Isoperimetric-type Inequalities for g-chordal Star-shaped Sets in $\mathbb{R}^n$

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ABSTRACT

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This paper generalizes certain existing isoperimetric-type inequalities from $\mathbb{R}^2$ to higher dimensions. These inequalities provide lower bounds for the n-dimensional volume and, respectively, surface area of certain star-shaped bodies in $\mathbb{R}^n$ and characterize the equality cases.

More specifically, we work with g-chordal star-shaped bodies, a natural generalization of equichordal compact sets. A compact set in $\mathbb{R}^n$ is said to be equichordal if there exists a point in the interior of the set such that all chords passing through this point have equal length. To justify the significance of our results, we provide several means of constructing g-chordal star-shaped bodies.

The method used to prove the above inequalities is further employed in finding new lower bounds for the dual quermassintegrals of g-chordal star-shaped sets in $\mathbb{R}^n$ and, more generally, lower bounds for the dual mixed volumes involving these star bodies. Finally, some of the previous results will be generalized to L^n-stars, star-shaped sets whose radial functions are n-th power integrable over the unit sphere $\mathbb{S}^{n-1}$.

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Contents

Notations

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R} \setminus \{0\}$ denotes the set of non-zero real numbers.
- $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
- $\mathbb{S}^{n-1}$ denotes the unit n-sphere, (i.e $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$).
- $\mathfrak{M}_X$ or just $\mathfrak{M}$ denotes the set of all measurable sets in X.
- $relint(S)$ denotes the relative interior of the set S.
- C^2 denotes the space of two times continuously differentiable functions.
- For any $x \in \mathbb{R}^n$, $\|x\|$ denotes the standard, Euclidean norm of x in $\mathbb{R}^n$.
- $L^p(\mu)$ denotes the space of measurable functions that are p-integrable; its norm is denoted by $\| \, . \, \|_p$.
- $L^\infty(\mu)$ denotes the space of essentially bounded measurable functions.
- $V(S)$ denotes the n-dimensional volume, i.e. the Lebesgue mesure, of the set $S \subset \mathbb{R}^n$.
- $A(S)$ denotes the $(n-1)$-dimensional volume, i.e. the $(n-1)$-Hausdorff measure, of the set $\partial S \subset \mathbb{R}^n$, otherwise known as the surface area.
- $l(S)$ denotes the perimeter of the set $S \subset \mathbb{R}^2$ which is the low-dimension equivalent to the surface area of S, $A(S)$, when $n = 2$.

Chapter 1

Introduction and Background

1.1 Introduction

The setting of this paper is the Euclidean n-space $\mathbb{R}^n$.

Definition 1.1.1 ([7]). *A set $S \subseteq \mathbb{R}^n$ is said to be star-shaped if there exists a point $P \in \overline{relint(S)}$ so that each line passing through P in $\mathbb{R}^n$ is intersecting S in a (possibly degenerate) line segment which joins at most two points other than P of the boundary ∂S.*

In fact, the point P can be viewed as a source of light from which each point on the boundary of S can be illuminated, hence the name of star-shaped.

Compact, star-shaped sets with non-empty interiors are called star bodies.

It follows from the above definition that a convex body (a convex compact set with non-empty interior) S must be star-shaped.

A star-shaped set S with P as the origin in $\mathbb{R}^n$ is uniquely characterized by its radial function $\rho : \mathbb{S}^{n-1} \to \mathbb{R}$, defined by

$$\rho_S(u) = \sup\{\lambda \geq 0 : \lambda u \in S\}. \tag{1.1}$$

Lemma 1.1.1. *If S is a star body with respect to $P \in \overline{relint(S)}$, then ρ_S, considered with respect to P as the origin, is a continuous function on $\mathbb{S}^{n-1}$.*

Proof. Assuming S is star-shaped with respect to $P \in \overline{relint(S)}$, it follows that the entire boundary of S is *visible* from P, that is for each direction $u \in \mathbb{S}^{n-1}$, there is a unique boundary point $x \in \partial S$, $x = P + \rho_S(u)\, u$, or as P is taken to be the origin, $x = \rho_S(u)\, u$.

Let u be an arbitrary point of $\mathbb{S}^{n-1}$ and let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^{n-1}$ be such that

$$u_n \to u \quad \text{as} \quad n \to \infty. \tag{1.2}$$

It follows from our assumption that, to each $u_n \in \mathbb{S}^{n-1}$, corresponds a unique boundary point x_n of S in the direction u_n given by

$$x_n = \|x_n\|_{\mathbb{R}^n}\, u_n = \rho_S(u_n)\, u_n = \max\{\lambda \geq 0 : \lambda u_n \in S\}\, u_n. \tag{1.3}$$

We will show that $\{x_n\}_{n\in\mathbb{N}}$ converges to a point x of the boundary of S. Consider the set of all such points $\{x_n\}_{n\in\mathbb{N}}$. Since S is compact, so is $\partial S = S \setminus relint(S)$ and there exists a subsequence of integers going to infinity $\{n_k\}_{k\in\mathbb{N}}$ such that

$$x_{n_k} \to x \quad \text{as} \quad k \to \infty, \tag{1.4}$$

where x is a boundary point. We will show first that $x = \lambda u$, where $\lambda = \lim_{k\to\infty} \|x_{n_k}\|_{\mathbb{R}^n} > 0$, in other words x is in the direction u. We may assume without any loss of generality that $\lambda > 0$. Otherwise, $\|x_{n_k}\|_{\mathbb{R}^n} \to 0$ and the conclusion is trivially satisfied. Indeed, if $x^{(i)}_{n_k}$ denotes the i-th component of x_{n_k}, then, by (1.4) and the fact that $u_n \to u$, we have $x^{(i)}_{n_k} = \|x_{n_k}\|_{\mathbb{R}^n} u^{(i)}_{n_k} \to \lambda u^{(i)}$.

Suppose now that $x_n \nrightarrow x$ as $n \to \infty$. Then, using the compactness of S and that of ∂S, we can, eventually, find another subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$, $x_{n_j} \to x'$, where $x \neq x' \in \partial S$. As both x and x' are distinct boundary points in the direction of u, we contradict the fact that S is star-shaped.

Thus, we may now consider that the sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ above is simply the sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x.

We claim that λ defined above is $\rho_S(u)$. This follows from the fact that $\rho_S(u)$ is reached on the boundary of S in the direction u and, as S is star-shaped, x is the unique boundary point of S in this direction.

As such, it follows that

$$\big|\rho_S(u_n) - \rho_S(u)\big| = \big|\|x_n\|_{\mathbb{R}^n} - \|x\|_{\mathbb{R}^n}\big| \leq \|x_n - x\|_{\mathbb{R}^n} \to 0 \quad \text{as} \quad n \to \infty. \tag{1.5}$$

Therefore, for every $u \in \mathbb{S}^{n-1}$, and every sequence $\{u_n\}_n \subset \mathbb{S}^{n-1}$ converging to u, we have that $\rho_S(u_n) \to \rho_S(u)$, i.e. ρ_S is continuous at u, and, as u was arbitrary, ρ_S is continuous on $\mathbb{S}^{n-1}$. □

In what follows, we consider S a compact star-shaped set in $\mathbb{R}^n$ whose boundary, ∂S, is an embedded, closed curve if $n = 2$, surface if $n = 3$ or respectively hypersurface, if $n \geq 4$, of class C^2. Let $g : (0, +\infty) \to \mathbb{R}$ be a continuous function. Denote by $d(A, B)$ the Euclidean distance between the points A and B in $\mathbb{R}^n$.

The following definitions are generalizations of the definitions of g-chordal points and, respectively, g-chordal curves introduced by Cieślak and Wołowik in [4].

Definition 1.1.2. *A point P in $\mathbb{R}^n$ is called a g-chordal point of S if S is star-shaped with respect to P and there exists a constant $c \in \mathbb{R}$ such that for any chord of S passing through P, joining the points P_1 and P_2 of ∂S, the following property is satisfied*

$$g(d(P, P_1)) + g(d(P, P_2)) = c. \tag{1.6}$$

For example, given any function g defined as above, the origin is a g-chordal point of a centered ball.

Definition 1.1.3. *If S has a g-chordal point P, then S is said to be a g-chordal star-shaped set of constant c.*

We will denote the class of all such star-shaped compact sets by $\mathcal{K}_{g,c}$.

In the context of this paper, the g-chordal point of a g-chordal star-shaped compact set is assumed to be at the origin O. We denote by $\mathcal{K}$ the class of all star-shaped compact sets whose boundaries are embedded in $\mathbb{R}^n$ and we parametrize the boundaries of such sets as graphs over the unit sphere

$$u \to \rho(u) \cdot u, \quad u \in \mathbb{S}^{n-1}. \tag{1.7}$$

Thus, $\mathcal{K}_{g,c}$ is the subclass of $\mathcal{K}$ of sets satisfying the following condition

$$g(\rho(u)) + g(\rho(-u)) = c, \quad u \in \mathbb{S}^{n-1}. \tag{1.8}$$

for some fixed function g and constant c.

1.2 Dual Mixed Volumes

Dual mixed volumes were introduced by Lutwak in [11] for convex bodies and later considered for star-shaped compact sets with non-empty interior (also called star bodies), as seen in Gardner's book [7], even though they are still attributed to Lutwak.

Definition 1.2.1 ([11]). *Let $S_1, ..., S_n$ be star bodies in $\mathbb{R}^n$. The dual mixed volume of $S_1, ..., S_n$, denoted by $\widetilde{V}(S_1, .., S_n)$, is given by*

$$\widetilde{V}(S_1, ..., S_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_{S_1} ... \rho_{S_n} \, d\mu_{\mathbb{S}^{n-1}}. \tag{1.9}$$

In particular, we use the notation $\widetilde{V}_j(S_1, S_2)$ to denote the dual mixed volume in which the set S_1 appears j times while the set S_2 appears $n-j$ times, i.e.

$$\widetilde{V}_j(S_1, S_2) = \widetilde{V}(\underbrace{S_1, ..., S_1}_{\text{j times}}, \underbrace{S_2, ..., S_2}_{\text{n-j times}}). \tag{1.10}$$

It is worth noting that if all star bodies are copies of the same set S, then the definition of the mixed volume resumes to the usual (Lebesgue) volume of the set $S \subset \mathbb{R}^n$.

Definition 1.2.2. *If $S_2 = \mathbb{B}_2^n$ is the usual Euclidean unit ball in $\mathbb{R}^n$ in (1.10), the resulting dual mixed volume of S_1 and S_2 is called the j-th quermassintegral of S_1, where $0 \leq j \leq n$ is an integer.*

Definition 1.2.3. *Let (X, Σ, μ) be a measure space and $p \in [1, \infty]$. Then, a measurable* *function f is in $L^p(\mu)$, for $p \in [1, \infty)$ if and only if*

$$\|f\|_p := \left(\int_S |f|^p \, d\mu \right)^{\frac{1}{p}} < \infty \tag{1.11}$$

For $p = \infty$, $f \in L^\infty(\mu)$ if and only if

$$\|f\|_\infty := ess\, sup\, |f| < \infty, \tag{1.12}$$

where $ess\, sup\, |f| = \inf\{M \in \mathbb{R} : \mu(\{t \in X : |f| > M\}) = 0\}$ is called the essential supremum of f.

Theorem 1.2.1 (Hölder's Inequality)**.** *Let (X, Σ, μ) be a measure space and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all measurable functions $f \in L^p(\mu)$ and $g \in L^q(\mu)$ on X, we have that $fg \in L^1(\mu)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{1.13}$$

In addition, if $p, q \in (1, \infty)$, the equality occurs if and only if there exists real numbers $\alpha, \beta \geq 0$ (not both 0) such that $\alpha|f|^p = \beta|g|^q$.

For the sake of completeness, we present the following inequality due to Lutwak and its proof.

Theorem 1.2.2 ([11])**.** *Let K, L be star bodies in $\mathbb{R}^n$. Then, for any integer $0 \leq j \leq n$*

$$\widetilde{V}_j(K, L)^n \leq [V(K)]^j [V(L)]^{n-j} \tag{1.14}$$

with equality if and only if there exists $c > 0$ such that $\rho_K = c\rho_L$ everywhere on $\mathbb{S}^{n-1}$.

Proof. For $j = 0$ or $j = n$, the inequality becomes trivially an equality. We may thus assume that $0 < j < n$.

Applying Hölder's inequality to the functions $f(u) = \rho_K^j(u)$, $g(u) = \rho_L^{n-j}(u)$ with $p = \frac{n}{j}$ and $q = \frac{n}{n-j}$, we obtain

$$\widetilde{V}_j(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K^j(u) \rho_L^{n-j}(u) \, d\mu_{\mathbb{S}^{n-1}}(u)$$

$$\leq \frac{1}{n}\left[\int_{\mathbb{S}^{n-1}} [\rho_K^j(u)]^{\frac{n}{j}}\, d\mu_{\mathbb{S}^{n-1}}(u)\right]^{\frac{j}{n}} \left[\int_{\mathbb{S}^{n-1}} [\rho_L^{n-j}(u)]^{\frac{n}{n-j}}\, d\mu_{\mathbb{S}^{n-1}}(u)\right]^{\frac{n-j}{n}}$$

$$= \frac{1}{n}\big[nV(K)\big]^{\frac{j}{n}}\big[nV(L)\big]^{\frac{n-j}{n}}$$

$$= \big[V(K)\big]^{\frac{j}{n}}\big[V(L)\big]^{\frac{n-j}{n}}. \tag{1.15}$$

The equality case follows from Hölder's equality case. That is, the equality holds if and only if there exists real numbers $\alpha, \beta \geq 0$ such that the following hold almost everywhere with respect to the surface area measure of $\mathbb{S}^{n-1}$

$$\alpha|\rho_K^j|^{\frac{n}{j}} = \beta|\rho_L^{n-j}|^{\frac{n}{n-j}} \iff \alpha\rho_K^n = \beta\rho_L^n \iff (\alpha/\beta)^{\frac{1}{n}}\rho_K = \rho_L. \tag{1.16}$$

However, the radial functions of K and L are both are continuous, hence the final claim follows. □

1.3 Isoperimetric Inequality

The classical isoperimetric problem dates back to antiquity. In the plane, the problem can be stated as either one of the following statements:

a. maximize the area enclosed by a closed curve of length L

b. minimize the length of a curve enclosing a region of area A.

Though the solution seems to be obviously a circle, it wasn't until 1838 that some progress was made towards proving this fact. It was Swiss Geometer Jakob Steiner who showed, using a method known today as *Steiner symmetrization*, that if a solution existed, it must be a circle. Steiner's proof, which we shall provide later, was deemed as incomplete because it did not prove the existence of a curve with maximal area. The first rigorous proof appeared in 1870 by Karl Weierstrass as a corollary of his Theory of Calculus of Several Variables.

Steiner symmetrization, the backbone of Steiner's proof of the Isoperimetric inequality, is a geometric method used to transform a compact set $K \subset \mathbb{R}^n$ into another compact set that is symmetric with respect to a hyperplane $H^{n-1} \subset \mathbb{R}^n$ passing through the origin. The resulting Steiner symmetral, denoted by $\mathcal{S}_H(K)$, will have equal n-dimensional volume as the original set K. We will take K to be a compact set and, for simplicity, let H be the rotated hyperplane H^{n-1} so that $x_n = 0$,(i.e. $H = \{(x', 0) : x' \in \mathbb{R}^{n-1}\}$). Then, for each $x \in H$, we will denote by G_x the line passing through x and perpendicular to H (i.e. $G_x = \{x + ye_n : y \in \mathbb{R}\}$ where e_n is the standard unit vector). Then, the length of the segment obtained by intersecting the line G_x with the set K is given by $m_x = |K \cap G_x|$. Each such segment will be replaced by an interval of the same length centered on H in order to obtain the symmetrized hypersurface. This means that

$$|K \cap G_x| = m_x = |S_H(K) \cap G_x|. \tag{1.17}$$

Moreover, the set $S_H(K)$ can be defined as follows

$$\mathcal{S}_H(K) = \{x + ye_n : x + ze_n \in K \text{ for some z and } -\frac{1}{2}m_x \leq y \leq \frac{1}{2}m_x\}. \tag{1.18}$$

Theorem 1.3.1. *Let $K \subset \mathbb{R}^n$ be a compact star-shaped set and let H be the hyperplane defined as above. The Steiner symmetrization satisfies the following properties:*

(i) Monotonicity with respect to inclusion: If $K \subseteq L$ then $S_H(L) \subseteq S_H(K)$.

(ii) Preserves convexity: If K is convex, so is $S_H(K)$.

(iii) Preserves volume/area: $V(K) = V(S_H(K))$.

(iv) Does not increase surface area/length: $A(K) \geq A(S_H(K))$.

Proof. The proof of (i) is obvious.

For (ii), we present an elementary geometric argument from [8] that uses trapezoids to show that Steiner symmetrization preserves convexity.

Assume that K is convex. Let $x, y \in S_H(K)$. Denote by l_x (and l_y) the line passing through x (and y, respectively) and orthogonal to H. Consider the convex trapezoid, $T \subset K$, obtained by taking the convex hull of the line segments $K \cap l_x$ and $K \cap l_y$. Since x and y belong to the convex set $S_H(T) \subset S_H(K)$, it follows that the line segment with endpoints at x and y is also contained in $S_H(K)$. That is, $S_H(K)$ is also convex. □

In order to prove (iii), we must first introduce Fubini's theorem. The following are some preliminary definitions to understand the assumptions of Fubini's theorem.

Definition 1.3.1. *A measure space (X, Σ, μ) is said to be σ-finite if X is the countable union of measurable sets with finite measure.*

Definition 1.3.2. *The Cartesian product $X \times Y$ is the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$, i.e. $X \times Y = \{(x, y) : x \in X, y \in Y\}$.*

We will denote by $\mathcal{S} \times \mathcal{T}$ the sigma algebra on $X \times Y$ generated by subsets of the form $S_1 \times T_1$ where $S_1 \in \mathcal{S}$ and $T_1 \in \mathcal{T}$. Then,

Definition 1.3.3. *A product measure, $\mu \times \nu$ is defined to be a measure on the space $(X \times Y, \mathcal{S} \times \mathcal{T})$ that satisfies the property*

$$(\mu \times \nu)(S_1 \times T_1) = \mu(S_1)\nu(T_1) \quad \forall S_1 \in \mathcal{S}, T_1 \in \mathcal{T}. \tag{1.19}$$

Definition 1.3.4. *A function $f : (X \times Y, \mathcal{S} \times \mathcal{T}) \to (Z, \mathcal{R})$ is said to be measurable if and only if*

$$f^{-1}(R_1) := \{(x, y) \in X \times Y : f(x, y) \in R_1\} \in \mathcal{S} \times \mathcal{T}, \forall R_1 \in \mathcal{R}. \tag{1.20}$$

Theorem 1.3.2 (Fubini's Theorem)**.** *Suppose X and Y are σ-finite measure spaces and $f(x, y)$ is a measurable function such that*

$$\int_{X \times Y} |f(x, y)| \, d(x, y) < \infty. \tag{1.21}$$

Then,

$$\int_X \left(\int_Y f(x, y) \, dy \right) dx = \int_Y \left(\int_X f(x, y) \, dx \right) dy. \tag{1.22}$$

Proof of (iii) from Theorem 1.3.1. Assume without loss of generality that H is the hyperplane in $\mathbb{R}^{n-1}$ such that $x_n = 0$. Recall that for each $x = (x_1, ..., x_{n-1}, 0) \in H$, we had defined G_x to be the line passing through x and perpendicular to H. It follows from (1.17) and Fubini's theorem that

$$\begin{aligned} V(K) &= \int_{(x_1,\ldots,x_{n-1})\in H} \left(\int_{x_n \in K\cap G_x} dx_n \right) dx_1 dx_2 \ldots dx_{n-1} \\ &= \int_{(x_1,\ldots,x_{n-1})\in H} m_x \, dx_1 dx_2 \ldots dx_{n-1} \\ &= \int_{(x_1,\ldots,x_{n-1})\in H} \left(\int_{x_n \in S_H(K)\cap G_x} dx_n \right) dx_1 dx_2 \ldots dx_{n-1} \\ &= V(S_H(K)). \end{aligned} \tag{1.23}$$

□

Property (iv) was first proved in full generality by De Giorgi in [5]. However, in order to prove the 2-dimensional isoperimetric inequality, it suffices to prove it for planar convex curves as we shall see later. So, we will now prove that the length of a simple, closed convex curve does not increase under Steiner symmetrization. This property along with the previous one is precisely how Steiner proved that of all simple closed curves with a given area A, the circle has the smallest perimeter. This is clear intuitively as the circle is the only figure that is unchanged under Steiner symmetrization (up to translation) for any choice of hyperplane since it is symmetric with respect to the origin in all directions.

To prove this, we will begin by proving a known fact about triangles.

Lemma 1.3.1. *Of all triangles with a fixed base and given area, the isosceles triangle has minimum perimeter.*

Proof. Let us consider the triangle ABC with base $\overline{AB}$ and height h. Draw a line L passing through the point C and parallel to the base $\overline{AB}$. Then, reflect the segment $\overline{CB}$ about the line L and denote the resulting reflected segment by $\overline{CB'}$. Note that $d(B, B') = 2h$ and $|\overline{CB}| = |\overline{CB'}|$.

Therefore, it suffices to minimize $|\overline{AC}| + |\overline{CB'}|$ which occurs when the points A and B' are joined by a straight line. This happens if and only if the angles of the triangle ABC

at A and, respectively B, are equal to each other i.e. ABC is an isosceles triangle, thus $|\overline{AC}| = |\overline{CB}|$. □

The following lemma, which follows from the previous one, will be used to prove property (iv).

Lemma 1.3.2 ([1]). *Of all trapezoids with parallel bases and altitude of fixed lengths, the isosceles trapezoid has minimum perimeter.*

Proof. Since the length of the bases is fixed, it suffices to minimize the length of the sides.

Let us consider the trapezoid $ABCD$ whose parallel bases $\overline{AB}$ and $\overline{CD}$ are at a fixed distance apart. Note that the area of the trapezoid is fixed since the length of the bases and the height of the trapezoid are fixed.

Assume without loss of generality that $|AB| < |CD|$. Draw two lines perpendicular to the bases starting at A and B. Assume without any loss of generality that they are intersecting the segment $\overline{CD}$ at the points P and Q, respectively. Consider the two resulting triangles APD and BQC on either end of the trapezoid. Those are the two triangles left if we were to remove the rectangle in the middle of the trapezoid. Joining those two triangles together, we obtain a larger triangle with fixed base and area and whose sides $\overline{AD}$ and $\overline{BC}$ coincide with the sides of the trapezoid. By lemma (1.3.1), this triangle will have minimum perimeter when its sides are congruent. That is, the sum of the length of the sides of the new triangle is smallest when the sides of the trapezoid are the same. A similar reasoning stands if P or Q are not both on the segment $\overline{CD}$, or if they are both outside the segment $\overline{CD}$. □

The following proof which can be found in [9] shows that perimeter does not increase under Steiner symmetrization.

Proof of (iv) of Theorem 1.3.1. Let $\{l_j\}_{1\leq j\leq k}$ be lines perpendicular to the hyperplane H and intersecting the boundaries of K and $S_H(K)$ at the points $Q_{j,1}$ and $Q_{j,2}$ and $Q'_{j,1}$ and $Q'_{j,2}$, respectively. It follows from (1.3.2) that for any two lines l_j and l_{j+1}, $1 \leq j \leq k-1$, the total length of the sides of the trapezoid $Q_{j,1}Q_{j,2}Q_{j+1,2}Q_{j+1,1}$ is greater or equal to that of the trapezoid $Q'_{j,1}Q'_{j,2}Q'_{j+1,2}Q'_{j+1,1}$, that is,

$$|Q_{j,1}Q_{j+1,1}| + |Q_{j,2}Q_{j+1,2}| \geq |Q'_{j,1}Q'_{j+1,1}| + |Q'_{j,2}Q'_{j+1,2}|. \tag{1.24}$$

Let P_{2k} and P'_{2k} be the polygons of $2k$ vertices formed by the points at which the lines $\{l_j\}_{1\leq j\leq k}$ intersect K and $S_H(K)$. It is clear from (1.24) that P_{2k} will have perimeter greater or equal to that of P'_{2k}, that is

$$l(P_{2k}) \geq l(P'_{2k}). \tag{1.25}$$

Moreover,

$$l(K) \geq l(P_{2k}). \tag{1.26}$$

Combining (1.25) and (1.26), we get

$$l(K) \geq l(P'_{2k}). \tag{1.27}$$

By letting k go to infinity, it has been established in [3] that the perimeter of the polygon P'_{2k} converges to that of the boundary of the convex set $S_H(K)$, i.e.

$$l(P'_{2k}) \xrightarrow[k\to\infty]{} l(S_H(K)). \tag{1.28}$$

Then, by (1.27) and (1.28), it follows that

$$l(K) \geq l(S_H(K)). \tag{1.29}$$

□

The isoperimetric problem was generalized to higher dimensions, though many of the methods used to prove the inequality in 2-dimensional Euclidean space no longer hold in higher dimensions. In 3-dimensional Euclidean space, under certain regularity conditions, by maximizing the volume with respect to the surface area, we obtain a ball. For even higher dimensions, the n-dimensional volume is maximized with respect to the $(n-1)$-dimensional surface area. In general, in order to make sense of the isoperimetric inequality, we must introduce the concept of (lower) Minkowski content which is a way to define the m-dimensional volume of an m-dimensional object, in our case the $(n-1)$-dimensional boundary of the domain, embedded in $\mathbb{R}^n$.

Definition 1.3.5 ([6]). *For a set $S \subset \mathbb{R}^n$ whose closure has finite Lebesgue measure and each integer m with $0 \le m \le n$ we define the m-dimensional **lower Minkowski content***

$$\mathcal{M}_*^m(S) = \liminf_{r \to 0^+} \frac{\mathcal{L}^n\{x : dist(x, S) < r\}}{\omega_{n-m} r^{n-m}} \tag{1.30}$$

where $\omega_{n-m} r^{n-m}$ is the volume of the $(n-m)$-ball of radius r and $\mathcal{L}^n$ is the n-dimensional Lebesgue measure.

Similarly, by taking the lim sup instead, we obtain the upper m-dimensional Minkowski content $\mathcal{M}^{*m}$. For sets which are sufficiently regular, these two values for will be equal and their common value is called the Minkowski content ($\mathcal{M}^m$).

Definition 1.3.6 ([6]). *A set E is said to be m-rectifiable if there exists a Lipschitz map $f : K \to E$ for some bounded subset K of $\mathbb{R}^m$ onto E.*

Theorem 1.3.3 ([6]). *If S is a closed m rectifiable subset of $\mathbb{R}^n$, then*

$$\mathcal{M}^m(S) = \mathcal{H}^m(S), \tag{1.31}$$

where $\mathcal{H}^m(S)$ is the m-dimensional Hausdorff measure of S.

Though the function $\mathcal{M}_*^{n-1}(\partial S)$ is not a measure, it coincides with the $(n-1)$-dimensional Hausdorff measure of the boundary of S, $\mathcal{H}^{n-1}(\partial S)$, when ∂S is $n-1$ rectifiable as in Theorem 1.3.3.

Theorem 1.3.4 ([6]). *Given a set $S \subset \mathbb{R}^n$ and $\mathcal{L}^n(\bar{S}) < \infty$, the isoperimetric inequality states that*

$$\mathcal{M}_*^{n-1}(\partial S) \ge n[\omega_n]^{\frac{1}{n}} \left[\mathcal{L}^n(\bar{S})\right]^{\frac{n-1}{n}}. \tag{1.32}$$

The equality occurs if and only if S is a ball in $\mathbb{R}^n$.

In our thesis, the $(n-1)$-dimensional Minkowski content of the boundary of S gives the surface area of S.

Note that Theorem 1.3.4 reduces to the following in $\mathbb{R}^2$:

Theorem 1.3.5. *Let $\gamma \subset \mathbb{R}^2$ be a simple closed curve whose length is L and the area of the region that it encloses is A. Then*

$$L^2 \geq 4\pi A. \tag{1.33}$$

Equality is reached if and only if the curve is a circle.

Proof. We will hint at yet another geometric proof. For the sake of simplification, we will denote by K the set bounded by the curve γ where A^* and L^* represent the area and perimeter, respectively, of the convex hull of K, denoted by $conv(K)$. We will begin by showing that the proof reduces to the proof for K convex set. Indeed, since $K \subset conv(K)$, it follows that $A \leq A^*$. Moreover, we know that the convexification of K reduces the perimeter (i.e. $L^* \leq L$). This is clear geometrically since the components of the boundary of K that lie in the interior of $conv(K)$ were replaced by a straight line segment so as to make the set convex, and thus reducing its perimeter. Therefore, the isoperimetric inequality will hold for K provided that it holds for $conv(K)$ since

$$L^2 - 4\pi A \geq (L^*)^2 - 4\pi A^*. \tag{1.34}$$

As for the equality case, since $L^2 = 4\pi A$ implies $(L^*)^2 = 4\pi A^*$ by (1.34), and we will show that the latter occurs when $conv(K)$ is a circle, then we must have that $K = conv(K)$, i.e. K is a circle too. Therefore, we may assume that γ is a simple closed convex curve. It remains to show that for a given area A, the smallest perimeter for γ is the perimeter of a circle. By applying Steiner's symmetrization to the set K about any line l_1, we obtain the Steiner symmetral $S_{l_1}(K)$, which we will denote by K_1. The set K_1 has clearly area equal to that of K and smaller, or equal, perimeter which we will denote by L_1. The equality happens if and only if K was symmetric with respect to the direction of the line l_1. Hence

$$L^2 - 4\pi A \geq L_1^2 - 4\pi A. \tag{1.35}$$

Applying Steiner's symmetrization again to the set K_1 about a different line l_2, we obtain the Steiner symmetral $S_{l_2}(K_1)$, which we shall denote by K_2. Similarly the set K_2 will have the same area as K and further reducing its perimeter to L_2, unless K_1 is symmetric with respect to the direction of the second line. Thus

$$L^2 - 4\pi A \geq L_1^2 - 4\pi A \geq L_2^2 - 4\pi A. \tag{1.36}$$

Continuing this reasoning, if K_i is not symmetric with respect to all directions, we may find a direction for which the Steiner symmetral of K_i with respect to that direction has equal area and strictly smaller perimeter than that of K_i. It can be shown that, for any convex set K, there exists a choice of directions so that K_i becomes a disk, thus symmetric with respect to every line through the origin and, consequently,

$$L^2 - 4\pi A \geq L_1^2 - 4\pi A \geq \ldots \geq L_i^2 - 4\pi A = 0. \tag{1.37}$$

A rigorous proof that repeated Steiner symmetrizations in different directions converges to a ball (in any dimension) can be found in [15]. □

Heuristically one can see that the same argument can be used to prove the isoperimetric problem in higher dimensions but in a class of *sufficiently regular surfaces bounding closed domains*, a restriction we want to emphasize. In this case, the n-dimensional ball is the only figure that is invariant (up to translation) under Steiner symmetrization for any hyperplane $H^{n-1} \subset \mathbb{R}^n$ passing through the origin.

1.4 Outline of the Thesis

We will start the next chapter by showing that there are infinitely many g-chordal star bodies (recall that the latter are star-shaped compact sets with nonempty interior) and we do so by presenting various ways to construct them. The methods presented are mostly due to Richard Gardner, [7]. We will then proceed to present several new isoperimetric type inequalities for g-chordal star bodies in $\mathbb{R}^n$.

The first result presented in this paper is a generalization to higher dimensions of a theorem by Cieślak and Wołowik for planar star-shaped bodies proved in [4]. It shall provide a lower bound for the n-dimensional volume of a g-chordal star body in $\mathbb{R}^n$ as long as g

meets certain requirements, namely that $g'(x) > 0$ and $g''(x) < 0$ for all $x \in (0, +\infty)$. The second condition can be relaxed as we will see later, in order to accommodate a larger class of functions.

Subsequently, we shall apply the general version of the isoperimetric inequality (1.3.4) to our first result in order to derive a lower bound for the surface area of such a star-shaped body. The resulting theorem is also a generalization of a theorem by Cieślak and Wołowik, [4]. However, our proof is different and more direct.

As we have seen in the section on dual mixed volumes, specifically Theorem 1.2.2, dual mixed volumes of star bodies are characterized by an upper bound. In fact, in general, one cannot expect to have lower bounds for the dual mixed volumes of arbitrary star bodies. However, we will use the lower bound on the function $\rho_j(u) = (\rho(u))^j + ((\rho(-u))^j$, on $u \in \mathbb{S}^{n-1}$, proved in the proof of our first main result in order to derive lower bounds for the dual mixed volumes of a particular class of star bodies. More specifically, we first consider the dual mixed volume of a g-chordal star body and a centrally symmetric star body. The lower bound will then follow naturally.

Furthermore, we consider the dual mixed volume of two star bodies, one that is g-chordal and the other, h-chordal. We then use Theorem 1.2.2 to derive a lower bound on their dual mixed volume.

The g-chordal star bodies associated to power functions $g(x) = x^i$, for $i = 1, \ldots, n-1$, are of particular interest in geometric tomography. Geometric tomography aims to reconstruct sets in $\mathbb{R}^n$ given the size of their projections or sections. If the sections of the set are obtained by the intersection with k-planes in $\mathbb{R}^n$, then the k-dimensional volume of the sections are given by integrating the radial function raised to the power k, namely $(\rho(u))^k = \rho^k(u)$, or respectively $(\rho^k(u)+\rho^k(-u))/2$, on the vectors u of the unit sphere $\mathbb{S}^{n-1}$ lying in that k-plane. Questions of unique determination of the set appear then naturally and, sometimes, such questions are settled via geometric inequalities. This motivates the interest of isoperimetric type inequalities for g-chordal sets.

Finally, we generalize the above results to L^n-stars, which are star-shaped sets whose radial function is n-th power integrable over the unit sphere $\mathbb{S}^{n-1}$ with the usual structure inherited from $\mathbb{R}^n$.

Chapter 2

g-chordal Star Bodies in $\mathbb{R}^n$

2.1 Existence of g-chordal Star Bodies

We shall begin this chapter by introducing a special case of g-chordal sets in $\mathbb{R}^2$ defined by Gardner in [7].

Definition 2.1.1. *Let $g(x) = x^i$ for some $i \in \mathbb{R} \setminus \{0\}$ and $S \in \mathcal{K}_{g,c}$. Then S is said to be i-equichordal with constant c and the g-equichordal point P of S is called an i-equichordal point.*

If $i = 1$, then P is called an *equichordal point* of S with constant c, while if $i = -1$, then P is called an *equireciprocal point* of S with constant c.

Definition 2.1.2. *Let $g(x) = \ln(x)$ and $S \in \mathcal{K}_{g,c}$. Then the g-chordal point of S with constant c is called an equiproduct point with contant e^c.*

Example 2.1.1 ([7])**.** *For any $i \in \mathbb{R} \setminus \{0\}$, the origin is an i-equichordal point of a centered ball.*

Another example among many provided by Gardner in [7] is the following construction of non-circular star bodies in $\mathbb{R}^2$ with the origin as an equichordal point

Example 2.1.2. *Let $\rho = \rho(\theta)$, $0 \leq \theta \leq \pi$, $0 < \rho(\theta) < 2$ be a curve in the upper half-plane that joins the points $(1, 0)$ and $(1, \pi)$. Furthermore, define $\rho(\theta) = 2 - \rho(\theta - \pi)$, for*

$\pi < \theta < 2\pi$. Then, the star-body in $\mathbb{R}^2$ determined by the radial function $\rho = \rho(\theta)$ has an equichordal point at the origin with constant 2.

The following formula of curvature will be used in the proof of the next theorem to prove the convexity of a planar curve.

Lemma 2.1.1. *Suppose $\rho = \rho(\theta)$ determines a planar curve in polar coordinates. Then, the curvature of the curve at the point (ρ, θ) is given by*

$$k = \frac{2(\rho'(\theta))^2 + (\rho(\theta))^2 - \rho(\theta)\rho''(\theta)}{[(\rho'(\theta))^2 + (\rho(\theta))^2]^{\frac{3}{2}}}. \tag{2.1}$$

Proof. Let us consider the position vector of the curve given in polar coordinates by

$$\overrightarrow{\gamma}(\theta) = \rho(\theta)(\cos(\theta), \sin(\theta)). \tag{2.2}$$

Let T be the unit tangent vector along γ defined by

$$T = \frac{\gamma'(\theta)}{\|\gamma'(\theta)\|}, \tag{2.3}$$

where $\gamma'(\theta) = \dfrac{d\gamma}{d\theta}$. The unit normal vector N is defined as

$$N = \frac{T'(\theta)}{\|T'(\theta)\|}. \tag{2.4}$$

The curvature, a measurement of the change of direction of the tangent line with respect to the distance travelled along the curve, is defined as the magnitude of the derivative of the tangent vector

$$k = \left\| \frac{dT}{ds} \right\|, \tag{2.5}$$

where s represents the arclength parameter of the curve γ. By the chain rule, we have that

$$T = \frac{d\gamma(\theta)}{ds} = \frac{d\gamma(\theta)}{d\theta} \cdot \frac{d\theta}{ds}. \tag{2.6}$$

Since the tangent has unit length (i.e. $T \cdot T = 1$), it follows from (2.6) that

$$\frac{d\theta}{ds} = \frac{1}{\|\gamma'(\theta)\|}. \tag{2.7}$$

Moreover, since the normal also has unit length (i.e. $N \cdot N = 1$), it follows from (2.4) that

$$\|T'(\theta)\| = T'(\theta) \cdot N. \tag{2.8}$$

Then, we will obtain an expression for $T'(\theta)$ by differentiating T with respect to s and using (2.6) and (2.7)

$$\begin{aligned}
\frac{dT}{ds} &= \frac{dT}{d\theta} \cdot \frac{d\theta}{ds} \\
&= \frac{d\theta}{ds} \cdot \frac{d}{d\theta}\left(\frac{d\gamma}{d\theta} \cdot \frac{d\theta}{ds}\right) \\
&= \frac{d\theta}{ds} \cdot \left(\frac{d^2\gamma}{d\theta^2} \cdot \frac{d\theta}{ds} + \frac{d\gamma}{d\theta} \cdot \frac{d^2\theta}{d\theta ds}\right) \\
&= \gamma''(\theta) \cdot \left(\frac{d\theta}{ds}\right)^2 \\
&= \frac{\gamma''(\theta)}{\|\gamma'(\theta)\|^2}.
\end{aligned} \tag{2.9}$$

Therefore,

$$T'(\theta) = \frac{\gamma''(\theta)}{\|\gamma'(\theta)\|}. \tag{2.10}$$

As such, we can rewrite (2.5) as

$$k = \left\|\frac{dT}{ds}\right\| = \left\|\frac{dT}{d\theta} \cdot \frac{d\theta}{ds}\right\| = \left\|\frac{dT}{d\theta} \cdot \frac{1}{\|\gamma'(\theta)\|}\right\| = \frac{T'(\theta) \cdot N}{\|\gamma'(\theta)\|} = \frac{\gamma''(\theta) \cdot N}{\|\gamma'(\theta)\|^2}. \tag{2.11}$$

By differentiating (2.2) with respect to θ, we obtain

$$\gamma'(\theta) = \rho'(\theta)(\cos(\theta), \sin(\theta)) + \rho(\theta)(-\sin(\theta), \cos(\theta)) = \rho'(\theta) \cdot \hat{u} + \rho(\theta) \cdot \hat{v}, \tag{2.12}$$

where $\hat{u} = (\cos(\theta), \sin(\theta))$ and $\hat{v} = (-\sin(\theta), \cos(\theta))$ are orthonormal vectors. By taking the second derivative, we obtain

$$\gamma''(\theta) = (\rho''(\theta) - \rho(\theta)) \cdot \hat{u} + 2\rho'(\theta) \cdot \hat{v}. \tag{2.13}$$

It follows from (2.12) that the unit normal vector can be written as

$$N = \frac{\rho'(\theta) \cdot \hat{v} - \rho(\theta) \cdot \hat{u}}{\|\gamma'(\theta)\|}. \tag{2.14}$$

Furthermore, from (2.12), we obtain the speed of γ

$$\|\gamma'(\theta)\| = \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2}. \tag{2.15}$$

Finally, from the definition of the curvature (2.11), as well as equations (2.10), (2.13), (2.14), and (2.15), we have that

$$\begin{aligned} k &= \frac{\left[(\rho''(\theta) - \rho(\theta)) \cdot \hat{u} + 2\rho'(\theta) \cdot \hat{v}\right] \cdot \left[\rho'(\theta) \cdot \hat{v} - \rho(\theta) \cdot \hat{u}\right]}{(\sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2})^3} \\ &= \frac{2(\rho'(\theta))^2 + (\rho(\theta))^2 - \rho(\theta)\rho''(\theta)}{[(\rho'(\theta))^2 + (\rho(\theta))^2]^{\frac{3}{2}}}. \end{aligned} \tag{2.16}$$

□

The following theorem provides a way of constructing non-spherical convex sets in $\mathbb{R}^n$ with an i-equichordal point. The idea behind the construction is to rotate a 2-dimensional non-circular compact convex set whose origin O is an i-equichordal point about an axis of symmetry passing through the origin. The resulting n-dimensional object will have O as an i-equichordal point as well.

Theorem 2.1.1 ([7]). *For each $i \in \mathbb{R} \setminus \{0\}$ and $n \geq 2$, there exists a non-spherical convex body in $\mathbb{R}^n$ with the origin as an i-equichordal point.*

For the sake of completeness, we present its proof due to Gardner.

Proof. Let K be a non-circular 2-dimensional convex body in the $\{x_1, x_n\}$-plane that is symmetric about the x_n-axis, and whose origin is an i-equichordal point. Let K' be the n-

dimensional convex body obtained by rotating K about the x_n-axis. Because the intersection of K' with any hyperplane H orthogonal to the x_n-axis is an $(n-1)$-dimensional ball and both u and $-u$ lie on the same 2-dimensional subspace containing the x_n-axis, we have that for any $u \in \partial K'$, some $u' \in \partial K$ such that $\rho_K(u) = \rho_{K'}(u')$ and $\rho_K(-u) = \rho_{K'}(-u')$. Therefore, K' also has an i-equichordal point at the origin.

It remains to show the existence of such 2-dimensional i-equichordal non-circular convex bodies. Define $f(\theta)$ to be a nontrivial function of class C^2 on $\mathbb{R}$ that is odd and 2π-periodic such that $f(\theta - \pi/2)$ is even. It follows that

$$f(\theta) = f\Big(\theta + \frac{\pi}{2} - \frac{\pi}{2}\Big) = f\Big(-\theta - \frac{\pi}{2} - \frac{\pi}{2}\Big) = -f(\theta + \pi) \tag{2.17}$$

for all $0 \leq \theta \leq 2\pi$. For now choose a fixed real number $c > 0$ such that $f(\theta) + c > 0$ for all θ, and define a radial function ρ to be

$$\rho(\theta) = (f(\theta) + c)^{\frac{1}{i}}. \tag{2.18}$$

Then, for all θ, we have

$$\rho^i(\theta) + \rho^i(\theta + \pi) = (f(\theta) + c) + (-f(\theta) + c) = 2c. \tag{2.19}$$

Moreover, for all θ,

$$\begin{aligned} \rho(\theta) = (f(\theta) + c)^{\frac{1}{i}} &= (-f(\theta + \pi) + c)^{\frac{1}{i}} \\ &= (-f(\theta + 2\pi - \pi) + c)^{\frac{1}{i}} \\ &= (-f(\theta - \pi) + c)^{\frac{1}{i}} \\ &= (f(-\theta + \pi) + c)^{\frac{1}{i}} \\ &= \rho(\pi - \theta). \end{aligned} \tag{2.20}$$

Therefore, the star body determined by the radial function $\rho(\theta)$ is symmetric with respect to the x_n-axis. Finally, to prove the convexity of K, we must show that the curvature of the boundary of K, given in polar form by

$$k = \frac{2(\rho'(\theta))^2 + (\rho(\theta))^2 - \rho(\theta)\rho''(\theta)}{[(\rho'(\theta))^2 + (\rho(\theta))^2]^{\frac{3}{2}}}, \tag{2.21}$$

is nonnegative. As such, it suffices to show that $\rho(\theta) - \rho''(\theta) \geq 0$ for our choice of $f(\theta)$ and c. Since we assumed that $f \in C^2$, we obtain

$$\rho(\theta) - \rho''(\theta) = (f(\theta)+c)^{\frac{1-2i}{i}} \Big[(f(\theta)+c)^2 - \frac{1}{i}\Big(\frac{1-i}{i}\Big)(f'(\theta))^2 - \frac{1}{i}(f(\theta)+c)f''(\theta)\Big],$$

which is non-negative for sufficiently large c. This is true since since the function f is of class C^2, thus $f'(\theta)$ and $f''(\theta)$ are bounded on $[0, 2\pi]$ and the term $(f(\theta)+c)^2$ grows faster to infinity than any linear term in c. Hence, if needed, we increase the fixed number c so that the curvature of the curve is positive, concluding so the proof of the theorem. □

Corollary 2.1.1 ([7]). *There exists a non-spherical convex body in $\mathbb{R}^n$ with the origin as an equiproduct point.*

Proof. The construction is identical to the one in the proof of Theorem 2.1.1 except that the body K is determined by the radial function as follows

$$\rho(\theta) = e^{\frac{f(\theta)}{c}}, \tag{2.22}$$

for all $0 \leq \theta \leq 2\pi$ and some appropriate real number c. Then,

$$\ln(\rho(\theta)) + \ln(\rho(\theta+\pi)) = \ln(e^{\frac{f(\theta)}{c}}) + \ln(e^{\frac{f(\theta+\pi)}{c}}) = \frac{f(\theta)}{c} + \frac{f(\theta+\pi)}{c} = 0. \tag{2.23}$$

The convexity of K, and the choice of c, will also follow from the calculation

$$\rho(\theta) - \rho''(\theta) = e^{\frac{f(\theta)}{c}} \left[1 - \frac{(f'(\theta))^2}{c^2} - \frac{f''(\theta)}{c}\right]. \tag{2.24}$$

The latter is positive if c is sufficiently close to zero. Therefore, K has an equiproduct point at the origin with constant 1 by Definition 2.1.2. □

Remark 2.1.1 ([7]). *The function $f(\theta) = a\sin(\theta)$, for a suitable choice of $a > 0$, satisfies* *the necessary properties stated above. For example, given $i = 2$, one may choose $a = \frac{1}{2}$ and $c = 1$.*

Proof. Clearly the function f here is odd, of class C^2, in fact it is smooth, and 2π-periodic. Moreover, $f(\theta - \frac{\pi}{2}) = -a\cos(\theta)$ is clearly even. □

2.2 Generating g-chordal Star Bodies via Ellipses

The following theorem provides another example of a 2-dimensional non-circular convex i-chordal body, for $i = -1$.

Theorem 2.2.1 ([7]). *An ellipse contains two equireciprocal points, one at each focus.*

Proof. Denote the eccentricity of the ellipse by e and assume that one focus is at the origin and the other is at the point $(0, -2c)$. Let a and b be the semimajor axis and semiminor axis of the ellipse, respectively, in the direction of the axes of coordinates. The eccentricity is given by the following equation

$$e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}. \tag{2.25}$$

Therefore,

$$\frac{1}{a^2} = \frac{e^2}{c^2}, \tag{2.26}$$

while

$$\frac{1}{b^2} = \frac{e^2}{c^2(1-e^2)}. \tag{2.27}$$

As such, the equation of the ellipse centered at $(0, -c)$ is given by

$$\frac{x^2}{b^2} + \frac{(y+c)^2}{a^2} = 1. \tag{2.28}$$

Equivalently,

$$\frac{e^2x^2}{c^2(1-e^2)} + \frac{e^2(y+c)^2}{c^2} = 1. \tag{2.29}$$

By converting equation (2.29) into a polar form by using $x = \rho\cos(\theta)$ and $y = \rho\sin(\theta)$, we obtain the radial function by which the ellipse is determined

$$\rho_{ellipse}(\theta) = \frac{c(1-e^2)}{e(1+e\sin(\theta))}. \tag{2.30}$$

It is easy to verify that this ellipse belongs to the class $\mathcal{K}_{g,c'}$ where $g(x) = \frac{1}{x}$ and $c' = \frac{2e}{c(1-e^2)}$ as

$$\frac{1}{\rho_{ellipse}(\theta)} + \frac{1}{\rho_{ellipse}(\theta+\pi)} = \frac{2e}{c(1-e^2)}. \tag{2.31}$$

Therefore, by Definition 2.1.1, we say that the ellipse has an equireciprocal point at the origin. By symmetry, it follows that $(0, -2c)$ is also an equireciprocal point. □

We will eventually use this theorem to generate more examples of g-chordal sets.

Before we do so, we will present another example of a 2-dimensional non-spherical g-chordal convex body with $g(x) = \ln(x)$.

Theorem 2.2.2 ([7]). *Let P be a point at a distance $d < r$ from the center of a disk of radius r. Then, P is an equiproduct point, with constant $(1-d^2)$.*

Proof. Assume, without loss of generality, that $r = 1$ and that the equation of the boundary of the disk is given by

$$(x-d)^2 + y^2 = 1 \tag{2.32}$$

where $d < 1$ and P is the origin. Letting $x = \cos(\theta)$ and $y = \sin(\theta)$, we obtain the following radial function in polar form

$$\rho(\theta) = \sqrt{1 - d^2\sin^2(\theta)} + d\cos(\theta). \tag{2.33}$$

We will express the above polar equation in terms of the slope t of the line passing through the origin and intersecting the boundary of the disk at angle θ. Since $\tan(\theta) = t$, we obtain the following two identities $\sin(\theta) = \frac{t}{\sqrt{1+t^2}}$ and $\cos(\theta) = \frac{1}{\sqrt{1+t^2}}$ using simple algebraic manipulations. Then, by substituting these two identities into (2.33), we obtain the following expression for ρ

$$\rho = \frac{(\sqrt{1+t^2(1-d^2)} \pm d)}{\sqrt{1+t^2}}. \tag{2.34}$$

Then,

$$\ln(\rho(\theta)) + \ln(\rho(\theta+\pi)) = \ln(\rho(\theta)\rho(\theta+\pi)) \tag{2.35}$$

$$= \ln\left[\frac{(\sqrt{1+t^2(1-d^2)}+d)}{\sqrt{1+t^2}}\frac{(\sqrt{1+t^2(1-d^2)}-d)}{\sqrt{1+t^2}}\right]$$

$$= \ln(1-d^2). \tag{2.36}$$

Therefore, the origin is an equiproduct point with constant $1-d^2$ by Definition 2.1.2. □

The method described in Theorem 2.1.1 can now be used to generate many more examples of i-chordal bodies. We already know from Theorem 2.2.1 that the ellipse has two equireciprocal points. Therefore, by rotating it about the x_n-axis as described in the aforementioned theorem, we obtain an n-dimensional ellipsoid which has the same two equireciprocal points. Likewise, from Theorem 2.2.2, we know that any point in the interior of a disk is an equiproduct point. Thus, any point in the interior of the n-ball, which is obtained by rotating the disk as in Remark 2.1.1, is also an equiproduct point.

Finally, a nontrivial example of an equichordal curve is given below.

Example 2.2.1. *The limaçon whose radial function is given by $\rho(\theta) = a\cos(\theta) + b$, $0 \leq \theta < 2\pi$ and $b > a$, has an equichordal point at the origin with constant $2a$.*

Note that for $b < a$, the limaçon is self-intersecting and, as such, is not considered.

Proof. First, we will check that property (1.8) is satisfied for $g(x) = x$:

$$(a + b\cos(\theta)) + (a + b\cos(\theta+\pi)) = a + b\cos(\theta) + a - b\cos(\theta) = 2a. \tag{2.37}$$

Note that the polar curve given in the form above is symmetric with respect to the x-axis. Therefore, the construction of Theorem 2.1.1 can be applied to yield star bodies in $\mathbb{R}^n$ whose origin is an equichordal point. In particular, the 3-dimensional object obtained by rotating the limaçon about the x-axis, which is shaped like an apple, has an equichordal point at the origin as well.

Finally, note that for $b = 0$, the limaçon reduces to a circle centered at $\left(\frac{a}{2}, 0\right)$.

Indeed, we will show that the polar equation of a circle centered at (h, k) is given by

$$r(\theta) = 2h\cos(\theta) + 2k\sin(\theta). \tag{2.38}$$

which will imply that the limaçon curve in the example is a circle when $b = 0$. To check that the radial function $r(\theta)$ is, in fact, consistent with that of a circle we will convert it to rectangular coordinates. Substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into (2.38), we get

$$r^2 = 2ar\cos(\theta) + 2br\sin(\theta), \tag{2.39}$$

$$\implies x^2 + y^2 = 2ax + 2by, \tag{2.40}$$

$$\implies x^2 + y^2 - 2ax - 2by = 0. \tag{2.41}$$

By completing the square, we get the following equation of a circle centered at (a, b)

$$(x-a)^2 + (y-b)^2 = a^2 + b^2. \tag{2.42}$$

□

Using the above examples, we will demonstrate another simple way of generating g-chordal star bodies.

Theorem 2.2.3. *Let $\rho_{ellipse}(\theta)$, $0 \leq \theta < 2\pi$, as in (2.30) and, for all $\theta \in [0, 2\pi]$, define $\rho(\theta) = \sqrt[p]{\rho_{ellipse}(\theta)}$. Then, the star body determined by the radial function $\rho = \rho(\theta)$ has two g-chordal points, one at O and the other at the point $(0, -2c)$, where $g(x) = -\dfrac{1}{x^p}$, $p \geq 1$.*

Proof. We already know from Theorem 2.2.1 that the ellipse has two equireciprocal points, one at the origin and the other at the point $(0, -2c)$. Therefore,

$$g(\rho(\theta)) + g(\rho(\theta+\pi)) = -\frac{1}{(\sqrt[p]{\rho_{ellipse}(\theta)})^p} - \frac{1}{(\sqrt[p]{\rho_{ellipse}(\theta+\pi)})^p} = -\frac{2e}{c(1-e^2)}$$

which implies that O and the point $(0, -2c)$ are g-chordal points of the star body determined by $\rho(\theta)$.

Moreover, the radial function $\rho(\theta)$ is positive, 2π-periodic and of class C^2 since the radial function of the ellipse, $\rho_{ellipse}(\theta)$, also is. $\square$

Note that the negative sign in the function g is not necessary. However, we conveniently put it so that $g'(x) > 0$ and $g''(x) < 0$ for all $x \in (0, +\infty)$ as these conditions will be necessary later on to prove the main results of this paper. We could have just as well picked $g(x) = \ln(x)$ and $\rho(\theta) = e^{\rho_{ellipse}(\theta)}$ so that property (1.8) is satisfied. Many other combinations are possible so long as g meets the necessary requirements and that the radial function is consistent with a star-shaped compact set whose boundary is an embedded hypersurface of class C^2. It's also worth nothing that, in general, one may have g-chordal sets for which the boundary need not be of class C^2; this condition is only necessary for the purposes of our theorems. In fact, later, we will even discuss star-shaped sets whose radial function is not continuous as in the case of L^n-stars.

Chapter 3

Main Results

3.1 Volume and Surface Area of g-chordal Star Bodies

Assume $g : (0, +\infty) \to \mathbb{R}$ is a twice-differentiable real function satisfying furthermore the following two conditions:

- $$g'(t) > 0 \quad \forall t \in (0, +\infty), \tag{3.1}$$

- $\exists j \in \mathbb{N}$ with $j \geq 1$ such that

$$g''(t) < \frac{(j-1)g'(t)}{t} \quad \forall t \in (0, +\infty). \tag{3.2}$$

We will say that g satisfies the **j-th condition** if (3.2) is satisfied for some $j \in \mathbb{N} \setminus \{0\}$.

Remark 3.1.1. *If $g''(t) < 0$ and $g'(t) > 0$ then (3.2) is satisfied $\forall j \in \mathbb{N}$.*

Note that the function $t \mapsto t^i,\ i > 0$, real, satisfies the above requirements for any $j > i$.

Let S be a g-chordal star body in $\mathbb{R}^n$ of radial function $\rho : \mathbb{S}^{n-1} \to (0, \infty)$ such that

$$g(\rho(u)) + g(\rho(-u)) = c, \quad \forall u \in \mathbb{S}^{n-1}, \tag{3.3}$$

for some real constant c.

We now introduce the following auxiliary real functions on $(0, +\infty)$:

$$\alpha(t) = g^{-1}(c - g(t)), \tag{3.4}$$

$$\Phi(t) = t^n + (\alpha(t))^n, \tag{3.5}$$

$$h(t) = \frac{t^{n-1}}{g'(t)}. \tag{3.6}$$

Theorem 3.1.1. *Under the assumptions (3.1), (3.2) and (3.3), the volume of the g-chordal star-shaped body S satisfies the inequality*

$$V(S) \geq \left[g^{-1}\left(\frac{c}{2}\right)\right]^n \omega_n, \tag{3.7}$$

where ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$. The equality holds if and only if S is an Euclidean n-ball of radius $g^{-1}\left(\frac{c}{2}\right)$.

Proof. We have, by equation (3.6), that

$$h'(t) = \frac{(n-1)t^{n-2}g'(t) - t^{n-1}g''(t)}{(g'(t))^2} > 0, \tag{3.8}$$

from which it follows that the function h is strictly increasing. Moreover, by equation (3.4), we have that $g(\alpha(t)) = c - g(t)$. Thus, taking the derivatives of both sides, we obtain

$$g'(\alpha(t))\alpha'(t) = -g'(t) \tag{3.9}$$

and, as g' is positive,

$$\Rightarrow \alpha'(t) = \frac{-g'(t)}{g'(\alpha(t))}. \tag{3.10}$$

By equation (3.5), we get that

$$\begin{aligned} \Phi'(t) &= nt^{n-1} + n(\alpha(t))^{n-1}\alpha'(t) \\ &= nt^{n-1} + n(\alpha(t))^{n-1}\left(\frac{-g'(t)}{g'(\alpha(t))}\right) \end{aligned}$$

$$
\begin{aligned}
&= ng'(t)\left[\frac{t^{n-1}}{g'(t)} - \frac{(\alpha(t))^{n-1}}{g'(\alpha(t))}\right] \\
&= ng'(t)[h(t) - h(\alpha(t))]. \qquad (3.11)
\end{aligned}
$$

To see the critical points of Φ, let $\Phi'(\widetilde{t}) = 0$ which implies, by (3.11), that $h(\widetilde{t}) = h(\alpha(\widetilde{t}))$. Since h is strictly increasing, the function h is also injective and

$$\widetilde{t} = \alpha(\widetilde{t}). \qquad (3.12)$$

By equations (3.4) and (3.12),

$$g^{-1}(c - g(\widetilde{t})) = \widetilde{t}, \qquad (3.13)$$

therefore, as g too is injective,

$$g(\widetilde{t}) = c - g(\widetilde{t}) \qquad (3.14)$$

$$\Rightarrow g(\widetilde{t}) = \frac{c}{2} \qquad (3.15)$$

$$\Rightarrow \widetilde{t} = g^{-1}\left(\frac{c}{2}\right). \qquad (3.16)$$

Now, by equations (3.10) and (3.12),

$$\alpha'(\widetilde{t}) = -\frac{g'(\widetilde{t})}{g'(\alpha(\widetilde{t}))} = -\frac{g'(\widetilde{t})}{g'(\widetilde{t})} = -1. \qquad (3.17)$$

Moreover, to study the nature of the critical point, we differentiate further equation (3.11), and use equations (3.12) and (3.17), to obtain

$$
\begin{aligned}
\Phi''(\widetilde{t}) &= ng''(\widetilde{t})\Big[h(\widetilde{t}) - h(\alpha(\widetilde{t}))\Big] + ng'(\widetilde{t})\Big[h'(\widetilde{t}) - h'(\alpha(\widetilde{t}))\alpha'(\widetilde{t})\Big] \\
&= ng'(\widetilde{t})\Big[h'(\widetilde{t}) + h'(\widetilde{t})\Big] \\
&= 2ng'(\widetilde{t})h'(\widetilde{t}) > 0. \qquad (3.18)
\end{aligned}
$$

Therefore, Φ attains its unique minimum at the point $\widetilde{t}$, and its minimum value is

$$\Phi(\widetilde{t}) = 2\widetilde{t}^n = 2\Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]^n. \tag{3.19}$$

Consequently,

$$\Phi(t) \geq 2\Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]^n \tag{3.20}$$

for arbitrary $t \in (0, +\infty)$. Therefore, by dividing the sphere $\mathbb{S}^{n-1}$ into the upper and lower hemispheres $\mathbb{S}^{n-1}_{\pm}$, we have

$$\begin{aligned}
n\mathrm{V}\,(S) &= \int_{\mathbb{S}^{n-1}} \rho^n(u)\, d\mu_{S^{n-1}}(u) \\
&= \int_{\mathbb{S}^{n-1}_{+} \cup\, \mathbb{S}^{n-1}_{-}} \rho^n(u)\, d\mu_{S^{n-1}}(u) \\
&= \int_{\mathbb{S}^{n-1}_{+}} \rho^n(u)\, d\mu_{S^{n-1}}(u) + \int_{\mathbb{S}^{n-1}_{-}} \rho^n(u)\, d\mu_{\mathbb{S}^{n-1}}(u) \\
&= \int_{\mathbb{S}^{n-1}_{+}} \rho^n(u)\, d\mu_{S^{n-1}}(u) + \int_{\mathbb{S}^{n-1}_{+}} \rho^n(-u)\, d\mu_{\mathbb{S}^{n-1}}(-u) \\
&= \int_{\mathbb{S}^{n-1}_{+}} \rho^n(u)\, d\mu_{S^{n-1}}(u) + \int_{\mathbb{S}^{n-1}_{+}} \rho^n(-u)\, d\mu_{\mathbb{S}^{n-1}}(u) \\
&= \int_{\mathbb{S}^{n-1}_{+}} [\rho^n(u) + \rho^n(-u)]\ d\mu_{\mathbb{S}^{n-1}}(u).
\end{aligned} \tag{3.21}$$

On the other hand, as $\alpha(t) = g^{-1}(c - g(t))$, we have that

$$g(t) + g(\alpha(t)) = c, \qquad \forall t \in (0, \infty), \tag{3.22}$$

and, since $g(\rho(u)) + g(\rho(-u)) = c$ for any $u \in \mathbb{S}^{n-1}$, it follows that

$$\alpha(\rho(u)) = \rho(-u). \tag{3.23}$$

Therefore,

$$\Phi(\rho(u)) = \rho^n(u) + \alpha^n(\rho(u))$$
$$= \rho^n(u) + \rho^n(-u). \tag{3.24}$$

Finally, from equations (3.24), (3.20) and (3.21),

$$\begin{aligned} n\mathrm{V}(S) &= \int_{\mathbb{S}^{n-1}_{+}} \rho^n(u) + \rho^n(-u)\, d\mu_{\mathbb{S}^{n-1}}(u) \\ &= \int_{\mathbb{S}^{n-1}_{+}} \Phi(\rho(u))\, d\mu_{\mathbb{S}^{n-1}}(u) \\ &\geq \int_{\mathbb{S}^{n-1}_{+}} 2\Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]^n\, d\mu_{\mathbb{S}^{n-1}}(u) \\ &= 2\Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]^n \frac{1}{2}\, n\omega_n \\ &= n\Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]^n \omega_n, \end{aligned} \tag{3.25}$$

which concludes the first part of the proof.

As for the equality case, assume first that S is a ball whose radial function $\rho(u)$ is given by the constant $g^{-1}\left(\frac{c}{2}\right)$. Then, it follows from (3.4), (3.5), and (3.25), that the equality holds since

$$\begin{aligned} \Phi(\rho(u)) = \Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]^n + \Big[\alpha\Big(g^{-1}\Big(\frac{c}{2}\Big)\Big)\Big]^n &= \Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]^n + \Big[g^{-1}\Big(c - g\Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]\Big)\Big]^n \\ &= 2\Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]^n. \end{aligned} \tag{3.26}$$

Conversely, assume that the equality case in (3.25) holds for every $u \in \mathbb{S}^{n-1}$, i.e.

$$\Phi(\rho(u)) = 2\Big[g^{-1}\Big(\frac{c}{2}\Big)\Big]^n, \quad \forall u \in \mathbb{S}^{n-1}. \tag{3.27}$$

Since Φ attains its unique minimum at the point $\widetilde{t} = g^{-1}\left(\frac{c}{2}\right)$, it follows that $\rho(u)$ coincides with the unique minimum $\widetilde{t}$. Therefore, S is the hypersphere determined by the constant radial function $\rho(u) = g^{-1}\left(\frac{c}{2}\right)$. □

Theorem 3.1.2. *Let S be a compact g-chordal star-shaped set in $\mathbb{R}^n$. Under the assumptions in (3.1), (3.2) and (3.3), the surface area of the boundary of S, denoted by $A(S)$, satisfies the inequality:*

$$A(S) \geq n\omega_n \left[g^{-1}\left(\frac{c}{2}\right)\right]^{n-1}. \tag{3.28}$$

Equality holds above if and only if S is an Euclidean ball in $\mathbb{R}^n$ of radius $g^{-1}\left(\frac{c}{2}\right)$.

Proof. By Theorem 1.3.4 and Theorem 3.1.1,

$$\begin{aligned} \mathrm{A}(S) &\geq n\mathrm{V}(S)^{\frac{n-1}{n}}\omega_n^{\frac{1}{n}} \\ &\geq n\left(\left[g^{-1}\left(\frac{c}{2}\right)\right]^n \omega_n\right)^{\frac{n-1}{n}} \omega_n^{\frac{1}{n}} \\ &= n\omega_n\left[g^{-1}\left(\frac{c}{2}\right)\right]^{n-1}, \end{aligned} \tag{3.29}$$

both implying also the equality case. □

3.2 Dual Mixed Volumes of g-chordal Star Bodies

The next two results are interesting because they provide lower bounds on the dual mixed volumes of certain star-shaped bodies. The classical inequalities for dual mixed volumes provide upper bounds on each j-th dual mixed volume of star-shaped bodies, $1 \leq j \leq n-1$, as it has been shown in the first chapter, but, for general star-shaped bodies, one cannot expect to have lower bounds.

Theorem 3.2.1. *Under the assumptions (3.1), (3.2), and (3.3), the dual mixed volume of a g-chordal star-shaped body $K \subset \mathbb{R}^n$ with constant c, and any centrally symmetric star-shaped* *body L, satisfies the inequality*

$$\widetilde{V}_j\,(K,L) \geq \left[g^{-1}\left(\frac{c}{2}\right)\right]^j \widetilde{V}_j(\mathbb{B}^n, L) \tag{3.30}$$

where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$.

Equality occurs if and only if K is an Euclidean ball in $\mathbb{R}^n$ of radius $g^{-1}\left(\frac{c}{2}\right)$.

Proof. Using equations (3.20) and (3.24),

$$\begin{aligned}
\widetilde{V}_j\,(K,L) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K^j(u)\rho_L^{n-j}(u)\,d\mu_{\mathbb{S}^{n-1}}(u) \\
&= \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} \rho_K^j(u)\rho_L^{n-j}(u)\,d\mu_{\mathbb{S}^{n-1}}(u) + \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} \rho_K^j(-u)\rho_L^{n-j}(-u)\,d\mu_{\mathbb{S}^{n-1}}(u) \\
&= \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} \rho_K^j(u)\rho_L^{n-j}(u)\,d\mu_{\mathbb{S}^{n-1}}(u) + \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} \rho_K^j(-u)\rho_L^{n-j}(u)\,d\mu_{\mathbb{S}^{n-1}}(u) \\
&= \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} (\rho_K^j(u) + \rho_K^j(-u))\rho_L^{n-j}(u)\,d\mu_{\mathbb{S}^{n-1}}(u) \\
&\geq 2\left[g^{-1}\left(\frac{c}{2}\right)\right]^j \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} \rho_L^{n-j}(u)\,d\mu_{\mathbb{S}^{n-1}}(u) \\
&= \left[g^{-1}\left(\frac{c}{2}\right)\right]^j \widetilde{V}_j(\mathbb{B}^n, L).
\end{aligned} \tag{3.31}$$

The equality case follows as in the equality case of Theorem 3.1.1. □

Theorem 3.2.2. *Assume $g : (0,+\infty) \to \mathbb{R}$ and $h : (0,+\infty) \to \mathbb{R}$ satisfy the j-th and $(n-j)$-th condition (3.1), respectively. Then, the dual mixed volume of $K \subset \mathbb{R}^n$, a g-chordal star body with constant c_1, and $L \subset \mathbb{R}^n$, an h-chordal star body with constant c_2 satisfies the inequality*

$$\widetilde{V}_j\,(K,L) \geq 2\omega_n\left[g^{-1}\left(\frac{c_1}{2}\right)\right]^j \left[h^{-1}\left(\frac{c_2}{2}\right)\right]^{n-j} - \left[V(K)\right]^{\frac{j}{n}}\left[V(L)\right]^{\frac{n-j}{n}}. \tag{3.32}$$

The equality case follows if each star-shaped body is an Euclidean ball of radii $g^{-1}\left(\frac{c_1}{2}\right)$ and $h^{-1}\left(\frac{c_2}{2}\right)$, respectively.

Proof. Using equations (3.20) and (3.24) as well as Theorem 1.2.2,

$$\begin{aligned}
\widetilde{V}_j(K,L) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K^j(u)\rho_L^{n-j}(u)\, d\mu_{\mathbb{S}^{n-1}}(u) \\
&= \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} \rho_K^j(u)\rho_L^{n-j}(u) + \rho_K^j(-u)\rho_L^{n-j}(-u)\, d\mu_{\mathbb{S}^{n-1}}(u) \\
&= \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} \big(\rho_K^j(u)+\rho_K^j(-u)\big)\big(\rho_L^{n-j}(u)+\rho_L^{n-j}(-u)\big)\, d\mu_{\mathbb{S}^{n-1}}(u) \\
&- \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} \rho_K^j(u)\rho_L^{n-j}(-u) + \rho_K^j(-u)\rho_L^{n-j}(u)\, d\mu_{\mathbb{S}^{n-1}}(u) \\
&= \frac{1}{n}\int_{\mathbb{S}_+^{n-1}} \big(\rho_K^j(u)+\rho_K^j(-u)\big)\big(\rho_L^{n-j}(u)+\rho_L^{n-j}(-u)\big)\, d\mu_{\mathbb{S}^{n-1}}(u) \\
&- \widetilde{V}_j(K,-L) \\
&\geq \frac{n\omega_n}{2}\frac{4}{n}\Big[g^{-1}\Big(\frac{c_1}{2}\Big)\Big]^j\Big[h^{-1}\Big(\frac{c_2}{2}\Big)\Big]^{n-j} - \big[V(K)\big]^{\frac{j}{n}}\big[V(L)\big]^{\frac{n-j}{n}} \\
&= 2\omega_n\Big[g^{-1}\Big(\frac{c_1}{2}\Big)\Big]^j\Big[h^{-1}\Big(\frac{c_2}{2}\Big)\Big]^{n-j} - \big[V(K)\big]^{\frac{j}{n}}\big[V(L)\big]^{\frac{n-j}{n}}. \qquad (3.33)
\end{aligned}$$

The equality case implies first, via Theorem 1.2.2, that K and L must be homothetic to each other and, as in the equality case of Theorem 3.1.1, each body must be an Euclidean ball. □

3.3 Extension of Results to L^n-stars

Definition 3.3.1 ([10]). *Let $p > 0$. A star-shaped set $S \subseteq \mathbb{R}^n$ is an L^p-star, if the radial function of S, denoted by ρ_S, is an L^p function on $\mathbb{S}^{n-1}$. If ρ_S is a positive and continuous function on $\mathbb{S}^{n-1}$, then S is called a star body.*

We will denote by ζ^n the set of all L^n-stars in $\mathbb{R}^n$ and by ζ_c^n the set of all star bodies in $\mathbb{R}^n$. The latter are precisely the star bodies we discussed previously as the continuity of the radial function implies the compactness of S. A star body is obviously an L^p-star for all $p \geq 1$. As with star bodies, which are compact star-shaped sets with nonempty interior, the formula for the volume of an L^n-star can be expressed as follows

$$vol(S) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_S^n \, d\mu_{\mathbb{S}^{n-1}}, \quad \forall S \in \zeta^n. \tag{3.34}$$

The dual mixed volumes introduced in the previous section were extended to L^n-stars by Klain in [10] who also introduced the L^n-stars. Two L^n-stars S_1 and S_2 are defined to be equal whenever $\rho_{S_1} = \rho_{S_2}$ almost everywhere on $\mathbb{S}^{n-1}$. As such, it follows from (1.9) that $\widetilde{V}$ is also well-defined on ζ^n since the Lebesgue integral is zero on sets of measure zero. Moreover, as Klain showed in [10], the inequality presented in Theorem 1.2.2 also holds when K and L are L^n-stars, that is, given $K, L \in \zeta^n$, then for any integer j, $1 < j < n$, we have that

$$\widetilde{V}_j(K, L)^n \leq [V(K)]^j [V(L)]^{n-j} \tag{3.35}$$

with equality if and only if there exists a constant $c > 0$ such that $\rho_K = c\rho_L$ almost everywhere on $\mathbb{S}^{n-1}$.

A simplistic way of thinking of g-chordal L^n-stars is to consider K a g-chordal star-shaped body, K' a different g-chordal shar-shaped body for the same g and constant c (this could be K rotated around the origin by a certain angle which will result in a copy of a rotated K not indentical to K in the ambient space). Then, we will construct an L^n-star by taking as its radial function on all, but one pair of opposite orthants, the radial function of K and, on the omitted pair of orthants, taking the radial function of K'. Different variations of this construction using combiatorially opposite orthants lead to other L^n-stars. Another, more trivial, construction is taking K to be a g-chordal star-shaped body and removing a chord passing through the origin. The resulting star-shaped set is g-chordal almost everywhere, that is, property (1.8) is satisfied for almost every $u \in \mathbb{S}^{n-1}$ and while its radial function is not continuous, it is bounded on $\mathbb{S}^{n-1}$ and, thus, L^n integrable on $\mathbb{S}^{n-1}$.

Based on the definition above, we may extend Theorem 3.1.1, Theorem 3.2.1 and Theorem 3.2.2 to L^n-star sets.

Theorem 3.3.1. *Under the assumptions (3.1), (3.2) and (3.3), the volume of the g-chordal L^n-star S satisfies the inequality*

$$V(S) \geq \left[g^{-1}\left(\frac{c}{2}\right)\right]^n \omega_n, \tag{3.36}$$

where ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$.

Equality occurs if and only if S is a sphere of radius $g^{-1}\left(\frac{c}{2}\right)$ a.e. in the sense that $\rho_S(u) = g^{-1}\left(\frac{c}{2}\right)$ u-almost everywhere with respect to the surface area measure of the unit sphere $\mathbb{S}^{n-1}$.

Proof. The proof of this inequality is identical to the proof presented for star bodies. The proof in question does not require the radial function ρ_S of S to be continuous so long as it is n-power integrable. Otherwise, the volume of the star-shaped set $S \subset \mathbb{R}^n$ characterized by ρ_S is not defined as per Definition 3.34. Therefore, the function Φ evaluated at $\rho_S(u)$, the radial function of the g-chordal L^n-star S where g satisfies the n-th-condition, is given by $\Phi(\rho(u)) = \rho^n(u) + \rho^n(-u)$ and is also bounded from below by $2\left[g^{-1}\left(\frac{c}{2}\right)\right]^n$. □

To extend Theorem 3.2.1 and Theorem 3.2.2, we must address another factor. If the radial functions of the L^n-stars considered are bounded, then the L^n-stars in question differ from star bodies strictly by the discontinuities of their radial functions in which case both proofs go as smoothly as before. However, L^n-stars can be unbounded, and we can envision the existence (although we did not prove it) of an unbounded g-chordal L^n-star for a function like $g(x) = \ln x$. The radial function can go to infinity while we approach some direction u_0, and go to zero in the opposite direction, such that (1.1.2), namely $\ln \rho_S(u) + \ln \rho_S(-u) = c$, is satisfied for all u. In the direction u_0, and its opposite, the radial function can be defined separately to match the needed constant.

In that case, if ρ_K^n and ρ_L^n are integrable on $\mathbb{S}^{n-1}$, we still need to know that ρ_K^j and ρ_L^{n-j}, where $1 \leq j \leq n-1$, are integrable on $\mathbb{S}^{n-1}$. While the integrability of $\rho_K^j \rho_L^{n-j}$ on $\mathbb{S}^{n-1}$ follows directly from Hölder's inequality as in the proof of Theorem 1.2.2, we need the following inclusion between L^p spaces of finite total measure (which can be applied for $\mathbb{S}^{n-1}$) to establish the former.

Proposition 3.3.1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. If $\mu(X) < \infty$, then*

$$1 \leq p < q < \infty \implies L^q(X) \subseteq L^p(X). \tag{3.37}$$

Proof. Let $f \in L^q(X)$, then by applying Hölder's inequality for $q/p > 1$ and its conjugate, we have

$$\left(\int_X |f|^p \, d\mu\right)^{1/p} = \left(\int_X |f|^{q \cdot \frac{p}{q}} \cdot 1 \, d\mu\right)^{1/p} \leq \left(\int_X |f|^q \, d\mu\right)^{1/q} \cdot \mu(X)^{\frac{q-p}{q}} < \infty, \tag{3.38}$$

thus $f \in L^p(X)$. □

Thus, as $\rho_K, \rho_L \in L^n(\mathbb{S}^{n-1})$, we have $\rho_K \in L^j(\mathbb{S}^{n-1})$ and, similarly, $\rho_L \in L^{n-j}(\mathbb{S}^{n-1})$. Therefore, we can proceed extending the aforementioned results.

Theorem 3.3.2. *Under the assumptions (3.1), (3.2), and (3.3), the dual mixed volume of $K \in \zeta^n$, a g-chordal L^n-star with constant c, and $L \in \zeta^n$, a centrally symmetric L^n-star satisfies the inequality*

$$\widetilde{V}_j\,(K, L) \geq \left[g^{-1}\left(\frac{c}{2}\right)\right]^j \widetilde{V}_j(\mathbb{B}^n, L). \tag{3.39}$$

Equality occurs if and only if K is a sphere of radius $g^{-1}\left(\frac{c}{2}\right)$ a.e. in the sense described earlier.

Theorem 3.3.3. *Assume $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ satisfy the j-th and $(n-j)$-th condition respectively as well as condition (3.1). Then, the dual mixed volume of $K \in \zeta^n$, a g-chordal L^n-star with constant c_1, and $L \in \zeta^n$, an h-chordal L^n-star with constant c_2, satisfies the inequality*

$$\widetilde{V}_j\,(K, L) \geq 2\omega_n \left[g^{-1}\left(\frac{c_1}{2}\right)\right]^j \left[h^{-1}\left(\frac{c_2}{2}\right)\right]^{n-j} - \left[V(K)\right]^{\frac{j}{n}} \left[V(L)\right]^{\frac{n-j}{n}}. \tag{3.40}$$

Equality holds if and only if $\rho_K = \rho_L$ almost everywhere on the unit sphere $\mathbb{S}^{n-1}$.

The proofs of the above theorems are very similar to the proofs of their corresponding results for star-shaped bodies including the equality cases which hold up to sets of measure zero on $\mathbb{S}^{n-1}$.

Bibliography

[1] Claudi Alsina; Roger Nelsen, *When less is More: Visualizing Basic Inequalities*, Mathematical Association of America, Washington, 2014.

[2] Marco Barchiesi; Filippo Cagnetti; Nicola Fusco, *Stability of the Steiner symmetrization of convex sets*, J. Eur. Math. Soc. 15, 2013, p.1245-1278.

[3] Étienne Baudrier; Mohamed Tajine; Alain Daurat, *Polygon estimation of planar convex-set perimeter from its two projections*, Discrete Appl. Math., vol. 161, no. 15, 2013, p. 2252-2268.

[4] Waldemar Cieślak; Józef Wołowik, *g-Chordal Curves*, Demonstratio Mathematica 42, 2009, p.413-424.

[5] Ennio De Giorgi, *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita*, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8), 1958, p.33-44.

[6] Herbert Federer, *Geometric Measure Theory*, Springer Verlag, New York Inc., 1969.

[7] Richard Harding Gardner, *Geometric Tomography*, Cambridge Univ. Press, 1995.

[8] Peter Gruber, *Convex and Discrete Geometry*, Springer Verlag, Berlin, 2007.

[9] Nicholas Kazarinoff, *Geometric Inequalities*, Random House, New York, 1961.

[10] Daniel Klain, *Star Valuations and Dual Mixed Volumes*, Adv. Math. 121, 1996, p.80–101.

[11] Erwin Lutwak, *Dual Mixed Volumes*, Pacific J. Math. 58, 1975, p. 531–542.

[12] Barret O'Neill, *Elementary Differential Geometry*, Elsevier, 2006.

[13] John Oprea, *Differential Geometry and Its Applications*, Mathematical Association of America, Washington, 2014.

[14] Rolf Schneider, *Convex Bodies: Brunn-Minkowski Theory*, Cambridge Univ. Press, 2013.

[15] Barry Simon, *Convexity: an analytic viewpoint*, Cambridge Univ. Press, 2011.