A New Approach to the Planar Fractional Minkowski Problem via Curvature Flows

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ABSTRACT

A New Approach to the Planar Fractional Minkowski Problem via Curvature Flows

Shardul Vikram

The L_p-Minkowski problem, a generalization of the classical Minkowski problem, was defined by Lutwak in the '90s. For a fixed real number $p \neq n$, it asks what are the necessary and sufficient conditions on a finite Borel measure on $\mathbb{S}^{n-1}$ so that it is the L_p surface area measure of a convex body in $\mathbb{R}^n$. For $p = 1$, one has the classical Minkowski problem in which the L_p surface area is the usual surface area of a compact set embedded in $\mathbb{R}^n$.

Under certain technical assumptions, the planar L_p-Minkowski problem reduces to the study of positive, π-periodic solutions, $h : [0, 2\pi] \to (0, \infty)$ to the non-linear equation

$$h^{1-p}(h'' + h) = \psi$$

for a given smooth, π-periodic function $\psi : [0, 2\pi] \to (0, \infty)$. In this thesis, we give a new proof of the existence of solutions of the planar L_p-Minkowski problem for $0 < p < 1$. To do so, we consider a parabolic anisotropic curvature flow on the space of strictly convex bodies K in $\mathbb{R}^2$ which are symmetric with respect to the origin. The case $0 < p < 1$ has been considered before by K.S. Chou and X.J. Wang, [5], by studying the corresponding Monge-Amprère type equation.

The connection between solutions to a parabolic equation, the flow, and a corresponding elliptic equation, the L_p-Minkowski problem, has been long conjectured by the specialists and this is yet another instance where it has been used.

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I would like to thank my advisor, Dr. Alina Stancu, for taking me as her student. I express my deepest gratitude for her patience during the course of my degree and for tolerating my shortcomings. Our meetings gave me a fresh insight into the topic and kept my interest alive. I learned a lot from our discussions about mathematics as I did attending her classes. This would not have been possible without her encouragement and guidance.

Contents

Chapter 1

Preliminaries

1.1 Convex Bodies

Definition 1.1.1 (Convex Set)**.** *A set $C \subset \mathbb{R}^n$ is convex if the line segment between any two points of the set lies in the set, that is $(1-\lambda)x + \lambda y \in C$ for any $x,\ y \in C$ and $0 \leq \lambda \leq 1$. The set is strictly convex if $(1-\lambda)x + \lambda y \in \operatorname{int} C$ for any $x,\ y \in C$, and $0 < \lambda < 1$.*

Definition 1.1.2 (Convex Body)**.** *A compact convex set in $\mathbb{R}^n$ with a nonempty interior is called a convex body.*

Let $C \subset \mathbb{R}^n$ be a closed convex set, and let x be a point on the boundary of C, denoted by ∂C.

Definition 1.1.3 (Support Hyperplane)**.** *A hyperplane, $H_C(x)$, is a support hyperplane of C at x if it touches C at x, and C lies entirely on one side of $H_C(x)$. More precisely, the support hyperplane at $x \in \partial C$ is defined as:*

$$H_C(x) = \{y \in \mathbb{R}^n : \langle u, y \rangle = \langle u, x \rangle\},$$

where u is a fixed outer unit normal vector of ∂C at x. Furthermore, the support hyperplane at a point may not be unique. However, if we fix $u \in \mathbb{S}^{n-1}$, then $H_C(x)$ of normal u, denoted

$H_C(u)$*, is unique.*

If C is not empty, then C has a *support hyperplane* at each point of ∂C. Conversely, if a set $C \in \mathbb{R}^n$ has a *support hyperplane* at each point of ∂C, then C is a convex set.

A support hyperplane, $H_C(x)$, or $H_C(u)$, separates the space into two closed half-spaces, one of which contains C. In particular, a convex body C can be expressed as an intersection of all half-spaces that contain C, namely $C = \bigcap_{u \in \mathbb{S}^{n-1}} H_C^-(u)$, where $H_C^-(u)$ is the *support half-space* containing C of outer normal u.

Definition 1.1.4 (Support Function)**.** *Another representation of a convex body C is via its support function $h_C : \mathbb{S}^{n-1} \to \mathbb{R}$ which is defined by*

$$h_C(u) = \sup\{\langle x, u\rangle : x \in C\}. \tag{1.1}$$

In other words, if the origin belongs to C, the support function $h_C(u)$ is the distance from the origin to the support hyperplane $H_C(u)$ of outer normal vector u at the boundary of C.

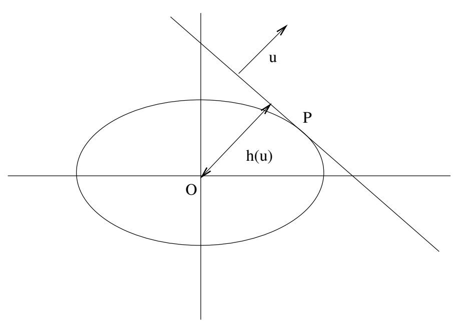

Figure 1.1: Support function h(u) of a convex body in direction of unit normal u.

Definition 1.1.5 (Hausdorff metric)**.** *Let $\mathcal{C}^n$ be a space of compact subsets of $\mathbb{R}^n$. The Hausdorff distance of two closed, convex sets $C, D \in \mathcal{C}^n$ is defined by:*

$$\delta(C, D) = \max\left\{\sup_{x\in C}\inf_{y\in D}|x-y|, \sup_{x\in D}\inf_{y\in C}|x-y|\right\} \tag{1.2}$$

or, equivalently by

$$\delta(C, D) = \min\{\lambda \geq 0 : C \subseteq D + \lambda B^n, D \subseteq C + \lambda B^n\}, \tag{1.3}$$

where B^n is the Euclidean unit ball in $\mathbb{R}^n$ centered at the origin.

Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$. Then $\mathcal{K}^n$ is a metric space with the *Hausdorff metric*, [30]. For convex bodies $K, L \in \mathcal{K}^n$, the Hausdorff distance between the two bodies is the same as the Hausdorff distance between their boundaries, i.e., $\delta(K, L) = \delta(\partial K, \partial L)$, which is essentially equivalent to sup norm of the difference between their support functions, $\delta(K, L) = \underset{u \in \mathbb{S}^{n-1}}{sup} \{|h_K(u) - h_L(u)|\}$.

For more details on convex bodies and their properties, see [30].

1.2 L_p-Minkowski Problem

Definition 1.2.1 (Surface Area Measure)**.** *Let ω a Borel subset of $\mathbb{S}^{n-1}$. Then the surface area measure of a convex body $K \in \mathbb{R}^n$ can be viewed as a measure on $\mathbb{S}^{n-1}$ such that $S(K, \omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of points on ∂K that have an outer unit normal in ω.*

Let μ be a finite Borel measure on $\mathbb{S}^{n-1}$. Given μ, the Minkowski problem asks what are the necessary and suffcient conditions on μ that guarantee that μ is the surface area measure of a convex body $K \in \mathbb{R}^n$, that is $\mathrm{S}(K, \cdot) = \mu$.

Let ω be a Borel subset of $\mathbb{S}^{n-1}$. The L_p surface area measure of a convex body $K \in \mathbb{R}^n$ that contains the origin, $\mathrm{S}_p(K, \omega)$, is equivalent to

$$\mathrm{S}_p(K, \omega) = \int_{u \in \omega} h_K^{1-p}(u)\, \mathrm{dS}(K, u), \tag{1.4}$$

where $h_K : \mathbb{S}^{n-1} \to (0, \infty)$ is the support function of K.

For a fixed real number $p \neq n$, the L_p-Minkowski problem asks what are the necessary and sufficient conditions on μ so that $\exists K \subset \mathbb{R}^n$ convex body with

$$\mu(\omega) = \mathrm{S}_p(K, \omega), \quad \forall \omega \subset \mathbb{S}^{n-1}. \tag{1.5}$$

The smooth L_p-Minkowski problem is equivalent to studying the existence and uniqueness of positive solutions to the following Monge-Ampère equation on the standard sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, [30] :

$$h^{1-p} \det(h_{ij} + h\delta_{ij}) = \psi, \tag{1.6}$$

where $p \neq n$, $\psi : \mathbb{S}^{n-1} \to (0, \infty)$ (also called *data*) is a given smooth density function of μ with respect to the spherical Lebesgue measure, h_{ij} is the covariant derivative of h with respect to an orthonormal frame on $\mathbb{S}^{n-1}$ and δ_{ij} is the Kronecker delta symbol.

In the planar form, given $\psi : \mathbb{S}^1 \to (0, \infty)$, and $p \neq 2$, the smooth L_p-Minkowski problem is equivalent to seeking positive solutions to the following equation:

$$h^{1-p}(h_{\theta\theta} + h) = \psi. \tag{1.7}$$

If there exists a positive solution h to this equation, then the convex body with the support function h is a solution to the L_p-Minkowski problem with given data.

The case $p = 1$, the classical problem, was solved for atomic measures by Minkowski [27].

Alexandrov, Fenchel and Jessen gave the complete solution to this problem for arbitrary measures, see [30] for an extensive discussion. Among those who contributed to establishing regularity for the Minkowski problem were Lewy [23], Nirenberg [28], Cheng and Yau [4], Pogorelov [29] and Caffarelli [2]. Lutwak generalized the problem and showed that there is an L_p equivalent of the surface area measure in [24], [25].

The case $p = 0$ is called the logarithmic Minkowski problem. The planar L_p-Minkowski problem for $p = 0$ and discrete measure was solved by Stancu [31], [32], [33]. Böröczky, et al., established the necessary and sufficient conditions for the existence of the solutions to the L_0-Minkowski problem in $\mathbb{R}^n$ [1].

For the case $p > 1$, a solution to the L_p-Minkowski problem, under the assumption that the measure μ is *even* (antipodal Borel sets of $\mathbb{S}^{n-1}$ have the same measure), was given by Lutwak [24].

The L_p-Minkowski problem without the assumption that the data was even, was studied by Guan and Lin [14], and by Chou and Wang [5].

Other planar L_p-Minkowski problems were studied by Umanskiy [35] for $p \neq 0$ with continuous and T-periodic ($0 < T < 1$) data, by Chen [3] for $-2 < p < 0$ with continuous, but not necessarily positive data and by Jiang [21] for $0 < p < 2$ with T-periodic ($T \leq \pi$) continuous data.

In the past, a substantial amount of work on the L_p-Minkowski problem was done for the case $p > 1$, where mixed volume inequalities can be used to show uniqueness of solutions, see [24]. This is not the case with $p < 1$, making the problem challenging.

Recent work on the discrete L_p-Minkowski problem for $0 < p < 1$ is done for *polytopes* by Zhu in [36]. A *polytope* is the convex hull of a finite set of points in R^n with a positive n-dimensional volume.

Important works on extensions and generalizations of the Minkowski problem are shown in [8], [13], [15], [17], [18], [20]. In addition, the solution to the even L_p-Minkowski problem led Lutwak, Yang, and Zhang [26] to extend the affine Sobolev inequality and obtain the L_p affine Sobolev inequality, and later enabled Cianchi et al. [7] to establish the affine Moser-Trudinger and the affine Morrey-Sobolev inequalities.

1.3 Geometric Flows

The study of geometric flows arose from the classical study of heat flow on n-dimensional surfaces, [19]. Many properties of the heat equation, like the maximum principle, regularity

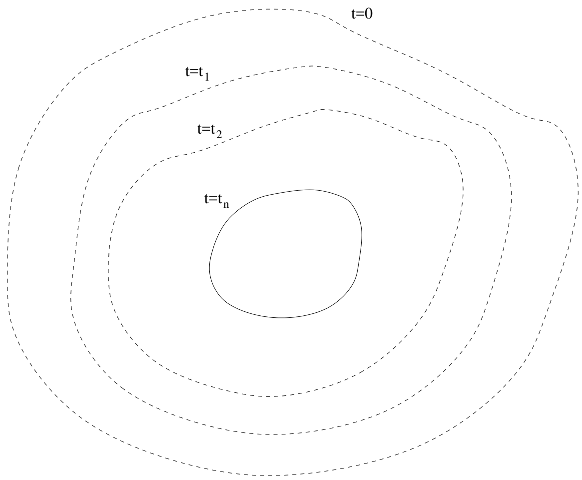

Figure 1.2: Curves evolving in time.

and convergence of solutions carry over to other parabolic equations describing what we call geometric flows.

The Mean Curvature Flow (MCF) is the flow along the negative gradient of the area functional where each point on the surface moves in the direction of the normal with a velocity equal to the mean curvature. The one-dimensional MCF is called the curve shortening flow (CSF), with the pointwise speed equal to the curvature at that point, [6], [9]. In general, the evolution of a plane smooth curve by a curvature flow is determined by a velocity which depends on the curvature of curve, k, and the direction of the normal, at each point on the curve.

The partial differential equation describing curvature flows on smooth planar curves can be written as:

$$X_t = \Psi(\theta) f(k) N, \tag{1.8}$$

where X is the position vector of the curve, N is the inner normal to the curve, $f : \mathbb{R} \to \mathbb{R}$ is an increasing, positive function, and Ψ is a smooth, strictly positive function of direction θ. The curve shortening flow is the case where $\Psi(\theta) = 1$ and f is the identity function. When Ψ is not a constant function, the flow is called anisotropic, otherwise we say that the flow is isotropic.

Grayson, [12], showed that any embedded, closed planar curve will become strictly convex under curve shortening flow. Gage and Hamilton, [9], [10], proved that after the flow becomes convex it shrinks to a point in finite time and in the process becomes circular. For dimensions $n \geq 2$, Huisken [19] showed that a convex hypersurface in $\mathbb{R}^{n+1}$ will shrink to a point approaching the shape of the sphere.

In this thesis, we study an anisotropic p-weighted planar curvature flow where $0 < p < 1$ on the space of compact, and convex curves. We show that the flow is parabolic and preserves convexity and central symmetry of solutions, as long as the flow exists. We study the long term existence of this flow and we show that, asymptotically, all solutions to the flow converge subsequentially, up to a dilation, to a solution of the L_p-Minkowski problem. Our approach relies on the monotonicity of a functional, called entropy of the flow, whose bounds from above and below lead to the compactness of solutions to the normalized flow. These two ingredients allow us to apply the Blaschke selection theorem and conclude the convergence of the evolving convex bodies to a solution of the planar fractional Minkowski problem.

Chapter 2

A new planar anisotropic flow

Let K be a compact, strictly convex body in $\mathbb{R}^2$, symmetric with respect to the origin. Let $x_K : \mathbb{S}^1 \to \mathbb{R}^2$ be the Gauss parameterization of its boundary γ_K, hence $x_K(z)$ is the position vector of the point on ∂K of outer normal z.

The support function of K, $h : \mathbb{S}^1 \to [0, \infty)$, is defined by $h(z) = \langle x(z), z \rangle$, for each $z \in \mathbb{S}^1$, [30]. We denote the curvature of the boundary by $k : \mathbb{S}^1 \to \mathbb{R}$.

Given the fixed convex body K, consider the following initial value problem:

$$\frac{\partial x(z,t)}{\partial t} = -\psi(z)\langle x(z,t), z\rangle^p k(z,t) z, \quad x(z,0) = x_K(z), \tag{2.1}$$

where $p \in (0, 1)$ is a fixed real number, and $\psi : \mathbb{S}^1 \to (0, \infty)$ is a smooth, even and a π-periodic function, $\psi(z) = \psi(-z)$.

In what will follow, we will work with the scalar form of the flow instead of its vector form given above. The scalar form is obtained from the inner product of equation (2.1) with the unit outer normal z, which does not depend on t :

$$\langle \frac{\partial x}{\partial t}, z \rangle = -\psi(z)\langle x(z,t), z\rangle^p k(z,t).$$

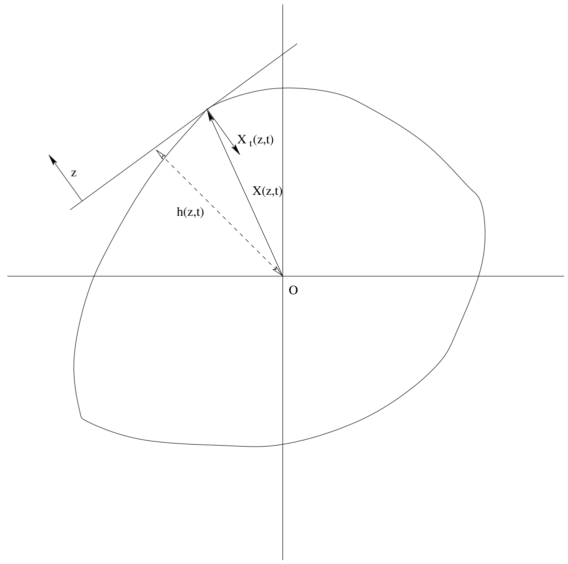

Figure 2.1: Vector parameterization of the curve at time t.

Thus, the scalar form of the initial value problem above is:

$$h_t(z,t) = -\psi(z)h(z,t)^p k(z,t), \quad h(z,0) = h_K(z), \tag{2.2}$$

where $h(z,t) = \langle x(z), z\rangle$, and $h_t(z) = \langle x_t(z), z\rangle = \dfrac{\partial h(z,t)}{\partial t}$.

Furthermore, all functions on $\mathbb{S}^1$ will be considered as 2π-periodic functions of θ on $[0, 2\pi]$ via the identification $z = (\cos\theta, \sin\theta)$. Expressing h, k, ψ as functions of θ and t, equation (2.2) becomes

$$h_t(\theta,t) = -\psi(\theta)h(\theta,t)^p k(\theta,t), \quad h(\theta,0) = h_K(\theta). \tag{2.3}$$

We call a solution to the flow (2.3) a family of smooth, convex, closed curves parameterized by time whose support function, at each time t is $h(\cdot,t) : \mathbb{S}^1 \times \{t\} \to \mathbb{R}$ and satisfies the above initial value problem. Similarly, we call a solution to the flow (2.1) a family of smooth, convex bodies parameterized by time whose boundary, at each time t is $x(\cdot,t) : \mathbb{S}^1 \times \{t\} \to \mathbb{R}$ and satisfies the vector valued initial value problem. It is well known in the theory of geometric flows that studying the scalar flow (2.3) is equivalent to the study of the vector valued flow (2.1) in the sense that, for any given initial convex body, the two corresponding solutions will differ by, at most, a tangential re-parameterization, [6].

We want to emphasize that, at $t = 0$, the curve is strictly convex, thus the curvature of ∂K is strictly positive, $k(\theta, 0) > 0, \ \forall\theta \in [0, 2\pi]$. It is worth noting that the curvature k is related to the support function h by

$$\frac{1}{k(\theta)} = h_{\theta\theta}(\theta) + h(\theta), \tag{2.4}$$

where $h_{\theta\theta}$ stands for the second derivative of h with respect to θ, [30]. For simplicity of writing, here and thereafter, we will drop the parameters θ and t for the support function, h, the curvature, k, and the weight function, ψ, unless their absence leads to ambiguity.

2.1 Parabolicity of the flow

To show that our flow is parabolic, we rely on finding its linearization. Let

$$\tilde{h}(\theta, t) = h(\theta, t) + \epsilon\phi(\theta, t), \tag{2.5}$$

where ϕ is an arbitrary positive, smooth function on $\mathbb{S}^1$, and ϵ is a sufficiently small real number. So, assuming that $\tilde{h}$ is a solution to the PDE (2.3), we have

$$(h + \epsilon\phi)_t = \frac{-\psi(h + \epsilon\phi)^p}{(h + \epsilon\phi)_{\theta\theta} + (h + \epsilon\phi)}.$$

Differentiating with respect to ϵ, and evaluating for $\epsilon = 0$, we obtain:

$$\frac{d(h + \epsilon\phi)_t}{d\epsilon}_{|\epsilon=0} = \frac{d}{d\epsilon}\left[\frac{-\psi(h + \epsilon\phi)^p}{(h + \epsilon\phi)_{\theta\theta} + (h + \epsilon\phi)}\right]_{|\epsilon=0}$$

$$\phi_t = \left[\frac{-\psi p(h+\epsilon\phi)^{p-1}\phi}{(h+\epsilon\phi)_{\theta\theta}+(h+\epsilon\phi)} + \frac{\psi(h+\epsilon\phi)^p}{((h+\epsilon\phi)_{\theta\theta}+(h+\epsilon\phi))^2}(\phi_{\theta\theta}+\phi)\right]_{|\epsilon=0}$$
$$\phi_t = \frac{-\psi p h^{p-1}\phi}{h_{\theta\theta}+h} + \frac{\psi h^p}{(h_{\theta\theta}+h)^2}(\phi_{\theta\theta}+\phi).$$

Thus, the linearization of the flow is

$$\phi_t = \psi h^p k^2 \phi_{\theta\theta} + (\psi h^p k^2 - \psi p h^{p-1} k)\phi. \tag{2.6}$$

As $\psi h^p k^2 > 0$ at time zero, the linearization of the flow is strictly parabolic and hence the flow is strictly parabolic. The short term existence of solutions to the flow (2.3) follows now from the strict parabolicity of the flow equation, [22].

Lemma 2.1.1. *The central symmetry of the evolving curves is preserved.*

Proof. The symmetry of the support function, $h(\theta,0)$, at time zero and that of $\psi(\theta)$ follows from the hypothesis,

$$h(\theta,0) = h(\theta+\pi,0)$$
$$\psi(\theta) = \psi(\theta+\pi)$$

and, from (2.4), so does the π-periodicity of curvature k at time zero,

$$k(\theta,0) = k(\theta+\pi,0).$$

As the curve evolves, at time t,

$$\begin{aligned} h_t(\theta+\pi) &= -\psi(\theta+\pi)h^p(\theta+\pi)k(\theta+\pi), \\ &= -\psi(\theta)h^p(\theta)k(\theta), \qquad (2.7) \\ &= h_t(\theta). \end{aligned}$$

Thus concluding that central symmetry is preserved for all time the flow exists. Later on, we will see that this property of the curves easily gives us the width of the curve in direction $\theta \in [0, 2\pi)$.

$\square$

2.2 Study of the flow on the interval of maximal time of existence

Proposition 2.2.1. *Let K be a strictly convex body in $\mathbb{R}^2$, symmetric with respect to the origin. Then the solution $h(\cdot, t)$ of $h_t = -\psi h^p k$ with h_K as initial data exists as long as $\min_{\theta \in [0,2\pi]} h(\theta, t) > 0$.*

The proof of the proposition is a consequence of the following lemmas.

Lemma 2.2.1. *The flow preserves convexity.*

Proof. We want to show that $k > 0$ for $t \geq 0$ as long as $\min_{\theta \in [0,2\pi]} h(\theta, t) > 0$. Let ω be the first time when $\min_{\theta \in [0,2\pi]} h(\theta, t) = 0$ and assume that no singularity occurs until ω. For $t < \omega$, let $\mathrm{G}(\theta, t) = \psi(\theta) h^p(\theta, t) k(\theta, t)$ and let $\Gamma(t) = \min_{\theta \in [0,2\pi]} \mathrm{G}(\theta, t)$. When G is minimum with respect to θ, we have $\mathrm{G}_\theta = 0, \mathrm{G}_{\theta\theta} \geq 0$. Consider the evolution equation of G:

$$
\begin{aligned}
\mathrm{G}_t &= p\psi h^{p-1} k h_t + \psi h^p k_t \\
&= -p\psi h^{p-1} k(-\psi h^p k) - \psi h^p k^2 (h_{t\theta\theta} + h_t) \\
&= -p\psi h^{p-1} k(-\psi h^p k) + \psi h^p k^2 [(\psi h^p k)_{\theta\theta} + (\psi h^p k)].
\end{aligned}
\tag{2.8}
$$

Thus, from the conditions on Γ,

$$
\begin{aligned}
\Gamma_t &\geq (\psi h^p k)^2 \left(k - \frac{p}{h}\right) \\
&\geq (\psi h^p k)^2 k - (\psi h^p k)^2 \frac{p}{h} \\
&\geq -\frac{p}{h} \Gamma^2.
\end{aligned}
\tag{2.9}
$$

Suppose $\exists t_* < \omega$ the first time when $\min_{\theta\in[0,2\pi]} k(\theta, t_*) = 0$. From the hypothesis, at $t = t_*$, we have that $\min\limits_{\theta\in[0,2\pi]} h(\theta, t) =: h^* > 0$. Thus, before time t_*, the function $\Gamma(t)$ is not zero. So, for $t \in [0, t_*)$:

$$\frac{\Gamma_t}{\Gamma^2} \geq -c, \tag{2.10}$$

where $c = \max\limits_{[0,2\pi]\times[0,t*]} \dfrac{p}{h} > 0$.

For all times, $0 \leq \forall t_1, t_2 < t_*$, we have

$$\begin{aligned} \int_{t_1}^{t_2} \frac{\Gamma_t}{\Gamma^2}\,\mathrm{d}t &\geq \int_{t_1}^{t_2} -c\,\mathrm{d}t, \\ -\frac{1}{\Gamma(t_2)} + \frac{1}{\Gamma(t_1)} &\geq -c\,(t_2 - t_1). \end{aligned} \tag{2.11}$$

In particular, if $t_1 = 0$, then

$$-\frac{1}{\Gamma(t_2)} + \frac{1}{\Gamma(0)} \geq -c\,t_2. \tag{2.12}$$

As $t_* < \omega$, then by letting $t_2 \to t_*$, we get

$$\lim_{t_2\to t_*}\left(-\frac{1}{\Gamma(t_2)} + \frac{1}{\Gamma(0)} \right) \geq -c\,t_*. \tag{2.13}$$

The left side of the inequality approaches $-\infty$, leading to a contradiction with $t_* < \omega$. Therefore, it must be that $t_* \geq \omega$ and $\min\limits_{\theta\in[0,2\pi]} k(t) > 0$ for $t \in [0, \omega)$. Hence the flow preserves convexity for all time until $\min\limits_{\theta\in[0,2\pi]} h(\theta, t) = 0$. $\square$

We will now show that no other type of singularity can occur prior to time ω.

Lemma 2.2.2. *For any fixed $\tau < \omega$, k is uniformly bounded above on $[0, \tau]$.*

Proof. We follow the method of Tso [34], to show that the curvature is bounded above.

Since $\tau < \omega$, $\exists \rho > 0$ such that $h \geq 2\rho$ on $[0, \tau]$. Consider

$$Y = -\frac{h_t}{h-\rho} = \frac{\psi h^p k}{h-\rho}. \tag{2.14}$$

Its derivative with respect to θ is:

$$Y_\theta = \frac{(\psi h^p k)_\theta}{h-\rho} - \frac{(\psi h^p k)}{(h-\rho)^2} h_\theta. \tag{2.15}$$

Since Y is a continuous function on a compact, $\exists\, (\theta_0, t_0)$ such that $Y(\theta_0, t_0) = \max\{Y(\theta, t) | (\theta, t) \in \mathbb{S}^1 \times [0, \tau]\}$.

Without any loss of generality, we may assume $t_0 > 0$. Then at (θ_0, t_0), we have:

$$Y_\theta = \frac{(\psi h^p k)_\theta}{h-\rho} - \frac{(\psi h^p k)}{(h-\rho)^2} h_\theta = 0. \tag{2.16}$$

Thus,

$$\frac{(\psi h^p k)_\theta}{h-\rho} = \frac{(\psi h^p k)}{(h-\rho)^2} h_\theta \tag{2.17}$$

and

$$\begin{aligned} Y_{\theta\theta} &= \frac{(\psi h^p k)_{\theta\theta}}{h-\rho} - 2\frac{(\psi h^p k)_\theta}{(h-\rho)^2} h_\theta - \frac{\psi h^p k}{(h-\rho)^2} h_{\theta\theta} + 2\frac{\psi h^p k}{(h-\rho)^3} h_\theta^2 \\ &= \frac{(\psi h^p k)_{\theta\theta}}{h-\rho} - \frac{\psi h^p k}{(h-\rho)^2} h_{\theta\theta}. \end{aligned} \tag{2.18}$$

At (θ_0, t_0), $Y_{\theta\theta} \leq 0$, which implies

$$\frac{(\psi h^p k)_{\theta\theta}}{h-\rho} \leq \frac{\psi h^p k}{(h-\rho)^2} h_{\theta\theta}. \tag{2.19}$$

Therefore,

$$
\begin{aligned}
Y_t &= \frac{(\psi h^p k)_t}{h-\rho} + \left(\frac{\psi h^p k}{h-\rho}\right)^2 \\
&= -\frac{\psi^2 p h^{2p-1} k^2}{h-\rho} + \psi h^p k^2 \frac{(\psi h^p k)_{\theta\theta}}{h-\rho} + \frac{\psi^2 h^{2p} k^3}{h-\rho} + \left(\frac{\psi h^p k}{h-\rho}\right)^2 .
\end{aligned}
\tag{2.20}
$$

At (θ_0, t_0), $Y_t \geq 0$, which further implies

$$
\frac{\psi^2 p h^{2p-1} k^2}{h-\rho} \leq \psi h^p k^2 \frac{(\psi h^p k)_{\theta\theta}}{h-\rho} + \frac{\psi^2 h^{2p} k^3}{h-\rho} + \left(\frac{\psi h^p k}{h-\rho}\right)^2 .
\tag{2.21}
$$

By equation (2.19),

$$
\begin{aligned}
\frac{\psi^2 p h^{2p-1} k^2}{h-\rho} &\leq \psi h^p k^2 \frac{\psi h^p k}{(h-\rho)^2} h_{\theta\theta} + \frac{\psi^2 h^{2p} k^3}{h-\rho} + \left(\frac{\psi h^p k}{h-\rho}\right)^2 \\
0 &\leq \left(\frac{\psi h^p k}{h-\rho}\right)^2 [k(h-\rho) - p + \frac{p}{h}\rho + 1 + k h_{\theta\theta}] \\
0 &\leq \left(\frac{\psi h^p k}{h-\rho}\right)^2 [k(h+h_{\theta\theta}) - p - k\rho + \rho\frac{p}{h} + 1].
\end{aligned}
\tag{2.22}
$$

Simplifying the equation above, we get

$$
0 \leq \left(\frac{\psi h^p k}{h-\rho}\right)^2 [2 - p - k\rho + \rho\frac{p}{h}].
\tag{2.23}
$$

As the first factor of the above equation is always positive, the second term must be greater than or equal to zero. This gives an upper bound on k as follows,

$$
0 \leq 2 - p - k\rho + p\frac{\rho}{h},
\tag{2.24}
$$

therefore

$$k \le \frac{1}{\rho}\,[2 - p(1 - \frac{\rho}{h})]. \tag{2.25}$$

Since $h \ge 2\rho$, then $\frac{1}{2} \ge \frac{\rho}{h}$ and $1 - \frac{\rho}{h} \ge \frac{1}{2}$. As $p \in (0,1)$, we have $p(1 - \frac{\rho}{h}) \ge \frac{p}{2}$ and $2 - p(1 - \frac{\rho}{h}) \le 2 - \frac{p}{2}$. This shows that $k \le \frac{2}{\rho}$.

□

Next we state the *Containment Principle* that is used to show that the flow exists for a finite time.

Proposition 2.2.2 (Containment Principle)**.** *Let γ_{in} and γ_{out} be two solutions to the scalar initial value problem* (2.2)*. Suppose at $t = 0$, $\gamma_{in}(\cdot, 0)$ lies inside the domain enclosed by $\gamma_{out}(\cdot, 0)$. Then $\gamma_{in}(\cdot, t)$ is contained in the domain of $\gamma_{out}(\cdot, t)$ $\forall t \in [0, \omega)$, where ω is the time when γ_{in} ceases to exist.*

Proof. At $t = 0$, $\gamma_{in}(\cdot, 0) \subset \gamma_{out}(\cdot, 0)$ and suppose that the curves intersect at some time t, $0 < t < \omega$. Then, at some time $t_0 < t$, the two solutions touch for the first time in the direction $\theta_0 \in [0, 2\pi)$, see figure 2.2. Since the curves are convex, in the direction θ_0 and at the point of tangency, the support function $h_{in}(\theta_0, t_0) = h_{out}(\theta_0, t_0)$ and $k_{in}(\theta_0, t_0) \ge k_{out}(\theta_0, t_0)$. Therefore, at (θ_0, t_0), we have:

$$\begin{aligned} |\frac{\partial h_{in}}{\partial t}(\theta_0, t_0)| &= | - \psi(\theta_0, t_0)\; h_{in}^p(\theta_0, t_0)\; k_{in}(\theta_0, t_0)| \\ &\ge |-\psi(\theta_0, t_0)\; h_{out}^p(\theta_0, t_0)\; k_{out}(\theta_0, t_0)| \\ &\ge |\frac{\partial h_{out}}{\partial t}(\theta_0, t_0)|. \end{aligned} \tag{2.26}$$

It could also be the case that the inequality above is strict. Since the speed of γ_{in} is greater than or equal to that of γ_{out}, the curve γ_{in} moves in faster, if not the same, than the curve γ_{out} along the direction of the inner normal. This shows that the inner curve will not

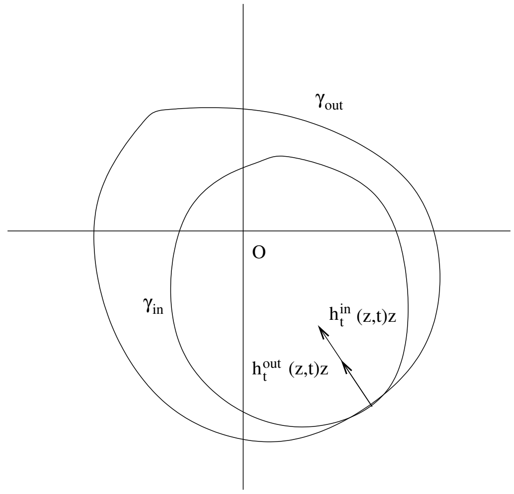

Figure 2.2: Two curves have velocity vectors in the same direction at the point where they touch.

leave the domain enclosed by the outer curve and the curves will either move simultaneously or become disjoint as they evolve after they touch.

□

Now, we will show that the flow exists for a finite time.

Lemma 2.2.3. *The total time of existence of the flow is finite.*

Proof. This follows by the use of the containment principle.

Let $\mathcal{F}(t)$ be the family of circles centered at the origin and parameterized by time and having radius $R(t)$ at time $t \geq 0$. At $t = 0$, let $\mathcal{F}(0)$ be a sufficiently large circle containing the initial curve C_0, see figure 2.3. The support function of the circle at time t is $R(t)$ and the curvature is $R(t)^{-1}$.

The circles are set to flow by the equation:

$$h_t = -(\min_{\theta} \psi)\ h^p k, \tag{2.27}$$

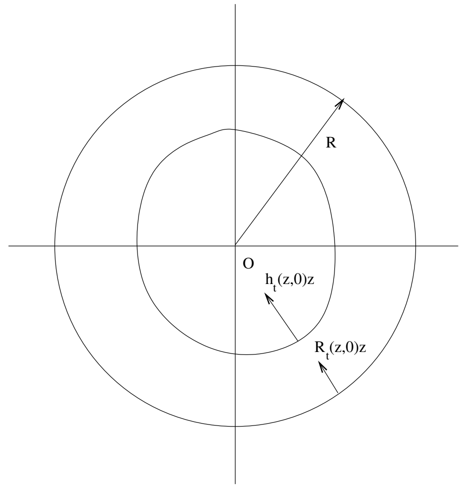

Figure 2.3: At time t=0, the initial curve contained in a circle.

or, equivalently,

$$\begin{aligned} R_t &= -(\min_\theta \psi)\ R^p k \\ &= -(\min_\theta \psi)\ R^{p-1}. \end{aligned} \tag{2.28}$$

Rearranging (2.28), we get

$$\frac{R_t}{R^{p-1}} = -\beta, \tag{2.29}$$

where $\beta = (\min_\theta \psi)$.

Recall the definition of ω, so that, for time $0 \leq t < \omega$, we have:

$$\int_0^t \frac{R_t}{R^{p-1}}\ \mathrm{d}t = \int_0^t -\beta\ \mathrm{d}t, \tag{2.30}$$

and, hence,

$$\frac{R(t)^{2-p}}{2-p} = -\beta\ t + \frac{R(0)^{2-p}}{2-p}. \tag{2.31}$$

Then there exists $t_0 < \infty$ such that $R(t_0) = 0$ which means that time of existence of the flow on the circles is finite. Due to the containment principle, the circles will contain the evolving curve with initial conditions C_0 as long as $t \leq \omega$, thus $\omega \leq t_0$.

$\square$

Chapter 3

Long term existence of the flow

3.1 The normalized flow

We will study the asymptotic behavior of the solution to our initial value problem at time ω. We re-scale the solution so that the normalized curves enclose constant area 1 and the origin remains inside the evolving curves. To do so, we define the support function of the normalized curve by:

$$\tilde{h}(\theta, t) = \frac{h(\theta, t)}{\sqrt{A(t)}}, \quad t \in [0, \omega), \tag{3.1}$$

thus the curvature becomes

$$\tilde{k}(\theta, t) = k(\theta, t)\sqrt{A(t)}, \tag{3.2}$$

where $A(t)$ is the area enclosed by the un-normalized curve at time t.

We perform a change of variable from $t \in [0, \omega)$ to a new time variable, $\tau \in [0, \infty)$, in the following manner

$$\tau = -\frac{1}{2} \ln \frac{A(t)}{A(0)}. \tag{3.3}$$

At $t = 0$, $A(t) = A(0)$, set to be 1, and $\ln \frac{A(0)}{A(0)} = 0$, so $\tau = 0$. As $t \to \omega$, the time when the flow collapses, the area $A(t)$ enclosed by the evolving curves goes to zero, so $\tau \to \infty$.

To find $\tilde{h}_\tau$ and $\tilde{k}_\tau$, we use the following relations, $\tilde{h}_\tau = \tilde{h}_t \frac{\mathrm{d}t}{\mathrm{d}\tau}$ and $\tilde{k}_\tau = \tilde{k}_t \frac{\mathrm{d}t}{\mathrm{d}\tau}$, where $\frac{\mathrm{d}t}{\mathrm{d}\tau} = -2\frac{A(t)}{A_t}$.

We calculate $\tilde{h}_t$ first:

$$\begin{aligned}
\tilde{h}_t &= \frac{h_t}{\sqrt{A(t)}} - \frac{1}{2}\frac{h}{A(t)^{3/2}}A_t \\
&= -\frac{\psi h^p k}{\sqrt{A(t)}} - \frac{1}{2}\frac{h}{A(t)^{3/2}}A_t \\
&= -\frac{1}{\sqrt{A(t)}}\left(\psi h^p k + \frac{1}{2}\frac{h}{A(t)}A_t\right).
\end{aligned} \tag{3.4}$$

The value of A_t is computed from

$$A(t) = \frac{1}{2}\int_{S^1} \frac{h}{k}\,\mathrm{d}\theta = \frac{1}{2}\int_{S^1} h(h_{\theta\theta} + h)\,\mathrm{d}\theta. \tag{3.5}$$

Differentiating with respect to t,

$$A_t = \frac{1}{2}\int_{S^1} h_t(h_{\theta\theta} + h) + h(h_{\theta\theta} + h)_t\,\mathrm{d}\theta. \tag{3.6}$$

For the term $h(h_{\theta\theta})_t$, the integral is:

$$\int_{S^1} h(h_{\theta\theta})_t\,\mathrm{d}\theta = h_{t\theta}h\big|_0^{2\pi} - \int_0^{2\pi} h_{t\theta}h_\theta\,\mathrm{d}\theta = -\int_0^{2\pi} h_{t\theta}h_\theta\,\mathrm{d}\theta, \tag{3.7}$$

where

$$\int_0^{2\pi} h_{t\theta}h_\theta\,\mathrm{d}\theta = h_t h_\theta\big|_0^{2\pi} - \int_0^{2\pi} h_t h_{\theta\theta}\,\mathrm{d}\theta = \int_{S^1} h_t h_{\theta\theta}\,\mathrm{d}\theta. \tag{3.8}$$

Substituting (3.8) into (3.6) we get,

$$\begin{aligned} A_t &= \frac{1}{2}\left(\int_{S^1} h_t(h_{\theta\theta} + h) \, \mathrm{d}\theta + \int_{S^1} h_t h_{\theta\theta} \, \mathrm{d}\theta + \int_{S^1} h h_t \, \mathrm{d}\theta \right) \\ &= \frac{1}{2}\left(2 \int_{S^1} h_t(h_{\theta\theta} + h) \, \mathrm{d}\theta \right) \\ &= \int_{S^1} \frac{h_t}{k} \, \mathrm{d}\theta. \end{aligned} \tag{3.9}$$

Therefore,

$$A_t = -\int_{S^1} \psi h^p \, \mathrm{d}\theta. \tag{3.10}$$

Finally, we evaluate $\tilde{h}_\tau$,

$$\begin{aligned} \tilde{h}_\tau &= \tilde{h}_t \, \frac{\mathrm{d}t}{\mathrm{d}\tau} \\ &= \left(-\frac{\psi h^p k}{\sqrt{A(t)}} - \frac{1}{2} \frac{h}{A(t)^{3/2}} A_t \right) \frac{-2A(t)}{A_t} \\ &= \tilde{h} + 2\sqrt{A(t)} \frac{\psi h^p k}{A_t} \\ &= \tilde{h} + 2 \frac{\psi h^p \tilde{k}}{-\int_{S^1} \psi h^p \, \mathrm{d}\theta} \\ &= \tilde{h} + 2 \frac{\frac{\psi h^p}{A(t)^{p/2}} \tilde{k} A(t)^{p/2}}{-A(t)^{p/2} \int_{S^1} \psi \frac{h^p}{A(t)^{p/2}} \, \mathrm{d}\theta}. \end{aligned} \tag{3.11}$$

Therefore,

$$\tilde{h}_\tau = \tilde{h} - \frac{\psi \tilde{h}^p \tilde{k}}{\frac{1}{2} \int_{S^1} \psi \tilde{h}^p \, \mathrm{d}\theta}. \tag{3.12}$$

We can now show that the area of the evolving normalized curve, with area 1 at time zero, remains constant for all time $\tau \in [0,\infty)$:

$$\begin{aligned}
\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tau} &= \int_{S^1} \frac{\mathrm{d}\tilde{h}}{\mathrm{d}\tau}\frac{1}{\tilde{k}}\,\mathrm{d}\theta \\
&= \int_{S^1} \left(\frac{\tilde{h}}{\tilde{k}} - \frac{\psi\tilde{h}^p}{\frac{1}{2}\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta}\right) d\theta \\
&= \int_{S^1} \frac{\tilde{h}}{\tilde{k}}\mathrm{d}\theta - \frac{1}{\frac{1}{2}\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta}\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta \\
&= 2\tilde{A}(\tau) - 2 = 0.
\end{aligned} \tag{3.13}$$

Next, we calculate the evolution of the curvature function, $\tilde{k}_\tau$:

$$\begin{aligned}
\tilde{k}_\tau &= \left(k_t\sqrt{A(t)} + \frac{1}{2}\frac{k}{\sqrt{A(t)}}A_t\right)\frac{-2A(t)}{A_t} \\
&= -2k_t\frac{A(t)^{3/2}}{A_t} - k\sqrt{A(t)} \\
&= -2k_t\frac{A(t)^{3/2}}{A_t} - \tilde{k}.
\end{aligned} \tag{3.14}$$

We differentiate (2.4) with respect to t to get k_t:

$$\begin{aligned}
k_t &= -k^2\Big(h_{t\theta\theta} + h_t\Big) \\
&= -k^2\Big((-\psi h^p k)_{\theta\theta} + (-\psi h^p k)\Big) \\
&= k^2\left((\psi\tilde{h}^p A(t)^{p/2}\frac{\tilde{k}}{\sqrt{A(t)}})_{\theta\theta} + (\psi\tilde{h}^p A(t)^{p/2}\frac{\tilde{k}}{\sqrt{A(t)}})_t\right) \\
&= \frac{k^2}{A(t)^{\frac{1-p}{2}}}\Big((\psi\tilde{h}^p\tilde{k})_{\theta\theta} + (\psi\tilde{h}^p\tilde{k})\Big) \\
&= \frac{\tilde{k}^2}{A(t)^{\frac{3-p}{2}}}\Big((\psi\tilde{h}^p\tilde{k})_{\theta\theta} + (\psi\tilde{h}^p\tilde{k})\Big).
\end{aligned} \tag{3.15}$$

So, we have

$$\tilde{k}_\tau = -\tilde{k} + \frac{-2A(t)^{\frac{p}{2}}}{A_t}\tilde{k}^2\Big((\psi\tilde{h}^p\tilde{k})_{\theta\theta} + (\psi\tilde{h}^p\tilde{k})\Big). \tag{3.16}$$

On further simplification of (3.16), we get the following expression for $\tilde{k}_\tau$:

$$\tilde{k}_\tau = -\tilde{k} + \frac{\tilde{k}^2}{\frac{1}{2}\int_{S^1} \psi \tilde{h}^p \, \mathrm{d}\theta} \Big((\psi \tilde{h}^p \tilde{k})_{\theta\theta} + (\psi \tilde{h}^p \tilde{k}) \Big). \tag{3.17}$$

3.2 The entropy of the normalized flow

We define the entropy, a monotone functional, of the un-normalized flow, $\mathcal{E}(t) : [0,\omega) \to \mathbb{R}$, by

$$\mathcal{E}(t) = \frac{1}{A^{\frac{p}{2}}} \int_{S^1} \psi h^p \, \mathrm{d}\theta. \tag{3.18}$$

The entropy of the normalized flow, $\tilde{\mathcal{E}} : [0,\infty) \to \mathbb{R}$, is

$$\tilde{\mathcal{E}}(\tau) = \int_{S^1} \psi \tilde{h}^p \, \mathrm{d}\theta. \tag{3.19}$$

We will show that the entropy, $\tilde{\mathcal{E}}(\tau)$, is monotone and uniformly bounded above and below for all $\tau \in [0,\infty)$.

Proposition 3.2.1. *The entropy of the normalized flow is monotone non-increasing for all time $\tau \geq 0$.*

Proof. Note that

$$\begin{aligned}
\tilde{\mathcal{E}}_\tau &= \int_{S^1} \psi p \tilde{h}^{p-1} \tilde{h}_\tau \, \mathrm{d}\theta \\
&= \int_{S^1} \psi p \tilde{h}^{p-1} \left(\tilde{h} - \frac{\psi \tilde{h}^p \tilde{k}}{\frac{1}{2}\int_{S^1} \psi \tilde{h}^p \mathrm{d}\theta} \right) \mathrm{d}\theta \\
&= 2p \int_{S^1} \left(\frac{1}{2} \psi \tilde{h}^p - \frac{\psi^2 \tilde{h}^{2p-1} \tilde{k}}{\int_{S^1} \psi \tilde{h}^p \mathrm{d}\theta} \right) \mathrm{d}\theta \\
&= \frac{2p}{\int_{S^1} \psi \tilde{h}^p \mathrm{d}\theta} \left[\frac{1}{2} \left(\int_{S^1} \psi \tilde{h}^p \, \mathrm{d}\theta \right)^2 - \int_{S^1} \psi^2 \tilde{h}^{2p-1} \tilde{k} \, \mathrm{d}\theta \right].
\end{aligned} \tag{3.20}$$

The normalized curve encloses an area of 1 for all $\tau \in [0, \infty)$, that is

$$\frac{1}{2}\int_{S^1} \frac{\tilde{h}}{\tilde{k}}\,\mathrm{d}\theta = 1. \tag{3.21}$$

$$\begin{aligned}
\tilde{\mathcal{E}}_\tau &= \frac{2p}{\int_{S^1}\psi\tilde{h}^p\mathrm{d}\theta}\left[\frac{1}{2}\left(\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta\right)^2 - \int_{S^1}\psi^2\tilde{h}^{2p-1}\tilde{k}\,\mathrm{d}\theta\right] \\
&= \frac{2p}{\int_{S^1}\psi\tilde{h}^p\mathrm{d}\theta}\left[\frac{1}{2}\left(\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta\right)^2 - \frac{1}{2}\int_{S^1}\frac{\tilde{h}}{\tilde{k}}\,\mathrm{d}\theta\int_{S^1}\psi^2\tilde{h}^{2p-1}\tilde{k}\,\mathrm{d}\theta\right] \\
&= \frac{p}{\int_{S^1}\psi\tilde{h}^p\mathrm{d}\theta}\left[\left(\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta\right)^2 - \int_{S^1}\frac{\tilde{h}}{\tilde{k}}\,\mathrm{d}\theta\int_{S^1}\psi^2\tilde{h}^{2p-1}\tilde{k}\,\mathrm{d}\theta\right]
\end{aligned} \tag{3.22}$$

By the Hölder's inequality, we have

$$\int_{S^1}\left(\sqrt{\frac{\tilde{h}}{\tilde{k}}}\right)^2\int_{S^1}\left(\psi\tilde{h}^p\sqrt{\frac{\tilde{k}}{\tilde{h}}}\right)^2 \geq \left(\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta\right)^2, \tag{3.23}$$

which gives us the following inequality:

$$\left(\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta\right)^2 - \int_{S^1}\frac{\tilde{h}}{\tilde{k}}\,\mathrm{d}\theta\int_{S^1}\psi^2\tilde{h}^{2p-1}\tilde{k}\,\mathrm{d}\theta \leq 0. \tag{3.24}$$

Therefore

$$\tilde{\mathcal{E}}_\tau \leq 0 \tag{3.25}$$

with equality if and only if

$$\frac{\tilde{h}}{\tilde{k}} = \lambda\psi^2\tilde{h}^{2p-1}\tilde{k}, \tag{3.26}$$

where λ is a nonnegative constant.

Rearranging the terms, we get the following condition in case of equality,

$$\frac{\tilde{h}^{1-p}}{\tilde{k}} = \sqrt{\lambda}\psi, \tag{3.27}$$

hence, up to rescaling, $\tilde{K}$ is a solution to the L_p-Minkowski problem which will be explained in detail in the following section. $\square$

Proposition 3.2.2. *The entropy of the normalized curve is uniformly bounded from both above and below for all time $\tau \geq 0$.*

Proof. Due to the previous proposition, $\tilde{\mathcal{E}}(\tau) \leq \tilde{\mathcal{E}}(0)$, so the entropy is bounded from above.

To obtain the lower bound, we proceed as follows:

$$\begin{aligned} \tilde{\mathcal{E}}(\tau) = \exp\left[\ln \tilde{\mathcal{E}}(\tau)\right] &= \exp\left[\ln\left(2\pi \int_{S^1} \psi \tilde{h}^p \, \frac{\mathrm{d}\theta}{2\pi}\right)\right] \\ &= \exp\left[\ln 2\pi\right] \cdot \exp\left[\ln\left(\int_{S^1} \psi \tilde{h}^p \, \frac{\mathrm{d}\theta}{2\pi}\right)\right]. \end{aligned} \tag{3.28}$$

By Jensen's inequality, we have

$$\begin{aligned} \tilde{\mathcal{E}}(\tau) &\geq 2\pi \exp\left[\int_{S^1} \ln\left(\psi \tilde{h}^p\right) \frac{\mathrm{d}\theta}{2\pi}\right] \\ &= 2\pi \exp\left[\int_{S^1} \left(\ln \psi + \ln \tilde{h}^p\right) \frac{\mathrm{d}\theta}{2\pi}\right] \\ &= 2\pi \exp\left[\frac{1}{2\pi}\int_{S^1} \ln\left(\psi\right) \mathrm{d}\theta\right] \cdot \exp\left[p \int_{S^1} \ln\left(\tilde{h}\right) \frac{\mathrm{d}\theta}{2\pi}\right]. \end{aligned} \tag{3.29}$$

Since ψ is independent of τ, the first integral is constant, and we get

$$\tilde{\mathcal{E}} = \tilde{C}_0 \exp\left[p \int_{S^1} \ln\left(\tilde{h}\right) \frac{\mathrm{d}\theta}{2\pi}\right]. \tag{3.30}$$

By Guan's [16] result in dimension 2, i.e. a logarithmic Minkowski inequality, the second integral is bounded below as $\int_{S^1} \ln \tilde{h} \, \frac{\mathrm{d}\theta}{2\pi} \geq \frac{1}{2} \ln \frac{\tilde{A}}{\pi}$. Consequently, we have

$$\tilde{\mathcal{E}}(\tau) \geq \tilde{C}_0 \exp\left[\frac{p}{2} \ln \frac{\tilde{A}}{\pi}\right]. \tag{3.31}$$

Since the area enclosed by the curve remains constant at any $\tau \in [0, \infty)$, the right hand side of the inequality is a constant, $\tilde{C}_1$, which is strictly positive. In other words,

$$\tilde{\mathcal{E}}(\tau) \geq \tilde{C}_0 \exp\left[\frac{p}{2}\ln\frac{1}{\pi}\right] = \tilde{C}_1 > 0. \tag{3.32}$$

Therefore the entropy is bounded below and above for all time by two positive constants, $\tilde{C}_1 \leq \tilde{\mathcal{E}}(\tau) \leq \tilde{C}_2$ where $\tilde{C}_2 = \tilde{\mathcal{E}}(0)$.

□

3.3 The non-degeneracy of solutions to the normalized flow

A fundamental result that we will use here is the Blaschke selection theorem.

Theorem 3.3.1 (Blaschke Selection Theorem). *[30] Let $\{K_j\}_j$ be a sequence of convex sets, in R^n, which are contained in a bounded set. Then there exists a subsequence $\{K_{j_k}\}_{j_k} \subseteq \{K_j\}_j$, and a convex set K, such that K_{j_k} converges to K in the Hausdorff metric.*

The non-degeneracy of solutions to the normalized flow is equivalent to the support function of the normalized curve being bounded from both above and below for all time.

Proposition 3.3.1. *The support function of the normalized curve, $\tilde{h}(\tau)$, is uniformly bounded from above and below for $0 \leq \tau < \infty$ by constants depending on the initial curve (conditions) only.*

Proof. By Proposition 3.2.2, the entropy of the curve, $\tilde{\mathcal{E}}(\tau)$, is bounded from above and below for each $\tau \in [0, \infty)$.

From the Mean Value Theorem, we have that for each $\tau \in [0, \infty)$, $\exists\ \theta_0(\tau) \in [0, 2\pi)$, such that

$$\tilde{\mathcal{E}}(\tau) = \int_{S^1} \psi \tilde{h}^p \,\mathrm{d}\theta = 2\pi(\psi\, \tilde{h}^p)(\theta_0(\tau)). \tag{3.33}$$

On the other hand, from the bounds on the entropy, we have

$$\tilde{C}_1 \leq 2\pi(\psi\, \tilde{h}^p)(\theta_0(\tau)) \leq \tilde{C}_2, \tag{3.34}$$

so

$$C_1 \leq \tilde{h}^p(\theta_0(\tau)) \leq C_2, \tag{3.35}$$

where C_1, C_2 are positive constants depending on $\tilde{C}_i$ $(i = 1, 2)$ and ψ.

The width of a convex body, K, in the direction θ is the distance between two supporting hyperplanes of K with outer unit normal parallel to θ, see figure 3.1.

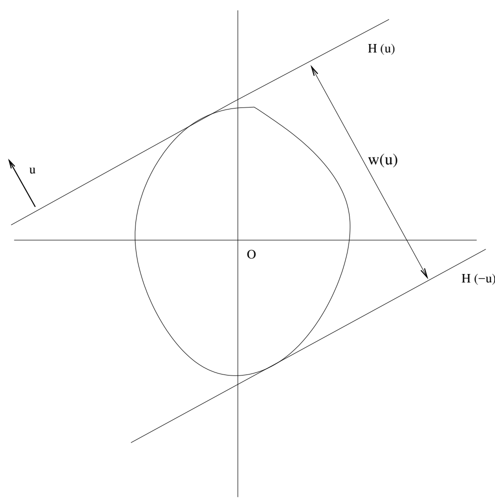

Figure 3.1: Width, w(u), of a convex body in direction of unit vector u.

Since the body $\tilde{\mathrm{K}}$ is centrally symmetric, its width in any direction is $\tilde{w}(\theta) = \tilde{h}(\theta) + \tilde{h}(\theta + \pi) = 2\tilde{h}(\theta)$. Thus, for any time τ, the width in the direction $\theta_o(\tau)$ is bounded above and below

$$\tilde{\lambda}_1 \leq \tilde{w}(\theta_0(\tau)) \leq \tilde{\lambda}_2, \tag{3.36}$$

where $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ are positive constants.

Next, we show that the body $\tilde{\mathrm{K}}(\tau)$ does not degenerate into an infinite line as $\tau \to \infty$. Let $\{\tau_j\}_j$ be an arbitrary sequence of times diverging to infinity and let $\left\{\mathrm{I}(\theta_0(\tau_j))\right\}_j$ be the family of segments centered at the origin, attaining the widths $\tilde{w}(\theta_0(\tau_j))$, thus whose lengths

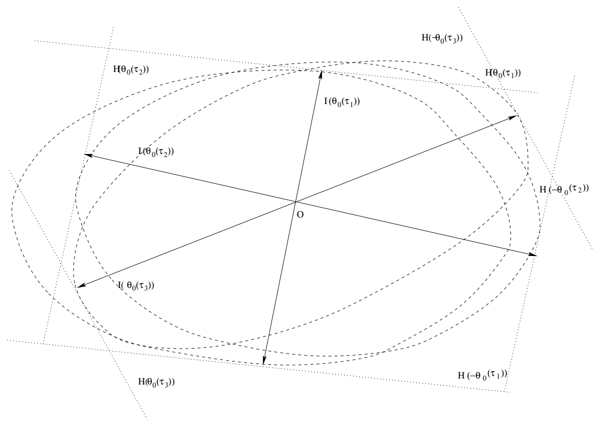

Figure 3.2: Segments, associated with convex bodies, attaining widths $\tilde{w}(\theta_0(\tau))$.

are bounded and lie in the interval $[\tilde{\lambda}_1,\tilde{\lambda}_2]$. So, each $\mathrm{I}(\theta_0(\tau_j))$ is associated with a $\tilde{K}(\tau_j)$ and has the length $\tilde{w}(\theta_0(\tau_j))$.

By Blaschke selection theorem, there exists a convergent subsequence $\left\{\mathrm{I}(\theta_0(\tau_n))\right\}_n$ that converges to an interval I_∞, where I_∞ is a segment associated with $\tilde{K}_\infty$, and whose length is in $[\tilde{\lambda}_1,\tilde{\lambda}_2]$.

Suppose that $\tilde{K}(\tau_n)$ do not remain in a compact set. Since the area enclosed is constant, $\tilde{K}(\tau_n)$ degenerates into an infinite line as $\tau_n \to \infty$ in some direction, say $\theta_\infty \in [0, 2\pi)$. Call $\theta_\infty^\perp$ the direction in $[0, \pi)$ perpendicular to θ_∞. This means that the width of the limit set is

$$\tilde{w}_{\tilde{K}_\infty}(\theta) = \begin{cases} 0 & \theta = \theta_\infty^\perp,\ \theta_\infty^\perp + \pi, \\ +\infty & \theta \neq \theta_\infty^\perp,\ \theta_\infty^\perp + \pi. \end{cases} \tag{3.37}$$

This means that the length of $\mathrm{I}_{\tilde{K}_\infty}(\theta)_{\theta\in[0,2\pi)}$ is either 0 or ∞. This is a contradiction as the length of $\mathrm{I}_{\tilde{K}_\infty}(\theta)_{\theta\in[0,2\pi)}$ is in $[\tilde{\lambda}_1,\tilde{\lambda}_2]$.

$\square$

Thus, the upper and lower bounds on the entropy yield a positive upper bound and a positive lower bound on the width of solutions to the flow.

Chapter 4

Applications to the L_p-Minkowski problem

4.1 Existence of solutions to the fractional Minkowski problem

The entropy of the normalized flow is non-increasing, that is $\dot{\tilde{\mathcal{E}}}_\tau \leq 0$. We will show $\limsup_{\tau\to\infty} (\dot{\tilde{\mathcal{E}}}_\tau) = 0$.

Proposition 4.1.1. *Let ω be the extinction time for the unnormalized flow. Then*

$$\limsup_{t\to\omega} \left(\frac{\mathcal{E}}{A^{\frac{p}{2}}}\right)_t = 0. \tag{4.1}$$

Proof. We first calculate $\mathcal{E}_t$,

$$\begin{aligned}
\mathcal{E}_t &= \left(\int_{S^1} \psi h^p \, \mathrm{d}\theta \right)_t \\
&= \int_{S^1} \psi p h^{p-1} h_t \, \mathrm{d}\theta \\
&= \int_{S^1} \psi p h^{p-1} (-\psi h^p k) \, \mathrm{d}\theta \\
&= -p \int_{S^1} \psi^2 h^{2p-1} k \, \mathrm{d}\theta.
\end{aligned} \tag{4.2}$$

Next, we evaluate $\left(\frac{\mathcal{E}}{A^{\frac{p}{2}}} \right)_t$:

$$\begin{aligned}
\left(\frac{\mathcal{E}}{A^{\frac{p}{2}}} \right)_t &= \frac{\mathcal{E}_t}{A^{\frac{p}{2}}} - \frac{p}{2} \frac{\mathcal{E}}{A^{\frac{p}{2}+1}} A_t \\
&= \frac{A_t}{A} \left[\frac{1}{A^{\frac{p}{2}}} \left(\frac{A}{A_t} \, \mathcal{E}_t - \frac{p}{2} \, \mathcal{E} \right) \right].
\end{aligned} \tag{4.3}$$

Substituting for terms, we get the following:

$$\begin{aligned}
\left(\frac{\mathcal{E}}{A^{\frac{p}{2}}} \right)_t = \frac{A_t}{A} \Bigg[\frac{1}{A^{\frac{p}{2}}} \Bigg(\frac{p}{2} \frac{\int_{S^1} \frac{h}{k} \, \mathrm{d}\theta}{\int_{S^1} \psi h^p \, \mathrm{d}\theta} \int_{S^1} \psi^2 h^{2p-1} k \, \mathrm{d}\theta & \\
- \frac{p}{2} \int_{S^1} \psi h^p \, \mathrm{d}\theta \Bigg) \Bigg] & \\
= \frac{p}{2} \frac{\int_{S^1} \psi h^p \, \mathrm{d}\theta}{A} \Bigg[\frac{A^{-\frac{p}{2}}}{\int_{S^1} \psi h^p \, \mathrm{d}\theta} \Bigg(\left(\int_{S^1} \psi h^p \, \mathrm{d}\theta \right)^2 & \\
- \int_{S^1} \frac{h}{k} \, \mathrm{d}\theta \int_{S^1} \psi^2 h^{2p-1} k \, \mathrm{d}\theta \Bigg) \Bigg]. &
\end{aligned} \tag{4.4}$$

Suppose that $\limsup_{t \to \omega} \left(\frac{\mathcal{E}}{A^{\frac{p}{2}}} \right)_t < 0$. Then $\exists \ \epsilon > 0$ such that for some $[t_0, \omega)$, with $0 < t_0 < \omega$, we have

$$\left(\frac{\mathcal{E}}{A^{\frac{p}{2}}} \right)_t < \frac{p}{2} \frac{\int_{S^1} \psi h^p \, \mathrm{d}\theta}{A} \left(-\frac{2\epsilon}{p} \right). \tag{4.5}$$

Integrating (4.5) from t_0 to ω, we obtain:

$$\lim_{t \to \omega} \int_{t_0}^{t} \left(\frac{\mathcal{E}}{A^{\frac{p}{2}}} \right)_t < \lim_{t \to \omega} \int_{t_0}^{t} \frac{p}{2} \frac{\int_{S^1} \psi h^p \, \mathrm{d}\theta}{A} \left(-\frac{2\epsilon}{p} \right), \tag{4.6}$$

and

$$\lim_{t\to\omega} \frac{\mathcal{E}}{A^{\frac{p}{2}}}(t) - \frac{\mathcal{E}}{A^{\frac{p}{2}}}(t_0) < \lim_{t\to\omega} \epsilon\Big(\ln A(t) - \ln A(t_0)\Big). \tag{4.7}$$

By Proposition 3.2.2, the normalized entropy is bounded below, hence the left-hand side of the above inequality is a finite real number. As $t \to \omega$, $\ln A(t) \to -\infty$, so the right-hand side of the inequality goes to $-\infty$, leading to a contradiction. □

Next, we express $\left(\frac{\mathcal{E}}{A^{\frac{p}{2}}}\right)_t$ as a function of normalized quantities $\tilde{h}$ and $\tilde{k}$:

$$\begin{aligned}\left(\frac{\mathcal{E}}{A^{\frac{p}{2}}}\right)_t &= \frac{p}{2A^{\frac{p}{2}+1}}\left[\left(\int_{S^1}\psi h^p \,\mathrm{d}\theta\right)^2 - \int_{S^1}\frac{h}{k}\,\mathrm{d}\theta \int_{S^1}\psi^2 h^{2p-1}k\,\mathrm{d}\theta\right]\\ &= \frac{p}{2A^{\frac{p}{2}+1}}\left[A^p\left(\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta\right)^2 - A^p\int_{S^1}\frac{\tilde{h}}{\tilde{k}}\,\mathrm{d}\theta \int_{S^1}\psi^2\tilde{h}^{2p-1}\tilde{k}\,\mathrm{d}\theta\right)\right]\\ &= \frac{p}{2A^{1-\frac{p}{2}}}\left[\left(\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta\right)^2 - \int_{S^1}\frac{\tilde{h}}{\tilde{k}}\,\mathrm{d}\theta \int_{S^1}\psi^2\tilde{h}^{2p-1}\tilde{k}\,\mathrm{d}\theta\right)\right].\end{aligned} \tag{4.8}$$

Let

$$\Psi = \left[\left(\int_{S^1}\psi\tilde{h}^p\,\mathrm{d}\theta\right)^2 - \int_{S^1}\frac{\tilde{h}}{\tilde{k}}\,\mathrm{d}\theta \int_{S^1}\psi^2\tilde{h}^{2p-1}\tilde{k}\,\mathrm{d}\theta\right)\right]. \tag{4.9}$$

Theorem 4.1.1. *Given ψ an even, π periodic and positive function, there exists a sequence of smooth convex bodies $\{K_j\}_j$, $j \nearrow \infty$, which converges to a smooth convex body whose support function satisfies, up to scaling,*

$$\tilde{h}^{1-p}(\tilde{h}_{\theta\theta} + \tilde{h}) = \psi.$$

Proof. As

$$\limsup_{t\to\omega}\left(\frac{\mathcal{E}}{A^{\frac{p}{2}}}\right)_t = 0, \tag{4.10}$$

then by (4.9), $\limsup\limits_{\tau\to\infty} \Psi = 0$. By Hölder's inequality, it follows that for some sequence of times $t_j \nearrow \omega$, as $j \nearrow \infty$, and thus, for some sequence of times $\tau_j \nearrow \infty$, as $j \nearrow \infty$, we have in the limit:

$$\left(\int_{S^1} \psi \tilde{h}^p \,\mathrm{d}\theta\right)^2 = \int_{S^1} \frac{\tilde{h}}{\tilde{k}} \,\mathrm{d}\theta \int_{S^1} \psi^2 \tilde{h}^{2p-1} \tilde{k} \,\mathrm{d}\theta. \tag{4.11}$$

In other words, the sequence of convex bodies with general term $K_j = \tilde{K}(\tau_j)$, $j \nearrow \infty$ satisfies the conditions of the Blaschke selection theorem and a subsequence of them, denoted for simplicity the same way, converges to a convex set $\tilde{K}$ satisfying the above equality.

Due to the smoothness of the functions involved, equality in this Hölder's inequality occurs if and only if

$$\frac{\tilde{h}}{\tilde{k}} = \lambda \psi^2 \, \tilde{h}^{2p-1} \tilde{k} \tag{4.12}$$

where λ is positive constant. Since the asymptotic shape of the normalized flow is non-degenerate, $\tilde{h}$ and $\tilde{k}$ are bounded from above and below for all time, so the equality is non-trivially satisfied, hence $\tilde{K}$ is a convex body.

Rearranging the terms, we get

$$\frac{\tilde{h}^{1-p}}{\tilde{k}} = \sqrt{\lambda} \psi. \tag{4.13}$$

By choosing $\mu = \lambda^{\frac{1}{4-2p}}$ and, rescaling again the limit body, $\tilde{K}$, by μ, we obtain a non-degenerate convex body $\tilde{K}$ satisfying $\tilde{h}^{1-p}(\tilde{h}_{\theta\theta} + \tilde{h}) = \psi$ as claimed. $\square$

4.2 Further directions

In this thesis, we considered a strictly convex, anisotropic and a centrally symmetric planar flow. We studied its longterm existence and showed that a solution to the L_p-Minkowski problem exists. The current work can be extended by studying the behavior of the flow in different settings and examining the existence of possible solutions to the L_p-Minkowski problem in each setting. We list two different settings for the extension of this work. One, we can study the flow in higher dimensions ($\mathbb{R}^n$, $n > 2$) and see if the higher dimensional flow exhibits the same properties as its planar counterpart. Two, we can relax some constraints on the flow, such as strict convexity and central symmetry. With these conditions relaxed, we can investigate the convexity preserving property of the flow; whether the flow contains the origin as long as it exists; and the difficulty of obtaining bounds on the width of the curves.

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