Analytical Structure of Stationary Flows of an Ideal Incompressible Fluid

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ABSTRACT

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The Euler equations describing the flow of an incompressible, inviscid fluid of uniform density were first published by Euler in 1757. One of the recent results of mathematical fluid dynamics was the discovery that the particle trajectories of such flows are real analytic curves, despite limited regularity of the initial flow (Serfati, Shnirelman, and others). Hence, the flow lines of stationary solutions to the Euler equations are real analytic curves. In this work we consider a two-dimensional stationary flow in a periodic strip. Our goal is to incorporate the analytic structure of the flow lines into the solution of the problem. The equation for the stream function is transformed to new variables, more appropriate for the further analysis. New classes of functions are introduced to take into account the partial analytic structure of solutions. This makes it possible to regard the problem as an analytic operator equation in a complex Banach space. The Implicit Function theorem for complex Banach spaces is applied to establish existence of unique solutions to the problem and the analytic dependence of these solutions on the parameters. Our approach avoids working in the Fréchet spaces and using the Nash-Moser-Hamilton Implicit Function Theorem used by the previous authors (Šverák&Choffrut), and provides stronger results.

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Contents

Chapter 1

Background

We start with a brief derivation of the equation of study, and introduce some relevant concepts and terminology. Consider some fluid filling a domain in $\mathbb{R}^n$, with mass density $\rho(x,t)$, velocity field $\boldsymbol{u}(x,t)$ and pressure $p(x,t)$. We pick some arbitrary blob of fluid (a fixed collection of fluid particles), then the blob will move around and deform as the fluid flows. At any moment in time it occupies a closed region $\Omega(t)$. Let b be some quantity B per unit volume that is carried by the flow. Then the total quantity of B in the blob is given by

$$B(t) = \int_{\Omega(t)} b(x,t)dV.$$

On the other hand, the rate of change of B carried by the blob is given by

$$\frac{dB(t)}{dt} = \int_{\Omega(t)} \frac{\partial b(x,t)}{\partial t}dV + \int_{\partial\Omega(t)} b(x,t)\boldsymbol{u}\cdot\boldsymbol{n}dS$$

where $\boldsymbol{n}$ is the outward pointing normal of the surface $\partial\Omega$. The above is known as Reynolds transport theorem for a material element. It can be understood in the following sense: at any moment in time, the rate of change of B in the blob is equal to the rate of change of B inside the volume it momentarily occupies plus the net flow rate of the quantity out of this volume. Applying the divergence theorem to the right most term we get

$$\frac{dB(t)}{dt} = \int_{\Omega(t)} \frac{\partial b}{\partial t} + \nabla\cdot(b\boldsymbol{u})dV. \tag{1.1}$$

Now we are ready to apply the conservation laws of physics to obtain the governing equations of a fluid flow. Conservation of mass tells us the mass of the blob is constant.

Hence

$$\frac{dm(t)}{dt} = \int_{\Omega(t)} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) dV = 0.$$

Note that our selection of blob was arbitrary, so the above equality must hold pointwise. We thus obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0.$$

Newton's second law of motion tells us that for any system (the blob), the rate of change of momentum is equal to the force acting on it. We denote the momentum components of the blob by q_i. Let us suppose there is no friction between the fluid particles. This corresponds to an inviscid fluid. In the absence of any external forces (such as gravity) acting on the fluid, the only force acting on the blob is the pressure from the surrounding fluid. Therefore the force acting on the blob is

$$\boldsymbol{F} = -\int_{\partial\Omega(t)} p \boldsymbol{n} dS = -\int_{\Omega(t)} \nabla p dV.$$

Newton's second law thus gives us

$$\int_{\Omega(t)} \frac{\partial(\rho u_i)}{\partial t} + \nabla \cdot (\rho u_i \boldsymbol{u}) + \partial_i p dV = 0.$$

As before the equality must hold pointwise, and together with the continuity equation we get the general Euler Equations

$$\begin{cases} \dfrac{\partial(\rho u_i)}{\partial t} + \nabla \cdot (\rho u_i \boldsymbol{u}) + \partial_i p = 0 \\ \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0. \end{cases}$$

Let us suppose that the fluid is incompressible and has uniform density. We can take $\rho \equiv 1$. We thus obtain the incompressible Euler equation, which can be written in vector notation as

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0 \\ \nabla \cdot \boldsymbol{u} = 0. \end{cases} \tag{1.2}$$

The first equation is known as the momentum equation, the second as the incompressibility condition. Together they govern the flow of a homogeneous, ideal (inviscid), incompressible fluid. For the rest of this work, by 'flow' we will always mean the flow of such a fluid, ie a solution to 1.2.

We will restrict ourselves to the study of stationary (steady state) flows of a two-dimensional fluid. By 'stationary' flows, we mean solutions of 1.2 which are time-independent: $\boldsymbol{u}(x,t) = \boldsymbol{u}(x)$. Two-dimensional flows have a particularly useful formulation in terms of the stream function and vorticity, which we now introduce. We can regard such a flow in three-dimension as $\boldsymbol{u}(x,y,z,t) = (u(x,y,t), v(x,y,t), 0)$. From vector calculus we know that a divergence free vector field can be written as the curl of some vector potential. Thus we can write $\boldsymbol{u} = \nabla \times \boldsymbol{\psi}$ where $\boldsymbol{\psi} = (0,0,\psi)$. The scalar $\psi(x,y,t)$ is called the stream function and it satisfies $(u,v) = (\psi_y, -\psi_x) = \nabla^\perp \psi$. Thus it is defined uniquely, up to an additive constant. We define the streamlines (or flow lines) of $\boldsymbol{u}$ as its integral curves at any moment in time.

Remark 1.1. The stream function is so called because its level lines correspond with stream lines. To see this, suppose at some fixed moment in time, the curve $\alpha(s)$ is a level line of ψ, ie $\psi(\alpha(s)) = c$. Then

$$\frac{d\psi(\alpha(s))}{ds} = \psi_x \frac{d\alpha_1}{ds} + \psi_y \frac{d\alpha_2}{ds} = 0$$

Therefore $\nabla\psi$ is perpendicular to $\frac{d\alpha}{ds}$. But $\nabla\psi = (-v, u)$ is perpendicular to (u,v), thus $\boldsymbol{u}(\alpha(s))$ and $\frac{d\alpha(s)}{ds}$ are parallel. Hence the flow points along $\alpha(s)$. Note that for a stationary flow, the flow lines coincide with the particle trajectories. Also note that the critical points of the stream function ($\nabla\psi = 0$) correspond to stagnation points of the flow (points at which velocity is zero).

We define the vorticity $\boldsymbol{\omega} = \nabla \times u$. For two-dimensional flows, only the z component is nonzero, thus vorticity can be regarded as a scalar defined by

$$\omega = -\Delta\psi. \tag{1.3}$$

Physically, $\boldsymbol{\omega}(x,y,t)$ represents the angular velocity of an infinitesimal blob centered on (x,y). By taking the curl of 1.2 we obtain the vorticity formulation of the two-dimensional Euler equation

$$\frac{\partial\omega}{\partial t} + \boldsymbol{u} \cdot \nabla\omega = 0. \tag{1.4}$$

We notice the following property, a flow is stationary if and only if $\boldsymbol{u} \cdot \nabla \omega = 0$. In other words, when ω is constant along flow lines. From 1.3, we see then that a stream function ψ defines a stationary flow if and only if it satisfies for some function F the equation

$$\Delta \psi = F(\psi). \tag{1.5}$$

We call this the equation of the stream function for a two-dimensional stationary flow, and it will be the main equation of study in this work.

The equations for the flow of an ideal, incompressible fluid were first published by Euler in 1757. To this day, they remain an active field of study, with many important questions still unanswered. The main problem of interest is to solve 1.2 for the flow $\boldsymbol{u}$ given an initial flow $\boldsymbol{u}_0$, keeping in mind the pressure p is also an unknown. Existence and uniqueness of solutions that are local in time, and as regular as the initial data goes back to classical works of Liechtenstein, Gunter, and later works of Ebin&Marsden, etc (see [6], [3], [7]). For two-dimensional flows, existence and uniqueness of global in time solutions was shown by Wolibner, Yudovich, etc (see [16], [17]). Existence and uniqueness of global in time solutions in three-dimensions remains an open problem. Starting with the works of Arnold (see [1], [4]) the Euler equations were shown to have an inherent geometric structure, by interpreting their solutions as geodesic flows on the group of volume preserving diffeomorphisms. One of the outstanding achievements in the study of ideal, incompressible flows was the discovery that despite limited regularity of initial flows, the particle trajectories are real analytic curves (Serfati [12], Shnirelman [13], and others). Hence, the flow lines of steady state solutions of the Euler equation are real analytic curves.

In the work of Šverák&Choffrut [14], they establish 'a geometric picture of the structure of the set of steady-states of Euler's equations'. They consider a two-dimensional flow on an annulus and find a local parametrization for the set of stationary solutions to the Euler equations. The main difficulty here is the fact that the set of stationary solutions in the Sobolev space is not a smooth manifold. It becomes smooth only in C^∞ which is a Fréchet space. Therefore, they were forced to use the Nash-Moser-Hamilton implicit function theorem for Fréchet spaces. This makes their results less natural and less complete.

Chapter 2

Introduction

In this work we seek to improve on the results of Šverák&Choffrut [14]. We consider the equation for the stream function for a stationary flow 1.5, along a periodic strip. Our goal is two-fold, to incorporate the analytic structure of the flow lines into the solution and to form a Banach space for the solutions. This will allow us to locally solve the equation using the Implicit Function Theorem in Banach spaces, thus avoiding the Nash-Moser Implicit Function Theorem in Fréchet spaces.

Consider a two-dimensional stationary flow along a periodic strip. The flow lines (which are analytic curves) correspond to the level lines of the stream function $\psi(x, y) = c$. In the case when the stream function has no critical points ($\nabla\psi \neq 0$), the flow lines have no stagnation points, never intersect and thus may be seen as graphs of some periodic functions $y = a(x, \psi)$ which are analytic in x for each fixed $\psi = c$. Consider the domain between two flow lines, say $\psi = 0$ and $\psi = 1$. Suppose these boundary flow lines correspond to graphs of some analytic functions $y = f(x) = a(x, 0)$ and $y = g(x) = a(x, 1)$ respectively. Then the stream function satisfies the following boundary value equation

$$\begin{cases} \Delta\psi(x, y) = F(\psi) \\ \psi(x, f(x)) = 0 \\ \psi(x, g(x)) = 1 \end{cases} \tag{2.1}$$

on the domain $\{(x, y) : x \in \mathbb{T}, f(x) \leq y \leq g(x)\}$. Notice that there is a trivial solution when $F = 0$ given by $\psi(x, y) = y$, $f(x) = 0$ and $g(x) = 1$. Such a solution corresponds to a constant flow along the strip, and every flow line between $\psi = 0$ and $\psi = 1$ is given by $y = a(x, \psi) = \psi$. We expect that when F is close enough to zero, no critical points of ψ are introduced, and therefore the flow lines remain graphs of functions $y = a(x, \psi)$. In our work, we will seek such solutions to 2.1. To incorporate the analytic structure of the flow lines, we face the problem that functions whose level lines are real analytic curves do not form a linear space. Thus the stream function ψ itself is not an appropriate object for our study. Instead we will consider the flow lines themselves. To do this we must transform the equation 2.1 from coordinates (x, y) to (x, ψ).

For clarity let's introduce new coordinates (ξ, η). Now consider the coordinate transformation

$$(x, y) \to (\xi, \eta) = (x, \psi(x, y)).$$

The inverse transformation is given by

$$(\xi, \eta) \to (x, y) = (x, a(\xi, \eta)).$$

Computing the Jacobian of the transformation we get

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_\xi & a_\eta \end{bmatrix}.$$

Taking the inverse, we find the Jacobian of the inverse transformation

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{a_\xi}{a_\eta} & \frac{1}{a_\eta} \end{bmatrix}.$$

We thus obtain equations for the partial derivatives

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{a_\xi}{a_\eta}\frac{\partial}{\partial \eta} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{1}{a_\eta}\frac{\partial}{\partial \eta}.$$

Taking second derivatives we get the expressions

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} - \frac{2a_\xi}{a_\eta}\frac{\partial^2}{\partial \xi \partial \eta} + \left(\frac{a_\xi}{a_\eta}\right)^2 \frac{\partial^2}{\partial \eta^2} + \left[\frac{a_\xi}{a_\eta}\left(\frac{a_\eta a_{\xi\eta} - a_\xi a_{\eta\eta}}{a_\eta^2}\right) - \left(\frac{a_\eta a_{\xi\xi} - a_\xi a_{\xi\eta}}{a_\eta^2}\right)\right]\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2}{\partial y^2} = \frac{1}{a_\eta^2}\frac{\partial^2}{\partial \eta^2} - \frac{a_{\eta\eta}}{a_\eta^3}\frac{\partial}{\partial \eta}.$$

Applying the Laplacian to $\psi(x,y) = \eta$ and writing our coordinates as $(\xi, \eta) = (x, \psi)$ we get

$$\Delta\psi = \frac{-a_{xx}}{a_\psi} + \frac{2a_x a_{x\psi}}{a_\psi^2} - \frac{(a_x^2+1)\, a_{\psi\psi}}{a_\psi^3} = \Phi(a). \tag{2.2}$$

Thus we have transformed 2.1, the equation of the stream function between two flow lines to the following boundary value problem

$$\begin{cases} \Phi\left(a(x,\psi)\right) = F(\psi) \\ a(x,0) = f(x) \\ a(x,1) = g(x) \end{cases} \tag{2.3}$$

on the domain $\Omega = \mathbb{T} \times (0,1)$. $\Phi(a)$ is a second order nonlinear differential operator, defined by a rational function of partial derivatives of a.

In this work, we consider the following problem. Suppose $F(\psi) \in H^{s-2}(0,1)$. We seek to construct a Banach space for the flow lines $a(x,\psi)$ that incorporates the 'partial analyticity' (analytic in x) of $a(x,\psi)$. We will do this using partial complex extensions of a, thus obtaining a complex Banach space for our solutions. Likewise we will construct an appropriate complex Banach space for the boundary data $f(x)$ and $g(x)$. It will be shown that in such a formulation, 2.3 becomes an analytic operator equation on complex Banach spaces. A local solution can be found by the Analytic Implicit Function theorem on complex Banach spaces, whose proof can be found in most text books of Nonlinear Functional Analysis (for example see [18], [10]):

Theorem 2.1 (Implicit Function Theorem on Banach Spaces)**.** *Let X, Y and Z be Banach spaces, and $A : X \times Y \to Z$ be a C^k mapping (k times continuously Fréchet differentiable). Suppose $(x_0, y_0) \in X \times Y$ such that $A(x_0, y_0) = 0$ and furthermore, $\frac{\partial A(x_0,y_0)}{\partial y} : Y \to Z$ is a Banach space isomorphism. Then there exists neighbourhoods $U \subset X$, $V \subset Y$ of x_0 and y_0 respectively, and a C^k map $\phi : U \to V$ such that $A(x, \phi(x)) = 0$ and $A(x,y) = 0$ if and only if $y = \phi(x)$ for all $(x,y) \in U \times V$. If X, Y and Z are complex Banach spaces and A is an analytic map, then ϕ is analytic.*

Let us briefly sketch our main result. We will construct a complex Banach space Y^s_σ of solutions $a(x,\psi)$ and space $X^{s-1/2}_\sigma$ for boundary functions $f(x)$ and $g(x)$. We define operators

γ_0 and γ_1 which restrict a to the boundary, ie $\gamma_0 a(x,\psi) = a(x,0)$ and $\gamma_1 a(x,\psi) = a(x,1)$. We then consider the map

$$A : X_\sigma^{s-1/2} \times X_\sigma^{s-1/2} \times H^{s-2}(0,1) \times Y_\sigma^s \to Y_\sigma^{s-2} \times X_\sigma^{s-1/2} \times X_\sigma^{s-1/2} \tag{2.4}$$

defined by

$$A\left(f, g, F, a\right) = \left(\Phi(a) - F, \gamma_0 a - f, \gamma_1 a - g\right). \tag{2.5}$$

By noticing that $\Phi(a) = 0$ when $a(x,\psi) = \psi$, we see that

$$A(0,1,0,\psi) = 0. \tag{2.6}$$

In other words, $a(x,\psi) = \psi$ is a solution to 2.3 for $f(x) = 0$, $g(x) = 1$ and $F(\psi) = 0$. This solution corresponds to a constant flow in a region of zero vorticity when the boundary flows are straight lines. After evaluating the derivative of Φ with respect to a at $a(x,\psi) = \psi$ we find that

$$\frac{\partial A(0,1,0,\psi)}{\partial a} : Y_\sigma^s \to Y_\sigma^{s-2} \times X_\sigma^{s-1/2} \times X_\sigma^{s-1/2} \tag{2.7}$$

is defined by

$$a(x,\psi) \to (-\Delta a(x,\psi), \gamma_0 a, \gamma_1 a). \tag{2.8}$$

In order to apply 2.1, we have to show this map is an isomorphism. The body of this thesis will be devoted to constructing the spaces Y_σ^s and $X_\sigma^{s-1/2}$, showing that A is well defined and an analytic operator on these spaces and showing that $\frac{\partial A(0,1,0,\psi)}{\partial a}$ is a Banach space isomorphism. This last step corresponds to solving the Dirichlet problem for the Poisson equation in our constructed spaces. The main result can then be obtained as an immediate application of the implicit function theorem 2.1.

Chapter 3

Partially Analytic Sobolev Spaces

In this chapter we introduce new spaces of functions, which we will use as the space of flow lines in our problem. These spaces should incorporate the analyticity of flow lines, have well behaved differential operators on them and have a Banach structure so that we may later use the implicit function theorem on Banach spaces to obtain our solution. The problem we face is the fact that real analytic functions do not form Banach spaces. Instead we will consider Sobolev functions that extend analytically to a chosen complex domain. The result will be a sort of complex Hardy space. The main tool of this chapter will be some variations of the Paley-Wiener theorem. We start be forming a space of single flow lines.

Let us remind of the Paley-Wiener theorems, which tell us that L^2 functions on the real line (or circle) can be complex analytically extended to various domains if their Fourier transforms satisfy some appropriate growth condition. The Paley-Wiener theorems then characterize the complex Hardy spaces, which are Banach spaces of complex analytic functions. We can adapt these well known results to include a Sobolev structure in such spaces.

We define the complex periodic strip as follows:

$$\mathbb{T}_\sigma = \{z = x + it : x \in \mathbb{T}, |t| < \sigma\} \tag{3.1}$$

where $\sigma > 0$. $\mathbb{T}$ can be identified with the interval $[0, 2\pi]$. Given a function $u(x)$ on the circle, we can write it as the Fourier series

$$u(x) = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikx}.$$

The following version of the Paley-Wiener theorem uses the Fourier series to characterize the subset of $H^s(\mathbb{T})$ that extends analytically to $\mathbb{T}_\sigma$. It will be useful to first prove the following lemmas.

Lemma 3.1. *There is a one-to-one correspondence between holomorphic functions (possibly taking values in a complex Banach space) on the periodic strip and holomorphic functions on the annulus.*

Proof. The map $z \to w = e^{iz}$ defines a holomorphic bijection between the periodic strip $\mathbb{T}_\sigma$ and the annulus $A_\sigma = \{e^{-\sigma} < |w| < e^{\sigma}\}$. Note $dw/dz \neq 0$ for any $w \in A_\sigma$. By the holomorphic inverse function theorem, at any w there exists an analytic map such that $z = F(w)$ in a neighbourhood of w. If $v(w)$ is holomorphic on the annulus, then the function $u(z)$ defined by $u(z) = v(e^{iz})$ is clearly holomorphic on $\mathbb{T}_\sigma$. On the other hand, suppose we are given $u(z)$ holomorphic on $\mathbb{T}_\sigma$. Then the map $v(w)$ defined by $v(w) = u(F(w))$ is holomorphic in the annulus. The result holds for functions taking values in a complex Banach space. □

Lemma 3.2. *If $u(z)$ is holomorphic on $\mathbb{T}_\sigma$ (possibly taking values in a complex Banach space) then it has the following representation*

$$u(z) = u(x + it) = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikz} = \mathcal{F}^{-1}_{k\to x}\left\{\hat{u}_k e^{-kt}\right\}$$

where $\hat{u}_k$ is understood to be the (complex Banach space valued) Fourier transform of $u(x + i0)$.

Proof. By the previous lemma, there exists $v(w)$ that is holomorphic on the A_σ such that $u(z) = v(e^{iz})$. Holomorphic functions on the annulus can be uniquely expressed as a Laurent series

$$v(w) = \sum_{k=-\infty}^{\infty} c_k w^k.$$

The coefficients (which may be Banach valued) are given by

$$c_k = \frac{1}{2\pi i}\oint_\gamma \frac{v(w)}{w^{k+1}} dw$$

where γ is a closed loop inside the annulus. If we denote $z = x + it$, and take γ to be a loop around the unit circle $|w| = 1$ then we get

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} \frac{v(e^{ix})}{e^{ikx}} dx = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ixk} dx = \hat{u}_k.$$

Then the desired result follows immediately:

$$u(z) = v(w) = \sum_{k=-\infty}^{\infty} \hat{u}_k w^k = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikz} = \mathcal{F}^{-1}_{k\to x}\left\{\hat{u}_k e^{-kt}\right\}.$$

□

Theorem 3.3 (Paley-Wiener Theorem for Sobolev functions)**.** *Suppose $u(x) \in H^s(\mathbb{T})$, where $s \geq 0$ is not necessarily a whole number. Let $\sigma > 0$. Then the following statements are equivalent:*

(i) $\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k e^{\sigma|k|}\} \in H^s(\mathbb{T})$

(ii) $u(x)$ extends to $u(z)$ holomorphic in the strip $\mathbb{T}_\sigma$ with

$$\sup_{|t|<\sigma} \|u(\cdot + it)\|_{H^s(\mathbb{T})} < \infty.$$

Proof. $\Longrightarrow$: We extend $u(x)$ to a complex function $u(z)$ by replacing the real variable x with complex variable $z = x + it$ in the Fourier series of $u(x)$. Thus define

$$u(z) = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikz} = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{-kt} e^{ikx} = \mathcal{F}^{-1}_{k\to x}\{\hat{u}_k e^{-kt}\}.$$

To see that the series is well defined for $z \in \mathbb{T}_\sigma$, note

$$|u(z)| \leq \sum_k |\hat{u}_k e^{ikz}| \leq \sum_k |\hat{u}_k e^{-kt}| \leq \sum_k |\hat{u}_k| e^{|t||k|}$$

$$= \sum_k |\hat{u}_k| e^{\sigma|k|} e^{(|t|-\sigma)|k|} \leq \|\mathcal{F}^{-1}_{k\to x}\left\{\hat{u}_k e^{\sigma|k|}\right\}\|_{L^2(\mathbb{T})} \|e^{(|t|-\sigma)|k|}\|_{\ell^2} < \infty$$

where in the second line we have used the Cauchy-Schwarz inequality. Since $|t| < \sigma$, $e^{(|t|-\sigma)|k|}$ is exponentially decaying and thus in ℓ^2. Thus the series is absolutely convergent. Similarly, by differentiating the Fourier series with respect to z, we get

$$|u'(z)| \leq \|\mathcal{F}^{-1}_{k\to x}\left\{\hat{u}_k e^{\sigma|k|}\right\}\|_{L^2(\mathbb{T})} \|k e^{(|t|-\sigma)|k|}\|_{\ell^2} < \infty.$$

We conclude that $u(z)$ is complex analytic in $\mathbb{T}_\sigma$. Furthermore, we have

$$\sup_{|t|<\sigma} \|u(\cdot + it)\|_{H^s(\mathbb{T})} = \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\left\{\hat{u}_k e^{-kt}\right\}\|_{H^s(\mathbb{T})}$$

$$\leq \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\left\{\hat{u}_k e^{|k||t|}\right\}\|_{H^s(\mathbb{T})} = \|\mathcal{F}^{-1}_{k\to x}\left\{\hat{u}_k e^{\sigma|k|}\right\}\|_{H^s(\mathbb{T})} < \infty.$$

The second line follows from the fact that the supremum over $|t| < \sigma$ corresponds to the limit as $|t| \to \infty$. By dominated convergence theorem, we can pass to the limit and obtain the desired result. Therefore the second statement of the theorem is satisfied.

$\Longleftarrow$: By 3.2 we can represent $u(z)$ as

$$u(x+it) = \sum_k \hat{u}_k e^{-kt} e^{ikx} = \mathcal{F}^{-1}_{k\to x}\left\{\hat{u}_k e^{-kt}\right\}.$$

The first statement of the theorem follows by:

$$\begin{aligned}
\|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k e^{\sigma|k|}\}\|_{H^s(\mathbb{T})} &= \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k e^{|k|t}\}\|_{H^s(\mathbb{T})} \\
&\leq \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k (e^{kt} + e^{-kt})\}\|_{H^s(\mathbb{T})} \\
&\leq \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k e^{-kt}\}\|_{H^s(\mathbb{T})} + \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k e^{-kt}\}\|_{H^s(\mathbb{T})} \\
&= \sup_{|t|<\sigma} \|u(\cdot + it)\|_{H^s(\mathbb{T})} + \sup_{|t|<\sigma} \|u(\cdot - it)\|_{H^s(\mathbb{T})} \\
&< \infty.
\end{aligned}$$

In the first line we have again used dominated convergence theorem to pull the limit $|t| \to \infty$ out of the norm. □

Note that it is immediately clear that if a function u satisfies the conditions of the above theorem, then it is well defined on the boundaries of the strip as a limit of H^s functions on horizontal sections of the strip (by the dominated convergence theorem). Thus we define

$$u(\cdot \pm i\sigma) = \lim_{t\to\pm\sigma} u(\cdot + it) \tag{3.2}$$

where this limit is understood as a limit in $H^s(\mathbb{T})$. We are now ready to properly define a space for individual analytic flow lines.

Definition 3.4. Define X^s_σ to be the space of functions satisfying 3.3. Thus we have two equivalent characterizations of this space, one real and the other complex.

(i)
$$X^s_\sigma = \left\{u(x) \in H^s(\mathbb{T}) : \mathcal{F}^{-1}_{k\to x}\{\hat{u}_k e^{\sigma|k|}\} \in H^s(\mathbb{T})\right\}$$
with norm
$$\|u\|^2_{X^s_\sigma} = \sum_{k=-\infty}^{\infty} (1+k^2)^s e^{2\sigma|k|}|\hat{u}_k|^2.$$

(ii) X^s_σ is the space of functions $u(z)$ that are complex analytic on the domain $\mathbb{T}_\sigma$ and satisfy
$$\sup_{|t|<\sigma} \|u(\cdot + it)\|_{H^s(\mathbb{T})} < \infty$$
with norm
$$\|u\|^2_{X^s_\sigma} = \sup_{|t|<\sigma} \|u(\cdot + it)\|^2_{H^s(\mathbb{T})} \cong \|u(\cdot + i\sigma)\|^2_{H^s(\mathbb{T})} + \|u(\cdot - i\sigma)\|^2_{H^s(\mathbb{T})}.$$

Remark 3.5. By viewing X^s_σ as a weighted ℓ^2 space, it is clear that X^s_σ is a (complex) Hilbert space.

Remark 3.6. We can define a mapping T on H^s by $T : u(x) \to \mathcal{F}^{-1}_{k\to x}\left\{\hat{u}_k e^{-\sigma|k|}\right\}$. Then the first part of the definition is equivalent to the statement that $T(H^s) = X^s_\sigma$. In other words, any function of X^s_σ can be seen as the result of taking a function in H^s and exponentially dampening its Fourier coefficients.

Remark 3.7. Note that when $s = 0$, X^0_σ corresponds to the usual complex Hardy space H^2 in the periodic strip. We can say in general, X^s_σ is a complex Hardy space in the strip with additional Sobolev structure.

Proposition 3.8. *X^s_σ is an algebra for $s > 1/2$. That is, for $u, v \in X^s_\sigma$ we have*
$$\|uv\|_{X^s_\sigma} \le C\|u\|_{X^s_\sigma}\|v\|_{X^s_\sigma}.$$

Proof. Let $u, v \in X^s_\sigma$. The product of two holomorphic functions is holomorphic, therefore the product $u(z)v(z)$ is holomorphic in $\mathbb{T}_\sigma$. Furthermore, on every fixed horizontal section $u(\cdot + it)v(\cdot + it)$ is a product of $H^s(\mathbb{T})$ functions up to and including the boundary $|t| = \sigma$.

Since $H^s(\mathbb{T})$ is an algebra for $s > 1/2$, then the product $u(z)v(z)$ is in $H^s(\mathbb{T})$ on each section up to and including the boundary. Therefore $u(z)v(z)$ satisfies second statement of 3.3 and is thus in X^s_σ. The desired inequality follows immediately from $\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s}$. □

Let $u(x,y) \in H^s(\Omega)$, where $\Omega = \mathbb{T} \times (0,1)$. We seek to characterize the subset of these functions which are 'partially analytic' (analytic in x) by proving a 'partial' analogue of theorem 3.3. To do so, we must make precise in what sense an $H^s(\Omega)$ function can be partially analytic. Let γ_x be an operator that restricts a function $u(\cdot,\cdot)$ on Ω to $u(x,\cdot)$ on the section $(0,1)$. It is a well known property of Sobolev spaces that such trace operators are bounded and surjective from $H^s(\Omega)$ to $H^{s-1/2}(0,1)$ (as long as $s > 1/2$), where, the fractional Sobolev space on $(0,1)$ can be defined by the Slobodetsky norm (see for example [9] and [5]). On the other hand, if a function in $H^s(\Omega)$ has additional regularity in x, then we expect the restriction $u(x,\cdot)$ to be in $H^s(0,1)$. Thus we claim the following definition:

Definition 3.9. We say that a function $u \in H^s(\Omega)$ is analytic in x if the map

$$x \to \gamma_x u : \mathbb{T} \to H^s(0,1)$$

is analytic (as a Banach valued function).

The following theorem confirms this result. Note that a function on Ω can be written as a partial Fourier series

$$u(x,y) = \sum_{k=-\infty}^{\infty} \hat{u}_k(y) e^{ikx}.$$

Theorem 3.10 (Partial Paley-Wiener Theorem for Sobolev Functions)**.** *Suppose $u(x,y) \in H^s(\Omega)$, where $s \ge 0$. Let $\sigma > 0$. Then the following statements are equivalent:*

(i) $\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{\sigma|k|}\} \in H^s(\Omega)$

(ii) *$u(x,y)$ extends to $u(z,y)$ complex analytic on $\mathbb{T}_\sigma$ in the sense that the map*

$$z \to u(z,\cdot) : \mathbb{T}_\sigma \to H^s(0,1)$$

is complex analytic (as a complex Banach valued map) and $u(z,y)$ satisfies

$$\sup_{|t|<\sigma} \|u(\cdot + it, \cdot)\|_{H^s(\Omega)} < \infty.$$

Proof. $\implies$: We extend u from $\Omega = \mathbb{T} \times (0,1)$ to $\mathbb{T}_\sigma \times (0,1)$ by

$$u(z,y) = u(x+it,y) = \sum_k \hat{u}_k(y)e^{ikz} = \sum_k \hat{u}_k(y)e^{-kt}e^{ikx} = \mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{-kt}\}.$$

To see that $u(z,\cdot) \in H^s(0,1)$ for $z \in \mathbb{T}_\sigma$, first note that statement one of the theorem directly implies that

$$\sum_k e^{2\sigma|k|}\|\hat{u}_k(y)\|^2_{H^s(0,1)} < \infty.$$

Then we have

$$\|u(x+it,\cdot)\|^2_{H^s(0,1)} = \sum_{p\le s}\int_0^1 |\partial_y^p u(x+it,\cdot)|^2 dy = \sum_{p\le s}\int_0^1 \left|\sum_k \hat{u}_k^{(p)}(y)e^{-kt}e^{ikx}\right|^2 dy$$

$$\le \sum_{p\le s}\int_0^1 \left(\sum_k \left|\hat{u}_k^{(p)}(y)e^{|k||t|}\right|\right)^2 dy = \sum_{p\le s}\int_0^1 \sum_{k,l}\left|\hat{u}_k^{(p)}(y)\right|\left|\hat{u}_l^{(p)}(y)\right| e^{|k||t|}e^{|l||t|}dy$$

$$\le \sum_{p\le s}\sum_{k,l} e^{|k||t|}e^{|l||t|}\|\hat{u}_k^{(p)}\|_{L^2(0,1)}\|\hat{u}_l^{(p)}\|_{L^2(0,1)} = \sum_{p\le s}\left(\sum_k e^{|k||t|}\|\hat{u}_k^{(p)}(y)\|_{L^2(0,1)}\right)^2$$

$$= \sum_{p\le s}\left(\sum_k e^{\sigma|k|}\|\hat{u}_k^{(p)}(y)\|_{L^2(0,1)}e^{(|t|-\sigma)|k|}\right)^2 \le \sum_{p\le s}\left(\sum_k e^{2\sigma|k|}\|\hat{u}_k^{(p)}(y)\|^2_{L^2(0,1)}\right)\left(\sum_k e^{2(|t|-\sigma)|k|}\right)$$

$$= \left(\sum_k e^{2\sigma|k|}\|\hat{u}_k(y)\|^2_{H^s(0,1)}\right)\left(\sum_k e^{2(|t|-\sigma)|k|}\right) < \infty.$$

Note we have used the Cauchy-Schwarz inequality between the second and third line, and again in the fourth line. Of course the right hand term of the last line is in ℓ^2 since $e^{(|t|-\sigma)|k|}$ is exponentially decaying when $|t| < \sigma$. Similarly, we also get

$$\|u_z(z,\cdot)\|^2_{H^s(0,1)} \le \left(\sum_k e^{2\sigma|k|}\|\hat{u}_k(y)\|^2_{H^s(0,1)}\right)\left(\sum_k |k|e^{2(|t|-\sigma)|k|}\right) < \infty.$$

Thus we have confirmed that the map $z \to u(z,\cdot)$ is complex differentiable from $\mathbb{T}_\sigma$ to $H^s(0,1)$ and therefore it is complex analytic. Furthermore, by the Fourier series representation of $u(x+it,y)$ we get

$$\|u(\cdot+it,\cdot)\|_{H^s(\Omega)} = \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{-kt}\}\|_{H^s(\Omega)} \le \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{|k||t|}\}\|_{H^s(\Omega)}.$$

By the dominated convergence theorem, we pass to the limit and get

$$\sup_{|t|<\sigma}\|u(\cdot+it,\cdot)\|_{H^s(\Omega)} = \sup_{|t|<\sigma}\|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{|k||t|}\}\|_{H^s(\Omega)} = \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{\sigma|k|}\}\|_{H^s(\Omega)} < \infty.$$

Thus we have shown $\|u(\cdot + it, \cdot)\|_{H^s} \leq M$ uniformly as $|t| \to \sigma$ and so we have a well defined $u(\cdot \pm i\sigma, \cdot) \in H^s(\Omega)$ on the boundary of $\mathbb{T}_\sigma$. This concludes the first direction of the proof.

$\Longleftarrow$: By 3.2, $u(z, \cdot)$ has representation

$$u(x + it, \cdot) = \sum_{k=-\infty}^{\infty} \hat{u}_k(\cdot) e^{-kt} e^{ikx} = \mathcal{F}^{-1}_{k \to x} \left\{ \hat{u}_k(\cdot) e^{-kt} \right\}.$$

Then the first statement of the theorem follows by the dominated convergence theorem:

$$\begin{aligned}
\|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{\sigma|k|}\}\|_{H^s(\Omega)} &= \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{|k|t}\}\|_{H^s(\Omega)} \\
&\leq \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)(e^{kt} + e^{-kt})\}\|_{H^s(\Omega)} \\
&\leq \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{-tk}\}\|_{H^s(\Omega)} + \sup_{|t|<\sigma} \|\mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{-kt}\}\|_{H^s(\Omega)} \\
&= \sup_{|t|<\sigma} \|u(\cdot + it, \cdot)\|_{H^s(\Omega)} + \sup_{|t|<\sigma} \|u(\cdot - it, \cdot)\|_{H^s(\Omega)} \\
&< \infty.
\end{aligned}$$

□

Using the above theorem, we can define a 'partial' complex Hardy space which will serve as the solution space of our problem.

Definition 3.11. Let $s \geq 0$. Define Y^s_σ to be the space of functions satisfying the preceding theorem. Thus we have two equivalent characterizations of this space, one real and the other complex.

(i)

$$Y^s_\sigma = \left\{ u(x, y) \in H^s(\Omega) : \mathcal{F}^{-1}_{k\to x}\{\hat{u}_k(y)e^{\sigma|k|}\} \in H^s(\Omega) \right\}$$

with norm

$$\|u\|^2_{Y^s_\sigma} = \sum_{p+q\leq s} \sum_{k=-\infty}^{\infty} (k^2)^p e^{2\sigma|k|} \|\hat{u}^{(q)}_k(\cdot)\|^2_{L^2(0,1)}.$$

(ii) Y^s_σ is the space of functions $u(z, y)$ on $\mathbb{T}_\sigma \times (0, 1)$ that are complex analytic on $\mathbb{T}_\sigma$ in the sense that the map

$$z \to u(z, \cdot) : \mathbb{T}_\sigma \to H^s(0, 1)$$

is complex analytic, and that satisfy

$$\sup_{|t|<\sigma} \|u(\cdot + it, \cdot)\|_{H^s(\Omega)} < \infty.$$

Y_σ^s has norm

$$\|u\|_{Y_\sigma^s}^2 = \sup_{|t|<\sigma} \|u(\cdot + it, \cdot)\|_{H^s(\Omega)}^2 \cong \|u(\cdot + i\sigma, \cdot)\|_{H^s(\Omega)}^2 + \|u(\cdot - i\sigma, \cdot)\|_{H^s(\Omega)}^2.$$

Remark 3.12. Completeness of Y_σ^s follows immediately from the completeness of H^s. If $u_n(x, y)$ is a Cauchy sequence in Y_σ^s then it converges to some function $u(x, y)$ in H^s. Likewise the sequence $v_n(x, y) = \mathcal{F}_{k\to x}^{-1}\left\{\hat{u}_n(k, y)e^{\sigma|k|}\right\}$ converges to some $v(x, y)$ in H^s. To see that $u(x, y) \in Y_\sigma^s$, note that since

$$\|\mathcal{F}_{k\to x}^{-1}\{\hat{u}_n(k, y)e^{\sigma|k|}\}\|_{H^s} \leq \|v\|_{H^s}$$

then by dominated convergence theorem

$$\|\mathcal{F}_{k\to x}^{-1}\{\hat{u}(k, y)e^{\sigma|k|}\}\|_{H^s} = \lim_{n\to\infty} \|\mathcal{F}_{k\to x}^{-1}\{\hat{u}_n(k, y)e^{\sigma|k|}\}\|_{H^s} < \infty.$$

Thus $u \in Y_\sigma^s$, hence Y_σ^s is complete and so it is a (complex) Hilbert space.

Remark 3.13. As in the case for X_σ^s, the first part of the definition of Y_σ^s is equivalent to the statement $T(H^s) = Y_\sigma^s$ where T maps $u(x, y)$ to $\mathcal{F}_{k\to x}^{-1}\left\{\hat{u}_k(y)e^{-\sigma|k|}\right\}$.

Remark 3.14. It should be noted that while $u(x + it, \cdot) \in H^s(0, 1)$ for $|t| < \sigma$, it fails to hold as $|t| \to \sigma$. In this limit we merely have $u(x + it, \cdot) \to u(x \pm i\sigma, \cdot)$ in $H^{s-1/2}(0, 1)$. This is clear because on the boundary of the domain of analyticity, $u(\cdot \pm i\sigma, \cdot)$ is an ordinary $H^s(\Omega)$ function (without additional regularity) and thus has the usual trace $H^s(\Omega) \to H^{s-1/2}(0, 1)$.

Proposition 3.15. *Y_σ^s is an algebra for $s > 1$. That is, for $u, v \in Y_\sigma^s$ we have*

$$\|uv\|_{Y_\sigma^s} \leq C\|u\|_{Y_\sigma^s}\|v\|_{Y_\sigma^s}.$$

Proof. Consider $u, v \in Y_\sigma^s$. Note that the map $z \to u(z, \cdot)v(z, \cdot)$ is well defined from $\mathbb{T}_\sigma$ to $H^s(0, 1)$ since the latter is an algebra for $s > 1/2$. Also we have by chain rule

$$\frac{d}{dz}[u(z, \cdot)v(z, \cdot)] = \frac{du}{dz}(z, \cdot)v(z, \cdot) + u(z, \cdot)\frac{dv}{dz}(z, \cdot) \in H^s(0, 1)$$

since $z \to u(z,\cdot)$ and $z \to v(z,\cdot)$ are complex analytic and $H^s(0,1)$ is an algebra. Therefore

$$z \to u(z,\cdot)v(z,\cdot) : \mathbb{T}_\sigma \to H^s(0,1)$$

is complex analytic. Furthermore, since $H^s(\Omega)$ is an algebra for $s > 1$, the product $u(\cdot + it,\cdot)v(\cdot + it,\cdot)$ on each section (up to and including the boundaries $|t| = \sigma$) is in $H^s(\Omega)$. Thus by the second statement of 3.10, the product $u(z)v(z) \in Y^s_\sigma$. The inequality follows directly from the equivalent result for $H^s(\Omega)$. □

Remark 3.16. It is worth briefly mentioning our first attempts at the problem, which were unsuccessful. We had initially considered the space $X^s_\sigma \otimes H^r(0,1)$ for our partially analytic functions. Despite the fact that this space also has a partial analyticity, it proved to be the wrong space for our study. Such a space is related to spaces of the form $H^s \otimes H^r$, which can be identified with the Sobolev spaces of dominating mixed derivatives (see [15], [11]) because they consist of functions whose derivatives up to $\partial^s_x \partial^r_y$ are in L^2. Such spaces are anisotropic in their Sobolev regularity, which causes problems when solving the Dirichlet problem for the Poisson equation. Note that $1 - \Delta : H^s \otimes H^r \to H^{s-2} \otimes H^{r-2}$ is not an isomorphism. The target space of the Laplacian instead must be chosen much more carefully. Even when the target space is constructed correctly (as the subset of H^{-2} whose distributional derivatives up to $\partial^s_x \partial^r_y$ are in H^{-2}), the Dirichlet problem remains unsolvable. The issue occurs at the boundary. We found that in order for the solution to lie in $H^s \otimes H^r$, the boundary functions should lie in $H^{s+r-1/2}$. In other words the boundary should be the trace of an H^{s+r} function, as opposed to a $H^s \otimes H^r$ function. One can conclude that such spaces of mixed dominating derivatives (and hence tensor products of Sobolev spaces) are unsuitable for the boundary value problems encountered in this work. The mistake in forming the space $X^s_\sigma \otimes H^r(0,1)$ was that in an attempt to introduce anisotropy in the sense of partial analyticity, we also introduced anisotropy in the Sobolev regularity. Ultimately, this problem was remedied by formulating the 'correct' space Y^s_σ that was used in this work. This space successfully incorporates partial analyticity, while remaining isotropic in Sobolev regularity.

Chapter 4

Dirichlet Problem

To be able to apply the Implicit Function theorem, we have to show that the map

$$u \to (\Delta u, \gamma_0 u, \gamma_1 u) : Y_\sigma^s \to Y_\sigma^{s-2} \times X_\sigma^{s-1/2} \times X_\sigma^{s-1/2} \tag{4.1}$$

is a Banach space isomorphism, which we show in this chapter.

Proposition 4.1. *The derivative operators ∂_x and ∂_y are well defined and bounded from Y_σ^s to Y_σ^{s-1} as long as $s \geq 1$. Likewise the Laplacian $\Delta : Y_\sigma^s \to Y_\sigma^{s-2}$ is well defined and bounded for $s \geq 2$.*

Proof. This follows immediately from the Sobolev structure of Y_σ^s. □

Proposition 4.2. *Let $s > 1/2$. Then the restriction to horizontal sections, defined by*

$$u \to \gamma_y u = u(\cdot, y) : Y_\sigma^s \to X_\sigma^{s-1/2}$$

is well defined and bounded.

Proof. For any $u \in Y_\sigma^s$ we can write $u(z, y) = \sum_k \hat{u}_k(y) e^{ikz}$, where $\hat{u}_k(y) \in H^s(0, 1)$. Thus it is clear that $\left\{e^{ikz}\right\} \otimes H^s(0, 1)$ is dense in Y_σ^s, where $\left\{e^{ikz}\right\}$ is the set of trigonometric polynomials. Furthermore since $C^\infty[0, 1]$ is dense in $H^s(0, 1)$, then $\left\{e^{ikz}\right\} \otimes C^\infty[0, 1]$ is dense in Y_σ^s. For any $v(z, y) \in \left\{e^{ikz}\right\} \otimes C^\infty[0, 1]$, the trace γ_y is well defined by evaluation at y, ie $\gamma_y v(\cdot, \cdot) = v(\cdot, y)$ and this trace is in $X_\sigma^{s-1/2}$. Since $\gamma_y : H^s(\Omega) \to H^{s-1/2}(\mathbb{T})$ is bounded for $s > 1/2$, we have $\|v(\cdot, y)\|_{X_\sigma^{s-1/2}} \leq C\|v\|_{Y_\sigma^s}$. Thus by density, we can extend γ_y to a bounded operator from all of Y_σ^s to $X_\sigma^{s-1/2}$. □

Corollary 4.3. *The map*

$$u \to (\Delta u, \gamma_0 u, \gamma_1 u) : Y^s_\sigma \to Y^{s-2}_\sigma \times X^{s-1/2}_\sigma \times X^{s-1/2}_\sigma$$

is bounded, provided $s \geq 2$.

Theorem 4.4.

$$u \to (\Delta u, \gamma_0 u, \gamma_1 u) : Y^s_\sigma \to Y^{s-2}_\sigma \times X^{s-1/2}_\sigma \times X^{s-1/2}_\sigma$$

is a Banach space isomorphism, provided $s \geq 2$.

Proof. We must show that 4.1 is invertible, which corresponds to solving the Dirichlet problem for the Poisson equation on the domain $\Omega = \mathbb{T} \times [0, 1]$ given by

$$\begin{cases} \Delta u(x, y) = f(x, y) \\ u(x, 0) = a(x) \\ u(x, 1) = b(x). \end{cases} \tag{4.2}$$

Here $f(x, y) \in Y^{s-2}_\sigma$ and $a(x), b(x) \in X^{s-1/2}_\sigma$. We must show the solution exists in $u \in Y^s_\sigma$. By expanding all functions as Fourier series in x we get the series of ODEs

$$\begin{cases} \hat{u}''_k(y) - k^2 \hat{u}_k(y) = \hat{f}_k(y) \\ \hat{u}_k(0) = \hat{a}_k \\ \hat{u}_k(1) = \hat{b}_k. \end{cases} \tag{4.3}$$

These ODEs can be solved explicitly. For the case when $k = 0$, straight integration yields

$$\hat{u}_0(y) = \int_0^y \int_0^\eta \hat{f}_0(t) dt d\eta + \left(\hat{b}_0 - \hat{a}_0 - \int_0^1 \int_0^\eta \hat{f}_0(t) dt d\eta \right) y + \hat{a}_0.$$

It can easily be seen that this term is in $H^s(0, 1)$. For general k, let's write the solution as $\hat{u}_k(y) = \hat{h}_k(y) + \hat{g}_k(y)$ where $\hat{h}_k$ is the solution to the homogeneous part (with non-zero boundary conditions) while $\hat{g}_k$ is the general part (with zero boundary conditions). Then we get

$$\hat{h}_k(y) = \frac{\hat{b}_k \sinh(ky) - \hat{a}_k \sinh[k(y-1)]}{\sinh(k)} \tag{4.4}$$

and

$$\hat{g}_k(y) = \int_0^1 G_k(y,t)\hat{f}_k(t)dt. \tag{4.5}$$

Here

$$G_k(y,t) = \begin{cases} \dfrac{\sinh[k(t-1)]\sinh(ky)}{k\sinh(k)} & y \le t \\ \\ \dfrac{\sinh(kt)\sinh[k(y-1)]}{k\sinh(k)} & y \ge t \end{cases} \tag{4.6}$$

is the Green's function. Then the solution to the Dirichlet problem is given by

$$u(x,y) = h(x,y) + g(x,y) = \sum \hat{h}_k(y)e^{ikx} + \sum \hat{g}_k(y)e^{ikx}. \tag{4.7}$$

Next we would like to show that $u \in Y_\sigma^s$. That is, show

$$\|u\|^2_{Y_\sigma^s} = \sum_{p+q\le s} \sum_k e^{2\sigma|k|} \left(k^2\right)^p \|\hat{u}_k^{(q)}(y)\|^2_{L^2(0,1)} < \infty.$$

First let's check $h(x,y)$. We have

$$\partial_y^q \hat{h}_k(y) = \frac{k^q \hat{b}_k \sinh(ky) - k^q \hat{a}_k \sinh[k(y-1)]}{\sinh(k)}$$

when q is even and

$$\partial_y^q \hat{h}_k(y) = \frac{k^q \hat{b}_k \cosh(ky) - k^q \hat{a}_k \cosh[k(y-1)]}{\sinh(k)}$$

when q is odd. Notice that,

$$\int_0^1 \sinh^2(ky)dy = \int_0^1 \sinh^2[k(y-1)]dy = \frac{\sinh(k)\cosh(k) - k}{2k}$$

and

$$\int_0^1 \cosh^2(ky)dy = \int_0^1 \cosh^2[k(y-1)]dy = \frac{\sinh(k)\cosh(k) + k}{2k}.$$

Therefore we get

$$\|\partial_y^q \hat{h}_k(y)\|^2_{L^2(0,1)} = \int_0^1 |\partial_y^q \hat{h}_k(y)|^2 dy \le (k^2)^q(\hat{a}_k^2 + \hat{b}_k^2)\frac{\sinh(k)\cosh(k) \pm k}{2k\sinh^2(k)}$$

and thus

$$\|\partial_y^q \hat{h}_k(y)\|^2_{L^2(0,1)} \approx \frac{(k^2)^q}{k}(\hat{a}_k^2 + \hat{b}_k^2).$$

It follows that

$$e^{2\sigma|k|}(k^2)^p\|\partial_y^q\hat{h}_k(y)\|^2_{L^2(0,1)} \approx e^{2\sigma|k|}(k^2)^{p+q-1/2}(\hat{a}_k^2+\hat{b}_k^2) \le e^{2\sigma|k|}(1+k^2)^{s-1/2}(\hat{a}_k^2+\hat{b}_k^2)$$

since $p+q \le s$. Summing over k,p,q we get

$$\|h(x,y)\|^2_{Y^s_\sigma} \le C\left(\|a(x)\|^2_{X^{s-1/2}_\sigma} + \|b(x)\|^2_{X^{s-1/2}_\sigma}\right). \tag{4.8}$$

Now let's consider the term $g(x,y)$. Starting with the equation

$$\hat{g}_k'' = k^2\hat{g}_k + \hat{f}_k$$

differentiating twice gives

$$\hat{g}_k^{(4)} = k^2\hat{g}_k'' + \hat{f}_k'' = k^2(k^2\hat{g}_k + \hat{f}_k) + \hat{f}_k'' = k^4\hat{g}_k + k^2\hat{f}_k + \hat{f}_k''.$$

In general differentiating an even number of times gives us

$$\hat{g}_k^{(2n)} = k^{2n}\hat{g}_k + \sum_{i=0}^{n-1} k^{2(n-1-i)}\hat{f}_k^{(2i)}. \tag{4.9}$$

Similarly an odd number of derivatives can be written

$$\hat{g}_k^{(2n+1)} = k^{2n}\hat{g}_k' + \sum_{i=0}^{n-1} k^{2(n-1-i)}\hat{f}_k^{(2i+1)} \tag{4.10}$$

Therefore to control the L^2 norm of $\hat{g}_k^{(q)}$ we need only to estimate the norms of $\hat{g}_k$ and $\hat{g}_k'$. Recall $\hat{g}_k$ satisfies

$$\begin{cases} \hat{g}_k'' - k^2\hat{g}_k = \hat{f}_k \\ \hat{g}_k(0) = \hat{g}_k(1) = 0 \end{cases}$$

on the interval $(0,1)$. Multiplying both sides of the ODE by $\overline{\hat{g}}_k$ and integrating over the interval gives

$$\int_0^1 \hat{g}_k''\overline{\hat{g}}_k dy - k^2\int_0^1 \hat{g}_k\overline{\hat{g}}_k dy = \int_0^1 \hat{f}_k\overline{\hat{g}}_k.$$

Integrating the first term by parts using that $\hat{g}_k$ vanishes on the end points and rearranging gives

$$\int_0^1 |\hat{g}_k'|^2 dy + k^2\int_0^1 |\hat{g}_k|^2 dy = -\int_0^1 \hat{f}_k\overline{\hat{g}}_k dy. \tag{4.11}$$

Applying Poincaré inequality:

$$\int_0^1 |\hat{g}_k|^2 dy \leq c \int_0^1 |\hat{g}_k'|^2 dy$$

and Cauchy-Schwarz inequality to 4.11 we get

$$(1+k^2)\int_0^1 |\hat{g}_k|^2 dy \leq c(\int_0^1 |\hat{f}_k|^2 dy)^{1/2}(\int_0^1 |\hat{g}_k|^2 dy)^{1/2}.$$

We thus obtain

$$(1+k^2)\|\hat{g}_k\|_{L^2(0,1)} \leq c\|\hat{f}_k\|_{L^2(0,1)}$$

and so

$$\|\hat{g}_k\|_{L^2}^2 \leq c\frac{\|\hat{f}_k\|_{L^2}^2}{k^4}. \tag{4.12}$$

Also notice from 4.11 we have

$$\int_0^1 |\hat{g}_k'|^2 dy \leq \int_0^1 |\hat{g}_k'|^2 + k^2|\hat{g}_k|^2 dy \leq (\int_0^1 |\hat{f}_k|^2 dy)^{1/2}(\int_0^1 |\hat{g}_k|^2 dy)^{1/2}.$$

It follows that

$$\|\hat{g}_k'\|_{L^2}^2 \leq \|\hat{f}_k\|_{L^2}\|\hat{g}_k\|_{L^2} \leq c\frac{\|\hat{f}_k\|_{L^2}^2}{k^2}. \tag{4.13}$$

We now have control of $\hat{g}_k$ and $\hat{g}_k'$, which we can use to control $\hat{g}_k^{(q)}$. Returning to 4.9 and 4.10 we find

$$\|\hat{g}_k^{(2n)}\|_{L^2}^2 \leq (k^2)^{2n-2}\|\hat{f}_k\|_{L^2}^2 + \sum_{i=0}^{n-1}(k^2)^{2n-2-2i}\|\hat{f}_k^{(2i)}\|_{L^2}^2$$

and

$$\|\hat{g}_k^{(2n+1)}\|_{L^2}^2 \leq (k^2)^{2n-1}\|\hat{f}_k\|_{L^2}^2 + \sum_{i=0}^{n-1}(k^2)^{2n-2-2i}\|\hat{f}_k^{(2i+1)}\|_{L^2}^2.$$

We can combine these two cases and write

$$\|\hat{g}_k^{(q)}\|_{L^2}^2 \leq \sum_{j=0}^{q-2} C_j \left(k^2\right)^{q-2-j} \|\hat{f}_k^{(j)}\|_{L^2}^2. \tag{4.14}$$

Multiplying both sides by $(k^2)^p$ and using that $p+q \leq s$ we get

$$\left(k^2\right)^p \|\hat{g}_k^{(q)}\|_{L^2}^2 \leq \sum_{j=0}^{s-2} C_j \left(k^2\right)^{s-2-j} \|\hat{f}_k^{(j)}\|_{L^2}^2.$$

Finally multiplying both sides by $e^{2\sigma|k|}$ and summing over k, p, q we obtain

$$\|g\|_{Y_\sigma^s} \leq C\|f\|_{Y_\sigma^{s-2}}. \tag{4.15}$$

Combining 4.8 and 4.15 we obtain the desired result

$$\|u\|^2_{Y^s_\sigma} \leq C\left(\|f\|^2_{Y^{s-2}_\sigma} + \|a\|^2_{X^{s-1/2}_\sigma} + \|b\|^2_{X^{s-1/2}_\sigma}\right). \tag{4.16}$$

Thus there is a bounded inverse map to 4.1 and so it is a Banach space isomorphism.

□

Chapter 5

Nonlinear Differential Operators

The goal of this chapter is to show that the nonlinear differential operator Φ defined in 2.2 is an analytic operator from Y^s_σ to Y^{s-2}_σ, in the neighbourhood of $a(x, \psi) = \psi$. To this end, we will need to first show such results on ordinary Sobolev spaces. Let's start by formulating this problem. Suppose A^s is some space of functions from a (real or complex) domain Ω to $\mathbb{K}$ (real or complex numbers), that are s times differentiable (in some informal sense). We denote mth order derivatives (possibly partial) by D^m. Given a function $f : \Omega \times \mathbb{K} \times \cdots \times \mathbb{K} \to \mathbb{K}$, we can define a nonlinear operator F on a set of functions $u_1, ..., u_N \in A^s$ by the composition

$$Fu = f\left(x, u_1(x), \cdots, u_N(x)\right).$$

Such operators are often referred to in literature as Composition operators, Substitution operators or Nemytskii operators (see [10]). We can in turn define an mth order nonlinear differential operator $F^{(m)}$ on $u \in A^s$ by the composition:

$$F^{(m)}u(x) = F\left(u(x), Du(x), \cdots, D^m u(x)\right) = f\left(x, u(x), Du(x), \cdots, D^m u(x)\right).$$

In other words, it is given by a composition of a linear differential operator

$$u(x) \to \left(u(x), \cdots, D^m u(x)\right) : A^s \to A^s \times \cdots \times A^{s-m}$$

with a nonlinear operator F. In practice we would like $F^{(m)}$ to map A^s to A^{s-m}. To do so, we need f to define the mapping

$$F : A^{s-m} \times \cdots \times A^{s-m} \to A^{s-m}.$$

For our purposes, it will be enough to consider the slightly less general operator $F(u_1, ..., u_N) = f(u_1(x), ..., u_N(x))$. That is, operators defined by functions that do not explicitly depend on the domain Ω. Starting with the simplest case, suppose $x \in \Omega \subset \mathbb{R}$ and $f : \mathbb{K} \to \mathbb{K}$ defines $Fu(x) = f(u(x))$. Computing the first few derivatives of $Fu(x)$ we get

$$DFu(x) = f'(u)(Du)$$

$$D^2Fu(x) = f'(u)(D^2u) + f''(u)(Du)^2$$

$$D^3Fu(x) = f'(u)(D^3u) + 3f''(u)(Du)(D^2u) + f'''(u)(Du)^3$$

By induction, we can write the general sth derivative as follows

$$D^s Fu(x) = \sum_{p=1}^{s} \sum_{\substack{(s_1+\cdots+s_p)\leq s \\ s_i \geq 1}} C^s_{s_1 \ldots s_p} (D^{s_1}u) \cdots (D^{s_p}u) f^{(p)}(u). \tag{5.1}$$

Remark 5.1. The formula 5.1 remains true if $x \in \mathbb{R}^n$ and $u(x)$ is a vector valued function; in this case the terms of 5.1 have an obvious tensor sense.

We are now ready to prove some properties of such composition operators on Sobolev spaces.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (or possibly a torus or toroidal along some dimensions) with sufficiently smooth boundary. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^s function. Suppose $s > n/2 + 1$ ($s > n/2$ suffices if n is odd) is a natural number. Then $f(u(x)) = Fu(x)$ defines a continuous mapping*

$$F : H^s(\Omega) \to H^s(\Omega).$$

Proof. First, to show F is well defined, we show $D^s Fu(x) \in L^2(\Omega)$. Suppose $s > n/2$. Then we have the inclusion $H^s(\Omega) \subset C(\overline{\Omega})$ ([2], [9]). The factor $f^{(p)}(u)$ in 5.1 is thus a composition of continuous functions and is therefore in $C(\overline{\Omega})$. Since the product of a continuous function with an L^2 function is L^2 (on compact domains), it suffices to show that in each term of 5.1, the product $(D^{s_1}u) \cdots (D^{s_p}u)$ is in L^2.

Our strategy will be to use the Sobolev Embedding Theorem (see ([2], [9] for more details) to embed each factor $(D^{s_i}u)$ into L^q_i with q_i high enough that their products are in L^2. First notice that because Ω is bounded, we have the inclusion $L^q \subset L^2$ when $q \geq 2$. We will show each $D^{s_i}u \in L^{q_i}$ such that $1/q_1 + \cdots + 1/q_p \leq 1/2$. Then the above mentioned inclusion combined with Hölder's inequality gives

$$\|(D^{k_1}u)\cdots(D^{k_p}u)\|_{L^2} \leq C\|(D^{k_1}u)\cdots(D^{k_p}u)\|_{L^q} \leq C\|D^{k_1}u\|_{L_{q_1}}\cdots\|D^{k_p}u\|_{L_{q_p}}.$$

Let's proceed by applying the Sobolev embedding theorem. Each factor $D^{s_i}u \in H^{s-s_i}$. If $s - s_i > k/2$ then this factor is continuous and can be factored out of the L^2 norm. If $s - s_i < n/2$ then $D^{s_i}u \in L^{q_i}$ with $1/q_i = 1/2 - (s - s_i)/n$. Finally in the critical case when $s - s_i = n/2$ (which can occur only if n is even), we instead use that $H^{s-s_i-1} \subset L^{q_i}$ with $1/q_i = 1/2 - (s - s_i - 1)/n = 1/n$.

First let's assume n is even. In each term of 5.1, there are p number of factors $D^{s_i}u$. Let's assume that p' is the number of factors with $s - s_i < n/2$, p'' is the number of factors in the critical case and p''' is the number of factors that are continuous. Then $p' + p'' + p''' = p$. Without loss of generality, let's order the indexes accordingly $(s_1, ..., s_{p'}), (s_{p'+1}, ..., s_{p'+p''}), (s_{p'+p''+1}, ..., s_{p'+p''+p'''})$. We have

$$s_1 + \cdots + s_{p'} + p''(s - n/2) + p''' \leq s_1 + \cdots + s_p = s$$

and therefore

$$s_1 + \cdots + s_{p'} \leq s - p''(s - n/2) - p''' \leq s - p''(s - n/2).$$

Now let's check the condition for Hölder's inequality (we can ignore the continuous factors):

$$\begin{aligned}\frac{1}{q} &= \frac{1}{q_1} + \cdots + \frac{1}{q_{p'+p''}} \\ &= \frac{p'}{2} - \frac{p's}{n} + \frac{s_1 + \cdots + s_{p'}}{n} + \frac{p''}{n} \\ &\leq \frac{p'+p''}{2} - \frac{s(p'+p''-1)}{n} + \frac{p''}{n}.\end{aligned}$$

We need $1/q \leq 1/2$, so set the last line above to be $\leq 1/2$. Rearranging for s we find that we need

$$s \geq \frac{n}{2} + \frac{p''}{p'+p''-1}.$$

Notice though that the fraction on the right is at most 2. The denominator is never zero because we assume at least two factors are not continuous (otherwise the result is trivial). Therefore we need $s \geq n/2+2$. Since n is even, this is equivalent to $s > n/2+1$.

When n is odd, the critical case does not occur. Thus take $p = p' + p'''$, where p''' is the number of continuous factors and p' is the number of factors that embed in some L^{q_i}. We have

$$s_1 + \cdots + s_{p'} + p''' \leq s_1 + \cdots + s_p = s$$

and therefore

$$s_1 + \cdots + s_{p'} \leq s - p''' \leq s.$$

Checking Hölder's inequality we have

$$\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_{p'}} = \frac{p'}{2} - \frac{p's}{n} + \frac{s_1 + \cdots + s_{p'}}{n} \leq \frac{p'}{2} - \frac{p's}{n} + \frac{s}{n}.$$

Setting the right side to be $\leq 1/2$ and rearranging for s we find $s \geq n/2$. Since n is odd, this corresponds to $s > n/2$.

To see that F is continuous, take a sequence in $H^s(\Omega)$ converging to u. Note the product of a sequence converging in L^2 and a uniformly converging sequence converges in L^2. Then by the Sobolev inequalities, each term of 5.1 converges in L^2 and so the expression converges in H^s to $DFu(x)$. □

Next we show that additional regularity of f carries forward to regularity of F.

Lemma 5.3. *Suppose f is in C^{s+r}. Then the map $F : H^s(\Omega) \to H^s(\Omega)$ is r times continuously differentiable. In particular, if f is smooth then F is smooth.*

Proof. Let $u, v \in H^s(\Omega)$. The Gâteaux derivative of F at u in direction v is given by

$$DF(u,v) = \frac{d}{dt}\bigg|_{t=0} F(u+tv) = f'(u)v(x).$$

By the previous lemma, $f'(u) \in H^s(\Omega)$ as long as $f \in C^{s+1}$. Since H^s is an algebra for $s > n/2$, the Gâteaux derivative is in H^s. Thus we see that the derivative map

$$DF(u,\cdot) : H^s(\Omega) \to H^s(\Omega)$$

is well defined, linear because it is a multiplication operator and has a bound on the norm

$$|||DF(u,\cdot)||| \le C\|f'(u)\|_{H^s}.$$

Consider the map $u \to DF(u,\cdot)$ from $H^s(\Omega)$ to $L\left(H^s(\Omega), H^s(\Omega)\right)$. It is clearly continuous by the results of the previous lemma, thus $DF(u,\cdot)$ is continuous in operator norm with respect to u. A map that has linear bounded Gâteaux derivative, continuous in operator norm is Fréchet differentiable (see [10]). Thus, $F : H^s \to H^s$ is continuously differentiable in the Fréchet sense.

Higher order derivatives work much the same way. The second Fréchet derivative of F at u, denoted by $D^2F(u,\cdot,\cdot)$ is given by the multiplication operator

$$(v, w) \to f''(u)vw : H^s(\Omega) \times H^s(\Omega) \to H^s(\Omega).$$

It is well defined and continuous as long as f'' is in C^s, that is $f \in C^{s+2}$. In general, we see that $D^rF(u,\cdot)$ is well defined and continuous so long as $f \in C^{s+r}$. In particular, if f is C^∞ then

$$F : H^s(\Omega) \to H^s(\Omega)$$

is a C^∞ map.

Finally note that in the case when F acts on a vector valued function $u(x)$, the derivative map changes from a multiplication operator to a 'dot product' operator

$$DF(u, v) = \nabla f(u) \cdot v = \sum_{j=1}^{N} \partial_j f(u)v_j.$$

Higher order derivatives are similar linear operators, summations over all components. The rest of the proof works in the same way.

□

Corollary 5.4. *Let f be a complex analytic function. Suppose $s > n/2+1$ ($s > n/2$ if n is odd) is a natural number. Then*

$$F : H^s(\Omega, \mathbb{C}) \to H^s(\Omega, \mathbb{C})$$

is an analytic operator.

Proof. By the previous lemmas, F is well defined and continuously complex differentiable between complex Banach spaces. By standard results of holomorphic maps on complex Banach spaces (see [8]), F is analytic. □

Corollary 5.5. *Let f be a complex analytic function. Suppose $s > 2$. Then*

$$F : Y^s_\sigma \to Y^s_\sigma$$

is an analytic operator.

Proof. Let $u(z, y) \in Y^s_\sigma$. Consider the map $z \to Fu(z, \cdot)$. It is given by the composition

$$z \to u(z, \cdot) \to f(u(z, \cdot)) : \mathbb{T}_\sigma \to H^s(0, 1) \to H^s(0, 1).$$

Thus it is a composition of complex analytic maps and is therefore complex analytic. Furthermore, for every $|t| \leq \sigma$ the map

$$u(\cdot + it, \cdot) \to Fu(\cdot + it, \cdot) : H^s(\Omega) \to H^s(\Omega)$$

is well defined and continuous (in fact analytic). Therefore

$$\sup_{|t|<\sigma} \|Fu(\cdot + it, \cdot)\|_{H^s(\Omega)} < \infty.$$

Thus $Fu(z, y)$ is in Y^s_σ and therefore the map $F : Y^s_\sigma \to Y^s_\sigma$ is well defined. Continuity of this map follows immediately from continuity of $F : H^s \to H^s$. As in the previous results, since Y^s_σ is an algebra, the map is continuously complex differentiable between complex Banach spaces and therefore analytic. □

Theorem 5.6. *Let $f : \mathbb{C} \times \cdots \times \mathbb{C} \to \mathbb{C}$ be a complex analytic function that defines an mth order nonlinear differential operator on Y^s_σ by*

$$F^{(m)}u = F(u, \cdots, D^m u) = f(u, \cdots, D^m u).$$

Suppose $s - m > 2$. Then

$$F^{(m)} : Y^s_\sigma \to Y^{s-m}_\sigma$$

is an analytic operator.

Proof. This follows immediately from the previous lemma. Since

$$u \to (u, \cdots, D^m u) : Y_\sigma^s \to Y_\sigma^{s-m} \times \cdots \times Y_\sigma^{s-m}$$

is continuous then the composition

$$u \to F(u, \cdots, D^m u) : Y_\sigma^s \to Y_\sigma^{s-m}$$

is continuous. The Fréchet derivative is given by

$$DF^{(m)}(u, v) = \sum_{j=1}^{m} \partial_j f(u, \cdots, D^m u) D^j v$$

with operator norm

$$|||DF^{(m)}(u, \cdot)||| \le C \sum_{j=1}^{m} \|\partial_j f(u, \cdots, D^m u)\|_{Y_\sigma^{s-m}}.$$

$F^{(m)}$ is complex differentiable and therefore analytic. □

Corollary 5.7. *Suppose $s > 4$. Then the map Φ defined in 2.2 is an analytic operator from Y_σ^s to Y_σ^{s-2} in a neighbourhood of $a(x, \psi) = \psi$.*

Proof. Note that Φ is defined by composition with a rational function, hence it is analytic everywhere except at its singularity, which occurs when $a_\psi(x, \psi) = 0$. Let U be a neighbourhood in Y_σ^s such that $a_\psi(x, \psi) \neq 0$ for all $a \in U$. Then by the previous theorem, $\Phi : U \to Y_\sigma^{s-2}$ is an analytic map. Thus we need only to confirm that there indeed exists such a neighbourhood of $\psi \in Y_\sigma^s$. Note that s is high enough that $a_\psi \in Y_\sigma^{s-1} \subset H^{s-1}(\Omega) \subset C(\overline{\Omega})$. There exists a constant C for which $\| \cdot \|_\infty \le C \| \cdot \|_{s-1}$. Take $\|a - \psi\|_{Y_\sigma^s} < 1/C$. Then

$$\|a_\psi - 1\|_\infty \le C\|a_\psi - 1\|_{s-1} \le C\|a - \psi\|_s \le C\|a - \psi\|_{Y_\sigma^s} < 1.$$

But $\|a_\psi - 1\|_\infty < 1$ implies that $a_\psi \in (0, 2)$ on $\mathbb{T} \times [0, 1]$, and is thus never zero. Therefore let U be the ball in Y_σ^s of radius $1/C$ centred on $a(x, \psi) = \psi$. Then $\Phi : U \to Y_\sigma^{s-2}$ is analytic. □

Chapter 6

Conclusion

We now have all the necessary tools to prove our main result.

Theorem 6.1 (Main Result)**.** *There exists $\epsilon > 0$, such that if $\|f\|_{X_\sigma^{s-1/2}} < \epsilon$, $\|g-1\|_{X_\sigma^{s-1/2}} < \epsilon$ and $\|F\|_{H^{s-2}} < \epsilon$ then the boundary value problem in 2.3 has a unique solution $a(x, \psi)$ which is parametrized analytically by f, g and F.*

Proof. Define the map

$$A : (f, g, F, a) \to (\Phi(a) - F \otimes 1(x), \gamma_0 a - f, \gamma_1 a - g) \,.$$

First note that the map

$$F(\psi) \to F(\psi) \otimes 1(x) : H^{s-2}(0, 1) \to Y_\sigma^{s-2}$$

is well defined, bounded linear operator. Then by 5.7, the map

$$(f, g, F, a) \to \Phi(a) - F \otimes 1(x)$$

$$X_\sigma^{s-1/2} \times X_\sigma^{s-1/2} \times H^{s-2}(0, 1) \times Y_\sigma^s \to Y_\sigma^{s-2}$$

is well defined and analytic (in a neighbourhood of $a = \psi$). By 4.2, the maps

$$\gamma_0, \gamma_1 : Y_\sigma^s \to X_\sigma^{s-1/2}$$

are bounded linear operators thus

$$(f, g, F, a) \to \gamma_0 a - f$$

$$(f, g, F, a) \to \gamma_1 a - g$$

$$X_\sigma^{s-1/2} \times X_\sigma^{s-1/2} \times H^{s-2}(0,1) \times Y_\sigma^s \to X_\sigma^{s-1/2}$$

are analytic. Therefore the map

$$A : X_\sigma^{s-1/2} \times X_\sigma^{s-1/2} \times H^{s-2}(0,1) \times Y_\sigma^s \to Y_\sigma^{s-2} \times X_\sigma^{s-1/2} \times X_\sigma^{s-1/2}$$

is well defined and analytic. At $(f, g, F, a) = (0, 1, 0, \psi)$, A is equal to zero and by 4.4 the map

$$\frac{\partial F(0, 1, 0\psi)}{\partial a} : Y_\sigma^s \to Y_\sigma^{s-2} \times X_\sigma^{s-1/2} \times X_\sigma^{s-1/2}$$

defined by

$$a \to (-\Delta a, \gamma_0 a, \gamma_1 a)$$

is a Banach space isomorphism. Thus by the analytic implicit function theorem 2.1, there exists a $X_\sigma^{s-1/2} \times X_\sigma^{s-1/2} \times H^{s-2}(0,1)$ neighbourhood U of $(f, g, F) = (0, 1, 0)$ for which 2.3 has a unique solution defined by a complex analytic map

$$(f, g, F) \to a : U \to Y_\sigma^s.$$

□

We have thus obtained a local parametrization for the set of flow lines on a periodic strip. This parametrization holds in the neighbourhood of a parallel flow with constant velocity in a straight channel, where the stream function has no critical points. Our solution incorporates the analytic structure of the flow lines and the resulting parametrization is also analytic. This work represents a step forward towards the goal of a global geometric description of the set of stationary solutions to the incompressible Euler equation. Further progress can be made by extending our results from the periodic strip to general domains. The case when the stream function has critical points needs to be addressed as well. Such a case presents new difficulties not encountered in this work, as the flow lines can no longer all be represented as graphs of functions. In our work, we have incorporated the real analytic structure of flow lines by considering only those whose complex singularities are at worst restricted to the boundary of our chosen complex strip. In general, a real analytic flow line can be extended to complex values until some singularity is reached. These complex singularities of real analytic flows merit investigation.

Chapter 7

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