Minimizing Direct Operating Cost for Turbojet and Turboprop Aircraft in Cruise

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Abstract

Minimizing Direct Operating Cost for Turbojet and Turboprop Aircraft in Cruise

Alexander Botros

Canada's greenhouse gas emissions increased by 20% between the years 1990 and 2014, and the aviation industry is a large contributor to this increase. The optimization of fuel consumption is therefore of paramount importance. This thesis focuses on minimizing the direct operating cost (DOC) for a cruising turbojet and turboprop aircraft. The DOC is a trade-off of fuel costs and time costs that are related by the cost index C_I. By determining DOC-optimal trajectories, aircraft may balance the need to arrive at their target destination in a timely fashion with the need to keep fuel emissions low. The main contribution of this thesis is a two-part approach to determining the DOC-optimal trajectories of a cruising turbojet and turboprop aircraft. For a turbojet, the first part of the proposed methodology is the derivation of an analytic expression for the optimal speed in terms of position and optimal initial speed, while the second part derives an analytic implicit definition of the optimal initial speed. For a turboprop, the first part of the proposed methodology is concerned with developing a suboptimal approximation for the DOC-optimal speed presented in terms of the weight of the aircraft and the optimal final speed. The second part presents a recursive algorithm by which the optimal final speed may be obtained. This thesis assumes that the aircraft cruises below its drag divergence Mach number at constant altitude. Numerical examples will illustrate the proposed methodologies.

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This work is dedicated to my Dad, Dr. Adel Botros. Life does not last long. Math does. Thank you for both.

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Nomenclature

α	Angle of attack $rads$
γ	Flight path angle $rads$
ρ	Air density $slug/ft^3$
c	Speed of sound, ft/s
C_0	Coefficient of parasitic drag
C_2	Coefficient of induced drag
C_D	Coefficient of drag
C_f	Cost of fuel $\$/lbs$
C_I	Cost index lbs/s
C_L	Coefficient of lift
C_t	Time cost $\$/s$
D	Force of drag lbs
f	Fuel flow rate lbs/s
g	Gravitational acceleration ft/s^2
h	Cruise altitude ft
J	Cost to go lbs

J^* Minimal cost to go lbs

J_W Sensitivity of cost to go to weight

J_W^* Sensitivity of optimal cost to go to weight

J_x Sensitivity of cost to go to position lbs/ft

J_x^* Sensitivity of optimal cost to go to position lbs/ft

L Force of lift lbs

S Wing reference area ft^2

S_{FC} Specific fuel consumption s^{-1} (turbojet) or ft^{-1} (turboprop)

T Thrust lbs

t Time from start of cruise s

t_f Final time at top of descent s

$t_{MR,f}$ Maximum range final cruise time s

V Direct operating cost lbs

v True airspeed ft/s

V^* Minimal direct operating cost lbs

v^* Direct operating cost minimizing cruise speed ft/s

v_f^* Optimal final cruise speed ft/s

V_{MR}^* Maximum range direct operating cost lbs

v_c* Optimal initial cruise speed ft/s

$v_{MR,c}$ Maximum range initial speed ft/st

$v_{MR,f}$ Maximum range final speed ft/s

v_{MR} Maximum range speed ft/st

v_{MR}^{W} Maximum range speed as a function of weight ft/s

v_{MR}^{x} Maximum range speed as a function of position ft/s

W Aircraft total weight lbs

W_c Initial cruise weight lbs

W_f Final cruise weight lbs

x Ground position from start of cruise ft

x_d Final cruise position ft

Chapter 1

Introduction

1.1 Motivation

Canada's total greenhouse gas (GHG) emissions experienced a 20% increase between the years 1990 and 2014 (613 to 732 mega-tonnes of carbon dioxide equivalent (CO_2 eq)) (see (*Greenhouse Gas Emissions* (2016))). According to Environment and Climate Change Canada (ECCC) (see *Greenhouse Gas Emissions* (2016)), the transportation sector was one of the primary contributors to this increase. In 2014, the Canadian government in cooperation with the Canadian aviation industry, released an action plan to reduce the GHG emissions due to air travel. The document included eight classifications of measures including fleet renewals, alternative fuels and more efficient air operations (see *Canada's Action Plan to Reduce Greenhouse Gas Emissions from Aviation* (2015)). Of the proposed measures, one of the least costly to implement is the amelioration of air operations which is the focus of this thesis, in particular, the computation of the cruising velocity.

Flying at high speeds increases the amount of fuel consumed during the flight which in turn increases GHG emissions as well as fuel related costs to the airline. However, flying at very low speeds increases the travel time for the passengers, crew and other time related costs. A truly beneficial question in light of the need to decrease GHG emissions while keeping costs low for airlines is: how slowly can an aircraft fly while maintaining a reasonable flight time? This question motivates the research presented here.Each gallon of jet fuel emits roughly 21.1 pounds of $C0_2$ (see *Carbon Dioxide Emissions Coefficients* (2015)), and it will be shown that by reconsidering air operations, over 2 million pounds per year of $C0_2$ can be saved for two round trips per day between Montreal and Toronto (see section 4.8).

1.2 Problem Overview and Thesis Contributions

A Flight Management System (FMS) is the brain of a modern aircraft. A FMS not only determines optimal trajectories by interfacing with navigation and performance databases, but it also guides the aircraft along those trajectories. The block diagram of a typical FMS is shown in Fig.1.1

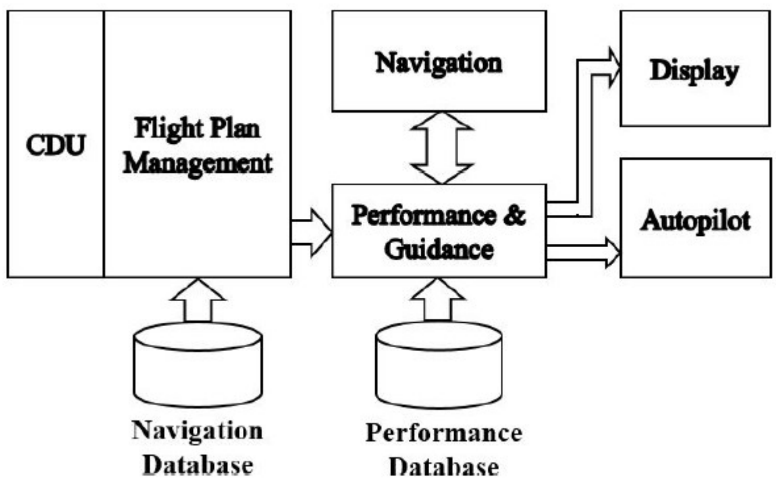

Figure 1.1: Block Diagram of a FMS (courtesy of Villarroel and Rodrigues (2016))

This thesis will focus on the functionality of the Performance and Guidance (PG) block of a FMS. During flight, the PG subsystem interfaces with the Flight Plan Management (FPM) subsystem to determine the position of the waypoints and the desired heading of the aircraft between those waypoints. The navigation block communicates the aircraft's position and heading to the PG block. Once the PG subsystem obtains the position and heading of the aircraft, it interfaces with the Performance Database (PD) to determine the speed at which the aircraft should travel. It is the mechanism of determining the optimal speed that defines the Economy Mode (ECON) problem which is the focus of this thesis. Two types of aircraft will be considered: Turbojet and Turboprop. The Turbojet and Turboprop ECON mode problems will be discussed in Chapters 3 and 4, respectively.

In Sorensen, Morello, and Erzberger (1979), the authors note that the ECON mode problem for cruise can be formulated as an Optimal Control Problem (OCP). The OCP involves minimizing the Direct Operating Cost (DOC) accrued by a cruising aircraft from the top of the aircraft's climb (TOC) at initial time $t = 0$ to the top of its descent (TOD) at final time $t = t_f$. The DOC is defined as the functional

$$DOC = \int_0^{t_f} (C_t + C_f f)dt \tag{1}$$

where C_t is the cost of one unit of cruise time, C_f is the cost of one unit weight of fuel, and f is the aircraft's fuel flow rate. Both C_t and C_f are assumed to be positive constants. Because $C_f > 0$ is constant, minimizing DOC is equivalent to minimizing V, where V is defined as

$$V = \frac{DOC}{C_f} = \int_0^{t_f} (f + C_I)dt \tag{2}$$

and where $C_I = C_t/C_f$ is a parameter called the cost index. The cost index is known to the pilot prior to flight, and is an input of a FMS. Let J denote the cost-to-go functional defined by

$$J(t) = \int_t^{t_f} (f + C_I)d\tau \tag{3}$$

The cost-to-go represents the cost accrued from time t to the final time. Therefore, $V = J(0)$.

The ECON mode problem for cruise is concerned with determining the optimal speed v and final time t_f for a given altitude that minimize (2) subject to the dynamics of the aircraft and mission constraints. This problem is one that involves the minimization of a cost functional (2) subject to the dynamics of the aircraft and as such, is an OCP. The objective of this thesis is to develop analytic expressions for v, W_f, and t_f for a cruising turbojet and turboprop aircraft.

By analyzing the resulting ECON mode OCP, the main contributions of this thesis are as follows:

(1) For turbojet aircraft, analytic expressions are presented for

- the DOC-minimal cruising speed of a turbojet aircraft in terms of position (see (129)),
- the optimal final cruise time (see (137)),
- the optimal final cruise weight (see (139)),
- the minimal DOC (see (140)),

- A near-optimal expression for the optimal cruising speed in terms of weight is also developed (see (119)),
- Algorithm 1 summarizes the turbojet ECON mode trajectory optimization techniques proposed in this thesis

Analytic expressions are important as they allow for the computation of sensitivities, which allow one to understand which physical variables play a role in conditioning the value of the variable of interest, and enable real-time implementations that are more efficient than numerical iterative algorithms. For example, computing the sensitivity of the optimal final time to changes in the cost index becomes trivial with an analytic expression for the optimal final time. It may be that decreasing the cost index saves a substantial amount of fuel at the expense of a minimal increase it the optimal final time. The importance of analytic expressions is addressed further in Mason (1990).

(2) For turboprop aircraft, analytic expressions are presented for

- A near-optimal approximation of the DOC-minimal cruising speed of a turboprop aircraft with error bound (see (221)),
- the optimal final cruise time (see (228)),
- the optimal final cruise weight (see (200)),
- the minimal DOC (see (229))

The contributions to the turboprop FMS of this thesis are, to the best of the author's knowledge, the only existing work in the open literature that derives an analytic expression for the optimal speed of a turboprop. As it was mentioned earlier, analytic expressions are important as they allow for the computation of sensitivities. Analytic expressions also capture the physics of the system and allow one to see the effect of a parameter on the system itself, and lead to efficient real-time implementations. Algorithm 4 summarizes the turboprop ECON mode trajectory optimization techniques proposed in this thesis. Using certified flight simulators courtesy of TRU Simulation and Training, the cost savings associated with flying at the speed proposed in this thesis (see (221)) versus the speeds suggested in the Pilot's Operating Handbook (see Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015)) is validated over 300 data points. The cost savings are higher than $\$8000$ for a 300 mile flight (roughly the distance between Montreal and Toronto) with a cost index of 16 lbs/s. At two flights between montreal and Toronto per day, this represents cost savings of more than $\$6,000,000$ per year.

1.3 Literature Survey

The application of optimal control techniques to the ECON mode problem is not a new field of study. Textbooks such as (Tewari (2011)), (Bryson and Ho (1969)) use Pontryagin's maximum principle to derive analytic expressions for the the speeds that minimize the fuel consumed, minimize the cruising time, maximize the range and rate of climb. They do not, however, address the ECON mode problem for cruise. In (Sorensen et al. (1979)) and (Erzberger and Homer (1980)), it is assumed that the optimal cruising speed is constant and that the change in weight due to fuel consumption is negligible. This assumption is equivalent to assuming that the optimal cost-to-go is insensitive to weight as was done in (Villarroel and Rodrigues (2016)). This will be discussed in Chapter 5. In (Miele (1959)), the author does not explicitly address the ECON mode problem for cruise, but the tools presented in the paper can be extrapolated to an analysis of the ECON mode problem for cruise. The result is identical to the analysis in (Villarroel and Rodrigues (2016)) and (Erzberger and Homer (1980)) and will be discussed in Chapter 5. In (Erzberger (1981)), the authors use Pontryagin's Maximum Principle to address the problem of trajectory optimization for the climb and descent stages of flight.

In (Diaz-Mercado, Lee, Egerstedt, and Young (2013)), the authors establish a linear quadratic cost functional which penalizes the control effort as well as the distance from the true trajectory to a reference trajectory. Thus, instead of the final position of the aircraft being a hard constraint, the authors penalize the distance from the desired final position to the actual final position. The authors assume that the final time is known and develop necessary conditions for optimality though no explicit analytic equation for the optimal speed is given.

The computational power of an on-board FMS has increased dramatically over the last two decades. It is therefore no surprise that a large body of research has emerged that takes advantage of the real-time computational capabilities of modern aircraft to generate optimal trajectories. Works such as (Waller (1990), Hagelauer and Mora-Camino (1998), H. Wu, Cho, Bouadi, Zhong, and Mora-Camino (2012), Wickramasinghe, Harada, and Miyazawa (2012)) have developed methods that rely on dynamic programming to generate optimal trajectories. While (Bonami, Olivares, Soler, and Staffetti (2013), Milam, Franz, and Murray (2002), Hok, Sridhar, and Grabbe (2012), Guijarro and Ruben (2015)) also provide computational methods for optimal trajectory generation, the methodologies are different from earlier works. The authors of (Bonami et al. (2013)) use mixed-integer nonlinear programming in their development of an optimal trajectory algorithm, while Hok et al. (2012) uses Pontryagin's maximum principle to develop algorithms that minimize flight time and fuel burn while considering the effect of wind and Salvador and Botez (2015) uses genetic algorithms to develop optimal flight trajectories. The authors of Guijarro and Ruben (2015) use Legendre's pseudospectral method to discretize the problem of obtaining a trajectory that minimizes fuel burned.

The range of an aircraft has also been considered in (Torenbeek (1997), Bert (1999)). Reference (Torenbeek (1997)) develops analytic expressions for the coefficients of lift and drag that minimize direct operating cost which are valid for turbojets, turboprops, and turbofans. The authors also provide an estimate for the range of the aircraft. Compressibility effects are considered in the development of their expressions. No analytic expression for the DOC-optimal speed is provided. In (Bert (1999)), the authors use a combination of optimal control techniques and empirical data to develop analytic expressions for the cruising range and endurance of a turboprop, turbofan or piston-propeller aircraft.

Computational algorithms, while useful, make the determination of sensitivities difficult. If, for example, one wishes to determine the effect of increased initial weight, or decreased frontal area on cruising speed, one would have to run numerous simulations to develop the relationship. However, if an analytic expression for cruising speed is known in terms of initial weight or frontal area, then obtaining the sensitivities becomes trivial. In (Almegren and Tourin (2014)), the author uses the Hamilton-Jacobi-Bellman equation to obtain optimal flight speeds and sink rate of a glider.

The problem differs from the ECON mode problem for a cruising aircraft in one crucial respect: the weight of a glider is assumed constant. Similar to (Diaz-Mercado et al. (2013)), the endpoint constraint on position is penalized in the cost functional instead of being a hard constraint.

There are several papers in the open literature that investigate analytic expressions for the DOC-optimal or fuel-optimal trajectories (see Erzberger and Homer (1980), S. Wu and Guo (1994), S. Wu and Shen (1993), Burrows (1983), Burrows (1982), Villarroel and Rodrigues (2016), Miele (1959)). Though the authors of these papers approach the problem in different ways, and use different cost functionals to define the OCPs, there is a strong similarity in the work. The similarity is strong enough to warrant study, and a comparison can be found in chapter 5.

In (Franco, Rivas, and Valenzuela (2010)), the authors investigate the DOC-minimizing thrust setting where final time is known and it is assumed that the aircraft cruises at constant altitude. Because the final time is known, the cost functional considered looks like (2) where $C_I = 0$. The authors of Franco et al. (2010) derive analytic expressions for the thrust setting π that minimizes the DOC of cruise. The analysis performed is not valid for free final time for two reasons: first, the free final time ECON mode OCP introduces the notion of the cost functional which does not apply if the final time is fixed, and second, the results of HJB and PMP analysis change depending on foreknowledge of final time.

Turbofan and all-electric optimal cruise speeds are considered in (Kapstov (2017)) in which optimal cruise speeds are are represented as the solution to quintic polynomial equations. The methodology used by the authors of (Kapstov (2017)) is similar to that employed by this thesis but on a system with different dynamics to those considered here. In (Morbidi, Cano, and Lara (2016), Candido, Galvao, and Yoneyama (2014), Ritz, Hehn, Lupashin, and D'Andrea (2011)), the authors apply optimal control techniques to the problem of performance optimization and energy management of Unmanned Aerial Vehicles (UAVs), and in (Traub (2011)), the maximum range and endurance optimal control problems for battery powered aircraft are considered.

To the best of the author's knowledge, the only previous work done on obtaining an analytic state-feedback solution to the ECON mode problem for a cruising jet was in reference Villarroel and Rodrigues (2016). The authors of Villarroel and Rodrigues (2016) considered the HJB equation that results from optimizing (2) subject to the dynamics of a jet. They did not, however, solve the resulting PDE, but did provide an approximate suboptimal control law that agreed with the well-known maximum range solution v_{MR} when $C_I = 0$. However, the approximation deviates from the expected optimal velocity (obtained using, for example, the shooting method) for larger values of C_I. The authors of Villarroel and Rodrigues (2016) also did not provide an upper bound for the error in their approximation.

To the best of the author's knowledge, no analytic expression exists in the open literature for the DOC-optimal cruising speed of turboprop aircraft. As it was mentioned earlier, the development of analytic expressions is of paramount importance when attempting to determine the sensitivities of the optimal speeds, final time, final weight, and minimal DOC to changes in aircraft and mission parameters. Though the expression presented in this thesis is a suboptimal approximation, it is (to the best of the author's knowledge) the only such analytic expression. Furthermore, no FMS exists for turboprop aircrafts. Optimization is done by referring to printed look-up tables like those in Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015), that include suggested true air speeds obtained by trial and error.

1.4 Thesis Structure

This thesis is organized as follows: Chapter 2 will present some preliminary information required to mathematically pose and analyse the ECON mode problem for turbojet and turboprop aircraft which is done in chapters 3 and 4, respectively.

The ECON mode problem for a turbojet will be addressed in Chapter 3 starting with the mathematical formulation of the problem and a brief overview of previous work. Chapter 3 presents expressions for the optimal cruising speed, the final time and weight at TOD, as well as the minimal DOC. These expressions all have one or two unknown arguments when $C_I > 0$: the optimal initial cruise speed v_c^* and the optimal final cruise speed v_f^*. The complete maximum range solution (when $C_I = 0$) is also derived in Chapter 3. When $C_I > 0$, however, the values of v_c^*, v_f^* must be determined in order to use the expressions presented. This chapter will also derive an analytic expression for v_f^* in terms of v_c^*, and presents an implicit expression for v_c^* as the solution to an algebraic equation. The methodology is summarized in Algorithm 1 which is in turn validated with a numerical example.

Chapter 4 begins with the mathematical formulation of the turboprop ECON mode problem for cruise. This chapter also presents a suboptimal approximation for the optimal speed when $C_I > 0$, the complete maximum range solution, and analytic expressions for the final cruise time at TOD, final weight and minimal DOC. As in the turbojet case, the expressions presented in Chapter 4 are in terms of one or two unknowns when $C_I > 0$: the optimal initial and final cruise speeds v_c^*, v_f^*. A recursive algorithm by which v_c^* and v_f^* can be determined will also be presented. The methodology of solving the Turboprop OCP is summarized in Algorithm 4 which is validated with a numerical example.

An in depth comparison of the work done previously by the authors of Erzberger and Homer (1980), S. Wu and Guo (1994), S. Wu and Shen (1993), Burrows (1983), Miele (1959), and Villarroel and Rodrigues (2016) can be found in Chapter 5. This comparison is performed in light of the results developed in chapters 3.2, 3 and 4 and thus must be presented after these chapters.

Chapter 6 draws the conclusions of this thesis.

Chapter 3 has been accepted for publication to the 56th IEEE Conference on Decision and Control in the following paper:

A. Botros, L. Rodrigues, ”State Feedback Optimal Solution for the ECON mode Velocity for a Cruising Turbojet”, 56th IEEE Conference on Decision and Control, December 12-15, 2017, Melbourne, Australia.

Chapter 2

Mathematical Preliminaries

The objective of this chapter is to present preliminary information required to properly formulate and analyse the ECON mode OCP. An OCP is characterized by two ingredients: the dynamics of the system under consideration, and a cost functional. The system dynamics describe how, for a given control input, the states of a system transition through time. The cost functional marks the ”cost” of these transitions. The cost functional for the ECON mode OCP is the expression V given in (2). The dynamics of a cruising Turbojet and Turboprop aircraft will be presented in section 2.1.

This chapter will also present tools used to analyse and solve OCPs including the HJB equation and PMP given in sections 2.2 and 2.3 respectively. These tools will be used to analyse the turbojet and turboprop ECON mode problems (see Chapters 3 and 4 respectively). Finally, section 2.4.3 will present the Weierstrass-Erdmann corner conditions.

2.1 Dynamic Model of a Cruising Aircraft

In order to properly formulate the ECON mode problem for a cruising turbojet or turboprop aircraft as an OCP, the flight dynamics of each type of aircraft must first be presented. The dynamic model for the longitudinal flight of a cruising aircraft is given by the following system of differential equations (see Tewari (2011), Hull (2007), Anderson (2016)):

$$\begin{aligned}
\dot{x} &= v\cos(\gamma) \\
\dot{h} &= v\sin(\gamma) \\
\dot{v} &= \left(\frac{g}{W}\right)(T\cos(\alpha) - D - W\sin(\gamma)) \\
\dot{\gamma} &= \left(\frac{g}{Wv}\right)(T\sin(\alpha) + L - W\cos(\gamma)) \\
\dot{W} &= \begin{cases} -S_{FC}T : \text{Turbojet model} \\ -S_{FC}Tv : \text{Turboprop model} \end{cases}
\end{aligned} \tag{4}$$

where the variables may be broken into four categories:

$$\begin{aligned}
\textbf{States} &= \begin{cases} x = \begin{cases} \text{Horizontal position;} \\ x(0) = 0,\ x(t_f) = x_d \end{cases} \\ v = \text{True airspeed; } v > 0 \\ h = \text{Altitude} \\ W = \begin{cases} \text{Weight;} \\ W(0) = W_c,\ W > 0 \end{cases} \\ \gamma = \text{Flight path angle} \end{cases} \\
\textbf{Control Inputs} &= \begin{cases} T = \text{Thrust} \\ \alpha = \text{Angle of attack} \end{cases} \\
\textbf{Aerodynamic Forces} &= \begin{cases} L = L(h, v, \alpha) = \text{Lift} \\ D = D(h, v, \alpha) = \text{Drag} \end{cases} \\
\textbf{Fuel Flow} &= \begin{cases} S_{FC} = \text{Specific fuel consumption} \\ f = -\dot{W} = \text{Fuel flow rate} \end{cases}
\end{aligned} \tag{5}$$

The aerodynamic forces acting on the aircraft are given by

$$L = \frac{1}{2}\rho S v^2 C_L \tag{6a}$$
$$D = \frac{1}{2}\rho S v^2 C_D \tag{6b}$$

where C_L, C_D are strictly positive constants representing the coefficient of lift and drag respectively, $\rho = \rho(h)$ is the air density, and S is the surface area of the wing. The assumptions for cruise will now be stated. These assumptions are identical to those made in Erzberger and Homer (1980), S. Wu and Guo (1994), S. Wu and Shen (1993), Burrows (1983), Burrows (1982), Miele (1959), and Villarroel and Rodrigues (2016) for a cruising aircraft:

- **Assumption 1**: The aircraft flies at constant altitude.

  Therefore, $\gamma = \dot{\gamma} = \dot{h} = 0$

- **Assumption 2**: The angle of attack α is small (this assumption is standard practice in performance analysis for commercial aircraft)

  Therefore, $\cos(\alpha) \approx 1, \sin(\alpha) \approx \alpha$

- **Assumption 3**: The altitude dictated by air traffic control is less than the maximum altitude, h_{max}.

  Therefore, $h(t) \leq h_{max}, \ \forall t \in [0, t_f]$.

- **Assumption 4**: The thrust and speed are within the flight envelope dictated by the engine and structural limits of the aircraft.

- **Assumption 5**: The component of the thrust that is perpendicular to the velocity vector is small in comparison to L and W.

  Therefore, $T\sin(\alpha) \approx T\alpha << L - W\cos(\gamma) = L - W$

- **Assumption 6**: The speed of the aircraft results in a Mach number that is below the drag divergence Mach number. Therefore, we need not consider drag due to the compressibility of air.

- **Assumption 7**: The density of air, ρ, is constant at constant altitude. The specific fuel consumption S_{FC} is constant at constant altitude for a Turbojet aircraft and inversely proportional to velocity for a Turboprop aircraft.
- **Assumption 8**: The aircraft is cruising steadily.

  Therefore, $\dot{v} \approx 0$
- **Assumption 9**: The aircraft is cruising in a straight line from one waypoint to another some known distance x_d away.

Under Assumptions 1-9, the dynamic model (4) reduces to (see Anderson (2016))

Turbojet	Turboprop
$\begin{cases} \dot{x} &= v \\ \dot{W} &= -S_{FC}T \\ L &= W \end{cases} \quad (7)$	$\begin{cases} \dot{x} &= v \\ \dot{W} &= -S_{FC}Tv \\ L &= W \end{cases} \quad (8)$
S_{FC} expressed as the change in weight per unit time per unit thrust	S_{FC} expressed as change in weight per unit time per unit power.

In the reduced models (7) and (8), the parameters x, W are the states, and v is the control input. Let us consider now the aerodynamic force of drag. Under Assumption 6, the coefficient of drag is modeled as

$$C_D = C_0 + C_2 C_L^2$$

where C_0, C_2 are the positive coefficients of parasitic and lift induced drag respectively. Solving (6a) for C_L and using $L = W$, we may rewrite the drag coefficient as

$$C_D = C_0 + C_2 \left(\frac{2L}{\rho S v^2} \right)^2 = C_0 + C_2 \left(\frac{2W}{\rho S v^2} \right)^2 \tag{9}$$

Thus the force of drag, according to (6b), is given by

$$D = \frac{1}{2} C_0 \rho S v^2 + \frac{2C_2 W^2}{\rho S v^2} \tag{10}$$

2.2 The HJB Equation

One of the two major approaches to solving OCPs is dynamic programming (see Athans and Falb (1966), Bellman (1963)) which is based on Bellman's Principle of Optimality (BPO). Dynamic programming leads to the Hamilton-Jacobi-Bellman equation. This section presents BPO and uses it to derive the HJB equation. An investigation into the necessity and sufficiency of the HJB equation for optimality will also be presented in this section.

Bellman's Principle of Optimality (see Bellman (1957))

Consider a process involving many stages of decisions. If a sequence of decisions constitutes an optimal policy then, regardless of the initial state and decisions, any sub-sequence must also constitute an optimal policy.

To see how the principle of optimality is used, consider the following general OCP:

$$\begin{aligned} J^*(x(t),t) &= \inf_{u \in U, t_f} \left[J(x(t),u(t),t,t_f) \right] \text{ where} \\ J(x(t),u(t),t,t_f) &= \phi(x(t_f),t_f) + \int_t^{t_f} L(x(\tau),u(\tau),\tau) d\tau \\ s.t.\ \ (x(t),u(t),t_f) \in A &= \left\{ (x,u,t_f)\ s.t. \left\{ \begin{array}{l} \dot{x}(t) = f(x(t),u(t),t),\ \forall t \in [0,t_f] \\ \Psi(x(t_f),t_f) = 0 \\ \Omega(x(t_0)) = 0 \end{array} \right. \right\} \end{aligned} \tag{11}$$

where J^* is a class C^1 function. Then, $J^*(x,t)$ can be rewritten as

$$J^*(x(t),t) = \inf_u \left[\int_t^{t+\Delta t} L(x(\tau),u(\tau),\tau)d\tau + \phi(x(t_f),t_f) + \int_{t+\Delta t}^{t_f} L(x(\tau),u(\tau),\tau)d\tau \right] \quad (12)$$

for some small $\Delta t > 0$ such that $t + \Delta t < t_f$. Note that the equation (12) represents a sequence of decisions made in two sub-sequences of time: from time t to $t + \Delta t$ and then from time $t + \Delta t$ to t_f. From BPO, it must hold that in order for J^* to be an optimal policy of decisions made from time t to time t_f, then the subset of decisions made from time $t + \Delta t$ to time t_f must also constitute an optimal policy. Therefore, (12) may be rewritten as

$$J^*(x(t),t) = \inf_u \left[\int_t^{t+\Delta t} L(x(\tau),u(\tau),\tau)d\tau + \inf_u \left(\phi(x(t_f),t_f) + \int_{t+\Delta t}^{t_f} L(x(\tau),u(\tau),\tau)d\tau \right) \right] \quad (13)$$

From the definition of $J^*(x(t),t)$ in (11), it must hold that

$$\inf_u \left(\phi(x(t_f),t_f) + \int_{t+\Delta t}^{t_f} L(x(\tau),u(\tau),\tau)d\tau \right) = J^*\big(x(t+\Delta t), t+\Delta t\big) \quad (14)$$

Therefore, (13) can be rewritten as

$$J^*(x(t),t) = \inf_u \left[\int_t^{t+\Delta t} L(x(\tau),u(\tau),\tau)d\tau + J^*\big(x+\Delta x, t+\Delta t\big) \right] \quad (15)$$

where $x + \Delta x = x(t + \Delta t)$. Because $J^* \in C^1$, we may take the first order Taylor series expansions of both terms inside the square brackets of (15) to obtain

$$\begin{aligned} J^*(x+\Delta x, t+\Delta t) &= J^*(x,t) + J_x^*(x,t)\Delta x + J_t^*(x,t)\Delta t + O(\Delta t^2) \\ \int_t^{t+\Delta t} L(x(\tau),u(\tau),\tau)d\tau &= 0 + L(x,u,t)\Delta t + O(\Delta t^2) \end{aligned} \quad (16)$$

Replacing (16) in (15) yields

$$J^*(x(t),t) = \inf_u \left[L(x(t),u(t),t)\Delta t + J^*(x(t),t) + J_x^*(x,t)\Delta x + J_t^*(x,t)\Delta t + O(\Delta t^2) \right] \quad (17)$$

Because $J^*(x(t),t)$ is independent of u, we may rewrite (17) as

$$= J^*(x(t),t) + \inf_u \left[L(x(t),u(t),t)\Delta t + J_x^*(x,t)\Delta x + J_t^*(x,t)\Delta t + O(\Delta t^2)\right] \tag{18}$$

Therefore, by (18),

$$\inf_u \left[L(x(t),u(t),t)\Delta t + J_x^*(x,t)\Delta x + J_t^*(x,t)\Delta t + O(\Delta t^2)\right] = 0 \tag{19}$$

Note that (19) must hold for any Δt sufficiently small and independent of u, by construction. Dividing both sides of (19) by Δt and taking the limit as $\Delta t \to 0$ while noting that $\dot{x} = f(x,u,t)$ yields

$$\inf_u \left[L(x,u,t) + J_x^*(x,t)f(x,u,t) + J_t^*(x,t)\right] = 0 \tag{20}$$

Because $J_t^*(x,t)$ is independent of u, (20) can be written as

$$\inf_u \left[L(x,u,t) + J_x^*(x,t)f(x,u,t)\right] + J_t^*(x,t) = 0 \tag{21}$$

Define the Hamiltonian, H, as

$$H(x(t),u(t),J_x,t) = L(x,u,t) + J_x f(x,u,t) \tag{22}$$

Then, (21) can be written as

$$J_t^* + \inf_u\{H\} = 0 \tag{23}$$

which is the celebrated **HJB equation**. From the derivation of the HJB equation, it is apparent that (23) is a necessary condition for the optimality of u^* provided that J^* is a class C^1 function. It will now be shown that under certain circumstances, the HJB equation is also a sufficient condition for the optimality of u^*.

Definition 1. *If the Hamiltonian H, as a function of the control input u, has a strong absolute minimum at u^*, i.e. if*

$$H(x,u^*,J_x,t) < H(x,u,J_x,t),\ \forall u \neq u^* \tag{24}$$

*then H is said to be **normal** and u^* is called the H**-minimal** control.*

Theorem 1 (Sufficient Condition)**.** *For the OCP given in (11), if*

(1) $\phi(x(t_f), t_f) = 0$

(2) t_f is free

(3) the Hamiltonian H, is normal with H-minimal control u^*

(4) (J^*, u^*) is a solution to the HJB equation (23) with boundary conditions $J^*(t_f) = 0$

then u^* is optimal.

Proof. Let

$$\tilde{L}(x, u, t) = J_t^*(x, t) + H(x, u, J_x^*, t) \tag{25}$$

Then

$$\tilde{L}(x, u, t) > \tilde{L}(x, u^*, t) \tag{26}$$

because u^* is the H-minimal control. Furthermore, because J^*, u^* are solutions to the HJB equation, it must hold that

$$\tilde{L}(x, u^*, t) = 0 \tag{27}$$

Consider that

$$\tilde{L}(x, u, t) = J_t^*(x, t) + J_x^* f + L = \dot{J}^* + L$$

from the definition of $\tilde{L}$. Therefore,

$$\begin{aligned} \int_t^{t_f} \tilde{L}(x, u, \tau) d\tau = \int_t^{t_f} \dot{J}^* + L(x, u, t) d\tau = \int_t^{t_f} L(x, u, t) d\tau + J^*(t_f) - J^*(t) \\ = \int_t^{t_f} L(x, u, t) d\tau - J^*(t) > 0 \end{aligned} \tag{28}$$

by the boundary condition $J^*(t_f) = 0$, and equations (26), (27). Similarly,

$$\int_t^{t_f} \tilde{L}(x, u^*, \tau) d\tau = \int_t^{t_f} \dot{J}^* + L(x, u^*, \tau) d\tau = \int_t^{t_f} L(x, u^*, \tau) d\tau - J^*(t) = 0 \tag{29}$$

by (27). Solving (29) for $J^*(t)$ and replacing the result in (28) yields

$$\int_t^{t_f} L(x,u,t)d\tau > \int_t^{t_f} L(x,u^*,\tau)d\tau$$

from which the result follows □

2.3 Calculus of Variations and PMP

Besides the HJB equation, the other major approach to solving OCPs is Pontryagin's Maximum Principle which contains transversality conditions, necessary conditions and an H-minimality condition. This section is devoted to providing a brief overview of the Calculus of Variations (COV), and a derivation of the necessary and transversality conditions of PMP. This section will also present the statement of the H-minimality condition.

2.3.1 Calculus of Variations: a Necessary Condition for Optimality

The purpose of this section is to use the calculus of variations to develop a necessary condition for the minimality of a cost functional. The procedure follows closely the development presented in (Liberzon (2012), Miele (1962), Bryson and Ho (1969)).

The perturbation of a continuously differentiable function $x(t)$ is a new continuously differentiable function $\hat{x}(t)$ defined by

$$\hat{x}(t) = x(t) + \epsilon\delta x(t)$$

where δx is a continuously differentiable function of t, and $\epsilon \in \mathbb{R}$ is assumed to be small. The total variation in the variable x, denoted dx, over an infinitesimal increment of time dt is given by

$$dx = \epsilon\delta x + \dot{x}dt \tag{30}$$

For an increment in time, the total variation dt is identical to the perturbation δt:

$$dt = \delta t \tag{31}$$

Let F be a functional $F : S \subseteq \mathbb{R}^n \to \mathbb{R}$.

Define the **first variation** of F at $s^*(t) \in S$ as a linear functional $\delta F|_{s^*} : S \to \mathbb{R}$ such that if $\hat{s} = s^* + \nu(t)\epsilon$ then

$$F(\hat{s}) = F(s^*) + \delta F|_{s^*}(\nu)\epsilon + o(\epsilon) \tag{32}$$

for all $\nu \in S, \epsilon \in \mathbb{R}$, and where $o(\epsilon)$ satisfies

$$\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0 \tag{33}$$

Suppose that it is desirable to determine a local minimum of F over a subset A of S. Then A is called the set of admissible trajectories of s. The vector ν is called an **admissible perturbation** if $\hat{s} \in A$ for all $|\epsilon|$ sufficiently small (note that this implies that $s^* \in A$). Suppose s^* is a local minimum of F over A, and let ν be any admissible perturbation, then by the definition of admissible trajectories, there exists $\tilde{\epsilon} > 0$ such that

$$F(\hat{s}) = F(s^* + \nu\epsilon) \geq F(s^*), \ \forall |\epsilon| \leq \tilde{\epsilon} \tag{34}$$

where $s^* + \nu\epsilon \in A$ for all $|\epsilon| \leq \tilde{\epsilon}$. The inequality (34) implies that if $G(\epsilon) = F(s^* + \nu\epsilon)$, then $G(\epsilon)$ has a local minimum at $\epsilon = 0$ over the ball $B_{\tilde{\epsilon}}$. Therefore, if s^* minimizes F over A, then $\epsilon = 0$ must minimize the scalar function $G : B_{\tilde{\epsilon}} \subseteq \mathbb{R} \to \mathbb{R}$ for each admissible perturbation ν. If $\epsilon = 0$ is a local minimum of the scalar function G over $B_{\tilde{\epsilon}}$, then it must hold from calculus that

$$\frac{dG}{d\epsilon}\Big|_{\epsilon=0} = \lim_{h \to 0} \frac{G(h) - G(0)}{h} = 0 \tag{35}$$

Replacing the definition of G in (35) yields

$$\lim_{h \to 0} \frac{F(s^* + \nu h) - F(s^*)}{h} = 0 \tag{36}$$

Recall that in the definition of the first variation of F, the equation (32) was said to hold for all

$\epsilon \in \mathbb{R}$. Therefore, from (32) and (33),

$$\delta F_{s^*}(\nu) = \lim_{\epsilon \to 0} \frac{F(s^* + \nu\epsilon) - F(s^*)}{\epsilon} \tag{37}$$

which is identical to (36). Therefore, if s^* is a local minimum of F over A, then

$$\delta F_{s^*}(\nu) = 0 \tag{38}$$

for all admissible perturbations ν.

Suppose now that $X = \{x, a, b\}$ and the functional F is given in integral form as

$$F(X) = \int_a^b f(x(t), t)dt$$

where f is continuously differentiable for all time $t \in T \subseteq \mathbb{R}$ where T completely contains (a, b). Let $X + \epsilon\delta X = \{x + \epsilon\delta x, a + \epsilon da, b + \epsilon db\}$[1] be an admissible perturbation of X such that $(a + \epsilon da, b + \epsilon db)$ is completely contained in T. Then,

$$\begin{aligned} F(X + \epsilon\delta X) - F(X) &= \int_{a+\epsilon da}^{b+\epsilon db} f(x + \epsilon\delta x, t)dt - \int_a^b f(x, t)dt \\ &= \int_a^b f(x + \epsilon\delta x, t) - f(x, t)dt + \int_b^{b+\epsilon db} f(x + \epsilon\delta x)dt \\ &- \int_a^{a+\epsilon da} f(x + \epsilon\delta x)dt \end{aligned} \tag{39}$$

Because f is continuously differentiable in T, it must hold that

$$\begin{aligned} f(x + \epsilon\delta x, t) &= f(x, t) + f_x(x, t)\epsilon\delta x + o(\delta x\epsilon) \\ \int_b^{b+\epsilon db} f(x + \epsilon\delta x)dt &= \left[f(x, t)\epsilon db + f_x(x, t)\epsilon^2\delta x db + o(\delta x\epsilon)\epsilon db\right]\Big|_{t=b} \\ -\int_a^{a+\epsilon da} f(x + \epsilon\delta x)dt &= \left[-f(x, t)\epsilon da - f_x(x, t)\epsilon^2\delta x da - o(\delta x\epsilon)\epsilon da\right]\Big|_{t=a} \end{aligned} \tag{40}$$

[1] In the definition of $X + \epsilon\delta X$, the perturbations of a and b are da, db respectively as opposed to $\delta a, \delta b$. This is because a, b are values of t, and therefore, by (31), $da = \delta a,\ db = \delta b$

Replacing (40) in (39), dividing the result by ϵ, taking the limit as ϵ approaches 0 and replacing the result in (38) yields

$$\begin{aligned}\delta F|_X(\delta X) &= \lim_{\epsilon\to 0}\frac{F(X+\epsilon\delta X)-F(X)}{\epsilon}\\ &= \int_a^b f_x(x,t)\delta x dt + f(x(b),b)db - f(x(a),a)da\end{aligned} \tag{41}$$

If $X^* = (x^*, a^*, b^*)$ minimizes $F(X)$, then by (41) and (38)

$$\delta F|_{X^*}(\delta X) = \int_{a^*}^{b^*}\Big(f_x(x,t)|_{x=x^*}\delta x\Big)dt + f(x^*(b^*),b^*)db - f(x^*(a^*),a^*)da = 0 \tag{42}$$

The result (42) will be used in the following section

2.3.2 PMP : Necessary, Transversality, and H-minimality Conditions

The goal of this section is to use the necessary condition (42) applied to a general OCP to develop the necessary and transversality conditions and to state the H-minimality condition that make up PMP. The proof of the H-minimality necessary condition has been omitted.

Consider the general OCP with boundary constraints given in (11), and define the total cost V and optimal total cost V^* as

$$\begin{aligned}V(x(t),u(t),t_f) &= J(x(t),u(t),t_0,t_f)\\ V^*(x(t)) &= \inf_{u,t_f}\Big(V(x,u,t_f)\Big)\end{aligned} \tag{43}$$

where t_0, the initial time is assumed fixed. Then the OCP (43) is a minimization problem of a functional V over trajectories $X = \big\{s = \big(x(t),u(t)\big), t_f\big\}$, where the set of admissible trajectories, A, is given in (11) and is defined by restrictions on the dynamics and boundary values of x , and the set of admissible controllers is the set of all controllers that generate trajectories in A. Suppose that $X^* = \Big\{s^* = \big(x^*(t),u^*(t)\big), t_f^*\Big\}$ solves (43). Let $\hat{X} = \{\hat{s}, \hat{t}_f\}$ denote a perturbation of X^* where

$$\begin{aligned}\hat{s} &= (x^* + \delta x, u^* + \delta u)\\ \hat{t_f} &= t_f^* + \delta t_f\end{aligned} \tag{44}$$

and assume that $\hat{X} \in A$. Define the **augmented cost function**, $\tilde{V}$ as

$$\begin{aligned}\tilde{V}(X) =& \left[\phi(x_f, t_f) + \zeta^T \Psi(x_f, t_f)\right] + \int_{t_0}^{t_f} \left(L(x, u, t) + \lambda^T f - \lambda^T \dot{x}\right) dt \\ =& \Phi(x_f, t_f) + \int_{t_0}^{t_f} H(x, u, \lambda, t) - \lambda^T \dot{x} dt\end{aligned} \tag{45}$$

where $\Phi(x_f, t_f) = \phi(x_f, t_f) + \zeta^T \Psi(x_f, t_f)$, H is the Hamiltonian defined in (22), and λ, ζ, called Lagrange multipliers, are of dimension $\dim(x)$ and $\dim(\Psi)$ respectively. Because it is assumed that $\hat{X} \in A$, it must hold that $\Psi(\hat{x}(\hat{t}_f), \hat{t}_f) = 0$, $\dot{\hat{x}} = f(\hat{x}(t), \hat{u}(t), t)$ and so $\tilde{V}(\hat{X})$ reduces to $V(\hat{X})$ for any admissible perturbation $\hat{X}$. Therefore, if X^* is optimal (and therefore admissible), then according to (38), it must hold that

$$\delta V_{X^*}(\hat{X}) = \delta \tilde{V}_{X^*}(\hat{X}) = 0 \tag{46}$$

for any admissible perturbation $\hat{X}$ of X^*. Thus (46) is a necessary condition for optimality on the augmented cost function (45). From (42), and assuming that t_0 is constant,

$$\begin{aligned}\delta \tilde{V}_{X^*}(\hat{X}) = \Big[&\Phi_x dx|_{t_f} + \Phi_t dt|_{t_f} + [H - \lambda^T \dot{x}]|_{t_f} dt_f \\ &+ \int_{t_0}^{t_f} H_x \delta x + H_u \delta u - \lambda^T \delta \dot{x} dt\Big]^*\end{aligned} \tag{47}$$

where the notation $[\cdot]^*$ means that $[\cdot]$ is evaluated along X^*. Now, using integration by parts and the identity (30):

$$\begin{aligned}-\int_{t_0}^{t_f} \lambda^T \delta \dot{x} dt &= -\lambda^T \delta x|_{t_f} + \lambda^T \delta x|_{t_0} + \int_{t_0}^{t_f} \dot{\lambda}^T \delta x dt \\ &= -\lambda^T dx|_{t_f} + \lambda^T \dot{x} dt|_{t_f} + \lambda^T dx|_{t_0} - \lambda^T \dot{x} dt|_{t_0} + \int_{t_0}^{t_f} \dot{\lambda}^T \delta x dt\end{aligned}$$

Note that $dt_0 = 0$ as t_0 is assumed fixed in the formulation of (43). Thus,

$$\delta\tilde{V}_{X^*} = \Bigg[\Phi_x dx|_{t_f} + \Phi_t dt|_{t_f} + [H - \lambda^T\dot{x}]|_{t_f} dt_f \\ -\lambda^T dx|_{t_f} + \lambda^T\dot{x}dt|_{t_f} + \lambda^T dx|_{t_0} \\ + \int_{t_0}^{t_f} H_x\delta x + H_u\delta u + \dot{\lambda}^T\delta x dt\Bigg]^* \tag{48}$$

$$= \Bigg[[\Phi_x - \lambda^T]dx|_{t_f} + [\Phi_t + H - \lambda^T\dot{x} + \lambda^T\dot{x}]dt|_{t_f} + [\lambda^T]dx|_{t_0} \\ + \int_{t_0}^{t_f} \left(H_x + \dot{\lambda}\right)\delta x + H_u\delta u dt\Bigg]^* \\ = \Bigg[[\Phi_x - \lambda^T]dx|_{t_f} + [\Phi_t + H]dt|_{t_f} + [\lambda^T]dx|_{t_0} \\ + \int_{t_0}^{t_f} \left(H_x + \dot{\lambda}\right)\delta x + H_u\delta u dt\Bigg]^* \tag{49}$$

This must be true for all possible admissible variations which implies that each term must be 0. To summarize, if X^* minimizes (43), then the following must hold along X^*:

$$\text{Transversality Conditions} : \begin{cases} \left([\Phi_x - \lambda^T]dx\right)|_{t_f} = 0 \\ \left([\Phi_t + H]dt\right)|_{t_f} = 0 \\ \left(\lambda^T dx\right)|_{t_0} = 0 \end{cases} \tag{50a}$$

$$\text{Necessary Conditions} : \begin{cases} H_x = -\dot{\lambda} \\ H_u = 0 \end{cases} \tag{50b}$$

From the definition of the Hamiltonian (22), the partial derivative of H with respect to λ is given by

$$H_\lambda(x, u, \lambda, t) = f(x, u, t) \tag{51}$$

Therefore, by (50b) and (51) the time derivative of the Hamiltonian along the optimal trajectory is given by

$$\begin{aligned}\dot{H}(x,u,\lambda,t)|_{X^*} &= \big[H_x\dot{x} + H_u\dot{u} + H_\lambda\dot{\lambda} + H_t\big]^* \\ &= \big[-\dot{\lambda}\dot{x} + f\dot{\lambda} + H_t\big]^* \\ &= H_t^*\end{aligned} \tag{52}$$

Therefore, along the optimal trajectory, the time rate of change of the Hamiltonian is given by the partial derivative of H with respect to time. If H does not depend explicitly on time, then (52) implies that H is a constant along the optimal trajectory. Furthermore, *if H does not depend explicitly on time and $\Phi_t = 0$, then by (50a)*

$$H|_{X^*} = 0 \tag{53}$$

The second part of PMP, called the H-minimality condition is stated as follows:

Theorem 2. *[H-minimality condition] If u^* minimizes the OCP (43), then for all admissible controllers u, it must hold that*

$$H(x^*, u^*, \lambda^*, t) \leq H(x^*, u, \lambda^*, t) \tag{54}$$

where $\lambda^* = \lambda|_{X^*}$.

The Transversality conditions and necessary conditions (50b) (50a) together with (54) form *Pontryagin's Maximum Principle*.

2.4 The Relationship Between HJB and PMP, Time and State

The objective of this section is to compare the techniques outlined in sections 2.2 and 2.3 for a certain class of OCPs, and to determine the relationship between the states of an OCP and time. It will first be shown that for time independent OCPs where the optimal cost-to-go is a class C^1 function, the HJB equation and PMP result in an identical PDE. Second, a method by which a time-based cost functional can be transformed into a state-based cost functional will be provided.

2.4.1 Equivalence of HJB and PMP for Time Invariant OCPs

This section will illustrate that the sufficient condition with respect to the HJB equation presented in Theorem 1, and PMP result in an identical system of PDEs when the OCP in question does not depend explicitly on time, where there is no final penalty on the states, where the Hamiltonian is a twice continuously differentiable function of u, and where the optimal cost-to-go is a class C^1 function.

Consider the general OCP in (11), and suppose that $J^* \in C^1$, $\phi(x(t_f), t_f) = 0$, that H is a twice continuously differentiable function of u, and that $\frac{\partial L}{\partial t} = \frac{\partial f}{\partial t} = 0$. Then from the definition of H in (22), it must hold that $\frac{\partial H}{\partial t} = 0$. The results of PMP are summarized in table 2.1

Result	**Condition**	**Equation**
$H\|_{X^*} = 0$	Transversality and necessary conditions	(53)
$H_u = 0$	Necessary condition	(50b)
$H(x^*, u^*, J_x^*, t) \leq H(x^*, u, J_x^*, t)$	H-minimality condition	(54)

Table 2.1: PMP results for OCP (11) when $J^* \in C^1$, $H_t = 0$

Therefore, from PMP, if X^* constitutes an optimal trajectory, then

$$
\begin{aligned}
H|_{X^*} = L(x^*, u^*) + J_x^* f(x^*, u^*) = 0 \\
H_u = L_u(x^*, u^*) + J_x f_u(x^*, u^*) = 0 \\
L(x^*, u^*) + J_x^* f(x^*, u^*) \leq L(x^*, u) + J_x^* f(x^*, u), \ \forall u \in U
\end{aligned}
\tag{55}
$$

From Theorem (1), (X^*, J^*) solves the general OCP (11) with $J^* \in C^1$, $\phi = 0$, and $L_t = f_t = 0$ if the results in table (2.2) hold.

Result	**Condition**	**Equation**
$J_t^* + \inf_u\{H\} = 0$	HJB equation	(23)
$H(x, u^*, J_x, t) < H(x, u, J_x, t), \ \forall u \neq u^*$	H is normal	(24)

Table 2.2: HJB results for OCP (11) when $J^* \in C^1$, $H_t = 0$

If H is a twice continuously differentiable function of u and there exists an admissible controller u^* such that

$$\begin{aligned} H_u|_{u=u^*} &= 0 \\ H_{uu}|_{u=u^*} &> 0 \end{aligned} \tag{56}$$

then H is normal with H-minimal controller $u = u^*$ and in particular,

$$H|_{X^*} = H(x^*, u^*, J_x^*, t) < H(x^*, u, J_x^*, t), \ \forall u \in U, \ u \neq u^* \tag{57}$$

Thus the necessary and H-minimality condition of PMP are recovered by the condition that H be normal in Theorem 1. The inequality (57) implies that

$$H|_{X^*} = \inf_{u \in U}\{H\}$$

Therefore the HJB equation (23) reduces to:

$$0 = J_t^* + \inf_u\{H\} = J_t^* + H|_{X^*} = J_t^* + L(x^*, u^*) + J_x^* f(x^*, u^*) = 0 \tag{58}$$

It will now be shown that there exists a solution J^* to the PDE (58) such that $J_t^* = 0$. Let

$$\begin{aligned} \tilde{t} &= t + s \\ \tilde{J}^* &= J^* + s \\ \tilde{x_1} &= x_1 \\ \tilde{x_2} &= x_2 \end{aligned} \tag{59}$$

where $s \in \mathbb{R}$. Then

$$\tilde{J}^*_{\tilde{x}_1} = \frac{\frac{\partial \tilde{J}^*}{\partial x_1} + \frac{\partial \tilde{J}^*}{\partial J^*} J^*_{x_1}}{\frac{\partial \tilde{x}_1}{\partial x_1} + \frac{\partial \tilde{x}}{\partial J^*} J^*_{x_1}} = \frac{0 + J^*_{x_1}}{1 + 0} = J^*_{x_1}$$

Similarly, $\tilde{J}^*_{\tilde{x}_2} = J^*_{x_2}$, and $\tilde{J}^*_{\tilde{t}} = J^*_t$. Therefore, the PDE (58) is invariant under the transformation (59) which implies that there must exist a solution J^* to (58) that does not depend explicitly on time. Therefore, $J_t^* = 0$, and the HJB equation (58) reduces to

$$H|_{X^*} = 0$$

and the Transversality and necessary condition (53) is recovered from the HJB equation.

To summarize, if $J^* \in C^1$, H is a twice continuously differentiable function of u, $\phi(x(t_f), t_f) = 0$, $L \in C^1$, $\frac{\partial L}{\partial t} = \frac{\partial f}{\partial t} = 0$, and if

$$\begin{aligned} H|_{X^*} &= 0 \\ H_u|_{u^*} &= 0 \\ H_{uu}|_{u^*} &> 0 \end{aligned} \tag{60}$$

Then the conditions of PMP and of Theorem (1) are simultaneously satisfied for the OCP defined in (11) with $\phi = f_t = L_t = 0$.

2.4.2 Time as a Function of X and Implications for a Class of Time Invariant OCPs

Consider a specific case of the general OCP defined in (11) defined by

$$\begin{aligned} J^*(x(t), t) &= \inf_{u, t_f} \left[J(x(t), u(t), t, t_f) \right] \text{ where} \\ J(x(t), u(t), t, t_f) &= \int_t^{t_f} \Big(L\big(x_2(\tau), u(\tau)\big) + K \Big) d\tau \\ s.t. \;\; (x_1(t), x_2(t), u(t), t_f) \in A &= \left\{ (x_1, x_2, u, t_f) \in \mathbb{R}^4 \; s.t. \begin{cases} \dot{x}_1 = G\big(x_2(t), u(t)\big) \\ \dot{x}_2 = -L\big(x_2(t), u(t)\big) \\ x_1(t_f) = x_{1,f} \\ \Omega(x_1(t_0), x_2(t_0)) = 0 \end{cases} \right\} \end{aligned} \tag{61}$$

where K is a constant in $\mathbb{R}$, and $G, L : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable for all $t \in [0, t_f^*]$ and x_1, x_2 are continuous, one-to-one functions of time. The assumption that x_1, x_2 are continuous and one-to-one functions of time implies that if $X^* = \{x_1^*, x_2^*\}$ denotes the optimal trajectory of the states, then time can be expressed as a continuous function of X^*. That is,

$$t = \Gamma(X^*) \tag{62}$$

Therefore, the cost function in (61) may be rewritten as

$$J^* = \int_t^{t_f} \Big(L\big(x_2^*(\tau), u^*(\tau)\big) + K\Big) d\tau = x_2^*(t) - x_2^*(t_f) + K\left(\Gamma_f - \Gamma(X^*)\right) \tag{63}$$

where $\Gamma_f = \Gamma(x_{1,f}, x_2^*(t_f))$. Then, according to (63),

$$\begin{aligned} J_{x_1}^* &= -K\Gamma_{x_1^*}(X^*) \\ J_{x_2}^* &= 1 - K\Gamma_{x_2^*}(X^*) \end{aligned} \tag{64}$$

From (50b), it must hold that

$$\dot{J}_{x_1^*} = H_{x_1^*}^* = \frac{\partial}{\partial x_1^*}\left(L(x_2^*, u^*) + K + J_{x_1} G(x_2^*, u^*) - J_{x_2} L(x_2^*, u^*)\right) = 0$$

Thus by (64),

$$\dot{J}_{x_1}^* = K\dot{\Gamma}_{x_1^*} = 0 \;\Rightarrow\; \dot{\Gamma}_{x_1^*} = 0 \;\Rightarrow\; \Gamma_{x_1^*} = \Gamma_1$$

for some real constant Γ_1. Therefore Γ may be separated as

$$\Gamma = \Gamma_1 x_1^* + \Gamma_2(x_2^*) \tag{65}$$

where the function Γ_2 is only in terms of x_2^*. Note that (62) and (65) imply that

$$\frac{d\Gamma}{dt} = \frac{dt}{dt} = \Gamma_1 G(X^*) - \Gamma_2'(x_2^*) L(X^*) = 1 \tag{66}$$

Furthermore, by (65), the equation (63) may be rewritten as

$$J^* = x_2^* - x_2^*(t_f) + K\Gamma_1(x_{1f} - x_1^*) + K\Big(\Gamma_2\big(x_2^*(t_f)\big) - \Gamma_2\big(x_2^*\big)\Big) \tag{67}$$

Recall the time invariant HJB equation (53). A secondary proof of (53) is obtained by taking the time derivative of both sides of the equation (67) and equating the result to the derivative of the original cost functional in (61). The result is

$$\dot{J}^* = -L(X^*) - K\Gamma_1 G(X^*) + K\Gamma_2'(X^*)L(X^*) = -L(X^*) - K \tag{68}$$

Note that by (65), and (64)

$$\Gamma_{x_1^*} = \Gamma_1 = -\frac{J^*_{x_1}}{K},\ \Gamma_{x_2^*} = \Gamma_2'(x_2^*) = \frac{1 - J^*_{x_2}}{K} \tag{69}$$

Replacing (69) in (68) yields

$$-L(X^*) - K = -L(X^*) + J^*_{x_1} G(X^*) + (1 - J^*_{x_2})L(X^*) \tag{70}$$

which holds if and only if

$$K + J^*_{x_1} G(X^*) + (1 - J^*_{x_2})L(X^*) = H|_{X^*} = 0 \tag{71}$$

which is exactly (53). Replacing (69) in (67) yields

$$\begin{aligned} J^* &= x_2^* - x_2^*(t_f) - J^*_{x_1}(x_{1f} - x_1^*) + \int_{x_2^*(t)}^{x_2^*(t_f)} (1 - J^*_{x_2}) dx_2 \\ &= -J^*_{x_1}(x_{1f} - x_1^*) - \int_{x_2^*(t)}^{x_2^*(t_f)} J^*_{x_2} dx_2 \end{aligned} \tag{72}$$

To summarize the results of this section, It is possible to rewrite the cost functional J^* in terms of time as an equivalent cost functional in terms of the states as

$$J^* = x_2^* - x_2^*(t_f) + K\Gamma_1(x_{1f} - x_1^*) + K\Big(\Gamma_2\big(x_2^*(t_f)\big) - \Gamma_2\big(x_2^*\big)\Big)$$

where, by (66),

$$\Gamma_1 G(X^*) + \Gamma_2'(x_2^*)L(X^*) = 1$$

The cost functional J^* may also be written as

$$J^* = -J^*_{x_1}(x_{1f} - x^*_1) - \int_{x^*_2(t)}^{x^*_2(t_f)} J^*_{x_2} dx_2$$

2.4.3 Theorem on Corner Points

This section will present the Weierstrass-Erdmann corner condition that will be used in the analysis of the turbojet and turboprop OCPs in chapters 3 and 4 respectively. The proof of the conditions is omitted (see Bryson and Ho (1969), Athans and Falb (1966)).

Theorem 3. *[Weierstrass-Erdmann corner condition] Suppose that u^* is the solution to (11). Suppose further, x^*, the trajectory associated with u^*, is continuous everywhere, but that u^* experiences a jump discontinuity at a finite number of times $\{t_i \in [0, t_f], i = 1, .., n\}$. Then the set $\{t_i\}$ are call corner points, and λ^*, the optimal costate, is continuous everywhere in $[0, t_f]$.*

Chapter 3

Flight Management System for a Turbojet in Cruise

The objective of this chapter is to formulate and solve the ECON mode problem for a cruising Turbojet aircraft. The resulting solution will include analytic expressions for the optimal cruise speed, final time, final weight and minimal DOC. The importance of obtaining analytic expressions was addressed in Chapter 1. Section 3.1 will use the cost functional (2) as well as the simplified dynamics of flight (7) to formulate the ECON mode problem for cruise as an OCP. Section 3.1 will also present the previous work done by the authors of Villarroel and Rodrigues (2016) in solving the ECON mode OCP for a Turbojet aircraft. Preliminary results will be presented in Section 3.2.

Three expressions for optimal speed v^* that solves the ECON mode problem for cruise are presented in Section 3.3. The first of the expressions provided is an approximation of v^* each in terms of the weight of the aircraft and the optimal final speed v_f^*. An upper bound on the error of this approximation is included. The remaining two expressions are exact, one describing v^* in terms of position for the case when $C_I = 0$ and the other in terms of position and v_f^* for the case when $C_I > 0$.

The value of v_f^* is unknown but is an argument of both of the expressions presented in this chapter when $C_I > 0$. Section 3.4 provides an analytic expression for v_f^* in terms of v_c^*, thus it is necessary to develop methods by which v_c^* can be obtained in order to use the expressions for v^*

detailed in section 3.3. In Section 3.5, an implicit analytic definition for v_c^* is given.

The approximations and exact expressions for v^* presented in this chapter are in terms of the weight of the aircraft and v_c^* (a constant) which must be approximated or obtained numerically (see section 3.5), while the expression for the optimal speed presented in Villarroel and Rodrigues (2016) is it terms of the aircrafts' weight and the unknown J_W^* (a function of time) which is the sensitivity of the optimal cost-to-go J^* to W. In other words, both the expressions presented here and the one detailed in Villarroel and Rodrigues (2016), are in terms of a state and an unknown that must be approximated. There are three major differences, however, between the analytic expressions given in this thesis and the one from Villarroel and Rodrigues (2016). First, the unknown v_c^* is a constant that must be computed only once and can be done prior to flight. Second, unlike the unknown J_W^*, an expression for v_c^* (though implicit) is provided. Finally, an upper bound on the error accrued when approximating v^* and v_c^* are provided.

This chapter will also present expressions for the optimal final cruise time t_f^*, optimal final weight W_f^*, and the minimal direct operating cost V^*. The results of this chapter will be validated with a numerical example in section 3.7. The type of aircraft (Airbus A320) used in the validation of the proposed speeds is identical to that used in Villarroel and Rodrigues (2016) to validate the author's results. The results proposed in Villarroel and Rodrigues (2016) as well as the results obtained using the shooting method were validated against real flight data in Villarroel and Rodrigues (2016).

3.1 Optimal Control Problem Formulation and Previous Work

The goal of this section is to pose the ECON mode problem for a cruising jet aircraft as an OCP, and to detail the work done thus far by the authors of Villarroel and Rodrigues (2016). The methodology in formulating the ECON mode OCP in section 3.1.1 follows very closely that presented in Villarroel and Rodrigues (2016).

3.1.1 OCP Formulation

For a cruising Turbojet aircraft, it is assumed that $T = D$. Under Assumptions 1-9, combining the cost functional (2) and the reduced dynamics (7) and (10), yields the following OCP:

$$
\begin{aligned}
V^*(x_0, W_0) = \inf_{v,t_f} &\int_0^{t_f} (S_{FC}D + C_I)dt \\
&s.t. \\
&\dot{x} = v \\
&\dot{W} = -S_{FC}D \\
&D = \frac{1}{2}C_0\rho S v^2 + \frac{2C_2W^2}{\rho S v^2} \\
&x(0) = 0,\ x(t_f) = x_d,\ W(0) = W_c \\
&v \in U = \{v : v > 0, \\
&v \text{ is piecewise continuous}\}
\end{aligned}
\tag{73}
$$

The position $x(t)$ and weight $W(t)$ are absolutely continuous functions of time. Note that the initial and final positions ($0, x_d$ respectively) are known, as are the initial weight (W_c) and time. However, the final weight W_f and final time t_f are left free. The speed v is the control input we wish to design.

The cost-to-go, $J = J(t, x_0, W_0, v)$, is defined as the total cost accrued from time t to t_f. It is given by the expression

$$
\begin{aligned}
J(t, x(t), W(t), v) &= \int_t^{t_f} (S_{FC}D + C_I)dt \\
&= (W(t) - W(t_f)) + C_I(t_f - t)
\end{aligned}
\tag{74}
$$

The *optimal* cost-to-go is given by

$$
J^*(t_0, x_0, W_0) = \inf_{v,t_f} J(t, x_0, W_0, v) \tag{75}
$$

The following section describes the suboptimal approximation to the solution of (73) presented in Villarroel and Rodrigues (2016). This suboptimal speed is (to the best of the authors' knowledge) the only existing analytic expression for the speed of a jet aircraft which minimizes DOC during cruise for $C_I = 0$.

3.1.2 Previous Work

In Villarroel and Rodrigues (2016), the authors prove that the optimal velocity that minimizes the OCP (73) is

$$v^*(W, J_W^*) = \sqrt{\frac{C_I + \sqrt{C_I^2 + 12(1 - J_W^*)^2 S_{FC}^2 C_0 C_2 W^2}}{(1 - J_W^*) S_{FC} C_0 \rho S}} \tag{76}$$

Where W denotes the weight of the aircraft at any time t, and J_W denotes the sensitivity of the optimal cost-to-go J^* to changes in weight. The authors of (Villarroel and Rodrigues (2016)) were also able to show that

$$\begin{aligned} &\dot{J}_W^* < 0 \; \forall t \in [0, t_f] \\ &J_W(t_f)^* = 0 \end{aligned} \tag{77}$$

and that if the Legendre-Clebsch sufficient condition $H_{vv} > 0$ holds, then $J_W^* < 1$. The equations (77) and the sufficiency condition $J_W^* < 1$ prompted the approximation $J_W^* \approx 0$, which resulted in the suboptimal cruising speed

$$v^*(W, J_W) \approx v_J(W) = \sqrt{\frac{C_I + \sqrt{C_I^2 + 12 S_{FC}^2 C_0 C_2 W^2}}{S_{FC} C_0 \rho S}} \tag{78}$$

The approximation $v^* \approx v_J$ works well for smaller values of C_I and reduces to the well known maximum range speed (Bryson and Ho (1969), Anderson (2016), Miele (1959))

$$v_{MR} = \left(12 \frac{C_2 W^2}{C_0 S^2 \rho^2}\right)^{1/4} \tag{79}$$

when $C_I = 0$. However, as C_I increases the difference in speeds between optimal and suboptimal regimes also increases, and the authors of Villarroel and Rodrigues (2016) were unable to determine a bound on the error incurred for their suboptimal speed. It was shown in Villarroel and Rodrigues

(2016) that $J_W \in [0,1)$. However, it must be noted that by (76),

$$\lim_{J_W \to 1^-} v^*(W, J_W) = \infty$$

Therefore, the optimal speed could conceivably be much greater than the suboptimal speed $v_J(W)$. Furthermore, the authors of Villarroel and Rodrigues (2016) did not provide an expression for the optimal final weight, final time or J^* all of which are required outputs of a FMS. In order to compute the final time, final weight, and DOC associated with $v_J(W)$, one must refer to numerical methods such as Euler's method which greatly increases the number of computations required. It will be shown in section 3.7, that resorting to Euler's method can require 29188 computations for every 1 computation involved in the proposed methods of this thesis.

3.2 Preliminary Results

This section will provide some preliminary findings required to prove the results in the remainder of the chapter. The following notation will be used

$$\begin{aligned} A_\beta(v) &= C_0 S S_{FC} \rho v (v_f^*)^2 + C_I v - \beta C_I v_f^*, \ \beta \in \mathbb{R} \\ B_\beta &= C_0 S S_{FC} \rho (v_f^*)^2 - \beta C_I, \ \beta \in \mathbb{R} \\ E_\beta(v) &= C_0 S^2 \rho^2 v^4 - \beta C_2 W_c^2, \ \beta \in \mathbb{R} \end{aligned} \tag{80}$$

where v_f^* is the optimal final (at TOD) cruise speed.

Lemma 4. *A minimizer v^* to the OCP in (73) exists.*

Proof. If $C_I = 0$, then the OCP in (73) reduces to the maximum range problem with minimizer v_{MR} given by (79). Suppose that $C_I > 0$. The Hamiltonian of the OCP in (73) is given by

$$H(x, W, v, J_x, J_W) = S_{FC} D (1 - J_W) + J_x v + C_I \tag{81}$$

where D is the drag given in (10), and J_W, J_x are the sensitivities of the cost-to-go to weight and position respectively. Noting that the Hamiltonian in (81) does not depend explicitly on time, and that the final time for the OCP in (73) is free, it must hold from equation (53) of the PMP that

$$H^* = S_{FC}D(1 - J_W^*) + J_x^* v^* + C_I = 0 \tag{82}$$

The PMP also states that H as a function of v must obtain a minimum at v^* (see Theorem 2). Thus, because H given in (81) is a class C^2 function of v for admissible control inputs, a necessary condition for optimality inside the feasible set is $H_v = 0$ along the optimal trajectory:

$$H_v^* = S_{FC}D_v(1 - J_W^*) + J_x^* = 0 \tag{83}$$

Solving (83) for J_W^*, replacing the result in (82) with D in (10) and rearranging terms yields

$$H^* = C_0\rho^2 S^2 v^{*4}(J_x^* v^* + 2C_I) - 4W^2 C_2(3J_x^* v^* + 2C_I) = 0 \tag{84}$$

Note that H^* is a continuous function of v. From the necessary conditions of the PMP (see 50b),

$$\begin{aligned} \dot{J}_x^* &= -H_x^* = 0 \\ \dot{J}_W^* &= -H_W^* = -\frac{4S_{FC}WC_2}{\rho S v^{*2}}(1 - J_W^*) \end{aligned} \tag{85}$$

implying that J_x^* is a constant (the expression for $\dot{J}_W^*$ will be used later). Furthermore, solving (84) for J_x^* yields

$$J_x^* = -\frac{2C_I\left(C_0 S^2\rho^2 v^{*4} - 4C_2W^2\right)}{v^*\left(C_0 S^2\rho^2 v^{*4} - 12C_2W^2\right)} \tag{86}$$

The expression (86) is well defined and negative. Indeed, in Rodrigues (2017), the authors show that

$$v^* > v_{MR} \text{ for } C_I > 0 \tag{87}$$

Replacing (79) in (87) yields

$$\begin{aligned} v^* > \left(12\frac{C_2W^2}{C_0S^2\rho^2}\right)^{1/4} \iff \\ v^{*4}C_0S^2\rho^2 > 12C_2W^2 \text{ because } v^* > 0 \end{aligned} \tag{88}$$

The result (88) together with $\dot{J}_x^* = 0$ and $C_0 S^2 \rho^2 v^{*4} - 4C_2 W^2 > C_0 S^2 \rho^2 v^{*4} - 12C_2 W^2$ imply that J_x^* given by (86) is a well defined negative constant.

Replacing $v = v_{MR}$ in (84) yields

$$H^*(v_{MR}^W) = 16 C_2 C_I W^2 > 0 \tag{89}$$

If a value $v_2 > 0$ can be found such that $H^*(v_2) < 0$, then by the continuity of H^* in v, there must exist a solution $v^* \in (v_{MR}, v_2)$ to $H^*(v^*) = 0$. Consider

$$v_2 = -\frac{2C_I(8AC_2W^2 + 1)}{J_x^*} \tag{90}$$

where $A > 0$. It was already shown that J_x^* is a negative constant, thus (90) implies that $v_2 > 0$. Noting that

$$\lim_{A \to \infty} -\frac{1048576 C_2^5 C_I^5 W^{10} C_0 \rho^2 S^2 A^5}{J_x^4} \times (H^*(v_2))^{-1} = 1$$

it may be concluded that there exists a sufficiently large value of A such that $H^*(v_2) < 0$ which finishes to proof. □

Lemma 5. *The time rate of change of the optimal speed $\dot{v}^*$ is given as a function v^* and W by*

$$\dot{v}^* = -\frac{8 S_{FC} W C_2 C_0 v^{*3} \rho S}{C_0 S^2 \rho^2 v^{*4} + 12 C_2 W^2} < 0 \tag{91}$$

Proof. Solving (83) for J_x^* and replacing the result in (82) with D given by (10) yields

$$H^*(v^*, W, J_W^*) = C_I - S_{FC}(1 - J_W^*) \frac{C_0 S^2 \rho^2 v^{*4} - 12 C_2 W^2}{2 v^{*2} \rho S} = 0 \tag{92}$$

Noting that $H^*(v^*, W, J_W^*) = 0$ for all time $t \in [0, t_f]$, it must hold that

$$\dot{H}^*(v^*, W, J_W^*) = \frac{\partial H^*(v^*, W, J_W^*)}{\partial W} \dot{W} + \frac{\partial H^*(v^*, W, J_W^*)}{\partial J_W^*} \dot{J}_W^* + \frac{\partial H^*(v^*, W, J_W^*)}{\partial v^*} \dot{v}^* = 0 \tag{93}$$

Replacing (92), (85), and the dynamics of W from (73) in (93) and solving the result for $\dot{v}^*$ yields (91). □

Lemma 6. *Let* $C_{I_c} = \frac{1}{2}B_0$ *where* B *is given by (80). Assuming* $C_I < C_{I_c}$*, the following equalities hold concerning the OCP defined in (73):*

(1) The sensitivity of the optimal cost with respect to position x*, is a constant given by*

$$J_x^* = -\frac{2}{3}\frac{B_{(-1)}}{v_f} \tag{94}$$

(2) The sensitivity of the optimal cost with respect to changes in weight is a continuous function of time for all $t \in [0, t_f]$ *and is given by*

$$J_W^* = \frac{(v - v_f)(C_0 S S_{FC} \rho v v_f - C_I)}{C_0 S S_{FC} \rho v^2 v_f} \tag{95}$$

and $J_W^* \in [0, 1)$.

(3) The optimal speed v^* *that minimizes the OCP (73) is unique and a continuous function of time for all time* $t \in [0, t_f]$.

(4) The optimal weight as a function of speed and the final speed v_f^* *is given by*

$$W^* = \frac{1}{6}\sqrt{3\frac{C_0}{C_2}} S \rho v^2 \sqrt{\frac{A_3(v^*)}{A_1(v^*)}} \tag{96}$$

Proof. Recall from the proof of Lemma 4 that J_x^* is a negative constant. Furthermore, $J_W^*(t_f) = 0$ from the PMP transversality conditions (see (50a)), as $W(t_f)$ is unspecified and the OCP in (73) has no terminal cost. Thus,

$$\begin{aligned} J_x^*(t_f) &= J_x^*(t) = J_x^* \\ J_W^*(t_f) &= 0 \end{aligned} \tag{97}$$

Evaluating (81) and (83) at the final time when $W = W(t_f) = W_f$, $v^* = v_f^*$, $J_W^*(t_f) = 0$, and D is given by (10) yields a system of two equations which may be solved for J_x^*, W_f resulting in J_x^* given by (94) and

$$W_f = \frac{\sqrt{3\rho S} v_f^*}{6\sqrt{C_2 S_{FC}}} \sqrt{C_0 S S_{FC} \rho v_f^{*2} - 2C_I} \tag{98}$$

which is well defined and positive for $C_I < C_{I_c}$. Therefore, in order to ensure that the final weight is strictly positive, it must hold that $C_I < C_{I_c}$.

Replacing (94) in equations (82) and (83) and solving the resulting system of two equations for J_W^* and W results in (95) and (96) respectively. It must now be shown that J_W^* is continuous and bounded by the interval $[0, 1)$, that v^* is continuous for all time $t \in [0, t_f]$, and that the expression (96) is well defined

To show that J_W^* is a continuous function of time for all $t \in [0, t_f]$, note that J_W^* presented in (95) is a continuous function of v^* for all possible values of v^*. Therefore, as a function of time, J_W^* has at most the same countable discontinuities that v^* has. Such discontinuities in v^* are called *corner points* and the continuity of J_W^* follows from Theorem 3.

In order to prove that $J_W^* \in [0, 1)$ that (96) is well defined, and that v^* is unique, it must first be shown that v^* is a continuous function of time for all $t \in [0, t_f]$. If $C_I = 0$, then $v^* = v_{MR}$ where v_{MR} is a continuous function of W given by (79). Because W is a continuous function of time, it may be concluded that v^* is continuous if $C_I = 0$. Assume that $C_I > 0$. Recall that v^* exists by Lemma 4. Because $v^* \in U$, it must hold that v^* is piecewise continuous in the interval $(0, t_f)$. Suppose v^* experiences a jump discontinuity at $t_1 \in (0, t_f)$. Suppose that $v^*(t_1^+) = v_R$, $v^*(t_1^-) = v_L$ where $v_R - v_L = \delta$ for $|\delta| > 0$. Then, because equation (92) must hold for all time,

$$\begin{aligned} &H^*(t_1^-) = H^*(t_1^+) = 0 \\ \iff &S_{FC}(1 - J_W^*(t_1^-))\frac{C_0 S^2 \rho^2 v^{*4}(t_1^-) - 12 C_2 W^2(t_1^-)}{2 v^{*2}(t_1^-)\rho S} \\ &= S_{FC}(1 - J_W^*(t_1^+))\frac{C_0 S^2 \rho^2 v^{*4}(t_1^+) - 12 C_2 W^2(t_1^-)}{2 v^{*2}(t_1^+)\rho S} \end{aligned} \tag{99}$$

Noting that J_W^*, W are continuous for all time in $(0, t_f)$, it must hold that $J_W^*(t_1^-) = J_W^*(t_1^+) = J_W^*(t_1), W(t_1^-) = W(t_1^+) = W(t_1)$. Therefore, (99) reduces to

$$0 = (1 - J_W^*)(v_L - v_R) \tag{100}$$

Which has two solutions: $J_W^*(t_1) = 1$ or $v_L = v_R$. If $J_W^*(t_1) = 1$, then evaluating equation (92) at time t_1 yields $C_I = 0$ which is a contradiction. Therefore

$$C_I > 0 \;\Rightarrow\; J_W^* \neq 1 \,\forall t \in [0, t_f] \tag{101}$$

and $v_L = v_R$ is the only solution to (100) which implies that v^* is continuous at t_1 and thus for all time $t \in [0, t_f]$.

Note that $\dot{v}^* < 0$ by Lemma 5, thus it must hold by the continuity of v^* that

$$v^* > v_f^*, \;\; \forall t \in [0, t_f) \tag{102}$$

Equation (102) implies that

$$0 > -C_0 S S_{FC} \rho v^* v_f^{*2} - C_I (v^* - v_f^*) \tag{103}$$

which in turn implies that

$$J_W^* < 1 \;\; \forall t \in [0, t_f] \tag{104}$$

for J_W^* given in (95). To show that $J_W^* > 0$, note that $J_W^* < 1$ together with $\dot{J}_W^*$ given by (85) imply that $\dot{J}_W^* < 0$ for all time $t \in [0, t_f]$. Furthermore, $\dot{J}_W^* < 0$ and $J_W^*(t_f) = 0$ (see (97)) and the continuity of J_W^* imply that $J_W^* > 0$ for all time $t \in [0, t_f)$.

Note that H^* is a class C^2 function of v^*. Therefore, to show the uniqueness of v^*, it suffices to verify the Legendre Clebsch condition $H_{vv}^* > 0$. From (81) and (10),

$$H_{vv}^* = S_{FC}\Big(C_0 \rho S + \frac{12 W^2 C_2}{\rho S v^{*4}}\Big)(1 - J_W^*) \tag{105}$$

Because it was just shown that $J_W^* \in [0, 1)$, equation (105) implies that $H_{vv} > 0$ and so v^* is unique.

Finally, note that (96) is well defined if $A_3(v^*) > 0$ as $A_1(v^*) > A_3(v^*)$. Furthermore,

$$A_3(v^*) = C_0 S S_{FC} \rho v^* v_f^{*2} + C_I v^* - 3 C_I v_f^* > 0 \;\iff\; v^* > \frac{3 C_I v_f^*}{C_0 S S_{FC} \rho v_f^{*2} + C_I}$$

Note that

$$\frac{3C_I v_f^*}{C_0 S S_{FC} \rho v_f^{*2} + C_I} < v_f^* \iff C_I < C_{I_c} \tag{106}$$

which holds by assumption. Equations (106) and (102) imply that

$$v^* \geq v_f^* > \frac{3C_I v_f^*}{C_0 S S_{FC} \rho v_f^{*2} + C_I} \tag{107}$$

and,

$$A_1(v^*) \geq A_3(v^*) > 0$$

thus (96) is well defined □

Lemma (6) introduced the constraint $C_I < C_{I_c}$. It may appear that this constraint limits the application of the lemma. In actuality, the constraint is a necessary condition for the existence of a solution to the OCP (73). This is illustrated in the following remark:

Remark 7. *Suppose that $C_I \geq C_{I_c}$. Solving (83) for J_x^*, replacing the result in (82), and evaluating at the final time when $W = W_f, v^* = v_f^*, J_W^* = 0$ (see (97)) yields*

$$C_I - \frac{S_{FC}(C_0 S^2 \rho^2 v_f^{*4} - 12 C_2 W_f^2)}{2\rho S v_f^{*2}} = 0 \tag{108}$$

If $C_I = C_{I_c} + \delta$ where $\delta \geq 0$, then (108) reduces to

$$\frac{6 C_2 S_{FC} W_f^{*2}}{\rho S v_f^{*2}} + \delta = 0$$

If $\delta > 0$ that is, if $C_I > C_{I_c}$ then this equation is unsolvable. If $\delta = 0$, that is, if $C_I = C_{I_c}$ then the only solution is $W_f^* = 0$ which cannot happen as $W > 0$. Therefore, if $C_I \geq C_{I_c}$ then the OCP (73) has no solution.

In chapter 2, an equivalent result of the HJB equation and PMP was noted provided that J^* was a class C^1 function of time. The following corollary shows that the optimal cost-to-go J^* for a cruising turbojet meets this condition

Corollary 1. *The optimal cost-to-go J^* given in (75) is a class C^1 function of time*

Proof. For any admissible controller v, the cost-to-go given by (74) is a continuous function of time as $W(t)$ is continuous. Furthermore,

$$\dot{J} = S_{FC}D + C_I$$

From the definition of drag in (10), D is a continuously differentiable function of both v and W for any admissible v. By the continuity of W and v^* (a result of theorem 6), it must hold that $\dot{J}^* = \dot{J}|_{v=v^*}$ is also be a continuous function of time for all $t \in [0, t_f]$. □

Lemma 8. *Let $C_{I_c} = \frac{1}{2}B_0$. Assuming $C_I < C_{I_c}$, the time rate of change of the optimum speed that minimizes (73) denoted $\dot{v}^*$ is given as a function of the optimal speed v^* by*

$$\dot{v}^* = -\frac{2S_{FC}v^*\sqrt{3C_0C_2}}{3}\frac{\sqrt{A_1(v^*)A_3(v^*)}}{A_2(v^*)} \tag{109}$$

Proof. Replacing W with (96) in (91) yields (109) □

Lemma 9. *Let $C_I > 0$, then it must hold that the optimal cruising speed v^* that minimizes the OCP defined in (73) is such that*

$$v^* > v_J > v_{MR},\ \forall t \in [0, t_f] \tag{110}$$

and

$$|v^*(W) - v_J(W)| \leq |v^*(W_c) - v_c^J| \tag{111}$$

where v_J is given by (78), $v_c^J = v_J(W_c)$, and v_{MR} is the maximum range speed defined in (79).

Proof. Let $A = (1 - J_W^*)$. From (76), it follows that

$$\frac{\partial v^*}{\partial A} = -\frac{1}{2}\frac{C_I\sqrt{C_I + \sqrt{12A^2C_0C_2S_{FC}^2W^2 + C_I^2}}}{\sqrt{C_0S_{FC}\rho SA^3(12A^2C_0C_2S_{FC}^2W^2 + C_I^2)}} < 0 \tag{112}$$

From Lemma 6, it must hold that

$$A = (1 - J_W^*) \in (0, 1] \tag{113}$$

therefore, $v^* > v^*|_{A=1} = v_J$ for v_J given by (78). Note now that from (78), it follows that

$$\frac{\partial v_J}{\partial C_I} = \frac{1}{2} \frac{\sqrt{C_I + \sqrt{12C_0C_2S_{FC}^2W^2 + C_I^2}}}{\sqrt{\rho S C_0 S_{FC}(12C_0C_2S_{FC}^2W^2 + C_I^2)}} > 0$$

Thus $v_J > v_J|_{C_I=0} = v_{MR}$ and (110) holds. To show (111), note that (76) implies:

$$\frac{\partial v^*}{\partial J_W^*} > 0 \tag{114}$$

which in turn implies that

$$\frac{\partial}{\partial J_W^*}\big(v^*(W) - v_J(W)\big) = \frac{\partial v^*}{\partial J_W^*} > 0 \tag{115}$$

By (50b),

$$\dot{J}_W^* = -H_W = -\frac{4S_{FC}WC_2(1 - J_W^*)}{\rho S v^{*2}} < 0 \tag{116}$$

as $J_W^* < 1$ by Lemma 6. Thus (111) follows from (115), (116), and the continuity of v^* (a result of Lemma 6). □

3.3 Expressions for v^*, t_f^*, W_f^*, V^*

This section will use the preliminary results of section 3.2 to propose three analytic expressions for the optimal speed v^* that minimizes the OCP (73). The first expression, presented in Theorem 10 is a suboptimal approximation of v^*, denoted v_1 and is given in terms of W and v_f^*. The speed v_1 is obtained by correcting the approximation v_J presented in Villarroel and Rodrigues (2016). The second expression for v^* presented in Theorem 12 of section 3.3.2 is the exact expression for v^* and is given in terms of x, v_f^* for the case when $C_I > 0$. The third expression is the maximum range solution (the case when $C_I = 0$) is presented in Theorem 14.

This section will also present analytic expressions for the optimal final time at TOD, the optimal final weight, and the minimal DOC.

3.3.1 Correction of $v_J(W)$

The approximation v_J proposed by the authors of Villarroel and Rodrigues (2016), and given in equation (78) is obtained from the exact expression (76) via the approximation $J_W^* \approx 0$. Consider the following equation for v^* that is equivalent to (76):

$$v^* = \sqrt{\frac{Q + \sqrt{12C_0C_2S_{FC}^2W^2 + Q^2}}{\rho SC_0S_{FC}}} \tag{117}$$

where

$$Q = \frac{C_I}{(1 - J_W)}$$

Then v_J is obtained from (117) via the approximation $Q \approx C_I$. Replacing J_W^* given in (95) into Q yields

$$Q = \frac{C_IC_0SS_{FC}\rho(v^*)^2v_f^*}{A_1(v^*)} \tag{118}$$

Replacing (118) in (117) and attempting to solve for v^* yields a fifth order polynomial equation for which no analytic solution can be found. In this section, an estimate, v_1, of the optimal speed v^* will be presented in theorem 10 that is obtained by replacing Q as it appears in (117) with a simplified expression $\tilde{Q}$.

Theorem 10. *Let $C_{I_c} = \frac{1}{2}B_0$. Assuming $C_I < C_{I_c}$, the speed v^* that minimizes the OCP (73) can be approximated using v_1 given in terms of W and the optimal final speed v_f^* by*

$$v^* \approx v_1(W, v_f^*) = \frac{\sqrt{2}v_f^*}{2}\sqrt{R + \sqrt{\frac{48C_0C_2S_{FC}^2W^2}{B_0B_1 + C_I^2} + R^2}} \tag{119}$$

where

$$R = \frac{C_IB_{(-1)}}{B_0B_1 + C_I^2} \tag{120}$$

The maximum error occurs at the initial time because

$$|v^* - v_2| \leq \left|v_c^* - v_2(W_cv_f^*)\right| \tag{121}$$

Proof. Let Q as it appears in (118) be approximated by

$$Q \approx \tilde{Q} = M_1(v^*)^2 + M_2$$

where M_1, M_2 are the solutions to the system of equations

$$\begin{cases} Q|_{v^*=v_f^*} & = \tilde{Q}|_{v^*=v_f^*} \\ \left(\frac{\partial Q}{\partial v^*}\Big|_{v^*=v_f^*} \right) & = \left(\frac{\partial \tilde{Q}}{\partial v^*}\Big|_{v^*=v_f^*} \right) \end{cases} \tag{122}$$

Solving (122) for M_1, M_2 yields

$$M_1 = \frac{C_I B_1}{2B_0(v_f^*)^2}, \quad M_2 = \frac{C_I B_{(-1)}}{2B_0} \tag{123}$$

Therefore,

$$Q \approx \tilde{Q} = M_1(v^*)^2 + M_2 = \frac{C_I}{2B_0(v_f^*)^2}\left(B_1(v^*)^2 + B_{(-1)}(v_f^*)^2\right) \tag{124}$$

Replacing (124) in (117) results in a biquadratic equation of v^* which can be solved resulting in a single positive real solution given by (119).

To show (121), note that from equation (117), it holds that $|v^* - v_2|$ varies directly with $|Q - \tilde{Q}|$. Thus it suffices to show that

$$|Q - \tilde{Q}| \leq \left|Q - \tilde{Q}\right|_{v^*=v_c^*}$$

Note that $Q - \tilde{Q} \leq 0$. Indeed,

$$Q - \tilde{Q} = -\frac{C_I(v^* - v_f^*)^2 B_{(-1)}}{2B_0(v_f^*)^2}\left(\frac{C_0 S S_{FC}\rho v^*(v_f^*)^2 - C_I v^* - C_I v_f^*}{C_0 S S_{FC}\rho v^*(v_f^*)^2 + C_I v^* - C_I v_f^*}\right) \tag{125}$$

From the definition of B_β,

$$\frac{C_I(v^* - v_f^*)^2 B_{(-1)}}{2B_0(v_f^*)^2} \geq 0$$

Furthermore, because

$$C_0 S S_{FC}\rho v^*(v_f^*)^2 + C_I v^* - C_I v_f^* > C_0 S S_{FC}\rho v^*(v_f^*)^2 - C_I v^* - C_I v_f^*$$

the result $Q - \tilde{Q} \leq 0$ follows if it can be shown that

$$C_0 S S_{FC} \rho v^* (v_f^*)^2 - C_I v^* - C_I v_f^* > 0$$

or equivalently, that

$$v^* > \frac{C_I v_f^*}{C_0 S S_{FC} \rho v_f^{*2} - C_I} \tag{126}$$

It is assumed that

$$\begin{aligned} C_I < C_{I_c} &= \frac{1}{2} C_0 S S_{FC} \rho v_f^{*2} \\ \Rightarrow \ v_f^* &> \frac{C_I v_f^*}{C_0 S S_{FC} \rho v_f^{*2} - C_I} \end{aligned} \tag{127}$$

Therefore, (126) follows from (127) and (102) and $Q - \tilde{Q} \leq 0$ holds. It will now be shown that $\frac{d(Q-\tilde{Q})}{dv^*} \leq 0$ which, together with $Q - \tilde{Q} \leq 0$ implies that the maximum value of $|Q - \tilde{Q}|$ occurs at the largest value of v^*. Recall from (102), that v^* is a monotonically decreasing function. Thus the larges possible value of v^* is v_c^*. Taking the derivative with respect to v^* of equation (125) yields

$$\frac{d(Q - \tilde{Q})}{v^*} = -\kappa(v^*) \big(B_1 B_{(-1)} v^* - C_I v_f^* (2 C_0 S S_{FC} \rho v_f^{*2} - C_I) \big) \tag{128}$$

where $\kappa(v^*) \geq 0$. By the monotonically decreasing nature of v^*, and the assumption that $C_I < C_{I_c}$,

$$v^* \geq v_f^* > \frac{C_I v_f^* (2 C_0 S S_{FC} \rho v_f^{*2} - C_I)}{C_0^2 S^2 S_{FC}^2 \rho^2 v_f^{*4} - C_I^2} \ \Rightarrow \ \big(B_1 B_{(-1)} v^* - C_I v_f^* (2 C_0 S S_{FC} \rho v_f^{*2} - C_I) \big) > 0$$

thus by (128), $\frac{d(Q-\tilde{Q})}{v^*} \leq 0$ which finishes the proof. □

Remark 11. *The approximation $v_1(W, v_f^*)$ reduces to the maximum range speed $v_{MR}(W)$ given in (79) when $C_I = 0$.*

3.3.2 Exact expressions for v^*, t_f^*, W_f^*, V^*

The objective of this section is to propose exact expressions for v^*, to determine the optimal final time at TOD, and to provide an expression for V^* the solution to (73). Begin by assuming that $C_I > 0$.

Assuming $C_I > 0$

Theorem 12. *Let $C_{I_c} = \frac{1}{2}B_0$. Assuming $0 < C_I < C_{I_c}$, the optimal speed v^* that minimizes the OCP (73) is given in terms of position and the optimal final speed v_f^* by*

$$v^*_{C_I>0}(x, v_f^*) = \frac{2}{3}\left(-2\frac{C_I}{J_x^*} + \sqrt{3C_2C_0S_{FC}^2\Delta(x)^2 + \frac{C_I^2}{(J_x^*)^2}}\right) \tag{129}$$

where

$$\Delta(x) = x_d - x + \frac{(v_f^*)^2}{2B_{-1}}\sqrt{\frac{3B_2S\rho}{C_2S_{FC}}} \tag{130}$$

and J_x^* is given by (94).

Proof. An expression for the time rate of change of the optimal velocity, $\dot{v}^*$, is equation (109) in lemma 8. Because the final position $x(t_f) = x_d$ is known, and the dynamics of x are $\dot{x} = v$, the following must hold

$$\begin{aligned}
x_d - x &= \int_t^{t_f} v dt = \int_{v^*}^{v_f^*} \frac{v}{\dot{v}} dv \\
&= -\frac{\sqrt{3}}{2S_{FC}\sqrt{C_0C_2}} \int_{v^*}^{v_f^*} \frac{A_2(v)}{\sqrt{A_1(v)A_3(v)}} dv \\
&= -\frac{\sqrt{3}}{2S_{FC}\sqrt{C_2C_0}} \frac{\sqrt{A_1(v)A_3(v)}}{B_{-1}}\Bigg|_{v^*}^{v_f^*} \\
&= -\frac{\sqrt{3}}{2}\sqrt{\frac{S\rho}{C_2S_{FC}}}\frac{\sqrt{B_2}}{B_{-1}}(v_f^*)^2 + \frac{\sqrt{3}\sqrt{A_1(v^*)A_3(v^*)}}{2S_{FC}\sqrt{C_0C_2}B_{-1}}
\end{aligned} \tag{131}$$

Solving (131) for v^* yields two solutions:

$$
\begin{aligned}
v^* = v_1(x) &= \frac{1}{3}\frac{6C_I v_f^* + \sqrt{12\Delta(x)^2 C_0 C_2 S_{FC}^2 B_{-1}^2 + 9C_I^2 (v_f^*)^2}}{B_{-1}} \\
v^* = v_2(x) &= \frac{1}{3}\frac{6C_I v_f^* - \sqrt{12\Delta(x)^2 C_0 C_2 S_{FC}^2 B_{-1}^2 + 9C_I^2 (v_f^*)^2}}{B_{-1}}
\end{aligned} \tag{132}
$$

which are well defined as $\Delta(x)$ is well defined and non-negative. Indeed $B_2 > 0$ for $C_I < C_{I_c}^*$ and $x \le x_d$ for all time $t \in [0, t_f]$. It will now be shown that $v_2(x)$ given in (132) is not a valid solution. From (102) v^* is a monotonically decreasing function of time. From the equation for $v_2(x)$ in (132),

$$
v_2 < 2\frac{C_I v_f^*}{B_{-1}} = 2\frac{C_I v_f^*}{C_0 S S_{FC} \rho (v_f^*)^2 + C_I} \tag{133}
$$

Equation (133) implies that $v_2 < v_f^*$. Indeed,

$$
2\frac{C_I v_f^*}{C_0 S S_{FC} \rho (v_f^*)^2 + C_I} < v_f^* \iff C_I < B_0 \tag{134}
$$

which holds as $C_I < C_{I_c} = \frac{1}{2}B_0$ and $B_0 > 0$. Thus $v_2 < v_f^*$ for all time $t \in [0, t_f]$ which contradicts the monotonically decreasing nature of v^*. Therefore, the unique optimal speed is given by $v_1(x)$. Note that by (94),

$$
J_x^* = -\frac{2B_{-1}}{3v_f^*} \Rightarrow \frac{B_{-1}}{v_f^*} = -\frac{3J_x^*}{2}
$$

and $v_1(x)$ reduces to (129). □

Theorem 12 provides the optimal speed v^* that minimizes the OCP (73) when $C_I \in (0, C_{I_c})$. The following theorem will provide the minimal cost as well as the optimal final time at TOD.

Theorem 13. *Let $C_{I_c} = \frac{1}{2}B_0$. Assuming $0 < C_I < C_{I_c}$, then the following hold*

- *Time as a function of velocity along the optimal trajectory is given by*

$$
t = \Psi(v^*) - \Psi(v_c^*) \tag{135}
$$

where

$$\Psi(v) = \frac{2\tanh^{-1}\left(\frac{2\sqrt{3}A_{3/2}(v)}{3\sqrt{A_1(v)A_3(v)}}\right) - \ln\left(A_2(v) + \sqrt{A_1(v)A_3(v)}\right)\sqrt{3}}{2\sqrt{C_0C_2}S_{FC}} \tag{136}$$

- *The optimal final time at TOD is given by*

$$t_f^* = \Psi(v_f^*) - \Psi(v_c^*) \tag{137}$$

- *Time as a function of position along the optimal trajectory is given by*

$$t = \Psi\big(v^*_{C_I>0}(x, v_f^*)\big) - \Psi(v_c^*) \tag{138}$$

where $v^*_{C_I>0}(x, v_f^*)$ is given by the equation (129).

- *The optimal final weight is given by (98) rewritten here:*

$$W_f^* = \frac{v_f^*}{6}\sqrt{\frac{3S\rho B_2}{C_2 S_{FC}}} \tag{139}$$

- *The minimal DOC is given by*

$$V^* = W_c - \frac{v_f^*}{6}\sqrt{\frac{3S\rho B_2}{C_2 S_{FC}}} + C_I\big(\Psi(v_f^*) - \Psi(v_c^*)\big) \tag{140}$$

Proof. The result (135) is obtained by noting that

$$t = \int_0^t d\tau = \int_{v_c^*}^{v^*} \frac{1}{\dot{v}(\tau)} d\tau \tag{141}$$

Replacing (109) in (141) and integrating yields (135). The results (137)-(138) follow from (135). To prove (140), evaluate the cost-to-go (74) along the optimal trajectory. The result is

$$J^*(t, x^*, W^*) = (W^* - W_f^*) + C_I(t_f^* - t)$$

Therefore,

$$V^* = J^*(0) = (W_c - W_f^*) + C_I(t_f^*)$$

Replacing (139) in V^* yields

$$V^* = W_c - \frac{v_f^*}{6}\sqrt{\frac{3S\rho B_2}{C_2 S_{FC}}} + C_I(t_f^*)$$

Replacing t_f^* given by (137) in V^* yields (140). □

It was shown in Corollary 1, that the optimal cost-to-go J^* is a class C^1 function of time. Furthermore, by Lemma 5 and the dynamics of x, W in (73) $\dot{x}, \dot{W} < 0,\ \forall t \in [0, t_f)$ which implies that $x(t), W(t)$ are one-to-one, continuous functions of time $\forall t \in [0, t_f)$. Therefore, by (72), it must hold that the optimal cost to go can be written as

$$J^*(t, x^*, W^*) = -J_x^*(x_d - x) + \int_{W_f^*}^{W^*} J_W^*(\tau) d\tau \tag{142}$$

Under the assumption made in Villarroel and Rodrigues (2016), that $J_W^* \approx 0$, and from the definition of J_x^* in (94), the optimal cost-to-go can be approximated as a function of x only as

$$J^* \approx J_{approx}(x^*) = \frac{2}{3}\frac{B_{(-1)}}{v_f^*}(x_d - x) \tag{143}$$

Assuming $C_I = 0$ (The Complete Maximum Range Solution)

Theorem 14. *If $C_I = 0$, let v_{MR} denote the speed that minimizes (73). Let $v_{MR,f}, v_{MR,c}$ denote v_{MR} evaluated at the final time and initial time respectively. Let J_{MR}^* denote the total DOC for the maximum range OCP ((73) for $C_I = 0$), and let $W_{MR,f}, t_{MR,f}$ denote the final weight and final* *time after flying at a speed of v_{MR}. Then, the following must hold*

$$v_{MR,c} = \sqrt{2\frac{W_c}{\rho S}\sqrt{3\frac{C_2}{C_0}}} \tag{144a}$$

$$v_{MR}(x) = v_{MR,c} - \frac{2\sqrt{3}}{3}\sqrt{C_0C_2}S_{FC}x \tag{144b}$$

$$v_{MR,f} = v_{MR}(x_d) \tag{144c}$$

$$W_{MR,f} = \frac{1}{6}\sqrt{\frac{3C_0}{C_2}}S\rho v_{MR,f}^2 \tag{144d}$$

$$t_{MR,f} = \frac{1}{2S_{FC}}\sqrt{\frac{3}{C_0C_2}}\ln\left(\frac{v_{MR,c}}{v_{MR,f}}\right) \tag{144e}$$

$$V_{MR}^* = W_c - W_{MR,f} \tag{144f}$$

Furthermore, if W_{min} is the minimum allowable weight of the aircraft, let $v_{MR,min} = v_{MR}(W_{min})$ and $x_{d,max}$ denote the maximum range of the aircraft. It must hold that

$$v_{MR,min} = \sqrt{2\frac{W_{min}}{\rho S}\sqrt{3\frac{C_2}{C_0}}} \tag{145a}$$

$$x_{d,max} = -\frac{1}{2}\frac{\sqrt{3}}{\sqrt{C_0C_2}S_{FC}}(v_{MR,min} - v_{MR,c}) \tag{145b}$$

Proof. The result (144a) holds by evaluating (79) at the initial time when $W = W_c$. Replacing $C_I = 0$ and $v^* = v_{MR}$ in (109) yields

$$\dot{v}_{MR} = -\frac{2}{3}\sqrt{3C_0C_2}S_{FC}v_{MR} \tag{146}$$

Consider the boundary constraint on position

$$\begin{aligned} x &= \int_0^t v_{MR}dt = \int_{v_{MR,c}}^{v_{MR}} \frac{v_{MR}}{\dot{v}_{MR}}dv = \int_{v_{MR,c}}^{v_{MR}} -\frac{3}{2\sqrt{3C_0C_2}S_{FC}}dv \\ &= -\frac{\sqrt{3}(v_{MR} - v_{MR,c})}{2\sqrt{C_0C_2}S_{FC}} \end{aligned} \tag{147}$$

Solving (147) for v_{MR} yields (144b) and (144c) follows from evaluating (144b) at the final time when $x = x_d$.

Substituting $C_I = 0$ in (98) yields (144d). To show (144e), consider the identity

$$\begin{aligned} t_{MR,f} &= \int_0^{t_{MR,f}} dt = \int_{v_{MR,c}}^{v_{MR,f}} \frac{1}{\dot{v}_{MR}} dv_{MR} \\ &= -\frac{1}{2S_{FC}}\sqrt{\frac{3}{C_0C_2}} \int_{v_{MR,c}}^{v_{MR,f}} \frac{1}{v_{MR}} dv_{MR} \\ &= \frac{1}{2S_{FC}}\sqrt{\frac{3}{C_0C_2}} \ln\left(\frac{v_{MR,c}}{v_{MR,f}}\right) \end{aligned} \tag{148}$$

which proves (144e). Equation (144f) follows from (74). The result (145a) follows from (79), and (145b) is a result of (147) and (145a). □

3.4 Determining Unknowns Part 1: v_f^*, W_f^* as Functions of v_c^*

The speeds that have been proposed in this chapter are summarized in Table 3.1.

Summary of Speeds from Section 3.3				
Methodology	**Expression**	**Arguments**	**Equation**	**Upper Error Bound**
From Villarroel and Rodrigues (2016)	v_J	W	(78)	$E_J \geq 0$
Correction of v_J	v_1	W, v_f^*	(119)	(121)
Using endpoint constraint $x(t_f) = x_d$ and assuming $C_I > 0$	$v_{C_I>0}^*$	x, v_f^*	(129)	0
Using endpoint constraint $x(t_f) = x_d$ and assuming $C_I = 0$	v_{MR}	x	(144b)	0

Table 3.1: Summary of Expressions for v in Section 3.3

Note that the two expressions proposed by this thesis for the case when $C_I \in (0, C_{I_c})$, namely $v_1, v_{C_I>0}^*$, have an unknown v_f^* in their argument. It is therefore necessary to develop methods by which v_f^* can be determined. This section provides analytic expressions for v_f^* and W_f^* in terms of the optimal initial cruising speed v_c^* thus reducing the problem of determining v_f^* to one of determining v_c^*. Section 3.5 will propose methods by which v_c^* can be determined.

Lemma 15. *If $C_I < C_{I_c}$, then the optimal final weight W_f^* that is associated with the optimal velocity v^* is given in terms of the optimal initial speed v_c^* by*

$$W_f^* = -\frac{C_I}{S_{FC} v_c^*} \frac{S_{FC} x_d E_4(v_c^*) - 2SW_c \rho (v_c^*)^3}{E_{12}(v_c^*)} \tag{149}$$

Proof. Evaluating (131) at the initial time when $x = 0, v = v_c^*$ yields

$$x_d = -\frac{\sqrt{3}}{2}\sqrt{\frac{S\rho}{C_2 S_{FC}}} \frac{\sqrt{B_2}}{B_{-1}} (v_f^*)^2 + \frac{\sqrt{3}\sqrt{A_1(v_c^*)A_3(v_c^*)}}{2S_{FC}\sqrt{C_0 C_2} B_{-1}} \tag{150}$$

From the definition of W in (96), we may rewrite (150) as

$$x_d = -\frac{3W_f^* v_f^*}{B_{-1}} + \frac{3A_1(v_c^*)W_c}{C_0 S S_{FC} \rho (v_c^*)^2 B_{-1}} \tag{151}$$

Solving (151) for W_f yields

$$\begin{aligned} W_f^* &= -\frac{x_d B_{-1}}{3v_f^*} + \frac{A_1(v_c^*)W_c}{C_0 (v_c^*)^2 S \rho S_{FC} v_f^*} \\ &= -\frac{x_d B_{-1}}{3v_f^*} + \frac{W_c (v_c^* B_{-1} - C_I v_f^*)}{C_0 (v_c^*)^2 S \rho S_{FC} v_f^*} \end{aligned} \tag{152}$$

To eliminate the dependence of (152) on v_f^*, consider the identity (See appendix A)

$$\frac{B_{-1}}{v_f^*} = \frac{3C_I}{v_c^*} \frac{E_4(v_c^*)}{E_{12}(v_c^*)} \Rightarrow B_{-1} = 3C_I \frac{v_f^*}{v_c^*} \frac{E_4(v_c^*)}{E_{12}(v_c^*)} \tag{153}$$

Replacing (153) in (152) yields the result of the Lemma □

Corollary 2. *If $C_I < C_{I_c}$, then the optimal final speed v_f^* is given in terms of the optimal initial speed by*

$$v_f^* = \sqrt{\frac{C_I}{\rho S C_0 S_{FC}} \left(1 + \sqrt{1 + \frac{12 C_0 C_2 (E_4(v_c^*) S_{FC} x_d - 2SW_c \rho v_c^{*3})^2}{v_c^{*2} E_{12}^2(v_c^*)}}\right)} \tag{154}$$

Proof. It was shown in the proof of Lemma 6 that $J_W(t_f) = 0$, therefore, by equation (76), it must hold that $v^*(t_f) = v_f^* = v_J(W_f^*)$. Furthermore, it was shown in Lemma 15 that The optimal final weight may be expressed in terms of the optimal initial speed by equation (149). Replacing (149) in $v_J(W_f^*)$ yields the result of the Corollary. □

3.5 Determining Unknowns Part 2: Expression for v_c^*

The expressions $v_1, v^*_{C_I>0}$ summarized in Table 3.1 have the unknown v_f^* in their arguments. The previous section provided expressions for W_f^* and v_f^* in terms of the optimal initial cruising speed v_c^*. This section provides an implicit definition of v_c^* as the solution to an algebraic equation, and illustrates how Newton's method may be implemented

Theorem 16. *Assuming $C_I < C_{I_c}$, then the following holds:*

(1) If $C_I = 0$, then the optimal initial speed $v_{MR,c}$ and the optimal final speed $v_{MR,f}$ are given by Theorem 14.

(2) If $C_I > 0$, then the optimal initial speed v_c^* is the solution to the nonlinear equation

$$C_I = f(v_c^*) \tag{155}$$

where

$$f(y) = \frac{1}{9}\frac{y^2 \rho S S_{FC} C_0 E_{12}^2(y)\left(2+\sqrt{\epsilon(y)+\left(\frac{E_{-12}(y)}{E_{12}(y)}\right)^2}\right)^2}{E_4^2(y)\left(1+\sqrt{\epsilon(y)+\left(\frac{E_{-12}(y)}{E_{12}(y)}\right)^2}\right)} \tag{156}$$

and

$$\epsilon(y) = \frac{12 C_0 C_2 S_{FC} x_d E_4(y)(S_{FC} x_d E_4(y) - 4 S W_c \rho y^3)}{y^2 E_{12}^2(y)} \tag{157}$$

Proof. Replacing (154) in the identity (153) and solving the result for C_I yields (155).

□

Remark 17. *For any given initial speed v_c, the function $f(v_c)$ given by (156) will return a value of C_I, say, $\hat{C}_I$ such that v_c is the optimal initial speed for an OCP defined by (73) with $C_I = \hat{C}_I$. It is* *therefore unsurprising that $f_y > 0$ for all $y > v_{MR,c}$. I.e, the larger the value of C_I, the larger the initial speed.*

Remark 18. *The value of $v_J(W_c)$ where $v_J(W)$ is given by (78), is the solution to (155) with $\epsilon(v_c^*) = 0$.*

Lemma 19. *Let*

$$\beta = \sqrt{2S\rho W_c\sqrt{C_0}} - C_0 S_{FC} S\rho C_2^{1/4} x_d \tag{158}$$

If

$$C_I < \sqrt{\frac{2^5 W_c C_2}{\rho S}\sqrt{\frac{C_0}{3}}} S_{FC}\beta \tag{159}$$

then

$$f'(y) > \frac{C_I}{v_c^*}, \quad \forall y > v_{MR,c} \tag{160}$$

where $f(y), v_{MR,c}$ are given by (156) and (144a) respectively.

Proof. Suppose (159) holds. Let

$$L(y,\delta) = \frac{S_{FC}\left(C_0 S^2\rho^2 y^4 + 12C_2 W_c^2\right)}{S\rho y^3} - \frac{C_I}{v_c^*} - \delta \tag{161}$$

where $\delta \geq 0$. Then, L is smooth in $(y,\delta) \in \mathbb{R}^+ \times \mathbb{R}$. The value of

$$\min_{y \geq v_{MR,c}} L(y,\delta) \tag{162}$$

will now be determined. From (161), it must hold that

$$L_y(y,\delta) = \frac{S_{FC}\left(C_0 S^2\rho^2 y^4 - 36C_2 W_c^2\right)}{S\rho y^4}$$

Thus, $L_y = 0$ for $y > 0$ if and only if

$$y = \tilde{y} = \sqrt{6\frac{W_c}{S\rho}\sqrt{\frac{C_2}{C_0}}}$$

Furthermore,

$$L_{yy}(\tilde{y}) = \frac{2\sqrt{6}}{3}\frac{S_{FC}(S\rho)^{3/2}C_0^{5/4}}{C_2^{1/4}\sqrt{W_c}} > 0$$

Thus $L(y,\delta) \geq L(\tilde{y},\delta)$. Let δ_1 be such that $L(\tilde{y},\delta_1) = 0$. Then, replacing $y = \tilde{y}, \delta = \delta_1$ in (161) and solving for δ_1 yields the unique solution

$$\delta_1 = \frac{4}{3}S_{FC}\sqrt{6W_cS\rho\sqrt{C_2C_0^3}} - \frac{C_I}{v_c^*} \tag{163}$$

Because $L_\delta(y,\delta) = -1 < 0$, $L(y,\delta) \geq L(\tilde{y},\delta)$, and L is smooth in (y,δ), it must hold that

$$L(y,\delta) \geq L(\tilde{y},\delta) > L(\tilde{y},\delta_1) = 0, \ \ \forall\delta < \delta_1, y > 0 \tag{164}$$

Suppose there exists $\delta_2 \geq 0$ such that

$$\delta_2 \in [0,\delta_1) \tag{165a}$$

$$f'(y) - \frac{C_I}{v_c^*} > L(y,\delta_2),\ \forall y > v_{MR,c} \tag{165b}$$

then, by (164),

$$f'(y) - \frac{C_I}{v_c^*} > L(y,\delta_2) > L(\tilde{y},\delta_1) = 0$$

which proves the result of the theorem. Thus it must now be shown that there exists a value δ_2 that meets the requirements (165a), (165b). Let

$$\begin{aligned}\delta_2 &= \left(\frac{S_{FC}\left(C_0S^2\rho^2y^4 + 12C_2W_c^2\right)}{S\rho y^3} - f'(y)\right)\Big|_{y=v_{MR,c}} \\ &= \frac{4}{3}S_{FC}^2S\rho\sqrt{3C_2C_0^3}x_d\end{aligned} \tag{166}$$

Then δ_2 satisfies the requirement (165a). Indeed, it is clear from (166) that $\delta_2 > 0$. Furthermore,

$$\begin{aligned}\delta_1 - \delta_2 = \frac{4}{3}S_{FC}\left(\sqrt{6S\rho W_c\sqrt{C_2C_0^3}} - \sqrt{3C_2C_0^3}SS_{FC}\rho x_d\right) - \frac{C_I}{v_c^*} > 0 \\ \iff \ \ v_c^* > \frac{\sqrt{3}}{4}\frac{C_I}{S_{FC}\sqrt{C_0}C_2^{1/4}\beta}\end{aligned} \tag{167}$$

where β is given by (158). Noting that $v_c^* > v_{MR,c}$ by Lemma 9, and by the definition of $v_{MR,c}$ given in (144a), the inequality (167) holds if

$$v_c^* > v_{MR,c} = \sqrt{2\frac{W_c}{\rho S}\sqrt{3\frac{C_2}{C_0}}} > \frac{\sqrt{3}}{4}\frac{C_I}{S_{FC}\sqrt{C_0}C_2^{1/4}\beta}$$

which in turn holds by (159). It must now be shown that δ_2 satisfies the requirement (165b). That is, that

$$\delta_2 > \frac{S_{FC}\left(C_0 S^2\rho^2 y^4 + 12C_2 W_c^2\right)}{S\rho y^3} - f'(y) \tag{168}$$

From the definition of δ_2 in (166), showing (168) is equivalent to showing that

$$\text{argmax}_{y\geq v_{MR,c}}\left(\frac{S_{FC}\left(C_0 S^2\rho^2 y^4 + 12C_2 W_c^2\right)}{S\rho y^3} - f'(y)\right) = v_{MR,c} \tag{169}$$

Consider f in (156) as a function of y and ϵ, then, by the definition of f in (156), the left hand side of the expression (169) can be written as

$$\begin{aligned}&\text{argmax}_{y\geq v_{MR,c}}\left(\frac{d}{dy}f\big(y,\epsilon(y)\big)|_{\epsilon(y)\equiv 0} - \frac{d}{dy}f\big(y,\epsilon(y)\big)\right)\\ =&\text{argmax}_{y\geq v_{MR,c}}(|\epsilon(y)|)\end{aligned} \tag{170}$$

Thus, the value of y that maximizes

$$\frac{d}{dy}f\big(y,\epsilon(y)\big)|_{\epsilon(y)\equiv 0} - \frac{d}{dy}f\big(y,\epsilon(y)\big)$$

is the value of y that maximizes $|\epsilon(y)|$. Finally, noting that

$$\begin{aligned}&\lim_{y\to v_{MR,c}^+}\left(\frac{d}{dy}f\big(y,\epsilon(y)\big)|_{\epsilon(y)\equiv 0} - \frac{d}{dy}f\big(y,\epsilon(y)\big)\right) = \delta_2\\ &\lim_{y\to v_{MR,c}^+}|\epsilon(y)| = \infty\end{aligned} \tag{171}$$

finishes the proof.

□

The following Theorem uses the results of Lemma 19 to develop an upper bound on the error incurred after k iterations of Newton's method on equation (155).

Theorem 20. *If the inequality (159) holds, let* $v_c^{N,k}$ *denote the result of* k *iterations of Newton's method applied to the definition of* v_c^* *in (155) with initial guess* $v_c^{N,0} = v_{MR,c}$ *where* $v_{MR,c}$ *is given in (144a). Let*

$$f(v_c^{N,k}) - C_I = \delta$$

where f *is given by (156). Then,*

$$\left|\frac{v_c^{N,k} - v_c^*}{v_c^*}\right| < \left|\frac{\delta}{C_I}\right| \tag{172}$$

Proof. by the smoothness of $f(v_c)$ in $v_c > v_{MR,c}$ and the intermediate value theorem, there must exist a point $m \in [\min(v_c^*, v_c^{N,k}), \max(v_c^*, v_c^{N,k})]$ such that

$$\begin{aligned} \frac{\delta}{v_c^{N,k} - v_c^*} &= f'(m) \\ \iff v_c^{N,k} - v_c^* &= \frac{\delta}{f'(m)} \\ &< v_c^* \frac{\delta}{C_I} \text{ by Lemma 19} \\ \therefore \left|\frac{v_c^{N,k} - v_c^*}{v_c^*}\right| &< \left|\frac{\delta}{C_I}\right| \end{aligned} \tag{173}$$

where

$$v_c^{N,k} = v_c^{N,k-1} + \frac{C_I - f(v_c^{N,k-1})}{f'(v_c^{N,k-1})}$$

for $k \in \mathbb{Z},\ k \geq 1$. □

3.6 Algorithm for Turbojet ECON mode Methodology for Cruise

This chapter has provided two novel expressions for v^* when $C_I > 0$, namely the approximation v_1 in terms of W and v_f^* and the exact solution $v_{C_I>0}^*$ in terms of x and v_f^* summarized in Table 3.1. Furthermore, the optimal final time at TOD, final weight at TOD and minimal DOC are provided in terms of v_f^*, and v_c^* in Theorem 13. An analytic expression for v_f^* in terms of v_c^* is given by (154), and an implicit definition of v_c^* is presented in (155). Finally, Theorem 20 shows that under certain circumstances, the error incurred by using Newton's Method to solve (155) can by easily bounded. The complete methodology proposed in this chapter is summarized in Algorithm 1:

Algorithm 1 Determining the optimal turbojet cruise trajectory

Require: $SET = \{C_I, C_0, C_2, S, S_{FC}, \rho, W_c, x_d\}$

```
if C_I=0 then
    Use Theorem 14 for the complete maximum range solution
else
    Determine v_c^* using Newton's Method (code found in Procedure 3 of Appendix B). Use Equation (172) to bound the error.
    Determine v_f^* from v_c^* using equation (154)
    if C_I ≥ C_{I_c} = (1/2)B_0 then No solution exists
    else
        Replace v_f^* in (129) to obtain v^*_{C_I>0}(x, v_f^*).    ▷ An expression in terms of x
        Replace v_f^* in (119) to obtain v_1(W, v_f^*).            ▷ An expression in terms of W
        Replace v_f^*, v_c^* in (137) to obtain t_f^*              ▷ Optimal final time at TOD
        Replace v_f^* in (139) to obtain W_f^*                      ▷ Optimal weight at TOD
        Replace v_f^*, v_c^* in (140) to obtain V^*                ▷ Minimal DOC
    end if
end if
```

3.7 Numerical Example

This section will follow the methodology proposed in Algorithm 1 to solve the ECON mode problem for a cruising jet. The methodology of Algorithm 1 will be validated against the work in Villarroel and Rodrigues (2016) which was itself validated against flight simulation data.

3.7.1 Worked Example

Consider the example of an Airbus A320 first proposed in Villarroel and Rodrigues (2016), with the following aircraft and mission parameters (see Rieck, Richter, and Holzapfel (2013)):

$C_I = 0.3674\ [lbs/s]$	$C_0 = 0.026659$	$C_2 = 0.038726$
$S_{FC} = 0.00012402\ [1/s]$	$S = 1319.6554\ [ft^2]$	$h_c = 30000\ [ft]$
$\rho = 0.00089068\ [slug/ft^3]$	$x_c = 0\ [ft]$	$x_d = 5016000\ [ft]$
$W_c = 127673\ [lbf]$	$\Lambda = 25\ [deg]$	$t/c = 0.108$

Table 3.2: A320 aircraft and mission parameters from Villarroel and Rodrigues (2016).

Following Algorithm 1, and noting that $C_I > 0$, we proceed to step 4.

Step 4

The value of v_c^* must be determined using Newton's Method (See Appendix B.2), and the percent error on the approximation $v_c^* \approx v_c^{N,k}$ must be bounded using Theorem 20.

Using three iterations Newton's method to solve (155) with an initial guess of $v_c^* = v_c^{N,0} = v_{MR,c} = 673.43$ where $v_{MR,c}$ is the maximum range speed given by (144a), yields a value denoted $v_c^{N,3}$ and given by

$$v_c^* \approx v_c^{N,3} = 748.81\ ft/s$$

According to Theorem 20, in order to bound the error in $v_c^{N,3}$, we must first test that (159) holds where β is given by (158). Replacing the example parameters in (159) yields $0.3674 < 2.97$. Thus, by Theorem 20, an upper bound on the percent error is given by

$$PE(v_c^{N,3}) = \left|\frac{v_c^{N,3} - v_c^*}{v_c^*}\right| \times 100 < \left|\frac{f(v_c^{N,3}) - C_I}{C_I}\right| \times 100 = \left(2.72 \times 10^{-8}\right)\%$$

where f is given by (156).

Step 5

Step 5 states that the value of v_f^* must be obtained from the determined value of v_c^* using equation

(154). According to (154), the optimal final speed associated $v_c^{N,3}$, is given by

$$v_f^* \approx v_f^{N,3} = vf(v_c^{N,3}) = 726.22\ ft/s$$

Step 6

Now that v_c^*, v_f^* have been computed, line 6 of Algorithm 1 states that we must test $C_I < C_{I_c}$:

$$C_{I_c} = \frac{1}{2}C_0 S S_{FC} \rho v_f^{*2} = \frac{1}{2}C_0 S S_{FC} \rho (726.22)^2 = 1.02$$

Therefore, $C_I = 0.03674 < C_{I_c} = 1.02$, and we may proceed with step 8 of Algorithm 1.

Step 8

To obtain an analytic expression for the optimal speed in terms of x, according to step 8, replace v_f^* in $v_{C_I>0}(x, v_f^*)$. Therefore,

$$v_{C_I>0}(x, v_f^{N,3}) = 220.79 + \frac{1}{3}\sqrt{1.91 \times 10^{-10}(1.12 \times 10^8 - x)^2 + 1.10 \times 10^5} \tag{174}$$

Step 9

An approximation of the optimal speed v^* expressed in terms of W may be obtained by replacing v_f^* in (119). Performing this substitution yields:

$$v^*(W) \approx v_1(W, v_f^{N,3}) = 513.52\sqrt{0.25 + \sqrt{2.13 \times 10^{-10} W^2 + 0.061}} \tag{175}$$

Steps 10, 11, 12

The remaining results of the algorithm are as follows:

$$\begin{aligned} t_f^* &= 6801.6\ s \ \text{ by line 10} \\ W_f^* &= 1.19 \times 10^5\ lbs \ \text{ by line 11} \\ V^* &= 11239.7\ lbs \ \text{ by line 12} \end{aligned} \tag{176}$$

This concludes the method proposed in Algorithm 1. The results presented above will now be validated.

Validation of Results

Four expressions for the optimal cruise speed, given in terms of position, will be compared. The first expression $v_{C_I>0}(x, v_f^{N,3})$ is the result replacing $v_f^* = v_f^{N,3}$ in (129). This expression is the result of following Algorithm 1, and can be found in (174). The second expression, $v_{C_I>0}(x, v_f^J)$ is the result of replacing $v_J(W_c)$ in (154) to obtain $v_f^J = v_f(v_J(W_c))$ where $v_J(W)$ is the speed proposed in Villarroel and Rodrigues (2016) and is given by (78). Replacing $v_f^* = v_f^J = v_f(v_J(W_c))$ in (129) results in:

$$v_{C_I>0}(x, v_f^J) = 219.24 + 0.33\sqrt{1.91 \times 10^{-10}(1.14 \times 10^8 - x)^2 + 1.01 \times 10^5} \tag{177}$$

The third expression that will be compared is $v_J(x)$ which is obtained directly from $v_J(W)$ in (78) using Euler's method. Finally, the fourth expression, denoted $v^*(x)$ is the theoretically optimal trajectory obtained using the shooting method.

The initial and final speeds of the four expressions are summarized in Table 3.3:

Expression	Initial speed v_c (ft/s)	Initial speed error (ft/s)	Final speed $v_f(v_c)$ (ft/s) (From (154))	Final speed error (ft/s)
$v_{C_I>0}(x, v_f^J)$	756.11	1.31	733.54	7.29
$v_{C_I>0}(x, v_f^{N,3})$	748.81	0.	726.26	0.
$v_J(x)$	746.86	1.95	726.27	0.01
$v^*(x)$	748.81	0	726.26	0

Table 3.3: Initial and final speed comparison for four speeds

The final time at TOD, final weight and DOC associated with the speed $v_{C_I>0}(x, v_f^{N,3})$ are given in (176) and were obtained by replacing the appropriate values in analytic expressions (137), (139) and (140) respectively. To obtain t_f, W_f and the DOC associated with $v_J(x)$, one must use Euler's method as no analytic expressions exist. Table 3.4 compares the final time, final weight and DOC for the four speeds considered here. Table 3.4 also considers the number of computations required to obtain t_f, W_f, and DOC considering that Euler's method would be employed to obtain these values for all expressions except $v_{C_I>0}(x, v_f^{N,3})$ and $v^*(x)$ which use analytic expressions (see 137, 139, and 140) and the shooting method respectively. Here, a computation is defined as the replacement of quantities in an expression and the evaluation of that expression.

Expression	Final Time (s)	Final Weight (lbs)	DOC (lbs)	Computations required
$v_{C_I>0}(x, v_f^J)$	6735.0	1.19×10^5	11240.9	204,322
$v_{C_I>0}(x, v_f^{N,3})$	6801.6	1.19×10^5	11239.7	7
$v_J(x)$	6801.6	1.19×10^5	11239.8	204,318
Shooting Method	6801.6	1.19×10^5	11239.7	2,044,490

Table 3.4: Final time, final weight, DOC, and computation time comparison for four speeds

Tables 3.3 and 3.4 imply that using the approximation $v^*(x) \approx v_{C_I>0}(x, v_f^{N,3})$ is not only more accurate than $v_{C_I>0}(x, v_f^J)$ and $v_J(x)$ (the speed proposed in Villarroel and Rodrigues (2016)), but also requires less computation time than any other method due to the existence of analytic expressions for t_f, W_f, and DOC. Figure 3.1 illustrates the four speeds considered here graphically.

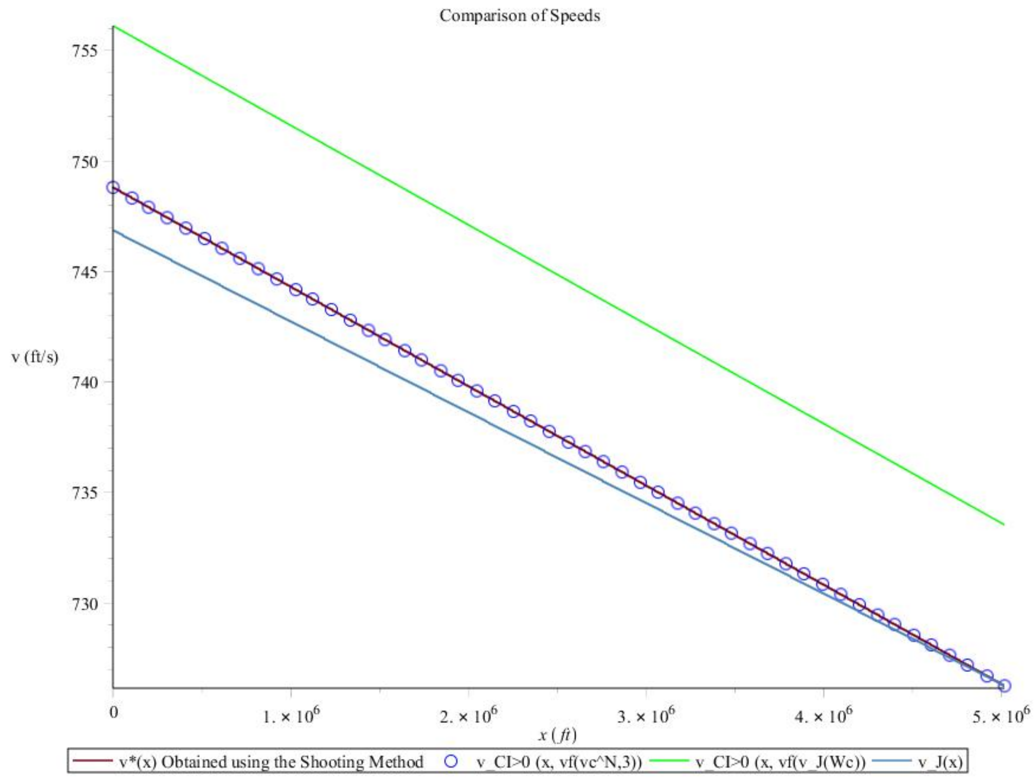

Figure 3.1: A comparison of $v_{C_I>0}(x, v_f^{N,3})$ given by (174), $v_{C_I>0}(x, v_f^J)$ given by (177), $v_J(x)$ proposed in Villarroel and Rodrigues (2016), and the optimal speed $v^*(x)$ obtained using the shooting method.

Figure 3.2 compares $v_1(W, v_f^{N,3})$ proposed by Algorithm 1 (see step 9) and given by (175) with $v_J(W)$ developed in Villarroel and Rodrigues (2016) and given by (78), and $v^*(W)$ obtained using the shooting method.

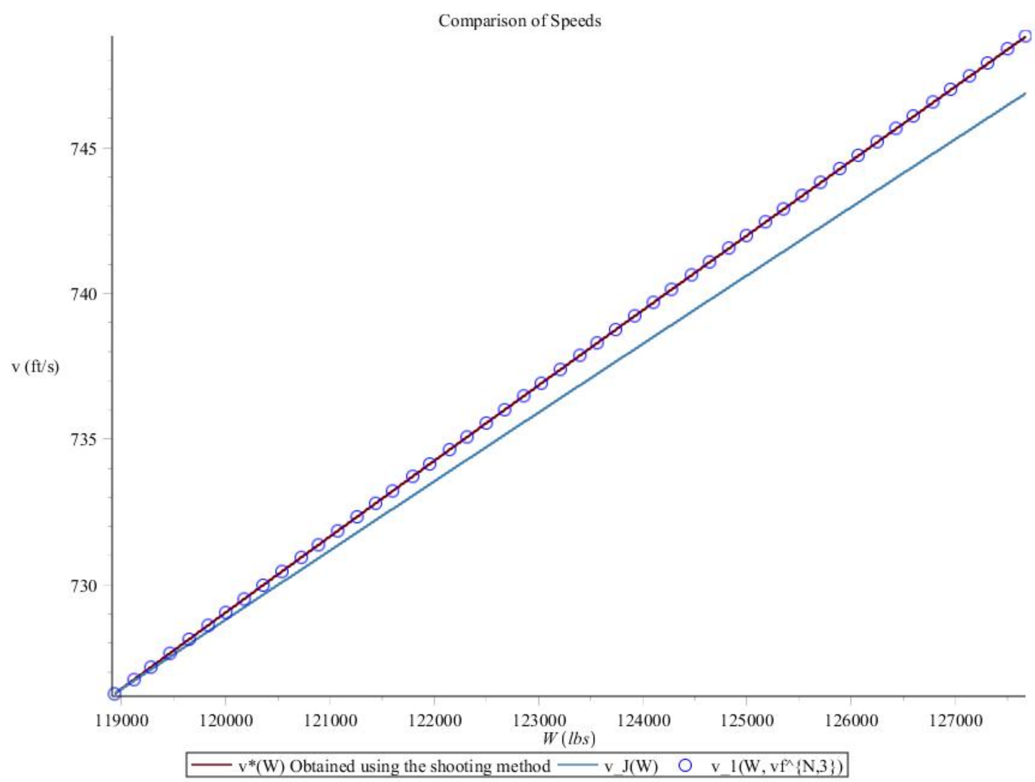

Figure 3.2: Theoretically optimal $v^*(W)$ obtained using the shooting method compared with $v_1(W, v_f^{N,3})$ (a result of Algorithm 1), and $v_J(W)$ (speed proposed in Villarroel and Rodrigues (2016))

Figure 3.2 implies that using the approximation $v^*(W) \approx v_1(W, v_f^{N,3})$ for v_1 given in (119) yields a more accurate approximation of the optimal speed than $v_J(W)$. Furthermore, as the initial weight W_c increases, the error incurred by making the approximation $v^* \approx v_J(W)$ grows almost linearly, while $v_1(W, v_f^{N,3})$ remains close to the optimal speed.

3.7.2 Case Study

To see the monetary impact of cruise speed, consider a single airline (see "A320-200" (2017)): Airbirlin has 62 Airbus A320s in its fleet and these jets cruise at a constant speed of 781 ft/s. Calculating the DOC with this speed and the parameters of the example yields a DOC of

$$V = 11278.29704$$

Thus $V - V^* = 38.59\ lbs$ for $V^* = 11239.7\ lbs$ given in (176). With 62 A320s in its fleet, at one trip per day and $\$0.24/lb$ as the price of jet fuel (see *Fuel Price Analysis* (2017)), the potential savings are $\$210,000$ per year for one class of jet alone by flying at a speed of $v^*(x, v_f^{N,3})$ instead of $781\ ft/s$.

3.8 Chapter Summary

This chapter has presented analytic expressions for

- the DOC-minimal cruising speed of a turbojet aircraft in terms of position (see (129)),
- the optimal final cruise time (see (137)),
- the optimal final cruise weight (see (139)),
- the minimal DOC (see (140)),

A suboptimal expression for the optimal cruising speed in terms of weight has also been developed (see (119)). An upper bound for the approximation is also provided. Algorithm 1 summarizes the turbojet ECON mode trajectory optimization techniques proposed in this thesis which were validated against the results of Villarroel and Rodrigues (2016) (which were in turn validated against the results of flight simulator data) in section 3.7.

Chapter 4

Flight Management System for a Turboprop in Cruise

The objective of this chapter is to formulate and solve the ECON mode problem for a cruising turboprop aircraft. Section 4.1 will use the cost functional (2) as well as the simplified dynamics of flight (8) to formulate the ECON mode problem for cruise as an OCP. Preliminary results will be presented in Section 4.2.

An exact equation for the optimal cruising speed in terms of position could not be determined for values of $C_I > 0$. However, the a complete solution for the maximum range problem (when $C_I = 0$) for a cruising turbojet is presented in section 4.3. The turbojet and turboprop OCPs are similar in many respects. An in depth comparison, including a transformation Φ between turbojet and turboprop OCPs can be found in section 4.4. The transformation Φ motivates an approximation for v^* the optimal speed that minimizes the ECON mode of a cruising turboprop. This approximation, denoted v_1 is based on the result for $v_1(x, v_f^*)$ presented in equation (119) of chapter 3 that has been modified using Φ. The expression v_1 is in terms of the weight W of the turboprop and the optimal final cruising speed at TOD v_f^* and can be found in section 4.5.

Like the results of chapter 3, in order to use the approximation presented in section 4.5, a method by which v_f^* may be obtained must be determined. Unfortunately, unlike v_f^* for the turbojet, no analytic expression was obtained for v_f^*. However, the transformation Φ in section 4.4 motivates the development of a recursive algorithm that may be employed to determine v_f^* and v_c^* the optimal initial cruising speed. The pseudocode for this algorithm is presented in section 4.6 and a Maple (TM) version of the code may be found in appendix B procedure 3.

The results of this chapter will be validated with a numerical example in section 4.8. In particular the algorithm presented in section 4.6 is timed to ensure that it is fast enough to be employed in real time. It is important to note that v_f^* is a constant that need only be calculated once before takeoff.

4.1 Optimal Control Problem Formulation

For a cruising turboprop aircraft, it is assumed that $T = D$. Under Assumptions 1-9, combining the cost functional (2) and the reduced dynamics (8) and (10), yields the following OCP:

$$\begin{aligned} V^*(x_0, W_0) = \inf_{v,t_f} \int_0^{t_f} (S_{FC}Dv + C_I)dt \\ s.t.\ \ \dot{x} = v \\ \dot{W} = -S_{FC}Dv \\ D = \frac{1}{2}C_0\rho S v^2 + \frac{2C_2W^2}{\rho S v^2} \\ x(0) = 0,\ x(t_f) = x_d,\ W(0) = W_c \\ v \in U = \{v : v > 0,\ v \text{ is piecewise continuous}\} \end{aligned} \tag{178}$$

The position $x(t)$ and weight $W(t)$ are absolutely continuous functions of time. Note that the initial and final positions (0, x_d respectively) are known, as are the initial weight (W_c) and time. However, the final weight W_f and final time t_f are left free. The speed v is the control input we wish to design.

The optimal cost-to-go is given by

$$\begin{aligned} J^*(t, x_0, W_0) = \inf_{v,t_f} J(t, x_0, W_0, v) \text{ where} \\ J(t, x, W, v) = \int_t^{t_f} (S_{FC}Dv + C_I)\, dt \end{aligned} \tag{179}$$

The set of admissible trajectories U is defined as the set of positive speeds v which are piecewise continuous. Note that the force of drag D given in (10) is a well defined continuous function of v for $v > 0$. Therefore, for any admissible speed v, the function $\dot{W} = -S_{FC}Dv$ is a well defined piecewise continuous function of time in the interval $[0, t_f]$ which is integrable over its domain. The cost-to-go (179) may thus be rewritten as

$$J = W(t) - W(t_f) + C_I(t_f - t) \tag{180}$$

Noting that $W(t)$ is continuous and W_c, C_I, t_f are constants, it must hold from (180), that $J(t)$ is also a continuous function of time for all time $t \in [0, t_f]$ and for any admissible controller v. Because v^* must be admissible, we may conclude that $J^*(t, x(0), W(0))$ is a continuous function of time provided v^* exists. The following section presents some preliminary results on the Turboprop OCP.

4.2 Preliminary Results

This section will present some preliminary results required to perform further analysis of the OCP (178). The results presented in this section are similar to the results presented in section 3.2 for the Turbojet. The extent to which the Turbojet and Turboprop OCPs are similar is investigated in the section 4.4. The following notation will be used:

$$\begin{aligned} F_{\beta,n,k,l}(v) &= 2C_0 S S_{FC} \rho v^{\beta} v_f^{*n} + kC_I v - lC_I v_f^*, \;\; \forall (\beta, n, k, l) \in \mathbb{R}^4 \\ G_\beta &= 2C_0 S S_{FC} \rho (v_f^*)^3 - \beta C_I, \;\; \forall \beta \in \mathbb{R} \end{aligned} \tag{181}$$

where v_f^* is the optimal initial and final speeds of cruise respectively.

Lemma 21. *A minimizer v^* to the OCP in (178) exists.*

Proof. If $C_I = 0$, then the OCP in (178) reduces to the maximum range problem. The speed $v_{MR}^W(W)$ known as the *maximum range speed* and given by

$$v_{MR}^W(W) = \sqrt{2 \frac{W}{S\rho} \sqrt{\frac{C_2}{C_0}}} \tag{182}$$

is the solution to the resulting OCP (see Miele (1959)). Suppose that $C_I > 0$. The Hamiltonian of the OCP in (178) is given by

$$H(x, W, v, J_x, J_W) = S_{FC} D v (1 - J_W) + J_x v + C_I \tag{183}$$

where D is the drag given in (10), J_W, J_x are the sensitivities of the cost-to-go to weight and position respectively. Noting that the Hamiltonian in (183) does not depend explicitly on time, and that the final time for the OCP in (178) is free, it must hold from Pontryagin's Maximum Principle (PMP) (see equation (53))

$$H^* = S_{FC} D v^* (1 - J_W^*) + J_x^* v^* + C_I = 0 \tag{184}$$

Because H given in (183) is a class C^2 function of v for admissible control inputs, a necessary condition for optimality inside the feasible set is $H_v = 0$ along the optimal trajectory (See equation (50b)):

$$H_v^* = S_{FC}(D_v v^* + D)(1 - J_W^*) + J_x^* = 0 \tag{185}$$

Solving (185) for J_W^*, replacing the result in (184) with D in (10) and rearranging terms yields

$$H^* = (2J_x^* v^* + 3C_I)(C_0 S^2 \rho^2 v^{*4} - 4C_2 W^2) + 8C_2 C_I W^2 = 0 \tag{186}$$

Note that H^* is a continuous function of v^*. From equation (50b) of the PMP,

$$\begin{aligned} \dot{J}_x^* &= -H_x^* = 0 \\ \dot{J}_W^* &= -H_W^* = -\frac{4 S_{FC} W C_2}{\rho S v^*}(1 - J_W^*) \end{aligned} \tag{187}$$

implying that J_x^* is a constant. Furthermore, solving (186) for J_x^* yields

$$J_x^* = -\frac{C_I(3C_0 S^2 \rho^2 v^{*4} - 4C_2 W^2)}{2v^*(C_0 S^2 \rho^2 v^{*4} - 4C_2 W^2)} \tag{188}$$

The expression (188) is well defined and negative. Indeed, it was shown in Rodrigues (2017), that

$$v^* > v_{MR}^W(W) \text{ for } C_I > 0 \tag{189}$$

Replacing v_{MR}^W given in (182) in (189) yields

$$\begin{aligned} v^* &> \sqrt{2\frac{W}{S\rho}\sqrt{\frac{C_2}{C_0}}} \iff \\ v^{*4} &> \frac{4W^2}{S^2\rho^2}\frac{C_2}{C_0} \text{ because } v^* > 0 \\ \Rightarrow\ v^{*4}S^2\rho^2C_0 &> 4W^2C_2 \end{aligned} \tag{190}$$

Therefore, noting that $3C_0S^2\rho^2v^{*4} > C_0S^2\rho^2v^{*4}$, the inequality (190) implies that

$$3C_0S^2\rho^2v^{*4} - 4C_2W^2 > C_0S^2\rho^2v^{*4} - 4C_2W^2 > 0$$

which in turn implies that J_x^* given in (188) is a well defined negative constant.

Replacing $v = v_{MR}^W$ in H^* given by (186) yields

$$H^*(v_{MR}^W) = 8C_2C_IW^2 > 0 \tag{191}$$

If a value $v_2 > 0$ can be found such that $H^*(v_2) < 0$, then by the continuity of H^* in v, there must exist a solution $v^* \in (v_{MR}^W, v_2)$ to $H^*(v^*) = 0$. Consider

$$v_2 = -\frac{C_I(8AC_2W^2 + 3)}{2J_x^*} \tag{192}$$

where $A > 0$. It was already shown that J_x^* is a negative constant, thus (192) implies that $v_2 > 0$. Noting that

$$\lim_{A\to\infty} -\frac{2048C_2^5C_I^5W^{10}C_0\rho^2S^2A^5}{J_x^4} \times (H^*(v_2))^{-1} = 1$$

for H^* given in (186), it may be concluded that there exists a sufficiently large value of A such that $H^*(v_2) < 0$ which finishes the proof. □

Lemma 22. *The time rate of change of the optimal speed $\dot{v}^*$ is given by*

$$\dot{v}^* = -\frac{8S_{FC}WC_2C_0v^{*4}\rho S}{3C_0S^2\rho^2v^{*4} + 4C_2W^2} < 0 \tag{193}$$

Proof. Solving (185) for J_x^* and replacing the result in (184) yields

$$H^*(v^*, W, J_W^*) = C_I - (1 - J_W^*)\frac{S_{FC}(C_0S^2\rho^2v^{*4} - 4C_2W^2)}{\rho Sv^*} = 0 \tag{194}$$

Noting that equation (194) holds for all time $t \in [0, t_f]$, and since $H^*(v^*, W, J_W^*)$ is differentiable it must hold that

$$\dot{H}^*(v^*, W, J_W^*) = \frac{\partial H^*(v^*, W, J_W^*)}{\partial W}\dot{W} + \frac{\partial H^*(v^*, W, J_W^*)}{\partial J_W^*}\dot{J}_W^* + \frac{\partial H^*(v^*, W, J_W^*)}{\partial v^*}\dot{v}^* = 0 \tag{195}$$

Replacing $\dot{J}_W^*$ given by (187), H^* given by (194), and the dynamics of W from (178) in (195) and solving the result for $\dot{v}^*$ yields (193). □

Lemma 23. *Let $C_{I_c} = \frac{1}{2}G_0$ where G is given in (181). Assuming $C_I < C_{I_c}$, the following equalities hold concerning the OCP (178).*

(1) The sensitivity of the optimal cost with respect to position, is a constant given by

$$J_x^* = -\frac{G_{(-1)}}{2v_f^*} \tag{196}$$

(2) The sensitivity of the optimal cost with respect to changes in weight is a continuous function of time for all $t \in [0, t_f]$ and is given by

$$J_W^* = \frac{(v^* - v_f^*)\left(2C_0SS_{FC}\rho v^{*2}v_f^* + 2C_0SS_{FC}\rho v^*v_f^{*2} - C_I\right)}{2\rho SC_0S_{FC}v^{*3}v_f^*} \tag{197}$$

with $J_W^* \in [0, 1)$.

(3) The optimal speed v^* is a continuous function of time for all time $t \in [0, t_f]$ and is unique.

(4) The optimal weight as a function of speed is given by

$$W^* = \frac{v^{*2} S\rho}{2}\sqrt{\frac{C_0}{C_2}}\sqrt{\frac{F_{1,3,1,3}(v^*)}{F_{1,3,1,1}(v^*)}} \tag{198}$$

Proof. Recall from the proof of Lemma 21 that J_x^* is a negative constant. Furthermore, from the PMP transversality conditions (see equation (50a)), $J_W^*(t_f) = 0$ because $W(t_f)$ is unspecified and the OCP in (178) has no terminal cost. Thus,

$$\begin{aligned} J_x^*(t_f) &= J_x^*(t) = J_x^* \\ J_W^*(t_f) &= 0 \end{aligned} \tag{199}$$

Evaluating (184) and (185) at the final time when $W = W(t_f) = W_f$, $v^* = v_f^*$, $J_W^*(t_f) = 0$, and D is given by (10) yields a system of two equations with the solution J_x^* given by (196) and

$$W_f = \frac{1}{2}\sqrt{\frac{S\rho v_f^*}{C_2 S_{FC}}(C_0 S S_{FC}\rho v_f^{*3} - C_I)} \tag{200}$$

which is well defined and positive for $C_I < C_{I_c}$. Therefore, in order to ensure that the final weight is strictly positive, it must hold that $C_I < C_{I_c}$.

Replacing (196) in equations (184) and (185) and solving the resulting system of two equations for J_W^* and W results in (197) and (198), respectively. It must now be shown that J_W^* is continuous and bounded and its value belongs to the interval $[0, 1)$, that v^* is continuous for all time $t \in [0, t_f]$, and that the expression (198) is well defined

To show that J_W^* is a continuous function of time for all $t \in [0, t_f]$, note that J_W^* presented in (197) is a continuous function of v^* for all possible values of v^*. Therefore, as a function of time, J_W^* has at most the same countable discontinuities that v^* has. Such discontinuities in v^* are called *corner points*. The continuity of J_W^* follows from Theorem 3.

In order to prove that $J_W^* \in [0, 1)$ and (198) is well defined, it must first be shown that v^* is a continuous function of time for all $t \in [0, t_f]$. Recall that v^* exists by Lemma 21. Because $v^* \in U$, it must hold that v^* is piecewise continuous in the interval $(0, t_f)$. Suppose v^* experiences a jump discontinuity at $t_1 \in (0, t_f)$. Suppose that $v^*(t_1^+) = v_R,\ v^*(t_1^-) = v_L$ where $v_R - v_L = \delta$ for

$|\delta| > 0$. Then, because equation (194) must hold for all time,

$$\begin{aligned}
&H^*(t_1^-) = H^*(t_1^+) = 0 \\
\iff &S_{FC}\big(1 - J_W^*(t_1^-)\big)\frac{C_0S^2\rho^2v^{*4}(t_1^-) - 4C_2W^2(t_1^-)}{v^*(t_1^-)\rho S} \\
=&S_{FC}\big(1 - J_W^*(t_1^+)\big)\frac{C_0S^2\rho^2v^{*4}(t_1^+) - 4C_2W^2(t_1^+)}{v^*(t_1^+)\rho S}
\end{aligned} \tag{201}$$

Noting that J_W^*, W are continuous for all time in $(0, t_f)$, it must hold that $J_W^*(t_1^-) = J_W^*(t_1^+) = J_W^*(t_1), W(t_1^-) = W(t_1^+) = W(t_1)$. Therefore, (201) reduces to

$$(v_L - v_R)\big(C_0S^2\rho^2v_Lv_R(v_L^2 + v_Lv_R + v_R^2) + 4C_2W(t_1)^2\big) = 0 \tag{202}$$

which has only one real root $v_L = v_R$. Therefore v^* is continuous at t_1 and thus for all time $t \in [0, t_f]$.

Recall from (197) that J_W^* is given by

$$J_W^* = \frac{(v^* - v_f^*)\big(2C_0SS_{FC}\rho v^{*2}v_f^* + 2C_0SS_{FC}\rho v^*v_f^{*2} - C_I\big)}{2\rho SC_0S_{FC}v^{*3}v_f^*} \tag{203}$$

which holds for all $t \in [0, t_f)$. Note that $\dot{v}^* < 0$ by Lemma 22, thus it must hold by the continuity of v^* that $v^* - v_f^* \geq 0$ for all time $t \in [0, t_f]$, which implies that

$$0 > -2C_0SS_{FC}\rho v^*v_f^{*3} - C_I(v^* - v_f^*) \tag{204}$$

This in turn implies that $J_W^* < 1$ for all time $t \in [0, t_f]$. Note that H^* is a class C^2 function of v^*. Therefore, to show the uniqueness of v^*, it suffices to verify the Legendre Clebsch condition $H_{vv}^* > 0$. From (183) and (10),

$$H_{vv}^* = S_{FC}(1 - J_W^*)\frac{3C_0S^2\rho^2v^{*4} + 4C_2W^2}{v^{*3}\rho S} \tag{205}$$

Because it was just shown that $J_W^* \in [0, 1)$, equation (205) implies that $H_{vv} > 0$ and so v^* is unique.

To show that $J_W^* > 0$, note that $J_W^* < 1$ together with $\dot{J}_W^*$ given by (187) imply that $\dot{J}_W^* < 0$ for all time $t \in [0, t_f]$. Furthermore, $\dot{J}_W^* < 0$ and $J_W^*(t_f) = 0$ (see (199)) and the continuity of J_W^* imply that $J_W^* > 0$ for all time $t \in [0, t_f)$.

Finally, note that (198) is well defined if $F_{1,3,1,3} \geq 0$ since, by the definition of F in (181), $F_{1,3,1,3}(v^*) < F_{1,3,1,1}(v^*)$. If $v^* \geq 3v_f$, then

$$F_{1,3,1,3}(v^*) = 2C_0 S S_{FC} \rho v^* v_f^{*3} + C_I v^* - 3C_I v_f > 0$$

If $v^* < 3v_f^*$, note that

$$\frac{\partial F_{1,3,1,3}(v^*)}{\partial C_I} = v^* - 3v_f^* < 0$$

Therefore, by the continuity of $F_{1,3,1,3}(v^*)$ in C_I, and the assumption $C_I < C_{I_c}$,

$$F_{1,3,1,3}(v^*) > F_{1,3,1,3}(v^*)|_{(C_I = C_{I_c})}$$

$$= 3C_0 S S_{FC} \rho v_f^{*3}(v^* - v_f^*) \geq 0 \tag{206}$$

since $v^* \geq v_f^*$ which follows from $\dot{v}^* < 0$ (see Lemma 22) and the continuity of v^* for all time $t \in [0, t_f]$. □

Lemma (23) introduced the constraint $C_I < C_{I_c}$. It may appear that this constraint limits the application of the lemma. In fact the constraint is a necessary condition for the existence of a solution to the OCP (178). This is illustrated in the following remark:

Remark 24. *Suppose that $C_I \geq C_{I_c}$. Solving (185) for J_x^*, replacing the result in (184), and evaluating at the final time when $W = W_f, v^* = v_f^*, J_W^* = 0$ (see (199)) yields*

$$C_I - \frac{S_{FC}(C_0 S^2 \rho^2 v_f^{*4} - 4C_2 W_f^2)}{\rho S v_f^*} = 0 \tag{207}$$

If $C_I = C_{I_c} + \delta$ where $\delta \geq 0$, then (207) reduces to

$$\frac{4C_2 S_{FC} W_f^{*2}}{\rho S v_f^*} + \delta = 0$$

If $\delta > 0$ that is, if $C_I > C_{I_c}$ then this equation is unsolvable. If $\delta = 0$, that is, if $C_I = C_{I_c}$ then the only solution is $W_f^* = 0$ which cannot happen as $W(t) > 0$ for all $t \in [0, t_f]$. Therefore, if $C_I \geq C_{I_c}$ then the OCP (178) has no solution.

4.3 The Maximum Range Solution

This section uses the results of Lemmas 21, 22, and 23 to determine the optimal trajectory that minimizes the OCP (178) when $C_I = 0$ and presents an equivalent expression for v_{MR}^W in (182) denoted by v_{MR}^x that is given in terms of position instead of weight. This section also presents expressions for the optimal final time at TOD, final weight, and minimal DOC when $C_I = 0$.

Theorem 25. *Let $C_I = 0$ for the OCP in (178). Then, the following must hold:*

(1) The optimal speed that minimizes (178) for $C_I = 0$ is given in terms of position by

$$v_{MR}^x(x) = v_{MR}^W(W_c)e^{-xS_{FC}\sqrt{C_2C_0}} \tag{208}$$

where $v_{MR}^W(W)$ is given by (182)

(2) The optimal final time at TOD for $C_I = 0$ is given by

$$t_{f,MR} = \frac{v_{MR}^W(W_c) - v_{MR}^x(x_d)}{v_{MR}^W(W_c)v_{MR}^x(x_d)\sqrt{C_0C_2}S_{FC}} \tag{209}$$

(3) The optimal final weight at TOD for $C_I = 0$ is given by

$$W_{f,MR} = \frac{1}{2}\Big(v_{MR}^x(x_d)\Big)^2 \rho S\sqrt{\frac{C_0}{C_2}} \tag{210}$$

(4) The minimal cost V^* that solves (178) for $C_I = 0$ is

$$V^* = W_c - W_{f,MR} \tag{211}$$

(5) Let W_{min} denote the minimal allowable weight of a turboprop. Then, the minimal allowable

speed is given by $v^W_{MR}(W_{min})$, and the maximum possible range is given by

$$x_{d,max} = \frac{1}{S_{FC}\sqrt{C_0 C_2}} \ln\left(\frac{v^W_{MR}(W_c)}{v^W_{MR}(W_{min})}\right) \tag{212}$$

Proof. The optimal initial speed $v^*(t=0)$ is given by $v^W_{MR}(W_c)$. From the dynamics of x in (178),

$$x = \int_0^t v dt = \int_{v^W_{MR}(W_c)}^{v^*} \frac{v}{\dot{v}^*(v)} dv \tag{213}$$

Replacing (198) in (193) yields an expression for $\dot{v}^*$ given in terms of v^* by:

$$\dot{v}^*(v^*) = -\sqrt{C_2 C_0} S_{FC} v^{*2} \frac{\sqrt{F_{1,3,1,1}(v^*) F_{1,3,1,3}(v^*)}}{F_{1,3,1,3/2}(v^*)} \tag{214}$$

Setting $C_I = 0$ in (214) yields

$$\dot{v}^*(v^*) = -\sqrt{C_0 C_2} S_{FC} v^{*2} \tag{215}$$

Replacing (215) in (213), evaluating the integral, and solving for v^* yields (208). The result (208) implies that the optimal final cruising speed v^*_f is given by $v^x_{MR}(x_d)$. To prove the result (209), note

$$t_f = \int_0^{t_f} 1 dt = \int_{v^W_{MR}(W_c)}^{v^x_{MR}(x_d)} \frac{1}{\dot{v}^*(v)} dv \tag{216}$$

Replacing (215) in (216) and evaluating the resulting integral yields (209). The result (210), follows from replacing $C_I = 0$ in (200). Setting $C_I = 0$ in (178) yields

$$V^* = W_c - W^*_f = W_c - W_{f,MR}$$

Replacing $W_{f,MR}$ given by (210) proves (211).

Finally, the result (212) follows from replacing (215) in (213) and evaluating the integral from the initial maximum range speed $v^W_{MR}(W_c)$ to the minimum possible final speed $v^W_{MR}(W_{min})$. □

4.4 Comparison of Turboprop and Turbojet OCPs

This section compares the turboprop and turbojet OCPs, and develops a transformation Φ between them. The transformation developed in this section motivates the suboptimal approximation of the optimal turboprop cruise speed presented in the following section.

A note on notation: This section will compare speeds and costs-to-go for turbojet and turboprop aircraft. To avoid confusion, supplemental notation will be used when comparing these values. This notation is summarized in table 4.1.

	Turbojet	Turboprop
Optimal speed	v_J^*	v_P^*
Optimal initial speed	$v_{J_c}^*$	$v_{P_c}^*$
Optimal final speed	$v_{J_f}^*$	$v_{P_f}^*$
Optimal cost-to-go	J_J^*	J_P^*
Optimal total cost	V_J^*	V_P^*

Table 4.1: Supplemental notation

Recall from equation (117), that the optimal cruising speed of a turbojet aircraft is

$$v_J^* = \sqrt{\frac{Q + \sqrt{12C_0C_2S_{FC}^2W^2 + Q^2}}{\rho S C_0 S_{FC}}}$$

where

$$Q = \frac{C_I}{(1 - J_{J_W}^*)}$$

and $J_{J_W}^*$ is given by (95). Unfortunately, solving the turboprop necessary condition (185) for $J_{P_x}^*$ and replacing the result in the turboprop HJB equation (184), does not yield a biquadratic in v_P^*. However, rearranging terms results in

$$v_P^* = \frac{\sqrt{2}}{2}\sqrt{\frac{R + \sqrt{16C_0C_2S_{FC}^2W^2 + R^2}}{\rho S C_0 S_{FC}}} \tag{217}$$

where

$$R = \frac{C_I}{v_P^*(1 - J_{P_W}^*)} \tag{218}$$

and $J_{P_W}^*$ is given by (197). Consider the transformation

$$\Phi : \big(S_{FC}, C_2, C_I, v_J^*\big) \to \left(S_{FC} v_{J_f}^*, \frac{C_2}{3}, \frac{C_I}{2}, v_P^*\right) \tag{219}$$

Then, replacing (95) in (117) and (197) in (217), it holds that

$$v_P^*(S_{FC}, C_2, C_I, v_{P_f}^*, W) = v_J^*\big(\Phi(S_{FC}, C_2, C_I, v_{J_f}^*), W\big) \tag{220}$$

Therefore, an optimal cruising speed for a turboprop in therms of weight may be obtained from the optimal cruising speed for a turbojet in terms of weight via a transformation of constant parameters. The equation (220) motivates the theorems presented in the following sections.

4.5 Approximation of the Optimal Speed

The purpose of this section is to present a suboptimal analytic expression for the optimal cruising speed of a turboprop aircraft that solves the OCP (178).

Theorem 26. *Let $C_{I_c} = \frac{1}{2}G_0$. Assuming $C_I < C_{I_c}$, the speed v^* that minimizes the OCP (178) can be approximated using v_1 given in terms of W and the optimal final speed v_f^* by*

$$v^* \approx v_1(W, v_f^*) = \frac{\sqrt{2}}{2} v_f^* \sqrt{\tilde{R} + \sqrt{\frac{64 C_0 C_2 S_{FC}^2 v_f^{*2} W^2}{G_0 G_1 + C_I^2} + \tilde{R}^2}} \tag{221}$$

where

$$\tilde{R} = \frac{G_{(-1)} C_I}{G_0 G_1 + C_I^2}$$

The maximum error of the approximation occurs at the initial time because:

$$|v^* - v_1(W, v_f^*)| \leq |v_c^* - v_1(W_c, v_f^*)| \tag{222}$$

Proof. The results of this theorem follow from Theorem 10 and equation (220) □

Remark 27. *The approximation $u_1(W, u_f^*)$ given in (221) reduces to the maximum range speed $u_{MR}^W(W)$ given in (182) when $C_I = 0$.*

In chapter 3, the optimal final cruising speed was given in terms of the optimal initial cruising speed by (154) which was derived from $v^*(W, J_W^*)$ in (76). The expression (76) was in turn derived (in Villarroel and Rodrigues (2016)) by solving the turbojet necessary condition (83) for J_x^* and replacing the result in the equation (82). The resulting equation was shown to be a biquadratic in v^* the solution to which is (76). Performing the same process for the turboprop case yields a fourth order polynomial equation in v^* that is not biquadratic. Though this equation can be solved for v^* in terms of W and J_W^*, the solution is too long to repeat. Furthermore, while the equation (131) can be solved for v resulting in $v^*(x)$ for given in (129) for a turbojet aircraft, the equivalent integral expression for a turboprop:

$$x_d - x = \int_0^{t_f} v^* dt = \int_{v_c^*}^{v_f^*} \frac{\tau}{\dot{v}^*(\tau)} d\tau \tag{223}$$

where $\dot{v}^*$ is given by (214), cannot be solved for v^*. Therefore, no exact expression of the optimal speed in terms of x can be found for a turboprop aircraft.

The approximation v_1 in equation (221) is given in terms of the state W, and the optimal final cruise speed v_f^*. Therefore, a method by which v_f^* can be found must be determined. The following section presents an implicit definition of v_f^* and v_c^* as the solutions to a system of two equations.

4.6 Determining v_f^*, v_c^*, t_f^*, W_f^*, and V^*

The objective of this section is to present a method by which v_f^* and v_c^* can be determined, and to use the values of v_f^*, v_c^* to determine the optimal final time at TOD t_f^*, the optimal final weight W_f^* and the minimal DOC V^* that solves (178).

Theorem 28. *The optimal initial and final cruise speeds that minimize the OCP (178) are defined* *implicitly as the solutions to the following system of equations:*

$$\frac{C_I}{2} = \frac{v_c^{*2} G_0 E_4^2 \left(2 + \sqrt{A_1 + \left(\frac{E_{-4}}{E_4}\right)^2}\right)^2}{v_f^{*2} E_{4/3}^2 \left(1 + \sqrt{A_1 + \left(\frac{E_{-4}}{E_4}\right)^2}\right)} \tag{224a}$$

$$G_0 = C_I(1 + \sqrt{1 + A_2}) \tag{224b}$$

where

$$\begin{aligned} A_1 &= \frac{4C_0 C_2 S_{FC} v_f^* x_d E_{4/3} \left(S_{FC} v_f^* x_d E_{4/3} - 4SW_c \rho v_c^{*3}\right)}{(v_c^* E_4)^2} \\ A_2 &= \frac{4C_0 C_2 \left(S_{FC} v_f^* x_d E_{4/3} - 2SW_c \rho v_c^{*3}\right)^2}{(v_c^* E_4)^2} \\ E_\beta &= C_0 S^2 \rho^2 v_c^{*4} - \beta C_2 W_c^2 \end{aligned} \tag{225}$$

Proof. The proof follows from equation (220), the proof of Corollary 2, and Theorem 16. □

Theorem 28 presents a system of equations which can be solved numerically to obtain values for v_c^* and v_f^*. The values of the constants v_f^*, v_c^* need only be computed once before flight. An algorithm written in Maple (TM) code is presented in Procedure 3 of Appendix B. The algorithm uses Newton's method to quickly solve the system of equations (224a), (224b) to within an acceptable error. The pseudocode for the Maple procedure is presented in Algorithm 3, which takes

$$\left(SET = \{C_I, C_0, C_2, S, S_{FC}, \rho, W_c, x_d\}, ERROR\right)$$

as its argument, and returns v_f^*, v_c^*.

Algorithm 3 Determine u_f^*, u_c^* for a Turboprop

Require: $SET = \{C_I < C_{I_c}, C_0, C_2, S, S_{FC}, \rho, W_c, x_d\}$, ERROR$> 0$

vfEQ := Equation (154);

vf[0] := 0; ▷ declare a first guess of the optimal final cruising speed of a turboprop

vf[1] := 1; ▷ declare a second guess of the optimal final cruising speed of a turboprop

i := 2;

while |vf[i-1]-vf[i-2]| >ERROR **do**

  TransitionSet[i] := $\{\frac{1}{2}C_I, C_0, \frac{1}{3}C_2, S,$ vf[i-1]$S_{FC}, \rho, W_c, x_d\}$ ▷ $\Phi(SET)$;

  vc[i] := Solve (155) for a turbojet with parameters = TransitionSet[i];

  vf[i] := vfEQ evaluated with v_c =vc[i], and parameters = TransitionSet[i];

  i := i+1;

end while

  return optimal initial turbojet speed = vc[i], optimal final turbojet speed = vf[i];

The following theorem presents expressions for the optimal final time, weight, and minimal DOC for a cruising turboprop in terms of v_c^*, v_f^*.

Theorem 29. *Let $C_{I_c} = \frac{1}{2}G_0$. Assuming $0 < C_I < C_{I_c}$, then the following hold*

- *Time as a function of the optimal speed v^* is given by*

$$t = \zeta(v^*) - \zeta(v_c^*) \tag{226}$$

where

$$\zeta(v^*) = -\frac{\sqrt{F_{1,3,1,1}(v^*)F_{1,3,1,3}(v^*)}}{2C_I S_{FC}\sqrt{C_2 C_0} v^* v_f^*} \tag{227}$$

- *The optimal final time at TOD is given by*

$$t_f^* = \zeta(v_f^*) - \zeta(v_c^*) \tag{228}$$

- *The optimal final weight at TOD is given by (200).*

- *The minimal DOC is given by*

$$V^* = W_c - \frac{1}{2}\sqrt{\frac{S\rho v_f^*(C_0 S S_{FC}\rho v_f^{*3} - C_I)}{C_2 S_{FC}}} + C_I\big(\zeta(v_f^*) - \zeta(v_c^*)\big) \tag{229}$$

Proof. Consider the identity

$$t = \int_0^t 1 d\tau = \int_{v_c^*}^{v^*} \frac{1}{\dot{v}^*(\tau)} d\tau \tag{230}$$

Replacing $\dot{v}^*$ with (214) in (230) and evaluating the integral yields (226). The result (228) follows from (226). Finally, (229) is obtained by replacing (200) and (228) in (180). □

4.7 Algorithm for Determining the ECON mode-Optimal Turboprop Cruise Trajectory

This chapter has provided a novel analytic expression $v_1(W, v_f^*)$ (see 221) for the optimal cruising speed of a turboprop aircraft that minimizes the OCP (178) and reduces to the maximum range speed $v_{MR}^W(W)$ in (182) when $C_I = 0$. Furthermore, an upper bound on the error of the estimate v_1 was provided in (222) in terms of v_c^*. This chapter has also provided the complete maximum range solution that solves the OCP (178) when $C_I = 0$ (see Theorem 25).

Expressions for the optimal final time, final weight and minimal DOC when $C_I > 0$ are given in terms of v_c^*, v_f^* by Theorem 29. Theorem 28 and Algorithm 3 provide the means by which v_c^*, v_f^* can be determined. The complete methodology outlined in this chapter is summarized in Algorithm 4:

Algorithm 4 Determining the optimal turboprop cruise trajectory

Require: $SET = \{C_I, C_0, C_2, S, S_{FC}, \rho, W_c, x_d\}$

if C_I=0 **then**

Use Theorem 25 for the complete maximum range solution

else

Determine v_c^*, v_f^* using Algorithm 3.

if $C_I \geq C_{I_P} = \frac{1}{2}G_0$ **then** no solution exists

else

Replace v_f^* in (221) to obtain $v^*(W) \approx v_1(W, v_f^*)$

Test the error of the approximation $u_1(W, u_f^*)$ by replacing v_f^*, v_c^* in (222):

$$|v^*(W) - v_1(W, v_f^*)| < |v_c^* - v_1(W_c, v_f^*)|$$

Replace v_f^*, v_c^* in (228) to obtain t_f^* ▷ Optimal final time at TOD

Replace v_f^* in (200) to obtain W_f^* ▷ Optimal weight at TOD

Replace v_f^*, v_c^* in (229) to obtain V^* ▷ Minimal DOC

end if

end if

4.8 Numerical Example

This section uses Algorithm 4 to approximate the optimal cruising speed, final cruise time, final cruise weight and minimal DOC for the Super King Air 350C. The aircraft parameters for the King Air A350 were measured directly from flight simulation data obtained courtesy of *TRU Simulation and Training*.

It was mentioned in chapter 1, that no FMS exists for turboprop aircraft. In order to optimize flight performance, pilots refer to printed look-up tables that provide suggested true air speeds based on weight and altitude. This section will compare the speeds suggested by Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015) against the speeds proposed by Algorithm 4. The next section will detail how the aircraft parameters of the KA were collected and will introduce the suggested TAS at each altitude and weight provided by Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015). Once the aircraft parameters have been obtained, Section 4.8.2

will consider an example flight and analyse the cost of flying at the speed proposed in Algorithm 4 against the cost of flying at the speeds suggested in Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015).

4.8.1 Obtaining Aircraft Parameters and Suggested Speeds

An excerpt of the suggested speeds and resulting fuel flow rates of reference Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015) are summarized in Table 4.2

$W \rightarrow$	14,000 lbs		13,000 lbs		12,000 lbs		11,000 lbs	
Altitude (ft)	TAS (KTS)	f (LBS/HR)	TAS (KTS)	f (LBS/HR)	TAS (KTS)	f (LBS/HR)	TAS (KTS)	f (LBS/HR)
0	256	1064	257	1062	258	1062	258	1062
2000	261	1040	262	1040	263	1038	263	1038
4000	266	1016	267	1016	268	1016	268	1016
6000	272	994	272	994	273	994	274	994
8000	277	976	278	976	278	976	279	976
10,000	282	958	283	956	284	956	285	956
12,000	288	940	289	940	290	940	291	940
...								
35,000	273	468	281	470	287	472	292	474

Table 4.2: Excerpt of data available in Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015).

Note that a speed is suggested in Table 4.2 that is a function of altitude and weight, but that does not depend on a cost index. For each speed, a resulting fuel flow-rate, f, is also provided.

Using a flight simulator at TRU Simulation and Training, it was determined that the wing area of a King Air 350C is $S = 287.9346\ ft^2$. The simulation was run at a constant altitude of $10,000\ ft$ for 21 miles. After 21 miles, the aircraft climbed to $20,000\ ft$ over 15 miles. At 36 miles, the aircraft cruised again at a constant altitude of $20,000\ ft$ for an additional 15 miles. During the simulated flight, the following data was collected: t, W, C_D, ρ, v.

To obtain the parameters of the aircraft, it was noted that by (9), the coefficient of drag C_D is an affine function of ϵ given by

$$C_D = C_0 + C_2\epsilon \tag{231}$$

where

$$\epsilon = \left(\frac{2W}{\rho S v^2}\right)^2$$

Noting that ϵ is unit-less and may be obtained from the data collected, a plot of $C_D(\epsilon)$ was produced, and a linear trend-line was added. The slope of the trend-line is, by (231), the value C_2, and the $y-$intercept is C_0. Figure 4.1 illustrates the recorded coefficient of drag as a function of ϵ and the affine trend-line.

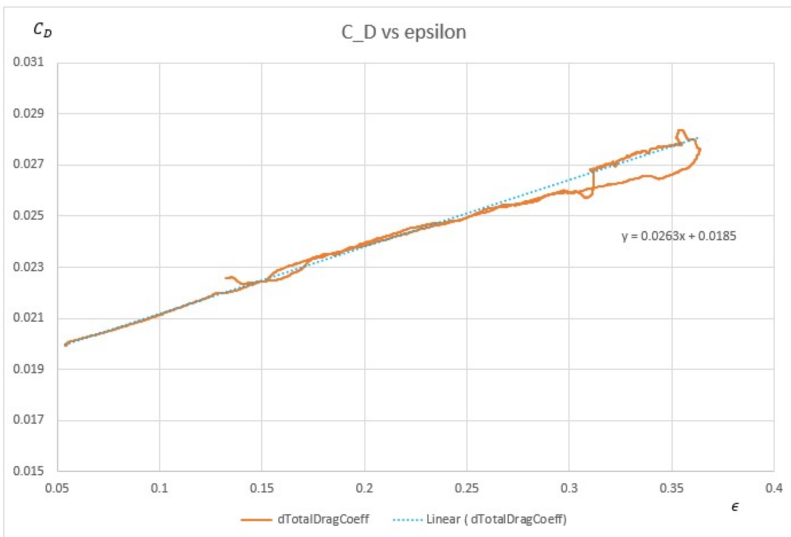

Figure 4.1: $C_D(\epsilon)$ (orange) and the linear trend-line (blue) over the entire simulated flight

Figure 4.1 suggests that the coefficients of parasitic and induced drag are relatively constant despite changes in altitude, and have the values

$$C_0 = 0.0185, \;\; C_2 = 0.0263 \tag{232}$$

From the dynamics of W in (178), the specific fuel consumption is given by

$$S_{FC}(h_c, W_i) = \frac{\dot{W}_i}{\left(\frac{1}{2}C_0 S \rho v_i^2 + \frac{2W_i^2 C_2}{\rho S v_i^2}\right) v_i}, \; i = 1..4 \tag{233}$$

where $W = \{14000, 13000, 12000, 11000\}$ and $\dot{W}_i, v_i$ are the resulting values of fuel flow rate and speed at altitude h_c associated with weight W_i in Table 4.2. Therefore, at each weight increment in Table 4.2, the value of S_{FC} may be determined as a function of h_c for C_0, C_2 given in (232). The results are summarized in Figure 4.2:

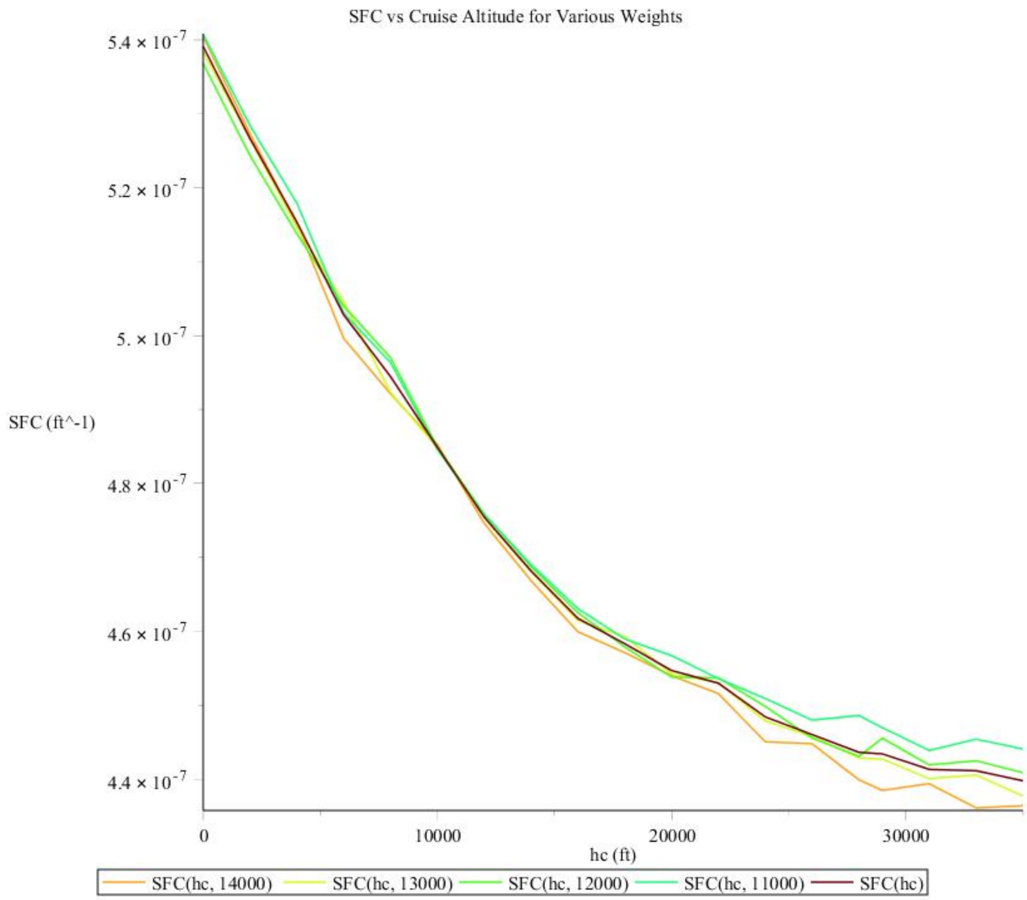

Figure 4.2: The value of S_{FC} calculated from (233) for $\dot{W}, v$ given in Table 4.2 for varying weights

Figure 4.2 suggests that S_{FC} is a function of altitude, but remains relatively constant at constant altitude (maximum of 1.7 % spread). The value of $S_{FC}(h_c)$ is taken to be the average S_{FC} at each altitude:

$$S_{FC}(h_c) = \frac{1}{4}\sum_{i=1}^{4} S_{FC}(h_c, W_i) \tag{234}$$

and is the red line in Figure 4.2. The values of $S_{FC}(h_c)$ are summarized in Table 4.3:

h_c (ft)	$S_{FC}(h_c)$ $(\times 10^{-7}\ ft^{-1})$
0	5.39
2000	5.267
4000	5.15
6000	5.03
8000	4.94
10,000	4.85
12,000	4.75
14,000	4.62
16,000	4.62
18,000	4.58
20,000	4.55
22,000	4.53
24,000	4.48
26,000	4.46
28,000	4.44
29,000,	4.43
31,000	4.41
33,000	4.41
35,000	4.40

Table 4.3: S_{FC} computed from (234) for various altitudes

To validate the aircraft model, the weight trajectory of the simulated flight is graphically compared (see Figure 4.3) to the theoretical (modeled) weight trajectory obtained from the dynamics of W in (178) where $S = 287.9346\ ft^2$, C_0, C_2 are given in (232), and S_{FC} is in (4.3). Note that Figure 4.3 suggests that the maximum error in the weight trajectories occurs at the final time and has a maximum value of 0.35%.

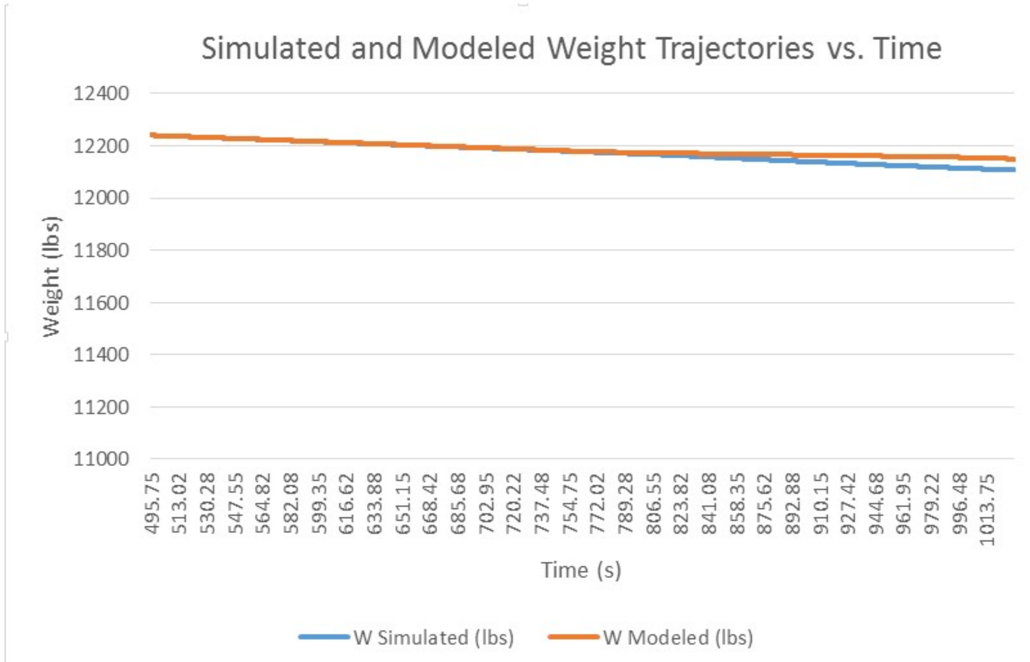

Figure 4.3: Simulated and Predicted weight trajectories over time

The parameters of the aircraft have been obtained and validated. The next section compares the DOC for two different speeds: the speed suggested in Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015), and the speed obtained using Algorithm 4.

4.8.2 Example Flight and Cost Savings

Consider a KA 350C with the following mission and aircraft parameters

$S = 287.9346\ ft^2$	$C_0 = 0.0185$	$C_2 = 0.0263$
$C_I = 0.1\ lbs/s$	$h_c = 10000\ ft$	$S_{FC} = 4.85 \times 10^{-7}\ ft^{-1}$
$\rho = 0.001756\ slug/ft^3$	$W_c = 14000$	$x_d = 1580000\ ft$

Table 4.4: King Air 350C aircraft parameters and example mission parameters

Note that the values C_0, C_2 are given by (232), and S_{FC} by (4.3) at 10000 feet. According to 4.2, the suggested TAS is $v = 282\ kts = 475.96\ ft/s$ and $\dot{W} = 958\ lbs/hr = 0.261\ lbs/s$. At this constant speed and fuel flow rate, the final time and weight are given by

$$t_{f,POH} = \frac{1580000\ ft}{475.96\ ft/s} = 3319.6\ s$$

with a final weight

$$W_{f,POH} = W_c + \dot{W} t_{f,POH} = 14000 - 0.261(3319.6) = 13116.61\ lbs$$

Therefore, by (180) for any cost index C_I, the DOC associated with the speed suggested in Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015) at 10000 feet is

$$DOC_{POH}(C_I) = W_c - W_{f,POH} + C_I t_{f,POH} = 883.38 + C_I 3319.6\ lbs \tag{235}$$

Thus, for a cost index $C_I = 0.1$, the DOC is given by

$$DOC_{POH}(0.1) = 1215.86\ lbs$$

Using Algorithm 4 results in

$$\begin{aligned} v_f^* &= 325.622\ ft/s \\ v_c^* &= 328.875\ ft/s \\ t_f &= 3943.436\ s \\ W_f &= 13571.833\ lbs \\ V^* &= DOH_{THEO} = 822.5101215.\ lbs \end{aligned} \tag{236}$$

Therefore, the savings is given by

$$DOC_{POH} - DOC_{THEO} = 393.350$$

If the cost of fuel is assumed to be $\$0.24/lb$ (See *Fuel Price Analysis* (2017)), this results in a savings of $\$94$. This cost savings are a substantial improvement over the POH for two reasons. First consider that for example the final position is around 300 miles (roughly the distance from Montreal to Toronto). Therefore, at only two round trips per day between Montreal and Toronto, the cost savings amounts to $\$137,240$ per year. Second, consider the amount of fuel saved: $-W_{f,POH} + W_{f,THEO} = 455.22lbs$. At $21.1lbs$ of CO_2 per gallon of jet fuel (see *Carbon Dioxide Emissions Coefficients* (2015)), and $6.71lbs$ of jet fuel per gallon of jet fuel (see *Handbook of Products* (2000)), the savings of 455.22 pounds of jet fuel is equivalent to 1499.32 pounds of $C0_2$ saved for a single trip from Montreal to Toronto. At two round trips per day, that equates to more than 2.1 million pounds of $C0_2$ saved every year for a single destination. The process of computing the cost savings $DOC_{POH} - DOC_{THEO}$ was repeated for varying values of the cost index and computed at varying altitudes for over 300 data points using a certified flight simulator courtesy of TRU Simulation and Training. The results are summarized for $C_I \in [0, 2]$ in Figure 4.4 and for higher values of C_I in Figure 4.5.

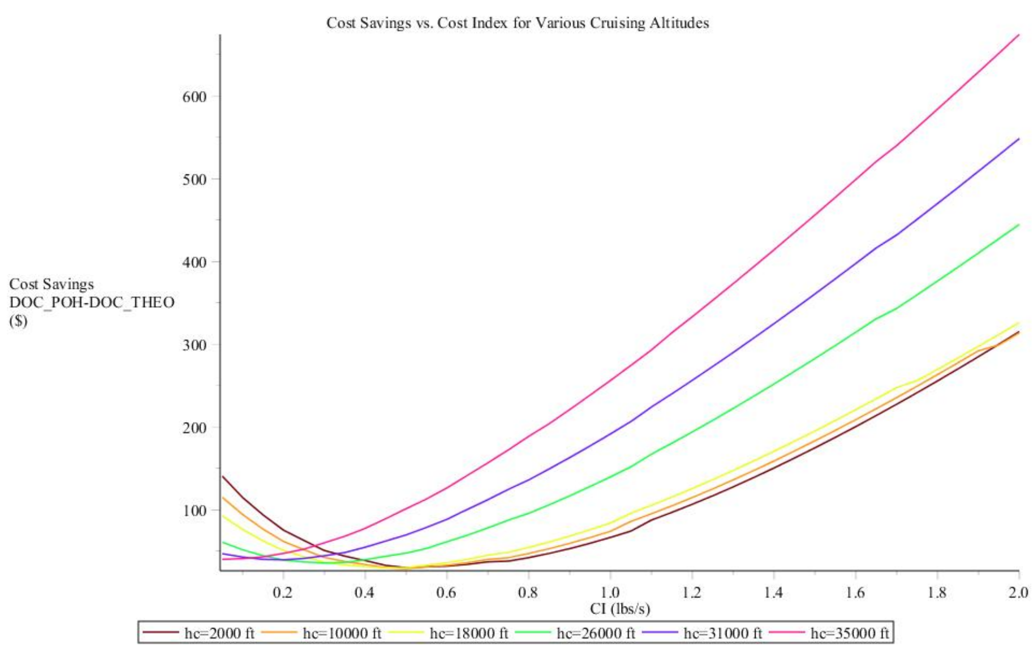

Figure 4.4: The savings in DOC $(DOC_{POH} - DOC_{THEO})C_f$ for $C_f = 0.24$ $\$/lbs$ at varying cruise altitudes and for various values of $C_I \in [0, 2]$

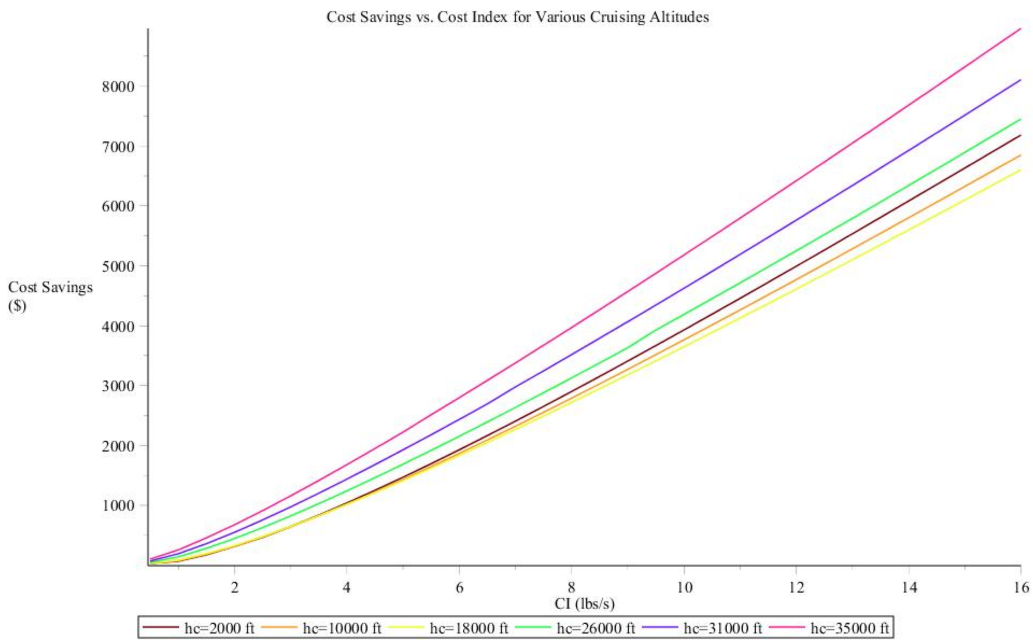

Figure 4.5: The savings in DOC $(DOC_{POH} - DOC_{THEO})C_f$ for $C_f = 0.24$ $\$/lbs$ at varying cruise altitudes and for various values of $C_I \in [0, 16]$

Figure 4.5 implies that for high values of C_I (say, $16 lbs/s$), and at high altitudes (35000 ftt), the cost savings can become as high as 8000 dollars for a 300 mile flight. Therefore, at two round trips per day, the cost savings are equivalent to more than $\$11680000$ saved each year for a single destination. Figures 4.4 and 4.5 suggest that the possible savings of cruising at speeds proposed in Algorithm 4 have the potential to be much cheaper than those suggested by Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015). It should be noted that at each cruising altitude, there is a value of C_I for which the speeds in Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015) are close to the optimal speeds. This is depicted as the minimum values of the cost savings curves in Figure 4.4.

4.8.3 Validation Against Shooting Method

According to Algorithm 4, an approximation for the optimal speed in terms of weight is given by

$$v^*(W) \approx v_1(W, 543.726) = 384.472\sqrt{0.944 + \sqrt{1.351 \times 10^{-9} W^2 + 0.890}}$$

This speed is compared graphically against the speed obtained using the shooting method in Figure 4.6. Table 4.5 compares the initial and final speeds of $v_1(W, 543.726)$ obtained using Algorithm 4 to $v^*(W)$, the theoretically optimal cruise speed obtained using the shooting method

Expression	Initial Speed (ft/s)	Initial Speed Error (%)	Final Speed (ft/s)	Final Speed Error (%)
$v_1(W, 543.726)$	546.215	7.8×10^{-3}	543.726	8.8×10^{-3}
$v^*(W)$	546.257	0.	543.679	0.

Table 4.5: Endpoint speeds of $v_1(W, 543.726)$ obtained using Algorithm 4 compared to $v^*(W)$, the theoretically optimal cruise speed obtained using the shooting method.

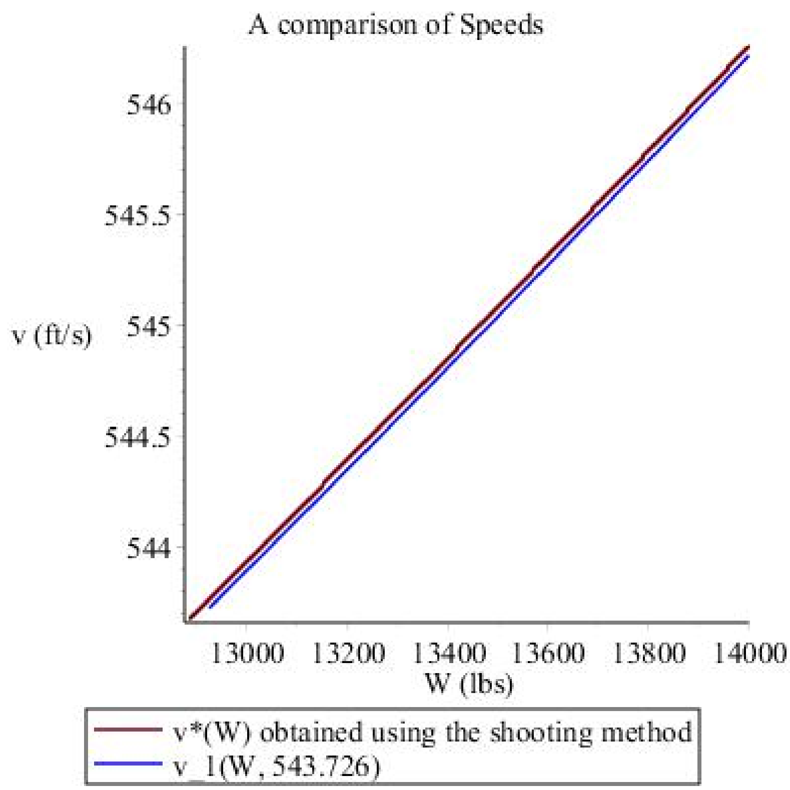

Figure 4.6: A comparison of the speed $v_1(W, 543.726)$ obtained using Algorithm 4 with the theoretically optimal speed $v^*(W)$ obtained using the shooting method.

Table 4.5 and Figure 4.6 suggest that the speed obtained using Algorithm 4 is close (within $8.8 \times 10^{-3}\%$) of the theoretically optimal speed $v^*(W)$ obtained using the shooting method.

4.9 Chapter Summary

This chapter has presented analytic expressions for

- A suboptimal approximation of the DOC-minimal cruising speed of a turboprop aircraft with error bound (see (221)),
- the optimal final cruise time (see (228)),
- the optimal final cruise weight (see (200)),
- the minimal DOC (see (229))

Algorithm 4 summarizes the turboprop ECON mode trajectory optimization techniques proposed in this thesis. To date (to the best of the author's knowledge) no FMS or analytic expression exists to address the ECON mode problem for a cruising turboprop aircraft. In order to optimize performance, pilots refer to look-up tables like those in Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015). The results of Algorithm 4 are validated against simulation results courtesy of TRU Simulation and Training in section 4.8. Figures 4.4, 4.5 illustrate the possible substantial savings of flying at speeds determined by 4 as opposed to those suggested in Beech Aircraft Corporation, Essco Aircraft Manuals and supplies (2015).

Chapter 5

A Comparison of Earlier Research

In the literature survey, it was noted that since the 1980s, six major contributors have advanced the goal of obtaining an analytic expression for the optimal speed that minimize (73), (178): Erzberger and Homer (1980), S. Wu and Guo (1994), S. Wu and Shen (1993), Burrows (1983), Burrows (1982), Miele (1959), and Villarroel and Rodrigues (2016). This chapter will use the tools outlined in section (2.4.2) and some of the results of chapters 3, and 4 to show that the work done by these major contributors yields equivalent results that differ from the results proposed in this thesis. This chapter uses the notation outlined in Table 4.1.

Begin by noting that the ECON mode optimal control problems (73) and (178) are specific cases of the more general OCP given in (61) where

Turbojet	Turboprop
$$\begin{cases} G &= v_J \\ L &= S_{FC}D \\ K &= C_I \\ u &= v_J \\ x_1 &= x \\ x_2 &= W \\ x_{1,f} &= x_d,\ x_{2c} = W_c \end{cases} \quad (237)$$	$$\begin{cases} G &= v_P \\ L &= S_{FC}Dv_P \\ K &= C_I \\ u &= v_P \\ x_1 &= x \\ x_2 &= W \\ x_{1,f} &= x_d,\ x_{2c} = W_c \end{cases} \quad (238)$$

Thus the OCPs in (73) and (178) are of the class investigated in section 2.4.2 and by (72), the optimal costs-to-go J_J^*, J_P^* given in (74) and (180) may be rewritten as

$$J_J^* = -J_{J_x}^*(x_d - x) - \int_W^{W_f^*} J_{J_W}^* \tag{239a}$$

$$J_P^* = -J_{P_x}^*(x_d - x) - \int_W^{W_f^*} J_{P_W}^* \tag{239b}$$

5.1 Work of Villarroel and Rodrigues (2016)

In order to obtain the suboptimal expression (78) from the optimal (76), the authors of (Villarroel and Rodrigues (2016)) make the assumption that $J_{J_W}^* \approx 0$ for all time $t \in [0, t_f]$. Under this assumption the cost functional (239a) reduces to

$$J_J^* = -J_{J_x}^*(x_d - x) \tag{240}$$

and the optimal cost to go is approximately an affine function of x with positive slope as $J_{J_x}^*$ is a negative constant by (94). Therefore, minimizing the cost-to-go if $J_{J_W}^* \equiv 0$, is equivalent to minimizing the slope $-J_x^*$. Similarly, under the assumption $J_{P_x}^* \equiv 0$, the cost-to-go (239b) reduces to

$$J_P^* = -J_{P_x}^*(x_d - x) \tag{241}$$

and minimizing the cost-to-go J_P^* is equivalent to minimizing the positive constant $-J_{P_x}^*$ (recall $J_{P_x}^*$ is a negative constant by (196)).

Replacing $J_{J_W}^* = 0$ in (82) and (83) and replacing $J_{P_W}^* = 0$ in (184) and (185) results in

$$S_{FC}D + J_{J_x}^* v_J^* + C_I = 0 \tag{242a}$$

$$S_{FC}D_{v_J} + J_{J_x}^* = 0 \tag{242b}$$

$$S_{FC}Dv_P + J_{P_x}^* v_P^* + C_I = 0 \tag{242c}$$

$$S_{FC}(D_{v_P}v_P + D) + J_{P_x}^* = 0 \tag{242d}$$

Solving (242b) for $J_{J_x}^*$ and (242d) for $J_{P_x}^*$ and replacing the results in (242a) and (242c) respectively yields a governing necessary condition for the turbojet and turboprop respectively. These equations are

$$C_0 S^2 S_{FC} \rho^2 v_J^{*4} - 2C_I S \rho v_J^{*2} - 12 C_2 S_{FC} W^2 = 0 \tag{243a}$$

$$C_0 S^2 S_{FC} \rho^2 v_P^{*4} - C_I S \rho v_P^{*} - 4 C_2 S_{FC} W^2 = 0 \tag{243b}$$

Note that the suboptimal velocity $v_J(W)$ in (78) was obtained by solving the biquadratic (243a) for v_J^* directly.

As stated above, the equations (243a), (243b) are a set of necessary equations for the optimality of v_J^*, v_P^* respectively and are thus one method of solving the turbojet and turboprop OCPs and will result in expressions $v_J^*(W), v_P^*(W)$. However, if $J_{J_W}^* \equiv 0, J_{P_W}^* \equiv 0$, a second set of constant solutions for v_J^*, v_P^* arise. Indeed, by (239a) and (239b), the turbojet and turboprop OCPs may also be solved by determining the speeds that minimize $-J_{J_x}^*, -J_{P_x}^*$ respectively. To this end, expressions for $J_{J_x}^*, J_{P_x}^*$ are determined by solving (242b) and (242d) for W, replacing the results in (242a) and (242c) respectively, and solving the resulting equations for $J_{J_x}^*$ and $J_{P_x}^*$ respectively:

$$J_{J_x}^* = -\frac{2}{3}\frac{C_0 S S_{FC} \rho v_J^{*2} + C_I}{v_J^*} \tag{244a}$$

$$J_{P_x}^* = -\frac{1}{2}\frac{2C_0 S S_{FC} \rho v_P^{*3} + C_I}{v_P^*} \tag{244b}$$

Note that (244a), (244b) are identical to (94), (196) where $v_{J_f}^*, v_{P_f}^*$ have been replaced with v_J^*, v_P^* respectively implying that $v_J^* = v_{J_f}^*, v_P^* = v_{J_P}^*, \quad \forall t \in [0, t_f]$ which in turn implies that v_J^*, v_P^* are constants. Therefore, if v_J^* minimizes $-J_{J_x}^*$, then

$$-\frac{\partial J_{J_x}^*}{\partial v_J^*} = \frac{2}{3}\frac{C_0 S S_{FC} \rho v_J^{*2} - C_I}{v_J^{*2}} = 0$$

Which has only one positive, real solution

$$v_J^* = v_{J,(J_W=0)}^* \triangleq \sqrt{\frac{C_I}{C_0 S S_{FC} \rho}} \tag{245}$$

To show that $v_{J,(J_W=0)}^*$ truly minimizes $-J_{J_x}^*$, take the second derivative of $J_{J_x}^*$ with respect to v_J^*

and evaluate the result at $v_J^* = v_{J,(J_W=0)}$ which yields

$$-\frac{\partial^2 J_{J_x}^*}{\partial v_J^{*2}}\Big|_{v_J^*=v_{J,(J_W=0)}} = \frac{4}{3}\frac{\left(C_0 S S_{FC}\rho\right)^{3/2}}{\sqrt{C_I}} > 0$$

Therefore, $v_{J,(J_W=0)}^*$, a constant, would minimize $-J_{J_x}^*$ and thus J_J^* if $J_{J_W}^* \equiv 0$. Similarly, it can be shown that the constant

$$v_P^* = v_{P,(J_W=0)}^* \triangleq \frac{(2C_I)^{1/3}}{2(C_0 S S_{FC}\rho)^{1/3}} \tag{246}$$

uniquely minimizes $-J_{P_x}^*$.

To summarize, if $J_{J_W}^*, J_{P_W}^* = 0$, then two solutions exist for each of the turbojet and turboprop OCPs. The first solutions are functions of weight and are determined by solving (243a), (243b) respectively. The second set of solutions are constants determined by minimizing $-J_{J_x}^*, -J_{P_x}^*$ and are given by (245) and (246) respectively. Note that if v_J^*, v_P^* are constants, then the necessary equation for turboprop optimality given by (243b) reduces to (243a) under the transformation Φ in (219). Recall that the mapping Φ transforms v_P^* into v_J^* when $J_{J_W}^* \neq 0, J_{P_W}^* \neq 0$. Therefore, it may be concluded that the mapping that relates turboprop to turbojet OCPs is preserved in the case when $J_{J_W}^* = J_{P_W}^* = 0$ if and only if v_P^*, v_J^* are constants.

The repercussions of the assumptions $J_{J_W}^* \equiv 0, J_{P_W}^* \equiv 0$ are summarized in the following table

	Turbojet	**Turboprop**
Assumption	$J_{J_W}^* \equiv 0$	$J_{P_W}^* \equiv 0$
$H\|_{X^*} = 0$ (53)	$S_{FC}D + J_{J_x}^* v_J^* + C_I = 0$	$S_{FC}Dv_P^* + J_{P_x}^* v_P^* + C_I = 0$
$H_u\|_{X^*} = 0$ (50b)	$S_{FC}D_{v_J} + J_{J_x}^* = 0$	$S_{FC}(D_{v_P}v_P^* + D) + J_{P_x}^* = 0$
Governing Equation	$S_{FC}(D - D_{v_J}v_J^*) + C_I = 0$ (243a)	$-S_{FC}(D_{v_P}v_P^{*2}) + C_I = 0$ (243b)
Cost-to-to	$J_J^* = -J_{J_x}^*(x_d - x)$	$J_P^* = -J_{P_x}^*(x_d - x)$
Speed that minimizes Cost-to-go	(245) or (78)	(246) or (221)

Table 5.1: Result of $J_{J_W}^* \equiv 0, J_{P_W}^* \equiv 0$

5.2 Previous Work in the Literature

In (Erzberger and Homer (1980), S. Wu and Guo (1994), S. Wu and Shen (1993), Burrows (1983)), Burrows (1982), the authors propose to minimize the cost functional (2) under the assumptions that the weight loss due to fuel burn is negligible, and that the energy during cruise is constant. The authors of the aforementioned work note that the cost-to-go can be decoupled into each stage of flight: climb, cruise, and descent, and, for the cruise portion, write the cost-to-go for cruise as

$$J_{cr} = \int_t^{t_f} -\dot{W} + C_I dt \equiv \int_t^{t_f} P dt = (x_d - x)\lambda \tag{247}$$

where J_{cr} denotes the cruise portion of the total flight cost, and λ denotes the cost of cruising at a given energy level E_c where

$$E_c = h + \frac{1}{2g}v^2$$

Under the change of variables

$$dt = \frac{dE}{\dot{E}} = \frac{W dE}{(T-D)V}$$

the authors reduce the states of the OCP to a single state $x = x_{up} + x_{dn}$ which is equal to the total flight distance minus the cruise distance. Therefore, under the change of variables, the Hamiltonian for the entire flight contains a single costate denoted $\psi(E)$ which is noted to be constant. It is also shown that

$$\psi(E) = \psi(E_c) = -\lambda$$

Thus the single constant costate is equal to the negative of the cruise costs, and the cruise costs are computed from

$$\lambda(E_c, v) = \frac{-\dot{W} + C_I}{v} \tag{248}$$

Thus

$$\psi = \frac{\dot{W} - C_I}{v} \tag{249}$$

The Hamiltonian of the cruising portion of the OCP (247) is

$$H = P + \psi_x v + \psi_W \dot{W}$$

where ψ_x, ψ_W are the costates. By (53), it must hold that

$$H = P + \psi_x v + \psi_W \dot{W} = 0$$

The equation is valid for turbojet and turboprop aircrafts. The assumption that the weight loss due to fuel burn is negligible, allows us to reduce the equation to

$$P + \psi_x v = 0$$

and rearranging terms yields

$$\psi_x = \frac{\dot{W} - C_I}{v} = \psi \tag{250}$$

Thus the costate ψ presented in the aforementioned papers is equal to the costate that would arise from the application of the HJB equation to (247) with the assumption that the weight loss due to fuel burn is negligible. The value of the costate in (250) is identically $J^*_{J_x}$ for a turbojet and $J^*_{P_x}$ for a turboprop. Indeed, let ψ_J, ψ_P denote the values of ψ for the turbojet and turboprop OCPs respectively, and let $\lambda_J = -\psi_J, \lambda_P = -\psi_P$. Then, for the dynamics in (73) and (178), the value of the costate for a turbojet and turboprop aircraft are thus

$$\psi_J = \frac{-S_{FC}D - C_I}{v}$$

$$\psi_P = \frac{-S_{FC}Dv - C_I}{v}$$

Therefore, minimizing (247) is equivalent to minimizing

$$\begin{aligned} \lambda_J = -\psi_J = \frac{S_{FC}D + C_I}{v} \\ \lambda_P = -\psi_P = \frac{S_{FC}Dv + C_I}{v} \end{aligned} \tag{251}$$

for the turbojet and turboprop, respectively. If v_J^* and v_P^* minimize λ_J and λ_P, respectively, then

$$(\lambda_J)_v = 0 \tag{252a}$$

$$\iff \quad C_0 S^2 S_{FC} \rho^2 v_J^{*4} - 2C_I S \rho v_J^{*2} - 12 C_2 S_{FC} W^2 = 0 \tag{252b}$$

$$(\lambda_P)_v = 0 \tag{252c}$$

$$\iff \quad C_0 S^2 S_{FC} \rho^2 v_P^{*4} - C_I S \rho v_P^{*} - 4 C_2 S_{FC} W^2 = 0 \tag{252d}$$

Therefore, the necessary conditions for minimizing the OCP (247) for a turbojet and turboprop under the assumption that the weight loss due to fuel burn is negligible are given by (252b) and (252d), respectively which are identical to the generating equations (243a) and (243b) used to develop the results of Villarroel and Rodrigues (2016). Furthermore, solving (252b), (252d) for W and replacing the results in λ_J, λ_P in (251), respectively yields $\lambda_J = -J^*_{J_x}$, $\lambda_P = -J^*_{P_x}$ for $J^*_{J_x}, J^*_{P_x}$ given in (244a), (244b). Therefore, the approach used by the authors of Erzberger and Homer (1980), S. Wu and Guo (1994), S. Wu and Shen (1993), Burrows (1983), and Burrows (1982) results in an identical minimization problem as that used in Villarroel and Rodrigues (2016), namely

$$\min_v -J_{J_x} = \min_v \lambda_J \quad \text{turbojet,}$$

$$\min_v -J_{P_x} = \min_v \lambda_P \quad \text{turboprop}$$

with identical necessary conditions (243a)≡(252b), (243b)≡(252d).

The results of this section are summarized in Table (5.2):

	Turbojet	**Turboprop**
Necessary Condition	(252b)≡(243a)	(252d)≡(243b)
Cost-to-to	$J_J^* = -J^*_{J_x}(x_d - x)$	$J_P^* = -J^*_{P_x}(x_d - x)$
Speed that minimizes Cost-to-go	(245) or (78)	(246) or (221)

Table 5.2: Result of $\dot{W} \equiv 0$

The authors of (Erzberger and Homer (1980), S. Wu and Guo (1994), S. Wu and Shen (1993), Burrows (1983), Burrows (1982)) do not provide an explicit expression for the DOC-optimal cruising speed, but as Table (5.2) suggests, the methodologies they employed and the necessary conditions they develop are identical to those in Villarroel and Rodrigues (2016).

5.3 Work of at Miele (1959)

The work of Miele (1959) was written before the development of the first FMS in the 1970s. It is therefore unsurprising that the ECON mode is not directly addressed. Instead the authors of Miele (1959) focus on performance in certain long standing modes using the method of Lagrange multipliers to tackle such aeronautical OCPs as the maximum range problem at a given altitude, and the maximum endurance problem. An ECON mode cost functional is not provided, but from maximum range at a given altitude OCP, one may develop the ECON mode problem that may be addressed by the methodologies presented in Miele (1959).

The author of Miele (1959) begins by establishing six equations that define the OCP in question. For a cruising aircraft in level flight, the first five of these equations are

$$\Phi_1 \equiv T(\pi, M, \beta) - D(\pi, M, L) = 0 \tag{253a}$$

$$\Phi_2 \equiv L - W = 0 \tag{253b}$$

$$\Phi_3 \equiv \pi - const = 0 \tag{253c}$$

$$\Phi_4 \equiv \theta = 0 \tag{253d}$$

$$\Phi_5 \equiv \omega = 0 \tag{253e}$$

where T is the thrust, M is the Mach number, β is a variable controlling the engine performance, L the lift, θ the inclination fo the velocity with respect to the horizontal axis, ω the inclination of the thrust with respect to the velocity vector, and π the ratio of the static pressure at the cruising altitude to the static pressure in the tropopause. The sixth equation is the cost functional to be maximized or minimized. In the case of maximum range at a given altitude, the cost functional provided is

$$\Psi_{MR} \equiv \frac{Ma(\pi)}{c(\pi, M, \beta)T(\pi, M, \beta)}$$

where a denotes the speed of sound, and c denotes the fuel consumption ($c = S_{FC}$ for a turbojet, $c = S_{FC}v_P$ for a turboprop). The value of cT is therefore the weight of fuel consumed. In the case of maximum range, where $C_I = 0$, the total DOC is exactly the fuel consumed and therefore, the maximum range OCP with cost functional Ψ_{MR} can be interpreted as the the maximization of the speed per unit DOC. Therefore, if $C_I > 0$, the ECON mode-minimizing cost functional can be interpreted as the minimal DOC per unit speed (i.e. the inverse if Ψ_{MR}) where the total DOC is now give by $c(\pi, M, \beta)T(\pi, M, \beta) + C_I = -\dot{W} + C_I$. We therefore define the ECON mode cost functional as

$$\Psi \equiv \frac{c(\pi, M, \beta)T(\pi, M, \beta) + C_I}{Ma(\pi)} = \frac{-\dot{W} + C_I}{v} \tag{254}$$

This cost functional is developed from the presentation of the maximum range cost functional, but is also equivalent to the familiar (251) used in section 5.2 and (240), (241) used in section 5.1. Let $\Phi_6 = \Psi$, and

$$z_1 = \pi,\ z_2 = M,\ z_3 = L,\ z_4 = \theta,\ z_5 = \beta,\ z_6 = \omega$$

Let A be a 6×6 matrix with

$$\left[A_{ij}\right] \equiv \left[\frac{\partial \Phi_i}{\partial z_j}\right]$$

Then a necessary condition for an extrema trajectory of Ψ with constraints $\Phi_1, ..., \Phi_5$ is (see Miele (1959))

$$|A| = 0$$

Constructing A and finding the determinant under the ideal engine case when $c_\beta = T_\beta = 0$, assuming cruise conditions when $T = D$, and replacing $c = S_{FC}$ for a turbojet and $c = S_{FC}v_P$ for a turboprop yields the now familiar equations (243a) for a turbojet, and (243b) for a turboprop. Therefore, analysis of the ECON mode problem for cruise using the methodology presented in (Miele (1959)) yields identical necessary conditions as those presented by the authors of (Erzberger and Homer (1980), S. Wu and Guo (1994), S. Wu and Shen (1993), Burrows (1983), Burrows (1982),Villarroel and Rodrigues (2016)).

Let

$$\Psi_J = \frac{S_{FC}D + C_I}{v_J} \tag{255a}$$
$$\Psi_P = \frac{S_{FC}Dv_P + C_I}{v_P} \tag{255b}$$

where Ψ_J, Ψ_P are derived from replacing $\dot{W}$ with $-S_{FC}D$ (for turbojet dynamics (7)) and $-S_{FC}Dv_P$ (for turboprop dynamics (8)), respectively in Ψ given by (254). Then,

$$\Psi_J = \lambda_J = -J_{J_x}$$

and

$$\Psi_P = \lambda_P = -J_{P_x}$$

by (251). Therefore, from the analysis of (Erzberger and Homer (1980)), it follows that Ψ_J, Ψ_P are constants and rearranging (255a) and (255b) and integrating yields:

$$\int_t^{t_f} \Psi v dt = \int_t^{t_f} -\dot{W} + C_I dt \\ \iff \\ \Psi(x_d - x) = J_{cr} \tag{256}$$

where J_{cr} is presented in (247) and applies to the turbojet and turboprop (for differing equations for $\dot{W}$). Therefore, the cost functional Ψ which is formed using the techniques outlined in (Miele (1959)) is identical to that presented in (Erzberger and Homer (1980)) and (Villarroel and Rodrigues (2016)), and we may replace the weight dynamics of the turbojet and turboprop in (256) to produce

$$J_J = \Psi_J(x_d - x) = -\psi_J(x_d - x) = -J_{J_x}(x_d - x) \tag{257a}$$
$$J_P = \Psi_P(x_d - x) = -\psi_P(x_d - x) = -J_{P_x}(x_d - x) \tag{257b}$$

To summarize,

	Turbojet	Turboprop
Necessary Condition	(252b)≡(243a)	(252d)≡(243b)
Cost-to-to	$J_J^* = -J_{J_x}^*(x_d - x)$	$J_P^* = -J_{P_x}^*(x_d - x)$
Speed that minimizes Cost-to-go	(245) or (78)	(246) or (221)

Table 5.3: Result of Miele (1959)

which is identical to Tables 5.1 and 5.2.

5.4 Analysis of $v_{J,(J_W=0)}^*, v_{P,(J_W=0)}^*$

The previous sections show that $v_{J,(J_W=0)}^*, v_{P,(J_W=0)}^*$ given in (245) and (246), minimize the costs-to-go

$$J_J^* = -J_{J_x}^*(x_d - x)$$

$$J_P^* = -J_{P_x}^*(x_d - x)$$

for the turbojet and turboprop, respectively which arise when $J_{J_W}^* = J_{P_W}^* = 0$. This section will show that the expressions $v_{J,(J_W=0)}^*, v_{P,(J_W=0)}^*$ not only validate the assumption $J_{J_W}^* = 0, J_{P_W}^* = 0$, but also verify the necessary conditions $H|_{X^*} = H_v|_{X^*} = 0$ (see 53, 50b).

Solving $J_{J_W}^* = 0, J_{P_W}^* = 0$ for $J_{J_W}^*, J_{P_W}^*$ given by (95), (197) respectively, yields two positive real solutions each that are summarized in Table 5.4.

Turbojet Solution for $J_{J_W}^* = 0$	**Turboprop Solution for $J_{P_W}^* = 0$**
$$\begin{cases} v_J^* &= v_{J_f}^*, \quad \text{or} \\ v_J^* &= \frac{C_I}{C_0 S S_{FC} \rho v_{J_f}^*} \end{cases} \qquad (258)$$	$$\begin{cases} v_P^* &= v_{P_f}^*, \quad \text{or} \\ v_P^* &= \frac{1}{2}\left(\sqrt{\frac{C_0 S S_{FC} \rho v_{P_f}^{*3} + 2C_I}{C_0 S S_{FC} \rho v_{P_f}^*}} - v_{P_f}^*\right) \end{cases} \qquad (259)$$

Table 5.4: Speeds in the Kernel of $J_{J_W}^*(v_J^*), J_{P_W}^*(v_P^*)$

Noting that $v_{J_f}^*, v_{P_f}^*$ are constants, we see that the solutions in (258) and (259) are also constants which implies that if $J^*_{J_W} = J^*_{P_W} = 0$, then $v^*_J = v^*_{J_f}$ and $v^*_P = v^*_{P_f}$ for all time $t \in [0, t_f]$. Therefore, from (258), $J^*_{J_W} = 0$ if

$$\begin{aligned} v^*_J = v^*_{J_f} &= \frac{C_I}{C_0 S S_{FC} \rho v^*_{J_f}} \\ \iff \quad v^*_J &= \sqrt{\frac{C_I}{C_0 S S_{FC} \rho}} = v^*_{J,(J_W=0)} \end{aligned} \tag{260}$$

and from (259), $J^*_{P_W} = 0$ if

$$\begin{aligned} v^*_P = v^*_{P_f} &= \frac{1}{2}\left(\sqrt{\frac{C_0 S S_{FC} \rho v^{*3}_{P_f} + 2C_I}{C_0 S S_{FC} \rho v^*_{P_f}}} - v^*_{P_f}\right) \\ \iff \quad v^*_P &= \frac{(2C_I)^{1/3}}{2(C_0 S S_{FC} \rho)^{1/3}} = v^*_{P,(J_W=0)} \end{aligned} \tag{261}$$

Therefore, the solution $v^*_{J,(J_W=0)}$ minimizes the the cost-to-go $J^* = -J^*_{J_x}(x_d - x)$ and verifies the assumption $J^*_{J_W} = 0$. Furthermore, the solution $v^*_{P,(J_W=0)}$ minimizes the the cost-to-go $J^* = -J^*_{P_x}(x_d - x)$ and verifies the assumption $J^*_{P_W} = 0$.

If $v^*_{J,(J_W=0)}$ is an optimal solution, as suspected, then it must hold that $v^*_{J,(J_W=0)}$ also verifies the equations $H = 0$ (see 82) and $H_{v^*} = 0$ (see 83). It was just shown that

$$J^*_W\left(v^*_{J,(J_W=0)}\right) = 0$$

Therefore, by (81)

$$H|_{v=v^*_{J,(J_W=0)}} = S_{FC} D + J^*_{J_x} v^*_{J,(J_W=0)} + C_I \tag{262}$$

$$H_v|_{v=v^*_{J,(J_W=0)}} = S_{FC} D_{v_J} + J^*_{J_x} = 0 \tag{263}$$

Solving (263) for W yields

$$W = \frac{v^{*3/2}_{J,(J_W=0)}}{2}\sqrt{\frac{S\rho(C_0 S S_{FC} \rho v^*_{J,(J_W=0)} + J^*_{J_x})}{C_2 S_{FC}}} \tag{264}$$

Replacing (264) in (262) results in

$$C_0 \rho S \big(v^*_{(J,J_W=0)}\big)^2 S_{FC} + \frac{3}{2} J^*_{J_x} v^*_{J,(J_W=0)} + C_I = 0 \tag{265}$$

Finally, replacing $v^*_J = v^*_{J,(J_W=0)}$ in $J^*_{J_x}$ given by (244a) and replacing the result in (265) yields $0 = 0$ as expected. Therefore, $v^*_{J,(J_W=0)}$ satisfies $J^*_{J_W} = 0, H = 0, H_v = 0$ and minimizes the cost to go $J^*_J = -J^*_{J_x}(x_d - x)$ that arises if $J^*_{J_W} = 0$. A similar argument can be made to show that $v^*_{P,(J_W=0)}$ satisfies $J^*_{P_W} = 0, H = 0, H_v = 0$ and minimizes the cost to go $J^*_P = -J^*_{P_x}(x_d - x)$ that arises if $J^*_{P_W} = 0$.

To summarize, though the assumptions $J^*_{J_W} = J^*_{P_W} = 0$ simplify the necessary conditions $H = 0, H_v = 0$ and, in the case of turbojet aircraft, yield a biquadratic equation (243a) that can be solved for v^*_J, they also lead to the existence of a second set of optimal constant solutions $v^*_{J,(J_W=0)}, v^*_{P,(J_W=0)}$ that have not been previously addressed in the open literature.

5.5 Summary of Earlier Research

As the above three sections suggest, the methodologies presented in (Erzberger and Homer (1980), S. Wu and Guo (1994), S. Wu and Shen (1993), Burrows (1983), Burrows (1982),Villarroel and Rodrigues (2016) , Miele (1959)) are equivalent in the case of an ideal engine and no wind. Of the authors presented here, only the authors of (Villarroel and Rodrigues (2016)) were able to determine an analytic expression for the optimal speed v_J for a cruising turbojet. This expression is obtained by solving the biquadratic equation (243a) which is shared by all papers presented in this section. Unfortunately, the equivalent expression for the turboprop (243b) is not biquadratic. Though a unique positive real solution to (243b) exists, it is too long to include and is suboptimal as it assumes that $J^*_{P_W} = 0$. It is for this reason that the analysis presented in chapter 4 is required. It is also important to note that the assumption $J^*_{J_W} = J^*_{P_W} = 0$, or equivalently, that $\dot{W} = 0$ gives rise to a second set of optimal speeds $v^*_{J,(J_W=0)}, v^*_{P,(J_W=0)}$ given in (245) and (246) respectively that are not addressed in the earlier research cited in this chapter.

The actual techniques employed by modern FMS to determine the ECON mod-optimizing cruise speed is confidential. However, it should be noted that Angelo Miele, the author of (Miele (1959)) was director of Astrodynamics and Flight Mechanics at Boeing. It is therefore suspected by the author of this thesis that the techniques employed by Boeing to determine the DOC-optimizing cruise speed is similar to the minimization of the cost functional common to all papers proposed in this chapter and summarized in Table 5.5.

	Turbojet	**Turboprop**
OCP	$J_J^* = \min_{v_J} J_J$	$J_P^* = \min_{v_P} J_P$
Cost-to-go from (Villarroel and Rodrigues (2016))	$J_J = -J_{J_x}(x_d - x)$	$J_P = -J_{P_x}(x_d - x)$
Cost-to-go from (Erzberger and Homer (1980))	$-\psi_J(x_d - x)$, $\psi_J = J_{J_x}$	$-\psi_P(x_d - x)$, $\psi_P = J_{P_x}$
Cost-to-go from (Miele (1959))	$\Psi_J(x_d - x)$, $\Psi_J = -J_{J_x}$	$\Psi_P(x_d - x)$, $\Psi_P = -J_{P_x}$
Speed that minimizes Cost-to-go	(245) or (78)	(246) or (221)

Table 5.5: Equivalence of Previous Work

The cost functionals summarized in Table 5.5 result in suboptimal cruise speeds. Recall from section 3.7, that number of computations required to determine the optimal final time, final weight, and DOC associated with these suboptimal speeds require up to 29188 times the number of computations involved in the proposed methods of this thesis.

Chapter 6

Conclusion and Future Work

This thesis has presented two novel expressions for the DOC-optimal cruising speed of a jet aircraft including one optimal and suboptimal expressions. Similarly, a suboptimal expression for the DOC-minimizing velocity of a turboprop has been derived. The techniques employ a combination of the HJB equation and PMP. Furthermore, expressions for the optimal final time, final weight, and minimal DOC are derived. Knowledge of these expressions is important as they are required outputs of an FMS and would have to be computed numerically if analytic expressions could not be found. The expressions provided in this thesis are in terms of the optimal initial and final speeds of the aircraft. For Turbojet aircrafts, the optimal final speed v_f^* is given in terms of the optimal initial speed, and the initial speed is defined implicitly. However, for a Turboprop aircraft, a novel algorithm presented here must be employed to obtain the values of the initial and final speeds u_c^*, u_f^*.

The merits of the contributions of this thesis are

- The speeds presented in chapter 3 are analytic expressions and provide a better view of the optimal speed than what exists in the open literature as they do not assume that the optimal DOC is insensitive to weight, or that the weight loss due to fuel consumption is negligible. It is shown that the suboptimal speed that exists in the open literature is necessarily lower than the optimal speed. Therefore, it could happen that though the suboptimal formulation predicts speeds lower than the Mach divergence speed, the true optimal speed lies above the Mach divergence speed. In this case, the definition of the drag force used in this paper and in (Villarroel and Rodrigues (2016)) is invalid.

- The suboptimal speed presented in chapter 4 is the first analytic expression for the optimal cruising speed of a turboprop aircraft in the open literature (to the best of the author's knowledge).

- The analytic expressions for the optimal final time at TOD, final weight and minimal DOC of cruising turbojet and turboprop aircraft are the first that do not assume that the optimal DOC is insensitive to weight, or that the weight loss due to fuel consumption is negligible (to the best of the author's knowledge).

The work of this thesis may be extended in future as follows:

- If the implicit definition for the optimal initial speed of a turbojet could be solved explicitly, that is, if (155) could be solved explicitly for v_c^*, then the complete explicit DOC-optimal trajectory of a turbojet would be known.

- The optimal initial and final speeds of a turboprop are presented in this thesis as the simultaneous solution to a system of two algebraic equations or as the result of a recursive algorithm. Determining explicit analytic expressions of these speeds is crucial to the development of a fully analytic expression for the DOC-optimal turboprop cruise speed.

- The analysis performed above does not take into account the climb or descent phases of flight. Though for long hall flights, these phases are small in comparison to the cruise phase, they are still important for the development of a full flight solution.

- It is also important to extend the results presented here to lateral flight.

- It is assumed that the optimal speed remains below the Mach divergence speed. This can be extended to transonic flight by adding a transonic drag term to the drag force in (10) and resolving the ECON mode cruise OCPs for a turbojet and turboprop.

- The analysis above does not consider the affects of wind or the efficiency of the engine. Using methods similar to those in (Miele (1959), Erzberger and Homer (1980)), it may be possible to extend the results presented in this thesis to take these factors into account.

Appendix A

Proof of Identity (153)

To prove Identity (153):

$$\frac{B_{-1}}{v_f^*} = \frac{3C_I}{v_c^*}\frac{E_4(v_c^*)}{E_{12}(v_c^*)}$$

note that it has already been seen in the proof of Lemma 6 that $\dot{J}_x^* = 0$. Therefore, it must hold that $J_x^*(0) = J_x^*(t_f)$ where $J_x^*(0)$ is the value of J_x^* evaluated at time $t = 0$ and is obtained by evaluating the necessary condition for optimality (83) at the initial time, and $J_x^*(t_f)$ is given by (94). Thus it must hold that

$$-\frac{S_{FC}E_4(1 - J_W(0)^*)}{\rho S(v^*)_c^3} = -\frac{2}{3}\frac{B_{-1}}{v_f^*} \tag{266}$$

where $J_W^*(0)$ is the function J_W^* evaluated at time $t = 0$. Replacing $J_W^*(0)$ with (95) evaluated at the initial time in (266) yields

$$\frac{B_{-1}}{v_f^*} = \frac{3}{2}\frac{E_4A_1(v_c^*)}{E_0v_c^*v_f^*} \tag{267}$$

Evaluating the expression for W^* given in (96) at the initial time when $v = v_c^*$ and rearranging terms, yields

$$\frac{1}{2}\frac{A_1(v_c^*)}{E_0v_c^*v_f^*} = \frac{C_I}{v_c^*E_{12}} \tag{268}$$

for $C_I > 0$. Combining (267) and (268) results in the identity (153)

Appendix B

Maple(TM) Code

The following procedures are written in Maple code. The first procedure in section B.1 solves an arbitrary equation where both sides of the equation are continuously differentiable in the variable to be solved using Newton's method. The second procedure found in section B.2 calls the first procedure to solve (155) for a set of aircraft and mission parameters. Section B.3 presents a procedure which returns the optimal initial and final cruise speeds of a turboprop aircraft. Finally, sections B.4 and B.5 present the shooting method procedures for a turbojet and turboprop respectively. The latter two procedures are used to validate the findings of this thesis.

B.1 Procedure 1: Newton's Method

```
###### QuickSolver Procedure.
###### Inputs   1) An equation of the form A=B (EQ)
######          2) The variable (var) in the equation A=B to be solved
######          3) An initial guess of the value of var that solves A=B (LB)
######          4) The acceptable percent error of [(A(var)-B(var))/A(var)] (ERROR)
######
###### Hidden Input 1) an initial value of counter, declared to be 0.
######           The value of counter is the number of iterations of Newton's method used.
######
###### Returns  1) The value of var such that [(A(var)-B(var))/A(var)]<=ERROR
######          2) The number of iterations used to compute the value of var
```

```
######
######  Notes: This procedure assumes that the input equation EQ has only
######         one non-numeric unknown (var) and that EQ is solvable
######         with Newton's method.

QuickSolver:=proc(EQ, var, LB, ERROR, counter:=0)
local L, R, func, PERERROR, newguess:


L:=lhs(EQ): # Decalre 'L' as the left hand side of equation EQ
R:=rhs(EQ): # Declare 'R' as the right hand side of equation EQ


func:=L-R: # Declare 'func' as the error L-R (a function of var)


PERERROR:=evalf(abs((subs(var=LB, L)-subs(var=LB, R))/subs(var=LB, L))):

# Declare 'PERERROR' as the percent error |(L(var)-R(var))/L(var)|

if  PERERROR<ERROR
then return (LB, counter): # Return initial guess LB and counter if PERERROR<ERROR


else
 newguess:=evalf(LB-subs(var=LB, func)/subs(var=LB, diff(func,var))):
# If PERERROR>ERROR, use Newton's method on func to determine a new guess (newguess)


 QuickSolver(EQ, var, newguess, ERROR, counter+1):
 #Invoke Quicksolver with LB=newguess,counter:=counter+1


end if:

end proc:
```

B.2 Procedure 2: Optimal Initial Jet Cruise Speed

```
###### vc_solver Procedure
###### Inputs 1) An input set 'SET_INPUT' containing values for
######  CI, C0, C2, SFC, S, rho, xd, Wc
######
###### Returns 1) Optimal initial speed for a cruising turbojet
```

```
######         2) The number of iterations of Newton's method used by QuickSolver
######         to obtain a maximum percent error ERROR=0.000001
######
###### Notes: This procedure assumes an initial position x(t=0)=0
######      To call the vc_solver procedure, first declare a set of the form:
######      SET:={CI=value, C0=value, C2=value, SFC=value, S=value, rho=value,
######            xd=value, Wc=value}
######      then call vc_solver(SET)

vc_solver:=proc(SET_INPUT)
local  EQvc, vcMR_SET, EQSET, vc_OUT, num_iters_out:


EQvc := (1/9)*C0*S*SFC*rho*(2*C0*S^2*rho^2*vc^5-24*C2*Wc^2*vc+
sqrt(12*C0^3*C2*S^4*SFC^2*rho^4*vc^8*xd^2+C0^2*S^4*rho^4*vc^10
-48*C0^2*C2*S^3*SFC*Wc*rho^3*vc^7*xd
-96*C0^2*C2^2*S^2*SFC^2*Wc^2*rho^2*vc^4*xd^2
+24*C0*C2*S^2*Wc^2*rho^2*vc^6+192*C0*C2^2*S*SFC*Wc^3*rho*vc^3*xd
+192*C0*C2^3*SFC^2*Wc^4*xd^2+144*C2^2*Wc^4*vc^2))^2*(C0*S^2*rho^2*vc^4
-12*C2*Wc^2)*vc/((C0*S^2*rho^2*vc^5-12*C2*Wc^2*vc
+sqrt(12*C0^3*C2*S^4*SFC^2*rho^4*vc^8*xd^2
+C0^2*S^4*rho^4*vc^10-48*C0^2*C2*S^3*SFC*Wc*rho^3*vc^7*xd
-96*C0^2*C2^2*S^2*SFC^2*Wc^2*rho^2*vc^4*xd^2
+24*C0*C2*S^2*Wc^2*rho^2*vc^6+192*C0*C2^2*S*SFC*Wc^3*rho*vc^3*xd+
192*C0*C2^3*SFC^2*Wc^4*xd^2+144*C2^2*Wc^4*vc^2))*(C0*S^2*rho^2*vc^4-4*C2*Wc^2)^2):

# Declare 'EQvc' as as f defined in (156)

vcMR_SET:=evalf(subs(SET_INPUT, sqrt(2)*3^(1/4)*sqrt(Wc)*C2^(1/4)/(C0^(1/4)*sqrt(S)*sqrt(rho)))):
# Declare 'vcMR_SET' as the maximum range speed(144a)for the parameters SET_INPUT

EQSET:=subs(vc=y, SET_INPUT, EQvc=CI):
 # Declare 'EQSET' as the equation EQvc=CI which is equation(155).

vc_OUT:=[QuickSolver(EQSET, y, vcMR_SET, 0.000001)][1]:
# Declare 'vc_OUT' as the first returned value of procedure QuickSolver with
# EQ=EQSET, var=y, LB=vcMR_SET, ERROR=.000001

   num_iters_out:=[QuickSolver(EQSET, y, vcMR_SET, 0.000001)][2]:
# Declare 'num_iters_out' as the number of iterations of Newton's method used.
# The value of num_iters_out is the second returned value of QuickSolver.
```

```
return  (initial_speed=evalf(vc_OUT), iterations=num_iters_out):

end proc:
```

B.3 Procedure 3: Optimal Initial and Final Turboprop Cruise Speed

```
###### vf_solver_PROP Procedure
######
###### Inputs 1) An input set 'SET_INPUT' containing values for
######           CI, C0, C2, SFC, S, rho, xd, Wc
######        2) An acceptable error ERROR>0 such that |uf[i]-uf[i+1]|<ERROR => Stop code
######           where uf[i] is the optimal cruising speed of a turboprop after i
######           iterations of vf_solver_PROP
######
###### Returns 1) Optimal initial cruising speed of a turboprop
######         2) Optimal final cruising speed of a turboprop
######         3) Number of iterations of required to obtain speed values.
######
###### Notes:  The procedure vf_solver_PROP assumes an initial position of 0
######         vf_solver_PROP calls vc_solver in its body.

vf_solver_PROP:=proc(SET_INPUT, ERROR)
local  vfEQ, vf, SET_IN_TRANS_1, SET_IN_TRANS_2, SET_IN_TRANS, vc, counter:



vf_EQ:=sqrt(CI+sqrt(12*C0*C2*CI^2*(SFC*xd*(C0*S^2*rho^2*vc^4-4*C2*Wc^2)-2*S*Wc*rho*vc^3)^2
       /(vc^2*(C0*S^2*rho^2*vc^4-12*C2*Wc^2)^2)+CI^2))/(sqrt(rho)*sqrt(S)*sqrt(C0)*sqrt(SFC)):

# Declare 'vf_EQ' as equation(154) the optimal final jet cruise speed
# in terms of the optimal initial jet cruise speed


vf[0]:=0:   # Declare an initial guess of the optimal turboprop final speed
vf[1]:=1:   # Declare a second guess of the optimal turboprop final speed
counter:=0: # Initiate a counter

SET_IN_TRANS_1:=subs(rhs(op(select(has, SET_INPUT, C2)))
               =1/3*rhs(op(select(has, SET_INPUT, C2))), SET_INPUT):

SET_IN_TRANS_2:=subs(rhs(op(select(has, SET_IN_TRANS_1, CI)))
```

```
            =1/2*rhs(op(select(has,SET_IN_TRANS_1, CI))), SET_IN_TRANS_1):
 #'SET_IN_TRANS_2' is SET where C2, CI have been replaced
  with Φ(C_2), Φ(C_I), for Φ in (219)

for i from 2 while abs(vf[i-1]-vf[i-2])>ERROR do
 SET_IN_TRANS[i]=subs(rhs(op(select(has, SET_IN_TRANS_2, SFC)))
                =vf[i-1]*rhs(op(select(has, SET_IN_TRANS_2, SFC))), SET_IN_TRANS_2):
 #SET_IN_TRANS[i] is  Φ(SET)

 vc[i]:=rhs(vc_solver(SET_IN_TRANS[i])[1]):  # Call vc_solver on the set  Φ_2^{-1}(SET)
 # to obtain the initial cruise speed for a jet.

 vf[i]:=evalf(subs(vc=vc[i], SET_IN_TRANS[i], vf_EQ)):
 # obtain the optimal final cruising speed of the jet from vf_EQ

 counter:=counter+1:

end do:

return (initial_speed=vc[counter], final_speed=vf[counter], iterations=counter):

end proc:
```

B.4 Procedure 4: Turbojet Shooting Method

```
###### Shooting_Method Procedure
######
###### Inputs  1) An input set 'SET_input' containing values for
######           CI, C0, C2, SFC, S, rho, xd, Wc
######          2) A guess of the optimal initial cruise speed 'v0_guess'
######          3) An acceptable increment in time 'time_step' in seconds
######          4) A value of the allowable error 'error_allowed'.  The
######             final value of Jw is computed for an initial speed.
######             The procedure is recursive and will correct the initial speed
######             until the value of the final Jw is within error_allowed
######             of 0.
######
###### Prints 1) Optimal initial speed
```

```
######        2) Optimal final speed
######        3) Optimal final weight
######        4) Final value of Jw (within error_allowed of 0)
######        5) Optimal initial value of Jw
######        7) Final position
######        8) Optimal final cost (lbs)
######        9) Optimal final time
######
###### Returns 1) An array 'OPT_TRAJ' where OPT_TRAJ[i]=[W[i], v[i]] at time i
######
###### Notes 1) This procedure assumes an initial position of x=0
######       2) This procedure assumes an initial time of t=0

Shooting_Method:=proc(SET_input, v0_guess, time_step, error_allowed)

local v_true, DRAG_out, Jwtc, x, W, v, DRAG, Jw, counter, i,
    epsilon, Jwtcnext_gen, OPT_TRAJ, j, plot1, new_v, Cost:


v_true := subs(SET_input, sqrt((CI+sqrt(12*(1-Jw_in)^2*C0*C2*SFC^2*W_in^2+CI^2))
/((1-Jw_in)*SFC*C0*rho*S))):

 # Declare 'v_true' as the optimal speed in terms of W, Jw given by (76).


DRAG_out:= subs(SET_input, (1/2)*C0*rho*S*(v_in)^2+2*C2*(W_in)^2/(rho*S*(v_in)^2)):

# Declare 'DRAG_out' as the equation for drag(10)


Jwtc := subs(SET_input, (C0*S^2*SFC*rho^2*v_in^4-2*CI*S*rho*v_in^2-12*C2*SFC*W_in^2)
/(SFC*(C0*S^2*rho^2*v_in^4-12*C2*W_in^2))):

# Declare 'Jwtc' as the initial value of Jw for a given initial speed.  The
# expression Jwtc is derived by solving(83) for Jx, replacing
# the result in (82), solving for Jw and evaluating the result
# at the initial time



x[0]:=0:  # Set the initial value of position to 0
W[0]:=subs(SET_input, Wc):  # Set the initial value of weight to Wc in SET_input

v[0]:=v0_guess:   # Set the initial speed as the guess v0_guess
DRAG[0]:=subs(v_in=v[0], W_in=W[0], DRAG_out):  # Replace v[0], W[0] in DRAG_out
```

```
Jw[0]:=subs(v_in=v[0], W_in=W[0],  Jwtc):       # Replace v[0], W[0] in Jwtc
counter:=0:

  for i from 1 while x[i-1]<=subs(SET_input, xd) do

x[i]:=x[i-1]+v[i-1]*time_step:
W[i]:=W[i-1]-subs(SET_input, SFC)*DRAG[i-1]*time_step:
Jw[i]:=Jw[i-1]+subs(SET_input, ((Jw[i-1]-1)*4*SFC*C2*W[i-1]
/(rho*S*(v[i-1])^2))*time_step):

   # The time derivative of Jw is obtained from (50b) which
```

states that $\dot{J}_W^* = -H_W = \frac{4S_{FC}C_2W(1-Jw)}{\rho S v^2}$

```
v[i]:=subs(Jw_in=Jw[i], W_in=W[i], v_true):
DRAG[i]:=subs(v_in=v[i], W_in=W[i], DRAG_out):
counter:=counter+1:

end do:
epsilon:=Jw[counter-1]:  # The final value of Jw is the error 'epsilon'


if evalf(abs(epsilon))<=error_allowed then
  OPT_TRAJ:=Array(1..counter-1):  #Construct the optimal trajectory in terms of W
  for j from 1 to counter-1 do

OPT_TRAJ[j]:=[W[j], v[j]]:

  end do:
  Cost:=subs(SET_input, (Wc-W[counter-1])+CI*(counter-1)*time_step):
  # The value of 'Cost' comes from (74)

print(v_initial=v[0], v_final=v[counter-1], W_final=W[counter-1],
Jw_initial=Jw[0], Jw_final=Jw[counter-1], x_final=x[counter-1],
COST=Cost, tf=(counter-1)*time_step):
# Print all of the desired results

  return OPT_TRAJ:


else

Jwtcnext_gen:=Jw[0]-epsilon:     #Correct the initial Jw based on the error epsilon


new_v:=subs(Jw_in=Jwtcnext_gen, W_in=W[0], v_true):
# Correct initial speed for corrected Jw


Shooting_Method(SET_input, new_v,time_step, error_allowed):
# Call the procedure again with corrected initial speed as v0_guess.
```

```
end if:
end proc:
```

B.5 Procedure 5: Turboprop Shooting Method

```
###### Shooting_Method_PROP Procedure
######
###### Inputs  1) An input set 'SET_IN' containing values for
######           CI, C0, C2, SFC, S, rho, xd, Wc
######          2) A guess of the optimal initial cruise speed 'v0_guess'
######          3) An acceptable increment in time 'time_step' in seconds
######          4) A value of the allowable error 'error_allowed'.  The
######             final value of Kw is computed for an initial speed.
######             The procedure is recursive and will correct the initial speed
######             until the value of the final Kw is within error_allowed
######             of 0.
######
###### Prints 1) Optimal initial speed
######        2) Optimal final speed
######        3) Optimal final weight
######        4) Final value of Kw (within error_allowed of 0)
######        5) Optimal initial value of Kw
######        7) Final position
######        8) Optimal final cost (lbs)
######        9) Optimal final time
######
###### Returns 1) An array 'OPT_TRAJ' where OPT_TRAJ[i]=[W[i], v[i]] at time i
######
###### Notes 1) This procedure assumes an initial position of x=0
######       2) This procedure assumes an initial time of t=0

Shooting_Method_PROP:=proc(SET_IN, v0_guess, time_step, error_allowed)

local DR, vdot, Kw_IN, v, x, W, Kw, DRAG, counter, epsilon, Kwtcnext_gen, v0_NEW,
     i, OPT_TRAJ, j, Cost:


DR:=(1/2)*C0*rho*S*v^2+2*W^2*C2/(rho*S*v^2):
```

```
# Declare 'DR' as the equation for drag(10)

vdot:=-8*C0*C2*v^4*W*S*SFC*rho/(3*C0*S^2*rho^2*v^4+4*C2*W^2):

# Declare 'vdot' as the time derivative of v in terms of v (214)

Kw_IN:=(C0*S^2*SFC*rho^2*v^4-4*C2*SFC*W^2-CI*S*rho*v)/(SFC*(C0*S^2*rho^2*v^4-4*C2*W^2)):

# Declare 'Kw_IN' as the value of Kw in terms of v and W.  The
# expression Kw_IN is derived by solving(185) for Kx, replacing
# the result in (184), and solving for Kw

v[0]:=v0_guess:
x[0]:=0:
W[0]:=subs(SET_IN, Wc):
Kw[0]:=subs(v=v[0], W=W[0], SET_IN, Kw_IN):
DRAG[0]:=subs(v=v[0], W=W[0], SET_IN, DR):

counter:=0:

    for i from 1 while x[i-1]<subs(SET_IN, xd) do

x[i]:=evalf(x[i-1]+v[i-1]*time_step):
v[i]:=v[i-1]+subs(v=v[i-1], W=W[i-1], SET_IN, vdot)*time_step:
W[i]:=W[i-1]+subs(v=v[i-1], W=W[i-1], SET_IN, -SFC*DR*v)*time_step:
Kw[i]:=subs(v=v[i], W=W[i], SET_IN, Kw_IN):
counter:=counter+1:

# Update x, v, W, Kw using ẋ = v, v̇ =vdot, Ẇ = −S_FC Dv
# and Kw is given by Kw_IN
end do:

epsilon:epsilon:=Kw[counter-1]:  # The final value of Kw is the error 'epsilon'

if evalf(abs(epsilon))<=error_allowed then
  OPT_TRAJ:=Array(1..counter-1):  #Construct the optimal trajectory in terms of W
     for j from 1 to counter-1 do

OPT_TRAJ[j]:=[W[j], v[j]]:

     end do:
  Cost:=subs(SET_IN, (Wc-W[counter-1])+CI*(counter-1)*time_step):
  # The value of 'Cost' comes from (74)
```

```
print(v_initial=v[0], v_final=v[counter-1], W_final=W[counter-1],
Kw_initial=Kw[0], Kw_final=Kw[counter-1], x_final=x[counter-1],
COST=Cost, final_time=(counter-1)*time_step):
# Print all of the desired results
```

return `OPT_TRAJ:`

else

```
Kwtcnext_gen:=Kw[0]-epsilon:  # Correct the initial value of Kw

v0_NEW:=max(op(select(is, remove(has, [solve(subs(W=Wc, SET_IN, Kw_IN)
=Kwtcnext_gen, v)], I), positive))):
# Solve Kwtcnext_gen=Kw_IN, that is, the corrected value of Kw equals Kw_IN
# for v to obtain the new corected initial speed.  Select the maximum positive
# real solution

Shooting_Method_PROP(SET_IN, v0_NEW,time_step, error_allowed):
# Call the procedure again with corrected initial speed as v0_guess.
```

end if:

end proc:

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