Affine Integral Quantization on a Coadjoint Orbit of the Poincaré Group in $(1+1)$-space-time Dimensions and Applications

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Abstract

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In this thesis we study an example of a recently proposed technique of integral quantization by looking at the Poincaré group in $(1+1)$-space-time dimensions, denoted $\mathcal{P}_+^\uparrow(1,1)$, which contains the affine group of the line as a subgroup. The cotangent bundle of the quotient of $\mathcal{P}_+^\uparrow(1,1)$ by the affine group has the natural structure of a physical phase space. We do an integral quantization of functions on this phase space, using coherent states coming from a certain representation of $\mathcal{P}_+^\uparrow(1,1)$. The representation in question corresponds to the "zero-mass" or "light-cone" situation, which when restricted to the affine subgroup gives the unique unitary irreducible representation of that group. This representation is also the one naturally associated to the above mentioned coadjoint orbit. The coherent states are labelled by points of the affine group and are obtained using the action of that group on a specially chosen vector in the Hilbert space of the representation. They satisfy a resolution of the identity, which can be computed using either the left or the right Haar measure of the affine group. The integral quantization is done using both choices and we obtain a relationship between the two quantized operators corresponding to the same phase space function.

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Dedication

To My Supervisor S. Twareque Ali

Contents

Chapter 1

Introduction

In this thesis we study the method of integral quantization and apply the procedure to quantize observables defined on a particular phase space, arising as a coadjoint orbit of the Poincaré group, $\mathcal{P}_+^\uparrow(1,1)$, in $(1+1)$-space-time dimensions. In physics, the full Poincaré group (in a four-dimensional space-time), is the the group of all relativistic transformations and space-time translations. The group we study here is much simpler, since it only holds on a two-dimensional model of space-time. We look at a particular coadjoint orbit of this group, as phase space of our system and obtain coherent states labelled by points of this space. These coherent states are constructed using the unitary irreducible representation of $\mathcal{P}_+^\uparrow(1,1)$ which is associated to this coadjoint orbit. It turns out that when this representation is restricted to the affine subgroup of $\mathcal{P}_+^\uparrow(1,1)$, one obtains the unique unitary irreducible representation of the affine group. An integral quantization of the classical observables on this phase space is carried out using these coherent states, now considered as coherent states of the affine group, or equivalently, as coherent states of $\mathcal{P}_+^\uparrow(1,1)$, modulo its affine subgroup. In doing this we follow the general theory for constructing such coherent states as outlined, for example, in [19]. The corresponding quantized operators live on the Hilbert space of the representation. Since the coherent states satisfy two different resolutions of the identity, depending on whether the left or the right Haar measure of the affine group is used, two different though equivalent quantizations are therefore obtainable. We also find a relationship between the operators corresponding to the two quantizations.

The canonical technique of quantization was formalized, by physicists and mathematicians, following the birth of quantum mechanics in the early part of the twentieth century (between 1900 and 1925). Since then several different techniques of quantization have been developed, the literature for which is vast. Reviews of some of these techniques may be found, for example, in [21], [16] or [22]. Briefly, the problem is as follows: classical mechanics lives on a phase space, Γ, and is defined by the symplectic geometry of this space. This is an even dimensional differential manifold and comes equipped with a two-form, Ω. The classical observables are the functions on the phase space Γ, which obey the Poisson bracket relation. In quantum mechanics (QM), observables are self adjoint operators on a Hilbert space $\mathcal{H}$, which obey the corresponding commutations relations. Quantization is the process of making this transition from the classical to the quantum observables. Briefly, the goal of a general theory of quantization is two-fold:

1. To understand how a quantum system is related to its original classical counterpart (the classical limit), and;
2. To understand how to pass from a classical system to its related quantum counterpart (quantization).

The original concept of quantization, usually referred to as canonical quantization, going back to Weyl [4], Von Neumann [8], and Dirac [15] consists in assigning to the observables of classical mechanics, which are real-valued functions $f(p, q)$ of $(p, q) \in \mathbb{R}^n \times \mathbb{R}^n := \Gamma$, on the phase space, self-adjoint operators Q_f on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ in a way that is discussed further in Chapter 2 Section 2.1 (following [16]).

Let $q_i, p_i,\ i = 1, 2, ..., n$ be the canonical position and momentum coordinates, respectively, of a free classical system with n degrees of freedom. Then their quantized counterparts, $\hat{q}_i,\ \hat{p}_i$ are to be realized as operators on the Hilbert space $\mathcal{H} =$

$L^2(\mathbb{R}^n, dx)$ by

$$(\hat{q}_i\psi)(x) = x_i\psi(x) \quad (\hat{p}_i\psi)(x) = -i\hbar\frac{\partial}{\partial x_i}\psi(x) \tag{1.1}$$

on an appropriately chosen dense set of vectors ψ on $\mathcal{H}$. This simple procedure is known as canonical quantization. The Stone-von Neumann *uniqueness theorem* [8] states that, up to unitary equivalence, this is the only representation which realizes the canonical commutation relations (CCR):

$$[\hat{q}_i, \hat{p}_i] = i\hbar\delta_{ij} \quad i, j = 1, 2, 3, ..., n \tag{1.2}$$

irreducibly on a separable Hilbert space.

Recently, a general technique of quantization, termed *integral quantization,* has been elaborated in [5], [9] and the book [19], Chapter 11. This method is not necessarily based on a phase space and has the theory of quantum measurements as its backdrop. In [6] the role that operator-valued measures (OVM) can play in the above quantization technique has been described. One needs to start here with a measure space, which is the space consisting of the possible outcomes of a given physical measurement. This space equipped with a measure, which could, for instance, be a probability measure or a measure coming from a group or one of its cosets.

In this thesis we obtain some specific results using this technique of quantization. Once a measure space has been fixed and an OVM on it, a classical observable (which is now a random variable or real-valued function on this measure space) is quantized by integrating the function with respect to the OVM. Of course, crucial to this procedure is the choice of the OVM so that a sufficiently large class of random variables yield proper self adjoint operators after quantization. If the physical system has an underlying symmetry group, it generally reflects itself in a covariance property of the OVM. Thus, for example, using coherent states of the symmetry group, one could build an OVM, which would then incorporate this covariance property. Coherent state quantization based on the positive operator valued measure (POVM) where the integral quantization based on the OVM so that the integral quantization has constructed in a more general sense.

1.1 Main contributions

The main contribution of this thesis consists of two parts:

The first part is the explicit computation of some results, appearing in the literature and which we use here. Specifically, we compute the formula for the adjoint and coadjoint actions of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$ (see Section 4.1.2 and 4.1.3). We also make explicit the connection between the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$ and the affine group, which appears as a subgroup of $\mathcal{P}_+^\uparrow(1,1)$ (see Section 4.1.5 and 4.2). We then construct the particular coadjoint orbit of $\mathcal{P}_+^\uparrow(1,1)$, which serves as our physical phase space on which we then do a quantization (see Section 4.1.5). We also figure out the generators of the affine group G_{aff}^+ and their representations on the Hilbert space $L^2(\mathbf{R}^{>0}, \frac{dx}{x})$ (see proposition 4.3.1).

The second part consists of some specific results. We quantize a number of real valued functions (see Table (4.2) and (4.3)) and obtain a relationship between the two quantized operators corresponding to the left and right Haar measures (see Lemma 4.4.2). Finally, we identify the observables for which the quantization is canonical (see Proposition 4.4.1). We also explicitly compute a particular reproducing kernel in the Hilbert space $L^2(\mathbf{R}^{>0}, \frac{dx}{x})$ (see Proposition 4.4.2).

1.2 Organization of the thesis

The rest of this thesis is organized as follows:

In Chapter 2, we introduce to the integral quantization, a procedure based on OVM's and resolutions of the identity. We also revisit the general construction of the covariant integral quantization procedure in detail.

In Chapter 3, we give three examples of covariant integral quantization, following [5] and [9]. based on the Weyl-Heasienberg, affine and $SU(2)$ groups.

Chapter 4 is devoted to applying the method of integral quantization on the Poincaré group in (1+1)-dimensional space-time and looking at some of its applications to QM. For this purpose, we examine the properties of the Poincaré group in

$(1+1)$-dimensional space-time and the affine group. We then apply the technique of integral quantization to the group $\mathcal{P}_+^\uparrow(1,1)$ and obtain the quantized operators and identify those for which the quantization is canonical.

Finally, in Chapter 5 we conclude with a discussion on the merits of the method of integral quantization and indicate some possibilities for future work.

In appendix A we include some general background material on group theory. The mathematical structure of classical and quantum mechanics is discussed in appendix B. Appendix C incorporates all the mathematical computations related to Chapter 4.

Chapter 2

Quantization

The material in this Chapter is adapted from the article entitled, "Integral quantizations with two basic examples" [5]. We study here integral quantization, a procedure based on OVM's and resolutions of the identity. We also revisit the general construction of the covariant integral quantization procedure in some detail with the help of group representation theory which can play an important role in encoding the covariance property.

2.1 Quantization

In this section we look at the general definition of quantization as used in mathematics and physics. Generally, quantization is viewed as a way to associate to a certain algebra, $\mathcal{A}_{cl}$ of classicical observables, an algebra, $\mathcal{A}_{qt}$, of quantum observables. The definition of quantization that we adopt here follows that of Ref. [16], Section 1.1.

The original concept of quantization (nowadays usually referred to as canonical quantization), going back to Weyl [4], von Neumann [8], and Dirac [15], consists in assigning to the observables of classical mechanics, which are real-valued functions $f(p, q)$ of $(p, q) \in \mathbb{R}^n \times \mathbb{R}^n := \Gamma$ the phase space, self-adjoint operators Q_f on the

Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ in such a way that

(i) the correspondence $f \mapsto Q_f$ is linear as a map

$$Q : C(\Gamma) \mapsto \mathcal{A}(\mathcal{H}) \tag{2.1}$$

where, $C(\Gamma)$ is a vector space of appropriately chosen, in general complex valued, functions $f(x)$ on the phase space Γ and $\mathcal{A}(\mathcal{H})$ is a vector space of linear operators $Q_f \equiv A_f$ on some Hilbert space $\mathcal{H}$;

(ii) the function $f = 1 \in C(\Gamma)$ (i.e., constant everywhere) is mapped to the identity operator: $Q_1 = I \in \mathcal{A}(\mathcal{H})$;

(iii) for any function $\phi : \mathbb{R} \mapsto \mathbb{R}$ for which $Q_{\phi \circ f}$ and $\phi(Q_f)$ are well defined, $Q_{\phi \circ f} = \phi(Q_f)$; and

(iv) the operators Q_{p_j} and Q_{q_j} corresponding to the coordinate functions p_j, q_j $(j = 1, 2, ..., n)$ are given by

$$Q_{q_j}\psi = q_j\psi \, , Q_{p_j}\psi = -\frac{ih}{2\pi}\frac{\partial\psi}{\partial q_j} \text{ for } \psi \in L^2(\mathbb{R}^n, d\mathbf{q}) \tag{2.2}$$

The condition (iii) is usually known as the *von Newmann rule*. The domain of definition of the mapping $Q : f \mapsto Q_f$ is called the space of *quantizable observables*, which one would of course like to make as large as possible. Ideally, it should include at least the infinitely differentiable functions $C^\infty(\mathbb{R}^{2n})$, or some other convenient function space. The parameter h, on which the quantization map Q also depends, is usually a small positive number, identified with the *Planck constant*.

2.2 Integral quantization

In this section we look at the general mathematical construction of integral quantization and its covariance properties using group representation theory.

2.2.1 General setting of integral quantization

In the current section we present an original approch to quantization based on OVM's, under the generic name of the integral quantization. The so-called Berezin-Klauder-Toeplitz quantization, and more generally coherent state quantization, is one kind of integral quantization. The method starts out with a space (not necessarily the phase space), $\mathcal{X}$ and/or $C(\mathcal{X})$, and depends on various mathematical properties of it, such as a measure, a group structure, a topology, a manifold structure, etc. While the method fulfills the conditions (i) and (ii), mentioned above, and real valued function in $\mathcal{X}$ are mapped to self-adjoint operators, the commutation relations of these operators are not necessarily canonical, in the sense that the conditions (iii) and (iv) are not *a priori* required to be fulfilled.

We now consider a complex Hilbert space $\mathcal{H}$ and a classical measure space $(\mathcal{X}, \nu)$, where the measure ν will play a crucial role. Normally, we take the measure space $\mathcal{X}$ to be a locally compact topological space and the map (2.1) is taken to be continuous with respect to this topology. Let $\mathcal{X} \ni x \mapsto M(x) \in \mathcal{L}(\mathcal{H})$ (i.e., $M(x)$ is an $\mathcal{X}$-labeled family of bounded operators on a Hilbert space $\mathcal{H}$), satisfying the resolution of the identity $I \in \mathcal{H}$, i.e.,

$$\int_{\mathcal{X}} M(x) d\nu(x) = I, \tag{2.3}$$

The operators $M(x)$ are of positive, unit trace. The convergence of the integral (2.3) is in the weak sense, i.e., for any two vectors ϕ, ψ in the Hilbert space $\mathcal{H}$ such that,

$$\int_{\mathcal{X}} \langle\phi|M(x)|\psi\rangle d\nu(x) = \langle\phi|\psi\rangle. \tag{2.4}$$

A density matrix is a trace-class operator that, in general, describes a quantum system in a mixed state (a quantum system in a pure state is given by a projection operator for a single state vector). Explicitly, suppose that a quantum system is found to be in the states $|\psi_i\rangle$ with probabilities p_i; then the density matrix for this system is $\hat{\rho} = \sum_i p_i|\psi_i\rangle$. We could take for the operator $M(x)$ above a density matrix $M(x) = \rho(x)$ at each point x or possibly a more general operator. Then $M(x)d\nu(x)$ defines an OVM. For real or complex-valued functions $\phi(x) \in \mathcal{C}(\mathcal{X})$ an integral quantization is then formally defined by the linear map:

$$\phi \mapsto A_\phi := \int_{\mathcal{X}} M(x)\phi(x)d\nu(x). \tag{2.5}$$

We consider here all those functions ϕ such that the above operator $A_\phi \in \mathcal{A}(\mathcal{H})$ is defined on a dense subspace of $\mathcal{H}$. Now, if we take a positive unit trace operator ρ_0 then we can obtain a classical like formula. To do this we multiply both sides of equation (2.5) by ρ_0 and taking the trace in both sides we have the following calculations:

$$\begin{aligned} \rho_0 A_\phi &= \int_{\mathcal{X}} \rho_0 M(x)\phi(x)d\nu(x) \\ &= \int_{\mathcal{X}} \rho_0 \rho(x)\phi(x)d\nu(x). \end{aligned} \tag{2.6}$$

Therefore we have,

$$\begin{aligned} tr(\rho_0 A_\phi) &= \int_{\mathcal{X}} tr(\rho_0\rho(x))\phi(x)d\nu(x) \\ &= \int_{\mathcal{X}} \omega(x)\phi(x)d\nu(x) \end{aligned} \tag{2.7}$$

where, we have set $tr(\rho_0\rho(x)) = \omega(x)$ which can be thought of as an averaging of the original function f and is the probability distribution from $x \mapsto tr(\rho_0\rho(x))$ because of the resolution of the identity in equation (2.3). Physically, (2.7) is the expected value of the operator A_ϕ in the state ρ_0.

We now investigate the boundedness and positivity of $M(x)$ or one can analyze the operators A_ϕ, equation (2.5) in terms of the sesquilinear form,

$$\mathcal{H} \ni \psi_1, \psi_2 \mapsto B_\phi(\psi_1, \psi_2) = \int_{\mathcal{X}} < \psi_1|M(x)|\psi_2 > \phi(x)d\nu(x). \tag{2.8}$$

We now present how we can return to the classical limit form in $C(\mathcal{X})$ from the operators A_ϕ. This leads to the lower or covariant (Berezin) symbol $\check{\phi}(x)$ defined as,

$$A_\phi \mapsto \check{\phi}(x) := \int_{\mathcal{X}} \phi(x')tr(\tilde{\rho}(x)\rho(x'))d\nu(x'), \tag{2.9}$$

where, $X \ni x \mapsto \tilde{\rho(x)} \in \mathcal{L}^+(\mathcal{H})$ is another (or the same) $\mathcal{X}$- labelled family of positive unit trace operators. Provided the above integral (2.9) is well defined.

Hence combing the equation (2.5) and equation (2.9) we have the following map:

$$\phi \mapsto A_\phi \mapsto \check{\phi}(x). \tag{2.10}$$

The map (2.10) is the generalization of the so called *Segal-Bargmann transform*.

2.2.2 Covariant integral quantizations using Lie groups

We want the integral quantization to be covariant under certain symmetry groups and for this we need their representations on Hilbert spaces. A representation of a group G on a vector space V, over a field K, is a group homomorphism from G to $GL(V)$, the general linear group on V. So that a representation is a map $\rho : G \mapsto GL(V)$ such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \forall g_1, g_2 \in G$. We now explain in the sense in which group representation theory is important for building a group covariant integral quantization proposed by [5].

Let G be a Lie group with left Haar measure $d\mu(g)$ and let $g \mapsto U(g)$ be a unitary irrereducible representation of G in the Hilbert space $\mathcal{H}$. One uses an operator valued function U over the group G defined through the group representation. Consider now a bounded operator M on $\mathcal{H}$ and using the group representation $U(g)$, an operator valued function is defined by

$$g \mapsto M(g) := U(g)MU^\dagger(g). \tag{2.11}$$

The above operator valued function $M(g)$ is then used to quantize the real or complex valued functions on G. Consider now the formal operator R,

$$R := \int_G M(g)d\mu(g), \tag{2.12}$$

the integral being defined weakly. We must show that the above operator R must be a multiple of the identity. Multiplying both sides of (2.12) by $U(g_0)$ and $U^*(g_0)$ and from the left invariance of $d\mu(g)$ and using the role of the group representation we have the following calculations on the operator R.

$$\begin{aligned} U(g_0)RU^\dagger(g_0) &= \int_G U(g_0)M(g)U^\dagger(g_0)d\mu(g) \\ &= \int_G U(g_0)U(g)MU^\dagger(g)U^\dagger(g_0)d\mu(g) \\ &= \int_G U(g_0g)MU^\dagger(g_0g)d\mu(g) \\ &= \int_G M(g_0g)d\mu(g) \\ &= R, \end{aligned}$$

Hence , $U(g_0)\ R\ U^\dagger(g_0) = R$. This implies that $U(g_0)R = RU(g_0)$ and so that R commutes with all unitary operators $U(g)$, $\forall g \in G$.

At this point we invoke Schur's Lemma ([1], page 144):

Lemma 2.2.1. *A unitary representation U of G in $\mathcal{H}$ is irreducible if and only if the only operators commuting with all the U_x are scalar multiples of the identity.*

Using Schur's Lemma we have a new relation for the operator R, given by

$$R = c_M I, \text{ where } c_M = \int_G tr(\rho_0 M(g))d\nu(g). \tag{2.13}$$

The constant c_M, where the unit trace positive operator is chosen in order to make the integral converge, is a positive real number. If we take this constant c_M into the measure then the family of operators, $M(g)$ and equation (2.13) provide the resolution of the identity,

$$\int_G M(g)\frac{d\mu(g)}{c_M} = I \tag{2.14}$$

$$\int_G M(g)d\nu(g) = I, \text{ where, } d\nu(g) = \frac{d\mu(g)}{c_M}. \tag{2.15}$$

Thus, starting with (2.12), a general family of bounded operators, we can construct the resolution of the identity,

$$\int_G M(g)d\nu(g) = I, \tag{2.16}$$

and thus an integral quantization rule

$$\phi \mapsto A_\phi := \int_G \phi(g)\ M(g)\ d\nu(g). \tag{2.17}$$

We now look in more detail at the above procedure in the case of square-integrable unitary irreducible representations (UIR) U. To do this let us pick an admissible unit vector $|\eta\rangle$ i.e., one for which

$$c(\eta) := \int_G |\langle\eta|U(g)\eta\rangle|^2 d\mu(g) < \infty. \tag{2.18}$$

Consider the pure state $M := |\eta\rangle\langle\eta|$; then we can also recover the resolution of the identity using the family of *generalized coherent state vectors*,

$$|\eta_g\rangle = U(g)|\eta\rangle. \tag{2.19}$$

for the Lie group G. We again have the resolution of the identity,

$$\frac{1}{c(\eta)} \int_G |\eta_g\rangle\langle\eta_g| d\mu(g) = I \tag{2.20}$$

If we consider $\rho(g) = |\eta_g\rangle\langle\eta_g|$ and $d\nu(g) = \frac{d\mu(g)}{c(\eta)}$ then we have the resolution of the identity

$$\int_G \rho(g) d\nu(g) = I. \tag{2.21}$$

Similarly, the construction leading to (2.21) provides an integral quantization of complex valued functions $\phi(x)$ on the group G

$$\phi \mapsto A_\phi := \int_G \phi(g)\ \rho(g)\ d\nu(g). \tag{2.22}$$

Moreover, the quantized operators A_ϕ obey a covariance property. Indeed,

$$\begin{aligned} U(g')A_\phi U^\dagger(g') &= U(g') \int_G M(g)\phi(g) d\nu(g) U^\dagger(g') \\ &= \int_G U(g')M(g)U^\dagger(g')\phi(g)d\nu(g) \\ &= \int_G M(g'g)\phi(g)d\nu(g) \end{aligned}$$

Now, making the change of variable $g_0 = g'g$ and $\frac{dg_0}{dg} = g'$ and after some trivial manupulation we have $d\nu(g) = d\nu(g_0)$ and hence we have,

$$U(g')A_\phi U^\dagger(g') = \int_G M(g_0)\phi(g'^{-1}g_0)d\nu(g_0)$$

Therefore, the integral quantization (2.17) is covariant in the sense,

$$U(g')A_\phi U^\dagger(g') = A_{l(g')\phi}, \tag{2.23}$$

where the quantity $(l(g')\phi)(g_0) := \phi(g'^{-1}g_0)$ is the regular representation if $\phi \in L^2(G, d\mu(g))$. If we consider $M(g) = \rho(g)$ and choose $\tilde{\rho} = \rho$ we obtain the Berezin or heat kernel transform on G by using the equation (2.9) in equation (2.17).

$$\phi \mapsto \check{\phi}(g) := \int_X \phi(g')tr(\tilde{\rho}(g)\rho(g'))d\nu(g'). \tag{2.24}$$

Therefore, like (2.10) we also have the so called Segal-Bargmann transform for the Lie group G.

2.2.3 Covariant integral quantizatins using coset manifolds

Under the same group theoretical framework, we now assume that the representation U is not necessarily square-integrable in the sense of (4.61) over the whole Lie group G. In this situation, there exists a definition of square-integrability and covariant coherent states with respect to a left coset manifold $X = G/H$, where H is a closed subgroup of G equipped with a quasi-invariant measure ν [19] and $\lambda(g, .)$, the Radon-Nikodym derivative of the transformed measure $\nu_g, g \in G$ with respect to ν:

$$\lambda(g, x) = \frac{d\nu_g(x)}{d\nu(x)}. \tag{2.25}$$

Then $\lambda : G \times X \mapsto \mathbb{R}^+$ is a cocycle with the properties,

$$\lambda(g_1 g_2, x) = \lambda(g_1, x)\lambda(g_2, g_1^{-1}x) \text{ and } \lambda(e, x) = 1. \tag{2.26}$$

To do a covariant integral quantization on the above coset manifold, we define a Borel section $\sigma : X \mapsto G$ and let ν_σ be the quasi-invariant measure on X defined by

$$d\nu_\sigma(x) = \lambda(\sigma(x), x)d\nu(x); \tag{2.27}$$

where λ is defined by

$$\lambda(g, x)d\nu(x) = d\nu(g^{-1}x), \ (\ \forall g \in \ G). \tag{2.28}$$

Let U be a unitary representation of G in a Hilbert space $\mathcal{H}$ and consider n linearly independent vectors η^i , $i = 1, 2, ..., n$on $\mathcal{H}$. Then, similar to the equation (2.19) we again consider the following family of coherent state (CS) vectors:

$$\mathcal{Q}_\sigma = \{|\eta^i_{\sigma(x)}\rangle := U(\sigma(x))\,\eta^i,\ i = 1, 2, 3, ...,\ n;\ x \in X\}, \tag{2.29}$$

which we assume to be total in $\mathcal{H}$. We call this a family of *covariant CS* for U. We now consider the following bounded operator on the Hilbert space $\mathcal{H}$,

$$F = \sum_{i=1}^{n} |\eta^i\rangle\langle\eta^i|. \tag{2.30}$$

Again, using (2.29) and (2.30) we have the family of operator $F_\sigma(x)$ which is of course again bounded operator i.e.,

$$\begin{aligned} F_\sigma(x) &= \sum_{i=1}^{n} |\eta^i_{\sigma(x)}\rangle\langle\eta^i_{\sigma(x)}| \\ &= \sum_{i=1}^{n} |U(\sigma(x))\,\eta^i\rangle\langle U^\dagger(\sigma(x))\,\eta^i| \\ &= U(\sigma(x))\sum_{i=1}^{n} |\eta^i\rangle\langle\,\eta^i|\,U^\dagger(\sigma(x)) \\ &= U(\sigma(x))FU^\dagger(\sigma(x)). \end{aligned} \tag{2.31}$$

Using the $F_\sigma(x)$ in (2.31) we define the operator

$$R_\sigma := \int_X F_\sigma(x)\,d\nu_\sigma(x), \tag{2.32}$$

and assume that $R_\sigma, R_\sigma^{-1} \in \mathcal{L}(\mathcal{H})$. (The integral above is again assumed to hold in a weak sense, as in (2.4)). Then the representation U is said to be *square integrable mod*(H, σ), the vectors η^i , $i = 1, 2, ..., n$, are called *admissible mod*(H, σ) and the operator F a *resolution generator* for the representation.

Consider next the sections $\sigma_g(x) : X \mapsto G,\ g \in G$,

$$\begin{aligned} \sigma_g(x) &= g\,\sigma(g^{-1}x) \\ &= \sigma(g.g^{-1}x)\,h(g, g^{-1}x) \\ &= \sigma(x)\,h(g, g^{-1}x) \end{aligned} \tag{2.33}$$

where, h is the cocycle which is defined by

$$g\sigma(x) = \sigma(gx)\ h(g,x) \qquad \text{with} \qquad h(g,x) = \sigma(gx)^{-1}g\sigma(x) \in H. \tag{2.34}$$

Here, $h : G \times X \mapsto H$ is again a cocycle, with $h'(g,x) = [h(g^{-1},x)]^{-1}$ satisfying conditions similar to (2.26):

$$h'(g_1g_2,x) = h'(g_1,x)h'(g_2,g_1^{-1}x) \text{ and } h'(e,x) = e, \tag{2.35}$$

for all $g_1,\ g_2 \in G$ and all $x \in X$.

Like ν_σ we now let ν_{σ_g} be measure

$$d\nu_{\sigma_g}(x) = \lambda(\ \sigma_g(x),\ x)\ d\nu(x), \tag{2.36}$$

again constructed using (2.27) and the bounded operator $F_{\sigma_g}(x) = U_{\sigma_g}(x)FU^\dagger_{\sigma_g}(x)$. Using (2.26) and (2.31) with F as in (2.30) we have the following calculations,

$$\begin{aligned}
U(g)F_\sigma(x)U^\dagger(g) &= U(g)\sum_{i=1}^{n}|\eta^i_{\sigma(x)}\rangle\langle\eta^i_{\sigma(x)}|\ U^\dagger(g) \\
&= U(g)\ \sum_{i=1}^{n}|U(\sigma(x))\ \eta^i\rangle\langle U^\dagger(\sigma(x))\ \eta^i|\ U^\dagger(g) \\
&= U(g)\ U(\sigma(x))\sum_{i=1}^{n}|\eta^i\rangle\langle\ \eta^i|U^\dagger(\sigma(x))\ U^\dagger(g) \\
&= U(g)\ U(\sigma(x))FU^\dagger(\sigma(x))\ U^\dagger(g) \\
&= U(\sigma_g(t))FU^\dagger(\sigma_g(t)) \\
&= F_{\sigma_g}(t) \quad \text{where} \qquad t = gx,
\end{aligned} \tag{2.37}$$

with $U(g)U(\sigma(x)) = U(g\sigma(x)) = U(\sigma(gx)h(g,x)) = U(\sigma(t)h(g,g^{-1}t)) = U(\sigma_g(t))$ and and $U^\dagger(\sigma(x)).U^\dagger(g) = U^\dagger(\sigma_g(t))$, taking $gx = t;\ x = g^{-1}t$.

Next we see that

$$\begin{aligned}
U(g)R_\sigma U^\dagger(g) &= \int_X U(g)\; F_\sigma(x)\; U^\dagger(g)\; d\nu_\sigma(x) \\
&= \int_X U(g)\; \sum_{i=1}^{n} |\eta^i_{\sigma(x)}\rangle\langle\eta^i_{\sigma(x)}\; U^\dagger(g)\; d\nu_\sigma(x) \\
&= \int_X U(g)\; \sum_{i=1}^{n} |U(\sigma(x))\; \eta^i\rangle\langle U^\dagger(\sigma(x))\; \eta^i|\; U^\dagger(g)\; d\nu_\sigma(x) \\
&= \int_X U(g)\; U(\sigma(x)) \sum_{i=1}^{n} |\eta^i\rangle\langle\; \eta^i|U^\dagger(\sigma(x))\; U^\dagger(g)\; d\nu_\sigma(x) \\
&= \int_X U(g)\; U(\sigma(x)) F U^\dagger(\sigma(x))\; U^\dagger(g)\; d\nu_\sigma(x) \\
&= \int_X U(\sigma_g(t)) F U^\dagger(\sigma_g(t))\; d\nu_\sigma(g^{-1}t) \\
&= \int_X F_{\sigma_g}(t)\; d\nu_\sigma(g^{-1}t) \\
&= \int_X F_{\sigma_g}(t)\; d\nu_{\sigma_g}(t) := R_{\sigma_g}(t). \qquad (2.38)
\end{aligned}$$

The coset space X carries quasi invariant measures not necessarily invariant measures. In the case where X does not admit a left invariant measure R_σ need not commutes with all the operators $U(g),\ \forall g \in G$. In this situation to construct a resolution of the identity is not possible.

But, if U is square integrable mod$(H,\ \sigma)$, there is a general covariance property enjoyed by the positive operators defined by the weak integrals,

$$a_{\sigma_g}(\Delta) = \int_\Delta F_{\sigma_g}(x)\; d\nu_{\sigma_g}(x), \quad g \in G,\ \Delta \in \mathcal{B}(X). \qquad (2.39)$$

Using the cocycle prescribed above (2.33), it is easily verified that the covariance condition

$$U(g)a_{\sigma_g}(\Delta)U^\dagger(g) = a_{\sigma_g}(g\Delta), \quad U(g)R_\sigma U^\dagger(g) = R_{\sigma_g} \qquad (2.40)$$

holds. In the case where X admits a left invariant measure m and one takes $\nu = m$ and

$$U(h)FU^\dagger(h) = F\ ,\ h \in H, \qquad (2.41)$$

it is an immediate consequence of (2.33) that

$$U(g)R_\sigma U^\dagger(g) \quad = \quad R_\sigma. \qquad (2.42)$$

Hence, R_σ commutes with all operaotors $U(g)$, $\forall g \in G$.

Now, getting back to the Schur's Lemma (2.2.1) we have again $R_\sigma = C_F I$ under the condition that the representation U is irreducible. Then (2.20) allows us to have the following form of the resolution of the identity

$$\frac{1}{c_F}\int_X F_\sigma(x) d\nu_\sigma(x) = I. \tag{2.43}$$

Now, if we consider a unit trace positive operator, i.e., $\mathrm{tr}(F) = 1$, then the constant c_F is similar to the construction in equation (2.13) and

$$c_F = \int_X tr(F\ F_\sigma(x)) d\nu_\sigma(x). \tag{2.44}$$

Like in the previous construction, the resolution of the identity (2.43) allows us to write an integral quantization of complex-valued functions $\phi(x)$ on the left coset manifold X, i.e.,

$$\phi \mapsto A_\phi = \frac{1}{c_F}\int_X F_\sigma(x)\phi(x)\ d\nu_\sigma(x). \tag{2.45}$$

The covariance property of A_ϕ, with the assumption that U is square integrable mod (H, σ), can now be calculated. We have

$$U(g)A_\phi U^\dagger(g) \;=\; \frac{1}{c_F}\int_X U(g)F_\sigma(x)U^\dagger(g)\phi(x)\ d\nu_\sigma(x), \tag{2.46}$$

and making the change of variables $gx = t$ and using equation (2.37), (2.38) we get the general covariance relation,

$$\begin{aligned} U(g)A_\phi U^\dagger(g) &= \frac{1}{c_F}\int_X F_{\sigma_g}(t)\phi(g^{-1}t)\ d\nu_\sigma(g^{-1}t) \\ &= \frac{1}{c_F}\int_X F_{\sigma_g}(t)\phi(g^{-1}t)\ d\nu_{\sigma_g}(t) \\ &:= A^{\sigma_g}_{l(g)\phi}, \end{aligned} \tag{2.47}$$

where, $A^{\sigma_g}_\phi = \frac{1}{c_F}\int_X F_{\sigma_g}(t)\ \phi(t)\ d\nu_{\sigma_g}(t)$ and $(l(g)\phi)(t) := \phi(g^{-1}t)$.

So far, we have studied integral quantization on the Lie group G and the coset manifold $X = G/H$ and its covariance properties. Definitely, it is possible to establish similar results by replacing the operator F, by a more general bounded operator M provided integrability and weak convergence hold in the above expressions.

To summarize, in this Chapter we started with the definition of quantization and introduced the general construction of integral quantization with its covariance properties. Introducing the integral quantization, we can strongly argue that integral quantization scheme can analyze the quantization issue properly, it also analyzes the spectral properties of the operators A_ϕ using the functional properties of the lower symbol in equation (2.9). Integral quantization can also satisfy a covariance property under the assumption of irreducibility of the group representation and the boundedness property of the operator M on the Hilbert space $\mathcal{H}$. Finally, based on the machinery developed in this Chapter, we shall introduce three examples of covariant integral quantizations in Chapter 3 and then discover a further interesting application of covariant integral quantization in Chapter 4.

Chapter 3

Three examples of integral quantization

We illustrate the procedure of integral quantization with two examples based on the Weyl-Heasienberg group and the affine group in this Chapter. We also briefly look at the $SU(2)$ in this connection. The discussion is based on [5] and [9].

3.1 Weyl-Heisenberg covariant integral quantization

We now revisit the covariant integral quantization, based on the Weyl-Heisenberg group, G_{WH}. An arbitrary element $g \in G_{WH}$ is of the form $g = (\theta, q, p)$, $\theta \in \mathbb{R}$, $(q, p) \in \mathbb{R}^2$. We do a covariant integral quantization using a coset manifold of the group as described in Section 2.2.3. To follow that procedure, let us pick H as the phase subgroup Θ (the subgroup of elements $g = (\theta, 0, 0)$,$\theta \in \mathbb{R}$) and left coset space $X = G_{WH}/\Theta$ as the measure space. This measure space is identidfied as the euclidian plane or complex plane, $\mathbb{R}^2$ and the general element in it is parametrized by

(q,p). The invariant measure on this measure space, G_{WH}/Θ is $d\nu(q,p) = \frac{dq\,dp}{2\pi} = \frac{d^2z}{\pi}$. Consider now the section in the group G_{WH} which is:

$$\sigma : G_{WH}/\Theta \mapsto G_{WH}\ ,\ \sigma(q,p) = (0,q,p). \tag{3.1}$$

Let us choose a vector $\eta \in \mathcal{H}$ and then the family of Weyl-Heisenberg coherent state is the set,

$$\eta^s_{\sigma(q,p)} = U(\sigma(q,p))\eta^s \mid (q,p) \in G_{WH}/\Theta. \tag{3.2}$$

Then the operator integral (2.43) give us the following resolution of the identity,

$$\int_{G_{WH}/\Theta} |\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}| = I. \tag{3.3}$$

The coherent state $\eta^s_{\sigma(q,p)}$ are labeled by points in the homogeneous space $X = G_{WH}/\Theta$ through the action of the unitary operators $U(\sigma(q,p))$ of a UIR of G_{WH} on a fixed vector η. On this homogeneous space X, the statement of square- integrabilty of the UIR, U is satisfied with a resolution of the identity (3.3). Now, using the complex number, $z \in \mathbb{C}$ the above operator $U(\sigma(q,p))$ can be written into the following displacement operator,

$$\mathbb{C} \ni z \mapsto D(z) = e^{za^\dagger - \bar{z}a} = U(\sigma(q,p)), \tag{3.4}$$

where, $a, a^\dagger$ are the lowering and raising operators in the seperable (complex) Hilbert space $\mathcal{H}$ with the orthonormal basis $\{|e_n\rangle\}$ are defined by their action on this basis

$$a|e_n\rangle = \sqrt{n}|e_{n-1}\rangle,\ a|e_0\rangle = 0,\ a^\dagger|e_n\rangle = \sqrt{n+1}|e_{n+1}\rangle. \tag{3.5}$$

The Weyl-Heisenberg algebra is generated by the triple $\{a, a^\dagger, I\}$ obeying the canonical commutation relation $[a, a^\dagger] = I$. Let M be a bounded operator on the Hilbert space $\mathcal{H}$ with the weight function $\bar{w}(z)$ on the complex plane with the normalization condition $\bar{w}(0) = 1$, defined by,

$$M := \int_{\mathbb{C}} \bar{w}(z) D(z) \frac{d^2z}{\pi}. \tag{3.6}$$

Then, the family $M(z) := D(z)MD(z)^\dagger$ of displaced operators under the unitary action $D(z)$ satisfies the resolution of the identity

$$\int_{\mathbb{C}} M(z) \frac{d^2z}{\pi} = I. \tag{3.7}$$

We can easily see that the displacement operator $D(z)$ has the property $D(z)D(z')D(z)^\dagger = e^{z\bar{z}'-\bar{z}z'}$, leading to the useful formula

$$\int_{\mathbb{C}} e^{z\bar{z}'-\bar{z}z'} \frac{d^2z}{\pi} = \pi\delta^2(z). \tag{3.8}$$

Recall also the normalization condion $\bar{w}(0) = 1$ and that $D(0) = I$. The resolution of the identity (3.7) allows us now to quantize a function $\phi(z)$ on the homogeneous space G_{WH}/Θ,

$$\phi \mapsto A_\phi = \int_{\mathbb{C}} \phi(z)M(z)\frac{d^2z}{\pi} \tag{3.9}$$

Using equation (3.8) we have the lower symbol

$$\hat{\phi}(z) = \int_{\mathbb{C}} e^{z\bar{\xi}-\bar{z}\xi}\phi(\xi) \frac{d^2\xi}{\pi}. \tag{3.10}$$

Using the equation (3.10) we have the alternative expression of (3.9)

$$A_\phi = \int_{\mathbb{C}} \bar{w}(z)D(z)\hat{\phi}(z) \frac{d^2z}{\pi}. \tag{3.11}$$

In the following table we list some functions quantized using this procedure in [[5], [9]].

Table 3.1: Quantized version on $\mathcal{H}$ for the homogeneous space G_{WH}/Θ

Complex function	Operators in the Hilbert space $\mathcal{H}$
$\phi = z$	$A_z = a\bar{w}(0) - \partial_{\bar{z}}\bar{w}\|_{z=0} = a$
$\phi = \bar{z}$	$A_{\bar{z}} = a^\dagger\bar{w}(0) - \partial_z\bar{w}\|_{z=0} = a^\dagger$
$\phi = q = \frac{z+\bar{z}}{\sqrt{2}}$	$A_q = \frac{a+a^\dagger}{\sqrt{2}}$
$\phi = p = \frac{z-\bar{z}}{\sqrt{2}i}$	$A_p = \frac{a-a^\dagger}{i\sqrt{2}}$
$\phi = z^2$	$A_{z^2} = a^2\bar{w}(0) - 2a\partial_{\bar{z}}\bar{w}\|_{z=0} + \partial_{\bar{z}}^2\bar{w}\|_{z=0}$
$\phi = \bar{z}^2$	$A_{\bar{z}^2} = (a^\dagger)^2\bar{w}(0) + 2a^\dagger\partial_z\bar{w}\|_{z=0} + \partial_z^2\bar{w}\|_{z=0}$
$\phi = qp = \frac{z+\bar{z}}{\sqrt{2}}$	$A_{qp} = A_qA_p - \frac{i}{2} + \partial_{\bar{z}}^2\bar{w}\|_{z=0} - \partial_z^2\bar{w}\|_{z=0} - (\partial_{\bar{z}}\bar{w}\|_{z=0})^2 + (\partial_{\bar{z}}\bar{w}\|_{z=0})^2$
$\phi = q^2$	$A_{q^2} = (A_q)^2 + \frac{1}{2}[(\partial_z - \partial_{\bar{z}})^2\bar{w}]_{z=0} - \frac{1}{2}(\partial_z\bar{w}\|_{z=0} - \partial_{\bar{z}}\bar{w}\|_{z=0})^2$
$\phi = p^2$	$A_{p^2} = (A_p)^2 - \frac{1}{2}[(\partial_z + \partial_{\bar{z}})^2\bar{w}]_{z=0} + \frac{1}{2}(\partial_z\bar{w}\|_{z=0} + \partial_{\bar{z}}\bar{w}\|_{z=0})^2$
$\phi = \|z\|^2$	$A_{\|z\|^2} = a^\dagger a + \frac{1}{2} - \partial_z\partial_{\bar{z}}\bar{w}\|_{z=0} + a\partial_z\bar{w}\|_{z=0} - a^\dagger\partial_{\bar{z}}\bar{w}\|_{z=0}$

where, $A_qA_p = A_pA_q + i,\ \bar{w}(0) = 1$ with the CCR, $[A_q, A_p] = iI,\ A_qA_p - A_pA_q = i[a, a^\dagger]$ and therefore the CR becomes the CCR. Moreover, $|z|^2$ is the energy for the harmonic oscillator and $A_{|z|^2}$ is the quantum energy operator. Also minimum of the quantum potential energy $E_m = [\min(A_{q^2}) + \min(A_{p^2})]/2 = -\partial_z\partial_{\bar{z}}\bar{w}|_{z=0}$ and the ground state energy $E_0 = \frac{1}{2} - \partial_z\partial_{\bar{z}}\bar{w}|_{z=0}$. Then the difference between them is $E_0 - E_m = \frac{1}{2}$.

3.2 Affine or wavelet covariant integral quantization

We now look at the covariant integral quantization, based on the affine or wavelet group. Following [[5], [9]] the measure space (X, ν), where X is the upper half plane $\{(q,p)|\ p \in \mathbb{R},\ q > 0\}$ with the left invariant measure $dqdp$ and the multiplication rule,

$$(q,p)(q_0,p_0) = (qq_0, \frac{p_0}{q} + p),\ \text{with}\ q \in \mathbb{R}^{>0},\ p \in \mathbb{R}, \tag{3.12}$$

is viewed as as the affine group, $\mathrm{Aff}_+(\mathbb{R})$, of the real line, acting as $x \mapsto (q,p)x = p+qx$. This group has two non-equivalent UIR's, $U_\pm$ and both are square integrable. The UIR $U_+ \equiv U$ is realized in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx)$:

$$(U(q,p)\psi)(x) = \frac{e^{ipx}}{\sqrt{q}}\psi(\frac{x}{q}). \tag{3.13}$$

Picking a unit norm vector $\eta \in L^2(\mathbb{R}^{>0}, dx) \bigcap L^2(\mathbb{R}^{>0}, \frac{dx}{x})$ and using the equation (2.19) we have the following family of coherent states:

$$|q,p\rangle = U(q,p)|\psi\rangle. \tag{3.14}$$

Then the square integrabilty of the UIR U_+ and the admissibility condition on the η yield the resolution of the identity,

$$\int_X |q,p\rangle\langle q,p|\ \frac{dqdp}{2\pi c_{-1}}\ \text{, where } c_\gamma := \int_0^\infty |\psi(x)|^2 \frac{dx}{x^{2+\gamma}}. \tag{3.15}$$

The corresponding quantization, based on the resolution of the identity (3.15) is:

$$\phi \mapsto \int_X \phi(q,p)\ |q,p\rangle\langle q,p|\ \frac{dqdp}{2\pi c_{-1}}. \tag{3.16}$$

In [9] and [5] the following quantizations have been obtained:

$$A_p = P = -i\frac{\partial}{\partial x},\ A_{q^\beta} = \frac{c_{\beta-1}}{c_{-1}}Q^\beta,\ (Qf)(x) = xf(x). \tag{3.17}$$

This quantization is canonical for q,p with the commutation relation $[A_q, A_p] = \frac{c_0}{c_{-1}}\, iI$ and the quantization of the kinetic energy gives

$$A_{p^2} = -\frac{d^2}{dx^2} + KQ^{-2},\ \text{where } K = K(\psi) = \int_0^\infty (\psi'(u))^2\ u\ \frac{du}{c_{-1}}. \tag{3.18}$$

3.3 Covariant integral quantization on $SU(2)$

In this section we consider the group $SU(2)$ and taking the UIR U corresponding to spin one-half. Following the same procedure as a covariant integral quantization based on the $SU(2)$ may be worked out. Detailed discussions may be found in [9].

So far, in Chapter 3, we have elaborated on the general construction of the method of covariant integral quantization with two examples and one more example is mentioned briefly. Those examples showed that the integral quantization satisfies the covariance property under the assumption of the irreducibility of the group representation used and the boundedness property of the operator M. In the next Chapter, we shall explore an interesting example of covariant integral quantization and it's applications in signal analysis.

Chapter 4

A Poincaré covariant integral quantization in (1+1)-dimensional space-time and the affine group

In this Chapter, we examine the properties of the Poincaré group in (1+1)-dimensional space-time and its affine subgroup. We then show how to connect the Poincaré group in $(1+1)$-dimensional space-time to the affine group G_{aff}^{+}. Ultimately our goal is to quantize on a phase space. To do this we construct this phase space using $\mathcal{P}_{+}^{\uparrow}(1,1)$ so that we work on a particular coadjoint orbit, which is then identified as a phase space. Mathematically, this space looks like the cotangent bundle of a "light cone", i.e., the cotangent bundle of the open half line (giving the open half plane). On this phase space, we construct the coherent states and then using these we find the resulting quantization by using the fundamental tool of integral quantization, following the construction outlined in Chapter 2. We then develop the integral quantization for the Poincaré group and introduce a number of quantized operators, some of which quantize canonically. Finally, we obtain a relation between the quantized versions of observables, obtained using the left and right actions of the group on the phase space.

4.1 The Poincaré group

In this section, we look at some mathematical properties of the full Poincaré group $\mathcal{P}$ and then on $\mathcal{P}_+^\uparrow(1,1)$, the Poincaré group in a $(1+1)$-dimensional space-time. The Poincaré group is the group of inhomogeneous Lorentz transformations on the 4 dimensional space-time. Briefly, the general Poincaré group $\mathcal{P}$ is characterized by ten parameters, six parameters of the homogeneous Lorentz group $\mathcal{L}$ and four parameters for the translations. The group contains two useful subgroups; one is the homogeneous Lorentz group $\mathcal{L}$ with elements denoted $(\Lambda, 0)$ and the other subgroup $\mathcal{S}$ of four dimensional translations (I, b), which is also an abelian invariant subgroup of $\mathcal{P}$. Therefore, any Poincaré transformation can be written as the product of a pure translation and a homogeneous Lorentz transformation

$$(\Lambda, b) = (I, b)(\Lambda, 0). \tag{4.1}$$

The set of 4×4 real or complex matrices that satisfy the equation

$$\Lambda^T g \Lambda = g, \ \det(\Lambda) = 1, \ \text{where } g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.2}$$

is called the real and complex, homogeneous Lorentz group $\mathcal{L}(\mathbb{R})$ and $\mathcal{L}(\mathbb{C})$ respectively.

The multiplication in $\mathcal{P}$ of two transformations is

$$g, g' \in \mathcal{P} \implies g'g = (\Lambda', b')(\Lambda, b) = (\Lambda'\Lambda, \Lambda' b + b') \tag{4.3}$$

and clearly the unit element of the general Poncaré group is $(I, 0) \in \mathcal{P}$. The inverse element of general Poncaré group is:

$$(\Lambda, b)^{-1} = (\Lambda^{-1}, -\Lambda^{-1} b). \tag{4.4}$$

The above discussion shows that the Poincaré group, $\mathcal{P}$ is the semi-direct product of the two subgroups:

$$\mathcal{P} = \mathcal{S} \rtimes \mathcal{L}. \tag{4.5}$$

We next consider the Poincaré group in $(1+1)$-dimensional space-time which is also a semi-direct product of two similar subgroups (see equation (4.5)):

$$G = \mathcal{P}_+^{\uparrow}(1,1) = \mathbb{R}^2 \rtimes \mathrm{SO}_0(1,1). \tag{4.6}$$

Here $\mathrm{SO}_0(1,1)$ is connected part of $\mathrm{SO}(1,1)$. The group $\mathrm{SO}_0(1,1)$ is the proper Lorentz group in 1-time and 1-space dimensions. The subscript zero of $\mathrm{SO}_0(1,1)$ means that the forward direction of time is preserved under its action. We use Mackey's theory of induced representation to construct unitary irreducible representations (UIR) of $\mathcal{P}_+^{\uparrow}(1,1)$. According to this theory [2, 3], the UIR's of $\mathbb{R}^2 \rtimes \mathrm{SO}_0(1,1)$ are in one-to-one correspondence with the orbits of $\mathrm{SO}_0(1,1)$ in the dual space $\widehat{\mathbb{R}}^2$ (which we identify here with $\mathbb{R}^2$ itself). Each dual orbit contributes precisely one irreducible representation (we drop the one-dimensional representation arising from the trivial orbit $\{0\}$).

4.1.1 Matrix representations of the Poincaré group $\mathcal{P}_+^{\uparrow}(1,1)$

The elements of the Lorentz group $\mathrm{SO}_0(1,1)$ are generated by a single real number ϑ:

$$\mathbb{R} \ni \vartheta \mapsto \Lambda_\vartheta = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix}. \tag{4.7}$$

In this parametrization, $d\vartheta$ is the invariant measure, under both left and right actions and we choose this for the Haar measure on $\mathrm{SO}_0(1,1)$. We write a generic element of $\mathcal{P}_+^{\uparrow}(1,1)$ as (Λ, b), with $b = \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix} \in \mathbb{R}^2$ and Λ a matrix of the form (4.7). Then a group element of G can be written as

$$G \ni g = \begin{pmatrix} \Lambda_\vartheta & b \\ \mathbf{0}^T & 1 \end{pmatrix}; \qquad \mathbf{0}^T = (0,0)\ . \tag{4.8}$$

Therefore the dimension of the Poincaré group $\mathcal{P}_+^{\uparrow}(1,1)$ is 3 since it has three parameters ϑ; b_0 and $\mathbf{b}$. Explicitly we have the matrix form,

$$g = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta & b_0 \\ \sinh\vartheta & \cosh\vartheta & \mathbf{b} \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.9}$$

We can find the Lie algebra of the $\mathcal{P}_+^\uparrow(1,1)$ by differentiating with respect to the parameters of its one-parameter subgroups. The Lie algebra $\mathfrak{p}$, has three elements which are given by:

$$Y^0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.10}$$

We shall use the above lie algebra $\mathfrak{p}$ to find a lie subalgebra $\mathfrak{g}_{\text{aff}}$ of the affine subgroup G_{aff}^+ in Subsection 4.1.5.

4.1.2 Adjoint action for the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$

A general element of the Lie algebra $X \in \mathfrak{p}$ can be written as a linear combination

$$X = \theta Y^0 + \xi_1 Y^1 + \xi_2 Y^2 = \begin{pmatrix} 0 & \theta & \xi_1 \\ \theta & 0 & \xi_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X_q & x_p \\ 0^T & 0 \end{pmatrix}, \tag{4.11}$$

where $X_q := X_q(0,\theta) = \begin{pmatrix} o & \theta \\ \theta & 0 \end{pmatrix} = \theta\sigma_1,$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\theta \in \mathbb{R},\ x_p = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2$. Using the equation (A.18) the adjoint action of $\mathcal{P}_+^\uparrow(1,1)$ on p, given by $X \mapsto X' = (\Lambda, b)X(\Lambda, b)^{-1}$. Therefore,

$$\begin{aligned}
(\Lambda, b)X(\Lambda, b)^{-1} &= \begin{pmatrix} \Lambda_\vartheta & b \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} X_q & x_p \\ 0^T & 0 \end{pmatrix} \begin{pmatrix} \Lambda_\vartheta & b \\ \mathbf{0}^T & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \Lambda_\vartheta & b \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} X_q & x_p \\ 0^T & 0 \end{pmatrix} \begin{pmatrix} \Lambda_\vartheta^{-1} & -\Lambda_\vartheta^{-1} b \\ \mathbf{0}^T & 1 \end{pmatrix} \\
&= \begin{pmatrix} \Lambda_\vartheta & b \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} X_q\Lambda_\vartheta^{-1} & -X_q\Lambda_\vartheta^{-1} b + x_p \\ 0^T & 0 \end{pmatrix} \\
&= \begin{pmatrix} \Lambda_\vartheta X_q \Lambda_\vartheta^{-1} & -\Lambda_\vartheta X_q \Lambda_\vartheta^{-1} b + \Lambda_\vartheta\, x_p \\ 0^T & 0 \end{pmatrix} \\
&= \begin{pmatrix} \theta\sigma_1 & -\theta\sigma_1 b + \Lambda_\vartheta\, x_p \\ 0^T & 0 \end{pmatrix}
\end{aligned} \tag{4.12}$$

The above leads to the transformation $X_{q'} = \sigma_1\theta'$ and $x_{p'} = \Lambda_\vartheta\, x_p - b^T\sigma_1\theta$

$$\begin{pmatrix} \theta' \\ x_{p'} \end{pmatrix} = \begin{pmatrix} \theta \\ \Lambda_\vartheta\, x_p - b^T\sigma_1\theta \end{pmatrix}. \tag{4.13}$$

The adjoint action of the group element $g = (\Lambda_\vartheta, b)$ on the algebra $X = \begin{pmatrix} \theta \\ x_p \end{pmatrix}$ is then written in the matricial form

$$Ad(g)X = M(g)X = M(\Lambda_\vartheta, b)\begin{pmatrix} \theta \\ x_p \end{pmatrix},\; M(\Lambda_\vartheta, b) = \begin{pmatrix} 1 & \mathbf{0}^T \\ -b^T\sigma_1 & \Lambda_\vartheta \end{pmatrix}, \tag{4.14}$$

where, σ_1 is the 2×2 matrix defined by $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

4.1.3 Coadjoint action for the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$

We now compute the coadjoint action in the three parameter space of k' and γ'. Let $\mathfrak{p}^*$ denote the dual space of $\mathfrak{p}$ and Y^{0*}, Y^{1*}, Y^{2*} be the dual basis in $\mathfrak{p}^*$. A general element of the Lie algebra $X \in \mathfrak{p}$ can be written as a linear combination

$$X^* = \gamma Y^{0*} + k_0 Y^{1*} + \mathbf{k}Y^{2*}. \tag{4.15}$$

Using the equation (A.19) we now compute the coadjoint action in terms of the matrix $M^\sharp(\Lambda_\vartheta, b)$ acting on the variables γ, k. The coadjoint action is then:

$$Ad^\sharp(g)X^* = \begin{pmatrix} \gamma & k \end{pmatrix} M(g^{-1}) = M(g^{-1})^T \begin{pmatrix} \gamma \\ k \end{pmatrix},\; k = \begin{pmatrix} k_0 \\ \mathbf{k} \end{pmatrix} \tag{4.16}$$

where $M(g) = M(\Lambda_\vartheta, b)$ is defined in (4.14). So that the matrix $M(g^{-1})$ is easily obtained from $M(g)$ using the inverse of a group element $g^{-1} = (\Lambda_\vartheta^{-1}, -\Lambda_\vartheta^{-1}b)$:

$$M(g^{-1}) = M(\Lambda_\vartheta^{-1}, -\Lambda_\vartheta^{-1}b) = \begin{pmatrix} 1 & \mathbf{0}^T \\ b^T\sigma_1\Lambda_\vartheta^{-1} & \Lambda_\vartheta^{-1} \end{pmatrix}. \tag{4.17}$$

Folowing the coadjoint action (4.16), we have

$$\begin{pmatrix} \gamma' \\ k' \end{pmatrix} = \mathrm{Ad}^\sharp(\Lambda_a, b)\begin{pmatrix} \gamma \\ k \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}^T \\ b^T\sigma_1\Lambda_\vartheta^{-1} & \Lambda_\vartheta^{-1} \end{pmatrix}^T \begin{pmatrix} \gamma \\ k \end{pmatrix} = \begin{pmatrix} \gamma + b^T\sigma_1\Lambda_\vartheta^{-1}k \\ \Lambda_{\vartheta^{-1}}k \end{pmatrix}. \tag{4.18}$$

The above again leads to the following transformation

$$k' = \Lambda_{\vartheta^{-1}} k,\ \gamma' = \gamma + b^T \sigma_1 \Lambda_\vartheta^{-1} k = \gamma + b^T \sigma_1 k' \ \text{ where, } \ b^T = \begin{pmatrix} b_0 & \mathbf{b} \end{pmatrix}. \tag{4.19}$$

Using these adjoint and coadjoint actions of the Poincré group $\mathcal{P}_+^\uparrow(1,1)$, all the adjoint and coadjoint orbits of $\mathcal{P}_+^\uparrow(1,1)$ can now be calculated in the following Subsections which will be exploited to construct the phase space of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$.

4.1.4 Orbits and Induced representations of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$

We now look at the orbits of $\mathcal{P}_+^\uparrow(1,1)$ which obtained from different initial vectors or dual vectors. We consider those initial vectors to cover different cases. As Haar measure on $\mathcal{P}_+^\uparrow(1,1)$ we may take $d\mu_{\mathcal{P}_+^\uparrow}(\Lambda, b) = dbd\vartheta$, where, db being the Lebesgue measure on $\mathbb{R}^2$, and note that this group is unimodular. The following two cases that we may consider to find the orbits:

(i) Purely space like vector or purely time like vector ;

(ii) Mixed space-time like vector ; this also cover the trivial vector

Therefore, the dual orbits are conveniently labelled by elements of the set

$$X = \left\{ mv \mid v \in \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \},\ m \in \mathbb{R}^* \right\} \bigcup \left\{ \begin{pmatrix} \pm 1 \\ \pm 1 \end{pmatrix} \right\} \bigcup \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \tag{4.20}$$

by following [18], where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$; case (i) is the first set and case (ii) is the second set of vectors. The last five points represent Lebesgue-null sets in $\mathbb{R}^2$.
In mathematical sense, when a Lie group G acts on a set X (this process is called the group action). The orbit of $x \in X$ under G is the set

$$Gx := \{y = gx : g \in G\} \subset X. \tag{4.21}$$

We now compute the orbits of $SO_0(1,1)$ on which we can find the UIR group $\mathcal{P}_+^\uparrow(1,1)$, following the inducing construction of Mackey [2, 3]. For each orbit we shall have inequivalent representations. To find the orbits we cosider two different cases for the two set of vectors, X in (4.20).

Case (i) : Using definition (4.21) and the first set of vectors listed in (4.20),

$$\mathcal{O}_{v,m} = m\mathrm{SO}_0(1,1)v \ , \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \quad \text{or} \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{4.22}$$

the Lorentz group operates freely. For the purposes of this research, it will be enough to choose one orbit, with $v = (1,0)^T$ (T denoting matrix transpose) and a fixed $m > 0$. Then we have the following calculations:

$$m \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\vartheta \\ m\sinh\vartheta \end{pmatrix} . \tag{4.23}$$

We denote this orbit by $\mathcal{O}_m$

$$\mathbb{R} \ni \vartheta \mapsto k = \begin{pmatrix} k_0 \\ \mathbf{k} \end{pmatrix} := m\ \Lambda_\vartheta\ v \in \mathcal{O}_m \ . \tag{4.24}$$

Then using the equation (4.23) and (4.24), we have the following hyperbolic orbits in the upper half plane by the above parametrization, i.e.,

$$\begin{pmatrix} k_0 \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} m\cosh\vartheta \\ m\sinh\vartheta \end{pmatrix} ; \tag{4.25}$$

$$m^2\cosh^2\vartheta - m^2\sinh^2\vartheta = k_0^2 - \mathbf{k}^2 \implies k_0^2 = \mathbf{k}^2 + m^2 \ , \quad k_0 > 0 \ . \tag{4.26}$$

If we consider the second dual vector then we will also have hyperbolic orbits. Moreover, the important point on this orbit is the invariant measure, $d\vartheta = \dfrac{d\mathbf{k}}{k_0}$, which is the image of the Haar measure of $\mathrm{SO}_0(1,1)$, under this parametrization.

Case (ii) : Again using definition (4.21) and from the second set of vectors listed in (4.20), it will again be enough to choose any particular one of the dual orbits. But we are interested on the the photonic representation so that we take the orbit generated by the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

$$\mathcal{O}_0 = SO_0(1,1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \cosh\vartheta - \sinh\vartheta \\ -\cosh\vartheta + \sinh\vartheta \end{pmatrix} \tag{4.27}$$

for it we adopt the parametrization,

$$\mathbb{R}^{>0} \ni k_0 = e^{-\vartheta} \mapsto k = \begin{pmatrix} k_0 \\ \mathbf{k} \end{pmatrix} := \Lambda_{-\vartheta}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathcal{O}_0. \tag{4.28}$$

Then using the equation (4.27) and (4.28), we have the following orbit in the upper half plane by the above parametrization, i.e.,

$$\begin{pmatrix} k_0 \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} \cosh\vartheta - \sinh\vartheta \\ -(\cosh\vartheta - \sinh\vartheta) \end{pmatrix}, \quad k_0 = -\mathbf{k}\ . \tag{4.29}$$

This orbit is the half straight line through the origin in the upper half plane. Similerly, for the other dual orbits we will also have $k_0 = \mathbf{k}$. On those orbits of the $SO_0(1,1)$ invariant measure is again $\dfrac{d\mathbf{k}}{k_0}$.

Following the appendix (A.8), the Mackey's theory of induced representations [2, 3], the orbit $\mathcal{O}_m$ contributes the UIR U_m of $\mathcal{P}_+^{\uparrow}(1,1)$, which acts on the Hilbert space $L^2(\mathcal{O}_m, \dfrac{d\mathbf{k}}{k_0})$ in the manner

$$(U_m(\Lambda, b)f)(k) = e^{i\langle k\ ,\ b\rangle} f(\Lambda^{-1}k)\ , \qquad f \in L^2(\mathcal{O}_m, \frac{d\mathbf{k}}{k_0})\ , \tag{4.30}$$

where, $\langle , \rangle$ denoting the dual pairing between $\mathbb{R}^2$ and $\widehat{\mathbb{R}}^2$, which we take (following the physicists' convention) as $\langle k\ ,\ b\rangle = k_0 b_0 - \mathbf{k}\mathbf{b}$.

Again, the Mackey theory yields the unitary irreducible representation U_0, which acts on $L^2(\mathcal{O}_0, d\mathbf{k}/k_0)$ again in the manner (see (4.30))

$$(U_0(\Lambda, b)f)(k) = e^{i\langle k\ ,\ b\rangle} f(\Lambda^{-1}k)\ , \qquad f \in L^2(\mathcal{O}_0, d\mathbf{k}/k_0)\ , \tag{4.31}$$

which formally looks the same as the equation (4.30).

However, U_m and U_0 are inequivalent representations; physically U_m represents a mass-m particle and U_0 a mass-null particle (analogue of the photon). It will be useful to map them both unitarily to representations on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Making a change of variables in Lorentz group $SO_0(1,1)$ using $a = e^{\vartheta}$ in equation (4.7) we have

$$\Lambda_{\vartheta} = \frac{1}{2}\begin{pmatrix} e^{\vartheta}+e^{-\vartheta} & e^{\vartheta}-e^{-\vartheta} \\ e^{\vartheta}-e^{-\vartheta} & e^{\vartheta}+e^{-\vartheta} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+\frac{1}{a} & a-\frac{1}{a} \\ a-\frac{1}{a} & a+\frac{1}{a} \end{pmatrix} := \Lambda_a \ , \tag{4.32}$$

where $a \in \mathbb{R}^+$ and the inverse of Λ_a is $\Lambda_{\frac{1}{a}}$ with the properties

$$\begin{aligned} \Lambda_a \Lambda_{a'} &= \frac{1}{4}\begin{pmatrix} a+\frac{1}{a} & a-\frac{1}{a} \\ a-\frac{1}{a} & a+\frac{1}{a} \end{pmatrix}\begin{pmatrix} a'+\frac{1}{a}' & a'-\frac{1}{a}' \\ a'-\frac{1}{a}' & a'+\frac{1}{a}' \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} aa'+\frac{1}{aa'} & aa'-\frac{1}{aa'} \\ aa'-\frac{1}{aa'} & aa'+\frac{1}{aa'} \end{pmatrix} \\ &= \Lambda_{aa'} . \end{aligned} \tag{4.33}$$

In case (i), making the change of variables, $\mathbf{k} = \frac{1}{2}\left(-x + \frac{1}{x}\right)$, $k_0 = \frac{1}{2}\left(x + \frac{1}{x}\right)$, and using

$$\begin{aligned} \langle k, b\rangle &= k_0 b_0 - \mathbf{k}\mathbf{b} = \frac{1}{2}\left(x + \frac{1}{x}\right) b_0 - \frac{1}{2}\left(-x + \frac{1}{x}\right)\mathbf{b} \\ &= \frac{x}{2}(b_0 + \mathbf{b}) + \frac{1}{2x}(b_0 - \mathbf{b}) \end{aligned} \tag{4.34}$$

in the Hilbert space $L^2(\mathcal{O}_m, \frac{d\mathbf{k}}{k_0})$ of the representation (4.30) unitarily maps to the new Hilbert space $L^2(\mathbb{R}^{>0}, \frac{dx}{x})$ and the representation $U_m(\Lambda, b)$ to

$$(U_m(\Lambda_a, b)f)(x) = e^{i[\frac{x}{2}(b_0+\mathbf{b})+\frac{1}{2x}(b_0-\mathbf{b})]}\ f(ax) \ , \quad f \in L^2(\mathbb{R}^{>0}, \frac{dx}{x}) \ . \tag{4.35}$$

Similarly in case (ii), making the change of variables using,

$$\mathbf{k} = k_0 = \frac{x}{2} \ ,$$

the Hilbert space $L^2(\mathcal{O}_0, \frac{d\mathbf{k}}{k_0})$ of the representation (4.31) unitarily maps into the Hilbert space $L^2(\mathbb{R}^{>0}, \frac{dx}{x})$ and the representation $U_0(\Lambda, b)$ to

$$(U_0(\Lambda_a, b)f)(x) = e^{i\frac{x}{2}(b_0+\mathbf{b})}\ f(ax) \ , \quad f \in L^2(\mathbb{R}^{>0}, \frac{dx}{x}) \ . \tag{4.36}$$

We have the UIR representations (4.35) and (4.36) which can be identical under the condition $b_0 = \mathbf{b}$. This fact will be exploited to construct the covariant integral quantization using representation of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$.

4.1.5 Coadjoint orbits of Poincaré group $\mathcal{P}_+^\uparrow(1,1)$

We now study of the coadjoint orbits on $\mathcal{P}_+^\uparrow(1,1)$ and look at their relationship to the orbits appearing in (4.20). We now reparametrize $\mathcal{P}_+^\uparrow(1,1)$ by using (4.32) and a general element of $\mathcal{P}_+^\uparrow(1,1)$ as a 3×3 matrix,

$$(\Lambda_a, b) = \begin{pmatrix} \Lambda_a & b \\ \mathbf{0}^T & 1 \end{pmatrix}, \qquad \mathbf{0}^T = (0,0) \text{ and } b = \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix} \tag{4.37}$$

where Λ_a is as defined earlier in Section 4.1.1. The Lie algebra $\mathfrak{p}$ is generated by the triplet $\{Y^0, Y^1, Y^2\}$, given by (4.10) which satisfy the commutation relations,

$$[Y^0, Y^1] = Y^2, \qquad [Y^0, Y^2] = Y^1, \qquad [Y^1, Y^2] = 0. \tag{4.38}$$

The Lie subalgebra $\mathfrak{g}_{\text{aff}}$ of the affine subgroup G_{aff}^+ is generated by the two elements Y^0 and $T = \frac{1}{2}(Y^1 + Y^2)$, which satisfy the commutation relation,

$$[Y^0, T] = T \, . \tag{4.39}$$

The dual $\mathfrak{p}^*$ of the Lie algebra $\mathfrak{p}$ can be identified with $\mathbb{R}^3$. Writing an element in the dual as

$$\begin{pmatrix} \gamma \\ k \end{pmatrix}, \quad \text{with} \quad k = \begin{pmatrix} k_0 \\ \mathbf{k} \end{pmatrix} \in \mathbb{R}^2 \, , \quad \gamma \in \mathbb{R} \, , \tag{4.40}$$

following the Subsection 4.1.3, the coadjoint action it transforms as:

$$\text{Ad}^\sharp(\Lambda_a, b) \begin{pmatrix} \gamma \\ k \end{pmatrix} = \begin{pmatrix} \gamma + k_0 \mathbf{b} + \mathbf{k} b_0 \\ \Lambda_{a^{-1}} k \end{pmatrix} . \tag{4.41}$$

Using these relations, all the coadjoint orbits of $\mathcal{P}_+^\uparrow(1,1)$ in $\mathbb{R}^3 \simeq \mathfrak{p}^*$ can now be calculated. Indeed, it is easy to verify that these orbits are precisely the *cotangent bundles*, $\mathcal{O}_{v,m}^* = T^*\mathcal{O}_{v,m}$, of the orbits $\mathcal{O}_{v,m} = mSO_0(1,1)v$ defined in (4.20). In particular, $\mathcal{O}_m^* = T^*\mathcal{O}_m \simeq \mathbb{R}^2$ (the orbit under the coadjoint action of the vector $(\gamma, k)^T = (0,(1,0))^T$) and $\mathcal{O}_0^* = T^*\mathcal{O}_0 \simeq \mathbb{R}^{>0} \times \mathbb{R}$ (the orbit of the vector $(\gamma, k)^T = (0,(1,-1))^T$). Note that in both cases, the invariant measure on the coadjoint orbit, which in here is also the Kirillov 2-form, is $d\omega = \dfrac{d\mathbf{k}\, d\gamma}{k_0}$. Moreover the subgroup $H_m = (\mathbb{I}_2, (\mathbf{b}, 0))$, $\mathbf{b} \in \mathbb{R}$, of $\mathcal{P}_+^\uparrow(1,1)$, is the stability subgroup for the vector $(0,(1,0))^T$ defining the coadjoint orbit $\mathcal{O}_m^*$, while the subgroup $H = (\mathbb{I}_2, (\mathbf{b}, -\mathbf{b}))$, $\mathbf{b} \in \mathbb{R}$, is the stability subgroup for the vector $(0,(1,-1))^T$ defining the coadjoint orbit $\mathcal{O}_0^*$. In other words, $\mathcal{O}_m^* \simeq \mathcal{P}_+^\uparrow(1,1)/H_m$ and $\mathcal{O}_0^* \simeq \mathcal{P}_+^\uparrow(1,1)/H$.

4.2 The Affine group

In this section, we have presented the affine group by using the fact of affine transformation of the line, consisting of a dilation (or scaling) by $a \neq 0$ and a (rigid) translation by $b \in \mathbb{R}$. Then the action on $\mathbb{R}: \; x = (b,a)y = ay + b, \; y \in \mathbb{R}$ and inverse: $y = (b,a)^{-1}x = (-\frac{b}{a}, \frac{1}{a})x = -\frac{b}{a} + \frac{x}{a} = \frac{x-b}{a}$ and inverse: $y = (b,a)^{-1}x = \frac{x-b}{a}$ leads to the composition rule

$$(b,a)(b',a') = (b + ab', aa'). \tag{4.42}$$

The above composition rule becomes the same as (3.12) if we make the identification $a = \frac{1}{q}$ and $b = p$.

The matrix representation

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,, a \neq 0 \,, b \in \mathbb{R} \tag{4.43}$$

reproduces this composition rule. The inverse is given by the matrix

$$(b,a)^{-1} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{pmatrix} = (-\frac{b}{a}, \frac{1}{a}). \tag{4.44}$$

If we take a vector $\begin{pmatrix} x \\ 1 \end{pmatrix}$ then the action of the matrix (4.42) on this vector is given by

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ 1 \end{pmatrix}, \tag{4.45}$$

which exactly reproduces the action $x \to ax+b$ on $\mathbb{R}$. It is clear that this class of matrices constitute a group, called the full affine group which is denoted by G_{aff} but note that, it is not a connected group. So that, if we consider only those matrices from the above class of matrices where $a > 0$, then the above class of matrices forms a subgroup of G_{aff} which is denoted by G^{+}_{aff}. We will be working with G^{+}_{aff} group along with the dilations which emerges as a subgroup of Poincaré group in 1- space and 1-time dimensions. This affine subgroup help us to the formalism of the integral quantization technique.

4.3 Example: $(1+1)$ Poincaré covariant integral quantization (s) and Affine group

The formalism of the integral quantization procedure introduced in Chapter 2 are going to be used on the non-unimodular Lie group $G = G^{+}_{\text{aff}}$ which is a upper half plane of the two dimensional Euclidean plane which will be constructed in the current Section. Fortunately, G is a Lie group with Haar measure, $d\mu_\ell = \dfrac{d\mathbf{b}\, da}{a^2}$ and $d\mu_{\text{r}} = \dfrac{d\mathbf{b}\, da}{a}$ and with the map $g \mapsto U(g)$, the UIR of G in a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ which allow us to introduce the covariant integral quantization in this stage by following the Subsection 2.2.3.

4.3.1 Phase space analysis using the group background

We now look for the classical phase space which interestingly emerges from (4.35) and (4.36), are that they coincide on a certain subgroup of $\mathcal{P}^{\uparrow}_{+}(1,1)$ – the affine subgroup.

Indeed, consider the subgroup of $\mathcal{P}_+^\uparrow(1,1)$, which consists of elements of the type $(\Lambda_a, (\mathbf{b},\mathbf{b})^T)$, $\mathbf{b} \in \mathbb{R}$. This subgroup can be identified with the affine group G^+_{aff}, of the real line, which acts on an element $y \in \mathbb{R}$ in the manner, $y \mapsto ay+\mathbf{b}$, $(\mathbf{b}, a) \in G^+_{\text{aff}}$, and obeys the product rule $(\mathbf{b}, a)(\mathbf{b}', a') = (\mathbf{b} + a\mathbf{b}', aa')$ which is identical with (**??**). Restricted to this subgroup $G = G^+_{\text{aff}}$, both equations (4.35) and (4.36) yield the common expression

$$(U_{\text{aff}}(\mathbf{b}, a)f)(x) = (U_m(\Lambda_a, (\mathbf{b},\mathbf{b})^T)f)(x) = (U_0(\Lambda_a, (\mathbf{b},\mathbf{b})^T)f)(x) = e^{ix\mathbf{b}}\ f(ax)\ , \tag{4.46}$$

which is immediately recognized as a UIR of G^+_{aff}. Moreover, Group G^+_{aff} has two non-equivalent UIR, $U_\pm$ [19]. From the equation (4.46), the representation $U_+ = (U_{\text{aff}}(\mathbf{b}, a)f)(x) = e^{ix\mathbf{b}}\ f(ax)$, is carried on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, \frac{dx}{x})$ and the second one being carried on $L^2(\mathbb{R}^{<0}, \frac{dx}{x})$. It is worth to remark that both of these representations G^+_{aff} are *square integrable,* and in topologically, G^+_{aff} is homeomorphic to $\mathbb{R}^2 \times \mathbb{R}^{>0}$ and the lie subalgebra $\mathfrak{g}_{\text{aff}}$ of the affine subgroup G^+_{aff} is generated by the two elements Y^0 and $T = \frac{1}{2}(Y^1 + Y^2)$ prescribed in the Subsection 4.1.5.

Proposition 4.3.1. *The affine subgroup G^+_{aff} of $\mathcal{P}_+^\uparrow(1,1)$ has two one parameter unitary subgroups $(e^{iaX}\psi)(x) = \psi(ax)$; $(e^{ibY}\psi)(x) = e^{ibx}\psi(x)$, $\psi \in \mathcal{H}$ may be realized on the same Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ and produce two self adjoint operators $\{X, Y\}$, appear now as the infinitesimal generators of this representation and are realized as follows:*

$$(X\psi)(x) = -i\ x\frac{d\psi}{dx}\ ,\quad (Y\psi)(x) = x\psi(x), \tag{4.47}$$

The unitary representation of G^+_{aff} which also integrates the CR, $[Y, X] = ix$.

Proof. The proof is given in the Appendix C. □

It is indeed our duty to look the corresponding real valued function in our phase space G^+_{aff} for both of the generators $\{X, Y\}$, which will be exploited in Subsection 4.4.1. We shall also investigate other self adjoint operators and their real valued fuctions with thier CR in Section 4.4 by using the fundamental tool of the integral quantization. We are now ready to provide some more interesting informations of the affine subgroup G_{aff}^{+} which will be exploited again in significantly to construct the covariant integral quantization in 1- space, 1- time $\mathcal{P}_{+}^{\uparrow}(1,1)$.

Lemma 4.3.1. *To construct a quotient space or coset space, picking H as a phase subgroup of $\mathcal{P}_{+}^{\uparrow}(1,1)$ consisting of elements of the type $(\mathbb{I}_2, (\mathbf{b}, -\mathbf{b})^T)$, where $\mathbb{I}_2$ is the 2×2 identity matrix and $\mathbf{b}\in\mathbb{R}$. Then, H is a normal subgroup of $\mathcal{P}_{+}^{\uparrow}(1,1)$ which allowed us to say $\mathcal{P}_{+}^{\uparrow}(1,1)/H \simeq G_{\text{aff}}^{+}$ is quotient group.*

Proof. The proof is given in the Appendix C. □

Lemma 4.3.2. *Once we have the coset space or quotient group $\mathcal{P}_{+}^{\uparrow}(1,1)/H \simeq G_{\text{aff}}^{+}$ then we have the corresponding left coset decomposition,*

$$(\Lambda_a, (b_0, \mathbf{b})^T) = (\Lambda_a, \frac{b_0+\mathbf{b}}{2}(1,1)^T)\ (\mathbb{I}_2, \frac{b_0-\mathbf{b}}{2}\Lambda_{\frac{1}{a}}(1,-1)^T)\ . \tag{4.48}$$

and the right coset decomposition,

$$(\Lambda_a, (b_0, \mathbf{b})^T) = (\mathbb{I}_2, \frac{b_0-\mathbf{b}}{2}(1,-1)^T)\ (\Lambda_a, \frac{b_0+\mathbf{b}}{2}(1,1)^T)\ . \tag{4.49}$$

Proof. The proof is given in the Appendix C. □

Lemma 4.3.3. *Once we have the left and right coset decomposition on the coset space $\mathcal{P}_{+}^{\uparrow}(1,1)/H \simeq G_{\text{aff}}^{+}$ then we will have the following left and right section:*

$$\sigma_\ell : \mathcal{P}_{+}^{\uparrow}(1,1)/H \simeq G_{\text{aff}}^{+} \longrightarrow \mathcal{P}_{+}^{\uparrow}(1,1)\ , \qquad \sigma_\ell(\mathbf{b}, a) = (\Lambda_a, (\mathbf{b}, \mathbf{b})^T)\ . \tag{4.50}$$

$$\sigma_{\mathrm{r}} : H\backslash\mathcal{P}_{+}^{\uparrow}(1,1) \simeq G_{\text{aff}}^{+} \longrightarrow \mathcal{P}_{+}^{\uparrow}(1,1)\ , \qquad \sigma_{\mathrm{r}}(\mathbf{b}, a) = (\Lambda_a, (\mathbf{b}, \mathbf{b})^T)\ , \tag{4.51}$$

respectively and the projection for the left action:

$$\pi_\ell : \mathcal{P}_{+}^{\uparrow}(1,1) \longrightarrow \mathcal{P}_{+}^{\uparrow}(1,1)/H \simeq G_{\text{aff}}^{+}\ , \pi_\ell(\Lambda_a, (\mathbf{b}, a)^T) = (\Lambda_a, \frac{(\mathbf{b}+\mathbf{b})}{2}(1,1)^T)\ . \tag{4.52}$$

Correspondingly, we define the projection for the right action:

$$\pi_{\mathrm{r}} : \mathcal{P}_{+}^{\uparrow}(1,1) \longrightarrow H\backslash\mathcal{P}_{+}^{\uparrow}(1,1) \simeq G_{\text{aff}}^{+}\ , \pi_{\mathrm{r}}(\Lambda_a, (\mathbf{b}, a)^T) = (\Lambda_a, \frac{(\mathbf{b}+\mathbf{b})}{2}(1,1)^T)\ . \tag{4.53}$$

where, $(\Lambda_a, (\mathbf{b}, b)^T) \simeq (\mathbf{b}, \Lambda)$.

Remark 4.3.1. *The left coset decomposition (4.48) is easily seen to lead to the following left action of the Poincaré group on the homogeneous space* $\mathcal{P}_+^\uparrow(1,1)/H$

$$(\Lambda_{a'},(b_0',\mathbf{b}')^T)\;(\mathbf{b},a) = (\frac{b_0'+\mathbf{b}'}{2} + a'\mathbf{b}, a'a) \tag{4.54}$$

and the right action on the homogeneous space $H\backslash\mathcal{P}_+^\uparrow(1,1)$ *due to the fact of the right coset decomposition (4.49) is*

$$(\mathbf{b},a)(\Lambda_{a'},(b_0',\mathbf{b}')^T) = (\mathbf{b} + \frac{a}{2}(b_0'+\mathbf{b}'), a'a)\;. \tag{4.55}$$

Moreover, Lemma (4.3.1) and Lemma (4.3.2) produce a new restriction of the representations U_m *and* U_0 *in (4.35) and (4.36), respectively, to this subgroup which act as folows:*

$$(U_m(\mathbb{I}_2,(\mathbf{b},-\mathbf{b})^T)f)(x) = e^{i\frac{\mathbf{b}}{x}}f(x) \text{ and } (U_0(\mathbb{I}_2,(\mathbf{b},-\mathbf{b})^T)f)(x) = If(x)\;, \tag{4.56}$$

where, I being the identity operator on $L^2(\mathbb{R}^{>0},\frac{dx}{x})$.

The above facts will later be exploited for constructing the coherent states for $\mathcal{P}_+^\uparrow(1,1)$ in the following Subsection. It is worth remarking here that the representation U_0 turns out to be just the representation of the affine subgroup G_{aff}^+, trivially extended to all of $\mathcal{P}_+^\uparrow(1,1)$. Also, note that unlike $\mathcal{P}_+^\uparrow(1,1)$, the group G_{aff}^+ is non-unimodular so that the left invariant Haar measure under the action (4.54) being $d\mu_\ell = \dfrac{d\mathbf{b}\;da}{a^2}$ and the right invariant measure under the action (4.55) is clearly $d\mu_{\text{r}} = \dfrac{d\mathbf{b}\;da}{a}$ which will be used to figure out the standard coherent states.

4.3.2 Standard coherent states (CS) for left action and right action

We now proceed to build coherent states for the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$ using the two representations U_m and U_0 in (4.35) and (4.36). These will turn out to be basically the coherent states of the affine group G_{aff}^+, familiar from the theory of wavelets, in view of the fact that the affine group is a subgroup of $\mathcal{P}_+^\uparrow(1,1)$. Now, let f be an element on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$, the Hilbert space of the two representations U_m and U_0, in (4.35) and (4.36). Suppose that f satisfies the *admissibility condition* (implies that if $f(x)$ is smooth then $f(0) = 0$) for being a *mother wavelet* (see, for example, [19]), which is the first condition:

$$I(f) = 2\pi \int_0^\infty |f(x)|^2 \, \frac{dx}{x^2} < \infty \ . \tag{4.57}$$

Moreover, second condition (scalar product or inner product) $\int_0^\infty |f(x)|^2 \, \frac{dx}{x} < \infty$ implies finite energy of the complex valued functions $f(x)$. But this condition is automatically satisfied in the Hilbert space $\mathcal{H}$. Then the function f is called the *mother wavelet.*

To construct the family of coherent states set picking a vector $f \in \mathcal{H}$ and the section or the function prescribed in equation (4.50) in Lemma (4.3.3) defined on the group $\mathcal{P}_+^\uparrow(1,1)$ insist us to write in the following way,

$$|\eta_{\mathbf{b},a}\rangle = [I(f)]^{-\frac{1}{2}} U_{\text{aff}}(\mathbf{b}, a) f \tag{4.58}$$

and then, as is well-known from the theory of the continuous wavelet transform, the operator integral using the left Haar measure $d\mu_\ell = \dfrac{d\mathbf{b} \; da}{a^2}$ becomes:

$$\int_{\mathbb{R}\times\mathbb{R}^{>0}} |\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}| \, \frac{d\mathbf{b} \; da}{a^2} = I_{\mathcal{H}} \tag{4.59}$$

with $U_{\text{aff}}(\mathbf{b}, a)$ as defined in (4.46). Note that, by (4.50) $U_{\text{aff}}(\mathbf{b}, a)(x) = U_0(\sigma_\ell(\mathbf{b}, a))(x) = (U_m(\Lambda_a, (\mathbf{b}, \mathbf{b})^T)f)(x) = (U_0(\Lambda_a, (\mathbf{b}, \mathbf{b})^T)f)(x) = e^{ix\mathbf{b}} \, f(ax)$. Finally, the coherent states $\eta_{\mathbf{b},a}$ are labelled by the points $(\mathbf{b}, a)$ in the homogeneous space $\mathcal{P}_+^\uparrow(1,1)/H \simeq G_{\text{aff}}^+$ of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$, and they are obtained by the action of the unitary operators $U_0(\sigma_\ell(\mathbf{b}, a))$ of a UIR of $\mathcal{P}_+^\uparrow(1,1)$, on a fixed vector $f \in \mathcal{H}$. These standard CS vectors for the left actions are

$$|\eta_{\mathbf{b},a}\rangle(x) = [I(f)]^{-\frac{1}{2}} (U_{\text{aff}}(\mathbf{b}, a)f)(x) = [I(f)]^{-\frac{1}{2}} \, e^{ix\mathbf{b}} \, f(ax), \tag{4.60}$$

which satisfy the resolution of the identity equation (4.59) and the *square integrability,*

$$C(\eta) = \int_{G_{\text{aff}}^+} |\langle \eta | U(g) \eta \rangle|^2 d\mu(g) < \infty \tag{4.61}$$

of the UIR, with respect to the homogeneous space $\mathcal{P}_+^\uparrow(1,1)/H \simeq G_{\text{aff}}^+$. We now present a lemma that will prove the resolution of identity which is the fundamental requirement of the integral quantization using left action, provided that, the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$.

Lemma 4.3.4. *For all compactly supported smooth functions ϕ, ψ and smooth function η and the coherent state vector $\eta_{\mathbf{b},a}$ satisfy the square integrability condition (4.61) then it also satisfy the following integral relation:*

$$\int_{\mathbb{R}\times\mathbb{R}^{>0}} \langle\phi|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle \, \frac{d\mathbf{b}\, da}{a^2} = \langle\phi|\psi\rangle; , \text{ where, } \quad |\eta_{\mathbf{b},a}\rangle = [I(\eta)]^{-\frac{1}{2}} U_{\text{aff}}(\mathbf{b},a)\eta \quad (4.62)$$

Proof. The proof is given in Appendix C. □

Let us now again define the family of CS vectors like (4.60) but the current standard CS vectors are constructed for the right action on the homogeneous space $H\backslash\mathcal{P}_+^\uparrow(1,1) \simeq G_{\text{aff}}^+$ of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$. Then the standard CS vectors are

$$|\xi_{\mathbf{b},a}\rangle = [I(f)]^{-\frac{1}{2}} U_{\text{aff}}(\mathbf{b},a)^* f = [I(f)]^{-\frac{1}{2}} U_0(\sigma_{\text{r}}(\mathbf{b},a))^* f \ , \quad (4.63)$$

with f as in (4.57). Then one again has a resolution of the identity like (4.59), but for this time using the right Haar measure $d\mu_{\text{r}}(\mathbf{b},a) = \dfrac{d\mathbf{b}\; da}{a}$,

$$\int_{\mathbb{R}\times\mathbb{R}^{>0}} |\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}| \, \frac{d\mathbf{b}\; da}{a} = I_{\mathcal{H}} \ . \quad (4.64)$$

Note that,

$$\begin{aligned}(U_{\text{aff}}(\mathbf{b},a)^* f)(x) &= (U_{\text{aff}}(-\frac{\mathbf{b}}{a}, \frac{1}{a}) f)(x) \\ &= U_0(\sigma_{\text{r}}(\mathbf{b},a))^* f(x) \\ &= (U_m(\Lambda_a, (\mathbf{b},\mathbf{b})^T)^* f)(x) \\ &= e^{\frac{-ix\mathbf{b}}{a}} f(\frac{x}{a}). \end{aligned} \quad (4.65)$$

Finally, the CS $|\xi_{\mathbf{b},a}\rangle$ are labelled by the points $(\mathbf{b},a)$ in the homogeneous space $H\backslash\mathcal{P}_+^\uparrow(1,1) \simeq G_{\text{aff}}^+$ of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$, and they are obtained by the action of the unitary operators $U_0(\sigma_{\rm r}(\mathbf{b},a))$ of a UIR of $\mathcal{P}_+^\uparrow(1,1)$, on a fixed vector $f \in \mathcal{H}$. These standard CS vectors are

$$|\xi_{\mathbf{b},a}\rangle(x) = [I(f)]^{-\frac{1}{2}}(U_{\rm aff}(\mathbf{b},a)^* f)(x) = [I(f)]^{-\frac{1}{2}} e^{-ix\frac{\mathbf{b}}{a}}\, f(\frac{x}{a}), \tag{4.66}$$

which satisfy the resolution of the identity (4.64) in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ and the *square integrability* (4.61) of the UIR, with respect to the homogeneous space $H\backslash\mathcal{P}_+^\uparrow(1,1) \simeq G_{\rm aff}^+$. Proof of the resolution of the identity (4.64) is almost similar to Lemma (4.3.4).

4.3.3 Covariant integral quantization using U_0 and the left action

We now follow the procedure of the covariant integral quantization described in Chapter 2, Subsection 2.2.3 and the resolution of the identity (4.59) to introduce a Poincaré covariant integral quantization of the real-valued functions on $\mathbb{R}\times\mathbb{R}^{>0}$. Indeed, let ϕ be a real-valued function on $\mathbb{R}\times\mathbb{R}^{>0}$ such that the operator

$$\widehat{\phi}_\ell = \int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\, \frac{d\mathbf{b}\, da}{a^2}\ , \tag{4.67}$$

is well-defined (in the weak sense (see equation 2.4)). In that case, $\widehat{\phi}_\ell$ is just the Berezin-Toeplitz operator representing the quantized version of the "classical observable" ϕ. This quantization is easily seen to be covariant with respect to the representation U_0 of $\mathcal{P}_+^\uparrow(1,1)$ (see (4.36), in the sense that

$$U_0(\Lambda_{a'},b')\,\widehat{\phi}_\ell\, U_0(\Lambda_{a'},b')^* = \widehat{[{}_{(\Lambda_{a'},b')}\phi_\ell]}\ , \tag{4.68}$$

where ${}_{(\Lambda_{a'},b')}\phi$ is the transformed function

$$\begin{aligned} {}_{(\Lambda_{a'},b')}\phi(\mathbf{b},a) &= \phi((\Lambda_{a'},b')^{-1}(\mathbf{b},a)) \\ &= \phi((\Lambda_{a'}^{-1}, -\Lambda_{a'}^{-1}b')(\mathbf{b},a)) \\ &= \phi(-\frac{b'}{a'} + a'^{-1}\mathbf{b}, \frac{a}{a'}) \\ &= \phi(\frac{\mathbf{b}-b'}{a'}, \frac{a}{a'})\ ,‘ \end{aligned} \tag{4.69}$$

which easily follows by the equation (2.23), Chapter 2 and from (4.48), (4.56), (4.54) with the invariance of the measure $\frac{d\mathbf{b}\, da}{a^2}$.

4.3.4 Covariant integral quantization and coadjoint invariant quantization using U_0 and the right action

It is useful to carry out the above procedure using the right coset decomposition (4.49) in place of the left coset decomposition (4.48). This will lead to a right action on the homogeneous space $H\backslash\mathcal{P}_+^\uparrow(1,1) \simeq G_{\text{aff}}^+$, which in fact will turn out to be the coadjoint action (see (4.41)). Moreover, the resulting quantization will in fact be the *coadjoint invariant*. We have the right coset decomposition (4.49) and correspondingly we defined the section (4.51) and in analogy with (4.54) we get, using (4.49) and (4.51), the right action on the homogeneous space $H\backslash\mathcal{P}_+^\uparrow(1,1)$, (4.55), provided the Lemma (4.3.3) and Remark (4.3.1).

From the equation (4.41) it is immediately clear that the above action coincides with the coadjoint action on the orbit $\mathcal{O}_0^*$. Indeed, if $k = \begin{pmatrix} k_0 \\ \mathbf{k} \end{pmatrix}$ denotes a point on this orbit, then by (4.27) and (4.28),

$$k = \begin{pmatrix} k_0 \\ \mathbf{k} \end{pmatrix} = \Lambda_{\frac{1}{a}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a + \frac{1}{a} & -a + \frac{1}{a} \\ -a + \frac{1}{a} & a + \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \qquad (4.70)$$

So that by the equation (4.41) we have the following calculations,

$$
\begin{aligned}
\mathrm{Ad}^*(\Lambda_{a'}, b') \begin{pmatrix} \gamma \\ k_0 \\ \mathbf{k} \end{pmatrix} &= \begin{pmatrix} \gamma + a(b'_0 + \mathbf{b}') \\ \Lambda_{a'^{-1}} k \end{pmatrix} \\
&= \begin{pmatrix} \gamma + a(b'_0 + \mathbf{b}') \\ \frac{1}{2} \begin{pmatrix} a' + \dfrac{1}{a'} & -a' + \dfrac{1}{a'} \\ -a + \dfrac{1}{a'} & a' + \dfrac{1}{a'} \end{pmatrix} \begin{pmatrix} a \\ -a \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \gamma + a(b'_0 + \mathbf{b}') \\ a'a \\ -a'a \end{pmatrix} ,
\end{aligned} \tag{4.71}
$$

which should be compared to (4.55). As in (4.67), using the resolution of the identity (4.64) we can introduce the covariant integral quantization which follows as:

$$
{}_{\mathrm{r}}\widehat{\phi} = \int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a) |\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}| \, \frac{d\mathbf{b}\; da}{a} \; . \tag{4.72}
$$

This quantization is also covariant with respect to the representation U_0 of $\mathcal{P}_+^{\uparrow}(1,1)$, but now in the sense that

$$
U_0(\Lambda_{a'}, b') \; {}_{\mathrm{r}}\widehat{\phi} \; U_0(\Lambda_{a'}, b')^* = [\widehat{{}_{\mathrm{r}}\phi_{(\Lambda_{a'}, b')}}] \; , \tag{4.73}
$$

where ${}_{(\Lambda_{a'}, b')}[\phi]$ is the transformed function

$$
\begin{aligned}
\phi_{(\Lambda_{a'}, b')}(\mathbf{b}, a) &= \phi((\mathbf{b}, a)(\Lambda_{a'}, b')^{-1}) \\
&= \phi((\mathbf{b}, a)(\Lambda_{a'}^{-1}, -\Lambda_{a'}^{-1} b')) \\
&= \phi(\mathbf{b} - \frac{ab'}{a'}, \frac{a}{a'}) \\
&= \phi(\frac{a'\mathbf{b} - ab'}{a'}, \frac{a}{a'}) \; ,
\end{aligned} \tag{4.74}
$$

which again follows by the equation (2.23), Chapter 2 and from (4.48), (4.56), (4.56), (4.49), (4.55) with the invariance of the measure $\frac{d\mathbf{b}\; da}{a}$ under the right action. In view of (4.55) and (4.71) we see that this quantization is *coad-covariant*.

4.3.5 Phase space analysis of the representation U_0

It is useful to transfer the representation U_0 to Hilbert spaces $\mathcal{H}$ built on phase space functions. Recall that the phase space in this case is $\mathcal{O}_0^* = T^*\mathcal{O}_0 \simeq \mathbb{R} \times \mathbb{R}^{>0}$, with invariant 2-form

$$\Omega = \frac{d\mathbf{b} \wedge da}{a} \, . \tag{4.75}$$

This invariant 2-form will be exploited to construct the Hamiltonian vector fields X_f in the next Section.

With $\xi_{\mathbf{b},a}$ as in (4.59), we define the map $W_\xi : L^2(\mathbb{R}^{>0}, dx/x) \longrightarrow L^2(\mathbb{R}\times\mathbb{R}^{>0}, d\mathbf{b}\; da/a)$ which computing the wavelet transform of a signal $\phi \in L^2(\mathbb{R}^{>0}, dx/x)$ as,

$$(W_\xi \phi)(\mathbf{b}, a) = F_\xi(\mathbf{b}, a) = \langle \xi_{\mathbf{b},a} \mid \phi \rangle = \int_{\mathbb{R}^{>0}} e^{i\frac{b}{a}x}\, \overline{\xi(\frac{x}{a})}\, \phi(x)\, \frac{dx}{x} \, . \tag{4.76}$$

It follows from the resolution of the identity in (4.59). It easily follows that W_ξ is a linear isometry.

Let $\mathcal{H}_\xi \subset L^2(\mathbb{R} \times \mathbb{R}^{>0}, d\mathbf{b}\;\; da/a)$ be the range of this isometry, which is a closed subspace of $L^2(\mathbb{R} \times \mathbb{R}^{>0}, d\mathbf{b}\;\; da/a)$, and $\mathbb{P}_\xi$ the corresponding projection operator : $\mathbb{P}_\xi L^2(\mathbb{R} \times \mathbb{R}^{>0}, d\mathbf{b}\; da/a) = \mathcal{H}_\xi$. Then, for all $F \in L^2(\mathbb{R} \times \mathbb{R}^{>0}, d\mathbf{b}\; da/a)$

$$(\mathbb{P}_\xi F)(\mathbf{b}, a) = \int_{\mathbb{R}\times\mathbb{R}^{>0}} K_\xi(\mathbf{b}, a;\; \mathbf{b}', a') F(\mathbf{b}, a)\, \frac{d\mathbf{b}\; da}{a} \, , \tag{4.77}$$

where, $K_\xi(\mathbf{b}, a;\; \mathbf{b}', a') = \langle \xi_{\mathbf{b},a} \mid \xi_{\mathbf{b}',a'} \rangle$ is a *reproducing kernel*, with the properties

$$\begin{aligned}
K_\xi(\mathbf{b}, a;\; \mathbf{b}, a) &> 0 \, , \quad \forall (\mathbf{b}, a) \in \mathbb{R} \times \mathbb{R}^{>0} \, , \\
K_\xi(\mathbf{b}, a;\; \mathbf{b}', a') &= \overline{K_\xi(\mathbf{b}', a';\; \mathbf{b}, a)} \, , \\
K_\xi(\mathbf{b}, a;\; \mathbf{b}', a') &= \int_{\mathbb{R}\times\mathbb{R}^{>0}} K_\xi(\mathbf{b}, a;\; \mathbf{b}'', a'') K_\xi(\mathbf{b}'', a'';\; \mathbf{b}', a')\, \frac{d\mathbf{b}''\; da''}{a''} \, ,
\end{aligned} \tag{4.78}$$

which easily follow from (4.59). Moreover, the quantization rule (4.72) now becomes

$$_{\mathrm{r}}\widehat{\phi} = \mathbb{P}_\xi \widehat{\phi}_{\mathrm{mult}} \mathbb{P}_\xi \, , \tag{4.79}$$

where, $\widehat{\phi}_{\mathrm{mult}}$ is the operator of multiplication by the function ϕ on $L^2(\mathbb{R}\times\mathbb{R}^{>0}, d\mathbf{b}\; da/a)$:

$$(\widehat{\phi}_{\mathrm{mult}} F)(\mathbf{b}, a) = \phi(\mathbf{b}, a) F(\mathbf{b}, a) \, , \qquad F \in L^2(\mathbb{R} \times \mathbb{R}^{>0}, d\mathbf{b}\; da/a) \, . \tag{4.80}$$

We compute next the image of the representation U_0 under the map W_ξ. Writing $U_0^\xi(\Lambda_a, b) = W_\xi U_0^\xi(\Lambda_a, b) W_\xi^*$, its action on functions $F_\xi \in \mathcal{H}_\xi$ is computed to be (see (4.55))

$$(U_0^\xi(\Lambda_a, b)F_\xi)(\mathbf{b}', a') = F_\xi(\mathbf{b} + \frac{a'(b_0 + \mathbf{b})}{2}, a'a) = F_\xi((\mathbf{b}', a')(\Lambda_a, (b_0, \mathbf{b})^T)\ . \qquad (4.81)$$

This representation can naturally be extended as a unitary though highly representation, which we denote by $\widetilde{U}_0$ on the whole of $L^2(\mathbb{R} \times \mathbb{R}^{>0}, d\mathbf{b}\ da/a)$ simply as

$$\begin{aligned} (\widetilde{U}_0(\Lambda_a, b)F)(\mathbf{b}', a') &= F(\mathbf{b} + \frac{a'(b_0 + \mathbf{b})}{2}, a'a) \\ &= F((\mathbf{b}', a')(\Lambda_a, b)\ , \quad F \in L^2(\mathbb{R} \times \mathbb{R}^{>0}, d\mathbf{b}\ da/a)\ , \end{aligned} \qquad (4.82)$$

and then

$$U_0^\xi(\Lambda_a, b) = \mathbb{P}_\xi \widetilde{U}_0(\Lambda_a, b)\ . \qquad (4.83)$$

A complete decomposition of this representation into irreducibles can be carried out along the lines of Section 6.1.2 in [10]. In particular, if we take for ξ the vector

$$\xi(x) = \left[\frac{2^\nu}{\pi\Gamma(\nu + \frac{1}{2})}\right]^{\frac{1}{2}} x^{1+\frac{\nu}{2}}\ e^{-x}\ , \qquad \nu \geq 0 \qquad (4.84)$$

is smooth since $\xi(0)$; satisfies the admissibilty condition,

$$\begin{aligned} I(\xi) &= 2\pi \int_0^\infty |\xi(x)|^2 \frac{dx}{x^2} \\ &= 2\pi \int_0^\infty \left[\frac{2^\nu}{\pi\Gamma(\nu + \frac{1}{2})}\right] x^{2+\nu}\ e^{-2x} \frac{dx}{x^2} \\ &= 2\pi \left[\frac{2^\nu}{\pi\Gamma(\nu + \frac{1}{2})}\right] \int_0^\infty x^{2+\nu}\ e^{-2x} \frac{dx}{x^2} \\ &= 2\pi \left[\frac{2^\nu}{\pi\Gamma(\nu + \frac{1}{2})}\right] \int_0^\infty x^\nu\ e^{-2x} dx \\ &= 2\pi \left[\frac{2^\nu}{\pi\Gamma(\nu + \frac{1}{2})}\right] \int_0^\infty (t/2)^\nu\ e^{-t} dt/2 \\ &= \left[\frac{1}{\Gamma(\nu + \frac{1}{2})}\right] \int_0^\infty (t)^\nu\ e^{-t} dt \\ &= \left[\frac{\Gamma(\nu + 1)}{\Gamma(\nu + \frac{1}{2})}\right] \end{aligned} \qquad (4.85)$$

and so that it is mother wavelet. We now see from (4.76) that

$$\begin{aligned}
(W_\xi \phi)(\mathbf{b}, a) &= F_\xi(\mathbf{b}, a) \\
&= \int_{\mathbb{R}>0} e^{i\frac{b}{a}x}\, \overline{\xi(\frac{x}{a})}\, \phi(x)\, \frac{dx}{x} \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+\frac{1}{2})}\right]^{\frac{1}{2}} \int_{\mathbb{R}>0} e^{x(-\frac{1}{a}+\frac{i\mathbf{b}}{a})} x^{\frac{\nu}{2}}\, \phi(x)\, dx \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+\frac{1}{2})}\right]^{\frac{1}{2}} \int_{\mathbb{R}>0} e^{ix(\frac{\mathbf{b}}{a}+\frac{i}{a})} x^{\frac{\nu}{2}}\, \phi(x)\, dx \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+\frac{1}{2})}\right]^{\frac{1}{2}} a^{-(1+\frac{\nu}{2})} \int_{\mathbb{R}>0} e^{iz_1 x} x^{\frac{\nu}{2}}\, \phi(x)\, dx\,, \qquad z_1 = \frac{\mathbf{b}}{a} + \frac{i}{a} \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+\frac{1}{2})}\right]^{\frac{1}{2}} a^{-(1+\frac{\nu}{2})} f(z_1)\,,
\end{aligned} \tag{4.86}$$

$$\text{where, } f(z_1) = \int_{\mathbb{R}>0} e^{iz_1 x} x^{\frac{\nu}{2}}\, \phi(x)\, dx\,, \tag{4.87}$$

is a function which is holomorphic in the upper half plane.

4.3.6 Poisson brackets

From the invariant two form (4.75), we calculate the *Hamiltonian vector fields* X_f, corresponding to real-valued phase space functions f, using the usual relation $df = \Omega(X_f, \cdot)$.

We get,

$$X_f = a\frac{\partial f}{\partial \mathbf{b}}\, \frac{\partial}{\partial a} - a\frac{\partial f}{\partial a}\, \frac{\partial}{\partial \mathbf{b}}\,, \tag{4.88}$$

and for the Poisson bracket of two functions f, g,

$$\{f, g\} = -\Omega(X_f, X_g) = a\left[\frac{\partial f}{\partial a}\frac{\partial g}{\partial \mathbf{b}} - \frac{\partial g}{\partial a}\frac{\partial f}{\partial \mathbf{b}}\right]\,. \tag{4.89}$$

We now investigate the Poisson brackets of some real valued function in the phase space.

In particular, we have the following table for the Poisson brackets:

Table 4.1: Necessary Poisson brackets

Poisson brackets	Results
$\{a\mathbf{b}, a^{-1}\} = \{\log a, \mathbf{b}\} = \{a, a^{-1}\mathbf{b}\}$	1
$\{a, \mathbf{b}\} = \log a, a\mathbf{b}\}$	a
$\{\mathbf{b}, \frac{1}{a}\} = \{\log a, \frac{\mathbf{b}}{a}\}$	$\frac{1}{a}$
$\{\frac{\mathbf{b}}{a}, a^n\}$	$-na^{n-1}$
$\{\frac{\mathbf{b}}{a}, \frac{1}{a^n}\}$	$\frac{n}{a^{n+1}}$
$\{a\mathbf{b}, \mathbf{b}\}$	ab
$\{a^n, \mathbf{b}\}$	na^n
$\{a^{-n}, \mathbf{b}\}$	$\frac{n}{a^n}$
$\{a\mathbf{b}, \frac{\mathbf{b}}{a}\}$	$2\mathbf{b}$
$\{a^n, a\mathbf{b}\}$	na^{n+1}
$\{a\mathbf{b}, a^{-n}\}$	$\frac{n}{a^{n-1}}$

4.4 Quantization using the representation U_m and U_0

In the current section, we work out on the quantization scheme, based on the coadjoint orbit $\mathcal{O}_m^* = T^*\mathcal{O}_m$ using the irreducible representation $U_m(\Lambda, b)$ in (4.35) and the irreducible representation $U_0(\Lambda, b)$ in (4.36) which are form equivalent when restricted to G_{aff}^+ by (4.46). We use the coherent state vectors described in Section (4.3.2) and the integral quantization prescribed in equation (4.67) and (4.72) to find the self adjoint operators in Hilbert space $\mathcal{H}$.

Lemma 4.4.1. *If* $\forall\ a,\ b \in \mathbf{R}$ $\phi(\mathbf{b}, a)$ *is real valued function then the corresponding operator in the Hilbert space* $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ *is essentially self adjoint under the sufficient conditions compactly supported functions* ϕ *and infinitely differentiable function* η*.*

The classical and quantum connection is defined by the following lemma.

Lemma 4.4.2. *The two families of coherent state vectors (4.60) and (4.66) are related by* $\eta_{\mathbf{b},a} = \xi_{-\frac{\mathbf{b}}{a},\frac{1}{a}}$*, so that the two quantizations* $\phi \to \widehat{\phi_\ell}$ *and* $\phi \to {}_{\mathrm{r}}\widehat{\phi}$ *are also seen to be related in the following manner:*

$$ {}_{\mathrm{r}}\widehat{\phi} = \widehat{\breve{\phi}_\ell}, \quad \text{where,} \quad \breve{\phi}(\mathbf{b}, a) = \phi(-\frac{\mathbf{b}}{a}, \frac{1}{a}) \,. \tag{4.90} $$

One can see that proof of the above Lemma is trivial but it is very interesting result and the operators in Table (4.2) and (4.3) using left and right Haar measures respectively have followed the above relation.

4.4.1 Operators in $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ using left and right Haar measures

We now present the operators in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ by using the left and right Haar measures corresponding to the real valued function $\phi(\mathbf{b}, a)$, or the classical phase space G^+_{aff} prescribed in Sections 4.3.1 and 4.3.6.

Table 4.2: Operators in $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ using left Haar measure

Real valued function	Operattor
$\phi(\mathbf{b}, a) = a$	$(\widehat{\phi_{\ell a}}\psi)(x) = \frac{\|\eta\|^2}{I(\eta)} \frac{\psi(x)}{x}$
$\phi(\mathbf{b}, a) = b$	$(\widehat{\phi_{\ell b}}\psi)(x) = -i\frac{d\psi}{dx} + \frac{i}{2}\frac{\psi(x)}{x}$
$\phi(\mathbf{b}, a) = \frac{1}{a}$	$(\widehat{\phi_{\ell \frac{1}{a}}}\psi)(x) = [\frac{I_1(\eta)}{I(\eta)}]x\psi(x)$
$\phi(\mathbf{b}, a) = \frac{b}{a}$	$(\widehat{\phi_{\ell \frac{b}{a}}}\psi)(x) = -[\frac{I_1(\eta)}{I(\eta)}]ix\frac{d\psi}{dx}$
$\phi(\mathbf{b}, a) = ab$	$(\widehat{\phi_{\ell ab}}\psi)(x) = [\frac{\|\eta\|^2}{I(\eta)}](\frac{-i}{x}\frac{d\psi}{dx} + \frac{i}{x^2}\psi(x))$
$\phi(\mathbf{b}, a) = \log a$	$(\widehat{\phi_{\ell \log a}}\psi)(x) = [\frac{I_{\log}(\eta)}{I(\eta)}]\psi(x) - \log(x)\psi(x)$
$\phi(\mathbf{b}, a) = a^n$	$(\widehat{\phi_{\ell a^n}}\psi)(x) = [\frac{I_{-n}(\eta)}{I(\eta)}]\frac{1}{x^n}\psi(x)$
$\phi(\mathbf{b}, a) = \frac{1}{a^n}$	$(\widehat{\phi_{\ell \frac{1}{a^n}}}\psi)(x) = [\frac{I_n(\eta)}{I(\eta)}]x^n\psi(x)$

where,

$$ I(\eta) := 2\pi \int_0^\infty |\eta(t)|^2 \, \frac{dt}{t^2}; \qquad I_{\log}(\eta) := 2\pi \int_0^\infty \log(t)|\eta(t)|^2 \, \frac{dt}{t^2} \tag{4.91} $$

$$I_n(\eta) := 2\pi \int_0^\infty |\eta(t)|^2 \, \frac{dt}{t^{n+2}}; \tag{4.92}$$

Note that in the following Table we also have used the same notations.

Table 4.3: Operators in $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ using right Haar measure

Real valued function	Operators
$\phi(\mathbf{b}, a) = a$	$(\widehat{{}_{\mathrm{r}}\phi_a}\psi)(x) = [\frac{I_1(\eta)}{I(\eta)}]x\psi(x)$
$\phi(\mathbf{b}, a) = b$	$(\widehat{{}_{\mathrm{r}}\phi_b}\psi)(x) = [\frac{I_1(\eta)}{I(\eta)}]ix\frac{d\psi}{dx}$
$\phi(\mathbf{b}, a) = \frac{1}{a}$	$(\widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a}}}\psi)(x) = [\frac{\lVert\eta\rVert^2}{I(\eta)}]\frac{\psi(x)}{x}$
$\phi(\mathbf{b}, a) = \frac{b}{a}$	$(\widehat{{}_{\mathrm{r}}\phi_{\frac{b}{a}}}\psi)(x) = i\frac{d\psi}{dx} - \frac{i}{2}\frac{\psi(x)}{x}$
$\phi(\mathbf{b}, a) = ab$	$(\widehat{{}_{\mathrm{r}}\phi_{ab}}\psi)(x) = [\frac{I_2(\eta)}{I(\eta)}](ix^2\frac{d\psi}{dx} + \frac{i}{2}x\psi(x))$
$\phi(\mathbf{b}, a) = \log a$	$(\widehat{{}_{\mathrm{r}}\phi_{loga}}\psi)(x) = \log(x)\psi(x) - \frac{I_{\log}}{I(\eta)}\psi(x)$
$\phi(\mathbf{b}, a) = a^n$	$(\widehat{{}_{\mathrm{r}}\phi_{a^n}}\psi)(x) = [\frac{I_n(\eta)}{I(\eta)}]x^n\psi(x)$
$\phi(\mathbf{b}, a) = \frac{1}{a^n}$	$(\widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a^n}}}\psi)(x) = \frac{I_{-n}(\eta)}{I(\eta)}\,\frac{1}{x^n}\psi(x)$

Details of calculations are given in Appendix C. It might be possible to change the constant coefficients of the above opertors by a suitable state vector η. The interesting fact about the above operators is that we can analyze the Self-adjointness and canonical quantization which is discussed in Lemma (4.4.1) and Proposition (4.4.1) respectively. We are now ready to provide the real valued functions $\phi(\mathbf{b}, a)$ corresponding to the generators of the representation of affine subgroup G^+_{aff}.

Lemma 4.4.3. *The one parameter unitary groups e^{iaX} and e^{ibY} that generate two self adjoint operators are given by Proposition (4.3.1) in the Hilbert spcae $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ for the left Haar measure. There exist two real valued functions corresponding to these operators on the classsical phase space. The real valued functions $\phi(\mathbf{b}, a)$ corresponding to the operators X and Y for the left haar measure are following:*

$$\phi_{lX}(\mathbf{b}, a) = \frac{I(\eta)}{I_1(\eta)}\frac{b}{a} = c_1^{-1}\frac{b}{a}, \tag{4.93}$$

$$\phi_{lY}(\mathbf{b}, a) = \frac{I(\eta)}{I_1(\eta)} \frac{1}{a} = c_1^{-1} \frac{1}{a}. \tag{4.94}$$

Lemma 4.4.4. *The one parameter unitary groups e^{iaX} and e^{ibY} that generate two self adjoint operators are given by Proposition (4.3.1) in the Hilbert spcae $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ for the right Haar measure. There exists two real valued functions corresponding to this operators on the classsical phase space. The real valued functions $\phi(\mathbf{b}, a)$ corresponding to the operators X and Y for the right haar measure are following:*

$$_{\mathrm{r}}\phi_X(\mathbf{b}, a) = \frac{I(\eta)}{I_1(\eta)} \mathbf{b} = c_1^{-1} \mathbf{b}, \tag{4.95}$$

$$_{\mathrm{r}}\phi_Y(\mathbf{b}, a) = \frac{I(\eta)}{I_1(\eta)} a = c_1^{-1} a. \tag{4.96}$$

It is easily seen that two unitary operators e^{iaX} and e^{ibY} are not commutative and the commutation relation of the two unitary operators are $[e^{iaX}, e^{ibY}]\psi(x) = e^{iabx}\psi(ax) - e^{ibx}\psi(ax) = (e^{iabx} - e^{ibx})\psi(ax)$.

Definition 1. *Starting with a classical system (Γ, ν), find a Hilbert space $\mathcal{H}$ and a linear mapping Q from the smooth functions on $\mathcal{H}$ which for some appropriate subset of functions $f, g, ...$, maps the Poisson bracket to the commutator bracket. If this relation*

$$[Q_f, Q_g] = iQ_{\{f,g\}} \tag{4.97}$$

holds then the quantization is called the canonical quantization, where $[A, B]$ is the commutator bracket of the operators $A; B$ on $\mathcal{H}$ and $\{f, g\}$ is the Poisson bracket of the functions f, g on Γ.

4.4.2 Poisson brackets versus commutator relation

We now compare in Table (4.4), below, the Poisson bracket relations of real valued functions $\phi(\mathbf{b}, a)$, as given in Section 4.3.6, Table (4.1) and the commutation relations between the corresponding quantized operators given in Section 4.4.1, Tables (4.2) and (4.3). Moreover, the interesting fact about the operators in the Hilbert space

$\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$ is that we can analyze the CR and self-adjointness which play important roles in QM.

Table 4.4: Poisson brackets versus Commutation Relations (CR)

Poisson brackets	CR for left Haar measure
$\{a\mathbf{b}, a^{-1}\} = \{a, a^{-1}\mathbf{b}\} = 1$	$[\widehat{\phi}_{\ell a\mathbf{b}}, \widehat{\phi}_{\ell a^{-1}}] = [\widehat{\phi}_{\ell a}, \widehat{\phi}_{\ell a^{-1}\mathbf{b}}] = -i\frac{c\,c_1}{x}$
$\{\log a, \mathbf{b}\} = 1$	$[\widehat{\phi}_{\ell \log a}, \widehat{\phi}_{\ell\mathbf{b}}] = -i\frac{1}{x}$
$\{a, \mathbf{b}\} = \{\log a, a\mathbf{b}\} = a$	$[\widehat{\phi}_{\ell a}, \widehat{\phi}_{\ell\mathbf{b}}] = [\widehat{\phi}_{\ell \log a}, \widehat{\phi}_{\ell a\mathbf{b}}] = -i\frac{c}{x^2}$
$\{\mathbf{b}, \frac{1}{a}\} = \frac{1}{a}$	$[\widehat{\phi}_{\ell\mathbf{b}}, \widehat{\phi}_{\ell\frac{1}{a}}] = -ic_1$
$\{\frac{\mathbf{b}}{a}, \frac{1}{a}\} = \frac{1}{a^2}$	$[\widehat{\phi}_{\ell\frac{\mathbf{b}}{a}}, \widehat{\phi}_{\ell\frac{1}{a}}] = -ic_1^2\, x$
$\{a^2, \frac{\mathbf{b}}{a}\} = 2a$	$[\widehat{\phi}_{\ell a^2}, \widehat{\phi}_{\ell\frac{\mathbf{b}}{a}}] = -2i\frac{c_1\,c_{-2}}{x^2}$
$\{\mathbf{b}, \frac{\mathbf{b}}{a}\} = \frac{\mathbf{b}}{a}$	$[\widehat{\phi}_{\ell\mathbf{b}}, \widehat{\phi}_{\ell\frac{\mathbf{b}}{a}}] = -c_1\frac{d}{dx} + \frac{c_1}{2x}$
$\{\frac{\mathbf{b}}{a}, \frac{1}{a^2}\} = \frac{2}{a^3}$	$[\widehat{\phi}_{\ell\frac{\mathbf{b}}{a}}, \widehat{\phi}_{\ell\frac{1}{a^2}}] = -2ic_1\, c_2\, x^2$
$\{a^n, \frac{\mathbf{b}}{a}\} = na^{n-1}$	$[\widehat{\phi}_{\ell a^n}, \widehat{\phi}_{\ell\frac{\mathbf{b}}{a}}] = -nic_1\, c_n\, \frac{1}{x^n}$
$\{\frac{\mathbf{b}}{a}, \frac{1}{a^n}\} = \frac{n}{a^{n+1}}$	$[\widehat{\phi}_{\ell\frac{\mathbf{b}}{a}}, \widehat{\phi}_{\ell\frac{1}{a^n}}] = -nic_1\, c_n x^n$
$\{a\mathbf{b}, \mathbf{b}\} = ab$	$[\widehat{\phi}_{\ell a\mathbf{b}}, \widehat{\phi}_{\ell\mathbf{b}}] = -\frac{c}{x^2}\frac{d}{dx} + \frac{3c}{2}\frac{1}{x^3}$
$\{a^n, \mathbf{b}\} = na^n$	$[\widehat{\phi}_{\ell a^n}, \widehat{\phi}_{\ell\mathbf{b}}] = -ni\; c_{-n}\frac{1}{x^{n+1}}$
$\{\mathbf{b}, a^{-n}\} = \frac{n}{a^n}$	$[\widehat{\phi}_{\ell\mathbf{b}}, \widehat{\phi}_{\ell\frac{1}{a^n}}] = -ni\; c_n\; x^{n-1}$
$\{a, a\mathbf{b}\} = a^2$	$[\widehat{\phi}_{\ell a}, \widehat{\phi}_{\ell a\mathbf{b}}] = -i\frac{c^2}{x^3}$
$\{a\mathbf{b}, \frac{\mathbf{b}}{a}\} = 2\mathbf{b}$	$[\widehat{\phi}_{\ell a\mathbf{b}}, \widehat{\phi}_{\ell\frac{\mathbf{b}}{a}}] = -cc_1\frac{2}{x}\frac{d}{dx} + cc_1\frac{2}{x^2}$
$\{a^n, a\mathbf{b}\} = na^{n+1}$	$[\widehat{\phi}_{\ell a^n}, \widehat{\phi}_{\ell a\mathbf{b}}] = -ni\; c\; c_{-n}\frac{1}{x^{n+2}}$
$\{a\mathbf{b}, a^{-n}\} = \frac{n}{a^{n-1}}$	$[\widehat{\phi}_{\ell a\mathbf{b}}, \widehat{\phi}_{\ell\frac{1}{a^n}}] = -i\; n\; c\; c_n x^n$
$\{\log a, \frac{\mathbf{b}}{a}\} = \frac{1}{a}$	$[\widehat{\phi}_{\ell \log a}, \widehat{\phi}_{\ell\frac{\mathbf{b}}{a}}] = -i\; c_1$

Poisson brackets	CR for right Haar measure
$\{a\mathbf{b}, a^{-1}\} = 1$	$[\widehat{{}_{\mathrm{r}}\phi_{a\mathbf{b}}}, \widehat{{}_{\mathrm{r}}\phi_{a^{-1}}}] = i\ c_2\ c$
$\{\log a, \mathbf{b}\} = \{a, a^{-1}\mathbf{b}\} = 1$	$[\widehat{{}_{\mathrm{r}}\phi_{\log a}}, \widehat{{}_{\mathrm{r}}\phi_{\mathbf{b}}}] = [\widehat{{}_{\mathrm{r}}\phi_{a}}, \widehat{{}_{\mathrm{r}}\phi_{a^{-1}\mathbf{b}}}] = ic_1$
$\{a, \mathbf{b}\} = a$	$[\widehat{{}_{\mathrm{r}}\phi_{a}}, \widehat{{}_{\mathrm{r}}\phi_{\mathbf{b}}}] = ic_1^2\ x$
$\{\mathbf{b}, \frac{1}{a}\} = \frac{1}{a}$	$[\widehat{{}_{\mathrm{r}}\phi_{\mathbf{b}}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a}}}] = ic_1\ c\ \frac{1}{x}$
$\{\frac{\mathbf{b}}{a}, \frac{1}{a}\} = \frac{1}{a^2}$	$[\widehat{{}_{\mathrm{r}}\phi_{\frac{\mathbf{b}}{a}}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a}}}] = ic\ \frac{1}{x^2}$
$\{a^2, \frac{\mathbf{b}}{a}\} = 2a$	$[\widehat{{}_{\mathrm{r}}\phi_{a^2}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{\mathbf{b}}{a}}}] = 2ic_2\ x$
$\{\mathbf{b}, \frac{\mathbf{b}}{a}\} = \frac{\mathbf{b}}{a}$	$[\widehat{{}_{\mathrm{r}}\phi_{\mathbf{b}}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{\mathbf{b}}{a}}}] = c_1 \frac{d}{dx} - \frac{c_1}{2x}$
$\{\frac{\mathbf{b}}{a}, \frac{1}{a^2}\} = \frac{2}{a^3}$	$[\widehat{{}_{\mathrm{r}}\phi_{\frac{\mathbf{b}}{a}}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a^2}}}] = 2i\ c_{-2}\ \frac{1}{x^3}$
$\{a^n, \frac{\mathbf{b}}{a}\} = na^{n-1}$	$[\widehat{{}_{\mathrm{r}}\phi_{a^n}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{\mathbf{b}}{a}}}] = in\ c_n\ x^{n-1}$
$\{\frac{\mathbf{b}}{a}, \frac{1}{a^n}\} = \frac{n}{a^{n+1}}$	$[\widehat{{}_{\mathrm{r}}\phi_{\frac{\mathbf{b}}{a}}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a^n}}}] = i\ n\ c_{-n} \frac{1}{x^{n+1}}$
$\{a\mathbf{b}, \mathbf{b}\} = ab$	$[\widehat{{}_{\mathrm{r}}\phi_{a\mathbf{b}}}, \widehat{{}_{\mathrm{r}}\phi_{\mathbf{b}}}] = c_1c_2(x^2 \frac{d}{dx} + \frac{x}{2})$
$\{a^n, \mathbf{b}\} = na^n$	$[\widehat{{}_{\mathrm{r}}\phi_{a^n}}, \widehat{{}_{\mathrm{r}}\phi_{\mathbf{b}}}] = i\ n\ c_nc_1\ x^n$
$\{\mathbf{b}, a^{-n}\} = \frac{n}{a^n}$	$[\widehat{{}_{\mathrm{r}}\phi_{\mathbf{b}}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a^n}}}] = ni\ c_{-n}c_1\ \frac{1}{x^n}$
$\{a, a\mathbf{b}\} = a^2$	$[\widehat{{}_{\mathrm{r}}\phi_{a}}, \widehat{{}_{\mathrm{r}}\phi_{a\mathbf{b}}}] = ic_1c_2\ x^2$
$\{a\mathbf{b}, \frac{\mathbf{b}}{a}\} = 2\mathbf{b}$	$[\widehat{{}_{\mathrm{r}}\phi_{a\mathbf{b}}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{\mathbf{b}}{a}}}] = 2c_2\ x\ \frac{d}{dx}$
$\{\log a, a\mathbf{b}\} = a$	$[\widehat{{}_{\mathrm{r}}\phi_{\log a}}, \widehat{{}_{\mathrm{r}}\phi_{a\mathbf{b}}}] = i\ c_2\ x$
$\{a^n, a\mathbf{b}\} = na^{n+1}$	$[\widehat{{}_{\mathrm{r}}\phi_{a^n}}, \widehat{{}_{\mathrm{r}}\phi_{a\mathbf{b}}}] = i\ n\ c_n\ c_2\ x^{n+1}$
$\{a\mathbf{b}, a^{-n}\} = \frac{n}{a^{n-1}}$	$[\widehat{{}_{\mathrm{r}}\phi_{a\mathbf{b}}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a^n}}}] = i\ nc_2\ c_{-n} \frac{1}{x^{n-1}}$
$\{\log a, \frac{\mathbf{b}}{a}\} = \frac{1}{a}$	$[\widehat{{}_{\mathrm{r}}\phi_{\log a}}, \widehat{{}_{\mathrm{r}}\phi_{\frac{\mathbf{b}}{a}}}] = i\ \frac{1}{x}$

The following notations has been used in the above Table (4.4).

Table 4.5: Necessary terms and notations

Term	Notation
$\frac{\\|\eta\\|^2}{I(\eta)}$	c
$\frac{I_n(\eta)}{I(\eta)}$	c_n
$\frac{I_{-n}(\eta)}{I(\eta)}$	c_{-n}

Proposition 4.4.1. *If $f(b,a)$ is a real valued function which only depends on a and $g(b,a)$ is again a real valued function depends on a and linearly on b on the classical phase space G^+_{aff}. Then the Poisson brackets $\{f(b,a), g(b,a)\} = h(a)$ implies the canonical quantization.*

Proof. Let $f(b,a) = \sum_n d_n a^n$ and $g(b,a) = \sum_m e_m a^m b$ be two real valued functions on the classical phase space G^+_{aff}. Then the Poisson brackets for the two real valued functions is $\{\sum_n d_n a^n, \sum_m e_m a^m b\} = \sum_n \sum_m d_n e_m \; n \; a^{n+m} := h(a)$. From Table (4.3) $\widehat{{}_{\mathrm{r}}\phi_{a^{n+m}}} = [\frac{I_{n+m}(\eta)}{I(\eta)}]x^{n+m}$. We want to find the operator $\widehat{{}_{\mathrm{r}}\phi_{a^m b}}$ for the real valued function $\phi(\mathbf{b},a) = a^m b$ by letting two compactly supported functions ϕ, ψ and infinitely differentiable function η. We then have the following calculations

$$\begin{aligned}
&\langle\phi|\mathcal{A}_{\phi\mathrm{r}}|\psi\rangle \\
&= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_{\phi\mathrm{r}}|\psi)(x)\tfrac{dx}{x} \\
&= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}|\;\psi\rangle \; \tfrac{d\mathbf{b}\,da}{a}] \; \tfrac{dx}{x} \\
&= \tfrac{1}{[I(\eta)]^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)\langle\xi_{\mathbf{b},a}|\psi\rangle e^{\frac{i\mathbf{b}x}{a}}\eta(\tfrac{x}{a})\tfrac{d\mathbf{b}\,da}{a^2}]\tfrac{dx}{x} \\
&= \tfrac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a^m b \; \{\int_{x'=0}^\infty e^{\frac{-i\mathbf{b}x'}{a'}}\bar{\eta}(\tfrac{x'}{a})\psi(x')\tfrac{dx'}{x'}\}e^{\frac{i\mathbf{b}x}{a}}\eta(\tfrac{x}{a}) \; \tfrac{d\mathbf{b}\,da}{a}] \; \tfrac{dx}{x}
\end{aligned}$$

Since $\psi = 0$ at the boundaries of the integral so we are allowed to write the integral like $\frac{\partial}{\partial x'}\{\int_{x'=0}^\infty e^{\frac{-i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\} = 0$ and this implies: $b\int_{x'=0}^\infty e^{\frac{-i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x') \; \frac{dx'}{x'} = -i\int_{x'=0}^\infty e^{\frac{-i\mathbf{b}x'}{a}}\bar{\eta}\prime(\frac{x'}{a})\psi(x')\frac{dx'}{x'} - i\int_{x'=0}^\infty \; a \; e^{\frac{-i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi'(x') \; \frac{dx'}{x'} + i\int_{x'=0}^\infty \; a \; e^{\frac{-i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x') \; \frac{dx'}{x'^2}$

Using the above fact we have the following calculations:

$$\begin{aligned}
&= \tfrac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a^m \; \{\int_{x'=0}^\infty e^{\frac{-i\mathbf{b}x'}{a}}\bar{\eta}'(\tfrac{x'}{a})\psi(x')\tfrac{dx'}{x'}\}e^{\frac{i\mathbf{b}x}{a}}\eta(\tfrac{x}{a}) \; \tfrac{d\mathbf{b}\,da}{a}] \; \tfrac{dx}{x} + \\
&\tfrac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a^m \; \{\int_{x'=0}^\infty \; a \; e^{\frac{-i\mathbf{b}x'}{a}}\bar{\eta}(\tfrac{x'}{a})\psi'(x')\tfrac{dx'}{x'}\}e^{\frac{i\mathbf{b}x}{a}}\eta(\tfrac{x}{a}) \; \tfrac{d\mathbf{b}\,da}{a}] \; \tfrac{dx}{x} + \\
&\tfrac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \; a^m \; \{\int_{x'=0}^\infty \; a \; e^{\frac{-i\mathbf{b}x'}{a}}\bar{\eta}(\tfrac{x'}{a})\psi(x')\tfrac{dx'}{x'^2}\}e^{\frac{i\mathbf{b}x}{a}}\eta(\tfrac{x}{a}) \; \tfrac{d\mathbf{b}\,da}{a}] \; \tfrac{dx}{x} \\
&= 2\pi \; [-\tfrac{i}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^m \; \int_{x'=0}^\infty\{\tfrac{1}{2\pi} \; \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}{a}} \; d\mathbf{b}\}\bar{\eta}'(\tfrac{x'}{a})\psi(x')\eta(\tfrac{x}{a}) \; \tfrac{dx'}{x'} \; \tfrac{da}{a}] \; \tfrac{dx}{x} - \\
&\tfrac{i}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^m \; \int_{x'=0}^\infty\{\tfrac{1}{2\pi} \; \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}{a}} \; d\mathbf{b}\}\bar{\eta}(\tfrac{x'}{a})\psi'(x')\eta(\tfrac{x}{a}) \; \tfrac{dx'}{x'} \; da] \; \tfrac{dx}{x} + \\
&\tfrac{i}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^m \; \int_{x'=0}^\infty\{\tfrac{1}{2\pi} \; \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}{a}} \; d\mathbf{b}\}\bar{\eta}(\tfrac{x'}{a})\psi(x')\eta(\tfrac{x}{a}) \; \tfrac{dx'}{x'^2} \; da] \; \tfrac{dx}{x} \;] \\
&= 2\pi \; [-\tfrac{i}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^m \; \int_{x'=0}^\infty\{\tfrac{1}{2\pi} \; \int_{s=-\infty}^\infty e^{is(x-x')} \; a \; ds\}\bar{\eta}'(\tfrac{x'}{a})\psi(x')\eta(\tfrac{x}{a}) \; \tfrac{dx'}{x'} \; \tfrac{da}{a}] \; \tfrac{dx}{x} - \\
&\tfrac{i}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^m \; \int_{x'=0}^\infty\{\tfrac{1}{2\pi} \; \int_{s=-\infty}^\infty e^{is(x-x')} \; a \; ds\}\bar{\eta}(\tfrac{x'}{a})\psi'(x')\eta(\tfrac{x}{a}) \; \tfrac{dx'}{x'} \; da] \; \tfrac{dx}{x} + \\
&\tfrac{i}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^m \; \int_{x'=0}^\infty\{\tfrac{1}{2\pi} \; \int_{s=-\infty}^\infty e^{is(x-x')} \; a \; ds\}\bar{\eta}(\tfrac{x'}{a})\psi(x')\eta(\tfrac{x}{a}) \; \tfrac{dx'}{x'^2} \; da] \; \tfrac{dx}{x} \;] \\
&= 2\pi \; [-\tfrac{i}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^m \; \int_{x'=0}^\infty \delta(x-x')\bar{\eta}'(\tfrac{x'}{a})\psi(x')\eta(\tfrac{x}{a}) \; \tfrac{dx'}{x'} \; da] \; \tfrac{dx}{x} \; -
\end{aligned}$$

$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}>0} a^{m+1}\ \int_{x'=0}^\infty \delta(x-x')\ \bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ da]\ \frac{dx}{x} +$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}>0} a^{m+1}\ \int_{x'=0}^\infty\ \delta(x-x')\ \bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'^2}\ da]\ \frac{dx}{x}\ ]$
$= -\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0} a^m\ \bar{\eta}'(\frac{x}{a})\eta(\frac{x}{a})\ da]\psi(x)\ \frac{dx}{x^2} -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0} a^{m+1}\ \bar{\eta}(\frac{x}{a})\eta(\frac{x}{a})\ da]\psi'(x)\ \frac{dx}{x^2} +$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0} a^{m+1}\ \bar{\eta}(\frac{x}{a})\eta(\frac{x}{a})\ da]\psi(x)\ \frac{dx}{x^3}$
$= -\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0}\ \bar{\eta}'(t)\eta(t)\ \frac{dt}{t^{m+2}}]x^{m-1}\psi(x)\ dx -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0}|\eta(t)|^2\ \frac{dt}{t^{m+3}}]x\ x^m\ \psi'(x)\ dx +$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0}|\eta(t)|^2\ \frac{dt}{t^{m+3}}]x^{m-1}\psi(x)\ dx$
$= -\frac{(m+2)i}{2I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0}\ \bar{\eta(t)}\eta(t)\ \frac{dt}{t^{m+3}}]x^m\psi(x)\ \frac{dx}{x} -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0}|\eta(t)|^2\ \frac{dt}{t^{m+3}}]x\ x^{m+1}\psi'(x)\ \frac{dx}{x} +$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0}|\eta(t)|^2\ \frac{dt}{t^{m+3}}]x^m\psi(x)\ \frac{dx}{x}$
$= -\frac{im}{2I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0}\ \bar{\eta(t)}\eta(t)\ \frac{dt}{t^{m+3}}]x^m\psi(x)\ dx/x -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}>0}|\eta(t)|^2\ \frac{dt}{t^{m+3}}]x\ x^{m+1}\psi'(x)\ dx/x$
$= -\frac{im\ I_{m+1}(\eta)}{2I(\eta)}\ \int_0^\infty \bar{\phi}(x)x^m\ \psi(x)\ \frac{dx}{x} - \frac{iI_{m+1}(\eta)}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)x^{m+1}\ \psi'(x)\ \frac{dx}{x}$
$= \langle\phi|\widehat{{}_{\mathrm{r}}\phi_{ba^m}}\psi\rangle.$

Making the change of variables, $x/a = t,\implies\ a = \frac{x}{t}$ and $da = \frac{-xdt}{t^2}$. We also use $\int_{\mathbb{R}>0} \bar{\eta}'(t)\eta(t)\ \frac{dt}{t^3} = \frac{3}{2}\int_{\mathbb{R}>0}|\eta(t)|^2\ \frac{dt}{t^4}$ and $I_{m+1}(\eta) := 2\pi\int_{\mathbb{R}>0}|\eta(t)|^2\ \frac{dt}{t^{m+3}}$. Finally, we have $(\widehat{{}_{\mathrm{r}}\phi_{a^m b}}\psi)(x) = -i[\frac{I_{m+1}(\eta)}{I(\eta)}]\ (\frac{m}{2}x^m\psi(x) + x^{m+1}\ \frac{d\psi(x)}{dx})$. Then the commutator relation $[\widehat{{}_{\mathrm{r}}\phi_{f(a,b)}}, \widehat{{}_{\mathrm{r}}\phi_{g(a,b)}}] = \sum_n\sum_m d_n e_m\ i\ c_n c_{m+1}\ n\ x^{n+m} = \widehat{{}_{\mathrm{r}}\phi_{h(a)}}$. □

We now look for the reproducing kernel in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^>0, \frac{dx}{x})$ by the following proposition.

Proposition 4.4.2. *Let η be the state vector*

$$\eta(\mathbf{x}) = \left[\frac{2^\nu}{\pi\Gamma(\nu+\frac{1}{2})}\right]^{\frac{1}{2}} x^{1+\frac{\nu}{2}}\ e^{-x}\ , \qquad \nu \geq 0 \tag{4.98}$$

which satisfies the admissibility condition (4.57) and where, $I(\eta) = \left[\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})}\right]$. Then the reproducing kernel (see properties in the equation 4.78) K_η is

$$K_\eta(\mathbf{b}, a;\ \mathbf{b}', a') = \langle\eta_{\mathbf{b},a} \mid \eta_{\mathbf{b}',a'}\rangle = \frac{2^\nu}{i^{2+\nu}}\ \frac{(aa')^{1+\frac{\nu}{2}}(\nu+1)}{\pi(\bar{z}-z')^{2+\nu}} \tag{4.99}$$

with the properties given in Section (4.3.5) and the equation (4.78), where the coherent states under the left action are:

$$|\eta_{\mathbf{b},a}\rangle(x) = [I(\eta)]^{-\frac{1}{2}}(U_{\text{aff}}(\mathbf{b},a)\eta)(x) = [I(\eta)]^{-\frac{1}{2}}e^{ibx}\eta(xa). \tag{4.100}$$

Proof. Using (4.100), we have the reproducing kernel:

$$\begin{aligned}
K_\eta(\mathbf{b},a;\ \mathbf{b}',a') &= \langle \eta_{\mathbf{b},a} \mid \eta_{\mathbf{b}',a'}\rangle \\
&= \langle \eta | e^{-ib\hat{Q}} e^{ib'\hat{Q}} \eta\rangle \\
&= \langle \eta | e^{-ib\hat{Q}+ib'\hat{Q}} \eta\rangle \\
&= \langle \eta | e^{-ib\hat{Q}+ib'\hat{Q}} \eta\rangle \\
&= \frac{1}{I(\eta)}\int_0^\infty \bar{\eta}(a\mathbf{x})e^{i(b'-b)x}\eta((a'\mathbf{x})\frac{dx}{x} \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+1)}\right]\int_0^\infty (ax)^{(1+\frac{\nu}{2})}\ e^{-ax}e^{i(b'-b)x}(a'x)^{(1+\frac{\nu}{2})}\ e^{-a'x}\frac{dx}{x} \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+1)}\right]\int_0^\infty (aa')^{(1+\frac{\nu}{2})}\ e^{-ax}e^{i(b'-b)x}x^{(\nu+2)}\ e^{-a'x}\frac{dx}{x} \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+1)}\right](aa')^{(1+\frac{\nu}{2})}\int_0^\infty e^{-(a+a'+ib-ib')x}x^{(\nu+2)-1}\ dx \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+1)}\right]\frac{(aa')^{(1+\frac{\nu}{2})}}{A^{\nu+2}}\int_0^\infty e^{-t}\ t^{\nu+2-1}\ dt \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+1)}\right]\frac{(aa')^{(1+\frac{\nu}{2})}}{A^{\nu+2}}\Gamma(\nu+2) \\
&= \left[\frac{2^\nu}{\pi\Gamma(\nu+1)}\right]\frac{(aa')^{(1+\frac{\nu}{2})}}{(a+a'+ib-ib')^{\nu+2}}(\nu+1)\Gamma(\nu+1) \\
&= \frac{2^\nu}{i^{2+\nu}}\ \frac{(aa')^{1+\frac{\nu}{2}}(\nu+1)}{\pi(\bar{z}-z')^{2+\nu}}
\end{aligned} \tag{4.101}$$

where, $z = b+ia$; $z' = b'+ia'$ and making the change of variables, $(a+a'+ib-ib')x = Ax = t$, $\implies$ $dx = \frac{dt}{A}$ and $\text{Re}[(a+a'+ib-ib')] > 0$. This is a reproducing kernel $K_\eta(\mathbf{b},a;\ \mathbf{b}',a')$ which produce another Hilbert space (or a closed subspace) $L^2(\mathbb{R}\times\mathbb{R}^{>0}, d\mathbf{b}\ da/a^2)$, in the Hilbert space $L^2(\mathbf{R}^{>}0, \frac{dx}{x})$. □

So far, in chapters 2 - 4, we described the general construction for integral quantization with an interesting application of covariant integral quantization. Application in the sense of the action on signals, since we observe that G_{aff} and G^+_{aff} consists precisely of the transformations we apply to a signal: translation (time-shift) by an amount b and zooming in or out by the factor a. Hence, the group G_{aff} and G_{aff}^{+} relate to the geometry of the signals. In the next Chapter we present the conclusion of this thesis.

Chapter 5

Conclusion and Future Directions

The main results of this thesis are related to the integral quantization scheme and its covariance properties that we have been elaborated upon in the following way.

In Chapter 2, we introduced the integral quantization technique and revisited the general construction of the covariant integral quantization procedure in some detail.

In Chapter 3, we looked at three examples of covariant integral quantization, using the Weyl-Heasienberg group, the affine group and the group $SU(2)$.

Chapter 4 was devoted to appling integral quantization on the Poincaré group in (1+1)-dimensional space-time and working out some of its applications to QM. We developed the integral quantization for the group $\mathcal{P}_+^{\uparrow}(1,1)$ and introduced the quantized operators and identified the ones which quantized canonically. We also introduced a relationship between the quantized versions of the classical observables obtained using the left and right Haar measures. In future, we would concentrate on applying the method of affine quantization to other physical systems and try to find other kinds of covariant integral quantizations by considering suitable operator valued measures.

Finally, in Appendix A we included some general background material on group theory. The mathematical structure of classical and quantum mechanics has been discussed in Appendix B. Appendix C incorporates all the mathematical computations related to the results developed in the thesis.

Appendix A

Some elements of group theory

This appendix has been put mainly for some background knowledge regarding group theory and the induced representation method for semidirect product. Moreover, an elegant account of group theoretical preliminaries has been carried out in the appendix of the reference [10].

A.1 Group

A group is a set G on which there is defined a binary operation, usually called the *group multiplication or group product* mapping $G \times G$ to G, $(g, g') \mapsto gg'$, and obeying the following three axioms.

(G1) *Associativity:* for any $g_1, g_2, g_3 \in G$, one has to be satisfies $(g_1 g_2)g_3 = g_1(g_2 g_3)$.

(G2) *Existence of identity:* there exists a (necessarily unique) element $e \in G$ such that $ge = eg = e, \forall g \in G$.

(G2) *Existence of inverse:* every element $g \in G$ possesses a unique inverse $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

Moreover, the group G is said to be *abelian or commutative* if $g_1\, g_2 = g_2\, g_1,\ \forall g_1, g_2 \in G$.

A.2 Topological Group

A *topological group* is a group G which is topological space and has the property that the group's binary operation and the group's inverse function are continuous functions with respect to the topology. In mathematically, the map $(g, g') \mapsto gg'^{-1}$ from $G \times G$ to G is continuous or equivalently obeying the following two axioms.

(G1) *Group operations of product:* the map $(g, g') \mapsto gg'$ from $G \times G$ to G is continuous. Here, $G \times G$ is viewed as a topological space by using the product topology.

(G2) *Inverse:* the map $g \mapsto g^{-1}$ from G to G is continuous.

A.3 Locally Compact Group

A locally compact group is a topological group G for which the underlying topology is locally compact and Hausdorff. Many examples of groups that arise throughout mathematics are locally compact so that Locally compact groups are important. Now, every locally compact group G carries a left and a right invariant Haar measure, both unique up to equivalence.

A.4 Haar Measure

The Haar measure is a way to assign an "invariant volume" to subsets of locally compact topological groups and subsequently an integral of those groups. Consider a lie group G which carries a left and right invariant Haar measure. We will be denoted the left Harr measure by $d\mu$ or $d\mu_\ell$ and the right Haar measure by $d\mu_{\mathrm{r}}$ through out this thesis. If $d\mu_\ell = d\mu_\mathrm{r}$, the group is called *unimodular*. In general, they are different but equivalent measures; that is they have the same null sets. Thus, there exists a measurable function $\Delta : G \mapsto \mathbb{R}^+$ such that $d\mu_\ell(g) = \Delta(g)d\mu_\mathrm{r}(g)$, where the function Δ is called the *modular function* of the groups.

A.5 Unitary Irreducible Representations

A *linear representation* is a map U from a locally compact group G to the set of bounded linear functions on a separable Hilbert space $\mathcal{H}$ which satisfies

$$U(g_1 g_2) = U(g_1)U(g_2), \ \forall g_1, g_2 \in G \tag{A.1}$$

and the identity is given by

$$U(e) = I, \text{ where } e \in G \text{ and } I \in \mathcal{H}. \tag{A.2}$$

A *unitary representation* is such that U is unitary operator that is $U^* = U^{-1}$. An *irreducible representation* has no nontrivial invariant subspaces. A *unitary irreducible representation* is denoted UIR and it has to be satisfied both of those properties.

A.6 Direct Product

Let G be a group with identity element e and H and K subgroups satisfying the conditions: H and K are normal in G, $H \cap K = \{e\}$ and then we obtain the *direct product* $G = HK$. One can easily see that G is isomorphic / "essentially the same" to the direct product $H \times K$ which is the set of pairs (h, k) with the group law

$$(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2) \tag{A.3}$$

Then both H, identified with $\{(h, e)\ , h \in H\}$, and K (similerly identified) are invariant subgroup of $H \times K$.

A.7 Semiderct Product

We now relax the first condition, so that H is still normal in G but K need not be. Again, Let G be a group with identity element e and H and K subgroups satisfying the conditions: H is normal in G, $H \cap K = \{e\}$ and then we obtain the *semidirect product* $H \rtimes K = G$. We now introduce *outer semidirect products*.

Let G be a group with a normal subgroup H and the subgroup K. Let $\mathrm{Aut}(H)$ denote the group of all automorphisms of H. Then the map $\phi : K \mapsto \mathrm{Aut}H$ defined by $\phi(k) = \phi_k = khk^{-1}\ \forall k \in K$ and $h \in H$ is a group homomorphism. Then we obtain the *semidirect product group* $G = H \rtimes_\phi K$ of H and K with respect to ϕ is the set of pairs (h, k) with $h \in H$ and $k \in K$. Multiplication of elements in $H \rtimes_\phi K$ is determined by the homomorphism ϕ. The operation is

$$* : (H \times K) \times (H \times K) \mapsto H \rtimes_\phi K. \tag{A.4}$$

The multiplication on this set is given by the rule

$$(h_1, k_1)(h_2, k_2) = (h_1\phi_{k1}(h_2), k_1k_2),\ \text{for } h_1, h_2 \in H\ , k_1, k_2 \in K, \tag{A.5}$$

the identity element is $(1, 1)$, and inverse is given by

$$(h, k)^{-1} = (\phi_{k^{-1}}(h^{-1}), k^{-1}) \tag{A.6}$$

One also says that G is a semidirect product of K acting on H. Note that the symbol $\rtimes$ is a combination of the normal subgroup symbol $\lhd$ and the product symbol $\times$ and the notation tells one which of H and K is the normal subgroup. Given any two groups H and K (not necessarily subgroups of a given group G) and a group homomorphism ϕ defined above then we can construct a new group $H \rtimes_\phi K$, called the *(outer) semidirect product* of H and K with respect to ϕ.

A.8 Induced Representation of semidirect product

In this section, we generalize this setting and consider semidirect products of the type $G = V \rtimes S$, where V is an n-dimensional real vector space (n is assumed to be finite) and S is usually a subgroup of $GL(V)$. The induced representations of semidirect products that is introduced here follows [19], Section 9.2.4. For semidirect product groups of the type we are considering here, all irreducible representations arise as induced representations and correspond to orbits, $\mathcal{O}^*$ in dual space, V^* [3]. The existence of square integrable coherent states allow us to work out these representations in some detail. Let X denote the resulting left coset space,

$$X = G/H_0, \qquad \text{where, } H_0 = V_0 \rtimes S_o \text{ is a subgroup of G.} \tag{A.7}$$

We then have the invariant measure $d\nu$ on the orbit X and the Hilbert space $\mathcal{K} = L^2(X, d\nu)$.It also turn out that the CS will be labeled by points in X or equivalently, by points in $V_0 \times \mathcal{O}^*$, where $V_0 = T^*_{k0}\mathcal{O}^*$. These CS will be square integrable and the associated induced representations square integrable mod(H_0, σ) for appropriate sections σ.

Consider again the element $k_0 \in V^*$ is an initial vector, of which O^* is the orbit under S. The associated unitary character χ of the abelian subgroup V,

$$\chi(v) = exp[-i\langle k_0; v\rangle], \qquad v \in V = \mathbb{R}^n. \tag{A.8}$$

defines a one-dimensional representation of V. In particular, on the $\mathcal{P}^{\uparrow}_{+}(1,1)$ group $V = \mathbb{R}^2$ and $k_0 = (1,0)$ for the hyperboloid and $k_0 = (1,-1)$ for the half line.

Let $S \mapsto L(s)$ be a UIR of S_0, the stability subgroup of k_0, and carried by some Hilbert space $\mathcal{K}$. The UIR, χL, of $V \rtimes S_0$ carried by $\mathcal{K}$ is then:

$$(\chi L)(v,s) = exp[-i\langle k_0; v\rangle]L(s). \tag{A.9}$$

We want to induce a representation of $G = V \rtimes S$, which is induced from χL and it clearly $G/(V \rtimes S_0) \simeq \mathcal{O}^*$.

Now, we shall need the section,

$$\lambda : \mathcal{O}^* \mapsto G, \qquad \lambda(k) = (0, \Lambda(k)),\ k \in \mathcal{O}^* \tag{A.10}$$

From the coset decomposition: $(v,s) = (0,\Lambda(k))(\Lambda(k)^{-1}v, s_0)$, $(v,s) \in G$. For the $\mathcal{P}^{\uparrow}_{+}(1,1)$, Λ_k is the action on the hyperboloid or the half line. Since an arbitrary elemrnt $(v, s) \in G$ act on the left part which represents $\mathcal{O}^*$ or the hyperboloid or the half line and for $p \in \mathcal{O}^*$

$$(v, s)(0, \Lambda(p)) = (0, \Lambda(sp))\ (\Lambda^{-1}(sp)v,\ \Lambda^{-1}(sp)s\Lambda(p)), \tag{A.11}$$

that is the action of G on $\mathcal{O}^*$ is also given by $k \mapsto (v, s)k = sk$. Recall that we are assuming the measure $d\nu$ on $\mathcal{O}^*$ to be invariant under this action. We then obtain the following two cocycles:

$$h : G \times \mathcal{O}^* \mapsto V \rtimes S_0, \qquad h((v, s), p) = (\Lambda^{-1}(sp)v, \Lambda^{-1}(sp)s\Lambda(p)); \tag{A.12}$$

$$h_0 : S \times \mathcal{O}^* \mapsto S_0, \qquad h_0(s, p) = \Lambda^{-1}(sp)s\Lambda(p). \tag{A.13}$$

We now need to compute the cocycles for the inverse group element, we get

$$h((v, s)^{-1}, p) = (-\Lambda^{-1}(s^{-1}p)s^{-1}v, \Lambda^{-1}(s^{-1}p)s^{-1}\Lambda(p)); \tag{A.14}$$

where, $h_0(s^{-1}, p) = \Lambda^{-1}(s^{-1}p)s^{-1}\Lambda(p)$. Following the equation (A.9), UIR is then wriiten in the following way:

$$\begin{aligned} (\chi L)(h((v, s)^{-1}, p)) &= (\chi L)(-\Lambda^{-1}(s^{-1}p)s^{-1}v, \Lambda^{-1}(s^{-1}p)s^{-1}\Lambda(p)) \\ &= exp[-i\langle k; -\Lambda^{-1}(s^{-1}p)s^{-1}v\rangle]L(\Lambda^{-1}(s^{-1}p)s^{-1}\Lambda(p)) \\ &= exp[-i\langle k; v\rangle]L(h_0(s^{-1}, p)). \end{aligned} \tag{A.15}$$

Finally, we write the representation of G induced from χL and carried by the Hilbert space χL_U, we obtain the unitary irreducible representation

$$(\chi L_U(v, s)\phi)(k) = exp[i\langle k; v\rangle]L(h_0(s^{-1}, k))^{-1}\phi(s^{-1}k). \tag{A.16}$$

It should be noted that if the group G is abelian then $\phi(s^{-1}k)$ will be replaced by $\phi(k)$ i.e. no translation and no shift.

A.9 Adjoint and coadjoint action

The adjoint map is the map of G into its group of automorphisms (i.e. isomorphisms from G to G) defined by:

$$Ad_g \mapsto G;\ h \mapsto Ad_g h := ghg^{-1}. \tag{A.17}$$

The adjoint action is the action of a lie group G on its algebra g, it is defined by:

$$Ad(g)X = gXg^{-1}\ , g \in G;\ X \in g \tag{A.18}$$

The coadjoint action is the action of the group on the dual of its algebra. The coadjoint action is denoted by $Ad^{\sharp}$ and is defined by by the following way:

$$< Ad^{\sharp}(g)X_1^*, X_2 >=< X_1^*, Ad(g^{-1})X_2 > . \tag{A.19}$$

A.10 Adjoint, Self-adjoint and Symmetric Operators (general case)

A bounded linear operator T in Hilbert space $\mathcal{H}$ has a *bounded linear adjoint* A^* on $\mathcal{H}$ connected with T by the equation

$$\langle Tf, g\rangle = \langle f, T^*g\rangle \text{ for all } f \in \mathcal{H} \text{ and all } g \in \mathcal{H} \tag{A.20}$$

The proof of the existence of such an adjoint operator provided by Riesz representation theorem for bounded linear functionals on $\mathcal{H}$. Therefore, a bounded linear operator $T : \mathcal{H} \mapsto \mathcal{H}$ is said to be *self adjoint or Hermitian* if $T^* = T$ and we see that the formula (A.20) becomes

$$\langle Tf, g\rangle = \langle f, Tg\rangle \text{ for all } f \in \mathcal{H} \text{ and all } g \in \mathcal{H}. \tag{A.21}$$

But if the operators is unbounded or not defined on all of Hilbert space $\mathcal{H}$ then the proof breaks down. Let us see the adjoints in this case.

Suppose T is a linear operator in $\mathcal{H}$ where domain of T is denoted by $\mathcal{D}_T$ which is assumed a linear manifold. If for some fixed vector g, there is only one vector g^* satisfying the condition

$$\langle Tf, g\rangle = \langle f, g^*\rangle \text{ for all } f \in \mathcal{D}_T, \tag{A.22}$$

then it is legitimate to write this unique vector g^* as $g^* = T^*g$ and to consider T^* as a mapping which is, at least, well defined for g. Then $\mathcal{D}_{T^*}$ is again a linear manifold and the operator T^* with domain $\mathcal{D}_{T^*}$, defined by

$$T^* g = g^* \text{ for all } g \in \mathcal{D}_{T^*} \tag{A.23}$$

is linear and *adjoint* of T if $\mathcal{D}_T$ is everywhere dense in $\mathcal{H}$ *i.e.*, $\bar{\mathcal{D}}_T = \mathcal{H}$. Hence a linear unbounded operator T in $\mathcal{H}$ such that $\bar{\mathcal{D}}_T = \mathcal{H}$ is called *selfadjoint* if $T^* = T$ i.e., $\mathcal{D}_{T^*}$ =D_T. In addition, the operator T is called *symmetric* if $T^* \supset T$.

Appendix B

Mathematical Structure of Classical and Quantum Mechanics

This appendix introduce the mathematical structure of classical and quantum mechanics which provides a framework for the study on the field of quantization. As we have seen, quantum mechanics was formulated in order to overcome the inadequacy of classical mechanics in explaining a whole range of physical phenomena. Moreover, there are fundamental geometric similarities between quantum and classical mechanics. The contents of this appendix are taken from the notes [14] and [20].

B.1 Classical Mechanics

Consider a particle with configuration space $\mathcal{M}$, with n-dimensional manifold. Then the configuration space of a physical system, consisting of N particles, and subject to k holonomic constraints is $n = (3N - k)$-dimensional diferential manifold. Its phase space is the cotangent bundle $T^*\mathcal{M}$, which if one has $\mathcal{M} = \mathbb{R}^n$, is isomorphic to $\mathbb{R}^{2n}$. The phase space of the physical system is defined to be the manifold described by the local coordinates q^j, p_j and hence has the structure of a cotangent bundle. We shall use for it the notation, $\Gamma = T^*\mathcal{M} =$ phase space.

The collection of classical observables of the real vector space $C^\infty(\Gamma, \mathbb{R})$ which is denoted by $\Upsilon^0_{\mathbb{R}}$. Pointwise multiplication defines a bilinear map on $\Upsilon^0_{\mathbb{R}}$, which is both commutative and associative. In addition, the Poisson bracket $\{.,.\} : \Gamma \times \Gamma \mapsto \Gamma$ gives $\Upsilon^0_{\mathbb{R}}$ the structure of a real Lie algebra, with the Poisson bracket playing the role of the Lie bracket. The bracket is related to pointwise multiplication $\circ$ by the Leibniz rule:

$$\{f, g \circ h\} = \{f, g\} \circ h + g \circ \{g, h\}, \tag{B.1}$$

where, Poisson bracket, $\{,\}$, of two functions f, g of the phase space variables q, p. We define it as:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} = \sum_{i=1}^{n}[\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}]. \tag{B.2}$$

Therefore, our classical system comprises of a triplet $(\Upsilon^0_{\mathbb{R}}, \circ, \{,\})$.

B.2 Quantum Mechanics

A quantum system is typically of infinite dimension. For the sake of simplicity, we restrict to N-level systems, with $N < \infty$. Indeed, the point of departure for a derivation of quantum mechanics was the replacement of classical Poisson brackets by quantum commutator brackets. The set of observables $\Upsilon_{\mathbb{R}}$ is the real vector space $M_n(\mathbb{C})$ of Hermitian complex $N \times N$ matrices. A symmetric bilinear product $\circ$ on $\Upsilon_{\mathbb{R}}$ is given by

$$A \circ B = \frac{1}{2}(AB + BA). \tag{B.3}$$

Note that $\circ$ is no longer associative. We also define a Poisson bracket on $\Upsilon_{\mathbb{R}}$: $\{A, B\}_\hbar = \frac{1}{\hbar}(AB - BA)$, where, $\hbar \in \mathbb{R}\backslash\{0\}$ is Planck's constant.

Appendix C

Proofs related to Chapter 3

In this Appendix, we collect together the proofs of some of the results quoted in chapter 4.

Proof of Lemma 4.3.1 Stone's theorem [13] guarantees us to write the unitary operators in the form like (C.1) and (C.2). We now look for the two generators X and Y using the UIR,

$$(e^{iaX}\psi)(x) = \psi(ax); \tag{C.1}$$

$$(e^{ibY}\psi)(x) = e^{ibx}\psi(x). \tag{C.2}$$

Without loss of generality (C.1) and (C.2), we can assume that $A = e^{iaX}$ and so $(A\psi)(x) = \psi(ax)$, $a \in \mathbf{R}^{>\mathbf{0}}$, $B = e^{ibY}$ then $(B\psi)(x) = e^{ibx}\psi(x)$, $b \in \mathbf{R}$. The action of A on the vector ψ is defined in (C.1) which "scales or dilates" the original vectors in the hilbert space. The operator X can be obtained by the folowing calculations:

$$\frac{d}{da}(e^{iaX}\psi)(x) = \frac{\partial}{\partial a}\psi(ax)$$

Using $\frac{\partial}{\partial a}\psi(ax) = x\frac{d}{dx}\psi(ax)$ in the previous equation, we have $iX((e^{iaX}\psi)(x) = x\frac{d}{dx}\psi(ax))$. This implies $iX\psi(ax) = x\frac{d}{dx}\psi(ax)$ and taking $a = 1$ then we have

$iX\psi(x) = \sum_i x_i \frac{d}{dx_i}\psi(x))$ and finally,

$$X = -ix\frac{d}{dx}. \tag{C.3}$$

So that, the self adjoint operator X which is a sort of the composition of the multiplication operator and derivative operator.

Similerly, the action of B on the vector ψ is defined in (C.2) which "translate or time shift" the original vectors in the hilbert space. The operator Y can be obtained by the folowing calculations:

$$\frac{d}{db}((e^{ibY}\psi)(x) = \frac{d}{db}((e^{ibx}\psi)(x) \implies iY(e^{ibY}\psi)(x) = ix(e^{ibx}\psi)(x). \tag{C.4}$$

Now, taking $b = 0$ on the above equation then we get $iY\psi(x) = ix\psi(x))$ which is a multiplication operator,

$$Y\psi(x) = x\psi(x). \tag{C.5}$$

So, we have the Hilbert space $h = L^2(\mathbb{R}^{>0}, \frac{dx}{x})$ which constructed by the induced representations and orbits in Section 4.1.4 and the affine subgroup G^+_{aff} in Section 4.3.1. Finally, the self adjoint operators X and Y are defined by their action on the state vectors $\psi(x) \in \mathcal{H}$,

$$X\psi(x) = -i\,x\frac{d\psi}{dx}\ ;\ Y\psi(x) = x\psi(x) \tag{C.6}$$

It is very easy to check that $[Y, X] = ix$ and therefore the self adjoint operators $\{X, Y\}$ generate the Affine subalgebra. $\square$

Proof of Lemma 4.3.1 If G is a group and H is a subgroup of G, then H is a normal subgroup of G if $ghg^{-1} \in H$ for every $g \in G,\ h \in H$. Following definition

$g \in \mathcal{P}_+^\uparrow(1,1)$ and $h \in H$ then we have,

$$\begin{aligned}
ghg^{-1} &= (\Lambda_\vartheta, (b_0, \mathbf{b})^T)\ (\mathbb{I}_2, (\mathbf{b}, -\mathbf{b})^T)(\Lambda_\vartheta, (b_0, \mathbf{b})^T)^{-1} \\
&= (\Lambda_\vartheta, (b_0, \mathbf{b})^T)\ (\mathbb{I}_2, (\mathbf{b}, -\mathbf{b})^T)(\Lambda_\vartheta^{-1}, -\Lambda_\vartheta^{-1}(b_0, \mathbf{b})^T) \\
&= (\Lambda_\vartheta, (b_0, \mathbf{b})^T)\ (\mathbb{I}_2\Lambda_\vartheta^{-1}, (\mathbf{b}, -\mathbf{b})^T) - \mathbb{I}_2\Lambda_\vartheta^{-1}(b_0, \mathbf{b})^T) \\
&= (\Lambda_\vartheta, \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix})\ (\Lambda_\vartheta^{-1}, \begin{pmatrix} \mathbf{b} \\ -\mathbf{b} \end{pmatrix} - \begin{pmatrix} \cosh\vartheta & -\sinh\vartheta \\ -\sinh\vartheta & \cosh\vartheta \end{pmatrix} \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix}) \\
&= (\Lambda_\vartheta, \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix})\ (\Lambda_\vartheta^{-1}, \begin{pmatrix} \mathbf{b} \\ -\mathbf{b} \end{pmatrix} - \begin{pmatrix} b_0\cosh\vartheta - \mathbf{b}\sinh\vartheta \\ -b_0\sinh\vartheta + \mathbf{b}\cosh\vartheta \end{pmatrix}) \\
&= (\Lambda_\vartheta, \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix})\ (\Lambda_\vartheta^{-1}, \begin{pmatrix} \mathbf{b} - b_0\cosh\vartheta + \mathbf{b}\sinh\vartheta \\ -\mathbf{b} + b_0\sinh\vartheta - \mathbf{b}\cosh\vartheta \end{pmatrix}) \\
&= (\Lambda_\vartheta\Lambda_\vartheta^{-1}, \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix} + \Lambda_\vartheta \begin{pmatrix} \mathbf{b} - b_0\cosh\vartheta + \mathbf{b}\sinh\vartheta \\ -\mathbf{b} + b_0\sinh\vartheta - \mathbf{b}\cosh\vartheta \end{pmatrix}) \\
&= (\mathbb{I}_2, \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix} + \begin{pmatrix} \cosh\vartheta & \sinh\vartheta \\ \sinh\vartheta & \cosh\vartheta \end{pmatrix} \begin{pmatrix} \mathbf{b} - b_0\cosh\vartheta + \mathbf{b}\sinh\vartheta \\ -\mathbf{b} + b_0\sinh\vartheta - \mathbf{b}\cosh\vartheta \end{pmatrix}) \\
&= (\mathbb{I}_2, \begin{pmatrix} \mathbf{b}\cosh\vartheta - \mathbf{b}\sinh\vartheta \\ \mathbf{b}\sinh\vartheta - \mathbf{b}\cosh\vartheta \end{pmatrix}) \\
&= (\mathbb{I}_2, (\mathbf{b}_1, -\mathbf{b}_1)^T),
\end{aligned} \tag{C.7}$$

where, $\mathbf{b}_1 = \mathbf{b}\cosh\vartheta - \mathbf{b}\sinh\vartheta$ and $ghg^{-1} \in H$ for every $g \in G,\ h \in H$. Then, H is a normal subgroup of $\mathcal{P}_+^\uparrow(1,1)$ which allowed us to say $\mathcal{P}_+^\uparrow(1,1)/H \simeq G_{\text{aff}}^+$ is quotient group. □

Proof of Lemma 4.3.2 Considering the multiplication rule (4.3) of the Poncaré group we have the following calculations for left coset decomposition,

$$
\begin{aligned}
(\Lambda_a, \frac{b_0+\mathbf{b}}{2}(1,1)^T)(\mathbb{I}_2, \frac{b_0-\mathbf{b}}{2}\Lambda_{\frac{1}{a}}(1,-1)^T) &= (\Lambda_a, \Lambda_a \frac{b_0-\mathbf{b}}{2}\Lambda_{\frac{1}{a}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{b_0+\mathbf{b}}{2} \\ \frac{b_0+\mathbf{b}}{2} \end{pmatrix}) \\
&= (\Lambda_a, \Lambda_a \frac{b_0-\mathbf{b}}{2} \begin{pmatrix} a \\ -a \end{pmatrix} + \begin{pmatrix} \frac{b_0+\mathbf{b}}{2} \\ \frac{b_0+\mathbf{b}}{2} \end{pmatrix}) \\
&= (\Lambda_a, \frac{a(b_0-\mathbf{b})}{2}\Lambda_a \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{b_0+\mathbf{b}}{2} \\ \frac{b_0+\mathbf{b}}{2} \end{pmatrix}) \\
&= (\Lambda_a, \frac{a(b_0-\mathbf{b})}{2} \begin{pmatrix} \frac{1}{a} \\ -\frac{1}{a} \end{pmatrix} + \begin{pmatrix} \frac{b_0+\mathbf{b}}{2} \\ \frac{b_0+\mathbf{b}}{2} \end{pmatrix}) \\
&= (\Lambda_a, \begin{pmatrix} \frac{(b_0-\mathbf{b})}{2} \\ -\frac{(b_0-\mathbf{b})}{2} \end{pmatrix} + \begin{pmatrix} \frac{b_0+\mathbf{b}}{2} \\ \frac{b_0+\mathbf{b}}{2} \end{pmatrix}) \\
&= (\Lambda_a, \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix}) \\
&= (\Lambda_a, (b_0, \mathbf{b})^T)). \qquad (C.8)
\end{aligned}
$$

Similarly, for the right coset decomposition we have

$$
\begin{aligned}
(\mathbb{I}_2, \frac{b_0-\mathbf{b}}{2}(1,-1)^T)\,(\Lambda_a, \frac{b_0+\mathbf{b}}{2}(1,1)^T) &= (\mathbb{I}_2\Lambda_a, \frac{b_0-\mathbf{b}}{2}(1,-1)^T + \mathbb{I}_2\frac{b_0+\mathbf{b}}{2}(1,1)^T) \\
&= (\Lambda_a, \frac{b_0-\mathbf{b}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{b_0+\mathbf{b}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \\
&= (\Lambda_a, \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix}) \\
&= (\Lambda_a, (b_0, \mathbf{b})^T)). \qquad (C.9)
\end{aligned}
$$

This completes the proof of the left and right coset decomposition of the coset space or quotient group of the Poincaré group $\mathcal{P}_+^\uparrow(1,1)$. □

Proof of Lemma 4.3.3 We know both of the projection and section map so that the proof is trivial. So that we have the following calculations,

$$\begin{aligned}(\pi \circ \sigma_l)(\mathbf{b}, a) &= \pi\sigma_l(\mathbf{b}, a)) \\ &= \pi(\Lambda_a, (\mathbf{b}, \mathbf{b})^T) \\ &= (\Lambda_a, \frac{(\mathbf{b}+\mathbf{b})}{2}(1,1)^T \\ &= (\Lambda_a, (\mathbf{b}, \mathbf{b})^T) \\ &= (\mathbf{b}, a). \end{aligned} \tag{C.10}$$

Hence $(\pi \circ \sigma_l) = Id_{G_{\mathrm{aff}+}} \in \mathcal{P}_+^{\uparrow}(1,1)/H$ has proved that the left and right section and the left and right projection of the G_{aff}^{+} group. □

Proof of Lemma 4.3.4 We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the follwing proof:

$\int_{\mathbb{R}\times\mathbb{R}^{>0}} \langle\phi|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle \; \frac{d\mathbf{b}\;da}{a^2}$
$= \frac{2\pi}{2\pi I(\eta)} \; \int_{\mathbb{R}\times\mathbb{R}^{>0}}[\int_0^\infty \int_0^\infty \bar{\phi}(x)e^{i\mathbf{b}(x-x')}\eta(xa)\bar{\eta}(x'a)\psi(x') \; \frac{dx}{x} \; \frac{dx'}{x'}] \; \frac{d\mathbf{b}\;da}{a^2}$
$= \frac{2\pi}{I(\eta)} \; \int_{\mathbb{R}^{>0}}[\int_0^\infty \int_0^\infty \delta(x-x')\bar{\phi}(x)\eta(xa)\bar{\eta}(x'a)\psi(x') \; \frac{dx}{x} \; \frac{dx'}{x'}] \; \frac{da}{a^2}$
$= \frac{2\pi}{I(\eta)} \; \int_{\mathbb{R}^{>0}}[\int_0^\infty \int_0^\infty \delta(x-x')\bar{\phi}(x)\eta(xa)\bar{\eta}(x'a)\psi(x') \; \frac{dx'}{x'} \; \frac{dx}{x}] \; \frac{da}{a^2}$
$= \frac{2\pi}{I(\eta)} \; \int_{\mathbb{R}^{>0}}[\int_0^\infty \bar{\phi}(x)|\eta(xa)|^2\psi(x) \; \frac{dx}{x^2}] \; \frac{da}{a^2}$
$= \frac{2\pi}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)\psi(x) \; [\int_{\mathbb{R}^{>0}} |\eta(xa)|^2 \; \frac{da}{a^2}] \; \frac{dx}{x^2}$
$= \frac{2\pi}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)\psi(x) \; [\int_{\mathbb{R}^{>0}} |\eta(t)|^2 \; \frac{dt}{t^2}] \; \frac{dx}{x}$
$= \frac{I(\eta)}{I(\eta)} \; \int_0^\infty \bar{\phi}(x)\psi(x) \; \frac{dx}{x}$
$= \langle\phi|\psi\rangle$. Making the change of variables, $xa = t, \implies a = \frac{t}{x}$ and $da = \frac{dt}{x}$ and $\frac{da}{a^2} = \frac{xdt}{t^2}$ and $I(\eta) := 2\pi \; \int_0^\infty |\eta(t)|^2 \; \frac{dt}{t^2}$. □

C.1 Quantization using the left Haar measure

In this Appendix we present the proofs of the operators which has quoted in Table (4.2), Chapter 4. In briefly, the integral quantization of the real valued function

$\phi(\mathbf{b}, a) \in G^{+}_{aff}$ for the left haar measure is formally defined by the following linear map:

$$\phi(\mathbf{b}, a) \mapsto \mathcal{A}_\phi = \widehat{\phi}_\ell := \int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}| \, \frac{d\mathbf{b}\; da}{a^2} \;, \tag{C.11}$$

where, the operator A_ϕ is in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. It should be noted that the initial business of the Integral Quintization scheme has been depicted in Chapter (2) and three examples in the Chapter (3). We shall have to change the order of integration to construct the operators by integral quantization technique so that we consider the compact supports and smoothness property of the functions ϕ and ψ. We shall also have to use the following identity.

$$\frac{1}{2\pi}\int_{b=-\infty}^{\infty} e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}\, d\mathbf{b} = \delta(x - x'). \tag{C.12}$$

Quantization of $\phi(\mathbf{b}, a) = a$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{\phi}_{\ell a}$ for the real valued function $\phi(\mathbf{b}, a) = a$ by doing the following calculations: $\langle\phi|\mathcal{A}_\phi|\psi\rangle$

$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_\phi|\psi)(x)\ \frac{dx}{x}$

$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle\ \frac{d\mathbf{b}\, da}{a^2}]\ \frac{dx}{x}$

$= \frac{1}{[I(\eta)]^{1/2}}\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)\langle\eta_{\mathbf{b},a}|\psi\rangle e^{i\mathbf{b}\mathbf{x}}\eta(xa)\ \frac{d\mathbf{b}\, da}{a^2}]\ \frac{dx}{x}$

$= \frac{2\pi}{2\pi I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a\ \{\int_{x'=0}^\infty e^{-i\mathbf{b}x'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\}e^{i\mathbf{b}\mathbf{x}}\eta(xa)\ \frac{d\mathbf{b}\, da}{a^2}]\ \frac{dx}{x}$

$= \frac{2\pi}{2\pi I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ [\int_{\mathbb{R}^{>0}} a\int_{x'=0}^\infty\{\ \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}\ d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\frac{da}{a^2}]\ \frac{dx}{x}$

$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a\ \int_{x'=0}^\infty\{\ \frac{1}{2\pi}\int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}\ d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}$

$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ [\int_{\mathbb{R}^{>0}} a\int_{x'=0}^\infty \delta(x - x')\ \bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}$

$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ [\int_{\mathbb{R}^{>0}} a\bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a^2}]\ \frac{dx}{x^2}$

$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\ \int_{\mathbb{R}^{>0}} a\ \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{dx}{x^2}\ \frac{da}{a^2}]\frac{dx}{x}$

$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\ \int_{\mathbb{R}^{>0}} \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a}]\ \frac{dx}{x^2}$

$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t}]\psi(x)\ \frac{dx}{x^2}$

$= \frac{\|\eta\|^2}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x^2}$

$= \frac{\|\eta\|^2}{I(\eta)} \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x^2}$

$= \langle\phi|\widehat{\phi}_{\ell a}\psi\rangle$. Making the change of variables, $xa = t,\ \Longrightarrow\ a = \frac{t}{x}$ and $da = \frac{dt}{x}$ and $\frac{da}{a} = \frac{dt}{t}$ and $|\eta|^2 = 2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t}$. Finally, we have $(\widehat{\phi}_{\ell a}\psi)(x) = \frac{\|\eta\|^2}{I(\eta)}\ \frac{\psi(x)}{x}$. $\square$

Quantization of $\phi(\mathbf{b}, a) = b$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{\phi}_{\ell b}$ for the real valued function $\phi(\mathbf{b}, a) = b$ by doing the following calculations:

$$\langle\phi|\mathcal{A}_\phi|\psi\rangle$$

$$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_\phi|\psi)(x)\ \frac{dx}{x}$$

$$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle\ \frac{d\mathbf{b}\, da}{a^2}]\ \frac{dx}{x}$$

$$= \frac{1}{[I(\eta)]^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)\langle\eta_{\mathbf{b},a}|\psi\rangle e^{i\mathbf{b}x}\eta(xa)\ \frac{d\mathbf{b}\, da}{a^2}]\ \frac{dx}{x}$$

$$= \frac{2\pi}{2\pi I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \mathbf{b}\{\ \int_{x'=0}^\infty e^{-i\mathbf{b}x'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\}e^{i\mathbf{b}x}\eta(xa)\ \frac{d\mathbf{b}\, da}{a^2}]\ \frac{dx}{x}$$

Since $\psi = 0$ at the boundaries of the integral so we are allowed to write the integral like $\frac{\partial}{\partial x'}\{\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\} = 0$ and this implies :b$\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi(x')\frac{dx'}{x'} = -i\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi'(x')\ \frac{dx'}{x'} - ia\int_0^\infty e^{-ibx'}\bar{\eta}'(x'a)\psi(x')\frac{dx'}{x'} + i\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi(x')\frac{dx'}{x'^2}$

Using the above fact we have the following calculations:

$$= 2\pi[\frac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\ \int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty\{\ \frac{1}{2\pi} \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}d\mathbf{b}\}\bar{\eta}(x'a)\psi'(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x} +$$
$$\frac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty\{\ \frac{1}{2\pi} \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}d\mathbf{b}\}\bar{\eta}'(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x} +$$
$$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty\{\ \frac{1}{2\pi} \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'^2}\ \frac{da}{a^2}]\ \frac{dx}{x}]$$
$$= 2\pi[\frac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty \delta(x - x')\ \bar{\eta}(x'a)\psi'(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x} +$$
$$\frac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\ \int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty \delta(x - x')\ \bar{\eta}'(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x} +$$
$$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty \delta(x - x')\ \bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'^2}\ \frac{da}{a^2}]\ \frac{dx}{x}]$$
$$= 2\pi[\ \frac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(xa)\psi'(x)\eta(xa)\ \frac{da}{a^2}]\ \frac{dx}{x^2} +$$
$$\frac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\ \int_{\mathbb{R}^{>0}} \bar{\eta}'(xa)\psi(x)\eta(xa)\ \frac{da}{a}]\ \frac{dx}{x^2} +$$
$$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a^2}]\ \frac{dx}{x^3}\ ]$$
$$= \frac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\ 2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t^2}]\psi'(x)\ \frac{dx}{x} + \frac{-i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t)\ \frac{dt}{t}]\psi(x)\ \frac{dx}{x^2} +$$

$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t) \frac{dt}{t^2}]\psi(x) \frac{dx}{x^2}$
$= \frac{-iI(\eta)}{I(\eta)} \int_0^\infty \bar{\phi}(x)\psi'(x) \frac{dx}{x} + \frac{iI(\eta)}{2I(\eta)} \int_0^\infty \bar{\phi}(x)\psi(x) \frac{dx}{x^2} + \frac{iI(\eta)}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t) \frac{dt}{t^2}]\psi(x) \frac{dx}{x^2}$
$= -i \int_0^\infty \bar{\phi}(x)\psi'(x) \frac{dx}{x} + \frac{-i}{2} \int_0^\infty \bar{\phi}(x)\frac{1}{x} \psi(x) \frac{dx}{x} + i \int_0^\infty \bar{\phi}(x) \frac{1}{x}\psi(x) \frac{dx}{x}$
$= -i \int_0^\infty \bar{\phi}(x)\psi'(x) \frac{dx}{x} + \frac{i}{2} \int_0^\infty \bar{\phi}(x) \frac{1}{x}\psi(x) \frac{dx}{x}$
$= \langle \phi | \widehat{\phi}_{\ell b} \psi \rangle.$

Making the change of variables, $xa = t, \implies a = \frac{t}{x}$ and $da = \frac{dt}{x}$ and $\frac{da}{a^2} = \frac{xdt}{t^2}$. Also $I(\eta) := 2\pi \int_{\mathbb{R}^{>0}} |\eta(t)|^2 \frac{dt}{t^2}$ and finally, we have $(\widehat{\phi}_{\ell b}\psi)(x) = -i \frac{d\psi}{dx} + \frac{i}{2x} \psi(x)$. □

Quantization of $\phi(\mathbf{b}, a) = \frac{1}{a}$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{\phi}_{\ell \frac{1}{a}}$ for the real valued function $\phi(\mathbf{b}, a) = \frac{1}{a}$ by doing the following calculations:

$\langle \phi | \mathcal{A}_\phi | \psi \rangle$
$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_\phi|\psi)(x) \frac{dx}{x}$
$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle \frac{d\mathbf{b}\, da}{a^2}] \frac{dx}{x}$
$= \frac{1}{[I(\eta)]^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)\langle\eta_{\mathbf{b},a}|\psi\rangle e^{i\mathbf{b}x}\eta(xa) \frac{d\mathbf{b}\, da}{a^2}] \frac{dx}{x}$
$= \frac{2\pi}{[(2\pi I(\eta))]} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \frac{1}{a}\{\int_{x'=0}^\infty e^{-i\mathbf{b}x'}\bar{\eta}(x'a)\psi(x') \frac{dx'}{x'}\}e^{i\mathbf{b}x}\eta(xa) \frac{d\mathbf{b}\, da}{a^2}] \frac{dx}{x}$
$= \frac{2\pi}{[(2\pi I(\eta))]} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \frac{1}{a} \int_{x'=0}^\infty\{ \int_{b=-\infty}^\infty e^{i\mathbf{b(x-x')}}d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa) \frac{dx'}{x'} \frac{da}{a^2}] \frac{dx}{x}$
$= \frac{2\pi}{[(I(\eta))]} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \frac{1}{a} \int_{x'=0}^\infty\{ \frac{1}{2\pi} \int_{b=-\infty}^\infty e^{i\mathbf{b(x-x')}}d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa) \frac{dx'}{x'} \frac{da}{a^2}] \frac{dx}{x}$
$= \frac{2\pi}{[(I(\eta))]} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \frac{1}{a} \int_{x'=0}^\infty \delta(x-x')\, \bar{\eta}(x'a)\psi(x')\eta(xa) \frac{dx'}{x'} \frac{da}{a^2}] \frac{dx}{x}$
$= \frac{2\pi}{[(I(\eta))]} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(xa)\psi(x)\eta(xa) \frac{da}{a^3}] \frac{dx}{x^2}$
$= \frac{1}{[(I(\eta))]} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t) \frac{dt}{t^3}]\psi(x)\, dx$
$= \frac{I_1(\eta)}{[(I(\eta))]} \int_0^\infty \bar{\phi}(x)x\psi(x) \frac{dx}{x}$
$= \langle \phi | \widehat{\phi}_{\ell \frac{1}{a}} \psi \rangle.$

Making the change of variables, $xa = t, \implies a = \frac{t}{x}$ and $da = \frac{dt}{x}$ and $\frac{da}{a^3} = \frac{x^2dt}{t^3}$ and $I_1(\eta) = 2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t) \frac{dt}{t^3}$. Finally, we have $(\widehat{\phi}_{\ell \frac{1}{a}}\psi)(x) = [\frac{I_1(\eta)}{I(\eta)}]\, x\, \psi(x)$. □

Quantization of $\phi(\mathbf{b},a) = \frac{b}{a}$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{\phi}_{\ell \frac{b}{a}}$ for the real valued function $\phi(\mathbf{b},a) = \frac{b}{a}$ by doing the following calculations:

$$\begin{aligned}
&\langle\phi|\mathcal{A}_\phi|\psi\rangle \\
&= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_\phi|\psi)(x)\ \frac{dx}{x} \\
&= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x} \\
&= \frac{1}{[I(\eta)]^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)\langle\eta_{\mathbf{b},a}|\psi\rangle e^{i\mathbf{b}x}\eta(xa)\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x} \\
&= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}}\ \frac{\mathbf{b}}{a}\{\int_{x'=0}^\infty e^{-i\mathbf{b}x'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\}e^{i\mathbf{b}x}\eta(xa)\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x} \\
&= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \mathbf{b}\{\int_{x'=0}^\infty e^{-i\mathbf{b}x'}\bar{\eta}(x'a)\psi(x')\frac{dx'}{x'}\}e^{i\mathbf{b}x}\eta(xa)\frac{d\mathbf{b}\,da}{a^3}]\frac{dx}{x}
\end{aligned}$$

Since $\psi = 0$ at the boundaries of the integral so we are allowed to write the integral like $\frac{\partial}{\partial x'}\{\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\} = 0$ and this implies :b$\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'} = -i\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi'(x')\ \frac{dx'}{x'}\ - ia\int_0^\infty e^{-ibx'}\bar{\eta}'(x'a)\psi(x')\ \frac{dx'}{x'} + i\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'^2}$

Using the above fact we have the following calculations:

$$\begin{aligned}
&= 2\pi[\ \frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{i\mathbf{b(x-x')}}d\mathbf{b}\}\bar{\eta}(x'a)\psi'(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^3}]\ \frac{dx}{x} + \\
&\frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{i\mathbf{b(x-x')}}d\mathbf{b}\}\bar{\eta}'(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x} + \\
&\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{i\mathbf{b(x-x')}}d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'^2}\ \frac{da}{a^3}]\ \frac{dx}{x}\ ] \\
&= 2\pi[\ \frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty \delta(x-x')\ \bar{\eta}(x'a)\psi'(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^3}]\ \frac{dx}{x} + \\
&\frac{-i}{I(\eta)}\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty \delta(x-x')\ \bar{\eta}'(x'a)\psi(x')\eta(xa)\frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x} + \\
&\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty \delta(x-x')\ \bar{\eta}(x'a)\psi(x')\eta(xa)\frac{dx'}{x'^2}\ \frac{da}{a^3}]\ \frac{dx}{x}\ ] \\
&= 2\pi[\frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(xa)\psi'(x)\eta(xa)\ \frac{da}{a^3}]\ \frac{dx}{x^2} + \\
&\frac{-i}{[(I(\eta)]}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}'(xa)\psi(x)\eta(xa)\ \frac{da}{a^2}]\ \frac{dx}{x^2} + \frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a^3}]\ \frac{dx}{x^3}] \\
&= \frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t^3}]\psi'(x)\ dx + \\
&\frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t)\ \frac{dt}{t^2}]\psi(x)\ \frac{dx}{x} + \frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t^3}]\psi(x)\ \frac{dx}{x} \\
&= [\frac{-iI_1(\eta)}{I(\eta)}]\ \int_0^\infty \bar{\phi}(x)x\psi'(x)\ \frac{dx}{x} - [\frac{iI_1(\eta)}{I(\eta)}]\ \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x} + [\frac{iI_1(\eta)}{I(\eta)}]\ \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x} \\
&= [\frac{-iI_1(\eta)}{I(\eta)}]\ \int_0^\infty \bar{\phi}(x)x\psi'(x)\ \frac{dx}{x} + 0 \\
&= \langle\phi|[\frac{-iI_1(\eta)}{I(\eta)}]\ x\ \frac{d\psi}{dx}\rangle.
\end{aligned}$$

Making the change of variables, $xa = t,\ \Longrightarrow\ a = \frac{t}{x}$ and $da = \frac{dt}{x}$ and $\frac{da}{a^3} = \frac{x^2dt}{t^3}$.

Also, $I_1(\eta) := 2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t) \frac{dt}{t^3}$ and finally, we have $(\widehat{\phi}_{\ell \frac{b}{a}}\psi)(x) = -[\frac{I_1(\eta)}{I(\eta)}]\ ix\ \frac{d\psi}{dx}$ □

Quantization of $\phi(\mathbf{b}, a) = ab$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{\phi}_{\ell ab}$ for the real valued function $\phi(\mathbf{b}, a) = ab$ by doing the following calculations:

$$\langle \phi | \mathcal{A}_\phi | \psi \rangle$$
$$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_\phi|\psi)(x)\ \frac{dx}{x}$$
$$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle\ \frac{d\mathbf{b}\, da}{a^2}]\ \frac{dx}{x}$$
$$= \frac{1}{[I(\eta)]^{1/2}}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)\langle\eta_{\mathbf{b},a}|\psi\rangle e^{i\mathbf{b}x}\eta(xa)\ \frac{d\mathbf{b}\, da}{a^2}]\ \frac{dx}{x}$$
$$= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a\mathbf{b}\{\int_{x'=0}^\infty e^{-i\mathbf{b}x'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\}e^{i\mathbf{b}x}\eta(xa)\ \frac{d\mathbf{b}\, da}{a^2}]\ \frac{dx}{x}$$
$$= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \mathbf{b}\{\int_{x'=0}^\infty e^{-i\mathbf{b}x'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\}e^{i\mathbf{b}x}\eta(xa)\ \frac{d\mathbf{b}\, da}{a}]\ \frac{dx}{x}$$

Since $\psi = 0$ at the boundaries of the integral so we are allowed to write the integral like $\frac{\partial}{\partial x'}\{\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\} = 0$ and this implies :b$\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'} = -i\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi'(x')\ \frac{dx'}{x'} - ia\int_0^\infty e^{-ibx'}\bar{\eta}'(x'a)\psi(x')\ \frac{dx'}{x'} + i\int_0^\infty e^{-ibx'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'^2}$
Using the above fact we have the following calculations:

$$= 2\pi[\ \frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}\ d\mathbf{b}\}\bar{\eta}(x'a)\psi'(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x} +$$
$$\frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}\ d\mathbf{b}\}\bar{\eta}'(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}da]\ \frac{dx}{x} +$$
$$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}\ d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'^2}\ \frac{da}{a}]\ \frac{dx}{x}\ ]$$
$$= 2\pi[\ \frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty \delta(x-x')\ \bar{\eta}(x'a)\psi'(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x} +$$
$$\frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty \delta(x-x')\ \bar{\eta}'(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}da]\ \frac{dx}{x} +$$
$$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty \delta(x-x')\ \bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'^2}\ \frac{da}{a}]\ \frac{dx}{x}\ ]$$
$$= 2\pi[\ \frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(xa)\psi'(x)\eta(xa)\ \frac{da}{a}]\ \frac{dx}{x^2} +$$
$$\frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}'(xa)\psi(x)\eta(xa)da]\ \frac{dx}{x^2} +$$
$$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a}]\ \frac{dx}{x^3}\ ]$$
$$= \frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t}]\psi'(x)\ \frac{dx}{x^2} +$$
$$\frac{-i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t)dt]\psi(x)\ \frac{dx}{x^3} +$$

$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t}]\psi(x)\ \frac{dx}{x^3}\ ]$
$= \frac{-i\|\eta\|^2}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\psi'(x)\ \frac{dx}{x^2} + \frac{i\|\eta\|^2}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x^3}$
$= \frac{-i\|\eta\|^2}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\frac{1}{x}\psi'(x)\frac{dx}{x} + \frac{i\|\eta\|^2}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\frac{1}{x^2}\psi(x)\ \frac{dx}{x}$
$= \langle\phi|\frac{\|\eta\|^2}{I(\eta)}\ \frac{-i}{x}\ \frac{d}{dx}\psi(x)\rangle + \langle\phi|\frac{\|\eta\|^2}{I(\eta)}\ iQ^{-2}\psi\rangle$
$= \langle\phi|\widehat{\phi}_{\ell ab}\psi\rangle.$

Making the change of variables, $xa = t$, $\implies$ $a = \frac{t}{x}$ and $da = \frac{dt}{x}$. The second term on the above integral become zero since $\int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t)dt = 0$. Also, $\|\eta\|^2 := 2\pi\ \int_{\mathbb{R}^{>0}} |\eta(t)|^2\frac{dt}{t}$; and finally, we have $(\widehat{\phi}_{\ell ab}\psi)(x) = [\frac{\|\eta\|^2}{I(\eta)}]\ (\frac{-i}{x}\frac{d\psi}{dx} + \frac{i}{x^2}\psi(x)).$ □

Quantization of $\phi(\mathbf{b}, a) = \log a$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{\phi}_{\ell \log a}$ for $\phi(\mathbf{b}, a) = \log a$ by doing the following calculations:

$\langle\phi|\mathcal{A}_\phi|\psi\rangle$
$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_\phi|\psi)(x)\ \frac{dx}{x}$
$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x}$
$= \frac{1}{[I(\eta)]^{1/2}}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)\langle\eta_{\mathbf{b},a}|\psi\rangle e^{i\mathbf{b}\mathbf{x}}\eta(xa)\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \log a\{\int_{x'=0}^\infty e^{-i\mathbf{b}\mathbf{x}'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\}e^{i\mathbf{b}\mathbf{x}}\eta(xa)\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x}$
$= \frac{2\pi}{2\pi I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \log a \int_{x'=0}^\infty\{\int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}\ d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \log a \qquad \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \log a \int_{x'=0}^\infty \delta(x - x')\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \log a\bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a^2}]\ \frac{dx}{x^2}$
$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \log a \quad \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a^2}]\ \frac{dx}{x^2}$
$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\log(\frac{t}{x})\ \frac{dt}{t^2}]\psi(x)\ \frac{dx}{x}$
$= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\{\log t - \log x\}\ \frac{dt}{t^2}]\psi(x)\ \frac{dx}{x}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\log t\ \frac{dt}{t^2}]\psi(x)\ \frac{dx}{x} -$
$\frac{1}{[(I(\eta)]}\ 2\pi\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t^2}]\log(x)\psi(x)\ \frac{dx}{x}$
$= \frac{I_{\log(\eta)}}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x} - \frac{I(\eta)}{[(I(\eta)]}\ \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x}$
$= \frac{I_{\log(\eta)}}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x} - \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x}$

$= \langle\phi|\frac{I_{\log(\eta)}}{I(\eta)}\ I\psi\rangle - \langle\phi|\log(x)\ \psi\rangle$
$= \langle\phi|\widehat{\phi}_{\ell\log a}\psi\rangle.$

Making the change of variables, $xa = t, \implies a = \frac{t}{x}$ and $da = \frac{dt}{x}$ and $\frac{da}{a} = \frac{dt}{t}$ and $I_{\log}(\eta) = 2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\log t\ \frac{dt}{t^2}$. Finally, we have $(\widehat{\phi}_{\ell\log a}\psi)(x) = [\frac{I_{\log}(\eta)}{I(\eta)}]\ \psi(x) - \log(x)\psi(x)$. □

Quantization of $\phi(\mathbf{b}, a) = a^n$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{\phi}_{\ell a^n}$ for the real valued function $\phi(\mathbf{b}, a) = a^n$ by doing the following calculations:

$\langle\phi|\mathcal{A}_\phi|\psi\rangle$
$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_\phi|\psi)(x)\ \frac{dx}{x}$
$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)^{1/2}}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)\langle\eta_{\mathbf{b},a}|\psi\rangle e^{i\mathbf{b}x}\eta(xa)\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a^n\{\int_{x'=0}^\infty e^{-i\mathbf{b}x'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\}e^{i\mathbf{b}x}\eta(xa)\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^n \int_{x'=0}^\infty\{\int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}\ d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^n\ \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^n\ \int_{x'=0}^\infty \delta(x - x')\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^n\ \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a^2}]\ \frac{dx}{x^2}$
$= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} a^n\ \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{dx}{x^2}\ \frac{da}{a^2}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a^{-n+2}}]\ \frac{dx}{x^2}$
$= \frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t^{-n+2}}]\psi(x)\ \frac{dx}{x^{n+1}}$
$= \frac{I_{-n}(\eta)}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x^{n+1}}$
$= \langle\phi|\frac{I_{-n}(\eta)}{(I(\eta)}\ Q^{-n}\psi\rangle$
$= \langle\phi|\widehat{\phi}_{\ell a^n}\psi\rangle.$

Making the change of variables, $xa = t, \implies a = \frac{t}{x}$ and $da = \frac{dt}{x}$ and $\frac{da}{a} = \frac{dt}{t}$ and $I_n(\eta) := 2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t^{n+2}}$. Finally, we have $(\widehat{\phi}_{\ell a^n}\psi)(x) = [\frac{I_{-n}(\eta)}{I(\eta)}]\ \frac{1}{x^n}\psi(x)$. □

Quantization of $\phi(\mathbf{b},a)=\frac{1}{a^n}$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H}=L^2(\mathbb{R}^{>0},\frac{dx}{x})$. Then we have the operator $\widehat{\phi}_{\ell\frac{1}{a^n}}$ for the real valued function $\phi(\mathbf{b},a)=\frac{1}{a^n}$ by doing the following calculations:

$$
\begin{aligned}
&\langle\phi|\mathcal{A}_\phi|\psi\rangle\\
&=\int_0^\infty \bar{\phi}(x)(\mathcal{A}_\phi|\psi)(x)\ \frac{dx}{x}\\
&=\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}}\phi(\mathbf{b},a)|\eta_{\mathbf{b},a}\rangle\langle\eta_{\mathbf{b},a}|\psi\rangle\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x}\\
&=\frac{1}{[I(\eta)]^{1/2}}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}}\phi(\mathbf{b},a)\langle\eta_{\mathbf{b},a}|\psi\rangle e^{i\mathbf{b}\mathbf{x}}\eta(xa)\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x}\\
&=\frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}}\ \frac{1}{a^n}\{\int_{x'=0}^\infty e^{-i\mathbf{b}\mathbf{x}'}\bar{\eta}(x'a)\psi(x')\ \frac{dx'}{x'}\}e^{i\mathbf{b}\mathbf{x}}\eta(xa)\ \frac{d\mathbf{b}\,da}{a^2}]\ \frac{dx}{x}\\
&=\frac{1}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\ \frac{1}{a^n}\int_{x'=0}^\infty\{\int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}\\
&=\frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\ \frac{1}{a^n}\ \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{i\mathbf{b}(\mathbf{x}-\mathbf{x}')}\ d\mathbf{b}\}\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}\\
&=\frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\frac{1}{a^n}\ \int_{x'=0}^\infty\delta(x-x')\bar{\eta}(x'a)\psi(x')\eta(xa)\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x}\\
&=\frac{2\pi}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\ \frac{1}{a^n}\ \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a^2}]\ \frac{dx}{x^2}\\
&=\frac{2\pi}{(I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\ \frac{1}{a^n}\ \bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{dx}{x^2}\ \frac{da}{a^2}]\ \frac{dx}{x}\\
&=\frac{1}{I(\eta)}\int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}}\bar{\eta}(xa)\psi(x)\eta(xa)\ \frac{da}{a^{n+2}}]\ \frac{dx}{x^2}\\
&=\frac{1}{I(\eta)}\int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}}\bar{\eta}(t)\eta(t)\frac{dt}{t^{n+2}}]x^n\psi(x)\frac{dx}{x}\\
&=\frac{I_n(\eta)}{I(\eta)}\int_0^\infty \bar{\phi}(x)x^n\psi(x)\frac{dx}{x}\\
&=\langle\phi|\frac{I_n(\eta)}{I(\eta)}Q^n\psi\rangle\\
&=\langle\phi|\widehat{\phi}_{\ell\frac{1}{a^n}}\psi\rangle.
\end{aligned}
$$

Making the change of variables, $xa=t,\implies a=\frac{t}{x}$ and $da=\frac{dt}{x}$ and $\frac{da}{a^{n+2}}=\frac{x^{n+1}dt}{t^{n+2}}$ and $I_n(\eta):=2\pi\ \int_{\mathbb{R}^{>0}}\bar{\eta}(t)\eta(t)\frac{dt}{t^{n+2}}$. Finally, we have $(\widehat{\phi}_{\ell\frac{1}{a^n}}\psi)(x)=[\frac{I_n(\eta)}{I(\eta)}]x^n\psi(x)$. □

C.2 Quantization using the right Haar measure

In this Appendix we present the proofs of the operators which has quoted in Table (4.3), Chapter 4. In shortly, using the resolution of the identity we can introduce the integral quantization of the real valued function $\phi(\mathbf{b},a)\in G^+_{aff}$ by the following linear map for the right haar measure:

$$\phi(\mathbf{b},a) \mapsto \mathcal{A}_{\phi \mathrm{r}} = {}_{\mathrm{r}}\widehat{\phi} := \int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}| \, \frac{d\mathbf{b}\; da}{a} \, . \tag{C.13}$$

where, the operator $\mathcal{A}_{\phi \mathrm{r}}$ is in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. It should be noted that the initial business of the Integral Quintization scheme has been depicted in Chapter(2) with three examples.

Quantization of $\phi(\mathbf{b},a) = a$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator ${}_{\mathrm{r}}\widehat{\phi_a}$ for the real valued function $\phi(\mathbf{b},a) = a$ by doing the following calculations:

$$
\begin{aligned}
&\langle\phi|\mathcal{A}_{\phi \mathrm{r}}|\psi\rangle \\
&= \textstyle\int_0^\infty \bar{\phi}(x)(\mathcal{A}_{\phi \mathrm{r}}|\psi\rangle)(x)\frac{dx}{x} \\
&= \textstyle\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}|\ \psi\rangle\ \frac{d\mathbf{b}\; da}{a}]\ \frac{dx}{x} \\
&= \textstyle\frac{1}{[I(\eta)]^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)\langle\xi_{\mathbf{b},a}|\psi\rangle e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\frac{d\mathbf{b}\; da}{a}]\frac{dx}{x} \\
&= \textstyle\frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a\{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\; da}{a}]\ \frac{dx}{x} \\
&= \textstyle\frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a\int_{x'=0}^\infty\{\int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x'}-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x} \\
&= \textstyle\frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x} \\
&= \textstyle\frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a\int_{x'=0}^\infty \delta(x'-x)\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ da]\ \frac{dx}{x} \\
&= \textstyle\frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a\ \bar{\eta}(\frac{x}{a})\psi(x)\eta(\frac{x}{a})\ \ da]\ \frac{dx}{x^2} \\
&= \textstyle\frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \frac{x}{t}\ \bar{\eta}(t)\eta(t)\ \ \frac{x}{t^2}\ dt]\ \psi(x)\frac{dx}{x^2} \\
&= \textstyle\frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\frac{dt}{t^3}]\psi(x)dx \\
&= \textstyle\frac{I_1(\eta}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ x\ \psi(x)\frac{dx}{x} \\
&= \langle\phi|{}_{\mathrm{r}}\widehat{\phi_a}\psi\rangle.
\end{aligned}
$$

Making the change of variables, $\frac{x}{a} = t, da = \frac{-xdt}{t^2}$ and $I_1(\eta) = 2\pi\ \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\frac{dt}{t^3}$. Finally, we have $({}_{\mathrm{r}}\widehat{\phi_a}\psi)(x) = [\frac{I_1(\eta}{(I(\eta)}]\ x\ \psi(x)$. □

Quantization of $\phi(\mathbf{b},a) = b$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{{}_{\mathrm{r}}\phi_b}$ for the real valued function $\phi(\mathbf{b},a) = b$ by doing the following calculations:

$\langle\phi|\mathcal{A}_{\phi\mathrm{r}}|\psi\rangle$
$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_{\phi\mathrm{r}}|\psi)(x)\frac{dx}{x}$
$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}|\ \psi\rangle\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x}$
$= \frac{1}{[I(\eta)]^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)\langle\xi_{\mathbf{b},a}|\psi\rangle e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\frac{d\mathbf{b}\ da}{a^2}]\frac{dx}{x}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} b\{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x}$

Since $\psi = 0$ at the boundaries of the integral so we are allowed to write the integral like $\frac{\partial}{\partial x'}\{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\} = 0$ and this implies: $b\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\frac{x'}{a})\psi(x')\ \frac{dx'}{x'} = i\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}'(\frac{x'}{a})\psi(x')\frac{dx'}{x'} + i\int_{x'=0}^\infty\ a\ e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi'(x')\ \frac{dx'}{x'}\ - i\int_{x'=0}^\infty\ a\ e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\ \frac{dx'}{x'^2}$

Using the above fact we have the following calculations:

$= 2\pi\ [\frac{i}{2\pi I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}}\{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}'(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x} +$
$\frac{i}{2\pi I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}}\{\int_{x'=0}^\infty\ a\ e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi'(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x} -$
$\frac{i}{2\pi I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}}\{\int_{x'=0}^\infty\ a\ e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'^2}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x}\ ]$
$= 2\pi\ [\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}'-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}'(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x} +$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}'-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ da]\ \frac{dx}{x} -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}'-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'^2}\ da]\ \frac{dx}{x}\ ]$
$= 2\pi\ [\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}'(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x} +$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ da]\ \frac{dx}{x} -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'^2}\ da]\ \frac{dx}{x}\ ]$
$= 2\pi\ [\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\int_{x'=0}^\infty \delta(x'-x)\bar{\eta}'(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ da]\ \frac{dx}{x} +$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a\ \int_{x'=0}^\infty \delta(x'-x)\ \bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ da]\ \frac{dx}{x} -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a\ \int_{x'=0}^\infty\ \delta(x'-x)\ \bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'^2}\ da]\ \frac{dx}{x}\ ]$
$= 2\pi\ [\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}'(\frac{x}{a})\eta(\frac{x}{a})\ da]\psi(x)\ \frac{dx}{x^2} +$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a\ \bar{\eta}(\frac{x}{a})\eta(\frac{x}{a})\ da]\psi'(x)\ \frac{dx}{x^2} -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a\ \bar{\eta}(\frac{x}{a})\eta(\frac{x}{a})\ da]\psi(x)\ \frac{dx}{x^3}\ ]$

$= 2\pi\ [\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t)\ \frac{dt}{t^2}]\psi(x)\ \frac{dx}{x} + \frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}|\eta(t)|^2\ \frac{dt}{t^3}]\psi'(x)\ dx -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}|\eta(t)|^2\ \frac{dt}{t^3}]\psi(x)\ \frac{dx}{x}\ ]$
$= \frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}}|\eta(t)|^2\ \frac{dt}{t^3}]\psi(x)\ \frac{dx}{x} + \frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}}|\eta(t)|^2\ \frac{dt}{t^3}]\psi'(x)\ dx -$
$\frac{i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}}|\eta(t)|^2\ \frac{dt}{t^3}]\psi(x)\ \frac{dx}{x}$
$= \frac{iI_1(\eta)}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\psi'(x)\ dx$
$= \frac{iI_1(\eta)}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ x\ \psi'(x)\ \frac{dx}{x}$
$= \langle\phi|\widehat{{}_{\mathrm{r}}\phi_b}\psi\rangle.$

Making the change of variables, $x/a = t$, $\implies$ $a = \frac{x}{t}$ and $da = \frac{-xdt}{t^2}$. Also $\int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t)\ \frac{dt}{t^2} = \int_{\mathbb{R}^{>0}}|\eta(t)|^2\ \frac{dt}{t^3}$ and $I_1(\eta) := \int_{\mathbb{R}^{>0}} |\eta(t)|^2\ \frac{dt}{t^3}$. Finally, we have $(\widehat{{}_{\mathrm{r}}\phi_b}\psi)(x) = i[\frac{I_1(\eta}{(I(\eta)}]\ x\ \frac{d}{dx}\psi(x)$. □

Quantization of $\phi(\mathbf{b}, a) = \frac{1}{a}$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a}}}$ for the real valued function $\phi(\mathbf{b}, a) = \frac{1}{a}$ by doing the following calculations:

$\langle\phi|\mathcal{A}_{\phi\mathrm{r}}|\psi\rangle$
$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_{\phi\mathrm{r}}|\psi)(x)\frac{dx}{x}$
$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}|\ \psi\rangle\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)\langle\xi_{\mathbf{b},a}|\psi\rangle e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\frac{d\mathbf{b}\ da}{a^2}]\frac{dx}{x}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \frac{1}{a}\{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \frac{1}{a} \int_{x'=0}^\infty\{\int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}'-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \frac{1}{a} \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty \delta(x' - x)\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\ \bar{\eta}(\frac{x}{a})\eta(\frac{x}{a})\ \frac{da}{a}]\psi(x)\ \frac{dx}{x^2}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi\ \int_{\mathbb{R}^{>0}}\ \bar{\eta}(t)\eta(t)\ \ \frac{dt}{t}]\ \psi(x)\frac{dx}{x^2}$
$= \frac{|\eta|^2}{I(\eta)} \int_0^\infty \bar{\phi}(x)\psi(x)\ \frac{dx}{x^2}$
$= \frac{|\eta|^2}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ \frac{1}{x}\ \psi(x)\frac{dx}{x}$
$= \langle\phi|\widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a}}}\psi\rangle.$

Making the change of variables, $\frac{x}{a} = t, \implies a = \frac{x}{t}$ and $da = \frac{-xdt}{t^2}$. Finally, we have $(\widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a}}}\psi)(x) = [\frac{|\eta|^2}{I(\eta)}]\ \frac{1}{x}\psi(x)$. □

Quantization of $\phi(\mathbf{b}, a) = \frac{b}{a}$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{{}_{\mathrm{r}}\phi_{\frac{b}{a}}}$ for the real valued function $\phi(\mathbf{b}, a) = \frac{b}{a}$ by doing the following calculations:

$\langle\phi|\mathcal{A}_{\phi\mathrm{r}}|\psi\rangle$

$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_{\phi\mathrm{r}}|\psi)(x)\frac{dx}{x}$

$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}|\ \psi\rangle\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x}$

$= \frac{1}{I(\eta)^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)\langle\xi_{\mathbf{b},a}|\psi\rangle e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\frac{d\mathbf{b}\ da}{a^2}]\frac{dx}{x}$

$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \frac{b}{a}\ \{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x}$

Since $\psi = 0$ at the boundaries of the integral so we are allowed to write the integral like $\frac{\partial}{\partial x'}\{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\} = 0$ and this implies: $b\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\frac{x'}{a})\psi(x')\ \frac{dx'}{x'} = i\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}\prime(\frac{x'}{a})\psi(x')\frac{dx'}{x'} + i\int_{x'=0}^\infty\ a\ e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi'(x')\ \frac{dx'}{x'}\ - i\int_{x'=0}^\infty\ a\ e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\ \frac{dx'}{x'^2}$

Using the above fact we have the following calculations:

$= \frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \frac{1}{a}\ \{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}'(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x} +$

$\frac{i}{2\pi I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \frac{1}{a}\ \{\int_{x'=0}^\infty\ a\ e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi'(x')\frac{dx'}{x'}\}e^{\frac{i-\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x} -$

$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \frac{1}{a}\ \{\int_{x'=0}^\infty\ a\ e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'^2}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\ da}{a}]\ \frac{dx}{x}$

$= \frac{2\pi i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}'-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}'(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x} +$

$\frac{2\pi i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}'-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}\ ]\ \frac{dx}{x} -$

$\frac{2\pi i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x}'-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'^2}\ \frac{da}{a}\ ]\ \frac{dx}{x}$

$= \frac{2\pi i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}'(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a^2}]\ \frac{dx}{x} +$

$\frac{2\pi i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}\ ]\ \frac{dx}{x} -$

$\frac{2\pi i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'^2}\ \frac{da}{a}\ ]\ \frac{dx}{x}$

$= \frac{2\pi i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \int_{x'=0}^\infty \delta(x'-x)\bar{\eta}'(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}\ ]\ \frac{dx}{x} +$

$\frac{2\pi i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\ \int_{x'=0}^\infty \delta(x'-x)\ \bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ da]\ \frac{dx}{x} -$

$\frac{2\pi i}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}\ \int_{x'=0}^\infty\ \delta(x'-x)\ \bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'^2}\ da]\ \frac{dx}{x}$

$= \frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}'(\frac{x}{a})\eta(\frac{x}{a}) \frac{da}{a}]\psi(x) \frac{dx}{x^2} + \frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(\frac{x}{a})\eta(\frac{x}{a}) da]\psi'(x) \frac{dx}{x^2} -$
$\frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \bar{\eta}(\frac{x}{a})\eta(\frac{x}{a}) da]\psi(x) \frac{dx}{x^3}$
$= \frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t) \frac{dt}{t}]\psi(x) \frac{dx}{x^2} + \frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}}|\eta(t)|^2 \frac{dt}{t^2}]\psi'(x) \frac{dx}{x} -$
$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}}|\eta(t)|^2 \frac{dt}{t^2}]\psi(x) \frac{dx}{x^2}$
$= i\frac{I(\eta)}{2I(\eta)} \int_0^\infty \bar{\phi}(x)\psi(x) \frac{dx}{x^2} + \frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}}|\eta(t)|^2 \frac{dt}{t^2}]\psi'(x) \frac{dx}{x} - i\frac{I(\eta)}{I(\eta)} \int_0^\infty \bar{\phi}(x)\psi(x) \frac{dx}{x^2}$
$= -\frac{i}{2} \int_0^\infty \bar{\phi}(x)\psi(x) \frac{dx}{x^2} + i \int_0^\infty \bar{\phi}(x)\psi'(x) \frac{dx}{x}$
$= i \int_0^\infty \bar{\phi}(x)\psi'(x) \frac{dx}{x} - \frac{i}{2} \int_0^\infty \bar{\phi}(x)\frac{1}{x}\psi(x) \frac{dx}{x}$
$= \langle\phi|\widehat{{}_{\mathrm{r}}\phi_{\frac{b}{a}}}\psi\rangle.$

Making the change of variables, $x/a = t, da = \frac{-xdt}{t^2}$. We also use $\int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t) \frac{dt}{t} = \int_{\mathbb{R}^{>0}}|\eta(t)|^2 \frac{dt}{t^2}$ and $I_1(\eta) := \int_{\mathbb{R}^{>0}} |\eta(t)|^2 \frac{dt}{t^3}$. Finally, we have $(\widehat{{}_{\mathrm{r}}\phi_{\frac{b}{a}}}\psi)(x) = i\frac{d\psi}{dx} - \frac{i}{2x}\psi(x)$.

□

Quantization of $\phi(\mathbf{b}, a) = ab$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{{}_{\mathrm{r}}\phi_b}$ for the real valued function $\phi(\mathbf{b}, a) = ab$ by doing the following calculations:

$\langle\phi|\mathcal{A}_{\phi\mathrm{r}}|\psi\rangle$
$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_{\phi\mathrm{r}}|\psi)(x)\frac{dx}{x}$
$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}|\ \psi\rangle \frac{d\mathbf{b}\, da}{a}] \frac{dx}{x}$
$= \frac{1}{I(\eta)^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b}, a)\langle\xi_{\mathbf{b},a}|\psi\rangle e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\frac{d\mathbf{b}\, da}{a^2}]\frac{dx}{x}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} ab\ \{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a}) \frac{d\mathbf{b}\, da}{a}] \frac{dx}{x}$

Since $\psi = 0$ at the boundaries of the integral so we are allowed to write the integral like $\frac{\partial}{\partial x'}\{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\} = 0$ and this implies: $b\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\frac{x'}{a})\psi(x') \frac{dx'}{x'} = i\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}\prime(\frac{x'}{a})\psi(x')\frac{dx'}{x'} + i\int_{x'=0}^\infty a\, e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi'(x') \frac{dx'}{x'} - i\int_{x'=0}^\infty a\, e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x') \frac{dx'}{x'^2}$

Using the above fact we have the following calculations:

$= \frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a\ \{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}'(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a}) \frac{d\mathbf{b}\, da}{a}] \frac{dx}{x} +$
$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a\ \{\int_{x'=0}^\infty a\, e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi'(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a}) \frac{d\mathbf{b}\, da}{a}] \frac{dx}{x} -$
$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a\ \{\int_{x'=0}^\infty a\, e^{\frac{i\mathbf{b}x'}{a}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'^2}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a}) \frac{d\mathbf{b}\, da}{a}] \frac{dx}{x}$

$= \frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a \int_{x'=0}^\infty \{\frac{1}{2\pi} \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x'}-\mathbf{x})}{a}} \, d\mathbf{b}\}\bar{\eta}'(\frac{x'}{a})\psi(x')\eta(\frac{x}{a}) \frac{dx'}{x'} \frac{da}{a}] \frac{dx}{x} +$

$\frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a \int_{x'=0}^\infty \{\frac{1}{2\pi} \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x'}-\mathbf{x})}{a}} \, d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a}) \frac{dx'}{x'} da] \frac{dx}{x} -$

$\frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a \int_{x'=0}^\infty \{\frac{1}{2\pi} \int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x'}-\mathbf{x})}{a}} \, d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a}) \frac{dx'}{x'^2} da] \frac{dx}{x}$

$= \frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a \int_{x'=0}^\infty \{\frac{1}{2\pi} \int_{s=-\infty}^\infty e^{is(x'-x)} \, a \, ds\}\bar{\eta}'(\frac{x'}{a})\psi(x')\eta(\frac{x}{a}) \frac{dx'}{x'} \frac{da}{a}] \frac{dx}{x} +$

$\frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a \int_{x'=0}^\infty \{\frac{1}{2\pi} \int_{s=-\infty}^\infty e^{is(x'-x)} \, a \, ds\}\bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a}) \frac{dx'}{x'} da] \frac{dx}{x} -$

$\frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a \int_{x'=0}^\infty \{\frac{1}{2\pi} \int_{s=-\infty}^\infty e^{is(x'-x)} \, a \, ds\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a}) \frac{dx'}{x'^2} da] \frac{dx}{x}$

$= \frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a \int_{x'=0}^\infty \delta(x'-x)\bar{\eta}'(\frac{x'}{a})\psi(x')\eta(\frac{x}{a}) \frac{dx'}{x'} da] \frac{dx}{x} +$

$\frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^2 \int_{x'=0}^\infty \delta(x'-x) \, \bar{\eta}(\frac{x'}{a})\psi'(x')\eta(\frac{x}{a}) \frac{dx'}{x'} da] \frac{dx}{x} -$

$\frac{2\pi i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^2 \int_{x'=0}^\infty \delta(x'-x) \, \bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a}) \frac{dx'}{x'^2} da] \frac{dx}{x}$

$= \frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} a \, \bar{\eta}'(\frac{x}{a})\eta(\frac{x}{a}) \, da]\psi(x) \frac{dx}{x^2} +$

$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} a^2 \, \bar{\eta}(\frac{x}{a})\eta(\frac{x}{a}) \, da]\psi'(x) \frac{dx}{x^2} -$

$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} a^2 \, \bar{\eta}(\frac{x}{a})\eta(\frac{x}{a}) \, da]\psi(x) \frac{dx}{x^3}$

$= \frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t) \frac{dt}{t^3}]\psi(x) \, dx +$

$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} |\eta(t)|^2 \frac{dt}{t^4}]x \, \psi'(x) \, dx -$

$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} |\eta(t)|^2 \frac{dt}{t^4}]\psi(x) \, dx$

$= \frac{3i}{2I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} \bar{\eta(t)}\eta(t) \frac{dt}{t^4}]\psi(x) \, dx +$

$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} |\eta(t)|^2 \frac{dt}{t^4}]x \, \psi'(x) \, dx -$

$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} |\eta(t)|^2 \frac{dt}{t^4}]\psi(x) \, dx$

$= \frac{i}{2I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi \int_{\mathbb{R}^{>0}} \bar{\eta(t)}\eta(t) \frac{dt}{t^4}]\psi(x) \, dx +$

$\frac{i}{I(\eta)} \int_0^\infty \bar{\phi}(x)2\pi[\int_{\mathbb{R}^{>0}} |\eta(t)|^2 \frac{dt}{t^4}]x \, \psi'(x) \, dx$

$= \frac{iI_2(\eta)}{2I(\eta)} \int_0^\infty \bar{\phi}(x)x \, \psi(x) \frac{dx}{x} \frac{iI_2(\eta)}{I(\eta)} \int_0^\infty \bar{\phi}(x)x^2 \, \psi'(x) \frac{dx}{x}$

$= \langle\phi|\widehat{{}_{\mathrm{r}}\phi_{ba}}\psi\rangle.$

Making the change of variables, $x/a = t, da = \frac{-xdt}{t^2}, \int_{\mathbb{R}^{>0}} \bar{\eta}'(t)\eta(t) \frac{dt}{t^3} = \frac{3}{2} \int_{\mathbb{R}^{>0}} |\eta(t)|^2 \frac{dt}{t^4}$ and $I_2(\eta) := 2\pi \int_{\mathbb{R}^{>0}} |\eta(t)|^2 \frac{dt}{t^4}$. Finally, we have $(\widehat{{}_{\mathrm{r}}\phi_{ba}}\psi)(x) = i[\frac{I_2(\eta}{(I(\eta)}] \; (\frac{x}{2}\psi(x) + x^2 \frac{d\psi(x)}{dx})$. □

Quantization of $\phi(\mathbf{b}, a) = \log a$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{{}_{\mathrm{r}}\phi_{\log a}}$ for the real valued function $\phi(\mathbf{b},a) = \log a$ by doing the following calculations:

$$\begin{aligned}
&\langle\phi|\mathcal{A}_{\phi\mathrm{r}}|\psi\rangle\\
&= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_{\phi\mathrm{r}}|\psi)(x)\tfrac{dx}{x}\\
&= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}|\ \psi\rangle\ \tfrac{d\mathbf{b}\ da}{a}]\ \tfrac{dx}{x}\\
&= \tfrac{1}{I(\eta)^{1/2}}\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)\langle\xi_{\mathbf{b},a}|\psi\rangle e^{\frac{-i\mathbf{b}x}{a}}\eta(\tfrac{x}{a})\tfrac{d\mathbf{b}\ da}{a^2}]\tfrac{dx}{x}\\
&= \tfrac{1}{I(\eta)}\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \log a\ \{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\tfrac{x'}{a})\psi(x')\tfrac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\tfrac{x}{a})\ \tfrac{d\mathbf{b}\ da}{a}]\ \tfrac{dx}{x}\\
&= \tfrac{1}{I(\eta)}\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \log a\ \int_{x'=0}^\infty\{\int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x'}-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}(\tfrac{x'}{a})\psi(x')\eta(\tfrac{x}{a})\ \tfrac{dx'}{x'}\ \tfrac{da}{a}]\ \tfrac{dx}{x}\\
&= \tfrac{2\pi}{I(\eta)}\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \log a\ \int_{x'=0}^\infty\{\tfrac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}(\tfrac{x'}{a})\psi(x')\eta(\tfrac{x}{a})\ \tfrac{dx'}{x'}\ \tfrac{da}{a}]\ \tfrac{dx}{x}\\
&= \tfrac{2\pi}{I(\eta)}\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \log a\ \int_{x'=0}^\infty \delta(x'-x)\bar{\eta}(\tfrac{x'}{a})\psi(x')\eta(\tfrac{x}{a})\ \tfrac{dx'}{x'}\ da]\ \tfrac{dx}{x}\\
&= \tfrac{2\pi}{I(\eta)}\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \log a\ \bar{\eta}(\tfrac{x}{a})\eta(\tfrac{x}{a})\ da]\psi(x)\ \tfrac{dx}{x^2}\\
&= \tfrac{2\pi}{I(\eta)}\int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} [\log x - \log t]\ \bar{\eta}(t)\eta(t)\ \tfrac{dt}{t^2}]\ \psi(x)\tfrac{dx}{x}\\
&= \tfrac{1}{I(\eta)}\int_0^\infty \bar{\phi}(x)\ \log x\ [2\pi\int_{\mathbb{R}^{>0}} |\eta(t)|^2\ \tfrac{dt}{t^2}]\ \psi(x)\tfrac{dx}{x} -\\
&\tfrac{1}{I(\eta)}\int_0^\infty \bar{\phi}(x)\ [2\pi\int_{\mathbb{R}^{>0}} \log t\ |\eta(t)|^2\ \tfrac{dt}{t^2}]\ \psi(x)\tfrac{dx}{x}\\
&= \tfrac{I(\eta)}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ \log x\ \psi(x)\tfrac{dx}{x} - \tfrac{I_{\log}(\eta)}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ \psi(x)\tfrac{dx}{x}\\
&= \int_0^\infty \bar{\phi}(x)\ \log x\ \psi(x)\tfrac{dx}{x} - \tfrac{I_{\log}(\eta)}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ \psi(x)\tfrac{dx}{x}\\
&= \langle\phi|\widehat{{}_{\mathrm{r}}\phi_{\log a}}\psi\rangle.
\end{aligned}$$

Making the change of variables, $\frac{x}{a} = t, \implies a = \frac{x}{t}$ and $da = \frac{-xdt}{t^2}$ and $I_{\log}(\eta) := 2\pi\int_{\mathbb{R}^{>0}} \log t\ |\eta(t)|^2\ \frac{dt}{t^2}$. Finally, we have $(\widehat{{}_{\mathrm{r}}\phi_{\log a}}\psi)(x) = \log(x)\psi(x) - [\frac{I_{\log}(\eta)}{I(\eta)}]\ \psi(x)$. □

Quantization of $\phi(\mathbf{b},a) = a^n$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{{}_{\mathrm{r}}\phi_{a^n}}$ for the real valued function $\phi(\mathbf{b},a) = a^n$ by doing the following calculations:

$$\begin{aligned}
&\langle\phi|\mathcal{A}_{\phi\mathrm{r}}|\psi\rangle\\
&= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_{\phi\mathrm{r}}|\psi)(x)\tfrac{dx}{x}\\
&= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}|\ \psi\rangle\ \tfrac{d\mathbf{b}\ da}{a}]\ \tfrac{dx}{x}
\end{aligned}$$

$= \frac{1}{I(\eta)^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)\langle\xi_{\mathbf{b},a}|\psi\rangle e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\frac{d\mathbf{b}\,da}{a^2}]\frac{dx}{x}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} a^n \,\{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\,da}{a}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^n\ \int_{x'=0}^\infty\{\int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x'}-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^n\ \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^n\ \int_{x'=0}^\infty \delta(x'-x)\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ da]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} a^n\ \bar{\eta}(\frac{x}{a})\psi(x)\eta(\frac{x}{a})\ da]\ \frac{dx}{x^2}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi\int_{\mathbb{R}^{>0}}(\frac{x}{t})^n\ \bar{\eta}(t)\eta(t)\ \ \frac{x}{t^2}\ dt]\ \psi(x)\frac{dx}{x^2}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)x^n\ [2\pi\int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\frac{dt}{t^{n+2}}]\psi(x)\frac{dx}{x}$
$= \frac{I_n(\eta}{I(\eta)}\ \int_0^\infty \bar{\phi}(x)\ x^n\ \psi(x)\frac{dx}{x}$
$= \langle\phi|\widehat{{}_{\mathrm{r}}\phi_{a^n}}\psi\rangle.$

Making the change of variables, $\frac{x}{a} = t, \implies a = \frac{x}{t}$ and $da = \frac{-xdt}{t^2}$ and $I_n(\eta) := \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t^{n+2}}$. Finally, we have $(\widehat{{}_{\mathrm{r}}\phi_{a^n}}\psi)(x) = [\frac{I_n(\eta)}{I(\eta)}]\ x^n\psi(x)$. □

Quantization of $\phi(\mathbf{b},a) = \frac{1}{a^n}$

Proof. We start out by taking two compactly supported functions ϕ, ψ and infinitely differentiable function η in the hilbert space $\mathcal{H} = L^2(\mathbb{R}^{>0}, dx/x)$. Then we have the operator $\widehat{{}_{\mathrm{r}}\phi_{\frac{1}{a^n}}}$ for the real valued function $\phi(\mathbf{b},a) = \frac{1}{a^n}$ by doing the following calculations:

$\langle\phi|\mathcal{A}_{\phi\mathrm{r}}|\psi\rangle$
$= \int_0^\infty \bar{\phi}(x)(\mathcal{A}_{\phi\mathrm{r}}|\psi)(x)\frac{dx}{x}$
$= \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)|\xi_{\mathbf{b},a}\rangle\langle\xi_{\mathbf{b},a}|\ \psi\rangle\ \frac{d\mathbf{b}\,da}{a}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)^{1/2}} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \phi(\mathbf{b},a)\langle\xi_{\mathbf{b},a}|\psi\rangle e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\frac{d\mathbf{b}\,da}{a^2}]\frac{dx}{x}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}\times\mathbb{R}^{>0}} \frac{1}{a^n}\ \{\int_{x'=0}^\infty e^{\frac{i\mathbf{b}x'}{a'}}\bar{\eta}(\frac{x'}{a})\psi(x')\frac{dx'}{x'}\}e^{\frac{-i\mathbf{b}x}{a}}\eta(\frac{x}{a})\ \frac{d\mathbf{b}\,da}{a}]\ \frac{dx}{x}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \frac{1}{a^n}\ \int_{x'=0}^\infty\{\int_{b=-\infty}^\infty e^{\frac{i\mathbf{b}(\mathbf{x'}-\mathbf{x})}{a}}\ d\mathbf{b}\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \frac{1}{a^n}\ \int_{x'=0}^\infty\{\frac{1}{2\pi}\ \int_{s=-\infty}^\infty e^{is(x'-x)}\ a\ ds\}\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ \frac{da}{a}]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \frac{1}{a^n}\ \int_{x'=0}^\infty \delta(x'-x)\bar{\eta}(\frac{x'}{a})\psi(x')\eta(\frac{x}{a})\ \frac{dx'}{x'}\ da]\ \frac{dx}{x}$
$= \frac{2\pi}{I(\eta)} \int_0^\infty \bar{\phi}(x)[\int_{\mathbb{R}^{>0}} \frac{1}{a^n}\ \bar{\eta}(\frac{x}{a})\psi(x)\eta(\frac{x}{a})\ da]\ \frac{dx}{x^2}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)[2\pi\int_{\mathbb{R}^{>0}}(\frac{t}{x})^n\ \bar{\eta}(t)\eta(t)\ \ \frac{x}{t^2}\ dt]\ \psi(x)\frac{dx}{x^2}$
$= \frac{1}{I(\eta)} \int_0^\infty \bar{\phi}(x)\frac{1}{x^n}\ [2\pi\int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\frac{dt}{t^{-n+2}}]\psi(x)\frac{dx}{x}$

$= \frac{I_{-n}(\eta}{I(\eta)} \int_0^\infty \bar{\phi}(x)\ \frac{1}{x^n}\ \psi(x)\frac{dx}{x}$
$= \langle\phi|_{\mathrm{r}}\widehat{\phi_{\frac{1}{a^n}}}\psi\rangle.$

Making the change of variables, $\frac{x}{a} = t,\ \implies\ a = \frac{x}{t}$ and $da = \frac{-xdt}{t^2}$ and $I_{-n}(\eta) := 2\pi \int_{\mathbb{R}^{>0}} \bar{\eta}(t)\eta(t)\ \frac{dt}{t^{-n+2}}$. Finally, we have $(_{\mathrm{r}}\widehat{\phi_{\frac{1}{a^n}}}\psi)(x) = [\frac{I_{-n}(\eta)}{I(\eta)}]\ \frac{1}{x^n}\psi(x).$ □

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