Some Fluctuation Identities of Hyper-Exponential Jump-Diffusion Processes

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ABSTRACT

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Meromorphic Lévy processes have attracted the attention of a lot of researchers recently due to its special structure of the Wiener-Hopf factors as rational functions of infinite degree written in terms of poles and roots of the Laplace exponent, all of which are real numbers. With these Wiener-Hopf factors in hand, we can explicitly derive the expression of fluctuation identities that concern the first passage problems for finite and infinite intervals for the Meromorphic Lévy process and the resulting process reflected at its infimum. In this thesis, we consider some fluctuation identities of some classes of Meromorphic jump-diffusion processes with either the double exponential jumps or more general the hyper-exponential jumps. We study solutions to the one-sided and two-sided exit problems, and potential measure of the process killed on exiting a finite or infinite intervals. Also, we obtain some results to the process reflected at its infimum.

1 INTRODUCTION

Lévy processes are stochastic processes with independent and stationary increments. The best known and most important examples are Poisson processes, Brownian motion, Cauchy processes, and more general the stable processes. They are prototypes of Markov processes (actually they form the class of space-time homogeneous Markov processes) and of semi-martingales. Historically, the first researches go back to the late 20's with the study of infinitely divisible distributions. Their general structure had been gradually discovered by de Finetti, Kolmogorov, Lévy, Khintchine and Itô. After the pioneer contribution of Hunt in the mid-50's, the spreading of the theory of Markov processes and its connection with abstract potential theory has had a considerable impact on Lévy processes. Many important properties of sample paths of Lévy processes have been noted by Getoor (1961), Rogozin (1972), and others. Further developments in this setting were made quite recently by Bertoin (1996), Barndorff et al (2001) Doney and Kyprianou (2006), Sato (2013) and others.

Lévy processes play an important role in several fields of science, such as in physics, for the study of tribulence, laser cooling and quantum field theory; in engineering, for the study of networks, queues and dams; in economics, for continuous time-series models; in actuarial science, for the calculation of insurance and re-insurance risk; in risk Gerber-Shiu theory, for the study of risk models; and of course, in mathematical finance, for the stock price in the market. A comprehensive overview of several applications of Lévy processes can be found in Prabhu (1998), in Barndorff et al (2001), in Pistorius (2003), in Kyprianou et al (2005), and in Kyprianou (2006).

In mathematical finance, Lévy processes are becoming extremely fashionable because they can describe the observed reality of financial markets in a more accurate way than models based on Brownian motion (though it is a very basic case of Lévy processes). In the real world, we observe that asset price processes have jumps or spikes, and risk managers have to take them into consideration. Models that accurately fit return distributions are essential for the estimation of profit and loss distributions. Similarly, in the risk-neutral world, we observe that implied volatilities are constant neither across strike nor across maturities as stipulated by the model of Black and Scholes (1973). Therefore, traders need models that can capture the behavior of the implied volatility smiles more accurately in order to handle the risk of trades. Lévy processes provide us with the appropriate tools to adequately and consitently describe all these observations, both in the real and in the risk-neutral world.

One of the most obvious and fundamental problems that can be stated for a Lévy process, particularly in relation to its role as a modelling tool, is the distributional characterization of the time at which a Lévy process first exits either an infinite or a finite interval together with its overshoot and undershoot beyond the boundary of the interval. The theory of Lévy processes forms the cornerstone of an enormous volume of mathematical literature which supports a wide variety of applied and theoretical stochastic models. As a family of stochastic processes, Lévy processes are now well understood and the exit problem has seen many different approaches dating back to the 1960s. Namely, Getoor (1961) and Rogozin (1972) were the pedal for other researchers to study more on the exit problems of Lévy processes.

Nonetheless, Lévy process is still a large field to study. It is classified into different classes such as, subordinators (an increasing Lévy process), spectrally negative Lévy processes (process with no positive jumps), jump-diffusion processes and so on. Subordinators and spectrally negative Lévy processes had attracted lots of reseachers, and their fluactuation identities had been established (for more details see Bertoin (1996), Avram et al (2004), Chiu and Yin (2005), Doney and Kyprianou (2006), and Baurdoux (2009)). On the other hand, jump-diffusion processes have been noticed recently due to its special application in finance (an introduction of jump-diffusion models can be found in Tankov and Voltchkova (2009)). Starting with seminal paper of Merton (1976) and up to the present date, various aspects of jump-diffusion models have been studied in the academic finance community. In the last decade, the research departments of major banks started to accept jump-diffusions as a valuable tool in their day-to-day modeling. The interest to jump-diffusion models in finance is gradually increasing because models using Brownian motion does not accurately describe the path of the stock price. Also, the continous models need to be replaced by dicontinous models due to the presence of jumps in observed prices.

Despite the maturity of this field of study, it is surprising to note that, until very recently before the discovering of Wiener-Hopf factorization, there were fewer than a handful of examples for which explicit analytical detail concerning the first exit problem could be explored. Given the closeness in mathematical proximity of the first exit problems to the characterization of the Wiener-Hopf factorization, one might argue that the lack of concrete examples of the former was a consequence of the same being true for the latter. The landscape for both the Wiener-Hopf factorization problem and the first exit problems has changed quite rapidly in the last ten years with the discovery of a number of new mathematically tractable families of Lévy processes. Kyprianou (2006) and Kyprianou et al (2011) have successfully solved the exits problems of some classes of Lévy processes. The Meromorphic class of Lévy process is one case of jump-diffusion processes. It can be simply understood as a Lévy process whose Lévy measure has a density with respect to the Lebesgue measure of an infinite mixture sum of exponential random variable with different rates. The hyper-exponential jump-diffusion process is a special case of this class of Lévy process with a finite mixture sum. Thank to its special structure, the Wiener-Hopf factors can be expressed as rational functions of infinite degree written in terms of poles and roots of the Laplace exponent, all of which are real numbers. Therefore, the solutions to the first exit problems can be expressed explicitely.

Initially, Kou and Wang (2003) gave an explicit expression for the one-sided exit problems of the double exponential jump-diffusion processes, the special case of hyper-exponential jump-diffusion process, in which there are only upward and downward jumps following exponential distributions with distinct rates. Their approach used infinitesimal generator, Itô's formula and the resulting martingales to derive the result. Later on, Chen et al (2007) proposed another approach connecting ODE boundary problems to exit problems of schochastic processes. And in Chen et al (2013), they found an explicit expression for the two-sided exit problems for hyper-exponential jump-diffusion processes. Recently, Aoudia and Renaud (2014) studied a more general case for the mixed-exponential jump-diffusion model.

Furthermore, jump-diffusion processes with one or two reflecting barriers appear in many applications in economics, finance, queueing, mathmatical biology, and electrical engineering. Thus, it drawed attention from lots of researchers. Dong and Han (2015) proposed an explicit expression for the exit-problem of process reflected at its supremum for hyper-Erlang jump-diffusion process. Their approach basically used Itô's formula and martingales, but they had done extra work to derive the infinitesimal generator of the reflected process.

Indeed, in Kyprianou et al (2012) which concern the Meromorphic jump-diffusion processes, with the key of explicit Wiener-Hopf factors, they first studied explicit identities for the exponentially discounted first passage problem. Then they considered the more complicated two-sided exit problems. Inspired by a technique of Rogozin (1972), they solve the system of equations which characterize the discounted overshoot distribution on either side of the interval. Furthermore, they directly derived the potential measure on exiting infinite and finite intervals for both Meromorphic jump-diffusion process and the resulting process reflected at its infimum. At the end, they presented some numerical examples.

In this thesis, we first give a brief introduction to jump-diffusion processes and review various mathematical tools needed to apply these processes for option pricing and hedging. Particularly, we focus on some fluctuation identities for the jump-diffusion processes: the double exponential jump-diffusion processes for the simple case and the hyper-exponential jump-diffustion processes for the general case. These identities are for the one-sided and two-sided exit problems and potential measure of the processes. Furthermore, we prove some results on the process reflected at its infimum: the expression of potential measure and consequensely the joint density of overshoot and undershoot.

This thesis is organized as following: Section 2 presents some definitions and previous results. Section 3 considers some fluctuation identities of hyper-exponential jump-diffusion processes. Section 4 discusses some similar results for the reflected processes at its infimum. Then, we will give some numerical examples in Section 5. Section 6 and Section 7 are for future study and appendix, respectively.

2 DEFINITIONS AND PREVIOUS RESULTS

2.1 Lévy processes

In this section, we introduce some basic concepts of Lévy processes.

Definition 2.1 *(Lévy process) A process $X = \{X_t : t \geq 0\}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is said to be one-dimentional Lévy process taking real value if it possesses the following properties:*

(i) The paths of X are $\mathbb{P}$-almost surely right-continuous with left limit.

(ii) $\mathbb{P}(X_0 = 0) = 1$.

(iii) For $0 \leq s \leq t, X_t - X_s$ is equal in distribution to X_{t-s}.

(iv) For $0 \leq s \leq t, X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

From the definition above, it is difficult to see just how rich the class of Lévy processes is. De Finetti (1929) introduced the notion of infinitely divisible distributions and showed that they have an intimate relationship with Lévy processes.

Definition 2.2 *We say that a real-valued random variable, Θ, has an infinitely divisible distribution if, for each $n = 1, 2, ...,$ there exits a sequence of i.i.d. random variables $\Theta_{1,n}, ..., \Theta_{n,n}$ such that*

$$\Theta \stackrel{d}{=} \Theta_{1,n} + ... + \Theta_{n,n},$$

where $\stackrel{d}{=}$ is equality in distribution.

Alternatively, we could have expressed this relation in terms of probability laws. That is to say, the law μ of a real-valued random variable is infinitely divisible if, for each $n = 1, 2, ...,$ there exits another law μ_n of a real-valued random variable such that $\mu = \mu_n^{*n}$. (Here μ_n^{*n} denotes the n-fold convolution of μ_n.) So, one way to establish whether a given random variable has an infinitely divisible distribution is via its characteristic exponent. Suppose that Θ has characteristic exponent $\Psi(u) := -\log \mathbb{E}(e^{iu\Theta})$, defined for all $u \in \mathbb{R}$. Then Θ

has an infinitely distribution if, for all $n \geq 1$, there exits a characteristic exponent of a probability distribution, say Ψ_n, such that $\Psi(u) = n\Psi_n(u)$, for all $u \in \mathbb{R}$. The full extent to which we may characterise infinitely divisible distribution is described by the characteristic exponent Ψ and an expression known as the Lévy Khintchine formula.

Theorem 2.1 *(Lévy-Khintchine formula)([20]) A probability law, μ, of a real-valued random variable is infinitely divisible with charateristic exponent (Lévy exponent) Ψ.*

$$\int_{\mathbb{R}} e^{i\theta x}\mu(dx) = e^{-\Psi(\theta)}, \quad or \quad \Psi(\theta) := -\log \mathbb{E}(e^{i\theta X}) \quad for\ \theta \in \mathbb{R},$$

if and only if there exists a triple (a, σ, Π), where $a, \sigma \in \mathbb{R}$, and Π is a measure concentrated on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$, such that

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}(1 - e^{i\theta x} + i\theta x\mathbf{1}_{|x|<1})\Pi(dx),$$

for every $\theta \in \mathbb{R}$. Moreover, the triple (a, σ^2, Π) is unique.

We also want to introduce the Laplace exponent which is defined as

$$\psi(\theta) := -\log \mathbb{E}(e^{-\theta X_t}),$$

so

$$\psi(\theta) = a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}}(1 - e^{\theta x} + \theta x\mathbf{1}_{|x|<1})\Pi(dx).$$

Now consider

$$\overline{X}_t = \sup_{x \leq t} X_s \quad \text{and} \quad \underline{X}_t = \inf_{x \leq t} X_s,$$

then by Duality Lemma in [20], the pairs $(\overline{X}_t, \overline{X}_t - X_t)$ and $(X_t - \underline{X}_t, -\underline{X}_t)$ have the same distribution in $\mathbb{P}$. Also, we define

$$\overline{G}_t = \sup\{s < t : \overline{X}_s = X_s\} \quad \text{and} \quad \underline{G}_t = \sup\{s < t : \underline{X}_s = X_s\}.$$

Then we have the following Wiener-Hopf factorization theorem which plays an essential role in developing fluctuation results of Lévy processes. The Wiener-Hopf factorization may be found as a common reference to a multitude of statements concerning the distributional decomposition of the path of any Lévy process, when sampled at an independent and exponentially distributed time.

Theorem 2.2 *(Wiener-Hopf factorization)([20]) Suppose that X is any Lévy process other than a compound Poissson process. As usual, denote by e_q an independent and exponentially distributed random variable with parameter $q > 0$.*

(i) The pairs

$$(\overline{G}_{e_q}, \overline{X}_{e_q}) \quad and \quad (e_q - \overline{G}_{e_q}, \overline{X}_{e_q} - X_{e_q})$$

are independent and infinitely divisible, yielding the factorization

$$\frac{q}{q - iv + \Psi(\theta)} = \Psi_q^+(v, \theta).\Psi_q^-(v, \theta),$$

where $\theta, v \in \mathbb{R}$,

$$\Psi_q^+(v, \theta) = \mathbb{E}(e^{iv\overline{G}_{e_q} + i\theta\overline{X}_{e_q}}) \quad and \quad \Psi_q^-(v, \theta) = \mathbb{E}(e^{iv\underline{G}_{e_q} + i\theta\underline{X}_{e_q}}).$$

Here, the pair $\Psi_q^+(v, \theta)$ and $\Psi_q^-(v, \theta)$ are called the Wiener-Hopf factors.

(ii) Via analytical extension, the Wiener-Hopf factors may be identified from the Laplace transforms

$$\Psi_q^+(v, \theta) = \mathbb{E}(e^{iv\overline{G}_{e_q} + i\theta\overline{X}_{e_q}}) = \frac{\mathcal{K}(q, 0)}{\mathcal{K}(q + \alpha, \beta)} \quad and \quad \Psi_q^-(v, \theta) = \mathbb{E}(e^{iv\underline{G}_{e_q} + i\theta\underline{X}_{e_q}}) = \frac{\hat{\mathcal{K}}(q, 0)}{\hat{\mathcal{K}}(q + \alpha, \beta)},$$

where $\alpha, \beta \in \mathbb{C}^+$.

(iii) The Laplace exponent $\mathcal{K}(\alpha, \beta)$ and $\hat{\mathcal{K}}(\alpha, \beta)$ may also be identified in terms of the law of

X by

$$\mathcal{K}(\alpha,\beta) = k\exp\left(\int_0^\infty \int_{(0,\infty)} (e^{-t} - e^{-\alpha t - \beta x})\frac{1}{t}\mathbb{P}(X_t \in dx)dt\right),$$

$$\hat{\mathcal{K}}(\alpha,\beta) = \hat{k}\exp\left(\int_0^\infty \int_{(-\infty,0)} (e^{-t} - e^{-\alpha t + \beta x})\frac{1}{t}\mathbb{P}(X_t \in dx)dt\right),$$

where $\alpha, \beta \geq 0$*;* k *and* $\hat{k}$ *are strictly positive constants.*

(iv) By setting $v = 0$ *and taking limits as* q *tends to zero in (i), we obtain*

$$k\hat{k}\Psi(\theta) = \mathcal{K}(0, -i\theta)\hat{\mathcal{K}}(0, i\theta).$$

We give an example of the Wiener-Hopf factorization for the simple Lévy process, the Brownian motion.

Recall that if X_t is a standard Brownian motion, then it has the characteristic exponent $\Psi(\theta) = \frac{\theta^2}{2}$, for $\theta \in \mathbb{R}$. Then

$$\frac{q}{q - iv + \frac{\theta^2}{2}} = \frac{\sqrt{2q}}{\sqrt{2q - 2iv} - i\theta}.\frac{\sqrt{2q}}{\sqrt{2q - 2iv} + i\theta}.$$

From part (iii) of the theorem above, we can identify

$$\mathcal{K}(\alpha,\beta) = \hat{\mathcal{K}}(\alpha,\beta) = \sqrt{2\alpha} + \beta,$$

for $\alpha, \beta \geq 0$. The fact that both $\mathcal{K}$ and $\hat{\mathcal{K}}$ have the same expression is obvious by symmetry.

We now introduce some fluctuation identities of Lévy processes. The first one are the one-sided and two-sided exit problems. In general, the exit problems of any Lévy process are problems concerning the Laplace transform of τ_a^+ and $X_{\tau_a^+}$. For example,

$$\text{One-sided exit problem:} \qquad \mathbb{E}_x\big(e^{-q\tau_a^+}, X_{\tau_a^+} \in dy\big),$$

$$\text{Two-sided exit problem:} \qquad \mathbb{E}_x\big(e^{-q\tau_a^+}, \tau_a^+ < \tau_0^-\big),$$

where $q \geq 0$, $\tau_a^+ = \inf\{t \geq 0 : X_t > a\}$, and $\tau_0^- = \inf\{t \geq 0 : X_t < 0\}$ with the convention $\inf\{\emptyset\} = \infty$. The exit problems for spectrally negative Lévy processes are very well-known and fully discovered (for details, see Kyprianou (2006)). The solutions are expressed in terms of scale function $W^{(q)}(x)$ and $Z^{(q)}(x)$ (for more details on scale functions see Biffis and Kyprianou (2010)) which are defined as follow.

Definition 2.3 *For any $q \geq 0$, we have $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is characterised as a strictly increasing and continuous function on $[0, \infty)$ with Laplace transform satistifies*

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q} \quad for\ \beta > \Phi(q),$$

where $\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$ is the right invese of the Laplace exponent $\psi(\theta)$. Then we let $Z^{(q)}(x)$ be

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y)dy, \quad for\ x \in \mathbb{R}.$$

Another useful tool to study the overshoot and undershoot distributions at the first passage within an interval is the potential measure of the process.

Definition 2.4 *For $q \geq 0$, a q-potential measure of any Lévy process X killed on exiting $[0, a]$ when issued from x is defined as*

$$U^{(q)}(a, x, dy) := \int_0^\infty e^{-qt} \mathbb{P}_x\big(X_t \in dy, \tau > t\big)dt, \tag{1}$$

where

$$\tau := \tau_a^+ \wedge \tau_0^- = \inf\{t \geq 0 : X_t < 0\ or\ X_t > a\}.$$

Now, if for each $x \in [0, a]$, a density of $U^{(q)}(a, x, dy)$ exists with respect to Lebesque measure, then it is called the potential density and denoted by $u^{(q)}(a, x, y)$. Also, we want to introduce a q-potential measure of X killed on exiting $[0,\infty)$ when issued from x as

$$R^{(q)}(x,dy) := \int_0^\infty e^{-qt}\mathbb{P}_x\big(X_t \in dy, \tau_0^- > t\big)dt.$$

Then the q-potential density of X killed on exiting $[0,\infty)$ is $r^{(q)}(x,y)$. The expression of potential measure of spectrally negative Lévy processes had been discovered using the scale function (for more details, see Chapter 8 of Kyprianou (2006)). We also consider the fluctuation identities of Lévy process reflected at its infimum which defined as

Definition 2.5 *Given any Lévy process X_t, a resulting process reflected at its infimum is defined as*

$$Y_t := X_t - \underline{X}_t \wedge 0.$$

It can be showed that Y_t is also a Markov process (see Bertoin (1996)). So, we can use Markov properties to derive the fluctuation identities of process reflected at its infimum.

2.2 Meromorphic jump-diffusion processes

We start by giving the definition of a general Meromorphic class of Lévy processes.

Definition 2.6 *A Lévy process X_t is said to belong to the Meromorphic class (M_class) if and only if the Lévy measure $\Pi(dx)$ has a density with respect to the Lebesque measure, given by*

$$\pi(x) = \mathbb{I}_{\{x>0\}}\sum_{n\geq 1} p_n\eta_n^+ e^{-\eta_n^+ x} + \mathbb{I}_{\{x<0\}}\sum_{n\geq 1} q_n\eta_n^- e^{\eta_n^- x}, \tag{2}$$

where all the coefficients $p_n,\ q_n,\ \eta^+,\ \eta^-$ are positive, the sequences $\{\eta_n^+\}_{n\geq 1}$ and $\{\eta_n^-\}_{n\geq 1}$ are strictly inscreasing, and $\eta^+ \to +\infty$ and $\eta^- \to +\infty$ as $n \to +\infty$.

To ease the notations, we denote

$$\mathbb{E}[\ \cdot\ | X_0 = x] = \mathbb{E}_x[\ \cdot\ ] \qquad \text{and} \qquad \mathbb{E}[\ \cdot\ | X_0 = 0] = \mathbb{E}[\ \cdot\ ],$$

$$\mathbb{P}[\,\cdot\,|X_0 = x] = \mathbb{P}_x[\,\cdot\,] \qquad \text{and} \qquad \mathbb{P}[\,\cdot\,|X_0 = 0] = \mathbb{P}[\,\cdot\,].$$

The following theorem concerns about the Wiener-Hopf factorization of the Meromorphic processes which plays essential role in deriving expression of potential measure in later sections.

Theorem 2.3 *([21]) If X_t is a Meromorphic process and e_q is an independent exponential random variable with rate q, then*
(i) The Wiener-Hopf factors are given by

$$\phi_q^+(iz) = \mathbb{E}\left[e^{-z\overline{X}_{e_q}}\right] = \prod_{n\geq 1} \frac{1+\frac{z}{\eta_n^+}}{1+\frac{z}{\rho_n}}, \qquad \phi_q^-(-iz) = \mathbb{E}\left[e^{z\underline{X}_{e_q}}\right] = \prod_{n\geq 1} \frac{1+\frac{z}{\eta_n^-}}{1+\frac{z}{\rho_n^-}}. \tag{3}$$

(ii) For $x \geq 0$

$$\mathbb{P}(\overline{X}_{e_q} \in dx) = \bar{\boldsymbol{a}}(\eta^+,\rho)^T \times \bar{\boldsymbol{v}}(\rho,x)dx, \qquad \mathbb{P}(-\underline{X}_{e_q} \in dx) = \bar{\boldsymbol{a}}(\eta^-,\rho^-)^T \times \bar{\boldsymbol{v}}(\rho^-,x)dx, \tag{4}$$

where all roots $\{\rho_n, -\rho_n^-\}$ of equation: $\psi(z) = q$ are real and interlacing with the poles $\{\eta^+, -\eta_n^-\}$ as

$$\ldots - \eta_2^- < -\rho_2^- < -\eta_1^- < -\rho_1^- < 0 < \rho_1 < \eta_1^+ < \rho_2 < \eta_2^+ < \ldots,$$

$$\bar{\boldsymbol{a}}(\eta^+,\rho) = [a_0(\rho,\eta^+), a_1(\rho,\eta^+), a_2(\rho,\eta^+), \ldots]^T,$$

$$\bar{\boldsymbol{a}}(\eta^-,\rho^-) = [a_0(\rho^-,\eta^-), a_1(\rho^-,\eta^-), a_2(\rho^-,\eta^-), \ldots]^T,$$

$$\bar{\boldsymbol{v}}(\rho,x) = [\delta_0(x), \rho_1 e^{-\rho_1 x}, \rho_2 e^{-\rho_2 x}, \ldots]^T, \qquad \bar{\boldsymbol{v}}(\rho^-,x) = [\delta_0^-(x), \rho_1^- e^{-\rho_1^- x}, \rho_2^- e^{-\rho_2^- x}, \ldots]^T,$$

$$a_0(\eta^+,\rho) = \lim_{n\to+\infty} \prod_{k=1}^{n} \frac{\rho_k}{\eta_k^+}, \qquad a_n(\eta^+,\rho) = \left(1 - \frac{\rho_n}{\eta_n^+}\right) \prod_{k\geq 1}^{k\neq n} \frac{1-\frac{\rho_n}{\eta_k^+}}{1-\frac{\rho_n}{\rho_k}},$$

$$a_0(\eta^-,\rho^-) = \lim_{n\to+\infty} \prod_{k=1}^{n} \frac{\rho_k^-}{\eta_k^-}, \qquad a_n(\eta^-,\rho^-) = \left(1 - \frac{\rho_n^-}{\eta_n^-}\right) \prod_{k\geq 1}^{k\neq n} \frac{1-\frac{\rho_n^-}{\eta_k^-}}{1-\frac{\rho_n^-}{\rho_k^-}}.$$

(iii) For every $q > 0$

$$\mathbb{P}(X_{e_q} \in dx) = q\left[\mathbf{1}_{(x>0)} \sum_{n\geq 1} \frac{e^{-\rho_n x}}{\psi'(\rho_n)} - \mathbf{1}_{(x<0)} \sum_{n\geq 1} \frac{e^{-\rho_n^- x}}{\psi'(\rho_n^-)}\right] dx.$$

Remark ([21]) If the process X_t has a finite sequence $\{\eta_n\}_{n=1,2,\ldots,N}$ and either N or $N+1$ roots ρ_N, then all the formulas in the theorem above are still valid if we adopt notation $\eta_k = \infty$ for $k > N$ and $\rho_k = \infty$ for $k > N$ or (or $k > N+1$).

A basic example of Meromorphic process is that the process consists of only two types of jump: upward exponential jump and downward exponential jump, and this is called a double exponential jump-diffusion process, defined as below.

Definition 2.7 *A double exponential jump-diffusion process X_t is defined as*

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \qquad X_0 \equiv 0. \tag{5}$$

Here $\{W_t : t \geq 0\}$ is a standard Brownian motion with $W_0 = 0$. $\{N_t : t \geq 0\}$ is a Poisson process with rate λ, constant μ and $\sigma > 0$ are the drift and volatility of the diffusion part respectively, and the jump sizes $\{Y_1, Y_2 \ldots\}$ are independent and identically distributed random variables. The common density of Y is given by

$$f_Y(y) = \mathbb{I}_{\{y>0\}} p\eta_1^+ e^{-\eta_1^+ y} + \mathbb{I}_{\{y<0\}} q\eta_1^- e^{\eta_1^- y},$$

where p,$q \geq 0$ are constant, $p + q = 1$, and $\eta_1^+, \eta_1^- > 0$. And it has the following Laplace exponent

$$\psi(\theta) = \theta\mu + \frac{\sigma^2\theta^2}{2} + \lambda\left[\frac{q\eta_1^-}{\eta_1^- + \theta} + \frac{p\eta_1^+}{\eta_1^+ - \theta} - 1\right].$$

Now, for $q > 0$ consider the equation $\psi(\theta) = q$ which has exactly four real roots (the expression of the four roots is given in the appdendix of this paper) $\{\rho_1(q), \rho_2(q), \rho_3(q), \rho_4(q)\}$

interlacing with its two poles $\{-\eta_1^-, \eta_1^+\}$

$$\rho_4(q) < -\eta_1^- < \rho_3(q) < 0 < \rho_1(q) < \eta_1^+ < \rho_2(q).$$

This is easily seen by look at these limits

$$\lim_{\theta \to -\infty} \psi(\theta) = \lim_{\theta \to -(\eta_1^-)^+} \psi(\theta) = \lim_{\theta \to (\eta_1^+)^-} \psi(\theta) = \lim_{\theta \to +\infty} \psi(\theta) = +\infty,$$

$$\lim_{\theta \to -(\eta_1^-)^-} \psi(\theta) = \lim_{\theta \to (\eta_1^+)^+} \psi(\theta) = -\infty \qquad \text{and} \qquad \lim_{\theta \to 0} \psi(\theta) = -q < 0.$$

To ease the expression, we denote $\{\rho_1, \rho_2, \rho_3, \rho_4\}$ for $\{\rho_1(q), \rho_2(q), \rho_3(q), \rho_4(q)\}$. Hence, applying Theorem 2.3 and its Remark into our X_t process we obtain

$$\mathbb{P}(\overline{X}_{e_q} \in dx) = \left(1 - \frac{\rho_1}{\eta_1^+}\right)\left(\frac{\rho_2}{\rho_2 - \rho_1}\right)\rho_1 e^{-\rho_1 x} + \left(\frac{\eta_1^+ - \rho_2}{\eta_1^+}\right)\left(\frac{\rho_1}{\rho_1 - \rho_2}\right)\rho_2 e^{-\rho_2 x}, \qquad (6)$$

$$\mathbb{P}(-\underline{X}_{e_q} \in dx) = \left(1 + \frac{\rho_3}{\eta_1^-}\right)\left(\frac{\rho_4}{\rho_3 - \rho_4}\right)\rho_3 e^{\rho_3 x} + \left(\frac{\eta_1^- + \rho_4}{\eta_1^-}\right)\left(\frac{\rho_3}{\rho_4 - \rho_3}\right)\rho_4 e^{\rho_4 x}. \qquad (7)$$

A general version of double exponential jump-diffusion process is the hyper-exponential jump-diffusion process which consists of upward jumps with m distinct rates and downward jumps with n distinct rates.

Definition 2.8 *A hyper-exponential jump diffusion is defined as*

$$X_t = \mu t + \sigma W_t + \sum_{n=1}^{N_t} Y_n, \qquad X_0 \equiv 0.$$

Here $\{W_t : t \geq 0\}$ is a standard Brownian motion with $W_0 = 0$. $\{N_t : t \geq 0\}$ is a Poisson process with rate λ, constant μ and $\sigma > 0$ are the drift and volatility of the diffusion part respectively, and the jump sizes $\{Y_1, Y_2 ...\}$ are independent and identically distributed random *variables. The common density of Y is given by*

$$f_Y(y) = \mathbb{I}_{\{y>0\}} \sum_{i=1}^{m} p_i \eta_i^+ e^{-\eta_i^+ y} + \mathbb{I}_{\{y<0\}} \sum_{i=1}^{n} q_i \eta_i^- e^{\eta_i^- y}, \tag{8}$$

where $\sum_{i=1}^m p_i + \sum_{i=1}^n q_i = 1$, $p_i, q_i \geq 0$, $\eta_i^+, \eta_i^- \geq 0$. And it has Laplace exponent given by

$$\psi(\theta) = \theta\mu + \frac{\sigma^2\theta^2}{2} + \lambda \left[\sum_{i=1}^{m} \frac{p_i \eta_i^+}{\eta_i^+ + \theta} + \sum_{i=1}^{n} \frac{q_i \eta_i^-}{\eta_i^- - \theta} - 1 \right].$$

The following lemma concerns about the roots of the equation $\psi(x) = q$.

Lemma 2.1 *[18] Consider the hyper-exponential jump-diffusion process X_t, then the equation*

$$\psi(x) = q \qquad for\ all\ q > 0,$$

has exactly $S = m + n + 2$ distinct real roots interlacing with their poles as

$$0 < \rho_1 < \eta_1^+ < \rho_2 < ... < \rho_m < \eta_m^+ < \rho_{m+1},$$

$$0 < -\rho_{m+2} < \eta_1^- < -\rho_{m+3} < ... < -\rho_{m+n+1} < \eta_n^- < -\rho_{m+n+2}.$$

In addition, let the overall drift of the jump-diffusion process be

$$\psi'(0) = \mu + \lambda\left(\sum_{i=1}^{m} \frac{p_i}{\eta_i^+} - \sum_{i=1}^{n} \frac{q_i}{\eta_i^-} \right).$$

Then as $q \to 0$,

$$\rho_{m+2} \to \begin{cases} \rho_{m+2}^* & if\ \psi'(0) \geq 0, \\ 0 & if\ \psi'(0) < 0, \end{cases}$$

$$\rho_i \to \rho_i^* \qquad for\ i = m+3,\ m+4, ...,\ S,$$

where ρ_{m+2}^*, $\rho_{m+3}^*, ...\rho_S^*$ are the distinct real roots of the equation: $\psi(x) = 0$.

Again, using Theorem 2.3 we obtain the density of $\overline{X}_{e_q}$ and $-\underline{X}_{e_q}$ for hyper-exponential jump-diffusion process as below

$$\mathbb{P}(\overline{X}_{e_q} \in dx) = \bar{\boldsymbol{a}}(\eta^+, \rho)^T \times \bar{\boldsymbol{v}}(\rho, x)dx, \qquad \mathbb{P}(-\underline{X}_{e_q} \in dx) = \bar{\boldsymbol{a}}(\eta^-, \rho^-)^T \times \bar{\boldsymbol{v}}(\rho^-, x)dx, \quad (9)$$

where

$$\begin{aligned}
\bar{\boldsymbol{a}}(\eta^+, \rho) &= [a_1(\eta^+, \rho), a_2(\eta^+, \rho), ..., a_{m+1}(\eta^+, \rho)]^T, \\
\bar{\boldsymbol{a}}(\eta^-, \rho^-) &= [a_1(\eta^-, \rho^-), a_2(\eta^-, \rho^-), ..., a_{n+1}(\eta^-, \rho^-)]^T, \\
\bar{\boldsymbol{v}}(\rho, x) &= [\rho_1 e^{-\rho_1 x}, \rho_2 e^{-\rho_2 x}, ..., \rho_{m+1} e^{-\rho_{m+1} x}]^T, \\
\bar{\boldsymbol{v}}(\rho^-, x) &= [-\rho_{m+2} e^{\rho_{m+2} x}, -\rho_{m+3} e^{\rho_{m+3} x}, ..., -\rho_S e^{\rho_S x}]^T,
\end{aligned}$$

$$a_i(\eta^+, \rho) = \left(1 - \frac{\rho_i}{\eta_i^+}\right) \prod_{k \neq i, k=1}^{m+1} \frac{1 - \frac{\rho_i}{\eta_k^+}}{1 - \frac{\rho_i}{\rho_k}}, \qquad a_i(\eta^-, \rho^-) = \left(1 + \frac{\rho_i}{\eta_{i-m-1}^-}\right) \prod_{k \neq i, k=1}^{n+1} \frac{1 + \frac{\rho_i}{\eta_k^-}}{1 - \frac{\rho_i}{\rho_{k+m+1}}},$$

with the convention that $\eta_{m+1}^+ = \eta_{n+1}^- = +\infty$.

2.3 One-sided exit problems of hyper-exponential jump-diffusion processes

In this section, we present some known results for the one-sided exit problems of jump-diffusion processes. The first one is the one-sided exit problem of double exponential jump-diffusion processes proposed by Kou and Wang (2003). Their approach used the infinitesimal generator, Itô's formula and the resulting martingale to derive the result.

Theorem 2.4 *[18] Let X_t be the double exponential jump-diffusion process. Then for any $a, q > 0$, we have the following results concerning the Laplace transforms of $\tau_a^+ = \inf\{t \geq 0 :$*

$X_t > a\}$ *and* $X_{\tau_a^+}$

$$\mathbb{E}[e^{-q\tau_a^+}] = \frac{\eta_1^+ - \rho_1}{\eta_1^+}\frac{\rho_2}{\rho_2-\rho_1}e^{-a\rho_1} + \frac{\rho_2 - \eta_1^+}{\eta_1^+}\frac{\rho_1}{\rho_2-\rho_1}e^{-a\rho_2}, \tag{10}$$

$$\mathbb{E}[e^{-q\tau_a^+}, X_{\tau_a^+} - a > y] = e^{-\eta_1^+ y}\frac{(\eta_1^+ - \rho_1)(\rho_2 - \eta_1^+)}{\eta_1^+(\rho_2-\rho_1)}[e^{-a\rho_1} - e^{-a\rho_2}] \qquad for\ all\ y \geq 0, \tag{11}$$

$$\mathbb{E}[e^{-q\tau_a^+}, X_{\tau_a^+} = a] = \frac{(\eta_1^+ - \rho_1)}{\rho_2-\rho_1}e^{-a\rho_1} + \frac{(\rho_2 - \eta_1^+)}{\rho_2-\rho_1}e^{-a\rho_2}. \tag{12}$$

More general, Kyprianou et al (2012) derived the expression of one-sided exit problem of Meromorphic processes using the direct proof of Lemma 1 in [2].

Theorem 2.5 *([21]) Define a matrix $\boldsymbol{A}$ having entries*

$$a_{ij} = \begin{cases} 0 & if\ i = 0, j \geq 0, \\ a_i(\eta^+, \rho)b_0(\rho, \eta^+) & if\ i \geq 1, j = 0, \\ \frac{a_i(\eta^+,\rho)b_j(\rho,\eta^+)}{\eta_j^+ - \rho_i} & if\ i \geq 1, j \geq 1, \end{cases}$$

then for $a > 0$, $y \geq 0$, we have

$$\mathbb{E}\Big[e^{-q\tau_a^+}, X_{\tau_a^+} \in dy\Big] = \bar{\boldsymbol{v}}(\rho, a)^T \times \boldsymbol{A} \times \bar{\boldsymbol{v}}(\eta^+, y)dy, \tag{13}$$

where

$$b_0(\rho, \eta^+) = \frac{1}{\rho_1}\lim_{n\to+\infty}\prod_{k=1}^{n}\frac{\eta_k^+}{\rho_{k+1}}, \qquad b_n(\rho, \eta^+) = -\Big(1 - \frac{\eta_n^+}{\rho_n}\Big)\prod_{k\geq 1, k\neq n}\frac{1 - \frac{\eta_n^+}{\rho_k}}{1 - \frac{\eta_n^+}{\eta_k^+}}.$$

Using the theorem above, we can modify the matrix $\boldsymbol{A}$ to obtain the one-sided exit problems for our hyper-exponential jump-diffusion processes as following

$$\mathbb{E}\Big[e^{-q\tau_a^+}, X_{\tau_a^+} \in dy\Big] = \bar{\boldsymbol{v}}(\rho, a)^T \times \bar{\boldsymbol{A}} \times \bar{\boldsymbol{v}}(\eta^+, y)dy, \tag{14}$$

$$\mathbb{E}\Big[e^{-q\tau_a^-}, X_{\tau_a^-} \in dy\Big] = \bar{\boldsymbol{v}}(\rho^-, a)^T \times \bar{\boldsymbol{B}} \times \bar{\boldsymbol{v}}(\eta^-, y)dy, \tag{15}$$

where $\rho_1^- = \rho_{m+2},\ \rho_2^- = \rho_{m+3}, ...,\ \rho_{n+1}^- = \rho_S$, $\tau_a^- = \inf\{t \geq 0 : X_t < a\}$, $\times$ is the dot product, and

$$\bar{\boldsymbol{A}} = \begin{bmatrix} a_1(\eta^+,\rho)b_0(\rho,\eta^+) & a_1(\eta^+,\rho)b_1(\rho,\eta^+) & \ldots & a_1(\eta^+,\rho)b_m(\rho,\eta^+) \\ a_2(\eta^+,\rho)b_0(\rho,\eta^+) & a_2(\eta^+,\rho)b_1(\rho,\eta^+) & \ldots & a_2(\eta^+,\rho)b_m(\rho,\eta^+) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+1}(\eta^+,\rho)b_0(\rho,\eta^+) & a_{m+1}(\eta^+,\rho)b_1(\rho,\eta^+) & \ldots & a_{m+1}(\eta^+,\rho)b_m(\rho,\eta^+) \end{bmatrix},$$

$$\bar{\boldsymbol{B}} = \begin{bmatrix} a_1(\eta^-,\rho^-)b_0(\rho^-,\eta^-) & a_1(\eta^-,\rho^-)b_1(\rho^-,\eta^-) & \ldots & a_1(\eta^-,\rho^-)b_n(\rho^-,\eta^-) \\ a_2(\eta^-,\rho^-)b_0(\rho^-,\eta^-) & a_2(\eta^-,\rho^-)b_1(\rho^-,\eta^-) & \ldots & a_2(\eta^-,\rho^-)b_n(\rho^-,\eta^-) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1}(\eta^-,\rho^-)b_0(\rho^-,\eta^-) & a_{n+1}(\eta^-,\rho^-)b_1(\rho^-,\eta^-) & \ldots & a_{n+1}(\eta^-,\rho^-)b_n(\rho^-,\eta^-) \end{bmatrix}.$$

2.4 Two-sided exit problems of double exponential jump-diffusion processes

The following two theorems are from Xu (2010) concerning the two-sided exit problems of double exponential jump-diffusion processes. She mimicked the approach of Kou and Wang (2003) for the case of two-sided exit problems.

Theorem 2.6 *([29]) For any $q, a, b > 0$ and $\tau = \inf\{t \geq 0 : X_t < -a$ or $X_t > b\}$, then*

$$\mathbb{E}(e^{-q\tau}; X_\tau \geq b) = A_1 e^{-\rho_1 b} + A_2 e^{-\rho_2 b} + A_3 e^{-\rho_3 a} + A_4 e^{-\rho_4 a}, \tag{16}$$

where

$$A_1 = \frac{A_{11}}{\eta_1^+ A},\ A_2 = \frac{A_{21}}{\eta_1^+ A},\ A_3 = \frac{A_{31}}{\eta_1^+ A},\ A_4 = \frac{A_{41}}{\eta_1^+ A},$$

.

$$A := \frac{(\rho_2-\rho_1)(\rho_4-\rho_3)}{(\rho_1-\eta_1^+)(\rho_2-\eta_1^+)(\rho_3-\eta_1^+)(\rho_4-\eta_1^+)} + \frac{e^{-a\rho_2}}{(\rho_1-\eta_1^+)(\rho_2+\eta_1^-)}\left(\frac{(\rho_1+\rho_3)(\rho_2+\rho_4)}{(\rho_3+\eta_1^+)(\rho_4-\eta_1^-)}e^{-a\rho_3} - \frac{(\rho_1+\rho_4)(\rho_2+\rho_3)}{(\rho_4+\eta_1^+)(\rho_3-\eta_1^+)}e^{-a\rho_4}\right)$$
$$+\frac{(\rho_1+\rho_3)(\rho_2+\rho_4)}{(\rho_2-\eta_1^+)(\rho_4+\eta_1^+)(\rho_1+\eta_1^-)(\rho_3-\eta_1^-)}e^{-a(\rho_1+\rho_4)}+\frac{e^{-a(\rho_1+\rho_3)}}{(\rho_3+\eta_1^+)(\rho_1+\eta_1^-)}\left(\frac{(\rho_1+\rho_4)(\rho_2+\rho_3)}{(\rho_2-\eta_1^+)(-\rho_4+\eta_1^-)} + \frac{(\rho_2-\rho_1)(\rho_4-\rho_3)}{(\rho_4+\eta_1^+)(\rho_2+\eta_1^-)}e^{-a(\rho_2+\rho_4)}\right),$$

$$A_{11} := \frac{e^{-a\rho_2}}{\rho_2+\eta_1^-}\left(\frac{e^{-a\rho_4}\rho_4(\rho_2+\rho_3)}{(\rho_4+\eta_1^+)(\rho_3-\eta_1^-)} - \frac{e^{-a\rho_3}\rho_3(\rho_2+\rho_4)}{(\rho_3+\eta_1^+)(\rho_4-\eta_1^-)}\right) - \frac{\rho_2(\rho_3-\rho_4)}{(\rho_2-\eta_1^+)(\rho_4-\eta_1^-)(-\rho_3+\eta_1^-)},$$

$$A_{31} := \frac{e^{-a\rho_1}}{\rho_1+\eta_1^-}\left(\frac{e^{-a(\rho_4+\rho_2)}\rho_4(\rho_2-\rho_1)}{(\rho_4+\eta_1^+)(\rho_2+\eta_1^+)} + \frac{\rho_2(\rho_1+\rho_4)}{(\rho_2-\rho_1)(-\rho_4+\eta_1^-)}\right) + \frac{e^{-a\rho_2}\rho_1(\rho_2+\rho_4)}{(\rho_1-\eta_1^+)(\rho_4-\eta_1^-)(\rho_2+\eta_1^-)},$$

$-A_{21}$ is obtained from A_{11} by changing ρ_2 to ρ_1, and $-A_{41}$ is obtained from A_{31} by changing ρ_4 to ρ_3.

Theorem 2.7 *[29] For any $q, b, y > 0$, then*

$$\mathbb{E}(e^{-q\tau}; X_\tau - b > y) = B_1 e^{-\rho_1 b} + B_2 e^{-\rho_2 b} + B_3 e^{-\rho_3 a} + B_4 e^{-\rho_4 a}, \tag{17}$$

where

$$B_1 = \frac{e^{-y\eta_1^+}B_{11}}{\eta_1^+ A},\ B_2 = \frac{e^{-y\eta_1^+}B_{21}}{\eta_1^+ A},\ B_3 = \frac{e^{-y\eta_1^+}B_{31}}{\eta_1^+ A},\ B_4 = \frac{e^{-y\eta_1^+}B_{41}}{\eta_1^+ A},$$

$$B_{11} := \frac{-\rho_3+\rho_4}{(\rho_4-\eta_1^-)(-\rho_3+\rho_2)} + \frac{e^{-a(\rho_2+\rho_4)}(\rho_2+\rho_3)}{(\rho_2+\eta_1^-)(\rho_3-\eta_1^-)} - \frac{e^{-a(\rho_2+\rho_3)}(\rho_2+\rho_4)}{(\rho_2+\eta_1^-)(\rho_4-\eta_1^-)},$$

$$B_{31} := \frac{e^{-a\rho_2}(\rho_2+\rho_4)}{(\rho_4-\eta_1^-)(\rho_2+\eta_1^-)} + \frac{e^{-a\rho_1}(\rho_1+\rho_4)}{(\rho_1+\eta_1^-)(-\rho_4+\eta_1^-)} + \frac{e^{-a(\rho_1+\rho_2+\rho_4)}(-\rho_1+\rho_2)}{(\rho_1+\eta_1^-)(\rho_2+\eta_1^-)},$$

$-B_{21}$ is obtained from B_{11} by changing ρ_2 to ρ_1, and $-B_{41}$ is obtained from B_{31} by changing ρ_4 to ρ_3.

2.5 Two-sided exit problems of hyper-exponential jump-diffusion processes

The first passage functional of a process X_t is defined as

$$\Phi(x) = \mathbb{E}_x[e^{-q\tau} g(X_\tau)],$$

where $q \geq 0$, g is a nonnegative bounded measureable function, $X_0 = x$ a.s under P_x and τ is the exit time of X from a finite interval $I = (h_1, h_2)$, ie,

$$\tau = \inf\{t \geq 0 : X_t > h_2 \text{ or } X_t < h_1\}.$$

Then the following boundary value problem which admits at most one solution must be solved. Find $\Phi \in \mathcal{C}([h_1, h_2]) \cap \mathcal{C}^2((h_1, h_2))$ such that

$$\begin{cases} (\mathcal{L} - q)\Phi = 0 & \text{on } (h_1, h_2), \\ \Phi = g & \text{on } (-\infty, h_1] \cup [h_2, \infty), \end{cases} \tag{18}$$

where $\mathcal{L}$ is the infinitesimal generator of X given by

$$\mathcal{L}h(x) = \mu h'(x) + \frac{\sigma^2}{2} h''(x) + \lambda \int h(x+y) f(y) dy - \lambda h(x).$$

The following two theorems give the expression of the solution for the boundary value problem above.

Theorem 2.8 *([11]) Suppose that Φ is a bounded solution to the boundary value problem above, and on (h_1, h_2), $\Phi(x) = \sum_{i=1}^{S} Q_i e^{\rho_i x}$ for some constants Q_i ($S = m + n + 2$, is the total number of distict real roots of the equation $\psi(x) = q$). Then the constant vector $\boldsymbol{Q}$ satisfies the equation*

$$\boldsymbol{A}\boldsymbol{Q} = \boldsymbol{V}_g, \tag{19}$$

where matrix $\boldsymbol{A}$ *and elements of vector* $\boldsymbol{V}_g$ *are defined as*

$$\boldsymbol{A} = \begin{bmatrix} \frac{e^{\rho_1 h_2}}{\rho_1-\eta_1^+} & \cdots & \frac{e^{\rho_S h_2}}{\rho_S-\eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{e^{\rho_1 h_2}}{\rho_1-\eta_m^+} & \cdots & \frac{e^{\rho_S h_2}}{\rho_S-\eta_m^+} \\ \\ e^{\rho_1 h_2} & \cdots & e^{\rho_S h_2} \\ \frac{e^{\rho_1 h_1}}{\rho_1+\eta_1^-} & \cdots & \frac{e^{\rho_S h_1}}{\rho_S+\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{e^{\rho_1 h_1}}{\rho_1+\eta_n^-} & \cdots & \frac{e^{\rho_S h_1}}{\rho_S+\eta_n^-} \\ \\ e^{\rho_1 h_1} & \cdots & e^{\rho_S h_1} \end{bmatrix}, \quad \boldsymbol{V}_g(i) = \begin{cases} -\int_{h_2}^{\infty} g(y)e^{-\eta_i^+(y-h_2)}dy, & if\ 1 \le i \le m, \\ g(h_2), & if\ i = m+1, \\ \int_{-\infty}^{h_1} g(y)e^{\eta_{i-m-1}^-(y-h_1)}dy, & if\ m+2 \le i \le S-1, \\ g(h_1), & if\ i = S, \end{cases} .$$

Theorem 2.9 *([11]) Given a constant* $q \ge 0$ *and a nonnegative bounded function* g *on* $(h_1, h_2)^c$*, the function* $\Phi(x)$*, defined by the formula*

$$\Phi(x) = \begin{cases} \sum_{i=1}^{S} Q_i e^{\rho_i x}, & if\ x\ \in (h_1, h_2), \\ g(x) & if\ x\ \notin (h_1, h_2), \end{cases} , \tag{20}$$

solves the boundary value problem above. Additionally, $\Phi(x) = \mathbb{E}_x[e^{-q\tau} g(X_\tau)]$.

These two theorems will be used to derive the two-sided exit problems for special choice of (h_1, h_2) and the function $g(x)$ in the next section.

2.6 Some facts about reflected Brownian motion with drift

A linear Brownian motion is defined as

$$B_t = \mu t + \sigma W_t,$$

where W_t is a standard Brownian motion and the constant μ and $\sigma > 0$ are the drift and the volatility of the process. Then from [9] we obtain the following results for the reflected linear Brownian motion with drift defined as $|B_t|$ and $\beta_a = \inf\{t \geq 0 : |B_t| > a\}$

$$\begin{aligned}
\mathbb{P}_x\Big(|B_t| \in dy\Big) &= \frac{1}{\sqrt{2\pi t}}\Big(e^{-(\frac{y}{\sigma}-\mu t-x)/2t} + e^{-(\frac{y}{\sigma}+\mu t+x)/2t}\Big)dy, && (21)\\
\mathbb{P}_x\Big(|B_{e_q}| \in dy\Big) &= \frac{\lambda\sigma}{\sqrt{2\lambda\sigma^2+\mu^2}}\Big(e^{\mu(\frac{y}{\sigma}-x)-|\frac{y}{\sigma}-x|\sqrt{2\lambda+(\frac{\mu}{\sigma})^2}} + e^{\mu(\frac{y}{\sigma}+x)-|\frac{y}{\sigma}+x|\sqrt{2\lambda+(\frac{\mu}{\sigma})^2}}\Big)dy, && (22)\\
\mathbb{E}_x\Big[e^{-q\beta_a}\Big] &= \frac{e^{\mu x+x\sqrt{2q+(\frac{\mu}{\sigma})^2}} + e^{-\mu x-x\sqrt{2q+(\frac{\mu}{\sigma})^2}}}{e^{\frac{\mu}{\sigma}a+\frac{a}{\sigma}\sqrt{2q+(\frac{\mu}{\sigma})^2}} + e^{-\frac{\mu}{\sigma}a-\frac{a}{\sigma}\sqrt{2q+(\frac{\mu}{\sigma})^2}}} \qquad x \in [0, a]. && (23)
\end{aligned}$$

These facts of reflected Brownian motion with drift play a role in deriving the potential measure of hyperexponential jump-diffusion process reflected at its infimum in Section 4.

3 FLUCTUATION IDENTITIES OF HYPER EXPONENTIAL JUMP-DIFFUSION PROCESSES

3.1 Two-sided exit problems

Kyprianou et al (2012) had alrealy derived solutions to the two-side exit problems for the Meromorphic processes. However, their expressions are too general and not explicit. Furthermore, we have to solve the system of linear matrix equations. Therefore, in this section, we want to provide more explicit solutions for two-side exit problems.

From Theorems 2.8 and 2.9 in the previous section, we see that with an appropriate choice of function $g(x)$, one can easily derive solution to two-sided exit problems from an interval (a, b). As a result, we have the following corollaries regarding to two-sided exit problems from the upper level a. The first corollary is the two-sided exit problem resulting by either creeping or jump over the level a.

Corollary 3.1 *Consider the hyper-exponential jump-diffusion process X_t, given $x \in (0, a)$, let $q \geq 0$, and $\tau = \tau_a^+ \wedge \tau_0^-$. Then*

$$\mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau \geq a\}}\right] = \sum_{i=1}^{S} D_i^1 e^{\rho_i x} = \sum_{i=1}^{S} \frac{(-1)^{m+i+1}det(\boldsymbol{A_i^1})}{\sum_{j=1}^{S}(-1)^{m+j+1}e^{\rho_j a}det(\boldsymbol{A_j^1})}e^{\rho_i}x, \tag{24}$$

where $\boldsymbol{D^1} = \{D_1^1, D_2^1, ..., D_S^1\}$ is the unique vector from Theorem 2.8, and $det(\boldsymbol{A_i^1})$ is the determinant of sub-matrix obtained by deleting the ith column from the following matrix

$$
\boldsymbol{A^1} = \begin{bmatrix}
\frac{\rho_1 e^{\rho_1 a}}{\rho_1-\eta_1^+} & \frac{\rho_2 e^{\rho_2 a}}{\rho_2-\eta_1^+} & \cdots & \frac{\rho_S e^{\rho_S a}}{\rho_S-\eta_1^+} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\rho_1 e^{\rho_1 a}}{\rho_1-\eta_m^+} & \frac{\rho_2 e^{\rho_2 a}}{\rho_2-\eta_m^+} & \cdots & \frac{\rho_S e^{\rho_S a}}{\rho_S-\eta_m^+} \\
 & & & \\
\frac{1}{\rho_1+\eta_1^-} & \frac{1}{\rho_2+\eta_1^-} & \cdots & \frac{1}{\rho_S+\eta_1^-} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\rho_1+\eta_n^-} & \frac{1}{\rho_2+\eta_n^-} & \cdots & \frac{1}{\rho_S+\eta_n^-} \\
1 & 1 & \cdots & 1
\end{bmatrix}.
$$

Proof. Using Theorems 2.8 and 2.9 in introdution Section with the settings $(h_1, h_2) = (0, a)$ and $g(x) = \mathbf{1}_{\{x \geq a\}}$, one can easily check that the matrix equation $\boldsymbol{AQ} = \boldsymbol{V}_g$ has the form

$$
\begin{bmatrix}
\frac{e^{\rho_1 a}}{\rho_1-\eta_1^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_1^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S-\eta_1^+} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{e^{\rho_1 a}}{\rho_1-\eta_m^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_m^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S-\eta_m^+} \\
 & & & \\
e^{\rho_1 a} & e^{\rho_2 a} & \cdots & e^{\rho_S a} \\
\frac{1}{\rho_1+\eta_1^-} & \frac{1}{\rho_2+\eta_1^-} & \cdots & \frac{1}{\rho_S+\eta_1^-} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\rho_1+\eta_n^-} & \frac{1}{\rho_2+\eta_n^-} & \cdots & \frac{1}{\rho_S+\eta_n^-} \\
 & & & \\
1 & 1 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
Q_1 \\ \vdots \\ Q_m \\ Q_{m+1} \\ Q_{m+2} \\ \vdots \\ Q_{S-1} \\ Q_S
\end{bmatrix}
=
\begin{bmatrix}
-\frac{1}{\eta_1^+} \\ \vdots \\ -\frac{1}{\eta_m^+} \\ \\ 1 \\ 0 \\ \vdots \\ 0
\end{bmatrix}.
$$

We can write this system of linear equations in a matrix form as follow

$$\left[\begin{array}{cccc|c} \frac{e^{\rho_1 a}}{\rho_1-\eta_1^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_1^+} & \dots & \frac{e^{\rho_S a}}{\rho_S-\eta_1^+} & -\frac{1}{\eta_1^+} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{e^{\rho_1 a}}{\rho_1-\eta_m^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_m^+} & \dots & \frac{e^{\rho_S a}}{\rho_S-\eta_m^+} & -\frac{1}{\eta_m^+} \\ e^{\rho_1 a} & e^{\rho_2 a} & \dots & e^{\rho_S a} & 1 \\ \frac{1}{\rho_1+\eta_1^-} & \frac{1}{\rho_2+\eta_1^-} & \dots & \frac{1}{\rho_S+\eta_1^-} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\rho_1+\eta_n^-} & \frac{1}{\rho_2+\eta_n^-} & \dots & \frac{1}{\rho_S+\eta_n^-} & 0 \\ 1 & 1 & \dots & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} \frac{\rho_1 e^{\rho_1 a}}{\rho_1-\eta_1^+} & \frac{\rho_2 e^{\rho_2 a}}{\rho_2-\eta_1^+} & \dots & \frac{\rho_S e^{\rho_S a}}{\rho_S-\eta_1^+} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\rho_1 e^{\rho_1 a}}{\rho_1-\eta_m^+} & \frac{\rho_2 e^{\rho_2 a}}{\rho_2-\eta_m^+} & \dots & \frac{\rho_S e^{\rho_S a}}{\rho_S-\eta_m^+} & 0 \\ e^{\rho_1 a} & e^{\rho_2 a} & \dots & e^{\rho_S a} & 1 \\ \frac{1}{\rho_1+\eta_1^-} & \frac{1}{\rho_2+\eta_1^-} & \dots & \frac{1}{\rho_S+\eta_1^-} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\rho_1+\eta_n^-} & \frac{1}{\rho_2+\eta_n^-} & \dots & \frac{1}{\rho_S+\eta_n^-} & 0 \\ 1 & 1 & \dots & 1 & 0 \end{array}\right].$$

The second matrix is obtained by doing the matrix operations. That is, we want to make the last column become all zeros except for the $(m+1)^{th}$ entry, so for each row i from 1 to m, we multiply it by η_i^+ and then add to the $(m+1)^{th}$ row. Now, applying the Crammer's rule we obtain our desired solution for D_i^1

$$D_i^1 = Q_i = \frac{(-1)^{m+i+1} det(\boldsymbol{A_i^1})}{\sum_{j=1}^{S}(-1)^{m+j+1} e^{\rho_j a} det(\boldsymbol{A_j^1})},$$

where matrix $\boldsymbol{A^1}$ is defined as in the corollary. □

The second corollary concerns about two-sided exit problem resulted by jumping over the level a.

Corollary 3.2 *Given $x \in (0,a)$ and $q \geq 0$, then*

$$\mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau > a\}}\right] = \sum_{i=1}^{S} D_i^2 e^{\rho_i x} = \sum_{i=1}^{S} \frac{(-1)^{i+1} det(\boldsymbol{A_i^2})}{\sum_{j=1}^{S}(-1)^j \frac{\eta_1^+ e^{\rho_j a}}{\rho_j - \eta_1^+} det(\boldsymbol{A_j^2})} e^{\rho_i} x. \tag{25}$$

And hence

$$\mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau = a\}}\right] = \mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau \geq a\}}\right] - \mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau > a\}}\right] = \sum_{i=1}^{S}(D_i^1 - D_i^2)e^{\rho_i x}, \tag{26}$$

where $det(\boldsymbol{A_i^2})$ *is the determinant of the sub-matrix obtained by deleting the ith column from the following matrix*

$$\boldsymbol{A^2} = \begin{bmatrix} \frac{\rho_1 e^{\rho_1 a}(\eta_2^+ - \eta_1^+)}{(\rho_1-\eta_1^+)(\rho_1-\eta_2^+)} & \frac{\rho_2 e^{\rho_2 a}(\eta_2^+ - \eta_1^+)}{(\rho_2-\eta_1^+)(\rho_2-\eta_2^+)} & \cdots & \frac{\rho_S e^{\rho_S a}(\eta_2^+ - \eta_1^+)}{(\rho_S-\eta_1^+)(\rho_S-\eta_2^+)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\rho_1 e^{\rho_1 a}(\eta_m^+ - \eta_1^+)}{(\rho_1-\eta_1^+)(\rho_1-\eta_m^+)} & \frac{\rho_2 e^{\rho_2 a}(\eta_m^+ - \eta_1^+)}{(\rho_2-\eta_1^+)(\rho_2-\eta_m^+)} & \cdots & \frac{\rho_S e^{\rho_S a}(\eta_m^+ - \eta_1^+)}{(\rho_S-\eta_1^+)(\rho_S-\eta_m^+)} \\ e^{\rho_1 a} & e^{\rho_2 a} & \cdots & e^{\rho_S a} \\ \frac{1}{\rho_1+\eta_1^-} & \frac{1}{\rho_2+\eta_1^-} & \cdots & \frac{1}{\rho_S+\eta_1^-} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\rho_1+\eta_n^-} & \frac{1}{\rho_2+\eta_n^-} & \cdots & \frac{1}{\rho_S+\eta_n^-} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Proof. Similar to the proof of Corollary 3.1, we consider the settings $(h_1, h_2) = (0, a)$ and $g(x) = \mathbf{1}_{\{x>a\}}$, then we have

$$\boldsymbol{V}_g = (-\frac{1}{\eta_1^+}, -\frac{1}{\eta_2^+}, ..., -\frac{1}{\eta_m^+}, 0, ..., 0)^T.$$

Then the system of linear equations: $\boldsymbol{AQ} = \boldsymbol{V}_g$ can be reduced as following

$$\left[\begin{array}{cccc|c} \frac{e^{\rho_1 a}}{\rho_1-\eta_1^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_1^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S-\eta_1^+} & \frac{-1}{\eta_1^+} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{e^{\rho_1 a}}{\rho_1-\eta_m^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_m^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S-\eta_m^+} & \frac{-1}{\eta_m^+} \\ e^{\rho_1 a} & e^{\rho_2 a} & \cdots & e^{\rho_S a} & 0 \\ \frac{1}{\rho_1+\eta_1^-} & \frac{1}{\rho_2+\eta_1^-} & \cdots & \frac{1}{\rho_S+\eta_1^-} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\rho_1+\eta_n^-} & \frac{1}{\rho_2+\eta_n^-} & \cdots & \frac{1}{\rho_S+\eta_n^-} & 0 \\ 1 & 1 & \cdots & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} \frac{-\eta_1^+ e^{\rho_1 a}}{\rho_1-\eta_1^+} & \frac{-\eta_1^+ e^{\rho_2 a}}{\rho_2-\eta_1^+} & \cdots & \frac{-\eta_1^+ e^{\rho_S a}}{\rho_S-\eta_1^+} & 1 \\ \frac{\rho_1 e^{\rho_1 a}(\eta_2^+ - \eta_1^+)}{(\rho_1-\eta_1^+)(\rho_1-\eta_2^+)} & \frac{\rho_2 e^{\rho_2 a}(\eta_2^+ - \eta_1^+)}{(\rho_2-\eta_1^+)(\rho_2-\eta_2^+)} & \cdots & \frac{\rho_S e^{\rho_S a}(\eta_2^+ - \eta_1^+)}{(\rho_S-\eta_1^+)(\rho_S-\eta_2^+)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\rho_1 e^{\rho_1 a}(\eta_m^+ - \eta_1^+)}{(\rho_1-\eta_1^+)(\rho_1-\eta_m^+)} & \frac{\rho_2 e^{\rho_2 a}(\eta_m^+ - \eta_1^+)}{(\rho_2-\eta_1^+)(\rho_2-\eta_m^+)} & \cdots & \frac{\rho_S e^{\rho_S a}(\eta_m^+ - \eta_1^+)}{(\rho_S-\eta_1^+)(\rho_S-\eta_m^+)} & 0 \\ e^{\rho_1 a} & e^{\rho_2 a} & \cdots & e^{\rho_S a} & 0 \\ \frac{1}{\rho_1+\eta_1^-} & \frac{1}{\rho_2+\eta_1^-} & \cdots & \frac{1}{\rho_S+\eta_1^-} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\rho_1+\eta_n^-} & \frac{1}{\rho_2+\eta_n^-} & \cdots & \frac{1}{\rho_S+\eta_n^-} & 0 \\ 1 & 1 & \cdots & 1 & 0 \end{array}\right].$$

Here, the second matrix is obtained by first multiplying the first row by $-\eta_1^+$ and the others from 2^{th} row to m^{th} row by η_i^+. Then, for each row from 2 to m, we add it to the first row. Now, applying the Cramer's rule we obtain our desired solution for D_i^2

$$D_i^2 = Q_i = \frac{(-1)^{i+1} det(\boldsymbol{A_i^2})}{\sum_{j=1}^{S} (-1)^j \frac{\eta_1^+ e^{\rho_j a}}{\rho_j - \eta_1^+} det(\boldsymbol{A_j^2})},$$

where $det(\boldsymbol{A_i^2})$ is defined in the corollary. □

The following corallary considers the density of overshoot in the exit problem resulted by jumping over level a.

Corollary 3.3 *Given $x \in (0, a)$, $q \geq 0$, and $s > 0$, then*

$$\mathbb{E}_x \left[e^{-q\tau} \mathbf{1}_{\{X_\tau > a+s\}} \right] = \frac{1}{det(\boldsymbol{A})} \sum_{i=1}^{S} \sum_{j=1}^{m} (-1)^{i+j+1} \frac{e^{-\eta_j^+ s}}{\eta_j^+} det(\boldsymbol{A_j^i}) e^{\rho_i x}. \tag{27}$$

And hence

$$\mathbb{E}_x \left[e^{-q\tau}, X_\tau - a \in ds \right] = \sum_{i=1}^{S} \sum_{j=1}^{m} D_{i,j} e^{-\eta_j^+ s} e^{\rho_i x} = \frac{1}{det(\boldsymbol{A})} \sum_{i=1}^{S} \sum_{j=1}^{m} (-1)^{i+j+1} det(\boldsymbol{A_j^i}) e^{-\eta_j^+ s} e^{\rho_i x}, \tag{28}$$

where matrix $\boldsymbol{A}$ is defined as

$$\boldsymbol{A} = \begin{bmatrix} \frac{e^{\rho_1 a}}{\rho_1 - \eta_1^+} & \frac{e^{\rho_2 a}}{\rho_2 - \eta_1^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S - \eta_1^+} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{e^{\rho_1 a}}{\rho_1 - \eta_m^+} & \frac{e^{\rho_2 a}}{\rho_2 - \eta_m^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S - \eta_m^+} \\ e^{\rho_1 a} & e^{\rho_2 a} & \cdots & e^{\rho_S a} \\ \frac{1}{\rho_1 + \eta_1^-} & \frac{1}{\rho_2 + \eta_1^-} & \cdots & \frac{1}{\rho_S + \eta_1^-} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\rho_1 + \eta_n^-} & \frac{1}{\rho_2 + \eta_n^-} & \cdots & \frac{1}{\rho_S + \eta_n^-} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

and $det(\boldsymbol{A_j^i})$ *is the determinant of sub-matrix obtained by deleting the ith column and the jth row of matrix* $\boldsymbol{A}$*.*

Proof. Again with the settings $(h_1, h_2) = (0, a)$ and $g(x) = \mathbf{1}_{x>a+s}$ for $s \geq 0$, it is easy to check that

$$\boldsymbol{V}_g = (-\frac{e^{-\eta_i^+ s}}{\eta_i^+}, -\frac{e^{-\eta_2^+ s}}{\eta_2^+}, ..., -\frac{e^{-\eta_m^+ s}}{\eta_m^+}, 0, ..., 0)^T.$$

Then the system of equations: $\boldsymbol{AQ} = \boldsymbol{V}_g$ can be written as

$$\begin{bmatrix} \frac{e^{\rho_1 a}}{\rho_1-\eta_1^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_1^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S-\eta_1^+} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{e^{\rho_1 a}}{\rho_1-\eta_m^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_m^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S-\eta_m^+} \\ e^{\rho_1 a} & e^{\rho_2 a} & \cdots & e^{\rho_S a} \\ \frac{1}{\rho_1+\eta_1^-} & \frac{1}{\rho_2+\eta_1^-} & \cdots & \frac{1}{\rho_S+\eta_1^-} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\rho_1+\eta_n^-} & \frac{1}{\rho_2+\eta_n^-} & \cdots & \frac{1}{\rho_S+\eta_n^-} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} Q_1 \\ \vdots \\ Q_m \\ Q_{m+1} \\ Q_{m+2} \\ \vdots \\ Q_{S-1} \\ Q_S \end{bmatrix} = \begin{bmatrix} -\frac{e^{-\eta_1^+ s}}{\eta_1^+} \\ \vdots \\ -\frac{e^{-\eta_m^+ s}}{\eta_m^+} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Using Cramer's rule we obtain

$$Q_i = \frac{1}{det(\boldsymbol{A})} \sum_{j=1}^{m} (-1)^{i+j+1} \frac{e^{-\eta_j^+ s}}{\eta_j^+} det(\boldsymbol{A_j^i}).$$

□

Using the same idea, one can easily show the following three corollaries regarding to two-sided exit problems from lower level 0.

Corollary 3.4 *Given $x \in (0,a)$ and $q \geq 0$, then*

$$\mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau \leq 0\}}\right] = \sum_{i=1}^{S} C_i^1 e^{\rho_i x} = \sum_{i=1}^{S} \frac{(-1)^{S+i} det(\boldsymbol{A_i^3})}{\sum_{j=1}^{S}(-1)^{S+j} det(\boldsymbol{A_j^3})} e^{\rho_i} x, \tag{29}$$

where $det(\boldsymbol{A_i^3})$ is the determinant of the sub-matrix obtained by deleting the ith column of the following matrix

$$\boldsymbol{A^3} = \begin{bmatrix} \frac{e^{\rho_1 a}}{\rho_1-\eta_1^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_1^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S-\eta_1^+} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{e^{\rho_1 a}}{\rho_1-\eta_m^+} & \frac{e^{\rho_2 a}}{\rho_2-\eta_m^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S-\eta_m^+} \\ \\ e^{\rho_1 a} & e^{\rho_2 a} & \cdots & e^{\rho_S a} \\ \frac{\rho_1}{\rho_1+\eta_1^-} & \frac{\rho_2}{\rho_2+\eta_1^-} & \cdots & \frac{\rho_S}{\rho_S+\eta_1^-} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\rho_1}{\rho_1+\eta_n^-} & \frac{\rho_2}{\rho_2+\eta_n^-} & \cdots & \frac{\rho_S}{\rho_S+\eta_n^-} \end{bmatrix}.$$

Proof. The proof is similar to one above with the settings $(h_1, h_2) = (0, a)$ and $g(x) = \mathbf{1}_{\{x \leq 0\}}$. □

Corollary 3.5 *Given $x \in (0,a)$ and $q \geq 0$, then*

$$\mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau < 0\}}\right] = \sum_{i=1}^{S} C_i^2 e^{\rho_i x} = \sum_{i=1}^{S} \frac{(-1)^{m+i} det(\boldsymbol{A_i^4})}{\sum_{j=1}^{S}(-1)^{m+j} \frac{\eta_1^-}{\rho_j+\eta_1^-} det(\boldsymbol{A_j^4})} e^{\rho_i} x. \tag{30}$$

And hence

$$\mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau = 0\}}\right] = \mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau \leq 0\}}\right] - \mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau < 0\}}\right] = \sum_{i=1}^{S}(C_i^1 - C_i^2)e^{\rho_i x}, \tag{31}$$

where $det(\boldsymbol{A_i^4})$ is the determinant of the sub-matrix obtained by deleting the ith column from *the following matrix*

$$\boldsymbol{A^4} = \begin{bmatrix} \frac{e^{\rho_1 a}}{\rho_1 - \eta_1^+} & \frac{e^{\rho_2 a}}{\rho_2 - \eta_1^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S - \eta_1^+} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{e^{\rho_1 a}}{\rho_1 - \eta_m^+} & \frac{e^{\rho_2 a}}{\rho_2 - \eta_m^+} & \cdots & \frac{e^{\rho_S a}}{\rho_S - \eta_m^+} \\ \\ e^{\rho_1 a} & e^{\rho_2 a} & \cdots & e^{\rho_S a} \\ \frac{\rho_1(\eta_1^- - \eta_2^-)}{(\rho_1+\eta_1^-)(\rho_1+\eta_2^-)} & \frac{\rho_2(\eta_1^- - \eta_2^-)}{(\rho_2+\eta_1^-)(\rho_2+\eta_2^-)} & \cdots & \frac{\rho_S(\eta_1^- - \eta_2^-)}{(\rho_S+\eta_1^-)(\rho_S+\eta_2^-)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\rho_1(\eta_1^- - \eta_n^-)}{(\rho_1+\eta_1^-)(\rho_1+\eta_n^-)} & \frac{\rho_2(\eta_1^- - \eta_n^-)}{(\rho_2+\eta_1^-)(\rho_2+\eta_n^-)} & \cdots & \frac{\rho_S(\eta_1^- - \eta_n^-)}{(\rho_S+\eta_1^-)(\rho_S+\eta_n^-)} \\ \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Proof. The result can be obtained with the settings $(h_1, h_2) = (0, a)$ and $g(x) = \mathbf{1}_{\{x<0\}}$. □

Corollary 3.6 *Given $x \in (0, a)$, $q \geq 0$, and $s < 0$, then*

$$\mathbb{E}_x\left[e^{-q\tau}\mathbf{1}_{\{X_\tau < s\}}\right] = \frac{1}{det(\boldsymbol{A})}\sum_{i=1}^{S}\sum_{j=1}^{n}(-1)^{m+i+j+1}\frac{e^{\eta_j^- s}}{\eta_j^-}det(\boldsymbol{A_j^i})e^{\rho_i x}. \tag{32}$$

And hence

$$\mathbb{E}_x\left[e^{-q\tau}, X_\tau \in ds\right] = \sum_{i=1}^{S}\sum_{j=1}^{n}C_{i,j}e^{\eta_j^- s}e^{\rho_i x} = \frac{1}{det(\boldsymbol{A})}\sum_{i=1}^{S}\sum_{j=1}^{n}(-1)^{m+i+j}det(\boldsymbol{A_j^i})e^{\eta_j^- s}e^{\rho_i x}, \tag{33}$$

where matrix $\boldsymbol{A}$ is defined as in the corollary 3.3 and $det(\boldsymbol{A_j^i})$ is the determinant of submatrix obtained by deleting the ith column and the $(m+j+1)^{th}$ row of matrix $\boldsymbol{A}$.

Proof. The proof is similar to the one in Corollary 3.3 with the settings $(h_1, h_2) = (0, a)$ and $g(x) = \mathbf{1}_{\{x<s\}}$. □

3.2 One-sided exit problems

As mentioned in the introduction, Kyprianou et al (2012) also proposed the density of one-sided exit problems for the Meromorphic processes. Also, we have induced the expression for our hyper-exponential jump-diffusion processes. However, this density cannot be used to obtain the one-sided exit problem resulted by creeping over that level. Therefore, we want to use another appoarch to derive solutions to the one-sided exit problems.

The one-sided exit problems for the special case of the double exponential jump-diffusion process can be found in Kou and Wang (2003). To expand the result to the hyper-exponential jump-diffusion processes, we first derive the following theorem.

Theorem 3.1 *Given $x \in (-\infty, h_2)$, $h_2 \geq 0$, $g(x)$ is a nonnegative bounded measureable function, and let $\tau = \inf\{t \geq 0 : X_t \notin (-\infty, h_2]\}$, then the solution of the following boundary value problem*

$$\begin{cases} (\mathcal{L} - q)\Phi = 0 & \text{in } (-\infty, h_2), \\ \Phi = g & \text{on } [h_2, \infty), \end{cases}$$

is $\Phi(x) = \sum_{i=1}^{m+1} Q_i e^{\rho_i x}$ for the constant vector $\boldsymbol{Q} = [Q_1, Q_2, ..., Q_{m+1}]^T$ satisfies the equation

$$\boldsymbol{B}\boldsymbol{Q} = \boldsymbol{V}_g,$$

where the matrix $\boldsymbol{B}$ and elements of vector $\boldsymbol{V}_g$ are defined as

$$\boldsymbol{B} = \begin{bmatrix} \frac{e^{\rho_1 h_2}}{\rho_1 - \eta_1^+} & \cdots & \frac{e^{\rho_{m+1} h_2}}{\rho_{m+1} - \eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{e^{\rho_1 h_2}}{\rho_1 - \eta_m^+} & \cdots & \frac{e^{\rho_{m+1} h_2}}{\rho_{m+1} - \eta_m^+} \\ \\ e^{\rho_1 h_2} & \cdots & e^{\rho_{m+1} h_2} \end{bmatrix}, \qquad \boldsymbol{V}_g(i) = \begin{cases} -\int_{h_2}^{\infty} g(y) e^{-\eta_j^+ (y - h_2)} dy, & if\ 1 \leq i \leq m, \\ g(h_2), & if\ i = m+1 \end{cases}.$$

Here we assume the matrix $\boldsymbol{B}$ is invertible.

Proof. By mimicking the proof of Proposition 2.2 in [11], since $(\mathcal{L}-q)\Phi = 0$ on $(-\infty, h_2)$ and $\Phi(x) = \sum_{i=1}^{m+1} Q_i e^{\rho_i x}$, then for $x \in (-\infty, h_2)$, we have

$$\begin{aligned} 0 &= \frac{\sigma^2}{2}\Phi''(x) + c\Phi'(x) + \lambda \int \Phi(x+y)f(y)dy - (\lambda+q)\Phi(x) \\ &= \sum_{i=1}^{m+1} Q_i e^{\rho_i x}\left(\frac{\sigma^2}{2}\rho_i^2 + c\rho_i - \lambda - q\right) + \lambda \int \Phi(x+y)f(y)dy. \end{aligned} \tag{34}$$

Also,

$$\begin{aligned} &\int \Phi(x+y)f(y)dy \\ &= \int_{-\infty}^{\infty} \Phi(u)f(u-x)du \\ &= \int_{-\infty}^{h_2} \Phi(u)f(u-x)du + \int_{h_2}^{\infty} \Phi(u)f(u-x)du \\ &= \int_{-\infty}^{h_2-x} \Phi(x+y)f(y)dy + \int_{h_2}^{\infty} g(u)\sum_{j=1}^{m} p_j\eta_j^+ e^{-\eta_j^+(u-x)}du \\ &= \int_{-\infty}^{0} \Phi(x+y)f(y)dy + \int_{0}^{h_2-x} \Phi(x+y)f(y)dy + \sum_{j=1}^{m} p_j\eta_j^+ e^{\eta_j^+ x}\int_{h_2}^{\infty} g(y)e^{-\eta_j^+ y}dy \\ &= \int_{-\infty}^{0} \sum_{i=1}^{m+1} Q_i e^{\rho_i(x+y)} \sum_{j=1}^{n} q_j\eta_j^- e^{\eta_j^- y}dy + \int_{0}^{h_2-x} \sum_{i=1}^{m+1} Q_i e^{\rho_i(x+y)} \sum_{j=1}^{m} p_j\eta_j^+ e^{-\eta_j^+ y}dy \\ &+ \sum_{j=1}^{m} p_j\eta_j^+ e^{\eta_j^+ x}\int_{h_2}^{\infty} g(y)e^{-\eta_j^+ y}dy \\ &= \sum_{i=1}^{m+1} Q_i e^{\rho_i x} \sum_{j=1}^{n} q_j\eta_j^- \int_{-\infty}^{0} e^{(\rho_i+\eta_j^-)y}dy + \sum_{i=1}^{m+1} Q_i e^{\rho_i x} \sum_{j=1}^{m} p_j\eta_j^+ \int_{0}^{h_2-x} e^{(\rho_i-\eta_j^+)y}dy \\ &+ \sum_{j=1}^{m} p_j\eta_j^+ e^{\eta_j^+ x}\int_{h_2}^{\infty} g(y)e^{-\eta_j^+ y}dy \\ &= \sum_{i=1}^{m+1} Q_i e^{\rho_i x} \sum_{j=1}^{n} \frac{q_j\eta_j^-}{\rho_i+\eta_j^-} + \sum_{i=1}^{m+1} Q_i e^{\rho_i x} \sum_{j=1}^{m} \frac{p_j\eta_j^+}{\rho_i-\eta_j^+}\left(e^{(\rho_i-\eta_j^+)(h_2-x)} - 1\right) \\ &+ \textstyle\sum_{j=1}^{m} p_j\eta_j^+ e^{\eta_j^+ x}\int_{h_2}^{\infty} g(y)e^{-\eta_j^+ y}dy. \end{aligned} \tag{35}$$

From (34) and (35), we have

$$0 = \sum_{i=1}^{m+1} Q_i e^{\rho_i x}\left(\frac{\sigma^2}{2}\rho_i^2 + c\rho_i - \lambda - + \lambda\left[\sum_{j=1}^{n}\frac{q_j\eta_j^-}{\rho_i+\eta_j^-} + \sum_{j=1}^{m}\frac{p_j\eta_j^+}{\rho_i-\eta_j^+}\right]\right) + \lambda\sum_{j=1}^{m} p_j\eta_j^+ e^{\eta_j^+ x}\int_{h_2}^{\infty} g(y)e^{-\eta_j^+ y}dy$$

$$+\lambda\sum_{i=1}^{m+1} Q_i e^{\rho_i x}\sum_{j=1}^{m}\frac{p_j\eta_j^+}{\rho_i-\eta_j^+}e^{(\rho_i-\eta_j^+)(h_2-x)}.$$

Using the fact that $\psi(\rho_i) - q = 0$ for all i we obtain

$$\sum_{i=1}^{m+1} Q_i e^{\rho_i x}\sum_{j=1}^{m}\frac{p_j\eta_j^+}{\rho_i-\eta_j^+}e^{(\rho_i-\eta_j^+)(h_2-x)} = -\sum_{j=1}^{m} p_j\eta_j^+ e^{\eta_j^+ x}\int_{h_2}^{\infty} g(y)e^{-\eta_j^+ y}dy$$

which implies
$$\sum_{j=1}^{m} p_j\eta_j^+ e^{\eta_j^+ x}\sum_{i=1}^{m+1} Q_i\frac{e^{\rho_i h_2}}{\rho_i-\eta_j^+} = \sum_{j=1}^{m} p_j\eta_j^+ e^{\eta_j^+ x}\int_{h_2}^{\infty} -g(y)e^{-\eta_j^+(y-h_2)}dy.$$

This can be rewritten in a matrix form $\boldsymbol{BQ} = \boldsymbol{V}_g$ defined as in the theorem.

□

Theorem 3.2 *Given $q \geq 0$ and also a nonnegative bounded function g on $(-\infty, h_2)^c$, and*

$$\tau_{h_2}^+ = \inf\{t \geq 0 : X_t \notin (-\infty, h_2]\}.$$

Then

$$\mathbb{E}_x\left[e^{-q\tau_{h_2}^+} g(V_{\tau_{h_2}^+})\right] = \sum_{i=1}^{m+1} Q_i e^{\rho_i x}. \tag{34}$$

Proof. The proof is similar to the proof of Theorem 4.1 in [13] by replacing $\mathbb{R}_+$ with $(-\infty, h_2]$.

□

Now, we are ready to obtain the one-sided exit problems for hyper-exponential jump-diffusiion processes. The following three corollaries are on the one-sided exit problems from the upper level 0.

Corollary 3.7 *Given $x \in (-\infty, 0)$, then*

$$\mathbb{E}_x\left[e^{-q\tau_0^+}, X_{\tau_0^+} \geq 0\right] = \sum_{i=1}^{m+1} d_i^1 e^{\rho_i x} = \sum_{i=1}^{m+1} \frac{(-1)^{m+i+1} det(\boldsymbol{B_i^1})}{\sum_{j=1}^{m+1}(-1)^{m+j+1} det(\boldsymbol{B_j^1})} e^{\rho_i x}, \tag{35}$$

where $det(\boldsymbol{B_i^1})$ is the determinant of sub-matrix obtained by deleting the ith column from the following matrix

$$\boldsymbol{B^1} = \begin{bmatrix} \frac{\rho_1}{\rho_1-\eta_1^+} & \cdots & \frac{\rho_{m+1}}{\rho_{m+1}-\eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{\rho_1}{\rho_1-\eta_n^+} & \cdots & \frac{\rho_{m+1}}{\rho_{m+1}-\eta_n^+} \end{bmatrix}.$$

Proof. The proof is similar to the case of two-sided exit problems using the Theorems 3.1 and 3.2 above with the settings $h_2 = 0$ and $g(x) = \mathbf{1}_{(x\geq 0)}$. □

Corollary 3.8 *Given $x \in (-\infty, 0)$, then*

$$\mathbb{E}_x\left[e^{-q\tau_0^+}, X_{\tau_0^+} > 0\right] = \sum_{i=1}^{m+1} d_i^2 e^{\rho_i x} = \sum_{i=1}^{m+1} \frac{(-1)^{i+1} det(\boldsymbol{B_i^2})}{\sum_{j=1}^{m+1}(-1)^{j+1} \frac{\eta_1^+}{\rho_j-\eta_1^+} det(\boldsymbol{B_j^2})} e^{\rho_i x}. \tag{36}$$

And hence

$$\mathbb{E}_x\left[e^{-q\tau_0^+}, X_{\tau_0^+} = 0\right] = \mathbb{E}_x\left[e^{-q\tau_0^+}, X_{\tau_0^+} \geq 0\right] - \mathbb{E}_x\left[e^{-q\tau_0^+}, X_{\tau_0^+} > 0\right] = \sum_{i=1}^{m+1}(d_i^1 - d_i^2)e^{\rho_i x}, \tag{37}$$

where $det(\boldsymbol{B_i^2})$ is the determinant of sub-matrix obtained by deleting the ith column from the following matrix

$$\boldsymbol{B^2} = \begin{bmatrix} \frac{\rho_1(\eta_1^+-\eta_2^+)}{(\rho_1-\eta_1^+)(\rho_1-\eta_2^+)} & \cdots & \frac{\rho_{m+1}(\eta_1^+-\eta_2^+)}{(\rho_{m+1}-\eta_1^+)(\rho_{m+1}-\eta_2^+)} \\ \vdots & \ddots & \vdots \\ \frac{\rho_1(\eta_1^+-\eta_n^+)}{(\rho_1-\eta_1^+)(\rho_1-\eta_n^+)} & \cdots & \frac{\rho_{m+1}(\eta_1^+-\eta_n^+)}{(\rho_{m+1}-\eta_1^+)(\rho_{m+1}-\eta_n^+)} \\ \\ 1 & \cdots & 1 \end{bmatrix}.$$

Proof. With the settings $h_2 = 0$ and $g(x) = \mathbf{1}_{(x>0)}$, we can obtain our desired result. □

Corollary 3.9 *Given $x \in (-\infty, 0), s > 0$, then*

$$\mathbb{E}_x\left[e^{-q\tau_0^+}, X_{\tau_0^+} > s\right] = \frac{1}{det(\boldsymbol{B})}\sum_{i=1}^{m+1}\sum_{j=1}^{m}\frac{e^{-\eta_j^+ s}}{\eta_j^+}(-1)^{i+j+1}det(\boldsymbol{B_j^i})e^{\rho_i x}.$$

Thus

$$\mathbb{E}_x\left[e^{-q\tau_0^+}, X_{\tau_0^+} \in ds\right] = \frac{1}{det(\boldsymbol{B})}\sum_{i=1}^{m+1}\sum_{j=1}^{m}(-1)^{i+j+1}det(\boldsymbol{B_j^i})e^{-\eta_j^+ s}e^{\rho_i x}ds = \sum_{i=1}^{m+1}\sum_{j=1}^{m}d_{i,j}e^{\eta_j^+}e^{\rho_i x}ds,$$

where $det(\boldsymbol{B_j^i})$ is the determinant of sub-matrix obtained by deleting the jth row and the ith column from the following matrix

$$\boldsymbol{B} = \begin{bmatrix} \frac{1}{\rho_1-\eta_1^+} & \cdots & \frac{1}{\rho_{m+1}-\eta_1^+} \\ \vdots & \ddots & \vdots \\ \frac{1}{\rho_1-\eta_n^+} & \cdots & \frac{1}{\rho_{m+1}-\eta_n^+} \\ \\ 1 & \cdots & 1 \end{bmatrix}.$$

Proof. In this case, the settings are $h_2 = 0$ and $g(x) = \mathbf{1}_{(x>s)}$. □

Moreover, one can easily derive the following theorem using the same method as in the proof of Theorems 3.1 and 3.2. For the one-sided exit problems from below, we have the similar resulting theorem.

Theorem 3.3 *Given $x \in (h_1, \infty)$, let g, defined on $[h_1, \infty)$ for $h_1 \leq 0$, be a nonnegative bounded measurable function, and let $\tau = \inf\{t \geq 0 : X_t \notin [h_1, \infty)\}$. Then the solution of the following boundary value problem*

$$\begin{cases} (\mathcal{L} - q)\Phi = 0 & in\ (h_1, \infty), \\ \Phi = g & on\ (-\infty, h_1], \end{cases}$$

is $\quad \Phi(x) = \mathbb{E}_x\Big[e^{-q\tau_{h_1}^-} g(X_{\tau_{h_1}^-})\Big] = \sum_{i=m+2}^{S} Q_i e^{\rho_i x}.$

for the constant vector $\boldsymbol{Q}$ *satisfies the equation*

$$\boldsymbol{C}\boldsymbol{Q} = \boldsymbol{V}_g,$$

where the invertible matrix $\boldsymbol{C}$ *and elements of vector* $\boldsymbol{V}_g$ *are defined as*

$$\boldsymbol{C} = \begin{bmatrix} \frac{e^{\rho_{m+2}h_1}}{\rho_{m+2}+\eta_1^-} & \cdots & \frac{e^{\rho_S h_1}}{\rho_S+\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{e^{\rho_{m+2}h_1}}{\rho_{m+2}+\eta_n^-} & \cdots & \frac{e^{\rho_S h_1}}{\rho_S+\eta_n^-} \\ \\ e^{\rho_{m+2}h_1} & \cdots & e^{\rho_S h_1} \end{bmatrix}, \qquad \boldsymbol{V}_g(i) = \begin{cases} \int_{-\infty}^{h_1} g(y) e^{\eta_j^-(y-h_1)} dy, & if\ 1 \le i \le n, \\ g(h_2), & if\ i = n+1, \end{cases}.$$

Now, we are ready to state the following three corollaries concerning about one-sided exit problems from lower level 0.

Corollary 3.10 *Given* $x \in (0,\infty)$, *then*

$$\mathbb{E}_x\Big[e^{-q\tau_0^-}, X_{\tau_0^-} \le 0\Big] = \sum_{i=m+2}^{S} c_i^1 e^{\rho_i x} = \sum_{i=m+2}^{S} \frac{(-1)^{S-i} det(\boldsymbol{C^1_{i-m-1}})}{\sum_{j=m+2}^{S} (-1)^{S-j} det(\boldsymbol{C^1_{j-m-1}})} e^{\rho_i x}, \tag{38}$$

where $det(\boldsymbol{C^1_{i-m-1}})$ *is the determinant of sub-matrix obtained by deleting the ith column of the matrix*

$$\boldsymbol{C^1} = \begin{bmatrix} \frac{\rho_{m+2}}{\rho_{m+2}+\eta_1^-} & \cdots & \frac{\rho_S}{\rho_S+\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{\rho_{m+2}}{\rho_{m+2}+\eta_n^-} & \cdots & \frac{\rho_S}{\rho_S+\eta_n^-} \\ \\ \end{bmatrix}.$$

Proof. Apply the Theorem 3.3 above for $h_1 = 0$ and $g(x) = \mathbf{1}_{(x \le 0)}$. □

Corollary 3.11 *Given $x \in (0,\infty)$, then*

$$\mathbb{E}_x\left[e^{-q\tau_0^-}, X_{\tau_0^-} < 0\right] = \sum_{i=m+2}^{S} c_i^2 e^{\rho_i x} = \sum_{i=m+2}^{S} \frac{(-1)^{i-m} det(\boldsymbol{C^2_{i-m-1}})}{\sum_{j=m+2}^{S}(-1)^{j-m}\frac{\eta_1^-}{\rho_j+\eta_1^-} det(\boldsymbol{C^2_{j-m-1}})} e^{\rho_i x}. \quad (39)$$

And hence

$$\mathbb{E}_x\left[e^{-q\tau_0^-}, X_{\tau_0^-} = 0\right] = \mathbb{E}_x\left[e^{-q\tau_0^-}, X_{\tau_0^-} \le 0\right] - \mathbb{E}_x\left[e^{-q\tau_0^-}, X_{\tau_0^-} < 0\right] = \sum_{i=m+2}^{S} (c_i^1 - c_i^2) e^{\rho_i x}, \quad (40)$$

where $det(\boldsymbol{C^2_{i-m-1}})$ is the determinant of sub-matrix obtained by deleting the ith column of the matrix

$$\boldsymbol{C^2} = \begin{bmatrix} \frac{\rho_{m+2}(\eta_1^- - \eta_2^-)}{(\rho_{m+2}+\eta_1^-)(\rho_{m+2}+\eta_2^-)} & \cdots & \frac{\rho_S(\eta_1^- - \eta_2^-)}{(\rho_S+\eta_1^-)(\rho_S+\eta_2^-)} \\ \vdots & \ddots & \vdots \\ \frac{\rho_{m+2}(\eta_1^- - \eta_n^-)}{(\rho_{m+2}+\eta_1^-)(\rho_{m+2}+\eta_n^-)} & \cdots & \frac{\rho_S(\eta_1^- - \eta_n^-)}{(\rho_S+\eta_1^-)(\rho_S+\eta_n^-)} \\ & & \\ 1 & \cdots & 1 \end{bmatrix}.$$

Proof. Again, with the settings $h_1 = 0$ and $g(x) = \mathbf{1}_{(x<0)}$. □

Corollary 3.12 *Given $x \in (0,\infty), s < 0$, then*

$$\mathbb{E}_x\left[e^{-q\tau_0^-}, X_{\tau_0^-} < s\right] = \frac{1}{det(\boldsymbol{C})} \sum_{i=m+2}^{S} \sum_{j=1}^{n} \frac{e^{\eta_j^- s}}{\eta_j^-} (-1)^{i+j-m+1} det(\boldsymbol{C_j^{i-m-1}}) e^{\rho_i x}, \quad (41)$$

$$\mathbb{E}_x\left[e^{-q\tau_0^-}, X_{\tau_0^-} \in ds\right] = \frac{1}{det(\boldsymbol{C})} \sum_{i=m+2}^{S} \sum_{j=1}^{n} e^{\eta_j^- s} (-1)^{i+j-m} det(\boldsymbol{C_j^{i-m-1}}) e^{\rho_i x} ds \quad (42)$$

$$= \sum_{i=m+2}^{S} \sum_{j=1}^{n} c_{i,j} e^{\eta_j^- s} e^{\rho_i x} ds.$$

where $det(\boldsymbol{C_j^{i-m-1}})$ is the determinant of sub-matrix obtained by deleting the jth row and *the ith column from the following matrix*

$$\boldsymbol{C} = \begin{bmatrix} \frac{1}{\rho_{m+2}+\eta_1^-} & \cdots & \frac{1}{\rho_S+\eta_1^-} \\ \vdots & \ddots & \vdots \\ \frac{1}{\rho_{m+2}+\eta_n^-} & \cdots & \frac{1}{\rho_S+\eta_n^-} \\ \\ 1 & \cdots & 1 \end{bmatrix}.$$

Proof. Here, the settings are $h_1 = 0$ and $g(x) = \mathbf{1}_{(x<s)}$. □

3.3 Potential measures of jump-diffusion processes

Recall that for $q \geq 0$, a q-potential measure of any Lévy process X killed on exiting $[0, a]$ when issued from x is defined as

$$U^{(q)}(a, x, dy) := \int_0^\infty e^{-qt} \mathbb{P}_x\big(X_t \in dy, \tau > t\big) dt,$$

and a q-potential measure of X killed on exiting $[0, \infty)$ when issued from x as

$$R^{(q)}(x, dy) := \int_0^\infty e^{-qt} \mathbb{P}_x\big(X_t \in dy, \tau_0^- > t\big) dt,$$

where

$$\tau = \tau_a^+ \wedge \tau_0^- := \inf\{t \geq 0 : X_t < 0 \text{ or } X_t > a\}.$$

Although the potential measure of Meromorphic processes is given in Kyprianou et al (2012), that expression is not very handy because it is not explicit and we have to solve the system of linear matrix equations. In this section, we first want to derive the potential measure for double exponential jump-diffusion processes. Then we generalize the idea to the case of hyper-exponential. The approach in Kyprianou et al (2012) and our approach are quite similar except that in their approach, to derive $U^{(q)}(a, x, dy)$ they used the q-potential measure killed on exiting $(-\infty, a]$ and the discounted joint overshoot and undershoot distribution for the two-sided exit problem.

Theorem 3.4 *Suppose, for $q \geq 0$, $R^{(q)}(x, dy)$ is the q-potential measure of the double exponential jump-diffusion process X_t killed on exiting $[0, \infty)$, where $x, y \in [0, \infty)$. Then it has a density $qr^{(q)}(x, y)$ given by*

$$C_1(e^{\rho_3(x-y)} - e^{\rho_3 x - \rho_1 y}) + C_2(e^{\rho_4(x-y)} - e^{\rho_4 x - \rho_1 y}) + C_3(e^{\rho_3(x-y)} - e^{\rho_3 x - \rho_2 y}) + C_4(e^{\rho_4(x-y)} - e^{\rho_4 x - \rho_2 y}),$$

where

$$\begin{aligned}
C_1 &:= \left(1-\frac{\rho_1}{\eta_1^+}\right)\left(\frac{\rho_2}{\rho_1-\rho_2}\right)\left(1+\frac{\rho_3}{\eta_1^-}\right)\left(\frac{\rho_4}{\rho_4-\rho_3}\right)\left(\frac{1}{\rho_1-\rho_3}\right)\rho_3\rho_1,\\
C_2 &:= \left(1-\frac{\rho_1}{\eta_1^+}\right)\left(\frac{\rho_2}{\rho_1-\rho_2}\right)\left(\frac{\eta_1^-+\rho_4}{\eta_1^-}\right)\left(\frac{\rho_3}{\rho_3-\rho_4}\right)\left(\frac{1}{\rho_1-\rho_4}\right)\rho_1\rho_4,\\
C_3 &:= \left(\frac{\eta_1^+-\rho_2}{\eta_1^+}\right)\left(\frac{\rho_1}{\rho_2-\rho_1}\right)\left(1+\frac{\rho_3}{\eta_1^-}\right)\left(\frac{\rho_4}{\rho_4-\rho_3}\right)\left(\frac{1}{\rho_2-\rho_3}\right)\rho_2\rho_3,\\
C_4 &:= \left(\frac{\eta_1^+-\rho_2}{\eta_1^+}\right)\left(\frac{\rho_1}{\rho_2-\rho_1}\right)\left(\frac{\eta_1^-+\rho_4}{\eta_1^-}\right)\left(\frac{\rho_3}{\rho_3-\rho_4}\right)\left(\frac{1}{\rho_2-\rho_4}\right)\rho_4\rho_2.
\end{aligned}$$

Proof. For $x \in [0,\infty)$, we have

$$R^{(q)}(x,dy) = \int_0^\infty e^{-qt}\mathbb{P}_x\big(X_t \in dy, \tau_0^- > t\big)dt = \frac{1}{q}\mathbb{P}_x\big(X_{e_q} \in dy, \underline{X}_{e_q} \geq 0\big),$$

where the second equality followed by conditioning on e_q. Using translation invariant of X_t and the independence of $X_{e_q} - \underline{X}_{e_q}$ and $\underline{X}_{e_q}$ by Wiener-Hopf factorization theorem, we obtain

$$\begin{aligned}
R^{(q)}(x,dy) &= \frac{1}{q}\mathbb{P}(X_{e_q} \in dy-x, \underline{X}_{e_q} \geq -x)\\
&= \frac{1}{q}\mathbb{P}\Big((X_{e_q}-\underline{X}_{e_q}) + \underline{X}_{e_q} \in dy-x, -\underline{X}_{e_q} \leq x\Big)\\
&= \frac{1}{q}\int_{[x-y,x]}\mathbb{P}\big((X_{e_q}-\underline{X}_{e_q}) \in dy-x+z\big)\mathbb{P}(-\underline{X}_{e_q} \in dz)\\
&= \frac{1}{q}\int_{[x-y,x]}\mathbb{P}(\overline{X}_{e_q} \in dy-x+z)\mathbb{P}(-\underline{X}_{e_q} \in dz) \qquad \text{(by duality).}
\end{aligned}$$

The two probabilities above are introduced in the introduction section, so we can develop the expression for $R^{(q)}(x,dy)$ as follow

$$\left[\frac{1}{q}\int_{x-y}^{x}\left[\left(1-\frac{\rho_1}{\eta_1^+}\right)\left(\frac{\rho_2}{\rho_2-\rho_1}\right)\rho_1 e^{-\rho_1(y-x+z)}+\left(\frac{\eta_1^+-\rho_2}{\eta_1^+}\right)\left(\frac{\rho_1}{\rho_1-\rho_2}\right)\rho_2 e^{-\rho_2(y-x+z)}\right]\right.$$

$$\left.\times\left[\left(1+\frac{\rho_3}{\eta_1^-}\right)\left(\frac{\rho_4}{\rho_3-\rho_4}\right)\rho_3 e^{\rho_3 z}+\left(\frac{\eta_1^-+\rho_4}{\eta_1^-}\right)\left(\frac{\rho_3}{\rho_4-\rho_3}\right)\rho_4 e^{\rho_4 z}\right]dz\right]dy.$$

This shows that there exists a density, $r^{(q)}(x,y)$ for the measure $R^{(q)}(x,dy)$. With a simple integration, the above expression turns out to be

$$C_1(e^{\rho_3(x-y)}-e^{\rho_3 x-\rho_1 y})+C_2(e^{\rho_4(x-y)}-e^{\rho_4 x-\rho_1 y})+C_3(e^{\rho_3(x-y)}-e^{\rho_3 x-\rho_2 y})+C_4(e^{\rho_4(x-y)}-e^{\rho_4 x-\rho_2 y}).$$

□

Now, we can use $R^{(q)}(x,dy)$ to derive $U^{(q)}(a,x,dy)$ in the theorem below.

Theorem 3.5 *Suppose, for $q \geq 0$, $U^{(q)}(a,x,dy)$ is the q-potential measure of the double exponential jump-diffusion process X_t killed on exiting $[0,a]$, where $x,y\in[0,a]$. Then it has a density $u^{(q)}(a,x,y)$ given by*

$$r^{(q)}(x,y)-r^{(q)}(a,y)\sum_{i=1}^{4}(D_i^1-D_i^2)e^{\rho_i x}-\frac{1}{q}\sum_{i=1}^{4}D_{i,1}e^{\rho_i x}\left(\frac{C_1}{\eta^+-\rho_3}(e^{\rho_3(a-y)}-e^{-\rho_3 a-\rho_1 y})\right.$$

$$\left.+\frac{C_2}{\eta^+-\rho_4}(e^{\rho_4(a-y)}-e^{-\rho_4 a-\rho_1 y})+\frac{C_3}{\eta^+-\rho_3}(e^{\rho_3(a-y)}-e^{-\rho_3 a-\rho_2 y})+\frac{C_4}{\eta^+-\rho_4}(e^{\rho_4(a-y)}-e^{-\rho_4 a-\rho_2 y})\right),$$

where C_i and $r^{(q)}(x,y)$ are defined as in Theorem 3.4; $D_i^1, D_i^2,$ and $D_{i,1}$ are defined as in (24),(25), and (28) respectively.

Proof. Now, we have

$$\begin{aligned} qU^{(q)}(a,x,dy) &= \mathbb{P}_x(X_{e_q}\in dy, \tau > e_q) \\ &= \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}\geq 0, \overline{X}_{e_q}\leq a) \qquad\qquad (\text{since } \tau=\tau_a^+\wedge\tau_0^- > e_q) \\ &= \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}\geq 0) - \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}> 0, \overline{X}_{e_q}> a) \\ &= \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}\geq 0) - \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}> 0, \overline{X}_{e_q}> a, \tau\geq e_q) \end{aligned}$$

$$-P_x(X_{e_q}\in dy, \underline{X}_{e_q}> 0, \overline{X}_{e_q}> a, \tau < e_q).$$

Here in the last equality, we imposed an extra condition $\tau \geq e_q$ and $\tau < e_q$ in the second probability. Notice that the second probability above is zero due to the conflict of $\{\overline{X}_{e_q}\geq a\}$ and $\{\tau\geq e_q\}$. Thus, we obtain that

$$\begin{aligned} qU^{(q)}(a,x,dy) &= \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}\geq 0) - \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}> 0, \overline{X}_{e_q}> a, \tau < e_q) \\ &= \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}\geq 0) - \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}> 0, X_\tau = a, \tau < e_q) \end{aligned}$$

$$-\mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}> 0, X_\tau - a > 0, \tau < e_q)$$

$$= \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}\geq 0) - \mathbb{P}_a(X_{e_q}\in dy, \underline{X}_{e_q}\geq 0)\mathbb{P}_x(X_\tau = a, \tau < e_q)$$

$$-\int_{(0,\infty)} \mathbb{P}_x(X_\tau - a\in dz, \tau < e)\mathbb{P}_{z+a}(X_{e_q}\in dy, \underline{X}_{e_q}\geq 0),$$

where in the last equality we conditioned on $\{X_\tau - a\}$ and used the Markov property. Using $qR^{(q)}(x,dy) = \mathbb{P}_x(X_{e_q}\in dy, \underline{X}_{e_q}\geq 0)$ in Theorem 3.4, we obtain
$U^{(q)}(a,x,dy)$

$$\begin{aligned} &= R^{(q)}(x,dy) - R^{(q)}(a,dy)\mathbb{P}_x(X_\tau = a, \tau < e_q) - \int_{(0,\infty)} \mathbb{P}_x(X_\tau - a\in dz, \tau < e_q)R^{(q)}(z+a,dy) \\ &= R^{(q)}(x,dy) - R^{(q)}(a,dy)\mathbb{E}_x(e^{-q\tau}, X_\tau = a) - \int_{(0,\infty)} \mathbb{E}_x(e^{-q\tau}, X_\tau - a\in dz)R^{(q)}(z+a,dy). \end{aligned}$$

From here, we see that there exists a density $u^{(q)}(a,x,y)$ for the measure $U^{(q)}(a,x,dy)$. For the expectation in the second term and the third term of the second equality above, we use the two-sided exit problem results

$$\mathbb{E}_x(e^{-q\tau}, X_\tau = a) = \sum_{i=1}^{4}(D_i^1 - D_i^2)e^{\rho_i x} \quad \text{and} \quad \mathbb{E}_x(e^{-q\tau}, X_\tau - a \in dz) = e^{-\eta_1^+ z}\sum_{i=1}^{4} D_{i,1}e^{\rho_i x}.$$

Thus,

$$u^{(q)}(a,x,y) \;=\; r^{(q)}(x,y) - r^{(q)}(a,y)\sum_{i=1}^{4}(D_i^1 - D_i^2)e^{\rho_i x} - \sum_{i=1}^{4} D_{i,1}e^{\rho_i x}\int_{(0,\infty)} e^{-\eta_1^+ z} r^{(q)}(z+a,y).$$

Using the expression of $r^{(q)}(x,y)$ in Theorem 3.4, we can evaluate the integral above as

$$\int_{(0,\infty)} e^{-\eta_1^+ z} r^{(q)}(z+a,y) = \frac{1}{q}\int_0^\infty e^{-\eta_1^+ z}\Big(C_1(e^{\rho_3(z+a-y)} - e^{\rho_3(z+a)-\rho_1 y}) + C_2(e^{\rho_4(z+a-y)} - e^{\rho_4(z+a)-\rho_1 y})$$
$$+C_3(e^{\rho_3(z+a-y)} - e^{\rho_3(z+a)-\rho_2 y}) + C_4(e^{\rho_4(z+a-y)} - e^{\rho_4(z+a)-\rho_2 y})\Big)dz$$
$$= \frac{1}{q}\Big(\frac{C_1}{\eta_1^+ - \rho_3}(e^{\rho_3(a-y)} - e^{-\rho_3 a-\rho_1 y}) + \frac{C_2}{\eta_1^+ - \rho_4}(e^{\rho_4(a-y)} - e^{-\rho_4 a-\rho_1 y})$$
$$+\frac{C_3}{\eta_1^+ - \rho_3}(e^{\rho_3(a-y)} - e^{-\rho_3 a-\rho_2 y}) + \frac{C_4}{\eta_1^+ - \rho_4}(e^{\rho_4(a-y)} - e^{-\rho_4 a-\rho_2 y})\Big).$$

□

From here, we can generalize the idea to find the q-potential measure for hyper-exponential jump-diffusion process X_t killed on exiting $[0,\infty)$. That is

$$\begin{aligned} R^{(q)}(x,dy) &= \frac{1}{q}\int_{[x-y,x]} \mathbb{P}(\overline{X}_{e_q} \in dy - x + z)\mathbb{P}(-\underline{X}_{e_q} \in dz)) \\ &= \Big\{\frac{1}{q}\int_{x-y}^{x}\Big(\bar{\boldsymbol{a}}(\eta^+,\rho)^T.\bar{\boldsymbol{v}}(\rho, y-x+z)\Big)\Big(\bar{\boldsymbol{a}}(\eta^-,\rho)^T.\bar{\boldsymbol{v}}(\rho^-, z)\Big)dz\Big\}dy, \end{aligned}$$

where $\bar{\boldsymbol{a}}$ and $\bar{\boldsymbol{v}}$ are defined in Theorem 2.3. This shows that there exits a density $r^{(q)}(x,y)$ of the measure $R^{(q)}(x,dy)$. The density above can be expressed explicitely as

$$qr^{(q)}(x,y)=\sum_{i=1}^{m+1}\rho_i a_i(\eta^+,\rho)\sum_{j=m+2}^{S}\frac{a_j(\eta^-,\rho^-)\rho_j}{\rho_i-\rho_j}\left(e^{\rho_j(x-y)}-e^{\rho_j x-\rho_i y}\right). \tag{43}$$

Theorem 3.6 *Suppose, for $q\geq 0$, $U^{(q)}(a,x,dy)$ is the q-potential measure of a hyperexponential jump-diffusion process X_t killed on exiting $[0,a]$, where $x,y\in[0,a]$. Then it has a density $u^{(q)}(a,x,y)$ given by*

$$r^{(q)}(x,y)-r^{(q)}(a,y)\sum_{i=1}^{S}\left(D_i^1-D_i^2\right)e^{\rho_i}x-\frac{1}{q}\sum_{i=1}^{m}\sum_{j=m+2}^{S}\sum_{k=1}^{S}\sum_{l=1}^{m+1}P_{ijkl}e^{\rho_k x}(e^{\rho_j(a-y)}-e^{\rho_j a-\rho_l y}),$$

where

$$P_{ijkl}=\frac{D_{k,i}a_l(\eta^+,\rho)a_j(\eta^-,\rho^-)\rho_l\rho_j}{(\rho_l-\rho_j)(\eta_i^+-\rho_j)},$$

and $D_i^1,D_i^2,D_{i,j}$ are defined in (24),(25), and (28) respectively; and $r^{(q)}(x,y)$ is the q-potential measure of hyperexponential jump-diffusion process killed on exiting $[0,\infty)$.

Proof. Using the same reasoning as in the case of double exponential jump-diffusion process, we have

$$u^{(q)}(a,x,y)=r^{(q)}(x,y)-r^{(q)}(a,y)\mathbb{E}(e^{-q\tau},X_\tau=a)-\int_{(0,\infty)}\mathbb{E}_x(e^{-q\tau},X_\tau-a\in dz)r^{(q)}(z+a,y).$$

Again, the expectations above are the two-sided exit problems which can be expressed as

$$\mathbb{E}(e^{-q\tau},X_\tau=a)=\sum_{i=1}^{S}\left(D_i^1-D_i^2\right)e^{\rho_i}x \qquad \text{and} \qquad \mathbb{E}_x\left(e^{-q\tau},X_\tau-a\in dz\right)=\sum_{i=1}^{S}\sum_{j=1}^{m}D_{i,j}e^{-\eta_j^+z}e^{\rho_i x}.$$

So for the integral, we can rewrite it as

$$\frac{1}{q}\int_0^\infty\left[\sum_{i=1}^{m+1}\rho_i a_i(\eta^+,\rho)\sum_{j=m+2}^{S}\frac{a_j(\eta^-,\rho^-)\rho_j}{\rho_i-\rho_j}(e^{\rho_j(z+a-y)}-e^{\rho_j(z+a)-\rho_i y})\right]\cdot\left[\sum_{i=1}^{S}\sum_{j=1}^{m}D_{i,j}e^{-\eta_j^+z}e^{\rho_i x}\right]dz$$

$$= \frac{1}{q}\int_0^\infty \Big[\sum_{i=1}^{m+1}\sum_{j=m+2}^{S}\frac{a_i(\eta^+,\rho)a_j(\eta^-,\rho^-)\rho_i\rho_j}{\rho_i-\rho_j}(e^{\rho_j(a-y)}-e^{\rho_j a-\rho_i y})e^{\rho_j z}\Big].\Big[\sum_{i=1}^{S}\sum_{j=1}^{m}D_{i,j}e^{-\eta_j^+ z}e^{\rho_i x}\Big]dz$$

$$= \frac{1}{q}\int_0^\infty \Big[\sum_{i=1}^{m}\sum_{j=m+2}^{S}\sum_{k=1}^{S}\sum_{l=1}^{m+1}\frac{D_{k,i}a_l(\eta^+,\rho)a_j(\eta^-,\rho^-)\rho_l\rho_j}{\rho_l-\rho_j}e^{\rho_k x}\big(e^{\rho_j(a-y)}-e^{\rho_j a-\rho_l y}\big)\Big]e^{(\rho_j-\eta_i^+)z}dz$$

$$= \frac{1}{q}\sum_{i=1}^{m}\sum_{j=m+2}^{S}\sum_{k=1}^{S}\sum_{l=1}^{m+1}\frac{D_{k,i}a_l(\eta^+,\rho)a_j(\eta^-,\rho^-)\rho_l\rho_j}{(\rho_l-\rho_j)(\eta_i^+-\rho_j)}e^{\rho_k x}\big(e^{\rho_j(a-y)}-e^{\rho_j a-\rho_l y}\big).$$

□

4 FLUTUATION IDENTITIES OF PROCESSES REFLECTED AT ITS INFIMUM

Recall that given a hyper-exponential jump-difussion process X_t, a processes reflected at its infimum is defined as

$$Y_t := X_t - \underline{X}_t \wedge 0.$$

Then Y_t is a Markov process. By using the Markov property, we can derive the q-potential measure killed on exitting $[0, a]$ of this reflected process defined as

$$\Lambda^{(q)}(a, x, dy) := \int_0^\infty e^{-qt}\mathbb{P}_x(Y_t \in dy, t < \gamma_a)dt,$$

where $\gamma_a = \inf\{t \geq 0 : Y_t > a\}$.

Theorem 4.1 *([21]) For any $q \geq 0$, $x, y \in [0, a]$, then*

$$\Lambda^{(q)}(a, x, dy) = U^{(q)}(a, x, y) + \sum_{i=1}^{S} C_i^1 e^{\rho_i x}. \lim_{z\to 0}\Big(\frac{U^{(q)}(a, z, dy)}{1 - \sum_{i=1}^{S} C_i^1 e^{\rho_i z}}\Big). \tag{44}$$

Although this epxression looks nice, we still need to evaluate the limit and we do not know if this limit exists or not. Another way to derive the potential measure of this reflected process is to introduce two independent exponential random variables e_q and e_ζ with rates q and ζ respectively (e_q is also independent with the reflected process and e_ζ is the first jump time of the process). By conditioning on the arrival time between: e_q, e_ζ, and γ_a, we can turn the problem into the potential measure of reflected Brownian motion with drift. We first derive the potential measure of reflected Brownian motion with drift. Recall that a linear Brownian motion is defined as

$$B_t = \mu t + \sigma W_t,$$

where W_t is the standard Brownian motion, then the reflected linear Brownian motion with drift is defined as $|B_t|$. Also, we have $\beta_a = \inf\{t \geq 0 : |B_t| > a\}$.

Lemma 4.1 *Consider $|B_t|$, a reflected linear Brownian motion with drift, then given $x \in [0, a]$ the potential measure of $|B_t|$ which is defined as*

$$B^{(q)}(a, x, dy) := \int_0^\infty e^{-qt} P_x(|B_t| \in dy, t < \beta_a) dt,$$

can be expressed as

$$qB^{(q)}(a, x, dy) = \Bigg\{ \frac{\lambda\sigma}{\sqrt{2\lambda\sigma^2 + \mu^2}} \Big(e^{\mu(\frac{y}{\sigma} - x) - |\frac{y}{\sigma} - x|\sqrt{2\lambda + (\frac{\mu}{\sigma})^2}} + e^{\mu(\frac{y}{\sigma} + x) - |\frac{y}{\sigma} + x|\sqrt{2\lambda + (\frac{\mu}{\sigma})^2}} \Big)$$

$$- \frac{\lambda\sigma}{\sqrt{2\lambda\sigma^2 + \mu^2}} \Big(e^{\mu(\frac{y}{\sigma} - a) - |\frac{y}{\sigma} - a|\sqrt{2\lambda + (\frac{\mu}{\sigma})^2}} + e^{\mu(\frac{y}{\sigma} + a) - |\frac{y}{\sigma} + a|\sqrt{2\lambda + (\frac{\mu}{\sigma})^2}} \Big)$$

$$\times \frac{e^{\mu x + x\sqrt{2q + (\frac{\mu}{\sigma})^2}} + e^{-\mu x - x\sqrt{2q + (\frac{\mu}{\sigma})^2}}}{e^{\frac{\mu}{\sigma}a + \frac{a}{\sigma}\sqrt{2q + (\frac{\mu}{\sigma})^2}} + e^{-\frac{\mu}{\sigma}a - \frac{a}{\sigma}\sqrt{2q + (\frac{\mu}{\sigma})^2}}} \Bigg\} dy.$$

Proof. We have

$$\begin{aligned} qB^{(q)}(a, x, dy) &= \mathbb{P}_x\Big(e_q < \beta_a, |B_{e_q}| \in dy\Big) \\ &= \mathbb{P}_x\Big(|\bar{B}_{e_q}| < a, |B_{e_q}| \in dy\Big) \\ &= \mathbb{P}_x\Big(|B_{e_q}| \in dy\Big) - \mathbb{P}_x\Big(|\bar{B}_{e_q}| \geq a, |B_{e_q}| \in dy\Big) \\ &= \mathbb{P}_x\Big(|B_{e_q}| \in dy\Big) - \mathbb{P}_x\Big(|\bar{B}_{e_q}| \geq a, |B_{e_q}| \in dy, \beta_a < e_q\Big) \end{aligned}$$

$$\begin{aligned} &= \mathbb{P}_x\Big(|B_{e_q}| \in dy\Big) - \mathbb{P}_x\Big(|B_{e_q}| \in dy, \beta_a < e_q\Big) \\ &= \mathbb{P}_x\Big(|B_{e_q}| \in dy\Big) - \mathbb{P}_a\Big(|B_{e_q}| \in dy\Big)\mathbb{P}_x\Big(\beta_a < e_q\Big) \\ &= \mathbb{P}_x\Big(|B_{e_q}| \in dy\Big) - \mathbb{P}_a\Big(|B_{e_q}| \in dy\Big)\mathbb{E}_x\Big(e^{-q\beta_a}\Big). \end{aligned}$$

Using the results from Section 2.6 for the reflected linear Brownian motion with drift we obtain our desired result. $\square$

Now, we are ready to state the following theorem.

Theorem 4.2 *For any $q \geq 0$, $x, y \in [0, a]$, we have*

$$\Lambda^{(q)}(a, x, dy) = U^{(q)}(a, x, dy) + A_0 \sum_{i=1}^{S} C_i^1 e^{\rho_i x}, \tag{45}$$

where

$$A_0 = \frac{B^{(q+\zeta)}(a, 0, dy) + \int_0^a \left[\mathbb{P}\Big(Y_{e_\zeta} \in dx, e_\zeta < e_q\Big) U^{(q)}(a, x, dy)\right]}{1 - \mathbb{P}\Big(Y_{e_\zeta} = 0, e_\zeta < e_q\Big) - \sum_{i=1}^{S} \int_0^a \left[C_i^1 e^{\rho_i x} \mathbb{P}\Big(Y_{e_\zeta} \in dx, e_\zeta < e_q\Big)\right]},$$

$$\begin{aligned}
\mathbb{P}\Big(Y_{e_\zeta} = 0, e_\zeta < e_q\Big) &= \int_0^\infty \zeta e^{-(q+\zeta)t} \int_0^a \mathbb{P}(|B_t| \in dz) \mathbb{P}(z + J \leq 0) dt, \\
\mathbb{P}\Big(Y_{e_\zeta} \in dx, e_\zeta < e_q\Big) &= \int_0^\infty \zeta e^{-(q+\zeta)t} \int_0^a \mathbb{P}(|B_t| \in dz) \mathbb{P}(z + J \in dx) dt,
\end{aligned}$$

the density of J is

$$\begin{aligned}
f_J(x) &= \mathbb{I}_{\{x>0\}} \sum_{i=1}^{m} p_i \eta_i^+ e^{-\eta_i^+ x} + \mathbb{I}_{\{x<0\}} \sum_{i=1}^{n} q_i \eta_i^- e^{\eta_i^- x}, \\
\mathbb{P}_x\Big(|B_t| \in dz\Big) &= \frac{1}{\sqrt{2\pi t}} \Big(e^{-(\frac{z}{\sigma} - \mu t - x)/2t} + e^{-(\frac{z}{\sigma} + \mu t + x)/2t}\Big) dz.
\end{aligned}$$

$\zeta = \sum_{i=1}^m \eta_i^+ + \sum_{i=1}^n \eta_i^-$, *$B^{(q)}(a, 0, dx)$ and $U^{(q)}(a, x, y)$ are the q-potential measure of reflected linear Brownian motion with drift and the hyper-exponential jump-diffusion process respectively. J is the jump size of the process X_t. These integrals can be computed explicitly.*

Proof. Let

$$A_0 = \mathbb{P}\Big(e_q < \gamma_a, Y_{e_q} \in dy\Big).$$

For $x \in [0, a]$, we see that $Y_t = X_t$ if $t < \tau_0^- \wedge \tau_a^+$. So we can condition on the exit time

τ_0^-, τ_a^+ of process X_t

$$\begin{aligned} q\Lambda^{(q)}(a,x,dy) &= \mathbb{P}_x\Big(e_q < \gamma_a, Y_{e_q} \in dy\Big) \\ &= \mathbb{P}_x\Big(\tau_0^- < e_q < \tau_a^+, Y_{e_q} \in dy\Big) + \mathbb{P}_x\Big(e_q < \tau_0^- \wedge \tau_a^+, Y_{e_q} \in dy\Big) \\ &= \mathbb{P}_x\Big(\tau_0^- < \tau_a^+, \tau_0^- < e_q\Big)A_0 + \mathbb{P}_x\Big(e_q < \tau, X_{e_q} \in dy\Big) \\ &= A_0\mathbb{E}_x\Big[e^{-q\tau}\mathbf{1}_{\{X_\tau \le 0\}}\Big] + qU^{(q)}(a,x,dy) \\ &= A_0\sum_{i=1}^{S} C_i^1 e^{\rho_i x} + qU^{(q)}(a,x,dy), \end{aligned}$$

where in the last equality, we used the result of the two-sided exit problem (29). Now, for A_0 we denote the first jump time of the hyper-exponential jump-diffusion process as e_ζ, then e_ζ follows an exponential distribution with rate $\zeta = \sum_{i=1}^m \eta_i^+ + \sum_{i=1}^n \eta_i^-$. We also have that $Y_t = |B_t|$ if $t < e_\zeta < \tau_a^+$. Thus we can condition on the exit time β_a of reflected Brownian motion with drift and e_ζ to get

$$\begin{aligned} A_0 &= \mathbb{P}\Big(e_q < \gamma_a, Y_{e_q} \in dy\Big) \\ &= \mathbb{P}\Big(e_q < e_\zeta \wedge \gamma_a, Y_{e_q} \in dy\Big) + \mathbb{P}\Big(e_\zeta < e_q < \gamma_a, Y_{e_q} \in dy\Big) \\ &= \mathbb{P}\Big(e_q < e_\zeta, e_q < \beta_a, |B_{e_q}| \in dy\Big) + \mathbb{P}\Big(e_\zeta < e_q < \gamma_a, Y_{e_q} \in dy\Big) \\ &= q\int_0^\infty e^{-qt}P\Big(t < e_\zeta, t < \beta_a, |B_t| \in dy\Big)dt + \mathbb{P}\Big(e_\zeta < e_q < \gamma_a, Y_{e_q} \in dy\Big) \\ &= q\int_0^\infty e^{-qt}P\Big(t < e_\zeta\Big)\mathbb{P}\Big(t < \beta_a, |B_t| \in dy\Big)dt + \mathbb{P}\Big(e_\zeta < e_q < \gamma_a, Y_{e_q} \in dy\Big) \\ &= q\int_0^\infty e^{-(q+\zeta)t}\mathbb{P}\Big(t < \beta_a, |B_t| \in dy\Big)dt + \mathbb{P}\Big(e_\zeta < e_q < \gamma_a, Y_{e_q} \in dy\Big) \\ &= qB^{(q+\zeta)}(a,0,dy) + \mathbb{P}\Big(Y_{e_\zeta} = 0, e_\zeta < e_q,\ e_q < \gamma_a, Y_{e_q} \in dy\Big) \\ &\quad + \int_{(0,a]}\mathbb{P}\Big(Y_{e_\zeta} \in dx, e_\zeta < e_q,\ e_q < \gamma_a, Y_{e_q} \in dy\Big) \\ &= qB^{(q+\zeta)}(a,0,dy) + \mathbb{P}\Big(Y_{e_\zeta} = 0, e_\zeta < e_q\Big)A_0 + \int_{(0,a]}\Big[\mathbb{P}\Big(Y_{e_\zeta} \in dx, e_\zeta < e_q\Big)q\Lambda^{(q)}(a,x,dy)\Big] \end{aligned}$$

$$
\begin{aligned}
= \quad & qB^{(q+\zeta)}(a,0,dy)+\mathbb{P}\Big(Y_{e_\zeta}=0, e_\zeta<e_q\Big)A_0 \\
& + \int_{(0,a]} \Big[\mathbb{P}\Big(Y_{e_\zeta}\in dx, e_\zeta<e_q\Big)\Big(A_0\sum_{i=1}^{S} C_i^1 e^{\rho_i x} + qU^{(q)}(a,x,dy)\Big)\Big].
\end{aligned}
$$

Hence

$$
A_0 = \frac{qB^{(q+\zeta)}(a,0,dy) + q\int_{(0,a]}\Big[\mathbb{P}\Big(Y_{e_\zeta}\in dx, e_\zeta<e_q\Big)U^{(q)}(a,x,dy)\Big]}{1-\mathbb{P}\Big(Y_{e_\zeta}=0, e_\zeta<e_q\Big) - \sum_{i=1}^{S}\int_{(0,a]}\Big[C_i^1 e^{\rho_i x}\mathbb{P}\Big(Y_{e_\zeta}\in dx, e_\zeta<e_q\Big)\Big]}.
$$

For the two probabilities in the expression of A_0, we notice that at the jump time $Y_t = |B_t| + J$, where J is the jump size of the hyper-exponential jump-diffusion process. Then J follows the hype-exponential distribution defined in the introduction. We have

$$
\begin{aligned}
\mathbb{P}\Big(Y_{e_\zeta}=0, e_\zeta<e_q\Big) &= \int_0^\infty \zeta e^{-\zeta t}\mathbb{P}\Big(Y_t=0, t<e_q\Big)dt \\
&= \int_0^\infty \zeta e^{-\zeta t}\mathbb{P}(t<e_q)\mathbb{P}(Y_t=0)dt \\
&= \int_0^\infty \zeta e^{-(q+\zeta)t}\mathbb{P}(Y_t=0)dt \\
&= \int_0^\infty \zeta e^{-(q+\zeta)t}\int_0^a \mathbb{P}(|B_t|\in dz)\mathbb{P}(z+J\le 0)dt.
\end{aligned}
$$

Similarly, we can obtain

$$
\mathbb{P}\Big(Y_{e_\zeta}\in dx, e_\zeta<e_q\Big) = \int_0^\infty \zeta e^{-(q+\zeta)t}\int_0^a \mathbb{P}(|B_t|\in dz)P(z+J\in dx)dt.
$$

□

By Remark 4 and Remark 5 in [21], we can use the conpensation formula to obtain the following corollary.

Corollary 4.1 *Given* $q \ge 0, y \ge a, z \in (0,a)$ *and the stopping time* $\gamma_a = \inf\{t \ge 0 : Y_t > a\}$

and $\gamma_a^- = \inf\{t \geq 0 : Y_t < a\}$, *the joint density of overshoot and undershoot is*

$$\mathbb{E}_x\left[e^{-q\gamma_a}; Y_{\gamma_a} \in dy; Y_{\gamma_a^-} \in dz\right] = \left(U^{(q)}(a, x, dy) + A_0 \sum_{i=1}^{S} C_i^1 e^{\rho_i x}\right)\Pi(dy - z), \qquad (46)$$

where $\Pi(x)$ *is the Lévy measure with density given by*

$$\pi(x) = \mathbb{I}_{x>0} \sum_{i=1}^{m} p_i \eta_i^+ e^{-\eta_i^+ x} + \mathbb{I}_{x<0} \sum_{i=1}^{n} q_i \eta_i^- e^{\eta_i^- x}.$$

5 NUMERICAL EXAMPLES FOR DOUBLE EXPONENTIAL JUMP-DIFFUSION PROCESSES

In this section, we present some numerical examples for the double-exponential jump-diffusion process. Recall that the double-exponential jump-diffusion process is defined as

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

and the common density of Y is

$$f_Y(y) = I_{\{y \geq 0\}} p \eta_1^+ e^{-\eta_1^+ y} + I_{\{y<0\}} q \eta_1^- e^{\eta_1^- y}.$$

To ease the computation, we consider three sets of parameters

$$\text{Set 1: } \{\mu, \sigma, p, \eta^+, \eta^-, \lambda, q\} = \{1, 1, 0.5, 1, 1, 1, 0.5\}\ ,$$
$$\text{Set 2: } \{\mu, \sigma, p, \eta^+, \eta^-, \lambda, q\} = \{1, 1, 0.5, 1, 1, 1, 0.05\}\ \ ,$$
$$\text{Set 3: } \{\mu, \sigma, p, \eta^+, \eta^-, \lambda, q\} = \{-1, 1, 0.5, 1, 1, 1, 0.05\}.$$

Here, we consider the process within the interval $[0, 1]$. Note that the overall drift the process is

$$\psi'(0) = \mu + \lambda\left(\frac{p}{\eta_1^+} - \frac{q}{\eta_1^-}\right).$$

So Set 2 deffers from Set 1 in q (practically $1/q$ is the maturity time) while it deffers from Set 3 in the positive and negative overall drift μ. Using the theoritical results in previous sections, we have computed the two-sided exit problem from above (level 1), two-sided exit problem from below (level 0), one-sided exit problem at level 0, potential measure density $r^{(q)}(x, y)$ killed on exiting $[0, \infty)$, potential measure density $u^{(q)}(x, y)$ killed on exiting $[0, 1]$, and the potential measure $\lambda^{(q)}(x, y)$ killed on exiting $[0, 1]$ of the reflected process. We denote the graph as following:

$$
\begin{aligned}
TwosideExitAbovege1(x) &:= \mathbb{E}_x[e^{-q\tau}, X_\tau \geq 1] = \mathbb{P}_x[X_\tau \geq 1, \tau < e_q] \quad (red\ line), \\
TwosideExitAboveg1(x) &:= \mathbb{E}_x[e^{-q\tau}, X_\tau > 1] = \mathbb{P}_x[X_\tau > 1, \tau < e_q] \quad (green\ line), \\
TwosideExitAboveeq1(x) &:= \mathbb{E}_x[e^{-q\tau}, X_\tau = 1] = \mathbb{P}_x[X_\tau = 1, \tau < e_q] \quad (blue\ line).
\end{aligned}
$$

Then the two-sided exit problem from below 0, the one-sided exit problem from above 0, and the one-sided exit problem from below 0 are defined in the same manner. The graphs on the next page clearly show the effects that we would expect to see. In figure 1 for the exit problems of double exponential jump-diffusion process, the y-axis is the probability scaled from zero to one, and the x-axis represents the starting position of the process.

Looking at the graphs for the two-sided exit problem from above level one, they suggest that when x = 0, the three graph are zero. When x = 1, then $\mathbb{P}[X_\tau \geq 1, \tau < e_q]$ and $\mathbb{P}[X_\tau = 1, \tau < e_q]$ are exactly one while the $\mathbb{P}[X_\tau > 1, \tau < e_q]$ is exactly zero; this makes sense. Also, the graph of $\mathbb{P}[X_\tau \geq 1, \tau < e_q]$ are always higher than the others; and $\mathbb{P}[X_\tau = 1, \tau < e_q] = \mathbb{P}[X_\tau \geq 1, \tau < e_q] - \mathbb{P}[X_\tau > 1, \tau < e_q]$. We get the similar results for other types of exit problems.

Now, comparing the graphs from Set 1 and Set 2 (the time e_q in Set 2 is likely longer than that of Set 1), we see that the area under the curve in Set 2 is larger than that of Set 1. This is because the event $\{\tau < e_q\}$ likely occurs with longer time e_q. Besides, graphs in Set 1 and 2 with the positive overall drift are higher than the graphs in Set 3 with the negative overall drift. Whereas, when considering the exit problems from below (one-sided and two-sided), graphs in Set 1 and 2 are lower than the ones in Set 3. This is what we expect the graphs should be because with a positive overall drift, it is more likely for the process to exit from above, and with a negative overall drift, it is more likely for the process to exit from below.

For the potential measure density (figure 2), since Set 1 and Set 2 have the positive overall drift while it is the negative drift for the Set 3, this has a strong impact on the potential

Figure 1: One-sided and Two-sided exit problems of double-exponential jump-diffusion processes.

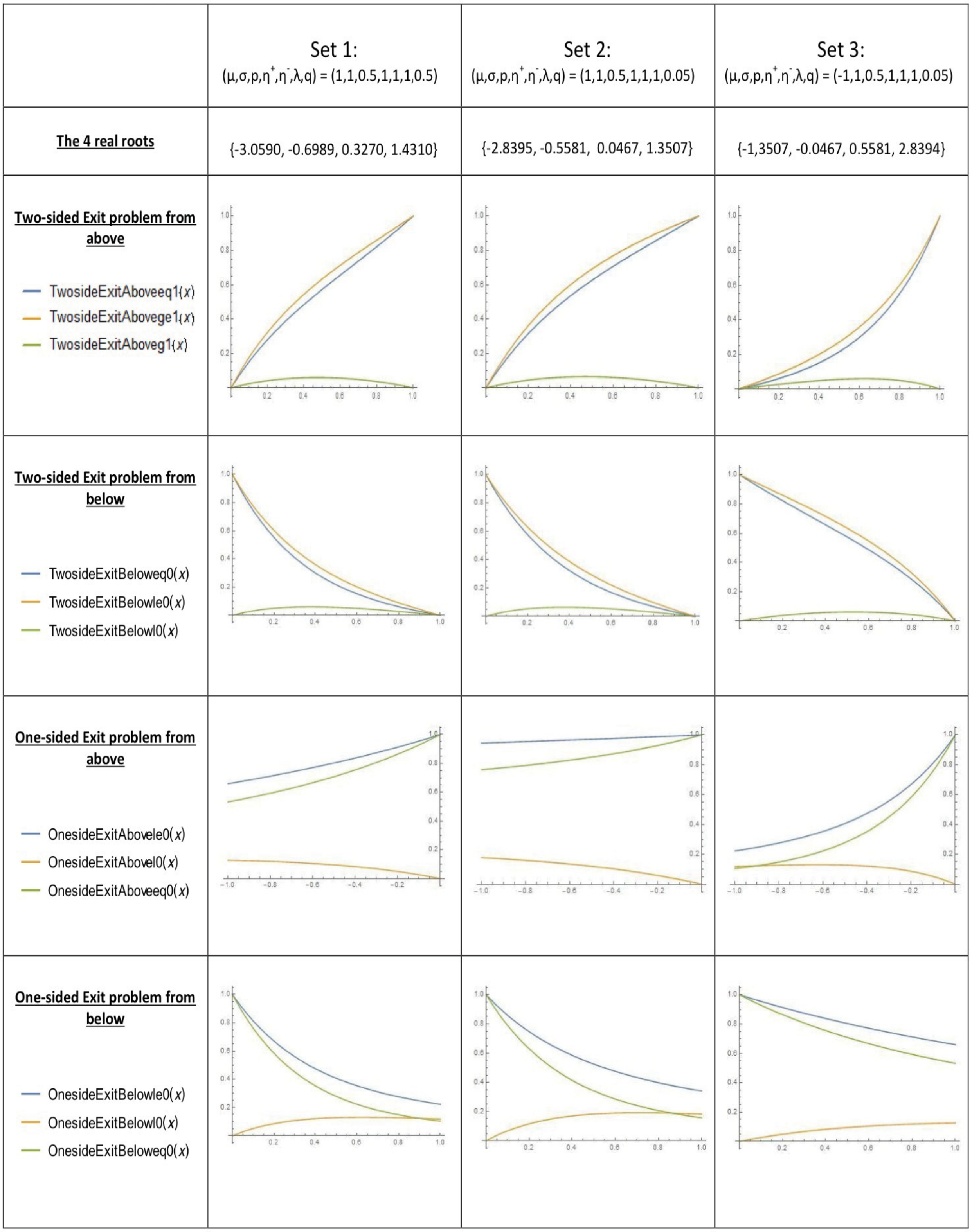
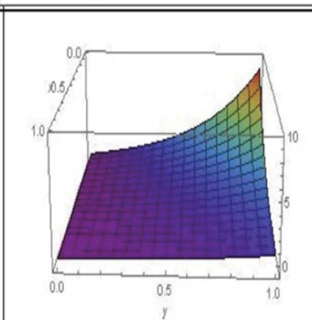
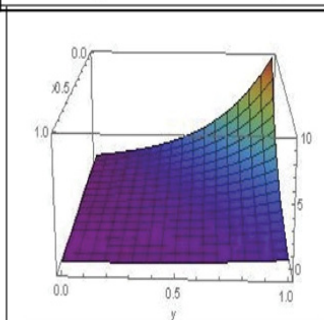
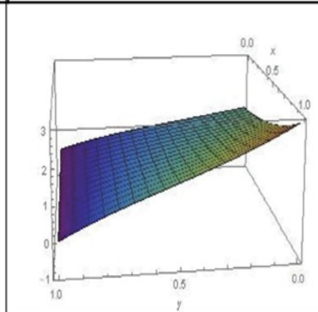
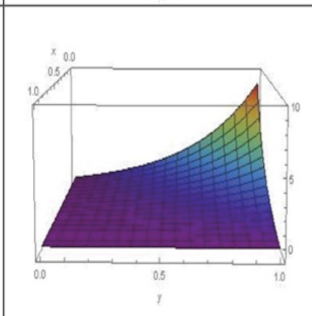
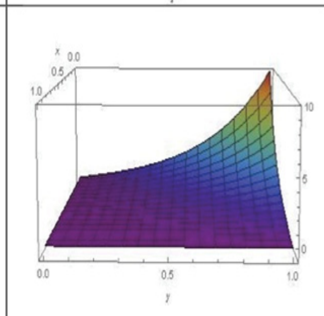
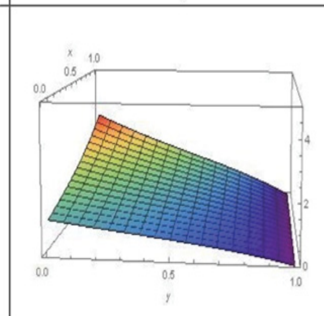
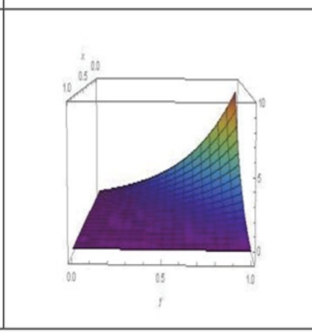
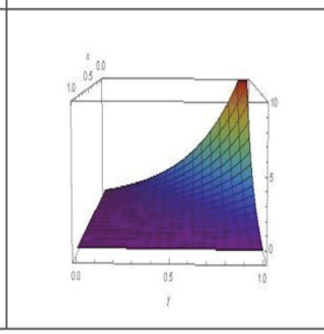
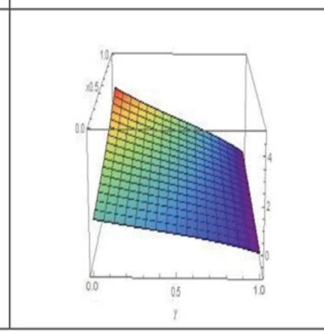


Figure 2: Potential measure of double-exponential jump-diffusion processes and reflected processes.

	Set 1 (positive drift)	Set 2 (positive drift)	Set 3 (negative drift)
Potential measure density $r^{(q)}(x,y)$ killed on exiting $[0,\infty)$ of the double-exponential jump-diffusion process.			
Potential measure density $u^{(q)}(x,y)$ killed on exiting $[0,1]$ of the double-exponential jump-diffusion process.			
Potential measure density $\lambda^{(q)}(x,y)$ killed on exiting $[0,1]$ of the double-exponential jump-diffusion process reflected at 0.			

measure density by comparing the graphs between Set 1 and Set 3. Intuitively, potential measure measures the total time that the process will stay within the interval $[0, \infty)$ for the case of $R^{(q)}(x, dy)$, and within the interval $[0, a]$ for the case of $U^{(q)}(a, x, dy)$, and up to an independent exponential time e_q for $\Lambda^{(q)}(a, x, dy))$. Therefore, with a positive overall drift (graphs for the first and second set) we see that the mass of q-potential measure concentrates on the right half interval $[0.5, 1]$ of y. While for the ovrerall negative drift (graphs in the third set), the mass concentrates on the left half interval $[0, 0.5]$. Also, by looking inside the set 1, we see that the effect of the starting point on potential measure is in consideration too. Particularly, when starting from 1, it is unlikely that during the time e_q the process will stay within $[0, 1]$ due to the overall positive drift, whereas it is more likely for the case starting from 0. Furthermore, by comparing the potential measures within a set, we see that the potential measure of reflected process has a larger mass than the others.

6 CONCLUSION AND FUTURE WORK

In this thesis, we have explicitly expressed the solutions to one-sided and two-sided exit problems as well as the q-potential measure of the hyper-exponential jump-diffusion process and the process reflected at its infimum. They are the key to obtain Laplace transform of occupation times for hyper-exponential jump-diffusion processes using the Poisson approach which had been applied in Li and Zhou (2014) for spectrally negative Lévy processes.

Occupation-time-related derivatives have been attracted much attention from investors and researchers. A defining charateristic of these contracts is an exercise payoff that depends on the time spent by the underlying asset in a predetermined region. Typically, the specification of the occupation regions involves flat barriers. In that sense, these contracts can be viewed as a generalized type of barrier option. In reality, many occupation-time-related options are based on a discrete time monitoring. In other words, such derivatives specify a series of reference dates. The occupation time is defined through the portion of monitoring dates in which the underlying price is below or above some level or between two levels. Some research is devoted to the study of such kind of options. However, the common feature of such research is that the underlying asset price is assumed to follow a geometric Brownian Motion model. Therefore, replacing geometric Brownian Motion by hyperexponential jump-diffusion process make it more accurate.

Cai et al (2010) proposed occupation times of double-exponential jump-diffusion process, so in future work, with these fluctuation identities in hand, we can expand the idea to a more general case for the hyper-exponential jump-diffusion process.

7 APPENDIX

From Appendix A in Cai et al (2010), the equation $\psi(\theta) = q$ can be reduced to

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

where

$$a_0 = 2q\eta_1^+\eta_1^- \quad a_1 = -2(\mu\eta_1^+\eta_1^- - \lambda p(\eta_1^+ + \eta_1^-) + \lambda\eta_1^+ + q(\eta_1^+ - \eta_1^-)$$

$$a_2 = -\sigma^2\eta_1^+\eta_1^- - 2(\mu(\eta_1^+ - \eta_1^-) - \lambda - q) \quad a_3 = 2\mu - \sigma^2(\eta_1^+ - \eta_1^-) \quad a_4 = \sigma^2.$$

And the four roots are given by

$$\rho_1(q) = -\frac{a_3}{4a_4} + \frac{\delta_1 - \delta_3}{2} \quad \rho_2(q) = -\frac{a_3}{4a_4} + \frac{\delta_1 + \delta_3}{2} \quad \rho_3(q) = \frac{a_3}{4a_4} + \frac{\delta_1 - \delta_2}{2} \quad \rho_4(q) = \frac{a_3}{4a_4} + \frac{\delta_1 + \delta_3}{2}.$$

where

$$\delta_1\sqrt{M_3 + N_0 + N_1} \qquad \delta_2 = \sqrt{M_4 - N_0 - N_1 - \frac{M_5}{4\delta_1}} \qquad \delta_3 = \sqrt{M_4 - N_0 - N_1 + \frac{M_5}{4\delta_1}},$$

$$M_0 = a_2^2 - 3a_1a_3 + 12a_0a_4 \quad M_1 = 2a_2^3 - 9a_1a_2a_3 + 27a_1^2a_4 + 27a_0a_3^2 - 72a_0a_2a_4 \quad M_2 = \sqrt{M_1^2 - 4M_0^3},$$

$$M_3 = \frac{a_3^2}{4a_4^2} - \frac{2a_2}{3a_4} \qquad M_4 = \frac{a_3^2}{2a_4^2} - \frac{4a_2}{3a_4} \qquad M_5 = \frac{4a_2a_3}{a_4^2} - \frac{8a_1}{a_4} - \frac{a_3^3}{a_4^3} \qquad M_6 = \sqrt[3]{M_1 + M_2},$$

$$N_0 = \frac{\sqrt[3]{2}M_0}{3a_4M_6} \qquad N_1 = \frac{M_6}{3\sqrt[3]{2}a_4}.$$

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