the inequality*

$$\bigvee_0^1 f^* \leq c \parallel f \parallel .$$

Before we state the next existence theorem, we illustrate the relation between the invariant measure and the fixed point.

Proposition 3.2.1. *Let $\tau : I \to I$ be non-singular. Then $P_\tau f^* = f^*$ if and only if the measure $\mu(A) = \int_A f^* d\lambda$ is τ-invariant, where $f^* \in \mathfrak{D}$ and $f^* \geq 0$.*

Proof:

If $P_\tau f^* = f^*$, then for any measurable set A,

$$\mu(A) = \int_A f^* d\lambda = \int_A P_\tau f^* d\lambda = \int_{\tau^{-1}A} f^* d\lambda = \mu(\tau^{-1}A)$$

Thus μ is τ-invariant.

If $\mu(A) = \mu(\tau^{-1}A)$, then

$$\int_A f^* d\lambda = \int_{\tau^{-1}A} f^* d\lambda = \int_A P_\tau f^* d\lambda.$$

Since A is arbitrary, we have $P_\tau f^* = f^*$. $\square$

So the existence of invariant density is equivalent to the existence of invariant measure. Straube (1981) proved the following theorem.

Theorem 3.2.2. *Let $(\Omega, \mathscr{B}, \lambda)$ be a measure space with normalized measure λ, f a nonsingular transformation of Ω into itself. Then*

1. *there exists an f-invariant normalized measure which is absolutely continuous with respect to λ*

   if and only if

2. *there exist $\delta > 0$, and α, $0 < \alpha < 1$, such that*

$$\lambda(E) < \delta \Rightarrow \sup_{k \in \mathbb{N}} \lambda(f^{-k}(E)) < \alpha, \qquad \forall E \in \mathscr{B}.$$

3.3 Representations of the Frobenius-Perron operator

3.3.1 For piecewise monotonic and expanding map

For a special class of piecewise monotonic transformations, the Frobenius-Perron operator has a convenient representation, which will be of great use in the sequel.

Definition 3.3.1. *Let $I = [a, b]$. The transformation $\tau : I \to I$ is called piecewise monotonic if there exists a partition of I, $a = a_0 < a_1 < ... < a_n = b$, and a number $r \geq 1$ such that*

(1) τ is a C^r function on (a_{i-1}, a_i), $i = 1, ..., n$, which can be extended to a C^r function on $[a_{i-1}, a_i]$, $i = 1, ..., n$ and

(2) $|\tau'(x)| > 0$ on (a_{i-1}, a_i), $i = 1, ..., n$.

If the condition (2) is replaced by $|\tau'(x)| \geq \alpha > 1$, then τ is called piecewise monotonic and expanding.

Let the transformation τ be piecewise monotonic on the partition $\mathscr{P} = \{a_0, a_1, ..., a_n\}$. Denote $\tau_{|[a_{i-1},a_i]}$ by τ_i, and $B_i = \tau([a_{i-1}, a_i])$, $i = 1, ..., n$. Then, for any measurable set $A \subset I$:

$$\tau^{-1}(A) = \bigcup_{i=1}^{n} \tau_i^{-1}(A \cap B_i).$$

It is obvious that sets $\{\tau_i^{-1}(A \cap B_i)\}_{i=1}^n$ are mutually disjoint. By Theorem 2.2.2 and Theorem 2.2.5, we can separate the integral and change the variable:

$$\begin{aligned}
\int_A P_\tau f(x) d\lambda &= \int_{\tau^{-1}(A)} f(x) d\lambda \\
&= \sum_{i=1}^{n} \int_{\tau_i^{-1}(A \cap B_i)} f(x) d\lambda \\
&= \sum_{i=1}^{n} \int_{A \cap B_i} f(\tau_i^{-1}(x)) |(\tau_i^{-1}(x))'| d\lambda \\
&= \sum_{i=1}^{n} \int_{A} f(\tau_i^{-1}(x)) |(\tau_i^{-1}(x))'| \cdot \chi_{B_i}(x) d\lambda \\
&= \int_A \sum_{i=1}^{n} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \cdot \chi_{B_i}(x) d\lambda
\end{aligned}$$

Since A is arbitrary, we can write

$$P_\tau f(x) = \sum_{i=1}^{n} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \cdot \chi_{\tau([a_{i-1},a_i])}(x) \tag{3.3}$$

Example 2.4.1 Let τ be the tent map as shown in Figure 3.1:

τ is a piecewise function on $[0, 1]$:

$$\tau(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2} \\ -2x + 2, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

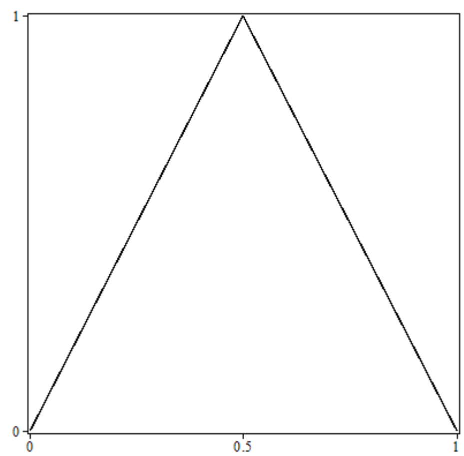


Figure 3.1: Tent map

We denote the left branch as τ_1 and τ_2 for the right one. Then

$$\tau_1'(x) = 2, \quad \tau_2'(x) = -2$$

$$\tau_1^{-1}(x) = \frac{1}{2}x, \quad \tau_2^{-1}(x) = -\frac{1}{2}(x-2).$$

So

$$P_\tau f = \frac{1}{2} f\left(\frac{x}{2}\right) + \frac{1}{2} f\left(1 - \frac{x}{2}\right)$$

We can see that $\rho(x) = 1$ is the invariant density for the tent map since $P_\tau \rho = \rho$.

3.3.2 Delta function representation

For the sake of future discussion, we introduce another form of the Frobenius-Perron operator.

Definition 3.3.2. *Let $(X, \mathscr{A})$ be any measurable space and let $x \in X$ be some point. Then $\delta_x : \mathscr{A} \to \{0, 1\}$, defined for $A \in \mathscr{A}$ by*

$$\delta_x(A) := \begin{cases} 0, & x \notin A, \\ 1, & x \in A, \end{cases}$$

is called Dirac delta measure at point x.

This is a measure intepretation of Dirac delta function. In late 1920's, P. A. M. Dirac derived the equation

$$\frac{d}{dx}ln(x) = \frac{1}{x} - i\pi\delta(x)$$

where δ satisfies the following properties:

1. $\delta(x) = 0$ for $x \neq 0$.
2. $\int_{-\infty}^{+\infty} \delta(x)dx = 1$.
3. For any function defined on $\mathscr{R}$, $f(y) = \int_{-\infty}^{+\infty} \delta(x-y)f(x)dx$. We call this sifting property.

This δ is now known as Dirac delta function. It has another property named compositon property, which we shall use later:

$$\int_{-\infty}^{+\infty} f(x)\delta(g(x))dx = \sum_i \frac{f(x_i)}{|g'(x_i)|},$$

where the sum extends over all roots of $g(x)$.

The delta function representation for the Frobenius-Perron operator is shown in the following theorem.

Theorem 3.3.1. *[7] For the piecewise differential and bijective map τ, Equation 3.3 and*

$$P_\tau f(x) = \int_0^1 \delta(x - \tau(y))f(y)dy$$

are equivalent.

Proof:

If τ is piecewise monotonic on partition $\{a_0, a_1, ..., a_n\}$, then

$$\begin{aligned}
&\int_0^1 \delta(x-\tau(y))f(y)dy \\
=&\sum_{i=1}^{n}\int_{a_{i-1}}^{a_i} \delta(x-\tau_i(y))f(y)dy \\
=&\sum_{i=1}^{n}\int_0^1 \delta(x-z)f(\tau_i^{-1}(z))|(\tau_i^{-1}(z))'|dz \\
=&\sum_{i=1}^{n} f(\tau_i^{-1}(x))|(\tau_i^{-1}(x))'| \\
=&P_\tau f(x).
\end{aligned}$$

□

3.3.3 Matrix Representation

In this section we focus on some definitions and theorems on matrices, which will be used in Chapter 5.

Definition 3.3.3. *A real $n \times n$ matrix $A = \{a_{ij}\}$ is called a right stochastic matrix if $a_{ij} \geq 0$ and each row sums to 1. It is called a left stochastic matrix is each column sums to 1.*

Theorem 3.3.2. *The largest eigenvalue of a stochastic matrix is 1.*

Proof:

Let $A = \{a_{ij}\}$ be a $n \times n$ stochastic matrix. We can just prove this for a right stochastic matrix since the transpose matrix has the same eigenvalues:

$$det(A^T - \lambda I) = det[(A - \lambda I)^T] = det(A - \lambda I).$$

Since each row of A sums to 1, $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the unit matrix. This shows that 1 is an eigenvalue of A. Suppose there exists $\lambda > 1$ and a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Let x_0 denote the largest entry of $\mathbf{x}$, then $\sum_{j=1}^{n} a_{ij}x_j \leq \sum_{j=1}^{n} a_{ij}x_0 = x_0$ for $1 \leq i \leq n$. Thus elements in $\lambda\mathbf{x}$ should not be larger than x_0. However, since

$\lambda > 1, \lambda x_0 > x_0$, so we have a contradiction. $\square$

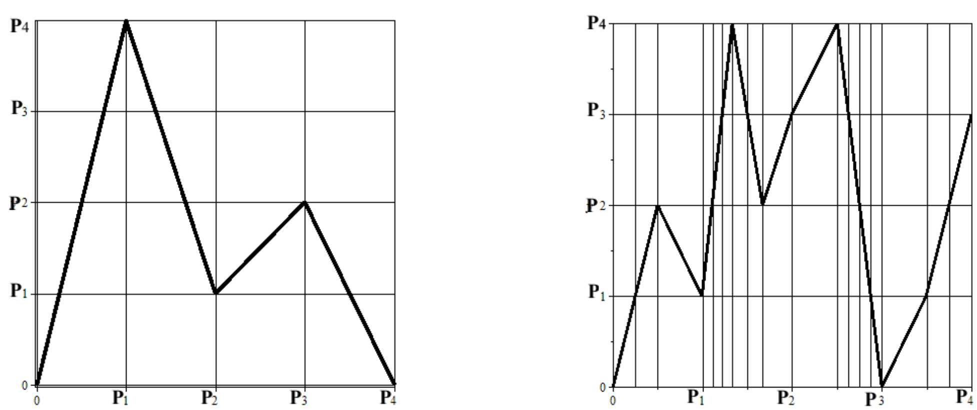
Let $\mathscr{P} = \{P_1, ..., P_N\}$ be a partition of $I = [a, b]$.

Definition 3.3.4. *A transformation $\tau : I \to I$ is called $\mathscr{P}$-Markov if $\tau|_{P_i}$ is monotonic and $\tau(P_i)$ is a union of intervals of $\mathscr{P}$.*

The following theorem was proved by Boyarsky and Haddad [1981].

Theorem 3.3.3. *[16] If a transformation τ is $\mathscr{P}$-Markov and piecewise linear and expanding, then any τ-invariant density is constant on intervals of $\mathscr{P}$.*

Definition 3.3.5. *A transformation $\tau : I \to I$ is called $\mathscr{P}$-semi-Markov if there exist disjoint intervals $Q_j^{(i)}$ such that for any $i = 1, ..., N$ we have $P_i = \bigcup_{j=1}^{k(i)} Q_j^{(i)}$, $\tau|_{Q_j^{(i)}}$ is monotonic, and $\tau(Q_j^{(i)}) \in \mathscr{P}$.*



(a) *$\mathscr{P}$-Markov* (b) *$\mathscr{P}$-semi-Markov*

Figure 3.2: Examples

Figure 3.2 gives us an example of a Markov map and a semi-Markov map. Note that every $\mathscr{P}$-Markov map is $\mathscr{P}$-semi-Markov, but the reverse is not necessary true.

Theorem 3.3.3 can be generalized to the semi-Markov case.

Theorem 3.3.4. *[16] Let τ be a $\mathscr{P}$-semi-Markov transformation, and $\tau|_{Q_j^{(i)}}$ is linear with slope greater than 1 for $j = 1, ..., k(i)$, $i = 1, ..., N$. Then any τ-invariant density is constant on intervals of $\mathscr{P}$.*

Proof:

Let $\mathscr{Q} = \{Q_j^{(i)}, j = 1, ..., k(i), i = 1, ..., N\}$ be a partition on interval I, then τ is $\mathscr{Q}$-Markov. Let f be a τ-invariant density. By Theorem 3.3.3, f is constant on interval $Q_j^{(i)}$. Denote the value of f on $Q_j^{(i)}$ by $f_j^{(i)}$. Fix $1 \le i_0 \le N$, and choose j_1, j_2 such that $1 \le j_1, j_2 \le k(i_0)$. The Frobenius-Perron equation for τ-invariant density gives us

$$f_{j_1}^{(i_0)} = \sum_{(i,j)} |(\tau_j^{(i)})'|^{-1} f_j^{(i)},$$

and

$$f_{j_2}^{(i_0)} = \sum_{(i,j)} |(\tau_j^{(i)})'|^{-1} f_j^{(i)}.$$

where $\tau_j^{(i)} = \tau|_{Q_j^{(i)}}$, and the sums extend over all pairs of (i, j) such that $\tau(Q_j^{(i)}) = P_{i_0}$. The right hand sides of both equations are equal, so $f_{j_1}^{(i_0)} = f_{j_2}^{(i_0)}$ and therefore f is constant on intervals of $\mathscr{P}$. $\square$

Next we introduce an important definition since it allows us to interpret an semi-Markov operator as a matrix.

Definition 3.3.6. *Let τ be a $\mathscr{P}$-semi-Markov piecewise linear transformation. The Frobenius-Perron matrix is $\mathbb{M}_\tau = (a_{ij})_{1 \le i,j \le N}$, where*

$$a_{ij} = \begin{cases} \sum_k |(\tau_k^{(i)})'|^{-1} & \textit{if } \tau(Q_k^{(i)}) = P_j, \\ 0 & \textit{otherwise}\ . \end{cases}$$

Let f be piecewise constant $f = \{f_1, f_2, ..., f_N\}$, then f is τ-invariant if and only if $f = f\mathbb{M}_\tau$, since we know, from Equation (3.3), that f is a fixed point of τ iff, for all j,

$$\sum_{i=1}^{n} \sum_k \frac{f(\tau_k^{(i)}(x)^{-1})}{|\tau'(\tau_k^{(i)}(x)^{-1})|} \cdot \chi_{\tau(Q_k^i)}(x) = f_j, \tag{3.4}$$

where the second summation runs over all subintervals of P_i such that $\tau(Q_k^{(i)}) = P_j$. This equation can be simplified by noticing that $f(\tau_k^{(i)}(x)^{-1}) = f_i$, so

$$\sum_{i=1}^{n} \sum_k \frac{f_i}{|\tau_k^{(i)'}|} = f_j,$$

$$\sum_{i=1}^{n} \left(\sum_{k} \frac{1}{|\tau_k^{(i)'}|} \right) \cdot f_i = f_j,$$

$$\sum_{i=1}^{n} a_{ij} \cdot f_i = f_j.$$

Hence Equation (3.4) is equivalent to $f = f\mathbb{M}_\tau$.

Chapter 4

Three solutions of IFPP

In this chapter, we will provide three solutions that solve the IFPP. They are devised by Grossman and Thomae (1977), Koga (1991), and Pingel, Schmelsher, Diakonos (1999). We will make a generalization for the third method.

4.1 Conjugation approach

This method was developed by Grossman and Thomae (1977), Mori (1981), Gyorgyi and Szepfalusy(1983), etc. As the name suggests, a conjugation transformation is used to construct symmetric maps with prescribed invariant density.

4.1.1 Symmetric density

Definition 4.1.1. *Two transformations $\tau : I \to I$ and $\sigma : J \to J$ on intervals I and J are called conjugate if there exists a one-to-one map (usually continuous) $u : I \xrightarrow{onto} J$ such that*

$$\tau(x) = u^{-1}(\sigma[u(x)]).$$

The map u is called the conjugation transformation.

Theorem 4.1.1. *Let σ and τ be conjugate: $\tau(x) = u^{-1} \circ \sigma \circ u(x)$, where u is a diffeomorphism. If $P_\sigma g = g$ then $P_\tau f = f$, where*

$$f = (g \circ u) \cdot |u'|.$$

Proof:

The fact that u is a diffeomorphism implies u is monotonic. By Equation (3.3),

$$P_{u^{-1}}g = \sum_{i=1}^{n}(g \circ u_i)|u_i'|\chi_{[a_{i-1},a_i]} = (g \circ u) \cdot |u'| = f.$$

Using the relation $P_{\phi\circ\psi}f = P_\phi \circ P_\psi f$, we obtain

$$\begin{aligned} P_\tau(f) = P_\tau(P_{u^{-1}}g) &= P_{u^{-1}\circ\sigma\circ u}(P_{u^{-1}}g) \\ &= P_{u^{-1}} \circ P_\sigma \circ P_u(P_{u^{-1}}g) \\ &= P_{u^{-1}} \circ P_\sigma(g) \\ &= P_{u^{-1}}(g) = f. \qquad \square \end{aligned}$$

Let $I = [0, 1]$ be the underlying space. To generate a symmetric map, we choose the tent map $t(x) = 1 - |2x - 1|$ in Example 2.4.1 as the map going to be conjugated. Let $u(x)$ be the conjugation transformation with $u(0) = 0, u(1) = 1, u'(x) > 0$ and apply it to $t(x)$:

$$\tau(x) = u^{-1} \circ t \circ u(x).$$

Recall that the invariant density of $t(x)$ is $\rho(x) = 1$. By Theorem 4.1.1, the invariant density of the transformed map $\tau(x)$ is

$$f(x) = (\rho \circ u) \cdot |u'(x)| = u'(x).$$

This relation gives us a way to derive $u(x)$ from the density function $f(x)$.

If the given f is symmetric, then $u(x) = \int_0^x f(t)dt$ has the property that $u(x) = 1 - u(1 - x)$, since

$$\begin{aligned} u(x) &= \int_0^x f(t)dt = 1 - \int_x^1 f(t)dt \\ &= 1 + \int_{1-x}^{0} f(1-z)dz \\ &= 1 + \int_{1-x}^{0} f(z)dz \\ &= 1 - \int_0^{1-x} f(z)dz \\ &= 1 - u(1-x) \end{aligned}$$

This guarantees the symmetry of τ:

$$\begin{aligned}\tau(1-x) &= u^{-1} \circ t \circ u(1-x) \\ &= u^{-1}(1 - |2[u(1-x)] - 1|) \\ &= u^{-1}(1 - |2[1-u(x)] - 1|) \\ &= u^{-1}(1 - |2u(x) - 1|) \\ &= \tau(x)\end{aligned}$$

In this way we are able to generate a symmetric map with symmetric invariant density.

Example 3.1.1 Let $f(x) = 2 - |2 - 4x|$ be the prescribed invariant density. We seek a symmetric transformation τ that generates $f(x)$.

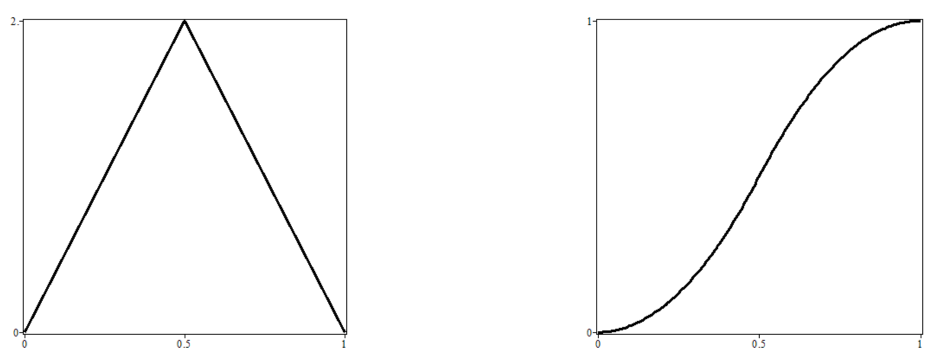
Observe that f is symmetric with respect to $x = \frac{1}{2}$, and it is piecewise linear:

$$f(x) = \begin{cases} 4x, & 0 \le x < \frac{1}{2} \\ -4x + 4, & \frac{1}{2} \le x \le 1. \end{cases}$$

Calculating $u(x) = \int_0^x f(t)dt$ for each piece, we obtain

$$u(x) = \begin{cases} 2x^2, & 0 \le x < \frac{1}{2} \\ -2x^2 + 4x - 1, & \frac{1}{2} \le x \le 1, \end{cases}$$

One can easily check that $u(x)$ satisfies $u(0) = 0$, $u(1) = 1$ and $u(x) = 1 - u(1-x)$.



(a) $f(x)$ (b) $u(x)$

Figure 4.1: The invariant density and the conjugation map

Now applying the conjugation process, $\tau(x) = u^{-1} \circ t \circ u(x)$, we get

$$\tau(x) = \begin{cases} \sqrt{2}x, & 0 \le x < \frac{1}{\sqrt{8}} \\ 1 - \sqrt{\frac{1}{2} - 2x^2}, & \frac{1}{\sqrt{8}} \le x \le \frac{1}{2}, \\ 1 - \sqrt{\frac{1}{2} - 2(1-x)^2}, & \frac{1}{2} < x < 1 - \frac{1}{\sqrt{8}}, \\ \sqrt{2}(1-x), & 1 - \frac{1}{\sqrt{8}} \le x \le 1 \end{cases}$$

The graph of $\tau(x)$ is shown in Figure 4.2.

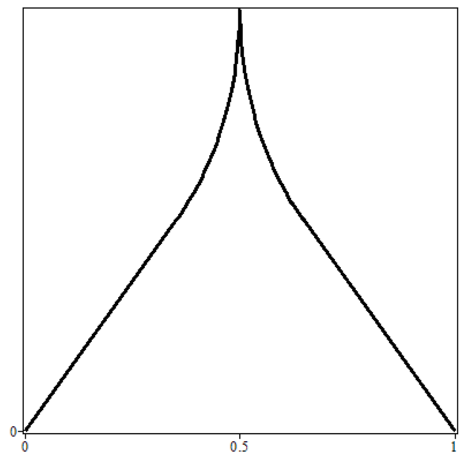


Figure 4.2: $\tau(x)$

4.1.2 Non-symmetric density

The above conjugation is appropriate for doubly symmetric maps — symmetric dynamical laws and symmetric invariant densities. To generate a symmetric map with non-symmetric density $f(x)$, we shall use the transformation

$$\tau(x) = U^{-1} \circ t \circ u(x) \tag{4.1}$$

where $U(x) = u(x) + v(x)$, and $u(x) = 1 - u(1-x)$, $v(x) = v(1-x)$, $v(0) = 0$.

Claim: The invariant density f is $U'(x)$.

Proof: Let $F(x) = \int_0^x f(x)dx$. Recall that for piecewise monotonic τ, the Frobenius-Perron equation is

$$f(y) = \sum_{x_i = \tau^{-1}(y)} \frac{f(x_i)}{|\tau'(x_i)|}.$$

Expressing $|\tau'(x_i)|$ as $|\frac{dy}{dx_i}|$, we can write :

$$f(y)|dy| = \sum_{x_i \in \tau^{-1}(y)} f(x_i)|dx_i| \tag{4.2}$$

In the case of a two-branch τ, integrating both sides of Equation (4.2) allows us to rewrite the Frobenius-Perron equation as

$$F(x) = F(\tau_l^{-1}(x)) + 1 - F(\tau_u^{-1}(x)),$$

where τ_l^{-1} and τ_u^{-1} denote the lower and upper branches of the inverse function τ^{-1}. If $U(x)$ satisfies

$$U(x) = U(\tau_l^{-1}(x)) + 1 - U(\tau_u^{-1}(x)), \tag{4.3}$$

then $f(x) = U'(x)$. Since $\tau^{-1} = u^{-1} \circ t^{-1} \circ U(x)$, and

$$t^{-1}(x) = \begin{cases} \frac{x}{2}, & 0 \le t^{-1}(x) < \frac{1}{2} \\ 1 - \frac{x}{2}, & \frac{1}{2} \le t^{-1}(x) \le 1, \end{cases}$$

we have

$$\tau_l^{-1} = u^{-1}\left(\frac{U}{2}\right)$$

and

$$\tau_u^{-1} = u^{-1}\left(1 - \frac{U}{2}\right).$$

In the following, we show that $\tau_u^{-1} = 1 - \tau_l^{-1}$. Let $z = u^{-1}(1 - \frac{U}{2})$, then

$$\begin{aligned} & u(z) = 1 - \frac{U}{2} \\ \implies\ & 1 - u(1-z) = 1 - \frac{U}{2} \\ \implies\ & u(1-z) = \frac{U}{2} \\ \implies\ & 1 - z = u^{-1}\left(\frac{U}{2}\right) \\ \implies\ & z = 1 - u^{-1}\left(\frac{U}{2}\right) \end{aligned}$$

The right hand side of Equation (4.3) becomes:

$$\begin{aligned}
&U\left(u^{-1}\left(\frac{U(x)}{2}\right)\right)+1-U\left(1-u^{-1}\left(\frac{U(x)}{2}\right)\right)\\
=&(u+v)\left(u^{-1}\left(\frac{U(x)}{2}\right)\right)+1-(u+v)\left(1-u^{-1}\left(\frac{U(x)}{2}\right)\right)\\
=&U(x)
\end{aligned}$$

The last equality uses the properties $u(x) = 1 - u(1 - x)$ and $v(x) = v(1 - x)$. This proves that $U(x) = \int_0^x f(t)dt$. $\square$

Let us now define

$$U_+(x) = \frac{1}{2}\int_0^x [f(t) - f(1-t)]dt \tag{4.4}$$

$$U_-(x) = \frac{1}{2}\int_0^x [f(t) + f(1-t)]dt. \tag{4.5}$$

Notice that $u(x) = U_-(x), \quad v(x) = U_+(x)$ is a decomposition for U, that is, $u(x) = 1 - u(1 - x)$, $v(x) = v(1 - x)$. Since

$$\begin{aligned}
U_-(x) &= \frac{1}{2}(\int_0^x f(t)dt + \int_0^x f(1-t)dt)\\
&= \frac{1}{2}(U(x) + \int_0^1 f(z)dz - \int_0^{1-x} f(z)d(z))\\
&= \frac{1}{2}(U(x) + 1 - U(1-x)),
\end{aligned}$$

we obtain the formula for the map

$$\tau(x) = U^{-1}[1 - |U(x) - U(1-x)|]. \tag{4.6}$$

Example 3.1.2 Suppose that $f(x) = 2x$ is the given invariant density. By substituting f into Equation (4.4) and (4.5), we get $u(x) = x$, $v(x) = x^2 - x$, and thus $U(x) = u(x) + v(x) = x^2$. Therefore $\tau(x) = \sqrt{1 - |2x - 1|}$. As shown in Figure (4.3), τ is a symmetric function.

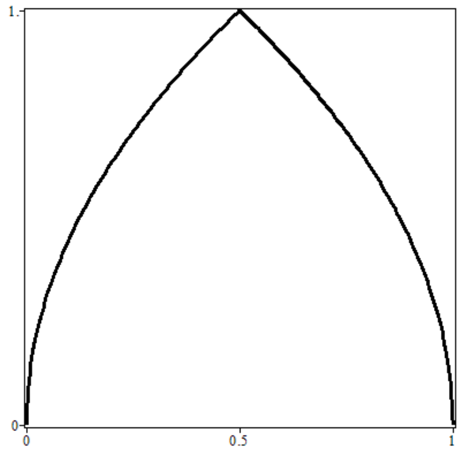


Figure 4.3: $\tau(x) = \sqrt{1 - |2x - 1|}$

4.2 Differential Equation approach

In 1991, Koga solved the Frobenius-Perron equation inversely for two kinds of maps by using differential equations on $I = [0, 1]$. The two types considered are

$$\tau(1 - x) = \tau(x), \qquad \text{(case 1)}$$

$$\tau(x + \frac{1}{2}) = \tau(x). \qquad \text{(case 2)}$$

The former represents the map symmetric about $x = \frac{1}{2}$, and the latter is the translationally symmetric map. Further, assume that the slope of $\tau(x)$ is positive in the interval $[0, \frac{1}{2}]$ and $\tau(0) = 0$.

Recall that Theorem 3.3.1 allows us to write the Frobenius-Perron equation as

$$P_\tau f(x) = \int_0^1 \delta(x - \tau(y)) f(y) dy$$

where δ is the Dirac delta function. f is the preassigned invariant density, so

$$f(x) = \int_0^1 \delta(x - \tau(y)) f(y) dy. \tag{4.7}$$

Now our task is to solve for τ from the above equation.

Substituting $\tau(x)$ into Equation (4.7), we obtain

$$f(\tau(x)) = \int_0^1 \delta(\tau(x) - \tau(y)) f(y) dy. \tag{4.8}$$

Recall that the Dirac delta function has the composition property, and it can be written as

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \tag{4.9}$$

where x_i's are roots of $g(x)$. Let $g(y) = \tau(x) - \tau(y)$. For case 1, the roots of $g(y)$ are $y = x$ and $y = 1 - x$. Also $|g'(x)| = |\tau'(x)| = |\tau'(1-x)| = |g'(1-x)|$. Substituting this into Equation (4.9) yields:

$$\begin{aligned}\delta(\tau(x) - \tau(y)) &= \delta(g(y)) \\ &= \frac{\delta(y-x)}{|g'(x)|} + \frac{\delta(y-(1-x))}{|g'(1-x)|} \\ &= \frac{\delta(x-y) + \delta(1-x-y)}{|\frac{d\tau(x)}{dx}|} \end{aligned} \tag{4.10}$$

Similarly, for case 2 we have

$$\delta(\tau(x) - \tau(y)) = \frac{\delta(x-y) + \delta(x + \frac{1}{2} - y)}{|\frac{d\tau(x)}{dx}|} \tag{4.11}$$

Substitute Equation (4.10) into (4.8) and by the sifting property of delta function, we get

$$\begin{aligned} f(\tau(x)) &= \int_0^1 \frac{\delta(x-y) + \delta(1-x-y)}{|\frac{d\tau(x)}{dx}|} f(y)dy \\ &= |\tau'(x)|^{-1}\left(\int_0^1 \delta(x-y)f(y)dy + \int_0^1 \delta(1-x-y)f(y)dy\right) \\ &= |\tau'(x)|^{-1}(f(x) + f(1-x)) \end{aligned}$$

Our assumption for $x \in [0, \frac{1}{2}]$ is $\tau'(x) > 0$. So for case 1,

$$\frac{d\tau(x)}{dx} = \frac{f(x) + f(1-x)}{f(\tau(x))} \tag{4.12}$$

In case 2,

$$\frac{d\tau(x)}{dx} = \frac{f(x) + f(x + \frac{1}{2})}{f(\tau(x))} \tag{4.13}$$

This is an ordinary differential equations for $\tau(x)$. To solve Equation (4.12), one can try to integrate both sides:

$$f(\tau(x))d\tau(x) = (f(x) + f(1-x))dx$$

$$\int_0^{\tau(x)} f(z)dz = \int_0^x (f(z) + f(1-z))dz \tag{4.14}$$

For Equation (4.13):

$$\int_0^{\tau(x)} f(z)dz = \int_0^x (f(z) + f(z + \frac{1}{2}))dz \tag{4.15}$$

Note here $x \in (0, \frac{1}{2})$. The behaviour on $(\frac{1}{2}, 1)$ can be determined by symmetry.

Example 3.2.1 Consider $f(x) = 2x$. To generate τ in case 1, substitute $f(x)$ into Equation (4.14):

$$\begin{aligned}
&\int_0^{\tau(x)} 2zdz = \int_0^x (2z + 2(1-z))dz \\
\Longrightarrow &\tau(x)^2 = 2x \\
\Longrightarrow &\tau(x) = \sqrt{2x}.
\end{aligned}$$

This is the left branch of τ. The right branch can be obtained by symmetry with respect to $x = \frac{1}{2}$. Figure 4.3 shows the graph.

To generate type 2, we use Equation (4.15) and yield the left branch of $\tau(x)$ as $\sqrt{2x^2 + x}$. We plot τ in Figure 4.4.

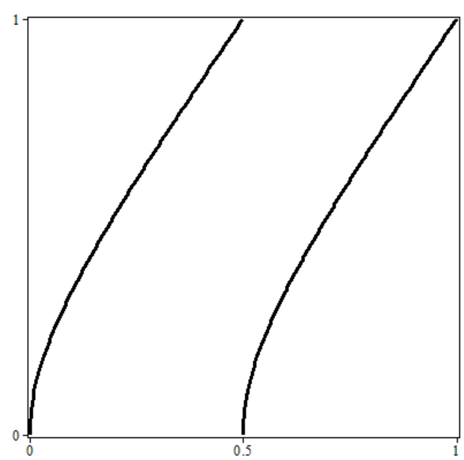


Figure 4.4: Case 2

4.3 Pingel's approach

4.3.1 Generating unimodal maps

In this section we introduce the approach devised by Pingel, Schmelsher and Diakonos (1999). They focused on constructing a map τ with one maximum, and each branch of τ covering the whole interval. Later we will see that this is a generalization of the previous two methods.

Let $F(x) = \int_0^x f(x)dx$. Recall from Section 3.1 that the Frobenius-Perron equation can be written as

$$f(y)|dy| = \sum_{x_i=\tau^{-1}(y)} f(x_i)|dx_i| \tag{4.16}$$

For unimodal τ, we label the preimages of y which are located in the left branch as x_L, and in the right as x_R (see Figure 4.5). Then Equation (4.16) becomes

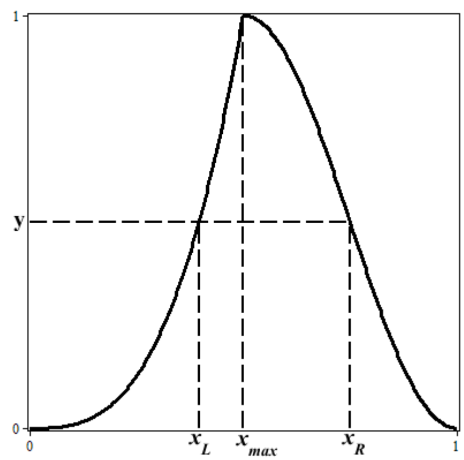


Figure 4.5: x_L and x_R

$$f(y)|dy| = f(x_L)|dx_L| + f(x_R)|dx_R| \tag{4.17}$$

Define a function $h_\tau(x)$ which maps the left preimage to the right one:

$$h_\tau : [0, x_{max}] \to [x_{max}, 1]$$

$$x_R = h_\tau(x_L), \tag{4.18}$$

where x_{max} is the position whose value of τ reaches the maximum. Note that h_τ is decreasing and differentiable except for a finite number of points, that is,

$$h'_\tau(x) < 0, x \in [0, x_{max}],$$

$$h_\tau(0) = 1,$$

$$h_\tau(x_{max}) = x_{max}.$$

Since x_R is a function of x_L, substituting h_τ in Equation 4.17, we obtain:

$$f(y)dy = f(x_L)dx_L - f(h_\tau(x_L))h'_\tau(x_L)dx_L \tag{4.19}$$

Let $\tau_L(x)$ denote the left part of τ. Integrating Equation 4.19 for $x \in [0, x_{max}]$ yields

$$\int_0^{\tau_L(x)} f(t)dt = \int_0^x [f(t) - f(h_\tau(t))h'_\tau(t)]dt.$$

Then

$$F(\tau_L(x)) = \int_0^x f(t)dt - \int_{h_\tau(0)}^{h_\tau(x)} f(t)dt,$$

$$F(\tau_L(x)) = F(x) - F(0) - F(h_\tau(x)) + F(h_\tau(0)).$$

Since $h_\tau(0) = 1, F(0) = 0, F(1) = 1$, we have

$$F(\tau_L(x)) = F(x) - F(h_\tau(x)) + 1, \tag{4.20}$$

$$\tau_L(x) = F^{-1}[F(x) - F(h_\tau(x)) + 1] \tag{4.21}$$

By substituting $h_\tau^{-1}(x)$ for x, one can obtain the right part $\tau_R(x)$. Thus

$$\tau(x) = \begin{cases} F^{-1}[F(x) - F(h_\tau(x)) + 1] & \text{if } 0 \leqslant x < x_{max} \\ F^{-1}[F(h_\tau^{-1}(x)) - F(x) + 1] & \text{if } x_{max} \leqslant x \leqslant 1 \end{cases} \tag{4.22}$$

$$= F^{-1}(1 - |F(x) - F(H_\tau(x))|), \tag{4.23}$$

where $H_\tau(x)$ is given by

$$H_\tau(x) = \begin{cases} h_\tau(x), & 0 \leqslant x < x_{max} \\ h_\tau^{-1}(x), & x_{max} \leqslant x \leqslant 1. \end{cases}$$

Comparing (4.6) and (4.14) with (4.22), we see that they are compatible in the case $h_\tau(x) = 1 - x$.

Example 3.3.1 Again $f(x) = 2x$, we use Pingel's method to generate a unimodal map τ on $[0, 1]$ that preserves f. We calculate $F(x)$

$$F(x) = \int_0^x f(t)dt = x^2$$

and set $h_\tau(x) = 1 - x$, so $x_{max} = \frac{1}{2}$. By Equation (4.22), we have

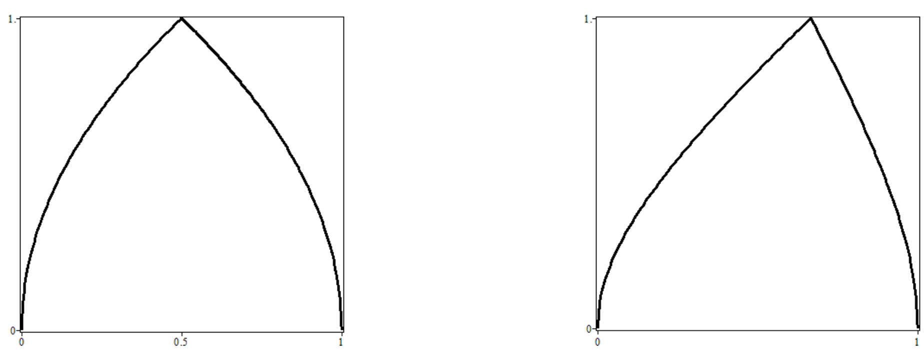
$$\tau(x) = \begin{cases} \sqrt{2x}, & \text{if } 0 \leqslant x < \frac{1}{2} \\ \sqrt{2 - 2x}, & \text{if } \frac{1}{2} \leqslant x \leqslant 1 \end{cases}$$

Since h_τ is symmetric, we obtain a symmetric τ(see Figure (4.6a)). Note that the result is the same as Example 3.1.2.

If we choose $h_\tau(x) = 1 - \frac{x}{2}$ and $x_{max} = \frac{2}{3}$, we get

$$\tau(x) = \begin{cases} \sqrt{\frac{3}{4}x^2 + x}, & \text{if } 0 \leqslant x < \frac{2}{3} \\ \sqrt{3x^2 - 8x + 5}, & \text{if } \frac{2}{3} \leqslant x \leqslant 1. \end{cases}$$

Its graph is shown in Figure (4.6b).



(a) τ with $h_\tau(x) = 1 - x$

(b) τ with $h_\tau(x) = 1 - \frac{x}{2}$

Figure 4.6: Example 3.3.1

4.3.2 Generating two-pieces maps

We can generalize Pingel's method to construct piecewise increasing maps with two pieces and each branch covering the whole interval. An example is shown in Figure (4.7).

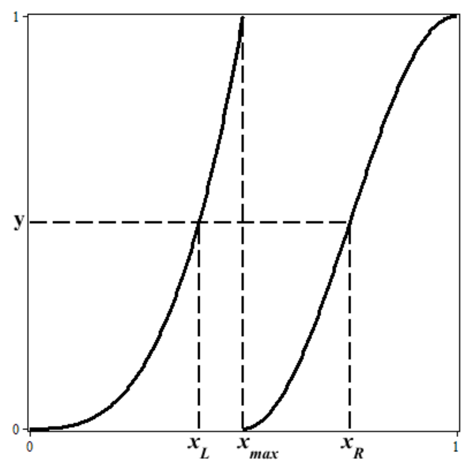


Figure 4.7: piecewise increasing τ

The only difference between our construction and Pingel's original one is in the function $h : [0, x_{max}] \to [x_{max}, 1]$ defined by Equation (4.18). It can be easily seen that

$$h'(x) > 0,$$

$$h(0) = x_{max},$$

$$h(x_{max}) = 1$$

in our case. Therefore, Equation (4.19) becomes

$$f(y)dy = f(x_L)dx_L + f(h_\tau(x_L))h'_\tau(x_L)dx_L,$$

and we obtain

$$\int_0^{\tau_L(x)} f(t)dt = \int_0^x [f(t) + f(h_\tau(t))h'_\tau(t)]dt,$$

$$F(\tau_L(x)) = \int_0^x f(t)dt + \int_{h_\tau(0)}^{h_\tau(x)} f(t)dt,$$

$$F(\tau_L(x)) = F(x) + F(h_\tau(x)) - F(x_{max}).$$

So

$$\tau_L(x) = F^{-1}[F(x) + F(h_\tau(x)) - F(x_{max})]$$

Notice here $F(x_{max})$ is a determined number as long as x_{max} is fixed. Hence we get a general form of τ:

$$\tau(x) = \begin{cases} F^{-1}[F(x) + F(h_\tau(x)) - F(x_{max})], & \text{if } 0 \leqslant x < x_{max} \\ F^{-1}[F(h_\tau^{-1}(x)) + F(x) - F(x_{max})], & \text{if } x_{max} \leqslant x \leqslant 1 \end{cases} \tag{4.24}$$

Again, we see that this is compatible with Koga's solution of case 2, provided that $h_\tau(x) = x + \frac{1}{2}$.

Example 3.3.2 Given $f(x) = 2x$, we want to construct a piecewise increasing map. In Example 3.3.1, we calculated that $F(x) = x^2$. Set $x_{max} = \frac{1}{3}$ and $h_\tau = 2x + \frac{1}{3}$. We see that $h_\tau(0) = \frac{1}{3}$, $h_\tau(\frac{1}{3}) = 1$. Therefore by Equation (4.24),

$$\tau(x) = \begin{cases} \sqrt{5x^2 + \frac{4}{3}x} & \text{if } 0 \leqslant x < \frac{1}{3} \\ \sqrt{\frac{5}{4}x^2 - \frac{1}{6}x - \frac{1}{12}} & \text{if } \frac{1}{3} \leqslant x \leqslant 1 \end{cases}$$

The graph of τ is shown in Figure (4.8).

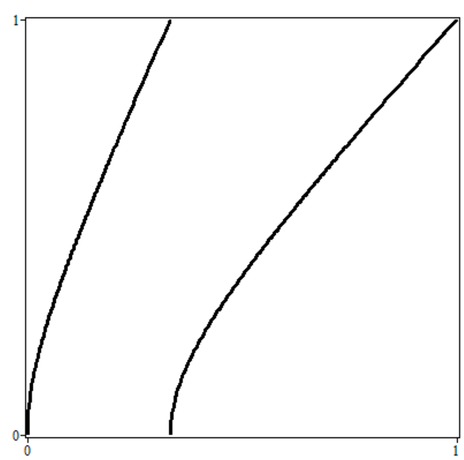


Figure 4.8: τ with $h_\tau(x) = 2x + \frac{1}{3}$

Chapter 5

Matrices-based Approach

Solving IFPP by matrix method has also been developed by various groups. In this chapter we will present two different matrix solutions from Rogers, Shorten, Heffernan[2004], Gora and Boyarsky[1993, 2007]. They all considered the case that the preassigned invariant density is piecewise constant. Basic concepts can be found in Section 3.3.3.

Beforehand, one shall see the relation between τ and its Frobenius-Perron matrix. Interpreting a $\mathscr{P}$-semi-Markov piecewise linear τ as a matrix was shown in Definition 3.3.6. The converse problem requires that we can construct τ. To do this, partition the unit interval into N equal subintervals $\{I_1, I_2, ..., I_N\}$. Let entry a_{ij} of the Frobenius-Perron matrix A denote the fraction of interval I_i being mapped into interval I_j. Then in the square $I_i \times I_j$, the slope of τ is $\pm\frac{1}{a_{ij}}$. For example, the map in Figure 5.1 is one possible interpretation of the transition matrix A. The first row of A corresponds to boxes B_1, B_2, B_3. More specifically, $a_{12} = \frac{1}{2}$ means that the probability of the transition from I_1 to I_2 is $\frac{1}{2}$. Setting the slope of τ equals to 2 (or -2) in box B_2 can satisfy this condition.

Generally we will start at the origin and draw the line segment end to end to yield a piecewise continuous map.

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{5} & \frac{3}{5} \end{pmatrix}$$

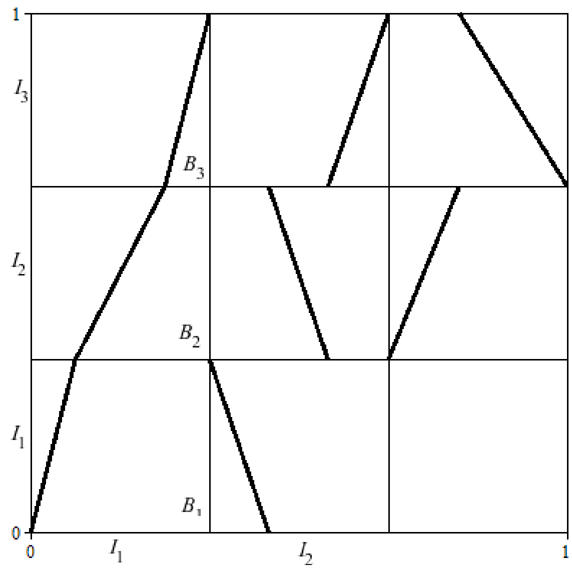


Figure 5.1: One possible τ for A

Therefore, to find a $\mathscr{P}$-semi-Markov piecewise linear solution of IFPP, we only need to construct its Frobenius-Perron matrix.

5.1 A solution based on stochastic matrices

In this section, the synthesis method developed by the group of Roger is discussed. This method is based on the theory of stochastic matrices, and the matrix used in the analysis of synchronised communication networks. The idea is to treat the prescribed piecewise constant density function as the eigenvector of a column stochastic matrix with eigenvalue 1, then the generated stochastic matrix represents a class of desired dynamic laws.

Consider the matrix A shown below. It originates in the work of synchronised communication networks:

$$A = \begin{pmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \beta_2 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \dots & \beta_n \end{pmatrix} + \frac{1}{Q} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \begin{pmatrix} 1-\beta_1 & 1-\beta_2 & \dots & 1-\beta_n \end{pmatrix}$$

where $Q = \sum_{i=1}^{n} \alpha_i$. More compactly,

$$A = \begin{pmatrix} \beta_1 + \frac{\alpha_1(1-\beta_1)}{Q} & \frac{\alpha_1(1-\beta_2)}{Q} & \dots & \frac{\alpha_1(1-\beta_n)}{Q} \\ \frac{\alpha_2(1-\beta_1)}{Q} & \beta_2 + \frac{\alpha_2(1-\beta_2)}{Q} & \dots & \frac{\alpha_2(1-\beta_n)}{Q} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_n(1-\beta_1)}{Q} & \frac{\alpha_n(1-\beta_2)}{Q} & \dots & \beta_n + \frac{\alpha_n(1-\beta_n)}{Q} \end{pmatrix}$$

Each column of A sums to 1, and every entry is positive when $\alpha_i \geq 0$ and $0 < \beta_i < 1, \forall i \in \{1, ..., n\}$, therefore A is a left stochastic matrix. By Theorem 3.3.2, A has a leading eigenvalue 1, and the corresponding eigenvector $\mathbf{x}_p$ is known as:

$$\mathbf{x}_p = \begin{pmatrix} \frac{\alpha_1}{1-\beta_1} \\ \frac{\alpha_2}{1-\beta_2} \\ \vdots \\ \frac{\alpha_n}{1-\beta_n} \end{pmatrix},$$

which we call the Perron eigenvector. One can easily check that $A\mathbf{x}_p = \mathbf{x}_p$.

So, if $\mathbf{x}_p$ is the prescribed density $f = \mathbf{x}_p$, then f is τ-invariant, where τ is a piecewise function determined by matrix A. To find the desired A for a given density $\mathbf{x}_p$, one simple way, for example, is to set all of the $\beta_i = 0.1$, and then α_i can be determined. Once A is fully determined, it can be translated into a map, since we can treat A as the transpose of a Frobenius-Perron matrix.

Example 4.1.1 Given density function f and make it equal to $\mathbf{x}_p$

$$f = \frac{4}{15}\begin{pmatrix} 2 \\ 3 \\ 1 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1}{1-\beta_1} \\ \frac{\alpha_2}{1-\beta_2} \\ \frac{\alpha_3}{1-\beta_3} \\ \frac{\alpha_4}{1-\beta_4} \\ \frac{\alpha_5}{1-\beta_5} \end{pmatrix} = \mathbf{x}_p$$

Let $\beta_i = 0.1$, $i = 1, 2, 3, 4, 5$. We set the above two vectors equal, then solve for α:

$$\alpha_1 = 0.48; \quad \alpha_2 = 0.72; \quad \alpha_3 = 0.24; \quad \alpha_4 = 1.44; \quad \alpha_5 = 0.72.$$

Then $\sum_{i=1}^{4} \alpha_i = 3.6$. Substituting in A, we get:

$$A = \begin{pmatrix} 0.22 & 0.12 & 0.12 & 0.12 & 0.12 \\ 0.18 & 0.28 & 0.18 & 0.18 & 0.18 \\ 0.06 & 0.06 & 0.16 & 0.06 & 0.06 \\ 0.36 & 0.36 & 0.36 & 0.46 & 0.36 \\ 0.18 & 0.18 & 0.18 & 0.18 & 0.28 \end{pmatrix}$$

A possible chaotic map corresponding to A is shown in Figure 5.2.

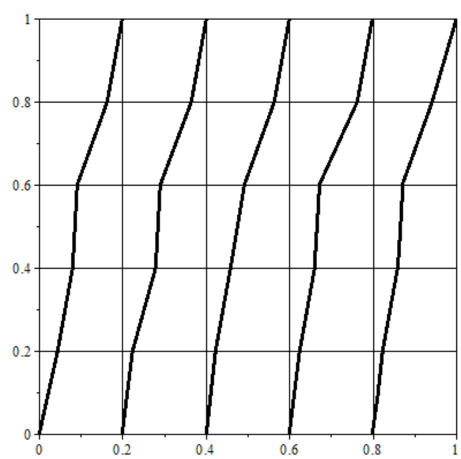


Figure 5.2: A possible map for A

5.2 The 3-band Matrix

Gora and Boyarsky devised this special matrix solution in 1993. They introduced the semi-Markov process, which we have presented in Section 3.3.3, and created a new class of matrix — 3-band matrix. They offered two simple ways to construct 3-band matrix for given piecewise constant density, and further proved the existence of semi-Markov transformation.

Definition 5.2.1. *An $\mathscr{R}$-semi-Markov piecewise linear transformation is called a 3-band transformation if its Perron-Frobenius matrix $\mathbb{M}_\tau = (p_{ij})$ satisfies: for any $1 \le i \le n, p_{ij} = 0$ if $|i-j| > 1$. That is:*

$$\mathbb{M}_\tau = \begin{pmatrix} p_{11} & p_{12} & 0 & \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots} & 0 \\ p_{21} & p_{22} & p_{23} & 0 & \multicolumn{2}{c}{\dots\dots\dots\dots\dots\dots} & 0 \\ 0 & p_{32} & p_{33} & p_{34} & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & p_{n-2,n-3} & p_{n-2,n-2} & p_{n-2,n-1} & 0 \\ 0 & \multicolumn{2}{c}{\dots\dots\dots} & 0 & p_{n-1,n-2} & p_{n-1,n-1} & p_{n-1,n} \\ 0 & \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots} & 0 & p_{n,n-1} & p_{n,n} \end{pmatrix}$$

The following theorem shows a simple relation between a 3-band transformation τ and its invariant density.

Theorem 5.2.1. *Let τ be a 3-band transformation with Perron-Frobenius matrix $\mathbb{M}_\tau = (p_{ij})$, and f be any τ-invariant density. Let f_i be the value of f on interval R_i, $i = 1, ..., n$. Then for any $2 \le i \le n$ we have*

$$p_{i,i-1} \cdot f_i = p_{i-1,i} \cdot f_{i-1} \tag{5.1}$$

Proof:

We will prove (5.1) by induction. Let $f = \{f_1, f_2, ..., f_n\}$. f is τ-invariant so $f = f \cdot \mathbb{M}_\tau$. The first equation is

$$f_1 \cdot p_{1,1} + f_2 \cdot p_{2,1} = f_1$$

Note that $\mathbb{M}_\tau$ is a right stochastic matrix, so $p_{1,1} = 1 - p_{1,2}$. Therefore,

$$p_{2,1} \cdot f_2 = p_{1,2} \cdot f_1.$$

Equation 5.1 is true for the case $i = 2$. Assume it is true for $2 \leq i < n$. Then the ith equation of the system is

$$f_{i-1} \cdot p_{i-1,i} + f_i \cdot p_{i,i} + f_{i+1} \cdot p_{i+1,i} = f_i \tag{5.2}$$

Substituting our induction hypothesis $p_{i-1,i} \cdot f_{i-1} = f_i \cdot p_{i,i-1}$ into Equation 5.2 yields

$$f_i \cdot p_{i,i-1} + f_i \cdot p_{i,i} + f_{i+1} \cdot p_{i+1,i} = f_i$$

$$f_{i+1} \cdot p_{i+1,i} = f_i \cdot (1 - p_{i,i-1} - p_{i,i})$$

$$f_{i+1} \cdot p_{i+1,i} = f_i \cdot p_{i,i+1}$$

This proves the theorem. $\square$

To prove the reverse: if we have the $p_{i,i-1}$ and $p_{i-1,i}$ that make the Equation (5.1) true, then ensuring the matrix is row stochastic by choosing each $p_{i,i}$, we get a 3-band map that preserve the given invariant density.

Example 4.2.1 Let $f = \frac{4}{15}(2, 3, 1, 6, 3)$, then

$$p_{21} \cdot f_2 = p_{12} \cdot f_1 \Rightarrow \frac{12}{15} p_{21} = \frac{8}{15} p_{12} \quad \Leftrightarrow \quad 3p_{21} = 2p_{12};$$

$$p_{32} \cdot f_3 = p_{23} \cdot f_2 \Rightarrow \frac{4}{15} p_{32} = \frac{12}{15} p_{23} \quad \Leftrightarrow \quad 1p_{32} = 3p_{23};$$

$$p_{43} \cdot f_4 = p_{34} \cdot f_3 \Rightarrow \frac{24}{15} p_{43} = \frac{4}{15} p_{34} \quad \Leftrightarrow \quad 6p_{43} = 1p_{34};$$

$$p_{54} \cdot f_5 = p_{45} \cdot f_4 \Rightarrow \frac{12}{15} p_{54} = \frac{24}{15} p_{45} \quad \Leftrightarrow \quad 2p_{54} = 1p_{45};$$

As an example, let $p_{21} = 0.2 \Rightarrow p_{12} = 0.3$, $p_{32} = 0.3 \Rightarrow p_{23} = 0.1$, $p_{43} = 0.1 \Rightarrow p_{34} = 0.6$, $p_{54} = 0.1 \Rightarrow p_{45} = 0.2$. To ensure the matrix is stochastic, we determine

$p_{11} = 0.7, \quad p_{22} = 0.7, \quad p_{33} = 0.1, \quad p_{44} = 0.7, \quad p_{55} = 0.9$. Thus, we get one desired τ with its Frobenius-Perron matrix:

$$\mathbb{M}_\tau = \begin{pmatrix} 0.7 & 0.3 & 0 & 0 & 0 \\ 0.2 & 0.7 & 0.1 & 0 & 0 \\ 0 & 0.3 & 0.1 & 0.6 & 0 \\ 0 & 0 & 0.1 & 0.7 & 0.2 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{pmatrix}.$$

The graph of τ is shown in Figure 5.3.

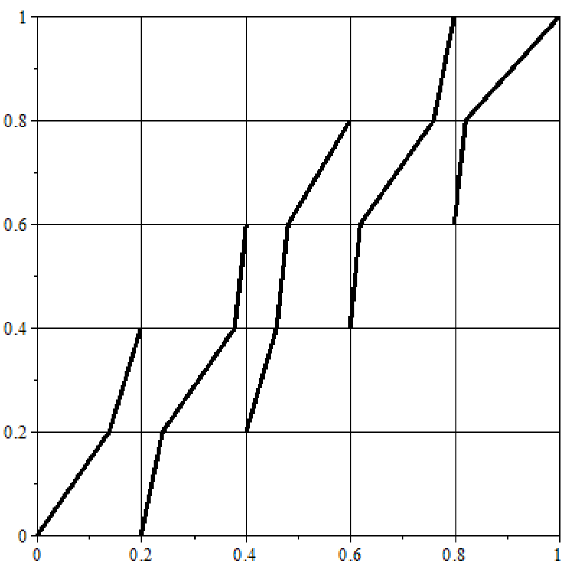


Figure 5.3: τ

A formal proof of the existence of 3-band τ is presented below.

Theorem 5.2.2. *Let $f = (f_1, ..., f_n)$ be a piecewise constant density on a partition $\mathscr{R}$ of $I = [a, b]$ into n equal intervals. Then there exists a 3-band piecewise expanding transformation τ such that f is τ-invariant.*

Proof:

To construct such a τ, we can make $g = [2 \cdot max(f_i : i = 1, ..., n)]^{-1} \cdot f$. Define a

Perron-Frobenius matrix $\mathbb{M}_\tau$ of some 3-band transformation τ as follows:

$$\begin{array}{c} 1 \\ \\ i \\ \\ n \end{array}\begin{pmatrix} 1-g_2, & g_2 & & & & & \\ & & \ddots & & & & \\ & & g_{i-1}, & 1-g_{i-1}-g_{i+1}, & g_{i+1} & & \\ & & & \ddots & & & \\ & & & & & g_{n-1}, & 1-g_{n-1} \end{pmatrix}$$

Check that $\mathbb{M}_\tau$ satisfies $g = g\mathbb{M}_\tau$, since for the i-th column, the right-hand side gives:

$$g_{i-1} \cdot g_i + g_i \cdot (1 - g_{i-1} - g_{i+1}) + g_{i+1} \cdot g_i = g_i.$$

Therefore, $f = f\mathbb{M}_\tau$. Each term of $\mathbb{M}_\tau$ is less than 1, so τ is piecewise expanding. Thus g and f is τ-invariant. $\square$

In fact the above proof gives us another way to construct the desired 3-band matrix. Note that the matrices generated by Theorem (5.2.1) and (5.2.2) may be different. In general, for each density function f, there exist infinitely many 3-band piecewise expanding transformations that preserve it. If the intervals for piecewise constant f is not equal, then we shall use the following theorem.

Theorem 5.2.3. *Let $\mathscr{P} = \{P_1, ..., P_n\}$ be a partition of $I = [a, b]$ into intervals and let the density $f = (f_1, ..., f_n)$ be constant on intervals of $\mathscr{P}$. Then there exists a $\mathscr{P}$-semi Markov piecewise linear and expanding transformation τ such that g is τ-invariant.*

Proof. Define $h : I \to I$ as follows:

$$h|_{P_i}(x) = e_0^{(i)} + \frac{b-a}{n \cdot m(P_i)}(x - l(P_i)),$$

where $e_0^{(i)} = a + (i-1)(b-a)/n$, $m(P_i)$ denote the length of P_i and $l(P_i)$ is the left-hand side endpoint of P_i, $i = 1, ..., n$. Figure (5.4) presents an example of h when $n = 4$. The function h is a piecewise linear homeomorphism and its Frobenius-Perron

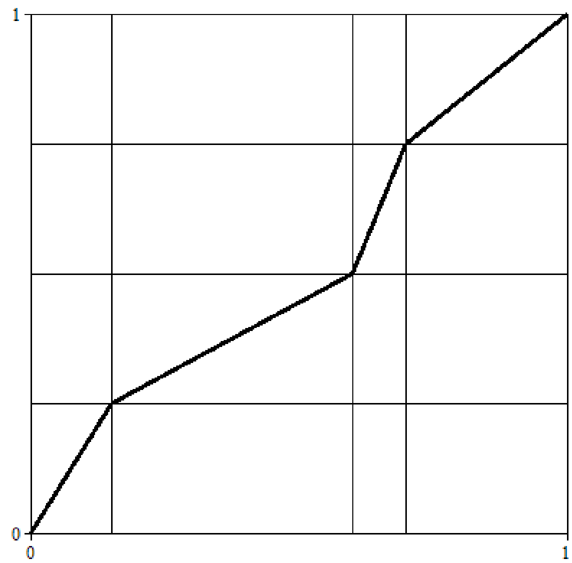


Figure 5.4: One example of $h(x)$

matrix we can calculate using the general definition, so we get a diagonal matrix $H = \{[n \cdot m(P_i)]/(b-a)\}_{i=1}^n$. Let $\mathscr{R}$ be the partition of I into n equal intervals, and g a piecewise constant function on $\mathscr{R}$:

$$g = (g_1, ..., g_n) = fH^{-1} = \left(\frac{f_i(b-a)}{n \cdot m(P_i)}\right)_{i=1}^n .$$

By Theorem (5.2.2), there exist a 3-band piecewise expanding map τ_0 such that $g = g\mathbb{M}_{\tau_0}$. Let $\tau = h^{-1} \circ \tau_0 \circ h$. See that τ is $\mathscr{P}$-semi-Markov, piecewise linear and expanding. Its Perron-Frobenius matrix $\mathbb{M}_\tau = \mathbb{M}_{h^{-1}}\mathbb{M}_{\tau_0}\mathbb{M}_h = H^{-1}\mathbb{M}_{\tau_0}H$, so

$$f\mathbb{M}_\tau = fH^{-1}\mathbb{M}_{\tau_0}H = g\mathbb{M}_{\tau_0}H = gH = f,$$

and therefore f is τ-invariant. $\square$

5.3 The $\mathscr{N}$-band Matrix

The 3-band matrix method is useful and has many good features, however, it has its limitations. When it comes to an invariant density on the partition with a huge number of subintervals, the 3 "bands" are so unremarkable. For exmaple, in a $1000 \times$

1000 matrix, each row and column can just have at most 3 non-zero elements, and this reduces the usability of the matrix to some extent. Therefore it is natural to seek a generalized method to tackle this problem. In 2006, Gora managed to generalize the 3-band matrix into an N-band version. It is shown in the rest of this section.

Definition 5.3.1. *An $\mathscr{N}$-semi-Markov piecewise linear transformation is said to be an $\mathscr{N}$-band transformation, $\mathscr{N} = 2s+1$, $s \leq N-1$, if its Frobenius-Perron matrix $M_\tau = (p_{ij})$, $1 \leq i, j \leq N$, satisfies the condition: $p_{ij} = 0$ if $|i-j| > s$.*

Note that the number $\mathscr{N}$ of the "band" can only be odd. Similar to Theorem 5.2.1, the $\mathscr{N}$-band matrix satisfies the following theorem.

Theorem 5.3.1. *Let T be a $\mathscr{N}$-band transformation on an N-element uniform partition $\mathscr{R}$, $\mathscr{N} = 2s+1$, with Frobenius-Perron matirx $M_\tau = (p_{ij})$. Let $f = (f_1, f_2, ..., f_N)$ be any probabilistic density with $f_i > 0, i = 1, ..., N$. If*

$$f_i \cdot p_{i,j} = f_j \cdot p_{j,i} \tag{5.3}$$

for any $1 \leq i, j \leq N$, then f is T-invariant.

A method for constructing an $\mathscr{N}$-band matrix with given invariant density is presented below.

Let $\mathscr{N} = 2s+1, s \leq N-1$. Fix s non-negative constant $c_1, c_2, ..., c_s$ with $c_1 + c_2 + ... + c_s \leq 1$ and other s constants $d_1, d_2, ..., d_s$ such that $0 < d_i < 1, \quad 1 \leq i \leq s$. Firstly we construct the first row and the first column.

- For the first row, we set $p_{1,1+i}$, $1 \leq i \leq s$ as follows:

	$c_i < \frac{f_{1+i}}{f_1}$	$c_i \geq \frac{f_{1+i}}{f_1}$
$p_{1,1+i}$	c_i	$d_i \cdot \frac{f_{1+i}}{f_1}$

  Note that $p_{1,1+i} \leq c_i$.

- For the first column, define $p_{1,1} = 1-(p_{1,2}+...+p_{1,1+s})$ and $p_{1+i,1} = p_{1,1+i}(\frac{f_1}{f_{1+i}})$. Note that $0 \leq p_{1+i,1} \leq 1$, $i = 0, ..., s$.

- Set $p_{1,j} = 0$ and $p_{j,1} = 0$ for $j > 1 + s$.

Now we construct the second row and the second column.

- The element $p_{2,1}$ has been defined.

- For the rest of the second row $p_{2,2+i}$, $i = 1, 2, ..., s - 1$, we define it as follows:

	$c_i(1 - p_{2,1}) < (1 - p_{2+i,1})\frac{f_{2+i}}{f_2}$	$c_i(1 - p_{2,1}) \geq (1 - p_{2+i,1})\frac{f_{2+i}}{f_2}$
$p_{2,2+i}$	$c_i(1 - p_{2,1})$	$d_i(1 - p_{2+i,1}) \cdot \frac{f_{2+i}}{f_2}$

  For $p_{2,2+s}$:

	$c_s(1 - p_{2,1}) < \frac{f_{2+s}}{f_2}$	$c_s(1 - p_{2,1}) \geq \frac{f_{2+s}}{f_2}$
$p_{2,2+s}$	$c_s(1 - p_{2,1})$	$d_s\frac{f_{2+s}}{f_2}$

  Note that $p_{2,2+i} \leq c_i(1 - p_{2,1})$, $i = 1, ..., s$.

- Now we can define the second column as $p_{2,2} = 1 - (p_{2,1} + p_{2,3} + ... + p_{2,2+s})$ and $p_{2+i,2} = p_{2,2+i} \cdot \frac{f_2}{f_{2+i}}$, $i = 1, ..., s$. Note that $0 \leq p_{2+i,2} \leq 1$, $i = 0, ..., s$.

Assume that the rows and columns with indices less than $k - 1$ have been defined. Next we should construct the kth row and the kth column.

- The elements $p_{k,j}$ has been defined for $j < k$.

- Define elements $p_{k,k+i}$, $i = 1, ..., s$:

	$c_i(1 - \sum_{j=1}^{k-1} p_{k,j}) < (1 - \sum_{j=1}^{k-1} p_{k+i,j})\frac{f_{k+i}}{f_k}$	else
$p_{k,k+i}$	$c_i(1 - \sum_{j=1}^{k-1} p_{k,j})$	$d_i(1 - \sum_{j=1}^{k-1} p_{k+i,j})\frac{f_{k+i}}{f_k}$

  Note that $p_{k,k+i} \leq c_i(1 - \sum_{j=1}^{k-1} p_{k,j})$ for $i = 1, 2, ..., s$.

- Define $p_{k,k} = 1 - \sum_{j=1}^{k+s}{}_{j \neq k} p_{k,j}$ and $p_{k+i,k} = p_{k,k+i} \cdot \frac{f_k}{f_{k+i}}$, $i = 1, ..., s$. Note that $0 \leq p_{k+i,k} \leq 1$, $i = 0, ..., s$.

- Set $p_{k,j} = 0$ and $p_{j,k} = 0$ for $j > k + s$.

The matrix generated from the above construction satisfies Equation (5.3.1). Therefore it preserves the given density f.

Example 4.3.1 Let $f = \frac{4}{15}(2, 3, 1, 6, 3)$, so $N = 5$. To build a 5-band matrix, choose $s = 2$. Following the above steps, fix $c_1 = 0.3$, $c_2 = 0.7$, $d_1 = 0.5$, $d_2 = 0.8$. So

- $c_1 = 0.3 < \frac{3}{2} = \frac{f_2}{f_1} \quad \Longrightarrow \quad p_{1,2} = c_1 = 0.3;$
- $c_2 = 0.7 > \frac{1}{2} = \frac{f_3}{f_1} \quad \Longrightarrow \quad p_{1,3} = d_2 \cdot \frac{f_3}{f_1} = 0.4;$
- $p_{1,1} = 1 - p_{1,2} - p_{1,3} = 0.3;$
- $p_{2,1} = p_{1,2} \cdot \frac{f_1}{f_2} = 0.2;$
- $p_{3,1} = p_{1,3} \cdot \frac{f_1}{f_3} = 0.8;$

Similarly we can determine the rest of rows and columns and finally obtain a 5-band matrix:

$$\begin{pmatrix} \frac{3}{10} & \frac{3}{10} & \frac{2}{5} & 0 & 0 \\ \frac{1}{5} & \frac{31}{150} & \frac{1}{30} & \frac{14}{25} & 0 \\ \frac{4}{5} & \frac{1}{10} & 0 & \frac{3}{100} & \frac{7}{100} \\ 0 & \frac{7}{25} & \frac{1}{200} & \frac{1001}{2000} & \frac{429}{2000} \\ 0 & 0 & \frac{7}{300} & \frac{429}{1000} & \frac{1643}{3000} \end{pmatrix},$$

Chapter 6

Conclusion

In this thesis we presented the inverse Frobenius-Perron problem and explained its solutions devised in different ways. The conjugation approach allows us to construct doubly symmetric maps and symmetric maps with non-symmetric densities. The differential equation approach considers even and translationally symmetric maps. Pingel's group gives us a more general way to generate a unimodal, while not necessarily symmetric, map. Another contribution of this thesis is generalizing Pingel's method to piecewise increasing maps. By comparing the above methods we found that results are compatible in some circumstance. The inverse problem can also be solved by matrix method. Each row of the matrix generated by Roger's method (Section 5.1) is either "full" or of only one element in diagonal, while the 3-band matrix or $\mathscr{N}$-band matrix can never be "full". So the their intersection contains only the unit matrix. The above methods tackle the inverse Frobenius-Perron Problem in different situations, and they have their own features. It is not easy to say any one of them is the best solution. The question of choosing which method depends on the kind of map we need.

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