Risk Optimization for Hybrid Pension Plans with ARMA and GARCH Investment Returns

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Abstract

The use of time series models in general, and conditional processes, in particular for modelling returns on investment, have been considered recently for pension plan funding. In this project, ARMA and GARCH models are applied to the rate of return of a hybrid pension plan. The first and second moments of the fund, contributions, and benefits are derived under both models. The aggregate risk and the optimal spread period of amortization are studied under different risk measures; Value at Risk, Coefficient of Variation, and Variance. All evaluations are done over finite as well as infinite time horizons. Finally, numerical illustrations under different investment strategies as well as different valuation interest Rates are proposed under GARCH model.

Keywords: Hybrid Pension Designs, Funding Methods, Risk Sharing, Yield Rate, ARMA$(1,1)$, GARCH$(1,1)$, Aggregate Risk, VaR, TVaR.

In memory of my parents

Febuary 2013

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"Nothing is impossible, the word itself says 'I'm possible' "

Audrey Hepburn

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Chapter 1

Introduction

Nowadays, pension plans have become important investments in an employee's life. Since the benefits received from a pension plan are one of the sources of income after retirement, great emphasis is put on optimal plan design. There are two common types of plans, defined benefit plans "DB" and defined contribution plans "DC". Beside these two plans an alternative design was introduced in the $1980's$ called "Hybrid Plan". The following sections give a brief definition of each plan.

1.1 Defined Benefit (DB) Plan

The simplest definition of this plan is that it determines pension benefits using a pre-defined formula. Examples of this plan are "Career Average" and "Final Salary" plans. Under this plan, the risks which might be due to investment, management, inflation, longevity, interest rates, or political decisions, fall on the employer who should maintain contributions at a level that covers all the benefits promised by the plan.

1.2 Defined Contribution (DC) Plan

Under this plan, each employee has been assigned an account, in which that employee contributes an amount from his/her salary each year. Likewise, employers may contribute the same amount or less in the employee's account. At retirement, the accumulated amount is used to purchase life annuities. In this plan, the employees decide how to invest their accounts, additionally, the risks described above are faced by the employees only. Finally, the benefits under this plan can not be predicted. The US 401(K) plan is an example of this design.

1.3 Hybrid Plan

This plan is neither a "DB" plan nor a "DC" plan, but it combines features of both schemes. For instance, a DC account might be set up for each participant to invest the contributions, however, the benefit might still be obtained using a DB formula. Moreover, the risks under this plan are shared by the employees and the sponsors, or it might be shared among participants, or between active participants and retirees. For this reason it is seen that recently employers are preferring to use this plan. In fact, there are many different types of hybrid schemes. Here, we mention the most common designs that have been studied in many research papers and used around the world.

1.3.1 Hybrid Pension Designs

- **Cash Balance Plan.** The Cash Balance Plan is also referred to as the "Shared Risk Plan" or "Retirement Balance Plan". This plan is more common in the US. In fact, each employee in this plan has a hypothetical account, as in DC plans, to which the employer contributes some amount, and promises to credit the accounts with a specific rate of return. Moreover, the benefit is promised, and paid as a lump sum at retirement. Contributions and investment earnings are not actually allocated to individual accounts as in the DC plan, but they remain in a single investment pool just as in the DB plan. In this plan, the employer is responsible for the risks at the beginning, then there are transferred to the members at retirement.

- **Sequential Hybrid.** The Sequential Hybrid plan is commonly known as "Nursery Scheme", since in this plan one specific type of pension benefit is used first, and then after a specified period, it is followed by a second pension arrangement. For instance, if an employee joins a company, he/she may be given DC benefits for a set number of years. After which, if he/she surpasses these years in a company then a DB plan is offered for any subsequent years of employment. In addition,"the trigger" to switch from DC to DB might depend on a period of service, or reaching a certain age. This type of plan is often used when the company has a high turnover of short-term staff. All the risks are shared between the plan sponsor and the members under this scheme.

- **Combination Plan.** Pension benefits in combination plans accrue in two parallel ways; a DC arrangement and a final salary arrangement. The member chooses one of the two pension benefits at retirement. The final salary is also subject to revaluations in this study. The risks are all shared between the plan sponsor and the members. The University of Victoria offers this type of plan to its employees.

- **Target Benefit Plan.** The target benefit plan can also be referred to as the "Pooled Variable Benefit Plan". It can have fixed or variable contributions, paid by the sponsor or by both the sponsor and employees, and a target benefit that is determined by a DB formula, to be expected but not promised. The contributions are pooled for investment purposes. The target benefit is not necessarily achieved by the plan sponsor. Moreover, when there is a deficit in the plan, the benefit may be decreased. Similarly, if there is a surplus, the benefits may be increased or the employer may save them to cover any future deficits. An example of a TB plan is The University of British Columbia pension plan. The "Variable Payment Plan", which was proposed by Khorasanee in [46] is another example of a target benefit plan. This plan depends on an allocation per employee equal to a fixed contribution plus or minus a share of the surplus or deficit arising from the benefit payments at time t.

- **Underpin Plan.** The underpin plan is also called "A Floor-offset Scheme", and the benefit is based on the best of the two schemes DB and DC. In other words, the DB is a "floor plan", and the DC is a "base plan". The DB plan provides a formula to define a guarantee minimum benefit level or floor. Then, the member will receive only the DC account balance, if the benefit that is provided by the DC plan is equal to or exceeds the floor plan benefit. However, if the benefit in the floor plan exceeds the DC annuity benefit, the DB plan will fill the gap and pay the difference.

There are other hybrid designs that have been discussed by different researchers such as the "Care Plan", or the "Lump Sum Final Salary"; for more details please refer to Wesbroom and Reayin [60].

Khorasanee in [45] introduced a new hybrid plan which is a modification of a DB plan that was studied by Dufresne in [20]. Under this new scheme, an adjustment parameter is added to the benefit payment at time t, then the benefit payment becomes a sum of the target benefit minus the adjustment of the unfunded liability, while the annual contribution is defined as the sum of the normal cost and the adjustment term of the unfunded liability. In this project, we mainly focus on this type of hybrid plan.

1.4 Types of Risks In Pension Plans

Risks under pension schemes have been studied extensively in the literature. In this section, a review of the main sources of risk are discussed below.

- **Investment Risk.** When the assets of the plan are invested in the market, the performance cannot be predicted, even if the investments are managed very well. Hence, the return on investment might be less than the expected rate of investment, then this leads to fund amounts that are insufficient to meet the benefits promised by the plan. This type of risk will be discussed in chapter 3.

- **Longevity Risk.** An important problem in many countries is the aging of their population. Based on the report of the World Bank the life expectancy will increase dramatically by 2045. Consequently, there will be a risk born from longevity, and it will impact on governments and employers who have to fund retirement and health obligations to the employees and retirees. Cairns, Blake and Dowd in [17] introduced a model where the longevity risk is a sum of a trend risk and a random variation risk. Moreover, they explain how longevity risk could be hedged if the plan member invests in a fund containing longevity bonds. So, governments should issue longevity bonds to help the private sector.
- **Interest Rate Risk.** When the benefit is a lump sum, converted to a life annuity at retirement, interest rate risk can occur at the time of purchasing the annuity. For instance, if the interest rate is low, then the cost of the annuity rises. Also, an increase in the interest rate affects the liability as well as the assets that sponsors hold. In other words, the liabilities will decrease, when the interest rate is increasing, at the same time as the price of assets will also decrease.
- **Inflation Risk.** Higher inflation rates will reduce the value of the benefit that will be received by the participants when he/she retires.

Furthermore, there are other risk factors in pension plan design that might be taken into account by sponsors and members, such as legislation risk, taxation risk, death and disability risk, in Chapter 3, we will discuss other of risk.

Before concluding this chapter, we review two actuarial cost methods that will help clearly illustrate pension valuation work, and then analyze pension plans in two different countries.

1.5 Actuarial Funding Methods

Actuarial funding methods for DB plans have been discussed in many books and articles. The funding methods are classified into two categories, namely "Accrued Benefit Funding Methods" and "Prospective Benefit Funding Methods". They differ from each other in their primary objective; the first category focuses on achieving a certain level of funding, and it attempts to establish and control the relation between the fund assets and the accruing liabilities. An example of this category is the "Projected Unit Credit" method and the "Current Unit Credit" method.

By contrast, the funding methods under the second category define a certain level of contributions, so the primary objective of these methods is to stabilize these contributions. Examples of these methods are "Entry Age" and "Attained Age" methods.

Moreover, there are other funding methods that cannot be classified as accrued benefit or prospective benefit, such as the "Pay as You Go" method, since the benefit is paid when it is due, and there are no periodic contributions. In this section, we discuss the most important methods that are used widely, "Entry Age" and "Projected Unit Credit".

- **Entry Age.** Entry Age is a common method in the US. In this method the normal cost component, which is defined as the level amount that is needed to fund the benefit over the employee's career, will have a present value equal to the present value of future benefits. For the actuarial liability component, two definitions can be used to express the liability, one is in a prospective way, where the liability is the difference between the present value of future benefits and the present value of future normal costs, and the second one is retrospective, where the liability is the present value of past normal costs. Under this method, the contributions are stable, and this target is the prime objective of the method.

- **Projected Unit Credit.** There is a tendency to apply this method in many countries, such as in the UK and Canada; it focuses on establishing and maintaining a connection between the fund assets and the accruing liabilities. Also, it allows to control the effect of future salary increases on the accrued benefits. Furthermore, in this method the actuarial liability is equal to the present value of the accrued liability. The normal cost under this plan equals the difference between the accrued benefit from one year to another.

Both of the above methods fall into a category that identifies and helps amortize the gains and losses. Interested readers in cost methods can refer to Anderson in [1].

Shapiro in [54] set some criteria for selecting cost methods, such as, adequacy, consistency, flexibility, robustness, but he mentions that no method can satisfy all these criteria in general.

In fact, Cairns in [14] mentions that the method that is used to calculate the liabilities and normal costs has an impact on the variability of the fund and contributions. More precisely, the most secure method is the one that produces the lowest liability, since the variance of the fund and contributions is expressed in terms of the squared liability.

1.6 Two Case Studies

Now, let us illustrate some examples of pension plans that are used in Saudi Arabia and Canada, by the private and the public sectors.

1.6.1 Pension Design In Saudi Arabia

Public Pension Plan

The Public Pension Agency (PPA) that administers the pension plan for Saudi employees in the civil and military sectors, was established under the name of Retirement Pension Department in 1958. Then, in 2004, the cabinet decided to transfer the Retirement Pension Department to a general organization that has an independent budget and management, and they called it the PPA.

In the civil sector, the employees who are entitled to receive retirement pensions based on the PPA law are:

1. Employees who reach the mandatory age of retirement of 60, although there is a debate in the Shura Council to increase it to age 62 to reduce the longivity risk for the pension fund.

2. Employees with at least 25 years of service, which is the minimum years of service, and then they leave employment due to any reason.

3. Employees with 20 years of service who request approval to retire.

4. Employees exposed to permanent disability or who die.

Then, the retirement pension is defined as follows for employees with at least the minimum years of service or who retire at age 60:

$$\frac{\textbf{Final Salary * Years of Service}}{40}.$$

If the employee completes the eligible 40 years of service, then he is entitled to receive the whole salary.

However, if the cause of decrement is either death or disability, but both causes are not due to work, then the pension is

$$40\% * \textbf{Final Salary}.$$

Notwithstanding, if either cause is due to work, then

$$80\% * \textbf{Final Salary}.$$

In the military scheme, the PPA determined that the following employees are entitled to receive retirement pensions:

1. Employees who reach the mandatory age of retirement, which is from age 44 to 58 according to their grade.

2. Employees with 18 years of services, which is the minimum years of service, and then they leave the employment due to any reason.

3. Employees with at least 15 years of service, and request approval to retire.

4. Employees exposed to permanent disability or who die.

Then, the retirement pension is obtained as follows for employees with the minimum years of service or who retire at age 60:

$$\frac{\textbf{Final Salary * Year of Services}}{35}.$$

Whenever the cause of decrement is either death or disability, but neither reason is due to work, then the pension is

$$70\% * \textbf{Final Salary}.$$

However, if the disability is temporary and due to work, then

$$80\% * \textbf{Final Salary}.$$

When death or permanent disability are due to work, then the pension is equal to the final salary.

If the pensioner dies, the pensioner's beneficiaries will receive his/her pension, and it should be distributed equally if they are three or more. However, if they are two, they are entitled to receive 75% of the pension, and when there is only one, then he/she is entitled to receive 50% of the pension.

Based on the PPA, the employees contribute 9% of their salaries, and the employer (Ministry of Finance) contributes the same rate of 9% in the civil sector and of 15% in the military sector, while the government contributes when there is any deficit.

In addition, the PPA's report also gives some details about the investments of the agency. It is mentioned that the agency has long-term investments in the stock markets and in real estate; these different investments are in the KSA and abroad. However, the report does not specify the total of investments in the Saudi market and in the external markets. PPA's investments are supported by reputable international specialists to help in setting long-term investment strategies and selecting experienced asset managers.

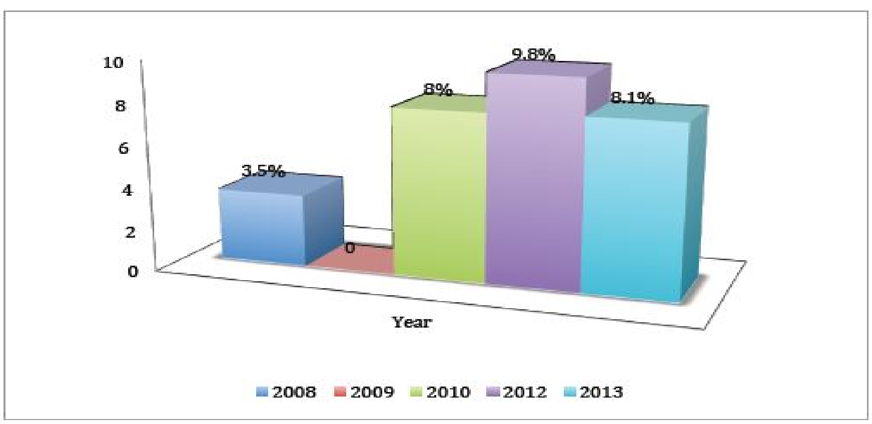

Figure 1.1: PPA's Returns on Investment (%) in the Last Six Years

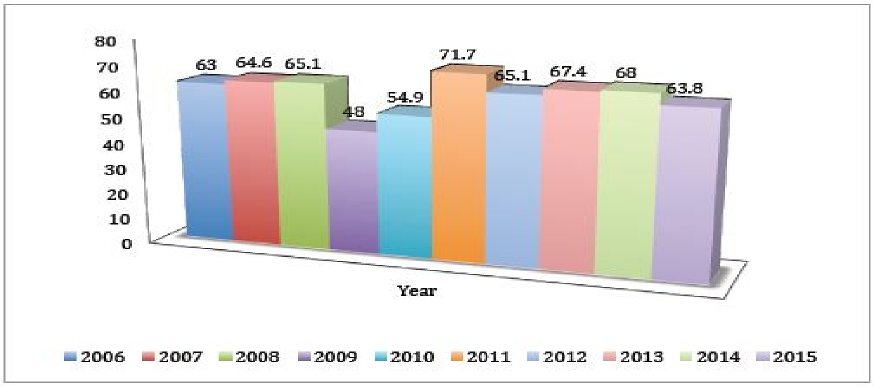

Figure 1.2: Investments (in Billions) by PPA in the Saudi Market

Private Pension Plan

The General Organization of Social Insurance (GOSI), which administers the benefits to retirees in the private sector, was established in 1969 to apply the Social Insurance law and to study the process of achieving a compulsory insurance coverage, collecting contributions from employers and employees, and paying benefits to retirees, disabled, and withdrawed members or to their beneficiaries. In fact, GOSI has an independent budget and management.

The GOSI defines the employees who are entitled to receive retirement pensions are as follows:

- Contributors who reach age 60 or more.
- Contributors with a period of contribution of at least 300 months, which is the minimum months of service.
- Contributors exposed to permanent disability, and with contribution periods of at least 12 months consecutive.

- Contributors who died, and who have contribution periods of at least 3 months consecutive.
- Contributors who are missing or absent, and are treated as dead.

Then, the pension retirement is calculated as follows:

a. The pension for the period prior to 1422H (2001):

$$\frac{\text{Average of the Last Two Salaries * Number of Previous Months of Contribution}}{600}.$$

In case there are dependents, a percentage is added on the pension in the amount of 10% for one dependent, 15% for two dependents, and 20% for three or more dependents.

b. The pension for the period after 1422 H (2001) is :

$$\frac{\text{Average of the Last Two Salaries * Number of Future Months of Contribution}}{480}.$$

Then, the sum of part (a) and part (b) is the retirement pension of the employee. However, if the total pension is less than $1,983.75$ Riyal, it will be raised to this amount.

Furthermore, if the contributors withdraw due to any reason that is not related to default, then they will receive a benefit until they find another job, that is what is called "Sanad", and it is a type of "Takaful", since all other contributors and employers contribute 2%. Moreover, the beneficiaries receive the pension in case of death of the retiree. Currently, the contribution rate that the employees pay to GOSI is 9% of their salary, and another 9% is paid by the employer.

The GOSI fund is invested by "Hassana Investment Company" that was established in 2009 to manage GOSI's investments and funds. All the contributions are invested in the financial market (stocks, bonds, sukuk, etc) and in real estate. Based on the yearly report of GOSI, the amounts invested reached 54 billion Riyal, and 5 billion

Riyal for real estate.

Some Disadvantages of the Pension Plan in Saudi Arabia.

1- The age of retirement in the public system is 60, and it has never been modified to keep up with the current changes in the ageing of the populations.

2- Lack of financial sustainability.

3- Very generous system, i.e, the maximum benefit is 100%.

4- Encourages early retirement since 60 is the latest age for retirement.

5- Lack of information about the investment strategy that is used with the plan assets.

1.6.2 Pension Plan In Canada

In the 1950s and 1960s, Canada adopted its current system of retirement income provision, and it has attracted widespread attention because of its design. It consists of three separate pillars which are intended to enable retirees to maintain a reasonable standard of living when they retire. These pillars are:

1. Old Age Security payments..
2. Canada Pension Plan, or the Quebec Pension Plan in Quebec.
3. Private retirement savings including registered pension plans (RPPs), and registered retirement savings plans (RRSPs) or other personal savings.

In what follows, we give a description of each pillar, and their eligibility rules. Then, we review how the pension funds are invested under this design.

Old Age Security Pension Plan

In 1951, the federal government introduced the Old Age Security (OAS) to provide a universal pension plan to all Canadians. Point of the fact, it is the cornerstone of Canada's retirement income system. It includes a basic pension, which goes to almost all citizens who are 65 or older, and who have lived in Canada for more than ten years.

As such OAS is Canada's largest public pension program.

In addition, if the retiree has little or no income other than the OAS pension at retirement, then he/she may be eligible for the Guaranteed Income Supplement (GIS). Allowance gives an additional monthly benefit to Canadians who are between 60 and 64, and who have a spouse or common-law partner who is receiving the GIS. If they are widows or widowers, then there is another additional benefit per month.

Canada Pension Plan

In 1965, the federal government reformed the public pension system, and they introduced the Canada Pension Plan (CPP), which was implemented as a complementary measure to OAS. CPP is offered throughout Canada, except in Quebec that has its own program called the Quebec Pension Plan (QPP) for workers in Quebec. The Canada Pension Plan pays a monthly retirement pension to employees who have contributed to the CPP. Furthermore, it provides benefits to the participants and to their children if they become severely disabled or die during their working years. A lump-sum death benefit is available to the participants estate when he/she dies.

The contribution amount depends on the participants earnings. Also, the amount of CPP benefits depends on several factors, for instance, how long a participant contributed to the plan, how much he/she contributed, and finally, the age at which they choose to begin receiving their CPP retirement pension which usually is between 60 and 70.

Private Retirement Savings

It consists of employment pension plans and individual retirement savings. To encourage savings for retirement, the government has created several plans that offer tax benefits to Canadians. These plans let people avoid or delay some of the tax they would pay otherwise.

For instance, Registered Retirement Savings Plans (RRSPs, tax-deferred accounts), Tax-Free Savings Account (TFSA), Non-registered Savings and Investments, and Basic Savings Accounts. Interested readers can refer to the "Financial Consumer Agency of Canada" web site (http://www.fcac-acfc.gc.ca/eng/Pages/home-accueil.asp).

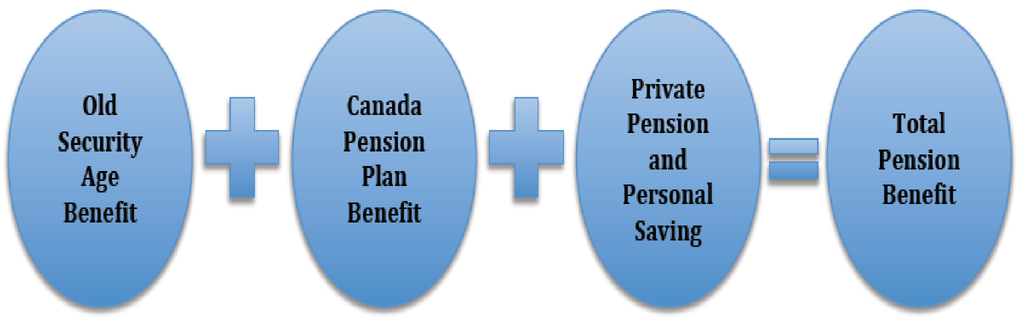

Figure 1.3: Summary of the Canada Pension Plan Design

CPP funds are invested by the Canada Pension Plan Investment Board (CPPIB) that emerged out of the realization in the 1990s, that the CPP fund was unsustainable. Primarily, this was because the Canada Pension Plan benefit payments were exceeding contributions and changing demographics were leading to fewer workers supporting a growing number of retirees.

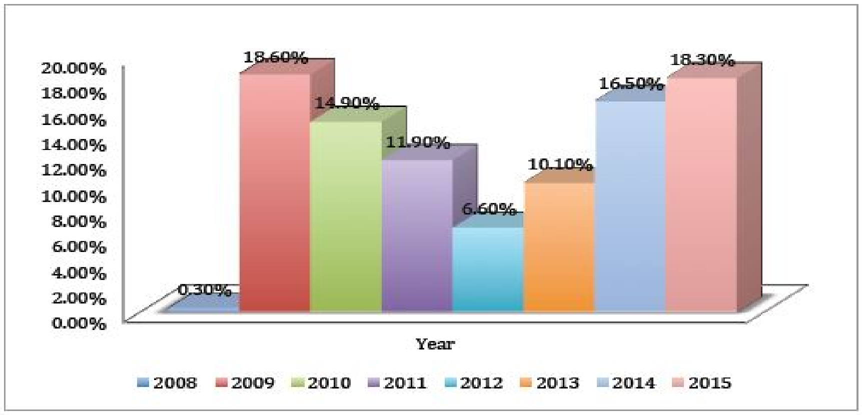

Figure 1.4: CPP Returns on Investment Over the Last Five Years

The CPPIB invests the funds of the CPP to help ensure its long-term sustainability, and it has offices in different cities around the world to help in managing its invesments internationally. Moreover, CPPIB invests in different ranges of asset classes, internal and external investments, such as real estate, public and private equities, and infrastructure. Furthermore, the current asset mix is as follows, 30.9% is in Public Equities, 18.6% in Private Equities, 34.0% in Fixed Income, and 16.5% in real estate.

The CPP fund ended its third quarter of fiscal year 2015 on December 31, 2014, with net assets of 238.8 billion dollars, and it is compared to 234.4 billion dollars at the end of the previous quarter. The investment return was 3.3% for the quarter, although CPPIB declared that the contribution rate reached 9.9%. Also, based on the CPPIB report “The CPP Fund is expected to grow significantly between now and 2022, and Canada’s Chief Actuary predicts that the CPP fund will grow to approximately 340 billion dollar by 2022”.

Investment Strategy of CPP

Based on the CPPIB report, the CPPIB investment strategy is comprised of three key elements; the CPP Reference Portfolio, Value-Adding Active Management and Total Portfolio Approach. The CPP reference Portfolio is the foundation of the investment strategy, with low cost and low complexity portfolio of the public market investments that can achieve the needed return for long-term, under this strategy. Value-Adding Active Management is the range of public and private market investment strategies that is employed to add value over the CPP reference portfolio returns. Then, Total Portfolio Approach is a principal element of the overall investment strategy. It determines the true underlying risk and return characteristics of each investment. This allows to manage the overall portfolio with more insight and precision. CPPIB now is investing 55% of the fund in global equities, 15% in Canadian equity, and 30% in Canadian Government Bonds.

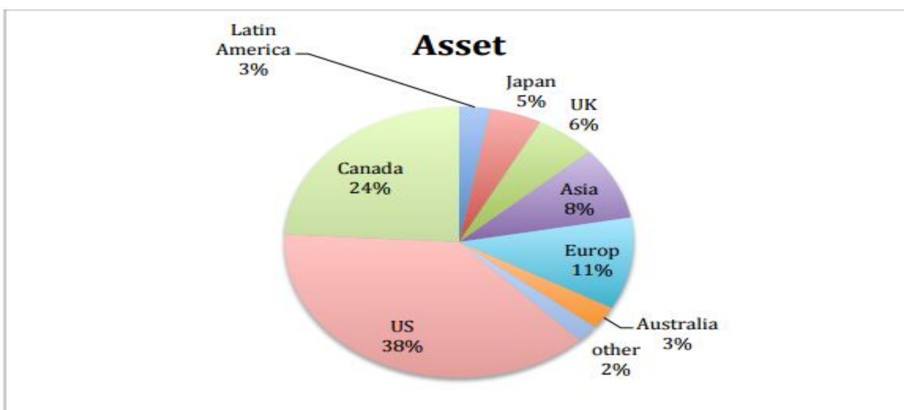

Figure 1.5: External Investment of CPPIB in 2015

Some Challenges Both Countries Face

1. Increased number of retirees compared to the number of employees.

2. Increased life expectancy, and this puts more pressure on pension plans.

3. Low long-term interest rates leads to a decline in pension funds.
4. In Canadian designs, Pillar 2 (CPP/QPP) provides lower benefits than in most other developed countries.
5. In Pillar 3, “People are not saving enough for retirement and if we let this go unchecked we are going to face a huge economic crisis.” Kathleen Wynne, Premier of Ontario, November 12, 2013.

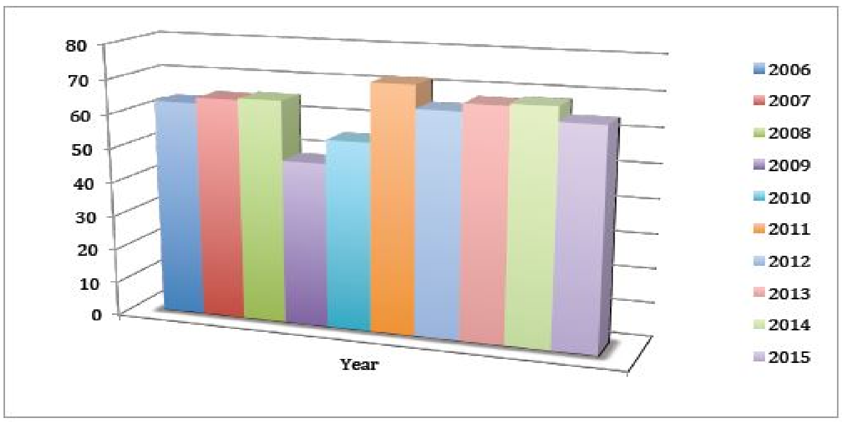

Figure 1.6: Investments (in Billions) by CPPIB in Canada

Chapter 2

Risk Measurement

For centuries, insurers and reinsurers have been selling risk coverages. Over time, they have been joined in this activity by banks and financial institutions. So it is not surprising that both groups face similar challenges; collecting and managing risks by looking for markets where these may be hedged or unbundled. However, when a market hedging of these risks does not exist, then a risk measurement is needed to allocate and evaluate performance. In this chapter we go over some important results in risk measurement, as well as some risk measures that are used most often in current research in actuarial science and finance.

2.1 Coherent Risk Measures

Many literature reviews defined the risk to be the variability in the future value of the position due to uncertainty. Or it might be the change in the position between two dates, and then whether those values are acceptable or unacceptable values.

Definition 2.1.1. *Artzneret al. in [2] defines a measure of risk, say ρ, as a mapping from the set of all risks $\mathbb{X}$ into a real number $\mathbb{R}$. Mathematically,*

$$\rho : X \to \mathbb{R}.$$

In other words, it is to determine a number $\rho(X)$ that quantifies the risk and can serve as a capital requirement. Further, if the value that is assigned by the measure is positive, then it is interpreted as the minimum extra cash the agent has to add to the risky position and to invest it to be acceptable. However, if it is negative, an equivalent cash amount can be withdrawn from the position without affecting its acceptability.

The functional form and fundamental properties of risk measures have been extensively studied in the actuarial literature since 1970. Here we list the most important properties commonly imposed on risk measures;

- Adding or (subtracting) an initial amount say α to the initial position and investing it, this will lead to a decrease or (an increase) in the risk measure by α. Mathematically, if ρ is a measure of the risk X, and α is a real number, then

  $$\rho(X + \alpha) = \rho(X) + \alpha.$$

  This property is called "translation invariance".

- The risk measure for two combined risks will not be greater than that of the risks measured separately. Mathematically, for all X_1 and X_2,

  $$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2).$$

  This property is called "sub-additivity", and it reflects the fact that there should be some diversification benefit from combining risks.

- Positive homogeneity: for a constant $\lambda \geq 0$, and a risk$X$,

  $$\rho(\lambda X) = \lambda \rho(X).$$

- Monotonicity: for two risks X and Y such that $X \leq Y$, then we have $\rho(Y) \leq \rho(X)$.
  It is clear that any risk measure should satisfy this property.

Sometimes risks increase in a non-linear way, and this leads to suggest the convexity property which is as following;

$$\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2).$$

This property means that the diversification does not increase the risk.

If the measure satisfies all the above conditions, then it is called a convex coherent risk measure, and if satisfies only the convexity property, then it is called a convex risk measure; for more details on convex risk measures see Follmer and Schied in [26].

In actuarial science, the first use of risk measures was in the development of premium principles to determine an appropriate premium to charge for an insurance. Since then, numerous risk measures have been used to determine not only the premium, but also the economic capital, that is, how much capital should an insurer hold to cover future liabilities. These risk measures are ranged from the most elementary to the most elaborate. In the following section, we illustrate some examples of the most commonly used risk measures in actuarial science and finance.

2.2 Value at Risk ($\mathbb{V}\mathrm{a}\mathbb{R}$)

The Value at Risk measure was actually in use by actuaries before it was reinvented for investment banking, and it is known as the quantile risk measure or quantile premium principle. In the last decade, $\mathbb{V}\mathrm{a}\mathbb{R}$ has become the established measure of risk exposure in financial service firms and has even begun to find acceptance in non-financial service firms. $\mathbb{V}\mathrm{a}\mathbb{R}$ also has roots in portfolio theory and a crude $\mathbb{V}\mathrm{a}\mathbb{R}$ measure was published in 1945; interested readers in the history of $\mathbb{V}\mathrm{a}\mathbb{R}$ can refer to Jorion in [44].

In fact, VaR was introduced to answer the following question; how much can we expect to lose in one day, week, year, with a given probability.

Definition 2.2.1. *The formal definition of* $\mathbb{V}a\mathbb{R}$ *measure is as follows;*

$$\mathbb{V}a\mathbb{R}_q(X) = \pi_q, \ \ such \ that \ \ P(X \leq \pi_q) = q.$$

This definition applies in the continuous case, however, for discrete and mixed distributed risks X, it is defined as follows;

$$\mathbb{V}\mathrm{a}\mathbb{R}_q(X) = \pi_q = min\{\pi_q : P(X \leq \pi_q) \geq q\},$$

where $0 \leq q \leq 1$ is the confidence level. Typical values of q range between 0.95 and 0.99. In general, to obtain $\mathbb{V}\mathrm{a}\mathbb{R}$, the underlying distribution should be known in advance.

As a matter of fact, $\mathbb{V}\mathrm{a}\mathbb{R}$ fails to be a coherent risk measure due to its lack of sub-additivity only, however, if risks follow a normal distribution and independent then $\mathbb{V}\mathrm{a}\mathbb{R}$ is sub-additive. Another drawback of the value-at-risk measurement is its inability to recognize an undue concentration of risks. In addition, it has the property that the $\mathbb{V}\mathrm{a}\mathbb{R}$ of a sum may be higher than the sum of the individual $\mathbb{V}\mathrm{a}\mathbb{R}$s. In such a case, diversification will lead to more risk being reported. Note that $\mathbb{V}\mathrm{a}\mathbb{R}$ is law invariant in a very strong sense; the distributions of X and Y do not need to be identical in order to imply $\mathbb{V}\mathrm{a}\mathbb{R}_q(X) = \mathbb{V}\mathrm{a}\mathbb{R}_q(Y)$. A certain local identity of the distributions suffices for this implication. In particular, random variables X with light tail probabilities and Y with heavy tail probabilities may have the same $\mathbb{V}\mathrm{a}\mathbb{R}$. This point is also one main criticism against $\mathbb{V}\mathrm{a}\mathbb{R}$ as a risk measure.

2.3 Conditional Value at Risk (CVaR)

Because of the above disadvantages of $\mathbb{V}\mathrm{a}\mathbb{R}$ as a risk measure, and also, the fact that the $\mathbb{V}\mathrm{a}\mathbb{R}$ does not give any information about the severity of losses beyond the $\mathbb{V}\mathrm{a}\mathbb{R}$ level, some alternative risk measures have been proposed. For instance, CVaR is a superior alternative to $\mathbb{V}\mathrm{a}\mathbb{R}$. It is also known as "Tail Value at Risk", "Expected

Shortfall", and "Tail Conditional Expectation", see Artzneret al. in [2], and Tasche [57]. Although, CVaR is still not used as a standard measure in the finance industry, CVaR is used more commonly in the insurance industry, and it has been used in credit risk evaluations; for more details see Embrechtset al. in [25].

Definition 2.3.1. *The CVaR measure is defined as below;*

$$TVaR_q(X) = CVaR_q(X) = \mathbb{E}[X|X > \mathbb{V}a\mathbb{R}_q(X)]$$

However, in general, CVaR is calculated as the weighted average of $\mathbb{V}$a$\mathbb{R}$ and losses exceeding $\mathbb{V}$a$\mathbb{R}$.

Pflug in [52] proved that CVaR is a coherent risk measure with the following properties; transition-equivalent, positively homogeneous, convex, monotonic w.r.t. stochastic dominance of order 1, and monotonic w.r.t. stochastic of order 2. Also, CVaR is more robust with respect to sampling error than $\mathbb{V}$a$\mathbb{R}$.

Furthermore, CVaR can be optimized and constrained with convex and linear programming methods, but $\mathbb{V}$a$\mathbb{R}$ is difficult to optimize. Moreover, CVaR may have a relatively poor out-of-sample performance compared with $\mathbb{V}$a$\mathbb{R}$ if tails are not modeled right. Hence, mixed CVaR can be a good alternative that gives different weights for different parts of the distribution; for more details see Uryasev in [55].

Additional risk measures can be used to measure risk. For instance, Markowitz in [48] and [49] was the first one who recognized the relationship between risk and reward and introduced the standard deviation as a measure of risk. However, he was also the first to suggest the semi-standard deviation as an alternative to deal with the standard deviation's symmetric nature. The coefficient of variation is a better measure of risk than the standard deviation and variance. However, it is not a coherent risk measure.

Pension schemes are faced with a number of different types of risk, and there is strong demand to measure such risks using a good risk measure. In the following chapter, some risks of pension plans are measured using the above measures.

Chapter 3

Model Structure and Assumptions

In this chapter, we describe the model structure of the hybrid plan that we use in this work. Then, we discuss the modifications that we apply to this plan, as well as, the notations used.

3.1 Risk Sharing in a Hybrid Pension Plan

Khorasanee in [45] introduced a model for pension plans that allows to share risk between employees and employers. The structure of this plan follows the **DB** plan that was proposed by Dufresne in [20]. Under Dufresne's plan, the surpluses and the deficits are amortized by adjusting the contribution income. So, the employers face all the risks. However, in Khorasanee's hybrid plan, it is assumed that surpluses and deficits are not only amortized by adjusting the contribution income, but also by adjusting the benefit outgo. So, the contribution risk is faced by the sponsor and the benefit risk is faced by the employee.

The subdivision of risks can be chosen arbitrarily by setting the amortization parameters in the benefit and contribution to whatever values the plan sponsor deems appropriate. Moreover, Khorasanee pointed out that the plan with lower aggregate risk, that is the sum of contribution and benefit risks, is more efficient. The first two moments of the fund and the benefit under this plan are derived, both when the time is finite and infinite. It is found that the expected benefit at a finite time differs from the target benefit by some fraction of the difference between the initial value of the fund and the liability.

In this project, we mainly focus on Khorasanee's hybrid plan, to which we apply the following modifications:

- The salary scale is constant, and equal to 1.
- The return on investment is modeled as a time series.
- The valuation rate of interest follows different scenarios.
- Contribution and benefit risks are measured using the value at risk ($\mathbb{V}\mathrm{a}\mathbb{R}$), and the coefficient of variation ($\mathbb{C}\mathrm{V}$).

3.2 Modeling a Hybrid Plan

In this section, we introduce the model assumptions, notation, and the mathematical model of a target benefit plan.

3.2.1 Assumptions

In the mathematical discussion, the following model assumptions are made:

1. All the actuarial assumptions are realized exactly, except for investment returns.
2. The population is stationary from the start.
3. There is no inflation on salaries, and no promotional salary scale.
4. The returns on investment are studied under two time series models;
    - An autoregressive moving average model, $\mathbf{ARMA}(1,1)$.
    - A generalized autoregressive conditional heteroskedasticity model,$\mathbf{GARCH}(1,1)$.
5. The valuation rate of interest is assumed to be fixed; and in general it is not necessarily equal to the expected rate of return, as is common in the literature.

6. The target benefit is considered to be constant, and equal to $\frac{2}{3}$ of the final salary which is equal to 1.
7. We consider only the case of active members.

3.2.2 Notation

The variables used in the model are defined as follows:

- B_t is the random benefit at time t.
- C_t is the random contribution rate at time t.
- F_t is the random fund level at time t.
- F_0 is the initial value of the fund, and it is assumed to be known.
- i_t is the random investment return between time $t-1$ and time t.
- $i = \mathbb{E}(i_t)$ is the expected rate of return i_t, assumed constant here.
- $\mathbb{V}\mathrm{ar}(i_t) = \sigma_i^2$, also assumed constant in time t.
- i_v is the known valuation interest rate.
- AL_t is the random actuarial liability at time t.
- NC_t is the random normal cost rate at time t.
- ${}^{T}NC_t$ is the random terminal normal cost at time t.
- TB is the constant target benefit.
- λ is the spread parameter for the plan. It is equal to $\frac{1}{\ddot{a}_{\overline{m}|}}$, where m is the number of years that the unfunded liability is spread into (spread period) for amortization.
- λ_c is the spread parameter for the contributions. It is equal to $p\lambda$, where $0 < p < 1$.
- λ_b is the spread parameter for the benefits. It is equal to $(1-p)\lambda$.

The spread method of amortization is used here, since our goal is to complete the amortization over a specific number of years m, and $\frac{1}{\ddot{a}_{\overline{m}|}}$ is calculated at rate i_v. Normally, it is assumed that $1 \leq m < \infty$, this implies that $d_v < \lambda \leq 1$, although it is possible to consider values of λ out of this range. Also, note that $\lambda_c + \lambda_b = \lambda$ in this study.

Moreover, the model is studied under a discrete time scale. Cairns in [15] considered Dufresne's model when the time is continuous.

The equations that explain the plan funding, contribution, and benefit are given as:

$$F_{t+1} = (1 + i_{t+1})(F_t + C_t - B_t), \qquad t = 0, 1, ... \tag{3.1}$$

$$C_t = NC_t + \lambda_c(AL_t - F_t), \qquad t = 0, 1, ... \tag{3.2}$$

$$B_t = TB - \lambda_b(AL_t - F_t), \qquad t = 0, 1, ... \tag{3.3}$$

Equation (3.1) can be re-expressed as follows, after substituting C_t and B_t from (3.2) and (3.3);

$$\begin{aligned} F_{t+1} &= (1 + i_{t+1})\big[(1 - \lambda)F_t + \lambda AL_t + NC_t - TB\big] \\ &= (1 + i_{t+1})\big[(1 - \lambda)F_t + R_t\big], \end{aligned}$$

where $R_t = \lambda AL_t + NC_t - TB$. From the last equation, we can obtain recursively the following formula for the fund. For $t = 1, 2, 3, ...$;

$$F_t = (1 - \lambda)^t \; \Pi_{j=1}^t (1 + i_j)F_0 + \sum_{s=0}^{t-1} (1 - \lambda)^{t-s-1} R_s \frac{\Pi_{j=1}^t (1 + i_j)}{\Pi_{j=0}^s (1 + i_j)},$$

by setting $\Pi_{j=1}^t (1 + i_j) = \mathrm{e}^{\Delta_t}$, and $\frac{\Pi_{j=1}^t (1+i_j)}{\Pi_{j=0}^s (1+i_j)} = \mathrm{e}^{\Delta_t - \Delta_s}$, with $i_0 = 0$, we get the final formula;

$$F_t = (1 - \lambda)^t \mathrm{e}^{\Delta_t} F_0 + \sum_{s=0}^{t-1} (1 - \lambda)^{t-s-1} R_s \mathrm{e}^{\Delta_t - \Delta_s}. \tag{3.4}$$

It is clear from equation (3.2) that the contribution rate at time t is the sum of the **normal cost** NC_t and the **adjustment** $\lambda_c(AL_t - F_t)$. Similarly, from equation (3.3) the benefit at time t is expressed as the difference between the **target benefit** TB and the **adjustment** $\lambda_b(AL_t - F_t)$. Also, the difference between the expected **liability** (AL_t) and the **actual fund** (F_t) is called the **unfunded liability**; if the difference is positive, in other words, AL_t is greater than F_t, then we have a deficit in the fund, and the plan contributions should be re-evaluated. However, if the difference is negative, AL_t is less than F_t, then we have a surplus in the fund; in this case the employers might use the excess to cover for future deficits, or, as assumed by the above equations, use it to decrease contributions C_t or augment benefits B_t, or both.

The terms AL_t and NC_t are obtained at every evaluation time, using one of the actuarial cost methods explained in Chapter 1.

Starting from (3.2), (3.3), and (3.4), the first two moments of the fund, the contribution, and the benefit are derived under the above assumptions, and then the optimal spread parameter is obtained in Chapter 5 when the time is both finite and infinite.

Chapter 4

Models for the Return on Investment

Modeling and analyzing financial time series is a complex scientific problem. This is not only due to the variability of the series in use or to the size of the data that is available, but also because of the stylized facts that exist in most financial data. Stylized facts were illustrated by Mandelbrot in [50], and some are mentioned in Section 3.3. Moreover, financial theory and empirical time series both contain an element of uncertainty.

The objective of this chapter is to provide some knowledge of financial time series; in particular asset returns, and introduce some time series models that are useful in Finance. We begin with a brief introduction of the rate of return. Then, follows a literature review of various stochastic models that have been used for rates of return for pension plans. We review some of the most useful time series models used widely in Finance. Finally, we fit two of these models to the rate of return in a pension plan.

4.1 Rate of Return $\mathbf{i}_t$

Most studies in Finance and Actuarial science model the returns on assets instead of the actual prices. This is because the rate of return is a complete and scale-free summary of an investment position, and the statistical properties of this quantity are more attractive and easier to handle than the price series. However, the continuous compound return or log-return has some advantage over the actual return; for example, the continuously compounded multi-period log-return is simply the sum of continuously compounded one-period log-returns, and the statistical properties of log returns are more tractable.

In Actuarial Science, in particular in the areas of Life Insurance and Pensions, authors consider the rate of return as independent and identically distributed (iid) random variables or as stochastic processes. For instance, Dufresne in [19]-[21], Haberman in [33], [36], Haberman and Vigna in [41], Haberman and Sung in [37], Zimbidis and Haberman in [62], Khorasanee in [45] all discuss the case of independent and identically distributed rates of return. However, time series models have also been used to model the return. For example, the first order of autoregressive model **AR**(1) was applied by Haberman in [32], [34], and [35], Haberman and Gerrad in [39], Cairns in [14], and Cairns and Parker in [16]. The **AR**(2) model is also considered by Haberman in [35]. Haberman and Wong in [38] studied the returns as moving averages **MA**(1) and **MA**(2) models.

4.2 Autoregressive Moving Average (ARMA) Model

ARMA models are used widely to describe data. This may due to the need of higher-order models with many parameters to explain the dynamic structure of data. Basically, **ARMA** models are linear processes which combine autoregressive and moving average terms into a single form. Although the chance of using an **ARMA**

model in Finance is low, the concept of **ARMA** models is highly relevant in volatility modeling, as we see in the next section.

Definition 4.2.1. *The log of return series* $\{\delta_t\, ; t \in \mathbb{Z}\}$ *is an* $\boldsymbol{ARMA}(p,q)$ *process if* $\{\delta_t\}$ *is stationary and if for every* t,

$$\delta_t - \phi_1\delta_{t-1} - .... - \phi_p\delta_{t-p} = \mu(1-\phi_1 - ... - \phi_p) + Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q},$$

where $\delta_t = \mathbf{ln}(1+i_t)$ *is the force of interest in the interval* $(t-1,t)$, $\{Z_t\} \sim$ *White Noise* $(0, \sigma_z^2)$, μ *is the long mean of the process, and the polynomials* $(1 - \phi_1 B - ..... - \phi_p B^p)$ *and* $(1 + \theta_1 B + ... + \theta_q B^q)$ *have no common factors, where* B *is back-shift operator such as* $BX_t = X_{t-1}$.

In this section, we study the simplest $\mathbf{ARMA}(1,1)$ model, and we assume that $\{Z_t\}$ are independent and identically distributed **normals** with zero mean and unit variance. Hence;

$$\delta_t - \phi\delta_{t-1} = \mu(1-\phi) + Z_t + \theta Z_{t-1}.$$

where $\phi + \theta \neq 0$. The conditions for stationary and the existence of an invertible solution of the model are that $|\phi| < 1$ and $|\theta| < 1$, respectively.

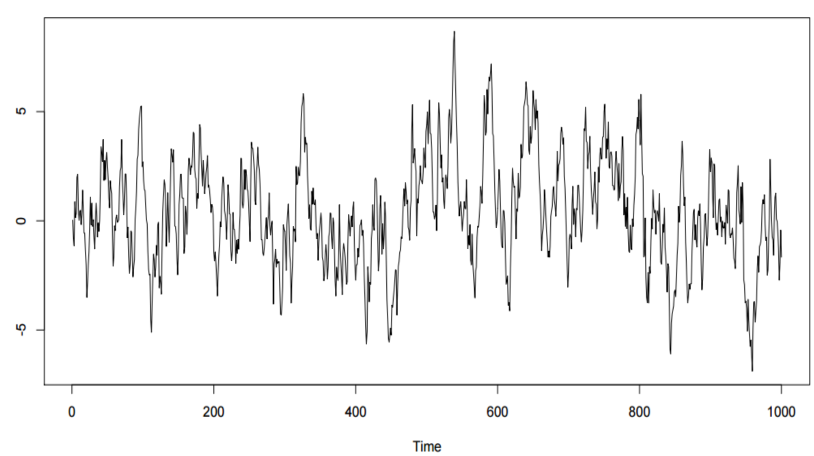

Figure 4.1: Simulated Data from $\mathbf{ARMA}(1,1)$ with $\phi = .9$ and $\theta = .1$

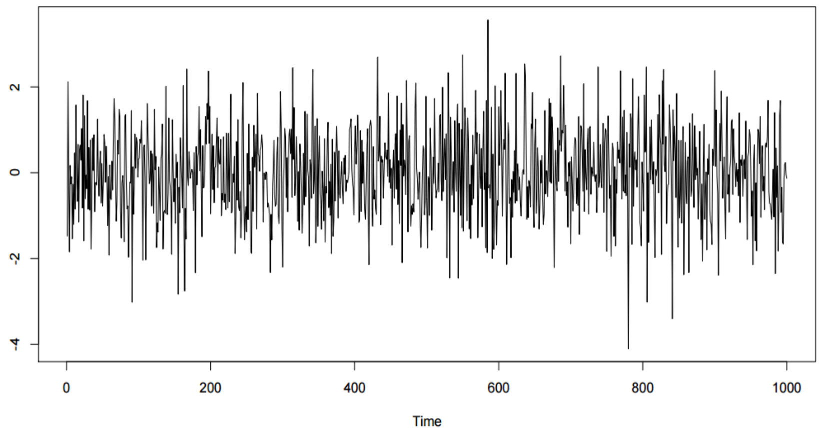

Figure 4.2: Simulated Data from **ARMA**$(1,1)$ with $\phi = -.5$ and $\theta = .5$

4.2.1 Marginal Moments under ARMA$(1,1)$

Here, the expected value, the variance, and the auto-covariance functions of δ_t are derived first. Similarly, the same quantities are obtained for the function Δ_t, since it is needed in the following chapter, and is defined as;

$$\Delta_t = \sum_{j=1}^{t} \delta_j.$$

Hence,

$$\mathbb{E}(\delta_t) = \mu \quad ; \text{for all } t \in \mathbb{Z}$$

To find the variance of the process, the **Yule-Walker** method is one possible approach that can be used to derive the auto-covariance of the process,

$$\gamma_\delta(m) = \gamma_\delta(t+m,t) = \mathbb{C}\text{ov}(\delta_{t+m},\delta_t) = \mathbb{E}[(\delta_{t+m}-\mu)(\delta_t-\mu)];$$

and it is generally defined for ARMA(p,q) as;

$$\gamma_\delta(m) - \phi_1\gamma_\delta(m-1) - \phi_2\gamma_\delta(m-2) - \ldots\ldots.. - \phi\gamma_\delta(m-p) = \sigma_z^2 \sum_{j=0}^{\infty} \psi_j\theta_{\{j+m\}} \quad ; \ m = 0,1,...,p$$

where $\psi_0 = 1$, $\psi_j = \sum_{k=1}^{p} \phi_k\psi_{j-k} + \theta_j$ for $j = 0,1,....$, and $\theta_j = 0$ for $j > q$. The above formula can be used then to find $\gamma_\delta(m)$ for different values of m. In case of

ARMA$(1,1)$ model, **Yule-Walker** is adapted as;

$$\gamma_\delta(m) - \phi\gamma_\delta(m-1) = \sigma_z^2 \sum_{j=0}^{\infty} \psi_j \theta_{\{j+m\}} \quad ; \; m = 0, 1$$

then, a system of two equations is generated, and it is used to find the variance of the process, which is derived as;

$$\mathbb{V}\text{ar}(\delta_t) = \gamma_\delta(0) = \frac{1 + 2\phi\theta + \theta^2}{1 - \phi^2}. \tag{4.1}$$

which is constant in time under this model.
Then we get;

$$\gamma_\delta(1) = \theta + \phi\gamma_\delta(0).$$

The auto-covariance function of the process at lag h is then defined recursively as;

$$\gamma_\delta(h) = \phi^{h-1}\gamma_\delta(1), \quad \text{for any } h \geq 2.$$

For Δ_t, the expectation and the variance are defined as follow;

$$\mathbb{E}(\Delta_t) = t\mu.$$

$$\mathbb{E}(\Delta_t - \Delta_s) = (t-s)\mu.$$

$$\mathbb{V}\text{ar}(\Delta_t) = \Big[\theta + \phi\gamma_\delta(0)\Big]\Big[t\phi + 2\sum_{i=1}^{t}\sum_{j=i+1}^{t} \phi^{j-i-1}\Big]. \tag{4.2}$$

For $s = 0, 1, 2, ..., t-1$, and assuming that $s > v$;

$$\begin{aligned} \mathbb{V}\text{ar}(\Delta_t - \Delta_s) &= \mathbb{V}\text{ar}(\sum_{i=s+1}^{t} \delta_i) = \mathbb{C}\text{ov}(\sum_{i=s+1}^{t} \delta_i, \sum_{j=s+1}^{t} \delta_j) = \sum_{i=s+1}^{t}\sum_{j=s+1}^{t} \gamma_\delta(i,j) \\ &= (\theta + \phi\gamma_\delta(0))\Big[(t-s)\;(\phi + \frac{2}{1-\phi}) - 2\frac{(1-\phi^{t-s})}{(1-\phi)^2}\Big]. \end{aligned} \tag{4.3}$$

$$\begin{aligned} \mathbb{V}\text{ar}(\Delta_t - \Delta_s + \Delta_t - \Delta_v) &= \mathbb{V}\text{ar}(\Delta_v - \Delta_s + 2(\Delta_t - \Delta_v)) \\ &= \mathbb{V}\text{ar}(\Delta_v - \Delta_s) + 4\mathbb{V}\text{ar}(\Delta_t - \Delta_v) + 4\mathbb{C}\text{ov}(\Delta_t - \Delta_s, \Delta_t - \Delta_v) \\ &= \sum_{i=s+1}^{v}\sum_{j=s+1}^{v} \gamma(i,j) + 4\sum_{i=v+1}^{t}\sum_{j=s+1}^{t} \gamma(i,j). \end{aligned} \tag{4.4}$$

Before concluding this section, we summarize some of the most common properties of **ARMA**$(1,1)$ processes:

- The stationarity condition of an **ARMA**$(1,1)$ model is the same as that of an **AR**(1) model, and the plot of the autocorrelation function **ACF** of an **ARMA**$(1,1)$ shows a pattern similar to that of an **AR**(1) model except that the pattern starts at lag 2.
- **ARMA** models are applied to model the conditional expectation of a process given the past information, however, in an **ARMA** model the conditional and marginal variance is constant.

4.3 Generalized Autoregressive Conditional Heteroskedasticity (GARCH)

We mentioned in the introduction of this chapter that financial series have stylized features or stylized facts, described by Mandelbrot in [50]. We review here some of the most commonly used in the financial literature.

Figure 4.3: Saudi Arabia TASI Index for 2011-2015 (from TASI website)

Figure 4.4: Canada S&P/TSX Toronto Stock Market Index for 2011-2015 (from S&P/TSX Toronto Stock Market website)

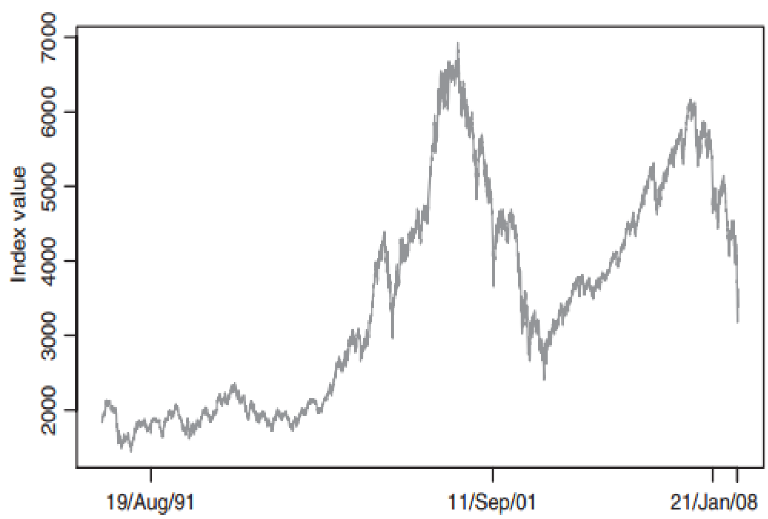

Figure 4.5: CAC 40 Index for the Period from March 1, 1990 to October 15, 2008 (from Francq and Zakoian in [27])

Properties of financial time series:

1. Generally financial time series are not stationary as it is clear from Figures 4.3, 4.4, and 4.5.

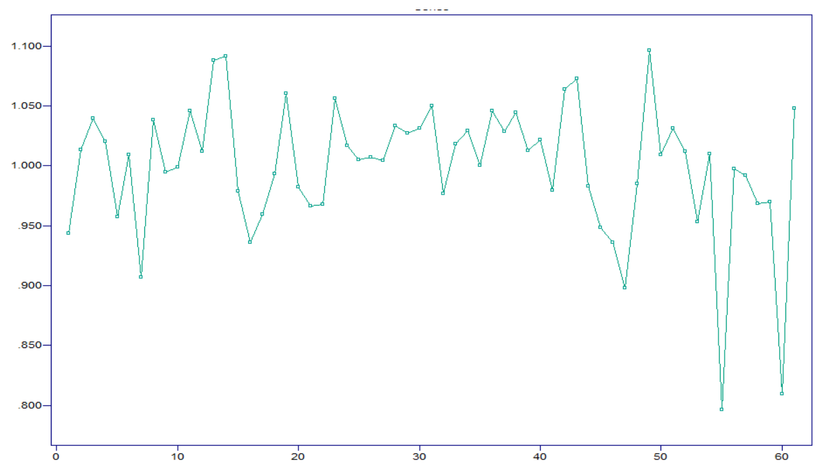

Figure 4.6: Saudi Arabia TASI Index for 2011-2015

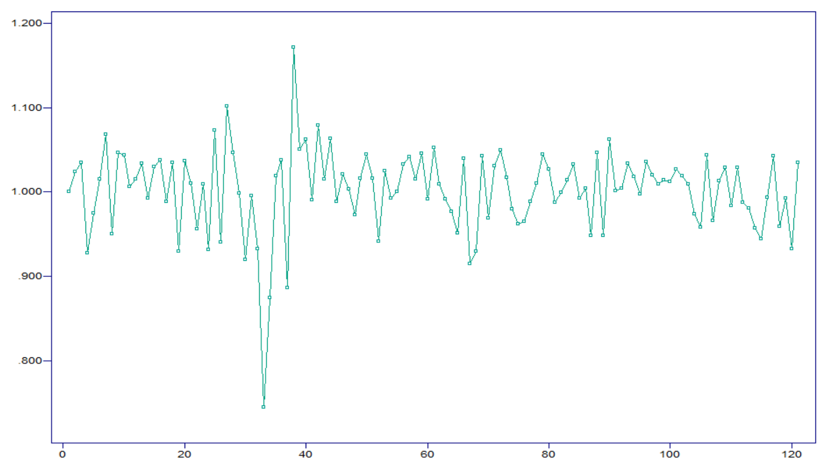

Figure 4.7: Canada S&P/TSX Toronto Stock Market Index for 2011-2015 (from S&P/TSX Toronto Stock Market website)

2. Some series display small autocorrelations, making them close to a white noise as illustrated in Figures 4.8, 4.9, and 4.10.

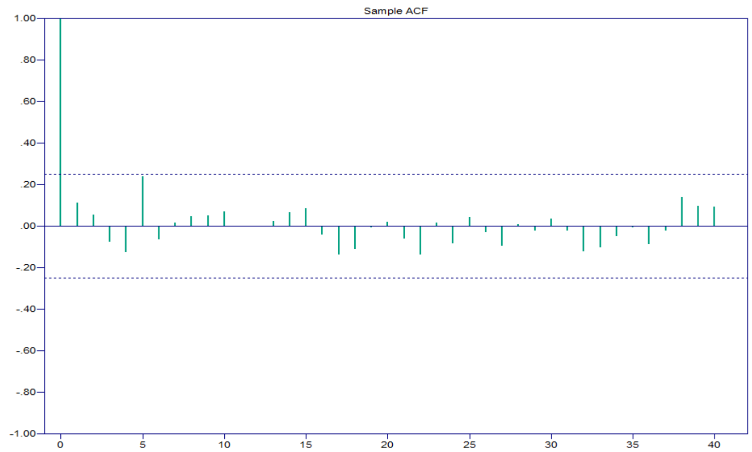

Figure 4.8: Sample Autocorrelations of Returns of the TASI (Feb. 23, 2013 to Feb. 23, 2016)

3. The square and absolute values of the processes show strong autocorrelations, see Figure 4.11.

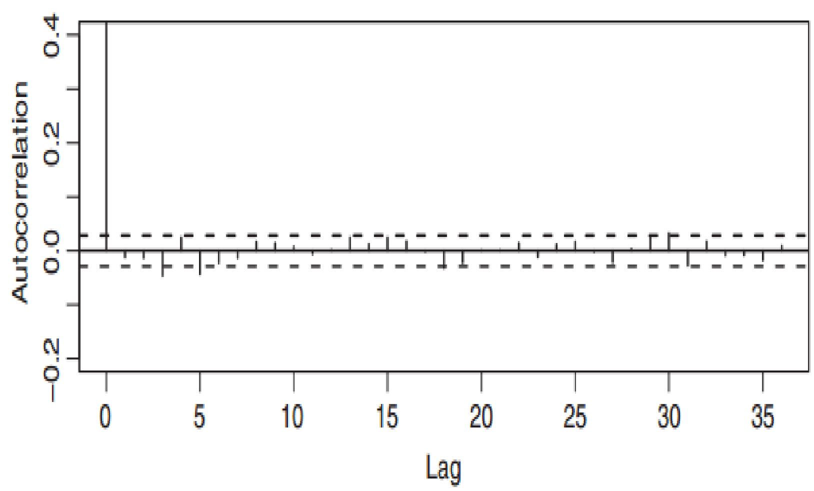

Figure 4.9: Sample Autocorrelations of Returns of the CAC 40 from January 2, 2008 to October 15, 2008 (from Francq and Zakoian in [27])

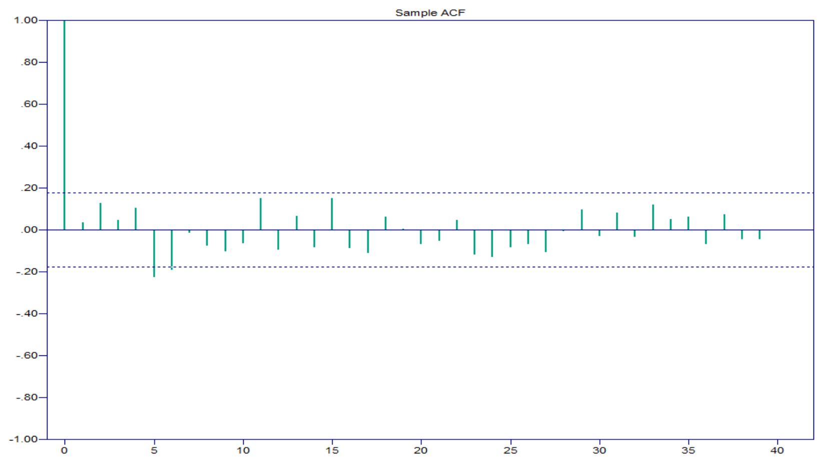

Figure 4.10: Sample Autocorrelations of Returns of Canada S&P/TSX Toronto Stock Market Index from Feb. 23, 2011 to Feb. 23, 2016

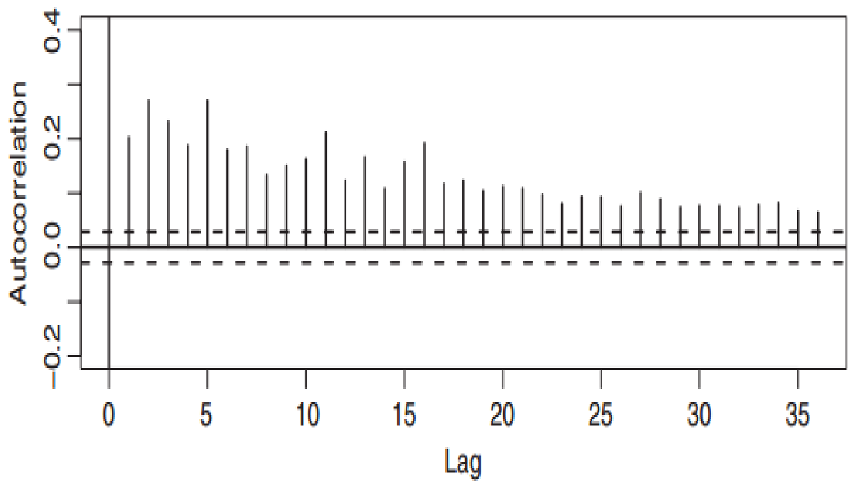

Figure 4.11: Sample Autocorrelations of Squared Returns of the CAC 40 from January 2, 2008 to October 15, 2008 (from Francq and Zakoian in [27])

4. Volatility clustering, which means large changes are followed by large changes, and small changes follow small changes. This is clear in the square and the absolute values of the series.
5. Financial series can have fat tails, so then they are called leptokurtic.
6. Calendar effects, holidays, days of the week, and other seasonal patterns, may have significant effects on the series.

For further explanations see Mandelbrot in [50] or Francq and Zakoian in [27].

As we see from these properties it is difficult to model financial time series, so there is a need for a stationary model that captures the main stylized facts of the series. Hence, the **ARMA**$(1,1)$ is not appropriate to fit the data because it assumes a constant variance, meaning that the conditional variance is time-invariant and contains no past information. So, conditional heteroscedasticity is preferred, together with the stationarity property.

Conditional heteroscedastic models were introduced to the Econometrics literature to account for the very specific nature of financial series, and they have been used extensively in research. In this section, we mainly focus on the generalized autoregressive conditional heteroscedastic **GARCH** model.

Engle in [23] introduced the autoregressive conditional heteroskedastic **ARCH** model to allow the conditional variance to vary over time, as a function of the past errors. Similarly to the extension of **AR** models to **ARMA** models; an extension of **ARCH** models is proposed by Bollerslev in [6] to allow for a more flexible lag structure, and permit a wider range of behavior, in particular, more persistent volatility.

Definition 4.3.1. *The process $\{\delta_t \ ; \ t \in \mathbb{Z}\}$ is called **GARCH**(p,q) if*

$$\delta_t = \Upsilon + \varepsilon_t,$$

where Υ is the long mean of the process under GARCH.

$$\varepsilon_t = \sigma_t Z_t,$$
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,$$

and Z_t as well as σ_t^2 are real processes such that:

- The volatility process σ_t^2 is measurable with respect to the history σ-field, denoted $\mathcal{F}_{t-1} = \sigma(\varepsilon_s; s < t)$. Hence, the volatility is a deterministic function of the past ε_t.

- $Z_t \sim$ iid random variables, and Z_t is independent of $\mathcal{F}_{t-1}$.
- $\alpha_0 > 0$, $\alpha_i > 0$, and $\beta_j > 0$ to guarantee that the conditional variance is non-negative.
- For $j = 1, ..., q$, if $\beta_j = 0$, the process is called ARCH(p) process.

For simplification, here the standard **GARCH**$(1,1)$ is considered, where Z_t follows a **normal** distribution with zero mean and unit variance. However, substantial research work has been devoted to the model when the error Z_t follows other distributions, such as the t-distribution, Z-distribution, gamma distribution, generalized Pareto distribution; see Bai, Russel, and Tiao in [4], Bollerslev in [7], and Lanne and Pentti in [47]. So, the conditional variance of the **GARCH**$(1,1)$ is given as;

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \tag{4.5}$$

where $\alpha_1 + \beta_1 < 1$ to ensure stationarity. Moreover, $\alpha_1 + \beta_1$ is the persistence measurement.

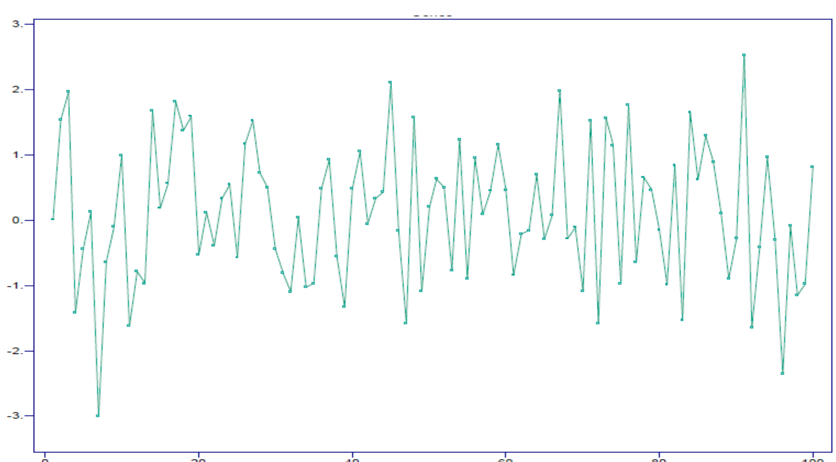

Figure 4.12: Simulated data from GARCH$(1,1)$ with $\alpha_0 = 1$, $\alpha_1 = .4$, and $\beta_1 = .5$,

As pointed out in the literature, the squared process $\{\varepsilon_t^2\}$ can be represented as an **ARMA**$(1,1)$ model by adding ε_t^2 to equation (4.5) to get;

$$\sigma_t^2 + \varepsilon_t^2 - \varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

$$\varepsilon_t^2 - (\alpha_1 + \beta_1)\varepsilon_{t-1}^2 = \alpha_0 + \nu_t^2 - \beta_1 \nu_{t-1}^2, \tag{4.6}$$

where $\nu_t^2 = \varepsilon_t^2 - \sigma_t^2 = \varepsilon_t^2 - \mathbb{E}(\varepsilon_t^2|\mathcal{F}_{t-1})$. Moreover, ν_t^2 is a white noise with mean zero and variance $\sigma_{\nu^2}^2$ that is given as;

$$\begin{aligned}\mathbb{V}\text{ar}(\nu_t^2) = \mathbb{E}(\varepsilon_t^2) - \frac{\alpha_0^2}{(1-\beta_1)^2} - \alpha_1^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \beta_1^i \beta_1^j \mathbb{E}(\varepsilon_{t-(i+1)}^2 \varepsilon_{t-(j+1)}^2) \\ + \frac{2\alpha_0\alpha_1}{1-\beta_1} \sum_{j=0}^{\infty} \beta_1^j \mathbb{E}(\varepsilon_{t-(j+1)}^2).\end{aligned}$$

Furthermore, **GARCH**$(1,1)$ processes can be represented as **ARCH**(∞), which allows to write σ_t^2 as function of the infinite sum of past ε_t^2 values in the form;

$$\sigma_t^2 = w_0 + \sum_{i=1}^{\infty} w_i \varepsilon_{t-i}^2,$$

where $w_0 = \frac{\alpha_0}{1-\beta_1}$, and $w_i = \alpha_1 \beta_1^{i-1}$.

4.3.1 Conditional Moments of the GARCH$(1,1)$ Model

Here, the conditional moments such as the first and the second moments of ε_t, δ_t, and Δ_t are listed;

$$\mathbb{E}(\varepsilon_t|\mathcal{F}_{t-1}) = 0.$$

$$\sigma_t^2 = \mathbb{V}\text{ar}(\varepsilon_t|\mathcal{F}_{t-1}) = \mathbb{E}(\varepsilon_t^2|\mathcal{F}_{t-1}).$$

$$\mathbb{C}\text{ov}(\varepsilon_t, \varepsilon_{t+h}|\mathcal{F}_{t-1}) = 0.$$

then by using the above moments, the moments of our functions are derived as below;

$$\mathbb{E}(\delta_t|\mathcal{F}_{t-1}) = \Upsilon.$$

$$\mathbb{V}\text{ar}(\delta_t|\mathcal{F}_{t-1}) = \mathbb{V}\text{ar}(\varepsilon_t^2|\mathcal{F}_{t-1}) = \sigma_t^2.$$

$$\mathbb{E}(\Delta_t|\mathcal{F}_{t-1}) = t\Upsilon.$$

$$\mathbb{V}\text{ar}(\Delta_t|\mathcal{F}_{t-1}) = \sum_{i=1}^{t} \sigma_i^2.$$

4.3.2 Marginal Moments of the GARCH$(1, 1)$ Model

Bollerslev in [6] states a theorem that helps find the even moments, and he provides sufficient conditions for the existence of the even order $2m$ moments, for $m = 1, 2....$ Note that, for the odd moments; that is of order $2m - 1$, are all equal to zero, due to symmetry.

Theorem 4.3.1. *For the* ***GARCH***$(1, 1)$ *process given above a necessary and sufficient condition for existence of the* $2mth$ *moment is*

$$\mu(\alpha_1, \beta_1, m) = \sum_{j=0}^{m} \binom{m}{j} a_j \alpha_1^j \beta_1^{m-j} < 1,$$

where $a_0 = 1$, $a_j = \Pi_{i=1}^{j}(2j - 1)$.

Then, the $2m$*-th moment can be expressed by the recursive formula;*

$$\mathbb{E}(\varepsilon_t^{2m}) = \frac{a_m \left[\sum_{n=0}^{m-1} a_n^{-1} \mathbb{E}(\varepsilon_t^{2n}) \alpha_0^{m-n} \binom{m}{m-n} \mu(\alpha_1, \beta_1, n) \right]}{1 - \mu(\alpha_1, \beta - 1, m)}.$$

Hence, the first four moments of ε_t are given as;

$$\mathbb{E}(\varepsilon_t) = \mathbb{E}(\sigma_t)\mathbb{E}(Z_t) = 0.$$

$$\mathbb{E}(\varepsilon_t^2) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}.$$

$$\mathbb{E}(\varepsilon_t^3) = 0.$$

$$\mathbb{E}(\varepsilon_t^4) = 3\,\frac{\alpha_0^2(1 + \alpha_1 + \beta_1)}{(1 - \alpha_1 - \beta_1)(1 - \beta_1^2 - 2\alpha_1\beta_1 - 3\alpha_1^2)}.$$

Then,

$$\mathbb{V}\text{ar}(\varepsilon_t) = \sigma_\varepsilon^2 = \gamma_\varepsilon(0) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}.$$

$$\mathbb{C}\text{ov}(\varepsilon_t, \varepsilon_{t-h}) = 0.$$

$$\text{Skewness}(\varepsilon_t) = 0.$$

$$\text{Kurtosis}(\varepsilon_t) = 3\frac{(1 + \alpha_1 + \beta_1)(1 - \alpha_1 - \beta_1)}{(1 - \beta_1^2 - 2\alpha_1\beta_1 - 3\alpha_1^2)}.$$

The marginal moments of our target functions;

$$\mathbb{E}(\Delta_t) = t\Upsilon.$$

$$\mathbb{V}\mathrm{ar}(\Delta_t) = t\,\frac{\alpha_0}{1-\alpha_1-\beta_1}.$$

For $s = 0, 1, ...t-1$, and $v > s$,

$$\mathbb{E}(\Delta_t - \Delta_s) = (t-s)\Upsilon.$$

$$\mathbb{V}\mathrm{ar}(\Delta_t - \Delta_s) = (t-s)\frac{\alpha_0}{1-\alpha_1-\beta_1}.$$

$$\mathbb{V}\mathrm{ar}(2\Delta_t - \Delta_s - \Delta_v) = (v-s)\frac{\alpha_0}{1-\alpha_1-\beta_1} + 4(t-v)\frac{\alpha_0}{1-\alpha_1-\beta_1}.$$

It is clear that the auto-covariance of the process ε_t is equal to zero which means that the random variables are uncorrelated although they are dependent, thus the process is weak white noise. As a consequence the square of the process is also considered in the literature, since the auto-covariance is not equal to zero;

$$\mathbb{C}\mathrm{ov}(\varepsilon_t^2, \varepsilon_{t+h}^2) = \gamma_{\varepsilon^2}(h) = (\alpha_1 + \beta_1)^{|h|}\gamma_{\varepsilon^2}(0),$$

where $h \in \mathbb{Z}$, and $\gamma_{\varepsilon^2}(0) = \mathbb{V}\mathrm{ar}(\varepsilon_t^2)$.

Although the marginal distribution of ε_t is not known explicitly, we can have an idea of the shape of the distribution from the marginal moments of the **GARCH** model. Hence, from the above moment formulas we can highlight some properties of the distribution:

- The skewness of the process is zero which means that the distribution is symmetric.
- The kurtosis of the process is greater than 3, and that means that the process is leptokurtic.
- When $0 < \alpha_1 < .06$ and $0 < \beta_1 < .9$, the kurtosis is approximately equal to 3.

Finally, it is interesting to mention the following additional properties of GARCH models that are considered in the Economics literature, to conclude this section:

1. The forecasting value of the conditional variance after multiple steps, say L, is obtained as;
$$\hat{\sigma}^2_{t+L} = \sigma^2_{\varepsilon_t} + (\alpha_1 + \beta_1)^L(\sigma^2_t - \sigma^2_\varepsilon), \tag{4.7}$$
where σ^2 is the unconditional variance of ε_t. Hence, as $L \to \infty$, $\hat{\sigma}^2_{t+L} \to \sigma^2_\varepsilon$.

2. From (4.7) we notice that $\alpha_1 + \beta_1$ determines how quickly the forecasting variance approaches the unconditional variance.

3. The initial value of the conditional variance can be set to the sample variance, to α_0, or to the unconditional variance.

4. There are other types of **GARCH** models that can be found in the literature such as Integrated GARCH (**IGARCH**), Fractional IGARCH (**FIGARCH**), Exponential GARCH (**EGARCH**). Some of these are reviewed in the next section.

4.4 Long-Memory Models

Long-memory processes are known to play an important role in many scientific disciplines and applied fields such as Physics, Geophysics, Hydrology, Economics, Finance or Climatology. Furthermore, they have been used in Economics since 1980, and there is a considerable evidence that this type of process successfully describes financial data such as forward premiums, inflation rates, and exchange rates. Currently, it is believed that long memory models are better candidates than other conditional heteroscedastic models to explain volatility in stock returns and exchange rates; for more details see Granger, Ding, and Engle in [30], Harvey in [40], Granger and Ding in [31].

Definition 4.4.1. *Given a discrete time series Y_t with autocorrelation $\rho(j)$ at lag j*

and spectral density function $f_y(\lambda)$, the process is long memory process if

$$lim_{n\to\infty} \sum_{j=-n}^{n} |\rho(j)|$$

is non-finite or diverges; or as $\lambda \to 0$, $f_y(\lambda)$ diverges.

Long-memory is also called strong memory, hyperbolic memory, long-range dependence or long-range correlation. Moreover, under this process the sample auto-correlations decline to zero hyperbolically rather than geometrically. However, if the sample autocorrelations of the process decay geometrically to zero, then the process is classified as a short memory process.

In this section, we go over some examples of time series models that exhibit long memory, and we review the most common features or characteristics of those models.

4.4.1 Autoregressive Fractionally Integrated Moving Average Model (ARFIMA)

When the data is stationary, and its autocorrelation function exhibits a rapidly decrease to zero, then **ARMA** models give a better fit. However, if the data does not exhibit these properties, then we need to difference the data until it produces new series that has **ARMA** characteristics. Thus, we say that the original series follows an autoregressive integrated moving average model (**ARIMA**).

In fact, **ARMA** and **ARIMA** processes are preferable models in time series, and this is due to their simplicity and flexibility. The definition of an **ARIMA** model is given below.

Definition 4.4.2. *If d is a nonnegative integer, then the series $\{X_t\}$ is an* ***ARIMA***(p, d, q) *process if $(1 - B)^d X_t$ is an* ***ARMA***(p, q) *process. Thus,*

$$\phi(B)(1 - B)^d X_t = \theta(B) Z_t,$$

where $\{Z_t\} \sim \mathbf{WN}(0, \sigma_z^2)$, $\phi(B) = 1 - \phi_1 B - .... - \phi_p B^p$, and $\theta(B) = 1 + \theta_1 B + ... + \theta_q B^q$. If $d = 0$, then the model reduces to an **ARMA** process. In other words, **ARMA** is a special case of the **ARIMA** process. In Econometrics, X_t is also called an integrated process of order d, or $\mathbf{I}(\mathbf{d})$.

Granger and Joyeux in [29], and Hosking in [43] introduced a model that is between an **ARMA** and **ARIMA**, and it allows d to be non-integer; this model is called autoregressive fractionally integrated moving average **ARFIMA**. In other words, it is defined as follows.

Definition 4.4.3. *If $d \in (\frac{-1}{2}, \frac{1}{2})$, the process $\{X_t\}$ is **ARFIMA** if*

$$\phi(B)(1 - B)^d X_t = \theta(B) Z_t,$$

where $(1 - B)^d$ is defined as an hypergeometric function;

$$(1 - B)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j + d)}{\Gamma(j + 1)\Gamma(d)} B^j.$$

Palma in [53] discusses the **ARFIMA** process and shows that it is stationary and invertible, if the roots of $\phi(B)$ and $\theta(B)$ lie outside the unit circle.

4.4.2 Fractionally Integrated Generalized Autoregressive Conditional Heteroscedastic (FIGARCH)

Under the **GARCH** model, we mentioned that $\alpha_1 + \beta_1$ measures the persistence, and as this sum approaches 1, the greater the persistence of shocks to volatility. So, if $\alpha_1 + \beta_1 = 1$, then the current shock persists indefinitely in conditioning the future variance. Hence, Engle and Bollerslev in [24] proposed an extension of **GARCH**, and called integrated GARCH (**IGARCH**).

Definition 4.4.4. *The IGARCH model is defined as;*

$$L(B)(1 - B)\varepsilon_t^2 = \alpha_0 + (1 - \beta_1 B)\nu_t^2,$$

where $L(B) = (1 - \alpha_1 B - \beta_1 B)(1 - B)^{-1}$, and $\alpha_1 + \beta_1 = 1$.

A key feature of the **IGARCH** model is that the impact of the past squared shock is persistent and the pricing of risky securities, including long-term options and future contracts, may show extreme dependence on the initial conditions. However, the unconditional variance under this model is infinite. Moreover, the forecasting of future conditional variance approaches the current conditional variance since $\alpha_1 + \beta_1 = 1$, see (4.7). Choudhry in [18] tested a real data made of stock returns from different countries, using **IGARCH** to check for the persistence of shocks to volatility.

Davidson in [22] mentions that both **GARCH** and **IGARCH** have short memory. In other words, it is shown that **GARCH** and **IGARCH** models have a memory which is much shorter than financial series generally exhibit. Breidt, Crato and de Lima in [10], Granger, Ding, and Engle in [30], Granger and Ding in [31], and Harvey in [40] all noticed the presence of long-memory in the autocorrelations of squared and absolute returns of various financial asset prices. Motivated by these observations, the fractional **IGARCH** (**FIGARCH**) was introduced.

The main feature of this model is that it provides a more flexible class of processes for the conditional variance that are better capable of explaining and representing the observed temporal dependencies in financial market volatility. In particular, the **FIGARCH** model exhibits a slow hyperbolic rate of decay for the lagged squared or absolute innovations in the conditional variance function.

Definition 4.4.5. *The process $\{\varepsilon_t\}$ follows a **FIGRACH** model if*

$$L(B)(1-B)^d\varepsilon_t^2 = \alpha_0 + (1-\beta_1 B)\nu_t^2,$$

where $0 < d < 1$.

The proof of stationarity in the general case of **FIGARCH**(p, d, q) is not yet available. Also, the process has an infinite unconditional variance. A good review of **FIGARCH** models can be found in Tayefi and Ramanathan in [58].

Finally, the interested reader can refer to Beran, Feng, Ghosh, and Kulik in [5] for more details. Baillic in [3] provides a good survey about long memory processes and fractional integration in Econometrics.

Chapter 5

Risk Measurement under Hybrid Pension Plan

Hybrid pension plans are subject to different type of risks; some of them were mentioned in Chapters 1 and 3. For instance, the **contribution risk** represents the size of deviations in the normal cost and is related to the stability of the plan. In the literature the contribution risk is usually measured by the variance of the contribution rate $\mathbb{V}\mathrm{ar}(C_t)$. Another type of risk is the **solvency risk**, when the pension fund can either meet its liabilities or the plan has to be wound up. In other words, this risk is an indicator of the financial stability of the plan. The variation in pension funds is measured in this case using the variance $\mathbb{V}\mathrm{ar}(F_t)$. For the employees, the stability and predictability of the benefit is important for financial planning. Hence, it is desirable to keep the variations in benefit in an acceptable range, and the variance $\mathbb{V}\mathrm{ar}(B_t)$ is often used as a measure; this type of risk is called **benefit risk**.

In this chapter, we study these risks under the proposed hybrid plan. The expected value and the variance of the fund, the annual contribution rate, and the annual benefit rate are derived when the rate of return follows an $\mathbf{ARMA}(1,1)$ and $\mathbf{GARCH}(1,1)$ process for $t \to \infty$ as well as $t < \infty$. Moreover, all the actuarial valuations of the plan are done when the valuation interest rate i_v is not necesarilly equal the expected rate of return $\mathbb{E}(i_t)$, taking into account the special case when $i_v = \mathbb{E}(i_t)$. Then, the aggregate risk, which is defined as the sum of the contribution risk and the benefit risk, is obtained under different risk measure and as t goes to infinity. Finally, the optimal spread parameter λ^* is considered by minimizing the aggregate risk.

5.1 Moments of the Fund, Contribution, and Benefit Levels

It is clear from (3.2) and (3.3) that the contribution and the benefit levels depend on the fund level. Hence, we start with the fund dynamics equation, to obtain the first two moments under each rate-of-return models.

For $t = 0, 1, 2, 3, ...$, equation (3.1) can be re-expressed as follows;

$$\begin{aligned} F_{t+1} &= (1 + i_{t+1})\big[(1 - \lambda)F_t + \lambda AL_t + NC_t - TB\big] \\ &= (1 + i_{t+1})\big[(1 - \lambda)F_t + R_t\big]. \end{aligned} \tag{5.1}$$

Since the population is stationary, we can drop t from R_t, AL_t, and NC_t, and assume that they reached their stationary constant values.

By taking the expectation on both sides in equation (5.1), we get the following:

$$\mathbb{E}(F_{t+1}) = q\mathbb{E}(F_t) + R(1 + i),$$

where $q = (1 + i)(1 - \lambda)$.

Then, we obtain recursively the following equation for $t = 1, 2, ...$;

$$\mathbb{E}(F_t) = q^t F_0 + R(1 + i)\frac{1 - q^t}{1 - q}. \tag{5.2}$$

For the variance of the fund, the formula is given as;

$$\mathbb{V}\mathrm{ar}(F_t) = b\sum_{j=1}^{t} a^{t-j}(\mathbb{E}(F_j))^2, \tag{5.3}$$

where $a = (1-\lambda)^2[(1+i)^2 + \sigma_i^2]$, and $b = \sigma_i^2(1+i)^{-2}$.

These formulas are derived under the assumption that the rates of return are independent and identically distributed random variables; the case studied by Dufresne in [20] and Khorasanee in [45]. However, here we are assuming that the rates of return follow a time series model, hence the independence assumption is not true any more.

5.1.1 Moments of the Fund Level under ARMA$(1,1)$

Recall equation (3.4) from Chapter 3;

$$F_t = (1-\lambda)^t \mathrm{e}^{\Delta_t} + R\sum_{s=0}^{t-1}(1-\lambda)^{t-s-1}\mathrm{e}^{\Delta_t-\Delta_s}.$$

Taking expectation of both sides, we have the following;

$$\mathbb{E}(F_t) = (1-\lambda)^t F_0\, \mathbb{E}(\mathrm{e}^{\Delta_t}) + R\sum_{s=0}^{t-1}(1-\lambda)^{t-s-1}\mathbb{E}(\mathrm{e}^{\Delta_t-\Delta_s}). \tag{5.4}$$

As we see from the above equation; $\mathbb{E}(\mathrm{e}^{\Delta_t})$ and $\mathbb{E}(\mathrm{e}^{\Delta_t-\Delta_s})$ are needed. It is known that under an **ARMA**$(1,1)$ model the marginal distribution of δ_t are normal; because the residual Z_t are assumed independent and identically **normal** distributed. Hence, e^{δ_t} has a **log-normal** distribution with mean and variance given as;

$$\mathbb{E}(\mathrm{e}^{\delta_t}) = \exp\{\mu + \frac{1}{2}\gamma_\delta(0)\},$$
$$\mathbb{V}\mathrm{ar}(\mathrm{e}^{\delta_t}) = (\mathrm{e}^{\gamma_\delta(0)} - 1)(\mathrm{e}^{2\mu+\gamma_\delta(0)}),$$

where $\gamma_\delta(0)$ is given in equation (4.1).

Then, for Δ_t and $\Delta_t - \Delta_s$, which are functions of δ_t, are also **normally** distributed. Hence, e^{Δ_t} and $\mathrm{e}^{\Delta_t-\Delta_s}$ follow a **log-normal** distribution, and their means are obtained as follow;

$$\begin{aligned}\mathbb{E}(\mathrm{e}^{\Delta_t}) &= \exp\Big\{t\mu + \frac{1}{2}(\theta + \phi\gamma_\delta(0))\big[t\phi + 2\sum_{j=1}^{t}\sum_{i=j+1}^{t}\phi^{i-j-1}\big]\Big\} \\ &= \eta_1^t\ \mathrm{e}^{-\xi(1-\phi^t)},\end{aligned} \tag{5.5}$$

$$\mathbb{E}(\mathrm{e}^{\Delta_t-\Delta_s}) = \exp\Big\{(t-s)\Big(\mu+\frac{1}{2}(\theta+\phi\gamma_\delta(0))(\phi+\frac{2}{1-\phi})\Big)\Big\}\exp\Big\{\frac{-(\theta+\phi\gamma_\delta(0))}{(1-\phi)^2}(1-\phi^{t-s})\Big\}$$
$$= \eta_1^{t-s}\ \mathrm{e}^{\xi(\phi^{t-s}-1)}, \tag{5.6}$$

where $\eta_1 = \exp\big\{\mu+\frac{1}{2}(\theta+\phi\gamma_\delta(0))(\phi+\frac{2}{1-\phi})\big\}$, and $\xi = \frac{(\theta+\phi\gamma_\delta(0))}{(1-\phi)^2}$.

Then, by substitution of equations (5.5) and (5.6) into equation (5.4), the following formula for the expectation of the fund is obtained;

$$\mathbb{E}(F_t) = (1-\lambda)^t F_0\ \eta_1^t\ \mathrm{e}^{-\xi(1-\phi^t)} + R\sum_{s=0}^{t-1}(1-\lambda)^{t-s-1}\eta_1^{t-s}\ \mathrm{e}^{\xi(\phi^{t-s}-1)}. \tag{5.7}$$

The second moment of the fund is derived by squaring both sides of equation (3.4) and then rearranging the terms;

$$\begin{aligned}\mathbb{E}(F_t^2) &= (1-\lambda)^{2t}\ F_0^2\ \mathbb{E}(\mathrm{e}^{2\Delta_t}) + 2R(1-\lambda)^t\ F_0\ \sum_{s=0}^{t-1}(1-\lambda)^{t-s-1}\mathbb{E}(\mathrm{e}^{\Delta_t+\Delta_t-\Delta_s})\\ &+ R^2\sum_{v=0}^{t-1}\sum_{s=0}^{t-1}(1-\lambda)^{t-s-1}\ (1-\lambda)^{t-v-1}\ \mathbb{E}(\mathrm{e}^{\Delta_t-\Delta_s+\Delta_t-\Delta_v})\\ &= (1-\lambda)^{2t}\ F_0^2\ \mathbb{E}(\mathrm{e}^{2\Delta_t}) + \frac{2\ R}{(1-\lambda)}\ \ F_0\sum_{s=0}^{t-1}(1-\lambda)^{2t-s}\mathbb{E}(\mathrm{e}^{2\Delta_t-\Delta_s})\\ &+ \frac{R^2}{(1-\lambda)^2}\Big[\sum_{s=0}^{t-1}(1-\lambda)^{2(t-s)}\mathbb{E}(\mathrm{e}^{2(\Delta_t-\Delta_s)}) + 2\sum_{v=1}^{t-1}\sum_{s=0}^{v-1}(1-\lambda)^{2t-s-v}\\ &\mathbb{E}(\mathrm{e}^{2\Delta_t-\Delta_s-\Delta_v})\Big].\end{aligned} \tag{5.8}$$

It is clear that the equation (5.8) depends on the terms $\mathbb{E}(\mathrm{e}^{\Delta_t})$, $\mathbb{E}(\mathrm{e}^{\Delta_t-\Delta_s})$, and $\mathbb{E}(\mathrm{e}^{2\Delta_t-\Delta_s-\Delta_v})$. Hence, we still need to find the last term.

For $v > s$, we have the following;

$$\begin{aligned}\mathbb{E}(\mathrm{e}^{\Delta_t-\Delta_s+\Delta_t-\Delta_v}) &= \exp\Big\{(t-s)\mu+(t-r)\mu+(\theta+\phi\gamma_\delta(0))H(t,v,s)\Big\}\\ &= \eta_1^{t-s}\ \eta_2^{t-v}\ \mathrm{e}^{-3\xi}\ \mathrm{e}^{2\xi\phi^{(t-s)}}\ \mathrm{e}^{2\xi\phi^{(t-v)}},\end{aligned} \tag{5.9}$$

where

$$H(t,s,v) = \frac{1}{2}\sum_{j=s+1}^{v}\sum_{i=s+1}^{v}\phi^{|i-j-1|} + 2\sum_{j=v+1}^{t}\sum_{i=s+1}^{t}\phi^{|i-j-1|}$$
$$= \frac{1}{2}(t-s)\Big(\phi+\frac{2}{1-\phi}\Big) + \frac{3}{2}(t-v)\Big(\phi+\frac{2}{1-\phi}\Big)$$
$$+ \frac{1}{(1-\phi)^2}\Big(2\phi^{t-s}+2\phi^{t-v}-3\Big).$$

and $\eta_2 = \exp\big\{\mu + \frac{3}{2}(\theta+\phi\gamma_\delta(0))(\phi+\frac{2}{1-\phi})\big\}$, while η_1 and ξ are give as before.

Then, by substitution of equations (5.5), (5.6), and (5.9) into (5.8);

$$\mathbb{E}(F_t^2) = (1-\lambda)^{2t}F_0^2\,\eta_1^{2t}\mathrm{e}^{-2\xi(1-\phi^t)} + \frac{2R\,F_0}{(1-\lambda)}\sum_{s=0}^{t-1}(1-\lambda)^{2t-s}\exp\Big\{(t-s)\mu+t\mu+$$
$$(\theta+\phi\gamma_\delta(0))H(t,s,0)\Big\} + \frac{R^2}{(1-\lambda)^2}\sum_{s=0}^{t-1}(1-\lambda)^{2(t-s)}\eta_1^{2(t-s)}\mathrm{e}^{-4\xi}\mathrm{e}^{4\xi\phi^{t-s}}$$
$$+\frac{2R^2}{(1-\lambda)^2}\sum_{v=1}^{t-1}\sum_{s=0}^{v-1}(1-\lambda)^{2t-s-v}\eta_1^{t-s}\ \eta_2^{t-v}\ \mathrm{e}^{-3\xi}\mathrm{e}^{2\xi\phi^{(t-s)}}\ \mathrm{e}^{2\xi\phi^{(t-v)}}. \tag{5.10}$$

The first and the second moments can be simplified as;

$$\mathbb{E}(F_t) = (1-\lambda)^t\ \eta_1^t\mathrm{e}^{-\xi(1-\phi^t)}F_0 + R\ \mathrm{e}^{-\xi}\eta_1\Big[\frac{1-(1-\lambda)^t\eta_1^t}{1-(1-\lambda)\eta_1} + S_k\Big],$$

where $S_k = \sum_{k=1}^{\infty}\frac{z^k\phi^k}{k!}\frac{1-(1-\lambda)^t\eta_1^t\phi^{kt}}{1-(1-\lambda)\eta_1\phi^k}$.

$$\mathbb{E}(F_t^2) = (1-\lambda)^{2t}\eta_1^{2t}\mathrm{e}^{-2\xi(1-\phi^t)}F_0^2 + \frac{2\ R\ F_0}{(1-\lambda)}\sum_{s=0}^{2t-s}(1-\lambda)^{2t-s}\exp\Big\{(2t-s)\mu$$
$$+(\theta+\phi\gamma_\delta(0))H(t,s,0)\Big\} + R^2\mathrm{e}^{-4\xi}\eta_1^2\Big[\frac{1-(1-\lambda)^{2t}\eta_1^{2t}}{1-(1-\lambda)^2\eta_1^2} + S_{k_1}\Big]$$
$$+\frac{2\ R^2\ \mathrm{e}^{-3\xi}\eta_1}{(1-\lambda)}\Bigg\{\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{(2\xi)^{k_2+k_3}\phi^{k_2}}{k_2!k_3!(1-(1-\lambda)\ \eta_1\phi^{k_2})}$$
$$\Big[(1-\lambda)^2\eta_1\eta_2\phi^{k_2+k_3}\Big(\frac{1-(1-\lambda)^{2(t-1)}\eta_1^{t-1}\eta_2^{t-1}\phi^{(k_2+k_3)(t-1)}}{1-(1-\lambda)^2\eta_1\eta_2\phi^{k_2+k_3}}\Big)$$
$$-(1-\lambda)^{(t+1)}\eta_1^t\eta_2\phi^{tk_2+k_3}\Big(\frac{1-(1-\lambda)^{t-1}\eta_2^{t-1}\phi^{k_3(t-1)}}{1-(1-\lambda)\eta_2\phi^{k_3}}\Big)\Big]\Bigg\},$$

where $S_{k_1} = \sum_{k_1=1}^{\infty} \frac{(4\xi\phi)^{k_1}}{k_1!}(\frac{1-(1-\lambda)^{2t}\eta_1^{2t}\phi^{k_1 t}}{1-(1-\lambda)^2\eta_1^2\phi^{k_1}})$.

Hence, simple substitution of equations (5.7) and (5.10) in the variance formula complete its deviation.

The expectations of the contribution and the benefit levels are given as;

$$\mathbb{E}(C_t) = NC + \lambda_c(AL - \mathbb{E}(F_t)), \tag{5.11}$$

$$\mathbb{E}(B_t) = TB - \lambda_b(AL - \mathbb{E}(F_t)), \tag{5.12}$$

where $\mathbb{E}(F_t)$ is given in equation (5.7), and the spread parameters λ_c and λ_b are chosen for contributions and benefits respectively. We will denoate by $\lambda = \lambda_c + \lambda_b$ the total spread parameters.

The variance of these cash flows is given as;

$$\mathbb{V}\text{ar}(C_t) = \lambda_c^2 \mathbb{V}\text{ar}(F_t), \tag{5.13}$$

$$\mathbb{V}\text{ar}(B_t) = \lambda_b^2 \mathbb{V}\text{ar}(F_t), \tag{5.14}$$

where $\mathbb{V}\text{ar}(F_t)$ is obtained from equations (5.7) and (5.10).

Remark 5.1.1. Ultimate Case ($\mathbf{t} \to \infty$)

In a stationary population it is interesting to study the behavior of the moments of the fund, when the time t goes to ∞.

For $(1-\lambda)\eta_1 < 1$, $\mathbb{E}(F_\infty)$ is derived as;

$$\mathbb{E}(F_\infty) = R\,\mathrm{e}^{-\xi}\eta_1\Big[\frac{1}{1-(1-\lambda)\eta_1} + S_k^*\Big], \tag{5.15}$$

where $S_k^* = \sum_{k=1}^{\infty} \frac{z^k\phi^k}{k!}\frac{1}{1-(1-\lambda)\eta_1\phi^k}$.

Existence $\mathbb{E}(F_\infty^2)$ require that $(1-\lambda)\eta_1 < 1$ and $(1-\lambda)^2\eta_1\eta_2 < 1$, thus;

$$\begin{aligned}\mathbb{E}(F_\infty^2) = R^2 e^{-4\xi}\eta_1^2\Big[\frac{1}{(1-(1-\lambda)^2\eta_1^2)} + S_{k_1}^*\Big] + \frac{2R^2 e^{-3\xi}\eta_1}{(1-\lambda)} \\ \Big\{\frac{1}{1-(1-\lambda)\eta_1}\big(\frac{(1-\lambda)^2\eta_1\eta_2}{(1-(1-\lambda)^2\eta_1\eta_2)}\big) + \sum_{k_3=0}^{\infty} S_{k_2} + S_{k_3}\Big\},\end{aligned} \tag{5.16}$$

where $S_{k_1}^* = \sum_{k_1=1}^{\infty} \frac{(4\xi\phi)^{k_1}}{k_1!}\big(\frac{1}{1-(1-\lambda)^2\eta_1^2\phi^{k_1}}\big)$, $S_{k_2} = \sum_{k_2=1}^{\infty} \frac{(2\xi)^{k_2+k_3}\phi^{k_2}}{(k_2)!(k_3)!(1-(1-\lambda)\eta_1\phi^{k_2})} \frac{(1-\lambda)^2\eta_1\eta_2\phi^{k_2+k_3}}{1-(1-\lambda)^2\eta_1\eta_2\phi^{k_2+k_3}}$, and $S_{k_3} = \sum_{k_3=1}^{\infty} \frac{(2\xi)^{k_3}}{k_3!(1-(1-\lambda)\eta_1\phi)}\big(\frac{(1-\lambda)^2\eta_1\eta_2\phi^{k_3}}{1-(1-\lambda)^2\eta_1\eta_2\phi^{k_3}}\big)$.

5.1.2 Moments of the Fund Level under GARCH$(1,1)$

In order to determine the first and second moments of the fund F_t under the **GARCH**$(1,1)$ model, we need to consider the terms $\mathbb{E}(e^{\Delta_t})$, $\mathbb{E}(e^{\Delta_t-\Delta_s})$, and $\mathbb{E}(e^{2\Delta_t-\Delta_s-\Delta_v})$ in equations (5.4) and (5.8).

As we mentioned in the previous chapter under **GARCH** models, when $0 \le \alpha_1 \le .06$ and $0 \le \beta_1 \le .9$, then the kurtosis of the model is approximately equal to 3. Also, the skewness of the model is equal to zero, so we can assume that δ_t follows normal distribution. Thus, following the same arguments of the previous section, it can be shown that

$$\begin{aligned}\mathbb{E}(e^{\Delta_t}) &= \exp\Big\{t\Upsilon + \frac{1}{2}t\gamma_\varepsilon(0)\Big\} \\ &= \kappa_1^t,\end{aligned} \tag{5.17}$$

$$\begin{aligned}\mathbb{E}(e^{\Delta_t-\Delta_s}) &= \exp\Big\{(t-s)\Upsilon + \frac{(t-s)}{2}\gamma_\varepsilon(0)\Big\} \\ &= \kappa_1^{t-s},\end{aligned} \tag{5.18}$$

$$\begin{aligned}\mathbb{E}(e^{2\Delta_t-\Delta_s-\Delta_v}) &= \exp\Big\{(t-s)\big(\Upsilon + \frac{1}{2}\gamma_\varepsilon(0)\big) + 3(t-v)\big(\Upsilon + \gamma_\varepsilon(0)\big)\Big\} \\ &= \kappa_1^{t-s}\kappa_2^{t-v},\end{aligned} \tag{5.19}$$

where $\gamma_\varepsilon(0) = \mathbb{V}\mathrm{ar}(\varepsilon_t) = \frac{\alpha_0}{1-\alpha_1-\beta_1}$, $\kappa_1 = \exp\{\Upsilon + \frac{1}{2}\gamma_\varepsilon(0)\}$, and $\kappa_2 = \exp\{\Upsilon + \frac{3}{2}\gamma_\varepsilon(0)\}$.

Then, the first and the second moments of the fund are given as below;

$$\begin{aligned}\mathbb{E}(F_t) &= (1-\lambda)^t\kappa_1^t\ F_0 + \frac{R}{(1-\lambda)}\sum_{s=0}^{t-1}(1-\lambda)^{t-s}\kappa_1^{t-s} \\ &= (1-\lambda)^t\kappa_1^t\ F_0 + R\ \kappa_1\ (\frac{1-(1-\lambda)^t\kappa_1^t}{1-(1-\lambda)\kappa_1}).\end{aligned} \tag{5.20}$$

$$\begin{aligned}\mathbb{E}(F_t^2) &= (1-\lambda)^{2t}F_0^2\kappa_1^{2t} + \frac{2\ R\ F_0}{(1-\lambda)}\sum_{s=0}^{t-1}(1-\lambda)^{2t-s}\kappa_1^{2t-s} + \frac{R^2}{(1-\lambda)^2} \\ &\sum_{s=0}^{t-1}(1-\lambda)^{2(t-s)}\kappa_1^{2(t-s)} + \frac{2R^2}{(1-\lambda)^2}\sum_{v=1}^{t-1}\sum_{s=0}^{v-1}(1-\lambda)^{2t-s-v}\kappa_1^{t-s}\kappa_2^{t-v} \\ &= (1-\lambda)^{2t}\kappa_1^{2t}F_0^2 + 2\ R\ F_0(1-\lambda)^t\kappa_1^{t+1}\Big(\frac{1-(1-\lambda)^t\kappa_1^t}{1-(1-\lambda)\kappa_1}\Big) \\ &+ R^2\kappa_1^2\Big(\frac{1-(1-\lambda)^{2t}\kappa_1^{2t}}{1-(1-\lambda)^2\kappa_1^2}\Big) + \frac{2R^2\ \kappa_1}{(1-\lambda)(1-(1-\lambda)\kappa_1)} \\ &\Big[(1-\lambda)^2\kappa_1\kappa_2\ \frac{1-(1-\lambda)^{2(t-1)}\kappa_1^{t-1}\kappa_2^{t-1}}{1-(1-\lambda)^2\kappa_1\kappa_2} - (1-\lambda)^{t+1}\kappa_1^t\kappa_2 \\ &\frac{1-(1-\lambda)^{t-1}\kappa_2^{t-1}}{1-(1-\lambda)\kappa_2}\Big].\end{aligned} \tag{5.21}$$

A simple substitution of equations (5.20) and (5.21) into the variance formula gives the variance.

Remark 5.1.2. Ultimate Case $(t \to \infty)$

As before we consider the behavior of the moments when t goes to ∞ under the **GARCH**$(1,1)$ model.

It is clear that $\mathbb{E}(F_\infty)$ and $\mathbb{E}(F_\infty^2)$ exist only if $(1-\lambda)\kappa_1 < 1$ and $(1-\lambda)^2\kappa_1\kappa_2 < 1$ respectively. Thus;

$$\mathbb{E}(F_\infty) = \frac{R\ \kappa_1}{(1-(1-\lambda)\kappa_1)}. \tag{5.22}$$

$$\mathbb{E}(F_\infty^2) = \frac{R^2\kappa_1^2}{(1-(1-\lambda)^2{\kappa_1}^2)} + \frac{2R^2\ \kappa_1^2\kappa_2(1-\lambda)}{(1-(1-\lambda)\kappa_1)(1-(1-\lambda)^2\kappa_1\kappa_2)}. \tag{5.23}$$

5.1.3 Comments on the Moments of Cash Flows

In general, it is noticeable that when $t \to \infty$, there is a difference between the first and the second moments of the fund under iid, ARMA$(1,1)$, and GARCH$(1,1)$ models. This difference comes from the exponential terms that are used in modeling δ_t.

Haberman in [32] and [35] considered an approximation for $\mathbb{E}(F_t)$ and $\mathbb{E}(F_t^2)$ under **AR**(1), **AR**(2), and **MA**(1) and studied the errors of approximation; so, following the same arguments for **ARMA**$(1,1)$ we get the following results:

$$\mathbb{E}(F_\infty) \approx \frac{R\mathrm{e}^{-\xi}\eta_1}{1-(1-\lambda)\eta_1},$$

$$\mathbb{E}(F_\infty^2) \approx \frac{R^2\mathrm{e}^{-4\xi}\eta_1^2}{1-(1-\lambda)^2\eta_1^2} + \frac{2R^2\mathrm{e}^{-3\xi}\eta_1^2\eta_2(1-\lambda)}{(1-(1-\lambda)\eta_1)(1-(1-\lambda)^2\eta_1\eta_2)}.$$

Also, under GARCH model, we choose the following range of the parameters;

$$0 < \alpha_0 \leq .00001 \quad ; 0 < \alpha_1 \leq .0001 \quad ; 0 < \beta_1 \leq .1,$$

$$1 - \frac{1}{\sqrt{\kappa_1\kappa_2}} < \lambda \leq 1,$$

to avoid any infringement of the conditions $(1-\lambda)\kappa_1 < 1$ and $(1-\lambda)^2\kappa_1\kappa_2 < 1$.

Finally, it is sufficient to consider the following range for λ, $\max\{d_v, 1-\frac{1}{\sqrt{\eta_1\eta_2}}\} \leq \lambda \leq 1$ under ARMA and $\max\{d_v, 1-\frac{1}{\sqrt{\kappa_1\kappa_2}}\} \leq \lambda \leq 1$ under GARCH, to avoid any negative values for the fund.

Models of $\mathbf{i_t}$	$\lim_{\mathbf{t}\to\infty} \mathbb{E}(\mathbf{F_t}) = \mathbb{E}(\mathbf{F}_\infty)$
iid	$(1-\lambda)(1+i) < 1 \quad \equiv \lambda > d = \frac{i}{i+1}$
ARMA$(1,1)$	$(1-\lambda)\eta_1 < 1 \equiv \lambda > 1-\eta_1^{-1} \equiv \mathrm{m}_{arma} < \frac{1}{\delta}\mathbf{ln}(\frac{\eta_1-1}{v\eta_1-1})$
GARCH$(1,1)$	$(1-\lambda)\kappa_1 < 1 \equiv \lambda > 1-\kappa_1^{-1} \equiv \mathrm{m}_{garch} < \frac{1}{\delta}\mathbf{ln}(\frac{\kappa_1-1}{v\kappa_1-1})$

Table 5.1: The Conditions of Existence the Limiting Value of the First Moment of the Fund under Different Models of $\mathbf{i_t}$

Models of $\mathbf{i_t}$	$\lim_{\mathbf{t}\to\infty}\mathbb{E}(\mathbf{F_t^2}) = \mathbb{E}(\mathbf{F_\infty^2})$
iid	$(1-\lambda)^2((1+i)^2+\sigma_i^2) < 1 \qquad \equiv \lambda > 1 - \frac{1}{\sqrt{((1+i)^2+\sigma_i^2)}}$
ARMA(1,1)	$(1-\lambda)^2\eta_1\ \eta_2 < 1 \equiv \lambda > 1-(\kappa_1\kappa_2)^{-.5} \equiv \mathrm{m}_{arma} < \frac{1}{\delta}\mathbf{ln}(\frac{\sqrt{\eta_1\eta_2}-1}{v\sqrt{\eta_1\eta_2}-1})$
GARCH(1,1)	$(1-\lambda)^2\kappa_1\kappa_2 < 1 \equiv \lambda > 1-(\kappa_1\kappa_2)^{-.5} \equiv \mathrm{m}_{garch} < \frac{1}{\delta}\mathbf{ln}(\frac{\sqrt{\kappa_1\kappa_2}-1}{v\sqrt{\kappa_1\kappa_2}-1})$

Table 5.2: The Conditions of Existence the Limiting Value of the Second Moment of the Fund under Different Models of $\mathbf{i_t}$

Remark 5.1.3. It is possible to rewrite R as;

$$R = AL(\lambda - d_v).$$

Thus, when $\lambda > d_v$ and $F_0 > 0$, we can say $\mathbb{E}(F_t)$ is positive for all t under ARMA and GARCH. However, if $\lambda = d_v$, then

$$\mathbb{E}(F_t) = v^t\eta_1^tF_0,$$

$$\mathbb{E}(F_t) = v^t\kappa_1^tF_0,$$

under ARMA and GARCH models respectively. Also,

$$\mathbb{V}\mathrm{ar}(F_t) = \mathbb{V}\mathrm{ar}(C_t) = \mathbb{V}\mathrm{ar}(B_t) = 0.$$

Moreover, when $F_0 = 0$, then the plan is funded on a Pay-as-You-Go basis, since $\mathbb{E}(F_t) = 0$, $\mathbb{E}(C_t) = NC$, and $\mathbb{E}(B_t) = TB$.

Under the **GARCH** model, the limiting value of the first moments can be expressed as

$$\mathbb{E}(F_\infty) = \frac{AL(\lambda - d_v)\kappa_1}{1-(1-\lambda)\kappa_1}.$$

Hence, if $\kappa_1 = \frac{1}{1-d_v} = 1 + i_v$, then $\mathbb{E}(F_\infty) = AL$, which means the fund reach its stationary status.

Models of $\mathbf{i_t}$	Assumpation	$\mathbb{E}(F_\infty)$
iid	$i_v = \mathbb{E}(i_t)$ $d_v < \lambda \leq 1$	AL
GARCH$(1,1)$	$i_v \neq \mathbb{E}(i_t)$ $\max\{d_v, 1 - \frac{1}{\sqrt{\kappa_1\kappa_2}}\} \leq \lambda \leq 1$ $\kappa_1 = 1 + i_v$	$\frac{(\lambda-d_v)\kappa_1}{1-(1-\lambda)\kappa_1}AL$ AL

Table 5.3: Difference Between F_∞ and its Stationary Status AL under Different Model of i_t

5.2 Aggregate Risk and Optimal Spread Parameter

In the previous section, we considered the first and the second moments of the fund F_t; it was noticeable that both moments function; and hence the variance; depend on the spread parameter $\lambda = \lambda_c + \lambda_b$. Thus it is important to study for which value of the spread parameter the variances of F_t, C_t, and B_t either increase or decrease. This value of the parameter is called the **optimal** spread parameter λ^* in [20], and it is obtained by minimizing the variance of the contribution level under his plan.

Khorasanee in [45] defined the optimal value of the spread parameter as the value that minimize the aggregate risk under the hybrid plan, where he defined the aggregate risk as the sum of the contribution and the benefit risks; because these were the main concern under his model.

In this section, we illustrate the results of Dufresne and Khorasanee about the optimal spread parameter λ^*. Then, we define the aggregate risk denoted as $\mathbb{A}\mathrm{gg}\mathbb{R}$, under different risk measures when $t \to \infty$. The optimal spread parameter λ^* is then obtained by minimizing $\mathbb{A}\mathrm{gg}\mathbb{R}$.

5.2.1 Analyzing the Optimal Spread Parameter

Dufresne in [20] set $\lambda = \frac{1}{\ddot{a}_{\overline{m}|}}$, and assumed that $\lambda \leq 1$; to guarantee the variance of F_t and C_t are not negative. He mainly focused on studying λ when $t \to \infty$. It was noticeable that when $\lambda > \frac{i}{1+i} = d$, which means paying more than the interest on the unfunded liability, results in F_t and C_t converging to their target values AL and NC respectively. This was true since it was assumed that $i_v = \mathbb{E}(i_t)$.

When $\lambda = 1$, which is the case when the whole unfunded liability is paid off at every valuation date, $\mathbb{V}\text{ar}(F_\infty)$ and $\mathbb{V}\text{ar}(C_\infty)$ are at their lowest and highest values respectively. Finally, he pointed out the following results about the optimal spread parameter λ^*:

for $y = (1+i)^2 + \sigma_i^2$ and $1 - \frac{1}{\sqrt{y}} < \lambda \leq 1$.

1. If $y \leq 1$, then there exists $\lambda^* < 1$ such as
   - for $1 - \frac{1}{\sqrt{y}} < \lambda < \lambda^*$, $\mathbb{V}\text{ar}(F_\infty)$ and $\mathbb{V}\text{ar}(C_\infty)$ decrease with increasing λ.
   - for $\lambda^* \leq \lambda \leq 1$, $\mathbb{V}\text{ar}(F_\infty)$ and $\mathbb{V}\text{ar}(C_\infty)$ decreases and increases (trade-off) respectively with increasing λ, where

$$\lambda^* = \begin{cases} 0 & y \leq 1 \\ 1 - \frac{1}{y} & y \geq 1 \end{cases}$$

2. If $y = 1$, then $\mathbb{V}\text{ar}(F_\infty)$ and $\mathbb{V}\text{ar}(C_\infty)$ decreases and increases with increasing λ.
3. The values in the region $1 - \frac{1}{\sqrt{y}} < \lambda < \lambda^*$, which is called inadmissible region, are higher compared to the values in the region $\lambda^* \leq \lambda \leq 1$, which is called admissible region.

Finally, Dufresne pointed out that for $i \geq 0$, the $\lambda^* > d(1+v)$. For more explanations see Dufresne in [20].

Khorasanee in [45] also derived the optimal λ^* under his hybrid plan by minimize the aggregate risks under the plan. Mathematically, his aggregate risk ($\mathbb{A}\mathrm{gg}\mathbb{R}$) is given by

$$\mathbb{A}\mathrm{gg}\mathbb{R} = \mathbb{S}\mathrm{D}(B_\infty) + \mathbb{S}\mathrm{D}(C_\infty),$$

where $\mathbb{S}\mathrm{D}(B_\infty)$ is the standard deviation of B_∞(benefit risk), and $\mathbb{S}\mathrm{D}(C_\infty)$ is the standard deviation of C_∞(contribution risk). Then, the optimal parameter of amortization is defined as;

$$\lambda^* = 1 - [\sigma^2 + (1+i)^2]^{-1},$$

which is equal to the same value as that of Dufresne's plan.

Finally, Khorasanee pointed out that if $\mathbb{A}\mathrm{gg}\mathbb{R}$ is minimized, then there is a trade-off between the benefit and contribution risks.

5.2.2 Aggregate Risk under Variance, Coefficient of Variation, and Value at Risk

Here the aggregate risk, hence the optimal spread parameter is derived under different risk measures $\mathbb{V}\mathrm{ar}$, $\mathbb{C}\mathrm{V}$, and $\mathbb{V}\mathrm{a}\mathbb{R}$, when the time t goes to infinity.

The reason behind using different risk measures and not only considering the variance is because Haberman in [35] pointed out that when the variance is minimized this leads to a decrease in the benefit and contribution levels. Moreover, we want to compare how the results vary under different risk measure. However, we know that applying value at risk measure requires knowledge the distributions of B_t and C_t. It is believed that B_t and C_t are linearly correlated, since Pearson's product-moment:

$$\mathbb{C}\mathrm{orr}(C_t, B_t) = \frac{TB - \lambda_b(AL - F_t), NC + \lambda_c(AL - F_t)}{\lambda_c \lambda_b \mathbb{V}\mathrm{ar}(F_t)} = 1.$$

In other words,

$$\mathbb{P}\{B_t = \zeta_1 C_t + \zeta_2\} = 1,$$

for some $\zeta_1$and $\zeta_2 \in \mathbb{R}$. So, knowing the distribution of either C_t or B_t is enough here.

For simplification, since C_t is a function of F_t which is itself a function of the sum of e^{Δ_t} and $\mathrm{e}^{\Delta_t-\Delta_s}$, and both functions are depenedent **log-normals** under **ARMA** and **GARCH** models, here, we can use the following argument; from the central limit theorem for dependent random variables see Hoeffding and Robbins in [42], as $t \to \infty$, the summation converges to **Gaussian** process. So, C_t has asymptotically Gaussian distribution.

Remark 5.2.1. Aggregate Risk under ARMA$(1,1)$ as $t \to \infty$

The aggregate risk under $\mathbb{V}$ar is given as;

$$\begin{aligned}\mathbb{A}\mathrm{gg}\mathbb{R} = \lambda^2 \frac{R^2\mathrm{e}^{-2\xi}}{(1-\lambda)^2}\Big[&\mathrm{e}^{-2\xi}\frac{(1-\lambda)^2\eta_1^2}{(1-(1-\lambda)^2\eta_1^2)} + \mathrm{e}^{-2\xi}S_{k_1^*}\\ &+ \frac{2\mathrm{e}^{-\xi}(1-\lambda)\eta_1}{(1-(1-\lambda)\eta_1)}\frac{(1-\lambda)^2\eta_1\eta_2}{(1-(1-\lambda)^2\eta_1\eta_2)} + 2\mathrm{e}^{-\xi}S_{k_2^*}S_{k_3^*}\\ &- \frac{(1-\lambda)\eta_1}{1-(1-\lambda)\eta_1} + S_{k^*}\Big].\end{aligned}$$

Then, by minimizing the above expression to find λ^* and setting the derivative of above expression, the roots can obtain numerically. If we set $R = AL(\lambda - d_v)$, then one of roots is obtained as follows;

$$\lambda^* = d_v = \frac{i_v}{1+i_v}.$$

However, the $\mathbb{A}\mathrm{gg}\mathbb{R}$ under the coefficient of variation $\mathbb{CV}$ is obtained as;

$$\mathbb{A}\mathrm{gg}\mathbb{R} = \lambda \mathbb{SD}(F_\infty)\Big[\frac{p(\zeta_1+\zeta_2\mathbb{E}(C_\infty)) + (1-p)\mathbb{E}(C_\infty)}{\mathbb{E}(C_\infty)(\zeta_1+\zeta_2\mathbb{E}(C_\infty))}\Big],$$

where $\mathbb{SD}(F_\infty)$ is given in the previous section with $\mathbb{E}(C_\infty)$.

Under $\mathbb{V}\mathrm{a}\mathbb{R}$ measure, the aggragate risk is defined as;

$$\mathbb{A}\mathrm{gg}\mathbb{R} = (1+\zeta_1)\mathbb{V}\mathrm{a}\mathbb{R}_q(C_\infty) + \zeta_2,$$

where $\mathbb{V}\mathrm{a}\mathbb{R}_q(C_\infty) = NC + \lambda_c[AL - \mathbb{E}(F_\infty)] + \lambda_c \mathbb{SD}(F_\infty) z_q$, and z_q is $100pth$ percentile of standard normal variable.

Using either the **Maple** or **Mathematica** software, one can obtain the critical values, hence, the minimum λ^* numerically.

Similarly, if we want to apply a coherent risk measure such as a **TVaR** the following equation is obtained;

$$\mathbb{A}\mathrm{gg}\mathbb{R} = (1 + \zeta_1)\mathrm{TVaR}_q(C_\infty) + \zeta_2,$$

where $\mathrm{TVaR}_q(C_\infty) = NC + \lambda_c[AL - \mathbb{E}(F_\infty)] + \frac{\lambda_c \mathbb{SD}(F_\infty)}{1-q} \frac{e^{\frac{-z_q^2}{2}}}{\sqrt{2\pi}}$.

Remark 5.2.2. Aggregate Risk under GARCH$(1,1)$ as $t \to \infty$

Following the arguments in the previous section, we can also obtain $\mathbb{A}\mathrm{gg}\mathbb{R}$ under the different risk measures in the GARCH case:

$$\mathbb{A}\mathrm{gg}\mathbb{R} = \frac{\lambda^2 R^2 \kappa_1^2}{(1-(1-\lambda)\kappa_1)} \Big[\frac{1}{(1+(1-\lambda)\kappa_1)} + \frac{2\kappa_2^2(1-\lambda)}{(1-(1-\lambda)^2\kappa_1\kappa_2)} - \frac{1}{(1-(1-\lambda)\kappa_1)} \Big].$$

After substitution of $R = AL(\lambda - d_v)$, one of the roots is given as;

$$\lambda^* = d_v = \frac{i_v}{1 + i_v}.$$

Similary one can obtain $\mathbb{A}\mathrm{gg}\mathbb{R}$ under $\mathbb{C}\mathrm{V}$ and $\mathbb{V}\mathrm{a}\mathbb{R}$.

Szyszkowicz and Yanikomeroglue in [56] propose a limit theorem for the sum of **log-normal** randoms variables that are positively correlated, and they show that the sum converges to a **log-normal** distribution. Hence, one can fit a **log-normal** for C_∞ instead of a normal distribution.

5.2.3 Comments on the Aggregate Risk and Spread Parameter

Here we highlight the most important points observed here for the aggregate risk when i_t follows different models including the iid case:

Models of $\mathbf{i_t}$	Assumption	$\mathbb{A}$gg$\mathbb{R}$ under $\mathbb{V}$ar, $\mathbb{C}$V, $\mathbb{V}$a$\mathbb{R}$	λ^*
iid	$i_v = \mathbb{E}(i_t); AL = F_0$	$\lambda^2 \frac{bAL}{1-a}$ $AL\sqrt{\frac{b}{1-a}}(\frac{p}{NC} + \frac{1-p}{TB})\lambda$ $[NC+\Phi^{-1}(p)AL\lambda_c^2 \frac{b}{1-a}](1+\zeta_1)+\zeta_2$	$1 - [(1+i)^2 + \sigma_i^2]^{-1}$
GARCH(1, 1)	$i_v \neq \mathbb{E}(i_t)$	$\lambda^2 \mathbb{V}\text{ar}(F_\infty)$ $\lambda SD(F_\infty)[\frac{p}{\mathbb{E}(C_\infty)} + \frac{1-p}{\mathbb{E}(B_\infty)}]$ $(1+\zeta_1)\text{VaR}(C_\infty) + \zeta_2$	Numerically

Table 5.4: $\mathbb{A}$gg$\mathbb{R}$ under Different i_t Models and the Optimal Spread Parameter

- It is clear that under the iid case, when λ^* is obtained under different risk measures, the values are all equal; which means this value is robust, and is not affected by the type of risk measure used. This might be due to the linear dependence between C_∞ and B_∞, and also the constraints that are proposed under this case.

- Under **ARMA** and **GARCH** models, $\mathbb{A}$gg$\mathbb{R}$ is defined in the intervals $1 - \frac{1}{\sqrt{\eta_1\eta_2}} < \lambda \leq 1$ and $1 - \frac{1}{\sqrt{\kappa_1\kappa_2}} < \lambda \leq 1$ respectively. Also, when $R = AL(\lambda - d_v)$, one of the roots of $\mathbb{A}$gg$\mathbb{R}$, which is equal to the discount rate d_v, is obtained under both model, which may due to the normality assumption under both models. $\mathbb{A}$gg$\mathbb{R}$ vanishes at this root, moreover, the same root is obtained under $\mathbb{C}$V and $\mathbb{V}$ar. Also, at $\lambda = 1$, $\mathbb{A}$gg$\mathbb{R}$ reaches to its lowest value which is zero.

- Under both models **ARMA** and **GARCH**, $\mathbb{A}$gg$\mathbb{R}$ for $\mathbb{V}$a$\mathbb{R}$ and TVaR can be easily analyzed using a software to check whether there is a minimum value or not.

5.3 Numerical Illustration

We set the following assumptions that are used in the numerical illustration.

- The population is stationary, and membrs join the plan at age 25 and retire at age 65.
- There are no decrements before retirement, and there is one active member at each age; this lead to have 40 members in the plan all the times.
- The pension fund is invested in two part of assets:
  1. $\alpha\%$ of the fund is invested in the risky asset such as stocks with a 5% mean rate of return and 20% standard deviation.
  2. $(1-\alpha)\%$ of the fund is invested in a free risky asset such as goverment bounds with 2% mean rate of return.

  where α is the proportion of the fund, and by choosing it the investment strategy is implemented.
- We focus on $i_v = i = \mathbb{E}(i_t)$, but we still consider the cases $i_v = .5i$ as well as $i_v = .9i$.
- Entry age method is used here, where AL and NC are given as;

$$TB = \frac{2}{3}\ddot{a}_{65}.$$

$$NC = 40 * \frac{2(1+i_v)^{-40}\ddot{a}_{65}}{3\ddot{a}_{40}}.$$

$$AL = \frac{(1+i_v)(TB - NC)}{i_v}.$$

- We consider the ultimate case when t goes to ∞ only.
- Here we only study **GARCH**, and we choose $\kappa_1 = 1.000006$ and $\kappa_2 = 1.000017$.
- We arbitrarily select $\lambda_c = .7\lambda$ and $\lambda_b = .3\lambda$.

X	$\mathbb{E}(i_t)$	$\mathbb{S}\mathrm{D}(i_t)$	NC	AL	TB
.2	.026	.04	12.076	365.866	21.348
.5	.035	.1	8.048	282.684	17.608
.8	.044	.160	5.447	223.233	14.855

Table 5.5: Hybrid Plan Parameters for different investment strategy

In this illustration we mainly focus on the strategy where $X = .5$, which is refered to neutral strategy.

5.3.1 Varying Spread Periods

m	$\mathbb{E}(F_\infty)$	$\mathbb{E}(C_\infty)$	$\mathbb{E}(B_\infty)$
1	273.1256387	14.73885288	10.91714712
5	238.0160363	14.73900034	10.91699966
10	200.404867	14.7391583	10.9168417
15	168.7358946	14.73929131	10.91670869
20	142.0704932	14.73940331	10.91659669
25	119.6182486	14.73949761	10.91650239
30	100.7135704	14.73957701	10.91642299
35	84.79599067	14.73964386	10.91635614
40	71.39356905	14.73970015	10.91629985
45	60.10891483	14.73974754	10.91625246
50	50.60741482	14.73978745	10.91621255
55	42.60731889	14.73982105	10.91617895
60	35.87139048	14.73984934	10.91615066

Table 5.6: The First Moments of the Cash Flows C_∞, B_∞, and F_∞, and When $X = .5$

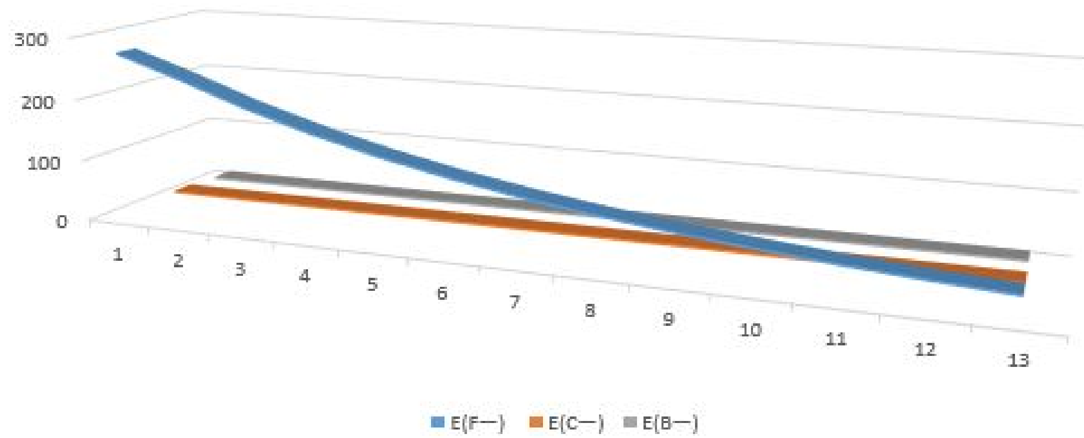

Figure 5.1: Expected Values of F_∞, C_∞, and B_∞

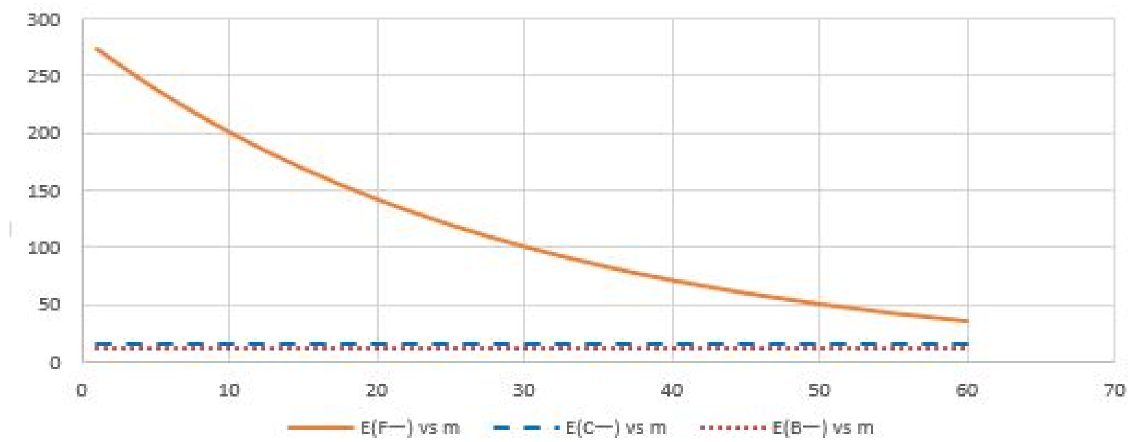

Figure 5.2: Expected Values of F_∞ vs m, C_∞ vs m, and B_∞ vs m

Remark 5.3.1. From Figures 5.1 and 5.2

- It is clear that when $m = 1$, the expected value of the fund tends to be close to its stationary values, however, as m increases $\mathbb{E}(F_\infty)$ starts to decease.
- For the the contribution rate, the target value NC is lower than the $\mathbb{E}(C_\infty)$ values. Also, $\mathbb{E}(C_\infty)$ fluctuates around 14.7 at different values of m.
- In case of $\mathbb{E}(B_\infty)$, its values are lower than TB at different values of m, and they also fluctuate around the value of 10.9.

- It is clear that all the three quantities show a trade-off; F_∞ and B_∞ decrease respectively with increasing m, however, C_∞ increases with increasing m.

m	$\mathbb{CV}(F_\infty)$	$\mathbb{CV}(C_\infty)$	$\mathbb{CV}(B_\infty)$	$\mathbb{A}\mathrm{gg}\mathbb{R}$
1	0	0	0	0
5	0.005033273	0.01217541	0.007044847	0.019220257
10	0.006867748	0.007593842	0.004394004	0.011987845
15	0.0080898	0.005438404	0.003146876	0.00858528
20	0.008989772	0.00412347	0.002386045	0.006509515
25	0.009682561	0.003224524	0.001865898	0.005090422
30	0.010229435	0.002570291	0.00148734	0.004057631
35	0.010668122	0.002075342	0.001200942	0.003276284
40	0.011023932	0.001691078	0.000978588	0.002669666
45	0.01131483	0.001387266	0.000802786	0.002190052
50	0.011554075	0.00114385	0.000661929	0.001805779
55	0.011751733	0.000946865	0.00054794	0.001494805
60	0.011915613	0.000786231	0.000454985	0.001241216

Table 5.7: The Coefficient of Variation of the Cash Flows C_∞, B_∞, and F_∞, and When $X = .5$

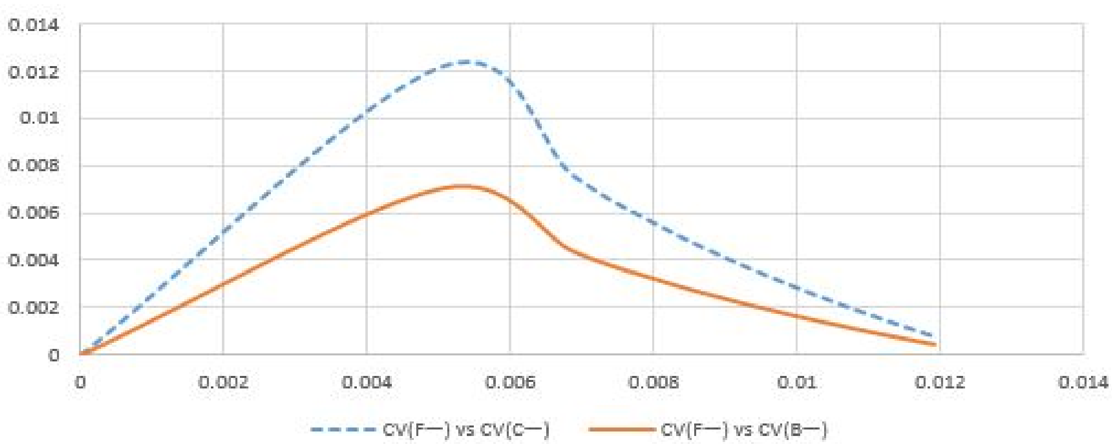

Figure 5.3: $\mathbb{CV}(F_\infty)$ vs $\mathbb{CV}(C_\infty)$ and $\mathbb{CV}(F_\infty)$ vs $\mathbb{CV}B_\infty$

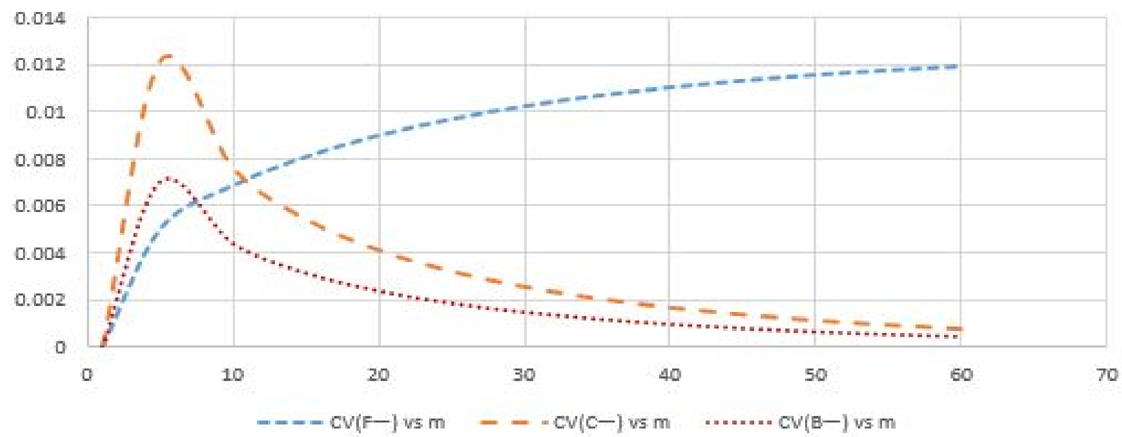

Figure 5.4: The Coefficient of Variation of F_∞ vs m, C_∞ vs m, and B_∞ vs m

Remark 5.3.2. From figures (5.3), and (5.4).

- It is clear that at $m = 1$, the coefficients of cash flows C_∞, B_∞, and F_∞ reach to their lowest value 0.
- In case of the fund, when $m > 1$, the variablility in the fund starts to increase, so $\mathbb{C}\mathrm{V}(F_\infty)$ is monotonically increasing with increasing m.
- For the contribution and benefit level, both quantities increase for $m > 1$, and then they decrease for $m > 5$. Hence, they are not monotonic functions.

5.3.2 Varying Risk Measures and Spread Periods

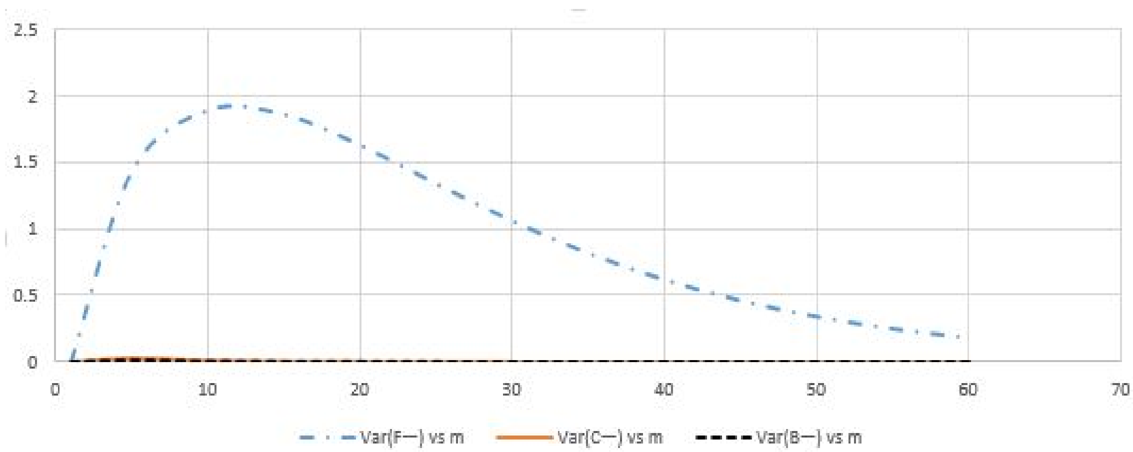

Figure 5.5: Ultimate values of the Fund, Contribution, and Benefit under Variance as Risk Measure

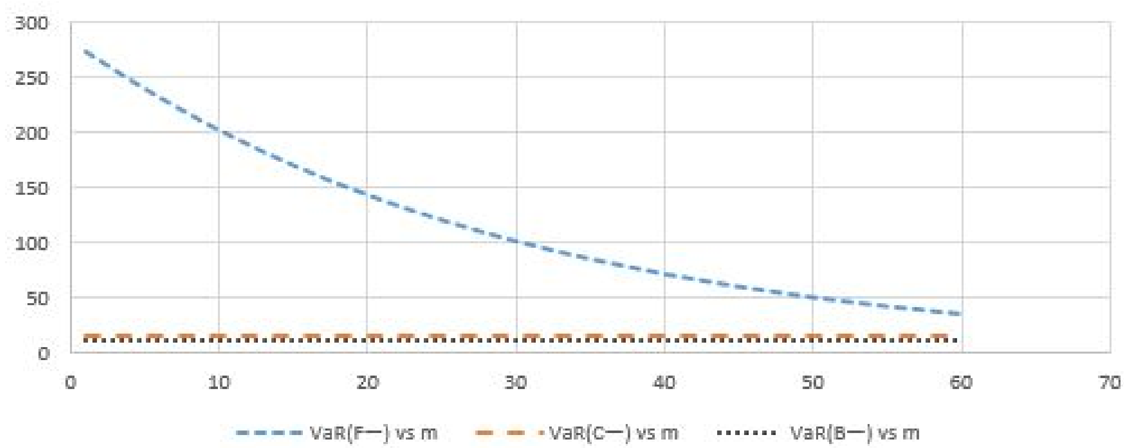

Figure 5.6: Ultimate values of the Fund, Contribution, and Benefit under Value at Risk Measure with Security Level $q = .9$

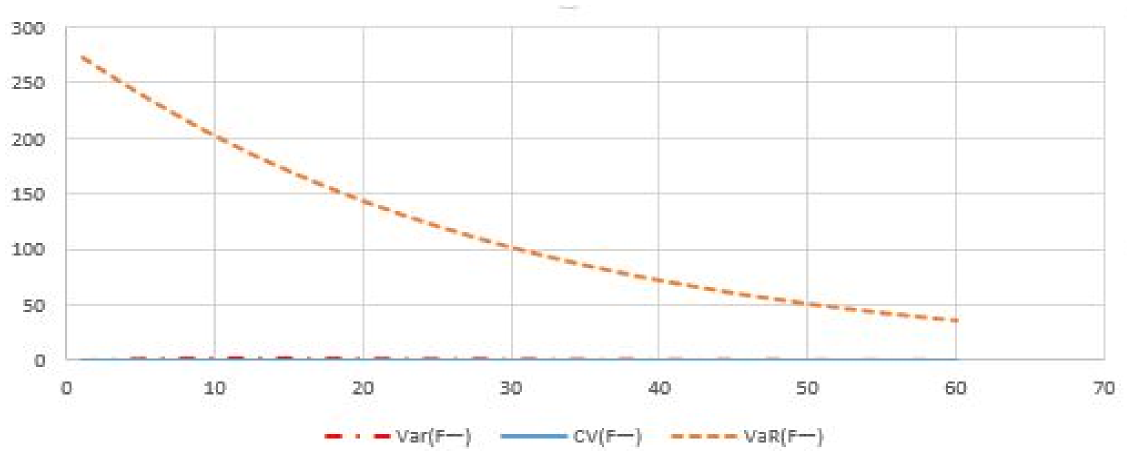

Figure 5.7: Fund under Different Risk Measures

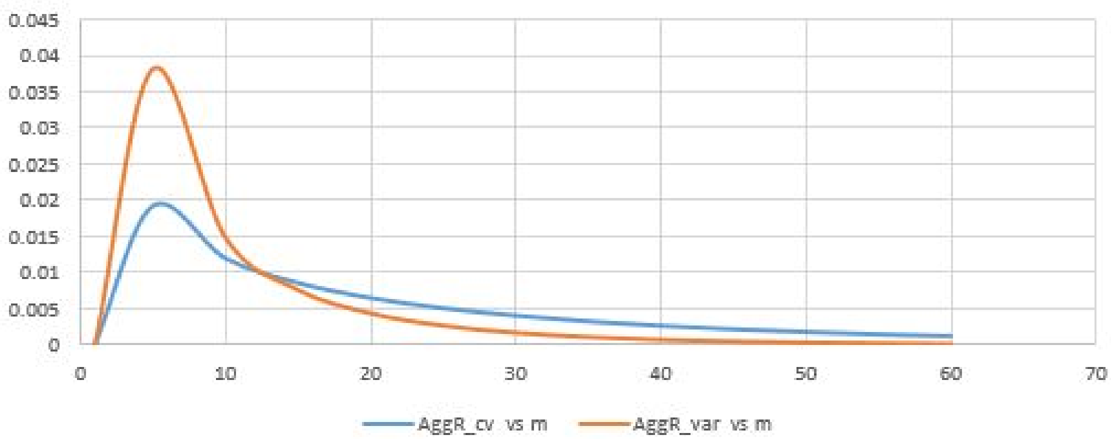

Figure 5.8: $\mathbb{A}\text{gg}\mathbb{R}$ vs m under Variance and Coefficient of Variation Measures

Remark 5.3.3. From Figures 5.5, 5.6, 5.7, and 5.8.

- It is clear that the fund under variance increase with increasing m, then it goes down again. Contributions and Benefit follow the same behaviour as the fund. It is noticeable after $m = 10$ all start to decrease.

- Under value at risk measure the fund declines with increasing the year of amorization. However, incase of contribution and benefit, both quantities increase for some years then start to drop again.

- Under the three different risk measures, the fund, contribution, and benefit have higher values under value at risk measure compare to other measures.

- at $m = 1$ the above quantities reasch to their lowest values under $\mathbb{V}\text{ar}$ and $\mathbb{C}\text{V}$, however under $\mathbb{V}\text{a}\mathbb{R}$ at $m = 1$, the fund reach to its highest value.

- aggregate risk under coefficient of variation measure has lower values compare to its value under the variance.

m	$\mathbb{C}\text{V}(F_\infty)$	$\mathbb{V}\text{ar}(F_\infty)$	$\mathbb{V}\text{a}\mathbb{R}(F_\infty)$
1	0	0	273.1256387
5	0.005033273	1.435203384	239.551872
10	0.006867748	1.894284676	202.1693223
15	0.0080898	1.863333234	170.4858754
20	0.008989772	1.631192098	143.7078397
25	0.009682561	1.341452767	121.1030751
30	0.010229435	1.061400571	102.0343418
35	0.010668122	0.818326425	85.95570577
40	0.011023932	0.619428592	72.40255159
45	0.01131483	0.462566168	60.98083145
50	0.011554075	0.341899663	51.35702825
55	0.011751733	0.250710355	43.24922892
60	0.011915613	0.182696054	36.41935522

Table 5.8: The Fund under Different Risk Measures

5.3.3 Varying Investment Strategies and Spread Periods

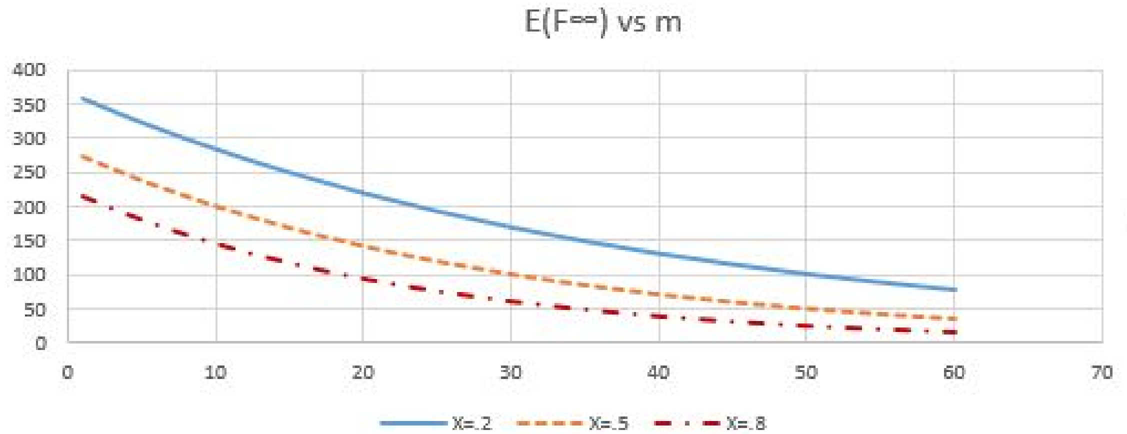

Figure 5.9: $\mathbb{E}(F_\infty)$ vs m under Different Investmet Strategies

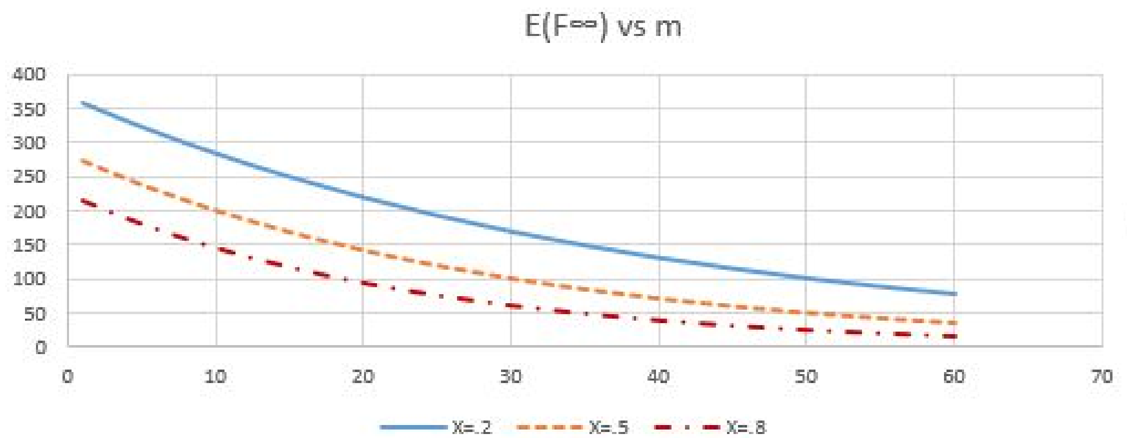

Figure 5.10: $\mathbb{C}\mathrm{V}(F_\infty)$ vs m under Different Investment Strategies

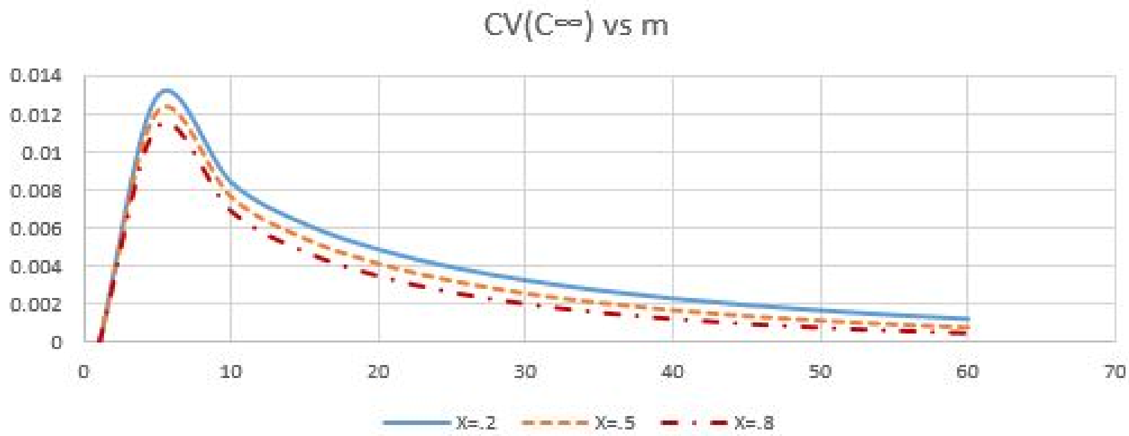

Figure 5.11: $\mathbb{CV}(C_\infty)$ vs m under Different Investment Strategies

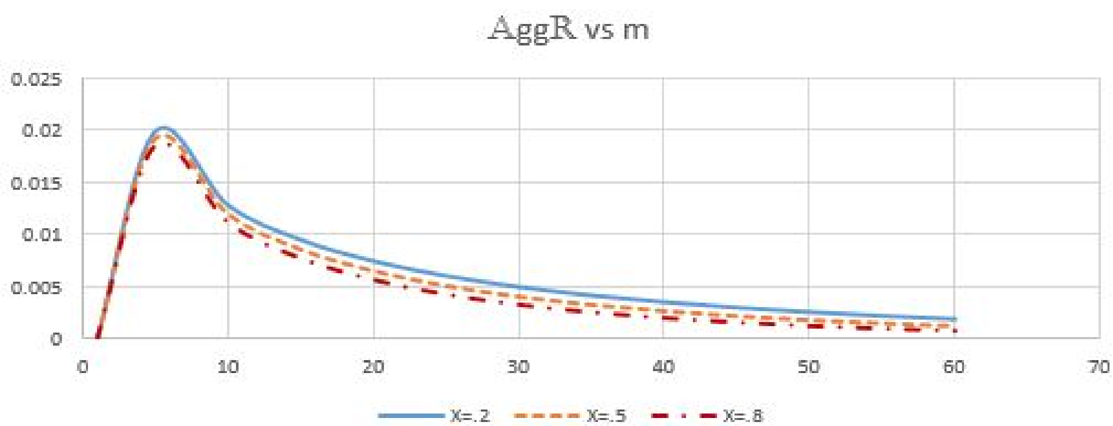

Figure 5.12: $\mathbb{A}\text{gg}\mathbb{R}$ vs m under Coefficient of Variation at Different Investment Strategies

Remark 5.3.4. From Figures 5.9, 5.10, 5.11, and 5.12.

- The risky strategy has lower $\mathbb{E}(F_\infty)$ values, however, under conservative and netural strategies the fund is close to its stationary limit with some difference that is smaller than other strategies.

- The variability of the fund is higher in the case of the conservative strategy, compared to the neutral and aggressive strategies, where the later shows lower variablility.

- Similarily under the contributions and the benefit levels, the risky strategy shows lower variability as well as lower $\mathbb{A}\mathrm{gg}\mathbb{R}$.

5.3.4 Varying Valuation Interest Rates and Spread Periods

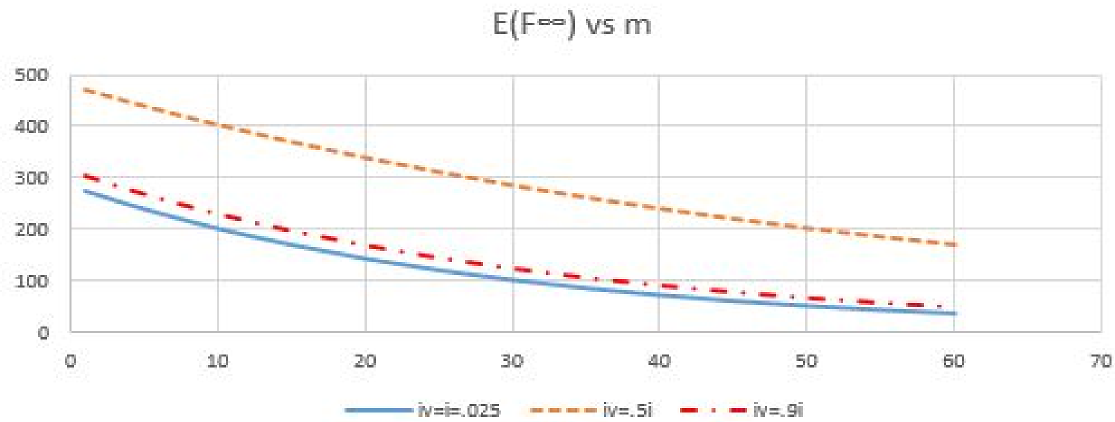

Figure 5.13: $\mathbb{E}(F_\infty)$ vs m at Different Valuation Rate i_v

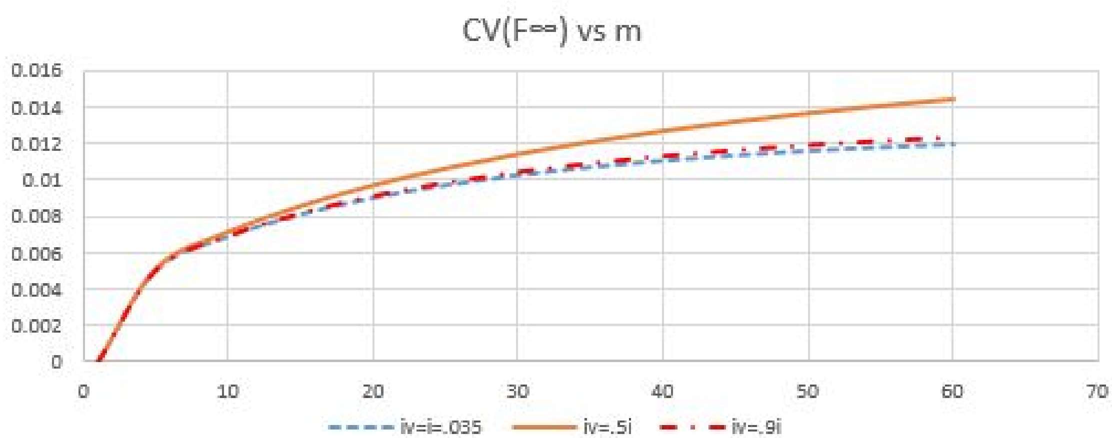

Figure 5.14: $\mathbb{C}\mathrm{V}(F_\infty)$ vs m at Different Valuation Rate i_v

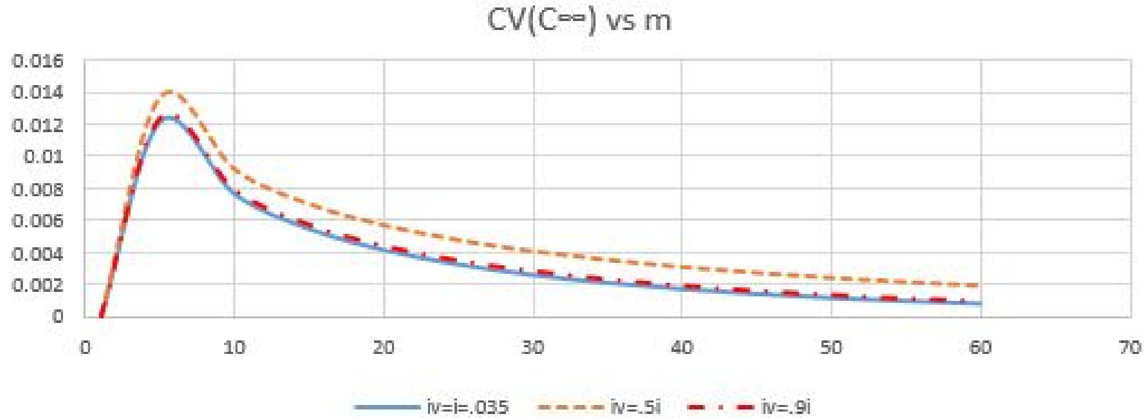

Figure 5.15: $\mathbb{CV}(C_\infty)$ vs m at Different Valuation Rate i_v

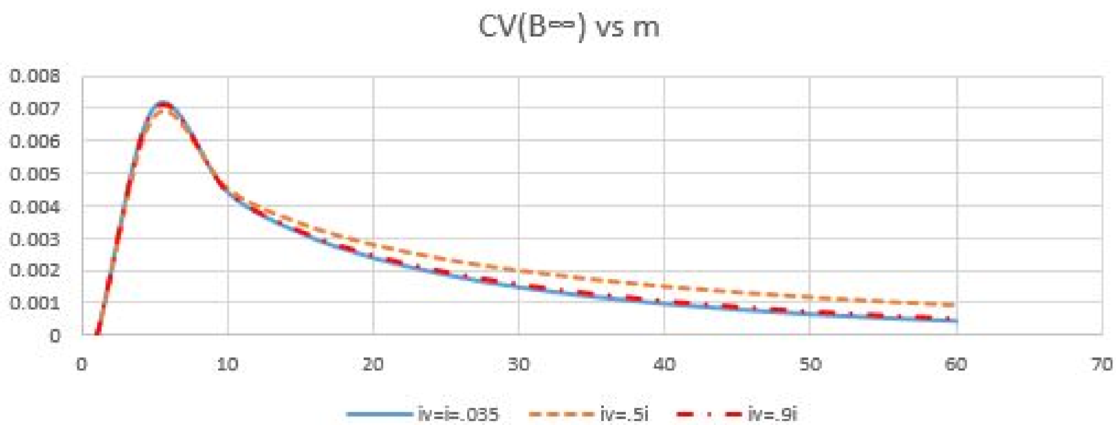

Figure 5.16: $\mathbb{CV}(B_\infty)$ vs m at Different Valuation Rate i_v

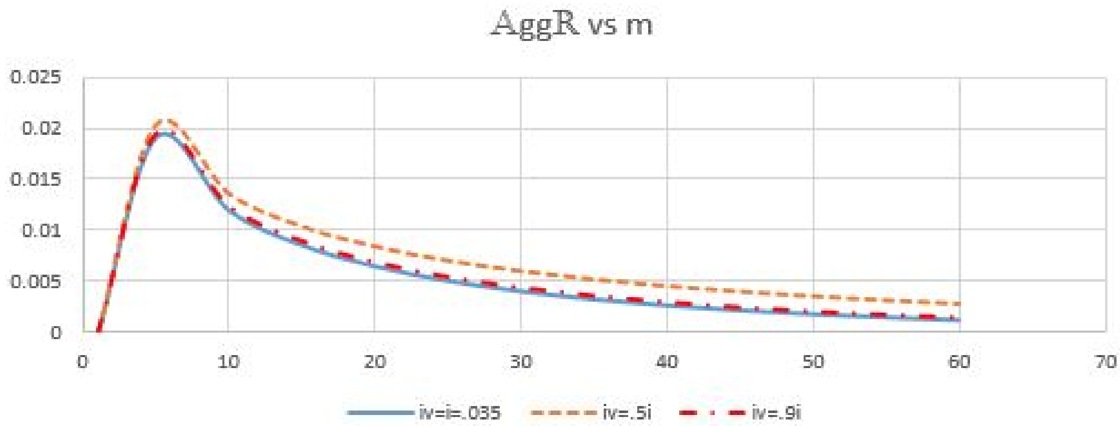

Figure 5.17: $\mathbb{A}\text{gg}\mathbb{R}$ vs m under Coefficient of Variation Measure at Different Valuation Rate i_v

i_v	NC	AL	TB
$.5i$=.0175	18.0035	477.2194	26.2112
$.9i$=.0315	9.4056	311.6549	18.9229

Table 5.9: Hybrid Plan Parameters for Different Valuation Rate

Remark 5.3.5. From the above Figures 5.13, 5.14, 5.15, and 5.16:

- It is clear that the lower expected value of F_∞ corresponds to higher valuation interest rates; this might due to lower NC and TB under the higher valuation rate. Also, F_∞ is closer to its AL under the smallest i_v.

- The variability in F_∞ is higher under the lower valuation rates, and this is true under B_∞.

- Similarly, $\mathbb{C}\text{V}(C_\infty)$ is lower under the higher valuation interest rate as well as $\mathbb{A}\text{gg}\mathbb{R}$

5.3.5 Varying Valuation Time and Spread Periods

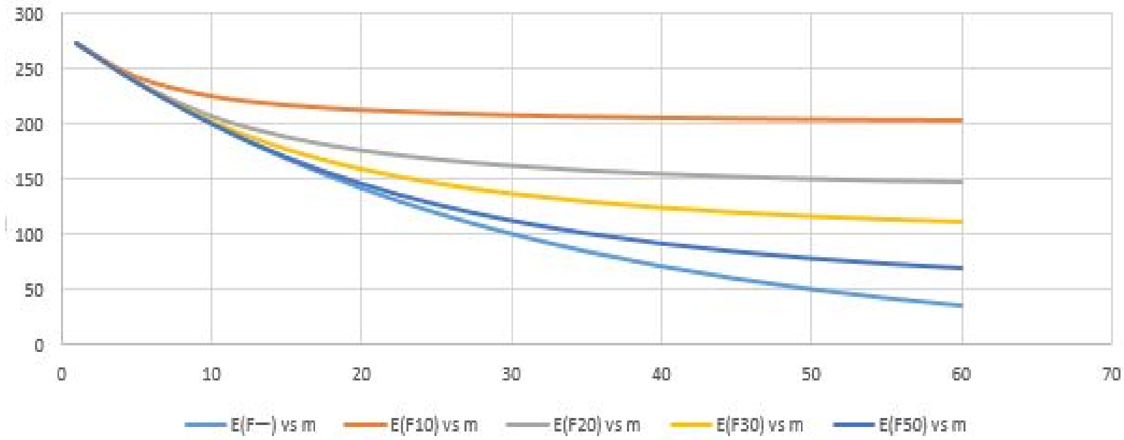

Figure 5.18: $\mathbb{E}(F_t)$ vs m at Different time t

Remark 5.3.6. From the above Figure 5.18:

- It is clear that F_t at $m = 1$has the same value at different valuation time.
- When the time increases, the expected fund move away from its stationary value as m increases.
- As valuation time increases, the values of $\mathbb{E}(F_t)$ decline with increasing m, and the level of decrease differe base on time t, as we go close to ∞, it drops dramatically.

Chapter 6

Conclusion

In this project, we review pension funding designs that are common in the literature. Then, we study in details the national pension schemes in Saudi Arabia and Canada, and the difficulties that face both schemes.

Risks under pension plans, in particular investment risk is one of many risks face pension plan, and this risk born when the rate of return does not meet its expected value. So, in this work we focus on studying the rate of return as time series models, and we fit models that are used widely in finance sector modifivated by Haberman in [32] and [35]. The expectation and the variance of the fund, contribution, and benefit levels are studied intensively under each model of rate of return that we propose for. The aggregate risk and the optimal spread parameter are discussed in details under different measure of risk.

It is noticeable that the ultimate value of F_t meets its stationary status AL under GARCH model when the parameter of GARCH model κ_1 set to be equal to 1 plus the interest rate. Also, we find out that the optimal spread parameter has no close form as in the *iid* case where we obtain the same optimal spread parameter under different risk measures.

Then a numerical illustration is proposed under GARCH model. We find out that when the optimal spread period is 30 since the variability in the fund above 30 is higher compare to below this value under coefficient of variation measure, also, when different investment strategies are applied, it is clear that the contributions and benefits decrease with increasing α; the proportion of investment in the asset; and the variability of the fund is lower under risky asset, whereas, it is higher under lower valuation interest rate. When, we change the measure, the aggregate risk has smaller values under the coefficient of variation compare to the variance.

One disadvantage of the pension model that we work on is that the surpluses and deficit are treated in the same way under the model, in other words there is no boundry for the benefit and the contributions, and this might lead to unrealistic results such as negative values of benefit and contributions.

6.1 Future Work

Here we propose some ideas that it might be interesting to be done for future work.

- One can assume that population is increasing (or declining) exponentially. Mathematically, the population has the following density function;

  $$l(x, t-x) = l_a \, e^{k(t-x+a)} \, s(x) = l_x \, e^{k(t-x+a)},$$

  where a is the hiring age (constant) of the new entrants, x is the attained age, and t is the time at attained age, k is the rate of increase (or decline) of the population.

  Then, AL_t and NC_t are obtained as below;

  $${}^{T}NC_t = TB \, l_r \, e^{k(t-r+a)} \, \bar{a}_r,$$

  $$NC_t = TBe^{k(t+a)-\delta_v r} l_r e^{\xi X(\xi)} \bar{a}_r,$$

  $$AL_t = {}^{T}NC_t \, \bar{a}_{r-X(\xi)}$$

where $X(\xi)$ is the geometric average age, $\xi = \delta_v - k$, and $r$is retirement age. Then one can easily substitute in the above formulas.

- It is interesting to consider **ARMA-GARCH** model since one of them model the conditional expectation and the other one model the conditional variance. Also, one can apply long-memory model for the rate of return because of its interesting properties that we mention them in Chapter 4.

- It is more realistic to work on salary scale that is not equal to one, so one can test how the model will be if the target benefit is function of the salary scale which is a function of random variables.

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Appendix A

Derivations

A.1 The Derivation of Equation (4.2)

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\Delta_t) &= \mathbb{V}\mathrm{ar}(\sum_{i=1}^{t}\delta_i) = \mathbb{C}\mathrm{ov}(\sum_{i=1}^{t}\delta_i, \sum_{j=1}^{t}\delta_j) = \sum_{i=1}^{t}\sum_{j=1}^{t}\mathbb{C}\mathrm{ov}(\delta_i,\delta_j)\\ &= \gamma_\delta(1)\sum_{i=1}^{t}\sum_{j=1}^{t}\phi^{|i-j-1|} = \gamma_\delta(1)\Big(t\phi + 2\sum_{i=1}^{t}\sum_{j=i+1}^{t}\phi^{j-i-1}\Big),\end{aligned}$$

A.2 The Derivation of Equation (4.3)

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\Delta_t-\Delta_s) &= \mathbb{V}\mathrm{ar}(\sum_{i=s+1}^{t}) = \sum_{i=s+1}^{t}\sum_{j=s+1}^{t}\mathbb{C}\mathrm{ov}(\delta_i,\delta_j)\\ &= \gamma_\delta(1)\Big((t-s)\phi + 2\sum_{i=s+1}^{t}\sum_{j=i+1}^{t}\phi^{j-i-1}\Big)\\ &= \gamma_\delta(1)\Big((t-s)\phi + 2\sum_{i=s+1}^{t}\sum_{u=1}^{t-u}\phi^{u-1}\Big); \quad \text{change variaable}\\ &= \gamma_\delta(1)\Big((t-s)\phi + 2\phi^{-1}\sum_{u=1}^{t-s-1}(t-s-u)\phi^{u}\Big)\\ &= \gamma_\delta(1)\Big((t-s)\phi + 2\phi^{-1}\phi^{t-s}\sum_{u=1}^{t-s-1}(t-s-u)\phi^{-(t-s-u)}\Big)\end{aligned}$$

$$= \gamma_\delta(1)\Big[(t-s)\phi + 2\phi^{-1}\phi^{t-s}\frac{\phi}{(1-\phi)^2}\Big(1-(t-s)\phi^{-(t-s-1)} + (t-s-1)\phi^{-(t-s)}\Big)\Big]$$
$$= \gamma_\delta(1)\Big[(t-s)\phi + \frac{2(t-s)}{1-\phi} - \frac{2(1-\phi^{t-s})}{(1-\phi)^2}\Big]$$
$$= \gamma_\delta(1)\Big[(t-s)(\phi + \frac{2}{1-\phi}) - \frac{2(1-\phi^{t-s})}{(1-\phi)^2}\Big],$$

A.3 The Derivation of Equation (4.4)

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(2\Delta_t - \Delta_s - \Delta_v) &= \mathbb{V}\mathrm{ar}(\Delta_v - \Delta_s + 2(\Delta_t - \Delta_v)) \\
&= \mathbb{V}\mathrm{ar}(\Delta_v - \Delta_s) + 4\mathbb{V}\mathrm{ar}(\Delta_t - \Delta_v) + 4\mathbb{C}\mathrm{ov}(\Delta_v - \Delta_s, \delta_t - \Delta_v) \\
&= \sum_{i=s+1}^{v}\sum_{j=s+1}^{v} \mathbb{C}\mathrm{ov}(\delta_i, \delta_j) + 4\sum_{i=v+1}^{t}\sum_{j=v+1}^{t} \mathbb{C}\mathrm{ov}(\delta_i, \delta_j) \\
&+ 4\sum_{i=s+1}^{v}\sum_{j=v+1}^{t} \mathbb{C}\mathrm{ov}(\delta_i, \delta_j) = \sum_{i=s+1}^{v}\sum_{j=s+1}^{v} \mathbb{C}\mathrm{ov}(\delta_i, \delta_j) \\
&+ 4\sum_{j=v+1}^{t}\Big(\sum_{i=v+1}^{t} \mathbb{C}\mathrm{ov}(\delta_i, \delta_j) + \sum_{i=s+1}^{v} \mathbb{C}\mathrm{ov}(\delta_i, \delta_j)\Big) \quad ;\text{(Fubini Theorem)} \\
&= \sum_{i=s+1}^{v}\sum_{j=s+1}^{v} \mathbb{C}\mathrm{ov}(\delta_i, \delta_j) + 4\sum_{i=s+1}^{t}\sum_{j=v+1}^{t} \mathbb{C}\mathrm{ov}(\delta_i, \delta_j) \\
&= \gamma_\delta(1)\Big[(v-s)(\phi + \frac{2}{1-\phi}) - \frac{2(1-\phi^{v-s})}{(1-\phi)^2} + 4\sum_{i=s+1}^{t}\sum_{j=v+1}^{t} \phi^{|i-j-1|}\Big] \\
&= \gamma_\delta(1)\Big[(v-s)(\phi + \frac{2}{1-\phi}) - \frac{2(1-\phi^{v-s})}{(1-\phi)^2} + 4\phi^{-1}\sum_{w=v-s}^{t-s-1}(t-s-w)\phi^w \\
&+ 4(t-v)\pi^{-1}\sum_{N=1}^{v-s-1}\phi^N + 4(t-v)\phi + 4\phi^{-1}\sum_{G=1}^{t-v-1}(t-v-G)\phi^G\Big]
\end{aligned}$$