Managing Consistency and Consensus in Group Decision-Making with Incomplete Fuzzy Preference Relations

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Abstract

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Group decision-making is a field of decision theory that has many strengths and benefits. It can solve and simplify the most complex and hard decision problems. In addition, it helps decision-makers know more about the problem under study and their preferences. Group decision-making is much harder and complex than individual decision-making since group members may have different preferences regarding the alternatives, making it difficult to reach a consensus.

In this thesis, we deal with three interrelated problems that decision-makers encounter during the process of arriving at a final decision. Our work addresses decision-making using preference relations. The first problem deals with incomplete reciprocal preference relations, where some of the preference degrees are missing. Ideally, the group members are able to provide preferences for all the alternatives, but sometimes they might not be able to discriminate between some of the alternatives, leading to missing values. Two methods are proposed to handle this problem. The first is based on a system of equations and the second relies on goal programming to estimate the missing information. The former is suitable to complete any incomplete preference relation with at least $n-1$ non-diagonal preference degrees whereas the latter is good to handle ignorance situations, where at least one alternative has not been given any preferences. The second problem deals with the theme of consensus. In a group decision-making situation, reaching an agreement

or consensus is important. A novel method based on Spearman's correlation to measure group ranking consensus is proposed. This method adopts the idea of measuring the monotonic degree among the decision-makers. Based on this method, a feedback mechanism is developed that acts as a moderator to guide the group into the consensus solution. The third problem deals with rank reversal. Our investigation leads to inconsistency of information and score aggregation method as the main causes of this phenomenon. However, obtaining a consistent preference relation is hard in practice. Thus, two score aggregation methods are proposed to handle rank reversal. The first method is used in case of replacement or addition of a new alternative in the alternative set. This method performs better than sum normalization aggregation method in avoiding rank reversal. The second method is used when an alternative is removed and has been proven to prevent rank reversal from occurring.

To my beloved parents,

To my dear wife, Heba,

To my sons, Jaber & Anees

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Aqeel A. Al-Salem

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List of Acronyms

AHP	Analytic Hierarchy Process
BK	Borda-Kendall
DC	Consistency degree
DEAHP	Data Envelopment analysis - Analytic hierarchy process
DM	Decision-maker
ELECTRE	ELimination and Choice Expressing Reality
GP	Goal Programming
I-IOWA	Importance induced ordered weighted averaging
IOWA	Induced ordered weighted averaging
MADM	Multi-attributes decision-making
MCDM	Multi-criteria decision-making
PROMETHEE	Preference Ranking Organization Method for Enrichment Evaluations
QGDD	Quantifier guided dominance degree
rcc	Rank correlation consensus
rccd	Rank correlation consensus degree
rsd	Rank similarity degree
RV	ranked vector of preference degrees
TOPSIS	Technique for Order of Preference by Similarity to Ideal Solution
V	Vector of preference degrees
WPM	Weighted Product Model
WSM	Weighted Sum Method

Chapter 1:

Introduction and Background

Decision-making can be done individually or by a group of decision-makers, also known as group decision-making. The group decision is a choice between at least two alternatives made by group members or by group's leader by consulting the members (Bedau & Chechile, 1984). Individual decision-making is difficult but group decision-making is even harder and complex due to involvement of multiple different preferences making the consensus difficult to reach. Furthermore, individual decisions in small organizations are usually done at lower managerial levels; however, in large organizations group decisions are commonly made at higher managerial levels (Lu et al., 2007).

Three preference representation formats are commonly used in group decision-making: preference orderings (where each individual ranks alternatives from best to worst), utility values (where an individual assigns utility values for alternatives such that the higher the value, the better is the alternative) and preference relations (Herrera-Viedma et al., 2014). Preference relations are based on pairwise comparisons where each pair of alternatives are compared at a time by an expert. Millet (1997) compared five different types of preference elicitation methods and concluded that preferences based on pairwise comparison are more accurate than the others. In group decision-making, some of the decision-makers may not be able to provide complete information about their preferences on the alternatives. That could be related to either the decision-maker not having enough knowledge about part of the problem or not being able to discriminate between some of alternatives (Herrera-Viedma et al., 2007b). Thus, the decision-maker gives incomplete preferences where some values are missing.

In group decision-making, two processes are employed: consensus and then selection. The selection process could be applied without adopting consensus through applying the preference relations provided by the decision-makers (Roubens, 1997). This could, however, lead to a solution that might not be accepted by some of the decision-makers, since it does not reflect their preferences (Saint & Lawson, 1994; Butler & Rothstein, 2007). Therefore, they might reject the solution. Thus, consensus is important before applying selection (Kacprzyk et al., 1992). For the selection process, some methods are known to exhibit rank reversal. Rank reversal occurs when a new alternative is added to (or removed from) a set of alternatives, which causes a change in the ranking order of the alternatives (Barzilai & Golany, 1994).

According to Lu et al. (2007), numerous types of decision-making methods can be used in group decision-making problems. Generally, each of these methods follows a rule. Among these rules are:

1. **Authority rule:** the leader of the group has the authority to make the final decision after holding an open discussion with the members of the group about the decision problem.
    a. Advantage(s): the method attains the final decision fast.
    b. Disadvantage(s): the method does not takes advantage of the strengths of the experts in the group.
2. **Majority rule:** the group decision is made based on the vote of the majority of the group.

   a. Advantage(s): clear voting rule (democratic participation) and generating fast final decision.
   b. Disadvantage(s): the decision may not be well executed because of inadequate discussions among the group.

3. **Negative minority rule:** the method is based on eliminating the most unpopular alternative one at a time through a vote until only one alternative remains.
   a. Advantage(s): good for situations with few experts (voters) and lots of ideas.
   b. Disadvantage(s): slow method and might lead to discomfort among decision-makers who are in favor of some eliminated alternatives.
4. **Ranking rule:** it is based on ranking of the alternatives by the experts. Such a method assigns a number for every alternative by all experts individually. Then the score of each alternative is aggregated. The alternative that has the highest score is selected.
   a. Advantage(s): includes voting procedure.
   b. Disadvantage(s): might result in a decision not supported by the group.
5. **Consensus rule:** consensus means full agreement by the group on the decision. The rule is based on reaching decision through discussions and negotiations until all the experts in the group understand and agree with what will be done.
   a. Advantage(s): the decision is supported by the group.
   b. Disadvantage(s): might be hard to reach consensus and is time consuming.

However, since it is hard and inconvenient to reach full and unanimous agreement among all the experts in the group, besides, a full agreement is not always necessary in practice. A soft consensus has been developed, which does not require a full agreement among the experts, relies mainly on consensus measure (Cabrerizo et al., 2010; Herrera-Viedma et al., 2014; Chiclana et al., 2013).

Based on these rules several methods have been developed to improve the processes of group decision-making. The most popular two are Delphi method and multi-voting technique (Lu et al., 2007). Delphi method was developed by Gordon and Helmer in 1953. The method is based on reaching consensus on an opinion without a need from the experts to set together. It could be through survey, questionnaires *etc*. Several applications of Delphi method have shown its effectiveness in dealing with complex decision problems. Multi-voting technique is used to attain group consensus fast by letting each expert rank the alternatives and collation of the expert's ranks into the group consensus.

1.1. A Brief Review

1.1.1. Preference relation preliminary knowledge

Definition 1.1 (Urena et al., 2015): A preference relation R is a binary relation defined on the set X and is characterized by a function $\mu_p: X \times X \to D$, where D is the domain of representation of preference degrees provided by the decision-maker.

Definition 1.2 (Urena et al., 2015): An additive preference relation P on a finite set of alternatives X is characterised by a membership function $\mu_p: X \times X \to [0,1]$, $\mu_p(x_i, x_j) = p_{ij}$ such that $p_{ij} + p_{ji} = 1 \; \forall i, j \in \{1, \ldots, n\}$. Furthermore:

$p_{ij} > 0.5$ indicates that the expert prefers alternative x_i to alternative x_j, with $p_{ij} = 1$ being the maximum degree of preference for x_i over x_j;

$p_{ij} = 0.5$ represents indifference between x_i and x_j; therefore, $p_{ii} = 0.5$.

Let $E = \{e_1, e_2, \dots, e_T\}$ be the set of decision-makers, and $w = \{w_1, w_2, \dots, w_T\}$ be the weight vector of decision-makers, where $w_k > 0, k = 1,2, \dots, T$ such that $\sum_{k=1}^{T} w_k = 1$. Then, $P^k = \left(p_{ij}^k\right)_{n \times n}$ is the judgment/preference relation of decision-maker $e_k \in E$ on the set of alternatives $X = \{x_1, x_2, \dots, x_n\}$.

1.1.2. *Incomplete preference relations*

The individuals in group decision-making come from different background and expertise and each has their motivations or goals. They look at the problem from different angles, yet all have to reach an agreement. Each individual is required to give preferences for a set of pre-determined alternatives. Since each individual has unique experience, he or she may not be able to give a preference degree for some of the alternatives. This could be related to number of reasons. First, they may not have enough knowledge about part of the problem or may not be able to give preferences degrees for some of the alternatives, or decide which alternative is better than the other. In this situation, they provide incomplete information (Herrera-Viedma et al., 2007b) or it can be simply due to time pressure (Xu, 2005a).

Definition 1.3 (Urena et al., 2015): A function $f\colon X \to Y$ is partial when not every element in the set X necessarily maps to an element in the set Y. When every element from the set X maps to one element of the set Y then we have a total function.

Definition 1.4 (Urena et al., 2015): A preference relation P on a set of alternatives X with a partial membership function is an incomplete preference relation.

A number of papers look at this problem. Xu (2005a) proposed two approaches to find the priority vector of an incomplete fuzzy preference relation based on a system of equations. The first approach uses the system of equations to generate the priority vector of an incomplete fuzzy preference relation. On the other hand, the second approach uses the provided information to estimate the unknown values and then generates the priority vector by the system of equations. Xu (2006) studied five types of incomplete linguistic preference relations, namely, incomplete uncertain linguistic preference relation, incomplete triangular fuzzy linguistic preference relation, incomplete trapezoid fuzzy linguistic preference relation, expected incomplete linguistic preference relation and acceptable expected incomplete linguistic preference relation. Then, based on some transformation functions, he converted them into the expected incomplete linguistic preference relations. He used the expected incomplete linguistic preference relations based on additive consistency to calculate the complete linguistic preference relations. Fedrizzi and Giove (2007) proposed a method based on a linear system to calculate missing values of an incomplete matrix of pairwise comparison. Chiclana et al. (2009) analyzed two methods for estimating missing values in incomplete fuzzy preference relation, one of them being Fedrizzi and Giove's (2007) method. They ended up with proposing a reconstruction policy for using both methods. Alonso et al. (2008) introduced an iterative procedure to estimate missing information for incomplete fuzzy, multiplicative, interval-valued and linguistic preference relations. Lee (2012) proposed an incomplete fuzzy preference relations method based on additive consistency and order consistency.

1.1.3. Consensus in group decision-making

Consensus in group decision-making can be interpreted in three ways (Herrera-Viedma et al., 2014). It could mean full agreement or a unanimous decision by the group members or reaching consensus by a moderator who facilitates the process of agreement, or it could mean attaining the consent, where some individuals might not completely agree but are willing to go with the majority opinion of the group. Consensus is the main goal in any group decision-making problem, since obtaining an acceptable solution by the group is important.

Xu (2009) proposed an automatic approach for reaching consensus in multi-attribute group decision-making. His approach was based on numerical settings, where each individual constructs a decision matrix. Then, these matrices are aggregated into one group-decision matrix. The method calculates the similarity measure between each individual matrix and the group decision matrix to determine the degree of consensus. A convergent iterative algorithm is introduced for individual matrices to reach the consensus. Sun and Ma (2015) proposed an approach for consensus using linguistic preference relations. They used consensus measure based on the dominance degree between group preference relation and individuals' preference relations. Zhang and Dong (2013) proposed an interactive consensus reaching process based on optimization to increase consensus of individuals and minimize the number of adjusted preference values. Guha and Chakraborty (2011) introduced an iterative fuzzy multi-attribute group decision-making technique to reach consensus using fuzzy similarity measures. In addition, their method considers the degrees of confidence of experts' opinions in the procedure. Herrera-Viedma et al. (2002) proposed a consensus model suitable for four different preference structures. Their model uses two consensus criteria: a consensus measure for measuring the degree of consensus between the experts, and a proximity measure to measure the difference between the preferences of individuals and the group preference relation.

1.1.4. Rank reversal

Our literature review on decision-making reveals that a number of methods suffer from the rank reversal phenomenon. These include Analytic Hierarchy Process (AHP) (Barzilai & Golany, 1994; Wang & Luo, 2009), Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) (Wang et al., 2007; Wang & Luo, 2009), ELimination and Choice Expressing Reality (ELECTRE), Preference Ranking Organisation Method for Enrichment Evaluations (PROMETHEE) (Frini et al., 2012; Mareschal et al., 2008), Data Envelopment analysis - Analytic hierarchy process (DEAHP), Borda-Kendall (BK) (Wang & Luo, 2009) and Weighted Sum Method (WSM)(Wang & Luo, 2009), to name a few.

The rank reversal issue has created concerns over the use of the affected methods, especially AHP. Rank reversal could be of two types: partial or total. Partial rank reversal happens to limited alternatives while other alternatives still have the same ordering. Suppose that the current ranking of three alternatives is $A_3 \succ A_1 \succ A_2$, which means that alternative A_3 is preferred over alternative A_1 and A_2 respectively. However, when another non-dominating alternative (A_4) is added, the ranking becomes: $A_1 \succ A_3 \succ A_4 \succ A_2$. Notice that alternative A_3 now becomes second while alternative A_1 is the first. This is called partial rank reversal. On the other hand, total rank reversal occurs when the whole ordering or ranking is reversed. In this case, the best alternative becomes the worst and the worst becomes the best $A_2 \succ A_4 \succ A_1 \succ A_3$ (Dymova et al., 2013; Garcia-Cascales & Lamata, 2012).

Many reasons behind the rank reversal in preference relation have been studied by researchers, especially in AHP. The three main reasons are inconsistency, preference relations aggregation method, and score aggregation method. Dodd et al. (1995) claimed that Saaty's AHP misses a form of inconsistency within its model, which makes the results doubtful. This claim somehow agrees with Stewart (1992) who stated that rank reversal is a consequence of the way the weights are elicited, ratio scales, and the eigenvector approach. Farkas et al. (2004) blamed inconsistency in pairwise comparison for this issue. Chou (2012) attributes rank reversal in AHP to the aggregation method, due to Saaty's ratio scale and the inconsistency of judgments.

Other researchers, like Schenkerman (1994), believed that rank reversal in AHP is caused by normalization and its scales seem arbitrary. He claimed that criteria weights are dependent on the alternatives measurements. Thus, any change in the number of alternatives and normalization imposes revision of the criteria weights. Other researchers such as Lai (1995) pointed out that rank reversal happens because of multiplying criteria weights by an unrelated normalized scale of performance ratings. Dyer (1990) claimed that the problem is not just rank reversal, but rather the AHP results' are arbitrary. This is because the criteria weights may not be right due to the normalization procedure.

1.2. Scope and Objectives

The scope of our work is limited to additive preference relations in group decision-making. In preference relations settings, decision-makers might not give complete information for their preference degrees on some of the alternatives. In fact, it is unrealistic in group decision-making to acquire all the knowledge about the problem and discriminate between the alternatives, especially if the set of alternatives is large (Urena et al., 2015). Thus, it is important and desirable to manage incomplete preference relations by estimating missing information (Urena et al., 2015). *Therefore, our first objective in the thesis is to capture decision-maker preferences in numerical settings, particularly additive preference relations, accurately by proposing a method that has the ability to handle incomplete preference relations.* Many papers have studied this issue in the general case where at least $n-1$ non-diagonal preference degrees are given; however, very few papers have studied the ignorance situations. In the ignorance situations, at least one alternative has not been given any preference degree. Thus, our goal is to handle these situations in preference relations.

In group decision-making, reaching a level of agreement between the group members is important even when each member differs from the others. Measuring consensus, aggregation of preferences, and ranking are considered as main issues to be solved in any group decision-making problem (Ben-Arieh & Chen, 2006). Therefore, it is very important to measure consensus degree between individuals and group preference relation to find the similarities. Consensus process in group decision-making involves aggregating individual preferences relations into a collective or group preference relation. Then a similarity measure is used to measure the degree of similarities between the individual matrices and the collective one. If the similarity is greater than or equal to a pre-defined threshold, then the collective preference relation is considered as consensus. Otherwise, the decision-makers with consensus degree below the threshold are asked to re-evaluate their preferences until the consensus degree reaches the acceptable level of similarity. Generally, consensus is an interactive and iterative process where decision-makers revise their preferences until they reach a manageable level of acceptance. *Thus, our second objective in this thesis is to* *develop a new consensus measure for group decision-making based on rank correlation, where similarity/distance functions are not the main measures to measure the coherence among decision-makers.* A number of papers, such as Perez et al. (2016), Cabrerizo et al. (2015), and Herrera-Viedma et al. (2014) reported that developing a new consensus measure is beneficial to overcome some drawbacks of similarity/distance functions. A study done by Chiclana et al. (2013) compares five different similarity/distance measures of consensus in group decision-making, namely, Manhattan, Euclidean, Cosine, Dice and Jaccard,. They found that different similarity/distance measures could generate significantly different results. Moreover, the chosen measure could affect the speed of convergence for consensus.

The last process in group decision-making is selection. The preference relations are known to have a phenomenon called rank reversal. This phenomenon is considered as an issue by some decision-makers. Thus, they might hesitate to rely on preference relations. For instance, recently Anbaroglu et al. (2014) chose to use Weighted Product Model (WPM) instead of well-known and widely used models such as AHP and WSM just because it does not suffer from any kind of rank reversal issues. Furthermore, they commented on the problem of rank reversal as “a serious limitation” of the multi-criteria decision-making (MCDM) field, which could mislead researchers from understanding the difference between examined alternatives. *Therefore, our third objective in this thesis is to solve the rank reversal issue by investigating and addressing its possible causes in additive preference relations.* The literature on preference relations, especially multiplicative preference relation, links this phenomenon to inconsistency of the data, the concept of pairwise comparison on which all preference relations are based, preference aggregation method, and score aggregation method. Currently, there is no complete study that investigates these possible reasons for rank reversal in preference relations. Leskinen and Kangas (2005) used a regression model to study the inconsistency of pairwise comparisons. They concluded that inconsistency could lead to rank reversal. However, this phenomenon does not occur when there is a single criterion and data is consistent. But, in multiple criteria, even if the data are consistent, the aggregation method or the arithmetic mean can result in rank reversals. Furthermore, to emphasize this issue we pointed out that it violates the *contraction consistency condition* $(condition\ \alpha)$ [mentioned by Pavlicic (2001), adopted from Amartya Sen] that states:

Contraction consistency condition: *If alternative A is the best in the set of alternatives S such that $A \in S$, then it has to be the best in every subset $E \subset S$ where $A \in E$.*

Our aim is to investigate the reasons behind this phenomenon so that they can be prevented or limited when not desired by the group.

1.3. Thesis Organization

In this thesis, we deal with three associated problems that decision-makers encounter during the process of reaching a final decision in a group decision-making setting. The three problems (challenges) are splits into three stages, where each stage relies on the stage before, as shown in Figure 1.1. In the first stage, we deal with incomplete reciprocal preference relations for missing information in general case and ignorance situations. In the second stage, we deal with the consensus process by proposing a novel consensus measure and feedback mechanism. For the third stage, we study the causes of rank reversal phenomena in preference relations. The rest of this thesis is organized based on these contributions, which are presented in Chapters 2, 3 and 4 respectively.

In Chapter 2, we propose two new methods for incomplete reciprocal fuzzy preference relations based on additive consistency. The first method is for the general case where at least $n-1$ non-diagonal preference degrees are given. This method is based on using a system of equations to estimate the values of missing preference degrees. The second method, based on goal programming, was designed specifically for ignorance situations. It can also be used to estimate values for the general case. In the goal programming model, the objectives are to minimize the errors between the missing preferences degrees and their estimations subject to all the missing preference degrees between 0 and 1.

In Chapter 3, we propose a new consensus measure based on rank correlation to address the consensus among decision-makers. We utilized Spearman's correlation to measure rank consensus on preference degrees between the decision-makers. Thus, we define ranked preference vector for each decision-maker and develop a new rank similarity degree measure. In addition to measuring rank consensus, we introduce a feedback mechanism to assist the group reach a consensus state.

In Chapter 4, we study the rank reversal in additive preference relations. We investigate the possible causes behind this phenomenon. The study is based on additive consistency. Thereby, we study the link of inconsistency, preference aggregation methods, score aggregation methods and their effect on generating rank reversal. We also propose two new score aggregation methods to handle this phenomenon when it is not desirable by the group. The first score aggregation method is used when a new alternative is added or replaced by the group. In this method, a consistency element on the aggregation method is used. The second method is used to prevent rank reversal when an alternative is removed from consideration.

Finally, in Chapter 5 we present the thesis conclusions, contributions, and our perspective for future works.

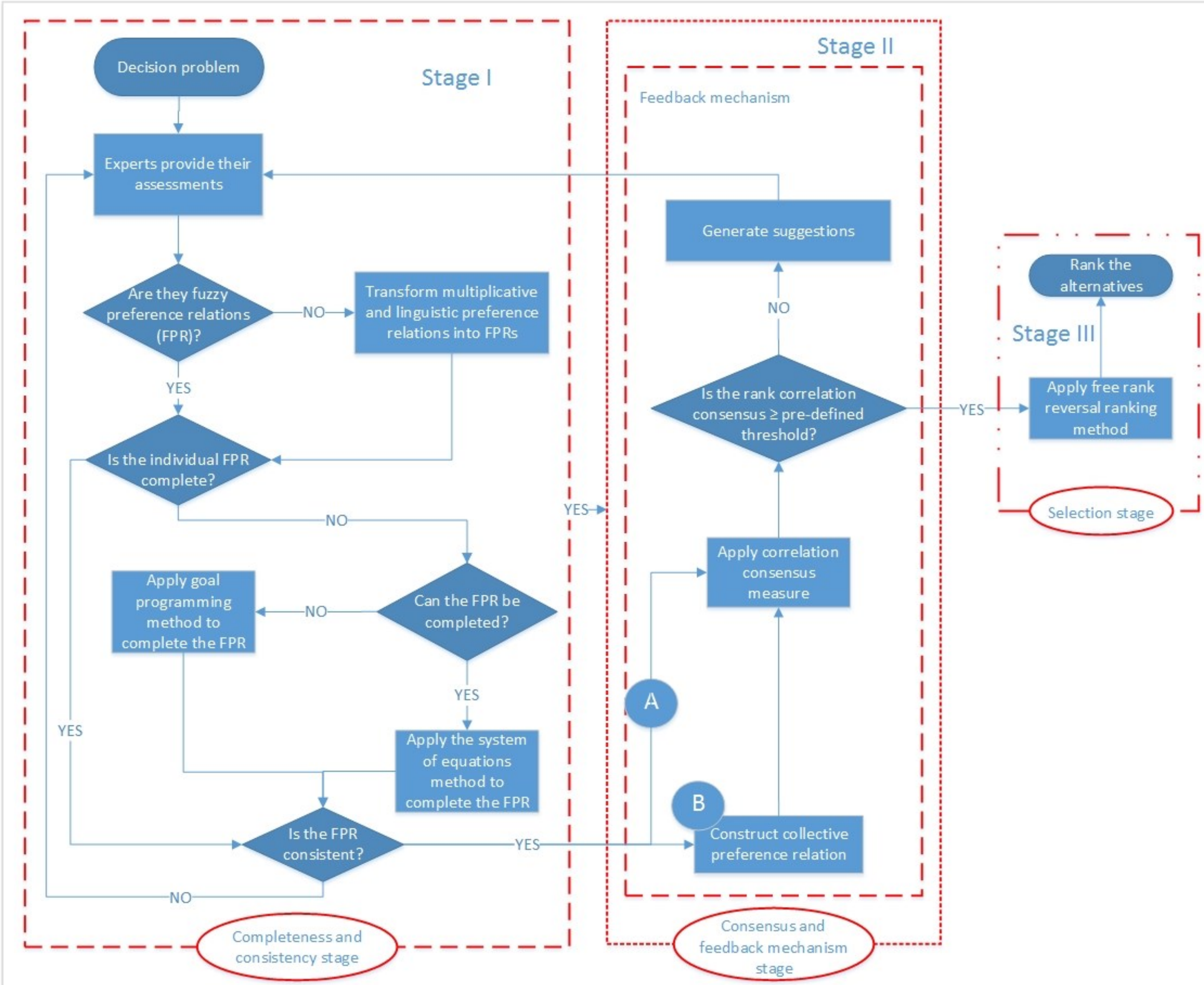

Figure 1.1: Relationship between the three stages

Chapter 2:

Two New Methods for Decision-Making with Incomplete Reciprocal Fuzzy Preference Relations Based on Additive Consistency

2.1. Introduction

Multi-attributes decision-making (MADM) involves making decisions among a set of alternatives with respect to a set of attributes/criteria by a committee of decision-makers. The main idea behind MADM is that the decision-maker (DM) usually faces a problem of selecting an alternative from a number of pre-determined choices, which need to be evaluated based on a number of criteria. Usually, there is no unique or best solution, as the solution is often reached through a compromise: typically, a trade-off between the criteria and decision-makers' preferences (Hwang & Yoon, 1981). Generally, MADM has issues with regards to the accuracy of decision-maker judgments, the consistency of the judgments, and the method to be used to find the solution (Easley et al., 2000).

In decision-making, there are three commonly used preference representation formats: preference orderings, where each individual ranks alternatives from the best to the worst, utility values, where an individual assigns utility values to alternatives such that the higher the value the better is the alternative, and preference relations (Herrera-Viedma et al., 2014). Preference relations are based on pairwise comparisons where each two alternatives are compared by an expert at a time. Millet

(1997) compared five different types of preference elicitation methods and concluded that preferences based on pairwise comparison are more accurate than the others.

Fuzzy preference relations are commonly used in decision-making for evaluating a set of alternatives with respect to a set of attributes. However, in some situations, decision-makers may not be able to provide complete information about their preferences on the alternatives. That could be due to the decision-maker not having enough knowledge about part of the problem or being unable to discriminate between some of the alternatives (Herrera-Viedma et al., 2007b) or it might be because of time pressure (Xu, 2005a). In fact, it is unrealistic for all decision-makers to acquire all the levels of knowledge of the whole problem and be able to discriminate between all the alternatives, especially if the set of alternatives is large (Urena et al., 2015). Thus, some decision-makers might not be able to provide information for some of the alternatives. In this case, it is important and desirable to manage incomplete preference relations by estimating the missing information (Urena et al., 2015). Moreover, in some cases, the decision-maker might not be able to give his/her assessments for at least one of the alternatives with respect to the others. This situation is called an ignorance situation and the alternative is called the ignorance alternative (Chen et al., 2014; Alonso et al., 2009). Most of the existing methods are not compatible with estimating unknown preference degrees for ignorance situations such as the model proposed by Herrera-Viedma et al. (2007a). Furthermore, according to Meng and Chen (2015), other methods try to assign fixed values for the ignorance alternative such as 0 or 0.5.

The main objectives of this work are to solve the problems associated with incomplete preference relations, namely, additive fuzzy preference relation, multiplicative preference relation, and linguistic preference relation, when some information is missing or when an ignorance situation is present by:

(1) *Proposing a method that has the ability to handle incomplete preference relations with high consistency rate, and also generate a perfect consistent matrix when at least each alternative is compared once.*

(2) *Proposing a method that solves the ignorance situation with a high consistency level without modifying or changing the decision-maker's preferences.*

The rest of the chapter is organized as follows: we present a brief preliminary knowledge on preference relations in section 2.2. Then a literature review on additive fuzzy preference relations is given in section 2.3 followed by proposed methodology in section 2.4. In section 2.5, the proposed methods are demonstrated with examples. In section 2.6, we validated the proposed methods. Finally, conclusions are given in section 2.7.

2.2. Preliminary Knowledge

In this section, we provide brief knowledge on three types of preference relations, namely, additive fuzzy preference relation, multiplicative preference relation and linguistic preference relation.

Definition 2.1 (Urena et al., 2015): A preference relation R is a binary relation defined on the set X that is characterized by a function $\mu_p: X \times X \rightarrow D$, where D is the domain of representation of preference degrees provided by the decision-maker.

2.2.1. Fuzzy additive preference relation

Definition 2.2 (Xu, 2007): A fuzzy additive preference relation P on a finite set of alternatives X is represented by a matrix $P = (p_{ij})_{n\times n} \subset X \times X$ with:

$p_{ij} \in [0,1],\ \ p_{ij} + p_{ji} = 1,\ \ p_{ii} = 0.5\ \ \forall i,j = 1,\dots,n.$

$$P = (p_{ij})_{nxn} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \dots & A_n \end{array} \\ \begin{bmatrix} 0.5 & p_{12} & \dots & p_{1n} \\ p_{21} & 0.5 & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & 0.5 \end{bmatrix} \end{array}$$

when $p_{ij} > 0.5$ indicates that the expert prefers alternative x_i over alternative x_j; $p_{ij} < 0.5$ indicates that the expert prefers alternative x_j over alternative x_i; $p_{ij} = 0.5$ indicates that the expert is indifferent between x_i and x_j, thus, $p_{ii} = 0.5$.

Furthermore, the additive preference relation $P = (p_{ij})_{n\times n}$ is additive consistent if and only if the following additive transitivity is satisfied (Meng & Chen, 2015; Urena et al., 2015; Herrera-Viedma et al., 2007a; Tanino, 1984):

$p_{ij} + p_{jk} = p_{ik} + 0.5 \qquad \forall ijk = 1,2,\dots,n.$

2.2.2. Multiplicative preference relation

Definition 2.3 (Saaty, 1980): A multiplicative preference relation A on the set $X = \{x_1, x_2, \dots, x_n\}$ of alternatives is defined as a reciprocal matrix $A = (a_{ij})_{n\times n} \subset X \times X$ with the following conditions:

$a_{ij} > 0,\ \ a_{ij}a_{ji} = 1,\ \ a_{ii} = 1,\ \ \forall ij = 1, 2, \dots, n.$

where a_{ij} is interpreted as the ratio of the preference intensity of the alternative x_i to x_j.

There are several numerical scales for the multiplicative preference relation, however, the most popular one is the 1-9 Saaty scale. $a_{ij} = 1$ means that alternatives x_i and x_j are indifferent; $a_{ij} > 1$ implies that alternative x_i is preferred to x_j. As the ratio of intensity of (a_{ij}) increases, the stronger is the preference intensity of x_i over x_j. Thus, $a_{ij} = 9$ means that alternative x_i is absolutely preferred to x_j.

The multiplicative preference relation $A = (a_{ij})_{n\times n}$ is called consistent if the following multiplicative transitivity is satisfied (Saaty, 1980):

$a_{ij} = a_{ik}a_{kj},\ \ a_{ii} = 1,\ \ \forall i, j = 1, 2, \dots, n.$

Chiclana et al. (2001) proposed a transformation function to transfer a multiplicative preference relation, $A = (a_{ij})_{n\times n}$, into a fuzzy preference relation, $P = (p_{ij})_{n\times n}$, as follows:

$$p_{ij} = \frac{1}{2}\left(1 + \log_9 a_{ij}\right)\ \ \forall i, j = 1, 2, \dots, n \qquad (2.1)$$

Moreover, if $A = (a_{ij})_{n\times n}$ is a consistent multiplicative preference relation, then the transformed $P = (p_{ij})_{n\times n}$ is an additive consistent fuzzy preference relation.

2.2.3. Linguistic preference relation

Definition 2.4 (Xu, 2005b): A linguistic preference relation L on the set $X = \{x_1, x_2, \dots, x_n\}$ of alternatives is represented by a linguistic decision matrix $L = (l_{ij})_{n\times n} \subset X \times X$ with

$$l_{ij} \in \bar{S},\ \ l_{ij} \oplus l_{ji} = s_0,\ \ l_{ii} = s_0,\ \ \forall ij = 1, 2, \dots, n.$$

where l_{ij} represents the preference degree of the alternative x_i over x_j. When $l_{ij} = s_0$, means that the decision-maker is indifferent between alternative x_i and x_j; $l_{ij} > s_0$ indicates that x_i is preferred over x_j.

Moreover, $L = (l_{ij})_{n\times n}$ is consistent when,

$$l_{ij} = l_{ik} \oplus l_{kj}\ \ \forall i, j, k = 1, 2, \dots, n.$$

Let $S = \{s_\alpha | \alpha = -t, \dots, -1, 0, 1, \dots, t\}$ be a linguistic label set with odd cardinality. Then s_α represents a possible value for a linguistic label. In addition, t is a positive integer number and $s_{-t}$and s_t are the lower and upper limits of linguistic labels, respectively, while s_0 represents an assessment of "indifference."

The linguistic label set has following characteristics (Xu, 2004, 2005b):

1. The set is ordered: $s_\alpha > s_\beta$ if and only if $\alpha > \beta$
2. There is the negation operator: $\text{neg}(s_\alpha) = s_{-\alpha}$

In addition, Xu (2004, 2005b) extended the discrete linguistic label set to a continuous set $\bar{S} = \{s_\alpha | \alpha \in [-q, q]\}$ to preserve all the information. In this extension q is a large positive integer such that $(q > t)$. In general, if $s_\alpha \in S$ then this represents the original linguistic label, otherwise, s_α is only the virtual linguistic label which appears only in operations.

Let s_α,$s_\beta \in \bar{S}$ and $\mu, \mu_1, \mu_2 \in [0, 1]$. Some operational laws introduced by Xu (2004, 2005b) are as follows:

1. $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$;
2. $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha$;
3. $\mu s_\alpha = s_{\mu\alpha}$;
4. $(\mu_1 + \mu_2)s_\alpha = \mu_1 s_\alpha \oplus \mu_2 s_\alpha$;
5. $\mu(s_\alpha \oplus s_\beta) = \mu s_\alpha \oplus \mu s_\beta$;

In addition, for any $s \in \bar{S}$, $I(s)$ represents the lower index of , e.g. if $s = s_\alpha \rightarrow I(s) = \alpha$ and it is called the gradation of s in $\bar{S}$. Likewise, we could get the inverse of $I(s)$: $I^{-1}(\alpha) = s_\alpha$.

An example of the linguistic label set is when $t = 3$, then S =\{s_{-3}= very low, s_{-2}= low, s_{-1}= slightly low, s_0=medium, s_1= slightly high, s_2= high, s_3= very high\}.

Sometimes, depending on the decision problem, experts provide their assessments on the linguistic preference relation using different granularity (multi-granularity). Thus, these granularities need to be unified. Dong et al. (2009) provided following transformation function for unifying multi-granularity into a common granularity (T):

$$l_{ij} = \frac{T-1}{T'-1} l'_{ij} \tag{2.2}$$

where T is the intended granularity (normal granularity) and T' is the granularity of $L'\left(l'_{ij}\right)_{n\times n}$.

Dong et al. (2009) and Xu (1999) propose a transformation function to transfer linguistic preference degree $\left(l_{ij}\right)$ into fuzzy preference degree based on linear scale function, as follows:

$$p_{ij} = 0.5 + \frac{I\left(l_{ij}\right)}{T-1} = 0.5 + \frac{I\left(l_{ij}\right)}{2t} \tag{2.3}$$

where T is the granularity of S.

2.3. Literature Review

Preference relations can be categorized into: numeric and linguistic preferences (Urena et al., 2015). The numeric preference relations are of five types: crisp preference relation, additive preference relation, multiplicative preference relation, interval-valued preference relation, and intuitionistic preference relation. On the other hand, there are two main methodologies for linguistic preference relation: linguistic preference relation based on cardinal representation and linguistic preference relation based on ordinal representation.

Many papers have been published about incomplete preference relation in decision-making. Xu (2005a) proposed two approaches to find the priority vector of an incomplete fuzzy preference relation based on a system of equations. The first approach uses the system of equations to generate the priority vector of an incomplete fuzzy preference relation. On the other hand, the second approach uses the provided information to estimate the unknown values and then generates the priority vector by the system of equations. Xu (2006) studied five types of incomplete linguistic preference relations, namely, incomplete uncertain linguistic preference relation, incomplete triangular fuzzy linguistic preference relation, incomplete trapezoid fuzzy linguistic preference relation, expected incomplete linguistic preference relation and acceptable expected incomplete linguistic preference relation. Then, based on some transformation functions, he converted them into the expected incomplete linguistic preference relations. He used the expected incomplete linguistic preference relations based on additive consistency to calculate the complete linguistic preference relations. Fedrizzi and Giove (2007) proposed a method based on a linear system to calculate missing values of an incomplete matrix of pairwise comparison. Chiclana et al. (2009) analyzed two methods for estimating missing values in incomplete fuzzy preference relation, one of them being Fedrizzi and Giove's (2007) method. They ended up with proposing a reconstruction policy for using both methods. Alonso et al. (2008) introduced an iterative procedure to estimate missing information for incomplete fuzzy, multiplicative, interval-valued and linguistic preference relations. Lee (2012) proposed an incomplete fuzzy preference relations method based on additive consistency and order consistency. Table 2.1 summarizes these approaches.

Table 2.1: Some approaches to solve incomplete preference relations

Author(s)	Method	Types of incomplete prefernce relation
Xu (2005a)	System of equations	Fuzzy preference relation
Xu (2006)	Expected incomplete linguistic preference relations based on additive consistency property	Uncertain linguistic, triangular fuzzy linguistic, trapezoid fuzzy linguistic, expected linguistic and acceptable expected linguistic
Fedrizzi and Giove (2007)	Linear system	Fuzzy, multiplicative
Alonso et al. (2008)	Based on additive consistency	Fuzzy, multiplicative, interval-valued and linguistic
Lee (2012)	Based on additive consistency and order	Fuzzy preference relations
Proposed method	Based on additive consistency	Fuzzy preference relations, multiplicative and linguistic

2.3.1. Incomplete preference relations

Definition 2.5 (Urena et al., 2015): A function $f: A \rightarrow Y$ is partial when not every element in the set A necessarily maps to an element in the set Y. When every element from the set A maps to one element of the set Y, we have a total function.

Definition 2.6 (Urena et al., 2015): A preference relation P on a set of alternatives A with a partial membership function is an incomplete preference relation.

The individuals in group decision-making come from different backgrounds or expertise and each has their motivations or goals in the problem, which might differ from the other members (Urena et al., 2015). Despite that, each individual might look at the problem from a different angle; they all have to interact to reach an agreement. Each individual is asked to give preferences on the set of pre-determined alternatives. However, since each individual has their own experience, they might not be fully aware of the problem and might not give their preferences degree for some of the alternatives. This could be related to number of reasons. They might not have enough knowledge about part of the problem, or they cannot discriminate between the alternatives. Then they do not give their preferences on those alternatives and provide incomplete information (Herrera-Viedma et al., 2007b) or it might be because of time pressure (Xu, 2005a).

In general, incomplete preference relation can be completed based on additive consistency if at least a set of $n-1$ nonleading diagonal preference values are known and each one of the alternatives are compared directly or indirectly at least once (Xu et al., 2013; Alonso et al., 2009; Herrera-Viedma et al., 2007a).

2.3.2. Research gaps

Despite the existing large number of publications on the incomplete preference relation problem, few of them discuss ignorance situations such as Alonso et al. (2009), Chen et al. (2014) and Meng and Chen (2015). Alonso et al. (2009) proposed five strategies for solving the ignorance situation. Two of these strategies are for individual, where the estimation of the missing information depends on the expert without relying on information from other members of the group. The other two are for social, where missing information of the ignorance alternative can be estimated from other members of the group. The last strategy is a hybrid of both individual and social strategies. Chen et al. (2014) solves the drawbacks of Lee's (2012) method. At the first stage, it assumes that the ignorance alternative is indifferent with respect to the other alternatives. Thus, its preference degrees are equal to 0.5. Then, based on this assumption, the method modifies the consistency, both the additive and the order consistency, of the matrix until it gets to the perfect consistency. Meng and Chen (2015) propose a goal programming method to find the priority vector for incomplete fuzzy preference relation based on additive consistency.

Chen et al. (2014) report that Alonso et al.'s (2009) method violates the property of additive consistency; thus they claim their method is more appropriate, as it satisfies the additive consistency and the order consistency. However, Chen et al.'s (2014) method does not preserve decision-maker preference degrees, at least for the ignorance situation. Nevertheless, all these methods are suitable for certain situations.

Moreover, most of the publications are based on comparing the alternatives directly without explicitly considering the attributes. Incorporating the attributes will make the alternatives evaluation process more accurate with deeper understanding of the differences between the alternatives. Although this could increase the number of preference relations, it will increase the confidence of the decision-maker about their assessments on the alternatives and thus the final solution.

2.4. Proposed Methodology

Transitivity is considered as the main part in defining consistency in decision-making. However, some might argue about its representation to real life individual behavior. One might argue that real life choices could be done in intransitive manner such that an individual might prefer apple over banana and banana over orange but orange over apple. This could be true if the decision was made based on comparing the alternatives directly. However, if a set of certain criteria or attributes has been defined first to draw a judgement on the alternatives, then intransitivity in preference relation under one attribute does not exist. For instance, if we set taste for comparing apple, banana and orange as an attribute, then if the individual prefers the taste of apple over the taste of banana and the taste of banana over the taste of orange, then certainly he or she prefers the taste of apple over the taste of orange. In MADM, the tradeoff between criteria makes maintaining the transitivity among the alternatives hard. However, transitivity within a criterion is a straightforward acquired property.

Therefore, the best adoption to additive consistency is to apply it in MADM concept. The general steps of the decision-making process in MADM as described by Howard (1991), Pohekar and Ramachabdran (2003), and Wang et al. (2009) are:

1. Defining the objectives;

2. Generating or choosing the criteria;
3. Identifying the alternatives;
4. Unifying criteria units through normalization;
5. Generating criteria weights;
6. Choosing and applying one of MADM methods; and
7. Selecting the best alternative.

Since we are dealing with preference relations, step 4 is not necessary.

2.4.1. System of equations method

Any incomplete additive preference relation with at least $(n-1)$ non-leading diagonal preference degrees can be completed by additive consistency. Additive consistency formulation, which is based on transitivity among preferences degrees, for known p_{ij} and p_{jk} and unknown p_{ik} is given by:

$$F^1: \quad p_{ik} = p_{ij} + p_{jk} - 0.5 \quad (i,j)\ and\ (j,k)\ are\ known \tag{2.4}$$

From this formulation, two other formulations can be generated based on the characteristics of reciprocal rule, $(p_{ij} + p_{ji} = 1)$, as follows:

$$F^2: \quad p_{ik} = p_{jk} - p_{ji} + 0.5 \quad (j,i)\ and\ (j,k)\ are\ known, \text{using } p_{ij} = 1 - p_{ji} \tag{2.5}$$

$$F^3: \quad p_{ik} = p_{ij} - p_{kj} + 0.5 \quad (i,j)\ and\ (k,j)\ are\ known, \text{using } p_{jk} = 1 - p_{kj} \tag{2.6}$$

Proposition 2.1: Given at least $(n-1)$ non-leading diagonal preference degrees, the additive preference relation can be completed for unknown preference degree p_{ik} by:

$$p_{ik} = \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right). \tag{2.7}$$

Proof: By taking the average of equations (2.4), (2.5) and (2.6) for unknown p_{ik} for n alternatives, the following equation is generated:

$$p_{ik} = \frac{1}{3n}\left[\sum_{j=1}^{n}(p_{ij} + p_{jk} - 0.5) + (p_{jk} - p_{ji} + 0.5) + (p_{ij} - p_{kj} + 0.5)\right]$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{j=1}^{n}(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5)$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{\substack{j=1 \\ i \neq j \neq k}}^{n}(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5) + \frac{1}{3n}(p_{ii} + p_{kk} + 4p_{ik} - 2p_{ki} + 1)$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{\substack{j=1 \\ i \neq j \neq k}}^{n}(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5) + \frac{1}{3n}(0.5 + 0.5 + 4p_{ik} - 2(1 - p_{ik}) + 1)$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{\substack{j=1 \\ i \neq j \neq k}}^{n}(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5) + \frac{1}{3n}(6p_{ik})$$

$$\Longrightarrow (3n)p_{ik} = \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + (6p_{ik})$$

$$\Longrightarrow 3(n-2)p_{ik} = \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

$$\Longrightarrow \boxed{p_{ik} = \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)} \blacksquare$$

The reciprocal rule implies that the matrix could be separated into two portions; upper triangular matrix and lower triangular matrix. Completing any portion will fulfill the other one. Thus, we will focus on completing the upper triangular matrix by using (2.7).

$$P = (p_{ij})_{n\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & \cdots & A_n \end{array} \\ \begin{bmatrix} 0.5 & p_{12} & p_{13} & \cdots & p_{1n} \\ 1-p_{12} & 0.5 & p_{23} & \cdots & p_{2n} \\ 1-p_{13} & 1-p_{23} & 0.5 & \cdots & p_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-p_{1n} & 1-p_{2n} & 1-p_{3n} & \cdots & 0.5 \end{bmatrix} \end{array}$$

Proposition 2.2: To complete the upper triangular matrix for an incomplete reciprocal additive preference relation with at least $(n-1)$ non-leading diagonal preference degrees, the following system of equations is applied:

$$p_{ik} = \frac{1}{n-2}\left[\sum\nolimits^{n}_{\substack{j=1 \\ i<j<k}} \left(p_{ij} + p_{jk} - 0.5\right) + \sum\nolimits^{n}_{\substack{j=1 \\ i>j<k}} \left(p_{jk} - p_{ji} + 0.5\right) + \sum\nolimits^{n}_{\substack{j=1 \\ i<j>k}} \left(p_{ij} - p_{kj} + 0.5\right)\right] \quad \forall\, i<k. \qquad (2.8)$$

Proof: j could fall in three positions between i and k: $i < j < k$, $i > j < k$ and $i < j > k$. Solve (2.7) for p_{ik} such that $i < j < k$:

$$p_{ik} = \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i<j<k}}^{n} \left(2p_{ij} + 2p_{jk} - (1 - p_{ij}) - (1 - p_{jk}) + 0.5\right)$$

$$\Rightarrow \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i<j<k}}^{n} \left(3p_{ij} + 3p_{jk} - 1.5\right)$$

$$\Rightarrow \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i<j<k}}^{n} 3\left(p_{ij} + p_{jk} - 0.5\right)$$

$$\Rightarrow \boxed{p_{ik} = \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i<j<k}}^{n} \left(p_{ij} + p_{jk} - 0.5\right) \ \ \forall\, i < j < k} \qquad (2.8.1)$$

Solve (2.7) for p_{ik} such that $i > j < k$:

$$p_{ik} = \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i>j<k}}^{n} \left(2(1 - p_{ji}) + 2p_{jk} - p_{ji} - (1 - p_{jk}) + 0.5\right)$$

$$\Rightarrow \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i>j<k}}^{n} \left(3p_{jk} - 3p_{ji} + 1.5\right)$$

$$\Rightarrow \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i>j<k}}^{n} 3\left(p_{jk} - p_{ji} + 0.5\right)$$

$$\Rightarrow \boxed{p_{ik} = \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i>j<k}}^{n} \left(p_{jk} - p_{ji} + 0.5\right) \quad \forall\, i > j < k} \tag{2.8.2}$$

Solve (2.7) for p_{ik} such that $i < j > k$:

$$p_{ik} = \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i<j>k}}^{n} \left(2p_{ij} + 2(1 - p_{kj}) - (1 - p_{ij}) - p_{kj} + 0.5\right)$$

$$\Rightarrow \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i<j>k}}^{n} \left(3p_{ij} - 3p_{kj} + 1.5\right)$$

$$\Rightarrow \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i<j>k}}^{n} 3\left(p_{ij} - p_{kj} + 0.5\right)$$

$$\Rightarrow \boxed{p_{ik} = \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i<j>k}}^{n} \left(p_{ij} - p_{kj} + 0.5\right) \quad \forall\, i < j > k} \tag{2.8.3}$$

Therefore, (2.7) can be rewritten as a system of linear of equations based on (2.8.1), (2.8.2) and (2.8.3) for all $i < k$ as follows:

$$p_{ik} = \frac{1}{n-2} \left[\sum_{\substack{j=1 \\ i<j<k}}^{n} \left(p_{ij} + p_{jk} - 0.5\right) + \sum_{\substack{j=1 \\ i>j<k}}^{n} \left(p_{jk} - p_{ji} + 0.5\right) + \sum_{\substack{j=1 \\ i<j>k}}^{n} \left(p_{ij} - p_{kj} + 0.5\right) \right] \quad \forall\, i < k \blacksquare$$

Thus, in the case of the decision-maker giving preference degrees only for $n-1$ values such that each pair of the alternatives are compared only once, (2.8) can be used to estimate the rest of the unknown values with perfect consistency.

Definition 2.7: Let $P = (p_{ik})_{n\times n}$ be a completed preference decision matrix from (2.8) and $P^e(p_{ik}^e)_{n\times n}$ be an estimated preference decision matrix from (2.7). Then the consistency degree (*DC*) between P and P^e is accepted if and only if $\boxed{CD(P,P^e) \geq \alpha}$ where $CD(P,P^e)$is the consistency degree between P and P^e, and α is the least accepted consistency that is defined by the expert(s).

Saaty (1980) suggested that α should be greater than or equal to 90%. In other words, the inconsistency degree should be less than or equal to 10%.

Thus, the consistency degree (similarity degree) between provided (or completed) matrix and the estimated one by additive consistency is:

$$CD(P,P^e) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} |p_{ik} - p_{ik}^e| \tag{2.9}$$

2.4.2. Goal programming model

Based on (2.8), multi-objective programming model is introduced. The model objectives are to find the errors between the missing preferences degrees and their estimations. Thus, the solution to the missing preferences degrees can be obtained by solving the following multi-objective programming model, where the objectives are to minimize the errors between the missing preferences degrees and their estimations subject to all missing preferences between 0 and 1.

$$(MOP)\ \min \varepsilon_{ik} = \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} \left| p_{ik} - \frac{1}{n-2} \left[\sum_{\substack{j=1 \\ i<j<k}}^{n} (p_{ij} + p_{jk} - 0.5) + \sum_{\substack{j=1 \\ i>j<k}}^{n} (p_{jk} - p_{ji} + 0.5) + \sum_{\substack{j=1 \\ i<j>k}}^{n} (p_{ij} - p_{kj} + 0.5) \right] \right|$$

$$s.t.\ p_{ik} \in [0,1] \quad i = 1,2, \dots n-1;\ \ k = 2,3, \dots n; i < k$$

$$p_{ki} = 1 - p_{ik} \quad i = 1,2, \dots n-1;\ \ k = 2,3, \dots n; i < k$$

The solution to the above multi-objective programming model is found by solving the following goal programming model.

$$(GP)\ \min z = \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} (d_{ik}^{+} + d_{ik}^{-})$$

$s.t.$

$$p_{ik} - \frac{1}{n-2} \left[\sum_{\substack{j=1 \\ i<j<k}}^{n} (p_{ij} + p_{jk} - 0.5) + \sum_{\substack{j=1 \\ i>j<k}}^{n} (p_{jk} - p_{ji} + 0.5) + \sum_{\substack{j=1 \\ i<j>k}}^{n} (p_{ij} - p_{kj} + 0.5) \right] - d_{ik}^{+} + d_{ik}^{-} = 0 \qquad i = 1,2, \dots n-1;\ \ k = 2,3, \dots n; i < k$$

$$p_{ik} \in [0,1] \qquad i = 1,2, \dots n-1;\ \ k = 2,3, \dots n; i < k$$

$p_{ki} = 1 - p_{ik} \quad i = 1, 2, \dots n-1; \;\; k = 2, 3, \dots n; i < k$

$d_{ik}^{+} * d_{ik}^{-} = 0 \qquad i = 1, 2, \dots n-1; \;\; k = 2, 3, \dots n; i < k$

$d_{ik}^{+}, d_{ik}^{-} \geq 0$

where d_{ik}^{+} is the positive deviation from the goal ε_{ik} and d_{ik}^{-} is the negative deviation from the goal ε_{ik}.

The model objectives are to minimize the deviations from the target of the goal subject to the same constraints as the multi-objective model. In addition to the errors between the missing preferences degrees and their estimations, the product of the positive and negative deviations from the goal should be equal to 0 and that all the decision variables are greater than or equal to 0.

2.4.3. Algorithm for group decision-making with incomplete fuzzy preference relations

In any group decision-making problem, there are usually two processes: A) consensus process and B) selection process. Since our focus is on completing the information, we will apply the selection process. The selection process consists of two phases, aggregation phase and exploitation phase as follows:

A. Aggregation phase

This phase constructs a collective preference relation by aggregating the provided preference relations by the decision-makers. The aggregation will be conducted by an importance-induced ordered weighted averaging (*I-IOWA*) operator.

Importance induced ordered weighted averaging (I-IOWA) operator

Importance-induced ordered weighted averaging *(I-IOWA)* is a modified aggregation operator of induced ordered weighted averaging (*IOWA)* proposed by Chiclana et al. (2007). This operator is based on transferring the original preference values into new values by using the decision-makers' importance degrees. First, we introduce the definition of (*IOWA)* as follows:

Definition 2.8 (Chiclana et al., 2007): An IOWA operator of dimension n is a function Φ_W: $(\mathbb{R} \times \mathbb{R})^n \rightarrow \mathbb{R}$, to which a set of weights or weighting vector is associated, $W = (w_1, w_2, \ldots, w_n)$, such that $w_i \in [0, 1]$ and $\sum_i^n w_i = 1$, to aggregate the set of second arguments of a list of *n* 2-tuples $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ according to the following expression:

$$P^c = \Phi_W(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle) = \sum_{i=1}^{n} w_i * p_{\sigma(i)}$$

being σ: $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ a permutation such that $\mu_{\sigma(l)} > \mu_{\sigma(l+1)}$ is the 2-tuple with $\sigma(i)$ the i^{th} highest value in the set $\{\mu_1, \mu_2, \ldots, \mu_n\}$.

The associated weights of the IOWA operator are obtained by

$$w_k = Q\left(\frac{S(k)}{S(n)}\right) - Q\left(\frac{S(k-1)}{S(n)}\right), \quad \forall k$$

where $k = \{1, 2, \ldots, n\}$, $S(k) = \sum_{l=1}^{k} \mu_{\sigma(l)}$, $S(n) = \sum_{l=1}^{n} \mu_l$, and σ is the permutation such that $\sigma(k)$ is the k^{th} largest value in the set $\{\mu_1, \mu_2, \ldots, \mu_n\}$.

The associated weights (w_k) are obtained by using a fuzzy majority concept and a fuzzy linguistic quantifier. There are several common fuzzy linguistic quantifiers such as all, most of, and as many as possible.

Definition 2.9 (Chiclana et al., 2007): If a set of experts, $E = \{e_1, \dots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \dots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \dots, P^m\}$, and each expert e_k has an importance degree, $\mu_I(e_k) \in [0, 1]$, then an I-IOWA operator of dimension n, Φ_W^I , is an IOWA operator whose set of order inducing values is the set of importance degrees.

B. Exploitation phase

By using the information of the collective preference relation, the alternatives are ranked from the best to the worst. The ranking of the alternatives will be obtained by using quantifier guided dominance degree (QGDD). This is used to quantify the dominance that one alternative has over the others, as follows:

$$QGDD_i = \Phi_W\left(p_{ij}^c,\ \forall j = 1, \dots, n\right) = \sum_{j=1}^{n} w_j * p_{\sigma(j)}^c$$

Where $\sigma(j)$ is the j^{th} highest value in the $\left(p_{ij}^c,\ \forall j = 1, \dots, n\right)$.

Thus, to solve any preference relation with incomplete information, different types of preference relation, i.e. linguistic or multiplicative preference relations, need to be transformed into a fuzzy additive preference relation. Then the following steps apply, see the flowchart in Figure 2.1:

1. If multiplicative or linguistic preference relations are provided, transfer them into fuzzy preference relation using (2.1) or (2.3) respectively.
2. Complete any incomplete fuzzy additive preference relation by using (2.8) or (GP) model.
3. Aggregate the decision-makers' preference relation into collective preference relation (P^C) by applying importance-induced ordered weighted averaging *(I-IOWA)* operator.
4. Rank alternatives based on quantifier guided dominance degree (QGDD) method.

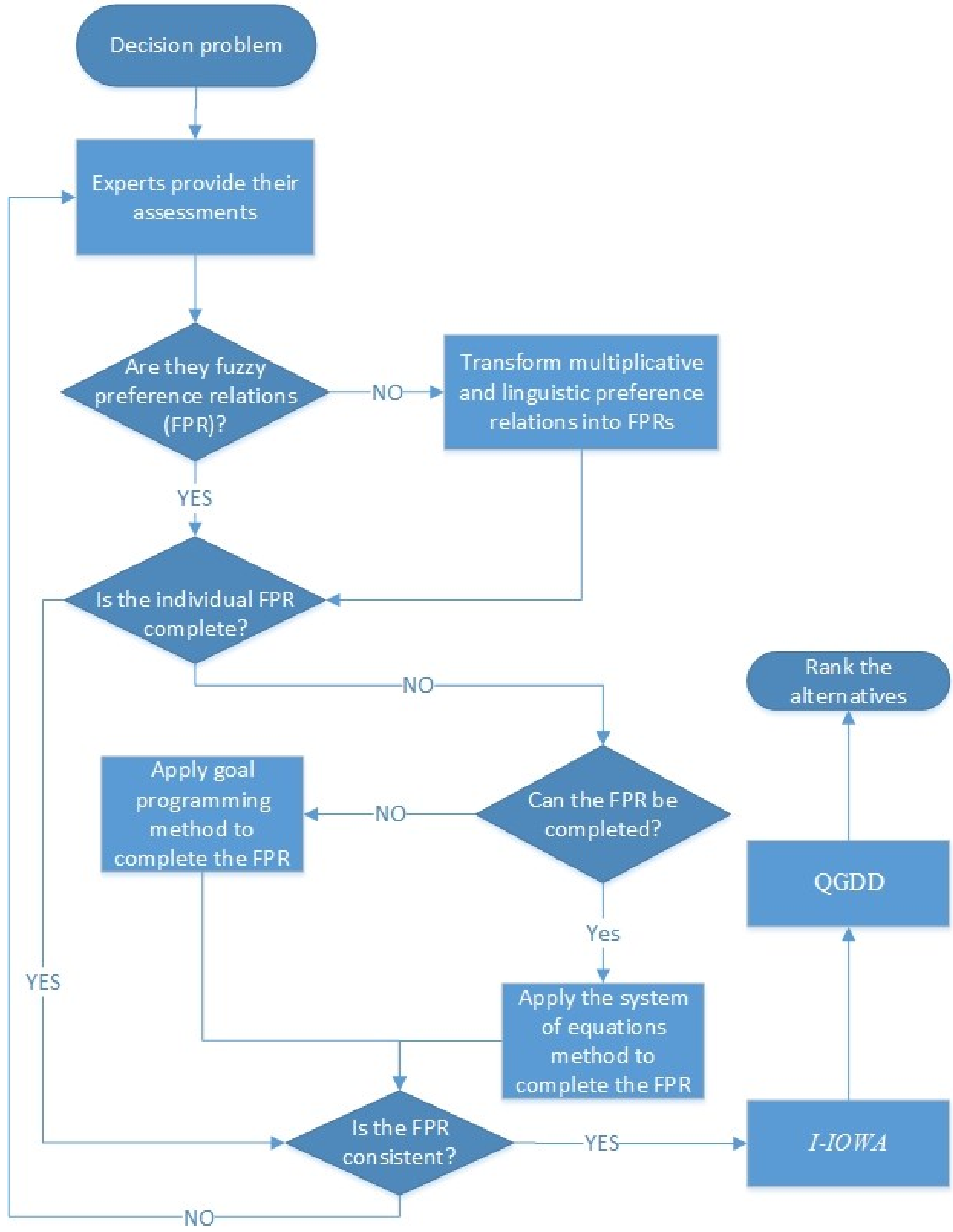

Figure 2.1: Solving incomplete preference relation flowchart

2.5. Numerical Examples

2.5.1. MADM under missing values

Suppose a decision-maker has to select one alternative from four pre-determined alternatives $A = \{A_1, A_2, A_3, A_4\}$ using four attributes $U = \{u_1, u_2, u_3, u_4\}$ with importance weights $\mu_U = (0.15, 0.2, 0.35, 0.3)$. The decision-maker provides the following assessments:

$$u_1: \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 0.5 & 0.9 & 1 & 0.8 \\ 0.1 & 0.5 & ? & ? \\ 0 & ? & 0.5 & 0.8 \\ 0.2 & ? & 0.2 & 0.5 \end{bmatrix} \end{array}, \quad u_2: \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 0.5 & 0.2 & 0.6 & 0.4 \\ 0.8 & 0.5 & ? & ? \\ 0.4 & ? & 0.5 & ? \\ 0.6 & ? & ? & 0.5 \end{bmatrix} \end{array}, \quad u_3: \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 0.5 & 0.6 & 0.6 & 0.3 \\ 0.4 & 0.5 & 0.4 & 0.3 \\ 0.4 & 0.6 & 0.5 & 0.3 \\ 0.7 & 0.7 & 0.7 & 0.5 \end{bmatrix} \end{array},$$

$$u_4: \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 0.5 & 0.3 & ? & 0.6 \\ 0.7 & 0.5 & ? & 0.5 \\ ? & ? & 0.5 & ? \\ 0.4 & 0.5 & ? & 0.5 \end{bmatrix} \end{array}$$

2.5.1.1. System of equations

By using the proposed system of equations method, the following estimation for preference relation under attributes (u_1) and (u_2) is obtained:

$$p_{23}^{u_1} = 0.5 * \left[\left(p_{13}^{u_1} - p_{12}^{u_1} + 0.5\right) + \left(p_{24}^{u_1} - p_{34}^{u_1} + 0.5\right)\right]$$

$$= 0.5 * \left[(1 - 0.9 + 0.5) + \left(p_{24}^{u_1} - 0.8 + 0.5\right)\right]$$

$$p_{24}^{u_1} = 0.5 * \left[\left(p_{14}^{u_1} - p_{12}^{u_1} - 0.5\right) + \left(p_{23}^{u_1} + p_{34}^{u_1} + 0.5\right)\right]$$

$$= 0.5 * \left[(0.8 - \ 0.9 - 0.5) + \left(p_{23}^{u_1} + 0.8 + 0.5\right)\right]$$

$$\Rightarrow p_{23}^{u_1} = 0.43, p_{24}^{u_1} = 0.57$$

$$\Rightarrow p_{32}^{u_1} = 1 - p_{23}^{u_1} = 0.57, p_{42}^{u_1} = 1 - p_{24}^{u_1} = 0.43$$

$$p_{23}^{u_2} = 0.5 * \left[\left(p_{13}^{u_2} - p_{12}^{u_2} + 0.5\right) + \left(p_{24}^{u_2} - p_{34}^{u_2} + 0.5\right)\right]$$

$$= 0.5 * \left[(0.6 - 0.2 + 0.5) + \left(p_{24}^{u_2} - p_{34}^{u_2} + 0.5\right)\right]$$

$$p_{24}^{u_2} = 0.5 * \left[\left(p_{14}^{u_2} - \ p_{12}^{u_2} - 0.5\right) + \left(p_{23}^{u_2} + p_{34}^{u_2} + 0.5\right)\right]$$

$$= 0.5 * \left[(0.4 - \ 0.2 - 0.5) + \left(p_{23}^{u_2} + p_{34}^{u_2} + 0.5\right)\right]$$

$$p_{34}^{u_2} = 0.5 * \left[\left(p_{14}^{u_2} - p_{13}^{u_2} + 0.5\right) + \left(p_{24}^{u_2} - p_{23}^{u_2} + 0.5\right)\right]$$

$$= 0.5 * \left[(0.4 - 0.6 + 0.5) + \left(p_{24}^{u_2} - p_{23}^{u_2} + 0.5\right)\right]$$

$$\Rightarrow p_{23}^{u_2} = 0.9, p_{24}^{u_2} = 0.7, p_{34}^{u_2} = 0.3$$

$$\Rightarrow p_{32}^{u_2} = 0.1, p_{42}^{u_2} = 0.3 \ p_{43}^{u_2} = 0.7$$

2.5.1.2. Missing value estimating using GP model

For preference relation under attribute u_4, the following GP model is constructed:

$$\min \mathrm{z} = d_{12}^{+} + d_{12}^{-} + d_{13}^{+} + d_{13}^{-} \ d_{14}^{+} + d_{14}^{-} + d_{23}^{+} + d_{23}^{-} + d_{24}^{+} + d_{24}^{-} + d_{34}^{+} + d_{34}^{-}$$

Subject to

$p_{12} = 0.3$

$p_{14} = 0.6$

$p_{24} = 0.5$

$p_{12} - 0.5\, p_{13} + 0.5\, p_{23} - 0.5\, p_{14} + 0.5\, p_{24} - d_{12}^{+} + d_{12}^{-} = 0.5$

$p_{13} - 0.5\, p_{12} - 0.5\, p_{23} - 0.5\, p_{14} + 0.5\, p_{34} - d_{13}^{+} + d_{13}^{-} = 0$

$p_{14} - 0.5 p_{12} - 0.5\, p_{24} - 0.5\, p_{13} - 0.5\, p_{34} - d_{14}^{+} + d_{14}^{-} = -0.5$

$p_{23} - 0.5\, p_{13} + 0.5 p_{12} - 0.5\, p_{24} + 0.5\, p_{34} - d_{23}^{+} + d_{23}^{-} = 0.5$

$p_{24} - 0.5\, p_{14} + 0.5\, p_{12} - 0.5 p_{23} - 0.5\, p_{34} - d_{24}^{+} + d_{24}^{-} = 0$

$p_{34} - 0.5\, p_{14} + 0.5\, p_{13} - 0.5\, p_{24} + 0.5\, p_{23} - d_{34}^{+} + d_{34}^{-} = 0.5$

$p_{ik} \in [0, 1] \qquad i = 1, 2, 3;\ \ k = 2, 3, \dots 4; i < k$

$p_{ki} = 1 - p_{ik} \quad i = 1, 2, 3;\ \ k = 2, 3, \dots 4; i < k$

$d_{ik}^{+} * d_{ik}^{-} = 0 \qquad i = 1, 2, 3;\ \ k = 2, 3, \dots 4; i < k$

$d_{ik}^{+}, d_{ik}^{-} \geq 0 \qquad i = 1, 2, 3;\ \ k = 2, 3, \dots 4; i < k$

This model derives $p_{13}^{u_4} = 0.9, p_{23}^{u_4} = 1, p_{34}^{u_4} = 0.1, p_{31}^{u_4} = 0.1, p_{32}^{u_4} = 0$ and $p_{43}^{u_4} = 0.9$.

Note that the same results are also generated by the GP model for these two matrices.

Thus, the complete preference relations are:

$$u_1: \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.9 & 1 & 0.8 \\ 0.1 & 0.5 & 0.43 & 0.57 \\ 0 & 0.57 & 0.5 & 0.8 \\ 0.2 & 0.43 & 0.2 & 0.5 \end{bmatrix} \end{array}, \quad u_2: \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.2 & 0.6 & 0.4 \\ 0.8 & 0.5 & 0.9 & 0.7 \\ 0.4 & 0.1 & 0.5 & 0.3 \\ 0.6 & 0.3 & 0.7 & 0.5 \end{bmatrix} \end{array}, \quad u_3: \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.6 & 0.6 & 0.3 \\ 0.4 & 0.5 & 0.4 & 0.3 \\ 0.4 & 0.6 & 0.5 & 0.3 \\ 0.7 & 0.7 & 0.7 & 0.5 \end{bmatrix} \end{array},$$

$$u_4: \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.3 & 0.9 & 0.6 \\ 0.7 & 0.5 & 1 & 0.5 \\ 0.1 & 0 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.9 & 0.5 \end{bmatrix} \end{array}$$

The consistency degrees, using (2.9), of these matrices are: $CD(P^{u_1}) = 83\%$, $CD(P^{u_2}) = 100\%$, $CD(P^{u_3}) = 93\%$ and $CD(P^{u_4}) = 90\%$. In this case, $CD(P^{u_1}) < 90\%$ the decision-maker might need to revise their judgments and update their preference relation. However, in this example we are going to pursue with this matrix as 83% is an acceptable consistency level.

2.5.1.3. *Aggregating and selecting processes*

The next step is to aggregate these attributes by using *I-IOWA* with the fuzzy linguistic quantifier *most of* defined by $Q(r) = r^{1/2}$, which gives the following weights vector $W = (0.59, 0.22, 0.11, 0.08)$. Thus, the collective fuzzy preference relation is:

$$P^U = \Phi_{most}(\langle 0.15, u_1\rangle, \langle 0.2, u_2\rangle, \langle 0.35, u_3\rangle, \langle 0.3, u_4\rangle)$$

$$P^U = 0.59 \cdot u_3 + 0.22 \cdot u_4 + 0.11 \cdot u_2 + 0.08 \cdot u_1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.58 & 0.63 & 0.35 \\ 0.42 & 0.5 & 0.46 & 0.37 \\ 0.37 & 0.54 & 0.5 & 0.34 \\ 0.65 & 0.63 & 0.66 & 0.5 \end{bmatrix} \end{array}$$

whose elements represent the preference of one alternative over another for *most of* the more important attributes.

Having (P^U), we apply the selection process with fuzzy quantifier *most of* defined by $Q(r) = r^{1/2}$, we get the associated weights vector $W = (0.5, 0.21, 0.16, 0.13)$. Then the quantifier guided dominance degrees (QGDD) of the alternatives are:

$$QGDD(A_1) = 0.5 \cdot 0.63 + 0.21 \cdot 0.58 + 0.16 \cdot 0.5 + 0.13 \cdot 0.35 = 0.563$$

$$QGDD(A_2) = 0.5 \cdot 0.5 + 0.21 \cdot 0.46 + 0.16 \cdot 0.42 + 0.13 \cdot 0.37 = 0.461$$

$$QGDD(A_3) = 0.5 \cdot 0.54 + 0.21 \cdot 0.5 + 0.16 \cdot 0.37 + 0.13 \cdot 0.34 = 0.479$$

$$QGDD(A_4) = 0.5 \cdot 0.66 + 0.21 \cdot 0.65 + 0.16 \cdot 0.63 + 0.13 \cdot 0.5 = 0.633$$

$$QGDD(A_4) > QGDD(A_1) > QGDD(A_3) > QGDD(A_2)$$

Thus, $A_4 \succ A_1 \succ A_3 \succ A_2$

2.5.2. Group decision-making with heterogeneous information

Suppose five decision-makers $E = \{e_1, e_2, e_3, e_4, e_5\}$ have to select one alternative from four pre-determined alternatives $A = \{A_1, A_2, A_3, A_4\}$. The decision-makers have the following importance degree vector $\mu_I(e_k) = \{0.15, 0.30, 0.1, 0.28, 0.17\}$. Suppose that the decision-maker e_1 provides an incomplete fuzzy additive preference relation while e_2 provides a complete one. Suppose that the decision-maker e_3 provides an incomplete linguistic preference relation while e_4 provides a complete one and the selected granularity of them are $T^{e_3} = 7$ and $T^{e_4} = 11$, respectively. Also, the decision-maker e_5 provides his assessments using incomplete multiplicative preference relation. The preference relations are as follows:

$$P^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.3 & ? & 0.6 \\ 0.7 & 0.5 & ? & 0.5 \\ ? & ? & 0.5 & ? \\ 0.4 & 0.5 & ? & 0.5 \end{bmatrix} \end{array} \qquad P^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.65 & 0.6 & 0.4 \\ 0.35 & 0.5 & 0.45 & 0.25 \\ 0.4 & 0.55 & 0.5 & 0.3 \\ 0.6 & 0.75 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

$$L^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} s_0 & s_1 & ? & s_1 \\ s_{-1} & s_0 & s_1 & ? \\ ? & s_{-2} & s_0 & s_0 \\ s_{-3} & ? & s_0 & s_0 \end{bmatrix} \end{array} \qquad L^4 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} s_0 & s_1 & s_3 & s_3 \\ s_{-1} & s_0 & s_2 & s_2 \\ s_{-3} & s_{-2} & s_0 & s_0 \\ s_{-3} & s_{-2} & s_0 & s_0 \end{bmatrix} \end{array}$$

$$A^5 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 1 & ? & 4 & 5 \\ ? & 1 & 3 & ? \\ \frac{1}{4} & \frac{1}{3} & 1 & ? \\ \frac{1}{5} & ? & ? & 1 \end{bmatrix} \end{array}$$

First, we transform $L^3 \rightarrow P^3$ and $L^4 \rightarrow P^4$ by using (2.3) with common granularity$(T = 9)$ by applying (2.2) and $A^5 \rightarrow P^5$ by using (2.1) into fuzzy additive preference relations, to get:

$$P^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.625 & ? & 0.625 \\ 0.375 & 0.5 & 0.625 & ? \\ ? & 0.375 & 0.5 & 0.5 \\ 0.375 & ? & 0.5 & 0.5 \end{bmatrix} \end{array} \qquad P^4 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.625 & 0.875 & 0.875 \\ 0.375 & 0.5 & 0.75 & 0.75 \\ 0.125 & 0.25 & 0.5 & 0.5 \\ 0.125 & 0.25 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

$$P^5 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & ? & 0.82 & 0.87 \\ ? & 0.5 & 0.75 & ? \\ 0.18 & 0.25 & 0.5 & ? \\ 0.13 & ? & ? & 0.5 \end{bmatrix} \end{array}$$

2.5.2.1. Missing value estimating using GP model

We apply the goal programming model to estimate the missing preference degrees for P^1, P^3 and P^5 to get the following complete preference relations:

$$P^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.3 & 0.9 & 0.6 \\ 0.7 & 0.5 & 1 & 0.5 \\ 0.1 & 0 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.9 & 0.5 \end{bmatrix} \end{array} \qquad P^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.625 & 0.6875 & 0.625 \\ 0.375 & 0.5 & 0.625 & 0.5625 \\ 0.3125 & 0.375 & 0.5 & 0.5 \\ 0.375 & 0.4375 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

$$P^5 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.57 & 0.82 & 0.87 \\ 0.43 & 0.5 & 0.75 & 0.8 \\ 0.18 & 0.25 & 0.5 & 0.55 \\ 0.13 & 0.2 & 0.45 & 0.5 \end{bmatrix} \end{array}$$

2.5.2.2. Aggregating and selecting processes

The next step is aggregating these preference relations by using *I-IOWA* with the fuzzy linguistic quantifier *most of* defined by $Q(r) = r^{1/2}$, which gives the following weights vector $W = (0.55, 0.21, 0.11, 0.08, 0.05)$. Thus, the collective fuzzy preference relation is:

$$P^C = \Phi_{most}(\langle 0.15, P^1\rangle, \langle 0.3, P^2\rangle, \langle 0.1, P^3\rangle, \langle 0.28, P^4\rangle, \langle 0.17, P^5\rangle)$$

$$P^C = 0.55 \cdot P^2 + 0.21 \cdot P^4 + 0.11 \cdot P^5 + 0.08 \cdot P^1 + 0.05 \cdot P^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.61 & 0.71 & 0.58 \\ 0.39 & 0.5 & 0.6 & 0.45 \\ 0.29 & 0.4 & 0.5 & 0.36 \\ 0.42 & 0.55 & 0.64 & 0.5 \end{bmatrix} \end{array}$$

whose elements represent the preference of one alternative over another for *most of* the more important decision-makers.

Having (P^C), we apply the selection process with fuzzy quantifier *most of* defined by $Q(r) = r^{1/2}$, to get the associated weights vector $W = (0.5, 0.21, 0.16, 0.13)$. Lastly, we calculate the quantifier guided dominance degrees (QGDD) for each of the alternatives:

$QGDD(A_1) = 0.640$, $QGDD(A_2) = 0.528$, $QGDD(A_3) = 0.430$, $QGDD(A_4) = 0.568$

$QGDD(A_1) > QGDD(A_4) > QGDD(A_2) > QGDD(A_3)$

Thus, $A_1 \succ A_4 \succ A_2 \succ A_3$

2.6. *Models Validation*

Example 2.1 (Herrera-Viedma et al., 2007b): Assume that a decision-maker provides the following incomplete fuzzy preference relation $P = (p_{ij})_{4\times4}$ for an attribute:

$$\begin{array}{cc} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 0.5 & 0.2 & 0.6 & 0.4 \\ 0.8 & 0.5 & ? & ? \\ 0.4 & ? & 0.5 & ? \\ 0.6 & ? & ? & 0.5 \end{bmatrix} \end{array}$$

In this example, there are 3 known preference degrees p_{12}, p_{13} and p_{14} in the upper triangular relation. To complete this matrix, 3 systems of equations were conducted to estimate the missing preference degrees, as follows:

$$p_{23} = 0.5 * [(p_{13} - p_{12} + 0.5) + (p_{24} - p_{34} + 0.5)]$$

$$p_{24} = 0.5 * [(p_{23} + p_{34} - 0.5) + (p_{14} - p_{12} + 0.5)]$$

$$p_{34} = 0.5 * [(p_{14} - p_{13} + 0.5) + (p_{24} - p_{23} + 0.5)]$$

Solving this system of equations results in $p_{23} = 0.9$, $p_{24} = 0.7$ and $p_{34} = 0.3$. These results match with Herrera-Viedma et al.'s (2007b) solution.

With regards to Example 2.1, the GP model generates same results as the system of equations method.

Sometimes, the provided preferences degrees are not consistent. Thus, trying to estimate the missing preference(s) will result in an inconsistent matrix or results that violate the additive consistency property.

Example 2.2 (Meng & Chen, 2015): Assume an expert provides the following incomplete preference relation:

$$\begin{array}{cc} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} 0.5 & 0.4 & 0.6 & ? \\ 0.6 & 0.5 & ? & 0.6 \\ 0.4 & ? & 0.5 & 0.6 \\ ? & 0.4 & 0.4 & 0.5 \end{bmatrix} \end{array}$$

By using the two proposed methods, we get: $p_{14} = p_{23} = 0.6$ and $p_{41} = p_{32} = 0.4$ with a consistency rate of 93.3%. These results are similar to Meng and Chen (2015).

Furthermore, the GP model can also estimate missing preferences of an ignorance situation. The ignorance situation rarely happens in a real life situation, but it could exist. Thus, having a model that could deal with this situation and produces a high successful consistency level is required. The next example represents this situation.

Example 2.3 (Chen et al., 2014; Meng & Chen, 2015): Assume there is an incomplete fuzzy preference relation $P = (p_{ij})_{4\times4}$ for $n = 4$ alternatives evaluated under one criterion, where the decision-maker has not provided any preference degrees for alternative 1 with respect to other alternatives, shown as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & ? & ? & ? \\ ? & 0.5 & 0.4 & 0.7 \\ ? & 0.6 & 0.5 & 0.8 \\ ? & 0.3 & 0.2 & 0.5 \end{bmatrix} \end{array}$$

By using the GP model, the following solution is generated:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 & 0 & 0.3 \\ 0.9 & 0.5 & 0.4 & 0.7 \\ 1 & 0.6 & 0.5 & 0.8 \\ 0.7 & 0.3 & 0.2 & 0.5 \end{bmatrix} \end{array}$$

Whereas, Chen et al.'s (2014) and Meng and Chen's (2015) solutions are as follows:

$$\text{Chen et al., (2014):} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.48 & 0.41 & 0.62 \\ 0.52 & 0.5 & 0.43 & 0.64 \\ 0.59 & 0.57 & 0.5 & 0.71 \\ 0.38 & 0.36 & 0.29 & 0.5 \end{bmatrix} \end{array}$$

$$\text{Meng and Chen (2015):} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.60 & 0.50 & 0.80 \\ 0.40 & 0.5 & 0.4 & 0.7 \\ 0.50 & 0.6 & 0.5 & 0.8 \\ 0.20 & 0.3 & 0.2 & 0.5 \end{bmatrix} \end{array}$$

The three solutions have a 100% consistency rate; however, Chen et al.'s (2014) solution is based on order consistency and their method modifies the decision-maker's preferences such that the output preference relation becomes 100% consistent. The solution of Meng and Chen (2015) does not modify the provided preferences degrees. However, Meng and Chen's (2015) model does not outperform our proposed model. For instance, consider the following example:

Example 2.4 (Meng & Chen, 2015): Assume there is an incomplete fuzzy preference relation $P = (p_{ij})_{4\times4}$ for $n = 4$ alternatives evaluated under one criterion, where the decision-maker has not provided any preference degrees for alternative 3 with respect to other alternatives, shown as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.3 & ? & 0.6 \\ 0.7 & 0.5 & ? & 0.5 \\ ? & ? & 0.5 & ? \\ 0.4 & 0.5 & ? & 0.5 \end{bmatrix} \end{array}$$

Meng and Chen's (2015) solution and our model solution are given as follows:

$$\text{Meng and Chen (2015):}\quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.3 & 0.43 & 0.6 \\ 0.7 & 0.5 & 0.52 & 0.5 \\ 0.57 & 0.48 & 0.5 & 0.44 \\ 0.4 & 0.5 & 0.56 & 0.5 \end{bmatrix} \end{array}$$

$$Our\ solution:\quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.3 & 0.9 & 0.6 \\ 0.7 & 0.5 & 1 & 0.5 \\ 0.1 & 0 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.9 & 0.5 \end{bmatrix} \end{array}$$

Meng and Chen's (2015) solution has an 85.6% consistency rate while our proposed model solution has a 90% consistency rate.

With regards to the ranking order of the alternatives, the three methods produce the same first ranking order by using the weighted arithmetic mean method; however, they differ slightly in the ranking order of the others. The proposed method tends to generate a high consistency level despite the nature of the problem under study. Tables 2.2 and 2.3 show the comparison between the proposed method and other methods for examples 2.3 and 2.4 respectively.

Table 2.2: Comparison of the three methods in example 2.3

	Matrix consistency	Ranking order
Meng and Chen (2015)	100%	A1=A3 A2 A4
Chen et al. (2014)	100%	A3 A2 A1 A4
Propose method	100%	A3 A2 A4 A1

Table 2.3: Comparison of two methods in example 2.4

	Matrix consistency	Ranking order
Meng and Chen (2015)	85.6%	A2 A3 A4 A1
Propose method	90%	A2 A1 A4 A3

2.7. Conclusions

In this chapter, two new methods to handle incomplete reciprocal fuzzy preference relations based on additive consistency have been proposed. The first is based on a system of equations. This method can deliver perfect consistency when $n-1$ non-leading diagonal preference values are given for each pair of alternatives. The second method is based on a goal programming concept. This method has the characteristics of the system of equations method in addition to estimating the information of ignorance alternative with a high consistency rate. Both methods are illustrated for multi-attributes/group decision-making problem and heterogeneous information cases. The proposed methods mainly focus on completing the upper triangular matrix (preference relation) by taking advantage of the additive transitivity properties.

Chapter 3:

New Consensus Measure for Group Decision-Making Based on Spearman's Correlation Coefficient for Reciprocal Fuzzy Preference Relations

3.1. Introduction

In group decision-making, reaching a level of agreement about the decision between the group members is important, even if each member has different goals or objectives about the alternatives. In fact, reaching consensus along with aggregation function and ranking method are considered as the main open-ended research problems in group decision-making (Ben-Arieh & Chen, 2006). Consensus is the main goal in group decision-making problems, since obtaining an acceptable solution by the group is important.

Therefore, it is very important to measure the consensus degree between the individuals of the group to find the degree of agreement among them. Consensus in group decision-making can be interpreted in three ways (Herrera-Viedma et al., 2014). It could mean full agreement or unanimous decision by the group members or reaching consensus by a moderator who facilitates the process of agreement, or it could mean attaining a consent in which some individuals might not completely agree but are willing to go with the opinion of the group.

Generally, two processes are employed in group decision-making: consensus and selection. The selection process could be applied without adopting a consensus process through applying the preference relations provided by the decision-makers (Roubens, 1997). However, this could lead to a solution that might not be accepted by some of the decision-makers since it does not reflect their preferences (Saint & Lawson, 1994; Butler & Rothstein, 2007). Therefore, they might reject the solution. Thus, it is important to reach a consensus before applying selection process (Kacprzyk et al., 1992).

Consensus in group decision-making involves aggregating individual preference relations into a collective or group preference relation. Typically, similarity/distance measures are used to measure the degree of similarities or consensus between the individuals and the individual and the collective preference relation. If the similarity is greater than or equal to a pre-defined threshold, then the collective preference relation is considered as consensus. Otherwise, the decision-makers with consensus degree below the threshold are asked to re-evaluate their preferences until the consensus degree reaches to the acceptable level of similarity. Generally, a consensus process is considered as an interactive and iterative process where decision-makers revise their preferences until they reach an acceptable level of agreement.

The remaining chapter is organized as follows: we start with some preliminary knowledge on preference relations in section 3.2. In section 3.3, we present a brief review on consensus from the literature. In section 3.4, we lay out the proposed consensus model including feedback mechanism. In section 3.5, we provide two numerical examples to show how the model works. Following that, in section 3.6 we validate the model performance in comparison to other existing models. Finally, in section 3.7 we present the conclusions.

3.2. Preliminary Knowledge

In this section, we provide brief knowledge on three types of preference relations, namely, additive fuzzy preference relations, multiplicative preference relation and linguistic preference relation.

Definition 3.1 (Urena et al., 2015): A preference relation R is a binary relation defined on the set X that is characterized by a function $\mu_p: X \times X \to D$, where D is the domain of representation of preference degrees provided by the decision-maker.

3.2.1. Fuzzy preference relation

Definition 3.2 (Xu, 2007): A fuzzy additive preference relation P on a finite set of alternatives X is represented by a matrix $P = \left(p_{ij}\right)_{n\times n} \subset X \times X$ with:

$p_{ij} \in [0,1],\ \ p_{ij} + p_{ji} = 1,\ \ p_{ii} = 0.5\ \ \forall i, j = 1, \dots, n.$

$$P = \left(p_{ij}\right)_{nxn} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \dots & A_n \end{array} \\ \begin{bmatrix} 0.5 & p_{12} & \dots & p_{1n} \\ p_{21} & 0.5 & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & 0.5 \end{bmatrix} \end{array}$$

when $p_{ij} > 0.5$ indicates that the expert prefers alternative x_i over alternative x_j; $p_{ij} < 0.5$ indicates that the expert prefers alternative x_j over alternative x_i; $p_{ij} = 0.5$ indicates that the expert is indifferent between x_i and x_j, thus, $p_{ii} = 0.5$.

Furthermore, the additive preference relation $P = (p_{ij})_{n\times n}$ is additive consistent if and only if the following additive transitivity is satisfied (Meng & Chen, 2015; Urena et al., 2015; Herrera-Viedma et al., 2007a; Tanino, 1984);

$$p_{ij} + p_{jk} = p_{ik} + 0.5 \qquad \forall ijk = 1,2,\dots,n.$$

3.2.2. Multiplicative preference relation

Definition 3.3 (Saaty, 1980): A multiplicative preference relation A on the set $X = \{x_1, x_2, \dots, x_n\}$ of alternatives is defined as a reciprocal matrix $A = (a_{ij})_{n\times n} \subset X \times X$ with the following conditions:

$$a_{ij} > 0,\ \ a_{ij}a_{ji} = 1,\ \ a_{ii} = 1,\ \ \forall ij = 1, 2, \dots, n.$$

where a_{ij} is interpreted as the ratio of the preference intensity of the alternative x_i to x_j.

There are several numerical scales for the multiplicative preference relation, however, the most popular one is the 1-9 Saaty scale. $a_{ij} = 1$ means that alternatives x_i and x_j are indifferent; $a_{ij} > 1$ implies that alternative x_i is preferred to x_j. As the ratio of intensity of (a_{ij}) increases, the stronger is the preference intensity of x_i over x_j. Thus, $a_{ij} = 9$ means that alternative x_i is absolutely preferred to x_j.

The multiplicative preference relation $A = (a_{ij})_{n\times n}$ is called consistent if the following multiplicative transitivity is satisfied (Saaty, 1980):

$$a_{ij} = a_{ik}a_{kj},\ \ a_{ii} = 1,\ \ \forall i, j = 1, 2, \dots, n.$$

Chiclana et al. (2001) proposed a transformation function to transfer a multiplicative preference relation, $A = (a_{ij})_{n\times n}$, into a fuzzy preference relation, $P = (p_{ij})_{n\times n}$, as follows:

$$p_{ij} = \frac{1}{2}\left(1 + \log_9 a_{ij}\right) \quad \forall i, j = 1, 2, \dots, n \tag{3.1}$$

$$a_{ij} = 9^{2p_{ij}-1} \quad \forall i, j = 1, 2, \dots, n \tag{3.1'}$$

Moreover, if $A = (a_{ij})_{n\times n}$ is a consistent multiplicative preference relation, then the transformed $P = (p_{ij})_{n\times n}$ is an additive consistent fuzzy preference relation.

3.2.3. Linguistic preference relation

Definition 3.4 (Xu, 2005b): A linguistic preference relation L on the set $X = \{x_1, x_2, \dots, x_n\}$ of alternatives is represented by a linguistic decision matrix $L = (l_{ij})_{n\times n} \subset X \times X$ with

$$l_{ij} \in \bar{S}, \;\; l_{ij} \oplus l_{ji} = s_0, \;\; l_{ii} = s_0, \;\; \forall ij = 1, 2, \dots, n.$$

where l_{ij} represents the preference degree of the alternative x_i over x_j. When $l_{ij} = s_0$, means that the decision-maker is indifferent between alternative x_i and x_j; $l_{ij} > s_0$ indicates that x_i is preferred over x_j.

Moreover, $L = (l_{ij})_{n\times n}$ is consistent when,

$$l_{ij} = l_{ik} \oplus l_{kj} \quad \forall i, j, k = 1, 2, \dots, n.$$

Let $S = \{s_\alpha | \alpha = -t, \dots, -1, 0, 1, \dots, t\}$ be a linguistic label set with odd cardinality. Then s_α represents a possible value for a linguistic label. In addition, t is a positive integer number and $s_{-t}$and s_t are the lower and upper limits of linguistic labels, respectively, while s_0 represents an assessment of "indifference."

The linguistic label set has following characteristics (Xu, 2004, 2005b):

1. The set is ordered: $s_\alpha > s_\beta$ if and only if $\alpha > \beta$
2. There is the negation operator: $\text{neg}(s_\alpha) = s_{-\alpha}$

In addition, Xu (2004, 2005b) extended the discrete linguistic label set to a continuous set $\bar{S} = \{s_\alpha | \alpha \in [-q, q]\}$ to preserve all the information. In this extension q is a large positive integer such that $(q > t)$. In general, if $s_\alpha \in S$ then this represents the original linguistic label, otherwise, s_α is only the virtual linguistic label which appears only in operations.

Let s_α,$s_\beta \in \bar{S}$ and $\mu, \mu_1, \mu_2 \in [0, 1]$. Xu (2004, 2005b) introduced some operational laws as follows:

1. $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$;
2. $s_\alpha \oplus s_\beta = s_\beta \oplus s_\alpha$;
3. $\mu s_\alpha = s_{\mu\alpha}$;
4. $(\mu_1 + \mu_2)s_\alpha = \mu_1 s_\alpha \oplus \mu_2 s_\alpha$;
5. $\mu(s_\alpha \oplus s_\beta) = \mu s_\alpha \oplus \mu s_\beta$;

In addition, for any $s \in \bar{S}$ then $I(s)$ represents the lower index of , e.g. if $s = s_\alpha \rightarrow I(s) = \alpha$ and it is called the gradation of s in $\bar{S}$. Likewise, we could get the inverse of $I(s)$: $I^{-1}(\alpha) = s_\alpha$.

An example of the linguistic label set is when $t = 3$, then $S =$ {s_{-3}= very low, s_{-2}= low, s_{-1}= slightly low, s_0=medium, s_1= slightly high, s_2= high, s_3= very high}.

Sometimes, depending on the decision problem, experts provide their assessments on the linguistic preference relation using different granularity (multi-granularity). Thus, these granularities need to be unified. Dong et al. (2009) provided following transformation function for unifying multi-granularity into a common granularity (T):

$$l_{ij} = \frac{T-1}{T'-1} l'_{ij} \tag{3.2}$$

where T is the intended granularity (normal granularity), $T'$is the granularity of $L'\left(l'_{ij}\right)_{n\times n}$.

Dong et al. (2009) and Xu (1999) propose a transformation function to transfer linguistic preference degree $\left(l_{ij}\right)$ into fuzzy preference degree based on linear scale function, as follows:

$$p_{ij} = 0.5 + \frac{I\left(l_{ij}\right)}{T-1} = 0.5 + \frac{I\left(l_{ij}\right)}{2t} \tag{3.3}$$

$$l_{ij} = I^{-1}\left(\left(p_{ij} - 0.5\right) \cdot (T-1) \right) \tag{3.3'}$$

where T is the granularity of S.

3.3. Literature Review

Consensus can be interpreted differently in the group decision-making field. It could mean full unanimous agreement, or it could be reaching consensus by a special individual or as a way for decision-making in multi-person settings. In general, consensus aims at attaining consent of the members of the group, which will lead to a decision that will benefit the entire group (Herrera-Viedma et al., 2014). Consensus is an iterative process that is composed of a number of rounds where decision-makers in each round are asked to revise their preferences to reach a consensus level by a facilitator or moderator. The facilitator's role is to gather all the information from the experts and apply some consensus measures to check if the group has reached a state of agreement or not. Therefore, the main step in the consensus process is to measure the consensus degrees of the experts.

Xu (2009) proposed an automatic approach for reaching consensus in multi-attribute group decision-making. His approach was based on numerical settings, where each individual constructs a decision matrix. Then these matrices are aggregated into one group decision matrix. The method calculates the similarity measure between each individual matrix and the group decision matrix to determine the degree of consensus. Moreover, he introduced a convergent iterative algorithm for individual matrices to reach the consensus. Sun and Ma (2015) proposed an approach for a consensus measure of linguistic preference relations. They used consensus measure based on the dominance degree to measure the consensus between group preference relation and individuals' preference relations. Zhang and Dong (2013) proposed an interactive consensus reaching process based on optimization consensus rules to increase consensus of individuals and minimize the number of adjustments of adjusted preference values. Guha and Chakraborty (2011) introduced an iterative fuzzy multi-attribute group decision-making technique to reach consensus by using a fuzzy similarity measure to measure consensus degree. In addition, their method considers the degrees of confidence of experts' opinions in the procedure. Herrera-Viedma et al. (2002), proposed a consensus model suitable for four different preference structures. Their model uses two consensus criteria: a consensus measure for measuring the degree of consensus between the experts and a proximity measure to measure the difference between the preferences of individuals and the group preference relation. Consensus and consistency measures have been used in the literature lately to guide the consensus process. For example, Herrera-Viedma et al. (2007a) proposed a consensus model based on consensus and consistency measures. They used two consensus measures: consensus degrees, to find the agreement of all experts, and proximity degrees, to find the agreement between the individuals and the group preference. Recently, Cabrerizo et al. (2010) proposed a consensus model for group decision-making in an unbalanced fuzzy linguistic setting using consistency and consensus measures. They used three different levels of consensus degrees: consensus degree on pairs of alternatives, consensus degree on alternatives, and consensus degree on the relation, in addition to proximity measures. Table 3.1 provides a summary of consensus measures.

Table 3.1: A brief literature of some consensus measures

Author(s)	Method based	Types of informatin
Sun and Ma (2015)	Dominance degree (similarity degree and deviation degree)	Linguistic preference relations
Zhang and Dong (2013)	Distance measure	Multi-attribute group decision making
Guha and Chakraborty (2011)	Similarity measure	Fuzzy multi-attribute group decision making
Cabrerizo et al., (2010)	Similarity measure and consistency measure	Unbalanced fuzzy linguistic
Xu (2009)	Distance measure	Multi-attribute group decision making
Herrera-Viedma et al., (2007)	Similarity measure and consistency measure	Fuzzy preference relation
Herrera-Viedma et al., (2002)	Dissimilarity measure	Preference ordering, fuzzy preference relation, multiplicative preference relation and utility function
Proposed Method	Rank correlation	Fuzzy preference relations, multiplicative and linguistic

3.3.1. Research gaps

Our goal is to measure the consensus degree among the experts differently by not relying directly on similarity/distance functions. As mentioned in number of papers such as Perez et al. (2016), Cabrerizo et al. (2015), and Herrera-Viedma et al. (2014), developing a new consensus measure is beneficial to overcome some drawbacks of similarity/distance functions. A study done by Chiclana et al. (2013) compares five different similarity/distance measures of consensus in group decision-making, namely, Manhattan, Euclidean, Cosine, Dice, and Jaccard. They found that different similarity/distance measures could generate significantly different results. Moreover, the chosen measure could affect the speed of convergence to consensus.

Furthermore, sometimes similarity/distance functions do not correctly reflect the agreement among the experts. For example, if a decision-maker provides his/her preferences on the alternatives by shifting (increasing/decreasing) other decision-maker preferences by 0.05, then the similarity/distance function will show that both decision-makers are not fully in agreement even though both prefer the same alternatives but with different intensities. Thus, we propose a new consensus measure based on Spearman's rank correlation coefficient. This new consensus measure calculates the rank correlation consensus between each pair of experts. Agreement on preference degrees ranks between pairs of decision-makers is measured. The new measure ranges from a value of 1 (perfect agreement among experts' rank preferences) to a value of -1 (total disagreement). The closer the rank correlation to 1, the more positive correlation between decision-makers' ranked preferences, which means that the preference degrees ranks are in the same direction. Conversely, the closer the rank correlation to -1, the more negative correlation between the decision-makers' ranked preferences, which means the preference degrees ranks are in the opposite direction. Moreover, based on this consensus measure we also propose a feedback mechanism to improve group consensus level for reciprocal preference relations.

3.4. A New Consensus Measure Based on Spearman's Rank Correlation Coefficient

We propose a new measurement of consensus based on Spearman's rank correlation. This new method utilizes the advantages of rank correlation coefficient among decision-makers' ranked preferences to measure the rank correlation consensus between each pair of experts. The new measurement is suitable for reciprocal preference relations.

The reason for choosing Spearman's correlation over Pearson's correlation is that we are interested in general monotonic relations rather than linear relations. Moreover, Spearman's correlation is a kind of qualitative measure whereas Pearson's correlation is a quantitative one. In addition, the proposed feedback mechanism relies on the decision-makers' input with respect to others and gives suggestions based on the rank of preference degrees rather than the valuations of other decision-makers. Thus, if the preference intensities among decision-makers vary, it does not matter as long as all the decision-makers have the same or almost the same rank preference degrees. This simply means that all prefer the same alternative but with different degrees. Therefore, trying to bring the preference intensities closer is not our main concern here. Making preference intensities almost the same could face resistance from decision-makers especially if their inputs are far from the original ones. Consequently, focusing on the preference ranks could make decision-makers more willing to accept changes, since this procedure, in general, asks them to rearrange their inputs in such a way that brings them closer to the rest of the decision-makers.

Definition 3.5: Let $V^k = \left(p_{12}^k, p_{13}^k, \dots, p_{1n}^k, p_{23}^k, \dots, p_{(n-1)n}^k\right)$ be a vector of preference degrees or intensities of decision-maker k of a reciprocal preference relation $P^k = \left(p_{ij}^k\right)_{n\times n}$, such that V^k represents the upper triangular relation with $\frac{n^2-n}{2}$ elements.

This vector represents the upper triangular relation of P^k provided by decision-maker k. We could also define another vector for the lower triangular relation in the same manner. However, that is not necessary since the reciprocal rule guarantees the same results if we apply any of the vectors. Then we rank the elements of the vector by using true rank scores to get ranked vector.

Definition 3.6: For every preference degree in V^k, there is a true rank score such that the preference degrees are given a score based on their intensity degree such that the largest preference degree is assigned a rank score of 1 and the smallest assigned a rank score of $\frac{n^2-n}{2}$,

$$RV^k = \left(\left(p_{12}^k, o_{12}^k\right), \left(p_{13}^k, o_{13}^k\right), \dots, \left(p_{(n-1)n}^k, o_{(n-1)n}^k\right)\right) = \left(o_{12}^k, o_{13}^k, \dots, o_{(n-1)n}^k\right)$$

where o_{ij}^{k} represents the true rank $\left(e.g: 1,2, \dots, \frac{n^2-n}{2}\right)$of p_{ij}^{k} among the elements of vector V^k.

Moreover, the ranked vector of the upper triangular relation (RV) and the ranked vector of the lower triangular relation (RV_*) of P have the following property:

$$o_{ij} + o_{ji} = \frac{n^2 - n + 2}{2} \tag{3.4}$$

where o_{ij} is the true rank on (RV) and o_{ji} is the true rank on (RV_*).

3.4.1. Spearman's rank correlation coefficient

The Spearman's rank correlation is a special case of Pearson's correlation coefficient (Chen & Popovich, 2002). Spearman's correlation measures the degree of monotonic relation between vector X and vector Y, while Pearson's correlation measures only the linear relationship (Hauke & Kossowski, 2011; Embrechts et al., 2002). Pearson's correlation treats real data in a quantitative way, whereas Spearman's correlation treats them to some extent in qualitative way (Hauke & Kossowski, 2011).

Spearman's correlation has the following properties (Embrechts et al., 2002):

1. Symmetry: $\rho(X,Y) = \rho(Y,X)$.
2. Normalization: $\rho(X,Y) \in [-1,1]$.
3. Comonotonic: $\rho(X,Y) = 1 \Leftrightarrow X, Y$.
4. Countermonotonic: $\rho(X,Y) = -1 \Leftrightarrow X, Y$.
5. For X strictly monotonic:$\rho(X,Y) = \begin{cases} \rho(X,Y) & X\ increasing \\ -\rho(X,Y) & X\ decreasing \end{cases}$

6. Uncorrelated (independent): $\rho(X,Y) = 0 \Leftrightarrow X, Y$.

The Spearman's correlation is a simplification of Pearson's correlation coefficient on the rank. The Pearson's correlation coefficient on the rank is defined as follows:

Definition 3.7: Given a sample of two n - dimensional vectors $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n)$, then for each vector, the variables are given true rank scores such that the largest variable is assigned a score of 1 and the smallest is assigned a score of $\frac{n^2-n}{2}$. Thus, having *RV* of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$,$\{(o_1^x, o_1^y), \dots, \left(o_{\frac{n^2-n}{2}}^x, o_{\frac{n^2-n}{2}}^y\right)\}$, the *Pearson correlation coefficient on the rank* is computed as

$$cor(\boldsymbol{x}, \boldsymbol{y}) = \rho^p = \frac{\sum_i^m (o_i^x - \overline{o}^x)(o_i^y - \overline{o}^y)}{\sqrt{\sum_i^m (o_i^x - \overline{o}^x)^2} \sqrt{\sum_i^m (o_i^y - \overline{o}^y)^2}} \tag{3.5}$$

where $\overline{o}^x = \frac{1}{m}\sum_i^m o_i^x$ and $\overline{o}^y = \frac{1}{m}\sum_i^m o_i^y$ are the arithmetic means of the true ranked scores, o_i^x and o_i^y are the true ranked of x_i and y_i, respectively.

In our case, *rank correlation consensus* (rcc) measure based on the *Pearson correlation coefficient on the rank of preference relation* is given by:

$$rcc(\boldsymbol{x}, \boldsymbol{y}) = \rho^p = \frac{\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^x - \overline{o}^x\right)\left(o_{ij}^y - \overline{o}^y\right)}{\sqrt{\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^x - \overline{o}^x\right)^2} \sqrt{\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^y - \overline{o}^y\right)^2}} \tag{3.5'}$$

where $\overline{o}^x = \frac{2}{n^2-n}\sum_i^{\frac{n^2-n}{2}} o_{ij}^x$ and $\overline{o}^y = \frac{2}{n^2-n}\sum_i^{\frac{n^2-n}{2}} o_{ij}^y$ are the arithmetic means of the true ranked scores, o_{ij}^x, and o_{ij}^y are the true ranked of RV^x and RV^y, respectively.

In the presence of tied ranks within a vector, the Pearson correlation coefficient formula is applied (Chen & Popovich, 2002). However, when there are no tied ranks, Spearman's rank correlation coefficient formula is used. To calculate Spearman's rank correlation coefficient, the *Pearson correlation coefficient* is simplified to (Chen & Popovich, 2002; Kendall & Gibbons, 1990):

$$cor(\boldsymbol{x}\,,\boldsymbol{y}\,) = \rho^s = 1 - \frac{6\sum_{i=1}^{m} d_i^2}{m^3 - m} \tag{3.6}$$

where d_i is the difference of the ranking of the two vectors and m is the number of elements or variables of the vector.

Proposition 3.1: To calculate the *rank correlation consensus* (rcc) measure based on Spearman's rank correlation coefficient with no tied ranks for (RV^k) and (RV^h), the following formula is equivalent to (3.6):

$$rcc^{kh} = \rho^s = 1 - \frac{48\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^k - o_{ij}^h\right)^2}{(n^3(1-n)^2 - 4n)(1-n)} \tag{3.7}$$

where $o_{ij}^k$and o_{ij}^h are the true ranked of V^k and V^h respectively and n is the number of alternatives.

Proof: from the ranked vector (RV^k) and (RV^h), the number of variables is $\frac{n^2-n}{2}$. Thus, $m = \frac{n^2-n}{2}$ by substituting this and $d_i = d_{ij} = o_{ij}^k - o_{ij}^h$ into (3.6) we get:

$$cor(RV^k, RV^h) = \rho = 1 - \frac{6\sum_{i=1}^{m} d_i^2}{m^3 - m}$$

$$= 1 - \frac{6\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^k - o_{ij}^h\right)^2}{\left(\frac{n^2-n}{2}\right)^3 - \left(\frac{n^2-n}{2}\right)} = 1 - \frac{6\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^k - o_{ij}^h\right)^2}{\frac{\left(n^2-n\right)^3}{8} - \frac{n^2-n}{2}}$$

$$= 1 - \frac{6\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^k - o_{ij}^h\right)^2}{\frac{\left(n^2-n\right)^3 - 4(n^2-n)}{8}} = 1 - \frac{48\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^k - o_{ij}^h\right)^2}{(n^2-n)^3 - 4(n^2-n)}$$

$$= 1 - \frac{48\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^k - o_{ij}^h\right)^2}{[(n^2-n)^2 - 4](n^2-n)} = 1 - \frac{48\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^k - o_{ij}^h\right)^2}{[n^2(n-1)^2 - 4](n-1)n}$$

$$= 1 - \frac{48\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^k - o_{ij}^h\right)^2}{[n^3(n-1)^2 - 4n](n-1)} \blacksquare$$

Note: The tied numbers are handled by midrank method. The midrank method is based on averaging the ranks that these tied numbers possess (Kendall & Gibbons, 1990). For instance, if the fifth and sixth numbers are tied, then each is assigned number $5\frac{1}{2}$, and if the third to the seventh are tied, each is assigned the number $\frac{(3+4+5+6+7)}{5} = 5$ and thus the next assigned rank number is 8 if it's not tied with others and so on.

Definition 3.8: The general rule of the midrank method is when there are z elements of tied ranks in V at l^{th} rank position, then the assigned number of each of these is $l + [(z-1) \cdot 0.5]$.

The *rank correlation consensus* (rcc) measure has a value range from -1 to 1. The value 1 means that both decision-makers' ranked preferences are the same (both are positively rank correlated). Whereas, -1 means that the two decision-makers' ranked preferences are opposite (both are negatively rank correlated). When (rcc) equals zero, there is no correlation between the two decision-makers' ranked preferences; both are independent. Ideally, the closer (rcc) to 1, the closer the decision-makers' ranked preferences are to the consensus and vice versa.

The rank correlation consensus measure could be mapped to the domain $[0,1]$ to have the rank correlation consensus degree $(rccd)$, as in the following definition:

Definition 3.9: The rank correlation consensus degree $(rccd)$ is a function $f: rcc \to [0,1]$, $f(rcc) = rccd = 0.5 \times (1 + rcc)$.

Thus, $rccd$ is interpreted as follows: when $rccd = 0.5$ then the two relations are rank independent where no rank correlation exists between the ranked preferences. For $rccd < 0.5$ the ranks of preference relations are negatively correlated (moving in the opposite direction), whereas for $rccd > 0.5$, the ranked preferences are moving in the same direction.

3.4.2. Rank similarity degree

From the ranked vector (RV) we could also find the rank similarity degree (rsd) between any pair of decision-makers' preference ranks. This degree shows how far the two preference rankings of a pair of decision-makers are from each other. The rank similarity degree value at 0 means there is no similarity at all and that one of the preference ranks is ranked first and the other is ranked last, while 1 means that both alternatives are ranked at the same position by both decision-makers.

Proposition 3.2: Given two ranked vector RV^k and RV^h , for a pair of decision-makers k and h $\in T = 1, \dots t$, the rank similarity degree (rsd_{ij}^{kh}) between each pair of preference ranks of the two decision-makers is given as follows:

$$rsd_{ij}^{kh} = 1 - \left| \frac{2\left(o_{ij}^k - o_{ij}^h\right)}{n(n-1) - 2} \right| \tag{3.8}$$

This formula has been derived by normalizing the difference of rank to the maximum possible difference on scores as follows: the maximum score a ranked vector (RV) could have is $\frac{n^2-n}{2}$ and the minimum is 1. Thus, the difference between the maximum score and the minimum is $\frac{n^2-n-2}{2}$. By dividing the difference of rank by this difference, we get $\frac{2\left(o_{ij}^k - o_{ij}^h\right)}{n(n-1)-2}$. Then, we take the absolute value of this normalization formula to prevent negative difference. Thus, the rank similarity degree is 1 minus the absolute value of the normalization formula.

3.4.3. Rank correlation consensus algorithm

The proposed consensus measure is based on rank correlation coefficient. This measure uses Spearman's rank correlation coefficient to measure rank correlation consensus coefficient for reciprocal preference relation $P^k = \left(p_{ij}^k\right)_{n \times n}$. To measure rank correlation consensus, the following steps are applied:

1. For each P^k, $k \in T = \{1, 2, \dots, t\}$ establish a preference vector $V^k = \left(p_{12}^k, p_{13}^k, \dots, p_{1n}^k, p_{23}^k, \dots, p_{(n-1)n}^k\right)$ from the preference relation (P^k) , which is provided by

the decision-maker k. This vector represents the preference intensities in the upper triangular relation of (P^k).

2. Give a true rank (e.g. $1,2,\dots,\frac{n^2-n}{2}$) for each element in V^k such that the largest preference degree is allotted score 1 and the smallest is allotted score $\frac{n^2-n}{2}$, then a ranked vector (RV^k) of $\frac{n^2-n}{2}$ elements can be established as follows:

   $$RV^k = \left(o_{12}^k, o_{13}^k, \dots, o_{(n-1)n}^k\right)$$

   where o_{ij}^k represents the true rank of p_{ij}^k with respect to the preference degrees of vector V^k.

3. For every pair of decision-makers, k and $h \in T$, calculate rank similarity degree $\left(rsd_{ij}^{kh}\right)$ on the vectors $RV^k$and RV^h:

   $$rsd_{ij}^{kh} = 1 - \left|\frac{2\left(o_{ij}^k - o_{ij}^h\right)}{n(n-1)-2}\right|$$

4. For every pair of decision-makers k and $h \in T$, calculate the rank correlation consensus coefficient $(rcc^{kh} = rcc^{hk})$ using (3.7) if there is no tied rank or (3.5′) if the ties exist with adopting the midrank method. The rank correlation consensus, $rcc \in [-1, 1]$, where -1 means strong negative rank correlation between decision-maker k and h, 0 no rank correlation, and 1 strong positive rank correlation. Ideally, the closer rcc is to 1 the better it is. This can be transformed into rank correlation consensus degree by:

$$rccd^{kh} = \frac{1}{2}(1 + rcc^{kh}) \in [0,1] \tag{3.9}$$

5. Calculate the rank correlation consensus for each decision-maker $e \in T$

$$rcc^{k} = \frac{\sum_{h=1,h\neq k}^{t} rcc^{kh}}{t-1} \tag{3.10}$$

6. Calculate experts' rank correlation consensus

$$rcc^{T} = \frac{\sum_{k=1}^{t} rcc^{k}}{t} \tag{3.11}$$

7. Calculate collective relation rank correlation consensus

$$rcc^{c} = \frac{\sum_{k=1}^{t} rcc^{ck}}{t} \tag{3.12}$$

By applying the rank correlation consensus measure, five types of measures are obtained as shown in Figure 3.1:

1. Between individuals' rank correlation consensus rcc^{kh}, which shows the similarity degree of the preferences ranks between decision-maker k and h.
2. Individual rank correlation consensus rcc^{k}, which represents the similarity degree of the preferences ranks of decision-maker k to other decision-makers.
3. Among decision-makers' rank correlation consensus rcc^{T}, which represents the similarity degree of the preferences ranks among them.

4. Between individuals and collective rank correlation consensus rcc^{ck}, which shows the degree of similarity of the preferences ranks between the collective relation c and the decision-maker k.

5. Collective relation' rank correlation consensus rcc^{c}, which represents the similarity degree of the preferences ranks between individuals and collective preference relation.

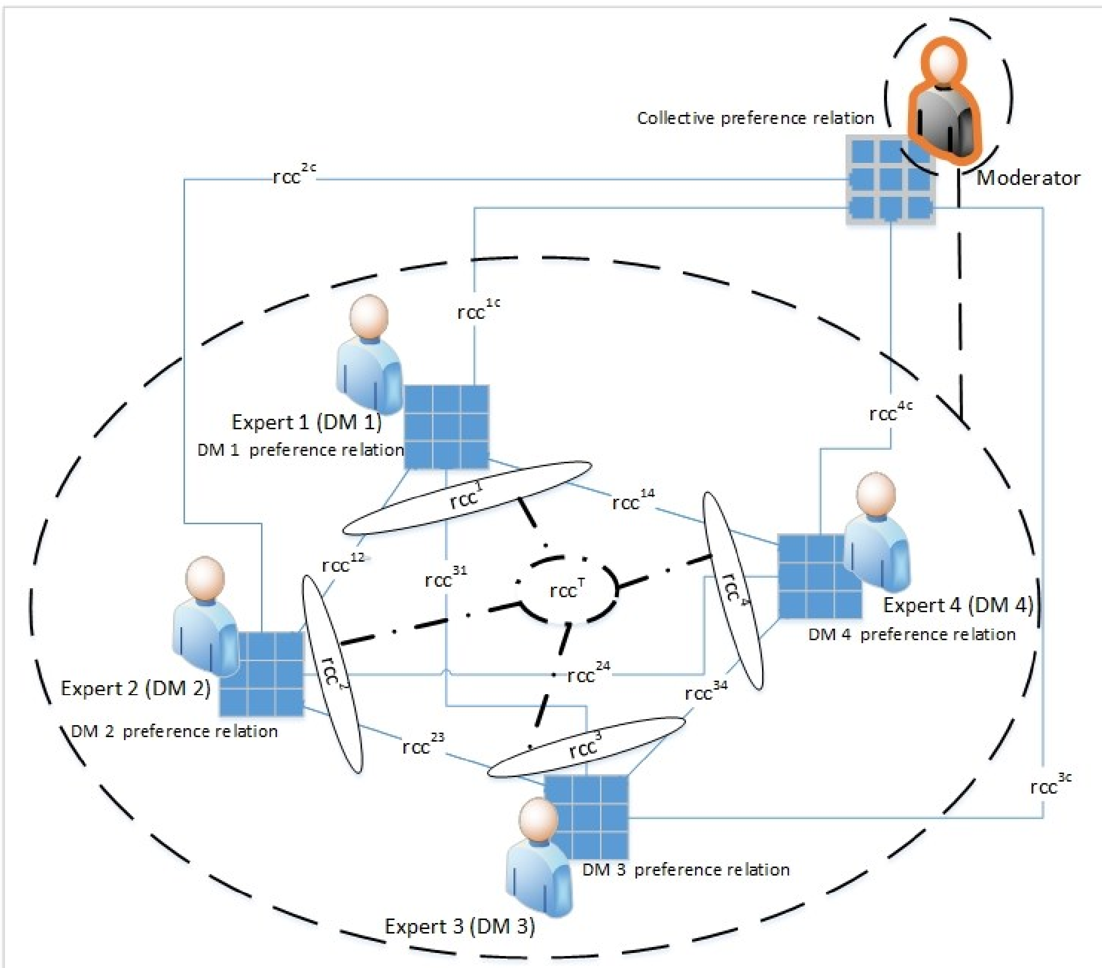

Figure 3.1: Rank correlation consensus types and relations

Note: Rank correlation consensus degree is not equivalent to consensus degree obtained by similarity/distance functions. In general, rank correlation consensus should be the one to rely on to attain the consensus level.

3.4.4. Feedback mechanism

The purpose of this mechanism is to help and guide decision-makers to improve their consensus level. The proposed feedback mechanism uses consensus results to help the experts with low consensus to improve their evaluations and thus their consensus level with regards to other decision-makers. This method uses other decision-makers' assessments, which usually have the best consensus in the group, to generate suggestions to the individuals who have fewer contributions to the consensus state. This feedback mechanism relies heavily on the rank similarity degrees between the experts.

Theorem 3.1: Let rcc^{kh} be the rank correlation consensus between preference relation of decision-maker k and preference relation of decision-maker h, then increasing rsd_{ij}^{kh} leads to increase rcc^{kh}.

Proof: From $rsd_{ij}^{kh} = 1 - \left|\frac{2\left(o_{ij}^k - o_{ij}^h\right)}{n(n-1)-2}\right|$, we get $\left|o_{ij}^k - o_{ij}^h\right| = \left(1 - rsd_{ij}^{kh}\right) \cdot \frac{(n(n-1)-2)}{2}$.

When no ties in ranks,

$$rcc^{kh} = 1 - \frac{48 \sum_{i=1}^{n-1} \sum_{j=2,i<j}^{n} \left(o_{ij}^k - o_{ij}^h\right)^2}{[n^3(n-1)^2 - 4n](n-1)}$$

$$= 1 - \frac{48\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(\left(1-rsd_{ij}^{kh}\right)\cdot\frac{(n(n-1)-2)}{2}\right)^2}{[n^3(n-1)^2-4n](n-1)} = 1 - \frac{12[(n(n-1)-2)]^2}{[n^3(n-1)^2-4n](n-1)}\cdot\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(1-rsd_{ij}^{kh}\right)^2$$

When there are ties in ranks,

$$rcc^{kh} < 1 - \frac{12[(n(n-1)-2)]^2}{[n^3(n-1)^2-4n](n-1)}\cdot\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(1-rsd_{ij}^{kh}\right)^2$$

Therefore,

$$rcc^{kh} \leq 1 - \frac{12[(n(n-1)-2)]^2}{[n^3(n-1)^2-4n](n-1)}\cdot\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(1-rsd_{ij}^{kh}\right)^2 \blacksquare$$

Moreover, the rank similarity degree has the following properties:

1. $rsd_{ij}^{kh} = rsd_{ij}^{hk}$

Proof: It is obvious from the absolute value in (3.8).

2. $rsd_{ij}^{kh} = rsd_{ji}^{kh}$

Proof: By substituting $o_{ji} = \frac{n^2-n+2}{2} - o_{ij}$, from (3.4), into $rsd_{ji}^{kh} = 1 - \left|\frac{2\left(o_{ji}^{k}-o_{ji}^{h}\right)}{n(n-1)-2}\right|$, we get

$$rsd_{ji}^{kh} = 1 - \left|\frac{2\left(o_{ji}^{k}-o_{ji}^{h}\right)}{n(n-1)-2}\right| = 1 - \left|\frac{2\left(\frac{n^2-n+2}{2} - o_{ij}^{k} - \frac{n^2-n+2}{2}+o_{ij}^{h}\right)}{n(n-1)-2}\right| = 1 - \left|\frac{2\left(o_{ij}^{h}- o_{ij}^{k}\right)}{n(n-1)-2}\right|$$

$$= rsd_{ij}^{hk} = rsd_{ij}^{kh} \blacksquare$$

The feedback mechanism can be conducted in two ways depending on the main concern: A) if the rank correlation consensus between decision-makers is the priority, or B) if the rank correlation consensus of the collective relation is important. Figure 3.2 shows the flowchart of the consensus process.

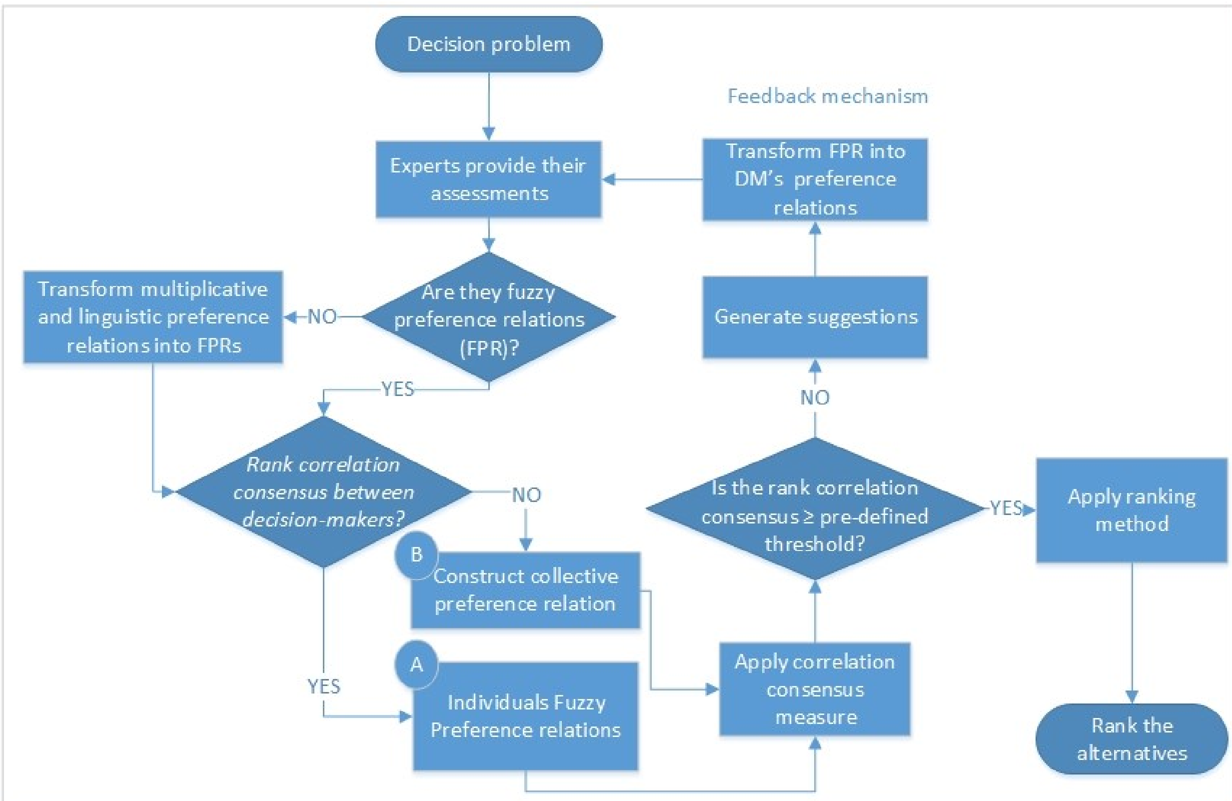

Figure 3.2: Consensus process

A. Feedback mechanism for rank correlation consensus between decision-makers:

1. Select the decision-maker who has the lowest rank correlation consensus (rcc^k) for him/her to review their judgments.

2. Once the decision-maker is selected (k), look at his/her rank similarities degrees $\left(rsd_{ij}^{kh}\right)$ with respect to other decision-makers. Find the lowest $\sum_{\substack{h=1 \\ h\neq k}}^{t} rsd_{ij}^{kh}$ to identify the element

of preference vector to modify and then find which decision-maker has the lowest rank similarity degree with respect to the identified vector element, $\min_{\forall h\in T}\{rsd_{ij}^{kh}\}$. For instance, if we have following rsd:

$rsd_{13}^{31} = 0.4$, $rsd_{13}^{32} = 0.2$, $rsd_{13}^{34} = 0.6$, and

$rsd_{12}^{31} = 0.9$, $rsd_{12}^{31} = 0.6$, $rsd_{12}^{34} = 0.3$,

then the decision-maker modifies p_{13}^{3} since $\sum_{\substack{h=1\\h\neq k}}^{4} rsd_{13}^{3h} = 1.2 < \sum_{\substack{h=1\\h\neq k}}^{4} rsd_{12}^{3h} = 1.8$, with respect to o_{13}^{2} position since $\min_{\forall h\in T}\{rsd_{13}^{kh}\} = rsd_{13}^{32}$.

3. Once the preference degree that needs to be changed is known $\left(p_{ij}^{k}\right)$, make the modification based on the position of the ranked element of decision-maker h who has the lowest rank similarity degree as follows:

   - If $o_{ij}^{k} > o_{ij}^{h} \Rightarrow p_{ij}^{k} \in [p^{k}\{o_{ij}^{h}\}, p^{k}\{o_{ij}^{h} - 1\}]$;
   - If $o_{ij}^{k} < o_{ij}^{h} \Rightarrow p_{ij}^{k} \in [p^{k}\{o_{ij}^{h} + 1\}, p^{k}\{o_{ij}^{h}\}]$

   where $p^{k}\{o_{ij}^{h}\}$ is the preference degree value of k at the rank position o_{ij}^{h}. Moreover,

   - If $p^{k}\{o_{ij}^{h} - 1\} = p^{k}\{0\} \Rightarrow p^{k} = 1$
   - If $p^{k}\{o_{ij}^{h} + 1\} = p^{k}\left\{\frac{n^2-n}{2} + 1\right\} \Rightarrow p^{k} = 0$
   - If $\{o_{ij}^{h}\}$ does not exist exactly in k then find where it lies such that p_{ij}^{k} should fall in the rank between $\{o_{ij}^{h}\}$and $\{o_{ij}^{h} + 1\}/\{o_{ij}^{h} - 1\}$. For example, if $p^{k}\{o_{ij}^{h}\} = p^{k}\{1.5\}$

but there is no rank position at 1.5 in k then we could approximate it to $p^k\{1\}$ or $p^k\{2\}$ depending on the other rank position $\{o_{ij}^h + 1\}/\{o_{ij}^h - 1\}$.

4. After the adjustment, recalculate the rank correlations consensus and repeat steps 1-3.
5. The process is finished when $rcc^T \geq \alpha$ and/or $rcc^{kh} \geq \beta$, where α and β are the agreeable consensus level between the experts and among the pair of experts respectively.

This feedback mechanism is built to improve rank consensus level of the experts without relying on the collective preference relation. Once the consensus level is attained then collective preference relation could be constructed.

B. *Feedback mechanism for rank correlation consensus on the collective preference relation*:

We could apply the feedback mechanism between the experts and the collective preference relation by modifying step 2 and 5 into:

Step 2'. Identify the lowest rank similarity degree $\left(rsd_{ij}^{ck}\right)$, that has the farthest rank between the rank position o_{ij}^c and o_{ij}^k.

Step 5'. The process is finished when $rcc^c \geq \alpha$ and/or $rcc^{ck} \geq \beta$, where α and β are the agreeable consensus level of the collective relation and between the experts and the collective relation respectively.

Thus, in this case we only need to know which preference degree to modify, since k is known from the rank correlation consensus $\left(\min_{\forall k \in T}\{rcc^{ck}\}\right)$ between RV^k and ranked collective vector RV^c.

3.5. Numerical Examples

3.5.1. Group decision-making example under homogeneous information

Suppose four decision-makers provide their assessments on four alternatives using fuzzy preference relations as follows:

$$P^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.50 & 0.38 & 0.20 & 0.28 \\ 0.62 & 0.50 & 0.32 & 0.40 \\ 0.80 & 0.68 & 0.50 & 0.58 \\ 0.72 & 0.6 & 0.42 & 0.50 \end{bmatrix} \end{array} \qquad P^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.50 & 0.38 & 0.25 & 0.33 \\ 0.62 & 0.50 & 0.37 & 0.45 \\ 0.75 & 0.63 & 0.50 & 0.58 \\ 0.67 & 0.55 & 0.42 & 0.50 \end{bmatrix} \end{array}$$

$$P^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.50 & 0.75 & 0.55 & 0.41 \\ 0.25 & 0.50 & 0.30 & 0.16 \\ 0.45 & 0.70 & 0.50 & 0.36 \\ 0.59 & 0.84 & 0.64 & 0.50 \end{bmatrix} \end{array} \qquad P^4 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.50 & 0.40 & 0.30 & 0.60 \\ 0.60 & 0.50 & 0.40 & 0.70 \\ 0.70 & 0.60 & 0.50 & 0.80 \\ 0.40 & 0.30 & 0.20 & 0.50 \end{bmatrix} \end{array}$$

A. Consensus measure

To find the rank correlation consensus among these experts (preferences), we apply the rank correlation consensus coefficient measure as mentioned above without relying on the collective relation.

1. $V^1 = (0.38, 0.20, 0.28, 0.32, 0.40, 0.58)$, $V^2 = (0.38, 0.25, 0.33, 0.37, 0.45, 0.58)$

   $V^3 = (0.75, 0.55, 0.41, 0.30, 0.16, 0.36)$, $V^4 = (0.40, 0.30, 0.60, 0.40, 0.70, 0.80)$
2. $RV^1 = (3, 6, 5, 4, 2, 1)$, $RV^2 = (3, 6, 5, 4, 2, 1)$

   $RV^3 = (1, 2, 3, 5, 6, 4)$, $RV^4 = (4.5, 6, 3, 4.5, 2, 1)$
3. The rank similarity degrees are summarized in Table 3.2.

Table 3.2: Rank similarity degrees between decision-makers

	Rank Similarity degrees					
	p^1/p^2	p^1/p^3	p^1/p^4	p^2/p^3	p^2/p^4	p^3/p^4
rsd_{12}	1	0.6	0.7	0.6	0.7	0.3
rsd_{13}	1	0.2	1	0.2	1	0.2
rsd_{14}	1	0.6	0.6	0.6	0.6	1
rsd_{23}	1	0.8	0.9	0.8	0.9	0.9
rsd_{24}	1	0.2	1	0.2	1	0.2
rsd_{34}	1	0.4	1	0.4	1	0.4

1 means that both decision-makers rank the associated preference degree in the same position. If the rank similarity degree is less than 1, more differentiation exists on the rank between both experts.

4. From step 2, we see that all of the decision-makers have no tied ranks except for decision-maker 4. Thus, we apply (3.7) to find the rank correlation consensus coefficient for rcc^{12}, rcc^{13} and rcc^{23}. Eq $(3.5')$ is used to find the rank correlation coefficient for rcc^{14}, rcc^{24} and rcc^{34}. For example,

$$rcc^{13} = 1 - \frac{48\sum_{i=1}^{n-1}\sum_{j=2,i<j}^{n}\left(o_{ij}^{k}-o_{ij}^{h}\right)^2}{(n^3(1-n)^2-4n)(1-n)} = 1 - \frac{48\sum_{i=1}^{3}\sum_{j=2,i<j}^{4}\left(o_{ij}^{1}-o_{ij}^{3}\right)^2}{(4^3(1-4)^2-4\times4)(1-4)}$$

$$= 1 - \frac{48\sum_{i=1}^{3}\sum_{j=2,i<j}^{4}\left(o_{ij}^{1}-o_{ij}^{3}\right)^2}{1680}$$

$$= 1 - \frac{48\left[(3-1)^2+(6-2)^2+(5-3)^2+(4-5)^2+(2-6)^2+(1-4)^2\right]}{1680}$$

$$= 1 - \frac{48[50]}{1680} = -0.429$$

Table 3.3 shows all the rcc^{kh}, rcc^{k} and rcc^{T}:

Table 3.3: Rank correlation consensus between DMs

	rcc^{kh}			
	e_1	e_2	e_3	e_4
e_1	1	1	-0.429	0.812
e_2	1	1	-0.429	0.812
e_3	-0.429	-0.429	1	-0.551
e_4	0.812	0.812	-0.551	1
rcc^k	0.461	0.461	-0.469	0.358
rcc^T	0.203			

From Table 3.3, we see that all the decision-makers have good positive rank correlation with each other except for decision-maker 3, which has the lowest rank correlation consensus ($rcc^1 = rcc^2 > rcc^4 > rcc^3$). Moreover, all the rank correlations consensus with decision-maker 3 are negative. Thus, to improve the consensus level among decision-makers, decision-maker 3 has to revise his/her assessments.

B. Feedback mechanism

1. We see that decision-maker 3 has the lowest $rcc^3 = -0.469$. Thus he/she is selected to revise his/her assessments.
2. Table 3.4 shows all rank similarity degrees of all decision-makers with respect to decision-maker 3:

Table 3.4: Rank similarity degrees between DM 3 and other DMs

	Rank Similarity degrees			
	p^1/p^3	p^2/p^3	p^3/p^4	Sum
rsd_{12}	0.6	0.6	0.3	1.5
rsd_{13}	0.2	0.2	0.2	0.6
rsd_{14}	0.6	0.6	1	2.2
rsd_{23}	0.8	0.8	0.9	2.5
rsd_{24}	0.2	0.2	0.2	0.6
rsd_{34}	0.4	0.4	0.4	1.2

Notice that $\min_{\forall h \in t}\left\{\sum_{\substack{h=1 \\ h \neq 3}}^{4} rsd_{ij}^{3h}\right\} = rsd_{13}^{3h} = rsd_{24}^{3h} = 0.6$, thus we pick any of them. For $\min_{\forall h \in t}\{rsd_{24}^{3h}\} \Longrightarrow rsd_{24}^{31} = rsd_{24}^{32} = rsd_{24}^{34} = 0.2$ for all decision-makers. That means the rank position for the decision-makers are the same, $o_{24}^1 = o_{24}^2 = o_{24}^4 = 2$.

3. Thus p_{24}^3 is the one to modify, $p_{24}^3\{o_{24}^3\} = p_{24}^3\{6\} \Rightarrow p_{24}^3\{2\}$. In this case: $o_{24}^3 > o_{24}^1$

   $\Longrightarrow p_{24}^3 \in [p^3\{2\}, p^3\{1\}]$

   $\Longrightarrow\ p_{24}^3 \in [0.55, 0.75]$

   Suppose that the decision-maker is willing to change his/her assessment for this preference degree from $p_{24}^3 = 0.16$ to $p_{24}^{3'} = 0.65$.

   This changes the result on new rcc^{kh}: $rcc^{1\prime} = 0.613$, $rcc^{2\prime} = 0.613$, $rcc^{3\prime} = -0.039$, $rcc^{4\prime} = 0.483$ and $rcc^{T\prime} = 0.418$.

4. Again $rcc^{3\prime}$ is the lowest one. The same steps are repeated. This time p_{34}^3 is the one to modify $p_{34}^3\{5\} \Rightarrow p_{34}^3\{1\}$. In this case; $o_{34}^3 > o_{34}^1$

   $\Longrightarrow p_{34}^3 \in [p^3\{1\}, p^3\{0\}]$

   $\Longrightarrow\ p_{34}^3 \in [0.75, 1]$

Suppose that the decision-maker decides to change it to $p_{34}^{3} = 0.8$. This results on a new rcc^{kh}: $rcc^{1'} = 0.842$, $rcc^{2'} = 0.842$, $rcc^{3'} = 0.640$, $rcc^{4'} = 0.705$ and $rcc^{T'} = 0.757$.

5. If these results satisfy the condition of the consensus level, then stop and construct the collective relation. However, if the consensus level is at 0.8, then carry on the process. Still decision-maker 3 has the lowest rank correlation consensus. This time $\min_{\forall h \in t} \left\{ \sum_{\substack{h=1 \\ h \neq 3}}^{4} rsd_{ij}^{3h} \right\} = rsd_{13}^{3h} = 1.8$, and $\min_{\forall h \in t} \{rsd_{13}^{3h}\} \Longrightarrow rsd_{13}^{31} = rsd_{13}^{32} = rsd_{13}^{34} = 0.6$. Thus $p_{13}^{3}$is the one to modify $p_{13}^{3}\{4\} \Rightarrow p_{13}^{3}\{6\}$. In this case; $o_{13}^{3} < o_{13}^{1}$

$\Longrightarrow p_{13}^{3} \in [p^{3}\{7\}, p^{3}\{6\}]$

$\Longrightarrow p_{13}^{3} \in [0, 0.3]$

Suppose that the decision-maker decides to change it to $p_{13}^{3} = 0.29$. This results in a new rcc^{kh}: $rcc^{1'} = 0.899$, $rcc^{2'} = 0.899$, $rcc^{3'} = 0.842$, $rcc^{4'} = 0.792$ and $rcc^{T'} = 0.858$, which attains the consensus level. Therefore the feedback mechanism is finished and the new $P^{3*}$is:

$$
P^{3^*} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}
\begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\
\begin{bmatrix} 0.50 & 0.75 & 0.29 & 0.41 \\ 0.25 & 0.50 & 0.30 & 0.65 \\ 0.71 & 0.70 & 0.50 & 0.8 \\ 0.59 & 0.35 & 0.2 & 0.50 \end{bmatrix} \end{array}
$$

The new rank correlation consensus for decision-maker 3 with the other decision-makers, each decision-maker's rank correlation consensus and experts' rank correlation consensus are shown in Table 3.5.

Table 3.5: The new rank correlation consensus

	rcc^{kh}			
	e_1	e_2	e_3*	e_4
e_1	1	1	0.886	0.812
e_2	1	1	0.886	0.812
e_3*	0.886	0.886	1	0.754
e_4	0.812	0.812	0.754	1
rcc^k	0.899	0.899	0.842	0.792
rcc^T	0.858			

Notice the improvement in the rank correlation consensus for the three types of correlations. They all become strongly positive correlated and more importantly, the expert's rank correlation consensus has increased.

If the collective preference relation is constructed using a weighted averaging operator with equal weights for the experts, the following preference relation is obtained:

$$P^c = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.50 & 0.48 & 0.26 & 0.41 \\ 0.52 & 0.50 & 0.35 & 0.55 \\ 0.74 & 0.65 & 0.50 & 0.69 \\ 0.60 & 0.45 & 0.31 & 0.50 \end{bmatrix} \end{array}$$

By using $A_i = \frac{2}{n^2} \sum_{j=1}^{n} P_{ij}^c$, which is equivalent to the sum normalization method, we get $A_3 \succ A_2 \succ A_4 \succ A_1$.

3.5.2. Group decision-making under heterogeneous information

Suppose four decision-makers provide their assessments. The first one prefers fuzzy preference relation and the second one uses multiplicative preference relation. The third and the fourth decision-makers provide linguistic preference relations with $T^3 = 5$ and $T^4 = 13$ respectively. The following assessments are obtained for the four alternatives:

$$P^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.6 & 0.8 & 0.9 \\ 0.4 & 0.5 & 0.7 & 0.7 \\ 0.2 & 0.3 & 0.5 & 0.5 \\ 0.1 & 0.3 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

$$A^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 1 & 4 & 6 & 7 \\ 1/4 & 1 & 3 & 4 \\ 1/6 & 1/3 & 1 & 2 \\ 1/7 & 1/4 & 1/2 & 1 \end{bmatrix} \end{array}$$

$$L^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} s_0 & s_1 & s_2 & s_0 \\ s_{-1} & s_0 & s_0 & s_2 \\ s_{-2} & s_0 & s_0 & s_1 \\ s_0 & s_{-2} & s_{-1} & s_0 \end{bmatrix} \end{array}$$

$$L^4 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} s_0 & s_2 & s_4 & s_6 \\ s_{-2} & s_0 & s_3 & s_5 \\ s_{-4} & s_{-3} & s_0 & s_5 \\ s_{-6} & s_{-5} & s_{-5} & s_0 \end{bmatrix} \end{array}$$

A. Consensus measure

Before measuring the rank correlation consensus, A^2, L^3 and L^4 need to be transferred into fuzzy preference relation using (3.1) and (3.3), respectively as follows:

$$P^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.82 & 0.91 & 0.94 \\ 0.18 & 0.5 & 0.75 & 0.82 \\ 0.09 & 0.25 & 0.5 & 0.66 \\ 0.06 & 0.18 & 0.34 & 0.5 \end{bmatrix} \end{array}$$

$$P^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.75 & 1 & 0.5 \\ 0.25 & 0.5 & 0.5 & 1 \\ 0 & 0.5 & 0.5 & 0.75 \\ 0.5 & 0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

$$P^4 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.67 & 0.83 & 1 \\ 0.33 & 0.5 & 0.75 & 0.92 \\ 0.17 & 0.25 & 0.5 & 0.92 \\ 0 & 0.08 & 0.08 & 0.5 \end{bmatrix} \end{array}$$

The collective preference relation is obtained by using weighted averaging operator with equal weights:

$$P^c = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.71 & 0.89 & 0.84 \\ 0.29 & 0.5 & 0.68 & 0.86 \\ 0.12 & 0.33 & 0.5 & 0.707 \\ 0.17 & 0.14 & 0.29 & 0.5 \end{bmatrix} \end{array}$$

1. $V^1 = (0.6, 0.8, 0.9, 0.7, 0.7, 0.5)$, $V^2 = (0.82, 0.91, 0.94, 0.75, 0.82, 0.66)$

   $V^3 = (0.75, 1, 0.5, 0.5, 1, 0.75)$, $V^4 = (0.67, 0.83, 1, 0.75, 0.92, 0.92)$

   $$V^c = (0.71, 0.89, 0.84, 0.68, 0.86, 0.707)$$

2. $RV^1 = (5, 2, 1, 3.5, 3.5, 6)$, $RV^2 = (3.5, 2, 1, 5, 3.5, 6)$,$RV^3 = (3.5, 1.5, 5.5, 5.5, 1.5, 3.5)$

   $RV^4 = (6, 4, 1, 5, 2.5, 2.5)$, $RV^c = (4, 1, 3, 6, 2, 5)$

Table 3.6 presents the results obtained by applying the rank correlation consensus measure between individuals and the collective relation:

Table 3.6: Rank correlation consensus between individuals and the collective

	rcc^{kh}				
	DM_1	DM_2	DM_3	DM_4	rcc^{ck}
DM_1	1	0.8676	-0.121	0.397	0.551
DM_2	0.8676	1	0.061	0.309	0.725
DM_3	-0.121	0.061	1	-0.061	0.717
DM_4	0.397	0.309	-0.061	1	0.290
rcc^k	0.381	0.412	-0.040	0.215	0.571
rcc^T	0.242				

B. Feedback mechanism

Based on these results, we find that decision-maker 4 needs to adjust their evaluation, $\min\{rcc^{ck}\} = rcc^{c4}$ to get the group closer to the rank consensus level. Thus, first $\min\{rsd_{ij}^{c4}\}$ is found to identify which preference degree is needed to modify, then step 3 is applied to provide suggestion to DM_4 with regards to this preference degree. The following summarizes the steps in the feedback mechanism to reach the target rank correlation consensus level:

Round 1. $\min\{rcc^{ck}\} = rcc^{c4} \Rightarrow \min\{rsd_{ij}^{c4}\} = rsd_{13}^{c4} = 0.4 \Rightarrow p_{13}^{4} \in [s_6, s_6] \Rightarrow p_{13}^{4\prime} = s_6 \Rightarrow rcc^{c^{(1)}} = 0.66.$

Round 2. $\min\{rcc^{ck}\} = rcc^{c1} \Rightarrow \min\{rsd_{23}^{c1}\} = rsd_{23}^{c1} = 0.5 \Rightarrow p_{23}^{1} \in [0,0.5] \Rightarrow p_{23}^{1\prime} = 0.5 \Rightarrow rcc^{c^{(2)}} = 0.7252.$

Round 3. $\min\{rcc^{ck}\} = rcc^{c4} \Rightarrow \min\{rsd_{12}^{c4}\} = rsd_{12}^{c4} = 0.6 \Rightarrow p_{12}^{4} \in [s_5, s_5] \Rightarrow p_{12}^{4\prime} = s_5 \Rightarrow rcc^{c^{(3)}} = 0.7563.$

Round 4. $\min\{rcc^{ck}\} = rcc^{c3} \Rightarrow \min\{rsd_{14}^{c3}\} = rsd_{14}^{c3} = 0.5 \Rightarrow p_{14}^{3} \in [s_1, s_2] \Rightarrow p_{14}^{3\prime} = s_2 \Rightarrow rcc^{c^{(4)}} = 0.9412.$

If $rcc^{c} = 0.9412$ is greater than or equal to the pre-assigned rank consensus level, then stop and start the selection process. Based on these modifications, the following new collective relation is obtained, using weighted averaging operator with equal weights:

$$P^{c\prime} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.77 & 0.93 & 0.96 \\ 0.23 & 0.5 & 0.63 & 0.86 \\ 0.07 & 0.38 & 0.5 & 0.71 \\ 0.04 & 0.14 & 0.29 & 0.5 \end{bmatrix} \end{array}$$

By using $A_i = \frac{2}{n^2} \sum_{j=1}^{n} P_{ij}^{c}$, we get $A_1 \succ A_2 \succ A_3 \succ A_4$.

3.6. Validation

We conducted some tests on the proposed method using some examples available in the literature. Our method shows almost the same conclusions to the other researchers' conclusions. However, we should mention that some of these methods are not based on reciprocal preference relations. In addition, the aggregation operators and selection methods are different, which play a significant role in the final results. Table 3.7 summarizes these findings.

Table 3.7: Proposed method vs. some available methods

	Authors		
	Zhang et al. (2016)	Pérez et al. (2014)	Wu and Xu (2012)
Consensus model based	Similarity	Similarity+expert weights	Distance
Aggregation Operator	Weighted Averaging	Not Given	Weighted Averaging
Group Members	4	4	5
No. of Alternatives	4	4	6
Selection Method	QGDD	Not Given	Sum Normalization
Alternatives Ranking	$A_2 \succ A_1 \succ A_3 \succ A_4$	$A_1 \succ A_2 \succ A_3 \succ A_4$	$A_3 \succ A_2 \succ A_1 \succ A_4 \succ A_6 \succ A_5$
Proposed Method Ranking	$rcc^c \approx 0.9$, $A_2 \succ A_1 \succ A_3 \succ A_4$	$rcc^c = 0.92$, $A_1 \succ A_2 \succ A_4 \succ A_3$	$rcc^c \approx 0.93$, $A_3 \succ A_2 \succ A_1 \succ A_4 \succ A_6 \succ A_5$

	Authors	
	Herrera-Viedma et al, (2007a)	Herrera-Viedma et al, (2002)
Consensus model based	Similarity+Consistency	ordinal/dissimilarity
Aggregation Operator	IOWA	S-OWA OR-LIKE
Group Members	4	8
No. of Alternatives	4	6
Selection Method	Not Available	QGDD
Alternatives Ranking	$A_2 \succ A_1 \succ A_4 \succ A_3^{*}$	$A_2 \succ A_3 \succ A_1 \succ A_5 \succ A_4 \succ A_6$
Proposed Method Ranking	$rcc^c \approx 0.9$, $A_2 \succ A_1 \succ A_4 \succ A_3$	$rcc^c \approx 0.77$, $A_3 \succ A_2 \succ A_1 \succ A_5 \succ A_4 \succ A_6$

QGDD-quantifier guided domonance degree, IOWA-induced ordered weighted averaging, * the ranking was generated by using weighted averaging method with equal weights using sum normalization by us

We point out that the examples of Zhang et al. (2016) and Wu and Xu (2012) are based on reciprocal relations while the others are not. Herrera-Viedma et al. (2002) started the problem with reciprocal preference relations; however, the aggregation operator does not maintain this property. It can be seen from Table 3.7 that the proposed method has similar results to the problems with reciprocal relations. Also, it performs great on problems that are not based on reciprocal relations.

We should mention that for the non-reciprocal problems in Table 3.7, the upper triangular relation of the problems were considered. The results are almost the same except for minor ranking orders for Pérez et al.'s (2014) and Herrera-Viedma et al.'s (2002) problems, which could be linked to the level of consensus to be achieved. Also, the effect of using different selection methods and aggregation operators in finding the solution should not be ignored.

3.7. Conclusions

In this chapter, we presented a consensus model based on Spearman's correlation. The proposed model does not rely directly on similarity/distance measures rather than on ranks of preference degrees on reciprocal preference relations. The novelty of this work lies in considering the coherence of decision-maker preference degrees ranks as a whole in comparison with rest of the group members. In addition, a feedback mechanism is proposed to play the role of the moderator to provide suggestions to the decision-maker who is not close in rank correlation consensus to the group members. The model was tested on several problems and the results proved the validity of the model. Two examples dealing with different information types are illustrated.

Chapter 4:

Investigating Rank Reversal in Preference Relation Based on Additive Consistency: Causes and Solutions

4.1. Introduction

Multi-Criteria Decision-making (MCDM) is a field with many strengths, among which is its ability to assist decision-makers in solving difficult decisions involving conflicting criteria and to help them learn more about their preferences. However, some methods are known to have a phenomenon called rank reversal. Rank reversal occurs when a new alternative is added to (or removed from) a set of alternatives, which causes a change in the ranking order of the alternatives (Barzilai & Golany, 1994). The literature on decision-making reveals that a number of methods suffer from this phenomenon. Some of them are Analytic Hierarchy Process (AHP) (Barzilai & Golany, 1994; Wang & Luo, 2009), Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) (Wang et al., 2007; Wang & Luo, 2009), ELimination and Choice Expressing Reality (ELECTRE), Preference Ranking Organisation Method for Enrichment Evaluations (PROMETHEE) (Frini et al., 2012; Mareschal et al., 2008), Data Envelopment analysis - Analytic hierarchy process (DEAHP), Borda-Kendall (BK) (Wang & Luo, 2009) and Weighted Sum Method (WSM)(Wang & Luo, 2009).

The rank reversal phenomenon has raised concerns against the use of affected methods, especially AHP. Rank reversal could be of two types: partial or total. Partial rank reversal happens to limited alternatives while other alternatives still have the same ordering. For example, suppose that the current ranking of three alternatives is $A_3 \succ A_1 \succ A_2$, such that alternative A_3 is preferred over alternative A_1 and A_2 respectively. However, when a new alternative A_4, which is not dominant, is introduced, the ranking could become $A_1 \succ A_3 \succ A_4 \succ A_2$. Notice that alternative A_3 now becomes second while alternative A_1 is first. This is called partial rank reversal. On the other hand, total rank reversal is the same as the partial rank reversal except that the whole ranking order is reversed. In this case, the best alternative becomes the worst and the worst becomes the best $A_2 \succ A_4 \succ A_1 \succ A_3$ (Dymova et al., 2013; Garcia-Cascales & Lamata, 2012).

Belton and Gear (1983) were the first to notice this phenomenon in AHP. Since then, the literature of MCDM has been in debate about the impact of this phenomenon, and the validity of the affected methods. Many researchers such as Dyer (1990), Schenkerman (1994), Perez (1995), and Leung and Cao (2001) criticized the exhibited methods, whereas researchers such as Saaty and Vargas (1984), Saaty (1987), Forman (1990), and Millet and Saaty (2000) argued for the legitimacy of this phenomenon.

To emphasize the phenomenon of rank reversal, we point the reader to the *contraction consistency condition* mentioned by Pavlicic (2001) adopted from Amartya Sen that states:

Contraction consistency condition: *If alternative A is the best in the set of alternatives S such that $A \in S$, then it has to be the best in every subset $E \subset S$ where $A \in E$.*

This phenomenon could drive some decision-makers away from using methods known to have rank reversal, even if they are well-known. For instance, recently Anbaroglu et al. (2014) chose to use the Weighted Product Model (WPM) instead of relying on well-known and widely used models such as AHP and WSM just because it does not suffer from any kind of rank reversal issues. Furthermore, they commented on the problem of rank reversal as "a serious limitation" of the MCDM field, which could lead researchers to misunderstand the difference between examined alternatives. Therefore, a need for handling this phenomenon is necessary, at least for the experts who are not in favor of it. The literature on preference relations, especially multiplicative preference relations, links this phenomenon to inconsistency of the data, the concept of pairwise comparison on which preference relations are based, the preference aggregation method, and the score aggregation method. To our knowledge, there is no complete study yet that investigates these three possible reasons for rank reversal in preference relations. There is one study, conducted by Leskinen and Kangas (2005), on the inconsistency of pairwise comparison based on a regression model. They concluded that inconsistency could lead to rank reversal. This phenomenon, however, does not occur when there is single criterion. But, in multiple criteria even if the data are consistent, the aggregation method (i.e. arithmetic mean) can result in rank reversals.

In this chapter, our goal is to investigate how inconsistency and aggregation methods could lead to rank reversal in preference relations.

The rest of the chapter is organized as follows: in section 4.2 we present some preliminary knowledge on preference relations. In section 4.3, we present a review of rank reversal literature regarding possible causes and attempts to solve rank reversal. In section 4.4, we study the possible causes of rank reversal in preference relation, namely, inconsistency of preference relation, aggregation operators, and score aggregation method and their link to rank reversal. In section 4.5, we propose score aggregation methods that have better performance than the sum normalization method in avoiding rank reversal. In section 4.6, we provide a numerical example. Finally in section 4.7, we present the conclusions.

4.2. Preliminary Knowledge

Definition 4.1 (Urena et al., 2015): A preference relation R is a binary relation defined on the set X that is characterized by a function $\mu_p: X \times X \rightarrow D$, where D is the domain of representation of preference degrees provided by the decision-maker.

Definition 4.2 (Xu, 2007): A fuzzy additive preference relation P on a finite set of alternatives X is represented by a matrix $P = (p_{ij})_{n\times n} \subset X \times X$ with:

$p_{ij} \in [0,1],\ \ p_{ij} + p_{ji} = 1,\ \ p_{ii} = 0.5\ \ \forall i, j = 1, \dots, n.$

$$P = (p_{ij})_{nxn} = \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & \dots & A_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_n \end{array} & \begin{pmatrix} 0.5 & p_{12} & \dots & p_{1n} \\ p_{21} & 0.5 & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & 0.5 \end{pmatrix} \end{array}$$

when $p_{ij} > 0.5$ indicates that the expert prefers alternative x_i over alternative x_j; $p_{ij} < 0.5$ indicates that the expert prefers alternative x_j over alternative x_i; $p_{ij} = 0.5$ indicates that the expert is indifferent between x_i and x_j, thus, $p_{ii} = 0.5$.

Furthermore, the additive preference relation $P = (p_{ij})_{n\times n}$ is additive consistent if and only if the following additive transitivity is satisfied (Meng & Chen, 2015; Urena et al., 2015; Herrera-Viedma et al., 2007a; Tanino, 1984):

$$F^1: \quad p_{ik} = p_{ij} + p_{jk} - 0.5 \tag{4.1}$$

Definition 4.3 (Saaty, 1980): A multiplicative preference relation A on the set $X = \{x_1, x_2, \dots, x_n\}$ of alternatives is defined as a reciprocal matrix $A = (a_{ij})_{n\times n} \subset X \times X$ with the following conditions:

$$a_{ij} > 0, \quad a_{ij}a_{ji} = 1, \quad a_{ii} = 1, \quad \forall ij = 1, 2, \dots, n.$$

$$A = \left(a_{ij}\right)_{nxn} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \dots & A_n \end{array} \\ \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 1 \end{pmatrix} \end{array}$$

where a_{ij} is interpreted as the ratio of the preference intensity of the alternative x_i to x_j.

There are several numerical scales for the multiplicative preference relation; however, the most popular one is the 1-9 Saaty scale. $a_{ij} = 1$ means that alternative x_i and x_j are indifferent; $a_{ij} > 1$ implies that alternative x_i is preferred to x_j. As the ratio of intensity of (a_{ij}) increases, the stronger is the preference intensity of x_i over x_j. Thus, $a_{ij} = 9$ means that alternative x_i is absolutely preferred to x_j.

The multiplicative preference relation $A = (a_{ij})_{n\times n}$ is called consistent if the following multiplicative transitivity is satisfied (Saaty, 1980):

$$a_{ij} = a_{ik}a_{kj}, \quad a_{ii} = 1, \quad \forall i, j = 1, 2, \dots, n.$$

The AHP method, which uses multiplicative preference relations, decomposes complex problems into a hierarchy consisting of several levels, where the top level represents the goal and the lower levels consist of criteria, sub-criteria and alternatives respectively. The elements in each level are compared with each other through pair-wise comparison by using scale of 1-9 to find their relative importance. Then the weight for each element is computed using the eigenvector method. The same technique is used at the lower level with respect to a higher level element to find their relative importance (Saaty, 1980).

4.3. Literature Review

The purpose of this section is to explore the literature of MCDM to investigate possible causes of rank reversal phenomena. We will then cover the attempts of researchers to solve this issue. Thus, two main subsections will be explored: the literature of rank reversal causes and attempts to fix rank reversal.

4.3.1. The literature on rank reversal causes'

The literature on multiplicative preference relations, especially AHP, discusses three possible reasons behind rank reversal, see Table 4.1: inconsistency, pairwise comparison, and aggregation method. Dodd et al. (1995) claimed that Saaty's AHP misses a form of inconsistency within its model, which puts its results under doubt. This claim somehow agrees with Stewart (1992), who stated that rank reversal is a consequence of the way the weights are elicited, ratio scales, and the eigenvector approach. Farkas et al. (2004) also blamed inconsistency in pairwise comparison for this issue. According to Paulson and Zahir (1995), judgmental uncertainty could also cause rank reversal. Chou (2012) referred the issue of rank reversal in AHP due to the aggregation method, Saaty's ratio scale, and the inconsistency of judgments. However, researchers like Karapetrovic and Rosenbloom (1999) refused to link the problem to inconsistency. They argued that there is no direct relation between the consistencies or inconsistencies of pairwise comparison matrices and the occurrence of rank reversal. They stated that each could be considered as a separate problem. Ishizaka and Labib (2011) agreed with them and reported that rank reversal is independent of the consistency of the data and priority method. Moreover, they believed this phenomenon could happen in any additive model.

Table 4.1: The causes' literature of rank reversal

Cause(s)	Author(s)
Inconsistency	Dodd et al. (1995); Farkas et al., (2004); Paulson and Zahir (1995)
Weights elicited method, Saaty's ratio scale and eigenvector approach	Stewart (1992)
Aggregation method, Saaty's ratio scale and inconsistency	Chou (2012)
AHP method and score aggregation	Ishizaka and Labib (2011)
Normalization and ratio scale	Schenkerman (1994)
Multiplying criteria weights by unrelated normalized scale	Lai (1995); Perez (1995)
AHP method	Dyer (1990); Triantaphyllou (2001)
Normalization methods	Rosenbloom (1997)
Eigenvalue method	Bana e Costa and Vansnick (2008)

Other researchers like Schenkerman (1994) believed that the rank reversal in AHP is caused by normalization, and its scales seem arbitrary. He claimed that criteria weights are dependent on the measurements of the alternatives. Thus, any change in the number of alternatives and normalization imposes revising of the criteria weights. Correspondingly, Ishizaka and Labib (2011) claimed that the rank reversal phenomenon is related to the method rather than modelling procedure and it may not be resolved because aggregation of the standardized units is not simply interpretable, which has been even disputed by French school. Lai (1995) pointed out that rank reversal happens because of multiplying criteria weights by unrelated normalized scale of performance ratings. Dyer (1990) claimed that the problem is not just rank reversal but the AHP results are arbitrary. This is because the criteria weights may not be right due to the normalization procedure. Triantaphyllou (2001) agreed with Dyer that in AHP or any additive variants of it, ranking is arbitrary often tends to generate rank reversal even if the data is perfectly consistent. According to Rosenbloom (1997), researchers tried to resolve this problem in AHP by proposing different normalization methods. Perez (1995) argued that the phenomenon of rank reversal is common in almost all of ordinal aggregation methods such as AHP. He claimed that rank reversal could be avoided if both criteria weights and performance ratings are generated from a common space of scales. On the other hand, Bana e Costa and Vansnick (2008) blamed the eigenvalue method. They stated that the priority vector violates a condition of order preservation, which makes use of AHP in decision-making very problematic.

4.3.2. Attempts to fix rank reversal

The rank reversal phenomenon in AHP was initially observed by Belton and Gear in 1983 after they discovered that introducing a new similar alternative to the existing ones could reverse the ranking of the alternatives. They proposed a modified normalization method to overcome the rank reversal issue in the original AHP, which is later known as a Revised AHP. The revised method differs from the original AHP prioritization method where each criterion is divided by the max value with respect to it for all the alternatives. Later on, this method came to be known as the ideal model. Afterwards, Schenkerman (1994) claimed that in methods such as Referenced AHP, normalization to maximum entry (ideal model), normalization to minimum entry, and linking pins avoid rank reversal only when the criteria are quantitative. On the other hand, Saaty (1987) linked rank reversal with the existence of near or similar copies within the set of alternatives. To solve this issue, either the set of alternatives has to be revised or more criteria need to be considered.

Saaty defines a near copy as an alternative that has close values within 10% for overall criteria. However, Dyer (1990) later criticized this suggestion.

Lootsma (1993), followed by Sheu (2004), claimed that using a geometric mean aggregation method in AHP helps to avoid rank reversal. Likewise, Ishizaka and Labib (2011) mentioned that using geometric mean in AHP prevents rank reversal since geometric mean in both row and column approaches produces the same results, unlike eigenvector methods. Barzilai and Golany (1994) stated that the rank reversal problem is related to the structure of AHP mainly through the additive aggregation rule. They argued that the multiplicative procedures such as the geometric mean and the weighted-geometric-mean aggregation rule are the solution. In fact, some studies have shown that multiplicative methods such as the weighted product model and the multiplicative AHP are immune against rank reversal (Wang & Triantaphyllou, 2008). Barzilai and Lootsma (1997) demonstrated that the multiplicative AHP method does not generate rank reversal by testing the method on Belton and Gear's (1983) example. Additionally, the multiplicative variants of the AHP tend to be more reliable and do not show any kind of rank reversal, which means they are perfect (Triantaphyllou, 2001). On the other hand, Buede and Maxwell (1995) pointed out that using geometric mean "will not eliminate rank reversal," contrary to removing normalization of the ratio scale, which guarantees immunity against rank reversal.

Farkas et al. (2004) developed an approach by determining the intervals for all possible occurrences of rank reversals. They demonstrated it for an example of a 3X3 matrix. Recently, Rodriguez et al. (2013) proposed a modification to the fuzzy AHP- TOPSIS method with a graphical approach for rank reversal detection and analysis. They claimed that this graphical approach increases the level of confidence in the results. However, they mentioned that the graphical approach is not suitable when large set of criteria are under consideration. Table 4.2 summarizes the attempts to avoid/solve rank reversal in AHP.

Table 4.2: Some attempts to solve AHP's rank reversal

Solution	Author(s)
Max normalization method	Belton and Gear (1983)
Exclude/remove near or similar copies of the alternatives	Saaty (1987)
Geometric mean aggregation method	Lootsma (1993); Sheu (2004); Ishizaka and Labib (2011); Barzilai and Golany (1994); Wang and Triantaphyllou (2008); Barzilai and Lootsma (1997)
Find the intervals of all rank reversals	Farkas et al. (2004)
Graphical approach	Rodriguez et al. (2013)

4.4. Mathematical Investigation of Rank Reversal Causes in Preference Relations

According to Chiclana et al. (2009), preference relations have three fundamental and hierarchical levels of rationality assumptions: 1) the first level requires indifference between any alternative x_i and itself, 2) the second level requires that if the decision-maker prefers x_i to x_j then they should not at the same time prefer x_j to x_i, and 3) the third level is related to transitivity among any three alternatives. There are a number of consistency properties in the literature. A few of these are: triangle condition, weak transitivity, max-min transitivity, max-max transitivity, restricted max-min transitivity, restricted max-max transitivity, multiplicative transitivity, and additive transitivity (Herrera-Viedma et al., 2004). Among these properties, additive and multiplicative transitivity are the most used and are equivalent to each other through a transformation function. The transitivity property is interpreted by the idea that the preference value of any two alternatives obtained directly by comparison should be equal to or greater than the preference value of an indirect alternative (intermediate alternative) that is between them (Herrera-Viedma et al., 2004). Furthermore, any property that enforces transitivity in the preferences is called a consistency property (Chiclana et al., 2009).

4.4.1. Additive consistency

From (4.1) two other formulations can be generated based on the characteristics of the reciprocal rule, $(p_{ij} + p_{ji} = 1)$, as follows:

- F^2: $p_{ik} = p_{jk} - p_{ji} + 0.5$ (4.2)
- F^3: $p_{ik} = p_{ij} - p_{kj} + 0.5$ (4.3)

Proposition 4.1: Let $P = (p_{ik})_{n\times n}$ be an additive preference relation, then for every preference degree on P we can find its estimation based on the additive consistency through:

$$p_{ik} = \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right). \tag{4.4}$$

Proof: by taking the average of equations (4.1), (4.2) and (4.3) for p_{ik} for n alternatives, the following equation is generated:

$$p_{ik} = \frac{1}{3n}\left[\sum_{j=1}^{n}\left(p_{ij} + p_{jk} - 0.5\right) + \left(p_{jk} - p_{ji} + 0.5\right) + \left(p_{ij} - p_{kj} + 0.5\right)\right]$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{j=1}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \frac{1}{3n}(p_{ii} + p_{kk} + 4p_{ik} - 2p_{ki} + 1)$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \frac{1}{3n}(0.5 + 0.5 + 4p_{ik} - 2(1 - p_{ik}) + 1)$$

$$\Rightarrow p_{ik} = \frac{1}{3n}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \frac{1}{3n}(6p_{ik})$$

$$\Rightarrow (3n)p_{ik} = \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + (6p_{ik})$$

$$\Rightarrow 3(n-2)p_{ik} = \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

$$\Rightarrow \boxed{p_{ik} = \frac{1}{3(n-2)}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)} \ \blacksquare$$

For a reciprocal additive preference relation, (4.4) can be re-written as:

$$p_{ik} = \frac{1}{(n-2)}\sum_{\substack{j=1 \\ i\neq j\neq k}}^{n}\left(p_{ij} + p_{jk} - 0.5\right) \quad (4.5)$$

Definition 4.4: Let $P = (p_{ik})_{n\times n}$ be a given reciprocal additive preference relation and $P^e = (p_{ik}^e)_{n\times n}$ be the estimated additive preference relation calculated by (4.5). Then the consistency degree of P is calculated by

$$CD(P, P^e) = 1 - \frac{2}{n(n-1)}\sum_{i=1}^{n-1}\sum_{\substack{k=2 \\ i<k}}^{n}|p_{ik} - p_{ik}^e| \quad (4.6)$$

Thus (4.4) is used to check the consistency degree of any reciprocal additive preference relation. When $CD(P, P^e) = 1$ then P is perfectly consistent; keeping in mind that p_{ik} is a preference degree or preference intensity of alternative i over alternative k.

The additive consistency implies dependency between alternatives, which is clear from the transitivity property. Thus, any change in the examined set of the alternatives implies a possible change on the preference degrees. This is correct, especially if the provided information is not perfectly consistent. To illustrate this, let us assume that the provided information for a set of n alternatives is perfectly consistent. Then, if we remove an alternative (h) from the set, $P^n \to P^{n-1} = (p_{ik})_{n-1 \times n-1}$, or if we add an alternative (h) to the set, $P^n \to P^{n+1} = (p_{ik})_{n+1 \times n+1}$. Therefore, the remaining preference degrees from P^n after n is modified can maintain their valuations only if (4.4) is satisfied. This can only happen if the original information and the new alternative are perfectly consistent.

Theorem 4.1: Based on additive consistency, a preference degree (p_{ik}^n) maintains its valuation after removing or adding an alternative h from n if

$$p_{ik}^{n-1} or\ p_{ik}^{n+1} = \frac{1}{3}(2p_{ih} + 2p_{hk} - p_{hi} - p_{kh}) + \frac{1}{6} \quad \forall\ ik \in n \tag{4.7}$$

Otherwise the preference relation P^nor P^{n+1} is not perfectly consistent.

Proof: from(4.4), $p_{ik}^n = \frac{1}{3(n-2)} \sum_{\substack{j=1 \\ i \neq j \neq k}}^{n} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$, when we remove h from n we get:

$$p_{ik}^{n-1} = \frac{1}{3(n-1-2)} \sum_{\substack{j=1 \\ i \neq j \neq k \\ n \setminus \{h\}}}^{n-1} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

$$= \frac{1}{3(n-3)} \sum_{\substack{j=1 \\ i \neq j \neq k \\ n \setminus \{h\}}}^{n-1} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

For $j \neq i \neq k$ and $n \setminus \{h\} = n - 1$, then

$$p_{ik}^{n-1} = \frac{1}{3(n-3)} \sum_{\substack{j=1 \\ i \neq j \neq k \\ n \setminus \{h\}}}^{n-1} \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right)$$

$$= \frac{1}{3(n-3)} \Big[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \cdots$$
$$+ \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)} + 0.5\right)\Big]$$

$$= \frac{1}{3(n-3)} \Big[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)\Big]$$
$$+ \frac{n-3}{6(n-3)}$$

For $j \neq i \neq k$ and $h \in n$, then

$$p_{ik}^{n} = \frac{1}{3(n-2)} \Big[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj} + 0.5\right) + \cdots$$
$$+ \left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)} + 0.5\right) + \cdots$$
$$+ \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)} + 0.5\right)\Big]$$

$$= \frac{1}{3(n-2)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)}\right) + \cdots \right.$$
$$\left. + \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)}\right)\right] + \frac{n-2}{6(n-2)}$$

Thus, the only way $p_{ik}^{n-1} = p_{ik}^{n}$ after removing h is if

$$\frac{1}{3(n-3)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)\right] + \frac{1}{6}$$
$$= \frac{1}{3(n-2)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)}\right)\right.$$
$$\left. + \cdots + \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)}\right)\right] + \frac{1}{6}$$

$$\Rightarrow \frac{1}{3(n-3)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)\right]$$
$$- \frac{1}{3(n-2)}\left[\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots \right.$$
$$\left. + \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)}\right)\right]$$
$$= \frac{1}{3(n-2)}\left(2p_{i(h)} + 2p_{(h)k} - p_{(h)i} - p_{k(h)}\right)$$

Since,

$$\left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n-1)} + 2p_{(n-1)k} - p_{(n-1)i} - p_{k(n-1)}\right)$$
$$= \left(2p_{ij} + 2p_{jk} - p_{ji} - p_{kj}\right) + \cdots + \left(2p_{i(n)} + 2p_{(n)k} - p_{(n)i} - p_{k(n)}\right)$$

Then,

$$\frac{1}{3(n-3)(n-2)}\left[\left(2p_{ij}+\ 2p_{jk}-p_{ji}-p_{kj}\right)+\cdots\right.$$

$$\left.+\left(2p_{i(n-1)}+\ 2p_{(n-1)k}-p_{(n-1)i}-p_{k(n-1)}\right)\right]$$

$$=\frac{1}{3(n-2)}\left(2p_{i(h)}+\ 2p_{(h)k}-p_{(h)i}-p_{k(h)}\right)$$

Multiply both sides by $3(n-3)(n-2)$, we get,

$$\left[\left(2p_{ij}+\ 2p_{jk}-p_{ji}-p_{kj}\right)+\cdots+\left(2p_{i(n-1)}+\ 2p_{(n-1)k}-p_{(n-1)i}-p_{k(n-1)}\right)\right]$$

$$=(n-3)\left(2p_{i(h)}+\ 2p_{(h)k}-p_{(h)i}-p_{k(h)}\right)$$

Thus,

$$\sum_{\substack{j=1\\ i\neq j\neq k\\ n\backslash h}}^{n-1}\left(2p_{ij}+\ 2p_{jk}-p_{ji}-p_{kj}\right)=(n-3)(2p_{ih}+\ 2p_{hk}-p_{hi}-p_{kh})$$

$$\frac{1}{3(n-3)}\sum_{\substack{j=1\\ i\neq j\neq k\\ n\backslash h}}^{n-1}\left(2p_{ij}+\ 2p_{jk}-p_{ji}-p_{kj}\right)=\frac{1}{3(n-3)}(n-3)(2p_{ih}+\ 2p_{hk}-p_{hi}-p_{kh})$$

$$\frac{1}{3(n-3)}\sum_{\substack{j=1\\ i\neq j\neq k\\ n\backslash h}}^{n-1}\left(2p_{ij}+\ 2p_{jk}-p_{ji}-p_{kj}+0.5\right)=\frac{1}{3}(2p_{ih}+\ 2p_{hk}-p_{hi}-p_{kh})+\frac{1}{6}$$

$$p_{ik}^{n-1}=\frac{1}{3}(2p_{ih}+\ 2p_{hk}-p_{hi}-p_{kh})+\frac{1}{6}$$

Also, we get the same conclusion when h is added to n ■

This shows how removing or adding an alternative could affect the entire information, especially when they are inconsistent. Thus, introducing new information implies a change in the original information, particularly if the new information is not consistent. Usually decision-makers do not revise their assessments based on the new information. In general, the decision-makers compare two alternatives at a time; however, when we consider the consistency of the information, all the alternatives need to be considered. So the decision-makers do not revise their previous assessments on a pair of alternatives if another alternative is removed or a new one is added. Moreover in real life, most decision-makers are not consistent in their opinions. Thus, how should we handle acceptably inconsistent information in a way to avoid rank reversal? Saaty (1980) suggested that the acceptable inconsistency degree should be less than or equal to 10%.

4.4.2. Aggregation methods

Aggregation methods or operators are used to aggregate individual preference relations into a collective one. For example, in group decision-making, the individuals' preference relations are aggregated into a collective preference relation. There are many aggregation operators in the literature; however, the most common one is the weighted averaging operator. The weighted averaging operator is defined as follows:

$$p_{ik}^{c} = \sum_{t=1}^{m} w_t \cdot p_{ik}^{t} \tag{4.8}$$

where w_t is the weight of decision-maker t such that $\sum_{t=1}^{m} w_t = 1$, p_{ik}^{t} is the given preference degree by decision-maker t, m is the number of decision-makers, and p_{ik}^{c} is the collective preference degree. The weighted averaging operator becomes an averaging operator when the decision-makers have equal weights.

Proposition 4.2: Let $P^t = (p_{ik}^t)_{n\times n}$ be a reciprocal additive preference relation given by a decision-maker t. When all P^ts are perfectly consistent then the collective preference relation is also perfectly consistent.

Proof: from (4.8)

$$p_{ik}^c = \sum_{t=1}^{m} w_t \cdot p_{ik}^t$$

and from (4.5) $p_{ik}^t = \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$ then

$$p_{ik}^c = \sum_{t=1}^{m} w_t \cdot p_{ik}^t = \sum_{t=1}^{m} w_t \cdot \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{t=1}^{m} w_t \cdot \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(\sum_{t=1}^{m} w_t \cdot p_{ij}^t + \sum_{t=1}^{m} w_t \cdot p_{jk}^t - \sum_{t=1}^{m} w_t \cdot 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(p_{ij}^c + p_{jk}^c - 0.5\right) = p_{ik}^e \blacksquare$$

Similarly, when an arithmetic mean operator is used, the consistency is also maintained.

$$p_{ik}^c = \frac{1}{m} \sum_{t=1}^{m} p_{ik}^t \qquad (4.8.1)$$

Proof: from (4.5) $p_{ik}^t = \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(p_{ij}^t + p_{jk}^t - 0.5\right)$ then

$$p_{ik}^{c} = \frac{1}{m}\sum_{t=1}^{m} p_{ik}^{t} = \frac{1}{m}\sum_{t=1}^{m} \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(p_{ij}^{t} + p_{jk}^{t} - 0.5\right)$$

$$= \frac{1}{(n-2)} \cdot \frac{1}{m}\sum_{t=1}^{m} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(p_{ij}^{t} + p_{jk}^{t} - 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(\frac{1}{m}\sum_{t=1}^{m} p_{ij}^{t} + \frac{1}{m}\sum_{t=1}^{m} p_{jk}^{t} - \frac{1}{m}\sum_{t=1}^{m} 0.5\right)$$

$$= \frac{1}{(n-2)} \sum_{\substack{j=1 \\ i\neq j\neq k}}^{n} \left(p_{ij}^{c} + p_{jk}^{c} - 0.5\right) = p_{ik}^{e} \blacksquare$$

Proposition 4.3: The constructed collective preference relation by arithmetic mean operator or weighted averaging operator gains the mean of the individuals' preference relations consistency degrees or the weighted averaging of the individuals' preference relations consistency degrees, respectively.

Proof: from (4.6)

$CD\left(P^{t}, P^{e(t)}\right) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} \left| p_{ik}^{t} - p_{ik}^{e(t)} \right|$, then for $t = 1,2, \ldots m$, we get:

$$CD\left(\sum_{t=1}^{m} P^{t}, \sum_{t=1}^{m} P^{e(t)}\right) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} \left| \sum_{t=1}^{m} p_{ik}^{t} - \sum_{t=1}^{m} p_{ik}^{e(t)} \right|,$$

$$CD\left(\frac{1}{m}\sum_{t=1}^{m} P^{t}, \frac{1}{m}\sum_{t=1}^{m} P^{e(t)}\right) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} \left| \frac{1}{m}\sum_{t=1}^{m} p_{ik}^{t} - \frac{1}{m}\sum_{t=1}^{m} p_{ik}^{e(t)} \right|,$$

$$\Rightarrow CD\left(\frac{1}{m}\sum_{t=1}^{m} P^{t}, \frac{1}{m}\sum_{t=1}^{m} P^{e(t)}\right) = CD\left(P^{c}, P^{e(c)}\right) = 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{\substack{k=2 \\ i<k}}^{n} \left| p_{ik}^{c} - p_{ik}^{e(c)} \right|,$$

This is also true for the weighted averaging operator■

For an inconsistent preference relation, removal or addition of an alternative h could play a significant role in altering the ranking order of the alternatives if h is the outbalance element among the alternatives.

To illustrate this, first we define the following score aggregation method, which is called the sum normalization method:

$$S_i = \frac{\sum_{k=1}^{n} p_{ik}}{\sum_{i=1}^{n}\sum_{k=1}^{n} p_{ik}} = \frac{2}{n^2}\sum_{k=1}^{n} p_{ik} \tag{4.9}$$

where S_i is the score of alternative i and $\sum_{i=1}^{n} S_i = 1$. The higher the score of an alternative, the better it is.

Theorem 4.2: Let the sum normalization method, equation(4.9), be the way to generate the ranking scores for the alternatives, then the following are true if alternative h is removed:

For $S_i^n > S_{i'}^n$ and $\sum_{k=1}^{n-1} p_{ik} \neq \sum_{k=1}^{n-1} p_{i'k}$ then,

$S_i^{n-1} > S_{i'}^{n-1}$ if and only if

$$\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih} \tag{4.9.1}$$

For $S_i^n > S_{i'}^n$ and $\sum_{k=1}^{n-1} p_{ik} = \sum_{k=1}^{n-1} p_{i'k}$ then

$$S_i^{n-1} = S_{i'}^{n-1} \tag{4.9.2}$$

$$p_{i'h} < p_{ih} \tag{4.9.3}$$

Proof: for (4.9.1),

$\sum_{k=1}^{n} p_{ik} = \sum_{k=1}^{n-1} p_{ik} + p_{ih}\ \forall i \in n$, substitute this into(4.9),

$S_i = \frac{2}{n^2}\sum_{k=1}^{n} p_{ik} = \frac{2}{n^2}[\sum_{k=1}^{n-1} p_{ik} + p_{ih}]$,

For $S_i^n > S_{i'}^n$ we get

$\frac{2}{n^2}[\sum_{k=1}^{n-1} p_{ik} + p_{ih}] > \frac{2}{n^2}[\sum_{k=1}^{n-1} p_{i'k} + p_{i'h}] \Rightarrow \sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih}$

Since $S_i^{n-1} = \frac{2}{(n-1)^2}\sum_{k=1}^{n-1} p_{ik}$ and $S_{i'}^{n-1} = \frac{2}{(n-1)^2}\sum_{k=1}^{n-1} p_{i'k}$, then $S_i^{n-1} > S_{i'}^{n-1}$ ■

However, when $S_i^n > S_{i'}^n$ but $\sum_{k=1}^{n-1} p_{ik} = \sum_{k=1}^{n-1} p_{i'k}$ then

$\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih} \Rightarrow p_{i'h} - p_{ih} < 0$ since $p_{i'h} < p_{ih} \Longrightarrow S_i^n > S_{i'}^n$ and since $\sum_{k=1}^{n-1} p_{ik} = \sum_{k=1}^{n-1} p_{i'k}$ then $S_i^{n-1} = \frac{2}{(n-1)^2}\sum_{k=1}^{n-1} p_{ik} = \frac{2}{(n-1)^2}\sum_{k=1}^{n-1} p_{i'k} = S_{i'}^{n-1}$, this completed the proof ■

This is also true when an alternative h is added. Therefore, rank reversal could occur when (4.9.1) and (4.9.3) are not satisfied.

Example 4.1: Suppose a decision-maker provides his assessments for one criterion on four alternatives using following reciprocal additive preference relation:

$$
\begin{array}{cc}
 & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} &
\begin{pmatrix}
0.5 & 0.55 & 0.62 & 0.65 \\
0.45 & 0.5 & 0.7 & 0.75 \\
0.38 & 0.3 & 0.5 & 0.85 \\
0.35 & 0.25 & 0.15 & 0.5
\end{pmatrix}
\end{array}
$$

Based on (4.6), the consistency degree of this preference relation is 82%. By using (4.9) the following ranking scores are generated:

$$A_2(0.3) > A_1(0.29) > A_3(0.254) > A_4(0.156)$$

However, when alternative A_4 is removed, the consistency degree increases to 87% with the following ranking scores:

$$A_1(0.371) > A_2(0.367) > A_3(0.262)$$

Notice that $A_1$and A_2 have been reversed. This is because A_4 was the outbalance element that differentiating between $A_1$and A_2. In fact, this happens because (4.9.1) is violated:

$\sum_{k=1}^{n-1} p_{1k} = 0.5 + 0.55 + 0.62 = 1.67, \sum_{k=1}^{n-1} p_{2k} = 0.45 + 0.5 + 0.7 = 1.65,$ $p_{14} = 0.65$ and $p_{24} = 0.75,$

$S_2^4(0.3) > S_1^4(0.29)$ Thus $S_2^3 > S_1^3$ only if

$\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih}$

But $\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} \ngtr p_{i'h} - p_{ih} \Longrightarrow -0.02 < 0.1.$

Theorem 4.3: For any perfectly consistent reciprocal preference relation, (4.9.1), (4.9.2), and (4.9.3) are satisfied by the additive consistency.

Proof: from (4.7)

$p_{ik}^{n-1} = \frac{1}{3}(2p_{ih} + 2p_{hk} - p_{hi} - p_{kh}) + \frac{1}{6}$ Then

$\sum_{k=1}^{n-1} p_{ik} = \frac{1}{3}(2p_{ih} + 2p_{h1} - p_{hi} - p_{1h}) + \frac{1}{6} + \frac{1}{3}(2p_{ih} + 2p_{h2} - p_{hi} - p_{2h}) + \frac{1}{6} + \cdots + \frac{1}{3}\left(2p_{ih} + 2p_{h(n-1)} - p_{hi} - p_{(n-1)h}\right) + \frac{1}{6}$, and

$\sum_{k=1}^{n-1} p_{i'k} = \frac{1}{3}(2p_{i'h} + 2p_{h1} - p_{hi'} - p_{1h}) + \frac{1}{6} + \frac{1}{3}(2p_{i'h} + 2p_{h2} - p_{hi'} - p_{2h}) + \frac{1}{6} + \cdots + \frac{1}{3}\left(2p_{i'h} + 2p_{h(n-1)} - p_{hi'} - p_{(n-1)h}\right) + \frac{1}{6}$, then

$\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} = \frac{(n-1)}{3}[(2p_{ih} - p_{hi}) - (2p_{i'h} - p_{hi'})]$, but for reciprocal relation $p_{hi} = 1 - p_{ih}$ then $\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} = (n-1)[p_{ih} - p_{i'h}]$

$\Rightarrow \sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} = (1-n)[p_{i'h} - p_{ih}]$

Thus $\sum_{k=1}^{n-1} p_{ik} - \sum_{k=1}^{n-1} p_{i'k} > p_{i'h} - p_{ih}$. When the left hand side of this equals to 0, which means $\sum_{k=1}^{n-1} p_{ik} = \sum_{k=1}^{n-1} p_{i'k} \Longrightarrow 0 > p_{i'h} - p_{ih} \Longrightarrow p_{i'h} < p_{ih}$ which satisfies (4.9.1), (4.9.2) and(4.9.3) ■

Example 4.2: Suppose a decision-maker provides his assessments for one criterion on four alternatives using following reciprocal additive preference relation:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0.5 & 0.41 & 0.62 & 0.66 \\ 0.59 & 0.5 & 0.71 & 0.75 \\ 0.38 & 0.29 & 0.5 & 0.54 \\ 0.34 & 0.25 & 0.46 & 0.5 \end{pmatrix} \end{array}$$

This preference relation is 100% consistent and yields following ranking scores using (4.9):

$$A_2(0.319) > A_1(0.274) > A_3(0.214) > A_4(0.193)$$

When A_4 is removed, the consistency degree is still 100%. Likewise, the ranking order is:

$$A_2(0.4) > A_1(0.34) > A_3(0.260)$$

There is no rank reversal because (4.9.1) is satisfied

$S_2^4 > S_1^4 \Rightarrow \sum_{k=1}^{n-1} p_{2k} - \sum_{k=1}^{n-1} p_{1k} > p_{1h} - p_{2h} \Rightarrow \{1.8 - 1.53\}0.27 > \{0.66 - 0.75\} - 0.09.$

$S_2^4 > S_3^4 \Rightarrow \sum_{k=1}^{n-1} p_{2k} - \sum_{k=1}^{n-1} p_{3k} > p_{3h} - p_{2h} \Rightarrow \{1.8 - 1.17\}0.63 > \{0.54 - 0.75\} - 0.21.$

$S_1^4 > S_3^4 \Rightarrow \sum_{k=1}^{n-1} p_{1k} - \sum_{k=1}^{n-1} p_{3k} > p_{3h} - p_{1h} \Rightarrow \{1.53 - 1.17\}0.36 > \{0.54 - 0.66\} - 0.12.$

4.5. Proposed Score Aggregation Methods

Based on these results, the only way to ensure ranking order is free of rank reversal in the preference relations is by ensuring that the preference relation(s) is perfectly consistent. However, to some extent this is hard to achieve in real world problems, especially in a group decision-making environment where there is a tradeoff between consistencies and consensus (Herrera-Viedma et al., 2007a). Thus, we need to handle rank reversal when it is not desirable by maintaining some guidelines that deal with the dependency of the data/information. Here we present three scenarios that are possible to happen to the set of alternatives during the decision process: a new alternative is introduced, an existing alternative is removed, or one alternative is replaced by a new one.

Note: This is only applied if the set of alternatives have been modified.

Proposition 4.4: The following formulation does better than the sum normalization method in avoiding rank reversal in reciprocal preference relations when a new alternative, h, is introduced:

$$\tilde{S}_i^{n+1} = \frac{2\sum_{k=1,h\notin n}^{n} p_{ik} + p_{ih} + p_{ih}^{e}}{(n+1)^2} \qquad \forall i \in n+1 \tag{4.10}$$

where p_{ih}^{e} is the estimated preference degree calculated by (4.5).

Proof:

When $P^{n+1}$is perfectly consistent, then $S_i^{n+1} = \tilde{S}_i^{n+1}$since $p_{ih} = p_{ih}^{e}$ thus

$\tilde{S}_i^{n+1} = \frac{2\sum_{k=1,h\notin n}^{n} p_{ik} + p_{ih} + p_{ih}^{e}}{(n+1)^2} = \frac{2\sum_{k=1,h\notin n}^{n} (p_{ik} + p_{ih})}{(n+1)^2} = \frac{2\sum_{k=1}^{n+1} p_{ik}}{(n+1)^2} = S_i^{n+1}$. However, when

P^n has an acceptable consistency degree but $h \in n+1$ is not, then the ranking generated by the sum normalization method might be affected by the information of h. Thus, integrating the values driven by the consistency property (4.5) and the provided ones for h will improve the consistency degree of P^{n+1}. The chances of rank reversal decreases as the consistency increases.

For $S_i^{n+1} > S_{i'}^{n+1}$ then

$\frac{2}{(n-1)^2}\sum_{k=1}^{n+1} p_{ik} > \frac{2}{(n-1)^2}\sum_{k=1}^{n} p_{i'k} \Rightarrow \sum_{k=1,h\notin n}^{n} p_{ik} + p_{ih} > \sum_{k=1,h\notin n}^{n} p_{i'k} + p_{i'h}$,

since $\sum_{k=1}^{n+1} p_{ik} = \sum_{k=1,h\notin n}^{n} p_{ik} + p_{ih}$ then

$\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} > p_{i'h} - p_{ih}$. Thus $S_i^n > S_{i'}^n$ and $S_i^{n+1} > S_{i'}^{n+1}$ only if $\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} > p_{i'h} - p_{ih}$.

However, with $\tilde{S}_i^{n+1} > \tilde{S}_{i'}^{n+1}$, after eliminating the constants in both sides we get,

$\Rightarrow \sum_{k=1,h\notin n}^{n} p_{ik} + \frac{p_{ih}+p_{ih}^e}{2} > \sum_{k=1,h\notin n}^{n} p_{i'k} + \frac{p_{i'h}+p_{i'h}^e}{2} \Rightarrow 2\sum_{k=1,h\notin n}^{n} p_{ik} - 2\sum_{k=1,h\notin n}^{n} p_{i'k} >$
$p_{i'h} + p_{i'h}^e - p_{ih} - p_{ih}^e$, but $p_{ih}^e = \frac{1}{(n+1-2)}\sum_{\substack{k=1\\k\neq i\neq h}}^{n+1} (p_{ik} + p_{kh} - 0.5)$ then

$\Rightarrow 2\left(\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k}\right) > p_{i'h} - p_{ih} - \frac{1}{(n-1)}\sum_{\substack{k=1\\k\neq i\neq h}}^{n+1} (p_{ik} + p_{kh} - 0.5) +$
$\frac{1}{(n-1)}\sum_{\substack{k=1\\k\neq i'\neq h}}^{n+1} (p_{i'k} + p_{kh} - 0.5)$,

$\Rightarrow 2\left(\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k}\right) > p_{i'h} - p_{ih} - \frac{1}{(n-1)}\sum_{\substack{k=1\\k\neq i\neq h}}^{n+1} (p_{ik} + p_{kh}) +$
$\frac{1}{(n-1)}\sum_{\substack{k=1\\k\neq i'\neq h}}^{n+1} (p_{i'k} + p_{kh})$,

$\Rightarrow 2\left(\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k}\right) > p_{i'h} - p_{ih} + \frac{1}{(n-1)}\sum_{\substack{k=1\\k\neq i\neq h}}^{n+1} (p_{i'k} - p_{ik})$,

but $\sum_{\substack{k=1\\k\neq i\neq h}}^{n+1} p_{ik} = \sum_{k=1,h\notin n}^{n} p_{ik} - 0.5$, thus

$\Rightarrow \frac{2n-1}{n-1}\left(\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k}\right) > (p_{i'h} - p_{ih}) \Rightarrow \sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} > \frac{n-1}{2n-1}(p_{i'h} - p_{ih}),$

When generating the ranking scores for P^{n+1} with sum normalization there is no rank reversal only if

$\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} > p_{i'h} - p_{ih},$

but with (4.10) there is no rank reversal only if

$\sum_{k=1,h\notin n}^{n} p_{ik} - \sum_{k=1,h\notin n}^{n} p_{i'k} > \frac{n-1}{2n-1}(p_{i'h} - p_{ih})$, so clearly (4.10) has a higher possibility to avoid rank reversal than sum normalization. In addition, (4.10) ensures maintaining the sum of the scores of the alternatives at 1, $\sum_{i=1}^{n+1} \tilde{S}_i = 1$■

Proposition 4.5: The following formulation does better than the sum normalization method in avoiding rank reversal in reciprocal preference relations when an alternative h is replaced by a new alternative h':

$$\tilde{S}_i^{n'} = \frac{2\sum_{k=1,h'\notin n-1}^{n-1} p_{ik} + p_{ih'} + p_{ih'}^{e}}{n^2} \qquad \forall i \in n \tag{4.11}$$

Proof:

Similar to the proof of the previous proposition.

Proposition 4.6: The following formulation prevents rank reversal from occurring in reciprocal preference relations when an alternative, h, is removed:

$$\tilde{S}_i^{n-1} = \frac{2(\sum_{k=1}^{n-1} p_{ik} + p_{ih})}{n^2 - 2\sum_{k=1}^{n} p_{hk}} = \frac{2\sum_{k=1}^{n} p_{ik}}{n^2 - 2\sum_{k=1}^{n} p_{hk}} \qquad \forall i \neq h \tag{4.12}$$

Proof:

$S_i = \frac{2}{n^2}\sum_{k=1}^{n} p_{ik} \Rightarrow n^2 S_i = 2\sum_{k=1}^{n} p_{ik}$, when $S_i^n > S_{i\prime}^n$ then

$n^2 S_i > n^2 S_{i\prime}$ Thus if we divide both sides by any constant greater than zero, the inequality will not be affected. Therefore, we divide both sides by $n^2 - 2\sum_{k=1}^{n} p_{hk}$ since $2\sum_{k=1}^{n} p_{hk}$ is always less than n^2, where n is the number of alternatives of the original problem. We get:

$$\frac{n^2 S_i}{n^2 - 2\sum_{k=1}^{n} p_{hk}} > \frac{n^2 S_{i\prime}}{n^2 - 2\sum_{k=1}^{n} p_{hk}} \Longrightarrow \frac{2\sum_{k=1}^{n} p_{ik}}{n^2 - 2\sum_{k=1}^{n} p_{hk}} > \frac{2\sum_{k=1}^{n} p_{ik}}{n^2 - 2\sum_{k=1}^{n} p_{hk}}$$

$\Rightarrow \tilde{S}_i^{n-1} > \tilde{S}_{i\prime}^{n-1} \ \forall i \neq h$, and this formulation also ensures maintaining the sum of the scores of the alternatives at 1, $\sum_{i=1}^{n-1} \tilde{S}_i = 1$ ■

4.6. Numerical Example

Suppose that four decision-makers provide their assessments (by fuzzy preference relations) on four alternatives as follows:

$$P^1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0.50 & 0.38 & 0.20 & 0.28 \\ 0.62 & 0.50 & 0.32 & 0.40 \\ 0.80 & 0.68 & 0.50 & 0.58 \\ 0.72 & 0.6 & 0.42 & 0.50 \end{pmatrix} \end{array} \qquad P^2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0.50 & 0.38 & 0.25 & 0.33 \\ 0.62 & 0.50 & 0.37 & 0.45 \\ 0.75 & 0.63 & 0.50 & 0.58 \\ 0.67 & 0.55 & 0.42 & 0.50 \end{pmatrix} \end{array}$$

$$P^3 = \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0.50 & 0.75 & 0.55 & 0.41 \\ 0.25 & 0.50 & 0.30 & 0.16 \\ 0.45 & 0.70 & 0.50 & 0.36 \\ 0.59 & 0.84 & 0.64 & 0.50 \end{pmatrix} \end{array} \qquad P^4 = \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0.50 & 0.40 & 0.30 & 0.60 \\ 0.60 & 0.50 & 0.40 & 0.70 \\ 0.70 & 0.60 & 0.50 & 0.80 \\ 0.40 & 0.30 & 0.20 & 0.50 \end{pmatrix} \end{array}$$

After several rounds of discussion, they reach an acceptable level of consensus, which results in the following collective preference relation, which has been aggregated by a weighted averaging operator assuming equal weights for decision-makers:

$$P^c = \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0.5 & 0.48 & 0.26 & 0.41 \\ 0.52 & 0.5 & 0.35 & 0.55 \\ 0.74 & 0.65 & 0.5 & 0.69 \\ 0.59 & 0.45 & 0.31 & 0.5 \end{pmatrix} \end{array}$$

This preference relation is 95% consistent. If we calculate the ranking score by the sum normalization method (4.9), then we get the following ranking order:

$A_3(0.323) \succ A_2(0.24) \succ A_4(0.231) \succ A_1(0.206)$.

A. Adding a new alternative

Now consider that the decision-makers introduce a new alternative A_5. Going through the consensus process, they end up with the following collective preference relation:

$$P^c = \begin{array}{c c} & \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \begin{pmatrix} 0.5 & 0.48 & 0.26 & 0.41 & 0.52 \\ 0.52 & 0.5 & 0.35 & 0.55 & 0.92 \\ 0.74 & 0.65 & 0.5 & 0.69 & 0.25 \\ 0.59 & 0.45 & 0.31 & 0.5 & 0.55 \\ 0.48 & 0.08 & 0.75 & 0.45 & 0.5 \end{pmatrix} \end{array}$$

The consistency degree of this preference relation has dropped to 78.5% and the new ranking order by (4.9) is:

$$A_2(0.227) \succ A_3(0.226) \succ A_4(0.192) \succ A_5(0.181) \succ A_1(0.174)$$

Notice that by introducing A_5, which is a non-dominant alternative, the ranking order for the first two alternatives has reversed. This is because the collective preference relation is not perfectly consistent and thus, there is no guarantee that (4.9.1) and (4.9.3) are satisfied.

However, if we apply(4.10), which relies on improving the consistency of the added alternative, we get the following ranking order:

$$A_3(0.251) \succ A_2(0.207) \succ A_4(0.19) \succ A_5(0.181) \succ A_1(0.171)$$

This ranking order is similar to the original problem except that alternative A_5 has been placed in its right ranking position among the alternatives.

B. Replacing an alternative

Now, let us consider that alternative A_2 has been replaced by $A_{2\prime}$ in the original problem. The collective preference relation is 81% consistent for the collective preference relation below:

$$P^c = \begin{array}{c} \\ A_1 \\ A_{2\prime} \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_{2\prime} & A_3 & A_4 \end{array} \\ \begin{pmatrix} 0.5 & 0.55 & 0.26 & 0.41 \\ 0.45 & 0.5 & 0.6 & 0.45 \\ 0.74 & 0.4 & 0.5 & 0.69 \\ 0.59 & 0.55 & 0.31 & 0.5 \end{pmatrix} \end{array}$$

The following are the ranking orders obtained by the sum normalization method (4.9) and the proposed formula(4.11), respectively:

Obtained by (4.9): $A_3(0.291) \succ A_2,(0.25) \succ A_4(0.244) \succ A_1(0.215)$

Obtained by (4.11): $A_3(0.314) \succ A_2,(0.25) \succ A_4(0.236) \succ A_1(0.2)$

Note that both methods generate the same ranking order but with different score values.

C. Removing an alternative

Consider Example 4.1 again,

$$\begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0.5 & 0.55 & 0.62 & 0.65 \\ 0.45 & 0.5 & 0.7 & 0.75 \\ 0.38 & 0.3 & 0.5 & 0.85 \\ 0.35 & 0.25 & 0.15 & 0.5 \end{pmatrix} \end{array}$$

Where the preference relation is 82% consistent and has the following ranking order, by (4.9):

$$A_2(0.3) > A_1(0.29) > A_3(0.254) > A_4(0.156)$$

We saw that when alternative A_4 is removed, the consistency degree increases but the ranking order has reversed between the first and the second:

$$A_1(0.371) > A_2(0.367) > A_3(0.262)$$

However, if we apply (4.12) we get the following ranking order:

$$A_2(0.356) > A_1(0.344) > A_3(0.3)$$

which is consistent with the ranking order of the original problem.

4.7. Conclusions

In this chapter, we have proved that consistency of the data/information is the main cause of rank reversal in preference relation. Also, we have shown that when the preference relations are perfectly consistent then neither a weighted averaging aggregation operator nor an arithmetic mean aggregation operator could cause rank reversal. This is also true for the score aggregation operator, particularly, the sum normalization method. However, when the preference relation is inconsistent, which is usually the case in real life decision problems, then the score aggregation operator could generate rank reversal when the set of alternatives is modified. Thus, we proposed modified score aggregation operators that could be used when a change in the set of alternatives is done. The proposed score aggregation operators integrate the consistency element to reduce the chances of rank reversal. We show that the proposed operators perform better than the sum normalization method in avoiding rank reversal when a change happens in the set of alternatives.

Chapter 5

Conclusions and Future Works

5.1. Conclusions and Contributions

The process of reaching a decision in group decision-making is complex and hard to achieve, especially if the decision problem is complicated. Usually, the members of the group differ in their expertise as well as come from different backgrounds with different goals and objectives. In addition, decision problems with different attributes or criteria, which usually are conflicting, make consensus hard to achieve. In this thesis, we have considered the most important processes of decision-making from the early stages where the decision-makers provide their preferences in the alternatives to the consensus process and reach a decision in the selection stage. Our scope was handling the challenges that the decision-maker usually faces under each of these three stages in preference relations format. Making decisions based on preference relation tends to be more accurate than other preference representation formats. However, this concept has its own challenges, as do the other decision-making representation formats.

In this thesis, we focus our work on solving the issues of the three main problems that decision-making could encounter. In the first stage, the stage of providing the information, we dealt with the problem of missing information and how to help the decision-maker complete their preference relation. In the second stage, we proposed a novel methodology to help the group members reach the consensus state. Lastly, in the third stage, the selection process, we addressed the issue of rank reversal in preference relation. Each of these issues has been solved separately, as presented in Chapters 2, 3 and 4 of this thesis.

For the first problem, we proposed two new methods to handle missing information in an incomplete reciprocal fuzzy preference relation. The solution was based on completing the missing information by relying on additive consistency through focus on completing the upper triangular relation. The first method, which is based on the system of equations, relies on the provided information from the decision-maker to estimate the preference degree(s) of ungiven one(s) using the additive consistency property. This method is suitable to complete any incomplete preference relation that has at least $n-1$ non-diagonal preference degrees. In the case that only $n-1$ non-diagonal preference degrees are given, this method guarantees a complete preference relation with perfect consistency. Nevertheless, even if more than $n-1$ non-diagonal preference degrees are given, the method provides an estimation of the missing preference degrees with better consistency than the existing methods. The second method, which also has the ability to match the first method's performance, is based on a goal programming model. This method was developed to handle ignorance situations, when the decision-makers are not able to provide their preferences for at least one alternative. To our knowledge, very few papers have worked on this situation. Unlike these papers, our approach does not require a modification to the decision-makers' preferences nor does it violate the reciprocal rule. Moreover, our comparison of these methods showed that our methods excel in terms of generation estimations with a better consistency level.

For the second problem, we proposed a novel method to measure consensus among the group during the consensus process. This process is a very important stage in any group decision-making, since attaining an agreeable solution by the group is very important. The consensus process is an iterative process where rounds of discussion need to be done until the group reaches an acceptable state of agreement about their preferences. Usually, this is conducted by a facilitator or a moderator. Our method, differs from the usual consensus measure trend in the literature, and relies on Spearman's rank correlation to measure the consensus among the decision-makers. Most of the consensus measures in the literature are based on similarity or distance functions, while our method takes advantage of Spearman's correlation ability to measure the monotonic degree to measure it between the members of the group. Thus, this method does not rely directly on similarity/distance functions, which sometimes might not reflect the actual consensus. Moreover, we proposed a feedback mechanism that acts as a moderator to guide the group members to the consensus level. With this feedback mechanism, the group could seek consensus without relying directly on the collective preference relation every time, unlike other methods, which require an update to the collective preference relation after each round of adjustments. The proposed method can guide the group members to the consensus either by using collective preference and decision-maker's preference relation or measuring the consensus directly from the decision-maker's preference relations. The feedback mechanism provides the decision-maker with a range of values based on their ranked preferences to the others that they need to choose from to improve the consensus. The method has been validated by applying it to several problems, and it shows good performance in terms of the results.

For the third problem, we investigated the rank reversal phenomena in additive preference relation. Many researchers have discussed this phenomenon; some claim that it is a legitimate outcome and others criticized the methods that allow it. Despite this division, our goal was to study what causes it to occur in preference relations and how to prevent it when it is not desirable. To the best of our knowledge, only Leskinen and Kangas (2005) have studied the inconsistency of preference relations based on a regression model. There is no study that investigates the three possible causes behind rank reversal. In Chapter 4, we investigated these three possible causes: inconsistency, preference relation aggregation, and score aggregation. Our investigation was based on examining the additive consistency in the preference relation. We have concluded that inconsistency is the main cause of this phenomenon. If the preference relation is perfectly consistent, then no rank reversal would happen. The problem with inconsistent information is that some alternatives are sometimes the outbalanced ones that have the largest intensities, which differentiate between two other alternatives. When they are eliminated or added, they create the difference between the alternatives when their scores are aggregated. The consistency drives by the transitivity control when the preference relation is perfectly consistent. Thus, no rank reversal occurred. With regards to aggregating the preference relations into the collective relation, we proved that this process does not incorporate directly into this phenomenon. Both the arithmetic mean operator and the weighted averaging operator maintain the mean of the individuals' preference relations consistency degrees or the weighted averaging of the individuals' preference relations consistency degrees, respectively. Basically, they just transfer the consistency degrees of the decision-makers' relations into the collective preference relation, whereas, a score aggregation method, namely, the sum normalization method, does incorporate preference relations in generating rank reversal. Since attaining a perfect consistency is hard, especially after the consensus process, we proposed two score aggregation methods to be used when any change happens to the set of alternatives. The first score aggregation method is used when a new alternative is introduced or when an existing one is replaced by a new one. We proved that this aggregation performs better than the sum normalization method in avoiding rank reversal. The second score aggregation method is used when an alternative is removed from consideration. We also guarantee that this method will not allow any rank reversal from occurring.

5.2. Future Works

The consensus work has been done under the assumption of cooperative environment, where the members of the group are willing to modify or adjust some of their assessments to get to the consensus state. For future work, we would like to test the new consensus measure on real group decision-making problems to validate it.

Also, we would like to extend our current work in incomplete preference relation to be based on multiplicative consistency. Multiplicative transitivity is considered as important as additive transitivity in the literature. Multiplicative consistency was introduced by Saaty in 1980 when he introduced the AHP method. We would like to develop a method based on this transitivity for ignorance situations.

Moreover, recently, a new extension of fuzzy sets called hesitant fuzzy sets has gained the attention of researchers. It has been introduced to deal with hesitant situations, where the decision-maker is undecided about their preferences. In this case, a hesitant fuzzy preference relation can be a preference relation where a preference degree could have more than one value because the decision-maker is not certain about which intensity value is the right one. It would be interesting to see how incomplete information in hesitant fuzzy preference relation would be completed or estimated.

In addition, it would be interesting to work on an approach that can detect the most inconsistent preference degree(s) within a preference relation and suggest a better estimation to it to improve the overall consistency. Sometimes, few inconsistent preference degrees can contribute heavily in reducing a preference relation consistency degree. Thus, knowing the most effective values and adjusting them will improve the consistency of the preference relation significantly.

It is known that consistency level and consensus level are moving in opposite directions in group decision-making; when one of them increases, the other one benefits. Nevertheless, both are important in any group decision-making decision problem. Most of the existing methods focus mainly on either one of them and less, if not neglecting, the other. Thus, incorporating a consistency measure within the consensus process will strengthen the output of this process, in addition to making the results robust for the next process, the selection process. There are some works in this direction; however, they are based on favoring one over the other. In our case, we are thinking of guiding the consensus process simultaneously toward consistency and consensus.

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