Reciprocity Law For Flat Conformal Metrics With Conical Singularities

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Abstract

Reciprocity Law For Flat Conformal Metrics With Conical Singularities

Lukasz Obara

We study an analogue of Weils reciprocity law for flat conical conformal metrics on compact Riemann surfaces. We give a survey of Troyanov's paper concerning flat conical metrics. The main result of this thesis establishes a relation among three flat conformally equivalent metric with conical singularities.

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Contents

1 Introduction

The main result of the present thesis is an elementary proof of an analogue of the classical Weil reciprocity law for flat conical conformal metrics on a compact Riemann surface. We establish a relationship between three conformally equivalent flat conical metrics. In particular, we prove the following:

Theorem 1.1: *Let $\mathcal{X}$ be a Riemann surface of arbitrary genus g, let ds_1, ds_2 and ds_3 be three conformal flat conical metrics on $\mathcal{X}$. Suppose that the metric ds_1 has conical points $R_1^1, \dots R_1^L$ with conical angles $2\pi(\alpha_1^1+1), \dots, 2\pi(\alpha_1^L+1)$, the metric ds_2 have conical points $R_2^1, \dots R_2^M$ with conical angles $2\pi(\alpha_2^1+1), \dots, 2\pi(\alpha_2^M+1)$, the metric ds_3 have conical points $R_3^1, \dots R_3^N$ with conical angles $2\pi(\alpha_3^1+1), \dots, 2\pi(\alpha_3^N+1)$, then one has the relation*

$$\prod_{i=1}^{N}\left(\frac{ds_1}{ds_2}(R_i^3)\right)^{\alpha_i^3}\prod_{i=1}^{L}\left(\frac{ds_2}{ds_3}(R_i^1)\right)^{\alpha_i^1}\prod_{i=1}^{M}\left(\frac{ds_3}{ds_1}(R_i^2)\right)^{\alpha_i^2}=1$$

This formula first appeared in [5] where it was proved using non elementary approaches from spectral theory of the Laplacian operator. Here we present a short elementary proof of this result. The structure of the thesis is as follows. In section 2, following the approach from Griffiths and Harris [4], we remind the reader of the classical Weil reciprocity law for meromorphic functions. In section 3 we discuss flat conical conformal metrics, introduce distinguished local parameters, and following the paper of Troyanov [9] we prove the existence of flat conical metrics. The latter uses some elements of Hodge theory that we briefly discuss. In section 4 we prove Theorem 1.1. We conclude by stating some open questions.

2 Weils Reciprocity Law

Definition 2.1: Let f be a meromorphic functions on a compact Riemann surface $\mathcal{X}$. Let P be a point on $\mathcal{X}$ we define $\text{ord}(f, P) = k$, where $k = -n$ if P is a pole of order n of the function f. $k = n$ if P is a zero of order n of the function f. And $k = 0$ if P is neither a zero or a pole of f

Definition 2.2: The divisor $\mathscr{D}$ is a finite formal linear combination of points

$$\mathscr{D} = \sum_{P_i \in \mathcal{X}} \alpha_i P_i$$

where the coefficient α_i are integers.

Remark: It will be convenient to denote the divisor as a product, instead of a sum. That is $\mathscr{D} = \prod_{P_i \in \mathcal{X}} P_i^{\alpha_i}$. At the same time, we set $P_i^0 = 1$ for all $P_i \in \mathcal{X}$.

Definition 2.3: The degree of the divisor is the sum of the order of α_i that is $\deg(\mathscr{D}) = \sum_{i=1}^{n} \alpha_i$

Theorem 2.1 (Weil Reciprocity law [4]): *Let f and g be meromorphic functions on the compact Riemann surface $\mathcal{X}$, with the divisor (f) disjoint from (g). Then*

$$\prod_{P \in \mathcal{X}} f(P)^{ord(g,P)} = \prod_{P \in \mathcal{X}} g(P)^{ord(f,P)} \tag{2.1}$$

Proof: Since the surface is compact the divisors (f) and (g) are discrete sets, hence the expression will be finite. We begin by dissecting the surface

$\mathcal{X}$ along its α_i and β_i cycles to the fundamental $4g$ polygon which we denote as Δ and can be seen in figure 1 .

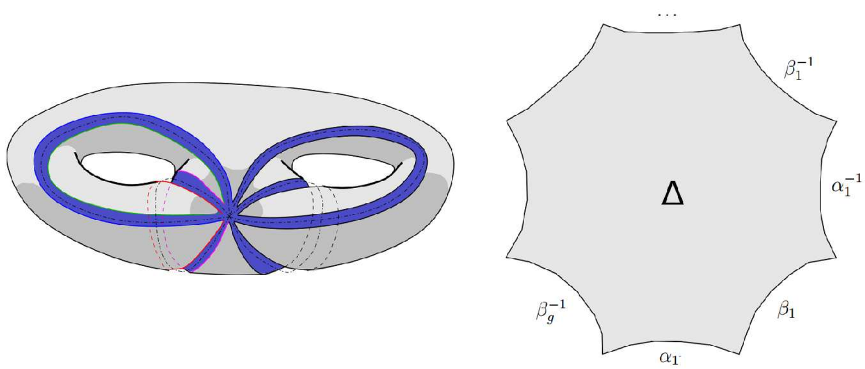

Figure 1: Genus 2 decomposition [1]

Let us denote $\{P_i\}$ the support of (f), and $\{Q_i\}$ the support of (g). Picking a point S_0 different from $\{P_i\}$ and $\{Q_i\}$, we draw smooth arcs δ_i from S_0 to $\{P_i\}$ disjoint except for their common base point S_0 and not containing any of the points $\{Q_i\}$. We repeat the same procedure for for the arcs from S_0 to $\{Q_i\}$.

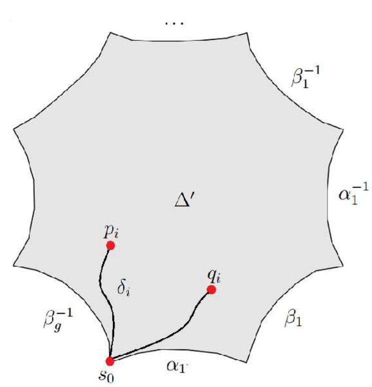

Figure 2: [1]

Let Δ' be the complement of the arcs δ_i in Δ. That is Δ' can be considered as the polygon with sides α_i, β_i and δ_i as drawn in the above figure. Δ' is simply connected, furthermore f is non-zero and holomorphic in Δ', we can choose a branch of the function $\log(f) \in \Delta'$. Let us now consider the meromorphic differential $\omega = \log(f) d\log(g) = \log(f)\frac{dg}{g} \in \Delta'$. We notice that $\frac{dg}{g}$ has a single pole with $\operatorname{ord}(g, Q_i)$ at each Q_i, so by the residue theorem we have

$$\begin{aligned} \int\limits_{\partial\Delta'} \omega &= 2\pi i \sum_{Q_i} \operatorname{Res}(\omega) \\ &= 2\pi i \sum_{Q_i} \operatorname{ord}(g, Q_i) \log\big(f(Q_i)\big) \end{aligned} \tag{2.2}$$

We will evaluate the points $P \in \alpha_i$, $P' \in \alpha_i^{-1}$ on $\partial\Delta'$ identified on $\mathcal{X}$

$$\log\big(f(P')\big) = \log\big(f(P)\big) + \int\limits_{\beta_i} d\log(f) \tag{2.3}$$

Therefore,

$$\int\limits_{\alpha_i + \alpha_i^{-1}} \omega = \left(\int_{\alpha_i} d\log(g)\right)\left(-\int_{\beta_i} d\log(f)\right) \tag{2.4}$$

Similarly,

$$\int\limits_{\beta_i + \beta_i^{-1}} \omega = \left(\int_{\beta_i} d\log(g)\right)\left(\int_{\alpha_i} d\log(f)\right) \tag{2.5}$$

For $P \in \delta_i,\ P' \in \delta_i^{-1}$ on $\partial\Delta'$ identified on $\mathcal{X}$

$$\log\big(f(P')\big) - \log\big(f(P)\big) = -2\pi i \ \mathrm{ord}(f, P_i) \tag{2.6}$$

So that

$$\int\limits_{\delta_i+\delta_i^{-1}} \omega = 2\pi i \ \mathrm{ord}(f, P_i) \int\limits_{S_0}^{P_i} d\log(g) \tag{2.7}$$

Hence,

$$\begin{aligned}\sum_i \int_{\delta_i+\delta_i^{-1}} \omega &= 2\pi i \Bigg(\sum \mathrm{ord}(f, P_i) \Bigg(\log\big(g(P_i)\big) - \log\big(g(S_0)\big) \Bigg) \Bigg) \\ &= 2\pi i \sum \mathrm{ord}(f, P_i) \log\big(g(P_i)\big)\end{aligned} \tag{2.8}$$

Now since $\sum\limits_{P_i} \mathrm{ord}(f, P_i) = 0$ we arrive at.

$$\begin{aligned}&2\pi i \Bigg(\sum \mathrm{ord}(g, Q_i) \log\big(f(Q_i)\big) - \sum \mathrm{ord}(f, P_i) \log\big(g(P_i)\big) \Bigg) \\ &\quad = \sum_{i=1}^{g} \Bigg(\Bigg(\int_{\alpha_i} d\log(f) \Bigg) \Bigg(\int_{\beta_i} d\log(g) \Bigg) - \Bigg(\int_{\alpha_i} d\log(g) \Bigg) \Bigg(\int_{\beta_i} d\log(f) \Bigg) \Bigg)\end{aligned} \tag{2.9}$$

$\int_{\alpha_i} d\log(f)$ is always an integer multiple of $2\pi i$, hence the term on the right hand side is an integral multiple of $(2\pi i)^2$, thus:

$$\sum \mathrm{ord}(g, Q_i) \log\big(f(Q_i)\big) - \sum \mathrm{ord}(f, P_i) \log\big(g(P_i)\big) \in 2\pi i \mathbb{Z} \tag{2.10}$$

Finally after taking exponents we arrive at

$$\prod_{P \in \mathcal{X}} f(P)^{\mathrm{ord}(g,P)} = \prod_{P \in \mathcal{X}} g(P)^{\mathrm{ord}(f,P)} \tag{2.11}$$

■

3 Flat surfaces with conical singularities

3.1 Isothermal coordinate

The material in this section can be found in [2]. Suppose that we have a real two-dimensional analytic manifold with Riemannian metric

$$ds^2 = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2 \tag{3.1}$$

where the functions E, F, G are such that $EG - F^2 > 0$. Through a change of coordinates the metric may be rewritten as $ds^2 = f(u, v)(du^2 + dv^2)$.

Theorem 3.1: *Let E, F, G - be real analytic functions of the real variables x and y. Then there exists new (real) local coordinates u and v, for the surface of which the induces metric 3.1 ds^2 takes the following form*

$$ds^2 = f(u, v)(du^2 + dv^2) \tag{3.2}$$

The coordinates (u, v) are referred to as isothermal coordinates.

Theorem 3.2: *Suppose we are given a metrized surface on which conformal co-ordinates are defined. Then the co-ordinate changes which preserve the conformal form of the metric (i.e. change it to another conformal form), are precisely those corresponding to complex analytic co-ordinate changes, and the composites of these with complex conjugation.*

If the coordinates (x, y) and (u, v) in a neighbourhood of P are conformal, then the transition between their complex forms $z = x + iy$ and $w = u + iv$ is carried out by the function $f(z, \bar{z})$, such that either f or $\bar{f}$ is holomorphic. The metric may be then be written in the form $ds^2 = g(z, \bar{z})|dz|^2$.

Lemma 3.1: *Given a surface in $\mathbb{R}^3$ equipped with the metric $ds^2 = f(u, v)(du^2 + dv^2)$, then the Gaussian curvature can be expressed as*

$$K = -\frac{1}{2f(u, v)} \Delta \ln \big(f(u, v) \big) \tag{3.3}$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the Laplacian operator in the coordinates (u, v)

If the metric can be written locally as $ds^2 = dx^2 + dy^2$ then we refer to that metric as flat. If we have a predetermined flat metric compatible with the structure of a holomorphic manifold, then in any local coordinate system our metric can be written in the form $|f(w)|^2|dw|^2$, where $f(w)$ is a holomorphic function.

3.2 Metrics with conical singularities

The results in this subsection are derived from the paper by Troyanov [9] and describe the local structure of a conic singularity.

Definition 3.1: The standard cone with total angle θ is defined to be $V_\theta := \{(r; t) : r \geq 0; t \in \mathbb{R}/\theta\mathbb{Z}\} \big/ (0; t) \sim (0; t')$ with metric

$$ds^2 = dr^2 + r^2 dt^2$$

Definition 3.2: The weight (residue) of V_θ is given by $\beta = \frac{\theta}{2\pi} - 1$

Proposition 3.1: *$\mathbb{C}$ equipped with the metric $ds^2 = |z|^{2\beta}|dz|^2$ is isometric to V_θ*

Proof: For $z = x + iy$ the isometry is given by

$$\begin{cases} x &= \left(\left(\frac{\theta}{2\pi}\right)r\right)^{\frac{2\pi}{\theta}} \cos\left(\frac{2\pi}{\theta}t\right) \\ y &= \left(\left(\frac{\theta}{2\pi}\right)r\right)^{\frac{2\pi}{\theta}} \sin\left(\frac{2\pi}{\theta}t\right) \end{cases}$$

■

Definition 3.3: We call a function $h : U \to \mathbb{R}$ (where U is an open subset of $\mathbb{C}$) harmonic with logarithmic singularity of weight β at a point $P \in U$ if

$$z \mapsto h(z) - \beta \log|z - P|$$

in harmonic in U.

Theorem 3.3: *Let U be an open subset of $\mathbb{C}$, if $h : U \to \mathbb{C}$ is harmonic with logarithmic singularity of weight $\beta > -1$ at $P \in U$. Furthermore, if U is equipped with the metric $ds^2 = e^{2h}|dz|^2$, then in a neighbourhood of $P \in U$ the metric is isometric to a neighbourhood of conical vertex V_θ*

We refer to the point P as a conic singularity of angle θ for the given metric ds^2

Proof: For simplicity we assume that $P = 0$. $h(z) - \beta \log|z|$ is a harmonic function, hence there exist a holomorphic function $g(z)$ in a neighbourhood of 0 such that

$$\Re\big(g(z)\big) = h(z) - \beta \log |z| \tag{3.4}$$

since this function this harmonic, we may write it as

$$e^{g(z)} = a_0 + a_1 z + a_2 z^2 + \mathcal{O}(z^3) \tag{3.5}$$

Where $a_0 \neq 0$. Let $b_k = \frac{\beta+1}{\beta+k+1} a_k$ then in a neighbourhood of 0 there exists a series $\sum\limits_k b_k z^k$ that converges to an analytic function f such that $f(0) \neq 0$. Let us consider the function $w = z f(z)^{\frac{1}{\beta+1}}$. We pick a branch of log near a neighbourhood of $f(0)$. We have

$$\frac{1}{\beta+1} w^{\beta+1} = \int_0^z t^\beta e^{g(t)} dt \tag{3.6}$$

. Hence:

$$w^\beta dw = z^\beta e^{g(z)} dz = e^{\log(z^\beta)} e^{g(z)} dz = e^{g(z)+\beta \log(z)} dz \tag{3.7}$$

So:

$$|w|^\beta |dw| = e^{\Re\big(g(z)+\beta \log(z)\big)} |dz| = e^{h(z)} |dz| \tag{3.8}$$

That is $ds^2 = |w|^{2\beta} |dw|^2$ ■

We refer to w as the distinguished local parameter.

Definition 3.4: Let $\{x_i\}_{i=1}^n$ be points on $\mathcal{X}$ with angles $\{\theta_i\}_{i=1}^n$, We say that a Riemann surface $\mathcal{X}$ possess a euclidean structure with conic singularities if $\mathcal{X}_0 = \mathcal{X} - \{x_i\}_{i=1}^n$ is euclidean and for which x_i admits an isometric neighbourhood to the vertex of the cone V_θ

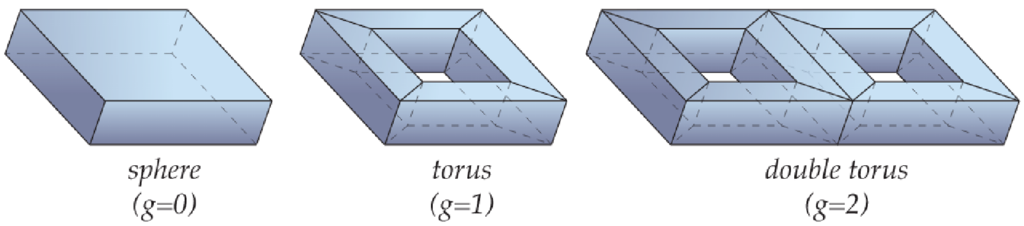

Figure 3: Examples

3.3 Elements of Hodge Theory

We give a brief overview of Hodge theory as found mainly in Springer [7] and other resources, [9],[3],[10], and [6].

Definition 3.5: Let $\omega = p(x,y)dx + q(x,y)dy$, we define the Hodge star operator to be:

$$\star\omega = -q(x,y)dx + p(x,y)dy$$

This is the only case that we will consider.

Proposition 3.2: *The operator $\star$ is independent of the choice of local co-ordinates*

Proof: We verify that $\star$ is well defined by showing that it is invariant under change of coordinates. Let $z = x + iy$ and $w = u + iv$. Let $z = \zeta(w)$, then $p := p\big(x(u,v), y(u,v)\big)$, similarly $q := q\big(x(u,v), y(u,v)\big)$. Under the change of coordinates we arrive at

$$\widetilde{p}(u,v) = p\big(x(u,v), y(u,v)\big)\frac{\partial x}{\partial u} + q\big(x(u,v), y(u,v)\big)\frac{\partial y}{\partial u} \tag{3.9}$$
$$\widetilde{q}(u,v) = p\big(x(u,v), y(u,v)\big)\frac{\partial x}{\partial v} + q\big(x(u,v), y(u,v)\big)\frac{\partial y}{\partial v}$$

substituting back into our differential we arrive at:

$$\star\omega = -\widetilde{q}\ du + \widetilde{p}\ dv = -\Big(p\frac{\partial x}{\partial v} + q\frac{\partial x}{\partial v}\Big)du + \Big(p\frac{\partial x}{\partial u} + q\frac{\partial x}{\partial u}\Big)dv \tag{3.10}$$

The change of coordinates is conformal, hence $\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}$ and $\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$. As such:

$$\star\omega = -\widetilde{q}\ du + \widetilde{p}\ dv = -\bigg(-p\frac{\partial y}{\partial u} + q\frac{\partial x}{\partial u}\bigg)du + \bigg(p\frac{\partial y}{\partial v} - q\frac{\partial x}{\partial v}\bigg)dv \tag{3.11}$$
$$= -q\bigg(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\bigg) + p\bigg(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\bigg) \tag{3.12}$$
$$= -q\ dx + p\ dy$$

■

Proposition 3.3: *If $\omega_1 = p_1dx + q_1dy$ and $\omega_2 = p_2dx + q_2dy$ locally, then the Hodge star operator has the following properties:*

(i) $\star(\omega_1 + \omega_2) = \star\omega_1 + \omega_2$

(ii) $\star(f\omega) = f \star \omega$

(iii) $\star\star\,\omega = -\omega$

(iv) $\omega_1 \wedge \star\omega_2 = \omega_2 \wedge \star\omega_1 = (p_1p_2 + q_1q_2)dx \wedge dy$

(v) $\omega \wedge \star\omega = (p^2 + q^2)dx \wedge dy$

We can then express Laplacian as :

$$\Delta f = d \star df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)dx \wedge dy \tag{3.13}$$

This can then be rewritten using complex coordinates as

$$\Delta f = d \star df = -2i\overline{\partial}\partial f \tag{3.14}$$

The Hilbert space $\mathscr{H}$ in will be working in is composed of measurable differentials ω on $\mathcal{X}$ which satisfy the conditions that $\omega \wedge \star\overline{\omega}$ be Lebesgue integrable and $||\omega||^2 = \iint_{\mathcal{X}} \omega \wedge \star\overline{\omega}$ be finite. If locally $\omega = pdx + qdy$, then $\omega \wedge \star\overline{\omega} = (p\overline{p} + q\overline{q})dx \wedge dy$. The requirement that ω be measurable implies that locally p and q are measurable. The requirement that $\omega \wedge \star\overline{\omega}$ be Lebesgue integrable implies that locally $|p|^2$ and $|q|^2$ are integrable [7].

Definition 3.6: We define the inner product of two differentials ω_1 and ω_2 to be

$$\langle \omega_1, \omega_2 \rangle = \iint_{\mathcal{X}} \omega_1 \wedge \star\overline{\omega_2} \tag{3.15}$$

Locally we have $\omega_1 = p_1dx + q_1dy$, $\omega_2 = p_2dx + q_2dy$ so that $\omega_1 \wedge \star\overline{\omega_2} = (p_1\overline{p_2} + q_1\overline{q_2})dx \wedge dy$

Theorem 3.4: *Let $L^2(\mathcal{X})$ be the set of equivalence classes of measurable* *differentials ω on $\mathcal{X}$ with $\omega \wedge \star\overline{\omega}$ integrable and $\iint_{\mathcal{X}} \omega \wedge \star\overline{\omega}$ finite. With the usual addition, multiplication by complex constants, and the inner product given by (3.15), $L^2(\mathcal{X})$ is a Hilbert space.*

Theorem 3.5: *Let $\mathscr{H}$ be a Hilbert space where $\mathscr{H}_1$ is a subspace of $\mathscr{H}$ and let $\mathscr{H}_1^{\perp}$ be its orthogonal complement. Then*

$$\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_1^{\perp} \tag{3.16}$$

that is, every element $x \in \mathscr{H}$ may be written in the form $x = x_1 + x_2$ where $x_1 \in \mathscr{H}_1$ and $x_2 \in \mathscr{H}_2$, and this representation is unique.

Definition 3.7: Let $\mathcal{X}$ be a Riemann surface and $L^2(\mathcal{X})$ the Hilbert space of measurable square integrable 1-forms, we define the following spaces as:

(i) $E = E(\mathcal{X})$ as the closure in $L^2(\mathcal{X})$ of the vector space of exact differentials df

(ii) $E^\star = E^\star(\mathcal{X})$ as the closure in $L^2(\mathcal{X})$ of the vector space of coexact differentials $\star df$

(iii) $H = E^{\perp} \cap (E^\star)^{\perp}$

Theorem 3.6 (Hodge Decomposition): *$E, E^\star$, and H are pairwise orthogonal and $L^2(\mathcal{X}) = H \oplus E \oplus E^\star$*

Proof: We begin by showing that E and $E^\star$ are orthogonal subspaces. Let $\omega \in E$ and $\gamma \in E^\star$ then in the norm of $L^2(\mathcal{X}), \omega = \lim_{n\to\infty} df_n$ and $\gamma = \lim_{n\to\infty} \star dg_n$

where $f_n g_n \in C_0^2$ and $\langle \omega, \gamma \rangle = \lim_{n \to \infty} \langle df_n, \star dg_n \rangle$. Since f_n and g_n have compact support then

$$\langle \omega, \gamma \rangle = -\iint_{\mathcal{X}} df_n d\overline{g_n} = -\iint_{\mathcal{X}} f_n dd\overline{g_n} = 0 \tag{3.17}$$

for all n, hence $\langle \omega, \gamma \rangle = 0$ and $E \perp E^\star$. Now $L^2(\mathcal{X})$ can be decomposed into orthogonal components, that is $L^2(\mathcal{X}) = (E \oplus E^\star) \oplus (E \oplus E^\star)^\perp$ We now show that $H = (E \oplus E^\star)^\perp$. If $\omega \in H$ and $\gamma_1 \in E, \gamma_2 \in E^\star$, then $\langle \omega, \gamma_1 + \gamma_2 \rangle = \langle \omega, \gamma_1 \rangle + \langle \omega, \gamma_2 \rangle = 0$ so that $H \subseteq (E \oplus E^\star)^\perp$. On the other hand if $\omega \in (E \oplus E^\star)^\perp$, then $\langle \omega, \gamma_1 + \gamma_2 \rangle = 0$ for all $\gamma_1 \in E$ and $\gamma_2 in E^\star$. In particular $\langle \omega, \gamma_1 \rangle = 0$ and $\langle \omega, \gamma_2 \rangle = 0$. Hence $\omega \in E^\perp \cap (E^\star)^\perp = H$ and $(E \oplus E^\star)^\perp \subseteq H$. Combining these gives the desired result. ∎

Lemma 3.2: *If $\omega \in C^1$ then*

(i) $d\omega = 0$ *if and only if* $\omega \in (E^\star)^\perp$

(ii) $d \star \omega = 0$ *if and only if* $\omega \in E^\perp$

Lemma 3.3: $\omega \in C^1 \cap H$ *if and only if* ω *is harmonic*

Proof: For if $\omega \in C^1$ and $d\omega = d \star \omega = 0$ which by lemma 3.2 implies that $\omega \in (E^\star)^\perp \cap E^\perp = H$. Conversely, if $\omega \in H \cap C^1$, then by lemma 3.2 $d\omega = 0$ and $d \star \omega = 0$, so ω is harmonic ∎

3.4 Quadratic differentials

We give a short survey of some basic facts related to quadratic differentials as found mostly in Strebel [8], in addition to Farkas and Kra [3].

Definition 3.8: A quadratic differential on a Riemann surface $\mathcal{X}$ is a map

$$\omega : T\mathcal{X} \to \mathbb{C}$$

satisfying the condition that $\omega(\lambda v) = \lambda^2 \omega(v)$ for all $v \in T\mathcal{X}$ and all $\lambda \in \mathbb{C}$.

We denote the vector space of quadratic differentials on $\mathcal{X}$ by $Q(\mathcal{X})$. We note that if $\omega \in Q(\mathcal{X})$ then $|\omega|^{\frac{1}{2}}$ defines a conformal metric. If $z : U \to \mathbb{C}$ is a coordinate chart defined on some open set $U \subset \mathcal{X}$, then ω is equal on U to $\omega_U(z)(dz)^2$ for some function ω_U defined on $z(U)$. We will be mostly interested in holomorphic or meromorphic quadratic differentials, for which the local functions ω_U is holomorphic or meromorphic. Suppose that two charts $z : U \to \mathbb{C}$ and $w : V \to \mathbb{C}$ on $\mathcal{X}$ overlap, then let $h := w \circ z^{-1}$ be the transition function. If ω is represented both as $\omega_U(z)(dz)^2$ and $\omega_V(w)(dw)^2$ on $U \cap V$, then $\omega_V(h(z))(h'(z))^2 = \omega_U(z)$, since $dw = h'(z)dz$ which characterises the change of coordinates [8].

Theorem 3.7: *The vector space $Q(\mathcal{X})$ of holomorphic quadratic differentials for a Riemann surface $\mathcal{X}$ of genus $g = 1$ has dimension 1 and for $g \geq 2$ has dimension $3g - 3$*

Proposition 3.4: *Let ω be a non-zero quadratic differential on compact*

Riemann surface $\mathcal{X}$. Then ω defines a euclidean structure with conic singularities on $\mathcal{X}$. If $\{x_i\}_{i=1}^n$ are the zeros of ω and have orders $\{m_i\}_{i=1}^n$ respectively then $\{x_i\}_{i=1}^n$ are the conic singularities with angles $\{(m_i+2)\pi\}_{i=1}^n$.

Corollary 3.1: *Given a Riemann surface $\mathcal{X}$ of genus g and $\omega \in Q(\mathcal{X}) - \{0\}$, then the number of zeros counting multiplicity of ω is equal to $4g-4$*

3.5 Existence of flat conical metrics

Definition 3.9: Given a Riemann surface $\mathcal{X}$, we say that two metrics are conformally equivalent if there exists a scalar function $h : \mathcal{X} \to \mathbb{R}$ such that

$$ds_1^2 = e^{2h} ds_0^2$$

Theorem 3.8 (Troyanov [9]): *Let $\mathcal{X}$ be a compact Riemann surface of genus g, where $x_1, \dots x_n \in \mathcal{X}$ and $\theta_1, \dots, \theta_n \in \mathbb{R}$ such that:*

$$\sum_{i=1}^{n} (2\pi - \theta_i) = 2\pi(2-2g) \tag{3.18}$$

Then there exists a flat conformal metric on $\mathcal{X}$ where the points $\{x_i\}_{i=1}^n$ are conic singularities with angle of θ_i. Furthermore, this metric is unique if we impose a normalisation condition.

Proof: **Uniqueness:** If ds and ds' are two such metric, then there is a harmonic function $h : \mathcal{X} \to \mathbb{R}$ such that $ds' = e^h ds$. Since the conical features between metrics match, the function h is non-singular, hence bounded on $\mathcal{X}$

and constant.

Existence: We first assume that $\mathcal{X}$ is fo genus $g > 0$. In this case, there is a non-zero quadratic differential ω. Let $y_1, \dots y_k$ be the zeros of ω with orders $m_1, \dots m_k$, respectively. The metric $ds_0^2 = |\omega|$ is then conformal on $\mathcal{X}$ with conic singularity at y_j and weight $-m_j/2$, hence:

$$\sum_{i=1}^{k} \frac{m_i}{2} = 2 - 2g = \sum_{j=i}^{n} \beta_j \tag{3.19}$$

where $\beta_j = \frac{\theta_j}{2\pi} - 1$. Consequently

$$\sum_{j=i}^{n} \beta_j + \sum_{i=1}^{k} -m_i/2 = 0 \tag{3.20}$$

By proposition 3.5 there exists a harmonic $h : \mathcal{X} \to \mathbb{R}$ with logarithmic singularities weight β_j at the points $\{x_j\}_{j=1}^{n}$ and with weight of $\frac{-m_i}{2}$ at points y_i where $ds^2 = e^{2h} ds_0^2$ is the desired metric.

The case when $g = 0$ is taken separately. If $g = 0$ then it is homeomorphic to the sphere, according Riemann's uniformization theorem there exists, up to isomorphism a unique conformal structure on $\mathcal{X} = \mathbb{C} \cup \{\infty\}$.

Let $a_i \in \mathbb{C} \cup \{\infty\}$, $a_i \neq \infty$. Furthermore, $\beta_i = \frac{\theta_i}{2\pi} - 1$ hence

$$\sum_{i=1}^{n} \beta_i = -2 \tag{3.21}$$

with

$$ds^2 = \left(\prod_{i=1}^{n} |z - a_i|^{2\beta_i}\right)|dz|^2 \tag{3.22}$$

As such ds_2 is a flat metric, where $\log|z - a_i|$ is a harmonic function and a_i is a conical point with angle θ_i at the vertex. For $a_i = \infty$, we perform the change of coordinate$z = \frac{1}{w}$, where $|dz|^2 = \frac{|dw|^2}{|w|^4}$ and

$$\prod_{i=1}^{n} |z - a_i|^{2\beta_i} = \prod_{i=1}^{n} |w|^{-2\beta_i}|1 - wa_i|^{2\beta_i} = |w|^4 \prod_{i=1}^{n} |1 - wa_i|^{2\beta_1} \tag{3.23}$$

Hence

$$ds^2 = \left(\prod_{i=1}^{n} |z - a_i|^{2\beta_i}\right)|dz|^2 = \prod_{i=1}^{n} |1 - wa_i|^{2\beta_i}|dw|^2 \tag{3.24}$$

This shows that the metric is flat when $w = 0$, that is when $z = \infty$ ■

Proposition 3.5 ([9]): *Let $\mathcal{X}$ be a compact Riemann surface, $x_1, \ldots x_n \in \mathcal{X}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $\sum_{i=0}^{n} \alpha_i = 0$. Then there exists a harmonic function with logarithmic singularities $h : \mathcal{X} \to \mathbb{R}$ with weight α_i at points $\{x_i\}_{i=1}^{n}$. Furthermore, if h and $\widetilde{h}$ are two such functions then they differ by a constant.*

Proof: **Uniqueness** If h and $\widetilde{h}$ are two harmonic functions with identical logarithmic singularities, then $h - \widetilde{h}$ is a harmonic functions without singularities, hence $h - \widetilde{h}$ is constant since $\mathcal{X}$ is compact.

Existence It will be shown that if $P, Q \in \mathcal{X}$ then there exists a function h, such that $h : \mathcal{X} \to \mathbb{R}$, where the function h is harmonic with logarithmic singularity of weight -1 at P, and 1 at q. For convenience we will assume that both P and Q lie within the same coordinate domain U, with local coordinate map z. Let D be a sub-domain containing both points P and Q, such that $\overline{D} \subset U$. We will denote the bump function by χ, that is $\chi : \mathcal{X} \to \mathbb{R}$, such that

$$\chi\big|_D = 1 \qquad\qquad \chi\big|_{\mathcal{X}-U} = 0 \tag{3.25}$$

We now define $f : U \to \mathbb{C}$ to be $f(z) = \chi(z)\log\left(\frac{z-q}{z-p}\right)$. We can now extend f to all of $\mathcal{X}$ by setting

$$f\big|_{\mathcal{X}-U} = 0 \tag{3.26}$$

Let us now consider the following 1-form

$$\begin{aligned} \zeta = df - i\star df &= (f_x dx + f_y dy) - i(-f_y dx + f_x dy) \qquad (3.27)\\ &= (f_x + if_y)dx - i(f_x + if_y)dy \\ &= (f_x + if_y)(dx - idy) \\ &= 2\frac{\partial f}{\partial \overline{z}} d\overline{z} \end{aligned}$$

We notice that $\zeta = 0$ on $D \cup (\mathcal{X} - U)$. By Theorem 3.6 we can rewrite ζ as:

$$\zeta = \underbrace{\omega_h}_{\text{harmonic}} + \underbrace{du}_{\text{exact}} + \underbrace{\star dv}_{\text{co-exact}} \tag{3.28}$$

let $\omega = df - du = \omega_h + i\star df + \star dv$, then $d\omega = d(df - du) = 0$ demonstrating that ω is a closed differential. Furthermore, $d\star\omega = d\star\omega_h - id^2 f - d^2 v = 0$ so that ω is co-closed, implying that ω is a harmonic differential. We can now define h to be $h : \Re(f-u) = \frac{(f-u)+(\overline{f}-\overline{u})}{2}$. Taking $d \star dh$ we arrive at:

$$d \star dh = \frac{d \star \left(d(f-u) + d(\overline{f} - \overline{u}) \right)}{2} = \frac{d \star (\omega + \overline{\omega})}{2} = 0 \tag{3.29}$$

By lemma 3.3 h is harmonic and has the desired logarithmic singularities. ■

The desired function will then be a linear combination of functions. That is there exist a function f such that $(f) = P - Q$. To see this we begin by connecting P and Q by a path and cover it with coordinate charts. Since the surface is compact we have a finite subcover. We are assuming that both P_i and Q_i lie within the same coordinate domain U_i

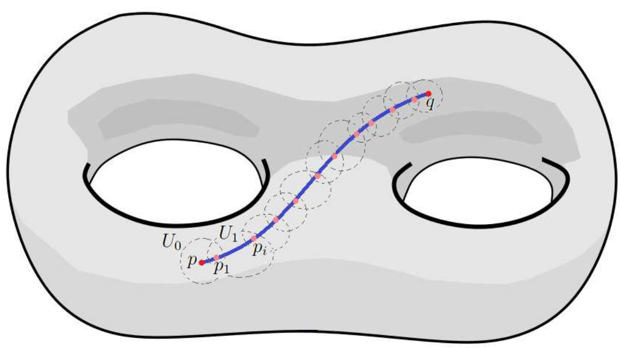

Figure 4: Path on genus 2 surface

$$p_1 \in U_0 \cap U_1 \text{ so that there exits a function } f_0 \text{ such that } (f_0) = p - p_1$$

$$p_2 \in U_1 \cap U_2 \text{ so that there exits a function } f_1 \text{ such that } (f_1) = p_1 - p_2$$

$$\vdots$$

$$p_n \in U_0 \cap U_n \text{ so that there exits a function } f_n \text{ such that } (f_n) = p_n - q$$

Taking the sum over all the functions we have

$$\left(\sum_{i=0}^{n} f_i \right) = p - \cancel{p_1} + \cancel{p_1} - \cancel{p_2} + \ldots + \cancel{p_n} - q = p - q \tag{3.30}$$

4 Main Results

We will now prove the theorem found in the introduction. In order to do so we need a few preliminary results. Let ds_0^2 and ds_1^2 be two conformally equivalent metrics. Then, their ratio $\frac{ds_0^2}{ds_1^2}$ is a well defined scalar function on $\mathcal{X}$. To get the value at the point $P \in \mathcal{X}$ one has to choose a local coordinate z near P for which is isothermal for both metrics where $ds_0^2 = g_0(z,\bar{z})|dz|^2$ and $ds_1^2 = g_1(z,\bar{z})|dz|^2$ and define $f := \frac{g_0(z,\bar{z})}{g_1(z,\bar{z})}\Big|_P$. This definition is independent of the choice of isothermal coordinate: if $z = \zeta(w)$ is a holomorphic change of variables then

$$g_0(z,\overline{z})|dz|^2 = g_0(\zeta(w),\overline{\zeta(w)})|\zeta'(w)|^2|dw|^2 = f_0(w,\overline{w})|dw|^2$$
$$g_1(z,\overline{z})|dz|^2 = g_0(\zeta(w),\overline{\zeta(w)})|\zeta'(w)|^2|dw|^2 = f_1(w,\overline{w})|dw|^2$$

where $f_0 = g_0\big(\zeta(w),\overline{\zeta(w)}\big)|\zeta'(w)|^2$ and $f_1 = g_1\big(\zeta(w),\overline{\zeta(w)}\big)|\zeta'(w)|^2$ analogously. Where the quotient $\frac{g_0(z,\overline{z})}{g_1(z,\overline{z})}$ prescribes the same numerical value is assigned to a given point regardless of which coordinate system is used. If both metrics are flat then f is harmonic with logarithmic singularities at the conical point of ds_0^2 and ds_1^2.

Lemma 4.1: *Let F and G be harmonic functions on a compact Riemann surface $\mathcal{X}$ such that F and G have logarithmic singularities* $\Big(\bigcup_p A_p\Big) \cup \Big(\bigcup_r C_r\Big)$

and $\left(\bigcup_q B_q\right) \cup \left(\bigcup_r C_r\right)$ *respectively, and* $A_p \neq B_q$*. Then*

$$\sum_q F(B_q)ord(G, B_q) = \sum_p G(A_p)ord(F, A_p) \tag{4.1}$$
$$+ \sum_r \lim_{x \to C_r} \Big(G(x)ord(F, C_r) - F(x)ord(G, C_r) \Big)$$

Proof: We begin by cutting out circles on the surface with centres at A_p, B_q and C_r with a small enough radii such that the circles will not overlap. Let Ω denote the surface $\mathcal{X}-$ {holes} and consider the differential form $\omega = F\frac{\partial G}{\partial z}dz + G\frac{\partial F}{\partial \bar{z}}d\bar{z}$.

Applying Stokes' theorem to the two-dimensional manifold with boundary Ω, we arrive at:

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega = \int_{\Omega} \frac{\partial F}{\partial \bar{z}} d\bar{z} \wedge \frac{\partial G}{\partial z} dz + \frac{\partial G}{\partial z} dz \wedge \frac{\partial F}{\partial \bar{z}} d\bar{z} \tag{4.2}$$
$$= \int_{\Omega} \left(- \frac{\partial F}{\partial \bar{z}} \frac{\partial G}{\partial z} + \frac{\partial G}{\partial z} \frac{\partial F}{\partial \bar{z}} \right) dz \wedge d\bar{z} = 0$$

It follows that the sum of the integrals ω with positively oriented boundary circle carved out is equal to 0. We now calculate these integrals separately and consider the radius of the circle corresponding to ρ as it tends to 0.

For the integral around the point B_q let us denote $\mathrm{ord}(G, B_q) = m_q$. We write $G(z, \bar{z}) = m_q \log |z - B_q| + g(z, \bar{z})$ where $g(z, \bar{z})$ is a bounded harmonic functions in a neighbourhood of B_q. Then:

$$\int_{S\rho(B_q)} F(z,\bar{z})\frac{\partial G}{\partial z}(z,\bar{z})dz = \int_{S_\rho(B_q)} F(z,\bar{z})\left(\frac{m_q}{2(z-B_q)} + \frac{\partial g}{\partial z}(z,\bar{z})\right)dz \quad (4.3)$$

Since $\frac{\partial g}{\partial z}$ is continuous, it is bounded in the neighbourhood of B_q and so limit the amount of $F(z,\bar{z})\frac{\partial g}{\partial z}(z,\bar{z})$, therefore the contribution of this term in the result of integration will tend to 0 as ρ tends to 0. Writing out the remaining integral we have:

$$\frac{m_q}{2}\int_{S_\rho(B_q)} \frac{F(z,\bar{z})}{z-B_q}dz = \frac{m_q}{2}\int_{S_\rho(B_q)} \frac{F(B_q) + \big(F(z,\bar{z}) - F(B_q)\big)}{z-B_q}dz \quad (4.4)$$

The value of $\frac{F(z,\bar{z})-F(B_q)}{z-B_q}$ is bounded in a vicinity of the point B_q, and therefore tends to 0 as $\rho \searrow 0$.

$$\frac{m_q}{2}\int_{S_\rho(B_q)} \frac{F(B_q)}{z-B_q}dz = \frac{m_q}{2} \times 2\pi i F(B_q) = \pi i m_q F(B_q) \quad (4.5)$$

For the second term in our differential ω we have:

$$\int_{S_\rho(B_q)} G(z,\bar{z})\frac{\partial F}{\partial \bar{z}}(z,\bar{z})d\bar{z} = \int_{S_\rho(B_q)} \big(m_q \log|z-B_q| + g(z,\bar{z})\big)\frac{\partial F}{\partial \bar{z}}(z,\bar{z})d\bar{z}$$

Since $\frac{\partial F}{\partial \bar{z}}(z,\bar{z})$ and $g(z,\bar{z})$ are bounded, the contribution of the product tends to 0. The remaining integral $\int_{S_\rho(B_q)} m_q \log|z-B_q|\frac{\partial F}{\partial \bar{z}}(z,\bar{z})d\bar{z}$ is bounded by $\mathcal{O}\big(\rho\log(\rho)\big)$ and thus also tends to 0.

Analogously, denoting $\operatorname{ord}(F, A_p) = n_p$ we find that:

$$\lim_{\rho\to 0}\int_{S_\rho(A_p)} F\frac{\partial G}{\partial z}dz = 0, \qquad \lim_{\rho\to 0}\int_{S_\rho(A_p)} G\frac{\partial F}{\partial \bar{z}}d\bar{z} = -\pi i n_p G(A_p) \tag{4.6}$$

The common points are a special case and require further analysis. For convenience, we choose coordinates in which $C_r = 0$. Let $\operatorname{ord}(F, 0) = n$ and $\operatorname{ord}(G, 0) = m$. By definition, the logarithmic singularity $F(z,\bar{z}) - n\log|z| = f(z,\bar{z})$ is a harmonic function that is bounded in a neighbourhood of 0, similarly $G(z,\bar{z}) - m\log|z| = g(z,\bar{z})$. He we have:

$$\int_{S_\rho} F\frac{\partial G}{\partial z}dz + G\frac{\partial F}{\partial \bar{z}}d\bar{z} \tag{4.7}$$

$$= \int_{S_\rho}\left(\big(n\log|z| + f(z,\bar{z})\big)\left(\frac{m}{2z} + \frac{\partial g}{\partial z}(z,\bar{z})\right)dz \right. \tag{4.8}$$

$$\left. + \big(m\log|z| + g(z,\bar{z})\big)\left(\frac{n}{2\bar{z}} + \frac{\partial f}{\partial \bar{z}}(z,\bar{z})\right)d\bar{z}\right)$$

Since $\frac{\partial g}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ are continuous the contribution of $\frac{\partial g}{\partial z}(z,\bar{z})f(z,\bar{z})$ and $\frac{\partial f}{\partial \bar{z}}(z,\bar{z})g(z,\bar{z})$ tend to 0. Hence the integral becomes

$$\int_{S_\rho}\left(\big(n\log|z| + f(z,\bar{z})\big)\frac{m}{2z}dz + \big(m\log|z| + g(z,\bar{z})\big)\frac{n}{2\bar{z}}d\bar{z}\right)$$

Since $nm\log|z|\big(\frac{dz}{2z} + \frac{d\bar{z}}{2\bar{z}}\big) = nm\log|z|\frac{\bar{z}dz + zd\bar{z}}{2z\bar{z}} = nm\frac{\log|z|}{2z\bar{z}}d(z\bar{z})$ about 0 integrated over the circumference of fixed radius, it remains to note that

$$\int_{S_\rho} f(z,\bar{z})\frac{m}{2z}dz + g(z,\bar{z})\frac{n}{2\bar{z}}d\bar{z} \to \pi i\big(mf(0) - ng(0)\big) = \lim_{x\to 0}\big(mF(x) - nG(x)\big)$$

Summing up all the limits and dividing by πi we obtain the desired expression ■

We now prove a result found in [5] and in the introduction.

Theorem 4.1: *Let $\mathcal{X}$ be a Riemann surface of arbitrary genus g and let ds_1, ds_2 and ds_3 be three conformal flat conical metrics on $\mathcal{X}$ Suppose that the metric ds_1 has conical points $R_1^1, \dots R_1^L$ with conical angles $2\pi(\alpha_1^1 + 1), \dots, 2\pi(\alpha_1^L + 1)$, the metric ds_2 have conical points $R_2^1, \dots R_2^M$ with conical angles $2\pi(\alpha_2^1 + 1), \dots, 2\pi(\alpha_2^M + 1)$, the metric ds_3 have conical points $R_3^1, \dots R_3^N$ with conical angles $2\pi(\alpha_3^1 + 1), \dots, 2\pi(\alpha_3^N + 1)$, then one has the relation*

$$\prod_{i=1}^{N}\left(\frac{ds_1}{ds_2}(R_i^3)\right)^{\alpha_i^3}\prod_{i=1}^{L}\left(\frac{ds_2}{ds_3}(R_i^1)\right)^{\alpha_i^1}\prod_{i=1}^{M}\left(\frac{ds_3}{ds_1}(R_i^2)\right)^{\alpha_i^2} = 1 \qquad (4.9)$$

Proof: Denote $F(x) = \frac{ds_1}{ds_2}(x)$ and $G(x) = \frac{ds_3}{ds_1}(x)$. Both quotients expressed in $F(x)$ and $G(x)$ define harmonic functions. Furthermore, the function $F(x)$ can be represented as e^f, where f is harmonic. The same can be said for the function $G(x)$, where it may be represented as e^g, where g is a harmonic function. Wherein f has a logarithmic singularity of weight a_i^1 at the point R_i^1 and another of weight $-a_i^2$ at the point R_i^2. Similarly, g has a logarithmic singularity of weight a_i^3 at the point R_i^3 and another of weight $-a_i^1$ at the point R_i^1.

Then, $\frac{ds_2}{ds_3}(x) = \frac{1}{G(x)F(x)}$ and rewriting (4.9) we arrive at:

$$\prod_{i=1}^{N} F(R_i^3)^{a_i^3} \prod_{i=1}^{L} \left(\frac{1}{G(R_i^1)F(R_i^1)} \right)^{a_i^1} = \prod_{i=1}^{M} G(R_i^2)^{a_i^2} \tag{4.10}$$

Because of the assumption that all the points R_j^i are distinct, the logarithmic singularities of f and g at the points R_i^1 cancel, the expressions $g(R_i^1)+f(R_i^1)$ and $G(R_i^1)F(R_i^1)$ are correctly defined. Hence $g(R_i^1) + f(R_i^1) = \lim_{x \to R_i^1} g(x) + f(x)$, and $G(R_i^1)F(R_i^1) = \lim_{x \to R_i^1} g(x)f(x)$. Taking the logarithm of (4.10) we get the identity from lemma 4.1.

■

5 Conclusion

The main research objective was to give an elementary proof of the reciprocity law for three flat conformally equivalent conical metrics found in [5].

In order to do accomplish this we describe the flatness criteria for a metric on a Riemann surface. We also introduce the distinguished local parameter in a neighbourhood of the conical vertex. Using components from Hodge theory we show that there exists flat metric with conical singularities on a Riemann surface. With the desired properties established we state and prove a reciprocity law for flat conformally equivalent metrics with conical singularities.

A case that was not considered was that of surfaces with boundaries, one might want to find an analogue of the reciprocity law for such occurrences. Furthermore, to develop a further understanding of conical points, one might wish study their behaviour through the machinery of θ-functions.

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