Designing Effective Lessons on Probability: A Pilot Study Focused on the Illusion of Linearity

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Abstract

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This thesis is a summary of a pilot study adopting the design experiment methodology investigating alternative approaches to the teaching of probability – the classical approach (called "Laplace approach" in our study) and the axiomatic approach (called "Properties approach") – and alternative pedagogical treatments of these approaches: with or without explicit refutations of common misconceptions by the teacher. Four short video-lessons were designed, focused on Bernoulli trials: "Laplace-exposition"; "Laplace-refutation"; "Properties-exposition" and "Properties-refutation ". The primary misconception addressed in the study was the erroneous application of proportionality models: the misconception called "illusion of linearity" in the context of probability. The design and layout of the videos' visuals and audio were informed by the principles of cognitive load theory in multimedia to ensure optimal learning potential. Each lesson was tried with one volunteer participant, a post-college student. Each participant listened to the video-lesson assigned to him or her, solved six related probability questions, responded to a questionnaire and was interviewed one-on-one by the researcher. Data in the form of participants' written responses, transcripts of the interviews and researcher's notes were used for a thorough qualitative analysis and to inform future iterations of the study.

Research suggests that the primary obstacles to probability education are misconceptions originating from cognitive biases and limitations. Often misconceptions in probability are attributed to intuitions. Intuitions are easily accessed and experience-based knowledge that can help or hinder individuals while problem solving. One question of this study was whether stating and refuting known misconceptions during a lecture-style lesson on Bernoulli trials (refutation treatment) promotes learning better than a lesson without the mention of misconceptions (exposition treatment).

This study also intended to illuminate the nature of probability misconceptions, as research suggests that attributing them to "intuitions" may be misleading. "Intuition", especially, "robust intuition", usually refers to a connected set of beliefs, but our observations suggest that students' predictions about probability of events are based on something much less homogeneous and connected. Participants in the study did not have consistent and robust intuitions about Bernoulli trials but rather a weaker form of basic knowledge called phenomenological primitives. Their intuitions would have to be

developed, perhaps through a frequentist approach to probability. Participants had a high distrust in their initial guesses – a symptom of the weakness or lack of “robust intuition” – and this played an important role throughout the problem solving process. The Properties approach seemed to have more success than the Laplace approach because of its computational simplicity and because participants may have naturally implemented the Properties approach when thinking of chance events. The approaches did not seem to alter the use of misconceptions. The refutation treatment did not have the expected outcome on learning. Participants subjected to it performed better than those subjected to the exposition treatment but this could be because they “knew” (were told) – not necessarily “understood” – that changes to the number of trials and the number of desired events affect the probability of the desired event in a non-proportional manner. Future iterations of this study would explore further the above conjectures about the nature of the basis of students’ predictions about probability and the appropriateness of different approaches to probability to build on this basis. Also, other common probability misconceptions would be explored to determine their cognitive origins in order to inform the design of instructional content.

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1 INTRODUCTION

This thesis is an account of a pilot study following the design experiment methodology (Brown, Collins, & Duguid, 1989; Cobb, diSessa, Lehrer, & Schauble, 2003) focused on the teaching of probability. In the course of the research, four lessons on probability were designed, with the aim of helping students overcome the common misconception of erroneously applying proportionality models to assess probabilities (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003; De Bock, Van Dooren, Janssens, & Verschaffel, 2002). In this preliminary phase of the design experiment, we sought answers to three main questions:

What are the sources of students' difficulties in learning probability?

What are the different approaches to teaching probability?

Of the 4 lessons designed mainly to address difficulties stemming from the "illusion of proportionality" misconception, which ones are to be definitely dismissed as reinforcing this misconception and perhaps causing more difficulties?

The content of the lessons revolved around comparing probabilities in random experiments based on the Bernoulli trials schema. The lessons differed by "approaches" based on two interpretations of probability – the classical, "Laplace" approach and the structural (axiomatic) approach – and two "treatments" called *exposition* and *refutation*. The refutation treatment differs from the exposition one only in that it presents and refutes common misconceptions about Bernoulli trials.

Many sources (Fischbein, 1975; Kahneman, 2011; Kahneman, Slovic, & Tversky, 1982; Greer, 2001) attribute the obstacles of learning probability to "intuitions" that give rise to several well-documented misconceptions that are inconsistent with probability theory. Misconceptions are notions about probability that are inherently biased or implicitly assumed, and can manifest themselves as paradoxes, fallacies or inconsistencies. They are often rooted in cognitive biases such as problem solving heuristics and other cognitive "shortcuts" that are uncritically applied by individuals (Myers, 2010).

A variety of common probability misconceptions could be explored using the format of this experiment, however the "illusion of linearity" (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003) applied to chance has been selected as the focus of the study. This misconception manifests itself in solving problems by the belief that variables are *always* related in a proportional manner, even when they are not. For example – that doubling the perimeter of a square doubles its area. Or that a person's height increases as a function of their age (De Bock, Van Dooren, Janssens, & Verschaffel, 2002). This tendency has been observed in many contexts, including probability problems related to Bernoulli trials. For example, it is common for individuals to believe doubling the number of trials doubles the chances to obtain the desired event.

Among the documented misconceptions, the illusion of linearity appears to be the most prominent and effective in its ability to misguide students – based on Van Dooren et al.'s (2003) results. An official ranking of the relative strengths and consistency of misconceptions has not been completed, to my knowledge. However, it appears that the rate of the tendency to apply proportional modelling in probability increases with age, likely owing to students' familiarity with fractions and long experience with solving problems based on simple linear relations in elementary school (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003; Fischbein & Schnarch, 1997). This makes the illusion of linearity a worthwhile misconception to address in teaching and research given the participants in the pilot study are of similar age (or older) as those in Van Dooren et al.'s study.

A supplemental goal of this study is to lend insight into the nature of probabilistic pre-conceptions for the purposes of informing the design of instructional material. The literature refers to these pre-conceptions as being based in "intuitions" (Fischbein, 1975) which assumes certain characteristics, and assumes certain cognitive origins (how they are constructed) (Myers, 2010; Muller, 2008). Generally, intuitions are characterized as being consistent and robust biases that steadily guide critical thought to inaccurate conclusions. Some probability misconceptions may be of this nature, but there is evidence (Muller, 2008) supporting that, in the field of educational studies in probability, "intuition" is a misleading term.

Among the many sources of difficulty in learning probability are the theoretical discrepancies implicit in the teaching of probability (Steinbring, 1991). These are likely rooted in the instructor's own biases and misconceptions, as they, too, are susceptible to the same errors in thought as students. For example, different interpretations of probability are often inadequately defined and simultaneously used.

This study, therefore, carefully separates the classical and axiomatic approaches to teach probability with a theoretically consistent framework, while informing some participants on common errors they might also commit. However, the effects of this teaching approach are dependent on the aforementioned nature of the misconceptions.

To justify the design of the study, *Section 2. Sources of difficulties in probability* identifies the primary obstacles in teaching and learning by isolating the relevant epistemological obstacles, cognitive obstacles and didactic obstacle.

In order to properly define the teaching approaches used in the teaching experiment, *Section 3. Approaches to Probability* outlines the four most common interpretations of probability: the classical interpretation, the frequentist interpretation, the subjective interpretation and the structural (or axiomatic) interpretation.

Section 4. Design of the teaching experiment discusses the factors that were considered in order to properly dissect the experimental objectives. It describes how cognitive load theory in multimedia informed the architecture of the video lessons, as well as the specific content to be taught. It describes the layout of the instructional material that helped produce the scripts for the videos.

Moreover, this section describes each question appearing in the questionnaire. It examines the specific aim of the question, the expected answer for each treatment and the anticipated errors.

This section also introduces the refutation statements as they appear in the lesson. These are the common misconceptions relevant to Bernoulli trials that are explicitly discussed and refuted.

Section 5. Implementation of the teaching experiment is an overview of how the teaching experiment was carried out, including the recruitment of the participants. Using the interviews as a primary source, participant profiles were constructed, which were also used to help interpret the participants' written justifications and thought processes. The profiles are found in *Section 5* with a table showing what video lesson was randomly assigned to which participant. Links to the video lessons are provided along with detailed scripts.

Section 6. Analysis of participants' responses completes a cross-examination of the participants' written solutions to probability questions they were given after the lessons and interview discussions. First, there is an analysis per participant, which examines the justifications and remarks of each individual for every question in the questionnaire. A brief summary of each participant's performance is given.

Next, the analysis is completed by examining the results per question in order to compare similarities and differences between participants. A summary of how the participants performed for a given question is provided.

Section 7. Discussion/ Remarks attempts to identify the potential patterns observed in the analysis and speculates on the underlying reasons certain results occurred.

Section 8. Conclusion provides a short discourse about the researcher's personal experiences with this study.

The *Appendix* contains the consent form, the questionnaire administered to the participants, the interview questions, the transcripts of the interviews performed and the scripts of the video lessons.

2 SOURCES OF DIFFICULTIES IN THE LEARNING OF PROBABILITY

This chapter examines three sources of difficulties (we will call them "obstacles") in the learning of probability: epistemological, cognitive and didactic obstacles.

Epistemological obstacles related to probability are to be found in the beliefs about mathematics and randomness and in ways of reasoning that mathematicians in the past had to change (or "overcome") in their efforts to develop a mathematical theory of probability. These obstacles are not to be viewed as negative factors only, hindering the development. Rather, these beliefs and ways of thinking were useful in identifying problems, constructing tentative solutions, provoking debates among mathematicians and thus leading to new concepts and improved solutions. Students learning probability theories today often encounter the same obstacles and their progress depends on how well they are able to overcome them, i.e., understand the shortcomings of their previous systems of beliefs and ways of reasoning and move on to more advanced understanding. Teaching that attempts to shelter students from encountering those obstacles or does not help students in overcoming them is assumed to be not effective (Bachelard, 1938/1983; Brousseau, 1997; Sierpinska, 1990; Sierpinska, 1994).

The cognitive sources of difficulty refer to the specific demands that probabilistic thinking imposes on the learner's mind. Learning probability requires a highly critical attitude towards some well-learned information and problem solving strategies, namely the use of automatic cognitive shortcuts that form biased representations and alternate conceptions about non-deterministic and uncertain events. Left to their own devices, these intuitions (Myers, 2010; Fischbein, 1987) can either help or hinder students when being taught new information- which requires what is called "effortful information processing" (Myers, 2010). Cognitive obstacles are the bedrock to many epistemological obstacles meaning that an examination of the former may illuminate the latter.

The didactic sources of difficulty are those that can be traced back to the ways of teaching probability in school. For example - the tendency to emphasize a particular approach to probability, using a limited type of examples, implicitly adopting a single or several conflicting interpretations of probability and implementing ill-defined or assumed concepts (Steinbring, 1991). Many didactic obstacles can be traced back to epistemological obstacles since the teacher, like the learner, is susceptible to the latter while attempting to transmit knowledge to the learner.

2.1 Epistemological obstacles in the historical development of probability theory

Epistemological obstacles related to probability consist of taken for granted beliefs about chance, and ways of reasoning about problems of estimating chances that may be useful in a limited area but lead to theoretical inconsistencies and difficulties in solving problems as soon as this area is transcended. These affect current students and instructors as it did the mathematicians and thinkers developing probability theory in history. They are obstacles to broadening one's concept of chance, and to learning to reason about it. Historically, theoretical development was heavily hindered by seemingly contradictory notions of "God" or nature, and of questions of morality in gambling. Attempts at formulating coherent definitions as integral parts of a general theory of probability were bogged down by paradoxes and fallacies which exposed the tenacity and inefficacy of certain beliefs and ways of thinking.

Eventually, mathematicians achieved some satisfactory theorizations of probability; these theorizations bear traces of the epistemological obstacles in the ways they chose of overcoming them. So, these obstacles still play an important role if one wishes to fully understand even the basics of probability.

2.1.1 The obstacle of metaphysical determinism – God does not play dice

This obstacle is blamed for the tardiness of the conceptualization of probability in the development of mathematics. The belief that "everything is minutely controlled by the Will of God… made impossible the construction of theoretical hypotheses from empirical data" and hindered the conceptualization of 'randomness' for the pagan philosophers (Borovcnik, Bentz, & Kapadia, 1991, p. 28). A belief in divine power is a belief in the existence of a higher order, meaning that, "[i]f events appear to occur at random, that is because of the ignorance of man and not in the nature of events" (Borovcnik, Bentz, & Kapadia, 1991, p. 28). The necessity to create formal rules for apparently random occurrences was non-existent because these occurrences were easily explained by God's mysterious workings.

Although, in the history of mathematics, this notion may have slowed the progress of any theoretical framework, it has not stopped it. But, in modern days, each student of probability is still faced with the question of the nature of events beyond human control: is their outcome determined or not? Belief in God's will and the predetermined path is still pervasive and for many, it is assumed.

Belief in God's will governing the world was an obstacle that hindered progress of not only probability theory, but of many other fields. One need only recall the paradigm shift of the 16th century's

Copernican revolution and how the new heliocentric model shook the foundations of religion (Chiasson & McMillan, 2008).

The obstacle of metaphysical determinism, whether nourished by the belief in the will of God or the limitations of human cognition, implies that "randomness" as such does not exist and therefore one can only speak of *subjective* randomness of uncertain events. Laplace believed in the limited human capacity to understanding the natural world (Borovcnik, Bentz, & Kapadia, 1991) , arguing that a hypothetical being aware of the initial state of any given system, such as our own universe and all its particles, could know with 100% certainty the arrangement of future states thereby eliminating uncertainty and randomness.

It is thought that the only intrinsically random processes – ones that are inherent to the structure of reality – reveal themselves in quantum physics. And so, paradoxically, accepting that randomness is intrinsic to the fabric of nature (as physics currently tells us) is not more convincing than believing in a divine power. This notion is akin to Pascal's Wager, in which it is in one's own best interest to live as if God existed, since the probability of eternal punishment in hell outweighs any advantage of believing otherwise (Popkin, 1999).

2.1.2 THE OBSTACLE OF PREDICTABILITY – PROBABILITY THEORY DOES NOT REMOVE UNCERTAINTY ABOUT FUTURE EVENTS

God need not be an integral part of one's 'model' of the universe in order to have epistemological difficulties with the notion of randomness. Einstein himself spent the last years of his life fervently denying the possibility that quantum physics was the best theory to describe reality – famously saying "God does not play dice."

Overall, the obstacle of predictability consists of the belief that the task of science is to predict future events with as much certainty as possible. Since probability theory cannot do that, it is useless or does not count as science, so one may question what the point of developing the theory is. Or they may feel that pursuing understanding of uncertainty which yields no knowledge of future events is a useless endeavor. "For a coin, the probability of ½ tells us nothing about which face will actually show up and if 'tail' will occur at the next throw of the coin. However, surprisingly, the probability constitutes some indirect knowledge for this specific toss" (Borovcnik, Bentz, & Kapadia, 1991, p. 29).

Overcoming this obstacle means accepting the fact that theoretical knowledge of probability is hypothetical, not certain. All theoretical claims are conditional, if … then, statements, but probabilistic statements are ‘doubly hypothetical’, per se. They do not carry the same certainty of a statement claiming ‘*if A then B’* but can at most claim ‘if A then B or C, etcetera’ with some assigned probability to each option. We arrive at a paradoxical feeling that we have set out to formalize some phenomenon of chance with the intention to understand it with more clarity yet the formulation does not guarantee that anything is known about future chance events.

Nowhere in mathematics or science is the hypothetical character of theoretical knowledge as salient as in probability theory. Theoretical thinking, in general, is very difficult for students and its hypothetical character is perhaps the most difficult to accept (Sierpinska, 2005; Sierpinska, Bobos, & Pruncut, 2011).

This notion falls within the purview of the incompatibility of scientific determinism and the inherent indeterminism of random events. This is similar to the epistemological struggle Einstein had with the unpredictable nature of quantum mechanics in a universe that he believed to be deterministic.

Probability theory and mathematical models for random events, unlike scientific models, cannot make predictions, and moreover, cannot be easily verified empirically. A deterministic scientific model produces reproducible results that attempt to falsify a theory. And all that is required to falsify the theory is a single experimental result. Probability theory is not falsifiable, per se, but can only be tested for statistical significance. A theoretical model for an experimental random event is deemed accurate within a range of values determined by random fluctuations, or deviations, from the mean. However, empirical results can fall out of this range and still fit the theoretical model. A single empirical trial is not enough to confirm or reject a theoretical model for random events as the trial result may indicate one of two things: (1) The experiment is correct and disagrees with the theory, or (2) The experiment has yielded a natural fluctuation from the unknown sample mean. With deterministic models, a single trial disagreeing with the theoretical model is enough to force a reevaluation of the model or the experiment.

Einstein could not accept randomness as an intrinsic property of nature. For the universe to be deterministic implies that all physical phenomena can be organized and *compressed* into equations. These equations were conditional statements, *if A then B*, that generalized and predicted some given phenomena. And that’s how science, up to that point, had conquered nature. Quantum physics dissolved this illusion, which meant that the long pursuit for understanding natural phenomena actually boiled down to the study of randomness and unpredictability- so phenomena that, to this day, cannot be compressed into a ‘theory of everything.’

But one does not have to delve into the physics of particle interactions to become dissatisfied and frustrated with the formulation of natural phenomena. A dice roll is perhaps only subjectively random due to human ignorance, but whether the randomness is intrinsic to the object or not, the outcome is still unpredictable. As Fischbein et al. (1991) wrote: "We see the process of learning and understanding as a reciprocal process of communication between a referent situation and the mathematical structure". But because of the 'doubly hypothetical' nature of chance it is difficult for students to build probabilistic concepts from the complex phenomena, just as it is for physicist to build quantum physical concepts for the phenomena of particle interactions.

It is a major cognitive achievement knowing that rational decisions about chance phenomena are even possible, given the impoverished nature of validation and evidence one receives after theorizing models to explain it (Greer, 2001).

This epistemological obstacle has deep roots in a cognitive obstacle – the limitations of the computational capacity of the human brain – and stems from man's difficulty to humbly accept these limitations and make do with shortcuts and half-chewed concepts concerning phenomena of chance. The issue of the difficulty of "checking" probabilistic statements empirically or using algebra as it can be done in the case of equations, for example, is very important and will be further discussed in sections to come.

Overcoming these aforementioned obstacles might see students hypothesizing models for various probabilistic scenarios, and performing rigorous experiments to critically observe if empirical results resemble their theory, as Shaughnessy (1977) suggested. This may build a better conceptual understanding of chance phenomena as a whole, as Freudenthal (1987) proposed.

2.1.3 THE OBSTACLE OF PROPORTIONALITY

The obstacle of proportionality is a well-documented and widely applied tendency (De Bock, Van Dooren, Janssens, & Verschaffel, 2002; Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003) to implement (often inappropriately) a model of linearity between two or more variables to describe change, relations or comparisons. This obstacle frequently appears in different fields and has transcended time as ancient philosophers (De Bock, Van Dooren, Janssens, & Verschaffel, 2002), mathematicians, students and teachers are readily susceptible to it. Van Dooren et al. (2003) often refers to the obstacle as linear thinking or proportional reasoning, and it is difficult to say whether it *hindered* the development of probability theory because it is believed that overcoming this obstacle was, in fact, an essential, if not, inevitable, component to its development. Regardless, the tendency to apply linear models, whether explicitly or implicitly, is one of the most persistent misconceptions in the learning of probability.

The issue is classified as an epistemological one because of its influence on probability development, however it is heavily rooted in cognition, for reasons that will be discussed later.

Linear models, besides being widely applicable, are intrinsically simple and easy to implement. Through primary and secondary school, students become adept at using linear models. The vast experience acquired with linearity from an early age often leads to an inclination to deal with numerical relations as though it were linear (Freudenthal, 1983).

Many students of various ages assume, for example, that an individual's height increases linearly as a function of age (from birth to 30 years old), and this line passes through the origin (Leinhardt, Zaslavsky, & Stein, 1990).

In geometry, many tend to believe that doubling the side lengths of a square will double the area. Or similarly, doubling the radius of a circle doubles its area (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). These sorts of obstacles have been recorded in the Plato's dialogue, *Meno,* (De Bock, Van Dooren, Janssens, & Verschaffel, 2002) but are also prominent in current students' thinking.

Overcoming the proportionality obstacle is difficult because the concepts of ratio and proportion are closely theoretically linked to probability although not in such a straightforward fashion as the obstacle may lead one to believe (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). Understanding of proportions and ratios is, indeed, critical to understanding probability (Piaget & Inhelder, 1951) and a weak understanding of the former negatively impacts the understanding of the latter (Ritson, 1998). Ritson even suggests that fractions be introduced to children within the context of probability instead of holding them apart as different topics. In fact, Lamprianou and Lamprianou (2002) say that the knowledge of comparing fractions is necessary for probabilistic reasoning. Moreover, probability, specifically conditional probability, is better understood by students and practitioners when probability notation and percentages are abandoned and natural frequencies are adopted (Hoffrage & Gigerenzer, 1995). This notion is aligned with the brain's natural tendency to automatically process and record frequencies (Myers, 2010), which is done with approximately the same fluency as reading in our native tongue (Myers, 2010). This suggests that learning can be made easier if discrete probability is dealt with using natural frequencies. Even Fischbein (1975) observed that preschool students have a so-called “intuition of relative frequency”. And so, given that students have a proportionality schema (a very general and influential one), which develops during the formal operational stage of a child (Piaget & Inhelder, 1951) and that the understanding of proportionality is critical to probability, a few notions become less surprising. The first of these notions is that linear reasoning is widely applied, that it is easily implemented and that the foundation of the epistemological obstacle is likely cognitive. The problem is that many students fail to consider the limitations of the linear model and the implication of their assumptions of linearity.

An important contributing factor to the illusion of linearity is the ease with which older students can reduce basic fractions. Section 2.2 on cognitive obstacles will argue that for experienced math students, reducing certain simple fractions is ‘intuitive’, just as driving a car becomes intuitive after some time of deliberation and practice. This is just a different side of the same coin, so to speak, to the schema of proportionality. But explicitly acknowledging it serves as further persuasion that linearity is influential in many situations.

There are various manifestations of experimental exercises that have probed the effects of the proportionality schema (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). To my knowledge, most of these problems are within the context of binomial probability. Now proportional thinking can occur in a qualitative form or a quantitative form. Generally, if we let n be the number of trials of an experiment, k be the number of successes required for a given event and p be the probability of success, then there are 4 types of common quantitative proportionality tendencies, and there are three types of common qualitative proportionality tendencies (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003).

In comparison tasks, there are two primary qualitative relations *'More A-more B' and 'Same A-same B'.* It is natural and correct to apply the *'More A-more B'* rule (the more height, the more area). “However, the 'Same A-same B' reasoning might occur too (figures share the same shape, so everything enlarges by the same factor), leading to an incorrect 'k times A-k times B' judgement (three times more height, so three times more area)*”* (De Bock, Van Dooren, Janssens, & Verschaffel, 2002).

It is believed that qualitative proportionality exerts an influence on its quantitative counterpart. Though not proven, it is thought that an incorrect numerical conclusion reached by quantitative proportionality is *reinforced* in its apparent correctness by qualitative statements that agree with it. For example, the misconception that rolling twice as many times yields twice the odds is reinforced by the notion that more trials implies more chances. Along with the ease with which the incorrect model is to apply, the combination of the two effects are strong. Below is a table describing the four quantitative proportionality errors observed by Van Dooren et al. (2003).

Table 1. Misconceptions about binomial probability resulting from the obstacle of proportionality

	Operation employed by proportionality model *b is an arbitrary integer* *p* is the probability of one success $P(X_n \geq k)$ is the probability of obtaining at least *k* successes in *n* trials.	**Example of Misconception**
1. Changing the number of trials *n*	*If* $P_1 = P(X_n \geq k)$ *then* $P(X_{bn} \geq k) = \boldsymbol{b} \times P_1$ *b* times more trials means the probability of k successes in *n* trials increases by a factor of *b*	Believing that rolling twice as many times doubles the probability of getting the same number of successes
2. Changing the required number of successes *k*	*If* $P_1 = P(X_n \geq k)$ *then* $P(X_n \geq bk) = \frac{\boldsymbol{1}}{\boldsymbol{b}} P_1$ *b* times more successes required means chances of success in *n* trials decreases by a factor of *b*	Birthday paradox (Freudenthal, 1973)
3. Changing the probability *p* of a single success	*If* $P_1 = P(X_n \geq k)$ *then* $P(X_n \geq k; b \times p) = b \times P_1$ *b* times more chances of a single success means the chance to obtain the same number of successes increases by a factor of *b*	Quantitative *p* item (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003)
4. Changing the number of trials *n* and the number of successes *k*	*If* $P_1 = P(X_n \geq k)$ *then* $P(X_{bn} \geq bk) = P_1$ *b* times more trials and *b* times more successes required does not change the chances	Cardano's dice problem (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003)

The misconceptions listed above are consequences of the obstacle of proportionality. The obstacle has disturbed mathematicians (and "reflective gamblers") in the past and its overcoming gave an important push to the development of probability theory.

The sixteenth century Italian mathematician Girolamo Cardano (1501-1506) is believed to have written the first systematic treatment of probability. Much of his experience and motivation derived from dice games wherein he attempted to devise mathematical approaches to beating the odds. Though he contributed greatly to the early development of probability theory, history notes some errors in his reasoning that can now be associated with linear thinking and the proportionality obstacle. Cardano pondered this question:

> *Two dice are rolled. To win, one has to obtain a double one – "double ace"[1] as it was called at the time. What is the minimum number of rolls of the two dice required for the chances of obtaining a double ace at least once to be greater than or equal to one-half?*

He correctly assessed that on a single roll, the chances of obtaining a double ace are as one to thirty-six (in today's language, the probability is $p = \frac{1}{36}$). His reasoning suggests that he believed that the chances of obtaining a double ace in n rolls of a pair of dice were n times larger (first misconception in Table 1). So, for $n \times \frac{1}{36} \geq \frac{1}{2}$, n must be at least 18. A similar reasoning can be found in today's students. When Freudenthal (1973) asked students "how many times must a dice be thrown to get an equal chance of at least one 6?" the students often answered "three": for $P(X_n \geq 1) = n \times \frac{1}{6} \geq \frac{1}{2}$, n must be at least 3.

Of course, anything beyond 36 rolls yields probability greater than 1. It is unknown if Cardano realized this, and if he had what he might have done to correct himself.

The proportionality obstacle was robust and can be found in the history of the next century. One famous story is that of Chevalier de Méré's gambling problems. Chevalier de Méré was a 17th century French writer, amateur mathematician and a notorious gambler (Freudenthal, 1973; Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). Chevalier de Méré knew that it was advantageous to bet on obtaining at least one ace in 4 rolls of a fair dice. He may have known it based on his experience with dice games, but he may have also reasoned based on the first misconception in Table 1: since there is one chance in six to obtain one ace in one roll of a dice, the chances must be 4 times bigger in 4 rolls: 4 x 1/6 = 2/3. This reasoning is incorrect but the result is valid[2]. He further reasoned that it must be equivalently advantageous to bet on obtaining at least one double ace in 24 rolls of two fair dice. His reasoning could be based again on the first misconception listed in Table 1: Since there is one chance

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[1] According to http://cours2.fsa.ulaval.ca/cours/mqt-21919/contenu/07_vad/exercices/paradoxe_mere.asp

[2] If the probability of success is 1/6, then $P(X_n \geq 1) = 1 - (\frac{5}{6})^n > \frac{1}{2}$ if $n > -\frac{\log 2}{\log\left(\frac{5}{6}\right)} \approx 3.8$. So the number of rolls n should be at least 4 for betting on getting at least one ace in rolling a fair dice.

out of 36 to get a pair of aces in one roll of a pair of dice, then the chances of getting at least one pair of aces in rolling a pair of dice 24 times should be 24 times bigger, i.e., $\frac{24}{36}$ or $\frac{2}{3}$. Or he might have reasoned by setting up a proportion between the one-ace scenario and the double ace scenario and applying the popular "rule of three":

$$4 \text{ rolls} \times \frac{1}{6} = \frac{4}{6} \text{ chance of obtaining at least one ace in 4 rolls.}$$

$$\text{If } \frac{4}{6} \text{ is advantageous, and } p = \frac{1}{36} \text{ when rolling two dice, then we have the proportion } \frac{4}{6} = \frac{\mathrm{k}}{36}$$

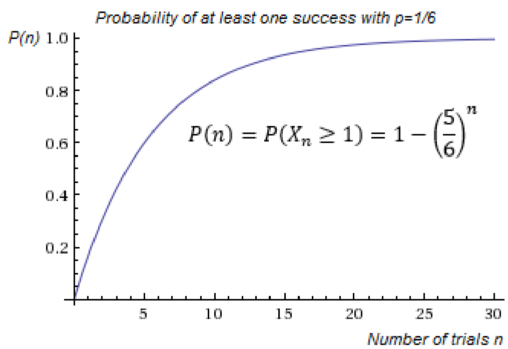
$$\text{Therefore } k = 24$$

$$\therefore \text{ It must be advantageous to roll 24 times in order to obtain at least one double ace.}$$

Contrary to his expectations, however, he was losing in such games[3]. He communicated his problem to the mathematician Blaise Pascal, who became interested in the problem and solved it thus contributing to overcoming the proportionality obstacle in the history of probability.

But every student has to overcome this obstacle in his or her own history of learning probability. Many students believe that rolling dice twice as many times doubles their chances of obtaining the desired event (special case of the first misconception in Table 1) (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). This is an example of quantitative proportionality where the number of trials is manipulated. Qualitatively it is acceptable to say 'more rolls, greater chance'. However, upon analysis, one can see that the binomial probability has a non-linear relation between the number of trials *n* and the probability *P* to obtain at least one success (*k*=1):

[3] If the probability of success is $p = \frac{1}{36}$ then $P(X_n \geq 1) = 1 - \left(\frac{35}{36}\right)^n > \frac{1}{2}$ holds for $n > -\frac{\log 2}{\log\left(\frac{5}{6}\right)} \approx 24.6$. Therefore at least 25 rolls of a pair of dice would be needed to make it worthwhile betting on at least one double ace in rolling a pair of dice.



Similarly, the probability *p* of success has a non-linear effect on the probability of k successes in $n > 1$ trials $P = P(X_n = k)$, yet the proportionality obstacle might lead one to believe that if the probability of success is increased a times, so does the probability P: it becomes aP. This is the third misconception in Table 1.

Resoundingly, problems centered on the simultaneous manipulation of the binomial variables *n* and *k* is among the most tempting cases of linearity for students according to Van Dooren et al. (2003) (misconception 4 in Table 1). Cardano's and de Méré's mistakes occurred in this kind of situations. Empirical evidence of this misconception has been provided by, among others, Fischbein and Schnarch (1997). As quoted in (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003), researchers gave this problem to grades 5-11 students:

> *The likelihood of getting heads at least twice when tossing three coins is smaller /equal/greater than the likelihood of heads at least 200 times out of 300 times.*

Summarizing the results of this research, Van Dooren at al. (2003) state:

> *It was found that a substantial number of students at each grade level argued that these probabilities are equal (most often justifying this by the equality of ratios 2/3 = 200/300). (Fischbein & Schnarch, 1997) found this misconception even increasing rather than decreasing with age (from 30% of the 5th graders to 80% of the 11th graders).*

This type of error could be attributed to the neglect of the effects of sample size (Kahneman & Tversky, 1972): students "do not acknowledge the relevancy of the law of large numbers, and fail to recognize the rule that more representative outcomes are obtained in larger samples" (ibid.). A more satisfying explanation of the error – because couched in terms of how students do think, and not in terms of how they do not think – however, is that "the subject's attention is… captured by the most *salient* data, that is, the schema of proportion" (Fischbein E. , 1999, p. 45). The two pairs of numerical values, 2 and 3, and 200 and 300, activate the proportionality schema such that $\frac{200}{300} = \frac{2}{3}$ and leading to the erroneous conclusion that the probabilities of these events are equal. The misleading simplicity of such an operation distracts from the absurdity of the conclusion that obtaining 200 heads in 300 trials is very unlikely ($P < 0.000001$).

Teaching the correct algorithm is likely an insufficient strategy to curb the reliance on linear thinking since the proportionality schema is probably engrained as intuition. The engagement of *intuitive tendencies* is cognitively quicker and more accessible than learned algorithms (Myers, 2010), so adjustments to didactics must follow.

2.1.4 THE OBSTACLE OF PRESUMED EQUIPROBABILITY – ELEMENTARY EVENTS ARE PROBABLY EQUIPROBABLE

The first ways of measuring probability were based on the assumption that, in today's terms, elementary events are assigned the same probability, or are "equiprobable." For example, in his article "Jouer" for Encyclopédie ou Dictionnaire Raisonné des Arts et Métiers, Diderot wrote:

> *Plusieurs Auteurs se sont exercés sur l'analyse des jeux; on en a un traité élémentaire de Huygens; on en a un plus profond De Moivre; on a des morceaux très - sçavans de Bernoulli sur cette matière. Il y a une analyse des jeux de hasard par Montmaur, qui n'est pas sans mérite.*
>
> *Voici les principes fondamentaux de cette science. Soit p le nombre des cas où une chose arrive; soit q le nombre des cas où elle n'arrive pas. Si la* ***probabilité de l'événement est égale dans chaque cas****, l'apparence que la chose sera est à l'apparence qu'elle ne sera pas, comme p est à q. (…) (Diderot, n.d.)* (our emphasis)[4]

A probabilistic problem is certainly simpler if the probabilities of the elementary events are equiprobable, but taking equiprobability for granted leads to errors. This obstacle might have been at the source of d'Alembert's reasoning about a game of coins in the article *Croix ou Pile* in the same *Encyclopédie*: a coin is tossed twice; to win, heads must turn up at least once. The question he considers is:

[4] Note that Diderot defines the odds, not the probability of an event.

> *On demande combien il y a à parier qu'on amènera croix en jouant deux coups consécutifs. (d'Alembert, Croix ou pile, 1754)* (How much should one bet on getting heads in two consecutive tosses of a coin?)

D'Alembert says that all authors claim that, since there are four possibilities, three winning and one losing, one should bet 3 against 1. If three tosses were allowed, there would be 8 possible outcomes, one only being the losing one; in this case, one should bet 7 against 1. But, he argues, it would make more sense to bet 2 against 1 in the case of 2 tosses and 3 against 1 in the case of 3 tosses, because if heads comes up at the first toss already, the game is won and the results of the subsequent tosses do not count.

> *Car… ne faut - il pas réduire à une les deux combinaisons qui donnent croix au premier coup? Car dès qu'une fois croix est venu, le jeu est fini, & le second coup est compté pour rien. Ainsi il n'y a proprement que trois combinaisons de possibles: Croix, premier coup. Pile, croix, premier & second coup. Pile, pile, premier & second coup. Donc il n'y a que 2 contre 1 à parier. De même dans le cas de trois coups, on trouvera Croix. Pile, croix. Pile, pile, croix. Pile, pile, pile. Donc il n'y a que 3 contre 1 à parier: ceci est digne, ce me semble, de l'attention des Calculateurs, & irait à réformer bien des règles unanimement reçûes sur les jeux de hazard. (d'Alembert, Croix ou pile, 1754)*

D'Alembert's take on this game might be explained by the equiprobability obstacle: he might have assumed that the three outcomes that decide on winning – heads first; tails then heads; no heads – are equiprobable, and that is why it is enough to bet 2 against 1, rather than 3 against 1 in the game. But heads first can come in two ways – HH and HT – and therefore this outcome is twice as probable as TH or TT. His error derives from the construction of an incorrect sample space.

Similarly, d'Alembert believed that the chances of obtaining at least one heads in 2 tosses of a coin was $\frac{2}{3}$. He argued that once a head occurs, there is no need for a second throw. Therefore the possibilities were H, TH, TT, yielding P(at least one heads)=$\frac{2}{3}$ (Gorroochurn, 2011).

The obstacle may have gained its strength from its institutionalization in the form of the basic assumption in the so-called "classical definition of probability." This definition, attributed to Pierre-Simon Laplace (Laplace, 1814), defines the probability of an event as the ratio of the number of cases favorable to it, to the number of all cases possible when we have no reason to suspect that any one of the cases is more likely to occur as another – assumption variably called "Principle of Insufficient Reason" or "Principle of Indifference" (Marinoff, 1994). Laplace was convinced of the deterministic character of physical phenomena; the impossibility of predicting some of them follows merely from human ignorance:

> *La courbe décrite par une simple molécule d'air ou de vapeurs, est réglée d'une manière aussi certaine que les orbites planétaires: il n'y a de différence entre elles, que celle qu'y met notre ignorance. La probabilité est relative en partie à notre ignorance, et en partie à nos connaissances. Nous savons que sur trois ou un plus grand nombre d'évènements, un seul doit arriver ; mais rien ne porte à croire que l'un d'eux arrivera plutôt que les autres. Dans cet état d'indécision, il nous est impossible de prononcer avec certitude sur leur arrivée. (…) La théorie des hasards consiste à réduire tous les évènements du même genre, à un certain nombre de cas également possibles, c'est-à-dire, tels que nous soyons* ***également indécis sur leur existence*** *; et à déterminer le nombre de cas favorables à l'événement dont on cherche la probabilité. Le rapport de ce nombre à celui de tous les cas possibles, est* ***la mesure de cette probabilité qui n'est ainsi qu'une fraction dont le numérateur est le nombre des cas favorables, et dont le dénominateur est le nombre de tous les cas possibles****. (Laplace, 1814, p. iv)* (our emphasis)

However this Principle of Insufficient Reason cannot be the sole criterion for the assignment of probabilities. In some cases it may be sufficient, but these cases are select. For example, Winkler (2013) devised a problem in which one must find the probability that a pencil with a pentagonal cross-section and the producer's logo on one of its sides, when rolled on a table, would stop with the logo facing up. The probability is 0, because this would require the pencil to stand on an edge. It is impossible for the pencil to land with any face up, for that matter, without defying the laws of physics regarding mass distribution in a gravitational field. However, by the Principle of Insufficient reason, the probability of $\frac{1}{5}$ seems to be a reasonable guess based on the pencil's symmetry. Although a dice can land with any of its faces up, the assumption that the outcomes 1, 2, 3, 4, 5, 6 in rolling a dice are equiprobable – that the dice is "fair" – must be treated as a theoretical hypothesis and not an empirical fact because there is no guarantee that a given material dice is made with perfect symmetry. Empirical testing of the fairness of a *particular* dice – by rolling it a large number of times – might corroborate or not the theoretical assumption. Fine (1973) recalls the fact that the first dice used in ancient times were made from bones of sheep and were quite irregular (Fine, 1973, p. 168), and he suggests that other reasons than the Principle of Indifference have decided that we now assume the outcomes of a the roll of a dice to be equiprobable:

> *It was only after many years of experimentation that the present-day cubical, symmetrical die evolved. We may speculate that this game of chance was developed, after much experience, to yield 'easily evaluated' equiprobable outcomes; it is this lengthy experience that may be elliptically invoked rather than the principle of indifference. (Fine, 1973, p. 168)*

This is just one of several examples of the limitations of the classical definition of probability (we will call it the "Laplace approach" in this thesis) detailed in (Fine, 1973, pp. 168-9) . Another example refers to theories of the behavior of particles. The Maxwell-Boltzmann theory assumes that all states are equally probable, which, however, was not confirmed by experiment. This theory assumed that the particles are distinguishable. When this assumption is dropped, the number of possible states changes:

> *The physically realistic statistical mechanics models of Bose-Einstein or Fermi-Dirac in effect amount to different identifications of the equally probable cases than that which may have seemed apparent to a classical probabilist; for example, in Bose-Einstein statistics we assume that the particles are indistinguishable and thereby combine the n! states arising from permutations of the n particles into one alternative.*
>
> *This illustrates the dilemma encountered in applications of the principle of indifference when very little is known about a set of alternatives A. Ignorance concerning A is not easily distinguished from ignorance concerning a new set of alternatives A' derived from A either by combining elements of A of by subdividing them. An argument from ignorance is not an empirical method and cannot be expected to yield true empirical conclusions. (Fine, 1973, p. 169)*

This kind of dilemma is familiar to mathematics educators as they discuss what should be "the correct" model of the random experiment of rolling two dice. Modelling the outcomes of rolling two dice using something like the Maxwell-Boltzmann's model where the dice are treated as distinguishable and can be in the same state produces 36 equally probable outcomes. Therefore the probability of each outcome is assigned the value $p = \frac{1}{36}$. (Figure 1)

	1	2	3	4	5	6
1	1-1	1-2	1-3	1-4	1-5	1-6
2	2-1	2-2	2-3	2-4	2-5	2-6
3	3-1	3-2	3-3	3-4	3-5	3-6
4	4-1	4-2	4-3	4-4	4-5	4-6
5	5-1	5-2	5-3	5-4	5-5	5-6
6	6-1	6-2	6-3	6-4	6-5	6-6

Figure 1. The "Maxwell-Boltzmann" model of rolling two dice. Dice are distinguishable and can be in the same state.

If, on the other hand, we assume that the dice are indistinguishable but can be in the same state (like the particles in the Bose-Einstein model), then there would be only 21 possible outcomes. If they are assumed to be equiprobable, each outcome would have an equal chance with $p = \frac{1}{21}$. (Figure 2)

	1	2	3	4	5	6
1	1-1	1-2	1-3	1-4	1-5	1-6
2	X	2-2	2-3	2-4	2-5	2-6
3	X	X	3-3	3-4	3-5	3-6
4	X	X	X	4-4	4-5	4-6
5	X	X	X	X	5-5	5-6
6	X	X	X	X	X	6-6

Figure 2. A "Bose-Einstein" model of rolling two dice.
Dice are indistinguishable and can be in the same state.

Fermi-Dirac utilizes the Pauli-Exclusion principle which says that particles cannot be in the same state, in addition to the conditions of the Bose-Einstein statistics. If we model the rolling of two dice this way, further six outcomes would be eliminated. In other words, the dice cannot come up with the same number. With the assumption of equiprobability, dice pairs would therefore have an equal chance of turning up with $p = \frac{1}{15}$, more than double the odds for any given outcome in the Maxwell-Boltzmann model! (Figure 3)

	1	2	3	4	5	6
1	X	1-2	1-3	1-4	1-5	1-6
2	X	X	2-3	2-4	2-5	2-6
3	X	X	X	3-4	3-5	3-6
4	X	X	X	X	4-5	4-6
5	X	X	X	X	X	5-6
6	X	X	X	X	X	X

Figure 3. The "Fermi-Dirac" model of rolling two dice.
Dice are indistinguishable and cannot occupy the same state.

Physical symmetry may be a decent *starting place* for the assignment of probability, but clearly this ideology fails quickly since other physical characteristics have important effects on the probability. As it will be discussed in the section on didactic obstacles, the Laplace approach serves as an estimator of odds but is too often implicitly assumed and automatically employed as probabilistic interpretation. That being said, other interpretations cannot stand alone as a complete description of chance phenomenon either (Section 3).

2.1.5 THE OBSTACLE OF INSUFFICIENT SPECIFICATION

The obstacle of insufficient specification is a consequence of probabilistic scenarios in which the events and sample space in question are ambiguous or ill-defined. It is considered an epistemological obstacle because it is inherent to probability theory. The obstacle hinges mostly on the fact that certain problem statements incorporate implicitly assumed definitions which may lead to paradoxical solutions. Such problems are few and far between but when they do arise they provoke much deliberation in current students, as they did in students and mathematicians throughout history (Marinoff, 1994).

One such problem is Bertrand's Chord Paradox. (Figure 4)

An equilateral triangle is drawn in a circle with radius *R*. A line is then drawn *randomly* through the circle. What is the probability that the segment *s* of the line in the circle is longer than the side *a* of the inscribed triangle?

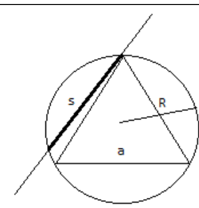
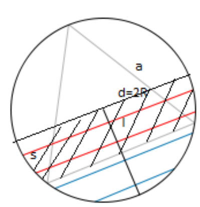


Figure 4. The Bertrand's Chord Paradox

A paradox arises because there are 3 seemingly equally acceptable but different answers for P(s>a). The three answers (in no particular order) are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$. These answers are reached using different methods to produce a random chord drawn through the circle. Figure 5 presents Bertrand's depiction of the answers (Jaynes, 1973), however the reader may skip it and continue on with the issues surrounding the obstacle itself.

Random radius method $P(s > a) = \frac{1}{2}$

Compare position *s* with diameter 2*R* perpendicular to *s.* If *s* falls within interval *I* where *I=R* then its length is greater than side *a*.



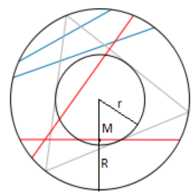
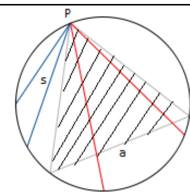
Random midpoint $P(s > a) = \frac{1}{4}$ A segment is uniquely determined by its midpoint *M* so we can focus only on *M*. If *M* is contained in the smaller circle *r* (where *r=R*/2) then P(s>a) =*area(r)/Area(R)=1/4.*	
Random endpoints $P(s > a) = \frac{1}{3}$ Consider the angle the segment sweeps out from point *P*. If the angle is in interval (60°,120°) then *s>a*.	

Figure 5. Different interpretations of the Bertrand's Chord Paradox.

So how could it be that there seemingly exists three possible solutions to the same problem? Many have indicated (Marinoff, 1994; Jaynes, 1973) that Bertrand's question is simply not well-posed in that the choosing of a random chord from a set of infinitely possible chords is misleading and imprecise, especially within the context of the classical approach to probability. Even an empirical approach via computer simulation yields solutions dependent on the choosing method of the random chord (Gardner, 1987).

Jaynes (1973) invokes *the Principle of Maximum Ignorance* in which he argues that one must not use information that has not been supplied by the problem statement itself- namely, the size and position of the circle. In other words, the ambiguity of Bertrand's puzzle can be rectified if one assumes solution that is scale and translation invariant. Under such constraints, the probability that chord *s* is greater than side length *a* is ½.

Of course this paradox is indicative of a larger epistemological issue that this paper reserves discussion for Didactic obstacles: Random is a poorly defined and understood term amongst students, teachers and laymen.

2.2 COGNITIVE OBSTACLES IN THE LEARNING OF PROBABILITY THEORY

The main focus of this section on cognitive obstacles can be reduced to the discrepancy between the nature of probability and the nature of problem solving mental schemas. The human brain gathers information from the outside world through sensory perception and some of it is processed and encoded to memory. Much information is simply not encoded. Some is forgotten and some is altered in memory. However, a portion of the information is either assimilated or accommodated to concepts. Concepts are mental groupings of similar objects, events, ideas and people (Myers, 2010). Assimilation "refers to the incorporation of new sensory experiences by existing cognitive structures without the alteration of those cognitive structures", and accommodation occurs when "existing schemas are adjusted to better match stimuli" (Muller, 2008). Assimilation and accommodation work in parallel to absorb and categorize information; organizing the world for us.

These concepts and the intake of information are constrained by the brain's processing power, and the limited amount of *cognitive resources* it can allocate to given experiences (Myers, 2010). Miller (1956) demonstrated that we can process approximately 7 (±2) bits of information (about the length of a phone number) at a time, meaning that anything beyond that is "missed", so to speak. Also, if we are 'cognitively occupied', we are more likely to make irrational judgements (Kahneman, Slovic, & Tversky, 1982; Kahneman D. , 2003; Shiv & Fedorkikhin, 1999) since additional resources are scarce. It also means that, in a world full of sensory stimuli, we are equipped with an *attentional spotlight* (Myers, 2010) with which we can consciously devote processing power to what we (subjectively) deem as relevant or important.

Not all information is processed equally, however. Information pertaining to *time*, *space*, *natural frequencies* and *well-learned* information are processed automatically (that is to say, very seamlessly) (Myers, 2010). Very little cognitive effort is required to recall a sequence of events (time), to read a word in your native language (well-learned), to track a location (space) or to keep track of how many times things happen (frequency). These are automatically processed. Effortfully processed information requires more resources to process (more attention) and is difficult to remember, but can be rehearsed through conscious repetition. Examples of effortful processed information are new concepts (concepts that cannot be assimilated to existing ones) and algorithms (such as formulas). For instance, the operation $1 + 1$ is more easily processed than the operation $(2 + 4)^3$, due in part to different relative complexities of the operations and because $1 + 1$ may be considered well-learned, and thus automatically processed (for a certain age).

2.2.1 Obstacle of Prediction Schemas and Intuitions

Concepts help us navigate the world via a process similar to Bayesian inferences. Concepts are updated by new information and serve as models of expectation of future events. Brains are advanced prediction machines which have evolved to minimize the amount of surprise or unpredictability experienced in a particular situation (Chennu, 2013; Myers, 2010). Given the brain's tendency to predict, it is not surprising that it struggles with probability. The brain is perpetually attempting to discern patterns in experiences by conditionality, such as if *A then B*, and evidently pattern recognition is more successful if the phenomenon being modelled is deterministic and consistent. The phenomenon of chance is neither.

Whether randomness is inherent to nature or subjective, due to ignorance, is irrelevant. The concepts constructed for inconsistent and non-deterministic phenomena will be, for the most part, inaccurate. They could only be accurate if the human brain encoded all specific details and conditions of the events it experienced thereby eliminating ignorance of said events and, consequently, unpredictability.

The concept for chance phenomena are therefore incomplete and severely misrepresent reality. These are the fundamentals of what mathematical education researchers (Borovcnik, Bentz, & Kapadia, 1991; Greer, 2001; Fischbein E. , 1975) who place the subject at the focus of didactics call Fischbein's primary and secondary intuitions. Also referred to as intuition and deliberation, or System 1 and System 2: All of these are used interchangeably.

Kahneman and Frederick (1982) define this dual thought system as a "collection of cognitive processes that can be distinguished by their speeds, controllability and the context on which they operate". The intuition is implicit, automatic, fast, accessible, inflexible and reliant on experience-based knowledge. It is a simple thinking strategy relying on superficialities and heuristics to make judgements and solve problems utilizing prior experiences to "predict outcomes based on what we're perceiving" (Kahneman, 2011). Intuitions require little cognitive effort and are the basis of many misconceptions in probability (among other fields) since intuitions are the recruitment of the mind's 'incomplete' schemas to make judgements on chance phenomena. Hence they also have a broad range of causes and effects. For example, at a certain age, the knowledge that a coin has $p = \frac{1}{2}$ constitutes information that is well-learned, automatically processed and intuitive. Being that intuitions are cognitive shortcuts, they come with a penalty that they are prone to error, making them a substantial obstacle to problem solving in mathematics.

Deliberation, as it pertains to problem solving, is the employment of rules, algorithms, novel concepts and formulas (effortfully processed information) that guarantees the solving of a particular problem.

Deliberation is slow, effortful, computational and rule-based. The dynamic between the two modes of thinking is described by Kahneman (2003):

> *"Intuitive decisions will be shaped by the factors that determine the accessibility of different features of the situation. Highly accessible features will influence decisions, while features of low accessibility will be largely ignored. Unfortunately, there is no reason to believe that the most accessible features are also the most relevant to a good decision." (Kahneman D. , 2003, p. 703)*

In other words, the character of judgement is dependent upon the salient features of the problem, and whether they are accessible to intuition, and also on the intrinsic complexity of the problem and the demands it imposes on deliberation.

Now, intuition can be a hindering factor in the process of critical thought or it can nudge us in the correct direction. An example of its successes is social intelligence: The ability to read and react to social contexts, the subtle cues of body language, facial expressions and the auditory dynamics of a person's voice. This intuition is powerful and serves as the backbone for the learning theory of social constructivism (Muller, 2008).

On a day-to-day basis, many of the same intuitions used to guide us along seamlessly can also serve as a deterrent in other contexts. There are several types of intuitions (Myers, 2010) but the two most formidable and relevant ones to probability learning are the *representative heuristic* and the *availability heuristic*.

The representative heuristic serves to "judge the likelihood of things in terms of how well they represent particular prototypes…which may lead us to ignore other relevant information" (Myers, 2010). Prototypes are a "mental image or best example that incorporates all the features we associate with a category" (Myers, 2010, p. 370). Manifestations of this heuristic appear in several forms. Students will believe that a fair coin tossed a number of times will produce the sequence (a) HTTHTT more readily than (b) HHHHHH because sequence *a* 'appears' more random and has an equal distribution of heads and tails – representing the salient features of fair coins and their theoretical population distribution. Another version of this heuristic is reflected in Rubel's (2007) Four Heads item where students tended to believe that tails is more likely to appear than heads after four consecutive heads were obtained. This misconception is widespread and is also referred to as the Gambler's fallacy. The sequence HHHHT better reflects the 'equal distribution' property better than HHHHH. However, it is noteworthy that some students conceded that HHHHH is more likely because it seemed consistent with what had already occurred. Students using what is called *positive recency* tend to refer to analogous scenarios as justification for their belief that a heads is more likely on the fifth toss. For example, they may report that, in a horse race, if it is known that a given horse has won four times in a row, why would one bet against it (Rubel, 2007)? In principle, the use of heuristics in probability can be detected but not necessarily predicted. Students seem to engage in reasoning that follows from the direction with which their subjective intuition has pointed them. This is a primary challenge posed for the correction of cognitive obstacles: How does one anticipate the use of a given heuristic?

The availability heuristic "operates when we base our judgments on how mentally available information is. Anything that enables information to 'pop into mind' quickly and with little effort – its recency, vividness, or distinctness – can increase its perceived availability, making it seem commonplace" (Myers, 2010, p. 375). The effects of this cognitive shortcut are broad, but we present only a few examples.

When asked if there were more ways to produce 2-person groups or 8-person groups from a total of 10 people (Fischbein & Schnarch, 1997), most students concluded that there were more possible combinations with 2-person groups, despite the two options being equal:

$$C_2^{10} = C_8^{10} = \frac{10!}{2!\,8!} = 40$$

Fischbein and Grossman (1997) posit that this misconception occurs as a consequence of a few factors. The first is that students, if not taught combinatorics and factorials, possess "combinatorics naiveté" with which they will employ simple binary algorithms to justify their solutions without being able to conceive of the appropriate algorithm. Secondly, when students imagine the number of possibilities for each option, groups of 2 are cognitively easier to manage than groups of 8 lending to the 'illusion' that there are, in fact, more of the latter. Hence, they engage the availability heuristic.

Analogously, one may feel that the pair (6, 6), when rolling two dice, is less likely to occur than (5, 6) because one associates the memory of waiting for such an outcome with its frequency. This intuition is correct, despite being based in the availability heuristic.

A secondary form of this heuristic involves the subjective importance of various events that have been encoded to memory over the encoding of non-important ones. Such events are then disproportionally represented and alter the perceived frequency of occurrence. For example, the minute odds of winning the lottery are negated by the recency of memories of lottery draws in which others have won millions of dollars. These events usually draw copious amount of attention in the media, and other bells and whistles, lending significance and memorability to nearly impossible events, while losses are kept soundlessly invisible (Myers, 2010).

Moreover, the salient characteristics of lottery numbers is that the winning sequence is usually devoid of a perceivable pattern. This convinces individuals that 'copying' said features when choosing their own numbers will increase their odds of winning. This has been observed by Fischbein and Schnarch (1997) and was explained using the representative heuristic, but the availability heuristic may equally play a role.

Similar to the aforementioned issue is the intuitive implementation of the proportionality schema that was discussed in the section on epistemological obstacles. The simplest and most cognitively accessible model for change and comparison is one of direct proportions (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003).

It is believed that proportional reasoning can be explained by (1) how natural frequencies are processed automatically – conducive to processing salient features of certain problems and (2) how the reduction of some basic fractions may constitute well-learned information, and thus be automatically processed. In addition, the correct algorithm requires combinatorics, which are largely not evident to the uninitiated. For these reasons, proportional reasoning is intuitively tempting, which is why it was selected as the focus of study for this thesis.

2.2.2 Obstacle of Fixation

There is a general form of intuitive tendencies which, to my knowledge, has not been addressed by mathematics education research. This is the tendency to implicitly rely on conventional and familiar probability 'examples' to model novel problems. Again, this may serve as an obstacle, or as a starting point for problem solving. The concern is that students are engaging a general cognitive obstacle called *fixation*- "the inability to see a problem from a fresh perspective" and difficulty to abandon a mental set which "predisposes us to how we think about a problem" (Myers, 2010, p.373). Two examples of this is the 'Boarding pass problem' (Winkler, 2013) and Father Smith and Son (Borovcnik, Bentz, & Kapadia, 1991). Here we will discuss only the Father Smith and Son problem.

> *Mr. Smith is known to have two children. He is seen in town and introduces the boy with him as his son. What is the probability that his other child is also male? (Borovcnik et al., 1991)*

Borovcnik et al. outline three possible solutions, which they deem a probability paradox:

(i) We assume boys and girls are roughly equally likely and independent in births. The information of who Mr. Smith is accompanied by is irrelevant. Therefore, p=1/2.

(ii) We examine the sample space of combinations in which two children are born {BB, BG, GB, GG} and deem that the information given narrows our choice (conditional probability) to {BB, BG, GB}. Therefore, p=1/3.

(iii) Preference is important. If Mr. Smith has no preference of which child should accompany him then the probability the other is a boy is p=1/2. If the children are BB or GG then he has no choice.

However, the question can be varied if we are told that Mr. Smith is seen with his younger boy. The new information is subtle but important. The designation "younger" means that of the sample space now must accommodate birth orders so that our options are {BB, GB}, yielding a $p = \frac{1}{2}$ for each. This is similar to the situation of preference since in reality, parents are more likely to leave the older child at home.

However, this obstacle may arise due to fixation, in that a situation borrowed from real-life occurrences has been over-idealized and implicitly modelled as if the set {B,G} is comparable to the two sides of a coin {H,T}. In the end, any of the proposed solutions could be valid; any one of them is a reasonable estimate of the chances. But the tendency to think of certain situations as idealized can deter an even more reasonable estimate.

There has not been research exploring mental fixation in relation to probability. Speculation suggests that an exposure to a wide array of problem types may also broaden the scope of problem solving strategies in probability, thereby helping students to acquire better intuitions from an inventory of various experiences and contexts.

2.2.1 Obstacle of Confirmation Bias in Learning

Intuitions and misconceptions are rarely explicitly addressed and contested in classrooms. There exists a cognitive obstacle known as *confirmation bias* in which individuals "seek evidence verifying ideas more eagerly than they seek evidence that might refute them" (Myers, 2010, p.375).

A student relying on intuitions in tandem with confirmation bias, increases the chances that their misconceptions will be reinforced (Muller, 2008). Muller showed that students who held common misconceptions about Newtonian mechanics often felt more confident about their beliefs even after being taught the correct material. However, students who were exposed to and refuted on common misconceptions reported lower confidence about their theoretical knowledge but with higher gains in post-test results.

So, a successful approach to probability may require a similar method, in which students engage in a *social acquisition* of knowledge (Borovcnik, Bentz, & Kapadia, 1991) by acknowledging their presumptions and showing them that they are incorrect.

Speculation suggests that a plausible method could introduce *cognitive caveats* warning students that their initial guesses should not be trusted. However, as mentioned, it is not certain if probability intuitions are as consistent and robust as intuitions for Newtonian physics. This is important because it illuminates whether an instructor should aim to build intuitions through experience or to focus on refuting them. As mentioned, this paper assumes that they are robust intuitions and that they can be refuted.

2.2.2 Obstacle of the Unknown Nature of Probabilistic Misconceptions

This obstacle concerns the potential directions teaching may take dependent upon the *origins* of probability misconceptions. Are misconceptions originating from intuitions or something else, called "phenomenological primitives" (diSessa, 1983)? Up until now, this paper has referred to biased 'intuitions' as the source of many difficulties in learning, however, as the Four Heads item (Rubel, 2007) demonstrated, some probabilistic scenarios can stimulate different avenues of reasoning suggesting, perhaps, that these intuitions are not consistent across students.

Now, much of this discussion has been inspired by the work of Muller (2008) wherein he demonstrated that students' misconceptions about Newtonian physics were a consequence of robust and consistent intuitions, whereas misconceptions about quantum physics were likely rooted in a more basic form of knowledge called phenomenological primitives (p-prims).

P-prims are small abstractions of simple and common phenomenon (diSessa, 1983). These are not structured and not internally-coherent conceptions, unlike intuitions observed for Newtonian physics, but rather little bits of disconnected knowledge that are only accessed when needed but never consciously examined or questioned (Muller, 2008). A person whose knowledge is dependent on p-prims may explain a given phenomenon with several inconsistencies and would, in fact, have many difficulties formulating their thoughts (diSessa, 1983).

What Van Dooren et al. (2003) labels 'qualitative proportional reasoning', diSessa (1983) labels as a 'more of this, more of that' p-prim, which may explain the pervasiveness of the misconception. Another example of a p-prim is "balance" which refers to the sense of equilibrium and stability, perhaps explaining the existence of the representativeness heuristic.

Now, intuitions about Newtonian physics are produced by experiences with the real world and these events that produce the experiences are consistently observed and reinforced giving rise to well-formed schemas for future predictions. On the other hand, quantum physics is “an abstract domain with which students have very little experience” and “It could be argued that alternative conceptions are less ingrained in student thinking because they have not been reinforced through repeated experience” (Muller, 2008, p. 161).

On the continuum of cognitively accessible concepts, there are p-prims of quantum physics (Muller, 2008), and on the other side of the continuum are the robust and reinforced intuitions of Newtonian physics. So where does probability lie on this spectrum? If students rely on p-prims when thinking about chance phenomena then didactics can focus on the construction of intuitions through empirical learning. If misconceptions are consequence of intuition then different approaches to teaching should be implemented, such as the refutation of common misconceptions, as Muller (2008) did, to improve the acquisition of correct knowledge by confronting confirmation bias head-on. The classification of the nature of probability misconceptions is unknown. In this study, we assume it to be an intuition, to remain aligned with Fischbein (1987).

2.2.3 Obstacle of Limited Cognitive Load

Cognitive load theory (CLT) is important to the didactics of mathematics because it offers a guideline to the design of instructional material in the classroom and multimedia (Pollock, Chandler, & Sweller, 2002; Muller, 2008), and may reveal what probabilistic interpretations (classical, empirical, etc.) are conducive to optimal learning.

The main structure of CLT is based on some fundamental ideas. The first fundamental idea, as mentioned, is that an individual can only process about 7 bits of information at a time in the working memory. The working memory is a ‘sketchpad’ for temporary recall of information (Myers, 2010). Some of that information can be stored in long-term memory into schemas by assimilation or accommodation for later retrieval. Motivation plays an important role, too. If the information is of high interest to an individual, it’s more likely to be retained in long-term memory. Repetition of information (conscious rehearsal) can also increase to chances of retention which, consequently, reduces the load on working memory (this is the notion of well-learned information).

Schemas are the library from which we retrieve generalizations such as heuristics and intuitions. Small alterations to long-term memory are feasible with conscious thought processes, inferring that conceptual understanding can be made regardless of its ingrained nature- but some effort is required (Muller, 2008).

CLT suggests that instructional design should be concerned with the transmittal of information within the constraints of the brains limited cognitive resources. Likewise, it should also be concerned with raising student interest to improve retention rates (but that goes without saying).

Now there are three types of cognitive load on working memory – all of which have an effect on learning: intrinsic cognitive load, extraneous cognitive load, germane cognitive load.

Intrinsic load is a “measure of the inherent difficulty of a subject area due to the number of interacting bits involved” (Muller, 2008, p.59). For example, calculus has a high intrinsic load since there are many interacting entities. Extraneous load refers to “the invested mental effort that does not result in learning” (Muller, 2008, p.59). Attempting to acquire information from a poorly written document consists of extraneous load since one must exert mental effort to sort out the relevant content. Germane load refers to “the mental effort used to form schemas and actively integrate new information with prior knowledge” (Muller, 2008, p.59).

Under CLT, the objective in teaching is to increase germane load and minimize extraneous load. Chandler et al. (1991) have outlined important guidelines for the didactics of any course, though they will not be discussed here. However, in section 4.2**,** guidelines on the instructional design of multimedia in reference to cognitive load theory are discussed.

Concerning the mental effort required to learn probability, one may hypothesize about the intrinsic loads of the different approaches to probability. For example, it is plausible that an axiomatic approach has more inherent complexity than the classical one. Assuming that equiprobability is intuitively understood when one considers a symmetric system, then employing simple representations of tree diagrams and sample spaces to develop fundamental theory (Guenther, 1968) may require less mental effort when introducing new concepts. However, depending on the objective of the instruction, an axiomatic approach, speculated to be slightly more complex, lends certain advantages that other approaches cannot begin to address; such as the problem of infinite sample spaces. That being said, the dependence on axioms to build a coherent theory can be seen as an extraneous load if the instructional objective is to develop concepts about chance phenomena and to understand how probability can successfully model reality.

2.3 DIDACTIC OBSTACLES IN THE LEARNING OF PROBABILITY THEORY

As the previous sections highlighted, there are obstacles in the understanding of probability that concern individual limitations. Moving away from these obstacles we find that there are some issues regarding the way with which probability knowledge is transmitted to the student. These issues vary with the age of the student, but the issues include the assumed organization of material as a purely mathematics-based topic for the construction of theoretical truth. Likewise, courses may focus on *intuitiveness* where overly-simplified and idealized phenomena are modelled. Between these two extremes are implicitly assumed interpretations of probability which then allows for theoretical inconsistencies. Moreover, the lack of communication between probability and statistics hinders the connection between theory and reality- especially when theory is idealized. Lastly, didactics often neglects the role of the individual whose misconceptions biasedly model phenomena and therefore require special attention.

2.3.1 Obstacle of Probability's Assumed Didactic Structure

Perhaps the most fundamental obstacle to learning probability that originates in its teaching is instructors' implicit assumption that probability can be taught effectively by following the same curricular format as other mathematics courses (Steinbring, 1991). Even more fundamental is the assumption that mathematics can be acquired in a linear fashion, whether through a *bottom-up* approach, from hands-on activities up to direct abstraction of concepts and generalization of empirical observations, or a top-down approach, from listening to an exposition of theory to its direct application in solving problems.

> *"We must, however, remember that a consistent presentation of mathematical knowledge does not necessarily yield the ideal structure for the teaching and learning of mathematics" (Steinbring, 1991, p. 136)*

By the time students have entered the classroom they have tacitly accumulated loads of experience with probability and are equipped with intuitions and presumptions on the concepts of randomness and chance events. Being acquired tacitly, the range of types and robustness of misconceptions are vast and dependent on a student's unique experiences; an important factor worth considering in teaching (Borovcnik, Bentz, & Kapadia, 1991).

However, the instructor's inherent desire to adhere to some assumed mathematical structure of consistency and "obsession with precision" (Steinbring, 1991, p. 138) neglects what Fischbein (1987) would refer to as the cognitive struggle students experience between intuitive tendencies to judge situations of uncertainty and the cognitively slow application of algorithms for problem solving (Myers, 2010).

Depending on the age of the student (Steinbring, 1991), the conventional objective of didactics is centered on either consistency or intuitiveness. For older students the focus is often on mathematical consistency, in which probability modelling must respect reality. They use and develop theory in order to communicate with the problem (Steinbring, 1991). This approach produces various partial modellings of reality for the subject:

> *"[…] the teacher must initially define basic concepts in order to be able to gradually introduce the further elements of knowledge into this framework […] the basic assumption seems to be that the axiomatic structure already contains […] the entire knowledge" (Steinbring, 1991, p. 137).*

Yet this approach fails because the theoretical knowledge is transmitted by the instructors who themselves are influenced by their own subjective domains of experience (Freudenthal, 1973), intuitions and misconceptions. Transmitted knowledge is, therefore, passed onto the student in an organization that the teacher has biasedly deemed appropriate, "hence, the teacher hands mathematics down to the pupil like a ready-made object in a simplified if piece-meal way" (Steinbring, 1991, p. 137).

However, the "didactic bias" is not unique to the approach of open or consistent mathematics as instructors have similar influences if their didactic methodology is centered on intuitiveness and concreteness. This approach is complementary to the consistency approach as it shifts the focus of learning onto the subject rather than the theory. This general approach constructs intuitions and the concepts surrounding chance events. Freudenthal (1973) explains that didactic structures focusing on developing conceptual understanding through simple games of chance, and the like, can encourage students to "acquire mental objects underlying the mathematical concepts which organize phenomena as he/she experiences." But this organization may be fraught with obstacles and misconceptions.

This notion is akin to the learning of Euclidean geometry compared to the learning of non-Euclidean geometry. Fischbein and Freudenthal would say that our intuitions on the concept of the former are well-developed and robust, simplifying the understanding of the theoretical structures; while the opposite is true for non-Euclidean geometry.

> *"Mathematical concepts, structures, and ideas serve to organize phenomena from the concrete world as well as from mathematics. By means of geometrical figures like triangles, parallelogram, rhombus, or square, one succeeds in organizing the world of contour phenomena" (Freudenthal, 1973, p. 28).*

One does not need to learn the theoretical construction of a given Euclidean shape; one understands with relative ease the possibilities of geometry due to their experience with the Euclidean reality; this is why many visual illusions work as well as they do.

It is unfruitful for the teacher to assume that an axiomatic structure that avoids surprises and attempts to be theoretically consistent can serve as the explicit source of 'truth' concerning probability.

Learning works by building concepts through assimilation and accommodation. It is, in a sense, 'cognitively economical' to build intuition of concepts through empirical and semantic learning. Model construction and activity-based learning actually improves student's theoretical understanding of probability (Shaughnessy, 1977). On the other hand, there seems to be no significant improvement in understanding, or reduction in the rate of misconceptions, for students taught via exposition of theory in lectures (Shaughnessy, 1977).

Winter (1983) suggests that the concern for clarity of concepts is conducive to providing many definitions and theorems (only some of which are practically applied), as well as using many specific terms unique to the field.

Clearly, such an approach which concerns itself with superfluous detail likely imposes an *extraneous load* (Miller, 1956), which is to say that students are attempting to merge their pre-conceptions of probability with transmitted theoretical knowledge, but are cognitively preoccupied by assimilating and accommodating seemingly irrelevant information.

This is not to say that all courses should focus on experimentation and concreteness. Shaughnessy (1977) developed probability theory with students as a means to construct models for given contexts while converging on the main probabilistic interpretations (classical and frequentist). In his study, students would estimate the odds of given unpredictable events and test it through repeated experimentation.

A drawback to this approach is its narrow scope due to physical limitations. Concrete examples tend to focus on cases of equiprobability, physical symmetry, discrete probability, finite sample spaces and a limited variety of scenarios. They rarely investigate continuous probability or events of everyday occurrence. The perceived scope and context of randomness is therefore biased and generally unrepresentative as it develops the concept centered on 'conventional' games such as dice, spinners, urns, etc. Additionally, *probabilistic interpretation* (classical, frequentist, subjective and structured) are often implicitly confounded in discussion and application (a point reserved for the next section).

The two teaching approaches are rarely developed simultaneously; older students are exposed to axiomatic approaches and younger students to empirical approaches (Steinbring, 1991). A combination of both approaches is Freudenthals' ideal, or what we refer to it as Freudenthals' '*Goldilocks concept'*:

> *"He believes that one should search for didactically fertile problems which are neither too simple, nor too special, nor too sharply formulated. It is such problems that compel the invention of concepts. For human learning is pre-programmed in such a way that a few examples will suffice; but it takes much less trouble to sprinkle a learner with a shower of examples than to search for one that really matters" (Borovcnik, Bentz, & Kapadia, 1991, p. 11).*

We build concepts the most effectively through experience (Myers, 2010). By doing so, we construct robust intuitions that serve as a foundation for formal mathematical development. This is not surprising as it mimics the process with which ancient mathematicians were motivated to develop probability theory from games (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003; Borovcnik, Bentz, & Kapadia, 1991).

An emphasis on the construction of theoretical knowledge through the subject's experience of reality has weaknesses though. These shortcomings are mainly cognitive but, essentially, they derive from "the impoverished nature of the feedback one gets after making judgements in probabilistic situations" (Greer, 2001, p. 21). For this reason, it may be difficult to see how probability theory successfully structures reality, since a prediction is not likely to immediately reveal itself as correct or incorrect.

This is why Fischbein's didactics places the learning subject in the focus. Any form of instruction is neither absolute nor effective if its emphasis is solely on the presentation of theory or solely on experimentation. The concepts about non-deterministic phenomena formed by our ever-predicting (Chennu, 2013), limited-resourced brains are highly biased and filled with cognitive shortcuts, giving rise to Fischbein's intuitions and the many misconceptions, paradoxes and fallacies that exist.

It is important, therefore, not to avoid mathematical inconsistency and to delve into the surprises of paradoxes and fallacies since they can be entertaining and can raise class interest and motivation, and therefore enhances learning (Borovcnik, Bentz, & Kapadia, 1991). Discussing inconsistencies cultivates critical thinking. This is essential since, anecdotally, students generally lack interest in probability. Moreover, learning where and how probability theory fails is an efficient way to see how it succeeds.

2.3.2 Obstacle of Ill-defined Basic Terminology

Possibly related to cognitive and epistemological issues, is a given individual's basic conceptions of chance and randomness. "Possibly related" because individuals amass plenty of experience with unpredictable events that, among the set of misconceptions, are incorrect notions about fundamental definitions such as *random* and *probability*.

Individuals misunderstand and misuse the words *random* and *chance*. Often these words are used to describe non-sequiturs (bumping into a friend downtown may be deemed as random despite being anything but that).

These words offer many conceptual difficulties implying they are appropriate for epistemological obstacles, but these obstacles have significant grounding in didactics since instructors, being didactically biased and/or lacking theoretical knowledge of the subject, rarely endeavor solid descriptions of these terms. The words "random" and "chance" are usually assumed to be self-explanatory and well-understood, instead of being explicitly explored theoretically.

Most probability courses aim to determine a clear and mathematically sufficient definition of probability in order to have a theoretical platform to build upon. This is achieved by introducing probability through the classical lens wherein physical symmetry determines equiprobability. The frequentist interpretation joined with the law of large numbers, are introduced to convince learners that random fluctuations in chance experiments stabilize and converge on the classical probability.

To elaborate this description, the notion of chance experiments and randomness are either assumed or developed. If developed, the conventional definitions are "self-referential" (Steinbring, 1991):

> *"Any experiment which cannot be predicted or determined, is dependent on chance. Experiments with several outcomes whose appearances cannot be forecast, are accordingly called an experiment. Another definition of chance experiment is that it is simply an experiment in which the results are dependent on chance." (Steinbring, 1991, p. 142)*

Rarely do instructors delve into what phenomena are explained by probability or discuss the knowledge that can be extracted from the quantitative assignment of probability. Besides having a narrow conception on the phenomena, there are "vicious circles" (Steinbring, 1991) interconnecting fundamental concepts, meaning that chance and randomness are not explicitly defined but rather mutually dependent and "characterized by their use and application" (Steinbring, 1991, p.142).

For example, in assigning probabilities, the classical definition is circular: it is based on the assumption of equiprobability which assumes the concept of probability as already understood.

Relative frequencies work to validate the classical interpretation and can be interpreted as "probability only if they refer to a stochastic object such as a stochastic collective" (Steinbring, 1991, p. 142). Interwoven into the fabric of this definition is the assumed existence of chance and independence of events. These assumed concepts are then developed by the theory for which it serves as its own theoretical foundation.

> *"One is able to speak of probability only if one knows the concept of chance experiments and randomness; vice versa, chance can only be understood and mathematically specified after the concept of probability has been introduced" (Steinbring, 1991, p. 142)*

Introducing probability using the limit of relative frequency confines the concept to an operationalized definition, meaning its mathematical meaning cannot be cultivated. But adopting the classical approach also reduces the concept of chance to empirical features as it relies on physical idealizations of symmetry to assume equiprobability.

Teaching cannot rely on explicit definitions only, since the circularities are unavoidable; especially if we are left to empirical idealizations. Fundamental to the basis of didactics should be basic statistics wherein students can learn to appreciate the methods to model and interpret random deviations from the theoretical models.

Chance, randomness and the various interpretations of probability should be developed mutually and treated as different but related forms of producing quantitative guesses about events that are unpredictable.

A description of randomness may utilize binary segments composed of '0' and '1' to distinguish patterns and unpredictability, and the instructor can offer epistemological caveats along the way. For example, the segment 101010 is a pattern because it can be compressed to a notation repeat (10). A caveat: This sequence of numbers may be deemed as non-random, but one cannot be too certain. The process of production for the segment could be, in fact, unpredictable, just as a coin toss can produce the sequence HTHTHT. The segment 111010 may have no discernable pattern and may appear random, however one must consider that an unknown algorithm could have produced this sequence just as an algorithm to calculate π produces an irrational and pseudo-random number.

Certainty about a given segment's random or non-random characteristics must be determined by repeated identical experiments. Just a few experiments will not suffice since the sequence attained could be part of a larger pattern, or it may simply be random- it cannot be determined *a priori* whether a sequence is random. A sequence is "judged to be random if it satisfies all known tests for randomness" (Steinbring, 1991). In other words, the definition of the term relies on statistics and the developmental level of the probability course. It may seem to the student that randomness is a type of *epistemological paradox.* However the discussion should not be avoided despite interfering with an instructor's desire for consistency. The struggles to define basic concepts reflect the notion that theoretical knowledge is not immediately validated by practice; hinting at the difficulties encountered at the cognitive level.

3 Approaches to probability

Until now, there have been frequent mentions of the approaches and interpretations of probability. The interpretation of probability encompasses the theoretical frameworks and philosophical views contained within the four main epistemological perspectives of chance: classical, empirical, subjective and structured. The approaches refer to the various ways probability can discussed and taught, as well as their underlying theoretical issues.

In previous sections on epistemological and didactic obstacles, theoretical inconsistencies and poorly defined concepts were addressed along with some propositions for their rectification. This section will explicitly define the four primary interpretations, though there are others (Borovcnik, Bentz, & Kapadia, 1991), as well as delving into their respective theoretical discrepancies.

As Steinbring (1991) indicated, basic ideas such as chance, chance experiments and randomness are nearly impossible to define without self-referentiality, thus giving rise to operationalized and mutually dependent definitions. Like “point” in geometry, and “time” in physics, probability’s axiomatic definition is inexact but it is generally understood and functional to a certain degree of theoretical accuracy.

Chance arises where there is either objective or subjective uncertainty, in which case, we refer to the probability of a given event occurring. The method with which we produce an estimate, or the probability of an event’s occurrence, depends on the information that is accessible and how one uses it. The various ways we can model a guess about an uncertain situation depends on the perspective.

3.1 Classical (Laplace, Theoretical) Interpretation

In probability education, the primary method to assign probabilities is by this approach. The probability of a “combined event is obtained by the fraction of outcomes favorable to this event in the sample space” (Smith, 2012). So if event A can occur in *m* ways out of a total of a total number of possible outcomes *n*, then we write: $P(A) = \frac{m}{n}$.

This approach is conceptually simple. It relies on the physical properties of a system, and it consists of an *a priori* assignment of values. There are a number of theoretical and practical limitations to this approach:

1. Equiprobability cannot be strictly defined (Steinbring, 1991).
2. Equiprobability relies on an idealization of physical properties (Steinbring, 1991).

3. Since this approach depends on a ratio of favorable outcomes to total number of outcomes, it cannot address infinite sample spaces.
4. Equiprobability is insufficient in assigning probabilities (i.e. Winkler's pencil (2013), Fine's Dice (Fine, 1973)- both mentioned in the section on epistemological obstacles). In other words, symmetry can occur in several ways.
5. Assigning a quantifying probability *a priori* paradoxically inherits some information from ignorance (Borovcnik, Bentz, & Kapadia, 1991).

3.2 Frequentist (Empirical) Interpretation

This interpretation consists of an *a posteriori* assignment of probability by repeated trials of identical and controlled experiments. The probabilities are not exact but constitute estimates by observed relative frequency that theoretically become more accurate as the number of trials *n* increases. The probability is, therefore, the limit approaching infinity to which the relative frequency tends.

In other words, an empirical assignment is at best an approximation. This interpretation, too, has theoretical and practical limitations.

1. It assumes that all experiments contributing to the relative frequency data are identical. In other words, it assumes symmetries through time and specific conditions which, theoretically, can never be verified.
2. The frequentist approach has a practical disadvantage. Random fluctuations may alter the relative frequency away from the theoretical probability even as *n* is large and increasing. There are no guidelines indicating the sufficient amount of trials required to assign probabilities.
3. Additionally, several experiments will likely end on different limiting frequencies. Which one is correct?

3.3 Subjective Interpretation

Probabilities are "evaluations of situations which are inherent to the subject's mind – not features of the real world around us which is implicitly assumed in the first two approaches" (Borovcnik, Bentz, & Kapadia, 1991, p. 41). This view is a 'personal' combination of classical and frequentist approaches. A subjectivist will utilize all information they deem fit including physical properties and/or prior data on a given situation.

1. Evidently, the largest issue is that the assignment of probability is inconsistent across individuals.
2. The method of combining various approaches is not defined, structured or consistent.
3. The perspective lacks internal coherence, in that the sum of probabilities of elementary events may end up being greater than 1.

Despite these limitations, the subjective interpretation resembles Bayesian statistics and the scientific method in that guesses are allowed to be updated by information the user deems as valid or invalid.

3.4 Structural Interpretation (Properties)

This formal approach is defined by a system of axioms which serve to develop a body of definition and theorems. “Probabilities are derived with no justification for their numerical values in any case of application” (Borovcnik, Bentz, & Kapadia, 1991), however the axiomatic perspective can be used to codify the other three interpretations.

The objectivist (classical, frequentist) interpret chance as inherent to nature and is predisposed to physical systems. The subjectivist interprets probability as a “degree of confidence in uncertain events” (Borovcnik, Bentz, & Kapadia, 1991, p. 42). The axiomatic approach provides rules that can be applied to the other approaches, while maintaining coherence and consistency so that probability can be dealt with rationally.

Kolomogrov’s axioms serve as the basis of this perspective:

Given an event E in a sample space S which is either finite with N elements or countably infinite with $N = \infty$ elements, then we can write:

$$S = \left(\bigcup_{i=1}^{N} E_i \right)$$

And a quantity $P(E_i)$, called the probability of an event E_i, is defined such that

1. $0 \leq P(E_i) \leq 1$
2. $P(S) = 1$
3. Additivity: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$ where E_1 and E_2 are mutually exclusive
4. Countable additivity: $P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$ for $n = 1,2 \ldots N$ where E_1, E_2...mutually exclusive

(Weisstein, n.d.)

Practically, for any given scenario to be probabilistically modelled, there is no way for the structuralist to decide whether an objectivist and subjectivist approach should be used, as it can only assign hypothetical probabilistic values.

4 THE DESIGN OF A TEACHING EXPERIMENT

This experiment adopted the design experiment format (Cobb, diSessa, Lehrer, & Schauble, 2003) and is intended to serve as a pilot study for future explorations. This format allows the relevant theory to be iteratively refined to obtain the most accurate representation of how learning occurs in the classroom and in the students mind. Following the guidance of Cobb et al. (2003), this research collected rich a varied data through several mediums: clinical interviews, audio recordings, researcher's notes and pre/post-tests. (It should be noted that pre-tests were not completed as it was shown to have a negative impact on the objectives of the research (Muller, 2008)).

- In the course of this pilot work, the team might also develop new methods for assessing aspects of student reasoning that need to be documented, given the purposes of the experiment.

4.1 OBJECTIVE OF STUDY

The objectives of this study are to illuminate the following questions that arose from the didactic, epistemological and cognitive obstacles:

- Does misconception-based instructional material promote better learning than expository material?
    - Do students invoke more mental effort for lessons that contain misconceptions?
- Is there a difference in learning outcomes if instruction is based on the Laplace approach (classical interpretation) or the Properties approach (based on axiomatic interpretation)?
    - Do students invoke more mental effort dependent on the probabilistic interpretation?
- What is the nature of pre-existing alternate conceptions in probability? Are they intuitions or diSessa's phenomenological primitives?

4.2 MATERIAL CONTENT OF VIDEO LESSONS

To fulfill the objectives of the study, five video lessons were produced- the fifth video was produced as a result of a mistake during the experiment. Initially, the intent was to produce four videos using two approaches to probability (Laplace and Properties). These were intended to be overlapped with two treatments: An expository lesson in which material was presented without discussion of common misconceptions and a refutation lesson where misconceptions are presented and refuted.

The scripts between the exposition and refutation treatments are identical except the refutation script dedicates short sections to the refutation of misconceptions. The misconceptions are listed in section 4.3.

The focus of the content material was on discrete probability – specifically Bernoulli trials – applied to rolling dice. The precise content of each lesson can be observed in Appendix B and C where there are two primary scripts- one for the Laplace approach and the other for the Properties approach. Within each of these scripts are specially denoted supplemental statements of misconceptions which are present only for the refutation treatments. Alongside the script are screenshots of the lessons.

Generally, both approaches contain essentially the same material except the Properties refutation lesson which contains two additional misconceptions that were not directly tested and serve as additional content to lengthen the video. Also, there are theoretical differences inherent to the approaches, but the lessons all aim at solving the same probabilistic problems.

4.2.1 Layout of Lesson Material

The video content begins with definitions of probability and uncertainty, and demonstrates how to calculate probabilities using dice within the framework of the given approach. The Laplace approach discusses how probability is determined by counting the number of all possible outcomes and comparing it to the number of outcomes that contribute to the desired event. Tree diagrams are introduced as a tool for counting the number of desired outcomes, the number of all possible outcomes and the relevant arrangements pertaining to the desired outcome. Next, the complementary event rule is introduced as an additional tool for calculation. Similarly, the Properties approach lesson begins with definitions of the same content as the Laplace approach. The multiplicative property is introduced to calculate the probability of an outcome that takes place in an experiment of *n* independent steps. The additive property is introduced to calculate the probability of more than one alternative outcome. Dice are used to exemplify these properties. Next, tree diagrams are introduced as a tool to count the relevant arrangements of a desired outcome.

After this brief introduction on calculations in probability, two examples involving dice are presented. The first asks to find $P(at\ least\ one\ 2\ in\ 3\ rolls)$, while the second $P(at\ least\ two\ 2's\ in\ 3\ rolls)$. The instructor/ narrator indicates that, between the examples, the required amount of successes has doubled, and the viewer is asked to ponder the new probability of the second example before the work is demonstrated. Both examples are solved using the respective tools of the given approaches. (Recall that the refutation treatments also instruct participants about some common misconceptions).

4.2.2 Design of Script and Video

Chronologically, the script was designed and written before the details of the video (images, diagrams, equations presented, etc.) were conceived. The script was designed to present the material in a simple and coherent way ensuring that no unnecessary material was taught. This was to minimize extraneous cognitive load for the learner. Moreover, great care was taken to make certain that instructional material was coherent and consistent with the probabilistic approach it attempted to present. It was vital that the information divulged in the lesson never implicitly crossed-over to other interpretations of probability.

The specifics of the video design encompass the details of the visuals, content, sounds and how all of these are organized to optimize learning potential. These specifications were informed by cognitive load theory in multimedia (CLTM) which offers a set of multimedia design principles (Mayer, 2001; Muller, 2008) that help maximize germane cognitive load and minimize extraneous cognitive load.

The principles of CLTM considered for the making of the video are the following (Mayer, 2010):

- Learning is enhanced when instruction is presented as words and images rather than words alone.
- Learning is enhanced when corresponding words and images are presented in close proximity.
- Learning is enhanced when corresponding words and images are presented simultaneously.
- Learning is enhanced when material extraneous to the learning outcomes is excluded.
- Learning is enhanced when words are presented as narration rather than as on-screen text.
- Learning is enhanced when narration is not duplicated as on-screen text when competing with dynamic visuals.

A section of the interview questions is dedicated to assessing the success of the video design and whether or not the video met the aforementioned principles. Some interview questions also concerned the length of the video, the participants' focus on the material and general interest.

The videos where produced using Microsoft PowerPoint, recorded using the "record screen" feature in PowerPoint, and clips were synthesized using Microsoft Movie Maker, then uploaded to YouTube as private videos (videos that cannot be searched for, but can only be accessed if given the link).

4.2.3 Measuring mental effort invested and confidence rating

It should be recalled that a pre-test was not administered since it was assumed to have unwanted effects on the objectives of this study. Muller (2008) observed that students had a tendency to devote more attention to parts of the lesson covering difficult material they encountered in the pre-test. These students saw gains in their post-test scores; but to the demise of the experimental objective which presented misconception-based lessons on Newtonian mechanics.

The first question on the questionnaire asks participants to self-assess their invested mental effort on a nine-point Likert scale. This serves as a direct measure of the cognitive load on participants during their lessons. There are other methods used to measure load (Muller, 2008) but the self-assessment is the most economic, most reliable and less invasive.

If cognitive load is high (high mental effort) and test results are good then it is likely the lesson induced a germane cognitive load, implying something meaningful was learned. If test results are poor and the cognitive load is high, then an extraneous load was induced. (A drawback to the measure of mental effort is that students have different degrees of prior knowledge, meaning different levels of mental effort are induced with meaningful learning). The interviews complement the information on mental effort and test results, and serve to assess whether something was learnt or not.

After each question, participants were asked to rate their confidence about their answer on a 7-point Likert scale. Accompanied by their written justifications and the interview discussion, their confidence level is an indicator of how the participant understands the given material. For example, high confidence and an incorrect answer with little written work suggests that the participant is relying on intuitive beliefs (Muller, 2008), while low confidence with written work and a correct answer suggests cognitive effort was required to implement what was learnt.

The details of the mathematical questions are provided in the following sections.

4.3 MISCONCEPTIONS ADDRESSED IN THE "REFUTATION" TREATMENTS

The misconceptions addressed in the refutation treatments are mainly centered on the erroneous application of proportionality models. Additional refutation statements were implemented so as to confront potential errors that were deemed common to the calculation of probabilities.

4.3.1 List of misconceptions and their refutations in the Laplace approach "Refutation" version

Misconception LR1: p(a SIX in n rolls) = n x p(a SIX in 1 roll)

From the Laplace Approach script: "{*Some people believe that the probability of getting at least one 6 in 3 tosses is ½; that their chances of winning are fifty-fifty. Here is how they reason. The probability of getting a 6 in a single roll of a fair die is 1/6. This is correct. But then they think that the probability of getting a 6 in THREE rolls will necessarily be THREE times bigger [display:* $3 \times \frac{1}{6}$ *]. Three times one-sixth is three-sixths which is the same as one-half [display calculation]. So they conclude that they have fifty-fifty chances of winning. But this reasoning is NOT correct. It is not only mathematically incorrect. It is DANGEROUS to reason that way. In fact, if you were to bet on winning in this game, you would likely lose your money. Be wary of applying linear thinking in probability.*}" (Laplace approach script, para 10)

Misconception LR2: number of possible outcomes in a sequence of n repetitions of a random experiment = n x number of possible outcomes in the random experiment (instead of (number of possible outcomes in the random experiment)^n)

From the Laplace Approach script: "{*Many believe that in order to calculate the number of all possible outcomes they must **add** the individual dice outcomes - not multiply. So they wouldn't even do this…. For example, with three dice rolls they would reason that they'd have 6 outcomes in the first dice roll plus 6 outcomes in the second dice roll plus another 6 outcomes in third dice roll- which adds up to 18 possible outcomes. So if we wanted to calculate the probability of some desired event we would have 18 in the denominator instead of the correct value which is 216. In other words, our probabilities would be very wrong. And betting on our calculations would lose us money.*}" (Laplace approach script, para 13)

Misconception LR3: The number of possible ways in which a sequence of events can occur does not depend on the order of these events in the sequence (arrangement neglect)

From the Laplace Approach script: "{*A common misconception is neglecting to consider the different arrangements in which the dice can land. Calculating the probability of obtaining at least two 6's requires two pieces- we have to know the number of ways to get exactly two 6's and the number of ways to get exactly three 6's. Getting exactly three 6's occurs only in one way, however, calculating the number of ways to get exactly two 6's, if we neglect the number of arrangements that two 6's are obtained in three dice rolls it means we are not considering 5 arrangements and another 5 arrangements so our final probability will only be 1 plus 5 over 216. Which comes out to approximately 0.027 which quite different than 0.074.*}" (Laplace approach script, para 24)

Misconception LR4: p(at least 2 x SIX in k rolls) = $\frac{1}{2}$ ***x p(at least 1 x SIX in k rolls)***

From the Laplace Approach script: {*The other misconception we should address is the following. Remember earlier when we found the probability of obtaining at least one 6 that was equal to 0.42. Here we have doubled the amount of 6's required to win and obtained 0.074. So doubling the amount of 6's does not halve the probability of obtaining at least one 6; it's not equal to 0.21 (half of 0.42) it's actually much less than* *that. So it's important to realize that proportions do not necessarily apply in probability.*} (Laplace approach script, para 25)

4.3.2 List of misconceptions and their refutations in the Properties approach "Refutation" version

Misconception PR0: Mathematics deals only with situations that are fully predictable

From the Properties approach script *{Many people think that mathematics deals only with situations whose outcomes are fully predictable, but probability theory is a proof that mathematics is open to discussing also situations whose outcomes are uncertain.}* (Properties approach script, para 3)

Misconception PR1: p(a SIX in n rolls) = n x p(a SIX in 1 roll)

From the Properties approach script *{Some people believe that this probability is ½; that their chances of winning are fifty-fifty. Here is how they reason. The probability of getting a 6 in a single roll of a fair die is 1/6. This is correct, as we have shown above. But then they think that the probability of getting a 6 in THREE rolls will necessarily be THREE times bigger:* $3 \times \frac{1}{6}$*. Three times one-sixth is three-sixths which is the same as one-half. So they conclude that they have fifty-fifty chances of winning in this game.[] But this reasoning is NOT correct. It is not only mathematically incorrect. It is DANGEROUS to reason that way. In fact, if you were to bet on winning in this game, you would likely lose your money.}* (Properties approach script, para 15)

Misconception PR2: In a sequence of a repeated random experiment (e.g., tossing a coin, or rolling a dice) previous outcomes influence the subsequent ones

From the Properties approach script *{Some people don't believe it but they are wrong. Dice have no memory of past events}* (Properties approach script, para 18)

Misconception PR3: The number of possible ways in which a sequence of events can occur does not depend on the order of these events in the sequence (arrangement neglect)

From the Properties approach script *{A common mistake is to neglect the arrangements in which an event can occur. For example, in order to properly calculate the probability of obtaining exactly two sixes- so these ones [on screen], we require the different arrangements of two sixes in 3 rolls. We may obtain 6 6 not 6, 6 not 6 6, or not 6 6 6. If we forget the possible arrangements of two 6's and one none 6 we are left only with the probability 5/216. We would have neglected these [on screen] arrangements, so they would not be part of the final calculation. So our final probability of obtaining at least two sixes in three rolls would actually be 6/216, which is approximately equal to 0.027. This is very different than 0.074 and obviously quite wrong}* (Properties approach script, para 26)

Misconception PR4: p(at least 2 x SIX in k rolls) = $\frac{1}{2}$ ***x p(at least 1 x SIX in k rolls)***

From the Properties approach script *{Some people believe that the probability of getting at least two 6's in three rolls should be two times smaller than the probability of getting at least one six in three rolls. But this is incorrect.}* (Properties approach script, para 27)

4.4 QUESTIONS TO TEST STUDENTS' UNDERSTANDING OF PROBABILITY AFTER THE TREATMENTS

4.4.1 Overview of the test questions

All questions, except Question 2, are related to Bernoulli trials with the success being a certain single (elementary) outcome O in a roll of a fair dice i.e., its probability $= \frac{1}{6}$. The questions ask to compare probabilities of getting at least (or exactly) k successes in n trials. Questions 1, 3 and 6 are about getting at least k successes, with k constant and n varying (Question 1); k varying and n constant (Question 3), and both k and n varying (Question 6). In Question 4, both k and n are constant and the probabilities of "at least k" and "exactly k" successes are to be compared. In each of these questions, the "success" outcome O is the same for both probabilities. If X_n denotes the number of successes in n such Bernoulli trials, and $\mathrm{P}(X_n \geq k)$ denotes the probability of at least k successes in n trials, then

Question 1 asks, which is larger: $\mathrm{P}(X_4 \geq 2)$ or $\mathrm{P}(X_5 \geq 2)$? The success is the outcome "3".

Question 3 asks, is it true that $\mathrm{P}(X_5 \geq 4) = \frac{1}{2} \times \mathrm{P}(X_5 \geq 2)$? The success is the outcome "5".

Question 6 asks, is it true that $\mathrm{P}(X_6 \geq 2) = \mathrm{P}(X_3 \geq 1)$? The success is the outcome "5".

Question 4 asks, which is larger, $\mathrm{P}(X_3 \geq 1)$ or $\mathrm{P}(X_3) = 1)$? The success is the outcome "2".

Question 5 is still about Bernoulli trials using a fair dice, but it asks to compare Bernoulli trials related to outcomes with different probabilities, $\frac{1}{2}$ and $\frac{1}{6}$. In one set of these Bernoulli trials, the success is an even number; its probability is $\frac{3}{6}$ or ½ . In the other – it is the single outcome "5". If Y_n denotes the number of times an even number is obtained in n trials, and X_n is the number of times a 5 is obtained, then Question 5 asks if it is true that $\mathrm{P}(Y_3 \geq 1) = 3 \times \mathrm{P}(X_3 \geq 1)$

Question 2 was not related to Bernoulli trials, but to the simultaneous rolling of two dice. It asked to compare the probability of getting a 5 and a 6 on the dice with that of getting two 6's.

All questions except question 2 addressed the issue of the "illusion of linearity" in probability. The role of this question was partly as a distractor; its other purposes will be explained in section 4.3.3.

4.4.2 Question 1 – Qualitative Comparison of Probabilities in two Bernoulli trials differing by the Number of Trials

4.4.2.1 Question Statement

I roll a fair die several times. The chance to have at least two times a three if I can roll four times is *(larger than, smaller than, equal to)* the chance to have at least two times a three if I can roll five times. *Select an answer in parentheses.* Show you reasoning. Justify your answer.

4.4.2.2 Mathematical representation of question statement

X_n:= number of 3's in n rolls of a fair dice

Is $P(X_4 \geq 2)$ greater, equal or smaller than $P(X_5 \geq 2)$

4.4.2.3 Aims of Question 1

The aim of this question was to see if participants' have an intuitive understanding of the qualitative proportionality rule.

Research suggests that, in general, I students have a good intuitive understanding of the "qualitative proportionality rule" (Van Dooren et al., 2003):

$$more\ rolls \Rightarrow higher\ probability$$

The justifications and solutions given by the participants can be mathematical, as taught in their respective lessons, or may be explanatory and qualitative. The expected solutions are detailed below. The purpose was to observe if participants engage in justifications for their answers to a question whose answer can be reasoned easily without the use of math and if they do, do they use anything from the different lesson treatments they listened to.

4.4.2.4 Expected Correct Solutions

Based on Van Dooren et al.'s research, participants are expected to easily deduce the answer intuitively (Van Dooren et al., 2003), arguing, for example, that one extra roll allows for more possibilities of achieving at least two 3's. Otherwise, students may draw one or two tree diagrams with branches ending in "3" and "not 3" (variations possible). The way they might use these diagrams would presumably vary between the two approaches.

4.4.2.4.1 Laplace Approach

Branches representing independent outcomes are not marked with the probabilities $\frac{1}{6}$ or $\frac{5}{6}$ in the Laplacean approach (this is done in the Properties approach), because probabilities of events are obtained by counting the total number of ways the event can occur and dividing it by the total number of outcomes. The trees are noticeably large, so some participants may start drawing the tree and then give up and look for a different way of thinking about the problem. They may opt for the intuitive approach mentioned above.

Students could also imagine the tree instead of drawing it in order to compute the probabilities. Imagining the bottom layer of the tree, they could conclude that the total number of outcomes in the case of rolling the dice four times is 6^4, and 6^5 in the case of rolling the dice 5 times. They could then visualize the branches with 2 (then 3, and then 4) 3's and 2, 1 and 0 non-3's respectively in the case of 4 rolls and determine the total number of arrangements of 3's and non-3's among the 4 rolls. The same procedure would be applied to the case of 5 rolls. A correct calculation should yield $P(at\ least\ 2\ threes\ in\ 4\ rolls) = \frac{171}{1296} \sim 0.1319$ and $P(at\ least\ 2\ threes\ in\ 5\ rolls) = \frac{1528}{7776} \sim 0.1962$ and thus one can conclude that obtaining at least two 3's is more likely if given 5 rolls.

$$P\left(at\ least\ two\ 3'\ s\ in\ 4\ rolls\right) = \frac{\#\ of\ ways\ to\ get\ at\ least\ 2\ \ 3'\ s\ in\ 4\ rolls}{\#\ of\ all\ possible\ outcomes}$$

$$= \frac{\#\ of\ ways\ to\ get\ 2\ \ 3'\ s + \#\ of\ of\ ways\ to\ get\ 3\ \ 3'\ s + \#\ of\ ways\ to\ get\ 4\ \ 3'\ s}{\#\ of\ all\ possible\ outcomes}$$

$$= \frac{6(5 \times 5 \times 1 \times 1) + 4(5 \times 1 \times 1 \times 1) + (1 \times\ 1 \times 1 \times 1)}{6 \times 6 \times 6 \times 6} \sim \mathbf{0.1319}$$

$$P\left(at\ least\ two\ 3'\ s\ in\ 5\ rolls\right) = \frac{\#\ of\ ways\ to\ get\ at\ least\ 2\ \ 3'\ s\ in\ 5\ rolls}{number\ of\ all\ possible\ outcomes}$$

$$= \frac{\#\ of\ ways\ to\ get\ two\ 3'\ s + \#\ of\ ways\ to\ get\ three\ 3'\ s + \#\ of\ ways\ to\ get\ four\ 3'\ s + \#\ of\ ways\ to\ get\ five\ 3'\ s}{6 \times 6 \times 6 \times 6 \times 6}$$

$$= \frac{10(5 \times 5 \times 5) + 10(5 \times 5) + 5(5) + 1(1)}{7776} = \mathbf{0.1962}$$

$$\therefore P\left(at\ least\ two\ 3\ '\ \ s\ in\ 5\ rolls\right) > P\left(at\ least\ two\ 3\ '\ \ s\ in\ 4\ rolls\right)$$

4.4.2.4.2 Properties Approach

Participants in the Properties approach, if they use tree diagrams at all, may produce two tree diagrams of 4 rolls and 5 rolls with branches representing the mutually exclusive outcomes of obtaining a 3 or a non-3 respectively marked with $p = \frac{1}{6}\ or\ \frac{5}{6}$. The mathematics should resemble the following, wherein the multiplicative property is used for the conjunction of outcomes and the additive property is used for the alternative outcomes:

$$P(at\ least\ two\ 3s\ in\ 4\ rolls)$$
$$= P\left(two\ 3'\ s\ OR\ three\ 3'\ s\ OR\ four\ 3'\ s\right)$$
$$= P\left(two\ 3'\ s\ AND\ two\ non\ 3\ '\ s\right) + P\left(three\ 3'\ s\ AND\ one\ non\ 3\right) + P\left(four\ 3'\ s\right)$$
$$= 6\left(\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\right) + 4\left(\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\right) + \left(\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\right) \sim \mathbf{0.1319}$$

$$P(at\ least\ two\ 3s\ in\ 5\ rolls)$$
$$= P\left(two\ 3'\ s\ OR\ three\ 3'\ s\ OR\ four\ 3'\ s\ OR\ five\ 3\ '\ s\right)$$
$$= P\left(two\ 3'\ s\ AND\ three\ non\ 3\ '\ s\right) + P\left(three\ 3'\ s\ AND\ two\ non\ 3\ '\ s\right) + P\left(four\ 3'\ s\ AND\ one\ non\ 3\right) + P(five\ 3'\ s)$$
$$= 10\left(\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\right) + 10\left(\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\right) + 5\left(\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\right) + \left(\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\right)$$
$$= \mathbf{0.1962}$$

$$\therefore P\left(at\ least\ two\ 3\ '\ s\ in\ 5\ rolls\right) > P\left(at\ least\ two\ 3\ '\ s\ in\ 4\ rolls\right)$$

4.4.2.5 Anticipated participants' errors

Since this question does not target any misconceptions and can be qualitatively assessed, no errors are expected. If the participant chooses to justify their solution using the material taught during the lesson then computation errors are plausible.

For example, since question 1 is the first question for which individuals will be applying what they've learned, it is anticipated that minor errors will be made such as missing a relevant arrangement of the dice or a calculation error that yields the incorrect numerical answer yet satisfies their qualitative intuition.

4.4.3 Question 2 – Qualitative comparison of probabilities of events in a simultaneous roll of two dice

4.4.3.1 Question Statement

Suppose one rolls two dice simultaneously. Which of the following has a greater chance of happening? Show your reasoning. Justify your answer. (a) Getting the pair 5-6. (b) Getting the pair 6-6 (c) Both have the same chance.

4.4.3.2 Mathematical representation of the question statement

X: Event where the pair (5, 6) is obtained with two dice rolled simultaneously.

Y: Event where the pair (6, 6) is obtained with two dice rolled simultaneously.

Is $P(X)\ larger\ than, equal\ to\ or\ smaller\ than\ P(Y)$?

4.4.3.3 Aims of Question 2

Question 2 differs from the other questions and lesson examples. The other questions and lesson examples are all about comparing probabilities of events that are conjunctions of events in Bernoulli trials, i.e., probabilities of the form *P*(at least *k* successes in *n* trials). Question 2 not only veers from the Bernoulli schema, but also does not attempt to inspect the application of mathematical linear reasoning.

The aim of this question was to test participants' attention to the different arrangements in which an outcome can occur, i.e., whether they have overcome the misconception LR3 (or PR3). Do they take into account the different arrangements in a situation that is not like the situations in the lessons? In other words, is their sensitivity to arrangements stable?

Research suggests that students tend to believe that rolling the pair (5, 6) in one roll of two dice is probabilistically equal to rolling the pair (6, 6). Some students believe that rolling the pair (5, 6) is more likely, which is what is expected, but their reasoning is arbitrary and can be traced to a form of availability heuristic wherein students implicitly and automatically associate the times they remember waiting for the often favored pair (6, 6) with its low probability (Fischbein & Schnarch, 1997).

There are a number of possible explanations as to why students tend to believe that any two pairs (a, b) or (a, a) of numbers are equally likely. One possible explanation includes a neglect of outcome arrangements due to subjective indistinguishability of the two dice. In other words, the student does not recognize a physical difference between the dice and so the dice are treated as identical (the Bose-Einstein model). If true, it may be possible to correct such misconception using visually different objects to eliminate subjective indistinguishability. An experiment can easily be organized to test this hypothesis.

The inclusion of question 2 in the study was to observe the effects of Laplace and Properties approach on the tendency to believe the outcomes are equal. It was thought that the use of tree diagrams would allow the students to overcome subjective indistinguishability by explicitly drawing out each event.

4.4.3.4 Expected correct solutions

In spite of the assumption that the dice were rolled simultaneously, the question is somewhat ambiguous. The symbol "(5, 6)" could be interpreted as an ordered pair as well as a set, and the question does not say whether the dice were distinguishable or not. So the most desired response to the question would be to discuss what would be the relation between the probabilities in different cases, for example, dice distinguishable or indistinguishable; outcomes coded as ordered pairs or sets.

If the dice were distinguishable, for example, one of the dice was red and the other – black, and the outcomes were coded as ordered pairs with the number on the red dice always recorded as first, and the number on the black one – as the second number, then the symbol "(5, 6)" means "5 on red and 6 on black" and the event X is made of one out of 36 different possible outcomes. If, on the other hand, the symbol "(5, 6)" is interpreted as a set then the event X is "5 on red and 6 on black or 6 on red and 5 on black" and so it is made of 2 of the 36 possible outcomes.

If, on the other hand, the dice were indistinguishable to the eye but would indeed be two independently existing objects then an event such as X, with two different numbers on the dice, would be possible. If they were indistinguishable not just to the human eye but "objectively" then they would be identical, which, from a certain philosophical point of view, could mean that they would behave in exactly the same way. In this case the outcome (5, 6) would be impossible no matter whether the symbol refers to an ordered pair or a set. In the case where the dice are indistinguishable but independent (so they are "objectively" distinguishable, or "distinguishable by God Almighty") , then, when we repeatedly roll the dice simultaneously, then getting a 5 on one and 6 on the other dice is a different outcome than 6 on the one and 5 on the other. So, again, the interpretation of the event X depends on whether "(5, 6)" represents an ordered pair or a set.

In the sections below we present possible reasoning only in the case of *dice distinguishable, "(5, 6)" represents a set*, for the two approaches.

4.4.3.4.1 Laplace Approach

Suppose we can distinguish the two dice (e.g., dice 1 and dice 2), and "(5, 6)", represents a set so that the outcome "5 on dice 1, 6 on dice 2" ($X1$) is distinguishable from "6 on dice 1, 5 on dice 2" ($X2$). Then we can represent the desired outcomes of the roll of the two dice simultaneously on a two-layered tree (Figure 6).

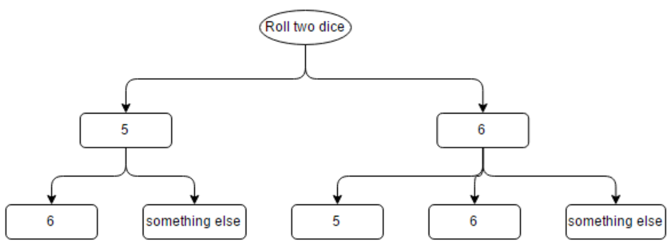


Figure 6. Tree diagram for Question 2 in the Laplace approach.

The Laplace approach yields:

	Number of ways to achieve desired outcome	Total number of outcomes	Probability
Pair (5,6)	$1+1=2$	$6 \times 6 = 36$	$\frac{2}{36}$
Pair (6,6)	1	$6 \times 6 = 36$	$\frac{1}{36}$

Thus concluding that the pair (5, 6) is more likely to occur than the pair (6, 6).

4.4.3.4.2 Properties Approach

In this approach, the tree diagram is supplemented with the probabilities of the outcomes, marked on the branches (Figure 7).

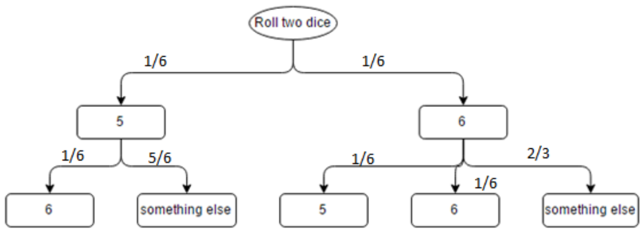


Figure 7. Tree diagram for Question 2 in the Properties approach.

Then we derive the probabilities of the events using the multiplicative property for conjunctions of outcomes (multiplying the probabilities along the branches) and the additive property for alternatives (adding the probabilities along a level).

Using the multiplicative property of probability, the probabilities $P(X1)$ and $P(X2)$ are both equal to $\frac{1}{36}$: $P(X1) = P(X2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.
Since the event X is an alternative of the two events $X1$ and $X2$: $X = X1\ OR\ X2$, then, by the additive property of probability, $P(X) = P(X1) + P(X2) = \frac{1}{36} + \frac{1}{36} = \frac{1}{18}$.
The event Y is the conjunction of events: 6 on one dice AND 6 on the other, both with probability $\frac{1}{6}$ so $P(Y) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. Therefore $P(X) > P(Y)$.

4.4.3.5 Anticipated participants' errors

Since this question targets participants' sensitivity to arrangements while assessing probabilities, the main anticipated error was that the second arrangement of the outcome (5, 6) would be neglected.

As explained in *Expected correct solutions,* the question can be interpreted as rolling the *ordered* pairs (5, 6) or (6, 6). However, if the question is interpreted as such, it indicates that the individual does not recognize the effects and importance of considering the possible orders for the probability; this is at the heart of the Fischbein and Schnarch's misconception. If interpreted 'correctly', it means the individual has the critical capacity to consider possible interpretations of the question and will therefore have the capacity to consider that (5, 6) can occur in two ways.

One who misinterprets the question likely believes that (5, 6) is equally likely to come up as (6, 6) when physically tossing two dice. The awareness that arrangements have a possible effect on chance need not be on their radar.

Refutation participants were exposed to this common error and should therefore succeed in the correct interpretation and calculation. Since Question 2 is of a different form than the previous material they've seen within this experiment it is possible that producing a tree diagram will not be immediately obvious; without the tree, which is a good tool for observing the possible sequences of events in this case, the second arrangement of (5, 6) may be missed.

For this reason, Properties participants may struggle to consider the second arrangement as well. The focus of their lesson was on the multiplicative and additive rules and the tree diagram may be perceived as a secondary, perhaps, optional tool. On the other hand, the Laplace approach places the tree diagram as an integral part of the practice.

4.4.4 Question 3 – Quantitative Comparison of Probabilities in Two Bernoulli trials Differing by the Number of Successes

4.4.4.1 Question Statement

I roll a fair dice several times. The chance to have at least four times a five if I roll five times is half as large as the chance to have at least two times a five if I roll five times. (*This is true, this is not true*). *Select an answer in parentheses.* Show you reasoning. Justify your answer.

4.4.4.2 *Mathematical representation of exercise statement*

X_n:= number of 5's in n rolls of a fair dice

True or false? $\mathrm{P}(X_5 \geq 4) = \frac{1}{2}\mathrm{P}(X_5 \geq 2)$

4.4.4.3 Aims of Question 3

The aim of Question 3 was to test participants' resistance to the "illusion of linearity" misconception in the particular case of a quantitative comparison of probabilities in Bernoulli trials that differed only by the number of successes, the number of trials being the same. We wanted to see if the approach (LA or PA) and treatment (Exposition or Refutation) a participant was subject to would have any bearing on their reasoning when engaged, this time, in a *quantitative* comparison of two probabilistic scenarios. In Question 1, the required comparison was qualitative, and, in principle, participants did not have to calculate the probabilities. This time, they had to and so it was expected that they will use some tools provided in the lessons to do the calculations.

4.4.4.4 Expected Correct Solutions

A mathematical argument was expected for all students. The Refutation students were expected to recall the refuted misconception and properly carry out the calculation with confidence.

4.4.4.4.1 Laplace Approach

Actual or imagined tree diagram with branches 5 and not 5 with 5 levels (for 5 rolls) can be used to determine the number of desired outcomes. The possible calculations are presented in the table below, but this does not mean that participants are expected to draw such table in their written work.

$$P(at\ least\ four\ 5's\ in\ 5\ rolls) = \frac{\#\ desired\ outcomes}{\#\ total\ outcomes} = \frac{\#\ ways\ for\ four\ 5's + \#ways\ for\ all\ 5's}{\#\ total\ outcomes}$$

$$= \frac{5(1 \times 1 \times 1 \times 1 \times 5) + (1 \times 1 \times 1 \times 1 \times 1)}{6 \times 6 \times 6 \times 6 \times 6}$$

$$P(at\ least\ four\ 5's\ in\ 5\ rolls) = = \frac{26}{7776} = 0.00334$$

$$P(at\ least\ two\ 5's\ in\ 5\ rolls) = \frac{\#\ desired\ outcomes}{\#\ total\ outcomes}$$

$$= \frac{\#\ ways\ for\ two\ 5's + \#\ ways\ for\ three\ 5's + \#\ ways\ for\ four\ 5's + \#ways\ for\ all\ 5's}{\#\ total\ outcomes}$$

$$= \frac{10(5 \times 5 \times 5 \times 1 \times 1) + 10(5 \times 5 \times 1 \times 1 \times 1) + 5(5 \times 1 \times 1 \times 1 \times 1) + (1)}{6 \times 6 \times 6 \times 6 \times 6}$$

$$P(at\ least\ two\ 5's\ in\ 5\ rolls) = = \frac{1526}{7776} = 0.96$$

$$\therefore P(at\ least\ four\ 5's\ in\ 5\ rolls) \neq \frac{1}{2} P(at\ least\ two\ 5's\ in\ 5\ rolls)$$

$$\therefore This\ statement\ is\ Not\ True.$$

4.4.4.4.2 Properties Approach

A tree can be drawn or imagined as in the Laplace Approach, but with branches marked with probabilities. Calculations could look as shown below, with oral references to the multiplicative and additive properties in the interviews.

$$P(at\ least\ four\ 5's\ in\ five\ 5\ rolls)$$

$$= P(four\ 5's\ OR\ five\ 5's\ )$$

$$= P(four\ 5's\ AND\ one\ non\ 5) + P(five\ 5's\ )$$

$$= 5\left(\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\ \right) + \left(\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\right)$$

$$P(at\ least\ four\ 5's\ in\ five\ 5\ rolls) = 0.00334$$

$$P(at\ least\ two\ 5's\ in\ five\ 5\ rolls)$$

$$= 1 - (\ P(no\ 5's\ OR\ one\ 5))$$

$$= 1 - (P(no\ 5's) + P(one\ 5))$$

$$= 1 - (P(no\ 5's\ ) + P(one\ 5\ AND\ four\ non\ 5's))$$

$$= 1 - \left(\left(\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\right) + 5\left(\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\right)\right)$$

$$P(at\ least\ two\ 5's\ in\ five\ 5\ rolls) = 0.196$$

$$\frac{1}{2}P(at\ least\ two\ 5's\ in\ five\ 5\ rolls) \neq P(at\ least\ four\ 5's\ in\ five\ 5\ rolls)$$

$$\therefore This\ statement\ is\ Not\ True.$$

4.4.4.5 Anticipated participants' errors

According to research on difficulties in probability (Van Dooren, 2003; Fischbein and Gazit, 1984; Fischbein and Schnarch, 1997), although students have good qualitative insight into probabilities of events in Bernoulli trials, their ability to quantify them is not as good. Even students with formal training in binomial probability and tree diagrams have a strong tendency to oversimplify and generalize binomial comparisons using proportionality models which are thought to have their roots in cognitive proportionality schemas and assumed equiprobability models

For this particular question, the “quantitative *k* item”, Van Dooren et al. (2003) report that students with formal training in probability were correct at an approximate rate of 38%: 20% better than students with no training.

In most cases in Van Dooren et al.’s study, students relied on the salient features of the problem statement and incorrectly applied a linear relation to conclude and justify their answers. With the implicit assumption that the outcomes of dice rolling are equiprobable, students incorrectly reasoned that the occurrence of a particular face of the dice *k* times in *n* rolls is twice as likely as its occurrence *2k* times in the same number *n* of rolls:

$$P(X_5 \geq 4) = \frac{1}{2} \times \ P(X_5 \geq 2)$$

The qualitative relation of this statement is intuitive in that it can be easily concluded that obtaining four 5’s is less likely than obtaining two fives. So upon reading, the statement seems plausible and the intuition will easily believe this to be true. However, it is the immediate cognitive action of applying qualitative *and* quantitative proportionality schemas to the salient numbers that tempts students to be vindicated in their intuitive urge and conclude that:

$$\times\, 2 \ the\ amount\ of\ desired\ events\ required\ to\ "win"$$
$$Qualitative\ reasoning\ says\ obtaining\ 4\ of\ the\ same\ event\ must\ be\ less\ likely$$
$$\therefore \frac{1}{2} P(2\ equally\ possible\ events\ in\ 5\ rolls) = P(4\ equally\ possible\ events\ in\ 5\ rolls)$$

In terms of possible errors, it was expected, in the present study, that Exposition participants would follow this type of erroneous justifications. Van Dooren et al. (2003) believed that proper education in the use of tree diagrams would have a positive effect. However it was believed that a student attempting to build a tree diagram would have been deterred by the number of rolls (the number of levels) the tree would require and hence would settle for other means of justification including being tempted by proportionality models. It was thought that without being warned against the linearity misconception, the Exposition participants would be more likely to consider the statement in the question to be true. Those treated with the refutation lesson are more likely to understand and implement the concept of non-linear change between variables.

Inadequate justifications are possible from all participants if they realize the form of Question 3 is similar to the pair of problems presented in the lesson. In Problem 2, $\Pr(X_3 \geq 2)$ was calculated with X_3 being the number of 6's in 3 rolls of a fair dice; in Problem 1, $\Pr(X_3 \geq 1)$ was calculated: number of rolls constant; number of successes variable. But only in the Refutation treatment was it pointed out that $\Pr(X_3 \geq 2) \neq \frac{1}{2}\Pr(X_3 \geq 1)$. So Refutation participants were not expected to fall into the trap of proportional reasoning in this question.

Though the students in Van Dooren et al. (2003) exhibited "quantitative negligence", in that their mathematical justifications effectively considered the probabilities $\Pr(X_n = k)$and not $\Pr(X_n \geq k)$ the participants in this study were expected to not to make this mistake. The video lessons gave examples of calculating binomial inequalities, so participants did not have to derive new methods to solve the test questions. The lesson notes that in order to correctly calculate $P(at\ least\ k\ successes\ in\ n\ rolls)$ one must be sure to consider *k, k+1,…,n* successes. So if students were to engage in proportional models, it was not certain whether they would also neglect to calculate probabilities of events with numbers of successes larger than *k.*

Concerning mathematical justification and calculation, Exposition participants were expected to arrive at the correct answer but were deemed more likely to neglect some of the possible arrangements of outcomes in the desired event (Misconceptions LR3 and PR3). Therefore, incorrect values could be quoted in their solutions despite concluding the statement was not true.

4.4.5 Question 4 – Qualitative Comparison of Probabilities of "at least $\boldsymbol{k}$" and "exactly $\boldsymbol{k}$" successes in Bernoulli trials

4.4.5.1 Question Statement

> I roll a fair dice several times. The chances to have *at least* once a two if I roll 3 times is (*larger than, smaller than, equal to*) the chance to have *exactly* once a two if I roll 3 times. *Select an answer in parentheses.* Show your reasoning. Justify your answer.

4.4.5.2 Mathematical representation of the question statement

X_n:= number of 2's in n rolls of a fair dice

Is $P(\boldsymbol{X_3} \geq \boldsymbol{1})\ more\ likely, equal\ to\ or\ less\ likely\ than\ P(\boldsymbol{X_3} = \boldsymbol{1})$?

4.4.5.3 Aims of Question 4

The aim of this question was to draw participants' attention to the difference between getting a certain *exact* number of successes and getting *at least* that number of successes. The lessons did not stress this difference (not even the Refutation treatments) but the worked out examples insisted on interpreting "at least k successes" as k, or $k+1$, …, or n successes. If, however, the participants ignored this difference (were "quantitatively negligent", in Van Dooren et al.'s terms) in previous questions, Question 4 gives them a chance to realize this mistake and go back to correct their responses if necessary.

In comparing these two Bernoulli trials, the participant will likely qualitatively evaluate the probability of "*at least k successes*" which requires an assessment of *P(k successes), P(k+1 successes)* up to *P(k=n successes),* which can be completed before any calculations are made, if they decide to do so.

The question is, will the participants have an intuitive grasp of the union of these events *P(k successes), P(k+1 successes), etcetera,* through there qualitative evaluation?

This specific question has not been posed by past research but it is a variation of Van Dooren et al. (2003). The answer is evident upon consideration of the mathematical formulation, however the participants of this study were not expected to construct the formal statement.

4.4.5.4 Expected Correct Solutions

4.4.5.4.1 Laplace Approach

A demonstrated conceptual understanding of the statement is enough to be deemed correct. For example, saying that the number of desirable outcomes in "at least 1 'two' in 3 rolls" includes the number of desirable outcomes in "exactly 1 'two' in 3 rolls", so the former must be a larger number than the latter. Since the number of all possible outcomes is the same in both cases, the probability of the former must be larger than the former.

4.4.5.4.2 Properties Approach

No calculation was expected in the Properties Approach, either, but the argument would be different. The event "at least 1 'two' in 3 rolls" is the same as "exactly 1 'two' OR exactly 2 'twos' OR exactly 3 'twos' in 3 rolls", so, by the additive property, its probability is a sum of three non-zero numbers, one of which is the probability of exactly 1 ‘two’ in 3 rolls. Therefore the probability of at least 1 ‘two’ is larger than the probability of exactly 1 ‘two’.

4.4.5.5 Anticipated participants’ errors

If this question is evaluated using the respective lesson material, then the solution and justification should only see a few computation errors. However, if evaluated exclusively qualitatively, then errors may emerge as a consequence of improper intuitive algorithms. For example, in qualitatively considering $P(k\ successes), P(k+1\ successes)$ up to $P(k=n\ successes)$, the participant may struggle with the application of the correct operation to estimate the odds of $P(X_3 \geq 1)$. Will they know intuitively that $P(X_3 \geq 1)$ can be qualitatively estimated by *P(k successes)+ P(k+1 successes)…+…P(k=n successes),* or might they implicitly believe that $P(X_3 \geq 1)$ is estimated by $P(k\ successes) \times P(k+1\ successes) \ldots \times \ldots P(k=n\ successes)$.

Such an error is not expected in their final calculation since the Properties approach and the Laplace approach are easily applied, but one-on-one interviews may reveal any potential struggles the participants had while deliberating. Perhaps they may believe that the higher *k* successes, such as *k+1…k=n* is so unlikely that it is negligible so that $P(X_3 \geq 1) \sim P(X_3 = 1)$.

4.4.6 Question 5 - Quantitative Comparison of Probabilities in two Bernoulli trials differing by the probability of the success

4.4.6.1 Question Statement

> I roll a fair dice several times. The chance to have at least once an even number if I roll three times is three times as large as the chance to have at least once a five if I roll three times. (*This is true, this is not true*). *Select an answer in parentheses*. Show your reasoning. Justify your answer.

4.4.6.2 Mathematical representation of the question statement

X_n:= number of 5’s in n rolls of a fair dice

Y_n:= number of even numbers {2, 4, 6} in n rolls of a fair dice

Is $3 \times P(X_3 \geq 1) = P(Y_3 \geq 1)$?

4.4.6.3 Aims of Question 5

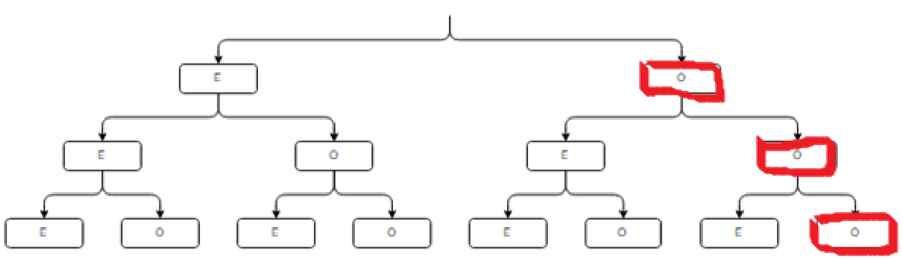
The aim of this question was to see how student's reasoning varies across lesson treatments when engaged in a quantitative comparison of two probabilistic scenarios that differ only by the probability *p* while holding the required number of successes *k* and the number of rolls *n* constant. The material up to this point in the experiment has held the probability of success *p* as constant throughout the problems ($p = \frac{1}{6}$), so participants were not primed to think about comparing scenarios with $p = \frac{3}{6}\ or\ \frac{1}{2}$ to $p = \frac{1}{6}$. This question is, therefore, a better concealed 'trap' for the application of linearity.

4.4.6.4 Expected Correct Solutions

4.4.6.4.1 Laplace Approach

In the lessons, Problem 1 consisted in calculating the probability of getting at least 1 'six' in 3 rolls of a fair dice. The probability was about 0.42. Participants were expected to realize that the probability of getting at least 1 "five" in 3 rolls is the same and not re-calculate it in solving Question 5. They were expected to work on the calculation of the probability of getting at least 1 even number in 3 rolls (and not jump to the erroneous conclusion that it must be 3×0.42, by proportionality). Here is how they could reason, using the Laplace approach, and thinking of there being 6 possible outcomes in rolling a dice, in symmetry to Problem 1 in the lesson.

In rolling a fair dice, 6 equally probable outcomes are possible, three of which are even numbers, and three are odd numbers. A tree diagram is easily producible to represent the possible outcomes in 3 rolls of the dice. For example, see Figure 8.



3x3x3 + 3x3x3 + 3x3x3 + 3x3x3 + 3x3x3 + 3x3x3 + 3x3x3 + 3x3x3

216 possible outcomes in total

P(at least 1 even in 3 rolls)= 1 - 27/216 = 189/216 = 0.875

Figure 8. Tree diagram for Question 5 in the Laplace approach

Noticing that 0.875 is not equal to 3 times 0.42, the answer would be “not true.”

4.4.6.4.2 Properties Approach

Assuming that the dice is fair, we conclude, as in the lesson, that all outcomes are equally probable, i.e., each is equal to 1/6. Getting an even number means getting 2 OR 4 OR 6, so, by the additive property of probability, $P(even\ number) = P(2) + P(4) + P(6) = 3 \times \frac{1}{6} = \frac{1}{2}$. Similarly, or by the complementarity property, $P(odd\ number) = \frac{1}{2}$. The experiment of rolling a fair dice 3 times and obtaining even or odd number can be represented by a tree diagram similar to the one in the Laplace approach, except that now the branches would be marked with probabilities (Figure 9).

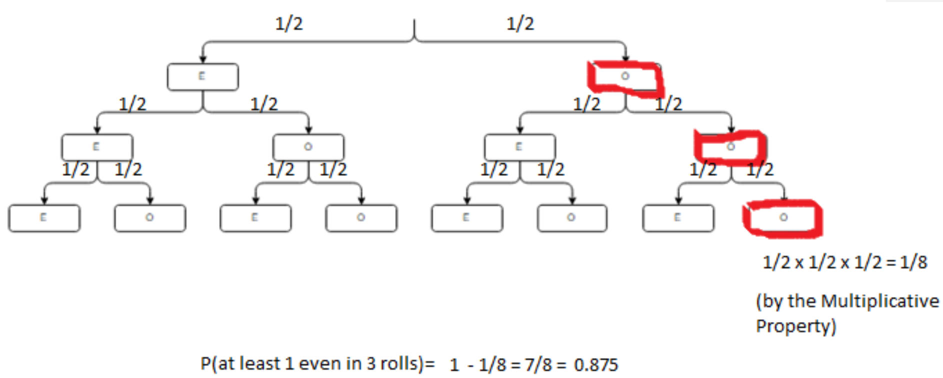


Figure 9. Tree diagram for Question 5 in the Properties approach.

From this, it would be concluded that since 0.875 is not equal to 3 times 0.42, the statement is false.

4.4.6.5 Anticipated participants’ errors

For this particular question, the ‘quantitative *p* item’ from Van Dooren et al. (2003), students with formal training in chance were correct at an approximate rate of 16.1%- only 0.2% better than student’s with no training.

In most cases, students relied on the salient features of the problem statement and incorrectly applied a linear relation to justify and conclude their answers. Here is the common erroneous reasoning:

$$p(obtaining\ an\ even\ number) = \frac{3}{6}$$

and

$p(obtaining\ a\ 2) = \frac{1}{6}$

$p(obtaining\ an\ even\ number) = 3 \times p(obtaining\ a\ 2)$

The above is true, but the erroneous reasoning in the form of proportionality models is extended to the following:

$$P(at\ least\ one\ even\ number\ in\ 3\ rolls) = 3 \times P(at\ least\ one\ 2\ in\ 3\ rolls),$$

since it can be qualitatively understood that obtaining at least one even number must have a higher probability than obtaining at least once a 2. The proportionality schema then decides the factor by which this statement is true (that factor being ‘3’).

Since participants were not explicitly taught or warned about the use of linearity in changing the probability of a single success two possible avenues of justification were possible: qualitative or quantitative with errors occurring in either.

A qualitative justification will more or less forgo the lesson and resemble the justification presented above. It is possible that during the interview the participant will reveal that they were detracted by the temptation to pursue linearity.

Otherwise, Laplace participants may carry through with the production of a tree diagram. The instructional material did not examine the method required to produce trees that were not asymmetrical (in the lesson the tree branches had weights of 1 desired outcome and 5 undesired outcomes; this new tree will have branches 3 desired outcomes or 1 desired even outcome). It is thought that this should not be an issue since symmetrical trees are easier to manage. However, should they chose to calculate the probability using the units of Even and Odd, then they must recognize how their sample space and the total number of outcomes has changed.

Properties participants are subject to the same errors as Laplace, except, as mentioned, these individual may view the tree diagram as secondary to the rules meaning their focus on integrating a novel scenario with the multiplicative and additive rules may encourage them to miss critical arrangements (given they forgo the tree).

4.4.7 Question 6 – Quantitative Comparison of Probabilities in two Bernoulli trials with Different Numbers of Trials and Successes

4.4.7.1 Question Statement

I roll a fair die several times. The chance to have at least two times a five if I can roll six times is equal to the chance to have at least once a five if I can roll three times. (*This is true, this is not true*). *Select an answer in parentheses*. Show your reasoning. Justify your answer.

4.4.7.2 *Mathematical representation of the question statement*

X_n:= number of 5's in n rolls of a fair dice

Is it true that $P(X_3 \geq 1)\ to\ P(X_6 \geq 2)$?

4.4.7.3 Aims of Question 6

Similarly to Question 5, the relation between the variables in question has not been seen by the participants. Moreover, because the question asks to compare scenarios within which k becomes $2k$ simultaneously to n becoming $2n$, a qualitative justification and inclination to solving the problem is more difficult than the previous questions.

As Van Dooren et al. indicate, students have good qualitative insight that:

(1) Increasing the amount of successes required to 'win' decreases the odds *P* and,
(2) Increasing the amount of trials increases the odds *P*.

Doing both simultaneously renders it difficult to conceptualize how the mathematical justification will turn out.

Now, comparing these two probabilistic scenarios hypothetically uses the same procedures as taught in the respective lessons however devising the relevant arrangements of 6 independent dice rolls is not straight forward nor is it computationally easy when one is not instructed on combinatorics. Thus, it was not expected that participants would carry through the proper algorithms.

The objective of this final question was to ascertain whether participants could reason the non-linear change that occurs between variables by experience of the previous questions. In that sense, the final answers may be qualitative, meaning that the participants may develop a hypothesis which can demonstrate to the researcher their acquired understanding of non-linearity and thus conclude (with low confidence) that obtaining at least one 5 in 3 rolls is not equal to obtaining two 5's in 6 rolls.

4.4.7.4 Expected Correct Solutions

In Problem 1 of the video lesson, the probability of getting at least 1 six in 3 rolls was calculated yielding about 0.42. The same probability appears in Question 6; participants are expected to realize that getting a five rather than a six makes no difference, and expected to not recalculate the probability, but use the value already obtained in Problem 1 in the lesson. The only calculation to be done in this question is the probability of getting at least 2 fives in 6 rolls (which is about 0.26). Participants cannot deduce this probability from the probability of getting at least 2 sixes in 3 rolls which was calculated in Problem 2 of the lesson, although they may be tempted to, using proportionality models. In the sections below, we show only the expected correct calculations of the probability of getting at least 2 fives in 6 rolls in the two approaches. The conclusion – e.g., "not true that the probabilities are equal because 0.42 ≠ 0.26" – is the same in both approaches and we shall not repeat it in the sections.

Generally, participants are expected to be deterred by the magnitude of the problem which may discourage them from producing calculations for a Bernoulli trial with *n=6* rolls. If this is the case, participants may attempt a qualitative justification explaining how rolling 6 times dilutes that probability *P* of obtaining at least two successes faster than rolling 3 times to obtain at least one success. They may reference other questions they've completed as they may have made observations on the non-linear effect *n ,k and p* have on the probability of the desired event. Justifications of this nature will likely be insufficient, but the interview will accompany their results to ascertain their understanding. Otherwise, the following calculations are expected.

4.4.7.4.1 Calculating the probability of getting at least 2 fives in 6 rolls using the Laplace Approach

If a calculation is used, complementary events yield the most efficient method but requires the knowledge that obtaining no 6's and obtaining one 6 is all one needs to solve the problem:

$$P(X_6 \geq 2) = 1 - ((P(X_6 = 0) + P(X_6 = 1)))$$

The calculation $P(X_3 \geq 1) = 0.42$ has been completed in previous exercises as well as the lesson so participants are not expected to recalculate this value.

The calculation can therefore resemble:

$$P(at\ least\ two\ 5'\,\mathrm{s}\ \mathrm{in}\ 6\ rolls) = 1 - (P(no\ 6's) + P(exactly\ one\ 6))$$

$$P(at\ least\ two\ 5'\,\text{s in}\ 6\ rolls) = 1 - \left(\frac{5^6 + 6(1 \times 5 \times 5 \times 5 \times 5 \times 5)}{46656}\right)$$
$$P(at\ least\ two\ 5'\,\text{s in}\ 6\ rolls) = 1 - \left(\frac{15625 + 18750}{46656}\right)$$
$$P(at\ least\ two\ 5'\,\text{s in}\ 6\ rolls) = 0.263$$
$$P(at\ least\ two\ 5'\,\text{s in}\ 6\ rolls) \neq P(at\ least\ one\ 6\ in\ 3\ rolls)$$
$$\therefore This\ statement\ is\ not\ true$$

4.4.7.4.2 Calculating the probability of getting at least 2 fives in 6 rolls using the Properties Approach

The property of complementary events was expected to be used. Participants were expected to break the complementary event into an alternative of two simpler ones and use the additive property, to which they would perhaps refer orally in the interviews. Multiplicative property would be used to calculate the probabilities of the simpler events.

$P(obtain\ at\ least\ two\ 5'\,\text{s in}\ 6\ rolls\ )$

$= 1 - \big(P(no\ 5's\ in\ 6\ rolls)\ OR\ P(exactly\ one\ 5\ in\ 6\ rolls)\big)$

$= 1 - (P(no\ 5's\ in\ 6\ rolls) +\ P(exactly\ one\ 5\ in\ 6\ rolls))$

$= 1 - \big(P(not\ 5\ AND\ not\ 5\ AND\ not\ 5\ AND\ not\ 5\ AND\ not\ 5\ AND\ not\ 5) +$
$P(not\ 5\ AND\ not\ 5\ AND\ not\ 5\ AND\ AND\ not\ 5\ AND\ not\ 5\ AND\ 5)\big)$

$= 1 - \left(\left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + 6 \times \left(\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)\right) \approx 0.26$

where the 6 is the number of ways one can arrange the success among 6 rolls.

4.4.7.5 Anticipated Errors

In Van Dooren et al. (2003), 31.4% of students with formal training in probability were correct compared to 22.4% of students with no training. Most errors occurred in the incorrect application of proportionality models to the salient features of the problem. Students had a tendency to believe that the probabilities of one success in 3 rolls (believed to be $\frac{1}{3}$) and of two successes in 6 rolls ($\frac{2}{6}$) were equal. Justifications included viewing *P(at least two successes in 6 rolls)* as a doubling of *P(at least one success in 3 rolls)*.

Similar incorrect insights are anticipated here as well, especially since all participants will be deterred to carry calculations through based on the perceived size of the computation and tree diagrams.

So, as mentioned, there are two main avenues of justification the participants may take: 'qualitatively' explaining the non-linearity between variables *n, k, p* with *P,* or they may be tempted by the proportionality of the salient features.

If a method resembling the respective lessons is undertaken then, Properties participants may forgo the diagram more readily than Laplace students since their formulae are quite useful without it. But participants using the Laplace approach are more dependent on the production of a tree diagram for counting the number of ways of the desired event can occur. That being said, the combinatorics of 2 successes in 6 rolls is the most difficult combinatorics calculation in the experiment; especially when one has not been taught the relevant formulae. Listing all the arrangements (without a tree) is, equivalently, a tedious task. If a calculation is attempted and a tree diagram is not produced or imagined, then it is expected that that student will miss important arrangements. But this error would not be by arrangement neglect, but by the overwhelming volume of the computation.

Refutation participants, being sensitized to non-linearity in probability, should fare very well on this question compared to Exposition. Exposition participants were expected to be more tempted to view the salient numbers as ratios which would have hindered their progress through the question.

Overall, if the computation is too large for an individual, then the grounds for justification will begin with the implicit justification that $\frac{1}{3} = \frac{2}{6}$.

5 Implementation of the Teaching Experiment

5.1 Organization of the Experiment

The selection of participants for the study was opportunistic. They were all personal connections to the researcher. They were asked by email if they wished to participate in the study and were given the choice to decline. The participants were required to be university students with backgrounds in science or mathematics (CEGEP), but were not currently pursuing mathematics in their higher education. This was to ensure that they were capable of critical thought while not being exposed to a formal course on probability. All participants have, at most, seen probability as part of a high school mathematics course, except one participant currently taking a statistics course at the university level. This was not believed to affect results.

Most of the contact between researcher and participant occurred via email. Only during the interview did they meet. After participants agreed to their involvement in the study, a consent form (Appendix A) was sent to them by email. Participants had the freedom to choose the date, time and location to watch the lesson. The freedom to choose these details was important, as it replicated an authentic learning situation (Muller, 2008). The lessons were randomly assigned and the links to the video lessons were sent to the participants 10 minutes before the agreed time.

Participants were informed to open the questionnaire file only after watching the lesson, and they were told to contact the researcher anytime during the test if they required clarification, and when they were finished. One-on-one interviews were conducted within an hour of test completion. The interviews lasted between 30 and 45 minutes and were recorded using VoiceRecorder while the researcher took notes on participant behavior. Transcripts of each interview are found in Appendix F-I.

The interview questions (Appendix E) were borrowed from Muller (2008), and were designed to assess interest, motivation, as well as engagement with the lesson's content. The interview illuminated their background experiences, prior knowledge and how they viewed probability and its effects on everyday life. Other questions aimed to see what stood out from the lesson, what was recalled, and to what accuracy. Also, it aimed to determine if the inclusion of misconceptions and the different probabilistic approaches caused the participant to engage with the material differently.

Finally, there were questions pertaining to the participants' written work and thought processes during the taking of the test. They were asked to recall their initial thoughts upon reading the question, and describe how their thoughts evolved into their final answer and justification, while struggles were identified and elaborated.

An error in editing the videos left two refutation statements (called LR3 and LR4) in a lesson intended to be the exposition treatment of the Laplace approach. This lesson was mistakenly shown to a participant, but instead of omitting this participant's results, it was kept and analyzed as an additional lesson, and a fifth participant was recruited.

Links to the video lessons are introduced in Table 2 by their respective titles (indicating the approach and treatment of the lesson).

Scripts for Lessons are in Appendices B and C**.** Background profiles for participants are in section 5.2.

Table 2: Links to the video lessons.

Lesson (approach and treatment)
Laplace Exposition https://www.youtube.com/watch?v=SHDmrlkSU48
Laplace Exposition Alternate https://www.youtube.com/watch?v=vIdfRLFfMzM
Laplace Refutation https://www.youtube.com/watch?v=vxvohyx9Vuk
Properties Exposition https://www.youtube.com/watch?v=gaVT-L25AjY
Properties Refutation https://www.youtube.com/watch?v=FlBnyPv1onM

5.2 PROFILES OF THE PARTICIPANTS

The profiles of each participant were assimilated from two primary sources. The first being specific interview questions (Appendix E) whose purpose was to gauge the participants' background knowledge, interests and opinions on probability as well as their opinions on the lesson they watched. The second source are researcher's notes gathered throughout the interview based on student-researcher interaction and non-verbal communication assessed by the researcher. The questionnaire inquired about the participant's self-rated investment of mental effort during the lesson. This information is presented in their profiles.

5.2.1 Participant treated with the Laplace Exposition lesson: "Lea"

Lea has taken grade 11 probability as part of a math class. She recalls that the material examined in this class was somewhat similar to the lesson material, except that she was certain that she had never seen the use of complementary events in the calculation of probabilities. Despite playing board games and visiting casinos, she reports that she has never explicitly used probability as part of a strategy and that she "goes for the ride." Her interest in probability is low but she would be more interested if she knew that she could acquire a great advantage in games: "Not that I'm not interested in probability but I feel like sometimes I like to control things and because I can't control it…. I don't know if that makes sense".

Moreover, she has been out of school for over a year and the researcher has detected that this has affected her overall confidence towards academic and mathematical matters. She gave the impression that she had something to prove to herself because she did not follow the same path as her peers after CEGEP; where she graduated from Science. She hoped to pursue mathematics in the future.

Throughout the interview on her solutions to the test exercises, she reports that her method to approaching problem solving is to consider many logical avenues. In general, her work is thorough and she thinks retroactively about her completed work.

Her self-rated mental investment is at 5 of 9: Neither low nor high mental effort invested.

5.2.2 Participant treated with the Laplace Exposition Alternate lesson: "Leah"

As mentioned, in this video lesson, refutations LR3 and LR4 were included by mistake.

Leah recalls taking a course on probability in high school but states that she has "repressed" her memories of it. When asked about her experience with probability she said that she does not like it. She is aware of the importance of probability in board games and card games but reports that she does not think of it. Additionally, she reports that she is not interested at all in the topic. This overall feeling she has towards probability is reflected in her solutions. She appears cognitively lazy and produces the minimum amount of justification in her written solutions. Overall, she did not seem engaged.

She graduated from CEGEP Science and is now pursuing Health Sciences.

Her self-rated mental investment is at 5 of 9: Neither low nor high mental effort invested.

5.2.3 Participant treated with the Laplace Refutation lesson: "Larry"

This lesson contained refutations of Misconceptions LR1, LR2, LR3 and LR4.

Larry graduated from CEGEP in Science and is now pursuing Exercise science at the university. In high school he recalls doing a chapter on probability for a grade 9 math course, however he does not remember what material was covered.

He is generally interested in math and finds that probability is a fun topic to discuss with his roommate who is majoring in mathematics.

Throughout the questionnaire and interview Larry seemed engaged with the material and problems. His overall attitude towards learning seems positive, which may play a part in his perception of not having invested much effort in watching the video lesson: 3 of 9 (low mental effort invested); he may be used to harder tasks in his studies.

5.2.4 Participant treated with the Properties Exposition lesson: "Peggy"

Peggy graduated from CEGEP Science and is now pursuing Health Sciences. She has taken probability as part of a math course in high school but only recalls that the material somewhat resembled that in the lesson and exercises.

In terms of her experience with probability she plays a game called Yahtzee in which five dice are rolled and one hopes to obtain either consecutive numbers or all the same number. After the first roll, one may decide to re-roll 1 to 5 dice to obtain the highest score. A third re-roll is permissible. Despite playing often, she does not engage in a mathematical strategy. She finds probability interesting yet claims a fairly low interest in it because of its counter-intuitiveness; an argument she reiterates throughout the interview.

Her self-rated mental investment is at 5 of 9: Neither low nor high mental effort invested.

5.2.5 Participant treated with the Properties Refutation lesson: "Perry"

This lesson contained refutations of Misconceptions PR0, PR1, PR2, PR3, and PR4.

Perry graduated from CEGEP Science and is now pursuing bio-chemical engineering. In CEGEP he took a statistics course where he recalls learning material similar to the Properties lesson. Namely, he has seen the union and intersection of events. However, what he recalls the most is how difficult probability was and how little he enjoyed it. He attributes the difficulty of the subject to its counter-intuitiveness and he claims he has always had difficulty with "*how to break down the events*" and "*considering all the possibilities.*"

In the interview he stated that dealing with dice is conceptually easier than relating probability to people and their potential decisions. For example, he discussed a type of problem where people are in an airport with given probabilities that they will take one of several flights. He expressed the view that problems involving inanimate objects are simpler than those with people.

Perry claims to have no personal experience with probability but he also plays Yahtzee where he claims he applies some mathematical reasoning in making decisions while playing.

When asked what he learnt while watching the video lesson he said “I think all the common mistakes that were pointed out in the video where ones I often make. Like the flawed logic that I often used”. Specifically, the Misconception PR4 that “rolling a one once over 3 times, is going to be half the chance of 2 ones in 6 rolls”, as he put it.

His self-rated mental investment is 7 of 9: High mental effort invested.

6 ANALYSES OF PARTICIPANTS' RESPONSES

The analysis is divided into two sections. The first examines written justifications and the interview comments of each participant for each question on the questionnaire. For each question there is a table presenting the participants' written work, formulas and diagrams for a given question, and there is a thorough examination of each part of the justification. For some questions, scans of their written work are missing because they could not be made visible enough for this document. Moreover, there is an additional interpretation analysis of the participant's solution that focuses on noted behaviors and dialogue during the interview.

The second part of the analysis examines the justifications and responses organized by question. In this section, summary tables are displayed as well as summaries per question.

6.1 LIST OF QUESTIONS

For ease of reference, the test questions are repeated here, again both exactly as they were written for the participants and in mathematical notation.

6.1.1 Question 1 as presented to participants

I roll a fair die several times. The chance to have at least two times a three if I can roll four times is *(larger than, smaller than, equal to)* the chance to have at least two times a three if I can roll five times. *Select an answer in parentheses.* Show your reasoning. Justify your answer.

6.1.1.1 Mathematical representation of Question 1

Which is larger: $\mathrm{P}(X_4 \geq 2)$ or $\mathrm{P}(X_5 \geq 2)$? The success is the outcome "3".

6.1.2 Question 2 as presented to participants

Suppose one rolls two dice simultaneously. Which of the following has a greater chance of happening? Show your reasoning. Justify your answer.

(d) Getting the pair 5-6.
(e) Getting the pair 6-6
(f) Both have the same chance.

6.1.2.1 Mathematical representation of Question 2

X: Event where the pair (5, 6) is obtained with two dice rolled simultaneously.

Y: Event where the pair (6, 6) is obtained with two dice rolled simultaneously.

Is $P(X)$ larger than, equal or smaller than (Y) ?

6.1.3 Question 3 as presented to participants

I roll a fair dice several times. The chance to have at least four times a five if I roll five times is half as large as the chance to have at least two times a five if I roll five times. (*This is true, this is not true*). *Select an answer in parentheses.* Show your reasoning. Justify your answer.

6.1.3.1 Mathematical representation of Question 3

X_n:= number of 5's in n rolls of a fair dice

True or false? $\mathrm{P}(X_5 \geq 4) = \frac{1}{2}\mathrm{P}(X_5 \geq 2)$

6.1.4 Question 4 as presented to participants

I roll a fair dice several times. The chances to have *at least* once a two if I roll 3 times is (*larger than, smaller than, equal to*) the chance to have *exactly* once a two if I roll 3 times. *Select an answer in parentheses.* Show your reasoning. Justify your answer.

6.1.4.1 Mathematical representation of Question 4

Is $\mathrm{P}(X_3 \geq 1)$ larger than, equal or smaller than $\mathrm{P}(X_3 = 1)$? The success is the outcome "2".

6.1.5 Question 5 as presented to participants

I roll a fair dice several times. The chance to have at least once an even number if I roll three times is three times as large as the chance to have at least once a five if I roll three times. (*This is true, this is not true*). *Select an answer in parentheses*. Show you reasoning. Justify your answer.

6.1.5.1 Mathematical representation of Question 5

Y_n is the number of times an even number is obtained in n trials

X_n is the number of times a 5 is obtained in n trials

Is it true that $\mathrm{P}(Y_3 \geq 1) = 3 \times \mathrm{P}(X_3 \geq 1)$?

6.1.6 Question 6 as presented to participants

I roll a fair die several times. The chance to have at least two times a five if I can roll six times is equal to the chance to have at least once a five if I can roll three times. (*This is true, this is not true*). *Select an answer in parentheses*. Show your reasoning. Justify your answer.

6.1.6.1 *Mathematical representation*

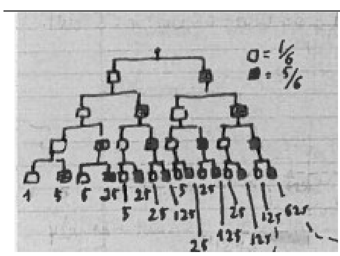
Is it true that $\mathrm{P}(X_6 \geq 2) = \mathrm{P}(X_3 \geq 1)$? The success is the outcome "5".

6.2 ANALYSIS BY PARTICIPANT

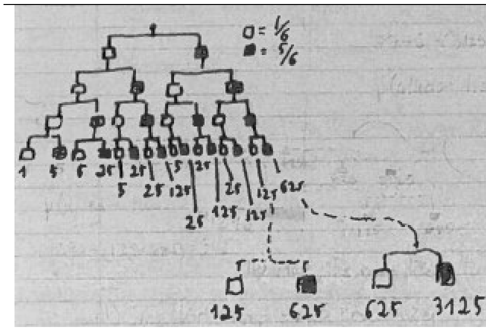
This section is an analysis of each participant's justification and solution to the questionnaire organized by participant.

6.2.1 Laplace Exposition- Lea

6.2.1.1 *Lea: Question 1*

Transcript – LA – E – "Lea" – Question 1	Comments
Scenario #1 # of possible outcomes = 6 x 6 x 6 x 6 = 1,296 at least two 3's = two 3's + three 3's + four 3's	Plans to calculate $\mathrm{P}(X_4 \geq 2)$ Evidence of using the interpretation of probability presented in the lesson (LA): calculates the number of all possible outcomes. Breaks down the event in question into simpler events. Mistake of imprecision: Uses "=" and "+" signs metaphorically; "+" stands for set union; "=" stands for equality of sets.
	Evidence of using techniques presented in the lesson: Draws a 4-layered tree with nodes in the shape of little white and black squares, representing "1/6" and "5/6" respectively. So – not the outcomes "3" and "not 3" but fractional expressions that could be interpreted as their probabilities. But she is not using these expressions as probabilities in the rest of her solution. She may be interpreting these expressions as "1 outcome out of 6" and "5 outcomes out of 6"; so these are records of

	numbers of times a three and a non-three can turn up. We see here another metaphorical use of the equals sign.
1296 – (125 + 625) = 546	Evidence of using techniques presented in the lesson: Attempts to use the complementary event rule but does not take into account all the possible arrangements of getting 3 non-threes; there are 4 of them, but she takes into account only one. She simply takes away the two rightmost numbers. She should have calculated 1296 – (625 + 4 x 125) = 171 Misconception LR3
p(at least two 3's) = $\frac{546}{1296} \approx 0.42$	Evidence of using the interpretation of probability presented in the lesson: Applies the classical definition of probability correctly
Scenario # 2 # of all possible outcomes = 6 x 6 x 6 x 6 x 6 = 7776	Now goes on to calculate $\mathrm{P}(X_5 \geq 2)$ Evidence of using the interpretation of probability presented in the lesson: Calculates the number of all possible outcomes.
At least two 3's = two 3's + three 3's + four 3's	Evidence of using techniques presented in the lesson: Breaks down the event into simpler ones; does not write the case of five 3's, but does take it into account in her calculations.
	Evidence of using techniques presented in the lesson: Does not draw the whole 5-layered tree but only extends the rightmost branch of the 4-layered tree. She plans to use the complementary event rule.

	
7776 – (3125 + 625 + 625) = 3400	This way she misses certain arrangements and her count of the desirable outcomes is incorrect. Misconception LR3
p(at least two 3's) = $\frac{3400}{7776} \approx 0.44$	Evidence of using the interpretation of probability presented in the lesson: Uses the definition of probability given in the lesson and divides the number 3400 of desirable outcomes that she got by the number of all outcomes.
Therefore it is more likely to obtain Scenario #2.	Answer correct: The probability she gets is still higher than the one she got for Scenario #1, so her conclusion is correct.
(1) smaller than → 5 certainty	Unclear what she means here
Self-rated confidence (1-7)	5

The participant's initial approach was to compare the salient numerical features of the problem as two ratios $\frac{2}{5}$ versus $\frac{2}{4}$, thus concluding that obtaining two 3's in 4 rolls was more likely. The participant then tried to carry out her calculations (maybe for the purposes of "justification") as learned in the lesson by employing one tree diagram to represent both events. She drew the 4-layered tree in full but for the 5 rolls case, she only extended the rightmost branch of the 4-layered tree to 5 rolls. She thought this was enough since she planned to use the complementary event rule. In the case of the 4 rolls she only used the rightmost branch. This way she missed some arrangements (Misconception LR3) which affected the values of the two probabilities but, incidentally, the relation between her numbers was correct so she correctly concluded that the probability of getting at least two 3's in 4 rolls was 'smaller than' the probability of getting at least two 3's in 5 rolls.

The participant reports that she was surprised by the result because, as she stated in the interview, she first believed that the answer was "larger than." However, on second thoughts, she accepted the result as more "logical": she qualitatively justified the correct answer by referring to her experience with dice and citing how "*non-logical*" her initial thoughts on the problem were:

> *Lea: Because if you have more chances to roll you have more chances to obtain that same ratio…um, not ratio but more times of obtaining the 'at least two times' part" (Lea)*

Furthermore, she expressed frustration over the fact that so much work must be done in order to prove something so intuitive:

> *Lea: […] then when I started putting it down on paper that's when it started getting messy. Yah, it's a lot easier to reason in your mind*

Albeit, had she not done the calculations her erroneous belief would not have been refuted.

6.2.1.2 Lea: Question 2

Transcript – LA – E – "Lea" – Question 2	Comments
Pair getting 6 – 6 $^{1}/_{36} \approx 0.028$ Pair getting 5 – 6 $^{2}/_{36} \approx 0.056$	Evidence of using the interpretation of probability presented in the lesson: divides the number of desirable outcomes by the number of all outcomes Evidence of using techniques taught in the lesson: considers different possible ways of getting the desired outcome after imagining (not drawing) the tree
Oral: "Because I guess timing wise, if we get really picky, one dice can touch the table like milliseconds before the other one."	Justifies why it makes sense to distinguish the dice
Therefore more likely to get 5 – 6 (or 6 – 5)	Answer
Self-rated confidence (1-7)	7

Initially the student believed that the pair (5, 6) was equally likely to occur as (6, 6), and she had indicated such as her final answer. It seems she felt the answer was obvious as no work was shown for this conclusion. No tree diagrams were drawn and no calculations were done.

In the interview she indicates that while she was drawing the tree diagram for Question 3, she realized that obtaining *two* 5's in *n* rolls occurred via different arrangements of the respective successes (5's) and failures (non-5's). This realization helps her refute the error she committed in Question 2; that she had neglected arrangements of (5, 6).

> *R: So you realized there's different orders to those 5's [in exercise 3]?*
>
> *Lea: Yah! Like when you're going down line…there's almost a mirrored image of something you can get here over here [pointing at tree diagram where the sequences {5,5,not 5, not 5, not 5} and {not 5, not 5, not 5, 5, 5}] every time, but not always.*

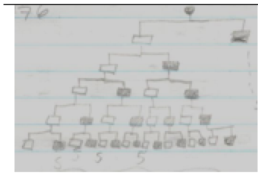
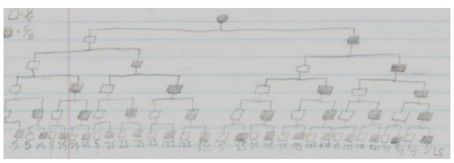
At this point, Lea returned to Question 2 and was confident (7/7) in her final conclusion stating (5, 6) is twice as likely as (6, 6). It was clear that at this point she was visualizing a small tree diagram because when asked about how the answer would change if the dice were, instead, rolled consecutively she immediately responded:

> *Lea: No. Because the tree diagram would have been the same, in my mind.*
>
> *R: Okay.*
>
> *Lea: Because I guess timing wise, if we get really picky, one dice can touch the table like milliseconds before the other one.*

Drawing the tree for Question 3 helped her realize the importance of arrangements independently and consequently she developed a sound conceptualization of simultaneous and consecutive rolls.

6.2.1.3 Lea: Question 3

Transcript – LA – E – "Lea" – Question 3	Comments
Oral: "[True] I think it's because in both situations you're rolling 5 times and in one you have almost… twice the amount that you're supposed to be obtaining".	First guess is that the statement is true.
# possible outcomes = 6 x 6 x 6 x 6 x 6 = 7776 p(at least four 5's) = $\frac{four\ 5's + five\ 5's}{7776}$ =	Evidence of using the interpretation of probability presented in the lesson: calculates the number of all outcomes; divides the number of desirable outcomes by the number

 $= \frac{25+1}{7776} = \frac{26}{7776} \approx 0.0033$	of all outcomes Evidence of using techniques taught in the lesson: considers different possible ways of getting the desired outcome after drawing the tree Mistakes: imprecision of language: “four 5’s” instead of “number of ways of getting four 5’s”
# possible outcomes = 6 x 6 x 6 x 6 x 6 = 7776 p(at least two 5’s) = $\frac{two\ 5's + three\ 5's + four\ 5's + five\ 5's}{7776}$  p(at least two 5’s) = 1 – p(one 5 + no 5) = 1 – 0.0033 = 0.9967 p(one 5 + no 5) = $\frac{1+25}{7776} = 0.0033$	Repeats the same reasoning for $\Pr(X_5 \geq 2)$, recalculating the number of all outcomes, re-drawing the tree, but making it larger, as if following a ritual. But eventually turns to the complementary event rule, thus departing from the way she calculated the previous probability. Evidence of using the interpretation of probability presented in the lesson: calculates the number of all outcomes, and plans to count the number of desirable outcomes Evidence of using techniques taught in the lesson: decomposes the desired event into simpler events; draws a tree, calculates the numbers of outcomes along each branch; uses the complementary event rule Misconception LR3: misses many arrangements Mistakes: uses equals sign instead of approximation sign
Therefore it is not true	Answer
Self-rated confidence (1-7)	4

A high extraneous cognitive load is suspected as Lea admits she had a lot of difficulty understanding what the question was asking:

Lea: Yah. Like….I don’t do well with, um, when words so closely worded together.

R: Like what?

Lea: [rereads question]. When I read that it becomes a big mess in my head but when I start laying it on paper it starts to make more sense, since I'm a visual person. Like the tree diagrams really helped me.

She was tempted to treat the problem as applying fraction to the salient numerical features:

Lea: […] chances to have 3 times a 5 if I roll 5 times. I think of fractions. And it doesn't really make sense initially in term of probability, I think of the fractions and it has nothing to do with it. Like I think visually I think of a fraction and it's like 'oh that makes sense'.

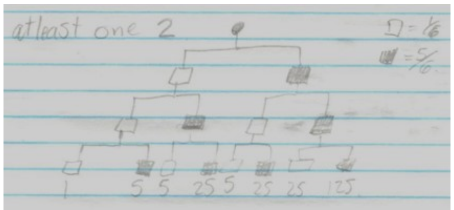
Moreover, Lea doubts her intuition: *"I don't believe it until I've done my work for it.*" This proved to be valuable as she was able to consider two different avenues to solving the problem: using complementary events rule, or the direct calculation.

She produced a large tree diagram and used the complementary events rule to complete the calculation, albeit missing some arrangements for *n=5* rolls, and arriving to the correct answer "this is not true."

In the interview for Question 2 she reflects on Question 3 which reveals that she did not expect the numerical answer to disagree with her intuition: "*I think it's because in both situations [in question 3] you're rolling 5 times and in one you have almost… twice the amount that you're supposed to be obtaining*"- suggesting an implicit belief that the relation *should* be proportional.

6.2.1.4 Lea: Question 4

Transcript – LA – E – "Lea" – Question 4	Comments
Oral: "For this question I didn't think about it… I just jumped into the work. Like at this point I knew how to prove my work and instead of thinking about what I thought was the right one I just went directly for the answer."	No first guess; no reflection, gets immediately into a routine that she has developed in solving Questions 1-3. Does not notice that the event "at least one 2 in 3 rolls" is isomorphic to "at least one 6 in 3 rolls" whose probability has been calculated in Problem 1 in the lesson and that she could reuse it without calculating.
# possible outcomes = 6 x 6 x 6 = 216	Evidence of using the interpretation of probability presented in the lesson: calculates the number of all outcomes and the number of ways each particular

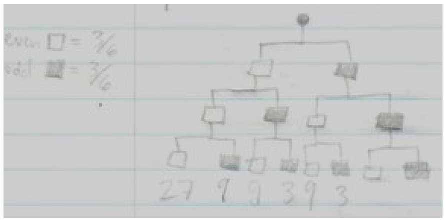
 p(at least one 2) = 1 – p(no 2's) = $1 - \frac{125}{216} \cong 0.42$ p(exactly one 2) = $\frac{25+25+25}{216} \approx 0.35$	outcome with 2's and non-2's could be obtained; calculates the probability of getting exactly one 2 using the "number of desirable outcomes / number of all outcomes" formula Evidence of using techniques taught in the lesson: draws a tree diagram; uses the complementary event rule
Therefore chances of rolling at least one 2 is greater than the chances of rolling exactly one 2.	Answer
Self-rated confidence (1-7)	6

Lea found this question intuitive and mathematically easy. She carried out the Laplace approach in her written justification and stated in the interview "*it says at least one 2, you can get one 2, two 2's and you can get three 2's. But the word 'exactly' in front, limits it to one option*", which demonstrates insight about the problem.

One may only speculate why she left her confidence at 6 of 7, since she did not display any signs of self-doubt.

6.2.1.5 Lea: Question 5

Transcript – LA – E – "Lea" – Question 5	Comments
[First guess] Oral: "I wouldn't feel confident saying it was the right answer just because it says 3 times as large. But I would say it's [at least once an even number] definitely a lot more chances than at least one 5 in 3 times."	First guess is a qualitative comparison: $\Pr(Y_3 \geq 1) \gg \Pr(X_3 \geq 1)$ No gut feeling about the exact number of times the former is greater than the latter.
# of possible outcomes = 6 x 6 x 6 = 216 p(even number) = 1 – p(odd number)	Starts calculating the probability of getting an even number at least once. Re-calculates, in a ritualistic

	manner, the number of all outcomes (this has been already calculated in the lesson and in Question 4). Writes that the probability of an even number is 1 minus the probability of an odd number, maybe identifying getting an odd number as complementary to getting an even number. Then draws the tree for getting even and odd numbers in three rolls of a dice. Counts some numbers of possible ways of getting the different outcomes, but not all and then gives up solving the problem. Evidence of using the interpretation of probability presented in the lesson: calculates the number of all outcomes and the number of ways some particular outcomes can be obtained. Evidence of using techniques taught in the lesson: draws a tree diagram; uses the complementary event rule Mistakes: there are 9 ways of getting the outcome even-even-odd (instead of 27); 9 ways of getting even-odd-even, 3 ways of getting even-odd-odd, etc.
Oral: "…in the video, it was one sixth and five sixths as the white and black boxes. … [I wrote, 27, 9, 9, 3…]. Then I stopped, because there was something in my head that was telling me… they were all going to be the same, so I was going to stop and erase it there and redo it… and every number in the bottom was going to be the same and I was thinking about it and it just didn't make sense… it made me feel uncomfortable knowing that at the bottom every number was the exact same…. But when I was thinking about it, it does	Realizes that there are 27 ways of getting any outcome and this is disturbing for her, because in the examples of multiple rolls experiences in the lesson the numbers of ways of getting different outcomes were always different. But she explains it to herself that the "equal numbers at the bottom" could make sense because the event and the complementary event are symmetrical ("mirrored" is the word she uses).

make sense because like it's mirrored."	
	No written answer
Self-rated confidence (1-7)	n/a

Lea could not finish this question. She became stuck calculating the probability of obtaining at least one even number in 3 rolls, despite having correctly produced the tree diagram with branches as even and odd and correctly calculating the number of all possible outcomes and reweighing each branch as $p = \frac{3}{6}$ (resembling the Properties approach).

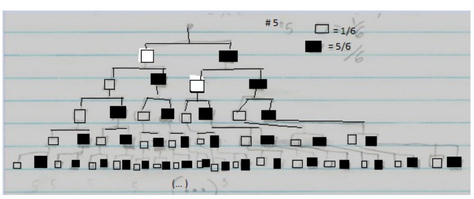
The bottom of her tree diagram incorrectly displays the numbers 27 9 9 3 9 3, as to imply the number of arrangements per branch. In the interview she pointed out that she began writing these numbers but:

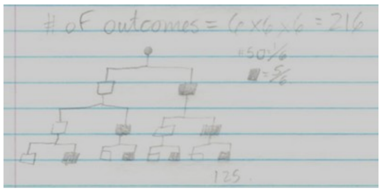
> *S: I stopped, because there was something in my head that was telling me it was…like they were all going to be the same, so I was going to stop and erase it there and redo it…and every number in the bottom was going to be the same and I was thinking about it and it just didn't make sense*
>
> *R: What didn't make any sense? That they're all the same or that they're not the same.*
>
> *S: That they were all going to be the same at the bottom each little box they were all going to be the same and it was going to be 27, 27, 27, 27…*
>
> *R: And you were uncomfortable with that idea?*
>
> *S: Yah.*
>
> *R: Why is that?*
>
> *S: Um…I just...um…*
>
> *R: Just tell me about your thought process. I'm curious.*
>
> *S: That was the word for it though…it made me feel uncomfortable knowing that at the bottom every number was the exact same. Um…'at least once an even number'… (Long pause). I don't know.*
>
> *R: Okay. [A long pause ensued, after which the researcher explained that her intuition was correct]*

During the interview she said she had a 'feeling' that the statement is true (with a confidence of 3 of 7), because she is sure that obtaining at least one even number is greater than obtaining at least one 5 within *n* rolls – demonstrating good qualitative reasoning. However she was not convinced of the quantitative relation being presented.

Lea demonstrated good qualitative insight of the relation presented in exercise 5, and also expressed doubt about the statement's validity. Despite rating her confidence at 3, her decision to do so implies she knows that there is something to be learned: "[…] *if there were some method to determine to show it is 3 times as large then I would take that method because I would be pretty confident in knowing that.*"

6.2.1.6 Lea: Question 6

Transcript – LA – E – "Lea" – Question 6	Comments
[First guess] Oral: "This one was another one where I didn't even bother thinking if I knew the answer before jumping into it."	No gut feeling.
# of possible outcomes = 6 x 6 x 6 x 6 x 6 x 6 = 46.656 Oral: "Well, this one's diagram frustrated me. I didn't even finish it. It was unfinished work as you can see the little dots." 	Evidence of using the interpretation of probability presented in the lesson: calculates the number of all outcomes Evidence of using techniques taught in the lesson: draws a tree diagram, but does not finish it.
p(at least two 5's) = 1 – p(no 5's + one 5) = 1 - $\frac{15{,}625+25}{46{,}656} = 0.66$	Turns to visualizing the relevant branches of the tree in her head. Evidence of using techniques taught in the lesson: uses the complementary event rule Misconception LR3 – omits some arrangements

 p(at least one 5) = 1 – p(no 5’s) = 1 - $\frac{125}{216}$= 0.42	Re-calculates $\Pr(X_3 \geq 1)$ Mistake: lack of reflective thinking; ritualistic application of procedures
Therefore answer is not true	No written answer
Self-Rated Confidence (1-7)	7

For this final Question, the student admits that she had no intuition and jumped into the calculation immediately: “*Oh yah. This one was another one where I didn’t even bother thinking if I knew the answer before jumping into it.*” From the previous questions she began to learn that she could not trust her intuition: *“[…] instead of thinking about what the answer could be, or trying to picture it in my head, I just jumped into the math, because even if I tried to picture it in my head it probably wouldn’t be right.*”

She carried out her calculation of $P(at\ least\ two\ 5's\ in\ 6\ rolls)$ using the complementary event rule $1-(P(no\ 5's)+P(one\ 5))$, however many of the arrangements for obtaining one 5 in 6 rolls were missed. She cited the difficulty in creating such a large tree and expressed interest in knowing simpler formulations to obtaining results without a tree diagram.

6.2.1.7 Summary

To reiterate what has been mentioned before in this participant’s profile, Lea is not that interested in probability and is currently not pursuing higher education, though she plans to do so. This latter fact has manifested itself through a feeling that she has ‘something to prove’ to herself. Off the record discussions revealed that she was unsure about her decision to temporally discontinue school, unlike many of her peers and friends. Moreover, she disclosed that she was weary about her mathematical capabilities because she had been out of school for over a year. These factors are likely the underpinning for some central characteristics observed in her analysis – one of such being for procedural approach to mathematics and ritualistic behavior.

Throughout the questionnaire she follows the Laplace approach closely but does not leave much room for critical analysis as she reproduces and recalculates tree diagrams that have been used before. Despite this, she is the only participant to have either used or orally referenced tree diagrams for every question, and she fared well, compared to others, when considering relevant arrangements (though she missed a few here and there).

Lea was the only participant to indicate the confusing construction of Van Dooren et al.'s (2003) questions, saying that she struggled with their interpretation. It is very possible that Van Dooren et al.'s questions require effort to extract the meaning of the question thus contributing to the extraneous load of the problem.

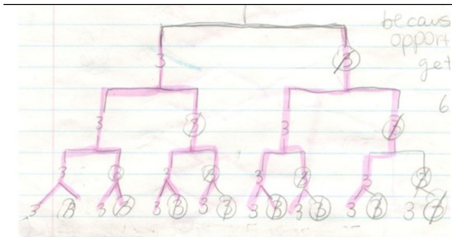
The most noteworthy point of Lea's result is her struggle on Question 5 wherein she discontinued the solution citing feelings of discomfort upon having an inclination that the required tree diagram would be symmetric and have branches of 27 outcomes each. Speculation for this reaction is elaborated in *Remarks and Conclusions*. It is thought that a low confidence in her abilities combined with the novel problem presented in Question 5 prevented her from confidently carrying the mathematics through; explaining her unease with the symmetric tree diagram. The expectation that the fifth question must be like the previous four is akin to Rubel's (2007) Four heads item and "positive recency" heuristic. Or it can be attributed to the discomfort felt knowing that the instructor has never mentioned how to approach this type of problem.

By Question 6, Lea explicitly noted that she had abandoned her initial thoughts about the problems because she had begun to notice that they were misleading.

6.2.2 Laplace Exposition Alternate- Leah

6.2.2.1 Leah: Question 1

Transcript – LA – E – Alternate – "Leah" – Question 1	Comments
$p(desired\ event) = \frac{\#\ of\ ways\ desired\ event\ may\ occur}{\#\ of\ possible\ outcomes}$ $0 \leq p(event) \leq 1$ no chance certain p(event) = 1 – p(complementary event)	Evidence of using the interpretation of probability presented in the lesson (LA): Appears to review the lesson and makes notes from it: writes the definition of probability and some other information about probability given in the lesson: probability is a number between 0 and 1, and the complementary event rule. There are

	also some operations that resemble those made in the lesson; she was probably reviewing the solutions of the problems in the lesson.
	Evidence of using techniques presented in the lesson: Draws a 4-layered tree with the outcomes marked as “3” and a crossed “3” in a circle *on the branches* – not in the nodes as it was done in the lesson. Highlights the branches corresponding to desirable outcomes. There are traces of numbers of possibilities corresponding to these highlighted branches; she has erased them after summing them up, correctly, to 171.
6 x 6 x 6 x 6 = 1296 $\frac{171}{1296} = 0.1319$	Evidence of using the interpretation of probability presented in the lesson: calculates the number of all possible outcomes (correctly), and divides the number of desirable event by the number of possible outcomes. Mistake of imprecision: Uses equals sign instead of the approximation sign to write the result.
	Evidence of using techniques presented in the lesson: Uses a tree to represent the possible outcomes in rolling a dice 5 times. Highlights in green the branches corresponding to the complementary event. Does not miss any arrangements.
$1 - \frac{6250}{7776} = 0.1924$	Evidence of using the interpretation of probability presented in the lesson: Calculates the probability of the complementary event using the classical definition given in the lesson. Answer almost correct: Mistake of imprecision: equality sign instead of approximation sign used.

Smaller than because you will have more opportunities to roll 2 threes if you have an extra roll.	Answer correct: Her final answer is correct but the reason she gives is inaccurate: she refers to rolling 2 threes instead of *at least* 2 threes. However, she did calculate probabilities of at *least* 2 threes in her work. The final justification she gives is not referring to the values of the probabilities but to a qualitative rule in which she seems to believe: the more opportunities, the higher the probability.
Self-rated confidence (1-7)	6

Leah found the question difficult because of the work that was required to prove a solution that was intuitively known to her:

> *R: So how did you find the question itself?*
>
> *Leah: At first they were pretty hard just because the trees are really tedious, I find. So they're really annoying so…ya. I found it challenging at first.*
>
> *R: Did you find this first question challenging or easy?*
>
> *Leah: A bit challenging.*
>
> *R: Is what you wrote here wat first came to mind?*
>
> *Leah: Ya, I kind of did it on first glance and circled the right answer then I went back and did the calculation.*
>
> *R: On first glance?*
>
> *Leah: It was my first impression or whatever.*

She produced two large tree diagrams and said in the interview that the lesson should have demonstrated how to deal with a larger number of trials. She appeared a bit discouraged or tired during the interview, probably because of the discrepancy between the work required and the intuitiveness of the answer "*more opportunities to roll 2 three's if you have an extra roll.*"

As the table (above) demonstrates, Leah uses the material taught in the lesson correctly. It is unknown as to why she carried out a calculation despite finding it tedious.

6.2.2.2 Leah: Question 2

Transcript – LA – E – Alternate – Question 2 – “Leah”	Comments
(c) both have the same chance because the probability of rolling a five and a six are the same. Any specific pair of numbers would have the same probability of occurring as any other pair of numbers Oral: ”… it was just like common sense. I mean you get the same probability… it’s a 1/6 chance for 5 and a 1/6 chance for 6 then the same thing here [pair 6,6]. So it’s like ‘Oh, it’s the same’.”	Misconception LR3: does not discuss the possibility of the dice being distinguishable and interpreting “(5, 6)” as representing 5 on one dice and 6 on the other or 6 on the one and 5 on the other. She ignores one of the possible arrangements in this case. No evidence of using LA: her justification is more suggestive of PA: she states the probability of getting a 5 and the probability of getting a 6 and notices that they are the same; in PA, this would lead to calculating the probability of getting two sixes by multiplying 1/6 by 1/6; since Leah ignores arrangements, she would calculate the probability of getting a 5 and a 6, also by multiplying 1/6 by 1/6.
Both have the same chance	Answer
Self-rated confidence (1-7)	7

With a confidence of 7, Leah seems to quickly and assuredly conclude the common misconception that the pair (5, 6) had equal chance of happening as the pair (6, 6). It was not unexpected that, treated with an exposition lesson, a student would make this mistake.

What is noteworthy is that she did not use the LA definition of probability taught in the lesson but instead reasoned as if using the Properties approach.

Leah reported in the interview that Question 2 was the easiest of the questions:

> *Leah: Oh, this one I knew like right off the bat because, I mean, it’s just kind of obvious.*
>
> *R: Okay, so you found this question was quite easy then?*

Leah: Ya, it's more like a trick question. Like if you were nervous or something you maybe would have chosen a different answer. I found it kind of hard to explain it in words but I tried my best. You get what I'm saying?

[…]

R: Did you try to incorporate some of the concepts used in the video?

Leah: Not really, it was just like common sense. I mean you get the same probability… it's a 1/6 chance for 5 and a 1/6 chance for 6 then the same thing here [pair 6,6]. So it's like 'Oh, it's the same'.

In conjunction with her written solution (above) it would imply that the notion that the pairs (5,6) and (6,6) are equally likely is more of an intuitive (and incorrect) belief than a result of cognitive deliberation.

Her written justification is mostly wrong and irrelevant. She argues that the pair (5, 6) has the same probability as (6, 6) because of the individual probabilities of obtaining a 5 or a 6, which are also equal. The student neglects the arrangements because perhaps considered it not worthwhile.

6.2.2.3 Leah: Question 3

Transcript – LA – E – Alternate – "Leah" – Question 3	Comments
This is not true because the ratios are not proportional to each other. The probability for rolling four times a five in five rolls is actually much lower than half the probability of rolling two times a five in five rolls. Oral: "Oh, this is the half one. Half as large or whatever. Well I knew that from the video because there was basically the same example, right? So I tried to do it out on paper but I knew the answer already."	Evidence of using the interpretation of probability presented in the lesson (LA): refers to "ratios" – weak evidence Evidence of using techniques presented in the lesson: says the question is similar to the problems in the video-lesson and that she tried to do the calculations for Question 3, but probably did not complete them. No trace of drawing a tree or calculation in her written work. Mistake: ("quantitative negligence") misinterprets the question as asking if the statement $\Pr(X_5 = 4) \neq \frac{1}{2}\Pr(X_5 = 2)$ is true or false
Not true	Answer

Self-rated confidence (1-7)	6

Given this Leah's response to Question 3, it is nearly impossible to ascertain whether she understood the concept of non-linearity between variables. Her initial guess and her final answer is that the statement is 'not true', which is plausible since she was presented with the refutation of the non-proportional relations.

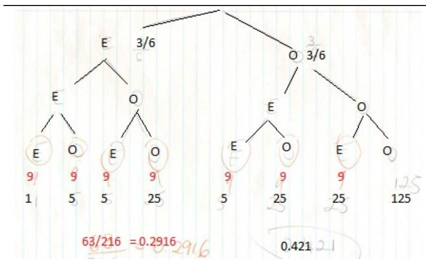
However, her written justification did not include any tree diagrams or any calculations. To conclude that she understands the concept seems unlikely especially since her argument is inconsistent, "*This is not true because the ratios are not proportional to each other.*" Some of her written justification echoes the lesson dialogue word for word "[…] *is actually much lower than half of the probability to roll two times a five in five rolls.*"

6.2.2.4 Leah: Question 4

Transcript – LA – E – Alternate – "Leah" – Question 4	Comments
Larger than because rolling at least once a two in three rolls gives options to roll two or three two's which would give larger probability than if you were limited to just rolling once a two.	No evidence of using the interpretation of probability presented in the lesson (LA): implicit use of the additive property of probability Evidence of using techniques presented in the lesson: decomposes the event "at least one two" into simpler events
Larger than	Answer
Self-rated confidence (1-7)	7

Leah intuitively knew that "*'at least'- you have more options so you have a larger probability. You have more than just exactly once a 2*", which demonstrates an understanding of the conceptual difference between 'at least *k* successes' and 'exactly *k* successes'. No tree diagrams or calculations were used however the expected solution was conceptual and did not require a detailed justification.

6.2.2.5 Leah: Question 5

Transcript – LA – E – Alternate – “Leah” – Question 5	Comments
This is not true because the probability of rolling an even number at least once is not the same as rolling a single number at least once. Therefore their probabilities will not be proportional.	This sounds illogical if taken literally: $p \neq 3 \times q$ because $p \neq q$. But maybe “not the same” refers to the events not being comparable? Maybe uses “non-proportionality of probabilities” as a universal argument for refuting any equation between probabilities or their multiples.
	Draws one tree for both events (5 is an odd number and its probability is 1/6) and counts the numbers of ways of getting each outcome, but makes a mistake in the case of even and odd numbers. Re-uses the probability 0.421 for at least one 5 in 3 rolls from the lesson or from an earlier question. Evidence of using the interpretation of probability presented in the lesson (LA): counts the numbers of ways of getting the different outcomes (with computational mistakes) Evidence of using techniques presented in the lesson: decomposes an event into simpler ones; uses a tree (one tree for both events) Mistakes: 9 ways instead of 27 of getting even and odd numbers in 3 rolls; so her probability of getting an even number at least once in three rolls is 63/216 instead of 189/216. This is not Misconception LR3 but misconception LR2.
In fact, the probability of rolling an even number	Her computational mistake leads her to

at least once in three rolls is less than that of rolling a five once in three rolls.	erroneous conclusion about the relation between the two probabilities. Mistake: omits "at least" in describing the second event
Not true	Answer
Self-Rated Confidence (1-7)	6

Before engaging in mathematical formulation, Leah believed, following qualitative reasoning, that obtaining at least one even number in 3 rolls would be more likely than obtaining at least one 5 in 3 rolls. This belief is expected among students of her age bracket (Van Dooren et al., 2003). The notion is intuitive to grasp for most students and it is not surprising that she believed the answer would have been "larger than."

However she noted that she was not sure about the statement which explains why she produced a tree diagram (she did not produce diagrams for most questions). Now, the diagram is correctly fabricated with branches of E's and O's representing evens and odds. Like Lea, she assigned weights of $p = \frac{3}{6}$ to each branch, a method reserved for the Properties approach.

When Leah denoted the number of outcomes per branch, however, she (presumably) applied the incorrect mathematical operator and wrote 9 (3+3+3) as opposed to 27 (3x3x3) possible outcomes-misconception LR2. A common error not refuted in her lesson.

Reusing $P(X \geq 1) = 0.42$ from previous questions, Leah concluded that obtaining at least one even number was probabilistically less than obtaining at least one 5 and she states: "I found this was the probability… so it kind of shocked me that it was less."

And so she correctly concluded "this is not true", with incorrect calculations and an incorrect mathematical conclusion derived from an incorrect assessment of her new tree diagram, after having the inclination that the opposite was true, and she rated her confidence at 6, which is very high considering that her intuition and her conclusion did not align.

Perhaps she has high confidence in her method combined with her lack of engagement that may have easily dismissed a strong intuitive tendency to reject the mathematical conclusion.

Her written justification reflects her lack of understanding the conclusion since "this is not true because the probability of rolling an even number at least once is not the same as rolling a single number at least once, therefore the probabilities will not be proportional" is a circular statement, in that no argument was made. At its core she begins her argument by saying "this is not true because it's not true."

6.2.2.6 Leah: Question 6

Transcript – LA – E – Alternate – "Leah" – Question 6	Comments
Oral: "I didn't have a diagram set-up but I figured it would be a lot less of a probability…. I just kind of figured. Like for this one [Question 1, where she drew a tree for 5 rolls]. The probabilities with 5 rolls, or something, it was way less. So I don't know, I just figured."	First guess is that it is not true that $\Pr(X_6 \geq 2) = \Pr(X_3 \geq 1)$.
Oral: "… there's just a lot more possibilities…	Leah may be referring to the fact that 6^6 is a lot more than 6^3 here. This would be a possible Evidence of using the interpretation of probability presented in the lesson (LA): weak, implicit only
Oral: "I guess like in the complementary rule, there's just more possibilities or something. I don't know how to explain…. There's more possibilities against not being able to get at least one or two times or whatever it is. Because like I noticed like this one goes on forever [referring to a branch in the tree diagram drawn for Question 1], you know, 5x5x5x5, you get so many big numbers that it lowers the probability."	But here she may be thinking of the complementary events, $X_6 \leq 1$ and $X_3 = 0$. (This would be Evidence of using techniques presented in the lesson) But it is not obvious, without calculation, that $\Pr(X_6 \leq 1) > \Pr(X_3 = 0)$ or that $\frac{5^6+6\times5^5}{6^6} > \frac{5^3}{6^3}$. So Leah may be just thinking that two outcomes account for $X_6 \leq 1$ and only one for $X_3 = 0$. Or she maybe just sees a vague analogy with the probabilities calculated in Question 1.
This is not true because it's much less likely to roll	This is the Answer.

two fives in six rolls than one five in three rolls. This is because the ratios are not proportional.	Reference to the refutation of Misconceptions LR1 and LR4 or to the instructor saying that "probabilities are not proportional" in the version of the lesson that Leah listened to. Uses "not proportional" as a universal explanation for saying "not true". Mistake: Is this just a slip of the tongue or "quantitative neglect" and misreading of the question as, is it true that $\Pr(X_6 = 2) = \Pr(X_3 = 1)$, or is the falsity of the latter equality considered a sufficient reason for the falsity of the equality actually given in the problem?
Self-rated confidence	6

No diagram or calculation was completed, and the participant's written argument has the same circular issue as her previous solution in that she writes "This is because the ratios are not proportional." She admits "I was getting kind of bored with the question too, just because they're very similar. It actually took me longer than I expected."

In her interview for question 6, Leah explains:

> *Leah: Oh, this is the equal chance one? This is the one where you roll 6 times?*
>
> *R: Yes.*
>
> *Leah: So I didn't have a diagram set-up but I figured it would be a lot less of a probability.*
>
> *R: How did you figure that?*
>
> *Leah: I don't know…I just kind of figured. Like for this one (for a large tree she drew). The probabilities with 5 rolls, or something, it was way less. So I don't know, I just figured.*
>
> *[…]*
>
> *R: When you say there's more possibilities, how do you reason that?*
>
> *Leah: There's more possibilities against not being able to get at least one or two times or whatever it is. Because like I noticed like this one goes on forever [referring to a branch in the tree diagram drawn for exercise 1], you know, 5x5x5x5, you get so many big numbers that it lowers the probability.*

This argument is accepted as a solution because the calculation was not expected for this question; only a qualitative description of how the probability is affected by the changing variables. However, Leah's confidence is quite high for a solution that does not contain a mathematical proof.

6.2.2.7 Summary

Leah has very low interest in probability and did not find the lesson easy nor challenging. Some items that stand out from the above analysis is that she is generally imprecise in her discussion of chance, and she tries to be efficient by providing minimal reasoning – though she was not very engaged with the material.

This last point does not hold true with Question 1 since she had good qualitative insight, yet she still produced all the necessary work aligned with the Laplace approach. Perhaps one may argue the contrary solely upon analysis of her transcript and written work but her tone of voice and body language during the interview communicated low interest and low commitment. These serve as a loose bedrock for the analysis.

The primary contention with Leah's solutions is the reliance on LR4 for justification: "this is not true because the ratios are not proportional." For questions 3, 5 and 6, this was the crux of her justification. Moreover, she used tree diagrams for Question 1 and Question 5 only. In general, she does not utilize the lesson material in her written work except for the reference to LR4. By question 6 she explains that the statement is not true and one wonders if she is regurgitating LR4 as an easy means to say "this is false', or whether she understands the non-linear relation between the variables *n, k, p* and the probability of the desired event *P*, but is sloppy in explaining her understanding.

Leah likely became tired by the experiment after the first question which may be why she showed very little work for the following questions and relied on LR4 (but not adequately demonstrating she understood what it meant).

For Question 2, she conceded that the answer was obvious (confidence of 7), but in the interview had difficulty explaining why. This is an indication that her answer was an intuitively held belief (Muller, 2008). On Question 6, she said she "figured that the probabilities were not going to be equal" – a statement that is as ambiguous as that for Question 2. However, she began to explain, albeit, not perfectly:

> *"There's more possibilities against not being able to get at least one or two times or whatever it is. Because like I noticed like this one goes on [referring to a branch in a diagram for another*

question] forever, you know, $5 \times 5 \times 5 \times 5$, *you get so many big numbers that it lowers the probability."*

Though her use of LR4 was insufficient for the other questions, it appears the refutation was not lost on her in Question 6, as she was able to explain how the probability *P* is diluted when multiplying the possibilities of obtaining the complementary events.

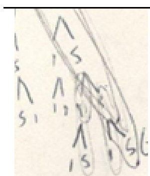
On Question 5, Leah produced a tree diagram which is a tool she used only for Question 1 (a question whose answer was intuitive to her). She miscalculated the number of arrangements (error LR3) maybe because she used "+" instead of "x" as the operation leading her to calculate $3 + 3 + 3 = 9\ outcomes$ instead of $3 \times 3 \times 3 = 27\ outcomes$.

This mistake led her to an erroneous conclusion that $P(at\ least\ one\ even\ number\ in\ 3\ rolls) < P(at\ least\ one\ 3\ in\ 3\ rolls)$ with a confidence of 6. Despite good qualitative proportional reasoning demonstrated in Q1, this conclusion should have been rejected on a qualitative level.

6.2.3 Laplace Refutation- Larry

6.2.3.1 Larry: Question 1

Transcript – LA – R – "Larry" – Question 1	Interpretation
at least 2 3's in 4 rolls at least 2 3's in 5 rolls	Takes note of the events whose probabilities are to be compared
	Evidence of using techniques taught in the lesson: draws a tree for 4 rolls Evidence of using the interpretation of probability presented in the lesson: writes the number of possibilities at the bottom of each stem in the tree (not the name of the outcome, as done in the lesson)
36 x 6 = 216 216 x 6 = 1296 $6^4 = 1296$	Evidence of using the interpretation of probability presented in the lesson: Calculates the number of all possible outcomes
625 + 125 = 750	Evidence of using techniques taught in the lesson:

$1-\frac{750}{1296}=0.42$	uses the complementary events rule Misconception LR3: misses three arrangements in calculating the possible ways of getting 3 non-3's in 4 rolls; takes only two rightmost branches into account, just as Lea, although this misconception has been mentioned in the lesson Larry listened to. However, he noticed his mistake in the interview, at the occasion of discussing Question 3. He did not redo the calculations but realized which arrangements he missed. Mistake of imprecision: equals sign used instead of approximation sign
	Evidence of using techniques taught in the lesson: draws a tree for 5 rolls, but not the complete tree; like Lea, Larry extends only two rightmost branches of the tree for 4 rolls to calculate the number of ways four non-3's can be obtained in 5 rolls. Misconception LR3
$\frac{5^5-2(5^4)}{6^5}=0.241$ $1-0.241=0.758$	Evidence of using the interpretation of probability presented in the lesson: Calculates the probability of getting all or four non-threes in 5 rolls by the ratio of the number of desirable outcomes to the number of all possible outcomes. Evidence of using techniques taught in the lesson: uses the complementary events rule Misconception LR3 Mistake of imprecision: "=" instead of "≈"
Smaller than	Answer: correct, but based on inaccurate reasoning
Self-rated confidence (1-7)	7

Larry had no problem concluding that rolling five times yields a higher probability. He noted:

> *"[…] like intuitively it makes sense right away. Like you're looking for the same thing but in the second [scenario] you have an extra roll. So it's like, of course, you're going to have a better chance for the second option"*

Despite having a strong intuitive sense for the answer, he engaged in calculations. A single tree diagram was produced along with appropriate usage of the complementary event rule to calculate both probabilities. He stated in the interview that he had never seen the use of complementary events in order to calculate probabilities.

Despite making a calculation error, the numerical results obtained were aligned with his intuitive guess and so no extra work is shown and he rates his confidence level at 7.

Also, despite being refuted on the misconception that arrangements are often missed in calculation, he missed 3 of the 4 arrangements of the outcome (non-3, non-3, non-3, 3) in 4 rolls and 3 of the 5 arrangements of the outcome (non-3, non-3, non-3, non-3, 3} out of 5 in 5 rolls. His interview reasoning follows:

> *"So in your two you already have… well in this whole branch you don't have a single three so I know that when you have an extra roll, assuming the same list of probabilities, like these two, it still doesn't matter, you still don't have it and this one you have one alternative way where you do actually get it. So basically you're splitting your final two alternatives, like you don't have it, you're almost going like you have a one quarter chance out of that four. Like getting one little extra chance"*

He was aware that there are more alternatives to a sequence with a single 3 only (as demonstrated in his work captured above), but only considered 2 of the 4 alternatives. His method was to first contemplate the complementary event of obtaining no 3's in 4 rolls and attempts to calculate the probability, but did so incorrectly. To calculate the probability of at least two 3's in 5 rolls, he then "tags on" the fifth roll at the end of the existing tree to multiply the amount of non-3's by 5, and hence obtain 5^5 sequences that are non-3's. In considering the outcome of four non-3's and one 3, he only considers how the probability is affected by the final two rolls saying that we can get (non-3, non-3, non-3, non-3, 3) and (non-3, non-3, non-3, 3, non-3) in those orders. Those outcomes, along with a sequence of five not 3's, contribute to the final probability. His reasoning is that the first 3 rolls before the last two, have already occurred. This causes him to commit misconception LR3, despite hearing a refutation of it in the lesson.

It should be noted that in discussing another problem later in the interview, Larry retrospectively realized he missed arrangements for this question.

6.2.3.2 Larry: Question 2

Transcript – LA – R – Question 2 – "Larry"	Comments
(5 or 6) = $\frac{1}{3} \cdot \frac{1}{6}^{this\ has\ to\ be\ the\ other\ number} = \frac{1}{18}$ vs $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$ Oral: "on my first die… this one could be either a 5 or a 6, so… you have a 2/6 chance of this happening…. Then… the next dice needs to be the opposite. So there is only 1/6 chance of getting that number. So then you multiply your 1/3 and your 1/6 to get 1/18. Versus if you are looking for 6 and 6 you will only have a 1/6 chance each time."	No evidence of using the approach taught in the lesson (LA): his justification is more suggestive of PA. He does not count the number of outcomes but operates directly on probabilities of the simpler events into which he has analyzed the event in question. He uses logical connectives, OR and AND. He does not quote the multiplicative property but he uses it.
Oral: "But it seemed odd, because I know that you are rolling simultaneously so I don't know if that changes anything…. [But] since it is simultaeous, the order does not necessarily matter… I am not looking for a 5 and then a 6. Because if I was looking for a 5 then a 6 then it would be the same chance…."	Is aware of the hypothetical character of his solution: that it hinges upon the assumption of the distinguishability of the two dice. Convinces himself that it makes sense to reason as he did by interpreting "simultaneously" as "order does not matter": so "(5, 6)" is interpreted not as an ordered pair, but as a set.
[the pair 5, 6] has a greater chance of happening.	Answer
Self-rated confidence (1-7)	4

Larry arrived at the correct solution quickly and without much trouble. He reports a low confidence in his answer for a few reasons which are mostly the researcher's speculation. First, he takes on a method of solving the problem that was not taught in his lesson. His written work follows the method of the Properties approach. Larry notes that from the beginning of thinking about the question he realized that he had to consider how the first dice can land on a 5 or a 6 and how the next toss must result in something different. Since this study is qualitative we may only suppose that his awareness of the importance of arrangements is associated with the explicit refutation made of the misconception. Though Larry did not draw nor mention tree diagrams for this particular exercise, we may speculate that the refutation played a role, or he visualized a two-layered tree.

Larry's approach began by assigning a probability $p = \frac{2}{6}$ to the event of rolling a 5 or a 6 on the first toss, after which he assigned a $p = \frac{1}{3}$ chance to the probability that the next toss will be different than the first.

He demonstrates cognitive flexibility as he follows the Laplace approach fairly diligently throughout the questionnaire but not only uses a different approach for Question 2, but engages concepts that were not covered in the lessons, namely conditional probability.

His thinking is sophisticated as it generalizes the probability of obtaining an the pair (5-6), saying that the second roll must be different than the first outcome

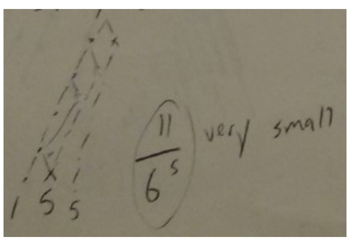
Moreover, there was little difficulty in comparing simultaneous rolls to consecutive rolls:

> *R: Would it be different if you rolled them one after another?*
>
> *L: Um...I suppose it wouldn't. Because the first one you are looking for either a 5 or a 6 and on the second one you are looking for the other one. So I guess not.*

It should be noted that when asked what content was part of the lesson that he had seen before, he implicitly referred to rules based in the Properties method.

6.2.3.3 Larry: Question 3

Transcript – LA – R – "Larry" – Question 3	Interpretation
Oral: [First guess] "I knew right away that there was a really slim chance.... You almost stated that in the video, no? Saying that it wasn't possible to just divide the odds by two."	First guess is not based on "intuition" but on memory of the refutation of proportionality made in the lesson (Misconceptions LR1 and LR4) Mistake: confuses the odds with the probability
4 5's in 5 rolls 2 times as in 4 rolls 0.42	Takes note of the statement to be assessed as true or false. Re-uses a former result. Mistakes:

	Misrepresents the statement: omits “at least” in writing the first event, “at least four 5’s in 5 rolls”; but may be just sloppiness because he considers five 5’s in his tree; the second event is read as referring to some outcome in 4 rolls, which is incorrect because, in this question, the number of rolls was constant. Since he writes the probability of this event as 0.42, which he obtained in Question 1 as the probability of getting at least two 3’s in 4 rolls, his second event was probably “at least two 5’s in 4 rolls.” He also reverses the coefficient of the relation and instead of ½, he takes 2. So the statement, whose truth he is assessing, is not the one given in Question 3 but, probably, “$\mathrm{P}(X_5 \geq 4) = 2 \times \mathrm{P}(X_4 \geq 2)$”
	Evidence of using techniques taught in the lesson: draws a part of a tree for 5 rolls; only those branches that he thinks correspond to “at least four 5’s in 5 rolls”; in the interview, he realizes he omitted four other relevant branches Evidence of using the interpretation of probability presented in the lesson: writes the number of possibilities at the bottom of the stem in the tree; calculates the probability by dividing the number he believes to be the number of desirable outcomes (1 + 5 + 5), by the number of all outcomes (6^5). Misconception LR3: ignores some arrangements in writing but realizes his mistake in the interview
Not true [because] $\frac{11}{6^5}$ [is] very small [compared to 0.42, so it cannot be 2 times 0.42]	Answer
Self-rated confidence (1-7)	7

The tree diagram Larry produced was simplified in that it dealt only with the outcomes he considered relevant. He correctly identified the number of all possible outcomes as 6^5, and used the tree to count the number of desired outcomes, though he did so incorrectly. That being said, in the interview, Larry quickly caught his mistake:

> *"Oh wait, no- there were more options than that. I screwed that up. Okay, well the number is going to be wrong […], so… missing some branches. I'm missing four branches. Regardless, it's still not true."*

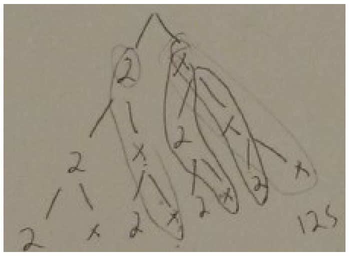
When asked why he seemed so sure that the statement was not true there was a long reflective pause before saying

> *"You almost stated that in the video, no? Saying that it wasn't possible to just divide the odds by two".*

Larry's intuitive reaction to deal with this question not to compare ratios, but to consider how to produce a relevant tree diagram.

6.2.3.4 Larry: Question 4

Transcript – LA – R – "Larry" – Question 4	Interpretation
Oral: [First guess] "I knew it was larger. Because 'exactly' is like the limiting factor here. I knew there were definitely going to be less options."	First guess is based on qualitative approximation.
1 2's in 3 rolls exactly 1 2 in three rolls	Takes note of the statement to be assessed as true or false.
Oral: "I just did the tree. And I think we already had the answer for the chances of getting at least one and now I just needed to look with the same tree where is it that it's exactly one."	Realizes the isomorphism of the problem of calculating the probability of getting at least one 2 in 3 rolls with Problem 1 in the lesson where the probability of getting at least one 6 in 3 rolls was calculated (it was 91/216 ~ 0.42). But still re-does the calculations in his written work.

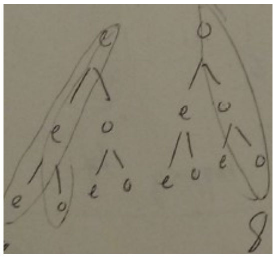
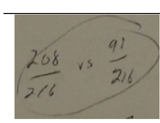
 $1-\frac{125}{6^3}$ vs $\frac{75}{6^3}$ $\frac{216}{216}-\frac{125}{216}$ $\frac{75}{216}$ $\frac{91}{216}$ vs $\frac{75}{216}$	Evidence of using techniques taught in the lesson: draws a tree; uses the complementary event rule Evidence of using the interpretation of probability presented in the lesson: writes the number of possibilities at the bottom of a stem in the tree; calculates the probability by dividing the number of desirable outcomes by the number of all outcomes
Larger than	Answer
Self-rated confidence (1-7)	7

Larry had no issues with this problem. His written justification followed the Laplace approach utilizing the complementary event rule to calculate the probability of obtaining exactly one 2 in 3 rolls, while he reused 0.42 for the second part of the question from previous questions.

In the interview he demonstrated an intuitive understanding and said it's the first thing to come to mind "*Because 'exactly' is like the limiting factor here. I knew there were definitely going to be less options.* "

6.2.3.5 Larry: Question 5

Transcript – LA – R – "Larry" – Question 5	Interpretation
once even 3 rolls at least 1 5, rolls	Takes note of the events at play in the statement to be assessed as true or false, with inaccuracies: "at least" missing in description of the first event (but he takes this into account in his calculations); does not take note of the quantitative relation between the probabilities of these events that has to be verified. This relation is not evokes anywhere in his written solution or oral account of it.

 Oral: "I did the tree again but everything was in E's and O's and everything was in ½ so at the end of it… it's very likely you're going to get at least one even. There's only one branch that it's all odds…. I didn't really do it, but each branch is 8, there's only 64 options, so there's a 7/8 chance that you will get even." $\frac{216}{216} - \frac{8}{216}$ $\frac{208}{216}$	Draws a tree for 3 rolls with two possible outcomes for each roll: odd and even number. Correctly estimates the probability of getting at least one even number to be 7/8 in his oral description of his solution, although the way he justifies it does not make sense. For a single roll, he is sometimes thinking of a probability space with outcomes {1, 2, 3, 4, 5, 6} three of which are even numbers, and sometimes – of one with outcomes {E, O}, and this may be the cause of his confusion. In the written work, he seems to believe that there are only 8 ways of getting an odd number in rolling 3 dice. He may be confusing the number of ways of getting an odd number on a dice with the number 2 of possible outcomes in an even-odd game. So instead of saying that there are 3 x 3 x 3 ways of getting an odd number, he says that there are 2 x 2 x 2 ways. Evidence of using the interpretation of probability presented in the lesson (LA): mentions "64 options", and writes a whole number (8) at the end of the odd-odd-odd branch, so appears to count the outcomes Evidence of using techniques taught in the lesson: draws a tree; uses the complementary event rule Mistake: incorrect calculation of the number of ways of getting odd-odd-odd
$\frac{91}{216}$ Oral: "Oh yah, I just use the same number here."	Realizes that the probability of getting at least one 5 in 3 rolls has already been calculated in Question 4 and re-uses it here.
 False	Answer: $\frac{208}{216}$ vs $\frac{91}{216}$ is circled in his written work so it intended as an answer, but it is not explicit what exactly is considered false. There are no traces of showing that 208/216 is not equal to 3 times 91/216

	and in the interview Larry says that he "didn't really do it" as if referring to calculations.
Self-rated confidence (1-7)	7

Larry found this problem easy and claimed that he had no immediate thoughts of what the answer would be. Instead, he began to re-evaluate the tree diagram since he realized, at the onset of reading the question, that a new tree would be required. This tree, he said, would not have branches of weightings 1 and 5, but 3 and 3 denoting the number of ways an even and an odd can occur upon rolling a dice. However, he mixes the probability of a single success as he refers to the branch weightings as 3 to 3, as well as saying that "everything was in E's and O's and everything was $\frac{1}{2}$". This was his third implicit reference to Properties interpretation of probability (the first being during the interview and the second in Question 2).

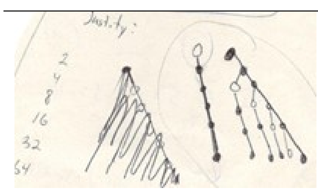
He correctly concluded the statement was not true and during the interview he correctly determined $P(at\ least\ one\ even\ in\ 3\ rolls) = \frac{7}{8}$ using complementary events rule. However, he did so with some calculation errors.

His written justification was centered on the tree he produced. He correctly determined that the number of all possible outcomes is 216 when trying to obtain at least one even number in 3 rolls. However, it is unclear what operation was used to conclude that each branch provided 8 unique sequences. In the interview he confirms this notion "each branch is 8, right? So 8 times 8…there's 64 options so there's a seven out of 8 chance that you will get an even*".*

His written justification has a slightly different approach in that he says there are 8 ways to obtain the complementary event (no even numbers in 3 rolls). Divided by the number of all possible outcomes, 216, we obtain $(no\ even\ numbers) = \frac{8}{216}$. The number of all possible outcomes, 216, was not recalculated; it was reused from the previous exercises. So his error is in obtaining an '8'. So Larry's numerical conclusion was incorrect, but the qualitative proportionality was correct in that $P(at\ least\ one\ even\ in\ 3\ rolls) > P(at\ least\ one\ 5\ in\ 3\ rolls)$.

It was not asked during the interview, but it is plausible that Larry's error stems from applying the correct operation, multiplication, but using an incorrect number, 2, to calculate the number of outcomes for that branch. So it is believed he found $2 \times 2 \times 2 = 8$ outcomes for one branch. It is uncertain why he made this error.

6.2.3.6 Larry: Question 6

Transcript – LA – R – "Larry" – Question 6	Interpretation
[First guess] Oral: "I would have thought they would have been pretty close. My inclination is that it was true but I wasn't sure. But the numbers ended being pretty close. So inclination was that its true, but you know, I didn't trust my gut."	First guess: True or almost true
2 5's in 6 rolls $\frac{91}{216}$ = 0.42	Takes note of the events at play in the statement to be assessed as true or false, with inaccuracies: "at least" missing in description of the first event (but he takes this into account in his calculations); does not describe the second event (at least one 5 in 3 rolls) but writes down its probability which he remembers from Problem 1 in the lesson or Question 4.
 5(3120) + 15600 $1 - \frac{31200}{46656} = 0.331$	Found that the last row of a tree for 6 rolls would have 64 items and did not feel like drawing it. So he decided to use the complementary event rule and just drew the branch for "no 5's" and those branches that he thought corresponded to "one 5". But they all start with "no 5 in the first roll" so he missed one outcome and instead of $6 \times 5^5 + 5^6 = 34{,}375$ outcomes he counted $5 \times 5^5 + 5^6 = 31{,}250$ except that he made a mistake in copying the result of 5^6 from the calculator and wrote 15,600 instead of 15,625. There is also a mistake in 5^5: 3120 instead of 3125. Evidence of using the interpretation of probability presented in the lesson (LA): appears to count the

	outcomes Evidence of using techniques taught in the lesson: draws a tree; uses the complementary event rule Misconception LR3: omission of one arrangement Mistake: computational mistakes
Not true	Answer
Self-rated confidence (1-7)	7

An incorrect numerical answer was obtained by Larry but only because he missed a single arrangement for one 5 in 6 rolls (he indicated there were 5 ways to obtain such an arrangement). He conceded to two things: "*I saw this and I just kind of wanted to get this done*" and that the question wasn't difficult but tedious because he was aware that there was likely an algorithm that forgoes large tree diagrams, the main difficulty he encountered in this problem. Despite his lack of motivation at this point, he still managed to complete a written solution in a sufficient way, unlike Leah.

He produced a modified tree diagram in which he denoted the ways a dice can land on no 5's or one 5 in 6 rolls. This method was successful since he calculated the number of all possible outcomes as 6^6, and realized that the entire tree was not necessary to count the arrangements of the desired event.

Moreover, Larry had no strong intuition towards any of the answers but said he was tempted to consider that the statement was true.

> *"I would have thought they would have been pretty close. My inclination is that it was true but I wasn't sure. But the numbers ended being pretty close. So inclination was that it's true, but you know, I didn't trust my gut."*

Despite this, he proceeded to develop his personalized method of solving while using the complementary event rule with a confidence of 7.

6.2.3.7 Summary

To recall, Larry is interested in mathematics and he found the lesson easy to follow (mental effort invested rated at 3 of 9); an unexpected result. Muller (2008) found mental effort was rated higher for students who watched a misconception-refutation-based lesson.

On a few occasions, Larry would miss arrangements but when he did, he would realize his mistake retroactively – a result perhaps associated with his exposition to LR3.

The refutation LR4 played a large role in Larry's justifications as he made explicit reference to it in the interview and never disclosed that he was tempted by linear modelling. He also did not overly rely on the statement uncritically on the occasions that he made reference to it (except, perhaps, in Question 3).

However, an issue with this format became obvious throughout his analysis as he seemed to overgeneralize the refutation statement, taking the meaning of the statement from a "linearity caveat", as it was meant to be, to an impossibility of linearity.

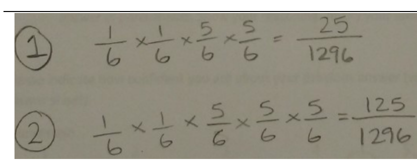
Larry followed the lesson closely by producing tree diagrams for each question except for Question 2 where he abandoned the approach presented in the lesson and followed a method resembling the Properties approach. Yet, he did not miss the second arrangement of the pair (5, 6). This may be attributed to the exposure to refutation statement LR3 warning about the neglect of arrangements.

By Question 6, Larry disclosed he was tired of the experiment especially since this final question was computationally demanding. Despite this, and his initial belief that the probabilities were equal, he produced a customized tree diagram but with computational mistakes.

Overall, Larry was confident but sloppy in his work as he made several minor calculation errors. Importantly, he exhibited distrust in his intuitive inclinations and carried through with proofs when needed.

6.2.4 Properties Exposition – Peggy

6.2.4.1 Peggy: Question 1

Transcript – PA – E – "Peggy"– Question 1	Interpretation
	Evidence of using techniques taught in the lesson: Uses the multiplicative property for calculating the probability of getting 2 threes in 4 rolls, and 2 threes in 5 rolls, but makes

	mistakes Misconception PR3 (=LR3): misses other possible arrangements of the threes in 4 and 5 rolls Mistake of replacing “at least 2 threes” by “exactly 2 threes”
smaller than because ${}^{25}/_{1296}$ vs ${}^{125}/_{1296}$	Answer correct, incidentally
Self-rated confidence (1-7)	6

Peggy was very doubtful of any intuitive inclination she had. It seems past experience with probability classes taught her to not trust intuition. She reported not having an intuition for this question and says she immediately engaged in calculating the probabilities. She does not draw a tree diagram. During the interview she concedes that she helped validate her mathematical justification by remembering her experience with dice games:

> “*Okay, if you were given the option of rolling four times to get two 3’s or five times, and I was like (nod of affirmation). The logic came after the math.*”

In fact, the answer became obvious to her only after reflecting on her past experience.

It is uncertain whether the calculation errors she made were as expected for an exposition participant or she was confident about her effort because her solutions aligned with her qualitative intuition.

Her errors were two-fold. The first was a demonstration of “quantitative negligence” in that the student devised a solution to a problem that was different from the one asked. Her solution appeared as an attempt to answer the question:

$P(X_4 = 2)$ Is greater, equal, or smaller than $P(X_5 = 2)$? (X_n is the number of 3’s in n rolls of a dice)

So Peggy did not consider the probabilities of obtaining three, four or five successes in the trials. She also neglected arrangements. Only a single arrangement for each probability was considered. In the end, her answer was correct but her mathematical justification was incomplete. However, as stated, her intuition about the problem matched the outcome of her calculations and so she deemed the solving process to be complete.

6.2.4.2 Peggy: Question 2

Transcript – PA – E – "Peggy" – Question 2	Comments
[Both have the same chance] because the odds are 1/36 for either pair Oral: "I just thought that it was just arbitrary what two numbers, like if you're aiming for any two numbers like it doesn't change the probability, I think."	Evidence of using the interpretation of probability presented in the lesson (PA): does not calculate the number of all possible outcomes but multiplies the probabilities directly; but evidence is very weak; she said in the interview that she did not do any calculations; she also speaks of "odds" not of probability, but the meaning of "odds" was not mentioned in the lesson Misconception PR3 (LR3): ignores the different possible arrangements in which the pair 5, 6 could occur.
Oral: [First guess] (6, 6) is less likely because (5, 6) "[two different numbers] seems more random [than the same two numbers],…, seems more like a typical guess"	Misconception (not addressed in the lessons): more typical ➔ more probable Might be thinking in terms of LA: p(two same numbers) = 6/36 = 1/6; p(two different numbers) = 1 – 1/6 = 5/6
Self-rated confidence (1-7)	7

The general response given by Peggy was very similar to that given by Leah. She reported a high confidence for the unjustified answer that both pairs have the same probability.

> *R: Did you have any initial thoughts?*
>
> *Peggy: Yah for sure. I think I just thought that it was just arbitrary what two numbers, like if you're aiming for any two numbers like it doesn't change the probability, I think.*
>
> *R: Actually, where is your work for number 2?*
>
> *Peggy: I didn't really do any. I just thought it was the same for both.*

Like Leah, Peggy forgoes the lesson. Deciphering the choice of words used in the interview, *"when you're <u>aiming</u> for any two numbers*" suggests that she was reflecting upon past experience with dice

(Yahtzee). She pointed out in the interview that while playing Yahtzee she relies on intuition more than logic.

During her interview for Question 4, she reported reflecting on her experience with dice and referred to Question 2. The discussion with her suggests that, despite her assertion during the interview on Question 2, she was, in fact, conflicted in her intuitive judgement of the problem:

> *Peggy: Oh, this is the type of situation where, even though that seems less likely [(6,6)] this seems more random [(5,6)], it's actually the same likelihood […].*
>
> *R: I'm curious. You said 6-6 seems less likely, and 5-6 seems more random?*
>
> *Peggy: I think through playing Yahtzee I know that that's not true. But I think, and I don't think I would be alone on this if you were to poll other people, I think there's something that seems more like 'Oh that has to be less likely'. Like what are the odds it's the same two numbers. Do you know what I mean?*
>
> *R: So what makes 5-6 more random?*
>
> *Peggy: Just the fact that it seems more like a typical roll, I guess. It's the least educate thought ever, you know what I mean?*
>
> *R: That's interesting.*
>
> *Peggy: Almost the same as like, in Yahtzee it's like 5 of the same number, and we think of that as impossible, like 5 sixes or whatever. But I remembered, it took some time to sink in, that it would be just as random if someone were to get 3,4,6,5 even though it seems more likely because its random numbers…I don't know it sounds really stupid.*

Peggy used the representative heuristic to explain her feeling that the pair (5, 6) is more likely than (6, 6) as indicated by the use of the word "typical." There are indications that the availability heuristic was influencing her judgement as well when she stated "Oh that has to be less likely. Like what are the odds it's the same two numbers. Do you know what I mean?"

Fascinatingly Peggy was very aware of her personal biases. And she was vocal about "not trusting her gut" which means she settled for the conclusion that the pairs were equally likely to occur.

> "[...] *in Yahtzee it's like 5 of the same number, and we think of that as impossible, like 5 sixes or whatever. But I remembered, it took some time to sink in, that it would be just as random* [correct usage of the word] *if someone were to get 3, 4, 6, 5 even though it seems more likely because it's random numbers… I don't know it sounds really stupid.*"

In her experience with dice five of the same numbers seems much less likely than 5 different numbers: A correct statement explained by the availability heuristic. Upon deliberation, she used the representative heuristic to argue 3, 4, 6, 5 appeared more likely because it's 'more random', but in reality she believed it to be equally likely to obtain 5, 5, 5, 5.

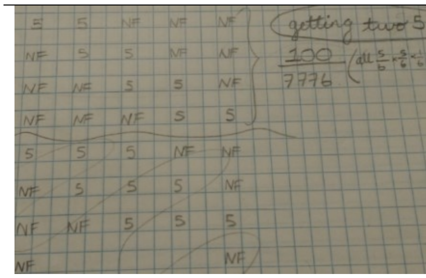
This is how she justified that statement is true. And it was done so by conflicting intuitions: heuristics versus doubt in her "gut reactions."

Though she may be answering a different question than the one asked, this demonstrates why the misconception exist. Students simply do not conceive of the different dice arrangements. They do not consider, for example, that the 3-4-6-5 can be obtained if the third dice lands on 3, and the first on 6. Though Peggy is correct in stating the outcomes are equal likely, her error was in her inability to properly imagine the exercise situation of rolling the unordered pair. This likely stems from subjective indistinguishability.

All of her justifications took place in her head and not on paper. She did not use any of the concepts from the video and rated her confidence at 7. She used cognitive biases to counter other biases (heuristics) where a single diagram would have been sufficient to illuminate the correct answer.

6.2.4.3 Peggy: Question 3

Transcript – PA – E – "Peggy"– Question 3	Interpretation
Oral: "I wanted to say that it was true, initially before doing any of that. But I've learned to never trust my gut."	First guess: True
5 AND 5 AND 5 AND 5 AND Not five = $\frac{5}{7776}$ 5 AND 5 AND 5 AND Not five AND 5 = " 5 AND 5 AND NF AND 5 AND 5 = " 5 AND NF AND 5 AND 5 AND 5 = " NF AND 5 AND 5 AND 5 AND 5 = " 5 AND 5 AND 5 AND 5 = $\frac{5}{7776}$ $\frac{26}{7776}$	Calculates $\mathrm{P}(X_5 \geq 4)$ Evidence of using the interpretation of probability presented in the lesson: uses the multiplicative and additive properties; does not count outcomes Evidence of using techniques taught in the lesson: Decomposes an event into simpler ones; uses the multiplicative property for calculating the probability of each possible arrangement of getting at least four 5's in 5 rolls and then the

	additive property to get the probability of getting at least four 5's in 5 rolls by adding those. Does not use the tree of all possible outcomes as shown in the lesson but creates a tabular representation of her own invention. Mistake: inaccuracy of notation: event = probability
 Oral: "So this was four times, and this was at least 2, and so when I calculated just getting a two it was more than double the odds without even calculating the 3 or 4 so I just stopped it."	Starts calculating $\mathrm{P}(X_5 \geq 2)$, but does not finish. Calculates the probabilities of some of the arrangements of two 5's in 5 rolls, notices that they will all be $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$ and estimates that $\mathrm{P}(X_5 \geq 2)$ will be greater than $\frac{100}{7776}$ which is already more than double $\frac{26}{7776}$, so the statement must be false. Mistake: "odds" instead of "probability"
(this is not true) because the odds of rolling two 5's is more than twice the odds of rolling four 5's without even adding on the odds of rolling three or four 5's	Answer correct
Self-rated confidence (1-7)	4

Peggy had an inclination to say the statement was true but was not too detracted by her intuition because of her awareness that probability is counter-intuitive: "*I've learned to never trust my gut.*" She proceeded to calculate the probabilities, loosely following the method presented in her lesson. She did not draw a tree diagram but instead determined all the relevant arrangements that obtained at least four 5's in 5 rolls. She then proceeded to calculate the probability of obtaining at least two 5's in 5 rolls but realized *"[…] when I calculated just getting a two it was more than double the odds without even calculating the 3 or 4 so I just stopped it."* She stated that she was unsure about her solution probably since her intuitive tendency to view the problem as a linear relation did not match the solution she derived, which explains her self-rated confidence of 4 (Muller, 2008).

Peggy decided to forgo the tree diagram on account of space and a desire for efficiency, which was reflected in her discontinuation of the calculation once she had sufficient evidence to conclude the statement as not true.

It should be noted that Question 3 was isomorphic to the relation presented in the video and yet this student did not make reference to it.

6.2.4.4 Peggy: Question 4

Transcript – PA – E – "Peggy"– Question 4	Interpretation
Oral: "[at least one 2 in three rolls] would be more likely because it's more vague… [exactly one 2] is way more precise…That is more precise, like you can only get this, whereas this is like more possibilities."	First guess: $\mathrm{P}(X_3 \geq 1) > \mathrm{P}(X_3 = 1)$
2 2 2 1/216 2 2 N2 5/216 * 2 N2 N2 25/216 N2 2 2 5/216 * N2 N2 2 25/216 N2 2 2 5/216 2 N2 2 5/216 * N2 2 N2 25/216 At least one 2 = $\frac{96}{216}$ Exactly one 2 = $\frac{75}{216}$	Writes out all possible arrangements of 2's and not 2's in a tabular form and notation of her own invention, but makes a mistake: the arrangement N2 2 2 appears twice. Writes the probability of each arrangement, probably using the multiplicative property and then adds up the probabilities corresponding to "at least two 2's" (additive property used implicitly). Gets 96/216 because of the above mistake: one 5/216 too many. Marks arrangements corresponding to exactly one 2 with a star and calculates the probability of getting exactly one 2. Evidence of using the interpretation of probability

	presented in the lesson: uses the multiplicative and additive properties; does not count outcomes Evidence of using techniques taught in the lesson: Decomposes an event into simpler ones; uses the multiplicative and additive properties implicitly Does not use the tree of all possible outcomes as shown in the lesson but creates a tabular representation of her own invention. Mistake: one arrangement repeated twice; the arrangement N2 N2 N2 missing; incorrect probability of getting at least one 2; event = probability
(larger than) because there are more possible combinations that result in at least one 2 than in exactly one 2 Oral: "I realized that this [rolling exactly one 2] was just a subset of that [rolling at least one 2]. And like that eliminates a bunch of numbers so obviously the probability is going to be lower."	Answer correct
Self-rated confidence (1-7)	5

With a self-rated confidence of 5, Peggy arrived at the correct answer after listing the relevant sequences of 2 and non 2, without a tree, and followed the Properties method. She stated that she struggled with this exercise and that she couldn't decide whether obtaining *exactly* one 2 would be probabilistically equal to or less than obtaining *at least* one 2.

In the interview she revealed that Q4 and Q2 were comparable and related:

> *"I think I was just thinking, when I first read the question, that like 'Oh, this is the type of situation where, even though that seems less likely [(6, 6)] this seems more random [(5, 6)], it's actually the same likelihood. I think that was my first thought just like 'Oh, I don't want to make any assumptions*

here because maybe like when I draw it out it will end up being the same combos but then I realized that didn't make sense. I guess, all that to say that I really wasn't sure about this question, at all"

She related the pair (6, 6) to obtaining exactly one 2 in 3 rolls and the pair (5, 6) to obtaining at least one 2 in 3 rolls. Though she did correctly evaluate the problem, her comparison is an important one to analyze.

Similar to Question 2, Peggy seemed to be influenced by at least two directions of thought before settling for her given justification. Her first intuition was correct in claiming that "[*exactly one 2]* is more precise, like you can only get this, whereas this [*at least one* 2] is like more possibilities." However, she was persuaded by her results from Question 2 wherein she had a strong intuition that the pair (5, 6) was more likely than (6, 6), yet she reasoned that they equal nonetheless.

Analogously, for Question 4, she felt that the constraint of "exactly" implies than it must be less likely than "at least", yet she remembered being challenged on her intuitions in Question 2 and wondered if her intuition has deceived her once again. In the end, she did reason correctly and concisely: *"[exactly] was just a subset of that [at least]. And like that eliminates a bunch of numbers so obviously the probability is going to be lower*", but this came only after entertaining two possible and contrary intuitions.

6.2.4.5 Peggy: Question 5

Transcript – PA – E – "Peggy"– Question 5	Interpretation
Oral: "… when I read it I thought 'Oh that totally makes sense [implying it's true]; there's one 5 out of the six, and there are 3 even numbers, yah sure, I like that statement, it's nice."	First guess: True that $\mathrm{P}(Y_3 \geq 1) = 3 \times \mathrm{P}(X_3 \geq 1)$
E E E E E O E O O O E E O O E	To calculate $\mathrm{P}(Y_3 \geq 1)$, Peggy decomposes the event "at least 1 even number in 3 rolls" into a conjunction of 7 simpler events, writing them in a tabular form instead of a tree form. Finds that the probability of each of these is $(\frac{3}{6})^3 = \frac{27}{216}$

O E O E O E $\frac{27}{216} \times 7 = \frac{189}{216}$ (Note: in Peggy's writing, the symbol $\cup E$ was used instead of "O")	(implicit use of the multiplicative property) and then calculates the probability of the compound event by multiplying this result by 7 (implicit use of the additive property). Does not use the complementary event property although this would have simplified the calculation. Evidence of using the interpretation of probability presented in the lesson: uses the multiplicative and additive properties; does not count outcomes Evidence of using techniques taught in the lesson: decomposition of an event into simpler ones; application of the multiplicative and additive properties
5 5 5 1/216 5 5 NF 5/216 5 NF 5 5/216 NF 5 5 25/216 NF 5 NF 25/216 NF NF 5 25/216 5 NF NF 25/216 $\frac{111}{216}$	Re-calculates $\mathrm{P}(X_3 \geq 1)$ for the event "at least one 5 in 3 rolls" although in question 4 she had calculated the probability of an isomorphic event – "at least one 2 in 3 rolls." Redraws the tree in tabular form. Makes a mistake in the fourth row: 25/216 instead of 5/216; as a result, $\mathrm{P}(X_3 \geq 1) = \frac{111}{216}$
(this is not true) intuitively it seems logical but my math told me that the odds were 189/216 and 111/216 respectively	Answer: correct; from the interview it is clear that "not true" refers to 189/216 not being equal to 3 times 111/216.
Self-rated confidence (1-7)	4

Peggy "*found this question pretty tough*" and that "*what [she] thought of right off the bat contradicted the math*"; a classic scenario where intuition and mathematical conclusions do not agree, yet the student, who, by nature, is doubtful of her intuitive abilities, settled for the mathematical justification.

A confidence level of 4 implies her preconceptions were refuted (Muller, 2008): "Okay, I'll believe it [the math], but I don't know why." And they were refuted not by explicit discussion of misconceptions (as the refutation students experienced), but by the mathematical tool she was equipped with in the lesson.

She reported that when she first read the question statement she thought "*I like that statement, it's nice*" suggesting she was tempted to apply a proportionality relation and it was difficult to abandon.

In the end, she calculated the respective probabilities using the Properties method correctly.

6.2.4.6 Peggy: Question 6

Transcript – PA – E – "Peggy"– Question 6	Interpretation
Oral: "it's just the same thing happening twice so the odds of the first thing are *x*, it should just be the same too… like more rolling opportunities, same chance it's going to happen again."	First guess: True that $\mathrm{P}(X_6 \geq 2) = \mathrm{P}(X_3 \geq 1)$ based on vague idea of proportionality
3 ROLLS 0.5138	Re-uses the value of $\mathrm{P}(X_3 \geq 1)$ from the previous question, with the mistake (111/216 instead of 91/216).
6 ROLLS 5 NF NF NF NF NF 5 5 5 5 5 one five = $\frac{3125}{46656}$ 0.06697	Peggy draws a table for all 6 of the possible arrangements of one 5 in 6 rolls, but then takes into account only one in calculating the probability of one 5 in 6 rolls: $\mathrm{P}(X_6 = 1) = \frac{5^5}{6^6}$, instead of $6 \times \frac{5^5}{6^6}$ (~0.40). Perhaps she was planning to use the complementary event property to calculate $\mathrm{P}(X_6 \geq 2)$. But then she would have to calculate also $\mathrm{P}(X_6 = 0) = \frac{5^6}{6^6}$ (~0.33). Then $\mathrm{P}(X_6 \geq 2) =$

	$1-(\mathrm{P}(X_6=1)+\mathrm{P}(X_6=0))$ which would be about 0.26. She does not do that; leaves the calculations unfinished. Evidence of using the interpretation of probability presented in the lesson: weak, appears to use the multiplicative property; does not count outcomes Evidence of using techniques taught in the lesson: decomposition of an event into simpler ones; application of the multiplicative property; beginning of using the complementary event property Misconception PR3 or just a mistake of forgetfulness: omits certain arrangements Mistakes: "odds" instead of "probability"; inconsistency in writing: event = probability; computational mistake carried over from Question 5
Oral: "So I ended up saying this is true, because I started doing the math and was like 'there has to be an easier way than doing that'. I was going to do it for one 5, two 5…and was like 'I don't know if you're expecting me to do that' so I ended up just going with my gut. So I put a low certainty."	Perhaps she was not going to use the complementary event property, since she says she was "going to do that for one 5, two 5." She may have believed that she did calculate the probability of getting one 5 in 6 rolls correctly, so the PR3 misconception or mistake would be there.
(this is true) because it is essentially two sets of the same occurrence	Answer
Self-rated confidence (1-7)	3

Peggy displayed some fatigue toward the experiment: "*I sort of flaked out*". She began to engage the Properties approach but ceased the procedure noting how much work it was going to be. With a confidence of 3 of 7 she remained with her intuitive inclination that the statement is true:

"[…] maybe slightly more founded than a guess, that it just seemed sort of, in my mind, like, it's just the same thing happening twice so the odds of the first thing are x, it should just be the same too… like more rolling opportunities, same chance it's going to happen again".

She attempted to think abstractly about the problem but decided that the statement was true using an argument centered around of proportional reasoning despite having correctly reasoned in the previous questions (with the exception of Question 2): "This is true because it's essentially two sets of the same thing."

It is evident that she did not conceive of the concept of non-linearity in binomial probability among changing variables. Her low confidence is attributed to her abandonment of the lesson and not engaging in a mathematical procedure. It was expected that the students could deduce how probabilities alter with changing variables even without a refutation of the misconception given in the lesson.

6.2.4.7 Summary

Generally, Peggy was interested in probability but is well aware of its counter-intuitiveness. She likely had the largest and most recent experience with dice games. Based on her persistent application of self-doubt in her intuitions, it seemed that her exposure to experimental probability was an important factor throughout the study. She rated her mental effort invested at 5 of 9.

The most noteworthy point of her behavior was that Peggy seemed the most tempted by common misconceptions ranging from using the representative heuristic, the availability heuristic and proportional modelling. Yet, she was also the most cautious about her intuitions and, as determined from the interview- she did not want to trust "gut reactions".

Peggy loosely followed the lesson, mainly employing the additive and multiplicative property (sometimes implicitly), but she never produced a tree diagram. She devised her own way of representing the situation, using a list of arrangements. This method seemed to be overall successful for her throughout the experiment. She made an error in Question 1, however, owed to misconception PR3 and considered only one arrangement for each event. She did not return to this question to fix the error despite successfully utilizing the same method for other questions. Also, she did not produce any calculations for Question 6 though she had considered potential solutions (as revealed in the interview). Additionally, she reasoned that $P(at\ least\ one\ 5\ in\ 3\ rolls) = P(at\ least\ two\ 5's\ in\ 6\ rolls)$ loosely resembling reasoning that follows linear modelling, but this was achieved with low confidence (3 of 7). For Questions 3-6 Peggy had the lowest confidence among participants.

6.2.5 Properties Refutation – Perry

6.2.5.1 Perry: Question 1

Transcript – PA – R – "Perry" – Question 1	Interpretation
$P(1\ or\ 2\ or\ 4\ or\ 5\ or\ 6) = P_1 + P_2 + P_4 + P_5 + P_6 = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{5}{6}$	Evidence of using techniques taught in the lesson: Breaks down the event "not 3" into an alternative of 5 outcomes. Evidence of using the interpretation of probability presented in the lesson: Calculates the probability of getting not-3 in one roll of a dice using the additive property of probability.
4 times $P(not\ 3\ and\ not\ 3\ and\ not\ 3\ and\ not\ 3) = \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{625}{1296}\cong 0.48$ 5 times $P(not\ 3\ and\ not\ 3\ and\ not\ 3\ and\ not\ 3\ and\ not\ 3) = \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{3125}{7776}\cong 0.403$	Evidence of using techniques taught in the lesson: appears to want to use the complementary event property, which he interprets as calculating the "chances to lose" and then concluding that the inverse inequality holds about the chances to win. erroneously takes the event of not getting a 3 at all in 4 rolls (resp., 5 rolls) to be complementary to the event of getting at least two 3's. Evidence of using the interpretation of probability presented in the lesson: uses the multiplicative property Misconception ~ PR3: omits other possible ways of "losing" or not getting at least two 3's in 4 (5) rolls
Smaller than	Answer: incidentally correct, but based on inaccurate reasoning
Self-rated confidence (1-7)	5

Perry had an initial guess that the answer was "larger than." When asked to explain his guess, his reasoning was unclear and actually seemed to support the belief that the answer was "smaller than":

R: What did you say your initial thoughts were?

Perry: I said it was going to be larger than.

R: Do you know what kind of reasoning you might have had in order to determine that?

Perry: I think because you're rolling more times the chances are greater. I think that's what I was going with.

He reported in the interview that had he not watched the lesson that he would have guessed the answer to be larger than. This error in thinking is likely similar to Lea in that the salient features of the problem were automatically compared as two fractions, which explains why he was surprised by the answer.

In retrospect, he understood that more rolls yields higher chances of success. His written solution refuted his misconception and it consisted of applying the complementary event rule.

6.2.5.2 Perry: Question 2

Transcript – PA – E – "Perry" – Question 2	Comments
[Both have the same chance] Oral: "A good quote from the video was 'Dice have no memory' so the chances of getting a 5 and a 6 are the same as getting a 6 then a 6."	Interprets the question as about rolling one dice twice and comparing the probability of first getting a 5 and then a 6, or first getting a 6 and then a 6. So for him, "(5, 6)" represents an ordered pair. His response that the probabilities are equal is correct under this interpretation. He does not fall into the trap of the Misconception PR2; if 6 is obtained in the first roll, it is not less likely to obtain another 6 on the second roll than another number. Evidence of using the interpretation of probability presented in the lesson (PA): only insofar as he evokes Misconception PR2

	mentioned in the lesson.
[After realizing that he misread the question, justifies, in a somewhat chaotic way, his answer in the case of two consecutive rolls referring, indirectly to the multiplicative property] Oral: "Using P of 6 AND 6… and [P] of 5 AND 6, which is 1/36, I guess. You have the addition rule for the [simultaneous rolling of two dice case] Because it's one event, you'd do a 6 AND 6 OR 6 AND 5…. [Drawing a tree would have helped] for the case of [two consecutive rolls of one dice] but I guess not for the simultaneous roll… because it's just one event, I think."	Evidence of using the interpretation of probability presented in the lesson (PA): refers to properties of probability, calling them "addition rule", and "multiplicative rule." Mistake: Thinks that he should be calculating P((6 & 6) or (6 & 5)) because rolling two dice simultaneously is "one event." No evidence of using the techniques shown in the lesson: no tree used, because it is just one event.
Self-rated confidence (1-7)	7

Perry's interpretation was that the dice were rolled consecutively. It should also be noted that Perry was the only one to be exposed to the misconception PR2: Dice have no memory of past events.

Perry showed no work, reported a 7 of 7 confidence and admitted that he completed the question in less than 30 seconds; this is indicative of an intuition-driven answer "*I just read it and thought 'Oh yah, that's it'*".

After rereading the question for the interview, he replied:

> *"I wrote that they were equally likely. A good quote from the video was 'Dice have no memory' so the chances of getting a 5 and a 6 are the same as getting a 6 then a 6" as well as stating "[…] just because I thought it was the independent kind of rolling I just considered the quote from the video "dice have no memory of past events" so every number had an equal chance every single roll".*

In Perry's justification, he referred to PR2 to justify that "*every number had an equal chance every single roll.*" Keep in mind that he believed the dice were rolled consecutively. Based on this evidence, it was not clear why he believed "no memory" justified equal likelihood of the events. The rest of the interview may have illuminated that:

R: Basically, would it make a difference if the question said 'rolling the dice simultaneously' or 'rolling one dice twice'.

Perry: Because you're using the different rolls aren't you? Using P of 6 and 6…and 5 and 6 which is 1/36, I guess. You have the addition rule for the other one.

R: The other one being simultaneously?

Perry: If you were rolling simultaneously versus the same dice twice you'd be using two different rules.

R: And which rule would be applied to simultaneously?

Perry: The multiplicative one. Because it's one event you'd do a 6 AND 6 or 6 AND 5. Versus the other one, which would be roll, then another roll.

R: Did you make a tree diagram for this one at all?

Perry: No I didn't

R: Do you think that would have helped?

Perry: Probably in this case, yah. Well, I guess not for the simultaneous one.

R: Why not?

Perry: Because it's just the one event I think. Whereas rolling a die then another die is two events.

R: So you're viewing rolling two dice simultaneously as one event?

Perry: I think so, yah.

Perry clearly confounds the mathematical term 'event' with its vernacular meaning. Moreover, he believes that the chronological order of the physical dice landing determines whether a tree diagram can be used.

6.2.5.3 Perry: Question 3

Transcript – PA – R – "Perry" – Question 3	Interpretation
Not true based on the arrangements of events shown in the video Oral: "I knew that the number of arrangements was going to be different…. Well, the next question was similar so I actually went through the different arrangements and stuff and I think once you lay them all out, and do the addition rule, and definitely changes	Evidence of using techniques taught in the lesson: does not actually use the technique of drawing a tree and splitting the events into simpler ones and using the multiplicative and additive properties, but indirectly refers to such things in the video and that he actually applied these techniques in Question 4. He

them a lot. Not in the way that I would expect…just considering…the logic I guess. I think definitely laying them out and doing the different rules does change them a lot. No, it was interesting, yes.	just signals these techniques using metonymy: “addition rule” instead of “multiplicative and additive properties”. Evidence of using the interpretation of probability presented in the lesson (PA): weak evidence; “addition rule” mentioned
Not true	Answer: correct, based on refutation of Misconception PR4 in the video
Self-rated confidence (1-7)	7

The written justification is insufficient in that it does not indicate a full understanding of the problem, solution or Properties approach. In the interview Perry cited the refutation PR4 in order to justify that the statement was not true, with a confidence of 7.

> *“Perry: Having seen the video, definitely, I was pretty sure it wasn’t going to be true.*
>
> *R: So you didn’t do work here either?*
>
> *Perry: For this one, no. I knew that the number of arrangements was going to be different.*
>
> *R: So how would you describe how the number of arrangements affects the probability?*
>
> *Perry: Well, the next question was similar so I actually went through the different arrangements and stuff and I think once you lay them all out, and do the addition rule, and definitely changes them a lot. Not in the way that I would expect…just considering…the logic I guess. I think definitely laying them out and doing the different rules does change them a lot. No, it was interesting.”*

He implied being surprised by the justification in Question 1 and how it conflicted with his intuition. He also said:

> *Perry: The video had laid out the number of arrangements and it was easy to see that more arrangements led to a different number, so that one [Q3] was easy. However, had I not seen the video I think this would have been a whole other story. If I hadn’t sat down to think about it, it would have been challenging because I would have considered the logic to be true.*

Perry relied on the video for justification, but vaguely demonstrated an understanding of why the probabilities were not equal.

6.2.5.4 Perry: Question 4

Transcript – PA – R – "Perry" – Question 4	Interpretation
[First guess] Oral: "I had a gut feeling for sure that rolling at least one two was going to be for sure greater than exactly one 2…. There's just so many more ways that you can have rolled than the other one."	First guess: $\mathrm{P}(X_3 \geq 1) > \mathrm{P}(X_3 = 1)$
P(2 and 2 and 2) = $\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{1}{216}$ * P(2 and not 2 and not 2) = $\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} = \frac{25}{216}$ P(2 and 2 and not 2) = $\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6} = \frac{5}{216}$ P(2 and not 2 and 2) = $\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} = \frac{25}{216}$ P(not 2 and 2 and 2) = $\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{5}{216}$ * P(not 2 and 2 and not 2) = $\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{5}{6} = \frac{25}{216}$ * P(not 2 and not 2 and 2) = $\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} = \frac{25}{216}$ $\sum = \frac{91}{216} \approx 0.42$ exactly $\frac{75}{216} \approx 0.35$	Evidence of using the interpretation of probability presented in the lesson (PA): uses the multiplicative and additive properties; does not count the outcomes Evidence of using techniques taught in the lesson: splits the events into simpler ones; does not draw a tree but tabulates the 7 outcomes included in the event "at least one 2" and marks with a star those that correspond to the event "exactly one 2"; uses the multiplicative and additive properties
Larger than	Answer: correct
Self-rated confidence (1-7)	7

Perry found this problem easy, but tedious to prove since he made an effort to diligently show his work. He said that the answer became "obvious" after he listed all the relevant arrangements out and marked which ones corresponded to the event of obtaining exactly one 2.

He decided to be detailed in his mathematical justification despite having the ability to describe the answer in words because "*the numbers surprised [him] on question 3, so [he] definitely wanted to see the numbers [for Q4*]."

6.2.5.5 Perry: Question 5

Transcript – PA – R – "Perry" – Question 5	Interpretation
[First guess] Oral: "… we were working with halves	First guess: The games are incomparable.

instead of one sixth…. we had almost like two separate incomparable games going on…. The combinations were different. I think it was harder to consider. It was harder to get an initial gut feeling about what it was going to be."	
P(at least one 5) = $\frac{91}{216} \approx 0.42$	Re-uses a result obtained earlier in an isomorphic situation (Question 4 and in the lesson) to write $\mathrm{P}(X_3 \geq 1)$.
P(even, even, even) = $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$ * P(even, even, odd) = $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{8}$ P(e, o, o) = $\frac{1}{8}$ P(e, o, e) = $\frac{1}{8}$ $\sum_{even} = \frac{4}{8} = \frac{1}{2}$	Calculates $\mathrm{P}(Y_n \geq 1)$ by decomposing the event $Y_3 \geq 1$ into an alternative of 4 instead of 7 sequences. Evidence of using the interpretation of probability presented in the lesson (PA): uses the multiplicative and additive properties; does not count the outcomes Evidence of using techniques taught in the lesson: splits the events into simpler ones; does not draw a tree but tabulates the outcomes included in the event "at least one even" (missing three outcomes), uses the multiplicative and additive properties Misconception PR3: misses some arrangements
not true	Answer: possibly correct, but refutation of the quantitative relationship is not sufficiently explicit; no evidence of numerical verification, but maybe it was obvious for the participant that ½ is not equal to 3 times 0.42.
Self-Rated Confidence (1-7)	7

We may only speculate about the reasons why Perry missed 3 arrangements when calculating the probability of obtaining at least one even number in 3 rolls despite being exposed to the misconception refutation PR3. He admitted being confused by the change of variable *p:*

> *Perry: It was also hard to think about, considering different, um…because we almost had like two separate incomparable games going on, you know? The combinations were different. I think it was harder to consider. It was harder to get an initial gut feeling about what it was going to be.*
>
> *R: What do you mean the games were different?*
>
> *Perry: … rolling just a single number versus rolling a combination. Or just multiple numbers.*

Perhaps the novel situation caused him to neglect the proper computation. Also, his final answer was $P = \frac{1}{2}$ which may seem correct given that this novel scenario deals with evens and odds.

6.2.5.6 Perry: Question 6

Transcript – PA – R – "Perry" – Question 6	Interpretation
[First guess] Oral: "I wrote not true…. Right off the bat I knew because the number of rolls that there would be more arrangements and the number would be completely different…. I think for sure, it was easy to know that it would be different, because there would be completely different sets of addition."	First guess: Not true Evidence of using the interpretation of probability presented in the lesson (PA): very weak and implicit in the reference to "addition" Evidence of using techniques taught in the lesson: very weak; mention of "arrangements"
Not true considering arrangements	This is Perry's answer and only trace of written work.
Self-Rated Confidence	6

In the interview, Perry said:

> *Perry: I think by the end of this assignment I found this question a lot easier.*
>
> *R: Why is that?*

Perry: I think it's because I was in the frame of mind of thinking about the arrangements and the number of times you'd be rolling and stuff. It's like I was warmed up.

R: Did you have initial thoughts when you first read it?

Perry: "Right off the bat I knew because the number of rolls that there would be more arrangements and the number would be completely different" [pointing to the probabilities on the questionnaire].

R: More arrangements for…?

Perry: For um…can you read the question again?

R: One times a five if I roll 3 times versus 2 times a five if I roll 6 times.

Perry: I think for sure, it was easy to know that it would be different, because there would be completely different sets of addition.

R: Because you're rolling…

Perry: Twice as many times.

The different sets of addition Perry refers to is the summation of alternatives denoted by the specification of the desired number k of successes in n rolls. Though his written work is largely insufficient, his interview justification suggests he understands how and why the probabilities ae not equal, but the evidence is weak.

He found he was in the "*correct frame of mind of thinking about the arrangements and the number of times you'd be rolling*" and he attributed the ease of this final question with the practice he accumulated with the previous questions. But he could have been used to the procedure for solving, by this question.

6.2.5.7 Summary

Perry found probability difficult, non-intuitive and not interesting. He rated his mental effort invested during the lesson a 7 of 9 (high mental effort). Also, he had experience playing dice games. He pointed out during the interview that he enjoyed the lesson because it discussed errors that he likely would make.

However, throughout the experiment he had tendency to miss important arrangements despite being exposed to PR3. He also did produce tree diagrams but often opted for a list of possible arrangements instead. Yet, the lists would still be incomplete (Question 5).

Perry referred to PR4 in Question 3 as part of his justification but, as Larry did, he inferred that relationships between variables are "never proportional." By Question 6, it is, once again, unclear about his level of understanding about the misconception of proportionality. During the interview, Perry said that he knew “right off the bat” that the two probabilities would not be equal based on the number of arrangements and the additive property. There is an indication that he began writing a calculation but settled for a short explanation referencing the number of arrangements.

In the interview there were hints that his understanding of the relation between probabilities not necessarily being proportional in Question 6 was deeper than his written response but this evidence is weak.

6.3 Analysis by Question

After each question is analyzed with respect to each participant, a summary is given exploring common themes and errors for the given question. Speculation is mostly withheld for the Discussion section.

6.3.1 Question 1

Participants’ behavior on Question 1 is summarized in Table 3.

Question 1	P(X4>=2) >, =, < than P(X5 >=2)?				
	Lea - LA-E	Leah - LA-E-Alt	Larry - LA-R	Peggy - PA-E	Perry - PA-R
First guess	larger because 2/4 > 2/5	smaller because there are more opportunities in 5 rolls	smaller because "if I can get an extra roll then I can get an extra chance at it"	[No first guess because does not trust her intuitions]	larger
First guess correct?	No	Yes	Yes	No	No
Final answer	smaller	smaller	smaller	smaller	smaller
Final answer correct?	Yes	Yes	Yes	Yes	Yes
Justifications given	Written: Calculation of the two probabilities Oral: More chances if you roll more times	Written: Calculation of the two probabilities Oral: More opportunities if you roll 5 times	Written: Calculation of the two probabilities Oral: More opportunities if you roll 5 times	Written: Calculation of the two probabilities Oral: you have more options with 5 rolls	Written: Calculation of the two probabilities Oral: "more rolls means more chances"
Justifications correct	Written: incorrect; Oral: correct	Written: correct, with small notational inaccuracies Oral: correct, but language not 100% accurate	Written: incorrect Oral: correct identification of the mistake in written calculation	Written: incorrect Oral: correct	Written: incorrect Oral: partly correct (calculation of the chances of losing not corrected in the interview)
Mistakes	Misconception LR3 (missing arrangements)	Mistakes of imprecision: "obtaining 2 threes" instead of "obtaining at least 2 threes"	Misconception LR3 (missing arrangements)	Misconception PR3 (misses some arrangements) Mistake of replacing "at least 2 threes" by "exactly 2 threes"	Misconception a bit more general than PR3; incorrect calculation of the complementary event
Evidence of using approach presented in lesson	Yes	Yes	Yes	Yes	Yes
Evidence of techniques presented in lesson	Yes: tree diagrams, complementary events rule	Yes: tree diagrams, complementary events rule	Yes: tree diagrams, complementary events rule	Yes: multiplicative property Tree diagram not used; additive property not used	Yes: complementary events rule, additive and multiplicative properties (does not use tree diagrams.)
Self-rated confidence	5 of 7	6 of 7	7 of 7	6 of 7	5 of 7

Table 3. Summary of participants' behavior on Question 1

Only Leah (LA-E-Alt) and Larry (LA-R) had correct "first guesses" (intuitions) about the question, both based on qualitative proportionality, that more rolls yields more chance of success. Another two participants, Lea (LA-E) and Perry (PA-R), had incorrect intuitions about the question, and Peggy (PA-E) switched off her intuition because she did not trust it. But eventually all participants used the qualitative proportionality argument to support their final answers, which were all correct. However, all written justifications consisted in the calculation of the two probabilities. Because of the ease with which the answer comes to mind, Laplace participants expressed frustration over the fact that sizeable tree diagrams were required to prove a statement that was otherwise fairly evident.

All participants except Leah made mistakes in their calculation of the probabilities. In all cases, the mistake was missing arrangements. Additionally, Leah and Peggy made the mistake of replacing "at least

2 threes” by “exactly 2 threes." Moreover, there was a general sloppiness with the precision of the language used for most participants, using “ $=$ ” instead of " $\approx$ ".

Tree diagrams were not attempted in the written work of Properties participants, while all LA participants did use trees. LA participants also all used the complementary rule. Both PA participants used the multiplicative property, but only Perry used also the additive property and the complementary event rule.

6.3.2 Question 2

Participants’ behavior on Question 2 is summarized in **Error! Reference source not found.** (following age).

Question 2	P(rolling 5-6) <,=,> P(rolling 6-6)?				
	Lea - LA-E	Leah - LA-E-Alt	Larry - LA-R	Peggy - PA-E	Perry - PA-R
First guess	(5, 6) and (6, 6) have the same chances of happening	(5, 6) and (6, 6) have the same chances of happening	(5, 6) has greater chances of happening than (6, 6)	(5, 6) is more likely than (6, 6)	Getting a 6 and then a 6 or getting a 5 and then a 6 have the same chance
First guess correct?	No	No	Yes	Yes	Incorrect despite misinterpretation
Final answer	(5, 6) is more likely than (6, 6)	(5, 6) and (6, 6) have the same chances of happening	(5, 6) has greater chances of happening than (6, 6)	Both have the same chance	P(6 AND 6 OR 6 AND 5) [=1/6 x 1/6 + 1/6 x 1/6 = 1/18]
Final answer correct?	Yes	No	Yes	No	Incorrect interpretation of the question: the two events are merged into one
Justifications given	Written: probability of getting pair (6, 6) is 1/36 ≈ 0.028; probability of getting pair (5, 6) is 2/36 ≈ 0.056 Oral: (describes how tree would look); there are two ways of getting the pair (5, 6); in spite of rolling the dice simultaneously, one could land a millisecond earlier [so it makes sense to distinguish the outcomes (5, 6) and (6, 5)	Written: probability of rolling a 5 and a 6 are the same Oral: the probability of getting a 5 is 1/6, and the same for 6	Written: P(5 and 6) = p. of 5 or 6 (= 2/6) x p. of getting the other number (= 1/6) = 1/18; probability of getting two 6's = p. of 6 x p. of 6 = 1/6 x 1/6 = 1/36 Oral: (same as written, but more elaborate) + evidence of awareness of the hypothetical character of reasoning which hinges on interpretation of "simultaneous" as implying that "order does not matter")	For the first guess: (5, 6) is more typical than (6, 6) For the final answer: Written: Calculation of 1/6 x 1/6 Oral: it is arbitrary what two numbers you are aiming at	Written: no justification Oral: reference to refutation of Misconception PR2 (dice have no memory) and additive and multiplicative properties
Justifications correct	Written: correct Oral: correct	Written: incorrect Oral: incorrect	Written: correct Oral: correct	Written: incorrect Oral: incorrect	Written: N/A Oral: incomplete or incorrect
Mistakes	None	Misconception LR3	None	Misconception PR3- Belief that more typical implies more probable (interprets "(5, 6)" as the event "different numbers" which includes more outcomes than "same number", so is more "typical"; better representative of the whole set of outcomes)	Misinterpretations of the question- First, it is about rolling one dice twice; second, merges two events into one
Evidence of using approach presented in lesson	Yes	No	No (evidence of PA, rather)	Yes (but weak)	Yes
Evidence of techniques presented in lesson	Yes	No	No	No	No
Self-rated confidence	7 of 7	7 of 7	4 of 7	7 of 7	7 of 7

Table 4. Summary of participants' behavior on Question 2

Lea and Leah's first guess was that the pair (5,6) is more likely than (6,6)- the common misconception LR3 to neglect arrangements. Lea explains her calculations orally by referencing tree diagram to justify why there is no difference in probability if one were to roll the dice simultaneously or consecutively. Her confidence was 7 of 7. Leah showed no evidence of using the Laplace approach or the techniques of the lesson as she concludes with 7 of 7 confidence that the pair (5,6) is equally likely to (6,6). She claims the answer was "obvious" and that not much time was allotted to thinking about the solution. Larry claimed that his initial thought about the solution was that the first roll could be a 5 or a 6. This lead Larry to a solution using Properties approach techniques where the probability of obtaining a 5 or a 6 on the first roll is considered, then the probability of obtaining an outcome different than the first is considered. His solution is correct, and his confidence is the lowest among the participants for this question (4 of 7). Recall that Larry was taught misconception LR3 on neglect of arrangements. Peggy's first guess was that (5, 6) had a greater chance than (6, 6) but she had difficulty explaining (in the interview) why this was her inclination. This is a symptom of reliance on the availability heuristic- a claim supported by her recent experience with dice games. But in being aware of misleading intuitions, she reasons that the outcome (5, 6) seems 'more typical'- a symptom of the use of the representative heuristic- thus leading her to the reflection that "it's arbitrary what two numbers you're aiming for" thus (6, 5) and (6, 6) are equally likely. Her confidence was 7 of 7.

Finally, Perry, also refuted on PR3, misinterpreted the question for his written solution saying that he understood the dice as being rolled consecutively. In which case the pairs are equally likely. This answer is incorrect despite his misunderstanding. Even if he understood the question as 'what is the probability of obtaining the *ordered* pair', it implies that he, like any other individual who believes the pairs are equally likely, did not consider the possibility of outcome order. For this interpretation, his confidence was 7 of 7.

6.3.3 Question 3

Summary of participants' behavior on Question 3 is given in Table 5.

Question 3	$P(X_5 \geq 4) \neq ½P(X_5 \geq 2)$?				
	Lea - LA-E	Leah - LA-E-Alt	Larry - LA-R	Peggy - PA-E	Perry - PA-R
First guess	TRUE	Not true	Not true	TRUE	Not true
First guess correct?	No	Yes	Yes	No	Yes
Final answer	Not true	(The same as first guess)	Not true: P(X5≥4)≠½P(X4≥2)	False : P(X5≥4)≠½P(X5≥2)	Not true
Final answer correct?	Yes	Yes	No (because he is not speaking of the same statement)	Yes	Yes
Justifications given	Written: calculation: probability of getting the pair at least four 5's in 5 rolls is about 0.0033 (correct); probability of getting at least two 5's is 0.9967 (incorrect)	Written: "ratios are not proportional to each other" (refers to refutation of LR4) Oral: the problem is basically the same as the example given in the lesson	Written: Calculation of one of the two probabilities; re-using the value of the other (incorrectly interpreted) Oral: reference to the authority of the lesson and analogy of Question 3 with problems discussed in the lesson.	Written & Oral: Calculation of the first probability and estimation of the second, noticing that 2xP(X5≥4)<<P(X5≥2)	Written: by reference to authority of the lesson, and similarity of the situation in Question 3 with the one presented in the video and refutation of Misconception PR4. Oral: by reference to similarity with Question 4 where participant did engage in calculations
Justifications correct	Written: partly correct [0.0033 is not half of 0.9967]	Incorrect; generalizes the existential assertion in the lesson ("ratios are not always proportional") to a universal assertion ("ratios are never proportional")	Written: incorrect Oral: incorrect	Written & oral: correct	Written & oral: insufficient and inaccurate: generalizing from "sometimes not proportional" to "never proportional"
Mistakes	Misconception LR3 : some arrangements are missed	Misinterprets "P(X5≥4)≠½P(X5≥2)" as "P(X5=4)≠½P(X5=2)"	Misconception LR3, but corrected orally in the interview Mistakes of imprecision: "=" instead of "≈"	Notational (event = probability) and terminological (odds = probability) inaccuracies	Terminological imprecision
Evidence of using approach presented in lesson	Yes	Yes, but weak and indirect (reference to the lesson)	Yes	Yes	Yes, but weak
Evidence of techniques presented in lesson	Yes	Yes, but weak and indirect (mentions she did some calculations as show in the lesson, but there is no trace of these in her written work)	Yes: tree diagrams	Yes	Not used, but evoked only
Self-rated confidence	4 of 7	6 of 7	7 of 7	4 of 7	7 of 7

Table 5. Summary of participants' behavior on Question 3

We observed some intrinsic issues with Van Dooren et al.'s (2003) question structure, in that it may have a substantial extraneous load. Lea indicated that extracting the meaning of the question was a challenge due to its wordiness and the amount of important information that must be held in the mind simultaneously (number of rolls, number of successes and the probability of success for each event). Moreover, Leah and Perry overgeneralized the refutation statement LR4/ PR4[5]. Leah wrote "ratios are never proportional" and Perry said probabilities were "never proportional".

Leah, Larry and Perry had correct intuitions that the statement was not true. These are the participants treated with refutation statements. However, Larry's written work indicates that he was solving a different question as he compared $P(X_5 \geq 4)$ to $P(X_4 \geq 2)$- perhaps a lapse of attention.

There were mistakes of imprecision with language and terminology observed in all participants except Lea. Leah orally justifies why $P(X_5 \geq 2)$ is not equal to $\frac{1}{2}P(X_5 \geq 4)$, instead of indicating 'at least'.

Peggy's initial answer was that the statement was true ($P(X_5 \geq 2)$=$\frac{1}{2}P(X_5 \geq 4)$), but used the Properties approach and techniques (no tree but used a list) to correctly conclude $2 \times P(X_5 \geq 4)$ is much less than $P(X_5 \geq 2)$.

Participants had good qualitative insight of the relation as ascertained from the interviews. Lea missed arrangements (LR3) of the outcome 'one success in 5 rolls' while using the complementary event to calculate $P(X_5 \geq 2)$. She obtained $P(X_5 \geq 2) = 0.9967$ which satisfies the qualitative intuition since she obtained $P(X_5 \geq 4) = 0.0033$ (which is correct), despite being the only participant to have assembled the most of a tree diagram for this question.

Larry produced an altered tree with only the relevant arrangements. The crux of Leah's justification relied on the refutation LR4 from the lesson. Larry only made reference to the refutation as a method to confirm his conclusion; not justify it, despite having statement LR4 in their lessons. The differences in the written work between Larry and Leah might be that Larry seemed more interested in probability than her.

Properties approach produced no trees. Peggy constructed a list of arrangements for both events, whereas Perry gave a weak justification referencing the lesson examples as well as referencing statement PR4, albeit incorrectly.

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[5] *p(at least 2 x SIX in k rolls) ≠* $\frac{1}{2}$ *x p(at least 1 x SIX in k rolls)*

The confidence ranged from 4 to 7, and the exposition treatment participants listed the lowest confidence- 4 of 7 for Lea and Peggy.

6.3.4 Question 4

Summary of participants' responses to Question 4 is given in Table 6.

Question 4	Is $P(X_3 \geq 1)>, =, <P(X_3=1)$?				
	Lea - LA-E	Leah - LA-E-Alt	Larry - LA-R	Peggy - PA-E	Perry - PA-R
First guess	No first guess	P(X3≥1)>P(X3=1)	P(X3≥1)>P(X3=1)	P(X3≥1)>P(X3=1)	P(X3≥1)>P(X3=1)
First guess correct?	N/A	Yes	Yes	Yes	Yes
Final answer	P(X3≥1)>P(X3=1)	(Same as first guess)	(same as first guess)	(same as first guess)	(same as first guess)
Final answer correct?	Yes	Yes	Yes	Yes	Yes
Justifications given	Calculation of both probabilities (correct)	There are more options in rolling at least one two than there are in rolling exactly one two.	Written: Calculation of both probabilities Oral: more options in "at least"	Written: Calculation of both probabilities Oral: more possible combinations for "at least one 2"	Written: by calculation of the two probabilities Oral: There's just so many more ways
Justifications correct	Yes	Yes	Yes	Yes	Yes
Mistakes	No mistakes but justifications involve unnecessary calculations (a shortcoming, not a mistake)	No mistakes	Misconception LR3, but corrected orally in the interview Mistakes of imprecision: "=" instead of "≈"	Notational (event = probability); Computational: one arrangement repeated twice in the enumeration	No mistakes
Evidence of using appr	Yes	No	Yes	Yes	Yes
Evidence of techniques	Yes	Yes (decomposition of an event into simpler ones)	Yes: tree diagrams; complementary event rule	Yes	Yes: decomposing the event into simpler ones; properties, additive and multiplicative; no tree but own representation (enumeration of outcomes in a tabular form)
Self-rated confidence	7 of 7	7 of 7	7 of 7	5 of 7	6 of 7

Table 6. Summary of participants' behavior on Question 4

This question (Question 4) had no particularly interesting results. Only minor computation errors were made (such as Peggy counting an arrangement twice). Again, some expressed frustration over the calculations required to prove something fairly evident.

All participants except Lea had good qualitative intuition of the relation between $P(X_3 \geq 1)$ and $P(X_3 = 1)$, and could explain in words why the former was greater than the latter.

Confidence levels were high, but Peggy indicated hers at 5 of 7. This is likely reflective of her mistrust in intuitions which she is vocal about. All Laplace participants had confidence levels of 7, while Perry's was 6.

6.3.5 Question 5

Summary of participants' responses to Question 5 is given in Table 7.

Question 5	$P(Y_3 \geq 1)=3xP(X_3 \geq 1)$?				
	Lea - LA-E	Leah - LA-E-Alt	Larry - LA-R	Peggy - PA-E	Perry - PA-R
First guess	P(Y3≥1)>P(X3≥1)	P(Y3≥1)>P(X3≥1)	(No first guess)	True that P(Y3≥1)>3xP(X3≥1)	Games are incomparable, so no first guess
First guess correct?	N/A (true but does not answer the question)	Guess true but does not respond to question	N/A	No	N/A
Final answer	No final answer	P(Y3≥1)<P(X3≥1)	False	Not true	Not true
Final answer correct?	N/A	No	Difficult to say what is false (perhaps referring to the probabilities not being equal)	Yes	Yes, but refutation of the quantitative relationship not explicit
Justifications given	Unfinished calculation of the first probability	Written: Calculation of probabilities Oral: "because the probability of rolling an even number at least once is not the same as rolling a single number at least once. Therefore their probabilities will not be proportional."	Written: Calculation of one probability and re-use of the other	Calculation of both probabilities	Written: by calculation of one probability and re-use of the second
Justifications correct	N/A	No	Incorrect, incomplete	Reasoning correct, but minor computational mistake	No
Mistakes	Incorrect calculation of some ways of getting particular outcomes but corrected orally	Computational (misconception LR2) and logical	Incorrect calculation of the number of ways of getting odd-odd-odd; imprecise answer	Computational (one probability miscalculated); un-reflective approach to problem solving (re-calculation of the probability of an isomorphic event)	Misconception PR3
Evidence of using approach presented in lesson	Yes	Yes	Yes	Yes	Yes
Evidence of techniques	Yes (decomposition of event into simpler ones, tree, complementary event rule)	Yes (decomposition of an even into simpler ones; tree)	Yes: tree diagrams; complementary event rule	Yes	Yes: decomposing the event into simpler ones; properties, additive and multiplicative; no tree but own representation (enumeration of outcomes in a tabular form)
Self-rated confidence	0 of 7 ("it was a 7 that I was certain my answer was wrong")	6 of 7	7 of 7	4 of 7	7 of 7

Table 7. Summary of participants' behavior on Question 5

Lea did not have an intuition of the quantitative validity of the question statement but did believe that obtaining at least one even in 3 rolls was more likely than obtaining at least once a 5 in 3 rolls. This is the same qualitative proportionality reasoning Leah had for this question.

Lea began to use the Laplace approach by building a tree diagram with 'even' and 'odd' branches, but discontinued the problem because she had a 'feeling' that the tree would be symmetric, despite having written different numbers at the bottom of the branches. Her confidence was 0 of 7.

Leah produced a tree diagram but committed misconception LR2 in adding the outcomes of the independent die rolls calculation, $3 + 3 + 3 = 9$, instead of multiplying them, $3 \times 3 \times 3 = 27$. This error lead to a conclusion $P(Y_3 \geq 1) < P(X_3 \geq 1)$[6], which Leah accepted noting that she was surprised by the result (despite having contrary qualitative intuition). Her confidence was 7 of 7.

Larry had no initial guess and went right into the work, he claimed. He used the approach and techniques as taught to him, but made errors in calculating the desired outcomes. His final answer implicitly concluded that $P(Y_3 \geq 1) > P(X_3 \geq 1)$, despite having the incorrect numerical value for $P(Y_3 \geq 1)$. His confidence was 7 of 7, likely because qualitative proportionality was satisfied. All Laplace approach participants had difficulties with this question, implying that the Laplace approach is difficult to apply when comparing the probability of a single success $p = \frac{1}{6}$ to $p = \frac{1}{2}$.

Peggy intuitively believed that $P(Y_3 \geq 1) = 3 \times P(X_3 \geq 1)$, however final answer and justification are correct with a confidence of 4 of 7 (the lowest confidence excluding Lea). Perry had no first guess and said in the interview that "the games are incomparable". He used the approach and techniques in his written work but commits misconception PR3 (despite being exposed to it in the lesson) and misses several arrangements in his calculation. He concluded $P(Y_3 \geq 1) = \frac{1}{2}$, and that the statement is not true with a confidence of 7 of 7.

All participants attempted to use the approaches and techniques taught to them in their lessons.

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[6] $Y\ is\ the\ event\ of\ obtaining\ an\ even\ number$
$X\ is\ the\ event\ of\ obtaining\ a\ 5$

6.3.6 Question 6

Summary of participants' responses to Question 6 is given in Table 8.

Question 6	$P(X_6 \geq 2) = P(X_3 \geq 1)$?				
	Lea - LA-E	Leah - LA-E-Alt	Larry - LA-R	Peggy - PA-E	Perry - PA-R
First guess	No first guess	It is not true that P(X6≥2)=P(X3≥1)	True or almost true	Maybe true P(X6≥2)=P(X3≥1)	Not true
First guess correct?	N/A	Yes	No	No	Yes
Final answer	Not true	Same as first guess	Not true	(Same as first guess)	Not true
Final answer correct?	Yes	Yes	Yes	No	Yes, but refutation of the quantitative relationship not explicit
Justifications given	Calculation of both probabilities, with mistakes	Written: "ratios not proportional", without making explicit what ratios are referring to. Oral: "more possibilities"	Written: Calculation of one probability and re-use of the other	Written: "it is essentially two sets of the same occurrence"	Written & oral: vague references to "different arrangements", "different sets of addition"
Justifications correct	No	No (not complete, not sufficiently explicit)	Incorrect	No	No
Mistakes	Misconception LR3	Quantitative neglect: inequalities replaced by equalities in the events to be considered	Misses one arrangement-misconception LR3; computational mistakes	Misconception PR3; odds" instead of "probability"; inconsistency in writing: event = probability; computational mistake carried over from Question 5	N/A
Evidence of using approach presented in lesson	Yes	Yes but very weak (mention of "possibilities")	Yes	Yes	Yes, but very weak
Evidence of techniques presented in lesson	Yes (decomposition of event into simpler ones, tree, complementary event rule)	Yes, but very weak (mention of complementary events)	Yes: tree diagrams; complementary event rule	Yes	Yes, but very weak
Self-rated confidence	7 of 7	6 of 7	7 of 7	3 of 7	6 of 7

Table 8. Summary of participants' behavior on Question 6

The statement of Question 6 may have been the most tempting of the questionnaire for participants to intuitively guess that the answer was "true." In other words, Leah, Larry and Peggy had an inclination that $P(X_6 \geq 2) = P(X_3 \geq 1)$.

Lea had no intuition about the statement and engaged the work immediately. She used the complementary event rule to calculate $P(X_6 \geq 2)$ and concluded, with misconception LR3, that

$P(X_6 \geq 2) = P(X_3 \geq 1)$ with a confidence of 7 of 7. In the interview she reasoned that $P(X_6 \geq 2)$ is "essentially two sets of the same occurrence"- an argument hinging on quantitative proportionality.

Leah intuitively believed that the statement was not true. Recall that she was exposed to misconception LR4 in her lesson. With a confidence of 6 of 7 she concludes that in her written work that "ratios are never proportional"- an insufficient justification. In her oral justification she makes weak references to the complementary event of obtaining at least two 6's in rolls, and describes:

> "*There's more possibilities against not being able to get at least one or two times or whatever it is. Because like I noticed like this one goes on [referring to a branch] forever, you know, 5x5x5x5, you get so many big numbers that it lowers the probability"*

The justification is insufficient, but recall that a sufficient qualitative justification was not expected from the participants.

Larry attempted to calculate $P(X_6 \geq 2)$ using a modified tree diagram but committed misconception LR3. His answer is correct but his justification is insufficient, with a confidence of 7 of 7.

Peggy did not accept that this question required a diagram (or list) denoting the arrangements of 6 rolls of a dice, so she settled for her intuitive answer- that $P(X_6 \geq 2) = P(X_3 \geq 1)$). Because she does not trust gut reactions and she abandoned the lesson (as indicated in her partially completed written work) her confidence was 4 of 7.

Perry, exposed to misconception PR4, intuitively believed that the statement was 'not true' claiming that by Question 6 "[he] was in the frame of mind of thinking about the arrangements and the number of times you'd be rolling and stuff"- a weak reference to how the number of rolls affects the probability.

He also states: "Right off the bat I knew because the number of rolls that there would be more arrangements and the number would be completely different". His answer was correct but his justifications, like the other participants, were insufficient, with a confidence of 6 of 7.

Moreover, all participants displayed signs of fatigue at this point tin the experiment.

7 Remarks and Conclusions

7.1 A comment on Van Dooren et al.'s (2003) question format

Intuitions are based on experience. When individuals roll dice in games, they probably do not model the situation in terms of Bernoulli trials and certainly not in the form of a $P(X_n \geq k)$ problem. When Peggy and Perry play Yahtzee, for example, and, in rolling the 5 dice involved in the game, they aim at getting at least three 6's, do they mentally construct the correct problem statement? Do they imagine that three 6's is sufficient, but so are four 6's or all 6's, and how these possibilities affect their chances? According to the interviews, it appears they do not. They "go with the flow", as Lea says within the context of casino gambling. They are retroactively aware that calculating the probabilities of the events in the categories they aim for can help them to choose a more advantageous strategy – but they do not do the calculations. They roll once, and, if they do not get the desired outcome, they roll a second time, and so on.

There is a feeling that, because probability is only an estimation of the chances of some non-deterministic event, it is beyond the power of our direct control. One may make calculated guesses but the actual result may not be consistent with the calculations. People may feel that "whatever happens happens" and that being formal about uncertainty, even in the most explicit scenarios such as board games, is a pointless task. We are, by nature, seeking the path of least resistance after all: the path that decreases the amount of mental effort invested. And being cognitively deliberate about non-determinism is felt as a somewhat a fruitless task – so why bother?

Van Dooren's questions do not reflect how individuals think about probability on an everyday basis and therefore a question about getting AT LEAST a certain number of successes is a foreign consideration, since it is not how they think about dice games and desired events. If the aim of Van Dooren et al's research was to reveal how students' think about Bernoulli trials, then formulating their questions in that format may not have been the best method. It is suggestive from this study's interview questions and results that individuals do not think about possible dice outcomes in terms of "*at least k successes",* but instead in terms of "*exactly k successes*."

But many will use erroneous justifications, I contend, because the "wordiness" of the questions imposes an extraneous cognitive load as students must invest a substantial amount of mental effort to decipher their meaning. As an initial problem solving strategy, students first notice the salient features and incorrectly set-up fractions to see how the 6 numbers presented in the question are related. If they were not taught combinatorics or adequately instructed on the use of tree diagrams to deal with 'at least k successes in n rolls' then they might use linear modelling.

7.2 Complexity of Intuitions and Heuristics

The implementation of heuristics and intuitive judgements when problem solving is more complex than initially believed. The participants of this study exhibited several cognitive shortcuts in their thinking processes that ultimately served as starting points for mathematical deliberations. Cognitive biases are often viewed in a negative light as they are interpreted as being major obstacles to students' learning and understanding. The following observations from the study are anecdotal but they paint a detailed picture of how students may think about probability and the various avenues of thought they may follow in their justifications.

Overall, the participants performed better than expected. Their final justifications and solutions did not seem affected, at least superficially, by erroneous thought processes and alternate conceptions. That being said, many mistakes were made; some of them were unexpected. A detailed analysis of the gamut of intuitions that exist and how they guide or distract a student is not irrelevant. Students are influenced by intuitions during many steps of the problem solving process: At the onset of reading a problem, throughout the solving process, upon arrival at a solution, and thinking retroactively about the problem. However, these intuitions are complex and vary in type. Understanding the different types of intuitions and the various stages of problem solving at which they occur illuminates the potential direction of probability education, its didactics and alternate experimental procedures.

7.2.1 The Effects of Social Intuition and Social Constructivism

Social cognitive theory "contends that learning in new situations often occurs through a borrowing or mimicking process" (Muller, 2008, p. 28). It is based on the innate human ability to read social contexts and is referred to as social intuition (Myers, 2010). Students use these intuitions to ease the process of learning through observation which saves the learner some effort (lower cognitive load). These ideas are the basis of social constructivism and cognitive apprenticeship (Brown, Collins, & Duguid, 1989) and they can be used in interpreting the results of this study.

For example, Question 1[7] was intuitively understood by the participants before completing any written work. This agrees with Van Dooren et al. (2003) findings about qualitative proportionality. It was expected, therefore, that the participants would construct a qualitative response to the question, as a mathematical justification is not required to demonstrate the truth in the statement. Moreover, the question asks about rolling a dice 5 times which the participant would have imagined would produce a large tree diagram.

Yet, most participants persisted with justifications following the approach showed in the lessons they watched. The participants were, indeed, mimicking the instructor's practice from the video lesson. This seems likely especially because the participants are not used to thinking about and implementing the inequality version of Bernoulli trials problems in real life scenarios. This explains why there were many errors with imprecise language.

Because Leah is not interested in probability and showed signs of frustration and fatigue after the first question, she became reliant on the minimum required justification which was to repeat the statement LR4 (doubling the amount of required successes does not halve the probability of obtaining the desired event) throughout.

In general, the refutation participants relied on the authority of the instructor by repeating the refutation statements as the sole justification to some problems. If they understand the refutation statements then they may deem that the statement is sufficient in justifying in their answer. But such a claim is impossible to verify at this point.

Similarly, Lea may have had difficulty with Question 5[8] due to a combination of low confidence in her mathematical abilities (disclosed to the researcher off-the-record) and because the lesson did not cover situations involving the changing of the variable p. She missed the "template solution" to mimic. All participants had difficulty with this question, but Lea may have required specific instruction on this situation in order for her to confidently assert and act on her intuition which was telling her the tree was symmetric. Moreover, Lea's symptoms of procedural thinking can equally be attributed to the confidence she places in the knowledge that she has acquired from the instructor. In addition to this, Lea (and perhaps the other participants as well) may have been experiencing positive recency, in that

[7] I roll a fair die several times. The chance to have at least two times a three if I can roll four times is (larger than, smaller than, equal to) the chance to have at least two times a three if I can roll five times.
[8] I roll a fair die several times. The chance to have at least two times a three if I can roll four times is (larger than, smaller than, equal to) the chance to have at least two times a three if I can roll five times.

the previous four question statements held p constant so that Question 5 was startling and "uncomfortable", even despite Lea's intuition. Perhaps her struggle was a combination of the two effects.

Social negotiation (Bauersfeld, 1983) was also observed during the interviews which raised concerns about the effects of the observer's paradox and the integrity of objective and authentic research; this is due largely to social intuitions and social cognitive theory. There are many subtleties to social interaction that are not captured by audio recordings and are limited only to what the interviewer can notice and note. These subtleties include body language, voice tone and choice of words, eye contact, just to name a few, and they play an important role in non-verbal communication between the interviewer and the interviewee.

This means that the verbatim record of the interviews does not necessarily represent the participants' objective thought processes as those are highly influenced by social cues whenever he/she divulges information to the researcher; whether the information is correct, incorrect or intriguing. During the interview with Perry, when he was asked about the application of the multiplicative and additive rule, moments of uncertainty were marked with non-verbal interaction involving more eye-contact while hesitantly answering the researcher's questions. This suggests he was tailoring his dialogue to the subtle and explicit reactions of the researcher – a socially intuitive process. Perhaps this interaction and interplay is an essential factor in the acquisition of knowledge but it is also indicative of the biased representation of a participant's thought processes. And these are only the subtleties that the participant brings to the surface and that the researcher consciously observes.

7.2.2 Intuition of doubt and mathematical reasoning as a refuter of intuitive belief

It is difficult to have accurate intuition about probability because it models random phenomena. An overlooked consequence of this, which became apparent throughout this study, was that participants were not only aware of probability's counter-intuitiveness but in many instances they exercised immediate self-doubt and distrust in their intuitive inclinations. This seemed especially true with Peggy who seemed to have the most experience with dice games in recent history.

With the exception of Question 2[9], it was often the case that participants admitted that they were tempted by various misconceptions yet they would demonstrate doubt in their intuitions. The

[9] Suppose one rolls two dice simultaneously. Which of the following has a greater chance of happening? Show you reasoning. Justify your answer. (a)Getting the pair 5-6 (b) Getting the pair 6-6 (c) Both have the same chance.

temptation to employ linear relations was sometimes followed by reflections such as: “this seems too good to be true”, “I learned never to trust my gut” or “the numbers have to speak first.”

This insight is useful because it suggests that self-doubt can be taught as an automatic thought process and as a critical step in problem solving. General education research on the existence of cognitive biases can serve as a caveat for unfounded judgements. Leron and Hazzan (2009) proposed that “the most important educational implication […] is the need to train people to be aware of the way S1 [intuition] and S2 [deliberation] operate, and to include this awareness in their problem-solving toolbox.”

Teaching self-doubt is aligned with the cognitive apprenticeship of model-building for situations involving uncertainty. In other words, science and applied mathematics, at their cores, hinge on the falsifiability of hypotheses. High level of confidence upon reaching a solution seems like an achievement, however it raises the likelihood that the justification will not be re-examined after the fact. A lower confidence in a solution implies there is a higher degree of doubt and a larger likelihood that the model will be re-examined. For a field like probability, uncertainty about one’s solution is important.

Participants had high levels of confidence in Question 2 but they also had the most incorrect answers with the least work shown. This problem was deceptive in that it appeared simple so participants assumed minimal work. In addition, the answer was highly intuitive (but incorrect). Had the video lesson explicitly encouraged self-doubt as an integral part of the solving practice it is plausible that Question 2 would have seen better results.

In Question 6[10], Peggy set a confidence of 4 of 7 because she could not conceive of the proper algorithm or method to solve the problem, and had relied on her intuitive inclination. The inclination was to automatically process the salient numerical features of the problem as two equivalent fractions (as a proportion). Despite saying that the answer seemed reasonable, she did not trust the solution because it relied heavily on intuition.

Another demonstration of the complex interplay between intuition and algorithmic deliberation is Lea’s justification and thought process for Question 5. Her result is a demonstration of how intuitive belief can be easily overcome by mathematical formulations. In fact, if a student believes that the work performed is correct or they trust the method used, then the work will refute the preconception (which is not

[10] I roll a fair die several times. The chance to have at least two times a five if I can roll six times is equal to the chance to have at least once a five if I can roll three times. (This is true, this is not true).

always a positive result). This reflects the difficulty to confirm or reject a probabilistic conclusion. Lea's conclusion, that $P(at\ least\ one\ even\ number\ in\ 3\ rolls)$ is smaller than $P(at\ least\ once\ a\ 5\ in\ 3\ rolls)$, should have been qualitatively rejected, but it was not. She noted being surprised by the answer indicating that she had believed the opposite to be true, but changed her belief based on the erroneous solution.

Others made calculation errors that misrepresented the quantitative relation between two probabilistic events. For example, in Question 1, missing arrangements led Larry to conclude that $P(X_4 \geq 2) = 0.42$, and that $P(X_5 \geq 2) = 0.758$. These results are incorrect, but since he got numbers between 0 and 1, and the former is less than the latter as predicted by his intuition, he has no reason why he should revise his calculations. In fact, for this particular question, Larry's confidence was at 7 of 7, likely because it agreed with the qualitative proportionality assessment. Moreover, after completing a large calculation involving a tedious tree diagram, the individual may not want to return to fix their computation.

In a very similar case, we now examine Perry's solution to Question 5 in which he missed several arrangements despite being exposed to a refutation of the misconception PR3 (arrangements do not matter). He wrote $P(at\ least\ one\ even\ number\ in\ 3\ rolls) = \frac{1}{2}$ which can be confirmed by the same qualitative intuitions as Larry had. However what is apparent is the implication that the event of obtaining at least one even is equiprobable with its complementary event: Obtaining no even numbers. This conclusion is absurd so one must ask why he settled for this answer and why his confidence was 7 of 7. It is possible that his answer was confirmed by the representative heuristic which would automatically assess the validity of his conclusion based on how it represents the individual characteristics of the possibilities. Namely that $P(even\ number\ in\ one\ rolls) = P(odd\ number\ in\ one\ rolls) = \frac{1}{2}$, so that $P(at\ least\ one\ even\ number\ in\ 3\ rolls) = \frac{1}{2}$, which may seem reasonable if that is the conclusion obtained through mathematical justification.

7.2.3 On the nature of probabilistic misconceptions

An issue that became evident throughout the analysis of the interviews was that intuitive judgments were not as consistent or robust as expected. Moreover, as demonstrated by Perry's Question 2 justification, the refutation of one misconception reinforced another. He referenced the independence of dice rolls to justify that the pair (6, 6) was equiprobable to (5, 6) using the representative heuristic of the individual dice's properties. Generally, when confronted with a problem, one immediately begins processing a solution; perhaps relying on salient features.

A larger study would be required, but it's suggestive that probabilistic intuitions are not consistent. Peggy, for example, had the most recent experience with dice games and her intuitions and misconceptions seemed more influential on her deliberation than others. On the other hand, Lea has had no recent experience with mathematics, let alone probability, which explains why, for Question 1, she was more tempted than others to view the problem as a comparison of two fractions. Her aptitude to think mathematically was less developed, so intuitively simple approaches were an effective starting point for problem solving. If any given group of students have different misconceptions derived from their subjective domains of experience, then, in tandem with an education on cognitive biases, group work could ensure that each student acts as a critique for other student's' cognitive biases. Although some biases are likely more universal or deeply rooted, it is suggestive that there is a decent variety of intuitions that are experienced at the onset of reading a problem. Theoretically, only a single student with an opposing conception is enough to refute the belief of a group, although practically there are other factors that influence an individual's decision to speak up. These factors concern the complex social dynamics of groups.

This issue suggests that on the spectrum of the strength of intuitions and pre-conceptions, probability likely lies closer to quantum physics then it does to Newtonian physics. Recall that misconceptions about Newtonian physics are fairly consistent and robust as compared to the whimsical and diverse misconceptions of quantum physics (Muller, 2008). Misconceptions about quantum physics are "less ingrained in student thinking because they have not been reinforced through repeated experience" (Muller, 2008), which led to the conclusion that quantum physics misconceptions are better described under the umbrella of diSessa's phenomenological primitives (p-prims). These are the "set of disconnected thoughts cobbled together to form answers when necessary" (Muller, 2008).

This notion is aligned with the idea that individuals simply do not think about probability or randomness in a critical or meaningful way outside the classroom – as suggested by the participants during the interview.

This may explain why refutation participants seemed to have higher confidence ratings than the exposition participants of the same approach – whereas Muller observed the opposite. Students taught using misconception-based Newtonian physics are exposed to their faulty thinking, refuted on their well-established concept, and they must relearn the concept of a given phenomenon and apply it. This lowers their confidence but posttest scores see gains (Muller, 2008). Newtonian misconceptions are reinforced through everyday interactions with the surrounding environment, which, by their deterministic nature, produces robust misconceptions. This is not the case with the illusion of linearity in probability. It is suggestive that proportionality applied to Bernoulli trials is not a robust and consistent misconception as observed in Newtonian physics, since chance phenomena are inherently inconsistent, hindering the construction of reinforced misconceptions.

If probabilistic intuitions are p-prims then they do not qualify as robust and coherent concepts, and so refuting them will not have the same effects as refuting pre-conceptions for Newtonian physics. Newtonian intuitions serve as an assumed notion of how certain phenomena work, whereas probabilistic p-prims serve as the first stepping stone, so to speak, to problem solving. However, as observed among the participants, p-prims can take thought processes down different “avenues” and some of these avenues lead to either erroneous conclusions or correct conclusions.

The refutation format may see better success with other misconceptions. For example, participants’ solutions to Question 2 (with the exception of Larry) had very little written work, high confidence levels and oral justifications were largely insufficient- in fact, Leah admitted that she had difficulty explaining why the pair (5,6) was equally likely to (6,6). These symptoms of certainty and poor justification are indicative of a decision hinged on belief and cognitive shortcuts, such as heuristics. The event of rolling two dice to obtain the pair (6,6) is likely more common place and familiar to individual than considering $P(at\ least\ k\ successes\ in\ n\ rolls)$.

The implication is that, if students apply p-prims to probability then probabilistic instruction has room for more empirical learning, so students can develop their intuitions as opposed to having misconceptions refuted. Peggy had strong intuitions due to her experience with dice games, but she was also very aware of probability’s counter-intuitiveness, which was an important asset to her critical thought.

This distinction supports Shaughnessy’s (1977) assertion that experimental-based courses encouraged better learning than lecture-based courses.

“All intuitions do not necessarily need to be eliminated; they need to be refined” (Pollatsek, Well, & Boyce, 1990), but if they can be refined it implies that they do need to be refuted. Refuting probabilistic misconceptions may have unwanted effects simply because they are not intuitions at all, but p-prims. Of course, this may not apply to other misconceptions in probability. A larger study could investigate that.

7.2.4 A new manifestation of representative heuristic?

In the introduction to Question 2, the notion of subjective indistinguishability was discussed as a possible explanation for the misconception attributed to Fischbein and Schnarch's compound and simple events problem (1997). Upon analysis, a better explanation may have emerged and it would appear that the representative heuristic plays a large role in the belief. Subjective indistinguishability would only reinforce the misconception.

Normally, the representative heuristic is defined as the extension of a population's characteristics to the characteristics of small sample.

A student of sufficient age may intuitively attribute a dice to its salient characteristics (while younger students may not have yet accumulated enough experience). For a dice, the salient characteristics that generalize it as a probabilistic object is that the result of its roll is random and any side has a $\frac{1}{6}$ chance to be obtained. These characteristics are learnt with experience and exposure, such that automatic associations are made between the object and these features (well-learned information).

In other words, these characteristics of a dice are "known" in an intuitive way. ("Known" is in quotations because the individual is confident that these properties are true, however objectively speaking they do not have to be).

For example, take the case of rolling two dice in Question 2. When a second dice is introduced, it takes critical thought to realize that the pair of dice can no longer be represented by the characteristics of the individual objects that comprise it; namely the characteristic of equiprobability. But this is difficult to realize because the dice appear identical (subjective indistinguishability).

This would explain why the participants who implemented tree diagrams correctly answered Question 2, and those who referred to the individual dice as a justification believed that the pairs were equally likely.

It is also conceivable that using visually dissimilar dice may be useful to correcting this conception.

In addition, participants may realize their erroneous reasoning by cognitive conflict if they are asked to list all outcomes that comprise the sample space of rolling two dice: Would they obtain 36 or 21 outcomes? Would they recognize the conflict of having 36 in the denominator, if 21 outcomes are listed?

7.2.5 The importance of intuition and heuristics

Despite all the obstacles and speculated complexities of the intuitions, it should be noted that intuitive insights to probability problem solving are an integral part of the solving process.

One needs intuitions (irrelevant of the domain of experience from which they were acquired) in order to begin down an avenue of deliberate problem solving, and the application of learned algorithms. Any problem solving or decision making is initially guided by an automatic thought process or reaction that may have basis in experience with that domain. In problem solving, the human brain is biased; especially when it comes to matters involving uncertainty. That is why one of the main goals of probability teaching should be to teach self-doubt as part of the problem solving strategy.

7.3 Laplace Approach versus Properties Approach

In view of the fact that this study is qualitative, and based on single participants' reactions to each approach-treatment, no conclusions can be drawn about whether the Laplace approach is better suited for the learning of probability than the Properties approach, or vise-versa. If the results of Question 2 are any indication, one conclusion is that future iterations of the design experiment should include a larger breadth of problems, since this problem produced interesting results, while diverging in style from the other problems.

7.3.1 Probabilistic interpretations are intuitively applied

Larry's solution to Question 2 was aligned with the Properties approach in spite of being taught the Laplace approach in the lesson. This suggests an important finding: Despite being taught using one approach, individuals may find other frameworks more intuitive or useful depending on the demands of the problem. In other words, just as students have certain cognitive biases during problem solving processes, they may also possess intuitive probabilistic interpretations that are dependent on the features of a problem. Hence, they can seamlessly move from one approach to another. Different frameworks are mathematically contradictory and some are more useful than others depending on the problem. But if solving probability problems is a model-building activity and the approaches are the tools for solving, then the problem itself is what sets the context in which some tools may be easier to implement, or they are quicker to call to mind as an intuitive reaction to the specific context. This highlights an additional obstacle analogous to intuitive cognitive biases. This obstacle is the intuitive, automatic and implicit implementation of probability approaches.

When asked what lesson material they had seen before, two Laplace approach participants (including Larry), referred to material that had not been explicitly discussed and was actually within the framework of Properties approach:

> *R: Were there things you already knew in the video?*
>
> *Larry: Yah when it came to studying the probability of a dice roll like I knew the 'multiply each component' and I knew how to divide which parts had more focus. Yah, 1/6x1/6x1/6 not the 1/12. Like I got that idea and I knew the tree – I would have been able to follow that.*

This supports the hypothesis that probabilistic interpretations applied to certain situations are intuitive due to being well-learned information, and thus, automatically applied. Moreover, it demonstrates the cognitive feature of confirmation bias in which the participants claim to have seen something from the lesson that existed in their mind only.

It is likely that, just as one should educate students on cognitive biases, one could also educate students on, not only the existence of different probabilistic interpretations, but on tendencies to implicitly assume and apply them.

As Steinbring (1991) notes, an obstacle in teaching is the implicit mixing of approaches and he states that the solution is to make each approach explicit. This seems logical because it would equip students with an array of problem solving tools that they can apply depending on the situation. They would also be aware of the approach's theoretical limitations.

7.3.2 Advantages and disadvantages to Laplace and Properties approach

Laplace students had a tendency to use tree diagrams more consistently than Properties participants who did not use trees at all. Properties participants seemed to abandon tree diagrams altogether and were more reliant on the probabilistic rules (additive and multiplicative properties), and seldom constructed tables to list relevant arrangements. It is suggestive that the tree diagrams were helpful in considering all relevant arrangements for the chances of rolling 5-6 and helping them understand that simultaneous rolls produce the same tree diagram as consecutive rolls. Not producing the diagram seemed to be connected to the participant's belief that 6-6 and 5-6 are equally likely. Leah did not draw a tree diagram on account that the answer was "obvious" and, as a result, her solution to Question 2 missed an arrangement. Likewise, Perry and Peggy never used the tree diagram as part of their justifications and missed the second arrangement of 5-6 in Question 2.

In fact, Perry demonstrated some confusion saying that consecutive rolls represented two different events and simultaneous rolls represent one event. He then concluded that simultaneous rolls could not be represented by a tree diagram. This reflects his understanding of 'event' as a chronological representation of process rather than a structure of logical possibilities. His understanding is the vernacular definition of event as specifying a common place and a common time.

Both approaches had missed arrangements. No conclusion can be drawn as to why either approaches would miss arrangements even if exposed to refutation statement LR3/ PR3. It could be that the computational demands of tracking all the arrangements 'manually' is fairly large- so, without the use of relevant formula. Missing arrangements is the most common misconception observed in this study. Although it does appear that when refutation participants missed arrangements, they missed less than exposition participants. Also, Larry missed arrangements for Question 1 and Question 4[11], but was the only one to catch his mistakes retroactively during the interview.

So why were Laplace participants more consistent than Properties participants in using tree diagrams? There are a few possibilities. Properties participants may view trees as redundant because the mathematical properties are interpreted as the primary tool of the approach. For the Laplace approach, tree diagrams are central to the solving process in that they are used to determine the number of desired outcomes, the number of all possible outcomes and the number of arrangements.

In the Properties approach, the tree diagram may be viewed as an unnecessary detour as it only yields partial information that is relevant to the solution. This may be why both Properties participants always opted out of tree diagrams and instead, sometimes, listed the desired outcomes.

If diagrams are perceived as being central to the Laplace solving process, it seems likely that individuals found it easier to begin the solving procedure immediately upon reading the problem statement because all one needs to do is begin drawing some lines on their paper. This could explain why the Laplace participants were more consistent in saying they had no initial thoughts about the problems (besides Peggy for Question 1).

There is an implicit belief that probability is part of mathematics and therefore must contain formulas and equations and etc. All Laplace participants hinted that they wished the lesson taught a formula to

[11] I roll a fair dice several times. The chances to have at least once a two if I roll 3 times is (larger than, smaller than, equal to) the chance to have exactly once a two if I roll 3 times.

replace the tree diagrams. Therefore, Properties participants may have felt that the multiplicative and additive properties are the only tools they require since it fits their conception of mathematics.

The Laplace approach is not without its faults. It has theoretical limitations in the number of trials it can be applied to. Participants expressed frustration and a sense of discouragement because, as the number of trials went beyond 4, the diagrams became practically unmanageable. Large diagrams seemed to miss arrangements even when the participants were aware of their importance.

Leah and Larry were aware of the existence of formulae that renders the tree diagram obsolete. They expressed a curiosity to know such formulae. Perry had voiced that he understood the limitations of what he had learned from the video since the diagrams could only deal with a certain number of cases. Because of these limitations, he stated that the section of the lesson devoted to tree diagrams was his least favorite.

It would have been interesting to survey all participants on their interest in probability after the questionnaire, instead of surveying their interest pre-questionnaire. The hypothesis is that interest in probability would decrease if participants completed a problem using the Laplace approach since the task of producing large diagrams may be seen as mundane and non-stimulating.

Another shortcoming in the Laplace approach can be identified in Question 5 wherein participants were asked to calculate the probability of obtaining at least one even number in 3 rolls. It should be noted that all participants had difficulty here.

Perry employed the correct method using the additive and multiplicative rules but missed arrangements. Lea could not finish this problem on feelings of discomfort while contemplating a symmetric tree. Leah correctly identified that each branch had 3 possible outcomes but possibly added the outcomes instead of multiplying. In the lessons that Leah and Lea watched, this misconception was not refuted (LR3), but Lea did not neglect arrangements and it is likely attributed to her drawing of a tree, where Leah produced none. Larry constructed the correct tree (evens versus odds) but still committed an error in counting the outcomes. One wonders why the Laplace participants struggled as they did on Question 5.

Here I present a hypothesis of a central difficulty with tree diagrams (as they were presented in the lesson) involving its salient visual features. If it is known that students are susceptible to qualitative proportional reasoning (more of this, more of that). It is not improbable that there may exist a cognitive dissonance when one tree branch represents more options then another branch of the same visual size. There is an implicit step to be made by an individual constructing a tree diagram whose outcomes for a single trial are (3, not 3) yet (3) represents one outcome and (not 3) "weighs more" with 5 outcomes. This could be the source of difficulty for the participants in the study who relied on tree diagrams and struggled on Question 5 when the branch weights changed. No instruction was given in the lesson on how a tree should be adjusted while changing the probability of a single success. This may explain why Properties participants, who did not rely on diagrams, seemed to perform better on Question 5. The next iteration of the design experiment could see the lesson produce diagrams with branch thickness proportional to the number of outcomes it represents.

7.4 EXPOSITION LESSONS COMPARED TO REFUTATION LESSONS

The primary question that arose from analyzing results between exposition and refutation treatments was: Did the explicit refutation of misconceptions help or hinder the participants' understanding in this study?

On one hand, those exposed to misconceptions on proportionality appeared to have understood the non-linear relation between (n, k) and P in Question 6: $\frac{k}{n} = \frac{k\prime}{n\prime}$ does not imply $P(X_n \geq k) = P(X_{n'} \geq k')$. The evidence does not appear in their written work but weakly appears in discussion during the interview. Their oral justifications are insufficient but what they do allude to suggests that the refutation format may have been effective.

Question 6 was not meant to be answered thoroughly, or by explicitly carrying out calculations. All participants were expected to be deterred by the number of trials ($n = 6$). And without the knowledge of combinatorics, the refutation participants were expected to qualitatively reason out an answer, but their confidence was expected to be low (as Muller (2008) observed). The exposition participants would have been equally deterred but were expected to be more tempted by proportionality.

What was observed, however, was unexpected. First, refutation participants seemed to over-rely on and overgeneralize the refutation statements (namely LR4/PR4) throughout the experiment. As explained, some of this can be attributed to general fatigue or a sense that regurgitating the statement as a justification was an "easy way out." In most cases, the refutation statement used as a justification was insufficient, and it is difficult to ascertain what these participants understood.

On the other hand, being that these regurgitated statements were overgeneralized from "not necessarily proportional" to "never proportional", it suggests that the participants took it upon authority of the instructor that such a statement is true without critical thought about its consequences or meaning. This may at least partially explain why high confidence was observed among refutation treatments.

But another factor is influential: the refutation format was designed to expose, refute and replace pre-conceptions that were assumed to be “intuitions”, and not, as it is now hypothesized, phenomenological primitives. In other words, the refutation format did not have the desired effect on probability learning but that’s because the refutation lesson mostly refuted “weak” misconceptions, i.e., misconceptions that are easily displaced by new concepts.

This can only be fortified by the notion that probability is counter-intuitive. The participants knew (to various degrees) that “gut-reactions”, “instincts”, “intuitions”, should not be trusted, which is a direct verbal indicator of their inherent confidence about their knowledge on probability. (Strong, robust and incorrect misconceptions would yield high confidence and certainty). So the word “intuition” as it relates to the illusion of linearity in probability is a bit of a misnomer. Especially if Van Dooren’s questions do not appeal to students’ experiences anyway.

So if refutation participants understood the notion of non-linearity it isn’t because the refutation was effective at displacing a misconception; it’s because they were simply learning something new. For such a statement to be understood thoroughly, perhaps students can be shown a graphical depiction of the relation.

7.5 REVALUATION OF THE DESIGN EXPERIMENT AND FUTURE INVESTIGATIONS

The contributions from this study to the existing body of knowledge on probability learning lies in the various ways the experiment can be redesigned to provide more fruitful insights. Indeed, one of the most significant suggestions from this study is that students are not necessarily equipped with intuitions about probability but rather phenomenological primitives, thus implying that a fertile basis for learning may be in the construction of intuitive conceptions through diverse sets of empirical probability. Such an approach requires greater resources, so the following is a compilation of design experiment iteration suggestions.

7.5.1 Abandoning Van Dooren’s (2003) question format and diversify problems on questionnaire

There is a concern that the exercises themselves were cognitively overwhelming, as Lea pointed out during the interview. In comparing two scenarios, there are 6 meaningful numbers to remember and organize in order to comprehend the problem. In other words, for a student to understand the demands of the problem, they must simultaneously hold in their minds the number of rolls per scenario, the number of desired events per scenario, the desired outcome per scenario as well as the technical detail contained in the condition of considering *at least* k successes. The participants managed well with these questions, but they were approximately two years older than the oldest participants of Van Dooren et al. (2003) study, implying maturity may have allowed them to cope with any possible extraneous cognitive load brought on by the internal reorganizing and understanding of the problem statement. Many students in Van Dooren et al. (2003) demonstrate quantitative negligence by posing solutions that neglected the specification of '*at least*'. However, this may not be surprising given the wordiness of the problems; such details may have been overlooked by the students.

Moreover, the questions appear to have been specifically designed to entice the automatic application of the schema of proportionality (in other words, the questions were tailored to be linearity 'traps'). In that respect, perhaps Van Dooren et al. (2003) obtained a biased representation of the pervasiveness of misconception. Not to say that the tendency to apply linear models does not extend to probability. But if one desires to draw meaningful and convincing conclusions about students' thinking it is better to diversify the problem types and limit the extraneous load of the problem. It should be clear that the only evidence insinuating that the problems had a high extraneous load comes from a single participant's comments. This is the reason why the next questionnaire should ask to self-assess the mental effort invested after each question.

Future iterations of the experiment would pose several different types of questions to observe how different probabilistic approaches are implemented. Perhaps certain approaches are preferred by students for given questions. Do different scenarios activate different avenues of thought? For example, if students are refuted on the use of blended approaches, will they struggle in the development of a justification? How would have Larry solved Question 2 if he was told that he could not use Properties approach? Are certain probabilistic approaches intuitively activated and engaged depending on the problem? With exercise diversity, a new direction in the investigation can be pursued.

Some questions on the questionnaire should have been open-ended. There were a number of issues that arose from this study's format. A correct answer could be obtained by incorrect mathematical justification and solution. The result is that participants can feel confident about erroneous work, and their confidence ensures that they will not be critical or doubtful. Similarly, a mistake made in previous questions could be reused by the participant leading to more erroneous conclusions.

Additional exercises may illuminate new questions about how students think about probability if taught using a single approach. For example, students can be asked to compare the probability of drawing balls from a vase of 100 white and 200 black, to the probability of drawing balls from a vase of 50 white and 100 black. Variations of this problem can further probe how and when students apply proportionality models, and whether or not they implement tree diagrams. Such problems could also compare how students approach conditional probability when drawing balls without replacement.

A few hypotheses were proposed to explain the difficulties encountered in Question 2. It is possible this issue can be corrected if images of two visually dissimilar dice were presented to counter subjective indistinguishability of dice. It is thought that the neglect of arrangements can be corrected if students can easily distinguish the objects that comprise the arrangements that contribute to the probability. To further understand this issue, students can be asked to judge whether it's advantageous to roll dice consecutively or simultaneously, if they wish to obtain the pair 5-6. Also, cognitive conflicts can be cultivated in the students if they are asked to write all possible outcomes in two dice rolls. If it is believed that 5-6 is equally likely to 6-6, then do they also believe that there are 21 possible outcomes as opposed to 36?

7.5.2 Modifications on interest levels and self-rated confidence

A problem not specific to probability is the importance of interest and its influence on engagement, justification and critical thinking. Future iterations of the experiment may administer a pre-lesson measure of interest and a pre-test/post-test measure of interest. Along with measures of confidence, one may observe a correlation between the two. Varying the types of questions asked between participants can serve as indicator as to what material students find the most engaging. And as mentioned, students can be asked about their self-rated mental effort after each question. The goal would be to paint an accurate picture of how confidence, interest, performance and rate of misconceptions vary before and after exercises are completed.

Interest in probability is important since students are likely equipped with p-prims, meaning that despite having several inconsistent or contradictory intuitions, the consequences of the beliefs are never consistently challenged and therefore rarely adjusted. One of the main objectives in teaching would be to make probability fun using various types of games, hands-on practices and model-building. Perhaps at more advanced levels, these sorts of empirical approaches would not be required since better conceptual and intuitive understandings would have been already constructed.

Finally, a second post-test may be administered some period of time after the lesson to determine what concepts were retained and if students use the approaches taught to them, or if they blend several approaches.

7.5.3 Modifications to video lessons and instructional material

7.5.3.1 Adjustments to interviews

The clinical interviews should be recorded with video as well as audio to capture the full dynamic interaction between participant and researcher. The concern, once again, is to acknowledge the effects of the observer's paradox. Eliminating such effects is difficult and an alternative is to have pairs (or groups) of students discuss questions together while being filmed. However, such an approach is not free of the effects of social-intuitions and group dynamics on critical thinking. A very large study could attempt to use both interview approaches for comparison.

7.5.3.2 Considerations about the visuals in the video lesson

In terms of the presentation of the video lesson, the next iteration would present tree diagrams with branch thicknesses roughly proportional to the number of outcomes they represent. This modification may have positive effects on the Laplace participants who struggled with producing the correct diagram for Question 5. Similarly, the importance of, not necessarily trees, but of a graphical representation of the structure of a probabilistic situation must be stressed to the Properties students.

7.6 PROPOSITION FOR A LARGER PROJECT

Ultimately, a larger project to investigate the optimization of probability learning in a classroom would adopt an approach similar to Shaughnessy's (1977) study but with some additional features informed by the research and analysis completed for this study.

These features are hypothesized to improve the small-group, model-building, activity-based approach tested by Shaughnessy by considering the additional epistemological, cognitive and didactic obstacles noted in this thesis. A larger study would adopt the design research methodology and be constructed upon the following pillars which are thought to be critical to the revision of probability learning.

7.6.1 Model-building, activity-based learning and didactically fertile refutations

When students engage in scientific method of discovery, they acquire a cognitive apprenticeship in which a probabilistic situation is theoretically modelled, outcomes are defined, justified guesses about their chances of occurring are put forth,and these models are tested by repeated and controlled trials. The situations to be modelled would range in complexity and applicability, and would be dependent on the specific content of the curriculum. These situations must also be diverse and unconventional; examining the role of probability and randomness in everyday situations so that students are encouraged to expand their conceptions of chance phenomena beyond what occurs in a classroom or with the typically used devices such as dice and coins. Modelling physical situations would be completed by developing simple probabilistic tools, possibly in the style that Guenther (1968) proposed. Tree diagrams are powerful visual representations that, if introduced alongside formal rules or properties, will optimize what is retained by appealing to some basic principles of cognitive load theory and the psychology of learning (Myers, 2010).

Commented [AS1]: "sample space" is a basic concept of probability, not a visual. Maybe you wanted to say "Tree diagrams OF sample spaces"?

A consequence of testing theoretical probabilistic models is that p-prims are built to become robust intuitions, and distrust in "gut-feeling" increases. Moreover, this strategy will exemplify the relation between the classical and frequentist approaches as well as the relation between probability theory and reality. Modelling systems that have no underlying physical symmetries can use subjective interpretations, but these must be tested too. Probability will adopt the role of making the 'best' guess about a situation that is non-deterministic. Any information pertaining to said situation contributes to that guess.

A possible explanation for the observed refutation-treatment participants' over-reliance on the lecturer's refutation of the "illusion of linearity" misconception and its over-generalization is that the refutation statements in the lesson were made on single, specific examples, thereby making it difficult to identify the essential features of the misconception. In other words, just as Freudenthal (1973) suggested about meaningful learning of mathematical knowledge, students require didactically fertile examples of how their alternate conceptions are inconsistent with probability theory. Along with the aforementioned modelling approach, instructors can encourage explicit testing and theorizing of misconceptions. Instead of the teacher explicitly refuting them with specific examples, students would have their beliefs refuted by deliberate experiences which would also improve their long-term retention (Myers, 2010). Again, this would be aligned with the general principle of challenging students' ideas rather than only reinforcing them – a principle valid for all domains of knowledge, not only probability.

7.6.2 Cognitive Caveats

Most individuals are fascinated by illusions. They are deceiving even upon learning about their true nature. Similarly, students should be challenged by paradoxes and fallacies as Borovcnik et al. (1991) proposes. These not only raise interest and cultivate discussion and critical thought, but they are also, in a sense, honest about the limitations of probability theory, as well as the human mind.

Following this train of thought, students should be informed about biases and heuristics to raise interest as well as awareness of self-limitations. The objective is to change the negative perception students have about probability from "I don't like this subject because it's counter-intuitive'" to "I like this subject because it's counter-intuitive, and here's why I do…."

7.6.3 Small-group work

Besides adopting a "cognitive apprenticeship" approach in the scientific method, students can exchange ideas about their thoughts on certain theoretical models, or why experimental results were unexpected. It is believed that informing students about their inherent cognitive biases will create an honest atmosphere wherein students feel that their misunderstandings about probability are not only acceptable but expected. Group work would therefore have students encouraged to detect and challenge each other's biases and misconceptions.

Moreover, if probabilistic misconceptions are weak intuitions, then it is likely that, within a group, different students would have different intuitive understandings as well as different incorrect intuitions. Students would be encouraged by the instructor to give their unique perspective and to freely question assumptions on the basis that everyone is susceptible to having incorrect beliefs in probability.

8 FINAL REMARKS

This section is dedicated to discussing some general and personal observations concerning the journey of writing this thesis.

It appears that the Properties approach is computationally easier to implement and may be more aligned with how students think about Bernoulli trials. Neither theoretical framework seemed to encourage or discourage the use of common misconceptions, except that Laplace participants had a tendency to miss arrangements due to the computational demands of the approach. The Laplace approach is appealing in its simplicity but is limited in its capacity and cumbersome. Additionally, it seems that the refutation treatment participants performed better than those subjected to the exposition treatment, although perhaps not for the expected reasons. Granted, the study assumed that the misconception of linear modelling derived from intuitions, and not phenomenological primitives. Those treated with the refutation lessons seemed to “know” rather than “understand” that changes to the number of trials and the number of desired events affect the probability of the desired event in a non-proportional manner; their justifications were largely insufficient.

Personally, it was a slow adjustment realizing that there are many dimensions to the study of mathematics education. Operating outside this fact presented many challenges to the researcher, who had to overcome some cognitive fixation and confirmation bias, which made it difficult to consider novel vantage points, as well as to confront implicit assumptions on chance phenomena, and participant results and behaviors.

That being said, I have come to appreciate the process of uncovering the subtle complexities in ideas that were previously assumed to be simple and innate. Similarly, there is no better feeling than drawing together seemingly unrelated ideas by one unifying notion. Rising from the many challenges that have germinated along the way is an overwhelming curiosity and a strangely satisfying dissatisfaction knowing that I have much to learn about probability in education.

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APPENDIX A – CONSENT FORM

CONSENT TO PARTICIPATE IN A STUDY OF PROBABILISTIC THINKING

This is to state that I agree to participate in a research experiment conducted by Jean-Marc Miszaniec, tel. 819-593-5703, email: jmmiszan@gmail.com, under the supervision of Dr. Anna Sierpinska of the Department of Mathematics and Statistics of Concordia University in Montreal.

A. PURPOSE

I have been informed that the purpose of the research is to test the effects of different approaches to teaching probability on students' probabilistic thinking.

B. PROCEDURES

I have been informed that I will participate in the study in the role of a student. The participation will consist in spending approximately 2 hours altogether on the following activities:

1. Watching a videotaped introductory lesson on probability (20-30 min). I am allowed to pause, rewind and re-watch some parts of the video during that time.
2. Working on a few exercises based on the lesson; I am allowed to re-watch all or some parts of the video while doing the exercises, but I am NOT allowed to look up any other sources (books, internet, etc.) (40-60 min)
3. Explaining my reasoning in doing the exercises, and sharing my impressions about the lesson in an interview with the researcher. (30-45 min)

C. RISKS AND BENEFITS

I am aware that I may experience some discomfort in the session due to frustration with the lesson or the exercises. I am also aware that my frustration and my reasons for it could be a means for the researcher to measure the effectiveness of the approach. Understanding how I felt during the experiment and what I found difficult or easy will contribute to improving the approach, in general, and devising better ways to teach probability.

D. CONDITIONS OF PARTICIPATION

- I understand that I am free to withdraw my consent and discontinue my participation at any time without negative consequences
- I understand that my participation in this study is confidential (i.e., the researcher will know, but will not disclose my identity)
- I understand that my conversations may be recorded for the purpose of research analysis
- I understand that the data from this study may be published, but my identity will not be disclosed in them

I HAVE CAREFULLY STUDIED THE ABOVE AND UNDERSTAND THIS AGREEMENT. I FREELY CONSENT AND VOLUNTARILY AGREE TO PARTICIPATE IN THIS STUDY.

NAME (please print): ________________________SIGNATURE: __________________________

If at any time you have questions about your rights as a research participant, please contact Monica Toca, Coordinator, Research Ethics, Concordia University, at (514) 848-2424, ext. 2425 or by email at Monika.Toca@concordia.ca.

Thank you for your participation.

APPENDIX B – SCRIPT OF THE LAPLACE APPROACH VIDEO LESSON

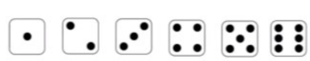
The script below contains the text for both the “Exposition” and the “Refutation” treatments. The text to be read only in the Refutation treatment is enclosed within curly brackets. The script contains also snapshots of the screen that was shown at the time when the words were spoken.

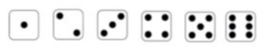
Words or phrases in bold are to be emphasized; words and phrases in capital letters are to be emphasized even more.

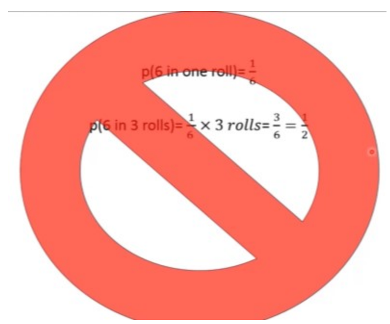
Paragraphs in the script have been numbered for ease of reference. Titles of episodes in the lesson were added to the script (in italics) to make the structure of the lesson clearer, but they were not read by the narrator. Moreover, the misconceptions addressed in the refutations are formulated in general terms and added to the script (in italics and red font) – these formulations were also not read by the narrator.

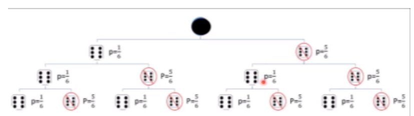
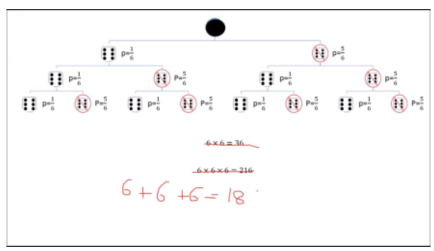
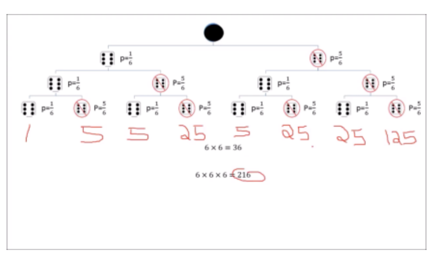
The exclamation mark (!) dictates the moment when a new image or text is added on screen (for the purposes of video production).

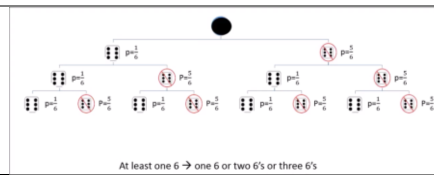
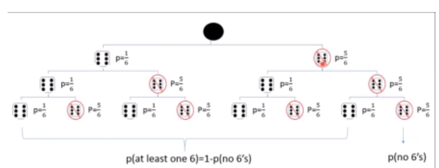
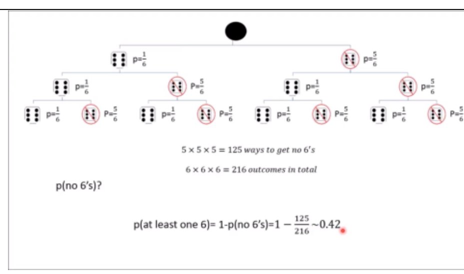
Double exclamation mark (!!) indicates that a whole new image or text replaces what was there before (the “slide” changes). (Only used if there is a slide change within a paragraph).

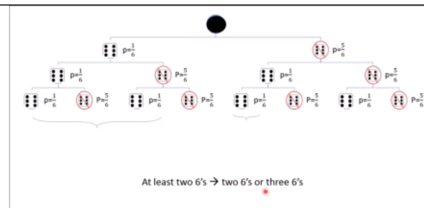
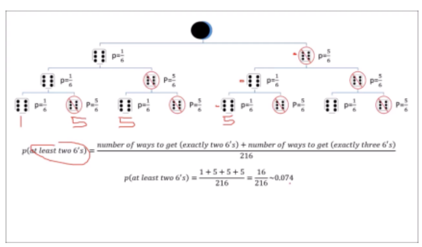
Script	Video snapshot
Introduction: Definition of probability theory and probability of an event	
1. In this short lesson, we will learn how to calculate probabilities of a few different dice game scenarios. The concepts presented in this lesson are not restricted to dice games. They can be applied to coins, cards, or anything that involves chance and randomness	Probability and Dice
2. Before we begin I will go over some definitions that are important for understanding probability.	
3. First of all, probability theory is a mathematical theory concerned with!uncertainty. To find the probability of some unpredictable and random event is to estimate the chances of its happening.	Uncertainty
4. !Let’s assume that in a game a certain number of outcomes are possible. For example, when you are rolling a dice, you can only get 1 or 2 or 3 or 4 or 5 or There are 6 possible outcomes. In a game, you are normally interested in only some of these outcomes – only some of these outcomes are !**desired**, because, for example, they make you win the game. We sometimes call these desired outcomes, a !**desired event**. Then we estimate the chances of this desired event to happen by the ratio of the number of the desired outcomes to the number of all possible outcomes. !We say that **the probability of the desired event is the number**	“desired” “desired event”  p(desired event)=$\frac{\text{number of ways desired event can occur}}{\text{number of all possible outcomes}}$

ways the desired event can occur divided by the number of all possible outcomes.	
5.The likelihood of a particular event among the set of possible outcomes is expressed by a !number from 0 to 1, where !0 represents an impossible event, and !1 represents the certainty of an event. !!	$0\leq p(event) \leq 1$ Probability(*Impossible* event)=0 Probability(*Certain* event)=1
For a fair dice, so a dice presumed to not have a favourable side to land on, we can say the !probability of obtaining 1 or 2 or 3 or 4 or 5 or 6 is 1. There is 6 ways to achieve the desired event from a total of 6 outcomes. In other words, it is certain that you will get something when rolling a dice. This is self-evident, perhaps, but equating certainty to the number 1 is important.	 p(1 or 2 or 3 or 4 or 5 or 6)= $\frac{number\ of\ ways\ desired\ event\ can\ occur}{number\ of\ all\ possible\ outcomes} = \frac{6}{6} = 1$
On the other hand, the probability of rolling a !7 is 0 since there is no way to obtain a 7 on a single dice so [p(7)=0/6=0]. Hence, it is impossible.	p(7)= $\frac{number\ of\ ways\ desired\ event\ can\ occur}{number\ of\ all\ possible\ outcomes} = \frac{0}{6}$ =0
6. The probability of rolling !a 6 on a dice is therefore 1 desired event divided by the number of all possible outcomes so p(6)=number of ways to get a 6/ number of all possible outcomes=1/6.	p(6)= $\frac{number\ of\ ways\ desired\ event\ can\ occur}{number\ of\ all\ possible\ outcomes} = \frac{1}{6}$ ~0.167
As an **alternative**, we may consider the number of ways a non 6 comes up. This is the !!complementary event. Two events are complementary if the outcomes that make them up cover **all the possible** outcomes.	Complementary event 
For example getting a !!6 in a roll of dice and getting a non-6 are complementary events. Also, getting an even number (2, 4 or 6) and getting an odd number (1, 3, 5) are complementary events.	Complementary event 
Probability theory tells us that once you know the probability of an event, it is easy to ! calculate the probability of the complementary event, by subtracting it from 1. For example, knowing that probability of getting a 6 is $\frac{1}{6}$ we can easily !calculate the probability of getting a non-6 in a roll of a dice!:	$probability\ of\ an\ event = 1 - probability\ of\ the\ complemetary\ event.$ $p(non\ 6) = 1 - \frac{1}{6} = \frac{5}{6}$ $p(non\ 6) = \frac{number\ of\ ways\ desired\ event\ can\ occur}{number\ of\ all\ possible\ outcomes} = \frac{5}{6}$

7. To see that this makes sense, we can calculate the probability of getting a non-6 directly: getting !non-6 amounts to getting 1 or 2 or 3 or 4 or 5: there are 5 desired outcomes.! So you see that the probabilities obtained directly and by the complementary event rule are the same.	
Problem 1: Probability of getting at least one 6 in three rolls of a dice	
8. So now let's calculate some probabilities. We will go over 2 different but related situations; 2 dice game scenarios. We will find the probability of obtaining certain outcomes when rolling a dice. We will assume that the dice is fair. (We won't bother verifying if the dice this is true.)	examples
9. Imagine a game that uses dice. For a particular round of this game, you are required to roll a die three times. You win if, at least once in the three rolls, a 6 turns up. Do you think it is ! likely for you to win? Or is it unlikely? !What is the probability of getting at least one 6 in three rolls of a fair die?	Likely or unlikely? p(at least one 6 in 3 rolls)?
Misconception LR1: p(a SIX in n rolls) = n x p(a SIX in 1 roll) 10. !!{Some people believe that the probability of getting at least one 6 in 3 tosses is ½; that their chances of winning are fifty-fifty. Here is how they reason. The probability of getting a 6 in a single roll of a fair die is 1/6. This is correct. But then they think that the probability of getting a 6 in THREE rolls will necessarily be THREE times bigger:! $3 \times \frac{1}{6}$ [display]. Three times one-sixth is three-sixths which is the same as one-half. So they conclude that they have fifty-fifty chances of winning [display calculation*]. But this reasoning is NOT correct. It is not only mathematically incorrect. It is DANGEROUS to reason that way. In fact, if you were to bet on winning in this game, you would likely lose your money. Be weary of applying linear thinking in probability.}	
11. Here is the correct reasoning. To calculate the probability of obtaining at least one 6 in 3 rolls we need to first consider the total number of outcomes and then we can discuss the desired events.	[blank]
12. To help us visualize the different outcomes obtained when rolling many dice we will use a !tree diagram [display].	Tree diagram

On the first roll we may obtain a 6 with a 1/6 chance, or we can get a non-6 with a 5/6 chance [display]. The second roll offers the same possibilities- one 6 with 1/6 chance or non 6 with 5/6- which can be obtained after getting a 6 on the first roll. Or we can get a 6 with 1/6 chance or a non 6 with 5/6 chance on the second roll after getting a non-6 n the first roll. So we build the tree this way because on the second roll we can get any of the 6 outcomes following any single one of the outcomes that was obtained on the first roll. All pairs of outcomes are accounted for such as (1,2), or (6,3), or (3,6) just to name a few.. That means after two rolls we have 6x6=36 options. So when you the dice a third time you have another 6 options which is 6x6x6=216 total outcomes.	
Misconception LR2: number of possible outcomes in a sequence of n repetitions of a random experiment = n x number of possible outcomes in the random experiment (instead of (number of possible outcomes in the random experiment)^n) 13. {Many believe that in order to calculate the number of all possible outcomes they must add the individual dice outcomes- not multiply. So they wouldn't even do this [on screen]. For example, with three dice rolls they would reason that they'd have !6 outcomes in the first dice roll plus 6 outcomes in the second dice roll plus another 6 outcomes in third dice roll- which adds up to 18 possible outcomes. So if we wanted to calculate the probability of some desired event we would have 18 in the !denominator instead of the correct value which is 216. In other words, our probabilities would be very wrong. And betting on our calculations would lose us money.}	
14. So lets imagine some possible outcomes after rolling the dice three times…We may have the outcome (1,6,6) which takes you down this [on screen] branch which is different than (6,6,1). Remember that 1 is grouped into the non 6's branch. We may also have a single (2,2,2)- all down the non 6's branch, or (6,6,6) and (1,3,4), different than the outcome (4,1,3) despite being down the same branch…notice however it is more likely to go down the non 6 branch since there is 5 ways to obtain a non 6 versus one way to get a 6. And in total there are 216 triplets. Now if you're not convinced that there are that many options- it's a	

<table>
<tr><td>little deceiving looking at this diagram, let's consider every branch. So obtaining 6,6,6 can only occur in one way. Obtaining 6,6 then non 6 occurs in 5 ways, since 6 occurs once, 6 occurs once and non 6 occurs 5 times. 6,non-6, 6 also 5 times. 6, non 6, non 6 yields 25 triplets so one way to get 6, 5 ways to get non 6 and another 5 ways to get non 6. Or we may get a non 6 on the first roll, which occurs in 5 ways, a 6 in one way and another 6 in one way so that's 5. Or we may obtain a non 6, 6 and non 6 which again is 25. Or nonc6, non 6 and 6 which is 25. Then we may get a non 6, followed by a non 6 then another non 6 which is 5x55 which is 125 different ways to get a triplet of non 6's. And if you add all these up [on screen] you actually get 216, the total amount of outcomes.</td><td></td></tr>
<tr><td>15. So we have 216 total outcomes, lets discuss the desired outcome, which is obtaining at least one 6 in 3 rolls.</td><td></td></tr>
<tr><td>16. First of all, !"at least one 6" means our desired event is obtaining exactly one, two or three 6's. All are important when stating 'at least'.</td><td>
</td></tr>
<tr><td>17. To calculate the probability of obtaining at least one 6 is easy when using the complimentary event rule, !p(no 6's) [display]. We can find the probability of the event of obtaining at least one 6 by finding the probability of obtaining no 6's, which is easier, and subtract from 1. So the !p(at least one 6) is equal to 1 minus p(no 6's). We do this instead of counting all the different ways of obtaining one 6, or two 6's or three 6's. All we have to do is calculate this [on screen- non 6's branch] and subtract it from 1.</td><td>
</td></tr>
<tr><td>18. Notice how many options we have to obtain no 6's at !all. The first roll yields 5 ways to roll no 6's. The second offers 5 ways to roll no 6's. And so does the third roll. That's !5x5x5=125 ways to get no 6's out of !216 outcomes [display]. The !probability of obtaining no 6's in 3 rolls is the number of desired outcomes divided by all possible outcomes [display]. In this case the probability of obtaining no 6's is 125/216. Therefore, p(at least one 6 in 3 rolls)= 1-p(no 6's)= 1-125/216=0.42.</td><td>
</td></tr>
</table>

Problem 2. Probability of getting at least 2 sixes in 3 rolls of a dice	
19. Okay, so now let's pretend that later in this game, you've realize that you've fallen behind and now you require at least two 6's in 3 rolls to win. !This is double the amount of 6's than before. Before, for at least one 6, we got a probability of 0.42. !Is it likely or is it unlikely for you to win? What is the probability of winning now that you require at least double the amount of 6's, so at least two 6's, with 3 dice rolls?	p(at least *two* 6's in 3 rolls)? Likely or unlikely?
20. We may begin again with the tree diagram representing the branching's of all possible outcomes [display]. The tree will look exactly the same as before since the total number of possibilities are the same…we are still rolling 3 dice and each dice each have 6 outcomes for a total of 216 outcomes. What we have changed are the specifications for the desired event. We now want !*at least two* 6's, which means the desired events are *exactly two* 6's [highlighted] and *exactly three* 6's [highlighted].	
21. How many ways can we obtain exactly three 6's or exactly two 6's? We can begin to count our options since the probability of at least two 6's will be the number of ways we can get exactly two 6's plus the number of ways we ca get exactly three 6's divided the total number of outcomes.	
22. Obtaining three 6's can occur only in one way. The first dice can be a 6 in one way, the second can be a six in one way and third too. So obtaining exactly three 6's occurs in one way out of 216 outcomes.	
23. What about obtaining exactly two 6's? Now this is a little more involved. The first dice can be a 6 in one way, the second dice can be a 6 in one way and the third must be a non 6 which occurs in 5 ways. So this particular sequence, 6, 6 non 6 occurs in 5 ways. But notice we can get exactly two 6's in 3 rolls in other arrangements. So for example, here [on screen], where there is a 6 in the first roll, a non 6 on the second roll and a 6 on the third roll, which occurs in 5 ways. Lastly, for exactly two 6's, we can get a non 6 in the first roll, then a 6 on the second roll and then a 6 on the first roll which also occurs in 5 different ways. SO that the ! probability of obtaining at least two 6's is equal to 1+5+5+5 over 216, which is equal to !16 over 216 which is approximately equal to 0.074. A pretty low probability that I would not want to bet on. So notice that in order to calculate exactly two 6's you had to consider all the possible	

<table>
<tr><td>arrangements in which two 6’s could land among the dice rolls.</td><td></td></tr>
<tr><td>Misconception LR3: The number of possible ways in which a sequence of events can occur does not depend on the order of these events in the sequence (arrangement neglect)

24. {A common misconception is neglecting to consider the different arrangements in which the dice can land. Calculating the probability of obtaining at least two 6’s requires two pieces- we have to know the number of ways to get exactly two 6’s and the number of ways to get exactly three 6’s. Getting exactly three 6’s occurs only in one way, however, calculating the number of ways to get exactly two 6’s, if we neglect the number of arrangements that two 6’s are obtained in three dice rolls it means we are not considering 5 arrangements and another 5 arrangements so our final probability will only be 1 plus 5 over 216. Which comes out to approximately 0.027 which quite different than 0.074.}</td><td>$p(at\ least\ two\ 6's) = \frac{number\ of\ ways\ to\ get\ (exactly\ two\ 6's) + number\ of\ ways\ to\ get\ (exactly\ three\ 6's)}{216}$

$p(at\ least\ two\ 6's) = \frac{1 + 5 + \cancel{5} + \cancel{5}}{216} = \frac{16}{216} \sim 0.074$

$\frac{1+5}{216} \sim 0.027$</td></tr>
<tr><td>Misconception LR4: p(at least 2 x SIX in k rolls) = $\frac{1}{2}$ x p(at least 1 x SIX in k rolls)

25. {The other misconception we should address is the following. Remember earlier when we found the probability of obtaining at least one 6 that was equal to 0.42. Here we have doubled the amount of 6’s required to win and obtained 0.74. So doubling the amount of 6’s does not halve the probability of obtaining at least one 6…it’s not equal to 0.21 (half of 0.42) it’s actually much less than that. So it’s important to realize that proportions to not necessarily apply in probability}</td><td>$p(at\ least\ two\ 6's) = \frac{1 + 5 + \cancel{5} + \cancel{5}}{216} = \frac{16}{216} \sim 0.074$ 0.42

$\frac{1+5}{216} \sim 0.027$</td></tr>
<tr><td colspan="2" align="center">Conclusion</td></tr>
<tr><td>26. Hopefully that was illuminating. Using tree diagrams to visualize probabilistic events is a very powerful way to break a problem down. Try it out and be more certain about uncertainty!</td><td>Thank you for taking a chance and participating</td></tr>
</table>

APPENDIX C – SCRIPT OF THE PROPERTIES APPROACH VIDEO LESSON

The script below contains the text for both the “Exposition” and the “Refutation” treatments. The text to be read only in the Refutation treatment is enclosed within curly brackets. The script contains also snapshots of the screen that was shown at the time when the words were spoken.

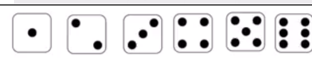
Words or phrases in bold are to be emphasized; words and phrases in capital letters are to be emphasized even more.

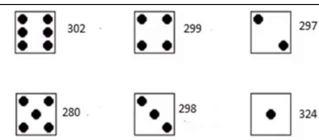
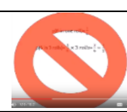
Paragraphs in the script have been numbered for ease of reference. Titles of episodes in the lesson were added to the script (in italics) to make the structure of the lesson clearer, but they were not read by the narrator. Moreover, the misconceptions addressed in the refutations are formulated in general terms and added to the script (in italics and red font) – these formulations were also not read by the narrator.

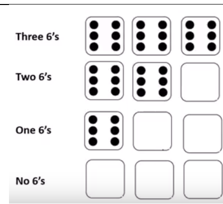
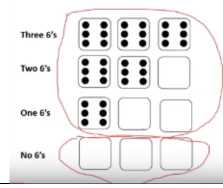
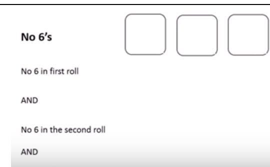
The exclamation mark (!) dictates the moment when a new image or text is added on screen (for the purposes of video production).

Double exclamation mark (!!) indicates that a whole new image or text replaces what was there before (the “slide” changes). (Only used if there is a slide change within a paragraph).

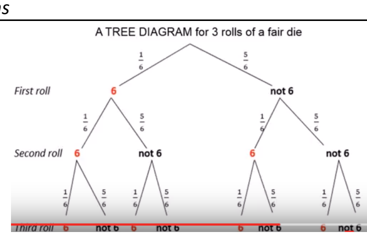
Script	Video snapshot
Introduction: Definition of probability theory and probability of an event	
1. In this short lesson, we will learn how to calculate probabilities of events that can happen in a few different dice games. The concepts presented in this lesson are not restricted to dice games. They can be applied to coins, cards, or anything that involves chance and randomness. 2. Before we begin I will go over some definitions that are important for understanding probability.	Probability and Dice
3. First of all, probability theory is a mathematical theory concerned with **uncertainty**. Yes, **uncertainty**. *Misconception PR0: Mathematics deals only with situations that are fully predictable* 4. {Many people think that mathematics deals only with situations whose outcomes are fully predictable, but probability theory is a proof that mathematics is open to discussing also situations whose outcomes are uncertain.}	Uncertainty
5. So, what is **probability**? Let me put it this way. To find the probability of some event is to **estimate** the **chances** of its happening.	
6. If we are personally **certain** that the event will **not** occur or that the event is objectively **impossible** (for example it is contradictory with some established fact) then we say that **its probability is 0**. For example, the probability of getting 7 in rolling a regular die is 0 because only 1, 2, 3, 4, 5 or 6 can be obtained.	Probability(*Impossible* event)=0 $p(7\ in\ a\ roll\ of\ a\ die) = 0$

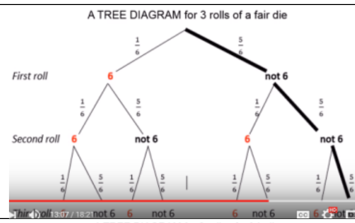
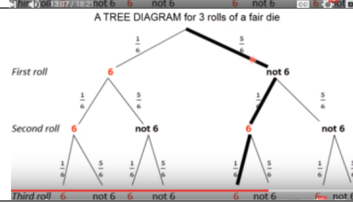
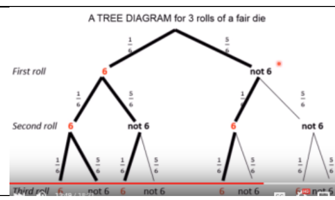
7. On the other hand, if we are **certain** that the event **will** occur, then we say that **its probability is 1.** For example, the probability of getting a 1, or a 2, or 3, or 4 or 5 or a 6 in a roll of a die is **1**, because these are **all** the possible outcomes.	Probability(*Certain* event)=1 $p(1\ OR\ 2\ OR\ 3\ OR\ 4\ OR\ 5\ OR\ 6\ in\ \ a\ roll\ of\ \ a\ die) = 1$
8. Probabilities of events whose occurrence or non-occurrence **cannot be determined** with certainty are expressed by numbers between 0 and 1. So, altogether, we can say that the probability of **any** event is 0 or 1 or a number between 0 and 1.	$0 \leq p(event) \leq 1$
9. The theory also assumes that probability of some **more complicated** events can be obtained from the probabilities of the **simpler** events of which they are made. In particular, if A and B are events that can occur in some situation, and we have estimated the chances of **A** occurring at, say, **three-quarters** and the chances of **B** occurring at, for example, **one-fifth** then the theory allows us to say that the chances of !!A **OR** B occurring are the **sum** of their individual probabilities; in this case, the sum of three-quarters and one-fifth: nineteen-twentieths.	$p(A) = \frac{3}{4}$ $p(B) = \frac{1}{5}$ $\boldsymbol{p(A\ OR\ B) = p(A) + p(B)}$ $p(A\ OR\ B) = \frac{3}{4} + \frac{1}{5} = \frac{3 \times 5 + 4 \times 1}{4 \times 5} = \frac{19}{20}$
10. We will call this property – that probabilities can be added – the **additive** property of probability.	ADDITIVE PROPERTY OF PROBABILITY $\boldsymbol{p(A\ OR\ B) = p(A) + p(B)}$
11. Let's see this property at work in a more **concrete** example. We roll a die. It is quite **uncertain** which face will show up on the top. [] All we can be **sure** of is that we can get 1 or 2 or 3 or 4 or 5 or 6. So we can say that the probability of getting 1 or 2 or 3 or 4 or 5 or 6 is 1. We **can't** predict **which** one of these 6 outcomes we will turn up if we roll a die. But we can **speculate** on what can happen. We can use some "**iffy**" arguments: what can happen under some **assumptions**.	 $p(1\ OR\ 2\ OR\ 3\ OR\ 4\ OR\ 5\ OR\ 6) = 1$
For example, we can assume that the die we are tossing is "**fair**", meaning that each of the possible six outcomes are **equally likely** to turn up during a trial: that the PROBABILITIES OF THOSE OUTCOMES ARE ALL EQUAL. That's what we mean when we say that a dice is "fair."	Fair dice $p(1) = p(2) = p(3) = p(4) = p(5) = p(6)$
12. This assumption of fairness and the additive property allow us now to calculate the **values** of these equal probabilities.	
If 1 is equal to the probability of getting any of the 6 outcomes, and these outcomes are equally likely,	$1 = \ p(1\ OR\ 2\ OR\ 3\ OR\ 4\ OR\ 5\ OR\ 6)$ $= p(1) + p(2) + p(3) + p(4) + p(5) + p(6)$ $= 6 \times some\ number$
then the probability of rolling any given number must be 1/6.	So the number must be $\frac{1}{6}$:

	$p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = \frac{1}{6}$
13. If you don't want to just **assume** that a die is fair and would like to verify if a given concrete die is **really** fair, you can roll the die **many, many** times, and check if each of the faces would turn up **approximately** as many times as any other; that is, if none were more favorable than the others.	
For example, if in 1800 rolls you got	In 1800 rolls of a dice:
a six 302 times, a five 280 times, etc., then perhaps you could be a bit more convinced that the dice is really fair. You could roll the die more times to be more sure. But this is very **tedious**. It is easier to just **assume** that a die is fair and REASON THEORETICALLY about probabilities.	
14. Another useful property of probability is the MULTIPLICATIVE PROPERTY. It says that the probability of A **and** B is equal to the probability of **times** the probability of B, provided that the events A and B are independent.	MULTIPLICATIVE PROPERTY $p(A\ AND\ B) = p(A) \times p(B)$ provided events A and B are independent.
To get a **sense** of this property, let us look at a situation in which it applies.	examples
Problem 1: Probability of getting at least one 6 in three rolls of a dice	
15. Imagine a game that uses a fair dice. For a particular round of this game, you are required to roll the dice **three** times. Suppose you win if a **6** turns up **at least once** in the three rolls. Do you think it is likely for you to win? Or is it unlikely? WHAT IS THE PROBABILITY OF GETTING AT LEAST ONE 6 IN THREE ROLLS OF A FAIR DICE?	Likely or unlikely? p(at least one 6 in 3 rolls)?
*Misconception PR1 = Misconception LR1: p(a SIX in **n** rolls) = **n** x p(a SIX in 1 roll)* 16. **{Some people believe** that this probability is ½; that their chances of winning are fifty-fifty.	
Here is how they reason. The probability of getting a 6 in a **single** roll of a fair die is 1/6. This is correct, as we have shown above.	p(6 in one roll)= $\frac{1}{6}$
But **then** they think that the probability of getting a 6 in THREE rolls will necessarily be THREE times bigger: $3 \times \frac{1}{6}$. Three times one-sixth is three-sixths which is the same as one-half. So they conclude that they have **fifty-fifty** chances of winning in this game.	p(6 in 3 rolls)= $\frac{1}{6} \times 3\ rolls = \frac{3}{6} = \frac{1}{2}$
But this reasoning is NOT correct. It is not only mathematically incorrect. It is DANGEROUS to reason that way. In fact, if you were to bet on winning in this game, you would likely lose your money.}	
17. Here is how we can **correctly** reason about this	**CORRECT REASONING**

problem.	
First, let's understand what can possibly happen in this game.	**WHAT CAN HAPPEN IN THIS GAME?**
You might get a 6 in all three rolls, or in 2 rolls only, or in 1 roll only or you might get no 6's at all.	
You win in the first three cases, these cases, and you lose in only one case, the fourth.	
This is certain. So we can say that, for sure, the probability of obtaining at least one six or no sixes is equal to 1.	$p(\mathit{at\ least\ one}\ 6\ \mathit{OR\ no}\ 6's) = 1$
By the ADDITIVE PROPERTY, this implies that the probability of winning the game and the probability of losing the game **add up to 1.**	$p(\mathit{at\ least\ one}\ 6) + p(\mathit{no}\ 6's) = 1$
So, if only we knew the probability of **getting no 6's** in the three rolls, we could calculate the probability of winning or getting at least one 6, by **subtraction**[12]:	$p(\mathit{at\ least\ one}\ 6) = 1 - p(\mathit{no}\ 6's)$
18. How can we find the probability of **not getting any 6's** in the three rolls?	$\boldsymbol{p(no}\ 6's) = ?$
Again, let's decompose this event into simpler ones. !No 6's means that we get no 6 in the! first roll **AND** no 6 in the second roll **AND** no 6 in the third roll.	
Note that this decomposition is **different** from the ones we did before. We say AND, not OR. These three simpler events must **all** happen; it's not like one can happen and another one not.	p(no 6's) = p(no 6 in 1^{st} roll AND no 6 in 2^{nd} roll AND no 6 in 3^{rd} roll)
Moreover, these events are quite **independent** of each other. What happens in the next roll does not depend on what happened in the previous one. Getting no 6 in the first roll does **not** increase or decrease our chances of getting a 6 on the second roll.[13]	Getting no 6 in the first roll does **not** increase or decrease our chances of getting a 6 on the second roll.

[12] It the Properties approach this is derived from the assumed properties of probability (axioms); in the Laplace approach, this is obtained by application of a "rule": the "complementarity rule" or procedure.
[13] It is impossible to avoid refutation of Misconception PR2 in the Properties Approach; it is implied in an explanation of independent events.

These events are independent.	These events are INDEPENDENT!
Misconception PR2: In a sequence of a repeated random experiment (e.g., tossing a coin, or rolling a dice) previous outcomes influence the subsequent ones 19. {Some people don't believe it but they are wrong. Dice have no memory of past events!}	
20. In this case, probabilities do NOT add up; they MULTIPLY. So the probability of get no 6 on the 1st roll **and** no 6 on the 2nd roll **and** no 6 on the 3rd roll is the probability of no 6 on the 1st roll **times** the probability of no 6 on the 2nd roll **times** the probability of no 6 on 3rd roll.	$p(no\ 6\ in\ 1st\ roll\ AND\ no\ 6\ in\ 2nd\ roll\ AND\ no\ 6\ in\ 3rd\ roll)$ $= p(no\ 6\ in\ 1st\ roll) \times p(no\ 6\ in\ 2nd\ roll) \times p(no\ 6\ in\ 3rd\ roll)$
So if we know the probability of not getting a 6 in a single roll then we can easily calculate the probability of getting no 6's in three rolls from this formula.	
The question is now, what is the probability of **not getting a 6 in a single roll**?	$p(not\ 6\ in\ a\ single\ roll) = ?$
21. Once again, we can **decompose** the event into simpler ones: not 6 means 1 OR 2 OR 3 OR 4 OR 5.	$p(not\ 6) = p(1\ OR\ 2\ OR\ 3\ OR\ 4\ OR\ 5)$
We can use the ADDITIVE PROPERTY to calculate this probability. So the p. of getting no 6's is equal to the p. of getting 1 plus p. of 2 plus p. of 3 plus p. of 4 plus p. of 5.Which is equal to 1 over 6 plus 1 over 6 (…) equals 5 over 6.	$p(not\ 6) = p(1) + p(2) + p(3) + p(4) + p(5)$ $= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$
Now, we can solve our problem really quickly. The probability of getting **no 6's** in **three** rolls is simply five-sixths times five-sixths times five-sixths: one hundred twenty five two hundred sixteenths.	$p(no\ 6's\ in\ 3\ rolls)$ $= p(no\ 6\ in\ 1st\ roll) \times p(no\ 6\ in\ 2nd\ roll) \times p(no\ 6\ in\ 3rd\ roll)$ $= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$
22. So the probability of **winning** in this game, that is, getting **at least one** 6 in 3 rolls is **one minus** this number: 1 mins 125/216 which is 91/216.	$p(at\ least\ one\ 6\ in\ 3\ rolls) = 1 - \frac{125}{216} = \frac{91}{216}$
23. A decimal approximation of this number is 0.42, which is **less** than the one-half that some people believe it is. It is not worth betting your money on winning in this game.	$\frac{91}{216} \approx 0.42 < 0.5$
Tree diagrams	
24. Some people find it useful to represent the outcomes and probabilities in games such the one we just discussed using the so-called **tree diagrams**. In the first roll, we can get a 6 or something else, with probabilities one-sixth and five-sixths. Whether we got 6 or not 6 in the first roll, we can get a 6 or something else, with probabilities one-sixth and five-sixths also in the second roll. The same can be said about the third roll.	

<table>
<tr><td>The event: not 6 in the 1st roll AND not 6 in the 2nd roll and not 6 in the 3rd roll has been highlighted by thick branches in this diagram now. The probability of this event is the product of the probabilities marked on the branches.</td><td>
</td></tr>
<tr><td>25. Similarly, the probability of getting not 6 in the first roll and 6 in the second and third rolls is the product of the probabilities on these branches: five-sixths times one-sixth times one-sixth. This is equal to five-two-hundred-sixteenths.</td><td>
</td></tr>
<tr><td colspan="2">Problem 2. Probability of getting at least 2 sixes in 3 rolls of a dice</td></tr>
<tr><td>26. Suppose now that we are betting on getting at least TWO 6's in three rolls.</td><td></td></tr>
<tr><td>The branches relevant to this event are highlighted now: 6 – 6 – 6 OR 6 – 6 – not 6 OR 6 – not 6 – 6 OR not 6 – 6 – 6.</td><td>
</td></tr>
<tr><td>We can use the multiplicative and the additive properties to calculate this probability. So the probability of getting 6 and 6 and 6 is one-sixth times one-sixth times one-sixth which is equal to one two hundred sixteenths.</td><td>$p(6\ AND\ 6\ AND\ 6) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$</td></tr>
<tr><td>And (reads the equation on the screen, stressing the "and's")</td><td>$p(6\ AND\ 6\ AND\ not\ 6) = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{216}$</td></tr>
<tr><td>And (reads the next equation on the screen, stressing the "and's" and the 5 in 5/6)</td><td>$p(6\ AND\ not\ 6\ AND\ 6) = \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{5}{216}$</td></tr>
<tr><td>And finally (reads the next equation, not stressing anything)</td><td>$p(not\ 6\ AND\ 6\ AND\ 6) = \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{5}{216}$</td></tr>
<tr><td>So the probability of getting at least two 6's in 3 rolls is (reads the equation on the screen).</td><td>So, $p(at\ least\ two\ 6's\ in\ three\ rolls)$
$= \frac{1}{216} + \frac{5}{216} + \frac{5}{216} + \frac{5}{216} = \frac{16}{216} \approx 0.074$</td></tr>
<tr><td>So the probability of getting at least two 6's in three rolls is very small. It's even less worth it to bet on at least two sixes than it is to bet on at least one six in three rolls.</td><td></td></tr>
<tr><td>Misconception PR3 = Misconception LR3: The number of possible ways in which a sequence of events can occur does not depend on the order of these events in the sequence
27. A common mistake is to neglect the different</td><td></td></tr>
</table>

arrangements in which an event can occur. For example, to calculate the probability of getting exactly two sixes…	
…so these ones…	$p(6\ AND\ 6\ AND\ not\ 6)=\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{5}{216}$ $p(6\ AND\ not\ 6\ AND\ 6)=\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{5}{216}$ $p(not\ 6\ AND\ 6\ AND\ 6)=\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{5}{216}$
… we require the different arrangements that two sixes can occur in 3 rolls. We may obtain 6, 6 and not 6, or 6, not 6 and 6, or not 6, 6 and 6. (These are underlined on the screen as the narrator speaks).	$p(6\ AND\ 6\ AND\ not\ 6)=\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{5}{216}$ $p(6\ AND\ not\ 6\ AND\ 6)=\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{5}{216}$ $p(not\ 6\ AND\ 6\ AND\ 6)=\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{5}{216}$
If forget about the different arrangements in which two sixes and a non six can occur we may end up only with one 5 over 216.	$p(6\ AND\ 6\ AND\ not\ 6)=\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{5}{216}$ $p(6\ AND\ not\ 6\ AND\ 6)=\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{5}{216}$ $p(not\ 6\ AND\ 6\ AND\ 6)=\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{5}{216}$
We've neglected these arrangements (crossed out on the screen). So they will not appear in the final calculation.	$p(6\ AND\ 6\ AND\ not\ 6)=\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{5}{216}$ $\cancel{p(6\ AND\ not\ 6\ AND\ 6)=\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}=\frac{5}{216}}$ $\cancel{p(not\ 6\ AND\ 6\ AND\ 6)=\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{5}{216}}$
So our final calculation would be 1/216 plus 5/216 which is 6/216, which is approximately 0.027 which is very different from point zero seven four, and obviously quite wrong.	$\frac{1}{216}+\frac{5}{216}+\cancel{\frac{5}{216}}+\cancel{\frac{5}{216}}=\cancel{\frac{16}{216}}\approx 0.074$ $\frac{6}{216}\sim 0.027$
*Misconception PR4 = Misconception LR4: p(at least **2** x SIX in k rolls) =* $\frac{1}{2}$ *x p(at least 1 x SIX in k rolls)* 28. {Some people believe that the probability of getting at least **two** 6's in three rolls should be **two times smaller** than the probability of getting at least one six in three rolls. But this is **incorrect**. The probability of getting at least 2 sixes in 3 rolls is not equal to the probability of getting at least one 6 in 3 rolls divided by 2. As we found, 0.074 is not equal to 0.42 divided by 2.}	$p(at\ least\ 2\ sixes\ in\ three\ rolls)$ ***IS NOT EQUAL TO*** $\frac{p(at\ least\ 1\ six\ in\ three\ rolls)}{2}$ $0.074 \neq \frac{0.42}{2}$
Conclusion	
29. Visualizing dice games using tree diagrams is useful to figure out how a **complex** event can be broken down into **simpler** events. But it can be cumbersome to actually draw the tree when the number of rolls is bigger. A tree for 3 rolls had 8 branches; a tree for 4 rolls would have 16 branches, and this only if we were interested in getting a 6. Still, it is useful to imagine the tree in one's head to help figure out such decomposition.	

30. Thank you for taking a chance and participating in the study.	Thank you for taking a chance and participating

Appendix D – Probability Questionnaire

Probability Exercises (6 exercises)

Required materials: Pen/ pencil, calculator, any blank paper to use as an answer sheet

On your answer sheet indicate the Section, the number of the question and your answer.

SECTION A

While I was watching the video, I invested (*write the corresponding number on your answer sheet*):

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
extremely low mental effort	very low mental effort	low mental effort	rather low mental effort	neither low nor high mental effort	rather high mental effort	high mental effort	very high mental effort	extremely high mental effort

SECTION B - Select an answer in parentheses where applicable.

(1) I roll a fair die several times. The chance to have at least two times a three if I can roll four times is *(larger than, smaller than, equal to)* the chance to have at least two times a three if I can roll five times. *Select an answer in parentheses.* Show you reasoning. Justify your answer.

Please indicate how confident you are about your previous answer (write a number from 1 to 7 on your answer sheet).

uncertain						certain
1	2	3	4	5	6	7

(2) Suppose one rolls two dice simultaneously. Which of the following has a greater chance of happening? Show you reasoning. Justify your answer.

(g) Getting the pair 5-6.
(h) Getting the pair 6-6
(i) Both have the same chance.

Please indicate how confident you are about your previous answer (write a number from 1 to 7 on your answer sheet).

uncertain						certain
1	2	3	4	5	6	7

(3) I roll a fair dice several times. The chance to have at least four times a five if I roll five times is half as large as the chance to have at least two times a five if I roll five times. (*This is true, this is not true*). *Select an answer in parentheses.* Show you reasoning. Justify your answer.

Please indicate how confident you are about your previous answer (write a number from 1 to 7 on your answer sheet).

uncertain						certain
1	2	3	4	5	6	7

(4) I roll a fair dice several times. The chances to have *at least* once a two if I roll 3 times is (*larger than, smaller than, equal to*) the chance to have *exactly* once a two if I roll 3 times. *Select an answer in parentheses.* Show your reasoning. Justify your answer.

Please indicate how confident you are about your previous answer (write a number from 1 to 7 on your answer sheet).

uncertain						certain
1	2	3	4	5	6	7

(5) I roll a fair dice several times. The chance to have at least once an even number if I roll three times is three times as large as the chance to have at least once a five if I roll three times. (*This is true, this is not true*). *Select an answer in parentheses*. Show you reasoning. Justify your answer.

Please indicate how confident you are about your previous answer (write a number from 1 to 7 on your answer sheet).

uncertain						certain
1	2	3	4	5	6	7

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(6) I roll a fair die several times. The chance to have at least two times a five if I can roll six times is equal to the chance to have at least once a five if I can roll three times. (*This is true, this is not true*). *Select an answer in parentheses*. Show your reasoning. Justify your answer.

Please indicate how confident you are about your previous answer (write a number from 1 to 7 on your answer sheet).

uncertain						certain
1	2	3	4	5	6	7

APPENDIX E – INTERVIEW QUESTIONS

The questions have 3 different focuses - some about the student, some about the particulars of the video lesson and some about the test and content of the lesson.

(1) About the participant

Have you ever taken a course on probability?

If yes, how does it compare to the video you watched?

What kind of experience do you have with probability?

What is your interest in probability?

(2) About particulars of the video

Did you learn something from the video? (Tell me something you already knew).

How was your concentration while watching the video?

Did you enjoy watching the video?

Do you think this video was a good learning tool?

Could you follow the explanations?

How was your focus on the information presented?

Was the video easy to follow?

What did you think of the length of this video?

Did you find the video dull or interesting?

What did you enjoy the most about the video?

What did you enjoy the least?

If you have any questions about material in the video (things that were not clear), please tell me.

(3) About test and lesson

In the video, we solved two dice problems. Would you have solved these differently? If so, can you show me?

(i) p(at least one 6 in 3 rolls)
(ii) p(at least two 6's in 3 rolls)

Ask for each test question:

How did you find this question? Was it challenging or easy? What made it challenging/ easy?

Is what you wrote the first thing that came to mind?

If not, what were your initial thoughts?

How did you decide that this was the best method?

Were you trying to use the concepts taught in the video?

If so, did you find those concepts useful? Was it easy to incorporate what was taught?

Note: Specific questions will be asked to challenge implicit assumptions in student's reasoning. These may entail assumptions about dice fairness, outcome fairness, randomness (if they begin to discuss it), identical looking dice and dice arrangement (this will be relevant when dealing with questions were tree diagrams are too large and they must rely on other methods to determine possible arrangements of outcomes).

Appendix F – Interview Transcript – Lea

Laplace Exposition (LEA)

CEGEP Science graduate. Will be pursuing studies in Mathematics.

S – Student

R- Researcher

Section 1

R: Have you ever taken a course on probability?

S: Not a full course on probability but I've had classes where learning probability was one of the criteria for passing that course.

R: What course would that be? What year? Do you remember?

S: Grade 11 was the last one.

R: Grade 11?

S: Yah.

R: Do you remember what kind of stuff you looked at?

S: Basic functions.

R: In terms of probability?

S: Really basic…like the first couple questions. If you have six faces on a die and you have to roll two times to get 6 and 6- like that's as basic as it gets.

R: How would that stuff that you learned in high school compare to the lesson you watched. Are they very different or quite similar?

S: They were similar in ways- I mean, with the math, but the way you used the formulas, I hadn't seen that before- probability of an event and the complementary event as well. The fact that you used the complementary event and just the formulas, I hadn't seen that.

R: Which formulas?

S: The… [Riffles through papers to find her notes], probability of a desired event and then the probability of at least one is equal to the probability of none of those numbers.

R: So you hadn't seen that done before?

S: I hadn't seen those done, no.

R: Ok. What kind of experience do you have with probability, in terms of, not necessarily with school but outside of school? A common example is with board games or gambling.

S: What have I been involved in?

R: What aspects of your life have you ever encountered probability?

S: Definitely gambling a bit. Obviously boardgames. I can't say I've ever been one of those sketchy people in an alley playing those dice games.

R: So when you're at the casino, or playing boardgames are you sort of consciously aware of probabilities? How do you approach those situations?

S: I guess I've never consciously approach those thinking of probability. I'm never really thinking like 'what are the chances'…as small as it's going to be of me actually winning this jackpot. And, like, I play the lottery but I know my chances are, like, slim to nothing. I still play…I guess I'm optimistic.

R: Do you ever employ strategy to win board games or something? Like how do you describe that experience? Are you in it to win it?

S: I just for the ride, I guess. Like…I were to go get a lottery ticket and chose my numbers instead of something random choosing it for me, I think…like, mentally I go in there thinking it's just equal chances. The chances are just so so slim of actually winning the lottery that I just kind of don't really care much that I'm going to win it.

R: Do you have method for choosing lottery numbers?

S: I thought I used to but I just let them (something inaudible) like a random chooser.

R: What is your general interest in probability?

S: In terms of…

R: Like are you interested in it?

S: Um…if I were to cheat the system…like if I wanted to cheat the system I would be interested in it. I guess because it's going to be happening regardless. Like the chances and the probability is something that going to be happening regardless…like I can't always change that so I just let it happen. Not that I'm not interested in probability but I feel like sometimes I like to control things and because I can't control it…I don't know if that makes sense.

R: I think so…

Section 2

R: About the video. Did you learn something in the video?

S: Yah.

R: Yes, that's right you told me. But I hadn't started recording yet. But if you can repeat what you learned or stuff that you already knew as well.

S: Well I had a basic idea of probability from grade 11 math. But the probability, the formula, the probability of a desired event is equal to the number of ways the desired event can occur over the number of all possible outcomes- just that specific formula I've never seen. Or the probability of at least one 6 occurring is equal to 1 minus the probability of no 6's occurring. Then getting a number from that. We used formulas that are way more general than that- I know it's the basic formula but we just didn't word it that way that's the only thing that's really different.

R: Did you find this as easier or harder?

S: Um, I found this was easier. I liked it this way...it’s a little better than what I can remember learning it before. I think it’s the way it was worded.

R: How was your concentration while watching the video?

S: Going to say like 92%.

R: 92% concentrated?

S: Yah, there’s like an 8% there.

R: For what reason?

S: Um, I don’t know. Sometimes I just get distracted. Not with a phone or a computer, I just look away from the computer and that’s just way I learn.

R: Did you enjoy watching the video?

S: Yah. It was 16 minutes of life that I didn’t throw away. Like I did enjoy it!

R: Ok!

S: Like you know there’s teachers who explain something forever and ever and ever and it takes 2 hours. But this 16 minute video was good and the visuals too.

R: Ok. SO you like the visual aspect and the fact that it wasn’t so long?

S: Yah exactly.

R: Ok. So you think this video was a good learning tool

S: Oh for sure! I would definitely use that again if ever I was in a probability class.

R: Or at the casino.

S: Or my lottery ticket numbers.

R: Yah, good luck with that. Did you follow the explanations?

S: Im normally one to take notes. Normally I take a lot of notes. But this isn’t very much for me.

R: How was your focus on the information being presented?

S: Oh, really high. Because like, um, everything in the video was condensed and there were no extra words or things that would trail off.

R: Did you find the video dull or interesting or something in between?

S: Um, it was interesting…as interesting as probability can be.

R: Ok.

S: Good visuals. I could understand everything. And everything was being written out and the same time.

R: What did you enjoy most the video?

S: Um, the diagrams were laid out and how at the bottom of each diagram they have the numbers written in. They weren’t just added I all at once. It went through and explained how each number was obtained. A lot of times you see diagrams and your kind of wondering where the numbers came from. But here they were being added as you were explaining it. I found that was simple.

R: Is there something you enjoyed the least?

S: No. Not really.

R: Do you have any questions about the material in the video that you need clarification on?

S: No…question 5 though. *laughs.

R: Question 5?

S: Evens and Odds.

R: Yah yah!

S: That's the one that got me.

R: Well we'll get there because we're onto the last section which is about the lesson.

Section 3

R: In the video, there were two dice problems solved. There was the probability of getting at least one 6 in 3 rolls and probability of at least 2 6's in 3 rolls- and that was solved a certain way in the video. If its possible, can you imagine how, prior to watching the lesson, you would have solved these problems differently?

S: How I would have solved either of these problems differently?

R: Yes.

S: It probably wouldn't have been the right answer.

R: Yah? How would you have approached it?

S: The first one was…

R: The probability of obtaining at least one 6 in 3 rolls.

S: I think I would draw a diagram and almost counted it out in my head.

R: Tree diagram? Or what kind of diagram?

S: Yah a tree diagram. But I definitely wouldn't have used the right math.

R: What kind of math?

S: I think I would have added the numbers together. The fractions…I don't know. I think I just would have got it wrong. Its tough to go back…and…

R: Yes…"forget everything you know"

S: *laughs*

Question 1

R: Alright, so concerning question 1 [repeats question].

S: What was my answer?

R: Sure!

S: Wait was that the question?

R: Yah what did you end up saying for this one?

S: I got smaller than. And I wasn't very certain about it.

R: Oh no?

S: I put 5 for certainty.

R: Wow, that super neat…very organized! [Looking at her answer sheet]. Um…how did you find this question?

S: Uh…I guess I found it okay. I didn't have full confidence in what my answer was. I wasn't sure if I had started doing the right way…like the right track. As I got more used to the way the questions were laid out I was like "ok, there's the first part, second part, and you have to calculate them separately. But now that I've done all of them (something inaudible). But starting with the first part being finding the all possible outcomes…

R: Was that the first thing that came to mind? When you read the question did you have a gut feeling about the answer?

S: [she rereads the question]. Oh yah! I remember thinking that it would probably be at least two times a 3 of I roll four times….I thought that would be way more likely than two times a three if I roll five times. But then it came out to be the opposite.

R: So you thought it was more likely to get two times a 3 in 4 rolls versus two times a 3 in five rolls?

S: Yah. As non-logical as that is.

R: Why is that non-logical?

S: Because if you have more chances to roll you have more chances to obtain that same ratio…um, not ratio but more times of obtaining the 'at least two times' part.

R: But you were saying that your gut reaction was to say that getting it in 4 rolls was larger than?

S: Yah the first time.

R: Can you say why you had that inclination?

S: As dumb as it sounds, when I first read it I thought 2 over 4 versus 2 over 5 and I went with that. But then I did the work.

R: Ok. How did you decide this was the best method?

S: I don't think this was the best method. I went back a few times and I think I was on question 3 and I was thinking about question 1 and I had to go back and I think I did it completely wrong.

R: Why is that?

S: I just wasn't sure if I was making it too complicated…like over thinking it, or if I should just go with the basics. Because I think there is more than one way you can do it. And when I was going through it I kind of bounced back a lot between one way you can do it versus another way you can do it, then I get both methods mixed up.

R: What's the other way that you would have done it?

S: Um…here I calculated…obviously what I calculated in the end, the probability of getting at least two 3's- so that two 3's, three 3's, four 3's…but you could have done it the other way….the one subtract….um..

R: The complementary.

S: Yah, I could have done that, and it would have been less work but then… just did that.

R: Did you find difficult….um…because obviously you had some kind of inclination- you had an intuition of what the answer should look like, um, yet there was so much work to be done to justify it. How does that make you feel?

S: Yah, that's what frustrated me a lot through it. Especially with the next question. Question 2. I felt I just explain it…

R: Like in words…

S: yah, but then when I started putting it down on paper that's when it started getting messy. Yah, it's a lot easier to reason in your mind.

Question 2

R: [rereads question]

S: I like this one.

R: I like that when you first think about it you think well obviously its answer C because they're both the same choice. And that's what I first wrote, and I moved on to question 3 and I started to think about it and I went back, erased my answer and thought "why did I do that?". Because when you're first thinking about it you're thinking they're both…um...there's one chance to get a 6 on the first dice…you're thinking of each ice individually…when you first roll that first dice, the chances of getting a 5 on that versus getting a 6 are equal and then when you roll a second dice the chances of getting a 6 or a 5 on that are just as equal, you know what I mean?

R: Yah.

S: But then when you put all the combinations together, that's what the answer is looking for. You can get dice 1 to have 6, and dice to have 6 and that's that. Or you can get dice 1 to have 6, and dice 2 to have 5, or dice 1 to have 5 and dice 2 to have 6. That made me go back and erase my answer.

R: So you realized that while doing question 3?

S: Yah!

R: What's question 3 about? Okay…so were you kind of multi-tasking in your mind?

S: Like when I'm doing math there almost every possible way that I could answer the question going through my mind and I'm always trying to pick one and I can never pick one because I can't make mind up. Always…it just comes out on the paper and it's really messy.

R: So there wasn't anything in particular about question 3 that gave you a hint about question 2?

S: I'd have to reread it [rereads]. I think it's because in both situations [in question 3] you're rolling 5 times and in one you have almost…twice the amount that you're supposed to be obtaining…something triggered it. I can't remember the exact thought process but something triggered to make me go back to number 2. But I think it was the fact that, in both situations you were rolling 5 times and in both situations you were trying to get a 5 but with different numbers…hmmm, it's a hard to explain.

R: Like you realize there's different orders to those 5's?

S: Yah! Like when you're going down line…there's almost a mirrored image of something you can get here over here [pointing at tree diagram] every time, but not always.

R: Okay, so you started doing question 3, and while making the tree you realized that these desired outcomes, which in this case was a 5, came in different orders then you realized that you forgot to account for [pointing to question 2]?

S: Yah! *laughs.

R: That's really cool.

S: But I don't think if question 3 had not been like that, that I would have been able to…well maybe later on I would have realized it.

R: You didn't make a tree diagram for question 2 then?

S: No. I could have I guess. It just made so much…like two became the easiest question for me.

R: Before you realized your mistake or after you realized it?

S: In both situations. It was the easiest one at first but afterwards it was still the easiest one

R: Did you think about making a tree diagram?

S: I think it crossed my mind but I thought why? But the way I explained in words made more sense to me but I should have just…hmmm, yah, it does say 'show your reasoning' and I did for every other one, except that one.

R: What was your reasoning here?

S: I just put the fractions…

R: Yah, that is reasonable.

S: I guess if it were an exam I'd get some points.

R: If the question said 'if one rolls two dice consecutively' so one after another. Does it change the answer?

S: No. Because the tree diagram would have been the same…in my mind.

R: Okay.

S: Because I guess timing wise, if we get really picky, one dice can touch the table like milliseconds before the other one.

R: Nice. That's very true! Even Einstein showed that simultaneous things don't actually occur at the same time. Alright let's talk about question 3!

Question 3

R: [rereads question]. How did you find this question?

S: I found this question second hardest for me…I mean, in terms of the ones I completed I think this one was the hardest. Again, I was using every possible way I could to get the answer going down different paths and merging them in wrong places.

R: What did you find hard? What did you struggle with the most?

S: [examines work she did]. Probably…I don't know.

R: Did you just have general trouble with it?

S: Yah. Like….I don't do well with, um, when words so closely worded together.

R: like what?

S: [rereads question]. When I read that it becomes a big mess in my head but when I start laying it on paper it starts to make more sense, since Im a visual person. Like the tree diagrams really helped me.

R: Did you a gut feeling about the question, what the answer might be?

S: …chances to have 3 times a 5 if I roll 5 times. I think of fractions. And it doesn't really make sense initially in term of probability, I think of the fractions and it has nothing to do with it. Like I think visually I think of a fraction and it's like 'ohhh that makes sense'.

R: So initially you would say 'this is true' without even doing anything?

S: Yah.

R: Do you find that your initial thought has an effect on how you work after?

S: No. Because I know it can be anything. I don't believe it until I've done my work for it.

R: Is that how you are in general?

S: Yah.

R: You're scientific.

S: Yah, My calculator has to speak before my brain.

R: Ok, so question 4…

S: What was the answer to question 3?

R: Oh, 'this is not true'

Question 4

R: [rereads question]

S: Oh yah…I like this one.

R: Oh yah? Why is that?

S: Maybe it's because I started to get the ball rolling again after not being in school for over a year now and finally not just doing basic math and I have to think about stuff. Now I was on question 4, my work was a little more laid out. It made more sense logically in my head, so I was pretty happy.

R: What did you end up saying?

S: The chances of rolling at least one 2 is larger than the chances of exactly one 2.

R: Okay. Was that the first thing that came to mind?

S: Um…for this question I didn't think about it…I just jumped into the work. Like at this point I knew how to prove my work and instead of thinking about what I thought was the right one I just went directly for the answer.

R: Yah. Di you find this question easy? Or challenging.

S: I wouldn't say easy, I was just getting used to the questions more…so this one came to me with more ease. But it wasn't 'easy'.

R: Um, does the answer you came up with surprise you? DO you agree with the result or do you feel like it makes sense.

S: I agree with it fully. I agree with the result. Because if it says at least one 2, you can get one 2, two 2's and you can get three 2's. But the word exactly in front, limits it to one option.

R: Okay, so you almost don't need math to justify this one.

S: Definitely not. So had I read this one and decided to think about what I could have done, without having to go through all the work, it would have been better, but I decided to do the work right away.

Question 5

R: So this is the one with the even numbers. [rereads question]. So um, I guess you found this question challenging because you said you didn't finish it.

S: I didn't finish the work for it. But I feel like I know the right answer.

R: What would you say?

S: [rereads question for herself]. I would say it's true.

R: Why would you say it's true? Or why would you say it's not 'not true', since I don't know what you're thinking.

S: Just because it says ‘at least’ once, so I guess because it doesn’t have exactly in front it opens up…’at least once an even number’…I mean I wouldn’t feel confident saying it was the right answer just because it says 3 times as large. But I would say it’s definitely a lot more chances than at least one 5 in 3 times.

R: So if you were just guessing the answer, what would you say your certainty is for that?

S: Like a 3…pretty low. Like I said I would need to calculate it.

R: So you have a feeling that it’s larger but you’re not convinced that its necessarily 3 times larger- it could be more or it could be less.

S: Yah and if there were some method to determine to show it is 3 times as large then I would take that method because I would be pretty confident in knowing (inaudible). Like you said.

R: What kind of struggle did you have?

S: The even and the odd numbers is what threw me off. I could have done the work for the second half ‘at least once a five in 3 rolls’, I guess because the order it was in, I started with the even and odd numbers and I didn’t really know where to go at a certain point and if my work was right so I just stopped.

R: So blank is even and the darkened one is odd [pointing at her work] and you gave each branching like a weight of 3, as in, there’s 3 even numbers and 3 odd numbers.

S: Yah, pretty much mirrored.

R: So, um, how did you determine these bottom numbers? [Bottom of her tree]

S: So that’s where I went wrong. Because I wasn’t sure if I was supposed to do, because I was focusing on even, so the white blocks, and I now you multiply them…but, um, I know you were supposed to multiply them…um, I don’t know how to explain it. In the video…um…like in the video, it was one sixth and five sixths as the white and black boxes. So what are the chances of getting a 6, its one over 6? And what are the chances of getting a non-6? Its 5 over 6. And in that situation, when there was, um…yah…I’m not able to explain it.

R: Well I see you wrote, 27, 9, 9, 3…

S: Then I stopped, because there was something in my head that was telling me it was…like they were all going to be the same, so I was going to stop and erase it there and redo it…and every number in the bottom was going to be the same and I was thinking about it and it just didn’t make sense.

R: What didn’t make any sense? That they’re all the same or that they’re not the same.

S: That they were all going to be the same at the bottom each little box they were all going to be the same and it was going to be 27, 27, 27, 27…

R: And you were uncomfortable with that idea?

S: Yah.

R: Why is that?

S: Um…I just...um…

R: Just tell me about your thought process. I’m curious.

S: That was the word for it though…it made me feel uncomfortable knowing that at the bottom every number was the exact same. Um…'at least once an even number'… (Long pause). I don't know.

R: Okay.

(Long pause).

R: Well, each number is the same in bottom, actually.

S: *laughs. What!?

R: It's okay, I could explain it to you after, I'm not really here to teach you anything..

S: No I do want to hear it

R: Okay, well, you're actually right, it is 27, and if you think about it, with each roll you have equal likelihood of getting an odd or an even so by the end of it why shouldn't all the odds be the same, right? There's 3 options, 3 option, 3 options, so that's 3x3x3 which is 27. [Going down another branch] 3 option, 3 options, 3 options is 27. 3, 3, 3, 27. So across the board you have 27 different options for each branch.

S: So why did that make me feel uncomfortable knowing that?

R: I don't know, it's interesting though. Maybe because in the lesson, it wasn't so pretty and it was kind of lopsided. You had a 125 over here and a 1 over here…

S: But when I was thinking about it, it does make sense because like it's mirrored.

R: Yes.

S: So they're all 27…

R: Another way you could have done it too, is to say even and odd [writes E and O] and not even put numbers in there necessarily, but have them representing 3 numbers each, and just have the E's and O's, so that in the end you just have one E, one E, one E. One E, one E, one O, and so on, so you have 8 different options in total.

S: Ah I see. Well I can definitely say that it was a 7 that I was certain my answer was wrong. I went to number 6 to get back to this one after, and I was like 7 that it's wrong.

Question 6

R: [rereads question].

S: Oh yah. This one was another one where I didn't even bother thinking if I knew the answer before jumping into it.

R: So you had no inclination.

S: I just went for the math.

R: OK. And how did you find it?

S: Well, this one's diagram frustrated me. I didn't even finish it. It was unfinished work as you can see the little dots. But then…I was trying to find the scenarios with one 5 and no 5's…so I just went down the line.

R: Using the complementary.

S: So I went down the line [branches] and just looked for each one…

R: … as opposed to writing everything out and being selective. That's clever. Um…so did you find this question hard easy or maybe tedious.

S: Yah! Tedious. I don't like when I can do something that involves math instead of having to write out a diagram I much prefer it, but because I'm a very visual person and I need a diagram like all the time. So I'm kind of cursing under my breath when I'm writing down the diagram because I like doing the math…that's why I quit half way, I didn't want to do another line with little boxes all stuck to each other and just, even though it would have helped.

R: Does the answer surprise you in any way? Or is it what you'd expect?

S: (pause). Um…I can see that. I got not true. Is that…

R: Yes, that's right. I was just curious because in the other questions, you would read the question and you saw two fractions and you'd compare two fractions….but in this case it doesn't sound like you did that.

S: Um…I was starting to get use to how the question was laid out. So for question 4 and 6 and 5 I guess too…I guess because they were the later questions, instead of thinking about what the answer could be, or trying to picture it in my head, I just jumped into the math, because even if I tried to picture it in my head it probably wouldn't be right. But I did that at the beginning because I was trying to not have to do the work. As to why I thought of the fractions…

R: For which question?

S: For the first two [I think she means one and three]. I, um, don't think I really have a reason. I've been away from school for so long…

R: Well that pretty much wraps it up. I'm just curious actually…you said you intended on going back to school. Do you know what kind of stuff you're interested in?

S: Well, I like calculus but I wouldn't be any good at it right now. It's been awhile.

R: I think you would do well. So you might go into math?

S: Yah, hopefully math. Still trying to figure out my plans specifically.

R: But something math and science?

S: Yah. Maybe marine biology, but there's a lot of memorization in that and I don't like memorizing.

Appendix G – Interview Transcript – Leah

Laplace Exposition Alternate (LEAH)

CEGEP Science graduate. Pursuing Health Science.

S- Student

R- Researcher

Section 1

R: Have you ever taken a course on Probability before?

S: Not a specific course, probably more like in high school.

R: What kind of stuff did you look at? Do you remember?

S: No, I repressed it.

R: What kind of experience do you have with probability?

S: I don't like it. I really don't.

R: Do you consider probability at all when your, for example, playing board games or something similar?

S: Not so much. Like I know it plays a part in it. But when I'm playing I don't usually associate it.

R: Okay. So you're not making explicit calculations. Are you aware of…

S: Not when I'm playing a card game with probabilities behind it. I know it plays a part in it but I don't really think of that.

R: Okay, that's fair. What kind of interest do you have in probability?

S: I'm not interested at all.

Section 2

R: Did you learn something from the video?

S: Kind of…I feel like I've learned it before but I feel like is was a refresher almost.

R: Okay. So tell me something you already knew.

S: Um..

R: Or would it be easier to answer: What did you learn?

S: Um, I think I already knew how to calculate probabilities like multiplying the probabilities together like the fractions together.

R: So that's something you already knew?

S: Yah.

R: So did you learn anything?

S: Not so much. It was more like a “refreshing” course.

R: Okay. How was your concentration during the video?

S: I think it was pretty good. I took a few notes and everything. Mostly just the equations. Didn’t really use them afterwards but it just kind of helped engrain it into my mind.

R: Did you enjoy watching the video?

S: Meh. It wasn’t bad. I didn’t hate it.

R: You can be honest.

S: It was educational.

R: Okay. So do you think it was a good learning tool?

S: Yes.

R: Could you follow the explanations fairly well?

S: Yes, it was very clear.

R: How was your focus on the information being presented?

S: What do you mean by that?

R: As in, was it easy to focus on the information?

S: Yes.

R: Do you have an opinion on the length of the video itself?

S: At first I thought ‘Oh, this is kind of long’ but it actually went by pretty quickly.

R: Oh, good to know. Would you the video was dull or interesting?

S: It wasn’t really either. It was informational.

R: Okay, so educational sums it up.

S: Yes, it wasn’t like ‘Oh, this is a fun video to watch’ but it wasn’t terrible either.

R: What did you enjoy the most about the video?

S: Um, I like how the diagrams were really helpful. So like, you had the tree diagrams.

R: Okay, so you like the visual aspect.

S: Yes, exactly!

R: What did you enjoy the least?

S: I don’t know. I don’t really have an opinion on that I guess. I didn’t really enjoy anything the least.

R: It’s okay to criticize.

S: I don’t know. I can’t really think of anything off the top of my head.

R: Do you have any questions about the material presented in the video that you would like to be clarified on?

S: No.

Section 3

R: Now for some questions about the content of the video. In the video there were two examples that were solved. There was calculating the probability of obtaining at least one 6 in three rolls. That was solved a certain way [in the video], would you have solved that differently?

S: No.

R: No? That's how you would have solved it?

S: Ya, it seems like the simplest way.

R: So there was also the second example: calculating the probability of obtaining at least two 6's in three rolls. Would you have solved that one differently?

S: Is that the one where you did the opposite? With the probability of not getting that. The 1 minus.

R: That would have been the first one. Calculating the probability of getting at least one 6 in three rolls we used the complementary event.

S: Ya. Okay. I think they're both pretty straight forward.

R: Okay, in that case lets go over some of the problems you did here.

S: Um, ya, its been a while since I did probability.

R: That's okay, that's okay!

S: I didn't want to do the tree diagrams because they'd be really long. I just tried to reason it out.

R: That's fine. If you didn't need them then you didn't need them.

S: I mostly do the calculations apart from it just to make sure.

Question 1

R: So how did you find the question itself?

S: At first they were pretty hard just because the trees are really tedious, I find. So they're really annoying so…ya. I found it challenging at first.

R: Did you find this first question challenging or easy?

S: A bit challenging.

R: Is what you wrote here wat first came to mind?

S: Ya, I kind of did it on first glance and circled the right answer then I went back and did the calculation.

R: On first glance?

S: It was my first impression or whatever.

R: Like your instinct?

S: Ya.

R: Were you trying to use any of the concepts taught in the video for this first question?

S: Ya, kind of, ya. The only thing- for the one in the video like um, it was only the 3 rolls so it was nice, but it would have been nice to offer an equation for the ones with, say, 6 rolls because it just gets so long to do the tree and gets really confusing I find. Like you can see I tried to map it out and stuff but it was really difficult.

R: And there is a method to do it…

S: I didn't remember it though.

R: That's okay it's not easy to remember if you haven't done it in a long time.

Question 2

R: Okay, so onto the second question.

S: Oh, this one I knew like right off the bat because, I mean, its just kind of obvious.

R: Okay, so you found this question was quite easy then?

S: Ya, its more like a trick question. Like if you were nervous or something you maybe would have chosen a different answer. I found it kind of hard to explain it in words but I tried my best. You get what I'm saying?

R: Yes, absolutely. So I basically going through the same basic format of questions for each exercise so the questions will get kind of repetitive…

S: Ya, my answers are kind of repetitive.

R: yes, the questions are very similar for sure. There's just very subtle differences. So for question number 2, did you try to incorporate some of the concepts used in the video?

S: Not really, it was just like common sense. I mean you get the same probability…it's a 1/6 chance for 5 and a 1/6 chance for 6 then the same thing here [pair 6,6]. So its like 'Oh, it's the same'.

R: Okay. Did you try making a tree diagram?

S: No I just kind of went for it.

Question 3

R: (Question restated)

S: Oh, this is the half one. Half as large or whatever. Well I knew that from the video because there was basically the same example, right? So I tried to do it out on paper but I knew the answer already.

R: So you said "this is not true" because the ratios are not proportional to each other.

S: So that makes sense, right? I used that answer a lot because it's the only way I find to word it properly.

R: So was this first thing that came to mind when you were doing this question?

S: Ya. Because I just watched the video so I knew.

R: So did you find it quite easy to reason through?

S: Its just like explaining yourself…like putting your ideas on paper that's the harder part then reasoning with math.

R: For this question, were you trying to use some of the concepts [from the video]?

S: Well ya, because its very similar to the video. So I kind of went with that.

R: Ok, so you followed that method?

S: Mostly yah.

R: Did you refer back to the video?

S: I didn't rewatch it.

R: Okay so you only watched it once?

S: Yah and I went back just to see the tree diagram and held it there.

Question 4

S: I think I did this one pretty well.

R: Ya? Well I'll be reading these later. I'm more curious about your thought processes went. So was this the first thing that came to mind?

S: Yah, well like "at least"- you have more options so you have a larger probability. You have more than just exactly once a 2.

R: You thought this question was more easy than challenging?

S: Yes. I think questions 2 and 4 were the easiest for me.

R: In this case, I suppose you didn't need to use a tree diagram at all?

S: No.

Question 5

S: This is the odd and even one.

R: Yes.

S: That one I didn't really have a first instinct on it. Like I mapped it out first, and it came out to be the opposite answer than what I would have expected [smaller than].

R: Ok, so you did have an instinct?

S: Ya I figured it would be like odd and even because you have more chance- a half chance of getting each but then it ended up being the opposite, like I did the math. Kind of hard to tell but like but this the normal probability of just getting one, right? [Referring to the probability of getting at least once a certain number]. Ya I just went back to the video because it was the same thing basically. I found this was the probability…so it kind of shocked me that it was less.

R: In this case, you did use the concepts- tree diagrams…

S: Yup.

R: What exactly where your initial thoughts when you first read the question?

S: It was that it would be..Oh...I wasn't sure if it would be 3 times as larger but I thought it would be larger so I wasn't sure if it was true in the first place but I just assumed it was going to be larger.

R: And how did you decide this was the best method to solve this question?

S: I don't know…I just kind of worked it out. I didn't give any other thoughts to other methods. There's basically just one method in the video?

R: Yup, that's right.

Question 6

S: Oh, this is the equal chance one? This is the one where you roll 6 times?

R: Yes.

S: So I didn't have a diagram set-up but I figured it would be a lot less of a probability.

R: How did you figure that?

S: I don't know…I just kind of figured. Like for this one (for a large tree she drew). The probabilities with 5 rolls, or something, it was way less. So I don't know, I just figured.

R: And so you said this is not true. Was that the first thing that came to mind?

S: Yah I figured they weren't going to be equal.

R: And how do you figure that?

S: I don't know, there's just a lot more possibilities. I was also getting kind of bored with the question too just because they're very similar. It actually took me a lot longer than I expected.

R: You said earlier, that you knew that there were more possibilities. I asked about your intuition about the answer…

S: Oh ya, I guess like in the complementary rule there's just more possibilities or something. I don't know how to explain.

R: When you say there's more possibilities, how do you reason that?

S: There's more possibilities against not being able to get at least one or two times or whatever it is. Because like I noticed like this one goes on [referring to a branch] forever, you know, 5x5x5x5, you get so many big numbers that it lowers the probability.

R: In this case, you did use concepts from the video.

S: Loosely. *laughs.

S: (comments on her confidence level since she put 6 for most) I'm pretty sure but there's still a little bit of doubt.

R: Just in general, was it quite easy to incorporate the concepts used in the video?

S: Yes.

R: But I suppose you also said the stuff that was presented were things that you already knew?

S: Yah. I feel like they teach that in high school.

R: Yes, close.

S: I feel like I've seen this before.

R: What exactly?

S: Like the tree diagrams.

R: Right. Usually at that level you're looking at flipping two coins or rolling two dice. Simpler questions.

Appendix H – Interview Transcript – Larry

Laplace Refutation (LARRY)

CEGEP Science graduate. Pursuing Exercise science.

S- Student

R- Researcher

Section 1

R: Have you ever taken a course on probability.

S: Not a course itself, no. But in the 9th grade we had chapter on it.

R: Do you remember what it went over?

S: Not specifically. I just remember we had a chapter on it and I can't remember other times that we covered it.

R: Do you have any kind of experience in probability?

S: I have a roommate who does probability. We've gone through discussions and he's shown me a couple things.

R: What kind of things did you discuss?

S: He basically did different problems that we discussed and how probability is conceptualized a little bit differently than we've done thus far in Math up to Cegep level. How it's like just its own different world, right? But I can't remember specifically.

R: Do you know if that kind of stuff helped you [for the exercises]?

S: I think so, a little bit yah. To get into the mindset because it is a bit different

R: So what you and your roommate have discussed in the past helped you a bit on this and these exercises?

S: Yes. Like I wasn't surprised by anything in your video.

R: Do you have any experience with board games and other similar things? I'm just wondering because have played board games, card games…

S: I guess I've never really analyzed it. I've never thought about it.

R: Can you describe to me what kind of interest you have in probability.

S: An edge at the casino? I can't think of too much of the top of my head.

R: I mean, if you don't like then you don't like. If you don't care for it, then you don' care for it.

S: Well I'm interested in math- it's not what I'm studying but I'm interested in it. So any math is cool to learn from the math perspective.

Section 2

R: Did you learn something from the video?

S: Yes. Um, when I have a set way to doing something I tend to follow that almost arbitrarily so just something simple like take your 1 and subtract it by what won't happen and you'll have what will happen. Like it's a basic concept but like I wasn't even thinking about doing that at first. I was just going to do all of what would happen. It's like putting you in the right direction I suppose.

R: Were there things you already knew in the video?

S: Yah when it came to studying the probability of a dice roll like I knew the 'multiply each component' and I knew how to divide which parts had more focused. Yah, 1/6x1/6x1/6 not the 1/12. Like I got that idea and I knew the tree- I would have been able to follow that.

R: How was your concentration while you watched the video?

S: Oh I was concentrated. I had no distractions. I put my phone away. I gave it my undivided attention.

R: Was it difficult to concentrate? Or was it easy?

S: I didn't find it difficult. I was able to follow along pretty easily. Like at one point I was trying to do the steps before you did and so it was a good pace.

R: Did you enjoy watching the video?

S: Yah, it had that Khan Academy feel.

R: So do you think this video is a good learning tool?

S: Yup. As a visual learner it always helps to see.

R: And could you follow the explanations pretty well?

S: Yah, it was good.

R: How was your focus in terms of the information being presented?

S: Yah that was good. When you asked in Section A, how much concentration was invested I was just a little bit intrigued by the question. I wasn't sure if it was like 'how difficult did I find it' , because I didn't find it very difficult, but I did give it my, like, I wasn't distracted. I gave it my undivided attention. So I put a 3, as in I wasn't mentally exerted, but I was invested.

R: Okay good to know. Um, what did you think of the length of the video. Yours was about 20 minutes.

S: Yah, 20 minutes is a little bit long, but you need it sometimes to get a point across.

R: What would you have preferred for an educational video?

S: Well I was just thinking like "would it help to divide 20 minutes, attention span-wise, if you don't have a 20 minute attention span to learn math I don't think you're going to learn very much math.

R: What do you mean divided?

S: Oh like parts 1,2 and 3. But if you need 20 minutes to teach something then I think its fine.

R: Did you find the video dull or interesting?...Or something in between?

S: Um, I'm going to have to say something in between. Like you presentation was good in the video. For me, anyway, I'm interested in math, but it's not my most go-to subject either. So yah, I would say somewhere in between.

R: What did you enjoy most about the video?

S: It was very accessible. The way you went through each example…you left no stone unturned. It was good that way. I felt anyone, despite the math background would be able to pick up on to and be able to do the exercises.

R: Is there something you enjoyed the least?

S: Hmm...Probably the same point. Sometimes when you go in depth a bit its, I'm not going to say insulting, but that's the one the point where I'm like 'okay, I get it' like when you're explaining, like I get that you have to explain every point in order to get other people to follow along but at the same time it's like when I'm sitting in a class and the teacher is going through, but it's like different learning styles. You just need to go at the base and follow along. But I always found sometimes the pace was too slow in general.

R: Is there something in particular?

S: I can't think of in particular. Like no…I don't know what I mean.

R: Was it condescending in a way?

S: I guess that's the way I was putting it but I didn't find it condescending. I just thought it was a slower pace. Sometimes with those videos, like if I'm watching Khan academy, that's a really quick pace but if I get stuck I can go back. I don't really know what I mean by that…

R: Okay, I think I understand. Do you have any questions about the material in the video?

Section 3

R: In the video lesson there were two examples. There was calculating the probability of obtaining at least one 6 in three rolls. Would you have calculated that probability differently then what was presented in the video?

S: To get one 6 in three rolls?

R: Yes.

S: Actually I had to see the example to get myself on the mind track to do the rest of the questions. I needed the example at first to be honest.

R: But would you have approached that differently? I don't know if you can imagine not seeing the lesson in the first place.

S: To get one 6 in three rolls?

R: Well, yes, at least one 6 in rolls.

S: I think I would have ended up doing the tree. I think that's what I would have gone to. What I would have done having not seen the video is 1, 2,3,4,5,6 and have a really long tree and go through every single tedious combinations. So I would have done a tree- that's what I would have gone to but it would have been really really long.

R: As opposed to in the video…

S: Where they have 6 and not 6. And you're almost categorizing your 1-5 as 'it doesn't matter what it is so long as it's not the one we are looking for- I won't have gotten that right away.

R: Same thing goes for the second example. It's very similar, the probability of getting at least two 6's in 3 rolls. Would you have done that one differently?

S: After having a certain strategy I would have followed that one. Whichever would have come up first I think?

R: Okay, you mean like in terms of…

S: System…

R: You mean making a tree diagram with 6 branches as opposed to making one with 6 and not 6?

S: Yah I would have taken the diagrams that I've made and just retraced it to see where I can get my two 6's.

R: oh, the same diagram used in the first example.

S: Yah. Like I think here [referring to exercises] I just drew one diagram and for the rest it's just based off that.

Question 1

R: Lets got through some of the questions. So how did you find this first question? Was it easy or challenging?

S: [rereads question]. Yah intuitively it makes sense, but to justify it that's where the real math occurs.

R: What makes sense?

S: Like if I can get an extra roll then I can get an extra chance at it, right? But then it's the justifying part that's the tricky part- the math, right?

R: So you thought this question was easy to answer but tougher to justify?

S: Um…easy to answer but tougher to justify? Yah like intuitively it makes sense right away. Like you're looking for the same thing but in the second you have an extra roll. So is like of course you're going to have a better chance for the second option. I think when I started justifying like I had this tree here [points to tree] and I knew that basically all of this counts like, you already got it so it doesn't really matter. So in your two you already have…well I this whole branch you don't have a single three so I know that when you have an extra roll assuming the same list of probabilities, like these two, it still doesn't matter, you still don't have it and this one you have one alternative way where you do actually get it. So basically you're splitting your final two alternatives, like you don't have it, you're almost going like you have a one quarter chance out of that four. Like getting one little extra chance.

R: The solution you came up here, was that the first thing that came to mind?

S: Yah. Because I already these already have two 3's- all of them. So I don't need to keep on going there.

R: How did you decide this was the best method to approach?

S: Just arbitrarily. I just went with it. I don't know if I have a system for it.

Question 2

R: The next question we are rolling two dice simultaneously.

S: This one I was actually, not as confident with my answer. Because the way my logic went was: Here's my two die. So on my first die...so if I'm choosing option (a) the way I looked at it, this one could either be a 5 or a 6, if we're looking at option (a) so you have a 2/6 chance for that happening so assuming it's a 5 or assuming its 6, then this one the next dice] needs to be the opposite- so 6 or 5 respectively. So there is only a 1/6 chance of getting that number. So then you multiply your 1/3 and 1/6 to get 1/18. Versus if you're just looking for 6 and 6 you will only have a 1/6 chance each time. So its 1/18 versus 1/36. But it seemed odd. Because I know that you're rolling them simultaneously I don't know if that changes anything.

R: As opposed to rolling them one after another?

S: I don't know! That was one of those tricky questions like I feel there is a trick to it.

R: Would it be different if you rolled them one after another?

S: Um...I suppose it wouldn't. Because the first one you are looking for either a 5 or a 6 and on the second one you are looking for the other one. So I guess not.

R: And was that [points to his solution] the first thing that came to mind when you were solving it?

S: I knew right away for the 5 or 6 that I needed to look…like it can either be 5 or 6 at first. Since its simultaneous the order doesn't necessarily matter right? I'm not looking for a 5 then a 6. Because if I was looking for a 5 then a 6 then it would be the same chance, right?

R: Yes.

Question 3

R: [rereads question]

S: Um...I knew right away that there was a really slim chance. So I took my tree, and just to avoid, drawing out all the options, I just took the true ones, I guess. So I knew to get four 5's you can either go all the way down this true section, where you have all four 5's. And then, you roll all four then the last roll you don't get it versus you don't get it the first time you do all the way through…Oh wait, no there were more options than that. I screwed that up. Okay, well the number is going to be wrong.

R: What are you missing?

S: I had this one is good, and this one is bad, and it could have been good the rest of the way here. And this one is good, this one is good and this one is bad and these are all good. So I missing some branches. I'm missing four branches. Regardless, its still not true..

R: Okay, and how do you figure its not true?

S: (long pause) You almost stated that in the video, no? Saying that it wasn't possible to just divide the odds by two.

R: Yah that's right.

S: Ok, well I got the numbers wrong but I'm still pretty confident.

R: When you read the question, what was your first instinct?

S: Once I started getting into the mentality of drawing the tree I think I just always been going for the tree. I don't know if there is a formula to just get it right away?

R: Well, there is a formula it's just a little more involved. It's not that much harder but it may make the video 10 minutes longer. But in terms, of whether this statement was true or not true, did you have an inclination towards one or the other?

S: Um...I had an inclination towards 'not true'.

Question 4

R: [rereads question]

S: So my answer was larger than, I was fairly confident. So yah, the same thing. I just did the tree. And I think we already the answer for the chances of getting at least one and now I just needed to look with the same tree where is it that it's exactly one. Now it's almost dependent on what you get- so you got your two right away and then it needs to be not a 2, not a 2. Then not a 2, not a 2, now it has to be a 2. Not 2, 2 and has to be a not 2. So there's really only these 3 options and then you multiply by your 5 5 5 5 your 5 5. So that was pretty cool.

R: So did you find that question to pretty easy then?

S: Yes.

R: Once again, considering the options you had, which were larger than, smaller than and equal to, did you an inclination towards one?

S: Um…(long pause). Yah I knew it was larger. Because "exactly" is like the limiting factor here. I knew there were definitely going to be less options.

R: And you knew the probability was going to change accordingly?

S: Yah. It was the only question that had 'exactly' involved. So that was the trick there.

Question 5

R: [reads question]

S: Once a five in 3 rolls…hmm. So I did the tree again but everything was in E's and O's and everything was in ½ so at the end of it, there's really…it's very likely you're going to get at least one even. There's only one branch that it's all odds. And I think, I mean I didn't really do it, but each branch is 8, right so. 8 times 8…there's only 64 options so there's a 7/8 chance that you will get even. Anyways, I just did that branch with the Es and Os instead and I don't know…Oh yah, I just use the same number here [recycling tree diagrams]

R: Did you find this question easy or challenging?

S: Well, I'm fairly confident with my answer, but I'm a little worried because you haven't told me if I'm right or wrong. I found the answer pretty easy to find.

R: Initial thoughts?

S: I didn't have an inclination right away. I just thought even and odds so I had to change the way I look at it. It's not 1 to 5, its 3 to 3. So right away I got the numbers, I didn't really have an inclination.

R: Okay, so you were thinking about how to rearrange your tree?

S: Yah, basically.

Question 6

R: [reads questions]

S: For the latter part of this question I just recycled my answer. And I saw this and I just kind of wanted to get this done. I didn't want to draw a number tree with 64 things on the bottom. And I don't know the formula and I don't have an alternative. But I figured if I just do- like substitute my tree with the ones that can't happen and do the 1 subtracted by all those options I can make my tree a lot simpler and faster because I'm looking for only at least two. So you're bound to get at least two, I think. But so, the black [dots], yah, I've simplified [the tree] a bit more, binary- black means no it wasn't a 5 and white means yes it was, so I'm only looking at things that didn't get at least two. So if you got a 5 right away in order for it to not be at least 5 it has to be all not 5. So that's the one option you can get from this tree [referring to smaller tree]. So now, I got it right this time, that's what I forgot over here [question 4], it's the other end of the tree that I didn't figure out yet. But if I used this method earlier it would have been better…So if doesn't work all the way through you didn't get it. Then it needs to be at least one each time. But before you got it on the last one, got it on the second last one, and the third last one and he fourth last one. But we only ever got one, and I did just the same like 5 to the power of 6 rolls and then…5 to the power 6 plus 5 to the power of 5. Sorry…5 to the power 5 times 5. So 3 times 5 to the power 6, over the total options, which was 6 to the power 6 and 1 minus that gave me my number. Then when I compared my number to my other answer I was like 'Oh that's smaller'. So I was able to do it quicker than I thought I would be able to do it, but it was almost lucky that I figured that, to get the part of the tree.

R: Did you find this question easy or challenging?

S: See because I didn't know the system, I wouldn't say it was challenging, but longer. But this I guess wasn't very much longer. I suppose it was a little more challenging because I had to do something different.

R: Did you have an inclination towards whether this true or not true at the beginning of your work?

S: I didn't have an inclination at all actually. I would have thought they would have been pretty close. My inclination is that it was true but I wasn't sure. But the numbers ended being pretty close. So inclination was that its true, but you know, I didn't trust my gut.

R: Why would you say they were close?

S: I guess intuitively, a hunch. Two 5's in 6 rolls or one 5 in 3 rolls. Yah, it's so….I would have guessed they were close.

R: It's tempting…

S: Yes it is.

R: But what made you decide to not follow your gut?

S: That was just a hunch. I wouldn't sit with that. I wanted to make sure I got the numbers that I was comfortable with and they didn't end up matching.

Appendix I – Interview Transcript – Peggy

Properties Exposition (PEGGY)

CEGEP Science graduate. Pursuing Dietician/ Health studies.

S- Student

R- Research

Section 1

R: Have you ever taken a course on probability?

S: No. I think my probability specific education is within High School. Like little sections- maybe a few weeks.

R: Do you remember what kind of stuff you guys looked at?

S: I want to say, pretty similar to the video. Like a lot of dice rolling, tree diagrams.

R: What kind of experience do you have with probability? That can be anything like board games or any kinds of that nature.

S: Oh okay, well I love Yahtzee. But it's mostly about intuition. I rarely apply logic to it. Like it has to be more like this way, you know what I mean?

R: Oh yah? What do you mean?

S: These questions were exactly like Yahtzee often, it made me think- well, I guess I could have realized this, but I could take out a pencil and paper and figure it out as opposed to being like 'I feel like it has to be more likely to get a full house then to get…' you know what I mean?

R: What's a full house? Is that the same as poker? No, eh?

S: I don't know much about poker.

R: Well, there's no cards involved in Yahtzee, is there?

S: No. Just dice.

R: I've never played Yahtzee.

S: No way! You would love it! You would probably really good at it. It's like two of one number, three of the other.

R: Oh okay! Yah? Cool.

S: Yah, and there's straights and stuff, you know what I mean? So yah, it made me think of that a lot.

R: So if you were aiming to roll three of one number, is it okay to roll 4 of one number?

S: Yah, so you have three rolls each turn, so yah, you can go for 3 of a kind, 4 of a kind, small straight, long straight, full house and then like each individual number. Trying to get the most…so the most 6's you can get the more points you can get type of thing.

R: Cool. I had no idea there was actually a game that had similar concepts.

S: Hmmm. Is there anything else?...Not really sure. I guess sometimes when I'm at stoplights.

R: Yah! I do that a lot. Like what are the chances of it turning green as soon as you get there, kind of thing?

S: Yah! Hmmm, there's probably 12 million more that I'm not thinking of.

R: That's okay. Can you describe to me what kind of interest you have in probability?

S: Um, like on a scale or what areas?

R: Sure, on a scale. Maybe 1 to 10.

S: Well I find it really neat to hear about it. I just feel like I never did really well in it. Like maybe I have a bad relationship with it. Like when someone says 'probability' I'm like 'uh', you know what I mean? But I shouldn't blame it for me being….

R: Well, it's a common sentiment. I don't know if I've ever heard anybody say "Oh I love this subject!'

S: Okay. But I find it super interesting. Especially watching someone else do a problem. You know, observing it is awesome but I do find it often completely counter-intuitive.

Section 2

R: Did you learn something from the video?

S: Yes, because it's been years since I've seen it, but there weren't any new concepts, per se. Like I think the additive and multiplicative properties- Like I'm sure I've seen that.

R: But is that what stood out to you in terms of new-ish information?

S: At this point in my life, but I'm sure in grade 9/10 math that I've seen it.

R: Okay, sure. Was there something that you already knew that was presented in the video?

S: The concept of tree diagrams. Um…I guess, with the trees, how they were like 6 and not 6.

R: So you've seen it done that way before? Looking at the event you are interested in, and then the other branch being the event that you aren't interested in?

S: Um, I'm fairly confident. When I was going through the problems and doing this type of stuff, all the possible combinations and stuff, like that felt super familiar to me.

R: Okay. How was your concentration while you were watching the video?

S: Oh that was super good. I liked how when there were really key phrases they would be written and I felt the visual went really well. Not too much of it, it was clean. It wasn't convoluted. I didn't find it hard to focus.

R: Did you enjoy watching the video?

S: Yes. Good amount of time too. Maybe double I would have been like ‘hmmmm okay, like…’ (implying too long). And there was a good amount of detail, even on the problem it wasn’t like ‘Okay, I get the picture’. Because like sometimes with Khan Academy, I love it but, I don’t know I guess it must be hard to find a good level of detail. But a similar video for a similar question I would have been skipping like ‘OK, I get the picture’.

R: Do you think this video would be a good learning tool?

S: Yes. I feel like in grade 9 and stuff I really relied on videos and I would have been super stoked to stumble upon that one.

R: Could you follow the explanations fairly well?

S: For sure.

R: How was your focus on the information that was being presented? Was it pretty easy to focus on the information content itself?

S: Oh yah.

R: Was it pretty easy to follow?

S: Yah. At the same time I didn’t feel like I was being baby talked. It was a really good balanced.

R: Did you find the video dull or interesting or something in between? You can be honest.

S: Very aligned with other educational videos I’ve watched. Maybe I wasn’t like ‘I can’t wait to tell everyone about it’.

R: Yah, it’s not a cat video.

S: No, but it was easy to concentrate.

R: Was there something you enjoyed the least?

S: No. And honestly I’m not just saying that. I don’t think there would be anything that I would change.

R: Do you have any questions about the material that was presented? Do you need any clarification?

S: No.

Section 3

R: In the video there were two examples. The first example was calculating the probability of obtaining at least one 6 in 3 rolls. Would you have calculated that differently than what the video had shown?

S: It so far in my memory I honestly wouldn’t have known how to approach that. That’s how ‘square one’ I was starting from.

R: Okay.

S: So no.

R: Is that the same for the second example?

S: Yah. I would have drawn a tree, I’m sure.

R: Yah, probably the most intuitive approach.

S: Yah.

R: Let's talk about some of these exercises here.

S: Okay.

Question 1

R: [rereads question 1]. What did you think of this question? Did you find it challenging or did you find it easy?

S: Um…I think for all the questions like at first I'd read very carefully and immediately start doing the math but a couple times it was like 'Oh, okay, well think about it'. So for this one, I did do math, but as soon as I did it I brought it back to Yahtzee and I was like 'okay, if you were given the option of rolling four times to get two 3's or five times, and I was like (affirmation nod). The logic came after the math.

R: What do you mean the logic came after the math? Oh you mean the intuitive process?

S: Yah, sorry, exactly. Well, geeze, depending if I'm right, I don't want to be too confident on any of these.

R: You kind of touched upon this already too, but what you wrote here, is it the first thing that came to mind when you read the question. So when you read the question, you saw you a different options- larger than smaller than equal to- is there something that kind of stood out initially, in terms of…

S: Like aside from doing this right away [shows work], you mean?

R: Yes. Was there something else came to mind before you put pencil to paper?

S: No…

R: How did you decide that this was the best method?

S: Well, after watching the video it was the only option I knew.

R: Okay. So were you trying to use the concepts that were taught in the video?

S: Yes. Well yes.

R: Did you find the concepts useful for answering the question?

S: Yah, like I said I wouldn't have known what else to do.

Question 2

R: [rereads question]. Did you find this question challenging or easy?

S: This was the question I felt most confident about my answer. Yah, probably the easiest. Unless I'm wrong.

R: Did you have any initial thoughts?

S: Yah for sure. I think I just thought that it was just arbitrary what two numbers, like if you're aiming for any two numbers like it doesn't change the probability, I think.

R: Actually, where is your work for number 2?

S: I didn't really do any. I just thought it was the same for both.

R: Okay. That's fine.

Question 3

R: For number 3… [rereads question].

S: I really wasn't sure intuitively about this one, because it seems like 'Oh yah, that makes sense!' but then when I just quickly did it, I felt like…um, what did I do here [looking at paper]? So this was four times, and this was at least 2, and so when I calculated just getting a two it was more than double the odds without even calculating the 3 or 4 so I just stopped it. I don't know.

R: So what did you end up answering finally?

S: That this is not true.

R: And did you have a gut-feeling about the answer in this case?

S: I wanted to say that it was true, initially before doing any of that. But I've learned to never trust my gut.

R: That's pretty smart.

R: So here [point to work], you are looking at different arrangements I guess?

S: This answer I ended up scrapping because I didn't know if it was very clear. So this was just getting at least 2 and so I said like tell you that before going on to at least threes. But all my math could be wrong…so.

R: So why did you decide to forgo the tree diagram? Lack of space or something else?

S: Yah I don't feel like I need to actually draw one, I just feel this in my mind is a written tree diagram. It just takes so much space.

R: Okay so you condensed it a bit.

S: Yah.

Question 4

R: [rereads question]. Did you find this question easy or challenging?

S: To be honest I really wasn't sure.

R: What made you so unsure about it?

S: Um, I guess when I really think about it, that's way more precise [exactly] but, and this how great I am at probability, back to this [question 2], I feel like I can almost relate the two questions. I don't know, I guess it's a lot more vague. Sometimes it's just hard to answer questions after I did this part now it's really clear to me, and I'm just trying to remember what I first thought before touching pencil to paper. I think I thought, maybe it's just a situation like that where it ends up being the same. Do you know what I mean?

R: Not exactly.

S: No, I know, this is unclear.

R: What was your inclination when you first read the question? What did you think the answer might be?

S: I think I thought ‘that’ would be more likely because it’s more vague.

R: That being ‘at least’?

S: Yah.

R: And what do you mean ‘vague’?

S: Like that is more precise, like you can only get this, whereas this is like more possibilities.

R: Okay, I understand.

S: So in the end, I was going to do both, but then I realized that this [exactly] was just a subset of that [at least]. And like that eliminates a bunch of numbers so obviously the probability is going to be lower.

R: And how do you figure that relates to number 2?

S: I think I was just thinking, when I first read the question, that like ‘Oh, this is the type of situation where, even though that seems less likely [(6,6)] this seems more random [(5,6)], it’s actually the same likelihood. I think that was my first thought just like ‘Oh, I don’t want to make any assumptions here because maybe like when I draw it out it will end up being the same combos but then I realized that didn’t make sense. I guess, all that to say that I really wasn’t sure about this question, at all.

R: I’m curious. You said 6,6 seems less likely, and 5,6 seems more random?

S: I think through playing Yahtzee I know that that’s not true. But I think, and I don’t think I would be alone on this if you were to poll other people, I think there’s something that seems more like ‘Oh that has to be less likely’. Like what are the odds it’s the same two numbers. Do you know what I mean?

R: So what makes 5,6 more random?

S: Just the fact that it seems more like a typical roll I guess. It’s the least educate thought ever, you know what I mean?

R: That’s interesting.

S: Almost the same as like, in Yahtzee it’s like 5 of the same number, and we think of that as impossible, like 5 sixes or whatever. But I remembered, it took some time to sink in, that it would be just as random if someone were to get 3,4,6,5 even though it seems more likely because its random numbers…I don’t know it sounds really stupid.

R: It doesn’t, no! Maybe later I can tell you about it, but it’s very interesting.

Question 5

R: [rereads question]. What did you think of this question? Easy of challenging?

S: I found it pretty tough. Maybe with like the other questions I wasn’t 100% sure, but once I did the math I would be like ‘Okay, that makes a lot of sense’. Whereas this one I found that what I thought off right the bat sort of contradicted the math. It wasn’t like, ‘Oh I see how I was thinking about it wrong’. It was more like ‘I guess I’ll trust the numbers’. I don’t know if that makes sense? I guess when I read it I thought 'Oh that totally makes sense [implying its true], there's one 5 out of the six, and there are 3 even numbers, yah sure, I like that statement, its nice. But then when I did the math, if I did it right, it was not three times as large as the 5. So I was like 'Okay, I'll believe it' but I don't really know why.

R: In this case, did you have any initial inclinations as to what the answer might be?

S: Almost like, 'too good to be true'.

R: It's funny how that works, eh? Sometimes on tests if the question is too easy you begin to doubt yourself.

S: Totally!

R: I don't trust my teacher-it's a human aspect. Like "I'm going to be betrayed here"

S: Yah yah! Totally.

Question 6

S: Yah, I sort of flaked out.

R: You flaked out?

S: Well, I'll tell you all about it.

R: [rereads question]. So what did you think of this one. Easy or tough, any initial thoughts?

S: Initial thoughts? Very similar to this one. And sort of what I ended up answering was that like, well…maybe slightly more founded than a guess, that it just seemed sort of, in my mind, like, it's just the same thig happening twice so the odds of the first thing are *x*, it should just be the same too…like more rolling opportunities, same chance it's going to happen again. So I ended up saying this is true, because I started doing the math and was like 'there has to be an easier way then doing that'. I was going to do it for one 5, two 5…and was like 'I don't know if you're expecting me to do that' so I ended up just going with my gut. So I put a low certainty.

R: Okay, why did you put a low certainty?

S: Just because with probability I never trust what I think makes sense.

Appendix J – Interview Transcript – Perry

Properties Refutation (PERRY)

CEGEP Science graduate. Pursuing Bio-Chemical Engineering. (Some experience with probability).

S- Student

R- Researcher

Section 1

R: Have you ever taken a course on probability?

S: I took an intro to stats class that had to do with some probability just last year in school.

R: Do you remember what kind of stuff you guys looked at? Was it similar to the stuff you looked at today?

S: It was slightly similar. We did a lot of coin stuff. A lot of coin and dice probability questions where would be P A U B…

R: P A U B?

S: Like event A union B. Kind of like the basic stuff. Really similar to what we did in the video.

R: Did you find that helped at all?

S: Well it was a pretty small section of the course and I always found that probability was difficult for me. Like I didn't want to take the time to learn it very well. So I always found it really difficult actually.

R: What specifically did you find difficult?

S: I think I find the thinking about it really counter intuitive for me. How to break down the events and stuff like that. And putting them into…well, obviously considering all the possibilities is something I always overlook.

R: In this course, you mean?

S: Exactly, yes.

R: Since you've learned a bit about probability before, how does that compare to the lesson you saw today? How was it similar? How was it different?

S: I think the types of questions you were asking in the worksheet were very similar to the types of questions I had on my first homework assignments. Not always with dice obviously, but stuff like it. In the course we did some real life scenarios, like people at airports. Like if they would take flight A, B or C. And I always found those questions difficult to consider. That's why I felt a little bit better about this assignment, just because it was easier to consider dice. Like easier to consider those outcomes.

R: Why do you think it's harder to imagine real life situations?

S: I'm not sure. I think maybe just because, in the dice rolls it's easy to consider like "you roll this or not this". But I often would have to consider 'they would take this route at this place at this airport' and stuff like that and would always have to back track a lot and consider all the options. With the dice it was just a bit more cut and dry, I think. It's easier to see all the outcomes, I think.

R: What kind of experience do you have with probability? Maybe not necessarily with courses, but do you have any other experience? For example with board games or casinos or maybe something less obvious than that?

S: I've only been to the casino once and I didn't even gamble, so I have no experience there. But just with board games and stuff. It's funny, I actually play Yahtzee, and so it's just stuff like that. I have a bit of experience with it. I guess it's just like really basic. I can't really think about any specific example…in real life other that.

R: I actually just learned about Yahtzee the other day and it turns its very similar to what the exercises were asking.

S: It is, yah!

R: Do you play often?

S: Yah. Often it's a game that me and my girlfriend play because she has the 5 dice kicking around at her place. So if we're doing nothing for the next half hour we'll just kind of play because it's easy to set up.

R: Do you ever approach the game with a mathematical mind? Do you analyze the probabilities? Do you have a strategy?

S: I think often yah. Because I know which kind of things I should be trying to get because I have a greater chance of getting them. Like 'Yahtzee' obviously is the hardest one to get with getting 5 in a row of the same numbers. There's just a lot easier ones to get, like three of a kind…you can often get those. Knowing a bit of the rules about it…you can definitely approach it with, or I often do like 'I have a greater chance of getting these combos' so you kind of go for those.

R: What kind of combos do you go for usually?

S: Often getting like the 1 through 6…if you roll 5 dice…so you get to roll the dice 3 times, right?

R: Like all five dice together 3 times?

S: Well, the first roll you roll all five then you kind of pick what you want to not roll and what you want to roll and you keep going, and at the end you have a total. There's combos where you can get like two 1's, then your score would be 2, then there's 6's and stuff. It's often going for things like that I find. So if you roll two 6's off the bat, so you pick up the other three and see how many more sixes I can get. Just stuff like that. I think you can think of it from a basic probability mind set and do pretty well.

R: And do you?

S: I try to, yah.

R: And how does it go?

S: Um…sometimes I win *laughs*. I don't always.

R: Well it is probability, you're going to win sometimes. So you kind of touched on this already but, what is your interest in probability?

S: Pretty low initially. I found in the class it was always the subject the most difficult for me to focus on. And simply because, I think, I wasn't very interested in it at all and it took a lot to motivate me for it. And I found it was often just tough to just sit down to do all the events, like calculating all the events and stuff. I think I just found tough to sit down and do that.

Section 2

R: Did you learn something from the video?

S: I think so, ya. I think all the common mistakes that were pointed out in the video where ones I often make. Like the flawed logic that I often used.

R: Which ones in particular?

S: I think…I can't remember the exact example but it was like, let's say it was rolling a one once over 3 times, is going to be half the chance of 2 ones in 6 rolls I think is often a trap I fall into and it's not true. I think it's often one I get into and then I sit down I calculate the events it's like 'No, you have to think about it'.

R: Is there something that you already knew that was presented in the video?

S: I think those were all ideas that I've seen before but there never things that I've actively used, other than that one class that I took before, I never really sat down and done work in that kind of mode. So…I think they were ideas that I've seen before.

R: What exactly have you seen before?

S: Well definitely the two rules, the addition rule and the multiplicative rule, I have seen those two for sure. And yah I've used those two before for sure.

R: How was your concentration while watching the video?

S: It was actually pretty high. I like sitting down to learn about math for a fun mindset. It was pretty easy for me to sit down for the 20 minutes or 15 minutes or so that the video took. I found it quite easy actually.

R: So did you enjoy watching the video?

S: I did!

R: Do you think this video is a good learning tool?

S: I actually like the videos where you can see the drawings and stuff, where the instructor is drawing on the screen and stuff. So as they're talking they can cross things out and highlight things. I like that kind of thing of learning and reviewing.

R: Like Khan Academy?

S: Exactly!

R: Could you follow the explanations in the video?

S: Yes, for sure!

R: How was your focus on the information being presented?

S: I think it was pretty good. I think I had prepared myself a little bit going into the video saying 'I gotta do this'. So I think that might have helped for sure. But I find math in a sense very cool to learn.

R: What did you think of the length of the video?

S: Super manageable. And it was good pace too. Nothing moved too fast or too slowly.

R: Did you find the video dull or interesting or something in between?

S: I'd say something in between, for sure. It was never really dull ever but it's not exactly the most thrilling thing to watch.

R: What did you enjoy most about the video?

S: The pace of it was the best. I find when it moves too fast or too slow for me to keep my attention. But I thought its pace was its greatest strength, at the speed it was the information was presented really clearly.

R: Did you ever pause, rewind or fast-forward?

S: Once the breaking down of the events occurred I went back to see the definition of the two rules once quickly to write those down before we jumped into the calculating, just to remind myself of what they were. The ‘Ands’ and ‘Ors’, I think.

R: What did you enjoy the least about the video?

S: Least? …I think probably the tree section because I know it really works only for small examples. It is a good tool to see I think but going through the whole tree was definitely kind of…the most uninteresting part.

R: So you were aware of its limited use?

S: And I think also how all the different possible routes were explained too…it was already stated earlier and to go through the whole tree again…

R: What did the video state earlier?

S: We were going through all the possible outcomes and I guess we had calculate one or two paths already and through the tree we went through it again. It had already been seen before I guess.

R: Do you have any questions about the material in the video?

S: I don’t think so, no.

Section 3

R: In the first example in the video we are looking at calculating the probability of obtaining at least one 6 in three rolls. That was solved a certain way in the video, would you have solved that a different way?

S: With the rules and stuff? Or using different logic?

R: Would you have approached it differently?

S: I don’t think so. I might have approached it in an incorrect way.

R: Would you know what that incorrect way would be?

S: I probably wouldn’t have considered all the options for sure and would have just kind of been ‘oh its 1/216’ or something like that.

R: When you play Yahtzee do you consider…like the different paths represents the different arrangements of the dice. Are you aware of the different arrangements when you play Yahtzee?

S: I don’t think as much so, just because the nature of the game. Just because you are making an initial roll and based on that you see what will happen next. Like just rolling at the same time a couple times. So I think it’s a little…I think every time I’m throwing dice, I’m just going with the initial throw. The only probability I’m considering is on that first roll based on what to roll. What to hope for.

R: What to hope for?

S: Exactly.

R: So with five dice, you roll two 6’s initially, which I guess is the highest pointage? Sixes is what you want essentially, right?

S: Depending on what you're going for, yah.

R: Say you had two 6's, what would you do in that scenario?

S: I would probably pick up the other three and try to keep rolling sixes.

R: What if you rolled a 3 and 4 that you put off to the side what would you do in that case?

S: I would probably try to roll for maybe something a bit safer…more 3's or more 4's, so I'd roll the rest few I would go for either more 4's or more 3's, to get more of one of those two numbers. Rather than go for the straights, like 1,2,3,4,5, or 1,2,3,4, I find those are a lot more difficult to obtain.

R: As opposed to obtaining groups of similar numbers?

S: Which I don't know if the chances are higher or lower now that I'm thinking about it…

R: Yah. It's tricky. Then there was the second example that was very similar: calculating the probability of at least two 6's in three rolls. Would you have approached that differently? I know it's tough to imagine how you would have done something prior to learning it now that you have learned it.

S: I don't think so…probably would have missed some of the options or not done some at all.

R: Okay, let's bring out your answers there.

Question 1

R: [rereads question 1]. What did you end up saying?

S: I wrote 'smaller than'.

R: Now was that the first thing that came to mind when you read the question?

S: I don't think so…I think I might have considered the opposite initially.

R: The opposite being?

S: That it would be larger than, but then the thought process I went through was that I considered what the chances of losing would be and I found that the chances of losing were higher than the 4 times roll, so I just considered the opposite.

R: Opposite?

S: Being if it were more likely to lose in one it would be more likely to win in the other. Which is, I don't actually if that was fair thinking. But that's what I went with.

R: Did you try to use the concepts learned?

S: Yah. I ended up doing the 'not 3' every single time [each roll] and then I got this number and then I went through 5 times what it would take to lose and I just got that. And if I considered if it were more likely to lose in this one then it's more likely to win in the other one.

R: You're doing using the complimentary event?

S: Yes.

R: What did you say your initial thoughts were?

S: I said it was going to be larger than.

R: Do you know what kind of reasoning you might have had in order to determine that?

S: I think because you're rolling more times the chances are greater. I think that's what I was going with.

R: You were obviously able to convince yourself otherwise, did you immediately get into the math?

S: I think I went with the initial conclusion, right off the bat, but then I thought "oh I should justify it.' So I sat down and tried to do a little bit of math anyways. Then I decided after seeing the numbers…initially I thought there would be no change because they were both very similar fractions. But once I got the decimal value it was clear.

R: Were you surprised by the answer?

S: A little bit, yah.

R: Did you find that the concepts taught in the video were useful for this question?

S: Oh yah!

R: Did you find it difficult to incorporate or pretty easy?

S: Once I learned them I thought it was pretty easy to sit down and do it. But had I not watched the video and just went for this then it would have been a different story.

R: How do you think it would have went?

S: I think I would have just logic my way through it. No math at all.

R: And what would that look like?

S: I don't even think I would have used math at all...just like 'oh yah, more rolls means more chances' I would have gone through it that way.

Question 2

R: [rereads question].

S: I wrote that they were equally likely. A good quote from the video was 'Dice have no memory' so the chances of getting a 5 and a 6 are the same as getting a 6 then a 6.

R: How do you figure that applies here? –Dice having no memory?

S: In that specific question?

R: Yah.

S: Oh sorry was the question to roll two dice? Not the same one twice?

R: It was simultaneously.

S: Oh yah...yah.

R: That’s okay.

S: I think I read it fast and it was the same dice.

R: So you roll once, and you pick it up and roll it a second time [clarifying his interpretation]?

S: Yah. I skipped that part of the question.

R: That’s okay. As long as I know that that’s what you read. It’s not a big deal. Would it make a difference if it were simultaneous?

S: um….No (seemed very unsure).

R: Why not?

S: Like getting a 5,6 or getting 6,6?

R: Basically, would it make a difference if the question said ‘rolling the dice simultaneously’ or ‘rolling one dice twice’.

S: Because you’re using the different rolls aren’t you? Using P of 6 and 6…and (5 and 6) which is 1/36, I guess. You have the addition rule for the other one.

R: The other one being simultaneously?

S: If you were rolling simultaneously versus the same dice twice you’d be using two different rules.

R: And which rule would be applied to simultaneously?

S: The multiplicative one. Because it’s one event you’d do a 6 AND 6 or 6 AND 5. Versus the other one, a roll which would be roll, then another roll.

R: Did you make a tree diagram for this one at all?

S: No I didn’t

R: Do you think that would have helped?

S: Probably in this case, yah. Well, I guess not for the simultaneous one.

R: Why not?

S: Because it’s just the one event I think. Whereas rolling a die then another die is two events.

R: So you’re viewing rolling two dice simultaneously as one event?

S: I think so, yah.

R: Did you find this question easy or challenging?

S: I think initially I just read it and skipped right over it because I just knew what I thought I knew what I thought had read. So I just went for it then moved on.

R: So did you have an intuition?

S: I think so, yah.

R: I don’t want to put words in your mouth, is that about right?

S: Yah, I probably did this question in less than 30 seconds for sure. I just read it and thought ‘Oh yah, that’s it’.

R: So you thought it was pretty easy, then?

S: Yah. But obviously I didn’t read the question properly.

R: That’s okay! So is this the work for two?

S: Yah, that’s it.

R: Did you use the concepts learned in the video for this question?

S: Um..just because I thought it was the independent kind of rolling I just considered the quote from the video “dice have no memory of past events” so every number had an equal chance every single roll.

Question 3

R: [rereads question]. Did you say this is true or not true?

S: I wrote not true.

R: Did you find this question easy or challenging?

S: It was easy having watched the video for sure.

R: Why is that?

S: The video had laid out the number of arrangements and it was easy to see that more arrangements led to a different number, so that one was easy. However, had I not seen the video I think this would have been a whole other story. If I hadn’t sat down to think about it, it would have been challenging because I would have considered the logic to be true.

R: So what were your initial thoughts when you read the question?

S: Having seen the video, definitely. I was pretty sure it wasn’t going to be true.

R: So you didn’t do work here either?

S: For this one, no. I knew that the number of arrangements was going to be different.

R: So how would you describe how the number of arrangements affects the probability?

S: Well, the next question was similar so I actually went through the different arrangements and stuff and I think once you lay them all out, and do the addition rule, and definitely changes them a lot. Not in the way that I would expect…just considering…the logic I guess. I think definitely laying them out and doing the different rules does change them a lot. No, it was interesting, yes.

Question 4

R: [rereads question]. What was your answer in the end?

S: I wrote larger than.

R: And did you find this question easy or challenging?

S: This one I found a little challenging because this is the first one that really sat down and did all the events considered that I thought were relevant. So I wrote down all the events where you would roll a two and then I set down…they were all considering if you had roll at least one two and all the different combinations that could work out and going through everything fraction for it. So I did that initially. And then I picked out the ones were you get exactly one two, and summed those separately. Even before having done the math I think it was obvious just having seen that there were 7 events- possible events, and only 3 for the other. I think it was definitely more obvious that one would be more likely than the other.

R: At what point did it become obvious?

S: Definitely when I had laid them all out then marked off the ones that had only had exactly one 2. It was just clear for sure. I knew the numbers surprised me on, I think, question 1, so I definitely wanted to see the numbers once I went through the layout, one versus the other.

R: And was this the first thing that came to mind?

S: I had a gut feeling for sure. That rolling at least one two was going to be for sure greater than exactly one 2. I definitely had a gut feeling about that for sure.

R: Could you explain why that would be true, just in English with some math- if you want- sprinkled in there?

S: I think, just because, um…just because you're rolling- the options are greater. The chance of you just doing…there's just so many more ways that you can have rolled than the other one. That was my definite thinking for sure.

Question 5

R: [rereads question]. What did you end saying for this question?

S: I wrote not true.

R: Did you find this question easy or challenging?

S: I think this one…um…I think this was a bit easier because we were working with halves. Or I thought we were working with halves instead of 1/6, for one of the parts. It was just easier to consider that. But it was also hard to think about considering different, um…because we had almost like two separate incomparable games going on. You know? The combinations were different. I think it was harder to consider. It was harder to get an initial gut feeling about what it was going to be.

R: What do you mean the games were different?

S: Whereas rolling just a single number versus rolling a combination. Or just multiple numbers.

R: Having multiple options? [in reference to 2,4,6 the evens].

S: The different odds. Halves versus sixths, it was a bit more challenging to get a gut feeling.

R: You're saying you didn't have a strong gut feeling towards this one?

S: I don't think so no.

R: But you thought this question was pretty easy to calculate?

S: I think so, yah. I think it was easy to compare them.

R: That's because you were dealing with halves?

S: Yah, the math is definitely quicker. Um...I think it was easier to arrange almost because it has evens and odds. Even though it's the same as rolling 5's and not 5's, almost. It's strange…I don't know. I think it was easier. But harder to get a gut feeling on it for sure.

Question 6

R: [rereads question]. Did you say this is true or not true?

S: I wrote not true.

R: Did you find it easy or challenging?

S: I think by the end of this assignment I found this question a lot easier.

R: Why is that?

S: I think it's because I was in the frame of mind of thinking about the arrangements and the number of times you'd be rolling and stuff. It's like I was warmed up.

R: Did you have initial thoughts when you first read it?

S: I think so. Right off the bat I knew because the number of rolls that there would be more arrangements and the number would be completely different.

R: More arrangements for…?

S: For um…can you read the question again?

R: One times a five if I roll 3 times versus 2 times a five if I roll 6 times.

S: I think for sure, it was easy to know that it would be different, because there would be completely different sets of addition.

R: Because you're rolling…

S: Twice as many times.