Mathematics for Engineers and Engineers' Mathematics

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A Thesis
in
the Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
For the degree of Master in the Teaching of Mathematics at
Concordia University
Montreal, Quebec, Canada

August 2015

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CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

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Entitled: Mathematics for Engineers and Engineers' Mathematics

and submitted in partial fulfillment of the requirements for the degree of

Master in the Teaching of Mathematics

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Date September 1, 2015

Abstract

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This thesis is composed of two parts. In the first part, the mathematics that engineering students and mathematics students are to be taught and expected to learn is identified by means of an analysis of the content of the courses each group of students has to take, and of the types of tasks each group is given in the final examinations of these courses. The aim is to determine if there are any significant differences between the education of the two groups.

In the second part, I demonstrate how professional engineers use mathematics to develop mathematical models that can be applied in solving tasks in their professional practice. Examples of mathematical models from the studies of statics, mechanics of materials, and structural analysis are presented, culminating in a discussion of the use of matrices in matrix structural analysis and the physical representation of eigenvectors and eigenvalues and what they mean to a structural engineer.

The comparison, analyses, and demonstrations are performed from an anthropological point of view using the Anthropological Theory of the Didactic (ATD). From this perspective it will be shown that the similarities between the mathematical praxeologies of engineers and mathematicians are limited principally to the tasks and techniques, while the differences are found in the level of the technology and theory.

Acknowledgements

I would not have been able to complete this thesis without the help and encouragement of many people.

First and foremost I must thank Dr. Anna Sierpinska for her vast knowledge, insightful remarks, guidance, and genuine interest in the life and career of a former professional civil engineer. I left the profession to pursue a degree in mathematics education in part because of my lifelong enjoyment of mathematics, but mainly because I had grown bored and unhappy with being an engineer. To have to revisit aspects of a career I had deliberately abandoned was challenging at times, but I enjoyed being able to look at it through the eyes of a mathematician. The idea for the topic of this thesis was entirely Anna's, and so I have her to thank for giving me that first push in the right direction, and for making sure I saw this project to its completion.

I owe thanks to former colleagues at the engineering firm CIMA+, most notably Peter Pietraszek and Marie-Claude Michaud, for providing with workplace documents for analysis demonstration purposes.

To all my Concordia classmates, thank you for lending me your ears, and for allowing me to vent some steam when the pressure built up. I am particularly thankful to Erin Murray, Steven Pileggi, and Dalia Challita. Thank you for helping me push through my writer's block, for your moral support and encouragement, and most importantly for your friendship.

Lastly, to my wife Stephanie, thank you for staying by my side through all of this. Your love means more to me than I could ever hope to show you. But I'll sure as hell try.

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1 INTRODUCTION

The purpose of this thesis is to compare the mathematics education of engineers with that of mathematicians, and to show how professional engineers use mathematics in practice. More specifically, in the first part of the thesis, the mathematics that engineering students and mathematics students are to be taught and expected to learn is identified by means of an analysis of the content of the courses each group of students has to take, and of the types of tasks each group is given in the final examinations of these courses. The aim is to determine if there are any significant differences between the education of the two groups. The second part of the thesis is a demonstration of how engineers use mathematics to develop mathematical models that can be applied in solving tasks in their professional practice, and how this application of mathematics differs from the interests of mathematicians. The comparison, analyses and demonstrations are performed from an anthropological point of view using the Anthropological Theory of the Didactic (ATD).

Before proceeding, a brief word must be said about the term “engineer.” The engineering profession has a lengthy and rich history, and as a result the term “engineer” is very broad. The profession comprises many disciplines whose foundational principles vary. Engineering encompasses the fields of civil engineering, mechanical engineering, chemical engineering, electrical engineering, and materials engineering. Within the past forty or so years there has been the emergence of relatively new fields such as computer engineering and software engineering. Within each of these fields we can find multiple sub-disciplines. A civil engineer, for example, may work as a structural engineer, municipal engineer, hydraulic and hydrological engineer, environmental engineer, or transportation engineer. For reasons of brevity, this thesis will restrict the meaning of the term engineer to civil engineers, and, in particular cases, only structural engineers and concepts of structural engineering may be referenced.

This topic is of particular interest to me since I earned my bachelor's degree in civil engineering, with a focus on structural engineering, from McGill University in 2003. After working as a project engineer for a private consulting firm, including three years as a professionally licensed engineer, I left the profession and returned to school in order to earn a graduate degree in mathematics education. I always had a particular fondness for mathematics, and enjoyed studying civil engineering due to its nature as an applied science involving plenty of mathematical formulae.

Changing direction of study and profession required taking mathematics courses for "generalist" mathematicians (undergraduate and graduate mathematics students). As a student in these courses I had the feeling of entering a very different culture from the one I had experienced as an engineering student and as a practicing engineer.

For example, engineers in practice will approximate the values of irrational numbers such as $\sqrt{2}$ and π as 1.414 and 3.14. Fewer or more decimal places may be used in the approximation depending on the context of the calculation they are being used for. It is also an accepted scientific practice to use numbers with an appropriate number of significant figures. Many of the numbers used in engineering calculations represent measurable quantities: in this case, numbers are accompanied by units. If a length is measured in millimeters with a ruler which has no finer unit than a millimetre, it would not make sense to say that the length is, say, $9\sqrt{2}\ mm$. For a mathematician, however, the number $9\sqrt{2}$ represents an abstract number, for example, a root of the equation $x^2 - 164 = 0$ and saying that the roots of this equation are 12.73 and -12.73 would be considered a serious inaccuracy. Units and their conversions are not usually the mathematician's concern.

I also noticed differences in the things that pique a mathematician's interest. While taking a course that involved second order logic, I noticed that mathematicians are mainly motivated by finding generalizations of mathematical properties and proving theorems. In a linear algebra course I was impressed by the level of importance placed on proving theorems, not just as a detail in the course lectures, but as tasks given on assignments and exams as well. Revisiting such tasks that I had not encountered since my days as a CEGEP student gave me a new appreciation for them. In being able to prove a theorem one can better understand its meaning and its purpose, which will lead to correctly applying the theorem as well.

I wished to see if more of these differences existed, and to understand their nature. This led me to identifying and collecting data about different institutions, namely the institutions of academic mathematics, mathematics courses in university programs for engineering and mathematics students, and the engineering profession itself. I then sought a theoretical framework that would help me to analyze and structure my observations of the data. The theoretical framework of the ATD models the mathematical knowledge of an institution in terms of units called praxeologies. Within an institution's mathematical praxeology there is a block of practical knowledge that contains a task to accomplish and a technique to accomplish it. Every praxeology also has a theoretical block that contains two levels of discourse. The praxeology's technology classifies the tasks into different types, describes the techniques for solving the types of tasks in more general terms, and justifies their use in performing the tasks, and there is also the theory which is a system of all formal arguments that justify the technology. This model, along the concept of institution, provides an appropriate framework for comparing the mathematical knowledge that engineering and mathematics students are expected to learn, and for demonstrating how mathematics is used in practice by professional engineers.

The structure of the thesis is as follows. In chapter 2 I will discuss the theoretical framework of the ATD in more detail, defining the concepts of praxeology, institution, and didactic and institutional transposition of mathematical knowledge. The chapter concludes with a justification of the chosen framework.

In the literature review in chapter 3, I will present the findings of several research papers on the mathematics education and the use of the mathematics in the workplace of various vocations and trades, including engineering. In this chapter an important distinction will be made between mathematical applications and the process of mathematical modelling, and the role that each plays in an engineer's education and practice. The concept of mathematics as a service subject is also addressed. Service mathematics courses teach mathematics to students whose main area of study is not academic mathematics itself. Engineering students fit this description.

Chapter 4 is titled "Mathematics for engineers", and it comprises an analysis of the mathematical tasks given to engineering students on the final exams of their required mathematics courses. These tasks are compared and contrasted with those given to mathematics students on final exams from comparable courses. Also included in the chapter is an institutional perspective on why engineers are required to learn the mathematics that they do.

In chapter 5, "Engineers' mathematics", particular attention is paid to how engineers use mathematics to develop mathematical models. Examples of mathematical models from the fields of statics, mechanics of materials, and structural analysis are presented, and in the discussion of each I also mention aspects that would be of interest to a mathematician. A specific model called the direct stiffness matrix, and the physical representation of eigenvalues and eigenvectors of a matrix, is discussed at length. In this chapter I also present and analyze workplace documents prepared by a professional engineer in the design of a structure.

At the outset of this project I had hoped to address a common concern among students in mathematics courses: "Why do we have to learn this?" The second part of this thesis is an attempt to answer this question by demonstrating how engineers use mathematics in models at the heart of their field.

Engineering students have to learn about eigenvalues because they can be used to solve important tasks in structural engineering.

On a larger scale, mathematics education researchers are interested in how various tradespeople use mathematics because it helps to broaden our view of what mathematics is, and what it means to have mathematical knowledge. ATD is useful in understanding how mathematics is used in practice and structuring our descriptions of this practice. This thesis is my contribution to this understanding.

2 THEORETICAL FRAMEWORK

The present chapter discusses the theoretical framework of this thesis, the Anthropological Theory of the Didactic (ATD) developed by Chevallard (1999). My principle sources of inspiration for choosing this framework are the works of Sierpinska et al. on sources of students' frustration in pre-university level pre-requisite mathematics courses (Sierpinska, Bobos, & Knipping, 2008), as well as Hardy's study on college students' perceptions of institutional practices regarding limits of rational functions (Hardy, 2009). In the following sections, I will discuss the basic concepts of the ATD – praxeology, institution, and didactic transposition – and then argue why this framework is suitable for the purposes of this study.

2.1 PRAXEOLOGY

The ATD is an epistemological framework in which the principal objects of study are institutionalized practices, and was developed as a means of describing the practice of teaching and learning mathematics (Chevallard, 1999). In this framework, the learning of mathematics is not something that occurs at an individual level, as in theories of cognition, but rather as a collective activity by the members of an institution. Furthermore, the mathematical knowledge itself is developed by institutions, not individuals. The concept of an institution is discussed is section 2.2.

The cornerstone of the ATD is the notion that institutional knowledge can be organized into units called *praxeologies*. Each praxeology is composed of two blocks: a practical block, called the *praxis*, and a theoretical block, called the *logos*.

In each praxeology, the practical block consists of a collection of tasks to be accomplished, and the techniques that are used to accomplish them. The praxis can be thought of as the "know how" portion of a praxeology; when given a mathematical task, knowing how to complete it is indicative of practical knowledge.

The theoretical block of the praxeology contains two levels of discourse: the technology and the theory. Technology classifies the tasks into different types, describes the techniques for solving the types of tasks in more general terms, and justifies their use in performing the tasks. Theory is a system of all formal arguments that justify the technology. Theory, in particular, provides the rationale underlying the classification of the tasks and techniques into particular types and makes explicit the assumptions and theoretical arguments that allow us to claim that the techniques "work", going beyond the experience of seeing them work in particular cases. Thus, the logos can be thought of as the "knowledge" block of the praxeology; having completed a mathematical task with a technique, the theoretical block provides the justification for the use of that technique.

In summary, the two blocks of a praxeology – [tasks & techniques] - [technology & theory] – suggest that the practices of mathematicians and engineers may differ not only in the types of tasks they perform, but in the nature of the theories they call upon to justify the techniques they use to accomplish those tasks.

2.2 Institutional perspective

While knowledge can be organized into praxeologies, the knowledge itself is created by human activity within institutions. This raises an important question: what is an institution? While the term is not explicitly defined in the ATD, the institutional perspective used by Sierpinska et al. (2008) remedies this by adopting a definition based on the work of Peters (1999) in the domain of Institutional Theory. According to Peters, there are four features that define an institution. As summarized in Sierpinska (2008) they are:

1. An institution is a structural feature of a society. The structure may be formal, requiring a legal framework, or an informal network of organizations.
2. An institution has some stability over time.

3. An institution constrains its participants through rules and norms.
4. Members of an institution share certain values and goals, and give common meaning to the basic actions of the institution.

The works of Hardy (2009), and, more recently, Castela (2015) further encapsulate these features by defining an institution as a stable social organization that offers a framework which allows repetitive interactions between individuals whose aim is to fulfill certain tasks. In the course of fulfilling its tasks, an institution takes purposeful collective action, subjecting its members to its expectations and regulating the members' actions through the use of rules, norms, and strategies (which Castela calls "rituals").

Rules are understood as explicitly stated regulations that must be followed, as breaking them will invoke sanctions against a member of the institution. In a university mathematics class, or the research mathematicians' community, a rule to be followed is that one must obey the axioms and theorems of mathematics. Not doing so results in mathematical contradiction and unfeasible results, or consequences such as the mathematics student failing an exam, or the research mathematician having his or her paper rejected in the review process. In engineering professional practice, an example of a rule that is enforced is the use of legally mandated design codes. The sanctions against professional engineers have a legal weight that those for mathematicians don't: criminal charges may be laid against the engineer who doesn't follow the appropriate design code. This is understandable, since not following the design code could result in the loss of lives.

Norms, on the other hand, are accepted customs that don't need to be explicitly stated, and not following them will not lead to sanctions. An example given in Hardy (2009) is a precept to use a certain technique to solve a type of limit task, common in college level calculus course: evaluating the limit of a function whose expression contains a radical in the denominator. The norm for solving such problems is to multiply the numerator and the denominator of the rational function by the conjugate of the expression that contains the radical. An example of a norm from engineering is drawn from the field of statics, the mechanics of rigid bodies, which engineering students study in their first year of university, and whose principles are used extensively in practice. When a known force, P, is applied to a horizontal beam, as in Figure 1, the forces are created in the supports at either end of the beam. The magnitude of those reaction forces, R_A and R_B, can be found using the fundamentals of statics: the sum of all forces must be 0, and the sum of the moments[1] of all forces about any point must also be 0.

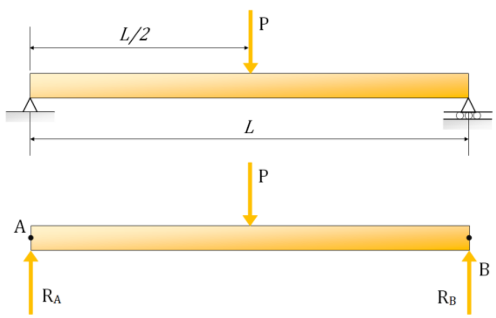


Figure 1 - Applied force and reaction forces on a beam (drawing is my own)

While the sum of the moments could be evaluated at any arbitrary point along the length of the beam, or even at any arbitrary point in space, the accepted norm is to evaluate the sum of the moments around one of the beam's endpoints. Evaluating the sum of the moments around point A, the moment of reaction force R_A is 0 since its line of action passes through the point. Thus, in the equation for the sum of the moments, there will only be one unknown, the force R_B.

In the research of Sierpinska et al. and Hardy, the institutions of study were college level mathematics courses being offered as pre-requisites. In this thesis the institutions considered are those of academic mathematics, mathematics courses in university programs for engineering and mathematics students, and the engineering profession.

[1] The moment of a force is a measure of its ability to generate a rotation about a point. The magnitude of a moment is the product of the magnitude of the force and the distance from the force's line of action to the point. A detailed explanation is offered in section 5.2.2.

2.3 Didactic and institutional transposition

The theoretical block of an institution's praxeology preserves the institution's activity as a practice and communicates it to others, so that they, too, can participate in it (Hardy, 2009, p. 344). In other words, the technology and theory not only serve to justify an institution's tasks and techniques, but also makes them teachable and learnable to others either within the same institution or within another. When one institution imports the praxeology of another with didactic intentions, then the knowledge of the praxeology undergoes didactic transposition (Chevallard, 1985; Castela & Romo Vasquez, 2011).

Consider the teaching of mathematics to engineering students. The mathematics that is taught originates in the institution of academic mathematics which, according to Castela (2015) has the status of the "reference point" for mathematical knowledge. However, the engineering students are not members of the institution of academic mathematics, but of the institution of mathematics courses in an engineering program. They are taught a certain didactic transposition of the academic mathematics.

But engineers also use mathematics in the workplace. When using mathematics in their professional practice, engineers adapt ("transpose") the praxeology of academic mathematics in order to use techniques and their associated technologies and theories in order to accomplish engineering tasks. But since the purpose isn't didactic, the mathematical knowledge is said to undergo institutional transposition (Castela & Romo Vasquez, 2011).

Referring again to Hardy (2009, p. 343), mathematical knowledge in an educational institution can take on different forms:

1. Scholarly knowledge, which is the wealth of knowledge that is produced by the professionals of an institution. An example of mathematical scholarly knowledge is, for example, the theory of vector spaces over an arbitrary field.

2. Knowledge to be taught, which is found in the curriculum documents of a course that seeks to impart some of the scholarly knowledge. The syllabus of an undergraduate Linear Algebra course may contain only a selection of results of the theory of finite dimensional real vector spaces.
3. Knowledge actually taught which can be found in a teacher's lecture notes and the tasks that are prepared for the students. In the Linear Algebra course from the previous example, the teacher may choose to refer students to the textbook for the proofs of theorems and only illustrate the theorems on examples in class.
4. Knowledge to be learned, which is interpreted by the students as the minimum amount of knowledge needed to complete the tasks. This knowledge can be deduced from assessment instruments such as assignments and exams. In the Linear Algebra course, it may sometimes be enough to know how to solve typical computational exercises to pass the course; reasoning based on the theorems introduced in the course to solve simple conceptual problems (e.g., of the "Show that…" type) is usually required to obtain a high grade.
5. Knowledge actually learned which is reflected in the students' responses to the assessments that they've been given.

The first part of this thesis focuses on the mathematical knowledge that is to be taught and to be learned by both engineering and mathematics students. Course descriptions and syllabi are used to determine the knowledge to be taught, and final exams are used to determine the knowledge to be learned.

2.4 Validation of the chosen framework

The chosen framework encapsulates all of the elements that are necessary for describing and analyzing the mathematics that engineers are expected to learn in their education – what I call mathematics for engineers (chapter 4) – and the mathematics used by engineers in their practice – referred to as engineer's mathematics (chapter 5). The knowledge in the institutions of interest can be modelled by praxeologies; the mathematics courses in engineering programs and the engineering profession itself both meet the requirements of being classified as institutions, and the knowledge imparted in both of these institutions is a result of the transposition, both didactic and institutional, of knowledge produced in the institution of mathematics.

The successful completion of a mathematics course in an engineering program requires that the students perform a number of mathematical tasks on assignments, tests, and exams. Each of these tasks can be accomplished using techniques that they are expected to learn. The technology and theories that justify the use of those techniques are found in the students' textbooks and in the discourse of the lectures they attend. When it comes to the profession of engineering, even large projects such as the design of a multi-storey building can be broken down into a series of smaller design tasks (design of the foundations, design of the structural framework made up of beams and columns and their connections, design of the concrete slabs for the floors, etc.), each with their own techniques. In this case the technology and theories that describe and justify the techniques are not only their mathematical soundness, but take the form of legally mandated design codes and manuals that an engineer must use to perform these design tasks (e.g., the National Building Code of Canada, the Handbook of Steel Construction, the Concrete Design Handbook, etc., are documents that have all been certified by the Canadian Standards Association, CSA). The contents of these documents, which include the appropriate formulae for design and analysis, have been developed and refined through decades of engineering science research.

Besides having knowledge that can be modelled by praxeologies, mathematics courses in engineering programs and the engineering profession itself both fit Peters' (1999) description of an institution. Engineering mathematics courses are a structural feature of accredited engineering programs which are entrenched in the higher education institution that they find themselves in. Since 1965 the accreditation of engineering programs has been overseen by the Canadian Engineering Accreditation Board (CEAB), though some programs pre-date its existence. For example, the Department of Civil Engineering and Applied Mechanics at McGill University was established in 1871, and the École Polytechnique, affiliated with Université de Montréal, opened its doors in 1873. The actions and behaviours of students in these programs are constrained by the rules, norms, and strategies put into place by the universities and the individual mathematics courses in the programs. Courses that teach pure mathematics must be taken as pre-requisites for several of the core engineering courses. Failure to pass a mathematics course can lead to sanctions that include academic probation or possibly expulsion from the program. All of the students in the engineering programs share a common goal and graduating and beginning their careers as engineers. But in order to undertake a career in that profession a minimum level of mathematical competence is required, not only at the behest of the university, but by the members of the profession as well. Thus, the mathematics courses are a welcome means to a desired end.

For its part, the engineering profession also fits Peters' (1999) description. The professional practice of all engineers is a formal structure of Canadian society. The practice is overseen by the constituent associations of Engineers Canada. The associations are the provincial and territorial engineering regulatory bodies that license professional engineers in their jurisdiction. In Quebec, the association is the *Ordre des ingénieurs du Québec* (OIQ), a professional order that was established by the Engineers Act of Quebec's National Assembly in 1974. Only members of the OIQ are legally allowed to practice engineering and refer to themselves by the exclusive title of Engineer (Eng), or *Ingénieur* (ing.). In other provinces, the term Professional Engineer, abbreviated P.Eng., is used. This gives engineers the same status of a profession as doctors and lawyers. Furthermore, to ensure that its members respect the shared values of the institution, professional orders such as the OIQ have instituted rules that require their members to take part in a minimum number of hours of professional and educational development activities each year, beyond the scope of their regular professional duties. Violating this rule leads to sanctions that include revocation of one's professional license. The shared values in question are found in the sense of ethics that engineers uphold. According to the OIQ's Code of Ethics, an engineer is required to respect, first and foremost, their obligations towards the safety of the general public. This primary value is also the result of engineers recognizing the effects of the profession's past failures.

Lastly, the mathematical knowledge that engineering students are expected to learn in their mathematics courses, and the knowledge that engineers use in their professional practice, are the result of transpositions, both didactic and institutional, as was discussed earlier. One question that this thesis will investigate is whether the mathematics that engineering students are expected to learn has been transposed in such a way that it is different from that which mathematics students are expected to learn. The key to answering this question may lie in the notion that engineering mathematics is applied mathematics (Blum & Niss, 1991), and relies heavily on the development and use of mathematical models. These concepts will be further explained in section 3.3.

3 LITERATURE REVIEW

This chapter presents and discusses the findings of several research studies in mathematics education on the nature of the mathematical content in courses offered to students in different educational settings, including mathematics taught to engineering students, mathematics taught in vocational education, the teaching of mathematical applications and modelling, mathematics as a service subject, and mathematics as it is used in the workplace. The final section illustrates an example of the teaching and use of geometry, measurement, and error in measurement at vocational schools.

The discussions in the sections that follow will be interspersed with comments that compare and contrast the findings in the literature with the educational and workplace mathematics of engineers based mainly on my experience. More detailed analysis of mathematics for engineering students and professional engineers will appear in chapters 4 and 5.

3.1 MATHEMATICS EDUCATION FOR ENGINEERS

The following section summarises the recent works of Castela and Romo-Vazquez and their studies in the mathematics education of engineering students at the Vocational Institute at the University of Evry in France (Castela, 2015; Castela & Romo Vasquez, 2011; Romo Vasquez & Castela, 2010). Using the framework of the ATD, they studied how mathematics is used by students in an engineering project design course, focusing on the use of Laplace transforms as a technique for completing the task of solving linear differential equations, and comparing how the technique is taught in mathematics textbooks with how it is taught in engineering textbooks.

This study showed that engineering students select the techniques they need for their tasks based on practicality. The chosen techniques were never justified by a technology and theory that relied solely on mathematical concepts and definitions, but on whether the techniques allowed them to quickly obtain the results of their calculations and whether or not the results made physical sense in the context of their design tasks. The students were allowed to use computer software for more difficult tasks, and while this allowed the students to explore some of the parameters of their designs, it had the effect of "black-boxing" the mathematics, making it less visible. But this did not seem important to the students. How the mathematics was validated was not as relevant as choosing the correct mathematical model and the values to input into that model. Ultimately the research found that the students' work itself could not be reduced to mathematics alone, since their tasks also required knowledge of physical and engineering sciences.

In the comparison of how Laplace transforms are taught in engineering and mathematics textbooks, it was noted that the technique taught in the engineering textbook was altered slightly from the traditional mathematical technique. The motivation behind the altered technique is that it results in functions whose arguments are formatted so that there is added value for the engineer. From the results of the transform, the engineer can determine properties of the function directly from its expression without having to do any algebraic manipulations or simplifications. Furthermore, while the mathematics textbooks focused on the comprehensive and accurate presentation of the technique using theorems and proofs, the engineering textbook gave a lower priority to proofs and instead correlated the technique with its use in a vocational context (Castela, 2015).

In terms of the ATD framework, the engineering textbook appropriated the praxeology from academic mathematics and augmented the theoretical block with a new description of the Laplace transform technique, and new justifications that are based on the practicality of the technique's application. If engineers' mathematics is different from that of mathematicians, the difference is most likely found in the theoretical block of the praxeology, as engineers complement mathematical praxeology with elements of engineering knowledge.

3.2 Mathematics in vocational education

A series of papers in *Educational Studies in Mathematics* (2014) explored the mathematics of specialized vocational disciplines, with the goal of characterizing and developing the vocational mathematical knowledge. The study examined the education of students who were learning to be electricians, practitioners in various pipe trades (welders & plumbers), laboratory technicians, and business school graduate students.

The studies involving the electrician and pipe trade apprentices were done at vocational schools in western Canada, while the others took place at schools in various European countries. This would be the equivalent to technical programs at a Quebec CEGEP for skilled trade workers, a construction certification from the *Régie de bâtiment du Québec* (RBQ), or other vocational and/or continuing education institutions. It should be mentioned, however, that engineering education is not vocational. In order to obtain a degree in engineering, a student from Quebec is required to do a two-year pre-university science program at CEGEP, followed by a three-year university education. As will be shown in chapter 4, mathematics courses are required in an engineer's education. So while the studies presently discussed are not representative of an engineer's situation, they can still offer insight into the transposition of academic mathematics for didactic purposes.

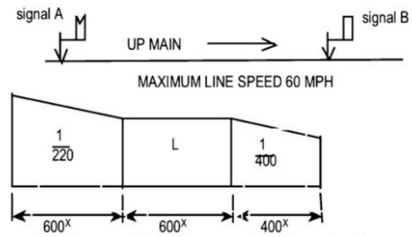
The general findings in the papers of LaCroix (2014) and Roth (2014) are that the mathematics taught to future practitioners of various trades is directly shaped by the immediate and practical requirements of the workplace, where it's more important to get the job done efficiently than to worry about theoretical rigour precision. The generalizations, formalities, and internal consistencies of academic mathematics are absent from the workplace. In the classroom, the instructors, who are themselves qualified and experienced tradespeople, are more concerned that their students get answers to problems that are “close enough”, than they are with the methods that are used to find the answers. Essentially, the old adage of “time is money”, prevalent in the workplace, is applied to their school work as well.

Electrician apprentices do learn the basic trigonometric functions (sine, cosine, tangent, and cotangent), in an effort to explain the demands of the electrical code, a legal document that is to be followed in doing electrical work. The code itself contains no mathematics, just general rules for the installation of electrical conduits. But, while the trigonometric functions justify the work methods that ensure the code is followed, on the job, trigonometry is replaced by “rules of thumb” that allow electrical conduits to be installed efficiently, cables passed through the conduits easily, and the tools and equipment of their trade to be used safely. Similarly, pipe trade apprentices who need to join two pipes at a specified angle eventually learn how to eyeball a fit that is “more or less” correct, without the rigorous use of trigonometry. This is in stark contrast to engineering design codes (for example: CSA-S6 – Canadian Highway Bridge Design Code; CSA-S16.1 – Limit States Design of Steel Structures; CSA-A23.3 – Design of Concrete Structures), which contain a multitude of mathematical formulae to be used in the design and analysis of steel and concrete structures. The mathematics found in these codes was developed and refined through extensive engineering research and testing of mathematical models.

Further evidence that the “why” of mathematics is less relevant than the “how” of the workplace can be found in the interactions between the students and their instructors. In the pipe trade training course, the instructor announced to his students that they would not use all of the mathematics that they were taught. The electrician apprentices for their part claimed that they did not see or use any trigonometry during the job site training portion of their course, and questioned the pertinence of learning it, even though it is a requirement for earning their certification. It is mentioned that the instructors “colluded” with their students to learn the mathematics simply for the sake of obtaining certification (Roth, 2014).

Coben and Weeks (2014) argue that mathematics education can be made more authentic and more meaningful for all vocational trades by exposing the students to more appropriate and authentic tasks and problems in which the mathematics is present, but no more explicitly than they are in the workplace. In other words, the problems given to students in the electrical and pipe trades should be designed to show exactly how trigonometry is helpful in accomplishing a task such as offsetting an electrical conduit, and how the rules of thumb they will eventually use can be derived from these same mathematical functions.

However, doing so doesn't guarantee an improved mathematical understanding. Wake (2014) presents vignettes of research studies in which students were brought to various workplaces and exposed to practical uses of the mathematics they were learning. In one study, college students following a pre-vocational engineering course visited the workplace of a practicing railway engineer. The railway engineer showed the students how to calculate the average downhill gradient (slope) of a railway track composed of three sections of track each with a different gradient. In discussions following the visit, it was revealed that the students believed that they could simply average the three values of slope, in the same way they had learned to find the average of a set of integers. The proper technique requires using the individual gradients to calculate the total change in elevation (rise or fall) of each of the three sections. The total change in elevation, Δy, is then divided by the total distance, Δx, along all three sections, and thus the average slope is found to be $S = \frac{\Delta y}{\Delta x}$. The railway engineer who explained the technique to the students used the chart in Figure 2 to perform his calculations in his actual practice.



SECTION	DISTANCE IN YARDS	GRADIENT	RISE (YDS) (DIST * GRAD)	FALL (YDS) (DIST * GRAD)
X - A	600	$\frac{1}{220}$	-	2.73
A-B	600	L	-	-
B-Y	400	$\frac{1}{400}$	-	1.00
TOTALS	1600			3.73

$\frac{1600}{3.73}$ = 428.95 THERFORE AVERAGE GRADIENT= $\frac{1}{429}$ FALLING

Figure 2 - Calculating the average gradient (slope) of a train track. (Wake, 2014)

For each section of track, labelled in column 1 as sections X-A, A-B, and B-Y, the distance in yards (column 2) and the gradient (column 3) are used to calculate the rise (column 4) or fall (column 5), which is the change in elevation. The engineer then calculates the ratio of the net rise and fall, 3.73 yards, to the total distance, 1600 yards, to calculate the average gradient of $\frac{1}{429}$. For the engineer who showed the students this procedure, the technique had become automatic and could be performed without any explanation. Because of this the students were unable to understand how or why the technique worked. Wake suggests that the workplace mathematics was too "black-boxed", made invisible by other aspects and practices of the workplace, and thus more difficult for the students to access. This confirms the findings of Castela and Romo Vazquez discussed in section 3.1.

Some of these findings are contrary to my experiences as a civil engineer. In practice, trigonometry is used frequently in a number of contexts: calculating loads on a structure, evaluating the safety of a roadway's curvature, determining the strength of the soil that a foundation will be built upon. Other mathematical tasks, such as solving differential equations, while used more infrequently in an office environment, are certainly necessary for deriving the formulas and creating the mathematical models that are used in practice. For example, the different formulae for calculating the deformation of a beam under a given load can be obtained by solving the differential equation:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

The details of this equation are presented in section 5.2.4. For a beam with the applied load shown in Figure 3, solving the differential equation results in the following formula for maximum deformation, Δy_{max}, the amount that the centre of the beam will sag due to the effect of the applied load, P:

$$\Delta y_{max} = -\frac{PL^3}{48EI}$$

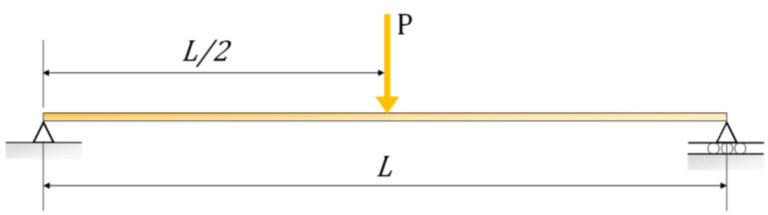


Figure 3 - Simply supported beam with concentrated load at centre (drawing is my own)

Furthermore, regarding the collusion between teachers and students in an effort to successfully pass the required courses (Roth, 2014), it has been my experience that the instructors in engineering education ensure that the required mathematics is well understood, and not simply glossed over for the sake of earning course credits. Perhaps this may highlight a difference between the institutions of vocational education and engineering education.

Noss (2001) quotes practicing engineers who say that they use at most 2% of the mathematics that they learned in school. But among the colleagues of these engineers are those who work as engineering "specialists", who use mathematics more frequently in their tasks. In my experience, engineers who claim to "use less" mathematics tend to be those who are responsible for project management as opposed to design and analysis, and spend less time working with mathematical models, and more time managing budgets and schedules. And while their mathematical tasks may be less advanced than those of the specialists, a project manager must be able to communicate with and understand what their specialists tell them. Thus, the mathematics in an engineer's education is not to be taken as lightly as it appears to be in some vocations.

3.3 Mathematical applications and mathematical modelling

Several papers in *Applications and modelling in learning and teaching mathematics* (Blum, Berry, Biehler, Huntley, Kaiser-Messmer, & Profke, 1989) discuss courses dedicated to the teaching of mathematical modelling in various countries including the UK, the United States, and Canada. These courses are offered to students of different disciplines and vocations including engineers, psychologists, mathematics teachers, and geologists.

It is important to understand the difference between mathematical applications and mathematical models. A mathematical application is the use of mathematics, its concepts, its objects, and its rules, to solve a problem in which some aspect is based in the real world (Blum & Niss, 1991; Galbraith, Henn, & Niss, 2007). Applications are useful in order to show students how mathematics can be used to solve every-day or otherwise relevant tasks. A mathematical model, on the other hand, is a mapping that transfers the objects that we wish to study from the real world into objects in a mathematical domain. Mathematical models are idealized representations of the real world. The task of modelling involves creating or designing an appropriate mathematical description of an extra-mathematical situation or phenomenon. However, applications and models are not completely distinct; in every application of mathematics there is an underlying mathematical model with built-in assumptions that are either explicit or implicit.

The following problem is an example of a mathematical application:

A rock is thrown from the edge of a cliff into the air. Its trajectory can be expressed by the function $H(t)$*, where* t *is the amount of time elapsed since the rock has been tossed, and* $H(t)$ *the distance, in feet, of the rock from the ground at time* t*:*

$$H(t) = -16t^2 - 32t + 100$$

For what value of t *is the height equal to 50 feet?*

This problem, which one could easily find in a pre-university level algebra course, is used to demonstrate an understanding of function notation and knowledge of techniques for solving quadratic equations (e.g., completing the square or using the quadratic formula). The function given in the problem is an example of a mathematical model. The quadratic function $H(t)$ is a mathematical object that represents the trajectory of real world object, a rock being thrown from a cliff. A modelling problem using the same real world situation would involve constructing the function $H(t)$ perhaps from some given data about the rock's trajectory. The result of the modelling problem would be the function itself, as opposed to a numerical value resulting from solving an equation.

For an example from civil engineering, consider the problem shown in Figure 4, which represents the real world situation of a crane lifting a mass of 2400 kg.

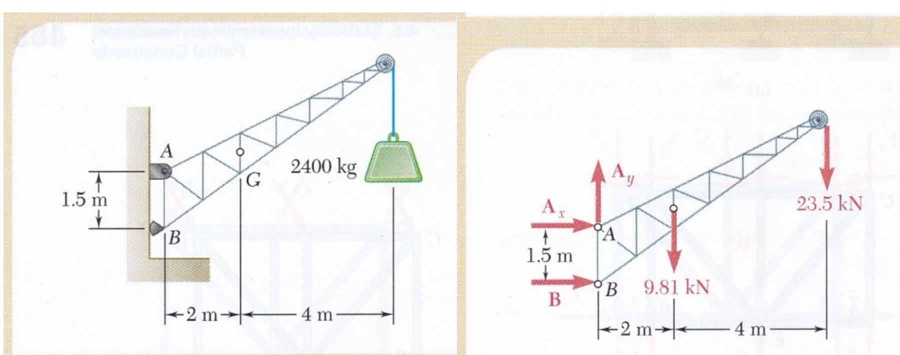


Figure 4 - Statics problem: a crane lifting a mass (Beer & Johnston, 2007, p. 166)

The crane is connected to a vertical wall by supports located at points A and B. The engineer's task consists of finding the magnitude and direction of the forces in two supports that keep the crane in place. The force at support B can be found by applying mathematics and using the following equation:

$$B(1.5\ m) - 9.81\ kN(2\ m) - 23.5\ kN(6\ m) = 0 \rightarrow B = 107.1\ kN$$

With this equation solved, the horizontal component of the force in support A can be found by solving the equation:

$$A_x + B = 0 \rightarrow A_x = -107.1\ kN$$

Lastly, the vertical component of the force in support A is determined by solving the equation:

$$A_y - 9.81\ kN - 23.5\ kN = 0 \rightarrow A_y = 33.3\ kN$$

Each of these equations is an example of an application of mathematics, since the numerical values and the symbols for unknown values all represent real world quantities. The underlying mathematical model that leads to these equations is that of a fundamental physical principle called static equilibrium which states that the sum of all forces and moments acting on a body must be zero. This model is expressed symbolically as:

$$\sum F = 0; \sum M = 0$$

This model was developed through a reformulation of Newton's first law of motion. The concepts of force, moments, and static equilibrium are discussed in further detail in chapter 5, but are included here solely to highlight the difference between a mathematical application and the underlying mathematical model.

Every mathematical model, including those used in engineering, is constructed with certain implicit or explicit assumptions about the physical objects it represents and the mathematical objects it uses to represent them. In the example of the rock being thrown from the cliff, the model (the function $H(t)$) most likely assumes that the force of gravity acting on the rock is constant, and that the rock encounters no resistance from the wind. These assumptions aren't explicitly stated in the application problem, but if one were tasked with creating the model, those assumptions would need to be made explicit. Another implicit assumption is the domain of the function $H(t)$. Since the independent variable is the time elapsed since a certain moment, the values of t are restricted to non-negative real numbers. Similarly, the mathematical model for the principle of static equilibrium assumes that the objects upon which the forces are acting are rigid, i.e., they do not bend or twist. When an engineer uses the principles of statics to solve such problems, this assumption is entirely implicit, even though it is not a true representation of reality.

In my experience as an engineering student, some early courses involved a combination of both application and modelling problems, particularly in courses that included laboratory sessions. In the laboratory, experiments were performed and data was analyzed in order to confirm the validity of established models. In more advanced courses, the textbooks and professors presented how certain models came to be developed; it was not up to the students to create the models as they were too complex. Once advanced models were learned and understood, application problems became more prominent, but the student had to determine which model was needed to solve the application problem.

An important question asked by the authors of selected readings in Blum (1989) is whether the mathematics that is used in creating mathematical models should be taught before or after students learn to model mathematically. According to Bkouche (1989), one can learn how to model, for example, with vectors before being taught vector theory. Consider students learning how to represent physical forces being applied to objects and the resulting accelerations (based on Newton's laws) with vector arrows, prior to learning the underlying mathematics of generalized vector spaces. The same could be imagined for learning and using rates of change of functions before learning the formal definition of the derivative. The mathematical theory can be taught after the students have learned to build models, as a means of unifying various representations, and reinforcing their models.

Alternatively to Bkouche's (1989) perspective, a polytechnic in the UK offers a "first course in mathematical modelling", which ensures that all of the necessary formal mathematics is taught to the students before they need to use it to create a mathematical model (Edwards & Hamson, 1989). Their philosophy is to learn the math before learning the model. The reasons for this method are that most situations that are modelled in this particular course have little to do with classical mechanics and physics, and that the mathematics is less inherent in the situations themselves. Examples of case studies include:

- Hospital corridor: can a bed be moved around a right-angle bend in a corridor?
- Order of play: how is a badminton match game order fixed when only 1 court is available?
- On the buses: how do buses become congested despite timetabling on a typical city route?
- Conifer trees: estimate the height against time as trees grow, allowing for seasonal growth.

Attempting to build a model for a real world situation in which the mathematics is not apparent can be difficult if one's knowledge of mathematical objects is limited. Learning about new mathematical objects can allow a student to see an otherwise benign real world situation with a new perspective (Edwards & Hamson, 1989).

Teague (1989) expresses an interesting thought when comparing pure mathematics with applied mathematics. In his view, much of the mathematics taught in secondary school, such as simplifying expressions or solving equations, is analogous to simply practicing finger exercises when learning to play the piano. In order to "play the music" of real mathematics, which includes creating and using mathematical models, one must be able to perform these manipulations, but there is more to playing music than the finger exercises.

3.4 MATHEMATICS AS A SERVICE SUBJECT

The concept of mathematics as a service subject contrasts with the literature on vocational mathematics education. Since engineering mathematics is not vocational mathematics, it more appropriately fits the definition of mathematics as a service subject, i.e., mathematics that is taught to students who are primarily engaged in studying other subjects. Engineers will use mathematics extensively in their work, but they will not be mathematicians. The rise of service mathematics was a response to a need. In the case of engineering, it is the need for engineering graduates to be mathematically competent (Howson, et al., 1988).

For the universities and other institutions of learning there is also the need to ensure that all of the students entering their programs have a common level of knowledge. At both McGill University and Concordia University, engineering students who are originally from outside of Quebec are required to extend their program by one year. During their first year, these students take courses in the fundamentals of physics, chemistry, and mathematics, including pre-university linear algebra and calculus. The mathematics courses in this year of study are service courses, intended to bring these students to the same level of knowledge as those who will enter the engineering program from CEGEP the following year.

The International Commission on Mathematical Instruction (ICMI) compiled selected papers on the teaching of mathematics as a service subject in an attempt to answer a few questions on the matter, including questions that are pertinent to this thesis, such as:

- Who teaches service mathematics courses?
- Are there differences in how service mathematics courses are taught and assessed, in comparison with mathematics courses for mathematicians?
- Do the students encounter any obstacles in the form of language or symbolism?

Another issue of concern is determining which topics should be taught in a service mathematics course. Howson et al. (1988) present two points of view to help address this question. The first approach is to consider mathematics as a tool that students use to solve concrete problems drawn from their own discipline. Abstract mathematical notions that are not directly related to relevant applications should be discarded. Alternatively, though not completely dissimilar in approach, mathematics can be viewed as a language, and the students should know how to read it, and communicate with it. Since much of the literature in their discipline is written in the language of mathematics, knowing how to read it will allow the students to learn more about their own subjects on their own.

The universities at Southampton (UK) and Orsay (Université Paris-Sud, France), for their part, have adopted similar attitudes. For these schools, service mathematics is meant to acquaint the students with the mathematical techniques that will be useful or essential in their core discipline courses. The teaching of service mathematics is meant to be done quickly, without the need to elaborate on the history or underlying theorems behind these techniques. In particular, they propose that engineers should be required to learn calculus, but not analysis. This is in fact presently the case for engineering students in Quebec.

When it comes to determining who should teach the service mathematics courses (those who teach in the discipline itself, or mathematicians and mathematics teachers), the majority of contributors appear to favour those from outside of mathematics. Teachers from the service discipline are aware of the profession's needs, and will also be able to properly structure the courses so that the mathematics that are needed can be taught immediately preceding a core course in which they will be needed. This can also provide uniformity in the use of language and symbolism between the mathematics and the core courses. Lastly, students in the service discipline will be more motivated, as they will feel more connected to a teacher from their field. A student from Cardiff (Wales, UK) offered the opinion that "engineering students should be taught by engineers, or at least by mathematicians who are based in the engineering faculty. The biggest single problem is motivation, and this is best achieved if the teaching is done by engineers who are respected by the students as engineers and who can draw examples to illustrate the mathematics from their own work."

The argument is strengthened by mentioning the additional work that would be required of a mathematician to teach to non-mathematicians. They would have to learn the language and symbolism of the service discipline, adapt it to a mathematical framework, provide mathematical analysis and techniques, and translate it back into the students' language.

So one could ask, is it easier for an engineer to teach the fundamentals of the mathematics that they use, or for mathematicians to provide engineering context to the mathematics that they teach?

In discussing how service mathematics courses should be taught, Howson (1988) states that there is nothing sacrosanct about the order in which mathematical topics are presented: fundamental concepts of a subject can be taught before showing students how to solve problems based on those concepts, or the concepts can be introduced while working through an example. In my own research on engineering programs at universities in Quebec (chapter 4) I found that some mathematics courses teach the concept of limit prior to infinite series, while others introduce the limit after teaching infinite series, thus supporting Howson's view.

Blum and Niss (1991) also characterize engineering mathematics as being different from vocational mathematics, and classify it as a service subject. In mathematics courses for engineers, the focus is not on the mathematics itself but on the other subjects in which mathematics will provide a service. In their research, the characterization of mathematics instruction is analyzed by considering the educational histories, the purpose, and the organizational framework of mathematics instruction. In terms of educational history, engineering mathematics is classified as a service subject, while mathematics taught to mathematicians or mathematics teachers is not. The purpose of mathematics instruction for engineers is to provide students with mathematical knowledge as it relates to other subjects in their field (mechanics, thermodynamics, structural analysis, etc.). Engineering mathematics is further characterized by the organizational framework in which it is found. In fact, it can be classified into two frameworks: it can be taught as a separate subject in mathematics courses, or as part of, and integrated into, one or more engineering courses. These separate frameworks can lead to a division of labour among the mathematics and engineering professors such that the mathematics courses may be entirely devoid of mathematical modelling, and are instead organized so as to present all of the mathematical concepts that will be needed in the subject being serviced, while the applications and modelling will take place in the extra-mathematical courses. The results of my research, presented in chapter 4, fit this description fairly well.

3.5 Mathematics in the Workplace

The edited volume *Education for Mathematics in the Workplace* (2000) presents a number of studies on mathematical knowledge and the use of mathematics at both school and work. The main theoretical framework of the book is based on the Anthropological Theory of the Didactic: that mathematics is a human activity that takes place in all kinds of contexts and situations; however, in the workplace, the mathematics is deeply entrenched in a given profession's activities, and is not always visible either to outside observers, or to the workers themselves (Noss, Hoyles, & Pozzi, 2000).

Evans (2000) discusses the problem that workers have in "transferring" their mathematical knowledge from school to the workplace, i.e., applying what they have learned at school to non-pedagogical contexts, such as their work environment. Of particular note is that there is no guarantee that transfer will even occur; in other words "book smarts" do not always transfer to "street smarts". For example, a student who is capable of properly subtracting fractions from whole numbers (e.g., performing the operation $24 - 3\frac{1}{4}$) using subtraction techniques in a classroom setting may not be able to perform the same task on a construction site where they are handed a two-foot long piece of lumber and asked to make it three and quarter inches shorter.

To say that mathematical knowledge can be transferred implies that it can be applied to similar problems but in different contexts. This is based on the assumptions that:

- Learning is the transmission of knowledge from a teacher to a student.
- The mathematical knowledge that is needed to solve a problem can be removed from the context of the problem. In the context of the ATD, this would mean that there exists mathematical knowledge that is independent of any task to accomplish, or technique to accomplish that task.
- A problem can be reduced to its mathematics, and that a mathematical strategy can be used to solve problems across different contexts.

But are these assumptions justified? Mathematics is learned in a socially constructed environment, with established and ongoing activities, social relationships, and language. Thus there are more factors at play than just the interaction between the teacher and the student. Furthermore, changing the environment in which a person finds themselves can change the mathematical strategies that they use. Recall the studies of Castela and Romo-Vazquez presented in section 3.1, and the finding that much of the students' work could not be reduced to mathematics alone. Referring to my earlier example of a construction worker cutting a two-foot long piece of lumber, by changing the context of the problem (subtracting a fraction from a whole number) from a school exercise to a work situation, the mathematical strategy of using techniques of subtraction becomes too time-consuming and ultimately useless; making the two-foot piece of lumber three and quarter inches shorter one simply has to measure $3\frac{1}{4}$ inches on a tape measure, and make the cut at that length. Determining the length of lumber that remains is not necessary. Moreover, since the construction site environment has additional pressures and stresses such as loud noises and having to work quickly and efficiently while following safety regulations, the mathematical knowledge of subtraction becomes only a small part of the total knowledge engaged when working.

Rather than thinking about how mathematical knowledge is transferred in the sense described by Evans, this thesis considers that mathematical knowledge undergoes institutional transposition. The difficulty in reconciling school mathematics with work mathematics may occur in the transposition that the mathematical knowledge undergoes between institutions.

In the same volume, Noss et al. (2000) present a study showing that professionals and practitioners (i.e., labourers) use mathematics differently than mathematicians. While mathematicians study mathematics for its own sake, professionals are driven by a pragmatic agenda: they study mathematics to solve problems external to mathematics. They must solve problems efficiently and effectively using mathematical knowledge and expertise, but their professional concerns take precedence over mathematical concerns. For the practitioners, a theorem must, first of all, “work” as a tool in solving their practical problems; for mathematicians, a theorem must be, first of all, “true” and then important for the advancement of a theory; it is better if it is also non-trivial. This brings to mind the studies discussed in section 3.2 involving electricians.

The theory of situated abstraction (a term coined by Noss) focuses on the similarities in the structures of “mathematics in learning” and “mathematics at work.” Other theories, such as situated cognition and policy-driven studies, tend to emphasize the differences between the two. From the perspective of situated abstraction, the authors performed an exploratory study to examine the relationship between practical, professional, and mathematical knowledge in three professions: commercial airline pilots, bankers, and nurses. Their goal was to locate the mathematics that was used at work, searching for “visible mathematics” in the day-to-day activities of the different professions. Visible mathematics could take the form of conventional mathematical symbolism and representation, or the use of concepts, strategies, and methods of a mathematics classroom. However, it was found that, in routine tasks, very little of the practitioners' mathematics is visible. Even the workers themselves failed to notice when they are using mathematics, or simply did not regard what they were doing as "mathematical." The visible mathematics that was found took the form of:

- Finding solutions in procedural ways, with the use of algorithms and spreadsheets to perform menial computations, particularly by bankers.
- Routine gathering and interpreting of data.
- Using "look up" methods, i.e., the use of tables and charts containing pre-calculated values for similar or general problems (rather than using analytic formulas or functions for calculating their values).

Look up methods are used to solve problems that are already well-understood, and can circumvent the need to work in the realm of school mathematics. But an understanding of the underlying mathematics that lead to the development of the look up tables is still a necessity for those who use them; otherwise the values in the tables may not be properly interpreted. For example, a pilot deciding whether or not to land in wintery conditions must quickly calculate the normal component of crosswinds acting on the aircraft. In Figure 5, the crosswind is represented by the red arrow, and its components are represented by the blue arrows. The normal component of the crosswind acts orthogonal to the direction of travel, and the amount of force it exerts on the plane is critical in determining whether it is safe to land.

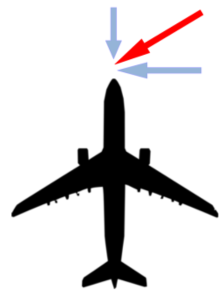


Figure 5 - Crosswind and its components (drawing is my own)

The generally accepted practice is to use lookup tables and rules of thumb. Calculating the normal component of the crosswind with proper trigonometric functions would be more accurate, but the accuracy they provide is unnecessary, given the size of the aircraft, and the speed at which it travels. The difference between the results obtained using trigonometry and those found by looking up tables is negligible, and using trigonometry requires too much of the pilot's time. Instead, a generally accepted technique is to estimate the value of a trigonometric function of an angle based on the known values of the more common angles (30, 45, 60 degrees). But, the pilot has the knowledge that this rule of thumb is based on using trigonometry for finding the components of vectors that are normal to the trajectory of the plane.

Rules of thumb are not generally found in engineering practice, but lookup tables are. An engineer can choose to look up the size of a steel beam with the desired strength if they know the forces that it must withstand, even though he or she has the ability to calculate the needed size with knowledge from school. Performing the calculations will not only be time consuming, but may result in finding a beam with dimensions (width and height) that are not even manufactured. The calculations may yield minimum required dimensions of x millimetres by y millimetres, but beams are mass produced, not custom-made, and their sizes are regulated by industry standards. Since the calculated dimensions are minimum requirements, the engineer can use a look up table to find the industry-standard beam whose dimensions are slightly larger than those calculated. Or, since the lookup table also contains the strengths of each of the standard beam sizes, the engineer can simply skip the calculations altogether and find a beam whose strength is greater than the applied forces.

Noss et al. (2000) points out that it is when "breakdown episodes" (conflicts in routine tasks) occur that mathematical knowledge may come to the fore, and become more visible. In these situations, workers are provoked into justifying their approach and the mathematical models on which they are based. For example, a nurse had to decide if their patient's blood pressure was too high for a drug to be administered. Statistically, the patient's blood pressure was considered to be too high for the drug to be administered, when compared with the average blood pressure of the population. But this particular patient's baseline blood pressure was also higher than normal. This created a conflict in the decision of whether or not to administer the drug, and the nurse was forced to re-evaluate their statistical model, and what it meant to have a "high" blood pressure. Similar conflicts arise in engineering as well. An engineer who specializes in designing single-storey steel structures for warehouses may be tempted to "recycle" an already-proven design on a new structure in a different location. But different locations have different geographic and topographic features. Perhaps the new structure is located in a city with higher winds, or in an area that is more prone to earthquakes. Such a scenario can create a breakdown episode, and force the engineer to develop an entirely new design.

3.6 Geometry, Measurement and Error

Geometry plays an important role in civil engineering. It is used in, among other fields, surveying, which is literally the measurement of the earth, transportation engineering, to determine the shapes and directions of roads and railways, and structural engineering, in determining the geometric properties of structural elements such as length, cross-sectional area, and moment of inertia. Furthermore, the construction of engineering projects requires technical drawings which rely on orthographic projections, idealized representations of real world objects, and the principles of descriptive geometry.

Mathematicians and engineers (as well as other tradespeople) have different motives for studying and using geometry. The mathematician seeks truth or the construction of consistent theoretical systems, while the engineer seeks practicality (Bessot, 2000). The underlying motives vary so greatly that Bessot remarks "a builder may well have problems finding solutions to practical problems in a traditional geometry course." The geometry of building construction, and civil engineering in general, is Euclidean, while the geometry of mathematicians refers to many theories, and include non-Euclidean geometries (hyperbolic and elliptic geometry), differential geometry, topology, etc. For the engineer, geometry is a description of the reality they work with: the axioms are treated as "facts." For the mathematician, a geometry is just one axiomatic theory among others; its axioms should be consistent with each other and they need not be a faithful or "true" model of some external reality.

In France, geometry courses as a structured entity in vocational schools have all but disappeared (Bessot, 2000). In its place are descriptive geometry and practical geometry which are used in the drawing, reading, and marking out of plans. Like mathematical models, plans are idealized representations of the real world that use two-dimensional shapes to represent three-dimensional objects: straight lines can represent various objects such as beams in a structure or water conduits under a roadway; rectangles and circles can be used to indicate multiple objects as well as the cross-sections of columns in a structure or valve chambers. Figure 6 shows the typical shapes, symbols, and text that are used on a municipal roadwork plan to indicate a water conduit (the black line) and its dimensions (AQ. Ø 250: *aquéduc diamètre 250 mm*; water conduit with 250 millimetre diameter), and a valve chamber[2] (C.V.: *chambre de vanne*; valve chamber).

[2] A valve chamber is a large concrete box that houses a valve. The valve is connected to the water conduit and can be opened or closed to either allow or impede the flow of water through the conduit. The valve can be accessed by entering the chamber through a manhole cover found in the roadway.

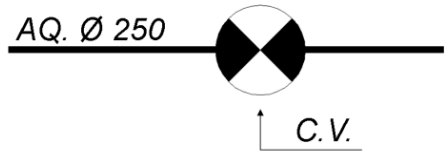


Figure 6 - Symbols for water mains and valve chambers (drawing is my own)

In mathematics, the solution to a geometric problem may require a formal proof of existence of an object or a relation between properties, a construction of an object satisfying certain conditions, and theoretical precision. In vocational and professional practices, a solution doesn't require the same exactitude. To a mathematician, the ratio of any circle's circumference to its diameter is always π; for an engineer, the decimal approximation of 3.14 may be "good enough" for many calculations. A project's construction plans need only be "sufficiently precise" in relation to desired tolerances to be considered acceptable. For example, connecting a steel beam to a column can be done as long as there is enough "wiggle room" for inserting the bolts. While the plan may state the required dimensions to the millimetre, variation in the fabrication of the steel pieces cannot be completely eliminated.

This is not to suggest that there isn't any amount of care taken, or precision required, when civil engineers use mathematics in their work of designing and executing engineering projects. In fact, the opposite is true – but it is a different precision and care about different aspects. Engineers raise questions about the nature and magnitude of errors and tolerances in calculations and in measurements. Eberhard (2000) discusses geometry and measurement as used by the students taking part in an actual construction project being overseen by their technical high school. At this school the students are being trained to become foremen and skilled workers, earning a *Brevet de Technicien*. The students use measuring techniques to transfer dimensions from a plan into physical markings on the ground, a process called marking out. They must then verify that their physical markings match the specified dimensions on the plan.

All measurements of the physical world contain an inherent error, even those made using state of the art technology. To mitigate these errors, the students in Eberhard's study are introduced to two techniques for measuring: partial dimensions, which are measured from point to point, and cumulated dimensions, which are measured from an origin to an endpoint. With partial dimensions (Figure 7 (top)), measuring can be done for any desired length by adding adjacent measurements, but the error in each of the measurements accumulates over the entire measured length. Cumulated dimensions (Figure 7 (bottom), on the other hand, are limited by the length of the tape being used to measure, but they require only a single measurement for multiple points of interest and provide fewer opportunities for error.

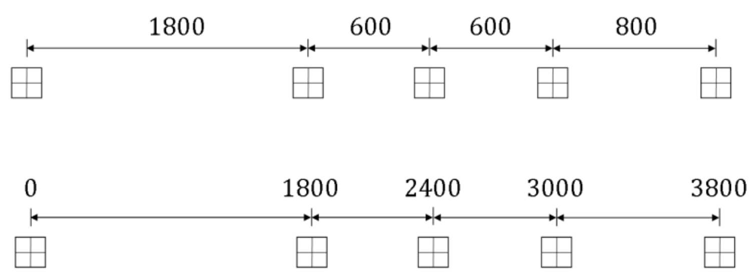


Figure 7 - Measuring with partial dimensions (top) and cumulated dimensions (bottom) (adapted from Eberhard (2000))

The exact transfer of measurements from a plan to markings on the ground is impossible, so the goal is to find a balance between the precision of a measurement and a tolerance for error in both the process of marking out and verifying the markings. As a practical example, consider a construction site that will install prefabricated components provided by other parties. When they arrive on site, they must fit properly with the components that were built in place. If the accumulated error in the parts already constructed is too great, then the prefabricated components may not fit, leading to delays and increased costs. Once again, time is money.

Since there is always error involved in measurement, we might well ask how the notion of error is taught in vocational schools. The general teaching of mathematics in France (as of 2000) does not include the topic of error in measurements, though it was taught in the 1960's (Eberhard, 2000). As it relates to the use of partial and cumulated dimensions, it could be explained that the latter technique is favourable if measurement error is thought of as either an uncertainty in measurement (the model of uncertainties), or as a random variable with a characteristic standard deviation (the model of probabilities). Eberhard (2000) states that "the algebraic theory of uncertainties or the probability theory of errors [could] work together as a basis for theoretical understanding", and could also help in showing why the use cumulated dimensions is preferred. But the teachers at schools awarding the *Brevet de technicien* are not required to teach the theoretical notion of error. In their view, using cumulated dimensions is preferred simply because it maximizes the efficiency of construction site. It is noted that the teaching of errors in measurement is found in a surveying textbook in a course for a *Brevet de Technicien* ***Supérieur***.

The notion of error in measurement is taught in some, but not all, engineering programs. In the engineering profession it arises in the fabrication of materials and construction of projects, but less so in the design process. It is usually the subject of contractual clauses that specify tolerances for error; clauses that surveyors, contractors, and fabricators are legally required to respect. Since it as a concept that is known in construction, it would be useful for engineers to have the knowledge of it as well, if only as a means of ensuring clear and precise communication between designers and builders.

The final topic of discussion in this section, which also relates to measuring and to communication between engineers and labourers, is the system of units used in measuring. In professional engineering design, there is an emphasis on the use of the metric system. But in the construction industry, use of the imperial system is still widespread, even in Canada. In a study of engineering technician apprentices, Ridgway (2000) discusses a "strong feeling" amongst employers at engineering technology companies of an apparent decline in basic number skills of their applicants, with the most frequent problem cited being a lack of awareness and familiarity with imperial units. In engineering education, many textbook application problems do make a point of presenting problems with imperial units, but any given problem is usually uses one system, either metric or imperial. Rarely are problems shown that require the use of both systems, and conversion between the two in order to find a solution.

4 MATHEMATICS FOR ENGINEERS

In this chapter I present details from the mathematics courses that a student must take in order to obtain a degree in civil engineering, and will compare and contrast them with the mathematics to be taught to, and expected to be learned by mathematics students.

The mathematical concepts to be taught to both groups of students were determined from course descriptions in university calendars and course outlines (syllabi). The mathematics that the students are expected to learn was determined from final examination questions (tasks). The bulk of the chapter is a comparative analysis of the tasks on final examinations from comparable courses. Comparisons of the necessary techniques to accomplish the given tasks are also analyzed. The analysis classifies the tasks according to their nature as either computational or conceptual, as well as by their mathematical content: are the tasks (1) purely mathematical, (2) mathematical applications, or do they require (3) mathematical modelling. Counting the number of tasks in each classification will allow me to quantify the differences, and similarities, between mathematics courses taken by engineers and those taken by mathematics students.

Prior to presenting the details of my research, I will first discuss the institutional reasons for engineering programs to teach the mathematics that they do, and will discuss the general education of engineering students in Quebec.

4.1 ACCREDITATION OF AN ENGINEERING PROGRAM

The education of an engineer is regulated by the Canadian Engineering Accreditation Board (CEAB), a standing committee of the national professional organization Engineers Canada. Established in 1965, the CEAB is responsible for accrediting engineering programs at higher education institutions in Canada. According to the CEAB, the purpose of accreditation is to inform the professional engineering organizations (such as the *Ordre des ingénieurs du Québec*) which programs are capable of producing graduates who are academically qualified to begin the process of becoming a professional engineer[3]. Among the requirements for a program to be accredited is an adequate curriculum with the goal that graduates demonstrate competence in university level mathematics, natural sciences, engineering fundamentals, and specialized engineering knowledge appropriate to the program (Canadian Engineering Accreditation Board, 2013).

The CEAB quantifies curriculum content using Accreditation Units (AU), with 1 AU corresponding to either one hour of lecture or two hours of laboratory or scheduled tutorial. An accredited engineering program must include, in its entirety, a minimum of 1,950 AU. A minimum of 195 AU, that is one-tenth of the entire program, and up to 225 AU, must be dedicated exclusively to mathematics courses. The topics covered in the mathematics courses must include linear algebra, differential and integral calculus, differential equations, probability, statistics, and numerical analysis (Canadian Engineering Accreditation Board, 2013).

The bulk of the program, a minimum of 900 AU, is to be dedicated to courses in engineering science and engineering design. The courses devised for engineering science must involve the "application of mathematics to practical problems through the development of mathematical or numerical techniques, modelling, simulation, and experimental procedures" (Canadian Engineering Accreditation Board, 2013). These courses may involve the development of mathematical or numerical techniques, modelling, simulation, and experimental procedures. Such courses are considered to be at the core of engineering education. For civil engineering programs they include courses such as statics, mechanics of materials, dynamics and thermodynamics. In other words, the core engineering courses are to include mathematical applications to engineering problems. This would not prevent mathematics courses from

[3] A bachelor's degree in engineering is only one of the requirements for becoming a professional engineer. Obtaining a license also entails acquiring a minimum amount of work experience, and passing a professional licensing exam.

including application problems, but modelling problems would most likely be left for the core engineering courses.

Engineering design, for its part, is meant to integrate mathematics, natural sciences, engineering sciences, and complementary studies in a creative, iterative, and open-ended design process, subject to constraints which may be governed by standards or legislation to varying degrees depending upon the discipline (Canadian Engineering Accreditation Board, 2013). The concept of engineering design and the design process is perhaps most closely associated with the professional practice of engineering and the execution of engineering projects.

At this point it is important to recall what is meant by mathematical applications and mathematical models. An application is the use of mathematics, its concepts, its objects, and its rules, to solve a problem in the real world (Blum & Niss, 1991; Galbraith, Henn, & Niss, 2007). Applications are useful in order to show students how mathematics can be used to solve relevant tasks. But in every application of mathematics there is an underlying mathematical model. A mathematical model is an idealized representation of the real world; it is a mapping of physical objects into objects of a mathematical domain. The task of modelling involves creating or designing an appropriate mathematical description of an extra-mathematical situation or phenomenon.

Based on the requirements of the CEAB, within an accredited engineering program I should expect to find courses that are dedicated solely to teaching mathematics, as well as engineering courses that have mathematical application and modelling problems embedded within them. Thus, I may not find tasks that involve mathematical applications and modelling in the mathematics courses themselves, since they would instead appear in the engineering courses.

The accreditation criteria of the CEAB reinforce the notion that engineering programs are institutions as defined by the chosen theoretical framework. And since the CEAB is overseen by the Engineers Canada, an umbrella organization that regulates the provincial professional associations, I can also claim that the institution of engineering education is situated within the institution of the engineering profession.

4.2 Accredited engineering programs in Quebec

The following universities in Quebec have civil engineering programs that are accredited by the CEAB. They are listed in chronological order the year of their accreditation.

1. McGill University – Faculty of Engineering – Civil Engineering and Applied Mechanics (1965)
2. Université Laval – Faculté des sciences et de génie – Génie civil (1965)
3. École Polytechnique, affiliated with l'Université de Montréal – Génie Civil (1965)
4. Université de Sherbrooke – Faculté de génie – Génie civil (1965)
5. Concordia University – Faculty of Engineering and Computer Science – Civil Engineering (1969)
6. École de technologie supérieure, affiliated with l'Université du Québec – Génie de la construction (1993)
7. Université du Québec à Chicoutimi – Département des sciences appliquées – Génie civil (2012)

For each of these programs, I analyzed the course descriptions found in the university calendars in an attempt to categorize the mathematical topics that they have in common and to determine the mathematical knowledge that is to be taught.

The faculties of engineering at Concordia and McGill are separated into engineering administrative departments, including their respective departments responsible for civil engineering (the Department of Building, Civil and Environmental Engineering at Concordia; the Department of Civil Engineering and Applied Mechanics at McGill). Some mathematics courses at these schools are common to multiple engineering departments; students from different engineering departments can enrol in them together. It is common for a calculus or differential equations class to be attended by civil, mechanical, and electrical engineering students. This could make the task of writing relatable application problems more difficult for the professors. Writing application problems, in general, is not easy for professors who are, for the most part, mathematicians and not engineers.

The Université du Québec is a network of universities located in various cities and regions across the province, including Montreal (UQAM), Trois-Rivières (UQTR), and Gatineau (UQO). Only the school located in Chicoutimi (UQAC) offers a degree in civil engineering, while degrees in other disciplines (mechanical, electrical, chemical, and industrial engineering) are offered elsewhere. The École de technologie supérieure (ÉTS) is an engineering- and technology-specific institution affiliated with the Université du Québec. While they do not offer a degree in civil engineering, they do have a program that leads to a bachelor's degree in construction engineering. The difference between the two degrees appears to be that the construction engineering program focuses on the management of civil engineering construction projects as opposed to engineering design and analysis. Some of its courses are entirely unique to this program, including a course on estimating the costs of construction projects and establishing project schedules. In my professional experience, these tasks were learned "on the job."

In all programs, after completing common core courses, students can choose to focus their studies on one of a number of sub-disciplines. In civil engineering, for example, a student can choose to study structural engineering, environmental engineering, or transportation engineering, among others. École Polytechnique de Montréal has a unique feature. Students enrolled in its program have the additional option of a thematic orientation called *mathématiques de l'ingénieur* (mathematics for engineers). The purpose of this orientation is to allow students to acquire knowledge in advanced applied mathematics, and broaden their abilities to model mathematically in order to solve engineering problems (École Polytechnique de Montréal). École Polytechnique is a renowned research facility, and as such it is important for them to offer proper training for students who wish to focus on research in engineering science in their graduate studies, hence the need for this orientation (École Polytechnique de Montréal, n.d.).

4.3 Mathematics in accredited civil engineering programs

The mathematics to be taught in the accredited engineering programs in Quebec can be grouped into seven subjects:

1. Pre-university linear algebra
2. Pre-university calculus
3. University level calculus
4. Differential equations
5. Probability and statistics
6. Numerical methods
7. Engineering geometry

I established these groupings by reading course descriptions found in the university calendars of the schools mentioned in the previous section and identifying commonalities in all of the required mathematics courses. The list of mathematics course names and complete courses descriptions for each program are included in appendix 8.2. Note that unlike calculus, there is no university level linear algebra course for engineers.

In the majority of the accredited programs identified in section 4.2, the instructors for subjects 1 through 4 are professors from the mathematics department, while subjects 5 through 7 are taught by members of the engineering faculty.

It is necessary to establish what is meant by pre-university (subjects 1 and 2) and university level mathematics (subjects 3 through 7). By pre-university mathematics I am referring to mathematical content that is also taught either in CEGEP (for students from Quebec) or in high school (for students from other provinces). Universities offer pre-university mathematics courses for various reasons. For one, students from outside of the province who move to Quebec to attend university may not have necessarily taken a course in linear algebra or calculus prior to starting their university education, while those who are from Quebec have. In Quebec, linear algebra and calculus are required to graduate from CEGEP, and are taken not only by students who want to be engineers, but also by those who will enter other science programs such as biology, physiology, and computer science. There is also a need, from the universities' perspective, to ensure that all of their students have the same level of mathematical knowledge before they begin courses in the engineering program. Some topics covered in pre-university calculus courses – differentiation and integration of single-variable functions, for example – are important and necessary concepts that are required in first year engineering courses such as statics and mechanics of materials. The pre-university courses are also necessary for students who were accepted into an engineering program on the condition that they retake the pre-university courses in order to improve their academic standing, or for students who return to school after a prolonged absence and lack the prerequisites to enter their desired program.

Because of these pre-university level courses, the first year of study at Concordia and McGill may differ for some students. For example, when I attended McGill, my first year of study included mathematics courses only at the university level. However, my classmates included many students from Ontario and other provinces that were in their second year at the school. They had spent the previous year, while I was finishing CEGEP, taking pre-university mathematics (and natural science) courses. This is referred to as year U0 (undergraduate year 0) at McGill, and EC (extended credit) at Concordia. Thus, for students from Quebec who are not required to retake pre-university level mathematics courses (subjects 3 to 7), more than half of their mathematics courses are taught by professors from the engineering faculty, as opposed to professors of mathematics.

4.3.1 Methodology for the analysis of final exams

In the sections that follow I present the results of my analysis of final exams from mathematics courses taken by engineering students, and from comparable courses taken by mathematics students, focusing only on the programs at Concordia and McGill. Final exams were chosen instead of textbooks since they give us the most insight into what mathematics students are expected to learn during the course.

One final exam from each course is analyzed. The engineering exams that I chose were similar to those that I wrote as an engineering student, and in my experience as a mathematics instructor final exams for mathematics courses do not change very much from one year to the next (at least in these universities). As a result of this I claim that the chosen exams represent institutionalized mathematical knowledge to be learned by the students of the respective courses.

The exams were analyzed by categorizing the tasks in two ways. First, the nature of each task is classified as either computational or conceptual. I make this distinction from the point of view of a mathematics educator, with the institution used as a reference model being that of mathematics courses in university programs. Various mathematics textbooks, including Lay (2016), describe the exercises contained within as either computational or conceptual. While computational tasks require routine calculations, conceptual tasks demand a bit more thought as well as a justification of their solutions. This particular distinction in ways of knowing and doing mathematics has been studied and discussed in mathematics education practically forever.

To identify the computational tasks on the final exams, I searched for keywords such as “find”, “evaluate”, or “compute”, in the problem’s text. Tasks that explicitly state which technique the student should use to solve the problem also fall into this category. Conceptual tasks were identified as those that required proving a given statement or showing an equality to be true or false (problems that used phrases such as “show that”, or “prove”), or demonstrating an understanding of mathematical concepts based on axioms and definitions. Any task that asked for an explanation of justification of a technique beyond its computational use is also considered a conceptual problem.

The tasks were also independently classified according to their content as either purely mathematical, an application problem, or a modelling problem. An application problem will make reference to some real world object or phenomena that is being investigated using mathematics, either explicitly or implicitly through the use of units. A modelling problem will require the construction of a mathematical model as opposed to just an application of an existing model. While modelling necessarily implies that real world objects or phenomena are involved, the task of modelling is more intensive and time-consuming than an application as it requires a mapping from the physical world into the language of mathematics. Perhaps as a result of these constraints, only two modelling tasks were found on all of the final exams that were analyzed, both for courses taken exclusively by engineering students. Tasks are considered purely mathematical if no real world context is given in the problem.

Each individual exam question can be comprised of multiple tasks, either explicitly identified as parts (a), (b), (c), etc., or not. For every final exam in my analysis, each identifiable task was appropriately categorized, and a table indicating the relative frequency of each type of task was constructed:

Table 1 - Sample table: Relative frequency of tasks on a final exam

	Computational	Conceptual	
Mathematical	_%	_%	_%
Application	_%	_%	_%
Modelling	_%	_%	_%
	_%	_%	

These tables offer a quantifiable measure of the different types of tasks given to engineering students versus those given to mathematics students. As a result of my analysis, I have found that the similarities between the mathematics for engineers and mathematics students outnumber the differences. Many exams in comparable courses feature the same types of tasks requiring the same techniques to solve them, use the same notation, and feature a similar number of purely mathematical and application problems.

The exams that were analyzed are included in their entirety in appendix 8.3.

4.3.2 Pre-university linear algebra

Topics taught in pre-university linear algebra include solving systems of linear equations, matrix algebra, evaluating determinants and inverses of square matrices, the geometry and algebra of vectors in $\mathbb{R}^n$, operations on vectors (dot product and cross product), the geometry of lines and planes, an introduction to vector spaces and subspaces, an introduction to linear transformations, eigenvalues and eigenvectors of matrices, and the diagonalisation of matrices using their eigenvectors and eigenvalues. This list is not written to suggest that linear algebra courses contain a random collection of disconnected topics, but rather to simply highlight what concepts the students are expected to learn in such a course.

The courses offered in this subject are MATH 204 – Vectors and Matrices at Concordia, and MATH 133 – Linear Algebra and Geometry, at McGill. I was unable to obtain any documentation about final exams for MATH 133. Concordia's MATH 204 is not reserved for engineering students. It is offered to students in many programs, including mathematics students who may need it as a prerequisite for their university level linear algebra courses MATH 251 – Linear Algebra I, and MATH 252 – Linear Algebra II.

I analyzed final exams from these two university level courses as well as from MATH 204. Since engineering students do not take university level linear algebra, it is assumed that the topics covered in MATH 204 represent all that engineers need to know of the domain of linear algebra. They are assumed not to need to know the theory of general vector spaces and linear transformations over arbitrary vector spaces that justifies, at the level of "theory" in the model of praxeology, the techniques and elements of "technology" learned in MATH 204.

The following tables show the relative frequencies of the different types of tasks found on the linear algebra final exams. The MATH 204 exam is dated December 2014, while the exams for MATH 251 and MATH 252 are dated December, 2013, and April 2013, respectively.

Table 2 - Relative frequency of tasks: MATH 204 - Vectors and Matrices (Concordia), December 2014

	Computational	Conceptual	
Mathematical	71%	29%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	71%	29%	

Table 3 - Relative frequency of tasks: MATH 251 - Linear Algebra I (Concordia), December 2013

	Computational	Conceptual	
Mathematical	39%	61%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	39%	**61%**	

Table 4 - Relative frequency of tasks: MATH 252 - Linear Algebra II (Concordia), April 2013

	Computational	Conceptual	
Mathematical	76%	24%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	76%	24%	

None of the tasks on any of the exams are application problems. All tasks contained only objects that are mathematical in nature (matrices, vectors, lines, and planes) without any physical meaning attached to them. As for the nature of the tasks, we can see that MATH 251 contains more conceptual problems (61%) than computational, but the proportions of computational tasks are about the same for MATH 204 and MATH 252 (71% and 76%, respectively).

The lack of applications could possibly be attributed to the fact that MATH 204 is a pre-requisite for multiple programs, or perhaps because the course is administered by the mathematics department.

Another reason is the lack of time. MATH 204 is a short, dense course, with many concepts and techniques that are new to the students. For engineering students this is the only linear algebra course they will take so everything they will ever need in their studies and work has to be included; there is very little time left for solving some more serious application problems. They may be briefly mentioned in general terms but that is usually all the instructor has the time for. The problem of time is exacerbated if, and this is often the case, the course examiner insists that all calculations be done by hand on the mid-term and final examinations. Much of the class time is then spent on computational tasks such as row reduction of matrices. This is a particularly boring and tedious activity but students insist on practicing it because every slight mistake in this process has dramatic consequences for the rest of the solution. The requirement of manual computation forces the choice of “easy” numbers (integers) for the entries of the matrices and vectors whose sizes are also chosen to be small. This is ironic because the introduction of the powerful methods of linear algebra in the course can hardly be justified by their application to systems of two equations in two unknowns or even three equations in five unknowns. Such cases can serve at most to illustrate the techniques and concepts, but can hardly motivate it.

Indeed, in the tasks given in MATH 204, the sizes of the objects are such that the computations are manageable. Question 4 (Figure 8) requires evaluating the determinant of a 4x4 matrix, and question 8 (Figure 9) asks the students to find the solution to a system of size 3x5, but all other matrices and systems of equations are limited to sizes 2x2 or 3x3. All of the vectors in the exam are from the vector space $\mathbb{R}^3$, as seen in question 5 (Figure 10).

4. (a) Evaluate the determinant of $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$.

Figure 8 - MATH 204 - Vectors and Matrices (Concordia), December 2014, question 4

8. Let $A = \begin{pmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix}$. Find a basis for the solution space of the homogeneous system $AX = 0$.

Figure 9 - MATH 204 - Vectors and Matrices (Concordia), December 2014, question 8

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5. (a) Let $u = (1, 4, 2)$, $v = (1, 1, 0)$. Find the orthogonal projection of u on v.

(b) Let $u_1 = (1, 1, 0), u_2 = (0, 1, 1), u_3 = (1, 0, 1)$. Find scalars c_1, c_2, c_3 such that $c_1u_1 + c_2u_2 + c_3u_3 = (1, 0, 0)$.

Figure 10 - MATH 204 - Vectors and Matrices (Concordia), December 2014, question 5

To solve question 4, students are expected to use cofactor expansion, a technique which is greatly simplified in this example by using a row operation to create a 0 in entry (1, 1). Solving question 8 doesn't require any row operations to be performed as the given matrix is already in reduced row echelon form. The students can use the given matrix to identify the two free variables, z and u, and find the general solution to the homogeneous system. The technique for solving part (a) of question 5 is to use the formula for the projection of one vector onto another, which the students are expected to memorize:

$$proj_v u = \frac{u \cdot v}{\|v\|^2} v$$

To solve part (b) of question 5, students must recognize that the given equation can be represented by the matrix-vector equation $U.c = b$, where U is the matrix whose columns are the vectors u_1, u_2, and u_3, c is the unknown vector $\langle c_1, c_2, c_3 \rangle$, and b is the vector $\langle 1,0,0 \rangle$. The task can then be solved using either row reduction, or by finding the inverse of the matrix U. All of these tasks are considered computational. Question 2 (Figure 11) is an example of a conceptual question from MATH 204.

2. Determine the values of a for which the system has no solution, exactly 1 solution or infinitely many solutions:

$$\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - (a^2 - 3)z &= a \end{aligned}$$

Figure 11 - MATH 204 - Vectors and Matrices (Concordia), December 2014, question 2

A complete solution to this task requires using Gaussian elimination to row reduce the augmented matrix of the system, and then justifying *why* certain values of the parameter a cause the system have no solution, exactly one solution, or infinitely many solutions by demonstrating an understanding of the conditions in which each of these situations arises. I make the claim of what constitutes a complete solution based on my experience as a mathematics instructor.

There are more tasks of a conceptual nature on the MATH 251 exam. Only two of the ten questions feature matrices with defined sizes and entries, and half of the questions involve arbitrary vector spaces and linear transformations.

Problem 4. Let V be a vector space and let $T : V \to V$ and $U : V \to V$ be two linear transformations.

1. Show that $T + U$ is also a linear transformation.
2. Show that aT is a linear transformation for any scalar a.
3. Suppose that T is invertible. Show that T^{-1} is also a linear transformation.

Problem 5. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation

1. State the Dimension Theorem for T.
2. Show that T is not 1-1.
3. Give an example for which T is onto. Give an example for which T is not onto. (In each case show that your example has the required property. Do not just give an example with no explanation).

Problem 6. Let $T : V \to W$ be a linear map. Suppose that it is one-to-one. Suppose that $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is a linearly independent subset of V. Let $\underline{w}_1, \ldots, \underline{w}_k$ be the images $\underline{w}_1 = T\underline{v}_1, \ldots, \underline{w}_k = T\underline{v}_k$. Show that $\underline{w}_1, \ldots, \underline{w}_k$ are linearly independent.

Figure 12 - MATH 251 - Linear Algebra I (Concordia), December 2013, questions 4, 5, and 6

Referring to Figure 12, problem 4 defines V as simply “a vector space”, while T and U are said to be “two linear transformations” on vectors in the space V. Similarly, in problems 5 and 6, T is “a linear transformation", or simply "a linear map" between two arbitrary vector spaces. What these vector spaces and transformations actually are is left unspecified, because it is unimportant to the given task, which is to verify properties that could be attributed to any transformation on the given vector space(s). Solving these problems involves constructing a logical proof using the definitions and axioms of vector spaces and linear transformations.

The tasks in MATH 252 lean more towards being computational in nature, but they differ from those in MATH 204 in the techniques required to accomplish them. Since MATH 252 is a university level linear algebra course, it features concepts and topics that engineering students simply don't learn in MATH 204. For example, consider the three questions in Figure 13.

7. Let $A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & -2 & -2 \\ -4 & -2 & 1 \end{bmatrix}$. The characteristic polynomial of A is $-(t+3)^2(t-6)$. Find an orthogonal matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$ (that is $Q^tAQ = D$). Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

8. Let $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. Show that A is a normal matrix. Find a unitary matrix U and a diagonal matrix D such that $U^{-1}AU = D$ (that is $U^*AU = D$). Find an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of A.

9. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 1 \\ -1 & 0 & 2 \end{bmatrix}$. Find an invertible matrix Q and a Jordan matrix J such that $Q^{-1}AQ = J$. Find bases β_1 and β_2 for the generalized eigenspaces K_{λ_1} and K_{λ_2} of A, respectively, such that $\beta = \beta_1 \cup \beta_2$ is a Jordan basis for A.

Figure 13 - MATH 252 - Linear Algebra II (Concordia), April 2013, questions 7, 8, and 9

In each of these questions the task is essentially the same: diagonalise the given matrix. However, different techniques of diagonalisation are requested: eigenvectors in question 7, unitary matrices in question 8, and a Jordan matrix in question 9. Of those three techniques, only diagonalisation using eigenvectors is found in the knowledge to be taught in MATH 204, as illustrated by question 10 (Figure 14).

10. Let $A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Figure 14 - MATH 204 - Vectors and Matrices (Concordia), December 2014, question 10

Though eigenvectors aren't specifically mentioned, it is the only diagonalisation technique presented to the students in MATH 204.

The following comment could be made about one's perception of the mathematical objects themselves. To an engineer (student or professional), a 4×4 matrix with integer entries, such as the one shown in Figure 8, is an abstract mathematical object. It may represent any number of real world objects or phenomena, though what those objects are remains unspecified in the context of the exam. But to a mathematician, this same matrix is a specific element from the set of all $m \times n$ matrices with real coefficients – it is an object that makes sense on its own without the need to relate it to something from outside the structure of mathematics. For a mathematician, an abstract matrix would be similar to that which is described in question 9 of the MATH 251 exam (Figure 15).

Problem 9. Let V be $M_{2\times 2}(\mathbb{R})$ the vector space of 2×2 matrices with real entries. Let $T : V \to V$ be defined by $T(A) = A + A^t$.

1. Show that T is a linear transformation.
2. Is T diagonalizable? If so, find a basis β of V consisting of eigenvectors of T and $[T]_\beta$, the matrix of T with respect to the basis β.

Figure 15 - MATH 251 - Linear Algebra I (Concordia), December 2014, question 9

In this problem, even though the size of the matrix is defined as 2×2, its entries can be any real number, not just integers, and the matrix itself is nothing more than an object in a vector space upon which a linear transformation is being applied.

This reflects the research of Sierpinska (2000) who described the difference between an "arithmetic" mode of thinking in linear algebra and a "structural" mode of thinking. The focus in university level linear algebra gears the students towards structural thinking with tasks involving the generalization of properties of linear transformations and abstract vector spaces, as well as more advanced computational techniques. Engineering students are not required to learn such material; rather, they learn the rules of manipulating numerical examples of linear algebra objects in order to solve problems that they will encounter in their core engineering courses, and thus tend to focus on an arithmetic mode of thinking.

4.3.3 Pre-university calculus

Topics taught in pre-university calculus include a review of functions and their graphs, functional notation, limits and continuity, derivatives and techniques for differentiating elementary functions, applications of differentiation (optimization, related rates, approximation using differentials), antiderivatives and definite integrals, techniques of integration, applications of integration (calculating arc lengths, areas and volumes), sequences and series, and Taylor series and power series. The functions considered in these courses include exponential, logarithmic, and trigonometric functions, but all are functions of a single real variable. Complex functions and functions with complex coefficients and variables are not included in these courses, and multivariable functions are taught in university level calculus, which is discussed in the next section.

At Concordia and McGill, pre-university calculus is broken up into two parts: differential calculus, and integral calculus. The courses offered in this subject are MATH 203 – Calculus 1, and MATH 205 – Calculus 2, at Concordia, and MATH 140 – Differential and Integral Calculus I, and MATH 141 – Differential and Integral Calculus II, at McGill. As with the courses for pre-university linear algebra, I was unable to obtain documentation from the McGill courses.

In general, both engineering students and mathematics students should be enrolled in university level calculus courses at the outset of their programs, but as with linear algebra, these pre-university calculus courses are offered as pre-requisites for students who are lacking the required credits, and for students from out of province. Since these courses are not taken exclusively by engineering students, I should not expect application or modelling problems necessarily to be related to engineering.

The following tables show the relative frequencies of the different types of tasks found on the final exams for MATH 203 and MATH 205, both dated December 2014.

Table 5 - Relative frequency of tasks: MATH 203 - Calculus I (Concordia), December 2014

	Computational	Conceptual	
Mathematical	68%	29%	**97%**
Application	0%	3%	3%
Modelling	0%	0%	0%
	68%	32%	

Table 6 - Relative frequency of tasks: MATH 205 - Calculus II (Concordia), December 2014

	Computational	Conceptual	
Mathematical	63%	37%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	63%	37%	

In general, about two thirds of the tasks on the exams that I analyzed are computational, and include routine computations such as evaluating limits, and differentiating and integrating arbitrary functions. The only application problem that was found on either exam is part (b) of question 7 (Figure 16), and as expected it is not related to engineering, but is rather a demonstration of related rates using the length and width of a rectangle. It is the use of the units of length (cm) and time (s) that categorizes this problem as an application: it is a problem about quantities, not abstract numbers or functions only.

[17] **7.** **(a)** Verify that the point (3,1) belongs to the curve defined by the equation $y^3 + x^3 - 2x^2y^2 = 10$, and find an equation of the tangent line to the curve at that point.

(b) The length of a rectangle is increasing at the rate of 8 cm/s and its width is increasing at the rate of 5 cm/s. When the length is 20 cm and the width is 12 cm, how fast is the area of the rectangle increasing at that instant?

(c) Use l'Hôpital's rule to evaluate the $\lim\limits_{x\to 0} \dfrac{e^{2x} + e^{-2x} - 2}{x \sin x}$.

Figure 16 - MATH 203 - Calculus 1 (Concordia), December 2014, question 7

I consider this question to be slightly more conceptual as opposed to computational, since it requires some thought in deciding what technique to use in order to solve it. Students who understand the concept of related rates wouldn't have difficulty spotting that this problem fits the description, especially given the use of the units centimetres *per second*, but it still requires deducing that the equation to differentiate is $A(t) = x(t)y(t)$.

The exponential and logarithmic functions found in these exams are all defined using base e, and there is also ample use of trigonometric functions, which are prevalent in the mathematical models used by engineers. The use of these functions is best illustrated by question 4 from MATH 203 (Figure 17), and questions 3 and 4 from MATH 205 (Figure 18).

[15] **4.** Find the derivatives of the following functions (you don't need to simplify your final answer, but you must show how you calculate it):

(a) $f(x) = \arctan x + (x^{3/2} + 2x^{-1/2})\sqrt{x}$

(b) $f(x) = \ln \dfrac{x^3}{x+3}$

(c) $f(x) = \dfrac{e^{-x} \tan x}{1 + e^x}$

(d) $f(x) = \ln[e^{x \sin x} + x \sin(e^x)]$

(e) $f(x) = (1 + \cos x)^{x^2}$ (use logarithmic differentiation)

Figure 17 - MATH 203 - Calculus 1 (Concordia), December 2014, question 4

[15] **3.** Find the following indefinite integrals:

(a) $\int x \ln(x+2)\,dx$ (b) $\int \frac{x}{x^2-4x+3}dx$ (c) $\int x\left(1+\frac{1}{\sqrt{x}}\right)^2 dx$.

[12] **4.** Evaluate the following definite integrals (give the exact values, do **not** approximate):

(a) $\int_0^{\pi/4} \frac{\sec^2(x)}{4+\tan^2(x)}\,dx$ (b) $\int_0^{\pi/2} \cos^3(x)\,\sin^5(x)\,dx$

Figure 18 - MATH 205 - Calculus 2 (Concordia), December 2014, questions 3 and 4

The tasks in Figure 17 can be solved using various techniques of differentiation including the product rule, the quotient rule, the chain rule, or any combination thereof, as well as logarithmic differentiation for part (e). Part (d) in particular requires multiple uses of the chain rule concurrently with the product rule. For the integration problems shown in Figure 18, students must know how to use the techniques of substitution, integration by parts, partial fraction decomposition, and trigonometric substitution. All of these problems are computational.

It should be noted that both exams also feature bonus questions (Figure 19) that ask to verify a statement about the chain rule (from MATH 203) and the average value of any single variable function (from MATH 205).

[5] **Bonus Question.** If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, use the Chain rule to derive the following formula for the second derivative:

$$\frac{d^2y}{dx^2} = \frac{d^2f}{du^2}\left(\frac{dg}{dx}\right)^2 + \frac{df}{du}\frac{d^2g}{dx^2}$$

[5] **Bonus Question.** It is known that for some continuous even function, $f(x) = f(-x)$, the average value of $f(x)$ on any given interval $[-a, a]$ $(a > 0)$ is equal to the length of the interval. Is this information sufficient to find $f(x)$? Find the function f if it is, otherwise explain why it is insufficient.

Figure 19 - MATH 203 - Calculus 1 (Concordia) and MATH 205 - Calculus 2 (Concordia), bonus questions

These questions reflect the fact that these courses are administered by the mathematics department.

The concepts learned in these courses are required knowledge for university level calculus which is discussed in the next section.

4.3.4 University level calculus

In university level calculus, students are introduced to multivariable and vector-valued functions, are taught (according to course descriptions) how to perform differential and integral calculus on such functions (e.g., partial differentiation, gradients, curl, line and surface integrals), and also learn about Lagrange multipliers and the theorems of Gauss, Green and Stokes. Thus, another appropriate name for the subject would be multivariable calculus.

In some course descriptions, such as at École Polytechnique, we find topics such as limits and continuity, approximations using differentials, and optimization. These are included in a review section at the beginning of the course since, unlike at McGill and Concordia, those programs don't offer pre-university level calculus courses for credit. The programs at École Polytechnique, Université Laval, and UQAT also introduce complex numbers and their representations in the complex plane in their calculus course descriptions. Lastly, McGill also introduces differential equations in their calculus course, though the bulk of that subject is reserved for a course dealing specifically with that topic.

The courses offered in this subject are ENGR 233 – Applied Advanced Calculus at Concordia, while MATH 262 – Intermediate Calculus, and MATH 264 – Advanced Calculus, are offered at McGill. Despite the course code ENGR, the course at Concordia is taught by a mathematics professor. The same is true for the two courses at McGill. However, all of these courses are offered exclusively to engineering students, and it is common for such courses to be attended by students of different engineering departments (civil, mechanical, electrical, etc.) at the same time. Comparable courses for mathematics students at Concordia are MATH 264 – Advanced Calculus I, and MATH 265 – Advanced Calculus II.

Besides listing the topics of study, the course outline for ENGR 233 also mentions that the "ability to identify, formulate and solve engineering problems", a competency from the List of Design Soft Skill

Competencies found in documentation by the CEAB, is relevant to the course. Analyzing the course's final exam should reveal whether this competency is in fact expected to develop in students.

Since there is only one calculus course for engineers at Concordia, some topics that are to be taught in the courses at McGill are not found in Concordia's ENGR 233 course outline. These include series and power series which are taught at the beginning of McGill's MATH 262, and the introduction to partial differential equations at the end of McGill's MATH 264. In essence, infinite series is a topic that students in ENGR 233 are expected to have learned prior to attending the course, and partial differential equations are taught to them in a separate differential equations course (section 4.3.5). Regardless, there is still considerable overlap in the topics on the course outlines for all of the courses offered to engineers and mathematics students at Concordia and McGill. Among them are vector geometry and vector-valued functions, differential and integral calculus of vector-valued functions and multivariable functions, line integrals and the theorems of Green and Stokes.

The following tables show the relative frequencies of the different types of tasks found on the final exams from the following courses: ENGR 233, dated December 2014; MATH 262, dated December 2010; MATH 264 for engineers at McGill, dated April 2007; MATH 264 for mathematics students at Concordia, dated December 2014; MATH 265, dated April 2014.

Table 7 - Relative frequency of tasks: ENGR 233 - Applied Advanced Calculus (Concordia), December 2014

	Computational	Conceptual	
Mathematical	78%	0%	**78%**
Application	11%	0%	11%
Modelling	0%	11%	11%
	89%	11%	

Table 8 - Relative frequency of tasks: MATH 262 - Intermediate Calculus (McGill), December 2010

	Computational	Conceptual	
Mathematical	69%	31%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	69%	31%	

Table 9 - Relative frequency of tasks: MATH 264 - Advanced Calculus (McGill), April 2007

	Computational	Conceptual	
Mathematical	100%	0%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	100%	0%	

Table 10 - Relative frequency of tasks: MATH 264 - Advanced Calculus I (Concordia), December 2014

	Computational	Conceptual	
Mathematical	65%	20%	**85%**
Application	15%	0%	15%
Modelling	0%	0%	0%
	80%	20%	

Table 11 - Relative frequency of tasks: MATH 265 - Advanced Calculus II (Concordia), April 2014

	Computational	Conceptual	
Mathematical	87.5%	12.5%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	87.5%	12.5%	

As seen in the tables, an overwhelming majority of the tasks on all of the exams are computational in nature, even in the courses taken by mathematics students. Examples of conceptual tasks that I did find are shown in Figure 20 and Figure 21.

6. (10 marks) If $u = f(x,y)$ where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right].$$

Figure 20 - MATH 262 - Intermediate Calculus (McGIll), December 2010, question 6

2. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ with $x(t) = t, y(t) = 1/t, z(t) = 0$ for $t > 0$. Compute $\kappa(t)$. Show that $\kappa(t) \to 0$ as $t \to 0, \infty$. Find the maximum value of $\kappa(t)$.

Figure 21 - MATH 264 - Advanced Calculus I (Concordia), December 2014, question 2

Question 6 from MATH 262 (Figure 20) does require some computation, but the task is not solved until the results of the computations are used to show that the given equality is true. Question 2 from MATH 264 for mathematics students (Figure 21) contains three tasks, two of which are computational. The conceptual task involves showing that $\kappa(t) \to 0$ as t approaches both 0 and ∞, which can be accomplished by evaluating the limit of $\kappa(t)$. It is worth noting that the task of computing $\kappa(t)$ is somewhat simplified since the formula for its evaluation was given at the top of the exam question sheet (Figure 22).

Formulaire

For a curve $\mathbf{r}(t)$ in $\mathbf{R}^3$, Arclength $s(t) = \int_a^t |\mathbf{r}'(u)| du, \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}, \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}, \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t), \kappa = |\frac{d\mathbf{T}}{ds}| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

Figure 22 - MATH 264 - Advanced Calculus I (Concordia), December 2014, formulas

Despite being reserved for engineering students, only one task on all the engineering courses' finals is an application, while one other is a modelling task. Both appear on the final for ENGR 233 at Concordia, though neither are related to the field of engineering. Question 3 (Figure 23) refers to a quarterback throwing a football, while question 4 refers to an insect on a heated metal plate.

3. A quarterback throws a football with the initial speed of $30m/s$ at an angle of $60°$ from the horizontal. Find the range of the football.

4. The temperature at a point (x, y) on a rectangular metal plate is given by $T(x, y) = 5+2x^2+y^2$. Determine the direction an insect should take, starting at (4, 2) in order to cool off as rapidly as possible.

Figure 23 - ENGR 233 - Applied Advanced Calculus (Concordia), December 2014, questions 3 and 4

To solve question 4, the student has to recognize that in order for the insect to "cool off as rapidly as possible" it would have to follow the path of the greatest decrease in temperature on the metal plate. Mathematically this is in the direction of the greatest decrease in directional derivative of the given function. This task was labelled as a conceptual due to the additional thought required in translating the problem from the English language into a mathematical concept. This task also requires creating a mathematical model. Recall that a mathematical model is a mapping of a real world concept into objects in the mathematical domain. In this problem, the direction in which the insect walks has to be *modelled* as a vector in the direction of the greatest decrease of the given function.

A similar question appears on the final exam from MATH 264 for mathematics students (Figure 24), asking for the direction of greatest decrease of a function said to represent the temperature at any given point in space. In this problem, the students are asked explicitly for the direction of greatest decrease of temperature, but this direction does not represent the trajectory or path taken by some physical object, such as an insect. As such it is not a modelling problem.

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4. Suppose that temperature at every point (x, y) is given by

$$T(x, y) = 10e^{-x^2-y^2}.$$

(a) Find the direction in which the temperature is *decreasing* most rapidly at the point $(1, -2)$, and give the rate of change in this direction.

(b) Find all directions in which the temperature is *not changing* at the point $(1, -2)$. answer.

Figure 24 - MATH 264 - Advanced Calculus I (Concordia), December 2014, question 4

The technique required for both questions is the same, and no additional justification or explanation are required of the mathematics students that aren't also expected of the engineering students. These two questions could have appeared on the exam for the other course without seeming out of place.

Two questions on the final exam for MATH 264 at McGill are tenuous applications at best (Figure 25). The given differential equations are said to be heat equations, but the task involves simply solving the equations using the requested techniques, and the solution is not dependent on the real world context of the equations themselves. Thus I labelled these tasks as mathematical.

Problem 5. Use separation of variables to solve the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= 3\frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad 0 < t < \infty, \\ u(0,t) &= u(\pi,t) = 0, && 0 < t < \infty, \\ u(x,0) &= \sin(x) - 6\sin(4x), && 0 < x < \pi. \end{aligned}$$

Problem 6. Use Fourier series to solve the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= 7\frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad 0 < t < \infty, \\ \frac{\partial u}{\partial x}(0,t) &= \frac{\partial u}{\partial x}(\pi,t) = 0, && 0 < t < \infty, \\ u(x,0) &= 1 - \sin(x), && 0 < x < \pi. \end{aligned}$$

Figure 25 - MATH 264 - Advanced Calculus (McGill), April 2007, questions 5 and 6

It is worth noting that the notation used on the exams given to engineering students is typical for courses that are taught by mathematics professors: functions are labelled f, g, or h, and all of their arguments are either x, y, z, or t. This is important because in engineering courses these symbols take on different meanings. Figure 26 shows a standard list of symbols that one would find in an engineering textbook.

E	Modulus of elasticity	u	Strain-energy density
f	Frequency; function	U	Strain energy; work
F	Force	v	Velocity
$F.S.$	Factor of safety	V	Shearing force
G	Modulus of rigidity; shear modulus	V	Volume; shear
h	Distance; height	w	Width; distance; load per unit length
H	Force	W, W	Weight, load
H, J, K	Points	x, y, z	Rectangular coordinates; distance; displacements; deflections
$I, I_x, \ldots$	Moment of inertia	$\bar{x}, \bar{y}, \bar{z}$	Coordinates of centroid
$I_{xy}, \ldots$	Product of inertia	Z	Plastic section modulus
J	Polar moment of inertia	α, β, γ	Angles
k	Spring constant; shape factor; bulk modulus; constant	α	Coefficient of thermal expansion; influence coefficient
K	Stress concentration factor; torsional spring constant	γ	Shearing strain; specific weight
l	Length; span	δ	Deformation; displacement
L	Length; span	ϵ	Normal strain

Figure 26 - List of symbols (Beer & Johnston, 1992, p. xvii)

This particular list shows that f is used to denote the value of a frequency as well as functions, h represents either a distance or the height of an object, the symbols x, y, and z, usually associated with arbitrary variables in mathematics, are used to denote physical distances and deformations (deflections) as well as rectangular coordinates, and t is used to represent a measure of an object's thickness. In many contexts t is also used to represent time. For an engineer outside of a mathematics class, each of these symbols is associated with a physical meaning in the real world.

Another similarity worth highlighting is questions that specify the use of Green's theorem and Stokes' theorem in the evaluation of integrals, and the use of i, j, and k in the notation for vectors in $\mathbb{R}^3$ (refer to problem 8 in Figure 27).

7. Use Green's theorem to to compute $\oint_C (x+y^2)\,dx + (2x^2-y)\,dy$ where C is the boundary of the region bounded by $x=0$, $x^2+y^2=1$, $x>0$. The integral is taken in counterclockwise direction.

8. Use Stokes' theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (x+2z)\mathbf{i} + (3x+y)\mathbf{j} + (2y-z)\mathbf{k}$ where C is the curve of intersection of the plane $x+2y+z=4$ with the coordinate planes, oriented counterclockwise if viewed from above.

Figure 27 - ENGR 233 - Applied Advanced Calculus (Concordia), December 2014, questions 7 and 8

Problem 3. Using Stokes' Theorem, compute the integral $\int_C (x^2-yz)dx + (2x+y^2-xz)dy + (z^2-xy)dz$, where C is the curve formed by the intersection of the sphere $x^2+y^2+z^2=25$ and the plane $z=4$, oriented counterclockwise (e.g. its projection into the (x,y)-plane is oriented counterclockwise).

Figure 28 - MATH 264 - Advanced Calculus (McGill), April 2007, question 3

[10] **5.** Use Green's theorem to evaluate

$$\int_C x^2 y dx - xy^2 dy,$$

where C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

[10] **6.** Use Stokes' theorem to evaluate

$$\int\int_s \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where

$$\mathbf{F}(x, y, z) = x^2yz\ \mathbf{i} + yz^2\ \mathbf{j} + z^3e^{xy}\ \mathbf{k}$$

and S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, oriented upward.

Figure 29 - MATH 265 - Advanced Calculus II (Concordia), December 2014, questions 5 and 6

Questions 7 and 8 in Figure 27 and problem 3 in Figure 28 are from engineering exams, while questions 5 and 6 in Figure 29 are from a mathematics exam. There is little difference in the mathematical content or the nature of the tasks in the questions. Both test students' memory of certain theorems and their ability to apply them to compute the values of given mathematical expressions, but they do not test their ability to choose a convenient mathematical technique or property to compute its values, which is what engineers often have to do in practice. This appears to be counter to the claim that is made in the ENGR 233 course outline about the competencies that are developed in the engineering student, particularly being able to identify engineering problems and the knowing how to solve them.

4.3.5 Differential equations

The knowledge to be taught in differential equations courses includes how to solve different types of ordinary (ODE) and partial (PDE) differential equations including separable equations, exact equations, homogeneous equations, second order and higher order linear equations with constant and undetermined coefficients, as well as systems of differential equations.

Applications of ODE's and their use in mathematical models related to mechanics and electrical circuits are mentioned in the course descriptions in some programs, such as the one at École de technologie supérieure (ÉTS), but the construction of mathematical models from given data is not. Engineering applications of ODE's include orthogonal trajectories, harmonic motion, and free and forced oscillations, while applications involving PDE's and boundary value problems include heat equations and problems of heat transfer, wave equations, and vibrations. The program at Université Laval is the only one whose curriculum mentions Sturm-Liouville theory and related problems, while Concordia has the only course that includes the study of eigenvalues and eigenvectors of linear systems of differential equations.

The courses offered in this subject are ENGR 213 – Applied Ordinary Differential Equations, and ENGR 311 – Transform Calculus and Partial Differential Equations, at Concordia; MATH 263 – Ordinary Differential Equations for Engineers, is offered at McGill. While engineering students at Concordia are required to take two courses including one in partial differential equations, students at McGill are only introduced to the topic of partial differential equations in their advanced calculus course, MATH 264.

As with the university level calculus courses, only students in an engineering program can enrol in these differential equations courses. However, these courses are multi-departmental; i.e., they are taken by students in different engineering departments. This could add to the difficulty of writing application problems that are relatable to all of the students in the class.

The comparable course for mathematics students at Concordia is MATH 370 – Ordinary Differential Equations.

The following tables show the relative frequencies of the different types of tasks found on the various final exams: ENGR 213, dated December 2014; ENGR 311, dated August 2009; MATH 263, dated December 2012; MATH 370, dated December 2014.

Table 12 - Relative frequency of tasks: ENGR 213 - Applied Ordinary Differential Equations (Concordia), December 2014

	Computational	Conceptual	
Mathematical	91%	0%	**91%**
Application	9%	0%	9%
Modelling	0%	0%	0%
	100%	0%	

Table 13 - Relative frequency of tasks: ENGR 311 - Transform Calculus and Partial Differential Equations (Concordia), August 2009

	Computational	Conceptual	
Mathematical	75%	12.5%	**87.5%**
Application	0%	12.5%	12.5%
Modelling	0%	0%	0%
	75%	25%	

Table 14 - Relative frequency of tasks: MATH 263 - Ordinary Differential Equations for Engineers (McGill), December 2012

	Computational	Conceptual	
Mathematical	67%	33%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	67%	33%	

Table 15 - Relative frequency of tasks: MATH 370 - Ordinary Differential Equations (Concordia), December 2014

	Computational	Conceptual	
Mathematical	69%	31%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	69%	31%	

A majority of the tasks on all of the exams are computational, but the students in the courses ENGR 311 and MATH 263 are given about as many conceptual tasks as mathematics students. Question 7(a) from MATH 263 is one example of such a conceptual (Figure 30). It involves finding the series solution for an arbitrary and undefined function $y(x)$.

7.) (a) State the general form of a series solution for $y(x)$ expanding about the point $x = 0$ and satisfying the initial conditions $y(0) = y'(0) = 0$.

Figure 30 - MATH 263 - Ordinary Differential Equations for Engineers (McGill), December 2012, question 7

Question 8(a) from the same exam (Figure 31) is also conceptual, since the task requires justifying a given statement using the definition of a regular singular point of a differential equation.

8.) (a) Show that $x = 0$ is a regular singular point of the equation

$$2x^2y'' - x(1+x)y' + y = 0.$$

Figure 31 - MATH 263 - Ordinary Differential Equations for Engineers (McGill), December 2012, question 8

The exam for ENGR 311 features a task that is conceptual but also set in the context of the real world, making it an application (Figure 32). The first part of the question is a computational task since it involves simply solving the differential equation.

4. a) Solve the following Heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - 3U$$

With the conditions

$$\frac{\partial U}{\partial x}(0,t) = \frac{\partial U}{\partial x}(1,t) = 0$$
$$U(x,0) = x(1-x)$$

b) What is the steady-state temperature?

Figure 32 - ENGR 311 - Transform Calculus and Partial Differential Equations (Concordia), August 2009, question 4

Part (b) however involves a bit more thought since it requires knowing the definition of steady-state temperature, and interpreting the result of the solution to the differential equation.

Examples of conceptual tasks on the exam for MATH 370 are seen in Figure 33. The first part of each of questions 3 and 4 asks the student to check whether the given equation is exact or not. This requires knowing the definition of an exact equation, and justifying that the given equation matches that definition. The second part of each question is considered a computational task.

3. Check whether the following equation is exact and if it is exact then solve it.

$$(2xy^3 + \frac{1}{y})dx + (3x^2y^2 - \frac{x}{y^2})dy = 0$$

4. Check whether the following equation is exact and if it is not exact then solve it, using an integrating factor $\mu(y)$.

$$(2\frac{x}{y} + 1)dx + \frac{x}{y}dy = 0$$

Figure 33 - MATH 370 - Ordinary Differential Equations (Concordia), December 2014, questions 3 and 4

As with the calculus course ENGR 233, the course outline for ENGR 213 lists the CEAB competencies that the course supposedly emphasizes and aims to develop in engineering students. These include the ability to use appropriate knowledge and skills to identify, formulate, analyze, and solve complex engineering problems in order to reach substantiated conclusions. Despite these assertions, nearly all of the tasks on the ENGR 213 final exam explicitly state which technique to use to solve the given differential equation, as seen in questions 3, 6, and 7 (Figure 34).

3. Using the integrating factor method find the general solution of the following differential equation:

$$x\frac{dy}{dx} - 4y - x^6e^x = 0$$

6. Find the general solution of the equation

$$y'' + 3y' + 4y = 3x + 2$$

by the method of undetermined coefficients.

7. Find the general solution of the equation

$$y'' - 4y = \frac{e^{2x}}{x}$$

by the method of variation of parameters.

Figure 34 - ENGR 213 - Applied Ordinary Differential Equations (Concordia), December 2014, questions 3, 6, and 7

These questions are all computational in nature. Question 3 asks the student to use the integrating factor method, while questions 6 and 7 state that the general solution is to be found using the methods of undetermined coefficients and variation of parameters, respectively. While these questions test the students' proficiency in each of the specified techniques, they do not test their abilities to identify which technique should be used to solve the problem. Similar problems on the final exam for MATH 263 at McGill, also computational in nature, do not explicitly state the method to be used, as it is to be inferred from the equation itself (Figure 35).

3.) Solve the initial value problem

$$(3y^6 - 3)dx + (6xy^5 + 2x^{-2}y)dy = 0, \qquad y(1) = 0.$$

You may leave your answer in implicit form.

4.) Find the general solution of

$$y'' - y = xe^x + e^{-x}.$$

5.) Find the general solution of

$$x^2y'' - 5xy' + 8y = 5x^4 \ln x.$$

6.) Solve the initial value problem

$$y^{(4)} - y = x^5, \qquad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

Figure 35 - MATH 263 - Ordinary Differential Equations for Engineers (McGill), December 2012, question 3 through 6

These questions would be a better test of an engineering student's ability to identify what type of problem they are being asked to solve.

Of the ten questions on the final exam for ENGR 213, only question 5 is an application (Figure 36). It is a problem in which a given second order differential equation is said to represent the motion of a mass and spring system.

5. The equation describing the motion of the mass-spring system is

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

where $k = 1N/m$ and $m = 1kg$. Find the position y of mass at an arbitrary time t if the initial position of the mass is $1m$ and the initial velocity is 0.

Figure 36 - ENGR 213 - Applied Ordinary Differential Equations (Concordia), December 2014, question 5

The equation is relatively simple to solve however since the spring constant, k, and the mass, m, in the problem are set equal to 1 N/m and 1 kg, respectively. Even though spring-mass systems are pertinent examples of how motion can be modelled using differential equations, the exams for the engineering courses are clearly lacking in application problems.

This would not be considered a modelling problem because all of the physical objects described in the problem have already been mapped into the mathematical domain. The position of the mass is already labelled as y, and this position is a function of time, t. Furthermore, no assumptions need to be made about the initial position and velocity of the mass since they are explicitly stated in the problem as being $1\ m$ and $0\ m/s$, respectively.

The overall similarity between the content of the exams for engineering and mathematics students is perhaps best illustrated by the following two questions. Figure 37 is taken from the exam for ENGR 213, while Figure 38 is from the exam for MATH 370 for mathematics students. The tasks are simply stated: find the general solution(s) of the following equation(s).

4. Find general solutions of the following equations:

(a) $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

(a) $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

Figure 37 - ENGR 213 - Applied Ordinary Differential Equations (Concordia), December 2014, question 4

5. Find the general solution of the equation

$$y^{(4)} - 2y'' + y = x^2 - 4$$

Figure 38 - MATH 370 - Ordinary Differential Equations (Concordia), December 2014, question 5

The only difference of note in these two problems is the different notations: Leibnizian (dy/dx) in the ENGR question, and Newtonian (y') in the MATH question. These notations aren't exclusive to either field though. The questions in Figure 30, Figure 34, and Figure 35 are all from engineering exams and use Newtonian notation, while the questions in Figure 33 are from a mathematics exam and use Leibnizian notation. A cursory glance at the required textbook by Boyce and DiPrima (1997) for the course MATH

263 at McGill reveals that the problems use Newtonian notation in a ratio of about 2:1 relative to Leibnizian notation.

All of these similarities may very well be a result of the fact that the differential equations courses for engineers are taught by mathematics professors, and thus there is a tendency to focus on the computational techniques of solving differential equations without the need to attach the equations to a physical situation of phenomenon. Engineering students do eventually see the practicality of differential equations in their engineering core courses, where they learn how the physical world can be modeled as required by their disciplines. Examples of this will be shown in Chapter 5.

4.3.6 Probability and statistics

Topics to be taught in probability and statistics courses include the axioms and concepts of probability theory and descriptive statistics, probability models and density functions of discrete and continuous random variables, statistical estimation with confidence intervals and hypothesis testing, linear regression and correlation, and statistical sampling and its use in quality control.

The courses offered in this subject are ENGR 371 – Probability and Statistics in Engineering, at Concordia, and CIVE 302 – Probabilistic Systems, at McGill. Both of these courses are administered by their respective civil engineering departments and taught by engineering professors. In the McGill course calendar, CIVE 302 is described as “an introduction to probability and statistics with applications to civil engineering design”, and as will be seen shortly, this is in fact reflected in the content of the course’s final exam. Comparable courses for mathematics students at Concordia are STAT 249 – Probability I, and STAT 250 – Statistics.

The following tables show the relative frequencies of the different types of tasks found on the various final exams: ENGR 371, dated April 2013; CIVE 302, dated April 2006; STAT 249, dated December 2011; STAT 250, dated December 2013.

Table 16 - Relative frequency of tasks: ENGR 371 - Probability and Statistics in Engineering (Concordia), April 2013

	Computational	Conceptual	
Mathematical	16%	4%	20%
Application	56%	24%	**80%**
Modelling	0%	0%	0%
	72%	28%	

Table 17 - Relative frequency of tasks: CIVE 302 - Probabilistic Systems (McGill), April 2006

	Computational	Conceptual	
Mathematical	12.5%	0%	12.5%
Application	75%	12.5%	**87.5%**
Modelling	0%	0%	0%
	87.5%	12.5%	

Table 18 - Relative frequency of tasks: STAT 249 - Probability I (Concordia), December 2011

	Computational	Conceptual	
Mathematical	39%	11%	**50%**
Application	50%	0%	**50%**
Modelling	0%	0%	0%
	89%	11%	

Table 19 - Relative frequency of tasks: STAT 250 - Statistics (Concordia), December 2013

	Computational	Conceptual	
Mathematical	42%	23%	**65%**
Application	12%	23%	35%
Modelling	0%	0%	0%
	54%	**46%**	

Once again a majority of the tasks are computational in nature. However, the engineering exams contain a significant number of applications, and considerably more than on the math exams (80% and 87.5% versus 50% and 35%). Moreover, the applications on the engineering exams are related to their field, particularly on the McGill exam.

Yet there is still one task that is both conceptual and mathematical in its nature and content on the exam for ENGR 371. Question 3(a) (Figure 39) asks the students to determine the value of a parameter in the function defining a probability density function.

3. Consider a continuous random variable X with the following pdf

$$f(x) = \begin{cases} mx & 2 < x < 3 \\ 0 & otherwise \end{cases}$$

a. Determine m. (3 marks)
b. Find the expected value of X. (3 marks)
c. Find the variance of X. (3 marks)

Figure 39 - ENGR 371 - Probability and Statistics in Engineering (Concordia), April 2013, question 3

Solving this task requires knowing that the area between the graph of a continuous random variable's probability density function and the x-axis must be 1. The value for m could thus be determined by solving the equation:

$$\int_2^3 m\, x\, dx = 1$$

Once the value of m is known, the other two tasks can be solved as well. The expected value, μ, and the variance, $V(x)$, can be found by evaluating the definite integrals:

$$\mu = \int_2^3 x(mx)dx$$

$$V(x) = \int_2^3 (x - \mu)^2 (mx)dx$$

The tasks in parts (b) and (c) are computational, not conceptual, and none of the tasks in this question are considered applications. Question 4 from the same exam (Figure 40) is also mathematical in its content, though it is entirely computational.

4. Let X be an exponential random variable with mean 4.

   a. Calculate the probability that Y>3. (3 marks)
   b. Find r such that P(X>r) =0.5. (3 marks)

Figure 40 - ENGR 371 - Probability and Statistics in Engineering (Concordia), April 2013, question 4

These tasks are easily solvable using basic antidifferentiation techniques. Since the problem states that the variable X has an exponential probability density function with an expected value of 4, part (a) can be solved by evaluating the following integral:

$$P(X > 3) = \int_{3}^{\infty} \frac{1}{4} e^{-\frac{1}{4}x} dx = \lim_{t \to \infty} \int_{3}^{t} \frac{1}{4} e^{-\frac{1}{4}x} dx$$

To answer part (b), a similar integral is used, but the following equation must be solved for the unknown value r:

$$\lim_{t \to \infty} \int_{r}^{t} \frac{1}{4} e^{-\frac{1}{4}x} dx = 0.5$$

It is expected of the students to know the definitions and mathematical formulae for the expected value and variance of several different kinds of probability distributions, including exponential distributions, since they are not provided on the exam’s question sheet.

Of the six questions on the ENGR 371 exam that are applications, two are framed in the context of engineering problems, while the remaining problems involve situations from commerce, school, medicine, and sports. Of the two engineering applications, one is merely a combinatorics problem where the objects being counted are professional engineers. Only question 7 (Figure 41) immerses the student in a situation where a decision needs to be made about the validity of an engineering product claim using statistical analysis.

7. Your employer develops energy efficient solutions for manufacturing sites. One of the key elements of these solutions is proper insulation. Your team leader has asked you to test a new product line, if the product line has a mean thermal insulation (TI) of at least 25 it will be used.

You receive 25 samples of insulating material and test the thermal isolation coefficient. Based on the testing you determine that the sample mean TI is 26.5. The TI is normally distributed with a standard deviation of 2. Show all of your work and state any reasonable assumptions.

a) Using a statistical test with a significance level of: ($\alpha = 0.05$) and based on the results given above, should this product line be used? (8 Marks)
b) If the true population mean was 26, determine β (type II error) with the same information given above. (8 Marks)
c) If the cost of using the product when it should have been rejected is $300,000 (due to recalls and refits), and the cost of not using the product when it should have been accepted is $100,000 (due to delays), determine the probability of having to pay each of the above costs and which is more likely to occur. (2 Marks)
d) How could you improve (reduce) the type I and type II errors? (2 Marks)

Figure 41 - ENGR 371 - Probability and Statistics in Engineering (Concordia), April 2013, question 7

Parts (a) and (d) of this problem are conceptual tasks that demand the students justify their decision using the results of their statistical analysis, as well as an explanation of how to reduce type I and type II errors in their analysis.

Similar questions are found on the final exam for CIVE 302, thus fulfilling the course's description as "an introduction to probability and statistics with applications to Civil Engineering design." This exam contains by far the largest quantity of engineering applications that I have found in my research. The most representative of these are problems 3 (Figure 42)and 6 (Figure 43) in which confidence intervals are to be established in order to make important decisions: which company to choose as a provider of construction materials, and whether or not the capacity of a batch of load bearing piles meets the minimum requirement based on nine test piles.

PROBLEM 03: **(10 marks)**

Two companies manufacture a composite material used as exterior cladding for buildings. Twenty-five samples from each company are tested in an abrasion test, and the amount of wear after 1000 cycles is observed. For company 1, the sample mean and standard deviation of wear are $\overline{X_1} = 20$ milligrams/1000 cycles and $s_1 = 2$ milligrams/1000 cycles, while for company 2 we obtain $\overline{X_2} = 15$ milligrams/1000 cycles and $s_2 = 8$ milligrams/1000 cycles.

a)- Does the data support the claim that the two companies produce material with different mean wear? Use $\alpha = 0.05$, and assume each population is normally distributed but their **variances are not equal**.

b)- What is the P-value of this test?

c)- Do the data support a claim that the material from company 1 has higher mean wear than the material from company 2? Use the same assumptions as in part (a).

Figure 42 - CIVE 302 - Probabilistic Systems (McGill), April 2006, question 3

PROBLEM 06: **(10 marks)**

The foundation for a building is designed with 100 piles based on an individual pile capacity of 80 tons. Nine test piles were driven at random locations into the supporting soil stratum and loaded until failure. The results are as follows:

Test Pile No.	Pile Capacity (tons)
1	82
2	75
3	95
4	90
5	88
6	92
7	78
8	85
9	80

The sample mean and standard deviation for pile capacity are respectively 85 tons and 6.76 tons, respectively.

a)- At the 5% significance level, perform a one-sided hypothesis test with the null hypothesis that the mean pile capacity is 80 tons.

b)- Establish the 98% confidence interval for the mean pile capacity, assuming that the standard deviation of the population is known (assume $\sigma = s$, where s is the sample variance).

c)- Determine the 98% confidence interval for the mean pile capacity assuming that the standard deviation of the population is unknown.

Figure 43 - CIVE 302 - Probabilistic Systems (McGill), April 2006, question 6

Note that in both of these problems, the real world concepts of material durability (figure 42) and pile capacity (figure 43) have already been modeled as statistical variables. In the former, the durability is measured by the amount of wear in the material, and labelled as X_1 and X_2 for the different materials. Labels aren't given explicitly in the latter problem, but by giving the student the sample mean, standard deviation, and explicitly stating the null hypothesis (in part (a)), the work of modelling the situation has already been done.

Some parts of these questions are computational, but those that involve making a decision and then justifying that decision are conceptual in nature. This type of statistical analysis is frequently performed by professional engineers involved in the construction of structures that are built on foundation support piles. Since it is too costly to test the strength of each individual pile, random tests are performed and the results analyzed statistically to determine whether or not there are any structural deficiencies.

The exams for mathematics students are decidedly more mathematical in their content than their engineering counterparts, but there are still a considerable number of applications. This may very well be due to the nature of the topics of probability and statistics themselves, and their historical development.

Indeed, the applications on the mathematics students' exams are set in a variety of real world situations including forming a committee from a group of men and women, the distribution of grades in professors' courses, the tossing of an unfair coin, the distribution of certain types of rare birds, and accidents occurring on a stretch of road. Three of these questions are shown in Figure 44. It is worth noting that these questions are all entirely computational in their nature.

3. Two professors taught a course. In the class of Professor X 10% of the students received an A grade for the course, and in the class of Professor Y 2% of the students received an A grade for the course. Of the students taking the course, 60% were in the class of Professor X and 40% were in the class of Professor Y. A randomly selected student did not receive an A grade, what is the probability that this student was in the class of Professor X?

[6 marks]

4. Suppose a coin with probability $p = 0.7$ of landing heads is tossed continually *until 2 heads* are obtained. Find the probability that

(a) the coin is tossed exactly 4 times;

(b) the coin is tossed 4 times or less. **[4 + 6 marks]**

5. The number of a certain type of rare bird seen each day from an observatory follows a Poisson distribution with mean 1. A particular observer looks each day and records the number of this type of birds he sees. However, to save time, if he sees 3 birds he records the number as 3 and makes no more observations that day. So the maximum number he records is 3, even if more birds arrive later in the day. If Y is the random variable which is the number of birds recorded by this observer, what is the mean and variance of Y?

[10 marks]

Figure 44 - STAT 249 - Probability I (Concordia), December 2011, questions 3, 4, and 5

Only two tasks on the exam for STAT 249 are conceptual, and one of them is nearly identical to the conceptual task given to students in ENGR 371, as it involves finding the value of a parameter in the joint probability density function of two random variables, X and Y (Figure 45).

9. Suppose (X, Y) have the joint density given by

$$f(x,y) = \begin{cases} \frac{x}{5} + ky, & 0 < x < 1 \text{ and } 1 < y < 5, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k.

(b) Calculate $P(X \geq \frac{1}{2} \mid Y = 2)$. **[5+5 marks]**

Figure 45 - STAT 249 - Probability I (Concordia), December 2011, question 9

The technique for solving this problem is similar to that taught to engineering students, except that engineers don't learn about join density functions of two variables.

The exam for STAT 250 is both more conceptual in its nature, and features fewer applications than the other three exams discussed thus far. Questions 6 and 7 (Figure 46) are posed in such a way that complete solutions to the given tasks require written explanations to be included in the students' answers.

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6. Let $Y_1, \ldots, Y_n$ be a sample from density $f(y) = \frac{1}{\beta}e^{-\frac{y}{\beta}}, y \geq 0$. Consider the estimator $\widehat{\beta} = n \min(Y_1, \ldots, Y_n)$.

a. (8 pts) Is $\widehat{\beta}$ an unbiased estimator of β? (*Justify your answer with a calculation*).

b. (4 pts) Is $\widehat{\beta}$ a consistent estimator of β? (*Justify your answer with a calculation*).

d. (4 pts) Is $\widehat{\beta}$ more efficient than the estimator $\overline{Y}$? (*Justify your answer with a comparison*).

---

7. Medical researchers were concerned that the new drug causes more variation than usually observed in patients' blood pressures. They ran a second study with in which they particularly focused on the sample variances of the two groups of measurements. In the second study they used a group of 14 patients taking the placebo and observed sample variance $s_1 = 12.7$, and a group of 10 patients taking the new drug and observed a sample variance of $s_2 = 26.4$

a. (2 pts) What is the null hypothesis, and what is the appropriate alternative hypothesis?

b. (2 pts) What statistic do you need to use for the hypothesis test, and what distribution does it have?

a. (6 pts) If we are allowing a type I error of 0.05, would you reject the null hypothesis based on the above data?

Figure 46 - STAT 250 - Statistics (Concordia), December 2013, questions 6 and 7

I also found in this exam topics and concepts that engineering students simply do not learn. As with the concept of joint density functions shown in the exam for STAT 249, question 1 of the exam for STAT 250 (Figure 47) asks the students to evaluate marginal and conditional densities of two random variables.

---

1. Let Y_1, Y_2 be random variables with joint density function

$$f(y_1, y_2) = \frac{1}{8}y_1 e^{-(y_1+y_2)/2} \quad \text{for } y_1 > 0, y_2 > 0$$
$$0 \quad \text{otherwise}$$

a. (4 pts) Calculate $P(Y_1 > 1, Y_2 > 1)$.

b. (4 pts) Find the marginal densities of Y_1 and of Y_2.

b. (4 pts) Find the conditional density of $Y_1|Y_2 = y_2$.

c. (2 pts) Are Y_1 and Y_2 independent? (*Justify your answer with an explanation*).

Figure 47 - STAT 250 - Statistics (Concordia), December 2013, question 1

Each task is mathematical in its content since the random variables are not said to represent any real world objects. While the first three tasks are computational, the fourth (mislabelled on the question sheet as part (c)) asks for a justification in the students' answers, making it a conceptual task.

Of the seven mathematical subjects in my research, probability and statistics is the only one in which the relative frequency of application problems on the final exams was greater than those that were purely mathematical.

4.3.7 Numerical methods

Topics to be taught in numerical methods include techniques for finding the roots of equations such as Newton's method, the secant method, and other iterative techniques, the approximation and interpolation of functions using linear and polynomial splines, numerical differentiation and integration of functions, solving differential equations by various techniques including Euler's method and the Runge-Kutta method, and finding solutions to initial-value and boundary value problems.

Though not explicitly identified in the curricula or course descriptions in every engineering program, an important feature in working with numerical methods is being able to evaluate the error that is inherent in the use of the various techniques. The inclusion of this topic arises from the fact that mathematical models are not always comprised of equations or functions with integer or rational coefficients or arguments; real numbers are more prevalent and errors due to the rounding of values can accumulate substantially. Computers are also more readily used, and while they are efficient at performing algorithms, they are still necessarily limited in their ability to represent and store real numbers. As a result, errors due to rounding or truncation can appear even when computers are used to solve problems (Gilat & Subramaniam, 2008). It is important for engineers, who use computers in their profession, to understand these errors and learn how to manage and minimize them.

The courses offered in this subject are ENGR 391 – Numerical Methods in Engineering, at Concordia, and CIVE 320 – Numerical Methods, at McGill. As with probability and statistics, these courses are overseen by the schools' respective civil engineering departments. A comparable course for mathematics students at Concordia is MATH 354 – Numerical Analysis, though the comparison is slightly more difficult since many topics taught to mathematics students, and thus techniques for solving tasks on their final exam, are not taught to engineering students. Examples of these include Neville's method and Stefensen's method for finding roots of an equation, Hermite interpolation polynomials, and dividend differences.

The following tables show the relative frequencies of the different types of tasks found on the various final exams: ENGR 391, dated December 2013; CIVE 320, dated December 2007; these are compared with two exams from MATH 354, the first dated December 2011, and another dated December 2012.

Table 20 - Relative frequency of tasks: ENGR 391 - Numerical Methods in Engineering (Concordia), December 2013

	Computational	Conceptual	
Mathematical	87.5%	0%	**87.5%**
Application	12.5%	0%	12.5%
Modelling	0%	0%	0%
	100%	0%	

Table 21 - Relative frequency of tasks: CIVE 320 - Numerical Methods (McGill), December 2007

	Computational	Conceptual	
Mathematical	46%	6%	**52%**
Application	42%	0%	42%
Modelling	6%	0%	5%
	94%	6%	

Table 22 - Relative frequency of tasks: MATH 354 - Numerical Analysis (Concordia), December 2011

	Computational	Conceptual	
Mathematical	65%	24%	**89%**
Application	11%	0%	11%
Modelling	0%	0%	0%
	76%	24%	

Table 23 - Relative frequency of tasks: MATH 354 - Numerical Analysis (Concordia), December 2012

	Computational	Conceptual	
Mathematical	83%	17%	**100%**
Application	0%	0%	0%
Modelling	0%	0%	0%
	83%	17%	

Both exams for engineering students (ENGR 391 and CIVE 320) feature more tasks that are computational in nature, though McGill's contains more applications, nearly as many application tasks as tasks that are purely mathematical. Furthermore, the CIVE 320 exam features the only other modelling problem that I found in my research. Lastly, the more recent MATH 354 exam (December 2012) does not include any applications, and the majority of its tasks are computational.

Indeed, the final exam for ENGR 391 features only one application that is tenuously related to engineering (Figure 48), asking for the rate of change of the stopping distance of a truck on a wet road from given discrete data points.

b. The following data is given for the stopping distance of a truck on a wet road versus the speed at which it begins breaking:

v (km/h)	20.0	40.5	62.5	80	100.5	125
d(m)	6	19	38	65	99	135

1. Calculate the rate of change of the stopping distance at a speed of 100.5 km/h using a two-point backward difference formula. (5 Marks)
2. Estimate the stopping distance at 125 km/h using the result from part 1) and a two-point central difference formula applied at the speed of 100.5 km/h. (5 Marks)

Figure 48 - ENGR 391 - Numerical Methods in Engineering (Concordia), December 2013, question 4 (b)

This problem can be solved using the two-point backward difference formula, a suggested technique of numerical differentiation. The solution to this problem would be written as follows:

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} = \frac{99 - 65}{100.5 - 80} = 1.658\ \frac{m}{km/_h}$$

The problem shown in Figure 48 is the second part of question 4. In the first part of question 4 (Figure 49), students are given the task of evaluating an integral and asked to solve it with the use of multiple techniques: analytically, using Simpson's method, and using four-point Gauss quadrature.

Question #4 [Numerical Differentiation & Integration] [25 marks]

a) Evaluate the Integral:

$$I = \int_0^{2.4} \frac{2x}{x^2+1}\,dx$$

Gauss Quadrature

$n = 4$:

$c_1 = 0.3478548;$ $x_1 = -0.86113631;$

$c_2 = 0.6521452;$ $x_2 = -0.33998104;$

$c_3 = c_2;$ $x_3 = -x_2;$

$c_4 = c_1;$ $x_4 = -x_1$

1. Analytically (2.5 Marks)
2. Using Simpson's 1/3 method, using 6 sub-intervals. (5 Marks)
3. Using four-point Gauss Quadrature (5 Marks)
4. Using the exact solution found in part a) evaluate the percent relative error associated with each of the approximations found in parts 2) and 3) (2.5 Marks)

Keep **3 decimals** in your calculations.

Figure 49 - ENGR 391 - Numerical Methods in Engineering (Concordia), December 2013, question 4 (a)

A question with a similar construction appears in the final exam for the McGill course CIVE 320 (Figure 50). In this problem, the maximum value of a function is to be found by using a golden-section search, quadratic interpolation, and Newton's method.

PROBLEM 3

Employ the following methods to find the maximum of

$$f(x) = 4x - 1.8x^2 + 1.2x^3 - 0.3x^4$$

(a) Golden-section search ($x_l = -2$, $x_u = 4$, $\varepsilon_s = 35\%$.
(b) Quadratic interpolation ($x_0 = 1.75$, $x_1 = 2$, $x_2 = 2.5$, iterations = 4).
(c) Newton's method ($x_0 = 3$, $\varepsilon_s = 10\%$.

Figure 50 - CIVE 320 - Numerical Methods (McGill), December 2007, question 3

The idea of accomplishing a task with multiple techniques is not something that is foreign to the engineering profession. Verifying calculations by independent methods is a technique that an engineer would use in practice, but since the given integral and function in the questions aren't related to any real world engineering problem, this notion may be lost on students who may instead see the task simply as repetitive manual labour. To an engineering student, the questions themselves would seem purely mathematical.

While there are considerably more tasks on the CIVE 320 final exam that are applications, none of them are immediately identifiable as being engineering problems. There are questions framed in the context of metabolism rates of various animals, evaluating the distance traveled given tabulated data of time and velocity, the tracking of an airplane via radar and the use of polar coordinates, and a differential equation that is said to represent a falling object such as a parachutist. In all cases, the technique required to solve the problem is explicitly stated in the question (

Figure 51).

PROBLEM 5

Employ inverse interpolation using a cubic interpolating polynomial and bisection to determine the value of x that corresponds to $f(x) = 0.23$ for the following tabulated data:

x	2	3	4	5	6	7
$f(x)$	0.5	0.3333	0.25	0.2	0.1667	0.1429

PROBLEM 7

Determine the distance traveled for the following data:

t, min	1	2	3.25	4.5	6	7	8	9	9.5	10
v, m/s	5	6	5.5	7	8.5	8	6	7	7	5

(a) Use the trapezoidal rule, **(b)** the best combination of the trapezoidal and Simpson's rules, and **(c)** analytically integrating second- and third-order polynomials determined by regression.

PROBLEM 9

Assuming that drag is proportional to the square of velocity, we can model the velocity of a falling object like a parachutist with the following differential equation:

$$\frac{dv}{dt} = g - \frac{c_d}{m}v^2$$

where v is velocity (m/s), t = time (s), g is the acceleration due to gravity (9.81 m/s^2), c_d = a second-order drag coefficient (kg/m), and m = mass (kg). Solve for the velocity and distance fallen by a 90-kg object with a drag coefficient of 0.225 kg/m. If the initial height is 1 km, determine when it hits the ground. Obtain your solution with **(a)** Euler's method and **(b)** the fourth-order RK method.

Figure 51 - CIVE 320 - Numerical Methods (McGill), December 2007, questions 5, 7, and 9

The task in the following problem, also from CIVE 320, involves determining an equation that relates metabolism rate to mass based on given data (Figure 52).

Determine an equation to predict metabolism rate as a function of mass based on the following data:

Animal	Mass, kg	Metabolism, watts
Cow	400	270
Human	70	82
Sheep	45	50
Hen	2	4.8
Rat	0.3	1.45
Dove	0.16	0.97

Figure 52 - CIVE 320 - Numerical Methods (McGill), December 2007, question 6

This is a modelling task. The mass and metabolism rate must both be modelled as variables, one of them labelled as the independent variable and the other as the dependent variable. Numerical techniques must then be used to find an appropriate relation between the two variables, be it linear, exponential, logarithmic, etc. A complete solution will require explicitly stating assumptions that were made in choosing which of these best models the relationship between the two variables.

There are two questions from ENGR 391 and MATH 354 (2012) that are so similar (Figure 53 and Figure 54) that it might be difficult to identify which question was given to engineering students and which was given to mathematics students without the captions. This is an apt example of just how similar mathematics for engineers can be to mathematics for mathematics students.

2. For the equation below, locate the positive solution p in an interval of as small length as you can and then compute p with the accuracy 10^{-5} using the Newton-Raphson method:

$$x^4 - 3\sin x = 0$$

Figure 53 - MATH 354 - Numerical Analysis (Concordia), December 2012, question 2

Question #1 [Solving Nonlinear Equations] [10 marks]

Obtain the first root above x = 0 for the following equation with accuracy of 4 digits
(Hint: use incremental search to locate the region of the root)

$$e^x - 2x^2 = 0$$

a. Use the method of False Position (5 Marks)
b. Use Newton Raphson method (5 Marks)

Figure 54 - ENGR 391 - Numerical Methods in Engineering (Concordia), December 2013, question 1

The only application found in either of the MATH 354 exams involves the use of discrete least squares approximation to estimate the temperature of a peach pie that is being cooled after being removed from an oven (Figure 55). One could argue that the given task could have also been given to mathematics students without the need to frame it in such a situation.

Problem 5. At the moment when a peach pie is taken out from an oven in a room, it is piping hot. It has been estimated that 2 minutes after this moment, the temperature of the pie is 94°C; 5 minutes after this moment its temperature is 87°C and 10 minutes after this moment its temperature is 80°C. The temperature of the pie at the time moment t is given by the formula

$$T(t) = (T_0 - T_r)e^{-0.02877\,t} + T_r,$$

where T_0 is the temperature of the pie at the initial moment (when it is taken out from the oven) and T_r is the constant temperature of the room.

(a) **[10 marks]** Use **discrete least squares approximation** in order to find T_0 and T_r.

(b) **[5 marks]** Then, estimate the temperature of the pie 20 minutes after the initial moment.

Figure 55 - MATH 354 - Numerical Analysis (Concordia), December 2011, question 5

Regardless, what is important is that discrete least squares approximation is not a technique learned by engineering students, and this is perhaps the only, and yet fairly important difference between the exams for ENGR 391 and MATH 354. Concepts such as inverse interpolation, Neville's method, Hermite interpolation polynomials and cubic continuous least squares approximation are found on the exam for mathematics students, but are absent from the engineering course content. This brings to mind the differences between pre-university linear algebra taken by engineering students and the linear algebra learned by mathematics students. The latter appeared to introduce more advanced topics and techniques that are simply not needed for the purposes of solving engineering problems. Perhaps the methods learned by engineering students in their courses are the only numerical methods they will need.

4.3.8 Engineering geometry

There are two broad subjects that could be considered part of engineering geometry: technical drawing, and surveying.

The technical drawing course at Concordia is CIVI 212 – Civil Engineering Drawing, and at McGill it is MECH 289 – Design Graphics. I was unable to obtain any exams from either of these courses. Despite the absence of course documentation, it can be stated that no equivalent courses are taken by mathematics students. The geometry of technical drawing and surveying is Euclidean, and serves only to represent the size, shape, and position of three-dimensional physical objects in the two-dimensional mediums of computer monitors and paper.

Topics covered in technical drawing courses include the fundamentals and standards of technical drawing, orthographic projections of objects onto plan, elevation, and section views, standards for indicating dimensions and measurements, and communication in engineering through the use of graphics. With this knowledge students learn how to create technical drawings both by hand and with the use of computer software, a process called CAD (computer assisted drafting).

Orthographic projections are used exclusively in engineering practice for the preparation of design and construction plans. In orthographic drawing, the images of objects in three-dimensional space are drawn by projecting lines of sight perpendicularly onto horizontal and vertical planes. The image in Figure 56 is taken from the design plan of a proposed bridge that will span (cross over) a river.

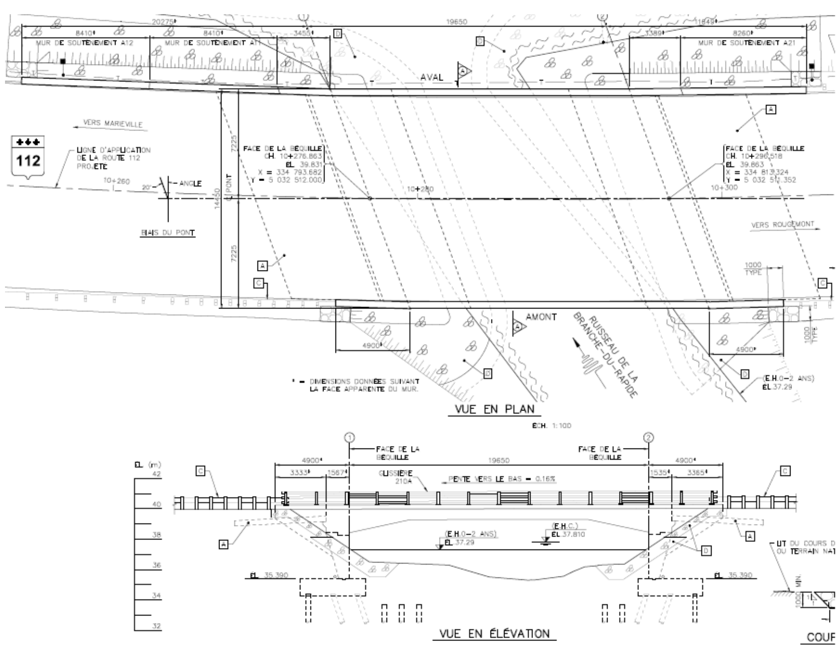

Figure 56 - Orthographic drawing of a bridge spanning a river (Used with permission of CIMA+)

The plan shows the result of the orthographic projections of the bridge onto a horizontal and a vertical plane. The upper portion of the plan is called the plan view (VUE EN PLAN). It is a projection of the bridge and its surrounding environment on a horizontal plane located directly above. Looking at this view, one must imagine hovering above the bridge looking down on it, but rather than having lines of sight merge at infinity (on the image's "horizon"), the lines of sight from each part of the image are perpendicular to the horizontal plane that they are projected onto.

The elevation view (VUE EN ÉLÉVATION), below the plan view, is a projection of the same bridge onto a vertical plane giving the viewer the impression of standing in the river facing the bridge. These two views appear on a structural plan in the same position they are shown in Figure 56, with the plan view on top, and the elevation view directly below. In technical drawing courses, engineers learn to get a sense of the entire three-dimensional shape of objects from these projections.

Indicated on these drawings are measurements such as lengths and distances between important points on the bridge and the terrain. The proposed elevations of the roadway and the bridge's foundations are also included. This technical drawing is part of a series of instructions that shows what the bridge is supposed to look like when it is built. The use of geometry is an efficient means of communicating those instructions.

The other subject in engineering geometry is surveying, which is taught in the courses BCEE 371 at Concordia, and CIVE 210 at McGill (both courses are simply titled "Surveying"). At both schools, the technical drawing course is a prerequisite to surveying. The surveying course is taught in a condensed three week session during the summer term. Students learn the elementary operations employed in engineering surveying: how to use, care for and adjust the technical instruments (levels, transits, and theodolites) that are used for linear distance and angular measurements between objects, how to design vertical and horizontal curves for roadways so that they fit into an existing terrain, how to perform calculations for earthworks projects, evaluating and managing the errors that are inherent in surveying measurement, and the application of surveying methods in field work.

An important aspect of surveying is the meticulousness that is needed when recording one's measurements in a field book. A surveyor's field book must contain all of the necessary data for preparing a technical drawing of the environment that was surveyed. During construction of a structure such as a bridge, the surveyor's measurements are also used to ensure that the construction is being done in accordance with the measurements on the plans. The students at the technical school discussed in the literature review were charged with a similar task (Eberhard, 2000).

In professional practice, an engineering surveyor will provide his field data, consisting of measured elevations, distances, and angular measures between points of interest to a technical draftsperson that will use the data to produce a technical drawing or plan.

Though I was unable to obtain any documentation in the form of a final exam for either surveying course, I do recall the final project in the surveying course that I took as a student at McGill. A parcel of land on Mount Royal near the University's campus was divided into adjacent regions that were roughly in the shape of chevrons approximately ten metres by twenty metres, though no two regions were exactly identical (Figure 57).

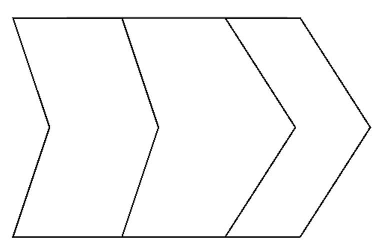

Figure 57 - Chevron-shaped parcels of land (drawing is my own)

The students were paired off, and two groups were assigned to each region. Topographic features such as large rocks, trees, and other points of interest including the regions' boundaries had to be precisely located and their elevations measured as accurately as possible. A technical plan of each region then had to be drafted by each independent pair of students. The plans of each region were then assembled to see if they fit together as they did in reality.

In this sense, engineering geometry closely fits the etymology of the subject, namely the measurement of the earth. But there is more to be said about an engineer's geometry beyond the topics of technical drawing and surveying. At its core, the geometry used by engineers is Euclidean, and is concerned with representing the three-dimensional physical objects as images on a two-dimensional plane. Euclidean geometry assumes that for all intents and purposes the world is flat, and remains so underneath any object or parcel of land that is being measured. It is essentially a very "local" geometry. In this geometry, the surface of the earth is a "horizontal plane", and "vertical planes" are orthogonal to it. Orthographic projection takes physical objects and projects their image onto three planes which are pairwise orthogonal to each other, with the horizontal plane being parallel to the surface of the earth.

Another aspect of this geometry is the use of pairwise orthogonal axes to describe the position and orientation of objects in the physical world. A structural engineer's task in the analysis and design of a structure focuses on the members that make up the structure. Each structural member has a position and orientation within the structure. In terms of geometry, a member can be thought of as a regular prism, such as the object shown in Figure 58.

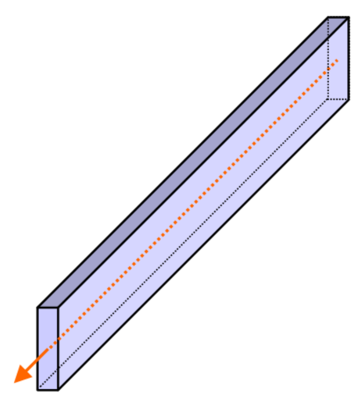

Figure 58 - Example of a structural member with a rectangular cross-section

The orange arrow identifies the orientation of the member's longitudinal axis. The length of the member in the direction of the longitudinal axis is called the axial length. The member's cross-section is the surface of intersection of the member and a vertical plane that is orthogonal to the longitudinal axis. For the member shown in Figure 58, the cross-section is a rectangle. A similar member with a circular cross-section would be identified by mathematicians as a "cylinder". Thus the cross-section is a two-dimensional geometric shape, and in designing a structural member, the engineer is concerned with many properties of these shapes such as their width, height, area, and moment of inertia (see section 5.2.2). Examples of cross-sectional shapes most commonly used in construction include W-shapes (wide-flange shapes) and S-Shapes (American standard shapes), both commonly referred to as "I-beams", as well as C-shapes (channels) and angles (Figure 59).

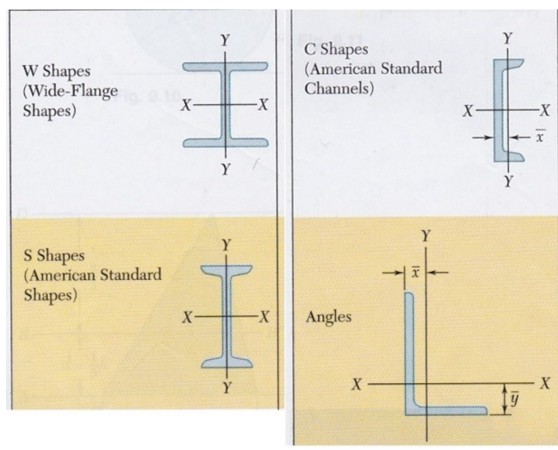

Figure 59 - Common geometric cross-sectional shapes for structural members (Beer & Johnston, 2007, p. 486)

It's worth noting that when a structural engineer speaks of a member's area, they may be referring either to the numerical value associated with the cross-section's measured area, or the cross-sectional shape itself, depending on the context.

In contrast, a mathematician's geometry is not strictly Euclidean. A mathematics student's study of geometry at Concordia will involve an introduction to geometric topology and differential geometry of surfaces, as opposed to the measurement of physical objects. The Concordia course MATH 480 –

Geometry and Topology, has listed as prerequisites courses in analysis and abstract algebra, hinting that the geometry in question involves more mathematical knowledge than an engineer needs.

4.4 Summary of Results

Despite the necessity to apply mathematics in their eventual professional careers, relatively few of the tasks given to engineering students on their final exams are application problems, with the sole exception being the courses in probability and statistics. The tables that follow show the results from compiling all of the tasks given to engineering students and those given mathematics students on their respective final exams for each subject.

Table 24 - Summary of the relative frequency of tasks: Pre-university linear algebra

	Engineering			Mathematics		
	Computational	Conceptual		Computational	Conceptual	
Mathematical	71%	29%	**100%**	54%	46%	**100%**
Application	0%	0%	0%	0%	0%	0%
Modelling	0%	0%	0%	0%	0%	0%
	71%	29%		**54%**	**46%**	

For pre-university linear algebra, we see that neither engineering nor mathematics students are given any application problems. However, mathematics students are given more tasks that are conceptual in nature than engineering students.

Table 25 - Summary of the relative frequency of tasks: Pre-university calculus

	Engineering		
	Computational	Conceptual	
Mathematical	65%	33%	**98%**
Application	0%	2%	2%
Modelling	0%	0%	0%
	65%	35%	

When the tasks of the two calculus courses are compiled, we see that the bulk of the questions are mathematical in nature, with an emphasis on computational tasks. The only application problem involved using related rates to determine the rate at which a rectangle's area was increasing.

Table 26 - Summary of the relative frequency of tasks: University level calculus

	Engineering			Mathematics		
	Computational	Conceptual		Computational	Conceptual	
Mathematical	78%	16%	**94%**	71%	18%	**89%**
Application	3%	0%	3%	11%	0%	11%
Modelling	0%	3%	3%	0%	0%	0%
	81%	19%		**82%**	18%	

In university calculus, we find nearly identical quantities of computational and conceptual tasks given to both groups of students. Surprisingly, it is mathematics students who are given a greater number of application problems, though only slightly more. This can perhaps be explained by the fact that the engineering students' calculus course is taught by mathematics professors.

Table 27 - Summary of the relative frequency of tasks: Differential equations

	Engineering			Mathematics		
	Computational	Conceptual		Computational	Conceptual	
Mathematical	76%	18%	**94%**	69%	31%	**100%**
Application	3%	3%	6%	0%	0%	0%
Modelling	0%	0%	0%	0%	0%	0%
	79%	21%		**69%**	31%	

The relative frequencies of the different types of tasks given to engineering and mathematics students are similar in theirs differential equations courses as well, though mathematics students are given a few more conceptual tasks.

Table 28 - Summary of the relative frequency of tasks: Probability and statistics

	Engineering			Mathematics		
	Computational	Conceptual		Computational	Conceptual	
Mathematical	14%	2%	16%	41%	18%	**59%**
Application	66%	18%	**84%**	27%	14%	41%
Modelling	0%	0%	0%	0%	0%	0%
	80%	20%		**68%**	32%	

Probability and statistics is the only subject in which we notice a significant difference in the types of tasks given to engineering and mathematics students. The emphasis for for the former is placed on computational application problems, while the latter have roughly half as many applications, and more tasks that are conceptual in nature.

Table 29 - Summary of the relative frequency of tasks: Numerical methods

	Engineering			Mathematics		
	Computational	Conceptual		Computational	Conceptual	
Mathematical	66%	3%	**69%**	72%	21%	**93%**
Application	28%	0%	28%	7%	0%	7%
Modelling	3%	0%	2%	0%	0%	0%
	97%	3%		**79%**	21%	

In numerical methods we again see a much greater emphasis placed on computational and application tasks on the engineering students' final exams. Both of these last two subjects are administered by the school's respective civil engineering departments.

Overall, the nature and content of the tasks given to engineering students is similar to those given to mathematics students. It could be argued that the differences in the mathematics education of an engineer and that of a mathematician begin to differ only after the second year of university study. The mathematics courses identified and discussed in this section are all taken in first two years of study in both engineering and mathematics programs. This raises the question: do things differ in subsequent years of study? The answer is an unqualified yes.

Engineering students complete their studies with courses in engineering sciences. For all intents and purposes, their mathematics education is finished after their second year. Mathematics students, however, learn about concepts and topics that for the most part remain foreign to engineers (both students and professionals). These include topics in mathematical logic, real and complex analysis, abstract algebra, and measure theory. These courses explore the inner workings of mathematics; they provide the “theory” in the academic mathematics praxeology.

Instead of diving deeper into the theory behind the rules of mathematics, engineers take the rules they have learned for manipulating mathematical objects – matrices and vectors, derivatives and integrals of functions, differential equations, numerical methods – and use them to solve problems in the engineering domain.

5 Engineers' mathematics

In chapter 4 it was shown that the types of mathematical tasks given to engineering students and the topics they are expected to learn in their mathematics courses are not noticeably different from those learned by mathematics students in the first two years of their university education. But the education of engineers isn't limited to mathematics classes. They must also learn about the physical sciences in order to be able to describe, understand, and predict physical phenomena. The goal of an engineer's mathematics education is to express these phenomena symbolically using mathematical models, and it is precisely this use of mathematics to model the physical world that differentiates an engineer's mathematics from that of the mathematician.

The present chapter will explore how engineers use mathematics to create models of the physical world in order to solve engineering problems, and will focus particularly on the models used in the field of structural engineering. Structural engineering comprises the tasks of design, analysis, construction, and maintenance of all manner of modern structures including conventional shelters such as houses, office buildings, and commercial warehouses, as well as specialized structures such as bridges, highway overpasses and interchanges, and hydroelectric dams.

The chapter begins with a brief explanation of two types of models: analytical and empirical. Established mathematical models, their assumptions, and how they were developed, are presented to students in their engineering textbooks. These models are then used by students in application problems. Examples of these models from the subjects of statics, mechanics of materials, and structural analysis will be shown in section 5.2. As they are presented, observations will be made about which aspects of the models are important to engineers, and which aspects a mathematician may see that an engineer might overlook.

Section 5.3 contains an in-depth examination of a mathematical model of matrix structural analysis. Included in this section is a discussion of the concept of eigenvalues and eigenvectors in the context of matrix structural analysis. My intention is to show the physical meaning of an abstract mathematical concept in an engineering context.

The final section of this chapter presents documents from a professional engineer's *dossier de calculs* that will illustrate how the mathematics used in professional practice rests upon the extensive research and refinement of the mathematical models developed by professional and research engineers.

5.1 Analytical models and empirical models

In chapter 4 I showed that mathematics courses taken during an engineer's education provide tasks that involve applications, but there were none that required the student to create a mathematical model. In the current chapter I will show that many of the mathematical models used by engineers are introduced in the textbooks of engineering core courses. I will present some of these models, explain how they are developed, and show that the tasks given to engineering students involve applications of these models.

Engineers make use of two different types of mathematical models: analytical models and empirical models. Analytical models are developed directly from foundational principles of physical sciences. In their physics courses, for example, engineering students learn about kinematics, the study of the motion of objects, without regard for the cause of the motion. The laws of how things fall were developed long before Newton described *why* things fall (Tipler, 1991). Kinematics relates three physical quantities: the displacement of an object, its velocity, and its acceleration. Displacement is the change in an object's position in space:

$$\Delta x = x_2 - x_1$$

Velocity is defined as the rate of change of displacement. This is expressed mathematically as:

$$v_{av} = \frac{\Delta x}{\Delta t}$$

for average velocity, and:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

for instantaneous velocity. Similarly, acceleration is defined as the rate of change of velocity, and is expressed as:

$$a_{av} = \frac{\Delta v}{\Delta t}$$

for average acceleration, and:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

for instantaneous acceleration. But, if the acceleration is constant (as it is for objects that are dropped or fall on Earth) then velocity varies linearly with time. This leads to the development of the four equations of kinematics that can be used to solve problems:

$$v = v_0 + at$$

$$\Delta x = \frac{1}{2}(v_0 + v)t$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$v^2 = {v_0}^2 + 2a\Delta x$$

Each equation can be used depending on the context of the problem that one wishes to solve, on what information is known, and what information is sought. The last equation would be used if one wants to know the final velocity, v, of an object as it hits the ground when dropped from a known height, Δx, without any regard to the time it took to fall (Tipler, 1991). These four equations are all derived through analytical means directly from the relationships between displacement, velocity, and acceleration.

An example of an analytical model used specifically by structural engineers is the various formulae for calculating the moment of inertia of common geometric shapes. These formulae, and how they are developed, will be shown and discussed in section 5.2.2.

Empirical models on the other hand are developed through experimentation, and are often refined with statistical analysis. Such models cannot be derived solely by analytical means. A well-known example of an empirical model is Hooke's Law which states that the force, F, required to stretch a spring is directly proportional to the length, X, that it is stretched: $F = kX$. This relationship is usually investigated in high school physics classes, and could not be determined based on the principles of mechanics alone. A similar relationship arises in the study of mechanics of materials, and is presented in section 5.2.3.

The design codes that structural engineers are obligated to follow in the design and analysis of structures are peppered with equations that were derived empirically. Historically, research engineers have always been interested in determining the effects that small changes in the shape, size, and configuration of structural members may have on a structure's performance. The only way to determine those effects is to perform controlled experiments and analyze the results. The result is the empirical models that are found in the design codes.

Recall that the use of these models to solve a problem results in an application of mathematics, but the development of the equations themselves in order to represent physical reality is done through the process of mathematical modelling. The mathematical models that are introduced throughout section 5.2 will be identified as either analytical or empirical in nature.

5.2 THE MATHEMATICS OF MECHANICS

Civil engineering is a multidisciplinary field encompassing many different kinds of engineering disciplines including environmental engineering, transportation engineering, and structural engineering among others. Prior to my career as a graduate student, I was employed as a licensed structural engineer, hence my interest in writing on the topic of this thesis. The remainder of this chapter will therefore focus solely on the mathematics used by structural engineers.

Structural engineering is an advanced discipline that stems from applied mechanics, particularly the fields of statics, which is the study of rigid bodies at rest, and the mechanics of materials, which is the study of deformable bodies[4]. All structures, be they houses, bus shelters, skyscrapers, or bridges, must be static, i.e., they must not move from their location after being exposed to some force pushing it or pulling on it. But they are not rigid bodies; they are subject to deformations due to the properties of the materials they are made from. A bus shelter may sway due to the gust of a strong wind (it will deform), but it should not move from where it was built (it will be static against the applied force of the wind).

While mechanics and mathematics may seem to be distinct subjects, historically there is very little separation between the two. Bkouche (1989) neatly summarizes the close relationship between the two fields and how each has allowed a better understanding of the other. It is in applying mathematics to the problems from which its concepts and theories were born, among which are physical problems, that we see its power. The formalisation of mathematics in the late nineteenth century removed it from its empirical origins in mechanics and physics. Prior to this, the lines between the two fields of study were blurred. Consider the works of Newton and Euler who are celebrated for their discoveries in both mathematics and mechanics. Newton's laws of motion are well-tested mathematical models that are generally accepted as being correct, and weren't superseded in their accuracy or generality until

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[4] Deformable bodies are objects whose shapes can be altered by stretching, compressing, bending, or twisting. In the field of statics, objects are assumed to be un-deformable, i.e., rigid.

Einstein's theory of relativity. Euler is also recognized for his contributions to mechanics as well and is remembered for models that he developed through experimentation and equations that that are named in his honour.

The sections that follow will present some mathematical models from statics, mechanics of materials, and structural analysis, along with an introduction to the underlying physical principles that they represent. Certain details will be highlighted to indicate the differences in what an engineer and a mathematician may notice about the mathematics used in the models. I begin by discussing the important concept of units, without which there would perhaps be no understanding of the measure of the physical world.

5.2.1 Units and numerical accuracy

Units are vital to a proper description of physical quantities. The number 10 by itself may or may not convey any meaning to the reader, but if one is asked to hold a toothbrush weighing 10 grams in one hand, and a bowling ball weighing 10 kilograms in the other, suddenly the number 10 can take on different meanings. It is worth noting that in the tasks from various exams shown in chapter 4, units of measure only appear in application questions, when objects from the physical world are involved.

In mechanics, there are three fundamental units of measure that are considered base units, i.e., they are defined arbitrarily, and are used to derive other units (Beer & Johnston, 2007). These are the units of time, length, and mass. Two systems are still in use to describe these units: the Système International (SI), also known as the metric system, and the U.S. Customary system, commonly referred to as the Imperial system. In the SI system, the base units for time, length, and mass are, respectively, the second (s) the metre (m) and the kilogram (kg).

A fourth unit that is considered fundamental to mechanics is that of force. Its unit of measure in the SI system is called the newton (N), and it is derived from the base units of time, length, and mass. A force of 1 N is the force that must be applied to a mass of 1 kg in order to make it move with an acceleration of 1 m/s^2:

$$1\ N = (1\ kg)\left(1\ \frac{m}{s^2}\right) = 1\ \frac{kg\ m}{s^2}$$

With these four units, other important units used in engineering can be derived as well, including:

- The pascal (Pa) for units of stress, also referred to as pressure, defined as applying a force of 1 newton over an area of 1 square metre:

$$1\ Pa = \frac{1\ N}{1\ m^2} = 1\ \frac{N}{m^2}$$

- The joule (J) for units of work, defined as the energy transferred when applying a force of 1 newton to an object over a distance of 1 metre:

$$1\ J = (1\ N)(1\ m) = 1\ N\ m$$

- The watt (W) for units of power, defined as 1 joule per second:

$$1\ W = \frac{1\ J}{1\ s} = 1\ \frac{J}{s} = 1\frac{N\ m}{s}$$

It is often the case that physical quantities can only be expressed by large numerical values of these units. To simplify numerical notation, a system of prefixes is used for various orders of magnitude of units (Figure 60).

Multiplication Factor	Prefix†	Symbol
1 000 000 000 000 = 10^{12}	tera	T
1 000 000 000 = 10^{9}	giga	G
1 000 000 = 10^{6}	mega	M
1 000 = 10^{3}	kilo	k
100 = 10^{2}	hecto‡	h
10 = 10^{1}	deka‡	da
0.1 = 10^{-1}	deci‡	d
0.01 = 10^{-2}	centi‡	c
0.001 = 10^{-3}	milli	m
0.000 001 = 10^{-6}	micro	μ
0.000 000 001 = 10^{-9}	nano	n
0.000 000 000 001 = 10^{-12}	pico	p
0.000 000 000 000 001 = 10^{-15}	femto	f
0.000 000 000 000 000 001 = 10^{-18}	atto	a

†The first syllable of every prefix is accented so that the prefix will retain its identity. Thus, the preferred pronunciation of kilometer places the accent on the first syllable, not the second.

‡The use of these prefixes should be avoided, except for the measurement of areas and volumes and for the nontechnical use of centimeter, as for body and clothing measurements.

Figure 60 - SI prefixes and multiplication factors (Beer & Johnston, 2007)

The most commonly used units of length, force, and stress are the kilometre ($1\ km = 10^3\ m$), the millimetre ($1\ mm = 10^{-3}\ m$), the kilonewton ($1\ kN = 10^3\ N$), the megapascal ($1\ MPa = 10^6\ Pa$), and the gigapascal ($1\ GPa = 10^9\ Pa$). The use of the prefixes *kilo*, *milli*, *mega*, and *giga* allows the numerical value of certain measures to be written more efficiently and therefore more easily understood.

It is interesting to note the recommendation that certain prefixes including *centi* and *deci* be avoided, especially for units of length. In engineering practice the preferred units for lengths and distances are the kilometre, the metre and the millimetre. Figure 61 is an extract from the same plan that was used to generate the image in Figure 56 (in section 4.3.8).

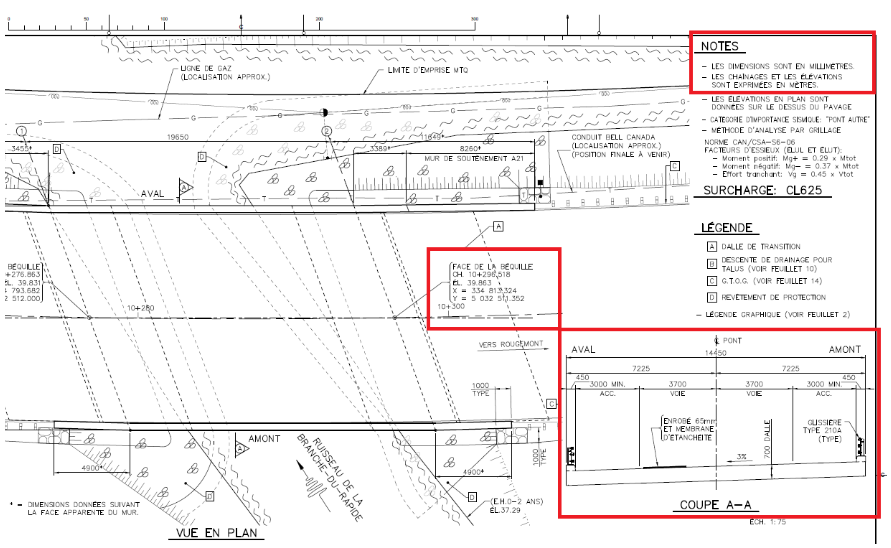


Figure 61 - General notes from an engineering plan (Used with permission of CIMA+)

It shows the plan view of a proposed bridge (VUE EN PLAN) as well as the bridge's cross section (COUPE A-A). The text in the red box in the upper right hand corner of the plan shows the general comments (NOTES) that apply to every numerical measurement on the plan. The text reads "The dimensions are in millimetres", and "Chainage distances and elevations are expressed in metres." With the inclusion of these notes, every number that appears on the plan is assigned an appropriate unit of measure.

The two red boxes in the lower half of Figure 61 are shown in larger scale inFigure 62 and Figure 63.

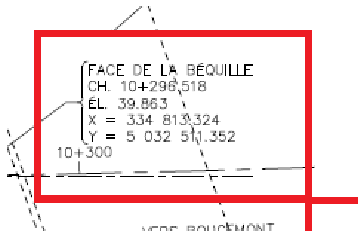


Figure 62 - Chainage and elevation of a bridge abutment

The numbers shown in Figure 62 indicate the chainage marker (CH.) and the elevation (ÉL.) of the roadway on the face of the bridge's abutment. According to the general notes, these measurements are in metres. This tells the engineers that the face of the bridge's abutment is to be located at the position 10296.518 metres from the beginning of the roadway, and the elevation of the roadway is to be 39.863 metres at this same spot.

Figure 63 shows the measurements of the bridge's cross-section. Since these are not chainage distances or elevations, these measurements all have units of millimetres. Thus, the total width of the bridge is 14450 mm broken up into two curbs of 450 mm each, two shoulders (ACC. for *accotement*) of 3000 mm, and two lanes (VOIE) of 3700 mm.

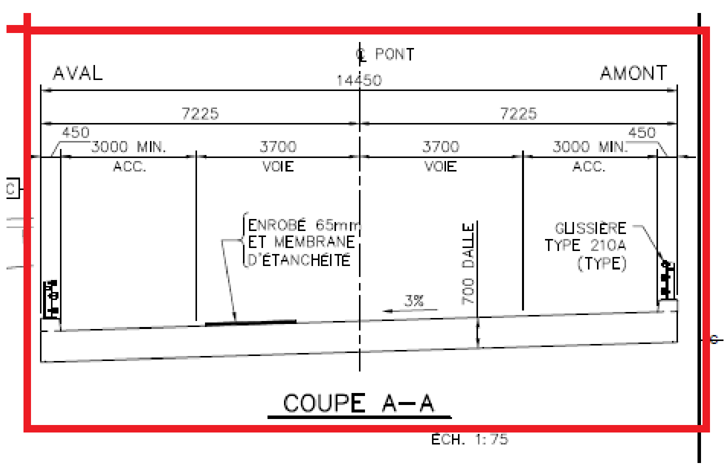


Figure 63 - Cross-section of a proposed bridge

Notice that when measurements are given in metres (Figure 62), there are three digits to the right of the decimal place. This means that the precision of the measurement is to the millimetre. As such, measurements whose units are millimetres have no digits following the decimal (Figure 63).

This raises questions about the level of precision that is expected of engineers in their use of numbers to represent physical quantities. The table of prefixes in Figure 60 is from a textbook section entitled "Systems of units" (Beer & Johnston, 2007). Nearly all introductory mechanics textbooks will include a section explaining the proper use of units, prefixes, and expressing quantities using scientific notation. An equally important section entitled "Numerical accuracy" discusses the use of significant figures when computing solutions to problems. In general, the numerical value of a solution cannot be more accurate than either (1) the accuracy of the given data, or (2) the accuracy of the computations performed (Beer & Johnston, 2007). Computed quantities are often truncated, removing digits that are considered to be beyond the accuracy of the known measured data. For example, while the circumference of a circle whose diameter is 1.00 m is precisely equal to π m, an engineer will express the circumference as 3.14 m. Any additional decimal places beyond the hundredths position would be considered more accurate than the measured value of the circle's diameter. If, however, more precise measurements are taken, and the diameter is measured to be 1.005 m, then the circumference can be expressed as 3.157 m. Similarly, the lengths and distances indicated on the plan in Figure 61 can be precise "to the millimetre" because the equipment that will be used to measure those same lengths during construction will have the same precision in their measuring capabilities.

This is an example of how engineers and mathematicians differ in their respective concepts of "precision." While mathematicians strive to achieve absolute precision in how they represent numbers, engineers seek practicality. While the symbol π is the only true way to represent the number that is exactly the ratio of a circle's circumference to its diameter, measuring a length of π metres is rather impractical since it requires a level of precision that cannot be achieved. Thus engineers use precision relative to the context of the physical problem which they are trying to solve.

Lastly, there is another unit of measure that I have neglected to mention thus far but that is important to every engineering project that a professional engineer will work on. That unit is the dollar ($). Structural engineers design and construct all manners of structure that much resist applied loads and have limited deformations, but they must also be built efficiently and economically. Professional engineers who work in the role of project manager are chiefly responsible for ensuring that the monetary aspects of the project are met.

5.2.2 Statics

Statics is the study of rigid bodies, i.e., objects that are assumed to remain un-deformed when acted upon by external forces. The principle of static equilibrium is at the heart of the subject. Static equilibrium is an interpretation of Newton's first law of motion which states that an object at rest will remain at rest unless it is acted upon by an external force. But the effects of any external force applied to an object can be countered by applying a second force that is equal in magnitude but opposite in direction to the first. Thus Newton's first law can be restated as: an object will be in static equilibrium if the net sum of the external forces acting upon it is 0.

Applying a force is akin to the act of pushing or pulling on an object. Imagine pushing a large block. When pushed with enough force the block moves in the same direction that the force is applied (Figure 64). Applying a second force of equal magnitude but in the opposite direction on the block will stop the block from moving; the block will be static.

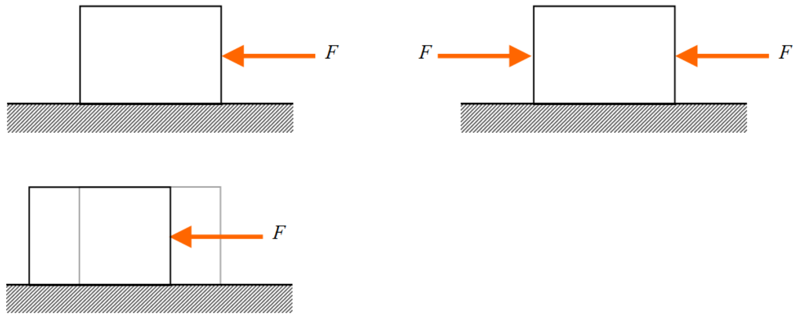


Figure 64 - Applying a force to a block and pushing it. Applying two forces: the block does not move (drawing is my own)

There is an important caveat: forces of equal magnitude do not always cancel each other out. Consider a book resting on a table, and the two forces applied to opposite corners (Figure 65). It is intuitive to see that these forces will cause the book to rotate. Even though they are equal in magnitude and opposite in direction, applying the forces in this manner does not leave the book in static equilibrium. The rotational potential of a force, or a pair of forces, is called a moment, M.

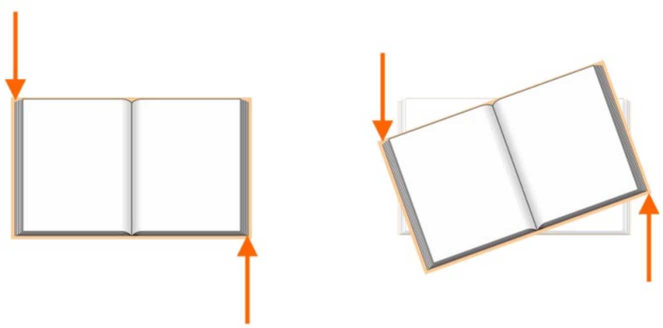


Figure 65 - Applying a force: rotating a book (drawing is my own)

To understand how moments are measured, imagine closing a door using only your index finger (Figure 66). If you place your finger near the door's extremity, far from the hinge, you only need to apply a small force, F, for the door to close. This is because the line of action of the force is located a relatively large distance, r, away from the axis about which the door rotates. The applied force and its distance from the axis of rotation work together to create the rotational force called a moment, M.

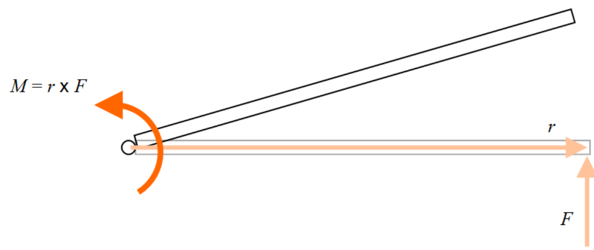


Figure 66 - Applying a moment to a door (1) (drawing is my own)

Moving your finger closer to the hinge reduces the distance, r (Figure 67). In order to close the door with the same rotational force, or moment, as before, you will now need to apply a much larger force to the door.

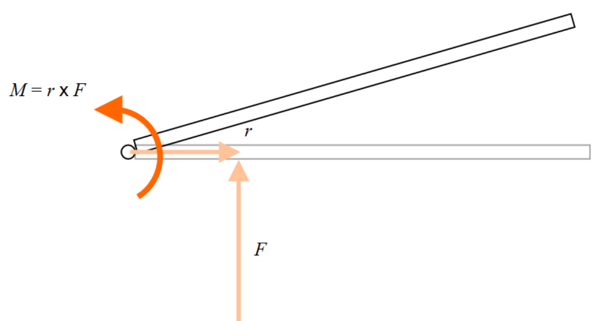


Figure 67 - Applying a moment to a door (2) (drawing is my own)

As a scalar, the magnitude of a moment is the product of the applied force and the distance from the line of action of the force to the axis of rotation. Thus, the unit of measure of the moment is the newton-metre, $N\ m$. As a vector, a moment is the cross product of the radius vector $\vec{r}$ defined from the point of rotation to the line of action of the force, and the force vector, $\vec{F}$. Due to its definition as a cross product, the line of action of M in Figure 66 and Figure 67 is directed out of the plane of the page, but its effects are drawn on the plane and represented by a curved arrow in the direction of its rotation. A positive moment will rotate an object in a counter-clockwise direction when viewed from overhead.

For a body to be in static equilibrium the sum of all forces and the sum of all moments acting upon it must be zero. The former prevents the body from being displaced in a plane, and the latter prevents it from rotating within the same plane. This is expressed mathematically as:

$$\sum F = 0\,; \sum M = 0$$

This is the mathematical model that was presented in section 3.3. Here I note that in expressing these requirements, engineers do not concern themselves with the indices or bounds in the sigma notation. In the context of Newton's first law and static equilibrium it is implied that the number of forces and moments is finite, and all those that are identified in a problem are to be included in the summation.

The concepts of forces and moments are two examples of how engineers use vectors in an analytical mathematical model. This particular model of static equilibrium allows engineers to solve problems that require finding unknown reaction forces in the supports that hold up a structure. Setting the sum of all forces and moments equal to 0 results in a system of equations that can be solved using techniques learned in linear algebra, with the solutions to the systems being the unknown forces that are sought.

A typical problem from a statics textbook is shown in Figure 68.

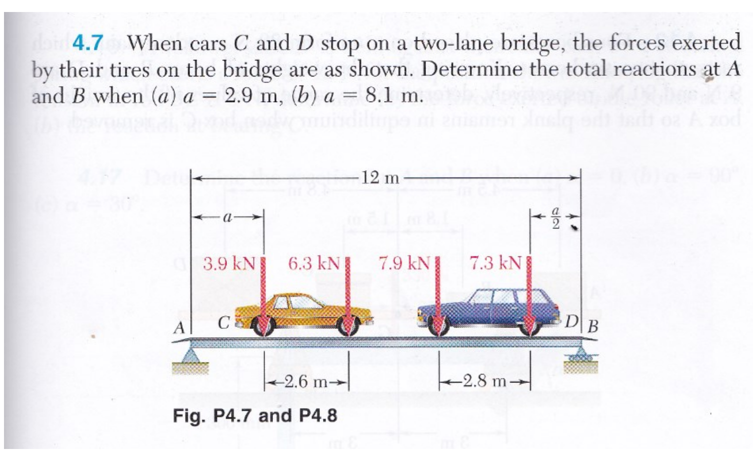


Figure 68 - Statics problem (Beer & Johnston, 2007, p. 173)

Letting R_A and R_B represent the forces at supports A and B leads the following system of two equations and two unknowns:

$$\sum F_y = 0 \rightarrow R_A + R_B - 3.9 - 6.3 - 7.9 - 7.3 = 0$$

$$\sum M_A = 0 \rightarrow R_B(12) - 3.9(a) - 6.3(a + 2.6) - 7.9\left(12 - \frac{a}{2} - 2.8\right) - 7.3\left(12 - \frac{a}{2}\right) = 0$$

Implicit in the derivation of these equations is the knowledge that forces and moments that act in opposite directions have opposite signs, just as vectors acting in opposite directions and equal length can be expressed as v and $-v$. Thus, if a force acting vertically upwards is taken to be positive (as is the case for the support reaction forces R_A and R_B), then those acting in the opposite direction are negative (as is the case for the applied forces of 3.9, 6.3, 7.9, and 7.3 kN). Notice as well the sudden use of subscripts on the symbols F_y and M_A. These indicate that the forces are being summed in the vertical direction (parallel to the standard $y - axis$) and that the moments are being summed about, or around, the point A, the leftmost point on the beam. While the general mathematical model of static equilibrium does not include these subscripts, they become necessary in particular applications to clarify the origins of the terms in each equation.

For the purposes of determining the reaction forces in the supports, it is implicitly assumed that the bridge is a rigid body. In reality the beam supporting the two cars in Figure 68 would deform according to the properties of the material it is made from. But since these deformations tend to be relatively small when compared to the size of beam itself, the assumption of rigidity is reasonable, and the model remains valid.

Another important mathematical model learned in statics is the one for evaluating the second moment of area, also known as the moment of inertia. The moment of inertia is a measure of an object's ability to resist being bent. Imagine being asked to stand on the two boards in Figure 69. Both have a rectangular cross-section, but the one on the left is resting on the cross-section's height while the other is resting on its base. It is not difficult to imagine that the board on the left would bend more easily than the board on the right.

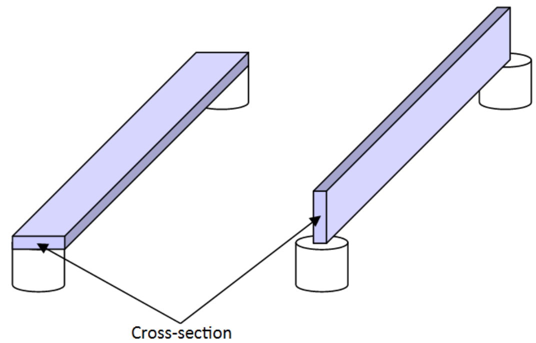


Figure 69 - Boards with rectangular cross-sections (drawing is my own)

The board on the right is more resistant to being bent because it has a larger moment of inertia. In Figure 70 the cross-sections of the two boards are shown (see section 4.3.8 for a discussion on the cross-section of a structural member).

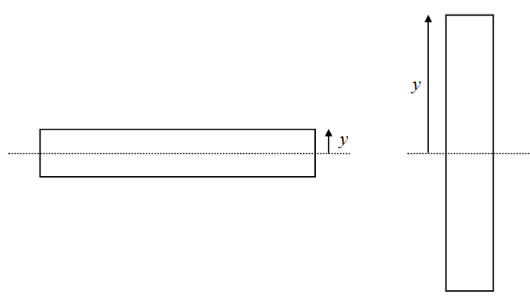


Figure 70 - Rectangular cross-sections (drawing is my own)

The dashed lines are passing through the centroid of each cross-section; these are referred to as the cross-section's horizontal centroidal axis. A centroidal axis divides a cross-section into two parts of equal area on either side of the axis. The vertical distance between the centroidal axis and the top of the cross-section is labelled y. Notice that the value of y is greater for the cross-section on the right of Figure 70, indicating that its cross-sectional area is located further away from its centroidal axis. Cross-sections whose areas are concentrated further away from the centroidal axis have greater moments of inertia.

Mathematically the moment of inertia, I, is evaluated using the integral:

$$I = \int y^2 dA$$

In this equation, dA is a differential area located somewhere in the cross-section, and y is its distance from the centroidal axis. The analysis that leads to the derivation of this model is beyond the scope of this thesis, but simply put, the integral is the sum of the product of each differential area (dA) with the square of its distance (y^2) away from the centroidal axis. Notice that doubling a differential area's distance from the centroidal axis increases its effects on the moment of inertia by a factor of 4. Thus, the further away the area is from the centroid, the larger the cross-section's moment of inertia.

An example of applying this model is as follows. Consider the rectangular cross-section in Figure 71 with the dimensions b and h. The differential area, dA, highlighted in dark blue, has dimensions $b\ dy$, and the sum of the differential areas is performed from the bottom of the cross-section, where $y = 0$, to the top, where $y = h$.

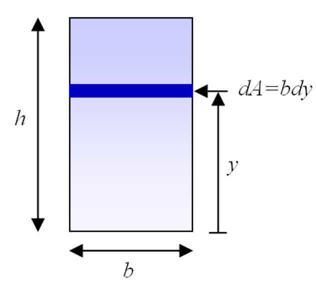


Figure 71 - Moment of inertia of a rectangular cross-section (drawing is my own)

The result of the integral then is:

$$I = \int y^2 dA = \int_0^h y^2\ b\ dy = b \int_0^h y^2 dy = \frac{1}{3} bh^3$$

The equation $I = \frac{1}{3}bh^3$ is recognized by engineers as the formula for the moment of inertia of a rectangular cross-sectional area. Of note is that the defining equation treats integral as being indefinite; there are no limits on the integral. However, when calculating the moment of inertia, the model requires establishing the limits and evaluating a definite integral.

This model, which is purely analytical in its origins, gives a mathematical justification for why the wood joists that support the floors of your home are aligned vertically, just like the board on the right in Figure 69. It also explains why shapes such as those shown in Figure 72 are standard in the construction industry.

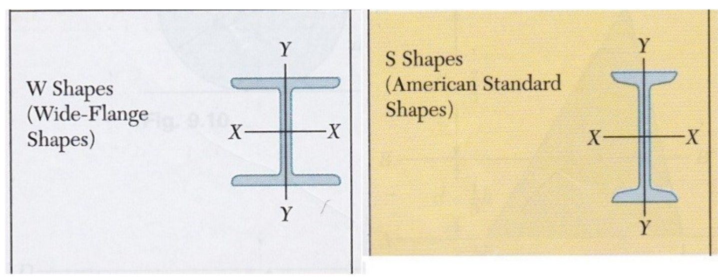


Figure 72 - Wide-flange and American standard cross sections (Beer & Johnston, 2007, p. 486)

The horizontal centroidal axes of these cross-section are located at mid-height (the axes labelled $X - X$), but the bulk of their areas are located at the bottom and the top of the sections, resulting in a very large moment of inertia for a relatively small amount of material, making these shapes very economical.

This model shows how engineers use integral calculus to model physical properties of real world objects. In a statics course, engineering students can be tasked with evaluating the moment of inertia of a given cross-sectional area composed of different shapes either by means of integration, or through more efficient techniques that they are taught (Figure 73).

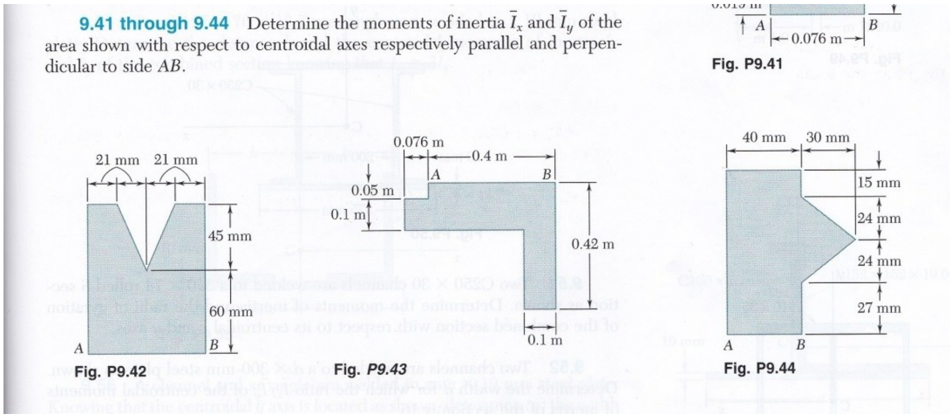


Figure 73 - Statics textbook problems: finding the moment of inertia of geometric shapes (Beer & Johnston, 2007, p. 493)

Solving such tasks is simplified by using formulas for the moment of inertia of standard geometric shapes such as rectangles, triangles, and circles (Figure 74).

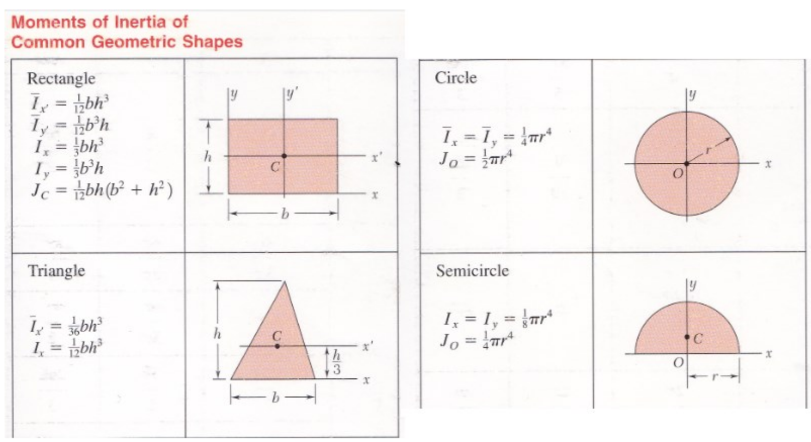
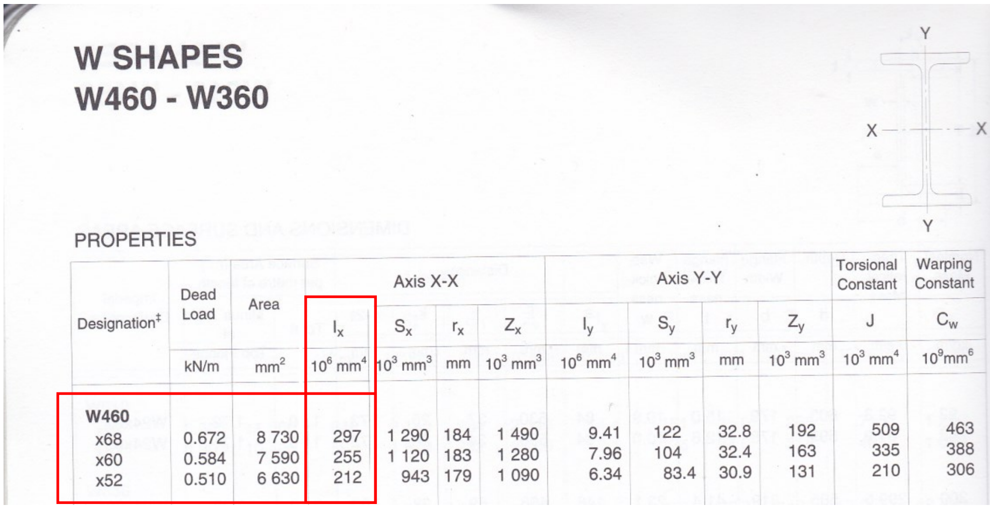


Figure 74 - Moments of inertia of common geometric shapes (adapted from (Beer & Johnston, 2007, p. inside back cover))

However, once one becomes a practicing engineer, even these tasks no longer become necessary since the moments of inertia of standard cross-sections have already been evaluated and the values compiled in tables found in design manuals. The table in Figure 75 is taken from the Handbook of Steel Construction (2000) and includes various geometric properties for different wide-flange shapes, standard in the design and construction of steel structures.



Designation‡	Dead Load	Area	Axis X-X				Axis Y-Y				Torsional Constant	Warping Constant
			I_x	S_x	r_x	Z_x	I_y	S_y	r_y	Z_y	J	C_w
	kN/m	mm^2	10^6 mm^4	10^3 mm^3	mm	10^3 mm^3	10^6 mm^4	10^3 mm^3	mm	10^3 mm^3	10^3 mm^4	$10^9 mm^6$
W460												
x68	0.672	8 730	297	1 290	184	1 490	9.41	122	32.8	192	509	463
x60	0.584	7 590	255	1 120	183	1 280	7.96	104	32.4	163	335	388
x52	0.510	6 630	212	943	179	1 090	6.34	83.4	30.9	131	210	306

Figure 75 - Properties and dimensions of structural shapes (Canadian Institute of Steel Construction, 2000, pp. 6-48)

From this table we read that for a structural member whose cross-sectional shape is designated W460x68 (in the column *Designation*), the moment of inertia is listed in the column I_x as 297 x 10^6 mm^4. A member whose cross-section is W460x52 has a moment of inertia of 212 x 10^6 mm^4.

5.2.3 Mechanics of materials

While statics allows an engineer to determine the resulting effects of forces that act upon a member, it cannot help in deciding if the member is strong enough to resist those forces. Whether or not a member will safely carry a load or break as a result of it depends not only on the magnitude of the force, but also on the material the member is made from, as well as the size and shape of the member itself, including its cross-section. The properties of the cross-section that are of particular interest are the area and the moment of inertia. The study of these properties and their effects on a member's ability to resist applied forces is called mechanics of materials (Beer & Johnston, 1992).

Mechanics of materials builds upon the concept of force from statics and explores the properties of deformable bodies. Even if an object is in static equilibrium, applying enough force will cause it to deform; it can stretch, compress, bend, twist, or any combination thereof. This is due simply to the physical and chemical properties of the material that the object is made of.

A concept that is introduced in mechanics of materials is that of stress: the amount of force applied per unit area. Symbolically, stress is represented by the lower case Greek letter sigma:

$$\sigma = \frac{F}{A}$$

Its unit of measure is the pascal (Pa), where $1\ Pa = 1\ N/m^2$. Due to their strength, measuring stress in materials such as steel and aluminum requires the use of orders of magnitudes such as the megapascal (MPa) or gigapascal (GPa).

Applying a force of equal magnitude to two objects with different cross-sectional areas will have different effects, since the object with the smaller area will be subjected to a larger stress (Figure 76). Here an engineer is required to understand direct and inverse relationships. While stress is directly proportional to force, it is inversely proportional to area.

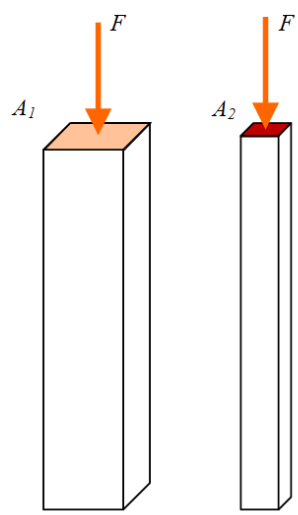


Figure 76 - Stress in a structural member (drawing is my own)

While both members in Figure 76 are subjected to the same applied force, F, since the area A_1 of the member on the left is greater than the area A_2 of the member on the right, the stress σ_1 in the member on the left is less than the stress σ_2 in the member on the right.

$$\sigma_1 = \frac{F}{A_1};\ \ \sigma_2 = \frac{F}{A_2};\ \ A_1 > A_2 \Rightarrow \sigma_1 < \sigma_2$$

If enough force is applied to an object, the stress it creates in the material will cause the object to deform. Deformations can be measured either as an absolute quantity, or as an amount relative to the original, un-deformed length of the object. Consider a steel rod of length 50 mm (Figure 77) to which we apply enough force until its length is stretched 50.75 mm.

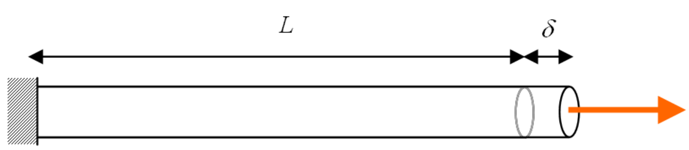


Figure 77 - Elongation a steel rod (drawing is my own)

The deformation can either be calculated in absolute terms, as follows:

$$\delta = 50.75\ mm - 50\ mm = 0.75\ mm$$

or as a ratio of the deformation to the original length. This relative deformation is called strain, a displacement per unit length, and it is represented by the lower case Greek letter epsilon:

$$\varepsilon = \frac{\delta}{L}$$

For the rod in Figure 77:

$$\varepsilon = \frac{\delta}{L} = \frac{0.75\ mm}{50\ mm} = 0.015$$

Since displacement and length are both measured with the same units, it would appear that strain is a unit-less quantity, but this is not the case. Though the units are often omitted, it is understood that the units of strain are mm/mm. Multiplication factors can be used in order to express strain in units such as mm/m (millimetres per metre), or $\mu m/m$ (micrometres per metre).

In section 5.1, Hooke's Law was mentioned as an example of an empirical mathematical model, relating weights attached to a spring and the resulting elongation. This relationship is also found between the applied stress and the resulting strain in certain materials. In laboratory sessions of their mechanics of material courses, engineering students perform a standard experiment on material samples. Stress is applied in increasing increments on the samples, and the resulting strain is measured. Plotting the results produces a graph similar to the one shown in Figure 78.

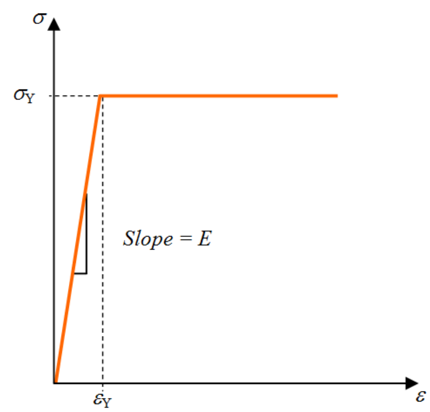


Figure 78 - Stress-strain diagram for a ductile material (drawing is my own)

A civil engineer would instantly recognize this drawing as the "stress-strain diagram" of a ductile material[5]. A mathematician would notice that the independent variable described in the experiment (stress, σ) is plotted on the ordinate (y-axis) while the dependent variable (strain, ε) is plotted on the abscissa (x-axis). The principal reason for this is to be able to express the relationship between stress and strain mathematically as:

$$\sigma = E\,\varepsilon$$

The coefficient of the linear relationship is called the modulus of elasticity, E. Graphically, the modulus of elasticity is the slope of the linear portion of the stress-strain diagram. Due to the nature of the units of stress and strain, it is more convenient to express the linear relation in this manner so that the units of the modulus of elasticity match those of stress, i.e., it has units of pascal, megapascal, or gigapascal.

[5] Ductility refers to a material's ability to yield under stress. A ductile material continues to deform without any increase in applied stress. Examples of ductile materials include steel, aluminum, and other metal alloys. Brittle materials on the other hand are prone to rupture, or breaking without yielding. Examples include glass, masonry, and cast iron.

The values σ_Y and ε_Y indicated on the axes in Figure 78 are referred to as the yield stress and yield strain of the material. If the applied stress remains below the value σ_Y, then the linear relation holds true, and the material is said to be elastic. Much like how a rubber band returns to its original shape after being stretched, the strain in elastic materials is eliminated once the applied stress is removed. Applying stresses greater than σ_Y causes excessive and permanent deformations in the material. At this point the material is said to become plastic. This is represented by the horizontal portion of the stress-strain diagram, indicating that strain, and therefore deformation, can increase without any corresponding increase in applied stress.

In general, an engineer seeks to design a structure so that the stresses in its structural members remain below the yield stress of the material they are made from. If this is achieved, then the members will not rupture. It is also important that the resulting deformations do not exceed the yield strain, as excessive deformations in a structure are unfavourable as well. Recall that the principle of static equilibrium depends on the assumption that the deformations of bodies are negligible. If this assumption is violated then a mathematical model at the heart of structural analysis becomes invalidated.

The modulus of elasticity appears in nearly every mathematical model that is used in structural analysis. The proper choice of material is vital to a structure's ability to resist the loads that are applied to it. And yet, the mathematics behind its development is remarkably simple: ratios of forces to areas (stress), deformations to lengths (strain), and the linear relationship found through empirical tests. It is the use of mathematics to describe these concepts that allows material properties such as elasticity to be understood.

The linear equation relating stress and strain via the modulus of elasticity is an empirical model that can be used for introductory application problems in mechanics of materials. But more advanced mathematics is required in order to develop concepts that are needed in structural analysis. Differential equations are used, for example, in a model for determining the maximum amount that a beam will deform when it is bent under an applied load (Beer & Johnston, 1992). A beam is a horizontal structural member, and its deformed shape is referred to as its elastic curve. While the details of its development are beyond the scope of this thesis, the following differential equation is used to relate the second derivative of the elastic curve to the bending moment (the bending force) in the beam, as well as the beam's modulus of elasticity and moment of inertia:

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

The function $y(x)$ whose second derivative is shown is the equation of the beam's elastic curve, with y representing the vertical distance that the beam is bent, and x the horizontal position along the length of the beam. Note that the symbols x and y are used to represent physical distances in the horizontal and vertical directions, and are not simply arbitrary variables. $M(x)$ is the magnitude of the bending moment at a position x along the length of the beam. Depending on what external forces are being applied to the beam, the bending moment at any position, x, along the length of the beam will vary. E and I are, respectively, the modulus of elasticity of the material the beam is made of and the moment of inertia of the beam's cross-section. The product EI is called the flexural rigidity of a member's cross-section.

Deriving this differential equation requires analyzing the curvature of the elastic curve $y(x)$. In the analysis, the assumption is made that dy/dx, the slope of the elastic curve, is very small compared to unity. As such, the actual differential equation:

$$\frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} = \frac{M(x)}{EI}$$

is simplified by neglecting the denominator in the term of the left hand side. This assumption is not unreasonable, as it is essentially the same assumption used in the principle of static equilibrium: deformations are considered small relative to the size of the object being deformed. If the slope of the elastic curve, dy/dx, is indeed negligible, then so is its square (text in read in the equation above). Thus the denominator on the left hand side is simply equal to 1.

If the magnitude and location of the forces applied to the beam are known, then the equation defining $M(x)$ can be determined from the principles of statics, and the general solution of the differential equation is shown to be (Beer & Johnston, 1992):

$$y(x) = \frac{1}{EI}\int_0^x dx \int_0^x M(x)dx + C_1 x + C_2$$

The constants C_1 and C_2 must be determined from the support conditions at the ends of the beam, and whether they are free to displace or rotate. Deflections and rotations can be set equal to 0 at either end depending on the type of support that holds the beam in place.

This model is shown in engineering textbooks to illustrate that a simply supported beam with a uniformly distributed loaded has a maximum displacement, y_{max}, at its mid-length of:

$$y_{max} = \frac{5\, w\, L^4}{384\, E\, I}$$

If an engineer is prudent with their use of units, the value of maximum displacement that results from this calculation will be in appropriate units such as millimetres. To accomplish this, the following units should be used for the other terms in the equation:

- The applied load, w: kN/mm
- The length of the beam, $L: mm$
- The modulus of elasticity, $E: GPa = kN/mm^2$
- The moment of inertia, $I: mm^4$

The formulae for maximum deflection under different loading configurations are all evaluated using this same differential equation. Many loading configurations can appear in different design situations, and so the solutions to the differential equations of the most common configurations have already been evaluated, and their results compiled in convenient tables that professional engineers can reference in design manuals, such as that shown in Figure 79, taken from the Handbook of Steel Construction (2000). While an engineer learns how to use the differential equation model as a student, their professional practice is made more efficient by not having to retrace the same calculations repeatedly.

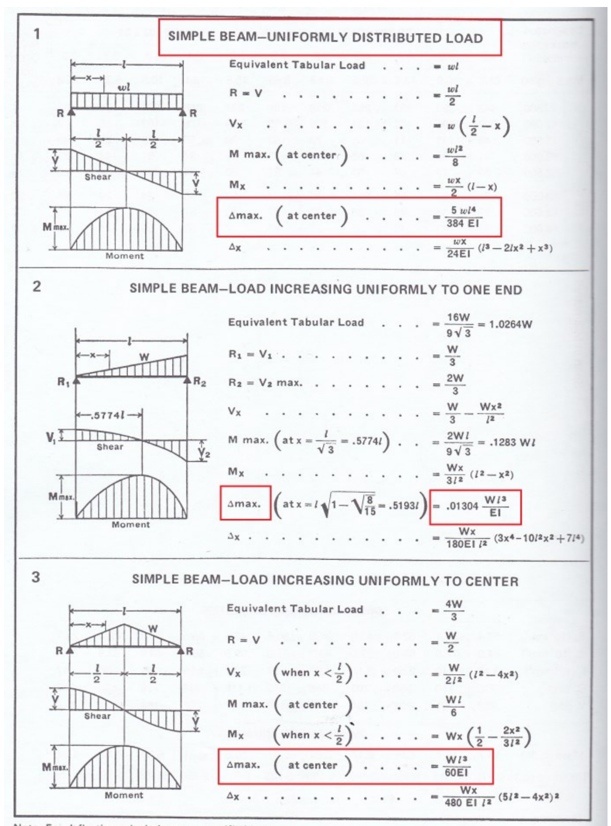


Figure 79 - Beam diagrams and formulae (Canadian Institute of Steel Construction, 2000, pp. 5-132)

The top section of Figure 79 shows the situation described in my example: a simple beam with a uniformly distributed load. That is, rather than having a single force applied at a unique point on the beam, there is an equal amount of force being distributed across its entire length. The weight of the beam itself is modelled in such a manner. The value labeled Δmax is the maximum displacement of the beam under the described loading condition. The bottom section of the figure shows that a simple beam, with a distributed load that increases uniformly from 0 at either end to some amount w at its centre, will experience a maximum deformation of:

$$\Delta \max = \frac{wL^3}{60EI}$$

Models such as these are essential in performing structural analysis, discussed in the next section.

5.2.4 Structural analysis

The goal of structural analysis is to predict the performance of a structure under prescribed forces, or loads. Loads can come from a variety of sources including the weight of the structure itself, from the use of the structure's occupants, and from other external effects such as wind, snow, earthquakes, and temperature changes. The performance characteristics that a structural engineer is interested in determining are the stresses in the structural members, the deflections or deformations of the structure, and the magnitude of the reaction forces in the structure's supports.

Structural analysis is one phase in the iterative process of structural design. The analysis phase allows the engineer to verify whether a design meets the criteria for safety (resistance to loads) and serviceability (limited deformation). If either of these criteria isn't met, the design is revised, and the analysis is performed anew.

A crucial step prior to the analysis is estimating the loads, or forces, that will act upon the structure. The National Building Code of Canada (NBCC) is the model code[6] upon which provincial jurisdictions base their own design and construction codes. Part 4 of the NBCC states that structural design be performed

[6] The word "model" in this case does not refer to a mathematical model, but is used in the sense of a guide. The various provincial building codes are written using the National Building Code of Canada as a model for their content.

using a method called limit states design. In limit states design, the factored resistance of a structure must be greater than the factored effect of the loads applied to the structure. This is expressed as:

$$Factored\ Resistance\ \geq Effects\ of\ Factored\ Loads$$

and alternatively as:

$$\frac{Factored\ Loads}{Factored\ Resistance} \leq 1$$

This inequality features prominently in design and analysis calculations performed by practicing engineers. Examples of its use will be shown in section 5.4.

The key to understanding limit states design lies in the use of the word "factored." In brief, the design process will over-estimate the magnitude of applied loads and under-estimate the strength of the structural members. This provides a margin of safety in the design of the structure. As the name implies, the process involves multiplying the values of calculated loads and resistances by factors. The terms "factored resistance" and "factored loads" can each be written symbolically as (Canadian Institute of Steel Construction, 2000):

$$\varphi R \geq \alpha_D D + \gamma\ \psi\ (\alpha_L L + \alpha_W W + \alpha_T T)$$

On the left hand side of the inequality, R is the resistance of a structural member. It is evaluated using models developed in mechanics of materials and in specialized courses on the design of structures. For the terms on the right-hand side of the inequality, the NBCC includes formulae and tables that are to be used to estimate the various loads that a structure must resist. Included among these are (National Research Council of Canada, 2010):

- Dead loads (D): the weight of the structure itself and all of the materials used in its construction
- Live loads (L): the load on an area of floor or roof depending on the intended use and occupancy; loads due to ice, snow, and rain are also considered live loads
- Wind loads (W): the load caused by wind blowing on the structure

- Loads induced the structure by changes in temperature (T)

The various coefficients α_i are load factors, while γ and ψ are the importance factor and the load combination factor, respectively. These factors have the effect of amplifying the prescribed loads that are applied to the structure in the analysis. The values assigned to them are chosen from pre-determined load combinations that are prescribed either by the NBCC, or the appropriate design manual. For example, in the Handbook of Steel Construction, clause 7.2.3 of the Limit States Design of Steel Structures states that the load factors α are to taken as:

- $\alpha_D = 1.25$
- $\alpha_L = 1.50$
- $\alpha_W = 1.50$
- $\alpha_T = 1.25$

Thus, the loads are automatically increased by either 25%, or up to 50% of those suggested by the NBCC. The value of ψ, however, may be less than 1. This would appear to have the effect of reducing the estimated load, but this is actually not the case. In the Handbook of Steel Construction, the value of ψ is taken to be 1.00 when the structure is analyzed under the effects of only one of the loads L, W, and T, but it can be taken to be 0.70 when the effects two of L, W, and T are analyzed. In other words, when analyzing the structure's performance against only live loads, L, 100% of the factored loads must be applied to the structure in the analysis. But, when analyzing its performance against live loads, L, and wind loads, W, only 70% of the factored loads need to be applied in the analysis. The underlying reason for this is due to statistical analysis. The load estimates for L and W provided by the NBCC are the maximum loads that the structure could expect to encounter in its life. The probability that the structure will have to resist the maximum values of L and W simultaneously is small; hence the estimated factored loads can be reduced. This is an example of how statistics are intrinsic to some empirical mathematical models used in the engineer's design process.

Engineers have several techniques available to accomplish the task of structural analysis. Since the advent of computer programming, techniques involving linear algebra and matrices have become prevalent. Such techniques are collectively referred to as matrix structural analysis. The most widely used analysis technique, called the direct stiffness method, is described in some detail in the following section since it can illustrate some different perceptions that engineers and mathematicians have of matrices as mathematical objects.

5.3 MATRIX STRUCTURAL ANALYSIS

In the previous sections I described the fundamental principles of structural engineering and how engineers use the mathematics they learn to create mathematical models that represent various physical concepts such as forces, stress and strain, the modulus of elasticity, and the moment of inertia. These principles gave way to a brief introduction to structural analysis and its goal of predicting the performance of a structure under the effects of applied loads. The purpose of these sections was also to lay the groundwork for the text that follows, which explains how matrix structural analysis is used as a technique to accomplish the task of predicting structural performance.

In preparation for my research for this thesis, a discussion about linear algebra and its practical uses prompted the following question: do matrix properties such as eigenvectors have any physical meaning to a structural engineer? The answer, as I will show, is yes, they do. After presenting the details of matrix structural analysis, I will discuss the physical representations of eigenvectors in this context. However, it will then be shown that, for a structural engineer, an eigenvector's physical representation holds no interest in the task of structure analysis, specifically because of what they represent. This is not to suggest that eigenvectors don't hold any meaning at all, it is simply an illustration of how the use of mathematics to model the physical world is context dependent.

5.3.1 The elastic stiffness matrix

Of particular interest in structural analysis is determining the displacements and rotations of a structure's joints due to the effects of applied forces. Joints are connection points between structural members; horizontal members are called beams and vertical members are called columns. Each individual member in a structure plays an important role in matrix structural analysis.

A member whose displacements are restricted to a two-dimensional plane is called a plane member. Each end of a member has the potential to move, or be displaced, in one of three directions in the plane: horizontally, vertically, or by rotating about the axis perpendicular to the plane. In Figure 80, the horizontal line represents a beam. The left and right ends of the beam are referred as to end 1 and 2, and the potential displacements of each end are labeled d_1 through d_6 as follows:

- d_1 and d_4: Horizontal displacements at ends 1 and 2 respectively
- d_2 and d_5: Vertical displacements at ends 1 and 2 respectively
- d_3 andd_6: Rotation about the axis perpendicular to the plane at ends 1 and 2 respectively

Figure 80 - Potential displacements of a beam (drawing is my own)

The convention in matrix structural analysis is to label the displacements first at one end of the member, starting with the horizontal displacement, then the vertical displacement, and lastly the rotation about the axis perpendicular to the plane. The displacements are then labeled at the other end in the same order (Kassimali, 1999).

Forces are labeled following the same convention (Figure 81). Note that the forces labeled F_3 and F_6 are moments. These forces contribute significantly (though not exclusively) to the creation of the rotational displacements d_3 and d_6.

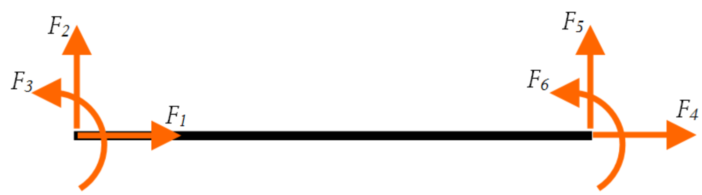


Figure 81 - Potential forces in a beam (drawing is my own)

The mathematical relationship between displacements and forces is linear, and the coefficient of this relationship is referred to as the member's stiffness. This relation can be represented mathematically by the matrix-vector equation:

$$\{\mathrm{F}\} = [\mathrm{K_e}]\,\{\mathrm{d}\}$$

where $\{F\}$ is the force vector whose entries are F_1 through F_6, $\{d\}$ is the displacement vector with entries d_1 through d_6, and $[K_e]$ is the member's elastic stiffness matrix. What an engineer wishes to determine from the analysis is the displacement vector, since, as discussed in section 5.2.4, the forces acting on the structure are known from the design process. The displacement vector is thus evaluated using the inverse of $[K_e]$ by the equation:

$$\{\mathrm{d}\} = [\mathrm{K_e}]^{-1}\,\{\mathrm{F}\}$$

Vectors $\{F\}$ and $\{d\}$ are of size $n \times 1$, where n is the number of displacements at the member ends, also called degrees of freedom. As seen in Figure 80, a plane member has 3 degrees of freedom at each end for 6 degrees of freedom in total. The vectors $\{F\}$ and $\{d\}$ for a plane member then are of size 6x1. For the matrix-vector equation to be compatible, the elastic stiffness matrix $[K_e]$ must therefore be of size 6×6 for a plane member.

The matrix equation can be thus expanded to show all of the elements of the vectors and the stiffness matrix:

$$\{\mathrm{F}\} = [\mathrm{K_e}]\{\mathrm{d}\}$$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}$$

In reality, the members in a structure aren't always restricted to movement in a plane. In a three-dimensional world, each member end can potentially be translated along, or rotated about, any of three independent axes. Each member would therefore have up to 12 degrees of freedom, making the stiffness matrix for a general structural member of size 12x12 (McGuire, Gallagher, & Zieman, 2000). Thus, an implicit assumption of the analysis of a plane frame is that the frame actually remains in the plane.

A structure is composed of several members connected at joints. The number of degrees of freedom for a structure is the total number of unrestrained displacements of the joints. Figure 82 depicts a structural plane frame made up of five members (three columns and two beams), with nine degrees of freedom (labelled d_1 to d_9).

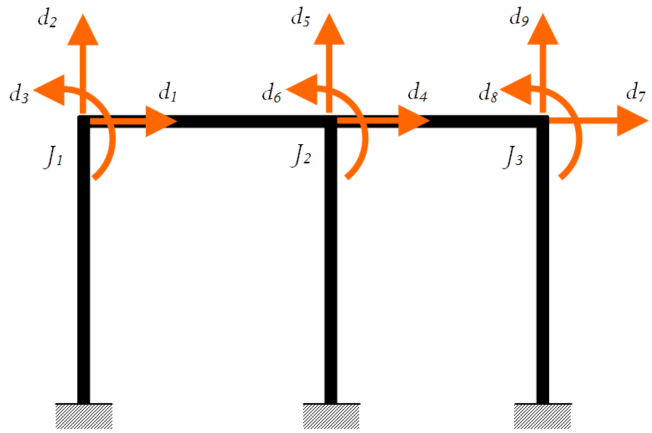


Figure 82 - Plane frame (drawing is my own)

The bases of the three columns are restrained by supports that connect the columns to the ground, restricting any potential movement. Thus, the degrees of freedom of those ends of the columns are eliminated, and are not counted among the degrees of freedom of the *structure*. But the joints at the top of the columns, labeled J_1, J_2, and J_3, are free to move within the plane. The displacement and force vectors for this structure would be of size 9x1, and its elastic stiffness matrix would be of size 9x9.

If this frame weren't restricted to the plane, then its stiffness matrix would be of size 18x18. It is perhaps obvious then why computer programs have become necessary for the computations involved matrix structural analysis. But the size of the matrix does not change the underlying mathematical relationship between forces and displacements, or the mathematical technique required to solve the matrix vector equation $\{F\} = [K_e]\{d\}$.

The values of each entry k_{ij} in the stiffness matrix $[K_e]$ are determined from the principles of static equilibrium and mechanics of materials – the basics of which were discussed in sections 5.2.2 and 5.2.3 – as well as principles known as structural compatibility and superposition. A structural member's physical and geometric properties of modulus of elasticity, E, cross-sectional area, A, moment of inertia, I, and length,L, all play a role in evaluating its stiffness.

Structural compatibility means that the displacement of a joint is shared by all of the members connected at the joint (Kassimali, 1999). If a joint connecting a column and a beam is displaced horizontally by 5 mm, then the ends of both the column and the beam are displaced horizontally by 5 mm. This may seem trivial, but stating this principle ensures that connected members in the real structure remain connected in the mathematical model as well. This principle could be considered a physical representation of the transitive property of equality: if the displacement of the column equals the displacement of the joint, and the displacement of the beam equals the displacement of the joint, then the displacement of the column equals the displacement of the beam.

The principle of superposition, for its part, states that the total displacement caused by a system of forces is equivalent to the sum of the displacements caused by each individual force applied separately (Kassimali, 1999). Physically this means that the displacement of a joint connecting a beam to a column is affected equally by the stiffness of both the beam and the column. In the matrix structural analysis model, this means that the stiffness of a joint is evaluated by simply adding the corresponding stiffness of all of the members connected at the joint.

With these principles defined, I can now present the values of the entries k_{ij} in the elastic stiffness matrix, $[K_e]$ for a member in a plane structure (Kassimali, 1999):

$$[K_e] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ \frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Each entry k_{ij} of is determined by answering the question “what force F_i creates a unit displacement in the direction of d_j, while keeping all other displacements 0?” For example, entry k_{32} has a value of $\frac{6EI}{L^2}$ because a rotational force, F_3, of this magnitude is required to create a vertical displacement of $d_2 = 1$, while keeping all other displacements 0.

Some typical values of E, A, and I are as follows:

- For structural steel, modulus of elasticity $E = 200\ GPa = 200\ \frac{kN}{mm^2}$
- For a member whose cross-section is the shape with designation W410x85:
    - $A = 10\ 800\ mm^2$
    - $I = 315\ \times\ 10^6\ mm^4$

The notation “W410x85” refers to a shape colloquially known as an “I beam”, but is technically called a “wide flange”, or W section. The value 410 refers to the nominal height of the cross section in mm; and 85 refers to the weight of the member per unit length, thus 85 kg/m.

For a member with these properties, and supposing a design length of 5 000 mm, the stiffness matrix is evaluated to be:

$$[K_e] = \begin{bmatrix} 540 & 0 & 0 & -540 & 0 & 0 \\ 0 & 11.8125 & 23625 & 0 & -11.8125 & 23625 \\ 0 & 23625 & 6.3\times10^7 & 0 & -23625 & 3.15\times10^7 \\ -540 & 0 & 0 & 540 & 0 & 0 \\ 0 & -11.8125 & -23625 & 0 & 11.8125 & -23625 \\ 0 & 23625 & 3.15\times10^7 & 0 & -23625 & 6.3\times10^7 \end{bmatrix}$$

A mathematician with a keen eye will notice that the columns of this matrix are linearly dependent; specifically the pairs of columns 1 & 4, and 2 & 5 are scalar multiples of each other. This matrix is singular. But this is no accident; it is a direct result of the principles of static equilibrium and mechanics of materials.

The entries in column 1 are the forces required to create a unit in the direction of the degree of freedom d_1, while keeping all other displacements 0 (Figure 83).

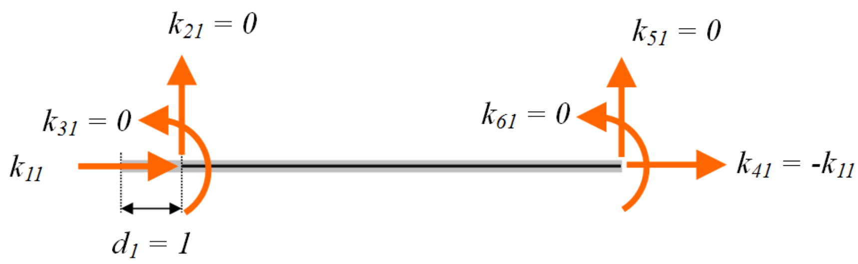


Figure 83 - Stiffness matrix entries for displacement d1 (drawing is my own)

The force k_{11} creates the desired displacement. At the same end, forces k_{21} and k_{31} are 0 since they don't create any displacement in the direction of d_1, and we wish to keep the displacements they do create equal to 0. For the same reason, forces k_{51} and k_{61} at the other end of the member are 0 as well. But a horizontal force k_{41} is needed in order to keep the member in static equilibrium. Without the force k_{41}, the member would simply be pushed away to the right, just like the block depicted in Figure 64 (section 5.2.2). The magnitude of the force k_{41} must be equal to that of k_{11}, but it must act in the opposite direction. Thus, $k_{41} = -k_{11}$. This creates the first column of $[K_e]$:

$$\begin{bmatrix} k_{11} & . & . & . & . & . \\ 0 & . & . & . & . & . \\ 0 & . & . & . & . & . \\ -k_{11} & . & . & . & . & . \\ 0 & . & . & . & . & . \\ 0 & . & . & . & . & . \end{bmatrix}$$

To create column 4, the same reasoning is used (Figure 84).

Figure 84 - Stiffness matrix entries for displacement d4 (drawing is my own)

In this case it is the force k_{44} that creates the desired displacement in the direction of d_4, while the forces k_{24}, k_{34}, k_{54}, and k_{64} are all 0; their actions neither create nor affect the displacement in the direction of d_4. As before, a horizontal force k_{14}, equal in magnitude but opposite in direction to k_{44}, is needed in to prevent a horizontal displacement at the other end and to keep the member in static equilibrium. Hence, $k_{14} = -k_{44}$. This creates the fourth column of $[K_e]$:

$$\begin{bmatrix} k_{11} & . & . & -k_{44} & . & . \\ 0 & . & . & 0 & . & . \\ 0 & . & . & 0 & . & . \\ -k_{11} & . & . & k_{44} & . & . \\ 0 & . & . & 0 & . & . \\ 0 & . & . & 0 & . & . \end{bmatrix}$$

The expression that represents the values of the forces k_{11} and k_{44}, namely the expression $\frac{EA}{L}$, is determined from the principles of mechanics of materials. Since the member's material and geometric properties are assumed to remain constant across its entire length, the forces needed to create the unit displacements at either end must be the same. This means that $k_{11} = k_{44}$. The result is that columns 1 and 4 become:

$$\begin{bmatrix} k_{11} & . & . & -k_{11} & . & . \\ 0 & . & . & 0 & . & . \\ 0 & . & . & 0 & . & . \\ -k_{11} & . & . & k_{11} & . & . \\ 0 & . & . & 0 & . & . \\ 0 & . & . & 0 & . & . \end{bmatrix}$$

So not only are columns 1 and 4 scalar multiples of each other, but because of the underlying physical principles of statics and mechanics of material, that scalar must necessarily be -1. The same reasoning holds for columns 2 and 5. The reasons for why the same is not true for columns 3 and 6 are beyond the scope of this thesis.

But now an interesting question must be asked: If the elastic stiffness matrix is singular, how can the displacement vector be calculated using the equation:

$$\{\mathrm{d}\} = [\mathrm{K_e}]^{-1}\{\mathrm{F}\}$$

since $[K_e]$ does not have an inverse?

The explanation is simple, but before presenting it, it should be noted that the singularity of $[K_e]$ is not something that most engineers would notice at first glance. For an engineer, a matrix is a tool that that is used in the technique to accomplish this particular task; the matrix itself is not the object of study.

The elastic stiffness matrix for a single structural member is singular, and so by definition it has no inverse. But the 6x6 matrix represents potential displacements in all directions, as if both ends of the member were completely unrestrained and free to move without bound. But this is not a true reflection of reality. Members in a structure are restrained; some are connected to other members at joints, while others are connected directly to ground via supports. At joints, the stiffness of other members prevents displacements from becoming too large, while supports effectively eliminate displacements altogether. This has an impact on the stiffness matrix, as will be shown shortly.

Supports connect a structural member to the ground and restrict displacements. Figure 85 depicts three different types of supports. Roller supports (Figure 85, left) allow horizontal displacements and rotations, but restrict vertical displacements; pinned supports (Figure 85, centre) allow rotations, but do not allow either horizontal or vertical displacements; fixed supports (Figure 85, right) do not allow displacements of any kind.

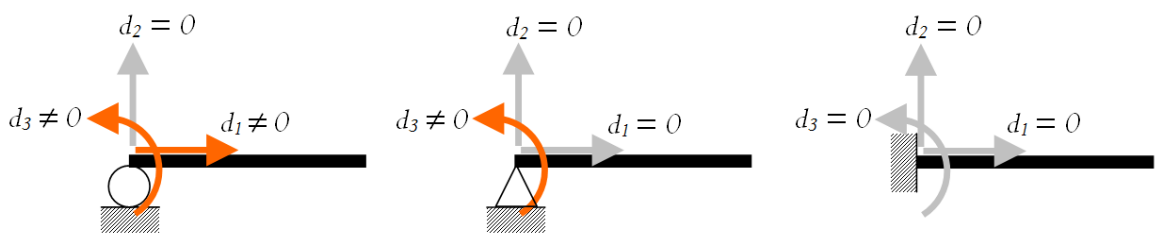


Figure 85 - Supports: Roller support (left), Pinned support (centre), Fixed support (right) (drawing is my own)

If the displacement at a support is 0, then its corresponding degree of freedom is removed from the structure. No amount of force will cause the member to move in the direction of the restricted displacement, and thus the row and column of the stiffness matrix that represents these forces and displacements are unnecessary to the analysis, and they are removed from the structure's stiffness matrix.

When all restricted degrees of freedom are removed from the stiffness matrix, the resulting matrix will be non-singular. In fact, a singular stiffness matrix necessarily implies that the structure it represents is unstable and at risk of having unbounded displacements which can lead to the structure's collapse. Hence, the mathematics of this technique of matrix structural analysis has built into it a way of ensuring that the engineer designs a stable structure that is properly supported.

Once a structure's stiffness matrix is properly evaluated and the load vector $\{F\}$ is assembled from the factored loads applied to the structure, the displacement vector can be calculated by using:

$$\{d\} = [K_e]^{-1}\{F\}$$

The following is an example of a displacement vector for a member in a plane frame resulting from an analysis using the direct stiffness method:

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix} = \begin{bmatrix} 3.0465\times 10^{1} \\ -3.9847\times 10^{-1} \\ -1.0225\times 10^{-2} \\ 2.9885\times 10^{1} \\ -4.5031\times 10^{-1} \\ -4.1687\times 10^{-3} \end{bmatrix} \begin{matrix} mm \\ mm \\ rad \\ mm \\ mm \\ rad \end{matrix}$$

The units are shown next to the displacement vector for clarity though it is the engineer's responsibility to interpret the displacements as shown in the vector. Computer software designed specifically for structural analysis can display the results visually so the engineer can see the deformed shape of the member. In Figure 86 we see the analysis results for a frame similar to that in Figure 82. In this structure, the five members are made of steel; the shape of the columns is W310x97, and the beams are W200x22. The beam on the right half of the frame is the member whose displacement vector is presented above. Its left end is translated horizontally to the right by 30.465 mm, vertically downwards by 0.39847 mm, and rotated clockwise by 0.010225 radians. Similarly, the right end of the beam is translated horizontally to the right by 29.885 mm, vertically downwards by 0.45031 mm, and rotated clockwise by 0.0041687 radians.

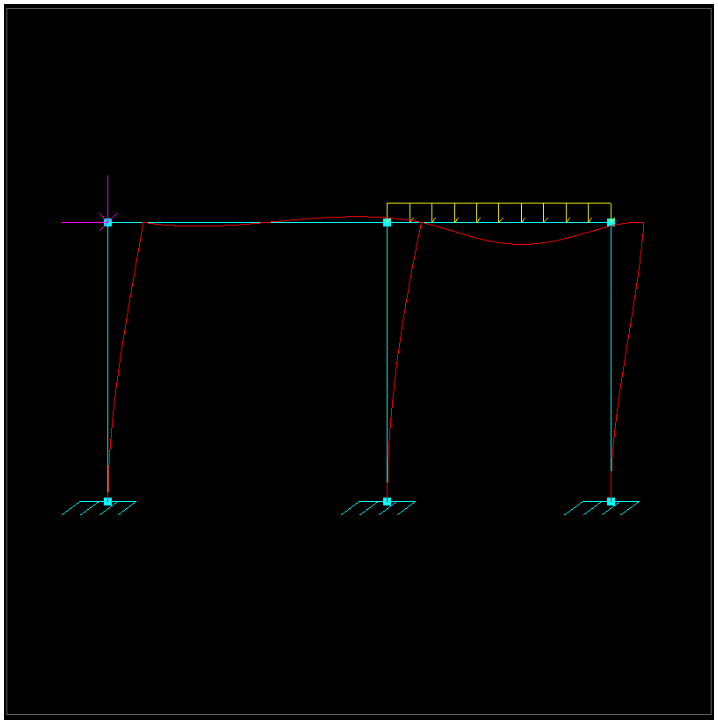


Figure 86 - Deformed shape of a plane frame (Software provided with (Kassimali, 1999))

Following this analysis the engineer would have to decide if these displacements are acceptable or not for the intended use of the truss. If the answer is no (and with horizontal displacements in excess of 30 mm this is a near certainty), then the members will be resized, the loads re-evaluated, and the structure subjected to a second analysis to verify the new displacements.

The complete report created by this software, including all of the data input and the output of the analysis is included in appendix 8.4.

5.3.2 The eigenvectors of an elastic stiffness matrix

Consider the structure in Figure 87, and suppose that the engineer has determined that loads applied to it are as shown. In order to simplify the demonstration that follows this particular structure is modelled so that its members do not experience any rotations at their ends. Such a structure is called a plane truss.

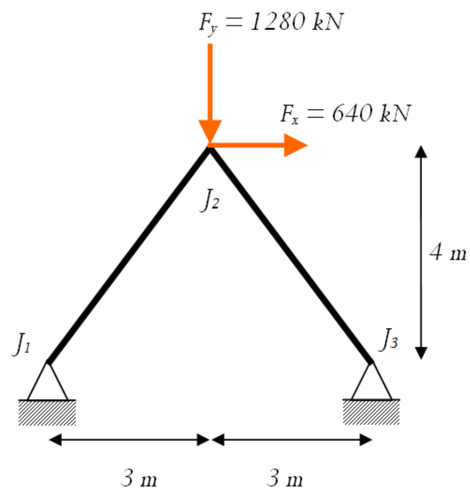


Figure 87 - Plane truss (drawing is my own)

Since the bottoms of the two members are connected to the ground by pinned supports at joints J_1 and J_3, their horizontal and vertical displacements are restricted, so they are not free to move. Only joint J_2 can be displaced by the applied forces. Thus, this structure has only two degrees of freedom: d_1 in the horizontal direction and d_2 in the vertical direction. Its elastic stiffness matrix will therefore be 2×2.

Suppose the truss members were designed with the following material and geometric properties:

- Member 1:
  - $E = 200\ GPa = 200\ kN/mm^2$
  - W410x85: $A = 10\ 800\ mm^2$

- Member 2:
  - $E = 200\ GPa = 200\ kN/mm^2$
  - W410x39: $A = 4\ 990\ mm^2$

Since the members are not subjected to rotations, their cross-sectional moments of inertia are not required. The stiffness matrix of this truss can be shown to be (Kassimali, 1999):

$$[K_{truss}] = \begin{bmatrix} 227.376 & 111.552 \\ 111.552 & 404.224 \end{bmatrix}$$

It is worth noting that the stiffness matrices of all stable structures are symmetric. This is by virtue of the principles of structural compatibility and superposition. With this stiffness matrix and the known loads, the displacements of joint J_2 can be calculated:

$$\{\mathrm{d}\} = [\mathrm{K_e}]^{-1} \{\mathrm{F}\}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 227.376 & 111.552 \\ 111.552 & 404.224 \end{bmatrix}^{-1} \begin{bmatrix} 640 \\ -1280 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 5.0523 \times 10^0 \\ -4.5608 \times 10^0 \end{bmatrix} \begin{matrix} mm \\ mm \end{matrix}$$

These results indicate that joint $\mathrm{J_2}$ will displace horizontally to the right by 5.052 mm, and vertically downwards by 4.561 mm. Figure 88 shows the deformed shape of the truss under the given applied loads (Kassimali, 1999).

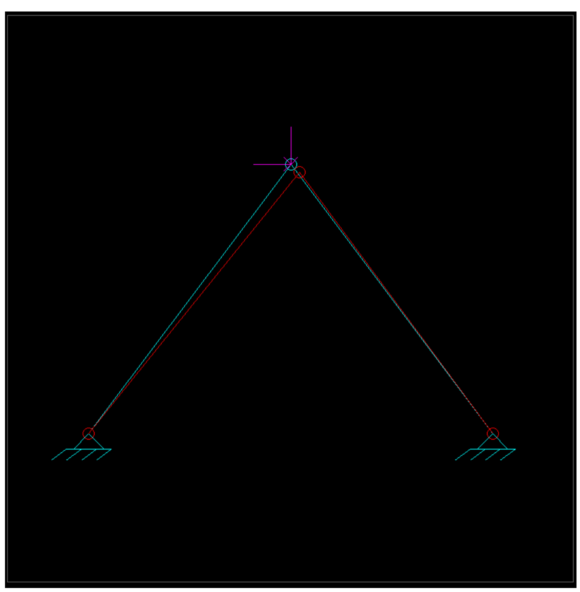


Figure 88 - Results from a structural analysis: deformed shape of a plane truss (Software provided with (Kassimali, 1999))

A mathematician might now be curious about what representation the eigenvectors of the matrix $[\mathrm{K}_{\mathrm{truss}}]$ might have. Using the mathematical software Maple, its eigenvalues, λ_{i}, and eigenvectors, v_{i}, are determined to be:

$$\lambda_1 = 173.453 \qquad \lambda_2 = 458.147$$

$$\mathrm{v}_1 = \begin{bmatrix} -0.900 \\ 0.435 \end{bmatrix} \quad \mathrm{v}_2 = \begin{bmatrix} -0.435 \\ -0.900 \end{bmatrix}$$

Notice that vectors v_1 and v_2 are unit vectors. This is not coincidental, especially considering how the entries of the stiffness matrix are obtained. Regardless, a set of eigenvectors could always be normalized.

Since these are eigenvectors of a stiffness matrix, $[\mathrm{K}_{\mathrm{truss}}]$, then physically they must represent displacements of the joint J_2. The forces that cause the displacements represented by the eigenvector v_1 are:

$$\mathrm{F}_1 = \mathrm{K}_{\mathrm{truss}} \mathrm{v}_1$$

Since v_1 is an eigenvector, this equation can be written as:

$$\mathrm{F}_1 = \lambda_1 \mathrm{v}_1$$

If we evaluate the Euclidean norm of each of these vectors we find that:

$$\|\mathrm{F}_1\| = \|\lambda_1 \mathrm{v}_1\|$$

$$\|\mathrm{F}_1\| = |\lambda_1| \|\mathrm{v}_1\|$$

Since the eigenvector v_1 is a unit vector, its Euclidean norm is 1, giving us:

$$\|\mathrm{F}_1\| = |\lambda_1|$$

From this we can see that the eigenvalues of a stiffness matrix tell us the magnitude of the eigenforces that create the unit eigendisplacements. The vector F_1 can be found using eigenvector v_1:

$$[F_1] = \begin{bmatrix} \mathrm{F}_{1x} \\ \mathrm{F}_{1y} \end{bmatrix} = \begin{bmatrix} 227.376 & 111.552 \\ 111.552 & 404.224 \end{bmatrix} \begin{bmatrix} -0.900 \\ 0.435 \end{bmatrix} = \begin{bmatrix} -156.165 \\ 75.488 \end{bmatrix} \begin{matrix} \mathrm{kN} \\ \mathrm{kN} \end{matrix}$$

Expressing vectors F_1 and v_1 in terms of their magnitude and direction yields:

$$\mathrm{F}_1 = 173.453 \text{ kN}, 154.20^{\circ}$$

$$\mathrm{v}_1 = 1 \text{ mm}, 154.20^{\circ}$$

Notice that the magnitude of F_1 is the eigenvalue λ_1. The interpretation of this is that applying a force with magnitude of 173.453 kN at an angle of 154.20° with the horizontal direction results in a displacement of 1 mm in the same direction as the applied force. This is the physical representation of the eigenvectors of a structure's elastic stiffness matrix. This situation is represented in Figure 89 (not to scale). The same interpretation exists for eigenvalue λ_2, and eigenvector v_2, along with its associated eigenforce, F_2. The direction of these latter vectors is orthogonal to the direction of vectors v_1 and F_1.

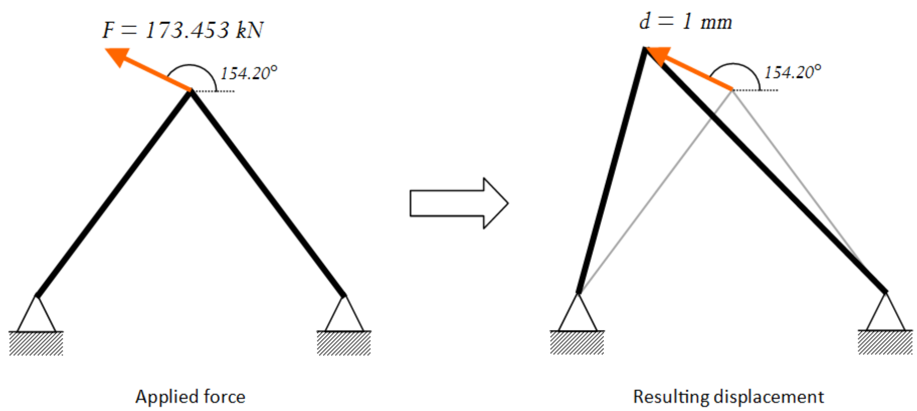


Figure 89 - Eigenvectors of a plane truss: applied force and resulting displacement are in the same direction (drawing is my own)

Though it is interesting to discover that eigenvectors in this model do have physical meaning, it is unfortunately irrelevant to the structural engineer performing this analysis. Recall that structural analysis is only one stage in the design process. Prior to analyzing a structure, the engineer must first ascertain the loads that will be applied to it. Thus, the forces acting on the structure are known ahead of time; their magnitudes and direction are pre-determined. It is the displacements that are sought, and whether or not they occur in the same direction as the net forces is of no consequence; what matters is their magnitude. Furthermore, the effect is difficult to demonstrate in the context of computer software since the applied forces must be input as component vectors acting in the horizontal and vertical directions. The force in Figure 89 would be input in to the software as two forces: a horizontal force with magnitude -156.165 kN, and vertical force with magnitude 75.488 kN. The output from the software, namely the displacement caused by the forces, would be displayed as two displacements in the same component directions. It's only in combining the components that the eigenvector effect becomes apparent, and this process can deter engineering students from the actual purpose of their analysis.

This particular example of eigenvalues was presented because it relates to a mathematical model that all civil engineering students learn in a required structural analysis course, and because it was an interesting exercise in examining an engineering model from a mathematics educator's perspective.

While the eigenvectors have no importance in this instance, there are cases where the eigenvalue of a matrix is important. In critical load analysis of columns, the largest eigenvalue of a stiffness matrix is used to determine the smallest force that will cause a structural column to buckle. The derivation of the model requires advanced engineering knowledge that is beyond the scope of this thesis (specifically non-linear structural analysis, which accounts for geometric changes in a structural member, i.e., changes in cross-sectional area or moment of inertia along the length of the member).

This kind of structural analysis is performed for members subjected to loads as shown in Figure 90. The column shown is being subjected to both a vertical load and a horizontal load. The horizontal load causes the top of the column to displace slightly. The vertical load is now being applied to the column a distance Δ away from the axis of the column. Thus, the load P now creates a moment about the bottom of the column causing it to rotate and deform even further, increasing the distance Δ. The creates a feedback loop known as the P-Δ effect.

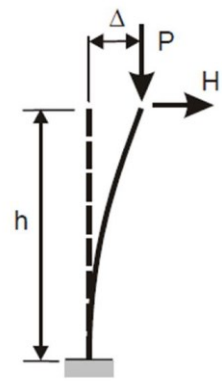


Figure 90 - Critical load analysis of a column - P-Δ effect

In the matrix analysis of this situation, the largest eigenvalue of the matrix corresponds to the smallest critical load, P, that can be applied to the column so that it remains stable. The eigenvector tells the engineer the column's final displacement. The other eigenvalues and eigenvectors play no role in the analysis since the engineer is looking to find the smallest critical load. Applying loads greater than this will cause the column to "buckle", i.e., to be unstable.

5.4 Mathematics in the engineer's workplace

The vocational mathematics of nurses, airline pilots, and construction workers has been studied in some detail in past mathematics education research (LaCroix, 2014; Roth, 2014; Coben & Weeks, 2014; Wake, 2014). These are all occupations that require some mathematics training and education, but once the practitioners reach the workplace much of the mathematics is hidden or black-boxed, and the workers describe the mathematics that they do use as rules of thumb. Many of them also take advantage of tabulated data.

To a certain extent, the same can be said for professional engineers. Design codes contain all of the formulae that engineers need to use in their design and analysis tasks, as well as tables of data with pertinent information that simplify many of the computations in the design process.

As a student, an engineer will learn how to calculate the factored resistance of a steel beam and determine whether or not their choice of beam will be able to resist the factored loads. But since the cross-sectional shapes used in steel construction are all standardised, their resistances for various lengths of beams have already been calculated and the results tabulated. Once an engineer knows the factored load that a beam must resist, they can look up an adequately sized cross-section size in a table that lists beam resistances.

Recall that not all practicing engineers are involved in the design process. Many are charged with project management, ensuring that a structure meets not only design specifications prescribed by code, but also those specified by the client, and that the project is completed on time and on budget. The most arduous mathematics used by these engineers is in the proper management of resources and personnel in an effort to complete a project efficiently. The use of tabulated data is thus a necessity for the designers to work efficiently.

5.4.1 *Dossier de calculs*

Returning to the tasks of a structural design engineer, another important outcome of structural analysis is the magnitude of the forces in the structure's supports. The forces applied to the structure are supported by the structural members and transferred into the ground via the supports. The supports therefore have to be designed to resist the loads that pass through them.

In this section, I present the details a work document called a *dossier de calculs* that was prepared by a practicing engineer for the design of a structural support. The document contains calculations that serve to determine the factored resistance of the support, and to verify that the resistance is greater than the factored loads, as prescribed by the limits states design model described in section 5.2.4. The calculations are written on a sheet graph paper with numbered rows and columns which will make identifying calculations in the images easier on the reader.

In Figure 91, the dimensions of the base plate – a thick rectangular plate made of steel on which the column will rest – are selected based on the known size of the column that it will support (row 7).

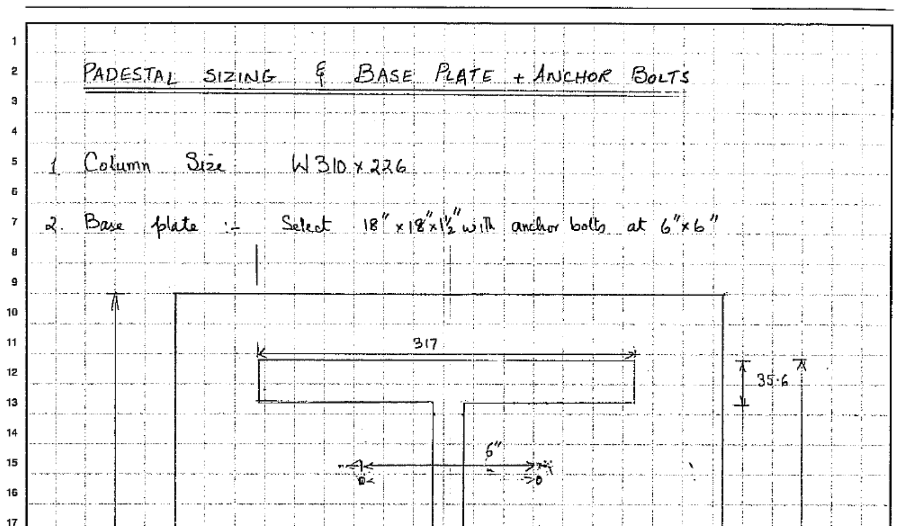
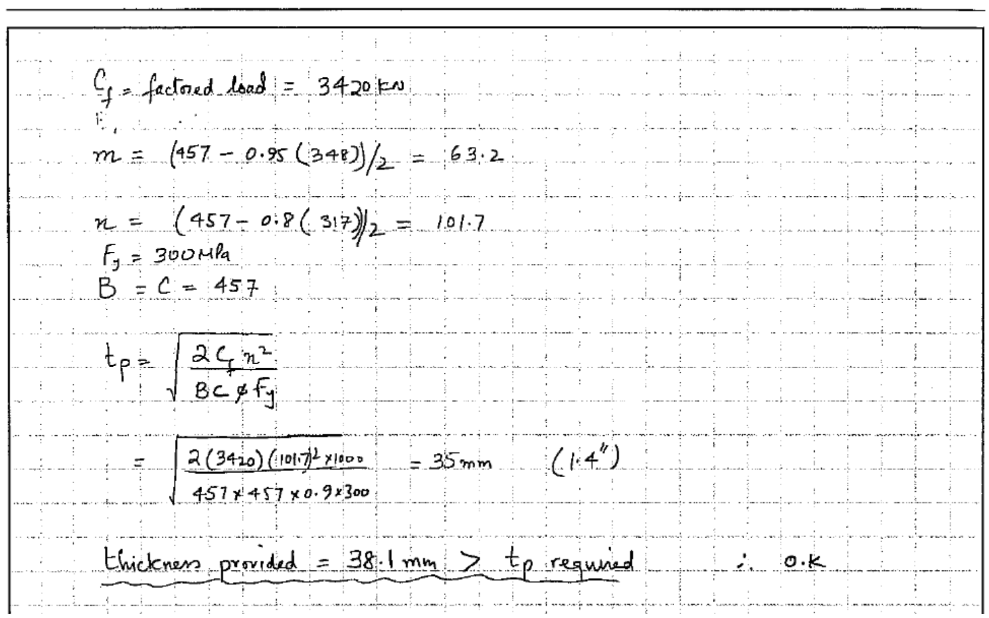


Figure 91 - Dossier de calculs: Base plate dimensions (Source: Obtained from a licensed engineer)

The selected dimensions are 18 inches x 18 inches x 1 ½ inches. Note that until this point, the units I have used have all been chosen from the SI system. However, in the construction industry, even in Canada, the Imperial system of measurement is still in frequent use. As a result, practicing engineers must be familiar with, and be able to convert measurements from both systems. In SI units, the base plate's dimensions are 457 mm x 457 mm x 38.1 mm.

In Figure 92, the engineer checks that the selected thickness of 38.1 mm is greater than the minimum thickness required by the design code.



C_f = factored load = 3420 kN

$m = (457 - 0.95(348))/2 = 63.2$

$n = (457 - 0.8(317))/2 = 101.7$

$F_y = 300 MPa$

$B = C = 457$

$$t_p = \sqrt{\frac{2 C_f n^2}{B C \phi F_y}}$$

$$= \sqrt{\frac{2(3420)(101.7)^2 \times 1000}{457 \times 457 \times 0.9 \times 300}} = 35 mm \quad (1.4'')$$

thickness provided = 38.1 mm > t_p required ∴ O.K

Figure 92 - Dossier de calculs: Checking base plate thickness (Source: Obtained from a licensed engineer)

The formula in row 10 determines the minimum required thickness of the base plate. It is an empirical equation (developed through research and experimentation) that requires knowing the factored load being transferred from the supported column into the base plate ($C_f = 3420\ kN$, row 2), the yield stress of the material that will be used to fabricate the base plate ($f_y = 300\ MPa$, row 7), and the dimensions of the base plate ($B = C = 457\ (mm)$, row 8). The term ϕ is a resistance factor for structural steel whose value is most frequently prescribed as 0.9. This factor takes into account the variability of material properties due to uncertainty in production and fabrication, and has the effect of under-estimating the strength of the material and the members made from it, as described in the method of limit states design (section 5.2.4).

In row 13 we see that the engineer has determined that a minimum thickness of 35 mm is required, but since the selected thickness of 38.1 mm is greater than the required minimum, the base plate is OK (row 16), and the calculations can proceed.

In a similar manner, the engineer checks that the concrete foundation pedestal upon which the base plate rests can also resist the force being transferred from the column (Figure 93).

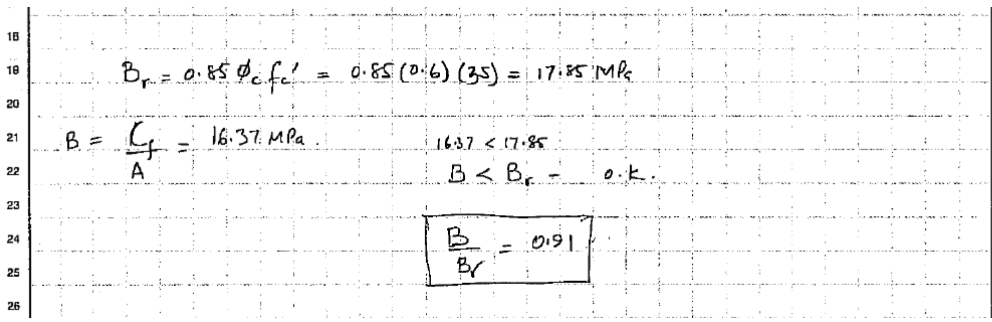


Figure 93 - Dossier de calculs: Checking concrete pedestal bearing stress (Source: Obtained from a licensed engineer)

In row 19, using another empirical formula that requires knowing the factored strength of the concrete to be used ($\phi_c f'_c$), the concrete pedestal is determined to have a resistive stress of $B_r = 17.85\ MPa$. Meanwhile, in row 21, the value of the stress that is transferred into the pedestal from the column and the base plate is found to be $B = 16.37$ MPa. This value is found by dividing the force transferred from the column ($C_f = 3\ 420\ kN$) by the area of the base plate that rests upon the pedestal ($A = 457mm \times 457mm = 208\ 849\ mm^2$). In performing this calculation, the engineer omitted the step of converting from units of kN/mm^2 to units of MPa by multiplying by 1 000.

With the bearing stress and bearing resistance calculated, the engineer expresses that the check is OK in two ways: (1) $B < B_r$ in row 22, and (2) $\frac{B}{Br} = 0.91(< 1)$ in row 24. Both inequalities are reminiscent of those presented in the discussion on limit states design in section 5.2.4.

For a final example, Figure 94 depicts the calculations verifying the resistance of the steel anchor bolts that connect the base plate to concrete pedestal.

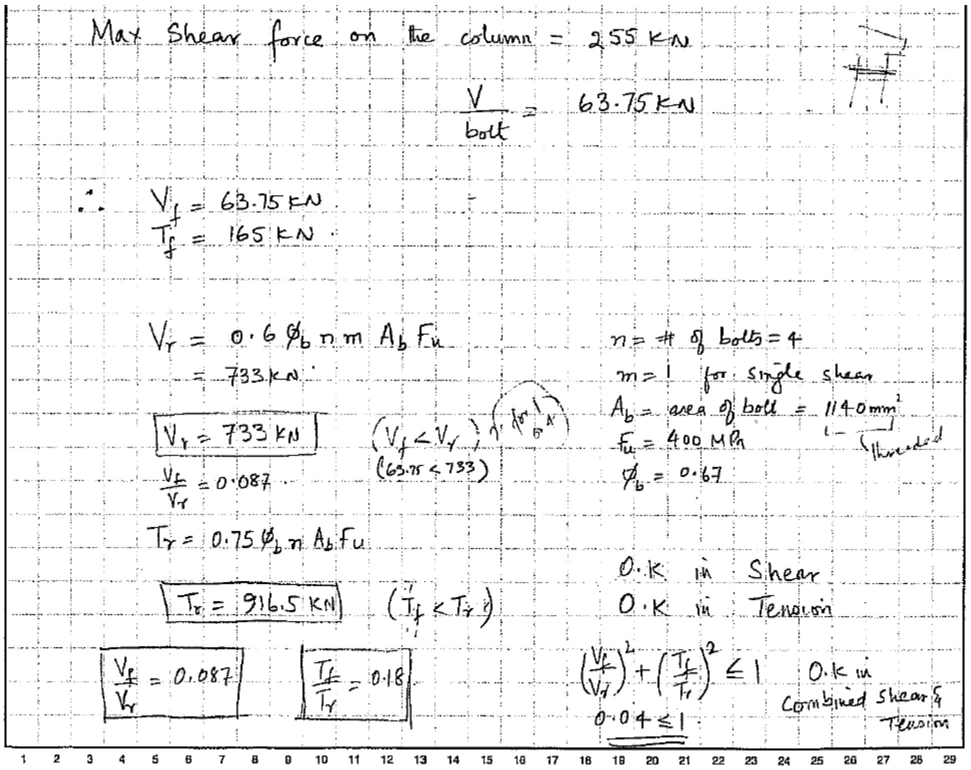


Figure 94 - Dossier de calculs: Checking anchor bolt resistance (Source: Obtained from a licensed engineer)

Rows 20 and 21 indicate the values of the factored loads in the column that the anchor bolts will have to resist. The first load is referred to as a shear force ($V_f = 63.75\ kN$), and the second is a tension force ($T_f = 165\ kN$). In rows 27 and 32 we find the corresponding factored resistances, the calculations for which are found in the rows directly above. The terms in the equations for shear resistance (V_r) and tension resistance (T_r) are material and geometric properties of the anchor bolts. As with the verification of the concrete pedestal's resistance, the resistance of the anchor bolts is compared to the factored load both in absolute terms ($V_f < V_r$; $T_f < T_r$), and as ratios that are less than 1 ($\frac{V_f}{V_r} = 0.087$; $\frac{T_f}{T_r} = 0.18$), and so the anchor bolts are deemed to be O.K. in combined shear and tension (row 34).

On the surface it would appear that the mathematics an engineer uses is limited to arithmetic operations, in particular multiplication, division, and evaluating roots. But what isn't visible is the underlying mathematical modelling that went into developing the formulas that were used for the calculations. Nor do we see the decision making and logic that went into the selection of the sizes of the base plate and the anchor bolts during the design. Much like the workplaces discussed in the research reviewed in chapter 3, where the mathematics was not always visible to the practitioner, the mathematics that is visible in the engineer's workplace does not tell the whole story of the profession. The engineer must decide which clauses of a design code apply and which don't. In terms of the engineering profession's praxeology, if performing the design is the task, then selecting the appropriate formulas from the handbook is part of the technique along with competent use of mathematics. The technology is found in the design codes themselves, in the text that describes when and how each formula is to be used, and the fields of engineering design, engineering science and engineering research that lead to the development of the design codes and their acceptance by the Canadian Standards Association and professional engineering societies are the theory.

6 CONCLUSIONS AND RECOMMENDATIONS

Before identifying and describing the differences between an engineer's and a mathematician's mathematics, I will first address the similarities that I found during my research. While the similarities in mathematical praxeologies of engineers and mathematicians are limited mainly to the tasks and techniques, differences are found in all levels of the praxeologies: tasks, techniques, technologies, and theories. Some of the differences are overt, such as in the standardised symbolism used by engineers. Others, while less obvious, are still describable in the terms of the ATD. A general view of an engineer's mathematical praxeology is presented towards the end of this chapter, which then concludes with a discussion on the limitations of this thesis, and proposals for future studies.

6.1 SIMILARITIES IN MATHEMATICAL PRAXEOLOGIES

The results of my analysis of final exams presented in chapter 4 show that there are some similarities in the mathematics expected to be learned by engineering students and mathematics students, though they are mainly limited to the tasks given to students on their final exams and the techniques that both groups of students can use to accomplish the tasks.

I did not intend to find these similarities, and in fact they surprised me to a certain extent. They were found, however, because I needed to identify what mathematics an engineering student is required to learn in order to obtain a degree, and wanted to compare it with the mathematics to be learned by mathematics students in comparable courses.

The most striking examples of similarities are found in problems from the differential equations and numerical methods courses, in particular the problems shown in Figure 52 - CIVE 320 - Numerical Methods (McGill), December 2007, question 6, and Figure 53 - MATH 354 - Numerical Analysis (Concordia), December 2012, question 2.

It could be argued that, to a certain extent, the technology in the mathematical praxeologies of the institutions of mathematics courses is similar as well. In the exams that were analyzed for chapter 4, I was unable to find any instances where the amount of justification of a technique seemed to be greater for mathematicians than it was for engineers. That is, the level of discourse for describing and justifying the use of certain techniques was about the same for both groups. In fact, many questions on engineering exams specifically asked the students to justify their answers. Furthermore, the fact that the exams for some mathematics courses taken exclusively by engineers feature questions that are entirely conceptual in nature and mathematical in content shows that engineers are expected to have some level of understanding of pure mathematics (for example, see Figure 39 - ENGR 371 - Probability and Statistics in Engineering (Concordia), April 2013, question 3, and Figure 40 - ENGR 371 - Probability and Statistics in Engineering (Concordia), April 2013, question 4).

But it should be noted that these similarities occur only in the first two years of university education, when the topics and concepts learned by both groups are essentially the same. After the first two years, the educational paths of the two groups diverge, and differences arise. These are discussed in the next section.

6.2 DIFFERENCES IN MATHEMATICAL PRAXEOLOGIES

For engineering students and mathematics students, the courses in linear algebra, calculus, probability and statistics, numerical methods, and differential equations are all required to be taken in the first two years of study. Considering the similarities in mathematical content in the first two years, the logical question to ask is “do things differ in the next two years of study?” The answer is an unqualified yes. The divergent educational paths of mathematicians and engineers are illustrated in Figure 95.

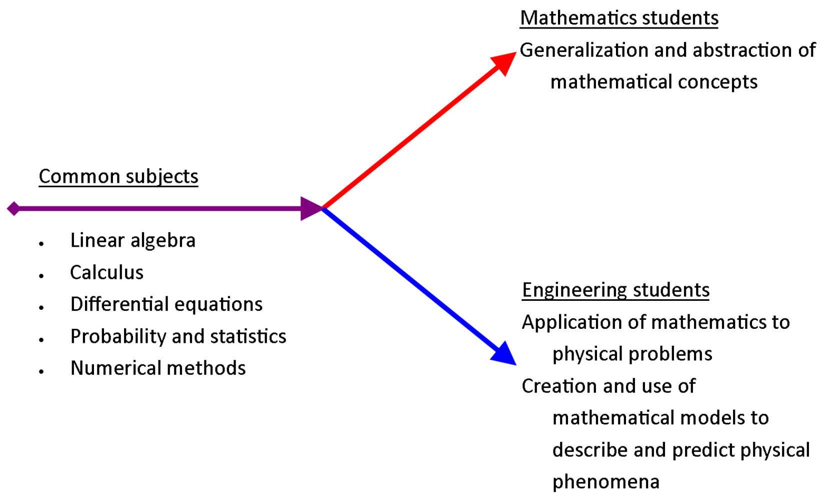


Figure 95 - Diverging educational paths of mathematics students and engineering students

During the first two years of university, both groups learn concepts from common subjects. But, in subsequent years, mathematics students learn concepts that engineers don't. Chief among these are topics in mathematical logic which can be used to prove theorems in subjects such as analysis, set theory, abstract algebra, measure theory, and number theory. Each of these topics aims at exploring how and why mathematics works the way it does.

To engineering students, and practicing engineers, these concepts are too general, and serve little to no practical purpose in solving problems based in the real world. Engineering students spend part of their first two years of university taking engineering courses as well as mathematics courses. In subsequent years they learn how to apply the mathematics they've learned to practical problems. In conjunction with the additional knowledge from their engineering courses, they learn to create and use mathematical models to describe and predict physical phenomena in various engineering disciplines.

The differences in the mathematics of engineers and that of mathematicians can be described through the lens of the ATD by considering (1) the different tasks and techniques in the divergent educational streams of both groups, and (2) the different technologies and theories that describe and justify the techniques that are taught in school and used in professional practice.

6.2.1 *Differences in tasks and techniques*

As previously mentioned, in their later university education, mathematics students learn concepts that engineering students don't. For mathematics students, the subject of linear algebra is taught from the perspective of linear transformations over abstract vector spaces; courses in logic introduce them to the concept of "mathematical thinking"; they learn how to prove the fundamental theorems of arithmetic, algebra, and calculus in real and complex analysis; and courses in abstract algebra, measure theory, and number theory generalize other mathematical concepts. These concepts equip the students with new techniques to accomplish new types of tasks, including, perhaps most importantly, proving mathematical theorems, a task that engineers simply never encounter in either their engineering courses or their professional practice.

Engineering students, on the other hand, use the techniques that they learned in their mathematics courses in conjunction with knowledge from the physical sciences to accomplish tasks that involve the creation and application of mathematical models to solve problems in engineering. Examples of this are shown in chapter 5, including the use of differential equations to model the deflection of a structural beam, and the necessity of statistical analysis in developing the model of limit states design.

6.2.2 *Differences in technologies and theories*

Arguably the most important differences in the mathematical praxeologies of engineers and mathematicians are found at the levels of technology and theory. Among these differences are the ones found in the standardised symbolism used by engineers, discussed in the next section.

6.2.2.1 Differences in standardised symbolism

Standardised symbolism belongs to the technology of the engineer's mathematical praxeology because it is codified knowledge that serves to communicate to each member of the institution of the professional practice of engineering how to properly communicate numerical information with each other.

Differences in standardised symbolism of engineers versus that of mathematicians include the ways in which real numbers are represented. While mathematicians prefer precision and use symbols such as π, $\sqrt{2}$, and $\frac{1}{2}$, engineers will often approximate the values of these numbers as 3.14, 1.414, and 0.5. The level of precision of the approximation will vary depending on the context of the task the number is being used for, and will depend on the accuracy of measurements or of the values in a given data set.

Also belonging to standardised symbolism is the use of units in application problems. When mathematicians encounter application problems they will use units as well, since it is the units that give the numbers physical meaning. But engineering science has led to the creation of new units of measure that are, for the most part, unknown to mathematicians. Examples of these are the newton for units of force, and the pascal for units of pressure.

Other examples mentioned in this thesis are the use of standard prefixes (Figure 60 - SI prefixes and multiplication factors) for units of measure. Some professional mathematicians may be surprised to learn that the preferred unit of measure for the size of small objects is the millimetre, and not the centimetre. Consider as well the sigma notation that omitted indices in the model of static equilibrium (section 5.2.2), and the integral that didn't include limits in the model of the moment of inertia (section 5.2.3). While a mathematician would certainly include the indices and limits, their omission is justified in the engineer's technology by the context in which the symbols are being used.

6.2.2.2 Differences in technologies in education

The research of Castela and Romo Vasquez proposes that engineers augment the technology of their mathematical praxeology with justifications based on practicality and efficiency, and not solely on mathematical consistency (Castela & Romo Vasquez, 2011). The theory of their praxeologies is also rooted in the natural and engineering sciences. Not everything an engineer does can be reduced to mathematics alone, thus, their technology and theory also includes knowledge from the physical sciences. This was exemplified by the study on the different methods of teaching Laplace transforms to engineering students (see section 3.1).

A comparable difference can be seen from the point of view of mathematicians. The new techniques that mathematics students learn in their later university courses augment the level of discourse, i.e., the technology, of the techniques that they learned in their earlier courses. Proving the fundamental theorem of calculus in an analysis course, for example, adds to the justification of its use in the calculus course taken earlier in their education. Engineers, however, never add this mathematical content to the technology of their mathematical praxeology.

While engineers augment their praxeologies with knowledge learned in physical sciences, mathematicians augment their own praxeologies with more mathematics. Thus, there is a difference in the technologies of the institutions of engineering education and mathematics education.

6.2.2.3 Differences in technologies and theories in professional practice

The differences in technologies and theories become more evident when we examine an engineer's professional practice. Evidence of the claim that not everything an engineer does can be reduced to mathematics alone was shown in section 5.4. The knowledge embedded in the engineer's *dossier de calculs* involves more than just the visible calculations using arithmetic and algebra. The engineer must decide which formulas necessarily apply to the structure he is designing, and which don't. These decisions must be justified, and doing so requires more than just mathematics. Furthermore, the formulas themselves that are used in the engineer's design calculations were developed through extensive empirical testing and mathematical modelling, and the creation of these models also requires knowledge from the physical sciences.

Recall from section 4.1 the accreditation requirements for an engineering program. The bulk of a program is to be dedicated to courses in engineering science and engineering design (Canadian Engineering Accreditation Board, 2013). While engineering design combines mathematics, natural sciences, and engineering sciences, it is an iterative process of applications of mathematical models. Engineering science, on the other hand, applies mathematics to practical problems through the development of mathematical models. In a sense, while engineering designers apply mathematical models, engineering scientists (i.e. academic engineers) create the mathematical models that are used in practice, including those that appear in design codes. In this way, the theory of the engineer's mathematical praxeology is also augmented by extra-mathematical concepts.

For a final example of how an engineer's mathematical praxeology includes more than just mathematics, consider the case of the London Millennium Pedestrian Bridge. Not long after it opened in the year 2000, the bridge began to sway excessively. Pedestrians began adjusting their footsteps to counter the effects of the swaying; this caused the bridge to sway even more creating a positive feedback loop. According to Noss (2001), the failure in the bridge's design wasn't in the mathematics or the mathematical models that the engineers used, but in the design code. The bridge was designed using existing techniques specified by the appropriate bridge design code. At the time of its design, though, lateral vibrations in pedestrian bridges were simply not a matter of consideration as it is generally vertical vibrations that are of more concern. In terms of the ATD, the existing techniques, while mathematically sound, failed when applied in the real world. Following extensive research on the cause of the vibrations (Dallard, et al., 2001), a new technique, in the form of an additional clause, was added to the design code. The new clause found in the British Standard on bridge live loading, BD 37, contained in volume 1 of the Design Manual for Roads and Bridges states:

> “Where the fundamental frequency of horizontal vibration is less than 1.5 Hz, special consideration shall be given to the possibility of excitation by pedestrians of lateral movements of unacceptable magnitude.” (Highways England, 2001)

The justification for this technique, at the level of technology and theory, is that it will now prevent bridges from suffering the same fate in the future. Its effects are justified for practical reasons, not only mathematical. A mathematical model needs to be more than internally logical and consistent on the mathematical level. It also has to be practical, operational, and serviceable; it has to actually work in practice. If a model doesn’t work it will become apparent in its application, and the model will then have to be refined.

6.3 A GENERAL VIEW OF AN ENGINEER’S MATHEMATICAL PRAXEOLOGY

In general, we can visualize the theoretical block of an engineer’s mathematical praxeology by considering the process of mathematical modelling, and the application of those models to problems based in the physical world (Figure 96).

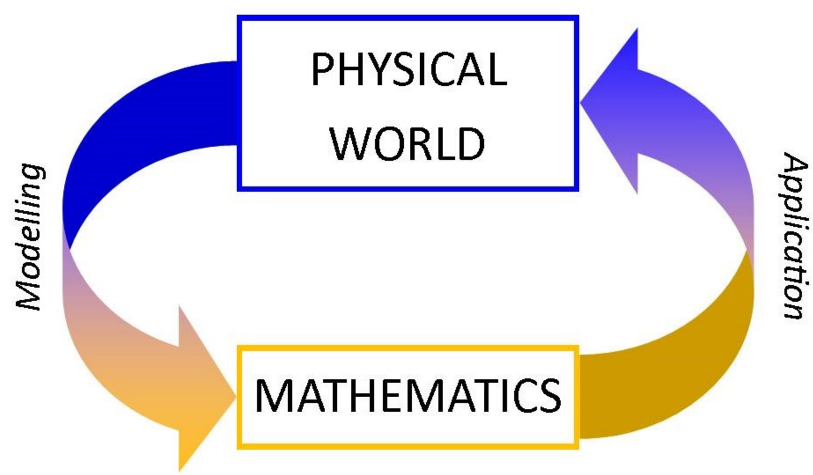


Figure 96 - Visualization of an engineer's mathematical praxeology

Modelling is the process that maps elements from the real world into the mathematical domain, while application problems involve using a mathematical model to solve a problem that involves extra-mathematical elements.

A mathematician's praxeology must be consistent entirely within the realm of mathematics. An engineer's mathematical praxeology, on the other hand, requires that the mathematics work when they are applied in the physical world. For a mathematician, a mathematical proof requires only the use of valid mathematics. For an engineer, the proof is in the successful application of mathematics.

6.4 LIMITATIONS OF THIS THESIS AND RECOMMENDATIONS FOR FUTURE STUDIES

The findings of this thesis are based on limited amounts of data. The statistics in section 4.3, for example, should be taken with a grain of salt. My analysis included exams from only two schools, and focused on a single school year for each course. While the final exams for most of these courses don't tend to change very much over a relatively short time span (the earliest exam that I referenced is eight years old at the time of writing), and the engineering exams that I analyzed didn't strike me as any different from those I that wrote as a student in the early 2000's, having more time to analyze exams from multiple years would certainly result in a more accurate representation of the relative frequencies of the different type of tasks.

For the subject of engineering geometry I had no exams to analyze, and my discussion of the topic was based largely on documents from professional practice and on past personal experience. Engineering surveying requires extensive use of trigonometry. Analyzing exams from surveying courses could offer a wealth of information into how engineers use trigonometry in practice, in particular how precise their calculations and approximations have to be in order to be considered "correct."

It would be pertinent to analyze final exams from engineering courses to see if, in fact, more modelling problems are given when compared with their mathematics courses. Since the exams analyzed for this thesis are clearly lacking in modelling problems, the engineering courses would almost certainly have to offer more of them in order to comply with the requirements of the Canadian Engineering Accreditation Board.

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8 APPENDICES

8.1 WEBSITES OF ACCREDITED ENGINEERING PROGRAMS IN QUEBEC

General information about the accredited civil engineering programs in Quebec can be found on the following university websites:

1. McGill University: www.mcgill.ca/engineering
2. Université Laval: www.fsg.ulaval.ca
3. École Polytechnique: www.polymtl.ca
4. Université de Sherbrooke: www.usherbrooke.ca/genie
5. Concordia University: www.encs.concordia.ca
6. École de technologie supérieure: www.etsmtl.ca
7. Université du Québec à Chicoutimi: http://programmes.uqac.ca/7480

8.2 LIST OF MATHEMATICS COURSES IN THE ACCREDITED ENGINEERING PROGRAMS IN QUEBEC

The tables in the sections that follow include the complete list of required mathematics courses in the seven accredited engineering programs in Quebec. The courses are grouped together by subject following the same order as section 4.3.

Some courses are listed in multiple subjects, e.g., the course MAT165 - Algèbre linéaire et analyse vectorielle at École de technologie supérieure is listed in both pre-university linear algebra and university level calculus. This is because there are a substantial number of topics in the course description that qualify the course for both subjects.

These course descriptions were used to define the different groups of mathematics subjects whose exams were analyzed in section 4.3.

8.2.1 Pre-university linear algebra

School	**Course Number**	**Course Title** ***Course Description***
McGill	MATH 133	Linear Algebra and Geometry *Systems of linear equations, matrices, inverses, determinants; geometric vectors in three dimensions, dot product, cross product, lines and planes; introduction to vector spaces, linear dependence and independence, bases; quadratic loci in two and three dimensions.*
Laval	-	-
Polytechnique	MTH1006	Algèbre linéaire *Plan et espace euclidiens. Vecteurs géométriques du plan et de l'espace. Produits scalaire, vectoriel et mixte. Droites et plans. Espaces vectoriels, sous-espaces vectoriels, indépendance linéaire, base, dimension. Bases orthogonales et orthonormales, procédé de Gram-Schmidt. Transformations linéaires, matrices et changement de bases. Noyau, image et rang. Systèmes d'équations linéaires homogènes, non homogènes et liens avec les matrices. Valeurs propres et vecteurs propres. Diagonalisation. Formes quadratiques et matrices symétriques. Applications à la géométrie : classification des équations du second degré (coniques et quadriques).*
Sherbrooke	GCI 100	Algèbre linéaire *Calcul matriciel : notation, opérations sur les vecteurs et les matrices, propriétés des opérations. Systèmes d'équations linéaires. Algorithme de Gauss-Jordan. Espace vectoriel : sous-espaces, indépendance linéaire, base, dimension, norme, orthogonalisation de Gram-Schmidt, interprétation géométrique. Déterminants. Vecteurs et valeurs propres : définitions, matrices diagonalisables, symétriques, à coefficients complexes, hermitiennes, unitaires et définies positives, interprétation géométrique, applications.*
Concordia	MATH 204	Vectors and Matrices *Algebra and geometry of vectors, dot and cross products, lines and planes. System of equations, operations on matrices, rank, inverse, quadratic form, and rotation of axes.*
ÉTS	MAT165	Algèbre linéaire et analyse vectorielle *Vecteurs, algèbre et géométrie vectorielle, produits scalaires, vectoriels et mixtes, fonctions vectorielles à une variable et applications. Transformations linéaires, matrices, déterminants, inversion de matrices, systèmes d'équations linéaires, valeurs propres et vecteurs propres. Fonctions à plusieurs variables, dérivées partielles, dérivées directionnelles, gradient; applications géométriques : courbes de niveaux, optimisation, plans tangents. Intégrales doubles et triples; applications : calcul de surfaces, volumes, centres de gravité, moments d'inertie. Champ vectoriel, divergence et rotationnel, intégrales de lignes et de surfaces; théorèmes de Green, Stokes et de la divergence.*

UQAC	8MAT142	Algèbre vectorielle et matricielle *Vecteurs géométriques: définition, addition, produit par un scalaire, combinaison linéaire de vecteurs parallèles et coplanaires, composantes d'un vecteur. Vecteurs algébriques: définition, opération sur ces vecteurs. Produit scalaire et applications. Produit vectoriel et applications. Le plan dans l'espace: équations vectorielle et algébrique du plan, vecteur normal à un plan, équation normale, angle de deux plans, distance entre deux plans parallèles, distance d'un point à un plan, équations paramétriques pour un plan.* *La droite dans l'espace: équations paramétriques et symétriques, droite d'intersection de deux plans non parallèles, distance d'un point à une droite, angle de deux droites, angle d'un plan et d'une droite, point d'une droite le plus rapproché d'un point donné, intersection d'une droite et d'un plan.* *Matrices: élément, format, addition, produit par un scalaire, produit des matrices, transposées, déterminants et calculs, inversions de matrices, matrices symétriques et orthogonales, valeurs et vecteurs propres, matrices diagonalisables. Systèmes d'équations linéaires: expression vectorielle et matricielle d'un système linéaire, matrice augmentée, méthode de Gauss.* *Notions de nombres et variables complexes: définition et justification des nombres complexes, représentation sur le plan complexe, formes polaire et cartésienne, égalité, inversion et conjugués. Addition, soustraction. Forme exponentielle. Multiplication et division. Racine. Fonctions d'une variable complexe: fonctions exponentielles et sinusoïdales.*

8.2.2 Pre-university calculus

School	**Course Number**	**Course Title** ***Course Description***
McGill	MATH 140	Calculus 1 *Review of functions and graphs. Limits, continuity, derivative. Differentiation of elementary functions. Antidifferentiation. Applications.*
	MATH 141	Calculus 2 *The definite integral. Techniques of integration. Applications. Introduction to sequences and series.*
Laval	-	-
Polytechnique	-	-
Sherbrooke	-	-
Concordia	MATH 203	Differential and Integral Calculus I *Functional notation. Differentiation of polynomials. The power, product, quotient, and chain rules. Differentiation of elementary functions. Implicit differentiation. Higher derivatives. Maxima and minima. Applications: tangents to plane curves, graphing, related rates. Approximations using the differential. Antiderivatives, definite integrals, area.*
	MATH 205	Differential and Integral Calculus II *Techniques of integration: substitutions, integration by parts, partial fractions. Improper integrals. Physical applications of the definite integral. Infinite series: tests for convergence. Power series, Taylor's theorem.*
ÉTS	MAT145	Calcul différentiel et intégral *Analyse : généralités sur les fonctions de R dans R; calcul différentiel : limites, dérivée, dérivée des fonctions élémentaires, règles de dérivation, étude de graphe, optimisation, etc. Calcul intégral : intégrales indéfinies, méthode d'intégration, utilisation des tables, intégrales définies, application (calcul d'aires, de volumes, de longueurs d'arc), méthodes numériques, intégrales impropres, etc. Suites et séries. Développements limités (Taylor, Maclaurin), évaluation de fonctions et d'intégrales définies à l'aide des séries.*
UQAC	8GMA102	Calcul différentiel et intégral *Rappels sur les ensembles et nombres réels. Valeur absolue, droite orientée, inéquations. Fonctions et graphes, fonctions élémentaires: puissances, exponentielles, logarithmiques, trigonométriques, hyperboliques, fonctions inverses et composées. Forme implicite. Lieux géométriques et les coniques. Représentations paramétriques. Définition d'une limite et ses propriétés. Calcul de limites de fonctions algébriques. Continuité d'une fonction et propriétés des fonctions continues. Dérivée: définition, existence, propriétés et calculs. Formules de dérivation, dérivation en chaîne, dérivation implicite. Différentielle. Applications des dérivées: extremums de fonctions, tracé d'une courbe, modélisation et optimisation, théorèmes des accroissements finis, limites des formes indéterminées: règle de l'Hôpital. Approximations d'une fonction par série. Applications au génie. Intégrales indéfinies. Intégrales définies:*

	définition et propriétés. Théorème fondamental du calcul. Applications: calcul des aires planes, des aires et volumes de révolution, centre de gravité, moment d'inertie, pression des fluides, travail, longueur d'arc. Intégration numériques. Intégrales impropres.

8.2.3 University level calculus

School	**Course Number**	**Course Title** ***Course Description***
McGill	MATH 262	Intermediate Calculus *Series and power series, including Taylor's theorem. Brief review of vector geometry. Vector functions and curves. Partial differentiation and differential calculus for vector valued functions. Unconstrained and constrained extremal problems. Multiple integrals including surface area and change of variables.*
	MATH 264	Advanced Calculus for Engineers *Review of multiple integrals. Differential and integral calculus of vector fields including the theorems of Gauss, Green, and Stokes. Introduction to partial differential equations, separation of variables, Sturm-Liouville problems, and Fourier series.*
Laval	MAT-1900	Mathématiques de l'ingénieur I *Calcul différentiel des fonctions de plusieurs variables: théorie et applications. Nombres complexes; polynômes. Équations différentielles du premier ordre et du premier degré; méthodes numériques. Équations différentielles du second ordre de types spéciaux. Équations différentielles linéaires d'ordre n à coefficients constants. Systèmes d'équations différentielles. Applications.*
	MAT-1910	Mathématiques de l'ingénieur II *Intégrales simples, calcul formel et numérique. Intégrales multiples, coordonnées curvilignes, applications. Calcul des champs de vecteurs. Intégrales sur les courbes et les surfaces: applications, circulation, travail, flux. Théorèmes fondamentaux: Stokes, Gauss; applications à la physique.*
Polytechnique	MTH1101	Calcul I *Suites infinies et séries. Séries entières. Approximations de Taylor. Analyse de l'erreur d'approximation par un polynôme. Nombres complexes. Fonctions de plusieurs variables. Courbes et surfaces de niveau. Limite et continuité. Dérivées de fonctions de plusieurs variables. Différentielle. Recherche des extrema avec ou sans contraintes. Méthode du gradient en optimisation. Multiplicateurs de Lagrange.*
	MTH1102	Calcul II *Intégrales multiples. Systèmes de coordonnées. Changements de variables. Courbes et surfaces paramétrées. Intégrales curvilignes : travail et circulation. Champs vectoriels, gradients et champs conservatifs. Théorème de Green. Intégrales de surface et de flux pour les cylindres, sphères et surfaces paramétrées. Divergence et théorème de divergence. Rotationnel et théorème de Stokes.*
Sherbrooke	GCI 101	Mathématiques I *Rappel des propriétés de l'intégrale simple. Dérivées partielles de fonctions de plusieurs variables, application à la géométrie dans R3.Coordonnéespolaires, cylindriques et sphériques. Techniques d'intégration des intégrales doubles et triples. Applications des intégrales*

		à la géométrie dans le plan et l'espace et à des problèmes reliés à la mécanique. Dérivée directionnelle, gradient d'une fonction scalaire, divergence et rotationnel d'un champ vectoriel.
Concordia	ENGR 233	Applied Advanced Calculus *This course introduces Engineering students to the theory and application of advanced calculus. Functions of several variables, partial derivatives, total and exact differentials, approximations with differentials. Tangent plane and normal line to a surface; directional derivatives; gradient. Double and triple integrals. Polar, cylindrical, and spherical coordinates. Change of variables in double and triple integrals. Vector differential calculus; divergence, curl, curvature, line integrals, Green's theorem, surface integrals, divergence theorem, applications of divergence theorem, Stokes' theorem.*
ÉTS	MAT165	Algèbre linéaire et analyse vectorielle *Vecteurs, algèbre et géométrie vectorielle, produits scalaires, vectoriels et mixtes, fonctions vectorielles à une variable et applications. Transformations linéaires, matrices, déterminants, inversion de matrices, systèmes d'équations linéaires, valeurs propres et vecteurs propres. Fonctions à plusieurs variables, dérivées partielles, dérivées directionnelles, gradient; applications géométriques : courbes de niveaux, optimisation, plans tangents. Intégrales doubles et triples; applications : calcul de surfaces, volumes, centres de gravité, moments d'inertie. Champ vectoriel, divergence et rotationnel, intégrales de lignes et de surfaces; théorèmes de Green, Stokes et de la divergence.*
UQAC	8MAP110	Calcul avancé I *Introduction aux équations différentielles : exemples, ordre d'une équation, équations linéaires. Équations différentielles linéaires d'ordre 1 : facteur intégrant, problème de valeur initiale, comportement à l'infini, représentation graphique, champ de directions.* *Les vecteurs de Rn et les vecteurs géométriques : repère cartésien, vecteur position d'un point, norme et distance, coordonnées polaires. Produits scalaire, vectoriel et mixte : propriétés, interprétations géométrique et physique (travail, moment vectoriel, flux). Projections scalaire et vectoriel d'un vecteur. Différentes équations d'une droite et d'un plan : paramétrique, normal-point et algébrique.* *Fonctions vectorielles d'une variable : courbes paramétrées, hélices circulaire et elliptique, cubique gauche, intersection d'un plan et d'un cylindre conique, trajectoire d'une particule, dérivée et règles de dérivation, vecteur tangent, intégrale définie, intégration et condition initiale, longueur d'arc, vecteurs vitesse et accélération, vitesse et accélération. Fonctions scalaires : relation entre variables, fonction de plusieurs variables et graphe, surface de révolution, les quadriques, courbes et surfaces de niveau, limite et continuité, dérivées partielles et dérivée le long d'une droite parallèle à un axe, dérivée directionnelle et dérivée le long d'une droite orientée, vecteur gradient et interprétation géométrique, variation optimale d'une fonction, dérivation des fonctions composées et dérivée le long d'une courbe orientée, plan tangent à une surface définie par une relation, plan tangent à une graphe et*

	approximation linéaire, dérivées partielles d'ordre supérieur, introduction à l'optimisation (extremums locaux, points critiques, test de dérivées secondes, ensemble fermé et borné, frontière, extremums globaux, multiplicateurs de Lagrange). Utilisation de la différentielle totale pour le calcul d'erreurs. Formules et séries de Taylor à une et deux variables : approximations d'une fonction. Introduction à la méthode des différences finies successives. Tables des différences, formules d'interpolation de Newton, solutions numériques. *Applications en ingénierie : principe de superposition des forces et des vecteurs de vitesse, les 3 lois de Newton, intégration de la deuxième loi de Newton et conditions initiales, vecteurs accélérations normale et tangentielle, topographie, équations de Laplace, de la chaleur et des ondes. Utilisations d'un logiciel de calcul.*
8MAP111	Calcul avancé II *Fonctions vectorielles de plusieurs variables : coordonnées cylindriques et sphériques, cylindres et solides cylindriques, sphères et boules, surfaces et solides paramétrés, taux de variation le long d'une courbe orientée et matrice jacobienne, plans tangents à une surface paramétrée. Intégrales multiples : rappel sur l'intégrale simple, principe de Cavalieri, intégrales doubles et triples, changement de variables, applications au génie, méthodes numériques (méthodes des rectangles, du trapèze et de Simpson). Intégration vectorielle: intégration de champs scalaire et vectoriel et interprétations, travail d'une force et circulation d'un champ vectoriel, intégrale d'une surface d'un champ scalaire et d'un champ vectoriel, flux d'un champ vectoriel, applications au génie. Théorèmes fondamentaux en analyse vectorielle : divergence et rotationnel, théorèmes de Green et de Stokes, champs conservatifs et potentiel scalaire, théorème de divergence, flux et divergence, champs solénoïdaux et potentiel vecteur, applications au génie. Fonctions d'une variable complexe : les nombres complexes (plan complexe, algèbre des nombres complexes), fonctions d'une variable complexe (limite, continuité, dérivation), équations de Cauchy-Riemann, fonctions analytiques, fonctions exponentielle et trigonométriques, fonction logarithmique et puissances complexes, intégration dans le plan complexe (intégration curviligne, théorème de Cauchy, principe de déformation des contours), formule intégrale de Cauchy, séries de Taylor (suites et séries de nombres complexes, série de puissances, séries de Taylor), zéros et pôles d'une fonction. Applications au génie. Utilisations d'un logiciel de calcul.*

8.2.4 Differential equations

School	**Course Number**	**Course Title** ***Course Description***
McGill	MATH 263	Ordinary Differential Equations for Engineers *First order ODEs. Second and higher order linear ODEs. Series solutions at ordinary and regular singular points. Laplace transforms. Linear systems of differential equations with a short review of linear algebra.*
Laval	GCI 2002	Mathématiques appliquées *Modélisation de problèmes appliqués par des équations aux dérivées ordinaires. Système d'équations aux dérivées ordinaires. Problème de Sturm-Liouville: définition et notions de fonctions orthogonales. Série de Fourier: fonctions paires et impaires, approximations. Équations aux dérivées partielles : séparation de variables et série de Fourier.*
Polytechnique	MTH1110	Équations différentielles ordinaires *Équations différentielles ordinaires. Équations d'ordre un : à variables séparables, exactes, linéaires, de Bernoulli. Équations linéaires d'ordre supérieur : ensemble fondamental de solutions, équations à coefficients constants (homogènes et non homogènes), équation d'Euler-Cauchy, oscillations libres et forcées. Systèmes d'équations différentielles d'ordre un : linéaires (homogènes et non homogènes), non linéaires (linéarisation et stabilité). Transformée de Laplace : propriétés et application aux équations linéaires non homogènes.*
Sherbrooke	GCI 103	Mathématiques II *Notions d'équations différentielles. Équations différentielles du 1er ordre : équations à variables séparables, exactes, équations linéaires, équations se ramenant au 1er ordre. Équations et systèmes d'équations différentielles linéaires à coefficients constants : opérateur D, solutions générales complémentaires et particulières. Transformée de Laplace : calcul de transformée, fonctions périodiques et avec délai. Équations différentielles partielles. Séries de Fourier. Applications.*
Concordia	ENGR 213	Applied Ordinary Differential Equations *This course introduces Engineering students to the theory and application of ordinary differential equations. Definition and terminology, initial-value problems, separable differential equations, linear equations, exact equations, solutions by substitution, linear models, orthogonal trajectories, complex numbers, form of complex numbers: powers and roots, theory: linear equations, homogeneous linear equations with constant coefficients, undetermined coefficients, variation of parameters, Cauchy-Euler equation, reduction of order, linear models: initial value, review of power series, power series solutions, theory, homogeneous linear systems, solution by diagonalisation, non-homogeneous linear systems. Eigenvalues and eigenvectors.*
	ENGR 311	Transform Calculus and Partial Differential Equations *Elements of complex variables. The Laplace transform: Laplace transforms and their properties, solution of linear differential equations with constant coefficients. Further theorems and their applications. The*

		Fourier transform: orthogonal functions, expansion of a function in orthogonal functions, the Fourier series, the Fourier integral, the Fourier transform, the convolution theorem. Partial differential equations: physical foundations of partial differential equations, introduction to boundary value problems.
ÉTS	MAT265	Équations différentielles *Origine et définition, famille de solutions, conditions initiales, équations différentielles du premier ordre : séparables exactes, linéaires. Applications : mouvement rectiligne, circuits électriques, etc. Équations différentielles linéaires à coefficients constants : solutions complémentaires (homogènes) et solutions particulières, méthode des coefficients indéterminés (variation des paramètres, opérateur inverse); applications : mouvement harmonique et circuits électriques. Transformées de Laplace en équations différentielles, applications, systèmes d'équations différentielles. Solutions d'équations différentielles par séries, méthodes numériques en équations différentielles. Séries de Fourier, résolutions d'équations différentielles par séries de Fourier.*
UQAC	8MAP120	Équations différentielles et séries de Fourier *Équations différentielles d'ordre deux ou plus : équations linéaires d'ordre deux à coefficients constants, réduction de l'ordre, principe de superposition, wronskien, méthode de variation de paramètres, coefficients indéterminés. Méthode numérique : solutionner des équations différentielles et systèmes d'équations différentielles à l'aide de la méthode d'Euler et de Runge-Kutta. Séries de Fourier : développement en série de Fourier, série de Fourier en cosinus, en sinus et exponentielles. Applications : redressement d'un signal alternatif, valeur efficace, identité de Parseval, système ressort-masse, équation des cordes vibrantes, équation de la chaleur dans une tige et de l'équation de Laplace. Méthode numérique : série de Fourier lorsque le signal est donné par un tableau de valeurs. Intégrale de Fourier : forme trigonométrique, forme exponentielle; transformée de Fourier : diverses transformées de Fourier, théorème de convolution. Méthode numérique : transformée de Fourier discrète à l'aide de la transformée de Fourier rapide (FFT). La transformée de Laplace : transformée de fonctions élémentaires, fonctions d'Heaviside et Dirac; propriétés élémentaires de la transformée, solutions de problèmes aux conditions initiales; les méthodes de décomposition des fractions partielles, transformée des fonctions causales périodiques, l'intégrale de convolution de deux fonctions, propagation de la chaleur dans une tige, équation des cordes vibrantes (longueur infinie). Utilisations d'un logiciel de calcul.*

8.2.5 Probability and statistics

School	**Course Number**	**Course Title** ***Course Description***
McGill	CIVE 302	Probabilistic Systems *An introduction to probability and statistics with applications to Civil Engineering design. Descriptive statistics, common probability models, statistical estimation, regression and correlation, acceptance sampling.*
Laval	STT-1900	Méthodes statistiques pour ingénieurs *Théorie des probabilités. Loi normale. Statistique descriptive. Lois échantillonnales. Estimation ponctuelle et par intervalle de confiance. Tests d'hypothèses. Analyse de la variance : expériences à un facteur, en blocs, à plusieurs facteurs et factorielles. Régression linéaire simple et multiple.*
Polytechnique	MTH2302C	Probabilités et statistique *Notions de probabilités : axiomes, probabilité conditionnelle, règle de Bayes, analyse combinatoire. Variables aléatoires : fonctions de répartition, de masse et de densité, espérance mathématique. Lois de probabilités discrètes et continues. Vecteurs aléatoires, distribution multi-normale, covariance et corrélation, théorème central limite. Probabilité d'événements extrêmes. Statistique : propriétés des estimateurs et distributions d'échantillonnage, moindres carrés, intervalles de confiance. Tests d'hypothèses : tests paramétriques et test d'ajustement. Analyse de décision. Régressions simple et multiple. Méthodes statistiques spatiales.*
Sherbrooke	GCI 102	Méthodes probabilistes en génie civil *Probabilités : concepts de base en probabilité. Lois de probabilité discrètes et continues. Moments et espérances. Distributions probabilistes uniforme, normale, binomiale, hypergéométrique, gamma et de Poisson. Statistiques : distributions empiriques. Mesures de tendance centrale et de dispersion. Distributions d'échantillonnage des moyennes (loi normale et du T de Student) et des variances (loi du Chi-carré et de Fisher). Estimation et tests d'hypothèse. Régression et corrélation.*
Concordia	ENGR 371	Probability and Statistics in Engineering *Axioms of probability theory. Events. Conditional probability. Bayes theorem. Random variables. Mathematical expectation. Discrete and continuous probability density functions. Transformation of variables. Probabilistic models, statistics, and elements of hypothesis testing (sampling distributions and interval estimation). Introduction to statistical quality control. Applications to engineering problems.*
ÉTS	MAT350	Probabilités et statistiques *Définition et axiomes de probabilité, règles d'union, d'intersection, d'addition et de multiplication, probabilité conditionnelle, loi de Bayes. Analyse combinatoire. Variables aléatoires discrètes et continues, distribution de probabilités standards. Mesures d'échantillonnage. Distribution des paramètres d'échantillonnage, combinaison des variables aléatoires, distribution du Khi-carré. Tests statistiques,*

		estimation, intervalle de confiance, tests sur la comparaison de deux populations. Régression linéaire, variance des résidus, tests statistiques et intervalles de confiance pour le paramètre du modèle.
UQAC	8GEN444	Statistiques de l'ingénieur *Distribution empirique et histogrammes. Dérivation expérimentale de la distribution gaussienne et exponentielle. Notion de probabilité. Fonctions et densités de probabilité. Aléas continus et discontinus. Densité de probabilité bidimensionnelle. Probabilité marginale et conditionnelle. Aléas indépendants. Approche bayésien. Espérance mathématique. Loi normale et loi uniforme. Simulation par la technique Monte Carlo de procédés stochastiques. Analyse combinatoire. Distribution binomiale, hypergéométrique, géométrique, Poisson. Calcul des probabilités à l'aide d'approximations. Distribution exponentielle. Introduction à la fiabilité.*

8.2.6 Numerical methods

School	**Course Number**	**Course Title** ***Course Description***
McGill	CIVE 320	Numerical Methods *Numerical procedures applicable to civil engineering problems: integration, differentiation, solution of initial-value problems, solving linear and non-linear systems of equations, boundary-value problems for ordinary-differential equations, and for partial-differential equations.*
Laval	MAT-2910	Analyse numérique pour l'ingénieur *Calcul numérique. Algèbre linéaire. Résolution de systèmes non linéaires. Approximation. Intégration et dérivation. Différences finies. Équations différentielles du premier ordre.*
Polytechnique	MTH2210A	Calcul scientifique pour ingénieurs *Interpolation, différentiation et intégration numérique. Discrétisation des équations différentielles. Résolution numérique des équations algébriques. Méthodes directes et itératives pour les systèmes d'équations algébriques linéaires et non-linéaires. Modélisation mathématique. Erreurs de modélisation, de représentation et de troncature.*
Sherbrooke	-	-
Concordia	ENGR 391	Numerical Methods in Engineering *Roots of algebraic and transcendental equations; function approximation; numerical differentiation; numerical integration; solution of simultaneous algebraic equations; numerical integration of ordinary differential equations.*
ÉTS	-	-
UQAC	8MAP110	Calcul avancé I *See Appendix 0*
	8MAP120	Équations différentielles et séries de Fourier *See Appendix 0*

8.2.7 Engineering geometry

School	**Course Number**	**Course Title** ***Course Description***
McGill	MECH 289	Design Graphics *The design process, including free-hand sketching; from geometry construction to engineering construction; the technology and standards of engineering graphic communication; designing with CAD software. The role of visualization in the production of engineering designs.*
	CIVE 210	Surveying *The construction and use of modern survey instruments; transit, level, etc.; linear and angular measurements and errors; horizontal and vertical curves; error analysis, significance of figures; use of computers and software; recent developments.*
Laval	GCI 1006	Dessin, plans et SIG pour l'ingénieur *Dessin technique et lecture de plan. Conventions du dessin technique. Devis: types de devis, sections de devis. Estimation. Préparation d'une soumission. Introduction au SIG. Notions de base de cartographie et de référence spatiale. Potentiels et limites des SIG en ingénierie.*
	GCI 1009	Dessin, plans et géomatique pour ingénieurs *Dessin technique: croquis et normes de base. Lecture de plans. Modélisation des informations sur le bâtiment (BIM). Devis: types de devis, sections de devis. Principes de base de la topométrie. Gestion de projets: estimation, préparation d'une soumission. Introduction aux systèmes d'information géographique (SIG): applications pratiques et limites.*
Polytechnique	MEC1515	DAO en ingénierie *Techniques de représentation graphique et numérique utilisées par les ingénieurs pour l'analyse et la définition de produits (composants ou bâtiments) selon les normes et les conventions établies. Projections orthogonales. Représentations tridimensionnelles. Projections en coupe. Technique du croquis. Conventions de cotation. Description de pièces normalisées ou commerciales. Réalisation de dessins de détail et de dessins d'assemblage. Création et modification de dessins d'ensemble ou de plans d'aménagement et de bâtiments. Interprétation et analyse de dessins. Lecture et recherche d'informations dans des catalogues industriels. Utilisation d'un logiciel de dessin assisté par ordinateur (DAO) pour la génération de dessins techniques. Introduction à la conception assistée par ordinateur (CAO). Travaux pratiques en laboratoires à l'aide du logiciel AutoCAD et introduction à CATIA.*
	CIV1101	Géométronique *Théorie des erreurs, précision, exactitude. Mesure linéaire, chaînage, instruments électroniques, modes opératoires, corrections. Nivellement différentiel, types, normes, précision. Nivellement trigonométrique topométrique, méthode stadimétrique. Plan laser, mesure goniométrique, instruments à dispositif optique, instruments électroniques, modes opératoires. Polygonation, levé topographique,*

		systèmes de coordonnées. Orientation, système arbitraire, magnétique, astronomique. Superficies et volumes. Topométrie routière, plans horizontal et vertical. Applications.
Sherbrooke	GCI 107	Communication graphique en ingénierie *Éléments de dessin technique et de croquis. Outils de dessin. Projections. Dessin à vues multiples. Coupes et sections. Cotations. Formats de papier et mise en pages. Apprentissage du logiciel AutoCAD par cours et tutoriels - commandes de base et avancées, introduction au dessin 3D. Lecture de plans dans différents domaines du génie. Éléments d'images numériques. Introduction à un logiciel de traitement des images.*
Concordia	CIVI 212	Civil Engineering Drawing *Fundamentals of technical drawing, orthographic projections, sectional views. Computer-aided drawing; slabs, beams, and columns; steel structures; building trusses and bridges, wood and masonry structures. Working drawing and dimensioning practice. Introduction to the design process.*
	BCEE 371	Surveying *Elementary operations employed in engineering surveying; use, care, and adjustment of instruments; linear and angular measurements; traversing; earthwork calculations; theory of errors; horizontal and vertical curves and curve layout; slope stakes and grades, application of surveying methods to city, topographic surveying, and introduction to advanced surveying techniques; use of digital computers in surveying calculations.*
ÉTS	-	-
UQAC	6DDG100	Sciences graphiques Rappels géométriques. Instruments de base. Tracés géométriques. Croquis et description de forme. Dessin à vues multiples. Coupes et sections. Conventions de représentation particulières. Vues auxiliaires. Cotation. Tolérances et ajustements. Classification des projections. Intersections et développements. Notions de lecture de plans. Introduction aux différents langages de dessin assisté par ordinateur (DAO et CAO). Les différentes notions du cours sont mises en application par des exemples et devoirs lors des séances de travaux dirigés.

8.3 FINAL EXAMS FROM MATHEMATICS COURSES AT MCGILL AND CONCORDIA

This appendix contains the complete final exams that were analyzed in section 4.3. They are organized by subject in the same order as previously presented. A table showing the classification of each of the tasks is presented before each exam.

8.3.1 Pre-university linear algebra

8.3.1.1 MATH 204

The classification of the tasks on the exam for Concordia's MATH 204 – Vectors and Matrices (December 2014) are shown in Table 30.

Table 30 - Classification of tasks: MATH 204 - Vectors and Matrices (Concordia), December 2014

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1		✓		✓		
2			✓	✓		
3		✓		✓		
4	a	✓		✓		
4	b	✓		✓		
5	a	✓		✓		
5	b	✓		✓		
6	a (i)		✓	✓		
6	a (ii)	✓		✓		
6	b (i)	✓		✓		
6	b (ii)	✓		✓		
7	a		✓	✓		
7	b	✓		✓		
8		✓		✓		
9	a		✓	✓		
9	b		✓	✓		
10		✓		✓		
TOTAL		**12**	**5**	**17**	**0**	**0**
Relative frequency		**71%**	**29%**	**100%**	**0%**	**0%**

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	204	A
Examination	**Date**	**Pages**
Final	December 2014	2
Instructors		**Course Examiners**
A. Sierpinska, S. Shi,		E. Cohen
S. Vassileva, R. Mearns		

Special Instructions

- ▹ Only approved calculators are allowed.
- ▹ Justify all your answers.
- ▹ All questions have equal value.

---

1. Use the Gauss-Jordan method to find all the solutions of the system:

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 2x_3 & = & 0 \\ -2x_1 & + & 5x_2 & + & 2x_3 & = & 1 \\ 8x_1 & + & x_2 & + & 4x_3 & = & -1 \end{array}$$

2. Determine the values of a for which the system has no solution, exactly 1 solution or infinitely many solutions:

$$\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 2 \\ 2x & - & 2y & + & 3z & = & 1 \\ x & + & 2y & - & (a^2-3)z & = & a \end{array}$$

3. Find the inverse of $A = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{pmatrix}$, if it exists.

4. (a) Evaluate the determinant of $A = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$.

   (b) Solve by Cramer's rule, when it applies:

$$\begin{array}{rcrcrcr} x_1 & & & + & 2x_3 & = & 6 \\ -3x_1 & + & 4x_2 & + & 6x_3 & = & 30 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 8 \end{array}$$

5. (a) Let $u = (1,4,2)$, $v = (1,1,0)$. Find the orthogonal projection of u on v.

   (b) Let $u_1 = (1,1,0), u_2 = (0,1,1), u_3 = (1,0,1)$. Find scalars c_1, c_2, c_3 such that $c_1u_1 + c_2u_2 + c_3u_3 = (1,0,0)$.

6. (a) Find the area of the triangle with vertices $(1,1,1)$, $(2,0,1)$, $(3,1,2)$. Find a vector orthogonal to the plane of the triangle.

   (b) (i) Find the distance between the point $(1,5)$ and the line $2x = 5y - 1$.

   (ii) Find the equation of the plane containing the points $(1,2,1), (2,1,1), (1,1,2)$.

7. (a) Let $u = (-1,0,2)$, $v = (2,-1,4)$, $w = (-1,1,-6)$ are the vectors linearly dependent or independent?

   (b) Find the parametric equations of the line in $\mathbb{R}^3$ passing through $(1,4,-5)$ and perpendicular to the plane $x - 3y + 2z = 4$.

8. Let $A = \begin{pmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix}$. Find a basis for the solution space of the homogeneous system $AX = 0$.

9. Find the standard matrices for the following 2 linear operators on $\mathbb{R}^2$:

   (a) a reflection about the line $y = x$.

   (b) a rotation counterclockwise of $30°$.

10. Let $A = \begin{pmatrix} -14 & 12 \\ -20 & 17 \end{pmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

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8.3.1.2 MATH 251

The classification of the tasks on the exam for Concordia's MATH 251 – Linear Algebra I (December 2013) are shown in Table 31.

Table 31 - Classification of tasks: MATH 251 - Linear Algebra I (Concordia), December 2013

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	(i)		✓	✓		
1	(ii)		✓	✓		
1	(iii)		✓	✓		
1	(iv)		✓	✓		
2	a (i)	✓		✓		
2	a (ii)	✓		✓		
2	b	✓		✓		
3	a	✓		✓		
3	b	✓		✓		
4	a		✓	✓		
4	b		✓	✓		
4	c		✓	✓		
5	a		✓	✓		
5	b		✓	✓		
5	c (i)		✓	✓		
5	c (ii)		✓	✓		
6			✓	✓		
7	a		✓	✓		
7	b (i)	✓		✓		
7	b (ii)	✓		✓		
7	b (iii)	✓		✓		
8	a (i)		✓	✓		
8	a (ii)	✓		✓		
8	b (i)		✓	✓		
8	b (ii)	✓		✓		
8	c (i)		✓	✓		
8	c (ii)	✓		✓		
9	a		✓	✓		
9	b (i)		✓	✓		
9	b (ii)	✓		✓		
9	b (iii)	✓		✓		
10	a		✓	✓		
10	b		✓	✓		
TOTAL		**13**	**20**	**33**	**0**	**0**
Relative frequency		**39%**	**61%**	**100%**	**0%**	**0%**

CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	251	All
Examination	**Date**	**Pages**
Final	December 2013	3
Instructors		**Course Examiners**
M. Bertola, C. Cummins, J. Brannlund		M. Bertola, C. Cummins

Special Instructions

- Ruled booklets to be used.
- **Only approved calculators are allowed.**

Problem 1. Let $V = M_{3\times 3}(\mathbb{R})$ and consider the following subspaces:

$$W_1 := \left\{ \begin{bmatrix} a & b & b \\ c & d & d \\ e & 0 & e \end{bmatrix} \;\; a,b,c,d,e \in \mathbb{R} \right\}, \qquad W_2 := \left\{ \begin{bmatrix} f & 0 & g \\ f & h & \ell \\ 0 & 0 & g \end{bmatrix} \;\; f,g,h,\ell,q \in \mathbb{R} \right\}$$

Find bases and dimensions of W_1, W_2, $W_1 + W_2$, $W_1 \cap W_2$.

Problem 2. Let $V = P_4$, the vector space of polynomials of degree at most 4 and with real coefficients. Let $W = \text{span}\{1 + x, 1 - x + x^3, x^3 + x^4, 2 - x^4\}$.

1. Find a basis of β of W. What is the dimension of W?
2. Extend β to a basis of V.

Problem 3. Let $T : P_2 \to \mathbb{R}^2$ be defined by $T(p(x)) = (p(5), p(3))$. Let $\beta = \{1, x, x^2\}$ and $\sigma = \{(1,0),(0,1)\}$, $\gamma = \{(1,1),(-1,1)\}$

1. Find the matrix representation $[T]_\beta^\sigma$.
2. Find an appropriate matrix Q such that $[T]_\beta^\gamma = Q[T]_\beta^\sigma$.

Problem 4. Let V be a vector space and let $T : V \to V$ and $U : V \to V$ be two linear transformations.

1. Show that $T + U$ is also a linear transformation.
2. Show that aT is a linear transformation for any scalar a.
3. Suppose that T is invertible. Show that T^{-1} is also a linear transformation.

Problem 5. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation.

1. State the Dimension Theorem for T.
2. Show that T is not 1-1.
3. Give an example for which T is onto. Give an example for which T is not onto. (In each case show that your example has the required property. Do not just give an example with no explanation).

Problem 6. Let $T : V \to W$ be a linear map. Suppose that it is one to-one. Suppose that $\{\underline{v}_1, \ldots, \underline{v}_k\}$ is a linearly independent subset of V. Let $\underline{w}_1, \ldots, \underline{w}_k$ be the images $\underline{w}_1 = T\underline{v}_1, \ldots, \underline{w}_k = T\underline{v}_k$. Show that $\underline{w}_1, \ldots, \underline{w}_k$ are linearly independent.

Problem 7.

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix}$$

(a) Is A invertible?

(b) Find the ranks of A, B and AB.

Problem 8. For each matrix A, determine whether or not A is diagonalizable. If A is diagonalizable, find a matrix Q and diagonal matrix D such that $Q^{-1}AQ = D$.

1. $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$.
2. $A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$.
3. $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$.

Problem 9. Let V be $M_{2\times 2}(\mathbb{R})$ the vector space of 2×2 matrices with real entries. Let $T : V \to V$ be defined by $T(A) = A + A^t$.

1. Show that T is a linear transformation.
2. Is T diagonalizable? If so, find a basis β of V consisting of eigenvectors of T and $[T]_\beta$, the matrix of T with respect to the basis β.

Problem 10. Let $T : V \to V$ be a linear transformation. Suppose v is an eigenvector for the eigenvalue λ;

1. for any positive integer m prove that v is an eigenvector of T^m corresponding to the eigenvalue λ^m;
2. assume now that T is invertible; show that v is an eigenvector of the inverse with eigenvalue λ^{-1}.

8.3.1.3 MATH 252

The classification of the tasks on the exam for Concordia's MATH 252 – Linear Algebra II (April 2013) are shown in Table 32.

Table 32 - Classification of tasks: MATH 252 - Linear Algebra II (Concordia), April 2013

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	(i)	✓		✓		
1	(ii)	✓		✓		
2		✓		✓		
3	a (i)	✓		✓		
3	a (ii)	✓		✓		
3	b (i)	✓		✓		
3	b (ii)		✓	✓		
4		✓		✓		
5	(i)	✓		✓		
5	(ii)	✓		✓		
6		✓		✓		
7	(i)	✓		✓		
7	(ii)	✓		✓		
8	(i)		✓	✓		
8	(ii)	✓		✓		
8	(iii)	✓		✓		
9	(i)	✓		✓		
9	(ii)	✓		✓		
10	a		✓	✓		
10	b		✓	✓		
11			✓	✓		
TOTAL		**16**	**5**	**21**	**0**	**0**
Relative frequency		**76%**	**24%**	**100%**	**0%**	**0%**

Department of Mathematics & Statistics

Course	Number	Section(s)
Math	252/4	All
Examination	**Date**	**Pages**
Final	April 2013	2
Instructors		**Course Examiner**
C. Cummins, F. Thaine		F. Thaine

Special Instructions: ▹ Only approved calculators are allowed.

Answer ten questions. All questions have equal value.

1. Calculate the orthogonal projection $\mathbf{u}$ of the vector $(2, 1, -1)$ on the subspace W of $\mathbb{R}^3$ spanned by the vectors $(1, -2, 1)$ and $(0, 1, -1)$, and find the vector $\mathbf{z} \in W^\perp$ such that $(2, 1, -1) = \mathbf{u} + \mathbf{z}$.

2. Let $V = \mathbb{R}^3$, the inner product space with the standard inner product. By means of the Gram-Schmidt process, find an orthogonal basis of V by using the basis $\{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$ as the starting basis.

3. Let $A = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 \end{bmatrix}$. Let $T = L_A$; that is

   $T : \mathbb{R}^4 \to \mathbb{R}^4$ is the linear operator given by $T(X) = AX$.

   a) Find a basis β of the T-cyclic subspace W of $\mathbb{R}^4$ generated by the vector

   $v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and find $[T_W]_\beta$, where $T_W : W \to W$ is the restriction of T to W.

   b) Find the characteristic polynomial of T_W and determine if T_W is diagonalizable.

4. Find the general solution to the following system of linear differential equations.

$$\begin{aligned} x' &= 2x \\ y' &= \quad y + 2z \\ z' &= x - 3y \end{aligned}$$

5. Use the least squares approximation to find the best fit with a linear function to the following data: $\{(-1,2),(0,1),(1,-2),(2,-3)\}$. Compute the error of the best fit.

6. Let $S = \{(0,1,-1),(1,i,1)\}$ in $\mathbb{C}^3$. Compute $S^{\perp}$.

7. Let $A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & -2 & -2 \\ -4 & -2 & 1 \end{bmatrix}$. The characteristic polynomial of A is $-(t+3)^2(t-6)$. Find an orthogonal matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$ (that is $Q^tAQ = D$). Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

8. Let $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. Show that A is a normal matrix. Find a unitary matrix U and a diagonal matrix D such that $U^{-1}AU = D$ (that is $U^*AU = D$). Find an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of A.

9. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 1 \\ -1 & 0 & 2 \end{bmatrix}$. Find an invertible matrix Q and a Jordan matrix J such that $Q^{-1}AQ = J$. Find bases β_1 and β_2 for the generalized eigenspaces K_{λ_1} and K_{λ_2} of A, respectively, such that $\beta = \beta_1 \cup \beta_2$ is a Jordan basis for A.

10. Let T be a self-adjoint operator on a, real or complex, inner product vector space V.

a) Prove that every eigenvalue of T is real. (Hint: use the fact that if T is normal and $Tv = \lambda v$, then $T^* = \bar{\lambda} v$.)

b) Let v and w be eigenvectors of T corresponding to distinct eigenvalues λ_1 and λ_2. Using only the definitions and part (a), prove that v and w are orthogonal. (Hint: prove first that $\lambda_1\langle v,w\rangle = \lambda_2\langle v,w\rangle$.)

11. Let V be a finite dimensional vector space, T a linear operator on V, and W a T-invariant subspace of V. Denote by T_W the restriction of T to W. Show that the characteristic polynomial of T_W divides the characteristic polynomial of T. (Hint: start with a basis of W, complete it to a basis β of V and study the form of the matrix $[T]_\beta$ of T with respect to β.)

8.3.2 Pre-university calculus

8.3.2.1 MATH 203

The classification of the tasks on the exam for Concordia's MATH 203 – Calculus 1 (December 2014) are shown in Table 33Table 30.

Table 33 - Classification of tasks: MATH 203 - Calculus 1 (Concordia), December 2014

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a	✓		✓		
1	b		✓	✓		
1	c (i)	✓		✓		
1	c (ii)	✓		✓		
1	c (iii)	✓		✓		
2	a	✓		✓		
2	b	✓		✓		
3		✓		✓		
4	a	✓		✓		
4	b	✓		✓		
4	c	✓		✓		
4	d	✓		✓		
4	e	✓		✓		
5	a		✓	✓		
5	b	✓		✓		
5	c	✓		✓		
6	a	✓		✓		
6	b		✓	✓		
7	a (i)	✓		✓		
7	a (ii)	✓		✓		
7	b		✓		✓	
7	c	✓		✓		
8	a	✓		✓		
8	b		✓	✓		
9	a		✓	✓		
9	b (i)	✓		✓		
9	b (ii)		✓	✓		
9	c (i)	✓		✓		
9	c (ii)		✓	✓		
9	d		✓	✓		
Bonus			✓	✓		
TOTAL		**21**	**10**	**30**	**1**	**0**
Relative frequency		**68%**	**32%**	**97%**	**3%**	**0%**

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	203	All
Examination	**Date**	**Pages**
Final	December 2014	3
Instructors:	Z. Ben Salah, A. Boyarsky, J. Brody, I. Gorelyshev, T. Hughes, P. Moore	**Course Examiner** A. Atoyan
Special Instructions:	Only approved calculators are allowed **Show all your work for full marks.**	

MARKS

[11] **1.** (a) Solve for x : $\ln(4x^2) + 2\ln(x) = 2\ln(6x)$.

(b) Sketch the graph of the function $f(x) = |(x-1)^2 - 4|$. (Suggestion: start from the graph of standard parabola, then use appropriate transformations.)

(c) Given the function $f(x) = \ln(1 + e^{2x})$, find the inverse function $f^{-1}(x)$, the range of $f(x)$ and the range of $f^{-1}(x)$

[7] **2.** Find the limit if it exists (**Do not use l'Hôpital's rule.**) :

(a) $\displaystyle\lim_{x \to -3} \frac{|x+3|}{x^2 + 2x - 3}$ (b) $\displaystyle\lim_{x \to \infty} \frac{2x(x^2 + \sqrt{1 + x^2 + 4x^4})}{1 + x^2 - 3x^3}$

[6] **3.** Find all horizontal and vertical asymptotes of the function

$$f(x) = \frac{9 + 2 \cdot 3^x}{3^x - 9}$$

[15] **4.** Find the derivatives of the following functions (you don't need to simplify your final answer, but you must show how you calculate it):

(a) $f(x) = \arctan x + (x^{3/2} + 2x^{-1/2})\sqrt{x}$

(b) $f(x) = \ln \dfrac{x^3}{x+3}$

(c) $f(x) = \dfrac{e^{-x} \tan x}{1 + e^x}$

(d) $f(x) = \ln(e^{x \sin x} + x \sin(e^x))$

(e) $f(x) = (1 - \cos x)^{x^2}$ (use logarithmic differentiation)

[12] 5. (a) Use the definition of derivative as the limit of difference quotient to find dy/dx for $y = \sqrt{5+x^2}$.

(b) Find the linearization $L(x)$ of the function $\tan(x)$ at $a = \pi/4$

(c) Use $L(x)$ found in (b) to approximate $\tan(x)$ at $x = \frac{\pi}{3} (= \frac{\pi}{4} - \frac{\pi}{12})$.

[7] 6. Let $f(x) = x^3 - 3x^2 - x - 3$.

(a) Find the slope m of the secant line joining the points $(2, f(2))$ and $(0, f(0))$.

(b) Find all points $x = c$ (if any) on the interval $[0,2]$ such that the rate $f'(c)$ of instantaneous change of $f(x)$ is equal to the slope m of the secant line in (a).

[17] 7. (a) Verify that the point (3,1) belongs to the curve defined by the equation $y^3 + x^3 - 2x^2y^2 = 10$, and find an equation of the tangent line to the curve at that point.

(b) The length of a rectangle is increasing at the rate of 8 cm/s and its width is increasing at the rate of 5 cm/s. When the length is 20 cm and the width is 12 cm, how fast is the area of the rectangle increasing at that instant?

(c) Use l'Hôpital's rule to evaluate the $\lim_{x\to 0} \dfrac{e^{2x} + e^{-2x} - 2}{x \sin x}$.

[11] 8. (a) Find the point (x_0, y_0) on the line $y + 2x = 2$ that is closest to the point $(5, 2)$.

(b) A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 3 - x^2$. Find the dimensions of such rectangle with the maximum possible area.

[14] 9. Given the function $f(x) = \ln(1 + x^2)$.

(a) Find the domain of $f(x)$, check for symmetry, and also find asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine intervals where the function is increasing, intervals where it is decreasing, and the local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine intervals where the function is concave upward, intervals where the function is concave downward, and the inflection points (if any).

(d) Sketch the graph of the function $f(x)$ using the information obtained above.

[5] **Bonus Question.** If $y = f(u)$ and $u = g(x)$, where f and g are twice differentiable functions, use the Chain rule to derive the following formula for the second derivative:

$$\frac{d^2y}{dx^2} = \frac{d^2f}{du^2}\left(\frac{dg}{dx}\right)^2 + \frac{df}{du}\frac{d^2g}{dx^2}$$

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8.3.2.2 MATH 205

The classification of the tasks on the exam for Concordia's MATH 205 – Calculus 2 (December 2014) are shown in Table 30.

Table 34 - Classification of tasks: MATH 205 - Calculus 2 (Concordia), December 2014

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a (i)		✓	✓		
1	a (ii)		✓	✓		
1	a (iii)	✓		✓		
1	b (i)	✓		✓		
1	b (ii)		✓	✓		
2	a	✓		✓		
2	b	✓		✓		
3	a	✓		✓		
3	b	✓		✓		
3	c	✓		✓		
4	a	✓		✓		
4	b	✓		✓		
5	a	✓		✓		
5	b	✓		✓		
6	a (i)		✓	✓		
6	a (ii)	✓		✓		
6	b	✓		✓		
6	c	✓		✓		
7	a	✓		✓		
7	b	✓		✓		
8	a		✓	✓		
8	b		✓	✓		
8	c		✓	✓		
9	a (i)		✓	✓		
9	a (ii)		✓	✓		
9	b	✓		✓		
Bonus			✓	✓		
TOTAL		**17**	**10**	**17**	**0**	**0**
Relative frequency		**63%**	**37%**	**100%**	**0%**	**0%**

CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	205	All
Examination	**Date**	**Pages**
Final	December 2014	2
Instructors:	J. Brody, R. Mearns, I. J. Pelczer, R. Wang	**Course Examiner** A. Atoyan
Special Instructions:	Only approved calculators are allowed. **Show all your work for full marks.**	

[10] **1.** (a) Sketch the graph of $f(x) = 2^{-x}$. Using partitioning of the interval $[-2, 2]$ into 4 subintervals of equal length, the definite integral $A = \int_{-2}^{2} f(x)\,dx$ can be approximated by either the leftpoint, or the midpoint, or the rightpoint Riemann sum. Explain which one of these three Riemann sums provides the best approximation for A, and calculate that Riemann sum.

(b) Find the derivative of the function $F(x) = 2e^{-x^2} - \int_{x^4}^{1} \sqrt{1+t}\, e^{-t}\, dt$, and determine whether $F(x)$ is increasing of decreasing at $x = 1$.

[11] **2.** Find the antiderivative $F(x)$ of the function $f(x)$ that satisfies the given condition:

(a) $f(x) = \dfrac{e^{-3x}}{(e^{-3x}+1)^3}$, $F(0) = 0$. (b) $f(x) = \tan^3 x$, $F\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}$.

[15] **3.** Find the following indefinite integrals:

(a) $\displaystyle\int x\ln(x+2)\,dx$ (b) $\displaystyle\int \frac{x}{x^2-4x+3}\,dx$ (c) $\displaystyle\int x\left(1+\frac{1}{\sqrt{x}}\right)^2 dx$.

[12] **4.** Evaluate the following definite integrals (give the exact values, do **not** approximate):

(a) $\displaystyle\int_0^{\pi/4} \frac{\sec^2(x)}{4+\tan^2(x)}\,dx$ (b) $\displaystyle\int_0^{\pi/2} \cos^3(x)\,\sin^5(x)\,dx$

[8] **5.** Evaluate the given improper integral or show that it diverges:

(a) $\displaystyle\int_{-\infty}^{0} x e^{-x^2}\,dx$ (b) $\displaystyle\int_{-2}^{2} \frac{dx}{(x+2)^{3/2}}$

[15] 6. (a) Plot the curve $y = \sqrt{4 - x^2}$, and the line $y = 2 - x$, and find the exact value of the area enclosed.

(b) Find the volume of a solid obtained by rotating the region bounded by the curve $y = \sqrt{3x}$ and the lines $y = 3$ and $x = 0$ about the axis $y = -1$.

(c) Find the average value of $f(x) = \sin^2(x)\cos^2(x)$ on the interval $[0, \frac{\pi}{2}]$.

[6] 7. Find the limit of the sequence $\{a_n\}$ as $n \to \infty$ or prove that it does not exist.

(a) $a_n = \dfrac{(-3)^{2n}}{1 + 3n^3 + 9^{n+1}}$ (b) $a_n = \dfrac{\sqrt{9 + n^2 + 4n^4}}{(n + 3n^{3/2})(4 + \sqrt{n})}$

[15] 8. Determine whether the series is divergent or convergent, and if convergent, then is it convergent absolutely or conditionally :

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{1+n}}{1+n^2}$ (b) $\displaystyle\sum_{n=0}^{\infty} e^{-n}(-3)^{n-1}$ (c) $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n \ln(n)}$

[8] 9. (a) Find (a) the radius of convergence and (b) the interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{n\,3^n}$$

(b) Derive the MacLaurin series of $f(x) = x^3 \ln(1 + 2x^2)$

(HINT: start with the series for $\ln(1 + z)$ where $z = 2x^2$).

[5] Bonus **Question.** It is known that for some continuous even function, $f(x) = f(-x)$, the average value of $f(x)$ on any given interval $[-a, a]$ $(a > 0)$ is equal to the length of the interval. Is this information sufficient to find $f(x)$? Find the function f if it is, otherwise explain why it is insufficient.

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8.3.3 University level calculus

8.3.3.1 ENGR 233

The classification of the tasks on the exam for Concordia's ENGR 233 – Applied Advanced Calculus (December 2014) are shown in Table 35.

Table 35 - Classification of tasks: ENGR 233 - Applied Advanced Calculus (Concordia), December 2014

Task	**Nature**		**Content**		
Question Part	Comp	Conc	Math	App	Mod
1	✓		✓		
2	✓		✓		
3	✓			✓	
4		✓			✓
5	✓		✓		
6	✓		✓		
7	✓		✓		
8	✓		✓		
9	✓		✓		
TOTAL	**8**	**1**	**7**	**1**	**1**
Relative frequency	**89%**	**11%**	**78%**	**11%**	**11%**

Concordia University
Faculty of Engineering and Computer Science
Applied Advanced Calculus - ENGR233/2
2014 Alternate Final Examination

Instructors: Dr.R.B.Bhat, Dr.D.Korotkin

Special instructions: Do all problems. Show relevant steps with intermediate results requiring in answering all the questions. Only Faculty approved calculators are allowed. No other aids are allowed. If you think that any data is missing, state your assumptions clearly and proceed with your answers. ALL PROBLEMS CARRY THE SAME WEIGHT

1. Find the volume of the parallelepiped with the following vectors forming the three edges: $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$
2. Find the parametric equations for the line of intersection of the planes $2x - 3y + 4z = 1$ and $x - y - z = 5$.
3. A quarterback throws a football with the initial speed of $30m/s$ at an angle of 60° from the horizontal. Find the range of the football.
4. The temperature at a point (x, y) on a rectangular metal plate is given by $T(x, y) = 5+2x^2+y^2$. Determine the direction an insect should take, starting at (4, 2) in order to cool off as rapidly as possible.
5. Evaluate $\int_C y dx - x dy$ where C is given by $x = 2\cos t$, $y = 3\sin t$, $0 \leq t \leq \pi$.
6. Evaluate the integral $\int_0^2 \int_{y^2}^4 \cos\sqrt{x^3} dx\, dy$ by reversing the order of integration.
7. Use Green's theorem to to compute $\oint_C (x + y^2)\, dx + (2x^2 - y)\, dy$ where C is the boundary of the region bounded by $x = 0$, $x^2 + y^2 = 1$, $x > 0$. The integral is taken in counterclockwise direction.
8. Use Stokes' theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (x + 2z)\mathbf{i} + (3x + y)\mathbf{j} + (2y - z)\mathbf{k}$ where C is the curve of intersection of the plane $x + 2y + z = 4$ with the coordinate planes, oriented counterclockwise if viewed from above.
9. Use divergence theorem to find the outward flux $\int\int_S (\mathbf{F} \cdot \mathbf{n}) dS$ of the vector field $\mathbf{F} = x^3\mathbf{i}+y^3\mathbf{j}+z^3\mathbf{k}$ where D is the region bounded by the sphere $x^2 + y^2 + z^2 = 1$.

8.3.3.2 MATH 262

The classification of the tasks on the exam for McGill's MATH 262 – Intermediate Calculus (December 2010) are shown in Table 36.

Table 36 - Classification of tasks: MATH 262 - Intermediate Calculus (McGill), December 2010

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a		✓	✓		
1	b (i)	✓		✓		
1	b (ii)	✓		✓		
2	a		✓	✓		
2	b	✓		✓		
3	a	✓		✓		
3	b	✓		✓		
4	(i)		✓	✓		
4	(ii)	✓		✓		
5	(i)	✓		✓		
5	(ii)	✓		✓		
6			✓	✓		
7		✓		✓		
8	(i)		✓	✓		
8	(ii)	✓		✓		
9		✓		✓		
TOTAL		**11**	**5**	**16**	**0**	**0**
Relative frequency		**69%**	**31%**	**100%**	**0%**	**0%**

McGill University
Faculty of Engineering
Final Examination
Math 262: Fall 2010

Intermediate Calculus

Examiner: Prof. N.Sancho
Assoc. Examiner: Dr. N. Dimitrov

Date: 17th Dec. 2010
Time: 2:00 – 5:00 pm

Instructions

Please answer all questions in the exam. booklet provided.
This is a closed book exam.
The exam. is worth 100 marks
Calculators are not permitted
Dictionaries are not allowed
This exam. is comprised of the cover page and 2 pages of questions

Final Examination Math 262 Fall 2010 17th December 2010

1.(a) (8 marks) Determine the values of x for which the series converge absolutely, converge conditionally or diverge:

$$\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{1/2} 4^n}$$

(b) (12 marks) If $C(x) = \int_0^x \left(\frac{1-\cos t}{t}\right) dt$.(i) Find the first three terms for C(x) about x = 0 (ii) Find C(0.2) with error $\leq 10^{-4}$.

2. (a) (4 marks) Show that the function $u(x,y) = e^x \sin y$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(b) (4 marks) If $f(x,y) = x^2 e^{y/x}$, find $\frac{\partial^2 f}{\partial y \partial x}$.

3. (10 marks) A particle P, moves along the curve defined by $\mathbf{r} = t^3 \mathbf{i} + \sqrt{\frac{3}{2}}\, t^2 \mathbf{j} + t\mathbf{k}$.

(a) Find the tangent and normal component of acceleration (scalar values)
(b) Find the length of the curve from t = 0 to t = 1.

4. (10 marks) Show that the equations $\begin{cases} xe^{v} - uz - \cos v = 2 \\ u\cos y + x^2 v - yz^2 = 1 \end{cases}$

can be solved for u and v as functions of x ,y and z near the point P_0 where (x, y, z) = (2, 0, 1) and (u, v) = (1, 0) and find $\left(\frac{\partial u}{\partial z}\right)_{x,y}$ at $(x, y, z) = (2, 0, 1)$.

5. (8 marks) Find the linear approximation of the function;
$f(x,y) = \sqrt{20 - x^2 - 7y^2}$ at (2, 1) and use it to approximate f(1.95, 1.08).

6. (10 marks) If $u = f(x,y)$ where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}\right].$$

7. (12 marks) Find the local maximum and minimum values and saddle points of $f(x,y) = (x^2 - 4y^2)(2 - x)$.

8. (10 marks) Sketch the region of integration for the following integral

$$\int_{y=0}^{1} dy \int_{x=y^2}^{1} \frac{y^3\,dx}{\sqrt{y^4 + x^2}}$$

Now reverse the order of integration and evaluate.

9. (12 marks) Find the volume of the solid region lying inside both the sphere $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 = 2ay$, above the xy-plane.

3

8.3.3.3 MATH 264 (for engineering students)

The classification of the tasks on the exam for McGill's MATH 264 – Advanced Calculus for engineering students (April 2007) are shown in Table 37.

Table 37 - Classification of tasks: MATH 264 - Advanced Calculus (McGill), April 2007

Task		Nature		Content		
Question	Part	Comp	Conc	Math	App	Mod
1		✓		✓		
2		✓		✓		
3		✓		✓		
4		✓		✓		
5		✓		✓		
6		✓		✓		
TOTAL		**6**	**0**	**6**	**0**	**0**
Relative frequency		**100%**	**0%**	**100%**	**0%**	**0%**

McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 264: Advanced Calculus

Examiner: Professor Jakobson Dmitry Jakobson | Date: Friday, April 13, 2007
Associate Examiner: Professor Laayouni | Time: 9.00 - 12:00

INSTRUCTIONS

Answer all six questions. Each question is worth 10 points. Please give a detailed explanation for each answer.
Non-programmable calculators are permitted. Translation dictionaries are permitted, regular dictionaries are not permitted. This is a closed-book exam

This exam comprises the cover and one page of questions, Printed Double-Sided

3

Problem 1. Compute

$$\int_C \left(x + y + \frac{y}{x^2 + y^2}\right) dx + \left(y - x + \frac{x}{x^2 + y^2}\right) dy,$$

where C is the ellipse $x^2/16 + 9y^2 = 1$.

Problem 2. Compute the outward flux of the vector field $\mathbf{F} = (y^2x - xz, yz + x^2y, e^x + \cos(y))$ across the surface S consisting of the paraboloid $z = x^2 + y^2, 0 \le z \le 4$, capped by the disk $D := \{(x, y, z) : z = 4, 0 \le x^2 + y^2 \le 4\}$.

Problem 3. Using Stokes' Theorem, compute the integral $\int_C (x^2 - yz)dx + (2x + y^2 - xz)dy + (z^2 - xy)dz$, where C is the curve formed by the intersection of the sphere $x^2 + y^2 + z^2 = 25$ and the plane $z = 4$, oriented counterclockwise (e.g. its projection into the (x, y)-plane is oriented counterclockwise).

Problem 4. Compute the surface integral

$$\iint_S (x^2 + y^2 + z^2) dA,$$

where S is the surface of the tetrahedron with vertices at

$$(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1).$$

Problem 5. Use separation of variables to solve the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= 3\frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad 0 < t < \infty, \\ u(0, t) &= u(\pi, t) = 0, && 0 < t < \infty, \\ u(x, 0) &= \sin(x) - 6\sin(4x), && 0 < x < \pi. \end{aligned}$$

Problem 6. Use Fourier series to solve the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= 7\frac{\partial^2 u}{\partial x^2}, && 0 < x < \pi, \quad 0 < t < \infty, \\ \frac{\partial u}{\partial x}(0, t) &= \frac{\partial u}{\partial x}(\pi, t) = 0, && 0 < t < \infty, \\ u(x, 0) &= 1 - \sin(x), && 0 < x < \pi. \end{aligned}$$

4

8.3.3.4 MATH 264 (for mathematics students)

The classification of the tasks on the exam for Concordia's MATH 264 – Advanced Calculus I for mathematics students (December 2014) are shown in Table 38.

Table 38 - Classification of tasks: MATH 264 - Advanced Calculus I (Concordia), December 2014

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	(i)	✓		✓		
1	(ii)	✓		✓		
2	(i)	✓		✓		
2	(ii)		✓	✓		
2	(iii)	✓		✓		
3	(i)	✓		✓		
3	(ii)	✓		✓		
3	(iii)	✓		✓		
3	(iv)		✓	✓		
4	a (i)	✓			✓	
4	a (ii)	✓			✓	
4	b	✓			✓	
5	(i)	✓		✓		
5	(ii)	✓		✓		
6		✓		✓		
7		✓		✓		
8			✓	✓		
9		✓		✓		
10	(i)	✓		✓		
10	(ii)		✓	✓		
TOTAL		**16**	**4**	**17**	**3**	**0**
Relative frequency		**80%**	**20%**	**85%**	**15%**	**0%**

CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	264	A & B
Examination	Date	Pages
Final	December 2014	3
Instructors		Course Examiners
H. Kisilevsky, J. Macdonald		H. Kisilevsky

Special Instructions

▷ Only approved calculators are allowed. Justify all your answers. All questions have equal value.

Formulaire

For a curve $\mathbf{r}(t)$ in $\mathbb{R}^3$, Arclength $s(t) = \int_a^t |\mathbf{r}'(u)|du$, $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, $\kappa = |\frac{d\mathbf{T}}{ds}| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

1. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ with $x(t) = t^3, y(t) = 3t, z(t) = t^4$. Give an expression for the normal plane to $\mathbf{r}(t)$ of the form

$$A(t)x + B(t)y + C(t)z = D(t).$$

At what point is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

2. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ with $x(t) = t, y(t) = 1/t, z(t) = 0$ for $t > 0$. Compute $\kappa(t)$. Show that $\kappa(t) \to 0$ as $t \to 0, \infty$. Find the maximum value of $\kappa(t)$.

3. Write a power series expansion around $x = 0$ of $f(x) = x - \ln(1 + x)$. Use it to evaluate the

$$\lim_{x \to 0} f(x)/x^2.$$

Give a power series for $\int f(x)/x^2$ around $x = 0$. What is its radius of convergence?

4. Suppose that temperature at every point (x, y) is given by

$$T(x, y) = 10e^{-x^2-y^2}.$$

(a) Find the direction in which the temperature is *decreasing* most rapidly at the point $(1, -2)$, and give the rate of change in this direction.

(b) Find all directions in which the temperature is *not changing* at the point $(1, -2)$. answer.

5. Let C be the curve defined by the parametric equations

$$\begin{aligned} x &= t^2 \\ y &= t^3 - t, \end{aligned}$$

where $t \in \mathbb{R}$. Find a point at which C intersects itself, and find the angle of this intersection.

6. Find and classify the critical points of the function

$$f(x, y) = 2x^3 - 6xy + 3y^2.$$

7. Find the minimum and maximum values of the function

$$f(x, y) = x^2 + y^2 - 2x - 5$$

on the region defined by the inequality

$$x^2 + 2y^2 \leq 16.$$

8. Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{\sqrt{n}}.$$

9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$yz = \ln(x + z).$$

10. Find the minimum and maximum distances from the origin $O = (0,0,0)$ to the surface of the ellipsoid given by

$$x^2 + y^2/4 + z^2/9 = 1.$$

Be sure to justify your answer.

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8.3.3.5 MATH 265

The classification of the tasks on the exam for Concordia's MATH 265 – Advanced Calculus II (April 2014) are shown in Table 39.

Table 39 - Classification of tasks: MATH 265 - Advanced Calculus II (Concordia), April 2014

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1		✓		✓		
2		✓		✓		
3		✓		✓		
4	(i)		✓	✓		
4	(ii)	✓		✓		
5		✓		✓		
6		✓		✓		
Bonus		✓		✓		
TOTAL		**7**	**1**	**8**	**0**	**0**
Relative frequency		**87.5%**	**12.5%**	**100%**	**0%**	**0%**

CONCORDIA UNIVERSITY

Department of Mathematics and Statistics

Course	Number	Sections
MATH	265/4	AA / B

Examination	Date	Time	Pages
Final	April 30, 2014	19:00 - 22:00	2

Instructors	Course examiner
D. Dryanov and J. Harnad	J. Harnad

Instructions. Answer all six numbered questions. The value for each part is indicated in square brackets in the margin (out of a possible total of 60). Only answer the **BONUS question** if you have time left at the end. Show all your steps. Write the complete solution on the **right hand** pages of your examination booklet. The left hand side is for your use for rough calculations, sketches, etc. and will **not** be read by the examiners. Only calculators authorized by the Department of Mathematics and Statistics may be used. Lined examination booklets will be provided. Other books, notes or recorded materials may not be used.

[10] **1.** Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$.

[10] **2.** Find the area of the part of the cone $z^2 = a^2(x^2 + y^2)$ between the planes $z = 1$ and $z = 2$.

[10] **3.** Evaluate the line integral

$$\int_C xy\,dx + y^2\,dy + yz\,dz$$

where C is the line segment from $(1, 0, -1)$ to $(3, 4, 2)$.

[10] **4.** Show that the vector field

$$\mathbf{F}(x, y, z) = \sin y\ \mathbf{i} + x\cos y\ \mathbf{j} - \sin z\ \mathbf{k}$$

is conservative and find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

[10] **5.** Use Green's theorem to evaluate

$$\int_C x^2 y\,dx - xy^2\,dy,$$

where C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

[10] **6.** Use Stokes' theorem to evaluate

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where

$$\mathbf{F}(x, y, z) = x^2yz\ \mathbf{i} + yz^2\ \mathbf{j} + z^3e^{xy}\ \mathbf{k}$$

and S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, oriented upward.

==

[5] **BONUS Question.** Compute the outward flux of

$$\mathbf{F}(x, y, z) = \frac{x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

through the ellipsoid

$$4x^2 + 9y^2 + 6z^2 = 36.$$

(Hint: show that $\nabla \cdot \mathbf{F}(x, y, z)$ vanishes if $(x, y, z) \neq (0, 0, 0)$ and use the divergence theorem to transform the integral from the ellipsoid to a more convenient surface also enclosing the origin.)

==

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8.3.4 Differential equations

8.3.4.1 ENGR 213

The classification of the tasks on the exam for Concordia's ENGR 213 – Applied Ordinary Differential Equations (December 2014) are shown in Table 40.

Table 40 - Classification of tasks: ENGR 213 - Applied Ordinary Differential Equations (Concordia), December 2014

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a	✓		✓		
1	b	✓		✓		
2		✓		✓		
3		✓		✓		
4	a	✓		✓		
4	b	✓		✓		
5		✓			✓	
6		✓		✓		
7		✓		✓		
8		✓		✓		
9		✓		✓		
TOTAL		**11**	**0**	**10**	**1**	**0**
Relative frequency		**100%**	**0%**	**91%**	**9%**	**0%**

Concordia University
Faculty of Engineering and Computer Science
Department of Mechanical and Industrial Engineering
Applied Ordinary Differential Equations - ENGR213/2
2014 Final Examination

Instructors: Alecsandru C, Dryanov D, Kokotov A, Korotkin D, Rossokhata N, Vatistas G H

Special instructions: Do all problems. Show all steps or else no marks will be given. Only Faculty approved calculators are allowed. No other aids are allowed. ALL PROBLEMS CARRY THE SAME WEIGHT

1. Find the general solution of the following equations by separation of variables:

$$(a) \qquad \frac{dy}{dx} = \frac{4-2x}{3y^2-5}$$

$$(b) \qquad x^2\frac{dy}{dx} = y - xy$$

2. Find the general solution of the following exact equation:

$$(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$$

3. Using the integrating factor method find the general solution of the following differential equation:

$$x\frac{dy}{dx} - 4y - x^6e^x = 0$$

4. Find general solutions of the following equations:

$$(a) \qquad \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

$$(a) \qquad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

5. The equation describing the motion of the mass-spring system is

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

where $k = 1N/m$ and $m = 1kg$. Find the position y of mass at an arbitrary time t if the initial position of the mass is $1m$ and the initial velocity is 0.

6. Find the general solution of the equation

$$y'' + 3y' + 4y = 3x + 2$$

by the method of undetermined coefficients.

7. Find the general solution of the equation

$$y'' - 4y = \frac{e^{2x}}{x}$$

by the method of variation of parameters.

8. Find the general solution of the system

$$\frac{dx}{dt} = x - y$$

$$\frac{dy}{dt} = y - 4x$$

by the method of your choice.

9. Use the power series method to find the solution of the initial value problem

$$y'' + 2xy' + y = 0\,, \qquad y(0) = 1,\;\; y'(0) = 0$$

(write the first four non-zero terms of the power series solution centred at $x = 0$).

8.3.4.2 ENGR 311

The classification of the tasks on the exam for Concordia's ENGR 311 – Transform Calculus and Partial Differential Equations (August 2009) are shown in Table 41.

Table 41 - Classification of tasks: ENGR 311 - Transform Calculus and Partial Differential Equations (Concordia), August 2009

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1		✓		✓		
2	a	✓		✓		
2	b		✓	✓		
2	c	✓		✓		
3		✓		✓		
4	a	✓		✓		
4	b		✓		✓	
5		✓		✓		
TOTAL		**6**	**2**	**7**	**1**	**0**
Relative frequency		**75%**	**25%**	**87.5%**	**12.5**	**0%**

ENGR 311 Final Exam
Summer 2009
Pierre Gauthier

1. Solve the following system of differential equations using Laplace Transforms

$$x' - 6x + 3y = 8e^t$$
$$-2x + y' - y = 4e^t$$

$$x(0) = 1$$
$$y(0) = 0$$

2. Given

$$f(x) = \begin{cases} \pi & -\pi \leq x \leq 0 \\ x & 0 \leq x \leq \pi \end{cases}$$

a) Find the Fourier Series
b) Sketch the graph from -7π,7π
c) Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2}$$

3. Solve the following wave equation

$$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}$$

With the conditions

$$U(0,t) = U(1,t) = 0$$
$$U(x,0) = 0$$
$$\frac{\partial U}{\partial x}(x,0) = 2\sin\left(\frac{3\pi}{4}x\right) - \frac{1}{4}\sin(2\pi x)$$

4. a) Solve the following Heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - 3U$$

With the conditions

$$\frac{\partial U}{\partial x}(0,t) = \frac{\partial U}{\partial x}(1,t) = 0$$
$$U(x,0) = x(1-x)$$

b) What is the steady-state temperature?

5. Solve the following Laplace Equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

With the conditions

$$U(0,y) = F(y) \text{ and } U(b,y) = G(y)$$
$$U(x,0) = U(x,a) = 0$$

8.3.4.3 MATH 263

The classification of the tasks on the exam for McGill's MATH 263 – Ordinary Differential Equations for Engineers (December 2012) are shown in Table 42.

Table 42 - Classification of tasks: MATH 263 - Ordinary Differential Equations for Engineers (McGill), December 2012

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	(i)	✓		✓		
1	(ii)		✓	✓		
2		✓		✓		
3		✓		✓		
4		✓		✓		
5		✓		✓		
6		✓		✓		
7	a		✓	✓		
7	b	✓		✓		
8	a		✓	✓		
8	b	✓		✓		
8	c		✓	✓		
9	a	✓		✓		
9	b	✓		✓		
9	c		✓	✓		
TOTAL		**10**	**5**	**15**	**0**	**0**
Relative frequency		**67%**	**33%**	**100%**	**0%**	**0%**

McGill University
Faculty of Engineering

December 2012
Final Examination

Ordinary Differential Equations for Engineers

Math 263

Thursday December 6, 2012

Time: 2pm-5pm

Examiner: Prof. A. Humphries

Associate Examiner: Dr. I. Karzhemanov

INSTRUCTIONS

1. Answer all questions in the exam booklets provided. Start each answer on a new page
2. All questions carry equal weight.
3. This a CLOSED BOOK exam.
4. Non-programmable calculators are permitted.
5. Translation dictionaries (English French) are permitted.

This exam comprises the cover page and two pages of questions, numbered 1 to 9.

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1.) Find the implicit solution of the initial value problem

$$y'e^y + \frac{x-1}{y} = 0, \qquad y(0) = 1.$$

Over which interval is the solution valid?

2.) Solve the Bernoulli equation

$$-2e^x y' + e^x y + y^3\cos(x) = 0.$$

3.) Solve the initial value problem

$$(3y^6 - 3)dx + (6xy^5 + 2x^{-2}y)dy = 0, \qquad y(1) = 0.$$

You may leave your answer in implicit form.

4.) Find the general solution of

$$y'' - y = xe^x + e^{-x}.$$

5.) Find the general solution of

$$x^2y'' - 5xy' + 8y = 6x^4\ln x.$$

6.) Solve the initial value problem

$$y^{(4)} - y = x^5, \qquad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

7.) (a) State the general form of a series solution for $y(x)$ expanding about the point $x = 0$ and satisfying the initial conditions $y(0) = y'(0) = 0$.

(b) Find the first four non-zero terms in the series solution of the initial value problem

$$y'' - xy = \frac{1}{6}x^3, \qquad y(0) = y'(0) = 0,$$

expanded about the ordinary point $x = 0$.

8.) (a) Show that $x = 0$ is a regular singular point of the equation

$$2x^2y'' - x(1+x)y' + y = 0.$$

(b) State and solve the indicial equation for a Frobenius solution

$$y(x) = x^r\sum_{n=0}^{\infty} a_n x^n, \quad x > 0$$

(c) State the recurrence relation satisfied by the coefficients a_n, and hence find two solutions of the form stated in (b). (Your answer should include an expression for the n^{th} coefficient in each series solution, not just a recursion relation.)

/PTO

9.) (a) Let $y(t)$ solve

$$y'' - y = \delta(t-\pi) + \mathcal{U}(t-2\pi)e^{-(t-2\pi)}, \qquad y(0) = 1, \quad y'(0) = 0.$$

Find an expression for $Y(s)$, the Laplace transform of $y(t)$.

(b) Find $y(t)$, the solution of the initial value problem stated in (a).

(c) Does $\lim_{t\to+\infty} y(t)$ exist? If so, what is it? If not, is $y(t)$ bounded as $t \to +\infty$?

Table of Laplace Transforms

function $f(t)$	Laplace transform $F(s)$
1	$1/s \quad (s>0)$
t^n	$n!/s^{n+1} \quad (s>0)$
e^{at}	$1/(s-a) \quad (s>a)$
$\sin at$	$a/(s^2+a^2) \quad (s>0)$
$\cos at$	$s/(s^2+a^2) \quad (s>0)$
$\sinh at$	$a/(s^2-a^2) \quad (s>\lvert a\rvert)$
$\cosh at$	$s/(s^2-a^2) \quad (s>\lvert a\rvert)$
$e^{-at}f(t)$	$F(s+a)$
$\mathcal{U}(t-a)$ or $\mathcal{U}_a(t)$ $(a \geq 0)$	$e^{-as}/s \quad (s>0)$
$\delta(t-a) \quad (a>0)$	e^{-as}
$\mathcal{U}(t-a)f(t-a)$ or $\mathcal{U}_a(t)f(t-a)$	$e^{-as}F(s)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{(n-1)}(0)$
$(-t)^n f(t)$	$F^{(n)}(s)$
$f * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$

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8.3.4.4 MATH 370

The classification of the tasks on the exam for Concordia's MATH 370 – Ordinary Differential Equations (December 2014) are shown in Table 43.

Table 43 - Classification of tasks: MATH 370 - Ordinary Differential Equations (Concordia), December 2014

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1		✓		✓		
2		✓		✓		
3	(i)		✓	✓		
3	(ii)	✓		✓		
4	(i)		✓	✓		
4	(ii)	✓		✓		
5		✓		✓		
6		✓		✓		
7		✓		✓		
8		✓		✓		
9			✓	✓		
10	(i)		✓	✓		
10	(ii)	✓		✓		
TOTAL		**9**	**4**	**13**	**0**	**0**
Relative frequency		**69%**	**31%**	**100%**	**0%**	**0%**

Concordia University	Course: Math	Number: 370

Examination	Date	Time	Pages
Final	December 2014	3 hours	3

Course Coordinator	Instructors
A. Kokotov	A. Kokotov

Each problem is worth 10 points. Solve all the problems.

1. Solve the differential equation
$$x(e^y - y') = 2$$
Hint: Use the relation $(e^{-y})' = -e^{-y}y'$, make an appropriate change of variable and reduce the equation to a linear one.

2. Solve the Bernoulli equation
$$y' = y^4 \cos x + y \tan x$$

3. Check whether the following equation is exact and if it is exact then solve it.
$$(2xy^3 + \frac{1}{y})dx + (3x^2y^2 - \frac{x}{y^2})dy = 0$$

4. Check whether the following equation is exact and if it is not exact then solve it, using an integrating factor $\mu(y)$.
$$(2\frac{x}{y} + 1)dx + \frac{x}{y}dy = 0$$

5. Find the general solution of the equation
$$y^{(4)} - 2y'' + y = x^2 - 4$$

6. Find the first six coefficients $a_0, a_1, \ldots, a_5$ of the power series expansion $y = \sum_{n=0}^{\infty} a_n x^n$ of the solution to the initial value problem
$$\begin{cases} y'' - x^2 y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

7. Solve the system of differential equations
$$\begin{cases} x' = y - 2x - 2z + 3 \\ y' = x - 2y + 2z - 1 \\ z' = 3x - 3y + 5z - 5 \end{cases}$$
(Hint: Show that $\lambda_1 = \lambda_2 = -1; \lambda_3 = 3$; PSNS is almost obvious; **mind the order of variables!**)

8. Solve the system of differential equations using the Laplace transform (other solutions will not be accepted!)
$$\begin{cases} x' = -y \\ y' = 2x + 2y \\ x(0) = y(0) = 1 \end{cases}$$

9. Find the function $f(t)$ if

$$\int_0^{\infty} f(t)e^{-pt}dt = \frac{1}{(p^2+1)^2}$$

10. Using the theorem on the integration of the image

$$\mathcal{L}(\frac{f(t)}{t}) = \int_p^{\infty} F(p)dp$$

if $\mathcal{L}(f(t)) = F(p)$, prove the equality

$$\int_0^{\infty} \frac{f(t)}{t}dt = \int_0^{\infty} F(p)\,dp$$

and then calculate the integral

$$\int_0^{\infty} \frac{e^{-2t}-e^{-t}}{t}dt$$

(Hint: do not get scared! This can be done in two lines!)

Table of Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$,	$s > a$
t^n , $n =$ positive integer	$\frac{n!}{s^{n+1}}$,	$s > 0$
t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$,	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$,	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$,	$s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$,	$s > \|a\|$
$\cosh(at)$	$\frac{s}{s^2-a^2}$,	$s > \|a\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$,	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$,	$s > a$
$t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}$,	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$,	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$,	$c > 0$
$\int_0^t f(t-x)g(x)\mathrm{d}x$	$F(s)G(s)$	
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	
$(-t)^n f(t)$	$F^{(n)}(s)$	

$$u_c(t) := \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} , \qquad c > 0$$

3

8.3.5 Probability and statistics

8.3.5.1 ENGR 371

The classification of the tasks on the exam for Concordia's ENGR 371 – Probability and Statistics in Engineering (April 2013) are shown in Table 44.

Table 44 - Classification of tasks: ENGR 371 - Probability and Statistics in Engineering (Concordia), April 2013

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a	✓			✓	
1	b	✓			✓	
2	a	✓			✓	
2	b		✓		✓	
2	c	✓			✓	
3	a		✓	✓		
3	b	✓		✓		
3	c	✓		✓		
4	a	✓		✓		
4	b	✓		✓		
5	a	✓			✓	
5	b	✓			✓	
5	c	✓			✓	
5	d	✓			✓	
5	e		✓		✓	
6	a	✓			✓	
6	b	✓			✓	
6	c		✓		✓	
7	a		✓		✓	
7	b	✓			✓	
7	c	✓			✓	
7	d		✓		✓	
8	a	✓			✓	
8	b	✓			✓	
8	c		✓		✓	
TOTAL		**18**	**7**	**5**	**20**	**0**
Relative frequency		**72%**	**28%**	**20%**	**80%**	**0%**

1. A company is forming an interdisciplinary team of six engineers. It should contain two computer engineers, two industrial engineers and two building engineers. The engineering department at that company has seven computer engineers, four industrial engineers and five building engineers.

   a. How many ways can the team of six be formed? (2 marks)
   b. Two of the building engineers are very junior. It is not appropriate to put two junior engineers on the same project. If we include the constraint that the two building engineers on the team cannot both be junior, how many ways are there to form the team? (3 marks)

2. 80% of all customers that visit a particular e-commerce site end up buying something. Assume that whether or not different customers buy something is independent of each other.

   a. What is the probability that three of the next five customers buy something? (3 marks)
   b. If Y is the number of customers to visit the website before four of them buy something, give a distribution for Y. Be sure to state its range and all parameters. (3 marks)
   c. What is the probability that it will take seven or less visits to the website, until four sales are made? (4 marks)

3. Consider a continuous random variable X with the following pdf

$$f(x) = \begin{cases} mx & 2 < x < 3 \\ 0 & otherwise \end{cases}$$

   a. Determine m. (3 marks)
   b. Find the expected value of X. (3 marks)
   c. Find the variance of X. (3 marks)

4. Let X be an exponential random variable with mean 4.

   a. Calculate the probability that Y>3. (3 marks)
   b. Find r such that P(X>r) =0.5. (3 marks)

5. A privately owned store operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-in and walk-in facilities are in use, and suppose that the joint probability distribution function of these random variables is described by

$$f(x,y)=\begin{cases}\frac{2}{3}(x+2y), & 0\le x\le 1,\ 0\le y\le 1\\ 0, & elsewhere\end{cases}$$

a) Find the marginal densities *f(x)* and *f(y)*. (6 marks)
b) Find the probability that the drive-in facility is busy less than one-half of the time. (3 marks)
c) Find the probability that the walk-in facility is busy more than 70% of the time given that the drive-in facility is used 10% of the time. (4 marks)
d) Find the covariance of X and Y. (5 marks)
e) Are the random variables X and Y independent? (Justify your answer) (2 marks)

6. Based on the history, TOEFL score is 550 in average and 30 in standard deviation. Assume that the population (scores) follows a normal distribution.

a) If 9 students in today's TOEFL exam have been selected at random, the final scores of those students are: 500, 520, 530, 540, 550, 550, 560, 570 and 600. Find the sample mean $\overline{X}$ and the sample variance S^2. (3 marks)
b) If 16 students are randomly selected, find $P\left(|\overline{X}-550|<15\right)$. (5 marks)
c) It is known that both the sample mean $\overline{X}$ and the sample variance S^2 are examples of unbiased estimators. Briefly explain why *minimum-variance unbiased estimators* are considered efficient estimators. (2 marks)

7. Your employer develops energy efficient solutions for manufacturing sites. One of the key elements of these solutions is proper insulation. Your team leader has asked you to test a new product line, if the product line has a mean thermal insulation (TI) of at least 25 it will be used.

You receive 25 samples of insulating material and test the thermal isolation coefficient. Based on the testing you determine that the sample mean TI is 26.5. The TI is normally distributed with a standard deviation of 2. Show all of your work and state any reasonable assumptions.

a) Using a statistical test with a significance level of: ($\alpha = 0.05$) and based on the results given above, should this product line be used? (8 Marks)
b) If the true population mean was 26, determine β (type II error) with the same information given above. (8 Marks)

c) If the cost of using the product when it should have been rejected is $300,000 (due to recalls and refits), and the cost of not using the product when it should have been accepted is $100,000 (due to delays), determine the probability of having to pay each of the above costs and which is more likely to occur. (2 Marks)

d) How could you improve (reduce) the type I and type II errors? (2 Marks)

8. In medical treatments it is very important to be aware of the dosage given to patients. Infusion pumps are tested regularly to insure that the dosage levels are properly regulated. You may assume that the dosage levels are normally distributed. A set of ten infusion pumps were tested and the dosage deliveries had a sample standard deviation of s = 0.1 doses/hour and a sample mean of 5 doses/hour assuming.

a) Compute the 95% confidence interval population for the population standard deviation based on the above data. (4 Marks)

b) Compute the 95% confidence interval for the population mean based on the above data with based on the above data. (4 Marks)

c) If the dosage level is more than six doses per hour the patient is in danger. Comment on whether the patient is at risk according to your findings. (2 Marks)

TOTAL: 90 marks

8.3.5.2 CIVE 302

The classification of the tasks on the exam for McGill's CIVE 302 – Probabilistic Systems (April 2006) are shown in Table 45.

Table 45 - Classification of tasks: CIVE 302 - Probabilistic Systems (McGill), April 2006

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a	✓		✓		
1	b	✓		✓		
1	c	✓			✓	
2	a	✓			✓	
2	b	✓			✓	
2	c	✓			✓	
3	a		✓		✓	
3	b	✓		✓		
3	c		✓		✓	
4	a	✓			✓	
4	b	✓			✓	
5	a	✓			✓	
5	b	✓			✓	
5	c	✓			✓	
6	a	✓			✓	
6	b	✓			✓	
6	c	✓			✓	
7	a	✓			✓	
7	b	✓			✓	
7	c	✓			✓	
8	a		✓		✓	
8	b	✓			✓	
8	c	✓			✓	
8	d	✓			✓	
TOTAL		**21**	**3**	**3**	**21**	**0**
Relative frequency		**87.5%**	**12.5%**	**12.5%**	**87.5%**	**0%**

McGill University
Faculty of Engineering
Department of Civil Engineering & Applied Mechanics
CIVE-302 – Probabilistic Systems

FINAL EXAMINATION - WINTER 2006

Examiner	Professor Luis E. Chouinard	Duration: 3 Hours
Associate Examiner	Professor V.T.V. Nguyen	Date: Tuesday, April 25, 2006

Name	
Student Number	

INSTRUCTIONS:

1. This is a closed book, closed notes examination.
2. Three (03) double-sided, hand-written crib sheets are allowed.
3. Standard faculty calculators are allowed.
4. Any other book or electronic device is not allowed.
5. This examination has eight (08) problems. You should attempt all of them.
6. Normal distribution, t-distribution, and Chi-squared distribution standard tables provided.
7. Write your answers in the space provided. If additional space is needed, use the reverse of the sheet, clearly indicating the question number.
8. Make suitable assumptions, where necessary, clearly stating them.
9. Do not detach any paper of this booklet.
10. Write your name on each page of this examination booklet.
11. This examination booklet has thirteen (13) pages this one included.

Good Luck!!

Problem	*Mark*
1	
2	
3	
4	
5	
6	
7	
8	
Final Examination Grade	
Midterm Examination Grade	
Assignments Grade	
Total Final Grade (%)	
Final Alpha Grade	

27

PROBLEM #1: (15 marks)

Water consumption, expressed in gallons per capita per day, is a major concern to city administrators in a growing desert city. To address the problem comprehensively, 5 cities of different population sizes were studied. The results are summarized in the following table, where X denotes the population size and Y denotes water consumption in gallons per capita per day. The scatter diagram of the tabulated data indicates that a linear relationship exists between X and Y.

City	x_i (x 10^5)	y_i	$x_i y_i$ (x 10^5)	x_i^2 (x 10^{10})	y_i^2 (x 10^4)
1	0.5	100	5.00	0.25	1.0000
2	2.0	110	22.00	4.00	1.2100
3	3.0	125	37.50	9.00	1.5625
4	5.0	130	65.00	25.00	1.6900
5	7.0	155	108.50	49.00	2.4025
Σ	17.5	620	238.00	87.25	7.8650

a)- Determine the linear regression equation of Y as a function of X.

b)- Determine the variance of the residuals.

c)- Suppose the population of a desert city will be 750,000 in year 2010. Assuming that the regression equation determined in (a) applies, determine the probability that the per capita water consumption will exceed 150 gallons per day?

28

PROBLEM 02: **(15 marks)**

The daily water levels (normalized to the respective full condition) of two reservoirs A and B are denoted by two random variables X and Y having the following joint PDF:

$$f(x, y) = (6/5).(x + y^2), \text{ where } 0 < x < 1 \text{ and } 0 < y < 1.$$

a)- Determine the marginal probability density function of the daily water level of reservoir A.

b)- What is the probability that reservoir A will be more than half full on a given day?

c)- Calculate the statistical correlation between the water levels in the two reservoirs.

PROBLEM 03: (10 marks)

Two companies manufacture a composite material used as exterior cladding for buildings. Twenty-five samples from each company are tested in an abrasion test, and the amount of wear after 1000 cycles is observed. For company 1, the sample mean and standard deviation of wear are $\overline{X_1} = 20$ milligrams/1000 cycles and $s_1 = 2$ milligrams/1000 cycles, while for company 2 we obtain $\overline{X_2} = 15$ milligrams/1000 cycles and $s_2 = 8$ milligrams/1000 cycles.

a)- Does the data support the claim that the two companies produce material with different mean wear? Use $\alpha = 0.05$, and assume each population is normally distributed but their variances are not equal.

b)- What is the P-value of this test?

c)- Do the data support a claim that the material from company 1 has higher mean wear than the material from company 2? Use the same assumptions as in part (a).

30

PROBLEM 04: **(10 marks)**

Because of spatial irregularities in surface geology, the depth, H, from the ground surface to bedrock may be modeled as a lognormal random variable with a median depth of 20 m and a COV of 30%. In order to provide satisfactory support capacity, a steel pile must be embedded 0.5 m into the bedrock.

a)- What is the probability that a 25-m-long pile will not be anchored satisfactorily into the bedrock?

b)- If a 25 m pile has been driven 24 m and has not encountered rock, what is the probability that an additional 2 m pile welded to the original length will be sufficient to anchor this pile satisfactorily into the bedrock?

31

PROBLEM 05: **(15 marks)**

The capacity of a building to withstand earthquake forces without damage follows a Gamma distribution with a mean of 2500 tons and a COV of 35%.

a)- If the building has survived a previous earthquake with a force of 1500 tons without damage, what is the probability that it can withstand a future earthquake with a force of 3000 tons?

b)- The occurrence of earthquakes with a force of 2000 tons constitute a Poisson process with an expected occurrence rate of once every 20 years. What is the probability that there will be no damage to the building over a life of 50 years?

c)- In a complex of five similar buildings, each with the same earthquake resistance capacity as described above, what is the probability that at least four of them will not be damaged under an earthquake force of 2000 tons? Assume that the occurrences of damage to the different buildings are statistically independent.

32

PROBLEM 06: **(10 marks)**

The foundation for a building is designed with 100 piles based on an individual pile capacity of 80 tons. Nine test piles were driven at random locations into the supporting soil stratum and loaded until failure. The results are as follows:

Test Pile No.	Pile Capacity (tons)
1	82
2	75
3	95
4	90
5	88
6	92
7	78
8	85
9	80

The sample mean and standard deviation for pile capacity are respectively 85 tons and 6.76 tons, respectively.

a)- At the 5% significance level, perform a one-sided hypothesis test with the null hypothesis that the mean pile capacity is 80 tons.

b)- Establish the 98% confidence interval for the mean pile capacity, assuming that the standard deviation of the population is known (assume $\sigma = s$, where s is the sample variance).

c)- Determine the 98% confidence interval for the mean pile capacity assuming that the standard deviation of the population is unknown.

33

PROBLEM 07: **(10 marks)**

The capacity of an isolated spread footing foundation under a column is modeled by a normal distribution with a mean of 300 kip and a coefficient of variation, COV, of 20%. Suppose that the column is subjected to a dead load of 100 kip and a live load of 150 kip.

a)- Calculate the probability of failure of the foundation under dead load only.

b)- Calculate the probability of failure of the foundation under the combined action of dead and live loads.

c)- If the probability of failure of the foundation needs to be limited to 0.001, and the dead load of 100 kips cannot be changed, what is the maximum amount of live load that can be applied to the foundation?

PROBLEM #8: (15 marks)

In any given year, the winter in a Midwest city can be cold (C) and wet (W). On the average, 50% of the winters in this city are cold and 30% of the winters are wet. Moreover, 40% of the cold winters are also wet. An unpleasant winter (U) is one when the weather is either cold or wet or both.

a)- Are the events C and W statistically independent? Justify.

b)- What is the probability of an unpleasant winter at any given year?

c)- What is the probability that the winter in any given year will be cold but not wet?

d)- If the winter in any given year is indeed unpleasant, what is the probability that it will be both cold and wet?

8.3.5.3 STAT 249

The classification of the tasks on the exam for Concordia's STAT 249 – Probability I (December 2011) are shown in Table 46.

Table 46 - Classification of tasks: STAT 249 - Probability I (Concordia), December 2011

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a	✓		✓		
1	b	✓		✓		
1	c	✓		✓		
2	a	✓			✓	
2	b	✓			✓	
3		✓			✓	
4	a	✓			✓	
4	b	✓			✓	
5		✓			✓	
6		✓		✓		
7	a	✓			✓	
7	b	✓			✓	
8	a (i)	✓		✓		
8	a (ii)		✓	✓		
8	b	✓		✓		
9	a		✓	✓		
9	b	✓		✓		
10		✓			✓	
TOTAL		**16**	**2**	**9**	**9**	**0**
Relative frequency		**89%**	**11%**	**50%**	**50%**	**0%**

Concordia University
Department of Mathematics & Statistics

Course	Number	Sections
STAT	249/2	All
Examination	**Date**	**Pages**
Final	December 2011	4
Instructors		**Course Examiner**
C. Cummins, A. Sen, W. Sun		A. Sen

Special Instructions:

- ▷ Show your work in sufficient detail and clearly identify your answer
- ▷ Approved calculators only.
- ▷ *A table of Normal (0,1) distribution is attached at the end of this booklet*

1. Events A and B are such that $P(A) = 0.3$, $P(B) = 0.6$.

   (a) If A and B are disjoint events, find $P(A \cup B)$.

   (b) If A and B are independent events, find $P(A \cup B)$.

   (c) Let C be a third event. Suppose that A, B and C are mutually independent events and that $P(C) = 0.2$. Find the probability that at least one of the three events A, B and C occurs. [3 + 3 + 4 marks]

2. From a group of 5 men and 6 women, a committee of 4 persons is formed by random selection.

(c) What is the probability that the selected committee consists of 3 men and 1 woman?

(b) If a 3 person subcommittee is selected at random from the selected 4 person committee, what is the probability that the 3 person subcommittee consists of 3 men? [6 + 8 marks]

3. Two professors taught a course. In the class of Professor X 10% of the students received an A grade for the course, and in the class of Professor Y 2% of the students received an A grade for the course. Of the students taking the course, 60% were in the class of Professor X and 40% were in the class of Professor Y. A randomly selected student did not receive an A grade, what is the probability that this student was in the class of Professor X?

[6 marks]

4. Suppose a coin with probability $p = 0.7$ of landing heads is tossed continually *until 2 heads* are obtained. Find the probability that

(a) the coin is tossed exactly 4 times;

(b) the coin is tossed 4 times or less. [4 + 6 marks]

5. The number of a certain type of rare bird seen each day from an observatory follows a Poisson distribution with mean 1. A particular observer looks each day and records the number of this type of birds he sees. However, to save time, if he sees 3 birds he records the number as 3 and makes no more observations that day. So the maximum number he records is 3, even if more birds arrive later in the day. If Y is the random variable which is the number of birds recorded by this observer, what is the mean and variance of Y?

[10 marks]

6. Suppose Y is a continuous random variable with p.d.f.: $f(y) = e^{-|y|}/2$ (i.e., $= e^{-y}/2$ if $y \geq 0$, $= e^{y}/2$ if $y < 0$). Find $P(Y \leq 5 \mid Y > 3)$ and $E(Y)$.

[6 + 4 marks]

7. Suppose Y, the score of a student in a test, has Normal ($\mu = 70$, σ^2) distribution.

(a) If the passing score is 50, what is the maximum value of σ for which at least 70% of the students will pass the test.

(b) Using the value of σ you found in Part (a), find $P(|Y - 70| > 5)$.

[6 + 4 marks]

8. Let the joint probability mass function of Y_1 and Y_2 be

$$P(Y_1 = y_1, Y_2 = y_2) = \frac{3y_1 - y_2}{60} \quad \text{for } y_1 = 0, 1, 2, 3;\ y_2 = 0, 1, 2.$$

(a) Find the marginal probability mass functions of Y_1 and Y_2. Are Y_1 and Y_2 independent?

(b) Calculate $P(Y_1 + Y_2 < 4 \mid Y_1 \geq 1)$.

[6+6 marks]

9. Suppose (X, Y) have the joint density given by

$$f(x, y) = \begin{cases} \frac{x}{6} + ky, & 0 < x < 1 \text{ and } 1 < y < 5, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k.

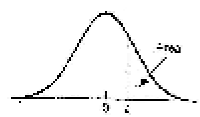
(b) Calculate $P(X \geq \frac{1}{2} \mid Y = 2)$.

[5+5 marks]

10. An accident occurs at a point X that is uniformly distributed on a road of length 80 m. At the time of the accident an ambulance is at a location Y that is also uniformly distributed on the road. Assuming that X and Y are independent, find the expectation of the absolute value of the distance between the ambulance and the point of the accident.

[8 marks]

Table 4 Normal Curve Areas
Standard normal probability in right-hand tail
(for negative values of z, areas are found by symmetry)



	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.00135									
3.5	.000 233									
4.0	.000 031 7									
4.5	.000 003 40									
5.0	.000 000 287									

From R. E. Walpole, *Introduction to Statistics* (New York: Macmillan, 1968).

8.3.5.4 STAT 250

The classification of the tasks on the exam for Concordia's STAT 250 – Statistics (December 2013) are shown in Table 47.

Table 47 - Classification of tasks: STAT 250 - Statistics (Concordia), December 2013

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a	✓		✓		
1	b	✓		✓		
1	c	✓		✓		
1	d		✓	✓		
2	a	✓		✓		
2	b (i)		✓	✓		
2	b (ii)	✓		✓		
3	a	✓		✓		
3	b	✓		✓		
3	c	✓		✓		
3	d		✓	✓		
4	a	✓			✓	
4	b		✓		✓	
4	c		✓		✓	
5	a	✓		✓		
5	b	✓		✓		
5	c	✓		✓		
6	a		✓	✓		
6	b		✓	✓		
6	c		✓	✓		
7	a		✓		✓	
7	b		✓		✓	
7	c		✓		✓	
8	a	✓			✓	
8	b	✓			✓	
8	c		✓		✓	
TOTAL		**14**	**12**	**17**	**9**	**0**
Relative frequency		**54%**	**46%**	**65%**	**35%**	**0%**

Department of Mathematics & Statistics

Course	Number	Section(s)
Statistics	250	AA
Examination	**Date**	**Pages**
Final	December 2013	2
Instructor		
L. Popovic		

Special Instructions:

- Answer all questions and show your work in clear steps leading to the final answer.
- No aids are allowed other than the provided sheets and an approved calculator.

1. Let Y_1, Y_2 be random variables with joint density function

$$f(y_1, y_2) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} & \text{for } y_1 > 0, y_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

a. (4 pts) Calculate $P(Y_1 > 1, Y_2 > 1)$.

b. (4 pts) Find the marginal densities of Y_1 and of Y_2.

b. (4 pts) Find the conditional density of $Y_1 | Y_2 = y_2$.

c. (2 pts) Are Y_1 and Y_2 independent? (*Justify your answer with an explanation*).

2. Let Y_1, Y_2, Y_3 be independent random variables with the same density $f(y) = e^{-y}, \ y > 0$.

a. (6 pts) Find the density of $U = \min(Y_1, Y_2, Y_3)$.

b. (4 pts) What variable is U? Use this to find $E(U)$ and $V(U)$. (*Do not use integration*)

3. Let $Y_1, \ldots, Y_n$ be a sample from the same density $f(y) = \lambda e^{-\lambda y}, y \geq 0$ representing the weights of items. Let $U = \sum_{i=1}^{n} Y_i$ be the sum total of the weights in the sample.

a. (4 pts) Find the density of U.

b. (2 pts) Find its mean $E(U)$ and variance $V(U)$. (*Do not use integration of the density of U*)

c. (4 pts) If $n = 100$ and $\lambda = 20$ estimate the probability $P(U > 5.5)$. (*Do not use integration*)

b. (2 pts) What approximation have you used in order to calculate the probability in part c.?

4. Two groups of patients of 30 individuals each were tested in a medical study designed to test the effectiveness of a new drug. The first group of patients received a placebo, while the second group of patients received the new drug, and their blood pressure was measured a few hours later. The sample mean and sample standard deviation of first group's pressure were $\bar{y}_1 = 167.1, s_1^2 = 24.9$, and for the second group they were $\bar{y}_2 = 140.9, s_2^2 = 17.6$.

a. (6 pts) Construct a 95% confidence interval for the true difference between the mean blood pressure after taking the placebo and after taking the new drug.

b. (2 pts) What statistic did you use to construct the CI and why?

b. (2 pts) Would you say it is likely that the new drug lowers the blood pressure by at least 15 points?

5. Let $Y_1, \ldots, Y_n$ be a sample from density $f(y) = \lambda e^{-\lambda y}, y \geq 0$.

a. (6 pts) Find the Maximum Likelihood Estimator (MLE) for λ.

b. (6 pts) Find a sufficient statistic for λ.

c. (6 pts) Find a Moment of Methods estimator for λ.

6. Let $Y_1, \ldots, Y_n$ be a sample from density $f(y) = \frac{1}{\beta}e^{-\frac{y}{\beta}}, y \geq 0$. Consider the estimator $\widehat{\beta} = n \min(Y_1, \ldots, Y_n)$.

a. (8 pts) Is $\widehat{\beta}$ an unbiased estimator of β? *(Justify your answer with a calculation).*

b. (4 pts) Is $\widehat{\beta}$ a consistent estimator of β? *(Justify your answer with a calculation).*

d. (4 pts) Is $\widehat{\beta}$ more efficient than the estimator $\overline{Y}$? *(Justify your answer with a comparison).*

7. Medical researchers were concerned that the new drug causes more variation than usually observed in patients' blood pressures. They ran a second study with in which they particularly focused on the sample variances of the two groups of measurements. In the second study they used a group of 14 patients taking the placebo and observed sample variance $s_1 = 12.7$, and a group of 10 patients taking the new drug and observed a sample variance of $s_2 = 26.4$

a. (2 pts) What is the null hypothesis, and what is the appropriate alternative hypothesis?

b. (2 pts) What statistic do you need to use for the hypothesis test, and what distribution does it have?

a. (6 pts) If we are allowing a type I error of 0.05, would you reject the null hypothesis based on the above data?

8. We have the following data on levels of safety Y (in percentages) as a function of pollution levels X:

$$(x, y) = \{(0.1, 100), (0.2, 95), (0.3, 85), (0.4, 65), (0.5, 55)\}$$

a. (6 pts) Give the equation for the line that best describes the dependence of Y on X.

b. (2 pts) Give an estimate of the level of safety when the pollution levels are at a record high of 0.9?

c. (2 pts) Do you think the equation should be used for the pollution level of 0.9?

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8.3.6 Numerical methods

8.3.6.1 ENGR 391

The classification of the tasks on the exam for Concordia's ENGR 391 – Numerical Methods in Engineering (December 2013) are shown in Table 48.

Table 48 - Classification of tasks: ENGR 391 - Numerical Methods in Engineering (Concordia), December 2013

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a	✓		✓		
1	b	✓		✓		
2	a (i)	✓		✓		
2	a (ii)	✓		✓		
2	b (i)	✓		✓		
2	b (ii)	✓		✓		
3	a	✓		✓		
3	b	✓		✓		
4	a (i)	✓		✓		
4	a (ii)	✓		✓		
4	a (iii)	✓		✓		
4	a (iv)	✓		✓		
4	b (i)	✓			✓	
4	b (ii)	✓			✓	
5	a	✓		✓		
5	b	✓		✓		
TOTAL		**16**	**20**	**14**	**2**	**0**
Relative frequency		**100%**	**0%**	**87.5%**	**12.5%**	**0%**

Concordia University
ENCS
Engineering & Computer Science

FACULTY OF ENGINEERING AND COMPUTER SCIENCE

COURSE		NUMBER	SECTION
Numerical Methods in Engineering		**ENGR 391**	**/4 (all)**
EXAMINATION	DATE	TIME	# of pages (including title page)
Final Exam	**Dec. 9, 2013**	**14 :00-17 :00**	**18**

PROFESSORS
Dr.D. Davis, Dr. P. Gauthier and Dr. A. Kaushal

MATERIALS ALLOWED - YES (One page **single-sided** crib sheet)

CALCULATORS ALLOWED - YES (Faculty approved calculators)

SPECIAL INSTRUCTIONS:

- Read carefully all the questions
- Total Marks – 100; Time 180 Minutes
- Closed Book Exam; **Single sided** 8 ½ x 11 formula sheet allowed
- You must **show all your steps** leading to the solution(s)
- Give the answers in the area provided.
- Please do not write in red (colour used for correction)
- Everything not readable will NOT be corrected

Good Luck!

MARKS

Name: ______________ __________________
Surname, Given names

I.D.: ___________________________

Signature: ______________________________

1

Question #1 [Solving Nonlinear Equations] [10 marks]

Obtain the first root above x = 0 for the following equation with accuracy of 4 digits
(Hint: use incremental search to locate the region of the root)

$$e^x - 2x^2 = 0$$

a. Use the method of False Position (5 Marks)
b. Use Newton Raphson method (5 Marks)

Question #2 [Systems of linear and Nonlinear equations] [25 marks]

a) Consider the following system of linear equations [*A*]{*X*} = {*B*}

$$\begin{bmatrix} 1 & 4 & 2 \\ 5 & 2 & 1 \\ 2 & 2 & 8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ 2 \end{Bmatrix}$$

If using decimals instead of fraction number, keep **3 decimals** in your calculations

1. Find the solution of this system of linear equations **using the LU decomposition** with **partial pivoting** (i.e. PA = LU). (10 Marks)
2. Find the first column of $[A]^{-1}$ (5 Marks)

b) Obtain the solution to the following nonlinear equations using Newton's Method of the form $\{X_{n+1}\} = \{X_n\} - [J(X_n)]^{-1}\{F(X_n)\}$

$$-x_1 + 2x_1^2 - 2x_1x_2 + x_2^2 = 1$$

$$x_1^2 - 2x_1x_2 - x_2 + x_2^2 = 0$$

Assume the starting vector $\{x_0\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$. Do **two iterations**. Compute the error **at each step** of the iteration using the $\|x\|_2$ norm; Use **4 decimals** in your calculations (10 Marks)

Question #3 [Curve Fitting] [20 marks]

a) If the following points are related by a formula of the type , $P(x) = Ae^{Mx}$
Find the best value of A and M; Keep **4 decimals** in your calculations. (10 Marks)
(Hint: Change the form to a linear equation and use least squares regression)

x_i	1	2	3	4
P_i	7	11	17	27

b) Use the Lagrange Interpolating Polynomial to approximate Cos (0.750) using the following values; (Note the given values are in radians) (10 Marks)

Cos(0.698) = 0.7661 Cos(0.733) = 0.7432 Cos(0.768) = 0.7193

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Question #4 [Numerical Differentiation & Integration] [25 marks]

a) Evaluate the Integral:

$$I = \int_0^{2.4} \frac{2x}{x^2+1}dx$$

Gauss Quadrature

$n = 4$:

$c_1 = 0.3478548;$ $x_1 = -0.86113631;$

$c_2 = 0.6521452;$ $x_2 = -0.33998104;$

$c_3 = c_2;$ $x_3 = -x_2;$

$c_4 = c_1;$ $x_4 = -x_1$

1. Analytically (2.5 Marks)
2. Using Simpson's 1/3 method, using 6 sub-intervals. (5 Marks)
3. Using four-point Gauss Quadrature (5 Marks)
4. Using the exact solution found in part a) evaluate the percent relative error associated with each of the approximations found in parts 2) and 3) (2.5 Marks)

Keep **3 decimals** in your calculations.

b. The following data is given for the stopping distance of a truck on a wet road versus the speed at which it begins breaking:

v (km/h)	20.0	40.5	62.5	80	100.5	125
d(m)	6	19	38	65	99	135

1. Calculate the rate of change of the stopping distance at a speed of 100.5 km/h using a two-point backward difference formula. (5 Marks)
2. Estimate the stopping distance at 125 km/h using the result from part 1) and a two-point central difference formula applied at the speed of 100.5 km/h. (5 Marks)

Question #5 [Ordinary Differential Equations] [20 marks]

a. Given the differential equation

$$\frac{dy}{dx} = \frac{4x}{y} - xy$$

Fill out the following table using the classical fourth- order Runge Kutta method; (10 Marks)
Keep **6 decimals** in your calculations.

Xi	Yi
0.00000	3.000000
0.10000	
0.20000	

14

b) Solve numerically using Euler's method the following second order ordinary differential equation with a step size, *h = 0.5* , from *t=1 to t=2 , for given* initial conditions:

(10 Marks)

$$\frac{d^2y}{dt^2} = 4y^3 \qquad y(1)=1/4 \text{ and } dy/dt(1)=0$$

Keep **5 decimal** places in your calculations

16

8.3.6.2 CIVE 320

The classification of the tasks on the exam for McGill's CIVE 320 – Numerical Methods (December 2007) are shown in Table 49.

Table 49 - Classification of tasks: CIVE 320 - Numerical Methods (McGill), December 2007

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1		✓		✓		
2		✓			✓	
3	a	✓		✓		
3	b	✓		✓		
3	c	✓		✓		
4	(i)	✓		✓		
4	(ii)		✓	✓		
5		✓		✓		
6		✓				✓
7	a	✓			✓	
7	b	✓			✓	
7	c	✓			✓	
8		✓			✓	
9	a	✓			✓	
9	b (i)	✓			✓	
9	b (ii)	✓			✓	
10	a	✓		✓		
10	b	✓		✓		
10	c	✓		✓		
TOTAL		**18**	**1**	**10**	**8**	**1**
Relative frequency		**94%**	**6%**	**52%**	**42%**	**6%**

McGill University
Faculty of Engineering

Numerical Methods

CIVE 320

Monday, December 17, 2007 – 9:00am – 12:00pm

Examiner: Dr. John HADJINICOLAOU

Associate Examiner: Professor Luc CHOUINARD

STUDENT NAME: ______________________________

MCGILL I.D. NUMBER: ____________________

INSTRUCTIONS:

- This is an ONLY **OPEN TEXTBOOK** examination.

  FACULTY STANDARD CALCULATOR permitted **ONLY**.

- Answer eight (8) from the ten (10) problems.

PROBLEM 1

Use the Gauss-Seidel method to solve the following system until the percent relative error falls below $\varepsilon_s = 5\%$.

$$10x_1 + 2x_2 - x_3 = 27$$
$$-3x_1 - 6x_2 + 2x_3 = -61.5$$
$$x_1 + x_2 + 5x_3 = -21.5$$

PROBLEM 2

An irreversible, first-order reaction takes place in four well mixed reactors (Fig. P)

$$A \xrightarrow{k} B$$

Thus, the rate at which A is transformed to B can be represented as

$$R_{ab} = kVc$$

The reactors have different volumes, and because they are operated at different temperatures, each has a different reaction rate:

Reactor	V, L	k, h^{-1}
1	25	0.075
2	75	0.15
3	100	0.1
4	25	0.1

Determine the concentration of A and B in each of the reactors at steady state.

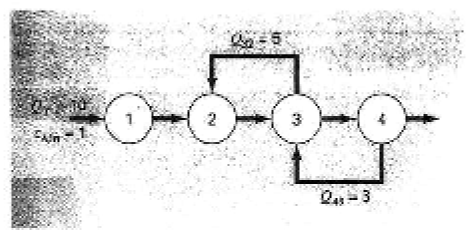

Figure P

PROBLEM 3

Employ the following methods to find the maximum of

$$f(x) = 4x - 1.8x^2 + 1.2x^3 - 0.3x^4$$

(a) Golden section search ($x_l = -2$, $x_u = 4$, $\varepsilon_s = 35\%$).
(b) Quadratic interpolation ($x_0 = 1.75$, $x_1 = 2$, $x_2 = 2.5$, iterations = 4).
(c) Newton's method ($x_0 = 3$, $\varepsilon_s = 10\%$).

PROBLEM 4

Find the minimum value of

$$f(x, y) = (x - 3)^2 + (y - 2)^2$$

starting at $x = 1$ and $y = 1$, using the steepest descent method with a stopping criterion of $\varepsilon_s = 1\%$. Explain your results.

PROBLEM 5

Employ inverse interpolation using a cubic interpolating polynomial and bisection to determine the value of x that corresponds to $f(x) = 0.23$ for the following tabulated data:

x	2	3	4	5	6	7
$f(x)$	0.5	0.3333	0.25	0.2	0.1667	0.1429

PROBLEM 6

Determine an equation to predict metabolism rate as a function of mass based on the following data:

Animal	Mass, kg	Metabolism, watts
Cow	400	270
Human	70	82
Sheep	45	50
Hen	2	4.8
Rat	0.3	1.45
Dove	0.16	0.97

PROBLEM 7

Determine the distance traveled for the following data:

t, min	1	2	3.25	4.5	6	7	8	8.5	9.3	10
v, m/s	5	6	5.5	7	8.5	8	6	7	7	5

(a) Use the trapezoidal rule, (b) the best combination of the trapezoidal and Simpson's rules, and (c) analytically integrating second- and third-order polynomials determined by regression.

PROBLEM 8

A plane is being tracked by radar, and data is taken every second in polar coordinates θ and r.

t, s	200	202	204	206	208	210
θ, (rad)	0.75	0.72	0.70	0.68	0.67	0.66
r, m	5120	5370	5560	5800	6030	6240

At 206 seconds, use the centered finite difference (second-order correct) to find the vector expressions for velocity $\vec{v}$ and acceleration $\vec{a}$. The velocity and acceleration given in polar coordinates are:

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \quad \text{and} \quad \vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

P.3

PROBLEM 9

Assuming that drag is proportional to the square of velocity, we can model the velocity of a falling object like a parachutist with the following differential equation:

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

where v is velocity (m/s), t = time (s), g is the acceleration due to gravity (9.81 m/s^2), c_d = a second-order drag coefficient (kg/m), and m = mass (kg). Solve for the velocity and distance fallen by a 90-kg object with a drag coefficient of 0.225 kg/m. If the initial height is 1 km, determine when it hits the ground. Obtain your solution with (a) Euler's method and (b) the fourth-order RK method.

PROBLEM 10

Solve the following differential equation from $t = 0$ to 1

$$\frac{dy}{dt} = -10y$$

with the initial condition $y(0) = 1$. Use the following techniques to obtain your solutions: (a) analytically, (b) the explicit Euler method, and (c) the implicit Euler method. For (b) and (c) use h = 0.1 and 0.2. Plot your results.

8.3.6.3 MATH 354 (2011)

The classification of the tasks on the exam for Concordia's MATH 354 – Numerical Analysis (December 2011) are shown in Table 50.

Table 50 - Classification of tasks: MATH 354 - Numerical Analysis (Concordia), December 2011

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	a	✓		✓		
1	b	✓		✓		
2	a		✓	✓		
2	b	✓		✓		
2	c		✓	✓		
2	d	✓		✓		
3	(i)	✓		✓		
3	(ii)	✓		✓		
4	a	✓		✓		
4	b		✓	✓		
5	a	✓			✓	
5	b	✓			✓	
6		✓		✓		
7	a	✓		✓		
7	b	✓		✓		
7	c	✓		✓		
Bonus			✓	✓		
TOTAL		**13**	**4**	**15**	**2**	**0**
Relative frequency		**76%**	**24%**	**89%**	**11%**	**0%**

Concordia University
Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	MAST 334/MATH 354	A, AA
Examination	**Date**	**Pages**
Final	December 2011	3
Instructors	**Time**	**Course Examiner**
D. Dryanov and E. Kritchevski	3 hours	D. Dryanov

Instructions: Department approved calculators are permitted. The problems are typed on 2 pages. Page 3 is a formula page. Evaluation out of 100.

Problem 1. Consider the function

$$I(x) = \int_1^x \sin[\ln(t)]\, dt.$$

(a) [**5 marks**] Find the best cubic polynomial approximation $P_3(x)$ to $I(x)$ near $x_0 = 1$.
(b) [**5 marks**] Give an upper bound for the approximation error $|I(1.25) - P_3(1.25)|$.

Problem 2. Given the non-linear equation

$$f(x) = 0, \quad \text{where} \quad f(x) = \ln(2 + x^2) - 5x^2.$$

(a) [**3 marks**] Show that the non-linear equation $f(x) = 0$ has a unique solution $p \in [0.5, 0.6]$.
(b) [**7 marks**] Use the bisectional method to find an approximation to the solution p of $f(x) = 0$ with 2 significant (true) digits. At each step of the bisectional method list the left boundary a_n, the right boundary b_n, the approximation p_n and give an upper bound for the relative error of the approximation p_n.
(c) [**8 marks**] Let

$$g(x) = x + \frac{f(x)}{3}.$$

Show that for any initial approximation $p_0 \in [0.5, 0.6]$ the fixed point method $p_{n+1} = g(p_n)$ converges to the solution p of $f(x) = 0$.
(d) [**5 marks**] Find an approximation p_n to the solution p of $f(x) = 0$ such that $|p_n - p| < 0.015$.

Problem 3 [10 marks]. Consider the function

$$f(x) = e^{x^2-x+1} - x^2 - 2.$$

The non-linear equation $f(x) = 0$ has a unique solution $p \in [1.1, 1.3]$. Use an inverse interpolation on the interpolation data $(1.1, f(1.1))$; $(1.2, f(1.2))$; $(1.3, f(1.3))$ and Neville's method in order to find an approximation to the exact solution p of $f(x) = 0$

Problem 4. **(a)** **[10 marks]** Construct *osculating* Hermite interpolation polynomial $H(x)$ for the function $f(x) = \sin(\pi x)e^{x^2-1}$ on the interpolation data $f(-1)$, $f'(-1)$, $f''(-1)$ and $f(1)$, $f'(1)$.

(b) **[5 marks]** Show that $H(x)$ is the unique algebraic polynomial of degree ≤ 4 satisfying the given interpolation data.

Hint. *If H_1 and H_2 are two polynomials of degree ≤ 4 that are solutions to* (a), *then show that the polynomial $P(x) = H_1(x) - H_2(x)$ must be identically zero.*

Problem 5. At the moment when a peach pie is taken out from an oven in a room, it is piping hot. It has been estimated that 2 minutes after this moment, the temperature of the pie is 94°C; 5 minutes after this moment its temperature is 87°C and 10 minutes after this moment its temperature is 80°C. The temperature of the pie at the time moment t is given by the formula

$$T(t) = (T_0 - T_r)e^{-0.0387t} + T_r,$$

where T_0 is the temperature of the pie at the initial moment (when it is taken out from the oven) and T_r is the constant temperature of the room.

(a) **[10 marks]** Use **discrete least squares approximation** in order to find T_r and T_0.

(b) **[5 marks]** Then, estimate the temperature of the pie 20 minutes after the initial moment.

Problem 6 **[8 marks]**. Compute the **cubic continuous least squares approximation** to the function $f(x) = |x|$ on the interval $[-1, 1]$ with a weight function $\omega(x) = 1$.

Problem 7. **(a)** **[8 marks]** Find $\Phi_3(x) = x^3 + ax^2 + bx + c$ in the orthogonal polynomial sequence $\Phi_0(x)$, $\Phi_1(x)$, $\Phi_2(x)$, $\Phi_3(x)$, ... on the interval $[-1, 1]$, with a weight function

$$\omega(x) = \frac{1}{1+x^2}.$$

(b) **[8 marks]** Construct the unique 3-nodes Gaussian quadrature formula of the form

$$\int_{-1}^{1} \frac{f(x)}{1+x^2}\,dx \approx A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3).$$

(c) **[3 marks]** Find an approximation to the definite integral

$$\int_{-1}^{1} \frac{\sin(x^2+x)}{1+x^2}\,dx$$

by using the quadrature formula obtained in **(b)**.

Bonus question **[3 marks]**. Is there a 3 nodes quadrature formula of the form given in **Problem 7** **(b)**, having a polynomial degree of precision (accuracy) 7? Explain.

Formula page:

$$\mathbf{P}_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k; \quad f(x) - \mathbf{P}_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}$$

$$|p - p_n| \le k^n \max(p_0 - a, b - p_0) \le k^n(b - a); \quad |p - p_n| \le \frac{k^n}{1 - k}|p_1 - p_0|$$

$$|g'(p)| < 1 \text{ convergent. } g'(p) = \cdots = g^{(m-1)}(p) = 0; \ g^{(m)}(p) \neq 0 \text{ order } m$$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}; \quad p_{n+1} = p_n - \frac{f(p_n)f'(p_n)}{[f'(p_n)]^2 - f(p_n)f''(p_n)}$$

$$p_{n+1} = p_n - m\frac{f(p_n)}{f'(p_n)}; \quad p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} - \frac{f''(p_n)f^2(p_n)}{2[f'(p_n)]^3}$$

$$\mathbf{P}_{0,1,\ldots,n}(x) = \frac{(x - x_0)P_{1,\ldots,n}(x) - (x - x_n)P_{0,1,\ldots,n-1}(x)}{x_n - x_0}$$

$$\begin{aligned}\mathbf{P}_3(x) = f(x_0) &+ f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)\end{aligned}$$

$$f(x) - \mathbf{P}_3(x) = \frac{f^{(4)}(\xi)}{4!}(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

$$\begin{array}{llccc}
x_0 & f(x_0) & & & \\
 & & f[x_0, x_0] = \frac{f'(x_0)}{1!} & & \\
x_0 & f(x_0) & & f[x_0, x_0, x_1] & \\
 & & f[x_0, x_1] & & f[x_0, x_0, x_1, x_1] \\
x_1 & f(x_1) & & f[x_0, x_1, x_1] & \\
 & & f[x_1, x_1] = \frac{f'(x_1)}{1!} & & \\
x_1 & f(x_1) & & &
\end{array}$$

$$\begin{aligned}\mathbf{H}_3(x) = f(x_0) &+ f[x_0, x_0](x - x_0) + f[x_0, x_0, x_1](x - x_0)^2 \\ &+ f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1)\end{aligned}$$

$$f(x) - \mathbf{H}_3(x) = \frac{f^{(4)}(\xi)}{4!}(x - x_0)^2(x - x_1)^2$$

free: $s''(x_0) = 0, s''(x_n) = 0$; clamped: $s'(x_0) = f'(x_0), s'(x_n) = f'(x_n)$

$$\min_{A,B,C} \sum_{i=0}^{n} \left[f(x_i) - (A(x_i - a)^2 + B(x_i - a) + C)\right]^2; \ a = \frac{x_0 + x_1 + \cdots + x_n}{n+1}$$

$$\mathbf{L}_0(x) = 1, \ L_1(x) = x, \ L_2(x) = x^2 - 1/3, \ L_3(x) = x^3 - 3x/5$$

$$\mathbf{T}_{n+1}(x) = 2x\,\mathbf{T}_n(x) - \mathbf{T}_{n-1}(x), \ \mathbf{T}_0(x) = 1, \ \mathbf{T}_1(x) = x$$

3

8.3.6.4 MATH 354 (2012)

The classification of the tasks on the exam for Concordia's MATH 354 – Numerical Analysis (December 2012) are shown in Table 51.

Table 51 - Classification of tasks: MATH 354 - Numerical Analysis (Concordia), December 2012

Task		**Nature**		**Content**		
Question	Part	Comp	Conc	Math	App	Mod
1	(i)	✓		✓		
1	(ii)		✓	✓		
2		✓		✓		
3		✓		✓		
4			✓	✓		
5	(i)	✓		✓		
5	(ii)	✓		✓		
6		✓		✓		
7		✓		✓		
8		✓		✓		
9		✓		✓		
10		✓		✓		
TOTAL		**10**	**2**	**12**	**0**	**0**
Relative frequency		**83%**	**17%**	**100%**	**0%**	**0%**

MATH-354 (MAST-334) NUMERICAL ANALYSIS

FINAL EXAM

Course examiner A. Shnirelman; instructors A. Shnirelman, E. Krichevsky

Date 10/12/2012. Duration 3 hours.

Solve as many problems as you can; each problem is worth 10%

1. Using the Taylor formula, find $\cos(1)$ with the accuracy 10^{-6}. How many first nonzero terms in the Taylor series is enough to achieve the desired accuracy?

2. For the equation below, locate the positive solution p in an interval of as small length as you can and then compute p with the accuracy 10^{-5} using the Newton-Raphson method:

$$x^4-3\sin x=0$$

3. Reduce the following equation to the fixed point problem and solve it by iterations with the accuracy 10^{-6}:

$$x^5-0.5x^3-2=0$$

4. Show that there is a unique solution p in the segment $[0.5,1]$ of the equation $x=0.8\tan x$ and find it with the accuracy 0,001 using the Stefensen's method.

5. Let $z_i=\cos\left(\frac{(2i+1)\pi}{2n}\right)$, $i=0,\dots,n-1$ be the roots of the Chebyshev polynomial $T_n(x)$. Taking $n=4$, and using the points z_i as interpolation points, find the interpolation polynomial $P(x)$ for the function $f(x)=e^x$ on the segment $[-1,1]$. Find the upper bound for the maximal error of approximation $|f(x)-P(x)|$ on $[-1,1]$.

6. Here is a fragment of a table:

x	$\sin x$
1.0	0.8414710
1.01	0.8468318
1.02	0.8521080
1.03	0.8572990

Using divided differences, find $\sin(1.0065)$; compare with the "exact" value given by your calculator.

7. Find a cubic spline $S(x)$ on the segment $-\pi/2 \le x \le \pi/2$ with free endpoint conditions such that $S(-\pi/2)=0$, $S(0)=1$, and $S(\pi/2)=0$.

8. Given a table of experimental data:

i	1	2	3	4	5	6	7	8	9
x_i	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
y_i	2.893	2.560	2.310	2.125	1.98	1.924	1.891	1.897	1.941

Find the best mean square approximation of these data by a function of the form $y=ae^x+be^{-x}$.

9. Find the cubic continuous mean square approximation to the function

$$f(x)=\begin{cases} x^2, & x\ge 0 \\ -x^2, & x<0 \end{cases}$$

on the segment $[-1,1]$ with the weight $w(x)=1$.

10. Using the composite Simpson method, find $\int_0^1 e^x dx$ with the accuracy 10^{-4} .

Compare with the "exact" value obtained with your calculator.

GOOD LUCK!

SOME USEFUL FORMULAS

1. Taylor formula:

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$ where ξ is between x_0 and x .

2. The interpolation error: if x_0 , ..., x_n are the interpolation nodes in a segment $[a,b]$, and $P(x)$ is the n -th degree polynomial such that $P(x_i)=f(x_i)$, then for some $\xi\in(a,b)$,

$$f(x)=P(x)+\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\ldots(x-x_n)$$

3. Simpson's Rule:

$$\int_{x_0}^{x_2} f(x)dx=\frac{h}{3}(f(x_0)+4f(x_1)+f(x_2))-\frac{h^5}{90}f^{(4)}(\xi)$$ where $x_0<\xi<x_2$

4. Composite Simpson Rule:

$$\int_{a}^{b} f(x)dx=\frac{h}{3}(f(x_0)+4f(x_1)+2f(x_2)+\ldots+4f(x_{n-1})+f(x_n))-n\frac{h^5}{90}f^{(4)}(\xi)$$

where $x_0=a$, $x_n=b$, $h=\frac{b-a}{n}$, $\xi\in(a,b)$; n is even.

8.4 STRUCTURAL ANALYSIS SOFTWARE REPORT

The appendix contains the report that is generated by the structural analysis software that is bundled with the engineering textbook Structural Analysis by Kassimali (1999). This software was used to generate the images of the deformed plane frame and plane truss in section 5.3. The report includes the data that was input into the software to describe geometry and material of the structural members as well as the applied forces, and the output of the analysis that the software performed using the direct stiffness method.

At the outset of the analysis, a decision must be made about what units will be used when inputting the structure's data. I chose to use the units of millimetres (mm) for distance and kilonewtons (kN) for force. The results of the analysis are therefore displayed in the same units. This makes interpreting the results of the analysis easier.

A detailed explanation of the information included in the report follows.

```
****************************
*    Computer Software     *
*           for            *
*   STRUCTURAL ANALYSIS    *
*     Second Edition       *
*           by             *
*     Aslam Kassimali      *
****************************
```

General Structural Data

Project Title : Plane Frame
Structure Type : Plane Frame
Number of Joints : 6
Number of Members : 5
Number of Material Property Sets (E) : 1
Number of Cross-Sectional Property Sets : 2

Joint Coordinates

Joint No.	X Coordinate	Y Coordinate
1	0.0000E+00	0.0000E+00
2	0.0000E+00	5.0000E+03
3	5.0000E+03	0.0000E+00
4	5.0000E+03	5.0000E+03
5	9.0000E+03	0.0000E+00
6	9.0000E+03	5.0000E+03

Supports

Joint No.	X Restraint	Y Restraint	Rotational Restraint
1	Yes	Yes	Yes
3	Yes	Yes	Yes
5	Yes	Yes	Yes

Material Properties

Material No.	Modulus of Elasticity (E)	Co-efficient of Thermal Expansion
1	2.0000E+02	0.0000E+00

Cross-Sectional Properties

Property No.	Area (A)	Moment of Inertia (I)
1	1.2300E+04	2.2200E+08
2	2.8600E+03	2.0000E+07

Member Data

Member No.	Beginning Joint	End Joint	Material No.	Cross-Sectional Property No.
1	1	2	1	1
2	3	4	1	1
3	5	6	1	1
4	2	4	1	2
5	4	6	1	2

Joint Loads

Joint No.	X Force	Y Force	Moment
2	1.5000E+02	-7.5000E+01	0.0000E+00

Member Loads

Member No.	Load Type	Load Intensity (W)
5	Uniform	1.000E-1

************** End of Input Data **************

****
*   Results of Analysis   *
****

Joint Displacements

Joint No.	X Translation	Y Translation	Rotation (Rad)
1	0.0000E+00	0.0000E+00	0.0000E+00
2	3.1373E+01	-1.1667E-01	-8.2185E-03
3	0.0000E+00	0.0000E+00	0.0000E+00
4	3.0465E+01	-3.9847E-01	-1.0225E-02
5	0.0000E+00	0.0000E+00	0.0000E+00
6	2.9885E+01	-4.5031E-01	-4.1687E-03

Member End Forces in Local Coordinates

Member	Joint	Axial Force	Shear Force	Moment
1	1	5.7402E+01	4.6148E+01	1.8835E+05
	2	-5.7402E+01	-4.6148E+01	4.2389E+04
2	3	1.9605E+02	2.0892E+01	1.4303E+05
	4	-1.9605E+02	-2.0892E+01	-3.8572E+04
3	5	2.2155E+02	8.2960E+01	2.4442E+05
	6	-2.2155E+02	-8.2960E+01	1.7038E+05
4	2	1.0385E+02	-1.7598E+01	-4.2389E+04
	4	-1.0385E+02	1.7598E+01	-4.5600E+04
5	4	8.2960E+01	1.7845E+02	8.4172E+04
	6	-8.2960E+01	2.2155E+02	-1.7038E+05

```
                  ==================
                  Support Reactions
                  ==================

Joint No.     X Force        Y Force       Moment
---------   -----------    -----------   -----------
    1       -4.6148E+01    5.7402E+01    1.8835E+05
    3       -2.0892E+01    1.9605E+02    1.4303E+05
    5       -8.2960E+01    2.2155E+02    2.4442E+05

*************** End of Analysis ***************
```

Joint coordinates

Joint coordinates are entered in millimetres. Joint 1 is chosen as the origin with coordinates (0, 0). Joint 6 is furthest away from the origin located at coordinates ($9000\ mm, 5000\ mm$). This indicates that the frame is 9 metres long and 5 metres high.

Material properties

The modulus of elasticity is entered using units of gigapascals (GPa) since $1GPa = 1kN/1mm$, making this consistent with our chosen units. The value that is entered is:

$$2.0000E + 02 = 200.00GPa = 200.00\ \frac{kN}{mm}$$

This is the modulus of elasticity of structural steel.

Cross-sectional properties

The values for cross-sectional area and moment of inertia are entered in units of mm^2 and mm^4, respectively Two values are entered for each property since two different shapes are used in the design. Cross-section 1 is the shape W310x97 and cross-section 2 is the shape W200x22. The entered value for the area of cross-section 1 is:

$$1.2300E + 04 = 12\ 300\ mm^2$$

Member data

This section indicates where each member begins and ends, and associates with each member a material and a cross-section. Member 2 begins at joint 3 and ends at joint 4, is assigned the properties of material 1, and the cross-sectional properties of the shape W310x97 (cross-section 1). Since only one modulus of elasticity was entered, only one material is defined in this analysis, and so all of the members are assigned material 1. Two cross-sections were defined though, so some members are assigned cross-section 1, while others are assigned cross-section 2.

Joint loads

The loads applied at joint 2 are entered using units of kilonewtons. The horizontal (X) force is:

$$1.5000E + 02 = 150.00\ kN$$

Member loads

The uniformly distributed load applied to member 5 is entered using units of kilonewtons per millimetre (kN/mm). While such loads are usually defined in units of kN/m, this would lead to an inconsistency in the units, and the results of the analysis would not be reliable. The entered value for the distributed load is:

$$1.000E - 1 = 0.1000 \frac{kN}{mm} = 100 \frac{kN}{m}$$

Joint displacements

Since the data was entered with consistent units, the results of the analysis are reliable and are displayed using the same units as the inputted data. Thus, the X translation of joint 4 is:

$$3.0465E + 01 = 30.465\ mm$$

The rotations are measured in radians. The rotation at joint 6 is:

$$-4.1687E - 03 = -0.0041687\ rad$$

Member end forces

The member end forces are the forces that are carried by the members. The results show the forces in units of kilonewtons and the moments in units of kilonewton-millimetres. The axial force in member 3 is:

$$2.2155E + 02 = 221.55\ kN$$

The moment in member 3 at joint 5 is:

$$2.4442E + 05 = 244\ 420\ kN\ mm = 244.420\ kN\ m$$

Support reactions

The support reactions are the forces that are transferred from the frame into the supports that connect it to the ground. These are the forces that the supports themselves must resist. At join 3, the support must resist a vertical (Y) force of:

$$1.9605E + 02 = 196.05\ kN$$

It must also resist a moment of:

$$1.4303E + 05 = 143\ 030\ kN \cdot mm = 143.030\ kN\ m$$