What algebra do Calculus students need to know?

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A Thesis

in

the Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements

for the degree of Master in the Teaching of Mathematics at

Concordia University

Montreal, Quebec, Canada

July 2017

CONCORDIA UNIVERSITY

School of Graduate Studies

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Entitled: What algebra do Calculus students need to know?

and submitted in partial fulfillment of the requirements for the degree of

Master in the Teaching of Mathematics

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Abstract

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Concordia University, 2017

Students taking a Calculus course for the first time at Concordia University are mature students returning to school after an extended period of time away from formal education, or students lacking the prerequisites to enter into a science, technology, engineering, or mathematics (STEM) related field. Thus, an introductory Calculus course is the gateway for many STEM programs, inhibiting students' academic progression if not passed. Calculus tends to be construed as a very difficult subject. This impression may be due to the fact that this course is taught in a condensed form, with limited class time, new knowledge (concept, type of problem, technique or method) introduced every week, and little practice time. Calculus requires higher order thinking in mathematics, compared to what students have previously encountered, as well as many algebraic techniques. As will be shown in this thesis, algebra plays an important role in solving problems that usually make up the final examination in this course. Through detailed theoretical analysis of problems in one typical final examination, and solutions produced by 63 students, we have identified the prerequisite algebraic knowledge for the course and the specific difficulties, misconceptions and false rules experienced and developed by students lacking this knowledge. We have also shown how the results of our analyses can be used in the construction of a "placement test" for the course – an instrument that could serve the goal of lessening the failure rate in the course, and attrition in STEM programs, by avoiding having underprepared students.

Acknowledgements

I would like to first and foremost express my sincerest gratitude to my supervisor Dr. Anna Sierpinska, for her continued guidance and supervision. This year presented itself with a few personal challenges, however with her dedication and positive outlook, we were able to complete this thesis. I would like to thank Drs. Nadia Hardy, and Harald Proppe for agreeing to take time out of their busy schedules to be a part of my thesis committee. Thank you to my boss, Dr. Jennifer McGrath for allowing me with enough flexibility at work to pursue a higher education. A thank you to my close family and friends for understanding when I was too busy with school work. I would like to thank my parents for always encouraging me to continue with school. Lastly, a great big thank you to my husband Robert, who has remained patient and supportive throughout all of my academic endeavors. Thank you!

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1 INTRODUCTION

"When a student first encounters algebra he moves from a world of specific numbers to a world of variables and reversible transformations" (Byers & Erlwanger, 1984). The switch from arithmetic calculations to algebraic manipulations can be very difficult for some students as it is the first time they are introduced to such abstract ideas. Further, they are initially taught to use letters to stand for unknowns (MacGregor & Stacey, 1997), and then are expected to eventually understand letters as variables, rather than unknowns or placeholders (Küchemann, 1981). We can acknowledge that when students are "doing" algebra, they may not be aware of the properties that justify the operations that can be performed on an expression to obtain an equivalent expression. This may lead to mistakes such as replacing $a(b + c)$ by $ab + c$ in an expression. Some students can successfully apply different algebraic techniques to solve problems, while others seem to form certain misconceptions or misunderstandings that persist throughout their studies. What does it mean for a student to be capable of performing algebraic manipulations successfully? What algebraic skills and knowledge do they possess? Does a student's high performance in algebra necessarily represent a correct understanding of algebra?

Following a college algebra course, students typically move on to a Calculus I course, where they are introduced to limits and derivatives. During my teaching experience of Calculus I for science students (MATH 203) in the summer of 2016, I started to think about whether it was important for students to "know" algebra in order to succeed in Calculus. The questions posed by students while I was solving a problem on the board, or during my office hours led me to the realization that a good number of students had difficulties with simple algebraic techniques, such as factoring a quadratic polynomial. I would take my time to explain why they could or could not perform certain operations, and would guide them to the appropriate properties learnt from past courses. These types of questions surprised me quite a bit. Students taking MATH 203 should have previously taken courses in algebra and functions (MATH 200 - Fundamental Concepts of Algebra, & MATH 201 - Elementary Functions). In these prerequisite courses, students should have learnt about basic properties of operations on real numbers, and functions. They should have learnt about factoring polynomials, taking the inverse of a one-to-one function, composition of functions, and about different kinds of functions, in particular, the rational, the exponential and the logarithmic functions. However, a good portion of the questions I received in this course were algebraic in nature, rather than Calculus content related. I thought to myself, "How will these students succeed in this course, as well as in future mathematics courses if they have difficulties with algebra? Can they truly succeed?"

On the one hand, a student may perform very well if given problems similar to ones previously encountered. But understanding *why* one could perform certain operations is a different story. For example, I would have students say "Move the constant to the other side" of an equality or inequality, and I would redirect their line of thinking to understanding why one could perform a specific operation. One does not simply *move* a term to one side of an equality or inequality, although the result of the operation one applies to both sides of the equality or inequality certainly gives the illusion of such movement. A few students confessed that they could not "do" algebra, and that finding the limit of a given function, or finding the derivative would be impossible for them to do, yet they seemed to have proper conceptual understandings of limits and derivatives. What is a teacher to do? If I did not try to address their mistakes by making the mistakes reflect in their grades, they would move on to the next course with the same algebraic misconceptions they came in with. Yet is it fair to penalize a student for their algebraic mistakes when their conceptual understanding of the content of the Calculus course appears to be correct?

Due to time constraints, I did not have enough time to go over a full algebra course. Therefore I tried my best to provide the correct reasoning for students, when asked algebraic questions. What about, however, all the students who did not ask me any questions, those that had initial misunderstandings of prerequisite material that were never addressed? I believe that these students would unfortunately exhibit these same misconceptions throughout the course, as well as towards the end of the semester. For this reason, we decided to analyze the final examination questions and students' solutions to determine what algebraic knowledge was necessary to solve them, and to what extent students appeared to possess this knowledge. We wanted to identify the algebraic mistakes and misconceptions about functions that were obstacles to students' correct solutions of the questions. Our plan was to construct, on this basis, items for a placement test to be administered at the beginning of the Calculus I course. The results of the test would be a measure of the students' preparedness for the particular Calculus I course in which the final examination was similar to the one that was used in the course I taught.

There is already a great body of literature regarding different algebraic errors and misconceptions held by students (Booth & Koedinger, 2008; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Payne &

Squibb, 1990), and how some of these misconceptions can persist over time (Booth, Barbieri, Eyer, & Paré-Blagoev, 2014).

There is also some debate as to whether algebra is *necessary* for a Calculus course for students who will not pursue a degree in mathematics but in other domains, e.g., in science or finance. Reality cannot always be modeled by compositions of elementary functions. For example, in finding the limit of a function, one can try to find or approximate the limit numerically, rather than using algebraic techniques. Also, the formal epsilon-delta definition of limits is not formally taught in this course. This decision of excluding the formal definition is consistent among many individuals, as it is believed to fall within the field of analysis, and is beyond the understanding of the majority of Calculus students (Thompson, Byerley, & Hatfield, 2013). However, even if the formal definition were taught, the majority of testing problems currently do not assess one's knowledge of this definition, rather most solutions to limit problems are solved using different algebraic techniques. Selden, Mason, and Selden (1989) reported that traditional calculus courses contained very few non-routine problems. The authors created a test containing five non-routine items, and administered it to C grade students. Most students were unable to do anything at all, thus the authors concluded that even though students pass a Calculus course, many are unable to solve non-routine problems. One such item was as follows:

> *Find values of a and b so that the line $2x + 3y = a$ is tangent to the graph of $f(x) = bx^2$ at the point where $x = 3$ (Selden, Mason, & Selden, 1989).*

Further, a study by Tallman, Carlson, Bressoud, and Pearson (2016) found that out of 150 Calculus I final exams, only 14.72% of items required some explanation by the students to demonstrate their understanding, while the majority of the exam items required only memory of procedures.

An interesting study by Hardy (2009), looked at questions in a common final examination of a College-Calculus course. It was determined that these examinations had not changed much over 6 years. The questions tended to be routine in nature, and followed unwritten norms. Reasons proposed by the author included that these examinations needed to assess a large number of students, and that the content was what the final examination committee determined to be the minimum knowledge expected to be learned by students. The author further suggests that students tend to make generalizations that are not mathematical in nature, based on their experience with routine problems. In her study, Hardy presented students with four non-routine questions about finding limits of functions, and found that students attempted these problems with unnecessary algebraic techniques. Three of the four problems could be solved with direct substitution, yet students factored the polynomials because, as one student pointed out, "they never gave me a problem that wasn't factorable". Here is one example: find $\lim_{x\to 2}\frac{x+3}{x^2-9}$. As teachers, we are trying our very best to teach our students the appropriate mathematical knowledge, yet based on the routine problems provided in the textbook, and final examinations, we are perhaps hindering our students with rote algebraic procedures.

Another interesting study, by Palmiter (1991), used a computer algebra system to teach integral Calculus to a group of students, eliminating the paper and pencil algebraic computations. These students were able to complete the course in half the amount of time compared to the traditional sections. They performed better on both the conceptual knowledge of the Calculus test, as well as on the computational test, in comparison to the traditional sections. Further, these students also performed better in subsequent traditional Calculus courses. Having de-emphasized algebraic manipulations, these students spent more time gaining conceptual knowledge of Calculus content. The results of this study are very intriguing, making me think about our current course outline, content, and assessments. Although algebra may not be *necessary* to learn Calculus, we must acknowledge that the institution in which a course is taught largely influences the course content and assessments. And at our university, the Calculus I course I taught has traditionally been strongly algebra based.

We are using the word “algebra” here as if it did not require an explanation. But this is an ambiguous term that we will discuss in the next section. In this thesis, we are including functions and their properties when referring to algebra.

This thesis has been organized into the following chapters. Chapter 2 contains the theoretical perspective of the Anthropological Theory of the Didactic, which we used to describe our influences in analyzing the students’ solutions of the final examination, keeping in mind different mathematical praxeologies, and institutional constraints. It also includes our definition of algebra, and what constitutes an algebraic activity. Chapter 3 contains our theoretical analysis of the final examination questions. Chapter 4 contains a detailed analysis of the students’ solutions to the final examination, illustrated by many examples of manifestations of different algebraic difficulties, misconceptions, and false rules in the solutions. This chapter concludes with a summary of the identified algebraic difficulties, misconceptions, and false rules. In chapter 5 we show examples of a placement test items constructed based on the results obtained in chapters 3 and 4. Lastly, in chapter 6 we provide our conclusions and recommendations stemming from this research and argue for the usefulness of testing students’ preparedness for an introductory Calculus course with a placement test.

2 Theoretical Perspective

The research reported in this thesis has been conceived from the perspective of the Anthropological Theory of the Didactic (Chevallard, 1999), abbreviated “ATD”. From this perspective, the Calculus I course and its algebraic prerequisites which are the object of our research are not regarded as abstract mathematical content but as an institutional practice, embedded in the broader institutional practices of the mathematics department where it is offered and the even broader institutional practices of a large comprehensive urban North American university, open to adults returning to study to re-orient their careers or finish their interrupted education. It is assumed that the course, as any institutional practice, has some tasks to fulfill, that it accomplishes these tasks using a set of routine techniques, and that it has developed a justificatory discourse for defending the tasks and the techniques against anybody who would want to question their choice or validity. If the task of the Calculus I course is to teach students knowledge deemed necessary to study science at the university level, then learning this knowledge is also seen as learning a certain institutional practice, characterized, again, by certain types of tasks, techniques for solving them and a particular justificatory discourse. In the particular Calculus I course that we are studying here, the types of tasks that characterize it as an institutional practice are ultimately defined by the sets of final examination problems, very similar from year to year. This is why we considered studying the final examination problems in the course as sufficient for sketching a portrait of the algebraic needs of the students.

Please note, that although we have kept in mind the perspective of ATD, we will not be using the technical terminology of the theory and its particular formalism in describing the final examination problems and students’ ways of solving them.

In this chapter, we first give some more details about ATD and its assumption of the institutional relativism of knowledge, and then discuss, from this perspective, the problems of determining what is meant by “algebra”. We conclude with a description of the meaning of algebra as assumed in this thesis.

2.1 The Anthropological Theory of the Didactic

The Anthropological Theory of the Didactic (ATD), developed by (Chevallard, 1999) and his collaborators (Barbé, Bosch, Espinoza, & Gascón, 2005), has been widely used as a theoretical framework for analyzing different aspects of mathematics education (Hardy, 2009; Sierpinska, Bobos, & Pruncut, 2011).

All educational activities, such as the preparation, implementation, and the learning of knowledge are thought of as institutionalized human practices, and it is assumed that such practices can be analysed into types of tasks the institution is called to solve, the often implicit techniques it uses to accomplish the tasks, the explicit methodology[1] it formulates to justify the techniques and teach them to novices to the practice, and the theories it invokes to justify the methodology. A description of a practice in such terms is called a "praxeology". Thus, a praxeology can be thought of as a model or theory of practice in a certain institution. For a detailed recent introduction to ATD see Bosch and Gascón (2014).

Hardy (2009) describes the practice of finding the limits of functions in a college Calculus course by several punctual mathematical praxeologies, related to particular types of such functions. One example provided was a mathematical task of evaluating the limit of a function of the following form: $\lim_{x \to c} \frac{\sqrt{P(x)} - Q(x)}{R(x)}$, where $P(x), Q(x),$ and $R(x)$ are polynomials. Further, $\sqrt{P(c)} - Q(c) = 0, R(c) = 0$, and $P(x) - [Q(x)]^2$ is a factor in $R(x)$. An example of this type of limit was provided as follows: $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x^2 - 16}$. Recognizing the indetermination of $\frac{0}{0}$ when substituting c in x, the common technique used by students was to multiply numerator and denominator by the conjugate of the numerator $(\sqrt{x} + 2)$. The numerator has a common factor with the denominator of $(x - 4)$. After simplifying, and replacing x by 4 the limit is found to be $\frac{1}{32}$. The explicit methodology used is as follows: "If two functions f and g agree in all but one value c then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$" (Hardy, 2009). Lastly, the theory justifying these steps would normally include a graph supporting the methodology used. The formal theory is an epsilon-delta proof however these types of proofs are not normally covered in such courses, thus students may not know that they even exist. Note that the techniques used by students may be their own way of solving a particular problem (or similar looking problems), without necessarily having a correct reasoning to back up each technique. For example, if given a limit including a radical expression without an indetermination, students may still apply the technique indicated above. Students see a radical expression and immediately think of multiplying by the conjugate. As Hardy (2009) pointed out, "...the *final examination institution* that I have studied has not - not even once in the last 6 years - included a problem of finding a limit involving radicals for which the rationalization technique would not apply." Hardy (2009) also commented on whether the direct substitution method used by students was

[1] In the French original version of ATD, what we call here "methodology" is called "technologie" in French. In papers written in English, "technologie" is often translated into "technology". In English, the word "technology" has a computer science and computer hardware connotation that it does not have in French. This is why we prefer to use the word "methodology" as a translation of "technologie", as better reflecting the intended meaning of the word (e.g., "technologie de la production horticole").

their attempt at finding the limit, finding the indetermination, or whether it was simply a first technique as part of a larger set of steps used by students. Lastly, another interesting technique used by students was to factor regardless of the function at hand, and one such student provided the following reason for doing so, "But I was taught, if you can factor, factor" (Hardy, 2009).

2.2 OPERATIONAL DEFINITION OF ALGEBRA & RELATED INSTITUTIONS

Although there may be a debate as to whether functions belong to a pre-university level algebra course or an “algebra with functions” course, in North America it is called “Pre-calculus”, and in this thesis we will define algebra to include functions and their properties.

We will adopt a definition of the content of a college algebra course proposed by the College Board. The College Board is a non-profit US organization which helps students transition into college, self-assess their knowledge levels, and prepare for various college examinations. Below is the content of a college algebra course as proposed by The College Board:

> ***1) Algebraic operations****: operations with exponents, factoring and expanding polynomials, operations with algebraic expressions, absolute value, and properties of logarithms;*
>
> ***2) Equations and inequalities****: linear, quadratic, and absolute value equations and inequalities, systems of equations and inequalities, and exponential and logarithmic equations;*
>
> ***3) Functions and their properties:*** *definition and interpretation, representations of functions (graphical, numerical, symbolic, and verbal), domain and range, algebra of functions[2], graphs and their properties (intercepts, symmetry, and transformations), inverse functions, as well a variety of functions, including linear, polynomial, rational, absolute value, power, exponential, logarithmic, and piecewise-defined functions (The College Board).*

[2] By “algebra of functions” we mean the operations of addition, subtraction, multiplication, division and composition of functions and their properties.

These topics are fairly consistent with the content in the prerequisite courses for MATH 203 (Differential & Integral Calculus I[3]), which are MATH 200 (Fundamental Concepts of Algebra) and MATH 201 (Elementary Functions) at Concordia University for the science or engineering stream. These topics are also consistent with the content in MATH 206 (Algebra & Functions), a college algebra course for candidates to a school of business.

Thinking about the content in a calculus course, is using the Product Rule of differentiation considered a calculus or an algebraic activity? Although we have provided the above definition for algebra, we also want to bring to your attention the definition of a mathematical activity as described by Drijvers (2011, pp. 8-9):

> *A mathematical activity becomes more 'algebraic' to the degree that it has more of the following characteristics:*
>
> *a) Implicit or explicit generalization taking place.*
>
> *b) Patterns of relationships between numbers and/or formulas are investigated.*
>
> *c) Problems are solved by applying general or situation-dependent rules.*
>
> *d) Logical reasoning is conducted with unknown or as yet unknown quantities.*
>
> *e) Mathematical operations are conducted with variables represented with letters. Formulas are created as a result.*
>
> *f) For numerical operations and relationships, special symbols are used.*
>
> *g) Tables and graphs represent formulas and are used to investigate formulas.*
>
> *h) Formulas and expressions are compared and transformed.*
>
> *i) Formulas and expressions are used to describe situations in which units and quantities play a role.*
>
> *j) Processes for solving problems contain steps that are based on calculation rules, but that do not necessarily have any meaning in the context of the problem.*

[3] Note that although MATH 203 is called Differential & Integral Calculus I, the Integral half is only covered in the second course (MATH 205 - Differential & Integral Calculus II).

Thus according to these characteristics, Drijvers (2011, p. 9) maintains that "Proving the product rule of differential calculus involves a combination of calculus and algebra. Using the product rule with known functions is an algebraic activity." Keeping these characteristics in mind, much of the routine problems found in a Calculus I final examination can be regarded as algebraic activities.

Students who take MATH 203 (Calculus I) in our university are generally students trying to enter a science or engineering undergraduate program, and are lacking prerequisite courses. These students are either mature students, having returned to school after an extended period away from formal education, or students who were not in a science stream in college (CEGEP[4] in Quebec) or in upper secondary classes in other Canadian provinces or in other countries. MATH 200 (Fundamental Concepts of Algebra) and MATH 201 (Elementary Functions) are courses equivalent to the higher levels of mathematics in secondary school, and MATH 203 and MATH 205 (Calculus II) are equivalent to the Science Calculus courses offered at CEGEP in Quebec. Therefore students taking these prerequisite mathematics courses (PMC) in University have several reasons for feeling frustrated (Sierpinska, Bobos, & Knipping, 2008). Sierpinska, Bobos, and Pruncut (2011) prepared and taught three different types of lectures about absolute value inequalities to mature students. In each of the lecture types, students' lack of algebraic knowledge became apparent in the misconceptions held about absolute value inequalities. Further, in learning a specific technique students were more interested in learning the steps, rather than learning *why* each step could be performed. Even though students taking PMC may have the required prerequisites, I believe that one source of their frustrations lies in their poor algebraic knowledge. Further, the content in these courses, as well as the typical questions and solutions in assessments are heavily influenced by different institutions.

Sierpinska, et al. (2008) thoroughly summarize the concept of institution as initially discussed by Peters (1999). Briefly, a social activity is characterized as an institution when meeting the following four criteria: 1) it is a formal or informal structural feature of society; 2) it is stable over time; 3) it is constrained by formal norms; and 4) its members share common values and goals. With this definition there are a number of institutions, and ultimately institutional constraints involved in the content and assessments of MATH 203 (Calculus I). The following institutions related to algebra influenced not only the content and assessments in this course, but also our analysis of the student solutions to the final

[4] In the province of Quebec, the post-secondary education system includes two years in CEGEP (an acronym for "*Collège d'enseignement général et professionnel*"). Students who complete a program in CEGEP obtain a Diploma of College Studies which is a requirement for admittance into University in Quebec, unless returning as a mature student at the age of 21.

examination: PMC, MATH 203, summer course, and course textbook. The PMC institution has been a formal feature in universities, being extremely stable over many years. These courses tend to be very intense, fast paced, and are taken with the common goal of obtaining a high grade, as they are an absolute requirement for admittance to a desired undergraduate science or engineering program. MATH 203 at Concordia University in and of itself is another institution. The content of this course, and typical textbook material have not changed over many years. When this course is taught throughout the fall and winter semesters, the final examinations are prepared by a course examiner and are very typical in the questions and algebraic solutions required. As I taught the course in the summer, I had a little more flexibility as I prepared the midterm and final examinations myself. However, I needed to follow the topics covered in the course outline, as well as have the assessments approved by the course examiner. Although I had this sense of freedom, I was strongly encouraged by other advisors to follow past examinations, choosing typical questions from the textbook, thus putting constraints on the questions and solutions I was able to choose from. Lastly, the textbook (Stewart, 2016) is yet another institution, containing similar questions, and requiring typical algebraic solutions which have remained stable over decades. The mathematical content in this textbook, as well as the explanations, cautions, and hints, were taken into account in our analysis of the student solutions to the final examination.

3 THEORETICAL ANALYSIS OF THE FINAL EXAMINATION QUESTIONS

The Science Calculus I August 2016 final exam closely resembles the previous final exams offered at Concordia University. As such, it is highly representative with regards to its layout and content. In the analysis that will follow, the typical solutions to each problem will be provided, thereby determining how students are expected to solve such an exam. The proposed solutions demonstrate how students' algebraic skills are inadvertently required to solve Calculus examination problems.

3.1 PROBLEM 1 – DOMAINS OF FUNCTIONS, COMPOSITION OF FUNCTIONS, INVERSE FUNCTIONS

In the more formal university mathematics, the definition of function is such that the "domain" of a function, its "codomain" and its "rule" are three independent entities that, taken as a system, constitute the function.

> *Let A and B be non-empty sets. By a function from A to B, written $f: A \to B$, we mean a relation from A to B [i.e., set of ordered pairs (a, b), with $a \in A, b \in B$] with the property that every element a in A is the first coordinate of exactly one ordered pair in f. (Chartrand, Polimeni, & Zhang, 2013, p. 216).*

Of course, the system $< A, B, f >$ must be consistent, but a function is not given if all three elements of this system are not given. For example, the rule $f(x) = x^2$ does not uniquely determine the function f. According to this general definition, the systems:

$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$
$f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, \quad f(x) = x^2$
$f: \mathbb{N} \to \mathbb{N}, \quad f(x) = x^2$
$f: \mathbb{C} \to \mathbb{C}, \quad f(x) = x^2$

represent different functions.

In pre-calculus and calculus courses, however, the rule of the function is the most important element of the system; the rule determines the domain and the codomain.

> *A function from a set D to a set Y is a rule that assigns a unique element $f(x) \in Y$ to each element $x \in D$. (Thomas, 2008, p. 2)*

In those courses, the sets D and Y are subsets of $\mathbb{R}$ and the rule has almost always the form of an algebraic expression. The way functions are talked about and named (e.g., "take the function $y = x^2$") conveys the conception that functions *are* algebraic expressions, just as they were for 18-19th century mathematicians such as Bernoulli, Euler, Lagrange or Cauchy (Sierpinska, 1992, p. 45). The domain of a function is understood as *the domain of the algebraic expression* that defines it[5]. This is sometimes called the "natural domain" of the function:

> *When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values, the so-called natural domain. (Thomas, 2008, p. 2)*

Such was the understanding of the domain of function assumed implicitly in the first problem of the final examination. In Problem 1, the students were required to find the [natural] domains of composite functions, given by algebraic expressions.

In the sense of natural domain, the domain of a composite function can be described in the following manner:

Given the composition $F(x) = f_1(f_2(f_3(\dots(f_{n-1}(f_n(x)))\dots))$ the natural domain of F is the set:
$D_F = \{x \in \mathbb{R} : x \in D_{f_n} \wedge\ f_n(x) \in D_{f_{n-1}} \wedge f_{n-1}\big(f_n(x)\big) \in D_{f_{n-2}} \wedge \dots \wedge\ f_2(f_3(\dots(f_n(x))\dots) \in D_{f_1}\}$.
For example, if $F(x) = h(g\big(f(x)\big))$ then $x \in D_F \Leftrightarrow x \in D_f \wedge f(x) \in D_g \wedge g\big(f(x)\big) \in D_h$.

More concretely, if $h(x) = x + 3$, $g(x) = \sqrt{x}$ and $f(x) = \frac{1}{1-x}$, then $F(x) = h\Big(g\big(f(x)\big)\Big) = \sqrt{\left(\frac{1}{1-x}\right)} + 3 = \frac{1}{\sqrt{1-x}} + 3$, and $D_h = \mathbb{R}$; $D_g = [0, +\infty)$; $D_f = \mathbb{R} - \{1\}$. According to the above definition of the domain of a composite function, $D_F = \left\{x \in \mathbb{R}: x \neq 1 \wedge \frac{1}{1-x} \geq 0 \right\} = (-\infty, 1)$.

In this example, the same set would be obtained by looking at the natural domain of the final algebraic expression for F. In general, however, finding the natural domain of a composition with the above definition and calculating the domain of the final algebraic expression do not yield the same set. For example, given $f(x) = \sqrt{x}$ and $g(x) = x^2$ then $D_f = [0, +\infty)$ and $D_g = \mathbb{R}$. If $F(x) = g\big(f(x)\big) = \left(\sqrt{x}\right)^2 = x$, looking solely at the final algebraic expression, one would conclude that $D_F = \mathbb{R}$. On the other hand, however, according to the above definition of the domain of a composite function,

[5] see, e.g., http://www.cengage.com/resource_uploads/downloads/1439049084_231926.pdf

$D_F = \{x \in \mathbb{R}: x \geq 0 \ \wedge \ x \in \mathbb{R}\} = [0, \infty)$ which is different from the domain of the final algebraic expression for F. Looking at another example, given $f(x) = \ln x$ and $g(x) = \ e^x$, then $D_f = (0, +\infty)$ and $D_g = \{x \in \mathbb{R}\}$. If $F(x) = g\big(f(x)\big) = e^{\ln x} = x$, looking solely at the final algebraic expression, one would conclude that $D_F = \mathbb{R}$. On the other hand, however, according to the above definition of the domain of a composite function, $D_F = \{x \in \mathbb{R}: x > 0 \ \wedge \ x \in \mathbb{R}\} = (0, \infty)$ which is different from the domain of the final algebraic expression for F.

The functions $f(x) = \sqrt{x+1}$ and $g(x) = 4x - 3$ in Problem 1a were such that the definition of the domain of a composite function and the domain of the final algebraic expression for both $g \circ f$ and $f \circ g$ yielded the same sets. So solutions where students found the formulas for the compositions of the given functions, and then determined the domains of the algebraic expressions defining them were considered correct, and were, in fact, expected.

In Problem 1b, students were asked to find a formula of the inverse of a given function and the domains of the function and its inverse.

Formally, the notion of the inverse function can be introduced via the concept of inverse relation. Every relation has an inverse but the relation which is the inverse of a function is not necessarily a function itself. Reasoning about the conditions necessary for the inverse of a functional relation to be a function leads to the conclusion that for a function to have an inverse, the function must be bijective (one-to-one and onto):

> *Theorem…: Let $f: A \to B$ be a function. Then the inverse relation f^{-1} is a function from B to A if and only if f is bijective. (…) (Chartrand, Polimeni, & Zhang, 2013, p. 229)*

Further, if $f: A \to B$ is a bijection, then a function $g: B \to A$ is the inverse of f if $g \circ f = id_A$ and $f \circ g = id_B$ (ibid., p. 230).

In Pre-calculus and Calculus I courses, the notion of function, as mentioned, is often identified with an algebraic expression which, for the purposes of introducing the notion of inverse, is interpreted as a sequence of operations on a variable: the function "does" something with the variable and the inverse function "undoes" it. In Thomas' Calculus, the notion of inverse function is first introduced informally:

> *A function that undoes, or inverts, the effect of a function f is called the inverse of f. Many common functions, though not all, are paired with an inverse. (Thomas, 2008, p. 47)*

Then, after the introduction of the concept of one-to-one functions, an official definition is given:

Definition: Inverse function

Suppose that f is a one-to-one function on a domain D with range R. The inverse function f^{-1} is defined by $f^{-1}(a) = b$ if $f(b) = a$. The domain of f^{-1} is R and the range of f^{-1} is D. (Thomas, 2008, p. 48)

Next, the procedure for "finding inverses" (ibid., p. 48) or "passing from f to f^{-1} (ibid., p. 50) is given:

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.

2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable. (Thomas, 2008, p. 50)

Thus, to find the domain of the inverse of a given function, students are expected to find the range of the given function, or follow the process above to find the algebraic expression for the inverse and then determine the domain of this algebraic expression, although applying the procedure without awareness of the conditions of existence of the inverse function could lead to mistakes. For example, applying the procedure to a function f given by the formula $y = \sqrt{x}$ would lead to the claim that the inverse function is $y = x^2$ and that the domain of the inverse function is $\mathbb{R}$, while the range of the given function f is $[0, \infty)$.

3.1.1 Problem 1a – Domains of composite functions involving linear and square root functions

The problem was given as follows:

Let $f(x) = \sqrt{x+1}$ and $g(x) = 4x - 3$. Find $g \circ f$ and $f \circ g$ and determine the domains of these composite functions.

Composition of functions requires a structural (rather than operational) understanding of algebraic expressions (Sfard, 1987), and understanding of letters as variables (rather than as unknowns or placeholders) (Küchemann, 1981).

Thus, looking at the function f students should see the square root function and by moving from outside to inside, see the structure as such: $f(x) = \sqrt{\blacksquare} = \sqrt{\blacksquare + 1} = \sqrt{x+1}$.

When looking at $g(x)$ students should first see the polynomial function and by moving from outside to inside, see the structure so:

$g(x) = \blacksquare = \blacksquare - 3 = 4\blacksquare - 3 = 4x - 3$

This way, the structural perception of the composition $g \circ f$ could be represented as:

$$g(\blacksquare) = g\big(f(x)\big) = g\left(\sqrt{x+1}\right) = 4\blacksquare - 3 = 4(\sqrt{x+1}) - 3$$

and $f \circ g$:

$$f(\blacksquare) = f\big(g(x)\big) = f\ (4x-3) = \sqrt{(\blacksquare)+1)} = \sqrt{(4x-3)+1)} = \sqrt{4x-2}$$

To find the domains of these composite functions, it is necessary to know what the symbol of square root means – the square root of a real number x is a non-negative real number y such that $y^2 = x$, and knowing that squares of real numbers are non-negative, conclude that square roots of negative numbers do not exist. So the natural domain of the square root function is the set of non-negative real numbers. With this understanding, students were expected to find the domain of the function $g \circ f$ by solving a linear inequality $x + 1 \geq 0$ and the domain of $f \circ g$ by solving $4x - 2 \geq 0$. Solving these inequalities requires operational knowledge of properties such as

If $a \geq b$ and $c > 0$, then $\frac{a}{c} \geq \frac{b}{c}$.

If $a \geq b$ then $a + c \geq b + c$, for any c.

The expected answers for the domains were: $D_{g \circ f} = [-1, +\infty)$, $D_{f \circ g} = \left[\frac{1}{2}, +\infty\right)$ or some equivalent expressions of these sets.

The above description of knowledge and reasoning involved in solving the Problem 1a uses standard mathematical language, such as would be expected to be used by university teachers. Students are not very likely to use such language or be aware of the definitions and algebraic properties they are using. Based on our experience listening to students' oral explanations, it is more realistic to expect students to think about the problem in the following way, using a kind of "school mathematical jargon":

"To calculate $g \circ f$, replace the x in $g(x)$ with the right side of $f(x)$. (…) Domain of a function is to solve for x any condition for the formula of the function to make sense. Values under the square root cannot be negative therefore make it ≥ 0. (…) To solve $x + 1 \geq 0$, move 1 to the right side. When you move 1 to the right side it becomes -1." (etc.)

3.1.2 Problem 1b – Domain of the inverse of a given composite function involving logarithmic and exponential functions

The problem was given as follows:

Find the domain of the function $f(x) = ln\ (e^x - 3)$*, the inverse function* f^{-1}*, and the domain of* f^{-1}*.*

The expected answers were: $D_f = (ln(3)\,,\infty)$, $f^{-1}(x) = \ln(e^x + 3)$, and $D_{f^{-1}} = (-\infty, \infty)$ or some equivalent expressions.

To find the domain of the function, and the domain of its inverse, it is necessary to know that the domain of the natural logarithmic function is $(0, +\infty)$, that its range is $\mathbb{R}$, and that $\ln x$ and e^x are inverse functions, so that $\ln e^x = x$ and $e^{\ln x} = x$. With this understanding, students were expected to find the domain of the function f by solving the inequality $e^x - 3 > 0$, and concluding that $D_f = (\ln 3\,, \infty)$. The domain of f^{-1} was expected to be found either by remembering that the domain of the inverse function is equal to the range of the function and that the range of the logarithmic function is $\mathbb{R}$, or by finding the formula $f^{-1}(x) = \ln(e^x + 3)$ and solving the inequality $e^x + 3 > 0$. The latter approach could lead students to an impasse upon arriving at the inequality $e^x > -3$, which some could be tempted to further process by writing $x > \ln(-3)$, concluding that $D_{f^{-1}} = (\ln(-3)\,, +\infty)$ instead at arriving at the conclusion that the domain f^{-1} was $\mathbb{R}$, which is the range of f.

Solving these inequalities requires operational knowledge of the property

If $a > b$ then $a + c > b + c$.

Students would usually understand this property in the form of allowable moves in processing an inequality rather than in the form of an implication such as above. This understanding would be enough to succeed in a first Calculus course such as the one studied here.

Given y as a function of x, in order to find the inverse function, one can express x as a function of y, as stated in the “inverse” procedure (Thomas, 2008, p. 50). With this understanding, students were expected to find the inverse of f by solving for y in terms of x.

Representing the given function with y instead of $f(x)$, they would write:

$y = \ln(e^x - 3)$

Given the function $f(x) = e^x$ is well-defined, if $a = b$ then $f(a) = f(b)$, therefore $e^a = e^b$, one can take the exponential function of both sides; in students' operational knowledge, this move would be expressed as "since there is a ln, let's put e in front of each side":

$e^y = e^{\ln(e^x-3)}$

Using operational knowledge of the property $e^{\ln(x)} = x$ would allow them to represent the above as:

$e^y = e^x - 3$

Using operational knowledge of the properties of equalities such as: If $a = b$ then $a + c = b + c$, and if $a = b$ then $b = a$, they would write:

$e^y + 3 = e^x$

$e^x = e^y + 3$

Given the function $f(x) = \ln x$ is well-defined, if $a = b$ then $f(a) = f(b)$, therefore $\ln(a) = \ln(b)$. In students' operational knowledge this step would be justified by "taking ln of both sides":

$\ln(e^x) = \ln(e^y + 3)$

Using the property $\ln e^x = x$, they could write:

$x = \ln(e^y + 3)$

Now they would be expected to interchange x and y (as prescribed by the second step of the "inverse function" procedure):

$y = \ln(e^x + 3)$

and write:

$f^{-1}(x) = \ln(e^x + 3)$.

The interchange of letters is not theoretically necessary; it was expected because it was part of the procedure that was taught. But students could as well express the inverse function as $f^{-1}(y) = \ln(e^y + 3)$. They could also verify the identities: $f(f^{-1}(y)) = y$ and $f^{-1}(f(x)) = x$ to check if their formula for the inverse function was correct, although such theoretical behavior was not expected of the students in this course.

$f^{-1}(f(x)) = f^{-1}(\ln(e^x - 3)) = \ln\left(e^{\ln(e^x-3)} + 3\right) = \ln(e^x - 3 + 3) = x$

$f(f^{-1}(y)) = f(\ln{(e^y + 3)}) = \ln\left(e^{\ln(e^y+3)} - 3\right) = \ln(e^y + 3 - 3) = y.$

3.2 Problem 2 – Limits of the ratio of functions containing absolute value and square root expressions

In finding the limit of a function, one can try to approximate the limit numerically, rather than using algebraic techniques, however students tend to use such techniques, hinted at by the expression. If the function contains a radical, students think of "rationalizing" the numerator or denominator, depending on the placement of the expression. If the function contains a binomial in the denominator, students proceed to factor the numerator to cancel it out. When encountering an absolute value expression, students believe that they need to find both one-sided limits, as routine problems suggest that the limit would not exist. Although this is not mathematical knowledge we wish our students to gain when finding limits, these types of limit problems tend to be quite algebraic in nature.

In taking a closer look at the textbook (Stewart, 2016), students are first introduced to limits via the problem of finding the tangent to the graph of a function $f(x)$ at a point x_0 . They are instructed to find values of a function for values of x, as x get closer and closer to x_0 on either side of it, and slopes of the secants passing through $\left(x_0, f(x_0)\right)$ and $(x, f(x))$. They construct a table of values. This first example shows the students that the limit of the slopes of the secant lines is the slope of the tangent line at x_0. As one progresses through the sections of the textbook, an intuitive definition of the limit is provided:

> *Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write $lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a (Stewart, 2016, p. 83).*

After providing this definition, the examples that follow instruct the students to make a table of values to guess the limit, similar to that of the tangent problem. Some pitfalls of this method are demonstrated in this section, and the author then promises a perfect method to come:

> *…some calculators and computers give the wrong values. In the next section, however, we will develop foolproof methods for calculating limits (Stewart, 2016, p. 87).*

The announcement of a "foolproof" method is very disturbing. After this point in the textbook, limit problems are reduced to applying limit laws and algebraic manipulations, and the students may believe that this is the best way, as it is the "foolproof" way. I fear that students reading these notes, may completely disregard the conceptual understanding of limits, as they only need to know the "foolproof" method. As promised by the author, students learn how to calculate limits using different limit laws in section 2.3 of the textbook. They are also shown the Direct Substitution Property (DSP):

> *If f is a polynomial or a rational function and a is in the domain of f, then*
>
> $lim_{x \to a} f(x) = f(a)$ *(Stewart, 2016, p. 97).*

Students tend to use this property as their first attempt at finding limits, even if a is not in the domain of f (this will be addressed in our analysis of the solutions to problem 2). These "foolproof" methods are only foolproof if the students have good algebraic knowledge. The book then provides some examples in which the DSP could not be used, i.e., the function is undefined at $x = a$. This is the precise location in the section, in which algebraic techniques are required. Students are shown to factor a numerator of a rational function in order to cancel a common term from the numerator and denominator, thus allowing the limit to be calculated via the DSP. They are told:

> *…we need to do some preliminary algebra. We factor the numerator as a difference of squares (Stewart, 2016, p. 98).*

Further, students are shown one example containing a square root expression in the numerator of a ratio of functions, and are told:

> *We cannot apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator… (Stewart, 2016, p. 99).*

Finally, they are provided with the following:

> ***Theorem:*** $lim_{x \to a} f(x) = L$ *if and only if* $lim_{x \to a^-} f(x) = L = lim_{x \to a^+} f(x)$ *(Stewart, 2016, p. 99).*

Following this theorem, a few examples of both one-sided limits are provided, demonstrating the necessity to calculate both one-sided limits for absolute value functions.

Thus, although the conceptual understanding of limits does not require the use of algebra, a good portion of the questions and solutions provided in the textbook necessitate the use of algebraic techniques. Therefore, a student who cannot "do" algebra will appear not to understand limits, by the nature of the questions asked.

3.2.1 Problem 2a – Limit of a function containing an indetermination at a point and an absolute value expression

The problem was given as follows:

Evaluate the limit if it exists, or explain why the limit does not exist ***(Do not use l'Hôpital's rule):*** $lim_{x \to -6} \frac{2x+12}{|x+6|}$

The approach introduced in the textbook suggests that the students notice the indetermination at $x = -6$, and therefore realize that the DSP could not be used. Note that some students did attempt the DSP at first (some correctly arriving at an indeterminate form, and others incorrectly arriving at some incorrect answer, which will be discussed in the next section). Further, students should recognize that the denominator is an absolute value expression, and know that given $f(x) = |x|$, $|x| = x \; for \; x \geq 0$ and $|x| = -x \; for \; x < 0$, resulting in $\frac{x}{|x|} = \begin{cases} 1 \; if \; x > 0 \\ -1 \; if \; x < 0 \end{cases}$. Applying this property to the given function, they were expected to conclude that the left-sided limit of the function is -2 and the right-sided limit is $+2$, and so the the limit at -6 does not exist.

Students not knowing or remembering the above property of the expression $\frac{x}{|x|}$ could still figure out the answer, by reasoning as follows, from the definition of absolute value alone: as x approaches -6 from the left, $x < -6$ and $(x + 6)$ is negative. Since the absolute value function **always** produces a positive value, this expression requires a negative sign in front of it, in order to remain positive: $-(x + 6)$. In finding the greatest common factor in the numerator (using distributivity: $ab + ac = a(b + c)$), one can then cancel out the common term in the numerator and denominator arriving at the limit of -2 as x approaches -6. Using the limit law of the limit of a constant, we arrive at the left-sided limit of -2. The right-sided limit is obtained in a similar manner, with a resulting limit of 2. Students recalling the theorem that $\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$ would conclude that since both one-sided limits are not equal, the $\lim_{x \to -6} \frac{2x + 12}{|x+6|}$ does not exist.

3.2.2 Problem 2b – Limit of a function containing an indetermination at a point and a square root expression

The problem was given as follows:

*Evaluate the limit if it exists, or explain why the limit does not exist (**Do not use l'Hôpital's rule**):* $lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2}$

The approach introduced in the textbook suggests that the students notice the indetermination at $x = 3$. Since $x = 3$ is not in the domain of the function, once again the DSP cannot be used. Again, for this problem some students attempted the DSP (some correctly arriving at an indeterminate form, and others incorrectly arriving at some incorrect answer).

The expected algebraic solution is as follows: since the Quotient Law could not be applied right away, as the limit of the denominator is 0, students were expected to "rationalize" the expression for the function, by multiplying numerator and denominator by the conjugate of the numerator. Stewart (2016) does not explain how to rationalize, as students should have learned this procedure in prerequisite courses. He does however mention rationalizing the numerator as the preliminary algebra required to solve a limit in a specific example (Stewart, 2016, p. 99).

The following definition came from a textbook used in a prerequisite course.

*Removing radicals in the denominator or the numerator of a fraction is called **rationalizing the denominator** or **rationalizing the numerator**, respectively. The procedure for rationalizing involves multiplying the fraction by 1 in a special way so as to obtain a perfect nth power (Ratti & McWaters, 2014, p. 15).*

A theoretical justification for this technique is the following algebraic property: If $b, c \neq 0$, then $\frac{a}{b} = \frac{ac}{bc}$. But the problem did not require students to justify the techniques they were using.

$$\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2} = \lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2} \cdot \frac{(\sqrt{x+6}+x)}{(\sqrt{x+6}+x)}$$

Students did not need to multiply out the binomials in the numerator, as these were the factors of the difference of squares ($(x+6) - x^2$). However, if they did not realize this, they would have used the FOIL (First, Outside, Inner, Last) method (Sullivan, 2016, p. 43).

$$= \lim_{x\to 3} \frac{\sqrt{x+6}\sqrt{x+6} + x\sqrt{x+6} - x\sqrt{x+6} - x^2}{(x^3-3x^2)(\sqrt{x+6}+x)}$$

Multiplying binomials is applying the distributivity property twice, however it is unclear how many students know this ($(a+b)(c+d) = a(c+d) + b(c+d) = ac + ad + bc + bd$), since college algebra textbooks tend to replace this theoretical justification by a “foolproof” FOIL “method”. (Sierpinska & Hardy, 2010)

Further, students need to know that the square of a square root of an expression is the expression. The algebraic knowledge required here is the following:

For any real number x, $\left(\sqrt{x}\right)^2 = x$ if $x \geq 0$ (Ratti & McWaters, 2014, p. 12).

$= \lim_{x\to 3} \frac{(x+6) - x^2}{(x^3 - 3x^2)(\sqrt{x+6} + x)}$

The greatest common factor amongst $-x^2$, x, and 6 is -1. Further, the greatest common factor among $x^2 - x - 6$ and $x^3 - 3x^2$ is $(x-3)$.

$= \lim_{x\to 3} -\frac{(x-3)(x+2)}{(x^2)(x-3)(\sqrt{x+6} + x)}$

Students would then divide numerator and denominator by $(x-3)$, or as they tend to know it as "cancel it out".

$= \lim_{x\to 3} -\frac{(x+2)}{(x^2)(\sqrt{x+6} + x)}$

As $a = 3$ is now in the domain of this function students could replace x by the value it is approaching and calculate the limit.

$= -\frac{((3)+2)}{(3^2)(\sqrt{3+6} + 3)} = -\frac{5}{54}$

Thus, the expected solution of this limit problem requires a good use of algebraic knowledge. Whether students have a good conceptual understanding of limits is not being assessed.

3.3 Problem 3 – Vertical and horizontal asymptotes

In order to find a vertical asymptote of a function $f(x)$, one has to find at what points a, a one-sided limit of the function is plus or minus infinity. To find horizontal asymptotes, one must find if the function has finite limits when $x \to +\infty$ or $x \to -\infty$.

Stewart provides the following intuitive definition of an infinite limit:

Let f be a function defined on both sides of a, except possibly at a itself. Then $lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a (Stewart, 2016, p. 89).

In this section of the textbook, Stewart provides a numerical approach to help students understand why the limit of the following function is infinity using the table method: $\lim_{x \to 0} \frac{1}{x^2} = \infty$. Students tended to have a little bit of difficulty with infinite limits, as in the example above, they have difficulty comprehending why we could divide by zero and obtain infinity. I had students ask me, "but how can this be possible, we are not allowed to divide by zero?" And I had to explain to them that we were looking at the behavior of the function as x tended to zero, but we were not concerned with what happens at zero, as indicated by the definition.

Following this intuitive definition, Stewart provided the following definition of vertical asymptotes:

*The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following holds: $lim_{x \to a} f(x) = \pm\infty$ or $lim_{x \to a^+} f(x) = \pm\infty$ or $lim_{x \to a^-} f(x) = \pm\infty$ (Stewart, 2016, p. 90).*

The section ends with the two examples that provide a numerical approach to finding limits. Note that both examples had vertical asymptotes at points in which the function was undefined. This poses a problem, as students tend to generalize that vertical asymptotes occur at every point in which the denominator is equal to zero. In finding the vertical asymptotes of a ratio function, we can look at points of indeterminacy to get an idea of *potential* vertical asymptotes. These points, when $x = a$, will either be vertical asymptotes or hole singularities. Thus, Stewart only providing two examples in which the exclusions of the domain resulted in vertical asymptotes can suggest false generalizations. A good number of students did not verify, or mention the infinite limits when finding the vertical asymptotes in the final examination.

Stewart provides a similar table method approach to limits at infinity, as well as the following intuitive definition of a limit at infinity:

Let f be a function defined on some interval (a, ∞). Then $lim_{x \to \infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large (Stewart, 2016, p. 127).

Following this intuitive definition, Stewart provided the following definition of horizontal asymptotes:

*The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either $lim_{x\to\infty} f(x) = L$ or $lim_{x\to-\infty} f(x) = L$.*

Stewart provides a few examples following this definition, and after an example of the limit of a rational function at infinity, Stewart provides the following advice:

...we need to do some preliminary algebra. To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator (Stewart, 2016, p. 130).

The author provides a procedure, a technique that students tend to memorize. Every time they see x approaching infinity they follow this procedure. As Sierpinska and Hardy (2010) point out, why aren't the authors showing our students how to reason? They bring attention to the "learning by example approach", and mention limits at infinity and how various textbook authors provide different techniques, and procedures dependent of the examples provided, rather than trying to teach our students how to reason. For rational functions, one can look at the leading coefficients of both the numerator and denominator and arrive at conclusions based on the highest degree without necessitating the use of algebra. Although Stewart provides the technique described above and uses it for some of his examples, he does offer some reasoning as well:

Find $lim_{x\to\infty} \frac{1}{x}$ ...Observe that when x is large, $\frac{1}{x}$ is small...; by taking x large enough, we can make $\frac{1}{x}$ as close to 0 as we please... (Stewart, 2016, p. 129).

Although the function provided in Problem 3 was not a rational function, similar reasoning such as the one provided by Sierpinska and Hardy (2010) could have been applied.

Problem 3 was given as follows:

Find all the horizontal and vertical asymptotes of the function $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$.

Students were expected to notice that this function is a ratio containing a radical expression. Although this function was not a rational function, similar reasoning as provided by the textbook could be applied in order to find the horizontal asymptotes. In order to find the vertical asymptote, students would look at points where $f(x)$ is not defined. This occurs when the expression in the denominator, $3x - 5 = 0$, that is when $x = \frac{5}{3}$. Students would then need to verify that one of the conditions in the definition holds, such as $\lim_{x \to \frac{5}{3}^{+}} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$. Students would then need to conclude that $x = \frac{5}{3}$ is a vertical asymptote.

A proportion of students did not verify any condition, rather stated that the vertical asymptote is $x = \frac{5}{3}$ solely based on the function being undefined at this point. For some students, finding vertical asymptotes is merely a procedure, by looking at when the denominator equals zero. An incorrect generalization, perhaps obtained by solving routine problems, in which the denominator equaling zero always provided vertical asymptotes. Despite having shown an example in class in which an asymptote did not occur, and reiterating the definition of vertical asymptotes, students still did not verify any conditions. An example of such a limit would be $\lim_{x \to 2} \frac{x^2-4}{x-2}$. This function is undefined when $x = 2$ however, $x = 2$ is not a vertical asymptote, it is a hole singularity represented by a hole in the graph of the line $y = x + 2$. Further, some students did verify a condition, found a limit as being infinity, and did not conclude that $x = \frac{5}{3}$ was the vertical asymptote. As though the limit equaling infinity was the answer to finding an asymptote.

In order to find the horizontal asymptotes, students needed to find whether the limits at infinity were equal to a specific value, i.e. the function approaches a finite limit as x tends to positive or negative infinity. The horizontal asymptote is the line with equation $y = L$. However, it can be expected that some students would leave the solution after writing that the limit of the function at plus or minus infinity is equal to L.

The expected solution, as suggested by the worked out examples and instructions in the textbook, is as follows. The same procedure as for rational functions is used: dividing numerator and denominator by the term in the denominator with the highest degree:

$$\lim_{x \to \infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to \infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x-5}{x}} = \lim_{x \to \infty} \frac{\sqrt{\frac{2x^2+1}{x^2}}}{3-\frac{5}{x}} = \lim_{x \to \infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}$$

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to -\infty} \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x-5}{x}} = \lim_{x \to -\infty} -\frac{\sqrt{\frac{2x^2+1}{x^2}}}{3-\frac{5}{x}} = \lim_{x \to -\infty} -\frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = -\frac{\sqrt{2+0}}{3-0} = -\frac{\sqrt{2}}{3}$$

Therefore the horizontal asymptotes are $y = \frac{\sqrt{2}}{3}$ and $y = -\frac{\sqrt{2}}{3}$.

Alternatively, students could have solved the problem as follows:

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty}\frac{\sqrt{x^2(2+\frac{1}{x^2})}}{x(3-\frac{5}{x})} = \lim_{x\to\infty}\frac{\sqrt{x^2}\sqrt{(2+\frac{1}{x^2})}}{x\,(3-\frac{5}{x})} = \lim_{x\to\infty}\frac{x\sqrt{(2+\frac{1}{x^2})}}{x\,(3-\frac{5}{x})} = \lim_{x\to\infty}\frac{\sqrt{(2+\frac{1}{x^2})}}{(3-\frac{5}{x})} = \frac{\sqrt{2}}{3}$$

$$\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty}\frac{\sqrt{x^2(2+\frac{1}{x^2})}}{x(3-\frac{5}{x})} = \lim_{x\to-\infty}\frac{\sqrt{x^2}\sqrt{(2+\frac{1}{x^2})}}{x\,(3-\frac{5}{x})} = \lim_{x\to-\infty}-\frac{x\sqrt{(2+\frac{1}{x^2})}}{x(3-\frac{5}{x})} =$$

$$= \lim_{x\to-\infty}-\frac{\sqrt{(2+\frac{1}{x^2})}}{(3-\frac{5}{x})} = -\frac{\sqrt{2}}{3}$$

Therefore the horizontal asymptotes are $y = \frac{\sqrt{2}}{3}$ and $= -\frac{\sqrt{2}}{3}$.

I believe that this second solution offers more insight into using the following algebraic knowledge: $\sqrt{x^2} = |x| = x \text{ } for \text{ } x \geq 0$ and $\sqrt{x^2} = |x| = -x \text{ } for \text{ } x < 0$.

Thus, although one's conceptual understanding of asymptotes may be correct, the procedures taught in the textbook make use of algebraic techniques to finding these limits. Again, in finding both the vertical and horizontal asymptotes, one is reduced to simple algebraic procedures. Although this is not the intention of the knowledge to be learned, this is what students appear to remember about asymptotes.

3.4 Problem 4 – Derivatives of various functions

Derivatives are introduced in the Stewart textbook after limits, as a special type of limit. This limit was previously seen with the tangent problem, and with the velocity problem. When trying to find the slope of the tangent line to a curve at a point $P(a, f(a))$, one looks at nearby points $Q(x, f(x))$ and calculates the slopes of secant lines PQ as point Q approaches point P. The limiting position of the secant lines PQ as Q approaches P is the tangent line. The following definition is provided:

> ***Definition*** *The* ***tangent line*** *to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with the slope $m = lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ provided that the limit exists (Stewart, 2016, p. 141).*

Equivalently, if we let $x = a + h$, the slope of the tangent line can be given as follows: $m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$. When trying to find the instantaneous velocity of an object at a point $P(a, f(a))$, one calculates the average velocities between point P, and point $Q(a+h, f(a+h))$, as the displacement of the object over time. In calculating these averages velocities over shorter and shorter time intervals (as we let Q approach P), the instantaneous velocity is given as follows: $v(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ (Stewart, 2016, pp. 142-143). At this point in the textbook, the author brings together the idea that the limits are the same in finding the slope of the tangent line to the graph of a function $y = f(x)$ at some point and in finding the instantaneous velocity of a moving object with the relationship between the path covered and time expressed by a function $y = f(x)$. He also points out that this type of limit $\lim_{\mathrm{h}\to 0}\frac{\mathrm{f(a+h)-f(a)}}{\mathrm{h}}$ arises whenever one calculates a rate of change. The derivative is then introduced as follows:

> ***Definition*** *The* ***derivative of a function f at a number a****, denoted by $f'(a)$, is*
>
> $f'(a) = lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ *if the limit exists (Stewart, 2016, p. 144).*

Alternatively, if we let $x = a + h$, an equivalent definition is $f'(a) = \lim_{\mathrm{x}\to\mathrm{a}}\frac{f(x)-f(a)}{x-a}$, which highlights the first interpretation of the derivative as the slope of a tangent line to the curve $y = f(x)$. Further, rates of change are introduced via the difference quotient, whereby the average rate of change of y with respect to x is given by $\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$, and again this can be interpreted as the slope of the secant line joining points $P(x_1, f(x_1))$, and $Q(x_2, f(x_2))$. If we would like to consider the instantaneous rate of change, we are looking at the limit of the slopes of the secant lines as Q approaches P, $\lim_{x_2\to x_1}\frac{f(x_2)-f(x_1)}{x_2-x_1}$. This limit is the derivative $f'(x_1)$. The second interpretation of the derivative is provided:

> *The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$ (Stewart, 2016, p. 146).*

Up until this point in the textbook, derivatives are presented as the derivative of a function at **a specific point a**. In the section that follows, the derivative is introduced **as a function**: $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, and the geometrical interpretations of the derivative of $f(x)$ are provided. Given a positive derivative (positive slope of $f(x)$) over an interval, the graph of $f'(x)$ lies above the x-axis ($f'(x)$ is positive) over that interval. For a negative derivative (negative slope of $f(x)$) over an interval the graph of $f'(x)$ lies beneath the x-axis ($f'(x)$ is negative). For derivatives equal to zero, horizontal tangent lines drawn to the curve $f(x)$ at some points will be zero(s) of $f'(x)$. Second, third, and higher order derivatives are then introduced, and the instantaneous rate of change of velocity with respect to time is provided as the acceleration $a(t)$, which is the derivative of the velocity function $v(t)$, and the second derivative of the position function $s(t)$. In the remaining sections on derivatives, different differentiation rules are provided such as the Power Rule, the Quotient Rule, the Chain Rule, etc.

Problem 4 entails finding the derivatives of five functions given by their algebraic expressions. According to Drijvers (2011, p. 9), finding the derivatives of functions is an algebraic activity. A conceptual understanding of derivative is not necessary in this problem. Further, students are told that they do not need to simplify the final answer. Therefore students' algebraic skills related to simplifying algebraic expressions are not assessed by this question. The competencies being assessed are whether the students know when and how to apply the different rules of differentiation. The algebraic skills engaged in solving this problem involve, first of all, a structural view of algebraic expressions (Sfard, 1987), analyzing the expression from its outermost form to the innermost details. In question 4a, for example, where the function is $f(x) = \frac{\sqrt[3]{x} - 2xe^x}{x}$, the students need to first recognize a quotient of two functions, and apply the rule of differentiation of such quotient. Next, they need to enter into the details of the two functions, recognize that the first is a sum of two functions, and the other a monomial, etc. This structural point of view is opposed to an operational one which would be enough if students were asked to calculate the value of the expression for some concrete value of the variable x.

Students with great memorization skills, and a good structural view of functions can obtain perfect scores on these derivatives, yet can have no conceptual understanding of derivatives. Students presented with a constant function will have memorized that the derivative of a constant is zero. When presented with a polynomial, one can apply the Sum and Difference Rules, and the derivative of each term may require the use of the Constant Multiple Rule, and the Power Rule. Presented with an exponential function, a trigonometric function, a product of functions, or a quotient of functions, students know to apply the appropriate differentiation rules. Presented with a composition of functions, students memorize that they will need to use the Chain Rule. As presented in Stewart:

> ***The Chain Rule*** *If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product $F'(x) = f'\big(g(x)\big) \cdot g'(x)$. In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ (Stewart, 2016, p. 198).*

Leibniz notation is quite elegant. It demonstrates the dependence of rates, and how rates of change multiply. In explaining the Chain Rule, it is customary to provide both Leibniz notation as well as prime notation. In the examples I provided to students, I explained the usefulness of decomposing the function into its elementary functions. One real life example of the Chain Rule that I find brings forward the idea of rates multiplying is the following. When landing or taking off in an airplane, one's ear tends to have a popping sensation. The reason for this is that the internal ear pressure system cannot keep up with the change of pressure, i.e., the rate of change of pressure with respect to time is too large $\left(\frac{dp}{dt}\right)$. The rate of change of pressure with respect to time is quite difficult to measure directly, however the rate of change of pressure with respect to altitude $\left(\frac{dp}{du}\right)$ is fairly easily obtained, as is the rate of change of altitude with respect to time $\left(\frac{du}{dt}\right)$. Thus a pilot changing altitude fairly quickly will in turn cause a large change in pressure with increasing (or decreasing) altitude. This results in a larger rate of change of pressure with respect to time $\left(\frac{dp}{dt} = \frac{dp}{du}\,\frac{du}{dt}\right)$ (Marsden & Weinstein, 1985, p. 116). This example, along with many others brings forward the idea that rates of change multiply. I have had students confess that Leibniz notation is too confusing. Thus they settle for the prime notation of the Chain Rule, and follow yet another procedure to finding the derivative:

> ***NOTE*** *In using the Chain Rule we work from the outside to the inside... We differentiate the outer function f [at the inner function $g(x)$] and then we multiply by the derivative of the inner function (Stewart, 2016, p. 199).*

Although I find that this procedure removes the elegance of the Chain Rule, it still requires students to recognize the overall structure of the expression. Working from the outermost to the innermost structures, one's ability to distinguish between the independent and dependent variables, as well as the parameters of an expression will determine which differentiation rules to apply, and in which order. Since all the functions in Problem 4 require the use of various differentiation rules and are not provided with any context, no slope, no instantaneous velocity, and no instantaneous rate of change interpretations are required, this problem is characterized as an algebraic activity. However, even though it is not testing any conceptual understanding of derivatives, this problem requires more than simple rote memorization.

Problem 4 was given as follows:

Find the derivatives of the following functions. (You don't need to simplify the final answer, but you must show how you calculate it):

Part a) $f(x) = \frac{\sqrt[3]{x} - 2xe^x}{x}$

Part b) $f(x) = e^{sin\, 2x} + sin\,(e^{2x})$

Part c) $f(x) = ln\left(\frac{e^x}{x+2}\right) + e^2$

Part d) $f(x) = sin\,(tan\sqrt{1 + x^3})$

Part e) $f(x) = x^{cosx}$ *(use logarithmic differentiation)*

Expected Solutions:

Part (a) can either be solved by applying the Quotient Rule, or can be simplified prior to taking the derivative. In simplifying the expression, an expected solution is as follows: $f(x) = \frac{\sqrt[3]{x} - 2xe^x}{x} = \frac{x^{\frac{1}{3}} - 2xe^x}{x} = x^{\frac{-2}{3}} - 2e^x$. Then by applying the Power Rule $\left[\frac{d}{dx}(x^n) = nx^{n-1}\right]$, and knowing the derivative of the natural exponential function $\left[\frac{d}{dx}(e^x) = e^x\right]$, one obtains the following derivative: $f'(x) = -\frac{2}{3}x^{\frac{-5}{3}} - 2e^x$.

Part (b) requires the use of the Chain Rule. In knowing the derivative of the natural exponential function, as well as the derivative of the sine function $\left[\frac{d}{dx}(\sin x) = \cos x\right]$, one would arrive at the following solution: $f'(x) = e^{\sin 2x} \cdot \cos(2x) \cdot (2) + \cos\,(e^{2x}) \cdot (e^{2x}) \cdot (2) = 2\,e^{\sin 2x}\cos(2x) + 2\,e^{2x}\cos\,(e^{2x})$.

Part (c) can also be solved in two ways. One can either take the derivative directly, apply the Chain Rule and Quotient Rule to the natural logarithmic term, or one can use logarithmic laws $[log_b(xy) = log_b x + log_b y$, and $log_b\left(\frac{x}{y}\right) = log_b x - log_b y]$ to rearrange the function prior to taking the derivative. One must know the derivative of the natural logarithm $\left[\frac{d}{dx}(\ln x) = \frac{1}{x}\right]$, and recognize e^2 as a constant in order to obtain the correct derivative of the function: $f(x) = \ln\left(\frac{e^x}{x+2}\right) + e^2 = \ln(e^x) - \ln(x+2) + e^2 = x - \ln(x+2) + e^2$ and $f'(x) = 1 - \frac{1}{x+2} + 0 = 1 - \frac{1}{x+2}$.

Part (d) also requires the use of Chain Rule. Students need to know the derivative of the trigonometric functions $\left[\frac{d}{dx}(\sin x) = \cos x\right.$, and $\left.\frac{d}{dx}(\tan x) = \sec^2 x\right]$, and how to use the power rule: $f'(x) = \cos\left(\tan\sqrt{1+\ x^3}\right) \cdot \left(\sec^2\left(\sqrt{1+\ x^3}\right)\right) \cdot \left(\frac{1}{2}\ (1+\ x^3)^{-\frac{1}{2}}\right) \cdot 3x^2$.

Lastly, in **Part (e)** students were specifically asked to use logarithmic differentiation, which, in the textbook, is given as follows:

> ***Steps in Logarithmic Differentiation***
>
> *1. Take natural logarithm of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.*
>
> *2. Differentiate implicitly with respect to x.*
>
> *3. Solve the resulting equation for y' $(Stewart, 2016, p. 221)$.*

In order to use logarithmic differentiation one must know how to differentiate implicitly. Stewart describes using implicit differentiation when trying to find the derivative of an implicit function which cannot easily be solved for y explicitly as a function of x (or in some cases even impossible to do so by hand). The process is described as follows:

> *...Instead we use the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y' (Stewart, 2016, p. 209).*

Stewart also provides a summary to help students determine between which differentiation rules to use depending on the exponents and bases provided:

> *In general there are four cases for exponents and bases:*
>
> *1.* $\frac{d}{dx}(b^n) = 0$ *(Constant base, constant exponent)*
>
> *2.* $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$ *(Variable base, constant exponent)*
>
> *3.* $\frac{d}{dx}\left[b^{g(x)}\right] = b^{g(x)}(ln b) g'(x)$ *(Constant base, variable exponent)*
>
> *4.* $\frac{d}{dx}[f(x)]^{g(x)}$*, logarithmic differentiation can be used. (Variable base, variable exponent) (Stewart, 2016, p. 221).*

If we had not explicitly asked students to use logarithmic differentiation they could have solved it using another method, namely writing $x^{cosx} = \left(e^{lnx}\right)^{cosx}$. An expected solution using logarithmic differentiation is as follows:

$f(x) = x^{cosx}$

$y = x^{cosx}$

$\ln \mathrm{y} = \ln x^{cosx}$ (step 1: taking natural logarithm of both sides)

$\ln \mathrm{y} = \cos x \ln x$ (using the law of logarithm: $log_b(x^r) = rlog_b x$, where r is any real number)

$\frac{y\prime}{y} = -\sin x \ln x + \frac{\cos x}{x}$ (step 2: differentiate implicitly with respect to x; apply the product rule)

$y' = y(-\sin x \ln x + \frac{\cos x}{x})$ (step 3: solve for y' by multiplying both sides with y)

$y' = x^{cosx}(-\sin x \ln x + \frac{\cos x}{x})$ (lastly, replace y with $y = x^{cosx}$)

3.5 Problem 5a – Equation of the tangent line

Throughout the course, a typical problem was that of finding the equation of the tangent to a curve at a specific point. Once the interpretation of the derivative was given as the slope of the tangent line (as previously discussed in section 3.4 of this thesis), the tangent line problem came up in every single section where new differentiation rules were presented. Although, in order to solve such a problem, students need to have a proper interpretation of the derivative as the slope of the tangent line, by the end of the semester this type of problem is nothing but routine. Its solution requires algebraic processing of expressions and solving equations, treating variables statically as representing hidden, unknown or arbitrary numbers rather than dynamically, as co-varying, in a way that is characteristic of thinking in Calculus (Thompson & Carlson, 2017).

Problem 5a was given as follows:

> *Verify that the point* $(2,1)$ *belongs to the curve defined by the equation*
>
> $x^2 + 2xy + 4y^2 = 12$*, and find the equation of the tangent line to the curve at this point.*

Even from the start, this problem suggests that the point $(2,1)$ needs to belong to the curve, as students are asked to verify that it does. If we had provided a problem in which one needed to find the tangent lines to a curve passing through a point not on the curve, I wonder how many students would proceed to finding the derivative of the function and then find the slope at the given point, even if the point did not belong to the curve? One such problem provided in their WebWork assignment was as follows:

> *Find the equations to both lines through the point* $(2,2)$ *that are tangent to the parabola* $y = x^2 + x + 5$.

However, in having experienced this non-routine problem students may have inquired as to why their answer was incorrect. Perhaps they would have learned the lesson and learned to use a different approach to finding tangents to curves from points not lying on the curve? However, many students were asking about this problem during class, claiming that the WebWork system was wrong.

In principle, verifying that a given point, given by its coordinates, belongs to a curve given by an equation requires understanding the equation as a condition that is "satisfied" by the coordinates of all points of the curve and only those. The notion of "condition" and of a condition being "satisfied" are non-trivial logical and algebraic ideas. But students may successfully answer this question by having learned to associate the instruction of "verifying if a point belongs to a curve" with plugging in the coordinates of the point into the variables in the equation. This activity can hardly be qualified as algebraic. It is a numerical activity.

Expected Solution to Problem 5a:

In verifying that a point belongs to a curve, students are required to obtain a solution satisfying the condition that the left hand side needs to be equal to the right hand side.

Given the point $(2,1)$, and the equation $x^2 + 2xy + 4y^2 = 12$:

$x^2 + 2xy + 4y^2 = 12$

$2^2 + 2(2)(1) + 4(1)^2 = 12$

$4 + 4 + 4 = 12$ (True, the point $(2,1)$ belongs to the curve)

Secondly, in order to find the equation of the tangent line, as previously mentioned students needed to interpret the derivative as the slope of a tangent line to the curve at a specific point. In this particular problem, students needed to recognize that the equation is implicit, thus needing to calculate the derivative implicitly. Below is the implicit differentiation of the equation with respect to x:

$x^2 + 2xy + 4y^2 = 12$

$2x + 2y + 2xy' + 8yy' = 0$

$y'(2x + 8y) = -2(x + y)$

$y' = -\frac{(x+y)}{(x+4y)}$

This processing of an equation according to rules is an algebraic activity.

In order to find the slope of the tangent line at the point $(2,1)$, one must evaluate the derivative at that point: $y' = -\frac{(2+1)}{(2+4(1))} = -\frac{1}{2}$, which is a numerical activity. Having found the slope of the tangent line as $m = -\frac{1}{2}$, one can use the point-slope form in order to find the equation of the tangent line as follows:

$y - y_1 = m\,(x - x_1)$

$y - 1 = -\frac{1}{2}\,(x - 2)$

$$y = -\frac{1}{2}x + 2$$

This is also an algebraic activity: recalling a formula, recognizing what the different letters stand for, distinguishing between variables and parameters, substituting numerical values for some parameters, processing an equation to isolate a variable.

Thus the equation of the tangent line to the curve, at the point $(2,1)$ is $y = -\frac{1}{2}x + 2$.

3.6 Problem 5b – Related rates

Stewart provides the following paragraph in the introduction to the related rates section, followed by typical examples in order to demonstrate the procedure:

> *In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time (Stewart, 2016, p. 245).*

It must be stressed, however, that all these problems require computing the rate of change of a quantity at the moment when the rate of change of the other, related quantity has a given numerical value. This turns solving these problems into an algebraic activity. The problems are "about numbers and static variables" (Thompson & Carlson, 2017) rather than about co-varying quantities and describing the character of this covariance. The quoted authors give examples of problems where covariational quantitative reasoning is necessary, e.g., the well-known bottle-filling problem asking to sketch a graph of the height of the water as a function of its volume (Carlson, 1998). The authors report that in Carlson's study, most students who performed highly in Calculus courses based on algebraic-procedural approaches such as Stewart's were unable to construct an appropriate graph.

In using the Chain Rule we are differentiating both sides of an equation implicitly with respect to time (t). Stewart also offers a step-by-step procedure, to solving the specific type of related rates problems:

> ***Problem Solving Strategy***
>
> *1. Read the problem carefully.*
>
> *2. Draw a diagram if possible.*
>
> *3. Introduce notation. Assign symbols to all quantities that are functions of time.*
>
> *4. Express the given information and the required rate in terms of derivatives.*
>
> *5. Write an equation that relates the various quantities of the problem. If necessary, use geometry of the situation to eliminate one of the variables by substitution.*
>
> *6. Use the Chain Rule to differentiate both sides of the equation with respect to t.*
>
> *7. Substitute the given information into the resulting equation and solve for the unknown rate (Stewart, 2016, p. 247).*

Students tend to have many difficulties with related rates problems. An analysis of these difficulties from the perspective of APOS theory can be found in (Tziritas, 2011). I assume a number of different reasons are at play. First, these problems require one to read multiple sentences and parse out the related information, which requires reading comprehension. Second, conceptual algebraic knowledge is required: distinctions among the various roles that letters can play: variables, unknown and known quantities, functions of other variables, functions of time. Lastly, these problems require students to set up equations that relate the quantities, and sometimes use geometry to eliminate one of the variables.

Problem 5b was given as follows:

> *A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4,2)$, the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?*

Expected Solution to Problem 5b:

The students needed to recognize this as a related rates problem, and to treat x and y as functions of time as well as functions of each other. In using the Chain Rule and differentiating implicitly with respect to time the following is the expected solution:

$$\frac{d}{dt}(xy) = \frac{d}{dt}(8)$$

$$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$$

Since we want to know how fast the x-coordinate is changing at point $(4,2)$, we solve for $\frac{dx}{dt}$.

$$\frac{dx}{dt} = -\frac{x}{y}\frac{dy}{dt}$$

We can now substitute the given quantities: the y-coordinate is decreasing at a rate of 3 cm/s, therefore we replace $\frac{dy}{dt}$ by -3, and since we want to know the rate of change of the x-coordinate at the point $(4,2)$ we replace x with 4, and y with 2:

$$\frac{dx}{dt} = -\frac{4}{2}(-3) = 6$$

Therefore the x-coordinate is increasing at a rate of 6 cm/s.

Students tend to make errors in writing out the units of the rate obtained, or forget them altogether. Also a common error includes students replacing the given quantities before differentiation despite warnings from the author:

> ***WARNING*** *A common error is to substitute the given numerical information (for quantities that vary with time) too early. This should be done only after differentiation (Stewart, 2016, p. 247).*

3.7 Problem 5c – L'Hospital's Rule

In evaluating limits of functions as x tends to a, sometimes we arrive at indeterminate forms such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when attempting the Direct Substitution Property. This occurs when both the function in the numerator and the function in the denominator tend to 0, or to ∞ (or $-\infty$). It is unclear whether we will obtain an infinite limit, a limit equal to zero, or a limit equal to some finite number. In such cases, a method known as l'Hospital's Rule can be used (also known as l'Hôpital's Rule):

> ***L'Hospital's Rule*** *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that*
>
> $lim_{x\to a} f(x) = 0$ *and* $lim_{x\to a} g(x) = 0$
>
> *or that* $lim_{x\to a} f(x) = \pm\infty$ *and* $lim_{x\to a} g(x) = \pm\infty$
>
> *(In other words, we have an indetermination form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.)*
>
> *Then*
>
> $lim_{x\to a} \frac{f(x)}{g(x)} = lim_{x\to a} \frac{f\prime(x)}{g'(x)}$ *if the limit on the right side exists (or is ∞ or $-\infty$)...*
>
> *It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule (Stewart, 2016, p. 306).*

The author also warns that when using l'Hospital's Rule, one differentiates the numerator and denominator separately, and not to confuse this with the Quotient Rule. The section in the textbook continues with different types of indeterminate forms such as indeterminate products $(0 \cdot \infty)$, differences $(\infty - \infty)$, and powers $(0^0, \infty^0, 1^\infty)$. Each can be written differently in order to obtain the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to be able to use l'Hospital's Rule. Limits of the type $(0 \cdot \infty)$ can be converted by writing the product as a quotient. Limits of the type $(\infty - \infty)$ can be converted by finding the common denominator between both terms. Limits of the types $(0^0, \infty^0, 1^\infty)$ can be converted by taking the natural logarithm of both sides of the function, or by writing the function as an exponential. In all such cases, students are expected to verify the conditions prior to using l'Hospital's Rule. In the previous final exams, as well as the one I wrote, students were told when to use the l'Hospital's Rule. Thus, no decision needed to be made by the students. If they were told to use l'Hospital's Rule, then they could assume that the functions at hand met all the conditions.

Problem 5c was given as follows:

Use the l'Hôpital's rule to evaluate the $lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - sin\, x}$

It is expected that students know when and how to use l'Hospital's rule, and when to stop using it. It is also expected that they should have a fairly good understanding of indeterminate forms. This problem, however, was already in the indeterminate form required to use l'Hospital's Rule, and the students were instructed to use the procedure. Thus, another algebraic activity.

Expected Solution of Problem 5c:

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \overset{H}{\Rightarrow} \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} \overset{H}{\Rightarrow} \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin\, x} \overset{H}{\Rightarrow} \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos\, x} = \frac{1+1}{1} = 2$$

3.8 Problem 6 – The Mean Value Theorem

The problem alludes to the Mean Value Theorem; the theorem guarantees the existence of a solution to the part (b) of the problem. Like the Intermediate Value Theorem, the Extreme Value Theorem, Fermat's Theorem, and Rolle's Theorem, the Mean Value Theorem is considered an existence theorem. It guarantees the existence of a number c under the following conditions: given a function $y = f(x)$, that is continuous on some closed interval $[a, b]$, and differentiable on the open interval (a, b), the Mean Value Theorem asserts that there will be at least one point $(c, f(c))$ in the interval (a, b), such that the instantaneous rate of change at c will be equal to the average rate of change over $[a, b]$. Geometrically the tangent line drawn at the point $(c, f(c))$ will be parallel to the secant line joining points $(a, f(a))$ and $\big(b, f(b)\big)$. Thus, the slope of the tangent line at that point will be equal to the slope of the secant line. Further, "the main significance of the Mean Value Theorem is that it enables us to obtain information about a function from the information about its derivative" (Stewart, 2016, p. 290). The section on the Mean Value Theorem in the textbook starts off with Rolle's Theorem as follows:

> ***Rolle's Theorem*** *Let* f *be a function that satisfies the following three hypotheses:*
>
> *1.* f *is continuous on the closed interval* $[a, b]$*.*
>
> *2.* f *is differentiable on the open interval* (a, b)*.*
>
> *3.* $f(a) = f(b)$
>
> *Then there is a number* c *in* (a, b) *such that* $f'(c) = 0$ *(Stewart, 2016, p. 287).*

Rolle's Theorem is followed by two examples, one of which requires proving that an equation has exactly one real root, via an argument by contradiction. Stewart also mentions that one of the main uses of Rolle's Theorem is to prove the Mean Value Theorem which was stated as follows:

> ***The Mean Value Theorem*** *Let f be a function that satisfies the following hypotheses:*
>
> *1. f is continuous on the closed interval $[a, b]$.*
>
> *2. f is differentiable on the open interval (a, b).*
>
> *Then there is a number c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b-a}$ or, equivalently,*
>
> $f(b) - f(a) = f'(c)(b - a)$ *(Stewart, 2016, p. 288).*

Students tend to confuse theorems, and complain that there are a lot of different theorems to remember. It is as though they are learning new, disconnected theorems each time. They are unable to see the relationships among them. For example, students have a hard time recognizing that Rolle's Theorem is a particular case of the Mean Value Theorem, in which $f(b) - f(a) = 0$, thus $f'(c) = 0$.

All this being said, however, this knowledge was necessary for the author of the exam question – it guaranteed the existence of a solution – but not for the student solving it. It was possible to solve problem 6 correctly without knowledge of the Mean Value Theorem. Only elementary algebraic knowledge was needed.

Problem 6 was given as follows:

> *Let $f(x) = x^3 - 3x + 2$*
>
> *Part a) Find the slope m of the secant line joining the points $(-2, f(-2))$ and $(2, f(2))$.*
> *Part b) Find all points $x = c$ (if any) on the interval $[-2, 2]$ such that $f'(c) = m$.*

Expected Solution to Problem 6a:

The students needed to recognize a different function notation, namely $f(-2)$ in order to plug in -2 for x in the equation as such:

$$f(-2) = (-2)^3 - 3(-2) + 2 = -8 + 6 + 2 = 0$$

$$f(2) = (2)^3 - 3(2) + 2 = 8 - 6 + 2 = 4$$

This alone is a numeric activity. It is enough to understand letters as placeholders (Küchemann, 1981).

The points joining the secant line are $(-2, 0)$ and $(2, 4)$. Thus, the slope formula gives the following slope: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{2-(-2)} = 1.$

This can be regarded as an algebraic and numeric activity, as students needed to recall the slope formula, and substitute the numerical values for (x_1, x_2) and (y_1, y_2).

Expected Solution to Problem 6b:

This problem is solvable. Given that the function is a polynomial, it is continuous over $\mathbb{R}$, thus continuous over $[-2, 2]$. It is differentiable over $\mathbb{R}$, thus differentiable over $(-2, 2)$. Then according to the Mean Value Theorem, there exists a number c such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

As mentioned above, however, the problem can be solved without being aware of the Mean Value Theorem's existence. One only needs to: 1) know how to differentiate polynomials (applying a general formula to a particular case which requires distinguishing variables from parameters but is essentially an algebraic activity) to calculate $f'(x) = 3x^2 - 3$; 2) be familiar with the meaning of the functional notation of the type "$f(x)$" to interpret the given equation $f'(c) = m$ as $3c^2 - 3 = m$; and 3) solve this equation for c, using the value for m obtained in part (a). The equation is a quadratic equation, thus the common error could be that of missing a solution, but it is a very simple quadratic equation.

Given $f(x) = x^3 - 3x + 2$

$f'(c) = 3c^2 - 3$

Since $m = 1$,

$3c^2 - 3 = 1$

Solving for c:

$3c^2 = 4$

$c = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$

Since both $\pm\frac{2}{\sqrt{3}}$ are in the interval $(-2, 2)$, then the answer is: $c = \pm\frac{2}{\sqrt{3}}$.

This is also another algebraic activity: finding the derivative of the function at hand, substituting m with 1 (as calculated in part a), and processing the equation to solve for c. In this particular problem, the values obtained for c fell in the interval $(-2, 2)$, thus students who did not verify whether the calculated values for c fell in the interval would have still obtained a correct answer.

3.9 Problem 7a – Definition of the derivative

In section 3.4, we discussed the different interpretations of derivatives, and how derivatives were defined as a special type of limit: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. When asked to find the derivative using the **definition of the derivative**, students are expected to use this limit. Calculation of the limit involves applying the theorem about the limit of a sum of functions, but the use of algebra is definitely required in this problem, specifically distributivity (from binomial expansions to factoring, etc.).

Problem 7a was given as follows:

Consider the function $f(x) = x^3 - 2x^2 + 1$

Use the definition of the derivative to find the formula for $f'(x)$*.*

Expected Solution to Problem 7a:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - 2(x+h)^2 + 1 - [x^3 - 2x^2 + 1]}{h} =$$

$$= \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3] - 2[x^2 + 2xh + h^2] + 1 - [x^3 - 2x^2 + 1]}{h} =$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 4x - 2h)}{h} = \lim_{h \to 0}(3x^2 + 3xh + h^2 - 4x - 2h) = 3x^2 - 4x$$

Applying this definition is not rote memorization, since substituting every x term by $(x + h)$ in $f(x + h)$ requires students to understand the structure of the function. Students tend to make mistakes of all sorts, from forgetting the " $+ 1$" in the $f(x + h)$ portion, to multiple errors with distributivity. Although applying the definition of derivative may be a Calculus activity, this problem requires much routine algebra to be solved correctly. There is no conceptual understanding of derivative that is necessary to complete this problem, other than that the derivative of a function is also a function. The algebraic goal of this problem is to expand the numerator, and factor an h term, with the goal of "cancelling it out" with the h in the denominator. This is done to remove the indeterminacy in the denominator, and to apply the Direct Substitution Property to obtain the limit, and ultimately the derivative. Additionally, once the students obtain their answer, they could have easily verified if it was correct using the memorized laws of differentiation. If they had done so, those that had made algebraic errors could have reviewed their solutions and corrected their mistakes.

3.10 Problems 7b and 7c – Linear approximations and differentials

Linear approximations to functions are useful when accuracy can be slightly overlooked. As example, a computer program can save hours or even days by approximating a complicated function near a point by a linear function. Other applications can be found in the theory of optics, where the linear approximations $\sin\theta \approx \theta$, and $\cos\theta \approx 1$ are used to design lenses (Stewart, 2016, pp. 251-254). As a geometrical approach, Stewart reminds the readers that when zooming in closer and closer to a point on a curve of a differentiable function, the graph looks more like that of a tangent line drawn at that point. Further, he mentions that given a function, we may be able to calculate $f(a)$, however we may have more difficulty in trying to calculate values near a; in fact this task may be impossible for some functions. A way to get around this issue is to use the values of a linear tangent line drawn at $(a, f(a))$, provided that the values are close to a. Stewart provides the following explanation to linear approximations:

> *...we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is $y = f(a) + f'(a)(x-a)$ and the approximation $f(x) \approx f(a) + f'(a)(x-a)$ is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is, $L(x) = f(a) + f'(a)(x-a)$ is called the **linearization** of f at a (Stewart, 2016, p. 252).*

Although it is called the linearization of f at a, it is nothing more than the equation of a tangent line at the point $(a, f(a))$. Thus finding the linearization entails the same procedure and algebraic skills as finding the equation of a tangent line to the graph of a function at a point where it is differentiable.

Differentials are then introduced in the textbook as a different notation and terminology for linear approximations.

> *If $y = f(x)$, where f is a differentiable function, then the **differential** dx is an independent variable; that is, dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation $dy = f'(x)dx$. So dy is a dependent variable; it depends on the values of x and dx...Let $P(x, f(x))$ and $Q(x+\Delta x, f(x+\Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding*

change in y is $\Delta y = f(x + \Delta x) - f(x)$. The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)dx = dy$. Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx (Stewart, 2016, p. 254).

Stewart provides an example, in which he compares Δy to dy, and demonstrates how the differential dy is a good approximation to Δy, and becomes more accurate when Δx is smaller. Further, dy appears to be easier to compute than Δy. To find a linear approximation to a curve using differentials, the linear approximation can be written as $f(a + dx) \approx f(a) + dy$, where $dy = f'(x)dx$. Finally, Stewart ends the section with an example using differentials to estimate errors that can occur with approximate measurements (Stewart, 2016, pp. 254-256).

Problem 7b was given as follows:

Consider the function $f(x) = x^3 - 2x^2 + 1$

Write the linearization formula for f at $a = 2$.

Expected Solution to Problem 7b:

$f(x) = x^3 - 2x^2 + 1$

$f(2) = (2)^3 - 2(2)^2 + 1 = 1$

$f'(x) = 3x^2 - 4x$

$f'(2) = 3(2)^2 - 4(2) = 4$

$L(x) = f(a) + f'(a)(x - a)$

$L(x) = f(2) + f'(2)(x - 2)$

$L(x) = 1 + 4(x - 2) = 4x - 7$

This is a numerical and algebraic activity. Students needed to recall the linearization formula, and that we can use a tangent line to approximate the curve when x is near a. There can be confusion in distinguishing between the variable and parameters, x, a, $f(a)$, and $f'(a)$. In $f(x)$ and in $f'(x)$ students need to evaluate the functions at a. In $L(x)$, x remains unchanged as the independent variable, while $f(a)$ and $f'(a)$ are substituted by their numerical values.

Problem 7c was given as follows:

Consider the function $f(x) = x^3 - 2x^2 + 1$

Find the differential dy *and evaluate it for the values* $x = 2$ *and* $dx = 0.2$*.*

Expected Solution to Problem 7c:

$y = x^3 - 2x^2 + 1$

$\frac{dy}{dx} = 3x^2 - 4x$

$dy = (3x^2 - 4x)\, dx$

By replacing $x = 2$ and $dx = 0.2$

$dy = (3(2)^2 - 4(2))\,(0.2) = (4)(0.2) = 0.8$

Although differentials entail an understanding of derivatives with the geometrical interpretation of dx and x, and dy and Δy, a student could solve this problem by applying a memorized formula. That being said, the ability to distinguish between variables from parameters in applying this formula is not a trivial matter. The algebraic activity was that of finding the derivative of the function and then knowing the differential dy is defined in terms of dx by $dy = f'(x)dx$. Further, if they had used Leibniz notation as above, multiplying the left and right-hand sides of the equation (in step 2) by dx results in the correct differential equation $dy = f'(x)dx$. In substituting the numerical values for x and dx, this part is simply a numerical activity. Thus once more this problem is an algebraic and a numerical activity.

3.11 Problem 8a – Absolute extreme values

In section 4.1 of the Stewart textbook (pp. 275-286), local and absolute extreme values are presented, along with Fermat's Theorem, the definition of critical numbers, and the Closed Interval Method. The section starts with different examples of optimization problems, in which the author mentions that optimization is one of the most important applications of Differential Calculus. Examples provided include the shape of a can that minimizes costs, the angle that blood vessels should branch in order to reduce the energy expended by the heart, etc. The author also mentions that these problems can easily be solved by reducing them to finding the extreme values (maximum and minimum values) of a function. The section starts with the definition of absolute extrema:

Definition *Let c be a number in the domain D of a function f. Then $f(c)$ is the*

absolute maximum *value of f on D if $f(c) \geq f(x)$ for all x in D.*

absolute minimum *value of f on D if $f(c) \leq f(x)$ for all x in D (Stewart, 2016, p. 276).*

This definition is provided alongside a graph of a function f, which demonstrates all extreme values, both the absolute and local. The definition of local extrema shortly follows:

Definition *The number $f(c)$ is a*

local maximum *value of f if $f(c) \geq f(x)$ when x is near c.*

local minimum *value of f if $f(c) \leq f(x)$ when x is near c.*

...if we say that something is true ***near*** *c, we mean that it is true on some open interval containing c (Stewart, 2016, p. 276).*

In the examples that follow, the author graphs extrema of functions, and points out that endpoints could only be absolute extreme values (if they satisfy the definition), and not local extrema. After providing such examples the author introduces a theorem, which guarantees the existence of extrema.

The Extreme Value Theorem *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$ (Stewart, 2016, p. 278).*

Following this definition are a couple of graphs demonstrating various types of functions and the existence of both absolute extrema on a closed interval. Also, the author provides examples in which functions do not have absolute extrema if a function fails to be continuous or if continuous but on an open interval. In both cases, students have difficulty understanding why at an open endpoint, a function cannot have an absolute extreme value. An example given in class was as follows: given a continuous increasing function in some open interval $(1, 2)$, the point $(1, f(1))$ appears to be an absolute minimum, even though $f(1)$ is not defined. I asked the students whether the function can be evaluated at $f(1)$, and they generally responded correctly in agreeing that this cannot be done. I then asked them to provide me with a number, say x_1 which is larger than 1, and close to 1 (but not equal to 1), such that $f(x_1)$ can be found. For whichever number $f(x_1)$ they provided, I could find a number $x_0 < x_1$ such that $f(1) < f(x_0) < f(x_1)$. A similar argument is provided for absolute maximum at an open endpoint.

Thus, absolute extrema cannot occur at endpoints. Understanding the definitions of absolute and local extrema is a calculus-related activity, and ties in with the concept of continuity.

Following these definitions, the author remarks that at local extreme points, the graph of the function that is differentiable at those points has horizontal tangent lines with slopes equal to zero, and provides the following theorem:

> ***Fermat's Theorem*** If f *has a local maximum or minimum at* c*, and if* $f'(c)$ *exists, then* $f'(c) = 0$ *(Stewart, 2016, p. 279).*

This theorem, if recalled incorrectly leads students to believe that whenever $f'(c) = 0$, the function has a local maximum or minimum: it is a common logical mistake to interpret an implication as an if and only if statement. Thus, they believe the converse of this theorem to be true. Also that if a derivative does not exist at a specific point, then no local extrema could be present. The author tries to caution against these misconceptions with two examples. One of them is the function $f(x) = x^3$ which has a horizontal tangent at the point $(0, 0)$, thus $f'(0) = 0$ however no local extreme value exists at that point. The second example was that of the absolute value function $f(x) = |x|$, which has a local minimum at $(0, 0)$, however is not differentiable at $(0, 0)$. The author does mention that we should start looking at the points in which $f'(c) = 0$ or where $f'(c)$ does not exist, for *potential extrema*. He provides the following definition:

> ***Definition*** *A* ***critical number*** *of a function* f *is a number* c *in the domain of* f *such that either* $f'(c) = 0$ *or* $f'(c)$ *does not exist (Stewart, 2016, p. 280).*

The section ends with the Closed Interval Method, which is given as a three-step procedure to finding extrema of a continuous function on a closed interval, hence the name. It is used to find local extrema that can occur at critical numbers, and absolute extrema which will occur at critical numbers, or at the endpoints. Thus this procedure guarantees finding absolute extrema of a continuous function on a closed interval.

> ***The Closed Interval Method*** *To find the absolute maximum or minimum values of a continuous function* f *on a closed interval* $[a, b]$*:*
>
> *1. Find the values of* f *at the critical numbers of* f *in* (a, b)*.*
>
> *2. Find the values of* f *at the endpoints of the interval.*

3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value (Stewart, 2016, p. 281).

Problem 8a was given as follows:

Find the absolute maximum and minimum values of $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ *on the interval* $[-2, 3]$.

Expected Solution to Problem 8a:

Using the Closed Interval Method, one can find the absolute maximum and minimum on the closed interval, given that the function is continuous on the interval. Since the function is a polynomial, it is continuous everywhere, hence it is continuous on $[-2, 3]$.

$f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

$f'(x) = 12x^3 - 12x^2 - 24x$

In order to find the critical numbers, one can use the distributive property to factor out the common term of $12x$, and then factor the quadratic $(x^2 - x - 2)$ into $(x - 2) \cdot (x + 1)$.

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1)$

Setting each factor equal to 0, the critical numbers are $x = -1$, $x = 0$, and $x = 2$. Note, no other critical points are found, as the derivative is defined for all values of x. The Closed Interval Method requires testing the critical numbers as well as the endpoints of the closed interval.

$f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 1 = 33$ (here is the absolute maximum)

$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 1 = -4$

$f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 + 1 = 1$

$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 1 = -31$ (here is the absolute minimum)

$f(3) = 3(3)^4 - 4(3)^3 - 12(3)^2 + 1 = 28$

Therefore, $(-2, 33)$ is the absolute maximum point, and $(2, -31)$ is the absolute minimum point of the function on the closed interval $[-2, 3]$. Evaluating the function at the critical points, and at the endpoints is a numerical activity.

3.12 Problem 8b – Optimization problem

Finding optimal solutions to problems dates back to the 17th century. Early historians have perhaps slightly misunderstood Fermat's work regarding his method (or methods) for finding maximum and minimum values. Nonetheless, his quest for optimal solutions was apparent in his research. Strømholm provides a good summary in explaining Fermat's methods, as well as the confusion of the early historians (1968). In Fermat's explanation of the foundations of his method(s) to Brûlart and Mersenne, he provides an example regarding dividing a segment which would produce the largest rectangle in a given area (Strømholm, 1968). Thus, finding optimal solutions to problems has been around for quite some time.

Optimization problems are introduced in the textbook as problems of maximizing and minimizing certain quantities. Thus "maximizing areas, volumes, and profits, and minimizing distances, times, and costs" (Stewart, 2016, p. 330). In the preceding sections, the students are taught how to find extreme values of a function. In this section they are introduced to a practical application of finding extreme values. Much like the related rates problems, students have much difficulty in parsing out the relevant information from a word problem, and in setting up the equations relating the knowns and unknowns. Stewart provides the following problem-solving steps tailored to Optimization Problems:

> ***Steps in Solving Optimization Problems***
>
> *1.* ***Understand the Problem*** *The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?*
>
> *2.* ***Draw a Diagram*** *In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.*
>
> *3.* ***Introduce Notation*** *Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols - for example, A for area, h for height, t for time.*
>
> *4. Express Q in terms of some other symbols from Step 3.*
>
> *5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables.*

Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of one variable x, say, $Q = f(x)$. Write the domain of this function in the given context.

6. Use the methods of Sections 4.1 and 4.3 to find the absolute maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 4.1 can be used (Stewart, 2016, pp. 330-331).

The section continues with a number of examples, demonstrating the steps above. Once a critical point is found, the author uses different ways to show that this point is an absolute extreme value. Recall that the converse of Fermat's Theorem is false; finding a point in which $f'(c) = 0$ does not immediately mean that the function will have a local maximum or minimum at that point.

One example that the author provides is one that includes a closed interval, thus the Closed Interval Method is used. Another example demonstrates a variation of the First Derivative Test which is used to find local extreme values (presented in the Section 4.3 of the textbook on graphing):

First Derivative Test for Absolute Extreme Values *Suppose that c is a critical number of a continuous function f defined on an interval.*

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f (Stewart, 2016, p. 333).

Alternatively, one could use the Second Derivative Test along with the Concavity Test with a slight variation to find the absolute extrema, provided that the second derivative exists, and is not equal to zero. The Second Derivative Test is given as follows:

The Second Derivative Test *Suppose that f'' is continuous near c.*

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c (Stewart, 2016, p. 297).

The Concavity Test is given as follows:

Concavity Test

(a) If $f''(x) > 0$ *for all* x *in* I*, then the graph of* f *is concave upward on* I.

(b) If $f''(x) < 0$ *for all* x *in* I*, then the graph of* f *is concave downward on* I *(Stewart, 2016, p. 296).*

Notice that if $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c. However this local minimum is also an absolute minimum if $f''(x) > 0$ for all x. The graph of f is concave upward over its entire domain, thus f has an absolute minimum at c. Also notice that if $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c. However this local maximum is also an absolute maximum if $f''(x) < 0$ for all x. The graph of f is concave downward over its entire domain, thus f has an absolute maximum at c.

Much like the related rates problems, students have great difficulties in setting up the equations that relate the quantities, as well as eliminate some of the variables in order to have the maximizing or minimizing function as a function of one variable. Students need to recall different geometric shapes and objects, as well as their perimeters, areas, and volumes, which are part of the secondary school curriculum. They need to be able to identify the known and unknown quantities, to set up the equations relating the quantities, and to differentiate the appropriate equation. Again students are required to have some conceptual algebraic knowledge such as being able to distinguish between the various roles that letters can play: variables, unknown and known quantities, and functions of other variables. The setting up of equations, eliminating some variables, and finding the critical point(s), are all algebraic activities. Substituting the critical point value to find the other unknowns is a numerical activity. The only activity which could be considered a calculus-related activity, is their conceptual understanding of absolute versus local extrema, and their verification that the critical point obtained is an absolute extreme value. In all of the optimization problems provided in the section, not one resulted in obtaining a critical point which was not the required absolute extreme value. This is somewhat expected as the goal of optimization problems is to maximize or minimize a certain quantity. Perhaps an example such as maximizing a volume, in which the materials to construct the object are infinite, would be useful. The critical point obtained could be that of an absolute minimum, and the graph of the function can be concave upwards, increasing without bound. In this scenario, the volume would not be able to be maximized, as the volume increases without bound. Most optimization problems are routine, however, and students know that the critical point obtained will be the absolute extreme value. They know it to be so, and as such some do not verify whether it is an absolute extreme value.

Problem 8b was given as follows:

> *A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the surface area.*

Expected Solution to Problem 8b:

This optimization problem requires students to know the formula for the volume of a box with a square base, as well as know the formula for the surface area. If students do not carefully read the problem they will include the top of the box in the surface area.

In terms of the Steps provided by the author:

Step 1: to understand the problem.

Step 2: to draw a diagram (the object being a rectangular box with open top is not drawn here).

Step 3: to introduce notation.

Let V represent the volume, S represent the surface area, h represent the height of the box, and x represent one side of the base of the box.

Step 4 and 5: to express the quantity to be minimized in terms of the other quantities, and to eliminate other variables.

$S = x^2 + 4xh$ (surface area to be minimized)

$V = x^2h$ and $x^2h = 32000$

Thus the variable to be eliminated will be h, where $h = \frac{32000}{x^2}$.

$S = x^2 + 4xh = x^2 + 4x\left(\frac{32000}{x^2}\right) = x^2 + \frac{128000}{x}$ (representing the surface area as a function of x alone),

Step 6: was to use the previously learnt methods to find the absolute minimum. Differentiating with respect to x:

$S'(x) = 2x - \frac{128000}{x^2}$

In order to find the critical numbers we let the derivative of the surface area equal to zero. Note that the derivative is undefined when $x = 0$, however $x = 0$ is not part of the domain of $S(x)$.

$2x - \frac{128000}{x^2} = 0$

$2x = \frac{128000}{x^2}$

$2x^3 = 128000$

$x^3 = 64000$

$x = 40$

Using the first derivative test for absolute extreme values, we can confirm that $x = 40$ is the absolute minimum.

Since $S'(x) < 0$ for $0 < x < 40$, the function is decreasing, and $S'(x) > 0$ for $x > 40$, then the function is increasing. Therefore $x = 40$ minimizes the surface area.

Lastly, we substitute the value of x into the following equation to solve for h.

$h = \frac{32000}{40^2} = 20$

Therefore the box has dimensions 40 cm x 40 cm x 20 cm.

3.13 Problem 9 – Curve sketching

In this problem, students use what they learned about first and second derivatives, and how these derivatives affect the shape of the graph of f. The information provided is used to graph the function. Section 4.3 of the textbook starts with the following test:

> ***Increasing/Decreasing Test***
>
> *(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.*
>
> *(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval (Stewart, 2016, p. 293).*

Stewart also offers a method for setting up a table in which the first column includes the intervals which were found by finding the critical numbers, and where the function is undefined. The next few columns include the factors of the first derivative, the second to last column is the sign of $f'(x)$, and the last column describes where f is increasing and decreasing. An example of this table will be shown in the solution to this problem. In looking at the constructed table, where one notices a change in the sign of $f'(x)$ coincides with local maxima and minima, provided that the points are critical numbers. Stewart offers the following test to determine local extrema:

> ***The First Derivative Test*** *Suppose that c is a critical number of a continuous function f.*
>
> *(a) If f' changes from positive to negative at c, then f has a local maximum at c.*
>
> *(b) If f' changes from negative to positive at c, then f has a local minimum at c.*
>
> *(c) If f' is positive to the left and right of c, or negative to the left or right of c, then f has no local maximum or minimum at c (Stewart, 2016, p. 294).*

If c is not a critical number, we can be dealing with vertical asymptotes, which are important, however a change in sign of $f'(x)$ at a vertical asymptote does not indicate a local extreme value. The concavity test, as well as the second derivative test have already been presented. Using the second derivative, a similar table to the first derivative can be prepared. Lastly, one definition remains:

> ***Definition*** *A point P on a curve $y = f(x)$ is called an* ***inflection point*** *if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P (Stewart, 2016, p. 297).*

Finally, the author provides guidelines for curve sketching, which summarizes all the important information required to sketch a curve, from the domain and intercepts, to the use of the second derivative to determine intervals of concavity. Note that a paraphrased version of the guidelines are presented, as all the tests have already been discussed throughout this thesis:

> ***Guidelines for Curve Sketching***
>
> ***A. Domain:*** *the domain of a function will help with sketching, as well as for finding critical points.*
>
> ***B. Intercepts:*** *the x and y intercepts will help with sketching. The author also mentions that finding the zeros of a function can sometimes be too difficult, thus it can be omitted.*

C. Symmetry: *if a function is even or odd, this will facilitate the sketch. For even functions, the graph is reflected about the y-axis; for odd functions, the graph is rotated about the origin by 180 degrees.*

D. Asymptotes: *vertical, horizontal, and/or slant asymptotes will help one determine what happens near points where the function is undefined, as well as what happens to the graph as x tends to positive and negative infinity.*

E. Intervals of Increase or Decrease: *steps include finding the first derivative of the function, finding the critical points, and using the Increasing/Decreasing Test.*

F. Local Maximum or Minimum Values: *steps include using the First Derivative Test or the Second Derivative Test.*

G. Concavity and Points of Inflection: *steps include computing the second derivative and using the concavity test. Further the points of inflection are determined as per the definition.*

H. Sketch the Curve: *Use all the information above to sketch the curve (Stewart, 2016, pp. 315-316).*

Problem 9 was given as follows:

Given the function $f(x) = 2 + 3x^2 - x^3$

Part a) Find the domain of the f and check for symmetry. Find asymptotes of f (if any).

Part b) Calculate $f'(x)$ and use it to determine intervals where the function is increasing, intervals where it is decreasing, and the local extrema (if any).

Part c) Calculate $f''(x)$ and use it to determine intervals where the function is concave upward, intervals where the function is concave downward, and the inflection points (if any).

Part d) Sketch the graph of the function $f(x)$ using the information obtained above.

Expected Solution to Problem 9a:

Since the function is a polynomial, it is defined for all values of x, therefore the domain is all real numbers. In order to check for symmetry, one must test if the function is even or odd.

$f(-x) = 2 + 3(-x)^2 - (-x)^3 = 2 + 3x^2 + x^3$

$f(-x) \neq f(x)$ therefore the function is not even

$f(-x) \neq -f(x)$ therefore the function is not odd

There are no vertical asymptotes as the function is defined for all values of x. There are also no horizontal asymptotes as the limits at infinity increase and decrease without bound. There are no slant asymptotes either.

$\lim_{x\to\infty} 2 + 3x^2 - x^3 = -\infty$

$\lim_{x\to-\infty} 2 + 3x^2 - x^3 = \infty$

Finding the domain of a function, checking for symmetry, and finding asymptotes can be all regarded as algebraic activities. We have already discussed that domains and functions are part of the definition of algebra. Checking for symmetry requires substituting x by $-x$. Thus, this is not simply numeric, students need to be able to view x as a variable and not a placeholder. Finally, in finding the asymptotes, we already discussed how finding these limits tends to be quite algebraic in nature. In this problem however, since we were dealing with a polynomial of degree 3, no asymptotes are found.

Expected Solution to Problem 9b:

$f(x) = 2 + 3x^2 - x^3$

$f'(x) = 6x - 3x^2 = 3x(2 - x)$

In order to find the critical numbers, set $f'(x) = 0$. This gives critical numbers at $x = 0$ and $x = 2$.

Intervals	$\boldsymbol{3x}$	$\boldsymbol{(2 - x)}$	$\boldsymbol{f'(x)}$	$\boldsymbol{f(x)}$
$(-\infty, 0)$	-	+	-	Decreasing on $(-\infty, 0)$
$(0,2)$	+	+	+	Increasing on $(0,2)$
$(2, \infty)$	+	-	-	Decreasing on $(2, \infty)$

Therefore the function is decreasing on intervals $(-\infty, 0)$ and $(2, \infty)$, and increasing on $(0,2)$. To find local extrema, the First Derivative Test can be used. If $f'(x)$ changes from positive to negative at a critical number c, then $f(x)$ has a local maximum at c. If $f'(x)$ changes from negative to positive at a critical number c, then $f(x)$ has a local minimum at c. Therefore at critical number $x = 0$ we have a local minimum, and at $x = 2$ we have a local maximum, where $f(0) = 2$ and $f(2) = 6$.

In this part, finding the derivative of a common function is an algebraic task. Using distributivity to find the factors of $3x$ and $(2 - x)$ to then find the critical numbers is yet another algebraic task. Understanding that the multiplication of the signs of the factors determine the sign of the derivative, requires the use of the following arithmetic rules:

If $a > 0$ and $b > 0$, then $(a)(b) = ab > 0$.

If $a > 0$ and $b < 0$, (or vice versa) then $(a)(b) = ab < 0$.

If $a < 0$ and $b < 0$, then $(a)(b) = ab > 0$.

Creating a table with the columns of the factors of the derivative was not necessary. Alternatively, students could instead chose a number in the interval $(-\infty, c)$ and evaluate it in $f''(x)$, and do the same for the other intervals.

Expected Solution to Problem 9c:

$f'(x) = 6x - 3x^2$

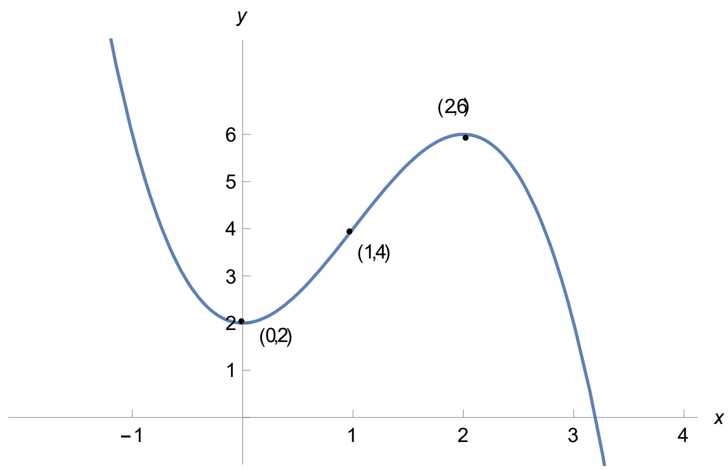
$f''(x) = 6 - 6x = 6(1 - x)$. The possible inflection point can occur when $x = 1$.

Intervals	$\boldsymbol{(1 - x)}$	$\boldsymbol{f''(x)}$	$\boldsymbol{f(x)}$
$(-\infty, 1)$	+	+	Concave up on $(-\infty, 1)$
$(1, \infty)$	-	-	Concave down on $(1, \infty)$

Therefore, the function is concave up on $(-\infty, 1)$, and concave down on $(1, \infty)$. Since there is a change in concavity and the function is continuous at $x = 1$, point $(1, 4)$ is the inflection point.

The algebra required in this part is the same as that of part b.

Expected Solution to Problem 9d:



Being able to sketch the graph of a curve requires the use of previously found information. If at any point the students make algebraic errors with the derivatives, with the intervals, or with the signs of the factors, they can obtain very different graphs. Plotting points is a skill students learn in secondary school, the only difference now is that they include local extrema, critical points and inflection points, which they found. Being able to sketch the curve based on the increasing and decreasing intervals, as well as concavity requires a little more effort. However, this too is an algebraic task, as graphs and their functions are part of the definition provided earlier for algebra.

3.14 Bonus Question – Points of Inflection

> ***Bonus Question*** *For what values of the constants a and b is* $(1, 3)$ *a point of inflection of the curve* $= ax^3 + bx^2$ *?*

Expected Solution to the Bonus Question:

Given $y = ax^3 + bx^2$, substituting x and y by $(1, 3)$ gives $3 = a + b$.

Taking the first derivative with respect to x of $y = ax^3 + bx^2$ we obtain $y' = 3ax^2 + 2bx$. Taking the second derivative we obtain $y'' = 6ax + 2b$.

Setting $6ax + 2b = 0$, and substituting x with 1, we obtain $6a + 2b = 0$ or $3a + b = 0$. Using a system of equations and solving for the unknowns we obtain:

$3a + b = 0$

$\underline{\ \ a + b = 3}$

$2a \qquad = -3$

Therefore $a = -\frac{3}{2}$ and $b = \frac{9}{2}$, and $y = -\frac{3}{2}x^3 + \frac{9}{2}x^2$.

To verify, $y'' = 6ax + 2b = -9x + 9 = -9(x - 1)$. Setting $y'' = 0$ we obtain $x = 1$.

Intervals	$\boldsymbol{-9}$	$\boldsymbol{(x - 1)}$	$\boldsymbol{f''(x)}$	$\boldsymbol{f(x)}$
$(-\infty, 1)$	-	-	-	Concave down on $(-\infty, 1)$
$(1, \infty)$	-	+	+	Concave up on $(1, \infty)$

Thus, since there is a change in concavity at $x = 1$, and the function is continuous there, $(1, 3)$ is the point of inflection to the curve $y = -\frac{3}{2}x^3 + \frac{9}{2}x^2$.

This problem required quite a bit of algebra, from applying formulas and recognizing the structure of algebraic expressions in finding the first and second derivatives, to using a system of equations. The numerical activities included substituting x with a numerical value. The only Calculus-related activity was determining the inflection point; however, this can also be regarded as algebraic as the inflection point is found by applying situation dependent rules, namely finding the second derivative, finding a change in concavity, verifying that the function is continuous at this point, and concluding that it is an inflection point.

4 ANALYSIS OF STUDENTS' SOLUTIONS TO THE FINAL EXAMINATION

This chapter contains an analysis of students' solutions to the final examination problems. Incorrect solutions were parsed for traces of students' difficulties, misconceptions, and false rules related to algebraic content. Throughout this chapter, these sources of errors will be coded with prefixes D- (difficulties), Mis- (misconceptions), and FR- (false rules). Solutions[6] to problems 1-3 were analyzed in great detail, as the content, and the majority of the errors were algebraic in nature. Problem 4 asked students to calculate the derivatives of five functions given by algebraic expressions. From an algebraic point of view, therefore, this problem required the ability to (a) correctly decode the structure of the algebraic expression of the function, and (b) correctly apply the appropriate differentiation rule to this expression. Students' solutions were analyzed for any manifestations of difficulty with these two abilities. In this thesis, our analysis of students' solutions to parts 4a-4c are presented in detail; solutions to parts 4d and 4e are only summarized as errors tend to be of the same nature. Analyzes of students' solutions to problems 5-8 are also briefly summarized by enumerating the algebraic difficulties, misconceptions, and false rules, as the nature of the algebraic mistakes was no different from those already exhibited in solutions to the previous problems. Problem 9 was not analyzed, as it required students to sketch the graph of a function, whereby the tasks required to do so were already assessed in previous problems in the final examination (except for the sketch). For example, taking the first derivative of a function, finding the critical points, and determining extrema were assessed in other problems.

When the incorrectness of solution could be attributed to the lack of such basic algebraic skills as the ability to correctly decode the structure of an algebraic expression, apply a formula, model a situation in terms of an equation or being able to discern which variable is a function of which other variable in a situation, we spoke of an "algebraic difficulty". When the incorrectness of the solution could be explained by a false or limited belief about algebraic concepts or processes (e.g., a belief that "inverse function" refers to a three-step procedure, or that "x" always represents a positive number), we spoke of "algebraic misconception". When the student appeared to incorrectly apply a rule or formula as if he or she believed in an alternative, faulty procedure, we generalized this faulty procedure as a "algebraic false rule". In the literature, false rules have sometimes referred to as "mal rules" (Payne & Squibb,

[6] Transcripts of complete students' solutions to problems 1-3 are uploaded to: [https://drive.google.com/open?id=0B2kYIbyY4SGRbTBIUmY2RTdyTm8]

1990) and identified as being infrequent and unstable. Thus, some of the false rules identified in our analysis of the students' solutions to the final examination could appear in only one student solution.

In sections 4.1 – 4.8, students' solutions to problems 1-8 are presented, with various level of detail, as explained above. Section 4.9 contains a succinct summary of algebraic difficulties, misconceptions, and false rules found in students' solutions. In chapter 5, we show how this information can be used in the construction of a Placement Test, to assess a student's readiness for a Calculus I course that uses final examinations of the type analyzed here.

Our analyses of problems 4-8 also led us to identify a number of false rules related to differentiation. They could not be used in a placement test; however, we mention them briefly in the section 4.10 as this information could be useful in the construction of an exit test, or perhaps a final examination.

4.1 Students' responses to Problem 1

In this research, we aimed at identifying students' needs in terms of algebraic skills and knowledge. Problem 1 asked students to combine and invert functions and to find domains of functions. We expected algebraic mistakes in the substitution of variables when combining functions, in solving the equation when calculating the inverse function, and in solving inequalities when determining the domains. Some such mistakes appeared, of course, but they were not the main issue in students who did not produce correct answers. Their problems could rather be explained by profound conceptual difficulties with the notion of domain of a function and with the logarithmic function.

As a reminder, Problem 1 was given as follows:

> *a) Let* $f(x) = \sqrt{x+1}$ *and* $g(x) = 4x - 3$. *Find* $g \circ f$ *and* $f \circ g$ *and determine the domains of these composite functions.*
>
> *b) Find the domain of the function* $f(x) = ln\,(e^x - 3)$*, the inverse function* f^{-1}*, and the domain of* f^{-1}*.*

In Table 1 we present the distribution of correct and incorrect answers (not necessarily correct reasoning) to this problem. Sixty three students wrote the examination.

Table 1. Distribution of correct and incorrect answers to Problem 1.

n=63	correct	incorrect	other	total
P1a				
$f \circ g$	60 (95%)	3 (5%)	0	63
$g \circ f$	61 (97%)	2 (3%)	0	63
$D_{f \circ g}$	45 (71%)	17 (27%)	1 (2%)	63
$D_{g \circ f}$	42 (66%)	20 (32%)	1 (2%)	63
P1b				
D_f	35 (56%)	22 (35%)	6 (9%)	63
f^{-1}	34 (54%)	23 (37%)	6 (9%)	63
$D_{f^{-1}}$	27 (43%)	28 (44%)	8 (13%)	63

Remarks: There are slight differences in the percentages between correct domains of $D_{f \circ g}$, and $D_{g \circ f}$. One idea is that since $f \circ g$ was a square root function with no other constants outside of the square root, some students happened to obtain the correct domain even with an incorrect reasoning. However, with $g \circ f$ this incorrect reasoning did not lead them to the appropriate answer (to be discussed in examples below). Also we can see that students had difficulty with the domain of logarithmic functions, and finding the inverse function. Also note, that among the 28 incorrect solutions for the domain of the inverse, 10 had a correct domain of an incorrect inverse.

Note that throughout this thesis, students will be referred to as he or she, and this designation is random.

4.1.1 Conceptions of domain

In our initial analysis of students' solutions to problem 1, a number of incorrect conceptions of domain became apparent. The problem was made of two parts; in part (a), domains of combinations of a square root function with a linear function were required. Part (b) asked to specify the domains of combinations of logarithmic, exponential and linear functions. We identified all students' conceptions of domain of function in their solutions to part (a), and then – to part (b). Then we verified whether each student exhibited the same (correct or incorrect) conception of domain of function in both parts of the problem. But we found little or no consistency; most students appeared to use one conception in part (a) and another conception in part (b). It was as if they had a conception of the domain of functions with the square root, and a conception of domain of logarithmic-exponential functions but no general conception of domain of function. Their knowledge of domain can be said to be "situated", or dependent on context (Lave & Wenger, 1991)[7]. Situated knowledge is very complex, dependent on details, both mathematical and specific of learning mathematics in a particular institution, and it might be partly to blame for some students' apparent incorrect recollection and confusion of properties relevant to identifying the domain. Also, obtaining a correct domain does not necessarily imply a correct conception of domain.

The first two compositions (Problem 1a) were the following two square root functions:

(1) $f(g(x)) = \sqrt{4x-2}$; and

(2) $g\big(f(x)\big) = 4\sqrt{x+1} - 3$.

The following statistics relate to correct conceptions of domain concerning functions (1) and (2). Of the 63 solutions, the correct expected conceptions of [the natural] domain were exhibited in 42 (~67%) solutions for *both* square root functions[8], in 1 solution for function (1) only, and in 1 solution for function (2) only. A total of 17 (~27%) solutions appeared to be based on incorrect conceptions of domain for *both* functions. Solutions of two students did not belong to any of the above categories (one did not attempt the problem; the other's solutions were incomprehensible).

In Problem 1b, students were asked to find the domains of the following logarithmic functions:

(3) $f(x) = \ln(e^x - 3)$; and

(4) $f^{-1}(x) = \ln(e^x + 3)$.

The following statistics relate to expected correct conceptions of [the natural] domain concerning functions (3) and (4). Of the 63 solutions, the correct conceptions of domain were exhibited in 27 (~43%) solutions for *both* logarithmic functions[9], in 11 (~18%) solutions for function (3) only, and in 3 solutions for function (4) only. A total of 11 (~18%) solutions were incorrect conceptions of domain for *both* functions. Several solutions either could not be categorized (2 solutions) or there was no attempt at a solution (3 solutions).

[7] See also the article and Jean Lave's talk at http://newlearningonline.com/new-learning/chapter-6/lave-and-wenger-on-situated-learning .
[8] The majority found the compositions $f \circ g$ and $g \circ f$ correctly (this will be discussed further).
[9] Only 34 (~54%) students found the inverse of f correctly (this will be discussed further).

The remaining 6 (~10%) solutions were a combination of a wrong conception of domain, an uncategorized domain, and no attempt at finding the domains for functions (3) and (4).

Six (~10%) students out of the 63 appeared to have an initial proper conceptual understanding of the domain for (3) and/or (4), but made algebraic mistakes in their calculations, and/or applied an incorrect logarithmic property which will be discussed in a later section. The issue with incorrect logarithmic properties also became apparent in finding the inverse. Of the 28 incorrect solutions for the domain of (4), 20 solutions were obtained because students identified the domain of the inverse function with the domain of the algebraic expression obtained through the "inverse function procedure" and the algebraic expression they obtained was incorrect. (The belief that the domain of the inverse function is the domain of the algebraic expression for the inverse function is coded as Mis-Dom3). Of these 20 solutions, 10 (~50%) solutions were correct domains for the incorrect inverses obtained.

In the following subsections, we describe and give examples of the correct (expected) and incorrect responses to problems about the domains of functions.

4.1.1.1 Expected Conception of Domain

All expected responses seemed to be based on the notion of domain of a function as the domain of the algebraic expression used in the formula of the function (the natural domain). Although, formally, this is a misconception (coded Mis-Dom1), it was assumed that students will use it in solving the problem: implicitly the notion of natural domain was intended in the formulation. Here are examples of responses that we categorized as representing a "correct" notion of domain.

Example of finding the domain of function (1)

Student #29 wrote:

$$f \circ g = f(g(x)) = \sqrt{(4x-3)+1}$$

$$\text{Domain } g(x) = \mathbb{R}$$

$$f \circ g: (4x-3)+1 \geq 0$$

$$4x-3 \geq -1$$

$$\frac{4x}{4} \geq \frac{2}{4}$$

$$x \geq \frac{1}{2}$$

$$\text{Domain of } f \circ g: x \in \left[\frac{1}{2}, \infty\right)$$

Note: The second and third lines of the solution above suggest that this student uses the correct conception of the domain of composite function $D_{f\circ g} = \{x \in \mathbb{R}: x \in D_g \wedge g(x) \in D_f\}$ which, applied to the given functions, gives $D_{f\circ g} = \left\{x \in \mathbb{R}: x \in \mathbb{R} \wedge g(x) \in D_f\right\} = \{x \in \mathbb{R}: (4x-3)+1 \geq 0\}$. She also does this for function (1).

Example of finding the domain of function (2)

Student #5 wrote:

$$\text{Domain of } g \circ f \rightarrow x+1 \geq 0$$
$$x \geq -1$$
$$\therefore x \epsilon [-1, \infty)$$

Note: Although this student obtained the expected answer based on the domain of the algebraic expression, which we have classified as "correct", the belief that the domain of a composition of functions is the domain of the final algebraic expression of the composite function is, formally, a misconception, coded as Mis-Dom2.

Example of finding the domain of function (3)

Student #7 wrote:

$$e^x - 3 > 0$$
$$e^x > 3$$
$$x > \ log_e 3$$
$$x > \ln\ 3$$
$$D_{f(x)} = \{x\ \in \mathbb{R} \mid\ \ x > \ln 3\}$$

Note: This student obtained the correct answer, implicitly using Mis-Dom2 (the expected misconception).

Example of finding the domain of function (4)

Student #11 wrote:

$$e^x + 3 > 0 \text{ always true } \therefore$$
$$\text{Domain } f^{-1}(x) = \mathbb{R}$$

Note: This student’s conception of the domain of the inverse function seems to be the domain of the algebraic expression for the inverse function (Mis-Dom3).

4.1.1.2 Incorrect Conceptions of Domain

In the analysis of the final exams, the incorrect conceptions of domain identified could be grouped into two main categories, and are coded as Mis-Dom4. Students either applied some condition to the *whole* algebraic expression representing the function, or to some *special elements* of that expression. Misconceptions about the domain of a function, coded as Mis-Dom, are summarized in section 4.9. Below are the different conceptions of domain identified.

4.1.1.2.1 Domain obtained by applying a condition on the whole expression representing the function

A few students solved the equation $f(x) = 0$, sometimes correctly, sometimes not, and claimed that $D_f = (a, \infty)$ where a is a solution of that equation. This method appeared in two solutions to finding the domains of the square root functions and in one solution to finding the domains of the logarithmic functions.

Example of finding the domain of function (2)

Student # 40 wrote:

$$g \circ f = 4\left(\sqrt{x+1}\right) - 3$$

$$(3)^2 = (4)^2(\sqrt{x+1})^2$$

$$9 = 16(x+1)$$

$$9 = 16x + 16$$

$$-7 = 16x$$

$$x = -\frac{7}{16}$$

$$\text{domain } \left(-\tfrac{7}{16}, \infty\right)$$

Note: The equation $g\big(f(x)\big) = 0$ is implicit between the first and the second lines of the solution.

Example of finding the domain of function (4)

The same student # 40 wrote:

[1] inverse $x = \ln e^y - 3$

[2] $x + 3 = \ln e^y$

[3] $x + 3 = \frac{\log y}{\log e}$

[4] $\frac{(x+3)(\log e)}{\log} = \log y$

[5] $y = (x+3)(\log e)$

[6] $= x$

[7] when $x = 0$ then $x = -3$

[8] Domain $(-3, \infty)$

Note: In the first line, the student appears to apply the second step of the "inverse function procedure": switches x and y. But he omits the brackets and treats the function as $(\ln(e^y)) - 3$, which leads to the equation in line 2. In line 3, the student appears to interpret the expression $\ln e^y$ as $\log_e y$, rather than $\log_e e^y$, and applies the change of base formula - the new basis is not specified. One would expect the student to multiply both sides of the equation in line 3 by $\log e$ and obtain $(x+3)\log e = \log y$. Instead, the equation obtained is the one in line 4, with the incomprehensible "log" in the denominator. Maybe, the student intended to divide both sides of the equation by $\log$, treating it as a variable but forgot to eliminate it on the right side. (The treatment of log as a variable is a misconception about notation coded as Mis-Notation2). He does so in line 5. It is unclear what the sixth line of the solution means, but his seventh line seems to be a solution of the equation $y = 0$ where y is as in the fifth line, i.e., $(x+3)\,(\log e)$. Since the solution is $x = -3$, the domain is $(-3, \infty)$. Although it appears from the above two solutions, that this student has a consistent conception of domain in Problem 1, his conceptions of domain in solutions to functions (1) and (3) are different. This inconsistency in the solutions of domain problems for similar and different types of functions is apparent in a number of students.

In a couple of other solutions, the domain of a function was obtained by solving the inequality $f(x) \geq 0$, correctly or incorrectly, and taking the domain to be the interval from the value obtained (inclusive) to infinity: $D_f = [a, \infty)$ where a is a solution of the inequality.

Example of finding the domain of function (1)

Student # 12 wrote:

$$f \circ g = \sqrt{4x-2}$$

$$\sqrt{4x-2} \geq 0$$

$$4x - 2 \geq 0$$

$$4x \geq 2$$

$$x \geq \frac{2}{4} = \frac{1}{2}$$

$$x \in \left[\frac{1}{2}, \infty\right)$$

Note: In this case, the final answer was correct but the conception used to obtain it was incorrect. Traces of the erroneous conception are in the second line of the solution.

Example of finding the domain of function (3)

The same student # 12 wrote:

$$f(x) = \ln(e^x - 3)$$
$$\ln e^x - \ln 3 > 0$$
$$x \geq \ln 3$$
$$x \in [\ln 3, \infty)$$

Note: Traces of the conception are in the second line. In passing from the first to the second line of the solution, the student seems to believe that logarithm of the sum is the sum of logarithms. (The assumption of linearity is a false algebraic rule coded as FR-Lin-Log). In the second line he writes a strict inequality but changes it to weak inequality in line 3. This student appears to have a fairly consistent misconception in his or her solutions of domain to all of Problem 1, namely making the whole expression greater than or equal to zero.

4.1.1.2.2 Domain obtained by applying a condition on a part of the expression representing the function

In one solution, a student obtained the domain of a function by equating to zero, not the whole function but a special element of the expression representing the function, solving for x (incorrectly) and concluding that the value obtained is the domain.

Example of finding the domain of function (4)

Student # 48 wrote:

$$f^{-1}(x) = \ln(e^x + 3)$$
$$e^x + 3 = 0$$
$$e^x = -3$$
$$\ln e^x = \ln -3$$
$$x = \ln -3$$
$$x = \ln 3$$

Note: The special element the student looks at is $e^x + 3$. The student is unaware of the inconsistency of the equation in line 3 and continues processing it to isolate x, applying the logarithm to both sides, in line 4 of the solution. Passing to line 5 is based on the correct identity $\ln(e^x) = x$, but it is unclear what belief about logarithms underlies the passage from line 5 to line 6. This student applied the same procedure to find the domain of function (3), suggesting a misconception about logarithmic functions. This student also demonstrates inconsistencies in conceptions of domain in his solutions for part (a), as he has different solutions for both square root functions.

In three other solutions, finding the domain of a function also started by equating to 0 a special element of the expression representing the function, and solving for x but now *excluding* the value obtained from $\mathbb{R}$ to obtain the domain.

This conception could explain the solution presented by student #42, whose answer to finding the domain of function (1) was $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$. Aside from this answer, no work was shown. The same type of solution was provided for function (2). The next example exhibits this conception more clearly.

Example of finding the domain of function (3)

Student # 56 wrote:

$$f(x) = \ln(e^x - 3)$$

this function is not defined for $e^x - 3 = 0$

$$e^x = 3$$

$$x = \ln 3$$

$$D_f: (-\infty; \ln(3))\ ; (\ln(3)\ ;\ \infty))$$

Note: The student excluded the solution of the equation $e^x - 3 = 0$ from $\mathbb{R}$, if we interpret the semi-colon sign in the last line of his solution as representing the union of sets. Although for function (4), this student appears to have a correct solution, and understanding.

We surmise that underlying this last conception of domain is a generalization of the conception of domain of rational functions, where students find where the denominator is undefined, by making it equal to zero and then excluding the value obtained from the set of all real numbers.

In five other solutions based on taking a condition on a special element of the expression for the function, the domain of a function was a solution of a weak inequality: the special element ≥ 0. There were three such solutions for the square root function(s) and two for the logarithmic functions.

Example of finding the domain of function (2)

Student #12 wrote:

$$\sqrt{x+1} \geq 0$$
$$x+1 \geq 0$$
$$x \geq -1$$
$$x \in [-1, \infty)$$

Note: Recall that function (2) was $(g \circ f)(x) = 4\sqrt{x+1} - 3$. So the special element the student takes to calculate the domain is $\sqrt{x+1}$. She then solves the weak inequality $\sqrt{x+1} \geq 0$. Line 2 of the solution is obtained by squaring both sides of the inequality in line 1. If the first line was not there, one could think that the student's conception was correct. The answer ends up being correct.

Example of finding the domain of function (3)

Student #30 wrote:

$$\text{domain of } f(x) = \ln(e^x - 3)$$
$$e^x - 3 \geq 0$$
$$e^x \geq 3$$
$$x \ln e \geq \ln 3$$
$$x \geq \ln 3$$

Note: The flaw in this solution is that the student solves a weak inequality instead of the strict one. This suggests either the above described misconception about the domain of functions in general or a misconception about the logarithmic functions. This student appears to have a good conception of domain for function (4), exhibiting inconsistencies in his conceptions of the domain of logarithmic functions.

A variation of the previous conception of domain appeared in seven other solutions: based on taking a condition on a special element of the expression for the function, the domain of a function was a solution of a *strict* (not weak) inequality: the special element > 0. All these solutions were about the square root function(s).

Example of finding the domain of function (1)

Student # 40 wrote:

$$f \circ g = \sqrt{4x - 2}$$
$$= \sqrt{2(x - 1)}$$
$$x - 1 > 0$$
$$\text{therefore } x > 1$$
$$\text{so domain is } (+1, \infty)$$

Note: The "special element" of the expression that the student looks at is not the whole expression under the square root (as it would be in a correct reasoning about the natural domain of the function) but only a part of it. Another aspect of this solution that suggests a misconception that a strict inequality is taken, not a weak one: the misconception is either about the domain of function in general or about the square root function. As you may recall (section 4.1.1.2.1), this student applied a condition to the whole expression for function (2), thus exhibiting inconsistencies in his or her conceptions of the domain of square root functions.

More frequent, 13 (~21%) solutions had the following conception: taking a special element of the expression, and finding the domain of this special element. This idea could explain six (~10%) solutions about the domains of the square root functions and seven (~11%) solutions about the domains of the logarithmic functions.

Example of finding the domain of function (1)

Student #45 wrote:

$$f \circ g = \sqrt{(4x - 3) + 1}$$
$$= \sqrt{4x - 3 + 1}$$
$$= \sqrt{4x - 2}$$
$$\text{D of } f \circ g = 4x - 3 \qquad x \in \mathbb{R}$$

Note: This student applied the same procedure for functions (1) and (2): she found the domain of a composite function, as being the domain of the input function.

Example of finding the domain of function (2)

Student #25 wrote:

$$g(f(x)) = 4(\sqrt{x+1}) - 3$$
$$= 4\sqrt{x+1} - 3$$
$$\text{Domain: } [0, \infty[$$

Note: This student obtains the same domain as well for function (1). As such, we hypothesize that this student does not take into account the function obtained, but rather remembers the domain of a square root function, namely $f(x) = \sqrt{x}$ as being $[0, \infty)$.

Example of finding the domain of function (3)

Student #2 wrote:

$$f(x) = \ln(e^x - 3)$$
$$\text{Domain of } f(x) \text{ All } \mathbb{R} \text{ numbers}$$

Three students did this for (3) or (4), and we hypothesize that they are either taking the domain of the function e^x alone, or have confusion with the range of logarithmic functions.

Example of finding the domain of function (4)

Student #16 wrote:

$$f^{-1}(x) = \ln(e^x + 3)$$
$$\text{Domain of } f^{-1}(x): (0, \infty)$$

Note: There is no mention of the range of the function f in the student's solution, so the student probably thinks of the domain of the inverse function as the domain of the expression for f^{-1} (Mis-Dom3). The formula for the inverse function (line 1) was calculated correctly, but the details of the formula seem, however, to play no part in the student's answer. He or she only notices "ln" (this is the "special element"), and writes the domain of the logarithmic function $f(x) = \ln(x)$ as the answer. Additionally, this student obtains the correct domain for function (3), exhibiting inconsistencies in his conception of the domain of logarithmic functions. Four students found the domain as $(0, \infty)$ for (3) or (4), and we hypothesize that they remember the domain of a logarithmic function $\ln(x)$, or they have a confusion with the range of e^x.

In two solutions about the domains of square root functions and in six of the logarithmic functions there appeared a very peculiar conception: the "special element" on which the students focused was some particular *constant* in the expression, and the domain was the set of real numbers greater than that constant.

Example of finding the domain of function (1)

Student #38 wrote:

$$f\big(g(x)\big) = \sqrt{(4x-3)} + 1$$

$$\text{Dom } f \circ g \ (-3, \infty)$$

Note: In this solution, the special element seems to be the number -3. The composition of functions in line 1 of the solution is also incorrect, which indicates this student's difficulties in decoding the structure of algebraic expressions, coded as D-Struc, on top of her problems with the notion of domain.

Example of finding the domain of function (2)

Student #4 wrote:

$$g \circ f = 4\sqrt{x+1} - 3$$

$$\text{Domain: } (-3, +\infty)$$

Note: Here, the composition of functions is correct but there is clearly a problem with the notion of domain. The special element the student focuses on is -3.

Example of finding the domain of function (3)

Student # 1 wrote:

$$e^x - 3 = 0$$

$$e^x = 3$$

$$e^x = 3$$

$$x = \ln\ 3$$

$$D\ (3, \infty) \text{ of } f$$

Note: The first line could be interpreted as representing one of the previous conceptions (equating a special element of the function's expression to 0), but the last line bears traces of the "focus on a constant" misconception.

4.1.1.3 Other Conceptions of Domains

The following responses to the problems about the domains of functions were difficult to interpret and could not be placed into any of the previous categories.

Example of finding the domain of functions (1 & 2)

After successfully obtaining the compositions of (1) and (2), student #3 wrote down the following:

$$\text{domain} = (\infty, -\infty) \cup (0, \infty)$$

Note: He did not indicate which function this domain referred to. Was it for function (1), (2), or perhaps both? For whichever function he intended this domain to be, the student is unaware of the redundancy of the second interval being contained in the first interval, that is if we interpret $(\infty, -\infty)$ to be $(-\infty, \infty)$.

Example of finding the domain of function (3)

Student #21 wrote:

$$e^x - 3 > 0$$
$$e^x > 3$$
$$x > \ln e$$
$$x > 1$$
$$D\, f(x)\text{: } (2;\ \infty) \text{ or } (1;\ \infty) \cup (x \neq 1)$$

Note: The first two lines suggests a correct expected conception of the domain of the logarithmic function as the set of all positive real numbers: so, if $f(x) = \ln(e^x - 3)$ then the domain of f should be the set of solutions of the equation $e^x - 3 > 0$ or $e^x > 3$. But things get muddled up starting from line 3. From line 2 to 3, it appears that this student miswrote e for 3. Line 5 bears evidence that this student does not understand interval notation. It appears that he thinks in terms of integers (since $x > 1$, then x must be at least 2), and perhaps means $[2, \infty)$ instead of $(2, \infty)$, confusing the meaning of open and closed brackets. If he understood the brackets conventionally, there would not be the need to exclude 1 in the third brackets of the last line. There is also confusion between union of sets and conjunction of conditions in the use of the operator ∪ in the last line of the above solution.

This student has a similar solution for function (4), although does not seem to think in terms of integers there. Throughout all of Problem 1, this student incorrectly uses interval notation.

Example of finding the domain of functions (3 & 4)

Student #6 wrote: Domain for both $f(x)$ and $f^{-1}(x)$: $1 < x < 1$

Note: This student obtains an incorrect formula for the inverse function, but the formulas for the functions play no part in the student's solutions for the domains. Nor does she look for the range of f to find the domain of the inverse function. She provides the same domain for both functions (3) and (4), seeming unrelated to the functions.

Example of finding the domain of function (4)

Student #3 did not complete finding the inverse, but claimed, without justification:

domain $(\infty, -\infty)$

Note: Was this student's notion of direction of the number line from right to left? Recall, this student was fairly consistent in his direction, having written one domain for functions (1) and (2) as $(\infty, -\infty) \cup (0, \infty)$. However, he exhibits inconsistencies in his conceptions as he uses a different approach for function (3).

Example of finding the domain of function (4)

After having found the correct inverse, student #53 wrote:

$$D_{f^{-1}(x)} \to e^x + 3 > 0$$
$$\to e^x > 3$$
$$\to \ln e^x > \underline{\ln -3}$$
Does not exist
$$\to f(x) = \ln(e^x - 3) \text{ is not invertible}$$

Note: This student incorrectly subtracted 3 from both sides in line 2, but appears to have -3 in line 3. He then notices the nonexistence of the number on the right side of the inequality in line 3 and becomes confused, then decides to claim that the function does not have an inverse.

4.1.2 Conceptions of composite functions

The majority, 60 (95%) of the students could correctly combine two functions. Three students made minor errors that could not necessarily be attributed to algebraic errors, although this still remains as a possibility. The following are examples of incorrect solutions.

Example of finding the composition function (1)

Student #38 wrote: $f \circ g = \sqrt{(4x-3)} + 1$

Note: It appears as though this student did not properly extend the root over the constant "1". This may very likely be an error due to inattention, although it might demonstrate difficulties with the structure of algebraic expressions (D-Struc). This student had the correct composition for function (2).

Example of finding the composition function (2)

Student #51 wrote:

$$\begin{aligned} g \circ f &= 4(\sqrt{x+1} - 3) \\ &= 4\sqrt{x+1} - 12 \end{aligned}$$

Note: It is possible that this student incorrectly positioned the second bracket. It could also be an algebraic mistake related to order of operations and brackets. This student had the correct composition for function (1).

Example of finding the composition function (2)

Student #58 wrote: $g \circ f = 4\,x\,\sqrt{x+1} - 3$

Note: It appears this student included an additional x, as though he was multiplying f with $4x$, and not replacing x by it. It is also possible that this student was writing 4 "times" $\sqrt{x+1}$. This student had the correct composition for function (1).

4.1.3 Conceptions of square root functions

4.1.3.1 Intended Conceptual Understanding of Square Root Functions

When students manipulate square root functions, we assume that they have an implicit knowledge of these functions. As stated previously in our a priori analysis of the final exam, the definition of square root functions is the following: $y = \sqrt{x}$ iff $y \geq 0$ and $y^2 = x$. Therefore $x \geq 0$. This definition is simple enough, although it contains a lot of information that some students appear to miss. One is that the domain of such functions requires the expression under the radical to be non-negative. We have already seen incorrect conceptions related to this in our analysis of the domain of functions (1) and (2). Interestingly, another incorrect conceptual understanding became apparent which will be discussed in the next section.

4.1.3.2 Incorrect Conceptual Understanding of Square Root Functions

Apart from the incorrect conceptual understanding of the domain that has previously been addressed concerning functions (1) and (2), two solutions seemed to be based on the belief in the linearity of square root functions, i.e. the students believed that the square root of the sum is the sum of the square roots (FR-Lin-Sqrt).

Example of linearity with function (1)

Student #24 wrote:

$$\begin{aligned} f \circ g &= \sqrt{(4x-3)+1} \\ &= \sqrt{4x-2} \\ &= 2x - \sqrt{2} \end{aligned}$$

Note: Traces of the conception are in the third line. Also, the student appears to drop the radical over the x, as though believing that $\sqrt{ax^b} = \sqrt{a} \cdot x^b$ (FR-Rad2). This student did not apply linearity to function (2).

Example of linearity with function (2)

Student #1 wrote:

$$\begin{aligned} g \circ f &= 4\left(\sqrt{x+1}\right) - 3 \\ &= 4\sqrt{x} + 4 - 3 \\ &= 4\sqrt{x} + 1 \end{aligned}$$

Note: Traces of the conception are in the second line. This student also applied linearity to function (1).

4.1.4 Conceptions of inverse functions

4.1.4.1 Intended Conceptual Understanding of Inverse Functions

Formally, the notion of the inverse function can be introduced via the concept of inverse relation. Every relation has an inverse but the relation which is the inverse of a function is not necessarily a function itself. Reasoning about the conditions necessary for the inverse of a functional relation to be a function leads to the conclusion that for a function to have an inverse, the function must be bijective (one-to-one and onto):

Theorem…: Let $f: A \to B$ be a function. Then the inverse relation f^{-1} is a function from B to A if and only if f is bijective. (…) (Chartrand, Polimeni, & Zhang, 2013, p. 229)

Further, if $f: A \to B$ is a bijection, then a function $g: B \to A$ is the inverse of f if $g \circ f = id_A$ and $f \circ g = id_B$ (ibid., p. 230).

As mentioned in section 3.1, formally, for the inverse f^{-1} of a functional relation $f: A \to B$ to be a function f must be *bijective*, so that $Range(f) = B$ and different elements of A are paired with different elements of B. In this course, bijectivity is not explicitly covered, however, the textbook used for the course does provide the following definition of inverse functions:

Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x \leftrightarrow f(x) = y$ for any y in B. (Stewart, 2016, p. 56).

So even if the function f is not onto, the function $f: A \to Range(f)$ is and the definition guarantees this condition. One-to-oneness is also assumed in the definition. However, the students-readers of the textbook do not seem to be expected to worry about the existence of the inverse function; their task is mainly to calculate the formula of the inverse of a function given by the authors, trusting that the answer to the task is a formula and not a phrase such as "the inverse does not exist". The textbook provides students with the following step-by-step instructions on how to perform such tasks, strongly resembling those of Thomas (2008, p. 50):

How to find the Inverse Function of a One-to-One Function f

STEP 1: Write $y = f(x)$.

STEP 2: Solve this equation for x in terms of y (if possible).

STEP 3: To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$. (Stewart, 2016, p. 58).

The possibility of non-existence of the inverse could be hinted at in STEP 2, in the note "if possible" but this is ambiguous. For example, if $f(x) = x^2$, then, in step 1, the student would write $y = x^2$ and in step 2 – he or she could write $x = \pm\sqrt{y}$ or $x = \sqrt{y}\ \ or\ \ x = -\sqrt{y}$. It is possible to solve the equation $y = x^2$ for x, yet the inverse function does not exist. The solution is not unique, which is a problem but the reader is not warned about this. A student may also interpret the phrase "if possible" as "if you can, or are able to". So if a student is unable to algebraically solve in terms of x this may lead to a misconception that the inverse would not exist. (Mis-Dom3, Mis-InvF1)

In the formal theory, it is stated that if f^{-1} is a function then $D_{f^{-1}} = Range(f) = B$, and $f^{-1}\big(f(a)\big) = a$ for all $a \in A$ and $f\big(f^{-1}(b)\big) = b$ for all $b \in B$. In the textbook for the course (Stewart, 2016), these conditions are stated under the title of "cancellation equations", prior to the step-by-step instructions mentioned above. Stewart's (2016, p. 58) explanation of these equations is similar to that of Thomas' (2008, p. 47) explanation of the inverse.

> *The first cancellation equation says that if we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started...Thus f^{-1} undoes what f does. The second equation says that f undoes what f^{-1} does (Stewart, 2016, p. 58).*

However, in the textbook, these equations are not used to verify if the function, obtained as a result of applying the steps, is the inverse of the given function in the example that follows the formulation of the steps, although it would be useful. If, in the final examination, the students had verified the following: $f^{-1}\big(f(x)\big) = f^{-1}(\ln{(e^x - 3)}) = \ln\big(e^{\ln(e^x-3)} + 3\big) = \ln(e^x - 3 + 3) = x$ and $f(f^{-1}(x)) = f(\ln{(e^x + 3)}) = \ln\big(e^{\ln(e^x+3)} - 3\big) = \ln(e^x + 3 - 3) = x$, this would have perhaps helped some students re-evaluate their work and find their errors. However, not one student did this, once they found the inverse of function (3). Students appear to only remember these steps (Mis-InvF1), going on to write out the steps in their solutions without knowing much else about inverse functions. The concept becomes reduced to a procedure for solving a certain type of task, and students do not feel responsible for knowing why this procedure works or even what is the meaning of the result it produces. In terms of the ATD framework, even the textbook suggests that students' knowledge of inverse functions may be confined within the boundaries of the "know-how" block of the mathematical praxeology of inverse functions; knowledge of the "know-why" block is under the responsibility of the authors of the book and instructors of courses using the textbook (Chevallard, 1999). Moreover, students are expected to know how to solve only certain types of tasks, for example, "Find the inverse of the given function", but not necessarily "Does the given function have an inverse? Why yes or why not?" or "Is the given function an inverse of the other given function? Justify your answer."

The textbook mixes exposition of mathematical theory with instructional advice, in the form of side notes and cautions which can be confusing for the learner. For example, the textbook cautions the readers with the following:

Do not mistake the -1 in f^{-1} for an exponent. Thus f^{-1} does not mean $\frac{1}{f(x)}$. The reciprocal $\frac{1}{f(x)}$ could, however, be written as $[f(x)]^{-1}$ (Stewart, 2016, p. 57).

In our oversimplification of content, we may be harming our students with the potential for misconceptions, and misunderstandings of procedures. For example, Stewart (2016) provides the steps to finding the inverse, followed by one example of a cubic function. As previously mentioned, the example does not verify the identities that a function and its inverse must satisfy. Students reading this section, and focusing only on the blue box containing the steps can then develop a rote knowledge of procedures. Encountering a function that was a little more difficult (i.e. a logarithmic function), some students were unable to solve in terms of one variable. If they were unable to follow the steps, but had a good understanding of inverse functions, they could have found the domain of the inverse by finding the range of the function. But this was not the case for those who couldn't follow the steps at finding the inverse. Thus, students could have a misconception that the inverse of a function is a step-by-step procedure to follow (Mis-InvF1). The setting of a simple problem in which a procedure is learned, can be very detrimental for students who develop a rote knowledge of procedures; as soon as the setting or problem differs, students are unable to apply the procedures learned (Star, 2004).

4.1.4.2 Incorrect Conceptual Understanding of Inverse Functions

Of the 63 students, 34 (~54%) correctly found the inverse function (4). Another 19 (~30%) appeared to have the general step-by-step procedure in mind, although either they did not complete finding the inverse, applied some incorrect logarithmic property, or confused the variables resulting in an incorrect inverse. These incorrect properties will be discussed in the next section. Four students did not attempt finding the inverse. The remaining students either had some incorrect conception (2 students), or an uncategorized conception of inverse (4 students).

4.1.4.2.1 The inverse of a function is obtained by changing the sign of each term

Example of finding the inverse function (4)

Student #6 wrote:

$$f(x) = \ln(e^x - 3)$$
$$f^{-1}(x) = -\ln(-e^x + 3)$$

Note: This student appears to believe that the inverse of a function is changing the sign of each term (Mis-InvF2).

4.1.4.2.2 The inverse of a function is obtained by interchanging x and y, moving terms around, and not isolating any variable.

Example of finding the inverse function (4)

Student #17 wrote:

$$f^{-1}:\ y = \ln\,(e^x - 3)$$
$$y = \frac{\ln e^x}{\ln 3}$$
$$f^{-1}:\ \ x = \frac{\ln e^y}{\ln 3}$$

Note: This student applies an incorrect logarithmic property from line 1 to 2 (false rule FR-Log2), and then interchanges x and y in line 3. It appears that they might have been following the steps to find the inverse, but may have skipped step 2 of the procedure, not solving for x.

Example of finding the inverse function (4)

Student #44 wrote:

[1] $f(x) = \ln(e^x - 3)$

[2] for $(f)^{-1} \rightarrow y = \ln(e^x - 3)$

[3] $y = \ln(e^x - 3)$

[4] Step 1 interchange x and y

[5] $x = \ln(e^y - 3)$

[6] Now using property of logarithm, we can write $x = \ln e^x$

[7] $\therefore \ln e^x = \ln\,(e^y - 3)$

[8] $\ln e^x - \ln(e^y - 3) = 0$

[9] $\ln\left(\frac{e^x}{e^y - 3}\right) = 0$

Note: This student is using the steps as described in the textbook not in the exact order, nor doing them all. In line 4, step 3 is called STEP 1, and STEP 2 appears to be skipped. He does not isolate a variable, i.e. y in this case as he interchanged the variables at the beginning. As you may recall, STEP 2 was to "Solve this equation for x in terms of y (if possible)." For this student it would be to solve the equation for y in terms of x (if possible). Perhaps he or she was unable to solve for y, therefore left the function as is.

4.1.4.2.3 Other Conceptions of Inverse

The following responses to the problem about the inverse of the function were difficult to interpret and could not be categorized.

Example of finding the inverse function (4)

Student #1 wrote:

[1] inverse of f^{-1} $f(x) = \ln\,(e^x - 3)$

[2] $y = \ln(e^x - 3)$

[3] $x = \ln(e^y - 3)$

[4] $e^x = e^y - 3$

[5] $\ln e^x =$

[6] $y = \frac{e^x - 3}{e^x}$

Note: The student starts by interchanging the variables in line 3, and progresses to line 4 adequately. At line 5, it is unclear what belief about logarithms underlies this line, nor the passage from line 5 to line 6.

Example of finding the inverse function (4)

Student #24 wrote:

$$f(x) = \ln(e^x - 3)$$
$$f'(x) = \ln(e^x - 3)$$
$$e = e^x - 3$$
$$e^x - e = -3$$
$$e(x - 1) = -3$$

Note: This student appears either to have notational confusion between f^{-1} and f' as seen in line 2, or believes that the derivative of a logarithm is the logarithm as it happens with exponential functions. If the former is the case then there could be some confusion about inverse functions in passing from line 1 to line 2. It is unclear what beliefs about logarithms underlie the passage from line 2 to 3. It is also unclear what beliefs about exponentials underlie passage from line 4 to line 5. This student doesn't seem to know the difference between taking the powers and multiplying.

Example of finding the inverse function (4)

Student #49 wrote:

$$f(x) = \ln(e^x - 3)$$
$$y = (e^{xln})e^{x(e^x-3)}$$
$$1 \cdot e^x$$

Note: From line 1 to line 2, it appears that this student views $\ln$ as a variable and not a function (Mis-Notation2), namely a variable multiplying $(e^x - 3)$. Also in line 2, it is unclear what beliefs about exponentials are present: the student applies the exponential to the right side of the equation, and believes that the exponential of the product of two terms is the product of the exponentials. He confuses the property of $e^{\ln 1} = 1$ with $e^{xln} = 1$ in line 3 (FR-Exp3), although it is unclear how he obtained the second term.

Example of finding the inverse function (4)

Student #62 wrote:

$$f(x) = \ln(e^x - 3)$$
$$y = \ln_e(e^x - 3)$$
$$e^y = e^x - 3$$
$$e^y \cdot y' = e^x - 3$$
$$f' \quad y' = \frac{e^x - 3}{e^y}$$

Note: In line 2, this student has a notational issue with the conventions of the natural logarithm. This student also appears to confuse the inverse procedure with implicit differentiation in line 4. It is unclear how her knowledge of procedures could have gotten confused.

4.1.5 Conceptions of logarithmic functions

4.1.5.1 Intended Conceptual Understanding of Logarithmic Functions

When students manipulate logarithmic functions, we assume that they have an implicit knowledge of these functions. As stated previously in our a priori analysis of the final exam, the definition of logarithmic functions is the following: $y = \ln(x)$ iff $e^y = x$ and $x > 0$. This definition contains a lot of information that some students appear to miss. One is that the domain of such functions requires the expression in the brackets to be positive. We have already seen incorrect conceptions related to this in our analysis of the domain of functions (3) and (4). After a more in depth analysis of these solutions, more incorrect conceptual understandings of logarithmic functions became apparent.

4.1.5.2 Incorrect Conceptual Understanding of Logarithmic Functions

Apart from the incorrect conceptual understanding of the domain that has previously been addressed concerning functions (3) and (4), nine (~14%) students also applied linearity to the logarithmic functions. These students believed that the logarithm of sums is the sum of logarithms (FR-Lin-Log).

4.1.5.2.1 The logarithm of a sum is the sum of logarithms

Example of using the logarithm of a sum as the sum of logarithms

Student #2 wrote:

[1] $f(x) = \ln(e^x - 3)$

[2] $y = \ln(e^x - 3)$

[3] $x = \ln(e^y - 3)$

[4] $\ln x = \ln e^y - \ln 3$

[5] $\ln x = y - \ln 3$

[6] $y = \ln x - \ln 3$

[7] $y = \ln\left(\frac{x}{3}\right)$

[8] $f'(x) = \ln\left(\frac{x}{3}\right)$

Note: This student is following the step-by-step instructions to find the inverse, although applies linearity in the passage from line 3 to line 4 (FR-Lin-Log). This student also adds a $\ln$ to the left side of the equation in line 4. This student does correctly apply a logarithmic property in the passage from line 6 to line 7, although in line 8 incorrectly writes f' instead of f^{-1}.

4.1.5.2.2 The logarithm of a difference is the quotient of logarithms or the logarithm of quotients

Four students applied these incorrect logarithmic properties in order to find the inverse:

Examples of logarithm of a difference as the quotient of logarithms

Student #13 wrote:

$$f(x) = y = \ln(e^x - 3)$$

$$y = \frac{\ln e^x}{\ln 3}$$

$$y(\ln 3) = \ln e^x$$
$$y(\ln 3) = x$$
$$\therefore f^{-1}(x) = x(\ln 3)$$

Note: Traces of the incorrect property can be seen from the passage of line 1 to line 2 (FR-Log2).

Student #59 wrote:

$$y = \ln(e^x - 3)$$
$$y = \ln e^x - \ln 3$$
$$y = \frac{\ln e^x}{\ln 3}$$
$$y = \frac{x}{\ln 3}$$
$$\ln 3 \cdot y = x$$
$$f^{-1} = \ln 3 \cdot x$$

Note: From line 1 to line 2 he applies linearity (FR-Lin-Log). From line 2 to line 3, we can see traces of the incorrect property (FR-Log2). The confusion faced by the student is understandable. He confuses the quotient of logarithms with the logarithm of the quotient.

Example of logarithm of a difference as the logarithm of a quotient

Student #42 wrote:

$$f(x) = \ln(e^x - 3)$$
$$\rightarrow f(x) = \ln \frac{e^x}{-3}$$

Note: Traces of the incorrect property can be seen from the passage of line 1 to line 2 (FR-Log4). Interestingly, this student keeps the minus sign in front of the 3.

4.1.5.2.3 Exponent and base confusion with logarithms

Instead of using the definition as $y = \log_{\mathrm{e}}(x)$ iff $e^y = x$ and $x > 0$, the following student confused the base with the exponent as $y = \log_{\mathrm{e}}(x)$ iff $y^e = x$ (FR-Log7).

Example of exponent and base confusion

Student #21 wrote:

$$f(x) = \ln(e^x - 3)$$
$$y = \ln(e^x - 3)$$

$$y = \log_e (e^x - 3)$$
$$y^e = e^x - 3$$
$$y^e + 3 = e^x$$
$$\ln(y^e + 3) = \ln e^x$$
$$\ln(y^e + 3) = x$$
$$\therefore y^{-1} = \ln(x^e + 3)$$
$$f^{-1}(x) = \ln(x^e + 3)$$

Note: Traces of the conception are in the fourth line.

4.1.5.2.4 Adding or dropping ln or e without justification

Eight (~13%) students either added or dropped a ln or an e from one side of the equation without justification. Procedural learning could lead to these errors. If math is a random set of rules that are not connected for students, they can easily confuse properties.

Examples of adding of dropping ln or e

Student #4 wrote:

$$\text{Inverse } y = \ln(e^x - 3)$$
$$y = \ln e^x - \ln 3$$
$$\ln(y + 3) = \ln e^x$$
$$\ln(y + 3) = x$$
$$f^{-1} = \ln (y + 3)$$

Note: This student applies linearity in passing from line 1 to line 2 (FR-Lin-Log). In passing from line 2 to line 3, it appears as though the student implicitly adds a ln in front of y and then again applies linearity to add them.

Student #38 wrote:

$$\text{Inverse} = \ln(e^x - 3)$$
$$= \ln(x - 3)$$
$$e^y = x - 3$$
$$x = e^y + 3$$
$$y^{-1} = e^x + 3$$

Note: In passing from line 1 to line 2, it appears that the student dropped the e and brought the exponent x down. One idea is that perhaps the student has some confusion with $\ln(e^x) = x$, or that $e^x = x$ (FR-Exp1). Further, she disregard the -3 as part of the expression in brackets.

4.1.5.2.5 Treating ln and e like variables

Six (~10%) students appeared to treat $\ln$ or e as though they were variables.

Examples of treating $\boldsymbol{ln}$ or $\boldsymbol{e}$ like variables

Student #42 wrote:

$$f^{-1}(x) \to x = \ln\,(e^y - 3)$$
$$\to x = \ln \frac{e^y}{-3}$$
$$-3x = y \ln e$$
$$y = -\frac{3x}{\ln e}$$

Note: An incorrect logarithmic property can be seen from line 1 to 2 (FR-Log4), as mentioned in (4.1.5.2.2). The placement of $\ln$ in line 2 without brackets appears as though it is multiplying the quotient. Further evidence of $\ln$ being treated as a variable is in the passing from line 2 to line 3: the denominator -3 is now multiplying the left-hand side of this equation. Implicitly, after multiplying both sides by -3, the numerator remains as $\ln e^y$, and became $y \ln e$ in line 3. Further, the student does not realize that $y \ln e$ is simply y.

Student #60 wrote:

[1] $y = \ln\,(e^x - 3)$

[2] $x = \ln\,(e^y - 3)$

[3] $x = \ln e^y - \ln 3$

[4] $x = e^y - 3$

[5] $x + 3 = e^y$

[6] make ln to loose e

[7] $\ln x + 3 = \ln y$

[8] isolate y

[9] $f^{-1}\colon \frac{\ln x + 3}{\ln} = y$

Note: This student drops both ln's in passing from line 3 to line 4, then incorrectly applies some property passing from line 5 to line 7. Finally, in passing from line 7 to line 9, demonstrates their treatment of ln as a variable by dividing the left-hand side of the equation by "ln" (Mis-Notation2).

4.2 STUDENTS' RESPONSES TO PROBLEM 2

Problem 2 asked students to evaluate the limits of two functions, both being ratios containing non-polynomial functions, with the value that x was approaching not included in the domain of the functions. One function contained an absolute value expression, and the other contained a radical expression. As previously mentioned, in both problems, in spite of the value a that x was approaching not being included in the domain, a number of students tried to calculate the limits by calculating the value of the function at a, as if applying the Direct Substitution Property. Some realized at that point that the function is not defined at a and tried something else. Others made mistakes in their calculations, obtained a number and left it as the limit of the function. Due to the fact that these functions contained radical and absolute value functions, which are notorious for being difficult for students, we did expect a number of algebraic mistakes. In part (a), students could ignore the absolute value. In part (b), mistakes could occur in multiplying the numerator and denominator by the conjugate radical expression. Their mistakes could represent an incorrect conception of the absolute value function, ignorance of the distributive property leading to the inability to factor, or the inability to multiply binomials, etc.

As a reminder, Problem 2 was given as follows:

Evaluate the limit if it exists, or explain why the limit does not exist ***(Do not use l'Hôpital's rule):***

a) $lim_{x\to -6} \frac{2x+12}{|x+6|}$ *b)* $lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2}$

In Table 2 we present the distribution of correct and incorrect answers to this problem.

Table 2. Distribution of correct and incorrect answers to Problem 2.

n=63	correct	incorrect	other	total
P2a Limit	44 (70%)	18 (28%)	1 (2%)	63
P2b Limit	29 (46%)	33 (52%)	1 (2%)	63

Remarks: For part (a), not all students calculated both one-sided limits. One reason could be that the students had an incorrect conception of absolute value functions, and as such did not attempt to find the left-hand limit. Also note, that a good proportion of students knew to rationalize the function for part (b) as a first step, however obtained an incorrect limit as they made various algebraic mistakes throughout.

The common mistakes amongst students will be grouped and described in following sections.

4.2.1 Application of the Direct Substitution Property (DSP) for parts a and b

In both parts, the value that x was approaching was not in the domain of the functions, yet four students applied DSP both to parts (a) and (b), four students to part (a) only, and two students to part (b) only. For the majority of these students this was a first "step", and as they obtained an indeterminate form, they proceeded to using other methods. This offers more support to our hypothesis that students learn a list of procedures in which they repeat, without fully understanding what they are doing. Their first step, regardless of the function provided is to apply the DSP. They were unable to recognize that a was not in the domain of $f(x)$. Also, the textbook does not indicate the DSP as a first step to finding limits, rather it is to be used to find a limit of a polynomial or a rational function, given that the value that x is approaching is in the domain of f, thereby foreshadowing the notion of continuity. Two interesting student solutions are provided below.

Examples of the DSP

Student #4 wrote for part (b):

$$\lim_{x\to 3}\frac{\sqrt{x+6}-x}{x^3-3x^2}=\lim_{x\to 3}\frac{\sqrt{x+6}-x}{x^2(x-3)}=\lim_{x\to 3}\frac{\sqrt{3+6}-3}{3^3(3-3)}$$
$$=\lim_{x\to 3}\frac{\sqrt{9}-3}{9\cdot 0}=\lim_{x\to 3}\frac{3-3}{0}=\frac{0}{0}\text{ Limit does not exist}$$

Note: This student was satisfied using the DSP, arriving at an indeterminate form, and concluding that the limit did not exist. This student does not recognize that $x=3$ is not part of the domain of the function, thus not allowing him to use the DSP. Perhaps the student believed that indeterminacy of the 0/0 form implies that the limit does not exist. Although the students were strictly informed not to use L'Hospital's Rule for these limits, this remark was a hint of sorts, because when arriving at an indeterminate form, L'Hospital's Rule is a systematic method for finding limits of these types of functions, provided the numerator and the denominator are continuous and differentiable in appropriate intervals. Thus, arriving at an indeterminate form does not imply that the limit does not exist.

Student #11 wrote for part (a):

$$\lim_{x\to-6}\frac{2x+12}{|x+6|}=\lim_{x\to-6}\frac{2\cancel{(x+6)}}{\cancel{|x+6|}}=\lim_{x\to-6}\frac{2\cancel{(-6+6)}}{\cancel{|-6+6|}}=2$$

Note: This student does not have an understanding of absolute value functions, nor of the DSP. Further, he is unaware that he is dividing by zero in the third step.

4.2.2 Calculating both one-sided limits for part a

Forty-three (68%) students recognized that this function contained an absolute value expression, and as such calculated both one-sided limits correctly. Our supposition is that they had a good enough conceptual understanding of absolute value functions. Only one student provided an explanation using a numerical approach. Eleven (~17%) students also included the definition of the absolute value function as part of their work.

Example of including the definition of the absolute value function

Student #10 wrote:

$$|x+6|=\begin{cases}(x+6)\ if\ x\geq-6\\-(x+6)\ if\ x<\ -6\end{cases}\ \text{So we have to check both sides}$$

Note: Most of the 11 students wrote out this definition instead of the elementary function definition, without noticing that the first piecewise defined domain needed to be $x>-6$, as the absolute value expression was in the denominator. Nonetheless, these students appeared to have a good notion of absolute value functions, as ten of them correctly found both one-sided limits.

Example of finding the correct limit in part (a)

Student #5 wrote:

$$\lim_{x\to-6^+}\frac{2x+12}{|x+6|}=\lim_{x\to-6^+}\frac{2\cancel{(x+6)}}{\cancel{(x+6)}}=\frac{2}{1}=2$$

$$\lim_{x\to-6^-}\frac{2x+12}{|x+6|}=\lim_{x\to-6^-}\frac{2\cancel{(x+6)}}{-\cancel{(x+6)}}=\frac{2}{-1}=-2$$

$$\text{Since}\lim_{x\to-6^+}\ \neq\ \lim_{x\to-6^-}\ \therefore\ \lim_{x\to-6}\ \text{DNE (does not exist)}$$

Some students calculated only one limit, of which 9 (~14%) students only calculated what we coded as the right-hand limit, and three students only calculated what we coded as the left-hand limit despite not having indicated them as such. This suggests that they believed that $|x| = x$, a manifestation of FR-Abs. Despite multiple explanations of the definition of the absolute value function, given $f(x) = |x|$, $|x| = x \; for \; x \geq 0$ and $|x| = -x \;\; for \; x < 0$, students seem very confused by the negative sign in front of the x (Mis-Notation1). Further, two students calculated both one-sided limits and did not provide a conclusion that the limit did not exist as both one-sided limits were not equal. This is either them taking for granted that the teacher knows what they are thinking, or that they follow a procedure and do not have a good understanding of what they are doing. There were no traces of reasoning based on knowledge that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist as the left-sided limit is -1, and the right-sided limit is 1. However, we cannot know for certain what the students were thinking. It does appear that most of the work followed that of a procedure. As previously mentioned, only one student wrote a paragraph explaining his reasoning which resembles that of the table approach in guessing a limit (a numerical approach).

Example of numerical approach in part (a)

Student #14 wrote:

> *The limits are* -2 *and* 2 *as* x *approaches* -6. *In* $\frac{2x+12}{|x+6|}$ *we replace* x *by a number close to* -6*, for instance* -6.00000001 *and find the limit* -2 *as* x *approaches* -6. *We repeat the process with* -5.9999999 *and find* 2 *as the limit since this time the numerator is positive while the denominator always is positive.*

Note: This student wrote a similar paragraph for part (b) as well. This explanation demonstrates his adequate understanding of absolute value functions. However, he mentions repeating a *process* for the other sided limit, making us wonder if this is yet another procedure learned (the table method). Also it is unclear whether this student has difficulty with algebra, and thus avoided it altogether.

4.2.3 Incorrect rationalization

As previously mentioned, 56 (~89%) students knew the first step of how to "rationalize", however only 29 (46%) students obtained the correct limit. The issues that arose were algebraic in nature and will be discussed in the following section.

Example of finding the correct limit in part (b)

Student #1 wrote:

$$\lim_{x\to 3}\frac{\sqrt{x+6}-x}{x^3-3x^2}=\lim_{x\to 3}\frac{\sqrt{x+6}-x}{x^2(x-3)}\cdot\frac{\sqrt{x+6}+x}{\sqrt{x+6}+x}=\lim_{x\to 3}\frac{(x+6)-x^2}{x^2(x-3)\left(\sqrt{x+6}+x\right)}$$

$$=\lim_{x\to 3}\frac{\cancel{(x-3)}(-x-2)}{x^2\cancel{(x-3)}(\sqrt{x+6}+x)}=\lim_{x\to 3}\frac{(-x-2)}{x^2(\sqrt{x+6}+x)}=\frac{-3-2}{3^2(\sqrt{3+6}+3)}=\frac{-5}{9(3+3)}=\frac{-5}{54}$$

One student started to rationalize the numerator correctly, but then crossed off his work and decided to multiply the numerator and the denominator by a "conjugate" of the binomial in the denominator:

Example of incorrect rationalization in part (b)

Student #32 wrote: $\lim_{x\to 3}\frac{\sqrt{x+6}-x}{x^3-3x^2}=\lim_{x\to 3}\frac{(\sqrt{x+6}-x)(x^3+3x^2)}{x^6-9x^4}$

Note: This student appears to remember vaguely that "multiplying by conjugate" was sometimes used in problems of finding limits but does not seem to see the purpose of the action, since what he obtains does not remove the indeterminacy and does not allow him to calculate the limit.

The remaining six (~10%) students did not attempt to rationalize. Student #14 offered a numerical method description similar to his answer for part (a). Student #4 was also previously mentioned as he applied DSP directly for both parts (a) and (b), and determined that the limit did not exist based on the fact that he obtained an indeterminate form. Three students' work were incomprehensible, and the remaining student's work is provided below.

Student #25 wrote:

$$\lim_{x\to 3}\frac{\left(\sqrt{x+6}-x\right)^2}{(x^3-3x^2)^2}=\lim_{x\to 3}\frac{x+6-x^2}{x^5-3x^4}=\lim_{x\to 3}\frac{x+6-x^2}{x^4(x-3)}=\lim_{x\to 3}\frac{(-x-2)\cancel{(x-3)}}{x^4\cancel{(x-3)}}=$$

$$\lim_{x\to 3}\frac{-x-2}{x^4}\lim_{x\to 3}\frac{-3-2}{3^4}=\lim_{x\to 3}\frac{-5}{81}$$

Note: The errors in this student's work are apparent starting from the first step. He seems to believe that $\lim_{x\to a} f(x) = \lim_{x\to a} f(x)^2$. In the second step, the student believes that squaring a binomial means squaring each term as such: $(a+b)^2=a^2+b^2$ (as seen in the numerator) (FR-Dist3). As for the denominator, the student applies the same reasoning, however also has an incorrect understanding of exponential laws. The student also does not square the 3 in the denominator (FR-Ops5). After this step, the student appears to have a good understanding of factoring. In the final step, the student does not take the limit of the constant.

4.2.4 Problems with associative, commutative, and distributive properties

Common school mathematical jargon, not only spoken but referred to as in many textbooks, coins the term "factoring", instead of engraining in the students that they are using the distributive property $(a(b + c) = ab + ac)$. When "factoring something out", or when "expanding" we are using the distributive property. Sierpinska and Hardy (2010) express the concern that the majority of textbooks do not emphasize the associative, commutative, and distributive properties. Rather textbooks provide mnemonics such as the FOIL method described earlier in this thesis. After a student has applied the FOIL method, they can verify their work using the "reverse FOIL technique". Again, this is simply the distributive property. The work by Booth et al. (2014) discusses many algebraic misconceptions (such as misconceptions with the negative sign) and the incorrect use of the associative, commutative, and distributive properties. In both parts of problem 2, errors with these properties became apparent. The main errors across students' solutions were that of the distributive property. Note: the main focus of this section is algebraic errors, thus only the relevant parts of the solutions are provided.

Examples of errors with the distributive property

Student #61 wrote for part (a): $2x + 12 = (x + 6)(x + 2)$

Note: This student knew that he needed to factor the numerator for problem 2a, yet as simple as it was, could not do so. It was as though the distributive property he used was $(ab + cd) = (a + c)(b + d)$ (FR-Dist13). If he used the FOIL method, he would have realized that these were not equivalent expressions.

For part (b), six (~10%) students had difficulty with "factoring", post rationalizing the numerator. The common error was that the leading term contained a negative sign in front, and we believe that students were unable to factor the polynomial as it was not in the form they were used to: $ax^2 + bx + c$, where $a > 0$.

Student #36 wrote: $(x + 6) - x^2 = -x^2 + x + 6 = (-x - 3)(x - 2)$

Note: The student does correctly apply the commutative property of addition. However, after "factoring" the student did not verify his work. In multiplying the binomials, applying the distributivity property twice, the student would have realized that something was incorrect. It was as though the distributive property he used was $-x^2 + x + ab = (-x - a)(x - b)$ (FR-Dist11).

In multiplying binomials for part (b), three students provided the following solution for the numerator:

Student #13 wrote: $\sqrt{x+6} - x \;\cdot\; \sqrt{x+6} + x = x + 6 - x$

Note: It is as though they were focusing primarily on the radical and forgot about multiplying the x terms.

In multiplying binomials for part (b), four students did not multiply both terms in the denominator by the conjugate, as though they have forgotten the brackets.

Student #46 wrote: $\frac{\sqrt{x+6}-x}{x^2(x-3)} \cdot \frac{\sqrt{x+6}+x}{\sqrt{x+6}+x} = \frac{-x^2+x+6}{x^2(x-3)\cdot\sqrt{x+6}+x}$

Note: The error (failure to apply the distributive law) can be seen in the second step. The student multiplies $x^2(x-3)$ by $\sqrt{x+6}$, while leaving the x term alone (FR-Dist2).

Lastly, one student provided an interesting version of the distributive property to the denominator.

Student #18 wrote: $(x^2)\left[\sqrt{x+6} + x\right] = x + 6x + x^3$

Note: This student appears to be applying some form of the distributive property to the radical. It is possible that he believes the square root to be linear as such $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ (FR-Lin-Sqrt). However, if this were so, he does not know his exponential laws. How would $(x^2)\sqrt{x+6} = x + 6x$?

4.2.5 Other errors

Five students incorrectly "cancelled out" a term from the numerator and denominator, apparently believing that the algebraic property: If $b, c \neq 0$, then $\frac{a}{b} = \frac{ac}{bc}$ applies also to addition: $\frac{a+c}{b+c} = \frac{a}{b}$ (FR-Frac3).

Student #29 wrote: $\frac{-x-2}{x^2(\sqrt{x+6}-x)} = \frac{-2}{x(\sqrt{3+6}-x)}$

Note: The student "cancels out" an x from the numerator and denominator, despite the numerator being a binomial. Also he replaces only one of the x terms by the value it is approaching in this step.

Despite rationalizing correctly, two students did not "factor", rather they suddenly divided all terms by the highest degree found in the numerator or denominator, as though they were confusing this with the technique for finding limits at infinity (this technique has been discussed earlier in this thesis).

Student #58 wrote:

$$\frac{x+6-x^2}{(x^3-3x^2)(\sqrt{x+6}+x)} = \frac{\frac{1}{x^2}+\frac{6}{x^3}-\frac{1}{x}}{\left(1-\frac{3}{x}\right)(\sqrt{x+6}+x)}$$

Note: This student divides all terms by x^3 which was the term with the highest degree in the denominator.

Lastly, seven (~11%) students made small arithmetic errors in their calculations such as $-(3+2) = -1$. Although we determined such errors to be small arithmetic errors, it is possible that this is yet another application of a false distributive rule: $-(a+b) = -a+b$ (FR-Dist1).

4.3 Students' responses to Problem 3

Problem 3 asked students to find all the horizontal and vertical asymptotes of a given function. This function was a ratio of two functions, with the square root of a quadratic polynomial in the numerator and a polynomial of degree one in the denominator. The problem was formulated as follows:

Find all the horizontal and vertical asymptotes of the function $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$.

As this function contained a radical expression, we did expect a number of algebraic mistakes to occur in finding both the vertical and the horizontal asymptotes. In finding potential vertical asymptotes, students could incorrectly solve for x, when setting the denominator equal to zero. Further, they could state the vertical asymptote solely based on the function being undefined at a point, without verifying a one-sided limit. For those that did verify one of the one-sided limits, we expected some algebraic mistakes. In finding horizontal asymptotes, mistakes could occur in dividing numerator and denominator by the term in the denominator with the highest degree (a procedure used for rational functions can be used here as well). Alternatively, students could incorrectly use the distributive laws to factor out the x term from the numerator. Their mistakes could stem from an incorrect conception of the absolute value function (e.g., claiming that $\sqrt{x^2} = x$ instead of $\sqrt{x^2} = |x|$) and/or from ignorance of the distributive property leading to the inability to factor.

In Table 3 we present the distribution of correct and incorrect answers to this problem.

Table 3. Distribution of correct and incorrect answers to Problem 3.

n=63	correct	incorrect	other	total
P3 - Horizontal Asymptotes				
Limit as $x \to \infty$	33 (52%)	17 (27%)	13 (21%)	63
Limit as $x \to -\infty$	19 (30%)	17 (27%)	27 (43%)	63
Asymptotes: $y = L; y = -L$	14 (22%)	20 (32%)	29 (46%)	63
P3 - Vertical Asymptote				
Limits as $x \to a^{\pm}$	2 (3%)	0	61 (97%)	63
Asymptote: $x = a$	58 (92%)	4 (6%)	1 (2%)	63

Remarks: Only 13 (~21%) students correctly found all asymptotes as the lines $x = \frac{5}{3}$, $y = \frac{\sqrt{2}}{3}$, and $y = -\frac{\sqrt{2}}{3}$. For vertical asymptotes, most students did not calculate a one-sided limit. One reason could be that the students made incorrect generalizations whereby they believed that vertical asymptotes are *always* found by making the denominator of a ratio of functions equal to zero, or that there is always an asymptote at a point of indeterminacy. For horizontal asymptotes, not all students verified both limits at infinity. Lastly, a number of students only calculated the limits, without stating the equations of the asymptotes. Note: the "other" column includes all students who omitted the respective portion of the problem, as well as those whose work was incomprehensible.

The common mistakes amongst students will be grouped and described in following sections.

4.3.1 Vertical asymptote - one-sided limits omitted

The majority, 58 (~92%) of the 63 students correctly identified the vertical asymptote by finding where the function was undefined; the point at which the denominator equaled to zero. Only one student continued the problem by finding both one-sided limits. Another student wrote out their thoughts, demonstrating their understanding of vertical asymptotes.

Examples of correctly finding the vertical asymptote

Student #26 wrote:

The vertical asymptote, f is not defined in $x = \frac{5}{3}$, then

$$\lim_{x \to \frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to \frac{5}{3}^+} \frac{\sqrt{2\left(\frac{25}{9}\right)+1}}{0^+} = \frac{\sqrt{\frac{50}{9}+1}}{0^+}$$

$$=> \lim_{x \to \frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$$

$$\lim_{x \to \frac{5}{3}^-} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x \to \frac{5}{3}^-} \frac{\sqrt{\frac{50}{9}+1}}{0^-} = -\infty$$

$$\lim_{x \to \frac{5}{3}^-} \frac{\sqrt{2x^2+1}}{3x-5} = -\infty$$

$$x \to \frac{5}{3}^- \quad => 3x-5 \ \to 0^-$$

then the horizontal asymptote are $x = \frac{5}{3}$

Note: Although this student wrote "horizontal asymptote" in his concluding statement, his work for the vertical asymptote, as well as work for the horizontal asymptotes prior to this work was correct. We believe he incorrectly wrote “horizontal”, but understood the difference based on his work. This student verified both one-sided limits, even though one would have sufficed.

Student #33 wrote:

Vertical asymptote: if $(3x - 5 = 0)$ so $f(x)$ approaches infinity

$3x - 5 = 0 => x = \frac{5}{3}$ vertical asymptote

Note: The student manages to find the limit, however did not show a calculation of it. In his explanation, he uses imprecise, ambiguous language, which might be a sign of insufficient conceptual distinctions between numerical solutions to equations and these equations. When he writes $3x - 5 = 0$ in the first line of his solution, does he mean “if x is a solution of the equation $3x - 5 = 0$”? It would have been more precise to say, “If x tends to the solution of the equation $3x - 5 = 0$ then $f(x)$ approaches infinity”. In the second line of his solution, is “$x = \frac{5}{3}$” a value of the solution to the equation “$3x - 5 = 0$” or is it the equation of a line, i.e., the set of points (x, y) such that $x = \frac{5}{3}$?

4.3.2 Vertical asymptote - problems with associative, commutative, and distributive properties

Four students obtained an incorrect vertical asymptote by having an incorrect understanding of vertical asymptotes, as well as making algebraic errors. One of these students attempted to rationalize the numerator. We suppose this was done as his recollection of procedures entails rationalizing every time he encounters a radical function. Lastly, one student did not attempt this portion of the problem.

Examples of errors in finding the vertical asymptote

Student #6 wrote:

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}\,\frac{\sqrt{2x^2-1}}{\sqrt{2x^2-1}}$$

$$f(x) = \frac{(2x^2-1)^2}{(3x-5)(\sqrt{2x^2}-1)}$$

Vertical Asymptotes

$$x = 0$$

Note: A number of algebraic errors are apparent in this student's work. In attempting to rationalize the numerator, this student changed the sign of the constant within the radical expression. In line 2 of his work, his difficulties are seen in the numerator. His knowledge of the rationalization process is flawed, as is his understanding of exponents. It is as though he believes that $\sqrt{a+b}\cdot\sqrt{a-b} = (a-b)^2$ (FR-Rad5). Further, in the denominator of line 2 he appears to remove the -1 from the radical expression. It is unclear if this is inattention. Lastly, he obtains a vertical asymptote of $x = 0$ without providing any information as to how he obtained it.

Student #24 wrote:

Vertical $\rightarrow \sqrt{2x^2+1}$

$$\left(\sqrt{2x^2+1}\right)^2 = 2x^2+1 = 0$$

$$= \sqrt{2x^2} = \sqrt{-1}$$

$2x = \sqrt{-1} \rightarrow$ Ø Vertical b/c $-$

Note: This student attempted to find the vertical asymptote by setting the expression in the numerator equal to zero, and trying to solve for x. This student has no conceptual understanding of vertical asymptotes. Rather, he incorrectly remembered a procedure in which he needed to solve for x after making an expression equal to zero. From line 3 to line 4, this student believes that $\sqrt{2x^2} = 2x$, as though the root does not apply to the coefficient (FR-Rad3). Lastly, since the numerator will never equal to zero, this student concludes that there is no vertical asymptote because of the negative expression under the radical.

Student #15 wrote:

> *To find vertical asymptotes, we look where the function could be undefined. We know $2x^2 + 1$ must be greater or $=$ to 0 for the function $f(x)$ to be defined as the argument under the square root must be 0 or positive. But x^2 will always be positive, $\therefore$ so will $2x^2$ and $2x^2 + 1$ hence the function $f(x)$ is defined everywhere (no vertical asymptotes).*

Note: This student starts with a correct understanding that a vertical asymptote can occur at points in which the function is undefined. However, this student looks solely at the numerator, disregarding the indeterminacy in the denominator.

Student #44 wrote:

For vertical asymptotes $=> 3x - 5 = 0$

$$3x = 5$$

$$x = \frac{3}{5}$$

$x = \frac{3}{5}$ is the vertical asymptote

Note: Perhaps this student made an inattention error, however, it is also possible that they believe the following: $ax = b$ is equivalent to $x = \frac{a}{b}$ (FR-Eq3). The student believes that there is always a vertical asymptote at a point of indeterminacy.

4.3.3 Horizontal asymptotes - limits as the asymptotes

A total of 19 (~30%) students correctly found both limits at infinity. Further, after obtaining these correct limits, only 10 (~16%) identified both horizontal asymptotes as either being the *lines* $y = \pm\frac{\sqrt{2}}{3}$, or wrote out that the horizontal asymptotes are $\pm\frac{\sqrt{2}}{3}$. It is as though some students believed that the horizontal asymptotes are the limits themselves and not lines. Another three students correctly identified both horizontal asymptotes without having shown their calculations of both limits at infinity. Nineteen (~30%) students did not attempt finding the limit as x tends to negative infinity, as though finding the horizontal asymptote as x tends to positive infinity was sufficient. Perhaps they confused the two definitions between vertical and horizontal asymptotes. As long as one of the limits at infinity produced a limit, then they may have believed that it would be sufficient in finding a horizontal asymptote. Six (~10%) students did not attempt finding either of the limits at infinity.

Examples of finding the horizontal asymptotes correctly

Student #15 wrote:

To find horizontal asymptotes:

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\lim_{x\to\infty}\frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{3x-5}$$

*$\sqrt{x^2} = \pm x$, but here as $x \to \infty$ we can consider it as positive

$$\therefore = \lim_{x\to\infty}\frac{\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{5}{x}\right)}=\frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}=\frac{\sqrt{2}}{3}$$

$f(x)$ has a horizontal asymptote $y = \frac{\sqrt{2}}{3}$ as it tends to positive infinity.

The process for the limit at $-\infty$ is similar, but there we can assume x to be negative

and so $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{-\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{5}{x}\right)} = -\frac{\sqrt{2}}{3}$

$f(x)$ has a horizontal asymptote $y = -\frac{\sqrt{2}}{3}$ as it tends to negative infinity.

Note: In the third line of this student's work, we find a common absolute value error. By convention, the square root of a number A is a non-negative number B such that $B^2 = A$. Thus, the square root of x^2 is $|x|$, and not $\pm x$ (FR-Sqrt4). In the fourth line, this student forgets to carry forward the limit symbol, however he correctly completes the problem.

Student #23 wrote:

$$f(x)=\frac{\sqrt{2\infty^2+1}}{3\infty-5} => \frac{\sqrt{2\infty^2}}{3\infty} => \frac{\sqrt{2}}{3}$$

$$f(x)=\frac{\sqrt{2(-\infty)^2+1}}{3(-\infty)-5} => \frac{\sqrt{2\infty^2}}{-3\infty} => -\frac{\sqrt{2}}{3}$$

$\therefore$ Horizontal asymptotes are $y = -\frac{\sqrt{2}}{3}$ as $x \to -\infty$, and $y = \frac{\sqrt{2}}{3}$ as $x \to \infty$.

Note: This student correctly identifies the horizontal asymptotes; however, he or she appears to treat ∞ as a number. He does not include the limit symbols in his work, yet his concluding statement appears to demonstrate his understanding of limits at infinity.

Lastly, 32 (~51%) students either made some algebraic error in finding one of the limits at infinity, or had an incorrect understanding of horizontal asymptotes. These errors will be discussed below.

4.3.4 Horizontal asymptotes - problems with associative, commutative, and distributive properties

In calculating the limits at infinity, 11 (~17%) students made errors with the associative, commutative, and distributive properties. As well, some arithmetic errors were apparent.

The following student decided to apply the Quotient Rule to the function $f(x)$ in order to calculate the limits at infinity. The student incorrectly obtains some of the derivatives, and makes a number of algebraic errors.

Student #49 wrote:

[1] horizontal Asymptote $\lim_{x \to \pm\infty}$

[2] $\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$

[3] $\lim_{x\to-\infty} \frac{d}{dx}\left[\frac{(\sqrt{2x^2+1})}{(3x-5)}\right]$

[4] $\to \lim_{x\to-\infty} \frac{(3x-5)\frac{d}{dx}[\sqrt{2x^2+1}]-\sqrt{2x^2+1}\frac{d}{dx}[3x-5]}{(3x-5)^2}$

[5] $\to \lim_{x\to-\infty} \frac{(3x-5)\cdot\frac{1}{2}(4x)-\sqrt{2x^2+1}\cdot(3)}{(3x-5)^2}$

[6] $\to \lim_{x\to-\infty} \frac{3\cdot(3x-5)\frac{d}{dx}\left[\frac{1}{2}\right]\cdot\frac{d}{dx}[4x]-\sqrt{2x^2+1}}{(3x-5)^2}$

[7] $\to \lim_{x\to-\infty} \frac{4\cdot3\cancel{(3x-5)}-\sqrt{2x^2+1}}{\cancel{(3x-5)}^2}$

[8] $\to \lim_{x\to-\infty} \frac{12-\sqrt{2x^2+1}}{3x-5}$

[9] $= \frac{12-\sqrt{2(0)^2+1}}{3(0)-5}$

[10] $= \frac{12-\sqrt{1}}{-5}$

[11] $x = -\frac{11}{5}$ horizontal asymptote

[12] $\lim_{x\to+\infty}$ cannot exist because function would be undefined

Note: The student starts by trying to find the limit as x tends to negative infinity. He then applies the Quotient Rule in line 4. Line 5 demonstrates that he knows the Quotient Rule, however in line 5 he incorrectly obtains the derivative of $\sqrt{2x^2+1}$. Post differentiating, in line 6, he appears to want to take the derivative of $\frac{1}{2}$ and multiply it by the derivative of $4x$, however this would require the use of the Product Rule. Further, in line 6, we can see his lack of knowledge of the commutative law. He decides to move the constant multiplying the second term in the numerator to the front of the first term. It is as though he believes that $a + bc = ca + b$ (FR-Com). In line 7, it appears that implicitly he knows the derivative of the constant $\frac{1}{2}$ is zero as it is nowhere to be found, however, the derivative of $4x$ appears. From line 7 to line 8, this student cancels the $(3x - 5)$ term from one of the terms in the numerator and from the denominator perhaps believing that the algebraic property: if $b, c \neq 0$, then $\frac{a}{b} = \frac{ac}{bc}$ applies also to addition: $\frac{ab+c}{bd} = \frac{a+c}{d}$ (FR-Frac1). Lastly, in line 9, the student replaces x with 0, however the limit to be calculated was as x tends to negative infinity. This is perhaps the student's vague recollection of problems of limits at infinity in which some terms go to zero, namely the $\lim_{x\to\infty} \frac{1}{x^r} = 0$, where $r > 0$ is a rational number. In line 11, he writes that the horizontal asymptote is $x = -\frac{11}{5}$, confusing the variables x and y. In his 12th line, he mentions that the limit at positive infinity cannot exist, however does not provide enough information to help us understand why he believes that the function would be undefined.

Two students attempted to square some part of the function in order to find a limit at infinity., demonstrating their difficulties in dealing with the radical expression.

Student #25 wrote:

$$\lim_{x\to\infty} f(x) \frac{(\sqrt{2x^2+1})^2}{(3x-5)^2}$$

$$\lim_{x\to\infty} f(x) \frac{2x^2+1}{3x^2-5^2}$$

$$\lim_{x\to\infty} f(x) \frac{\frac{2\cancel{x^2}}{\cancel{x^2}} + \cancel{\frac{1}{x^2}}}{\frac{3\cancel{x^2}}{\cancel{x^2}} - \cancel{\frac{25}{x^2}}}$$

$$\lim_{x\to\infty} f(x)\ \frac{2}{3}$$

Note: In line 2, this student appears to believe that $(a + b)^n = a^n + b^n$ (FR-Dist3), and forgets to distribute the square to the 3 (FR-Ops5). In line 3, he then divides all terms by the term with the highest degree, x^2, obtaining a limit of $\frac{2}{3}$. Throughout, this student leaves $f(x)$ as part of the limit symbol as if not recognizing that the limit he was trying to obtain was of the function to the right of $f(x)$.

Student #60 wrote:

[1] H.A.

[2] $\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}\ \frac{(2x^2+1)^2}{3x-5}$

[3] $\lim_{x\to\infty}\frac{(2x^2+1)^2}{3x-5}$

[4] $\lim_{x\to\infty}\left(\frac{2x^4+4x^2+1}{3x-5}\right) \div x$

[5] Anything divide by ∞ equal 0

[6] $\lim_{x\to\infty}\frac{2x^3+4x+\frac{1}{x}}{3-\frac{5}{x}} \to$

[7] $\lim_{x\to\infty}\frac{2x^3+4x}{3}$

[8] $\lim_{x\to\infty}\frac{\frac{2x^3+4x}{x}}{3}$

[9] $= \lim_{x\to\infty}\frac{2x^2}{3}$

[10] H.A. by higher exponent $\frac{2}{3}$

Note: "H.A." is the student's acronym for horizontal asymptote. Throughout this solution, the student omits equal signs between limits. In line 2, the student appears to forget to add a limit symbol to the second step, and appears to square only the numerator, after having removed the radical. Perhaps they believe that to remove a square root symbol over a radicand, one must square that expression as such: $\sqrt{x} = x^2$. This student does not have a good understanding of square root functions. Line 3 includes the previous limit with the limit symbol. From line 3 to line 4, the student incorrectly squares the numerator, leaving the coefficient in front of x^4 as 2 instead of 4 (FR-Ops5). In this step, he decides to divide by the term with largest degree of the denominator. This step is done correctly. He includes a statement that "Anything divide by ∞ equal 0" and moving from line 6 to 7, drops the terms that tend to 0. In line 8, he decided to divide the numerator by x, although it is unclear why he chose to do so. From line 8 to line 9, dividing $4x$ by x should produce 4, although it is nowhere to be found. In his concluding statement, the student claims that the horizontal asymptote is $\frac{2}{3}$ "by higher exponent", however by his last limit line, that limit would equal to infinity and not $\frac{2}{3}$, demonstrating his ignorance of limits. The "H.A. by higher exponent" statement makes us presume that he is incorrectly recalling the rational function technique for finding limits at infinity.

Three students attempted some form of the rationalization procedure. One successfully applied the correct procedure, and had good knowledge of the distributive property, whereas the other two made a number of errors.

Example of a correct rationalization procedure

Student #54 wrote:

[1] $\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\sqrt{2x^2+1}}{\sqrt{2x^2+1}}$

[2] $\lim_{x\to\infty} \frac{2x^2+1}{(3x-5)(\sqrt{2x^2+1})}$

[3] $\lim_{x\to\infty} \frac{2x^2+1}{3x\sqrt{2x^2+1}-5\sqrt{2x^2+1}}$

[4] $\lim_{x\to\infty} \frac{\frac{2x^2}{x^2}+\frac{1}{x^2}}{\frac{3x\sqrt{2x^2+1}}{x^2}-\frac{5\sqrt{2x^2+1}}{x^2}}$

[5] $\lim_{x\to\infty} \frac{2+\frac{1}{x^2}}{3\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}-5\sqrt{\frac{2x^2}{x^4}+\frac{1}{x^4}}}$ $\lim_{x\to-\infty} \frac{2+\frac{1}{x^2}}{-3\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}-5\sqrt{\frac{2x^2}{x^4}+\frac{1}{x^4}}}$

[6] $\frac{2+0}{3\sqrt{2+0}-0} = \frac{2}{3\sqrt{2}}$ $-\frac{2}{3\sqrt{2}}$

[7] Horizontal asymptote: $y = \frac{2}{3\sqrt{2}}$ and $y = -\frac{2}{3\sqrt{2}}$

Note: From line 1 to 2, the student correctly rationalizes the numerator. In line 3, he correctly applies the distributive property. In line 4, he decides to divide all terms by the term with the highest degree. In line 5, he correctly inserts the x^2 and x^4 terms into the radical expressions - this is where he starts the work for the limit at negative infinity as well. Although there are no signs equating the limits from line to line, this student has good algebraic knowledge, and obtains the correct limits.

Example of a wrong rationalization procedure

Student #24 wrote:

[1] Horizontal → $\frac{\sqrt{2x^2+1}}{3x-5}$

[2] $= (\sqrt{2\cancel{x^2}+1})^2$

$$\frac{\overline{\cancel{x^2}} \quad \overline{x^2}}{3x-5}$$

[3] $= \frac{2}{3x-5} \cdot \frac{3x+5}{3x+5}$

[4] $= \frac{6x+10}{3x-5}$

[5] $= 2x - 2 = 0$

[6] $= 2x = 2$

[7] $x = 1$

Note: This student does not include any limit symbols throughout his solution. In line 2, his equal sign is next to the numerator only, and he appears to only divide the numerator by x^2. He understands that $\frac{1}{x^2}$ will tend to zero as x tends to infinity. He appears to only square the radical which leaves him with a numerator of 2. In the third line, he decides to "rationalize" the denominator, missing the purpose of this technique. He correctly applies the distributive property in the fourth line for the numerator, however disregards the second binomial in the denominator. In line 5, it appears that he believes that the first terms divide, and the second terms divide as such: $\frac{ab+cd}{a+c} = b + d$ (FR-Frac2), demonstrating his ignorance of algebraic properties.

Four students made arithmetic errors, in which two claimed that a division by zero equaled zero. One example is provided below.

Example of diving by zero

Student #42 wrote:

$$\text{H.A} = \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{(2x^2+1)^{\frac{1}{2}}}{3x-5}$$

$$= \lim_{x\to\infty} \frac{x^2\left(2+\frac{1}{x^2}\right)^{\frac{1}{2}}}{x^2\left(\frac{3}{x}-\frac{5}{x^2}\right)}$$

$$= \frac{(2+0)^{\frac{1}{2}}}{0-0}$$

$$= \frac{\sqrt{2}}{0} = 0$$

Note: In line 2, the student factored an x^2 from the numerator, but when removing it from the expression with the exponent of $\frac{1}{2}$, does not distribute the exponent. This in turn causes him to divide the denominator by x^2 as well. In line 4, the student believes that dividing a number by zero will return a zero.

4.3.5 Horizontal asymptotes - problems with square root functions

Similar to the misconceptions provided in the student solutions to problem 1, four solutions seemed to be based on the belief in the linearity of square root functions, i.e. the students believed that the square root of the sum is the sum of the square roots.

Example of linearity of square root functions

Student #36 wrote:

$$\lim_{x\to\infty}\frac{(2x^2+1)^{\frac{1}{2}}}{3x-5}=\lim_{x\to\infty}\frac{2x^1+1^{\frac{1}{2}}}{3x-5}=\frac{2+\frac{1}{x}}{3-\frac{5}{x}}=\frac{0}{0}$$

Note: Prior to this solution, this student rationalized the numerator, however ended up with an indeterminate form of the type $\frac{\infty}{\infty}$. He then decided to proceed with this solution. The linearity belief can be seen in the second step (FR-Lin-Sqrt), however this student does not carry the exponent of $\frac{1}{2}$ to the coefficient 2 (FR-Rad3). In the third step, the student omits the limit symbol, and then, in the fourth concludes the solution of an indeterminate form of the type $\frac{0}{0}$. It is possible that the student confused the sum with the zero product property in which $ab = 0$ if either $a = 0$, $b = 0$, or both $a, b = 0$. He perhaps believes that $a + 0 = 0$, regardless of the value of a (FR-Zero).

For the students who attempted to find the limit as x tends to negative infinity, 10 (~16%) students had an incorrect understanding of square root and absolute value functions whereby they did not use the definition $\sqrt{x^2} = |x|$.

Example of a limit at negative infinity

Student # 10 wrote:

$$\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\lim_{x\to-\infty}\frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}=\frac{\sqrt{2+0}}{3}=\frac{\sqrt{2}}{3}$$

Horizontal asymptote at $y = \frac{\sqrt{2}}{3}$

Note: This student appears to divide numerator, and denominator by x, following the procedure for finding limits at infinity for rational functions. However he misses the fact that since x tends to negative infinity ($\sqrt{x^2} = |x| = -x \; for \; x < 0$), when dividing the numerator by x this term inserts itself under the radical by becoming x^2, however a negative sign must be placed in front of the radical, not to confuse it with the principal square root of x^2 (FR-Sqrt3).

Another misunderstanding of square root functions appeared in the solutions of three students who decided to drop the radical at some point in their solution.

Student #17 wrote:

$$\lim_{x\to\infty} \frac{(2x^2+1)^{\frac{1}{2}}}{3x-5} = \lim_{x\to\infty} \frac{(x^2)^{\frac{1}{2}}\left(\frac{2x^2}{x^2}+\frac{1}{x^2}\right)}{x\left(\frac{3x}{x}-\frac{5}{x}\right)}$$

$$\lim_{x\to\infty} \frac{\cancel{x}(2)}{\cancel{x}(3)} = \frac{2}{3}$$

Note: This student factors an x^2 term from the numerator, correctly applies the exponent of $\frac{1}{2}$ to this term, however appears to drop this exponent over the other terms. This student also understands that $\lim_{x\to\infty} \frac{1}{x^r} = 0$, where $r > 0$ is a rational number.

4.3.6 Horizontal asymptotes - rational function technique

Five students attempted to find the limits based on the coefficients of the terms with the highest degree. This technique was mentioned earlier, in reference to rational functions. Even though this function was not a rational function, the same reasoning could be applied. Two students used this reasoning correctly, without manipulating the function algebraically, whereas the other three were unable to obtain the correct answers.

Example of using the rational function technique correctly

Student #35 wrote:

$$\text{degrees: } f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

$$y = \frac{\sqrt{2}}{3}, y = -\frac{\sqrt{2}}{3}$$

Note: This student did not provide much work, although they did write "degrees" and circled the most important coefficients to look at $\left(\frac{\sqrt{2}}{3}\right)$. We can guess that this student implicitly knew the following definition $\sqrt{x^2} = |x|$, as they correctly identified both horizontal asymptotes as the lines $y = \pm\frac{\sqrt{2}}{3}$.

Example of using the rational function technique incorrectly

Student #8 wrote:

Horizontal

the higher exponent is 1, denominator and numerator has same exponent

so H.A. is the division of coefficient with the higher exponent.

$y = \frac{2}{3}$ horizontal asymptote

Note: This student appears to have a good understanding of the technique, however fails to recognize the radical over the numerator, thus obtaining an incorrect asymptote. Further, he only provided one asymptote. This omission may be a misunderstanding of horizontal asymptotes and the necessity to verify both limits at infinity, or a manifestation of the false rule $\sqrt{x^2} = x$ (FR-Sqrt3).

4.3.7 Horizontal asymptotes - other errors

This section lists a number of errors made by students in finding one of the limits at infinity. Two students applied l'Hospital's Rule, and could not produce the correct limit.

Example of l'Hospital's Rule

Student #46 wrote:

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{(2x^2+1)^{-\frac{1}{2}} \cdot 4x}{2(3)}$$

$$= \lim_{x\to\infty} \frac{4x}{6\sqrt{2x^2+1}}$$

$$= \lim_{x\to\infty} \frac{4}{\frac{6\sqrt{2x^2+1}}{x}} = 0$$

Note: The student applied l'Hospital's Rule correctly, however in the third line, he divides both numerator and denominator by x, as if he switches his method by dividing by the highest degree (technique used with rational functions). He does not offer enough work to demonstrate how his resulting limit is 0. Perhaps he views the denominator $\frac{6\sqrt{2x^2+1}}{x}$ as tending to infinity, and not as another indeterminate form. He could have tried to apply l'Hospital's Rule again in line 3, as one arrives once more at the indeterminate form of $\frac{\infty}{\infty}$, however this would bring him back to a function similar to $f(x)$ in which he would continuously obtain indeterminate forms of the type $\frac{\infty}{\infty}$ after applying l'Hospital's Rule.

The other student applied l'Hospital's Rule, however forgot to apply the Chain Rule in the second step, resulting in a limit of zero.

Student #63 wrote:

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\lim_{x\to\infty}\frac{\frac{1}{2}(2x^2+1)^{\frac{1}{2}}}{3}=\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{2x^2+1}}}{3}=\lim_{x\to\infty}\frac{1}{6\sqrt{2x^2+1}}=0$$

Two students obtained indeterminate forms for the limit at infinity. One student concluded that the indeterminate form of $\frac{\infty}{\infty}$ meant that the function contained no horizontal asymptote. The other student's work is shown below.

Student # 51 wrote:

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}$$

$$=\lim_{x\to\infty}\sqrt{2x^2+1}\cdot\lim_{x\to\infty}\frac{1}{3x-5}$$

$$=\lim_{x\to\infty}\sqrt{2x^2+1}\cdot\lim_{x\to\infty}\frac{\frac{1}{x}}{3-\frac{5}{x}}$$

$$=\ \infty\cdot 0$$

$$=0$$

$$=\lim_{x\to-\infty}\sqrt{2x^2+1}\cdot\lim_{x\to-\infty}\frac{1}{3x-5}$$

$$=+\infty\cdot 0=0$$

Horizontal asymptote: $y=0$

Note: This student obtains an indeterminate form of the type $\infty\cdot 0$ in line 4, yet believes that it equals to zero. He obtains the same value of zero for the limit at negative infinity.

The following student decided to set the numerator equal to zero, and to solve for x, concluding that there was no horizontal asymptote. He has no understanding of limits at infinity, and perhaps is confusing the procedures for finding horizontal and vertical asymptotes. He does however obtain the correct vertical asymptote.

Student #59 wrote:

$$2x^2 + 1 = 0$$
$$2x^2 = -1$$
$$x^2 = -\tfrac{1}{2} \text{ not defined}$$
$$F \text{ has not horizontal asymptotes}$$

Lastly, one student obtained a limit of infinity for the limit of the function as x tends to infinity. He did not provide work for this. He concluded that "The function admits no horizontal asymptotes". Although this answer is incorrect, he appears to have a good understanding of horizontal asymptotes, as a limit of infinity does indicate that a function increases without bound. Although he did not provide work for this answer, in his side scribbles he sheds some light in the difficulties he had in finding this limit.

Student #56 wrote:

$$\frac{\sqrt{2x^2+1}\,\sqrt{2x^2+1}}{3x-5}$$
$$\frac{4x^4+1}{(3x-5)\sqrt{2x^2+1}}$$
$$(3x\sqrt{2x^2+1} - 5\sqrt{2x^2+1})$$

Note: In his first line, he appears to rationalize the numerator, however forgets to multiply the denominator by $\sqrt{2x^2+1}$. In his second line, he then multiplies the denominator by $\sqrt{2x^2+1}$. He also believes that multiplying two radicals eliminates the root symbol, however, multiplies the inner expressions without applying the distributive property. It is as though he believes $\sqrt{a+b} \cdot \sqrt{a+b} = a^2 + b^2$ (FR-Rad4). In his third line, he only writes the expression for the denominator and does correctly apply the distributive property. No other work is shown after this point, however, since it appears that the numerator has the term with the highest degree the student may have concluded that the limit equaled infinity.

4.4 STUDENTS' RESPONSES TO PROBLEM 4

Problem 4 asked students to find the derivatives of five functions. Since students were told that they do not need to simplify their final answers, the question did not test their elementary algebraic skills, but mostly their ability to apply differentiation rules to moderately complicated algebraic expressions. This application, as mentioned in our a priori analysis of the question (section 3.4), requires the ability to recognize the structure of the expression. Thus, in our analysis of the students' solutions, we tried to assess whether students recognized correctly the overall structure of the expression. For problems that required the use of the Chain Rule, we attempted to determine whether students knew how many derivatives needed to be multiplying each other, as well as whether these derivatives were of the correct elementary function. We will not dwell on the correctness of the derivatives that students provided for the elementary functions. The most common difficulties were with applying the appropriate differentiation formula (D-App), and difficulties with the structure of the expressions (D-Struc).

As a reminder, Problem 4 was given as follows:

> *Find the derivatives of the following functions. (You don't need to simplify the final answer, but you must show how you calculate it):*
>
> ***Part a)*** $f(x) = \frac{\sqrt[3]{x} - 2xe^x}{x}$
>
> ***Part b)*** $f(x) = e^{\sin 2x} + \sin(e^{2x})$
>
> ***Part c)*** $f(x) = \ln\left(\frac{e^x}{x+2}\right) + e^2$
>
> ***Part d)*** $f(x) = \sin\left(\tan\sqrt{1 + x^3}\right)$
>
> ***Part e)*** $f(x) = x^{\cos x}$ *(use logarithmic differentiation)*

In Table 4 we present the distribution of correct and incorrect answers to this problem.

Table 4. Distribution of correct and incorrect answers to Problem 4.

n=63	correct	incorrect	other	total
P4 - Derivatives				
Part a	31 (49%)	30 (48%)	2 (3%)	63
Part b	37 (59%)	24 (38%)	2 (3%)	63
Part c	37 (59%)	23 (36%)	3 (5%)	63
Part d	36 (57%)	26 (41%)	1 (2%)	63
Part e	38 (60%)	24 (38%)	1 (2%)	63

Remarks: the "other" column includes all students who omitted the respective portion of the problem. It turns out that differentiation of a quotient of functions was the most difficult task. The common mistakes amongst students are grouped per problem, and described in the following sections.

4.4.1 Problems with part a

Reminder: $f(x) = \frac{\sqrt[3]{x} - 2xe^x}{x}$

This problem can be solved by applying the Quotient Rule directly, or the function's expression can be simplified prior to taking the derivative. For those that simplified first, we expected a number of errors as exponents and fractions are known for being difficult for students. Twenty-two (~35%) students chose to simplify prior to finding the derivative; 19 (~30%) did so correctly, two students did so incorrectly, while one did not complete the simplification.

Example of a correct simplification prior to taking the derivative

Student #5 wrote:

$$f(x) = \frac{(x)^{\frac{1}{3}}}{x} - \frac{2\cancel{x}e^x}{\cancel{x}}$$

$$f(x) = x^{-\frac{2}{3}} - 2e^x$$

$$f'(x) = -\frac{2}{3}x^{-\frac{5}{3}} - 2e^x$$

Examples of an incorrect simplification prior to taking the derivative

Student #4 wrote:

$$f(x) = \frac{\sqrt[3]{x} - 2xe^x}{x} = \frac{x^{\frac{1}{3}} - 2x \cdot x}{x}$$

$$= \frac{x^{\frac{1}{3}} - 2x^2}{x} = x^{\frac{1}{3}-\frac{3}{3}} - 2x^{2-1}$$

$$= x^{-\frac{2}{3}} - 2x$$

$$f'(x) = -\frac{2}{3}x^{-\frac{5}{3}} - 2$$

Note: The error in this student's work can be seen in the second step of the first line. He replaced e^x by x, and it is unclear whether this was inattention, or whether there is some misconception about the relationship between x and e^x (FR-Exp1).

Student #51 wrote:

$$f(x) = \frac{\sqrt[3]{x} - 2xe^x}{x}$$

$$= \frac{3x^{\frac{1}{2}} - 2xe^x}{x}$$

$$= 3x^{-\frac{1}{2}} - 2e^x$$

$$f'(x) = -\frac{3}{2}x^{-\frac{3}{2}} - 2e^x$$

Note: The error in simplification can be found in line 2; it is as if this student believes the following: $\sqrt[a]{x} = ax^{\frac{1}{2}}$ (FR-Rad1), or perhaps $\sqrt[a]{x} = ax^{\frac{1}{a-1}}$. Alternatively, the error could be attributed to inattention.

For the 34 (~54%) students who chose to use the Quotient Rule directly, 31 (49%) students applied the correct formula (including those that obtained the wrong derivative), two did so incorrectly, and one student reversed the derivatives of the numerator. Reversing the derivatives of the numerator in the Quotient Rule comes up also in other student solutions to problem 4. By reversing the derivatives of the numerator we mean the following: given differentiable functions f and g, $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f(x)\frac{d}{dx}[g(x)] - g(x)\frac{d}{dx}[f(x)]}{[g(x)]^2}$. This gives $-f'(x)$ instead of $f'(x)$. It is possible that students may have confused the Quotient Rule with the Product Rule, given that the order of functions provided by Stewart is as follows:

The Product Rule *If f and g are both differentiable, then*

$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + \ g(x)\frac{d}{dx}[f(x)]$ *(Stewart, 2016, p. 184).*

Although this is the Product Rule provided by Stewart, with the use commutativity I prefer the following equivalent rule for calculating the derivative of a product of two functions: the derivative of the first function multiplied by the second, plus the first function multiplied by the derivative of the second ($\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + \ f(x) \cdot \frac{d}{dx}[g(x)]$). I believe that this way offers an easier way to help students remember the Quotient Rule, as it is the derivative of the numerator multiplied by the denominator minus the numerator multiplied by the derivative of the denominator, all divided by the denominator squared.

Example of a correct use of the Quotient Rule

Student #12 wrote:

$$f(x) = \frac{(x)^{\frac{1}{3}} - 2xe^x}{x}$$

$$\frac{u'v - uv'}{v^2}$$

$$u = (x)^{\frac{1}{3}} - 2xe^x \qquad v = x$$

$$u = 2x \quad v = e^x \qquad v' = 1$$

$$u' = 2 \quad v' = e^x$$

$$u' = \frac{1}{3}(x)^{-\frac{2}{3}} - (2e^x + 2xe^x)$$

$$= \frac{\left[\left(\frac{1}{3}(x)^{-\frac{2}{3}} - (2e^x + 2xe^x)\right)(x)\right] - \left[\left((x)^{\frac{1}{3}} - 2xe^x\right)(1)\right]}{x^2}$$

Note: This student appears to be using the variables u and v which are functions of x, whereby u is the function in the numerator and v is the function in the denominator, obtaining a Quotient Rule of $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

Example of an incorrect use of the Quotient Rule

Student #21 wrote:

$$f(x) = \frac{\sqrt[3]{x} - 2xe^x}{x}$$

$$u = \sqrt[3]{x} - 2xe^x$$

$$u' = \frac{1}{3}x^{-\frac{2}{3}} - 2e^x - 2xe^x$$

$$v = x$$

$$v' = 1$$

$$f'(x) = \frac{\left(\frac{1}{3\sqrt[3]{x^2}} \cdot 1\right) - (\sqrt[3]{x} - 2xe^x)}{x^2}$$

Note: Although the student writes out his functions and derivatives correctly using u and v, in combining these functions, he does not state which formula he is using. In the first term of the last line, it appears that he forgot the remaining two terms of the u', as well he appears to be multiplying by v' instead of v. It is as though he was applying (with mistakes) an invented false rule: $\left(\frac{u}{v}\right)' = \frac{u'v'-u}{v^2}$. We are not providing codes for false rules related to differentiation for reasons explained in the introduction to this chapter.

Example of reversing the derivatives in the numerator of the Quotient Rule

Student #55 wrote:

$$f(x) = \frac{(x)^{\frac{1}{3}} - 2xe^x}{x}$$

$$f(x) = \frac{x^{\frac{1}{3}}}{x} - \frac{2xe^x}{x}$$

$$f'(x) = \frac{d}{dx}\frac{x^{\frac{1}{3}}}{x} - \frac{d}{dx}\frac{2xe^x}{x}$$

$$= \frac{x^{\frac{1}{3}}\frac{d}{dx}x - \frac{d}{dx}x^{\frac{1}{3}} \cdot x}{(x)^2} - \frac{\frac{d}{dx}\,x\ \cdot 2xe^x - \frac{d}{dx}\,2xe^x \cdot x}{(x)^2}$$

Note: This student's error can be seen in line 4, where he reverses the derivatives of the numerator of the Quotient Rule, for both quotients of the expression. The remaining part of his solution was correct. As well, this student uses the same incorrect Quotient Rule for part (c) of problem 4.

The remaining common errors seen for students who used the Quotient Rule include 12 (~19%) students who made distributive mistakes with a negative sign, seven (~11%) who misplaced brackets, or did not include them altogether, and others who made derivative errors with the Product Rule applied to $-2xe^x$, or not applying it, or with the Power Rule applied to $\sqrt[3]{x}$. Misplacing brackets is a symptom of not perceiving the structure of the expression correctly (D-Struc).

Example of a distributive error and a bracket error

Student #1 wrote:

$$\left[\frac{1}{3}x^{-\frac{2}{3}} - \left((2e^x) + (2xe^x)\right)(x) - (1)\left(x^{\frac{1}{3}} - 2xe^x\right)\right] \div x^2$$
$$\left(\frac{1}{3\sqrt[3]{x^2}} - (2e^x + 2xe^x)(x) - \sqrt[3]{x} - 2xe^x\right) \times \frac{1}{x^2}$$
$$\left(\frac{1}{3\sqrt[3]{x^2}} - 2xe^x + 2x^2e^x - \sqrt[3]{x} - 2xe^x\right) \times \frac{1}{x^2}$$

Note: In the first line, the bracket issue is apparent when multiplying the derivative of the numerator by the denominator x. In line 2, the student does not distribute the -1 to the last term of the expression. This studen makes the same distributive error in the third line by not distributing the -1 to the 3rd term of the expression. It is as if he believed in an invented false rule: $-(a+b) = -a + b$ (FR-Dist1).

Example of Product Rule error

Student #42 wrote:

$$f'(x) = \frac{\left(\frac{1}{3}x^{-\frac{2}{3}} - 2xe^x(x)(x)\right) - \left(\sqrt[3]{x} - 2xe^x\right)(1)}{x^2}$$

Note: It is unclear how the derivative of $-2xe^x$ became $-2xe^x(x)(x)$. Perhaps the last x in the expression was due to a bracket issue, however it is still unclear how the derivative of $-2xe^x$ would equal $-2xe^x(x)$: what is the student's idea of the Product Rule is not easily guessed here.

Lastly, there were three students who, in the expression of the function, replaced division by x by multiplication by x^{-1} and used the Product Rule. One student did this correctly, while the other two made mistakes with the Product Rule. Two students' work could not be categorized, and finally two students did not attempt this problem.

4.4.2 Problems with part b

Reminder: $f(x) = e^{sin(2x)} + sin(e^{2x})$

Looking at the outermost structure of the algebraic expression of this function, we notice a sum of two functions. Students were expected to apply the property whereby the derivative of a sum of functions is the sum of the derivatives. Each of these functions requires the use of the Chain Rule, as each is a composite function of three elementary functions. Sixty (~95%) students correctly assessed the outermost structure, adding the derivatives. One student applied the Product Rule instead, and two students did not attempt this problem.

Example of incorrect identification of the outermost structure of the function's expression (D-Struc)

Student #12 wrote:

$$f(x) = e^{\sin 2x} + \sin(e^{2x})$$
$$u = \ e^{\sin 2x} \qquad v = \sin(e^{2x})$$
$$u' = 2e^{\sin 2x} \quad v' = \cos(2e^{2x})$$
$$= \left(2e^{(sin2x)}\right)(\sin(e^{2x})) + \left(e^{(\sin 2x)}\right)(\cos(2e^{2x}))$$

Note: Although this function was a sum of two functions, this students appeared to use the Product Rule, where $(u+v)' = u'v + uv'$. Further, this student did not apply the Chain Rule to the composite functions. His derivative of $e^{\sin 2x}$ is $2e^{\sin 2x}$, as though the only important part of the exponent is the coefficient in front of the x variable. The student omits the derivative of the sine function, however knows that the derivative of an exponential function is that exponential function. His derivative of $\sin(e^{2x})$ is $\cos(2e^{2x})$, as though he took the derivative of the sine function, but evaluated it at the derivative of the inner function e^{2x}. This student does not have a good understanding of the Chain Rule, and has difficulty with the structure of algebraic expressions of functions and differentiation rules. This student seems to implicitly use the following invented false rules:

$(u+v)' = u'v + uv'$; $(e^{\sin(ax)})' = a \cdot e^{\sin(ax)}$; $(f\left(g(x)\right)' = f'(g'(x))$

With regards to the use of the Chain Rule, in differentiating each function in the sum, 52 (~83%) and 47 (~75%) students correctly multiplied three derivatives. These numbers, however, include students who made errors in differentiating the elementary functions. They demonstrated a correct understanding of the algebraic structure of the whole function. The number of students who moreover correctly calculated the derivatives of the elementary components of the first and second function in the sum were 47 (~75%) and 41 (~65%), respectively.

Example of a correct use of the Chain Rule

Student #1 wrote:

$$f(x) = e^{\sin 2x} + \sin\,(e^{2x})$$
$$= \frac{d}{dx}\left(e^{\sin 2x}\right) + \frac{d}{dx}(\sin(e^{2x}))$$
$$= (e^{\sin 2x} \cdot \cos 2x\ \cdot 2) + \cos(e^{2x})\, e^{2x} \cdot 2$$
$$f'(x) = \ e^{\sin 2x} 2\cos 2x + 2\cos(e^{2x})\, e^{2x}$$

Note: This student correctly demonstrates his knowledge of the Sum Rule in line 2. In the last line of his work, he uses commutativity to reorder his derivative, moving around the coefficients 2, although not with the same organization. One of the coefficients is in the middle of the exponential and cosine function of the first expression, whereas the other is at the front of the second expression.

Example of an incorrect use of the Chain Rule

Student #4 wrote:

$$f(x) = e^{\sin 2x} + \sin(e^{2x})$$
$$= e^{\sin 2x} + \sin(2x)$$
$$f'(x) = \sin 2x\, e^{x} + 2\sin x$$

Note: This student starts the problem by incorrectly “simplifying” the function from line 1 to line 2, by making $\sin e^{2x}$ equal to $\sin 2x$. In question 4a, this same student replaced e^{x} by x. So it is possible that the student believes in the false rule $e^{ax} = ax$ (FR-Exp2). In line 3, the student treats the exponent $\sin 2x$ as though it were a coefficient, and multiplies it by e^{x}. It is as though the student applied some form of the Power Rule whereby he "brings down" the coefficient. If he were truly applying the Power Rule, and assuming he treated $\sin 2x$ as a coefficient, then the derivative of $e^{\sin 2x}$ would have to be $\sin(2x)\, e^{\sin(2x)-1}$, however his exponential part of the derivative was e^{x} alone. For the second expression in the third line, $\sin 2x$ became $2\sin x$. The coefficient of 2 in front of the expression can be assumed to be the derivative of $2x$. This student did not take the derivative of the sine function, and further the inner expression became x post differentiating. Perhaps this student's belief about the Chain Rule is that once he obtains the derivative of the inner expressions, he multiplies those derivatives by the elementary functions, and not the functions at hand. His knowledge of the structure of functions is clearly lacking, and his knowledge of the Chain Rule is flawed. He believes that $e^{ax} = ax$, $\left(e^{\sin(ax)}\right)' = \sin\,(ax) \cdot e^{x}$, and $(\sin(ax))' = a \cdot \sin\,(x)$.

Interestingly, four students applied the Product Rule to the second function $(\sin(e^{2x}))$, as though the sine function was multiplying the exponential function (Mis-Notation2). Also, one of these students applied the Product Rule to each composite function of this problem.

Example of a Product Rule applied to a composite function

Student #23 wrote:

$$f(x) = e^{\sin 2x} + \sin(e^{2x})$$
$$f'(x) = \left(e^{\sin 2x}\right)(\cos 2x)\,(2) + \cos(e^{2x}) + \sin(2e^{2x})$$

Note: The Product Rule is applied to second expression $\sin\,(e^{2x})$, whereby the resulting derivative is $\cos(e^{2x}) + \sin(2e^{2x})$. This student appears to treat the trigonometric functions of cos and sin as though they were variables that can be multiplied with other terms, and not as functions of some variables (Mis-Notation2). What is interesting about this is that this student correctly obtains the derivative of the first expression, which also contains an exponential and a trigonometric function, however perhaps the placement of the functions (the structure) in the second expression confused the student. Implicitly, the students appears to apply the false rule $(\sin(e^{ax}))' = \cos(e^{ax}) + \sin\,(a \cdot e^{ax})$.

Other errors in solving this problem include students having difficulties with the derivative of the exponential and trigonometric functions. These were combined with incorrect uses of the Chain Rule, or not applying it altogether, as in the following example.

Student #6 wrote:

$$f(x) = e^{\sin 2x} + \sin(e^{2x})$$
$$f'(x) = e^{\cos 2x} + \cos(2e^{x})$$

Note: In the second line, it appears that this student believes that the derivative of a composite function is the derivative of each function in their respective position of the structure of the function. For example, $\left(g(x)^{h(x)}\right)' = g'(x)^{h'(x)}$, and $(f(g(x)))' = f'(g'(x))$. We have already seen the latter false rule at work in student #12's solution to this part of problem 4. Even if this is the student's belief, in the first expression he does not recognize that the innermost expression is $2x$. Further, in the second expression, he does appear to see the innermost expression of $2x$, however after differentiating, the innermost expression changes from $2x$ to x.

4.4.3 Problems with part c

Reminder: $f(x) = ln\left(\frac{e^x}{x+2}\right) + e^2$

This problem can be solved by simplifying the function using the logarithmic laws, prior to taking the derivative, or by directly applying the Chain Rule. For those that simplified first, we expected a number of errors as there tends to be confusion with the logarithmic laws, however 27 (~43%) students who simplified first, did so correctly. Only one student did so incorrectly.

Example of correct simplification

Student #50 wrote:

$$f(x) = \ln\left(\frac{e^x}{x+2}\right) + e^2$$
$$= \ln e^x - \ln(x+2) + e^2$$
$$= x - \ln(x+2) + e^2$$
$$f'(x) = 1 - \frac{1}{x+2}$$

Note: This student correctly applies the logarithmic laws in line 2, knows that $\ln e^x = x$ in line 3, and lastly that the derivative of e^2 is zero, as it is a constant.

Example of incorrect simplification

Student #4 wrote:

[1] $f(x) = \ln\left(\frac{e^x}{x+2}\right) + e^2$

[2] $= \ln\left(\frac{e^x}{x+2}\right) + \ln e^2$ [Implicit rule, false: (*) $\ln(x) = x$]

(side work)

[3] $\ln\left(\frac{e^x}{x+2}\right) = v$

[4] $\ln\left(\frac{e^x}{x+2}\right) = \ln v$ [Rule (*) applied again, to derive [4] from [3]]

[5] $e^v = \frac{e^x}{x+2}$ [Implicit rule, true: (**) If $\ln(x) = y$ then $e^y = x$, applied to [3]]

[6] $v = \frac{x}{x+2}$ [Implicit rule, false: (***) $e^x = x$, already used in parts a and b]

[7] $\ln e^2 = u$

[8] $\ln e^2 = \ln u$ [Rule (*) applied to [7] to derive [8]]

[9] $e^2 = e^u$ [Rule (*) applied to the right side of [8]; then rule {**)]

[10] $u = 2$ [Implicit rule, true: (****) $e^x = e^y$ then $= y$]

(back to the main body of solution, following [2])

[11] $= \frac{x}{x+2} + 2$ [Results [6] and [10] applied to [2]]

[12] $= \frac{x+2x+4}{x+2}$ [Correct algebraic processing of [11] : addition of ratios]

[13] $= \frac{3x+4}{x+2}$ [Correct algebraic processing of [12]: addition of monomials]

Note: This student appears to believe in two false rules about logarithmic and exponential functions: $\ln(x) = x$ (FR-Ln1) and $e^x = x$ (FR-Exp1). Perhaps he confuses these rules with the differentiation of the logarithmic and exponential functions.

For students who correctly simplified first, the common errors encountered were with the derivatives of the three elementary functions. Three students made an error with the derivative of $\ln e^x$, and one made an error with $\ln(x+2)$.

Examples of errors with the derivatives of elementary functions

Student #8 wrote:

$$f(x) = \ln\left(\frac{e^x}{x+2}\right) + e^2$$
$$\ln e^x - \ln(x+2) + 2e^{-1} \cdot 2'$$
$$\frac{1}{e^x} - \frac{1}{x+2} + 0$$
$$= \frac{1}{e^x} - \frac{1}{(x+2)}$$

Note: In passing from line 1 to 2, this student appears to be processing the expression using a correct logarithmic law ($\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$). However, the student also appears to take the derivative of e^2, as though it were the function x^2 and not a constant, and does it with errors. This student's use of the Power Rule is flawed. Assuming they believed that e was a variable and not a constant, the derivative, using the power rule would be $2e^{2-1}$, and not $2e^{-1} \cdot 2'$. However, the student may have been implicitly using the false rule $\left(e^{a(x)}\right)' = a(x)e^{a(x)-1} \cdot a'(x)$. Note that a number of students made errors with the derivative of e^2, which will be discussed further. From line 2 to line 3, the derivative of $\ln e^x$ was found to be $\frac{1}{e^x}$. This student appears to have missed applying the Chain Rule – a structural mistake (D-Struc). Further, he could have noticed that $\ln e^x = x$, and that the derivative of x would have been 1.

Student #47 wrote:

$$f(x) = \ln\left(\frac{e^x}{x+2}\right) + e^2$$
$$f(x) = \ln e^x - \ln(x+2) + e^2$$
$$f'(x) = \frac{1}{e^x} \cdot e^x - \frac{1}{x+2} \cdot 2 + 0$$
$$f'(x) = \frac{e^x}{e^x} - \frac{2}{x+2}$$

Note: The error can be seen in line 3. The student appears to apply the Chain Rule, and incorrectly multiplies by 2 instead of 1. This could be a mistake of inattention. In his final solution the student does not replace $\frac{e^x}{e^x}$ by 1; but then there was no need to simplify the final expression so this is not necessarily a sign of a conceptual problem in the student.

For the students who chose to apply the Chain Rule and the Quotient Rule, 27 (~43%) students applied the Chain Rule correctly, and four did so incorrectly. For the Quotient Rule, 26 (~41%) students did so correctly, and five did so incorrectly. Two students reversed the Quotient Rule as discussed in part (a).

Example of correct use of the Chain Rule & Quotient Rule

Student #9 wrote:

$$f(x) = \ln\left(\frac{e^x}{x+2}\right) + e^2$$
$$f'(x) = \left(\frac{x+2}{e^x}\right) \cdot \left(\frac{e^x(x+2) - e^x}{(x+2)^2}\right)$$

Example of incorrect use of the Chain Rule

Student #14 wrote:

[1] $f(x) = \ln\left(\frac{e^x}{x+2}\right) + e^2$

(side work)

[2] $a = \ln\left(\frac{e^x}{x+2}\right)$

[3] $b = \frac{e^x}{x+2}$

[4] $c = e^2$

[5] $d = e^x$

[6] $e = x + 2$

[7] $c' = 0$ therefore $f'(x) = a'$

[8] $a' = \ln\left(\frac{e^x}{x+2}\right) \cdot b'$

[9] $b' = \frac{d'e - e\prime d}{e^2}$

[10] $b' = \frac{e^x(x+2) - e^x}{(x+2)^2}$

[11] $d' = e^x$

[12] $e' = 1$

(back to main solution)

[13] $f'(x) = \ln\left(\frac{e^x}{x+2}\right) \cdot \frac{e^x(x+2) - e^x}{(x+2)^2}$

Note: This student seemed to have a good structural view in identifying all the elementary functions. His error occurred in the 8th line, in which he does not use the Chain Rule. He does not take the derivative of the outermost function $\ln\left(\frac{e^x}{x+2}\right)$. However, it could also be that this student believes that the derivative of a natural logarithmic function is the natural logarithm (an implicit false rule, $(\ln(f(x))' = \ln\big(f(x)\big) \cdot f'(x)$) , confusing the rules of differentiation of $\ln x$ with those of e^x.

Example of incorrect use of the Quotient Rule

Student #3 wrote:

$$f(x) = \ln\left(\frac{e^x}{x+2}\right) + e^2$$

$$f'(x) = \frac{\frac{e^x \cdot \ln e \cdot 1}{1}}{\frac{e^x}{x+2}}$$

Note: It appears that this student knew the derivative of the outermost function, as he divides by $\frac{e^x}{x+2}$, however it appears as though he does not apply the Quotient Rule to the inner function. Rather, he may believe that the derivative of a ratio of functions is the derivative of the numerator divided by the derivative of the denominator (the implicit false rule $\left(\frac{u}{v}\right)' = \frac{u'}{v'}$). In the numerator, he recalls the derivative of exponential functions as $b^x \ln b$, and he multiplies by 1, as though he is applying the Chain Rule to e^x.

Regardless of which method students used to start this problem, 10 (~16%) students applied some rule of differentiation for e^2 treating e as a variable and not a constant. Five students left the derivative of e^2 as e^2, as if treating e^2 like e^x. The derivative of e^x is e^x, therefore the derivative of e^2 must be e^2. Four students obtained $2e^2$, also appearing to treat it as an exponential function. However, they appeared to be applying the Chain Rule to multiply the expression by 2, even though there is no variable in the exponent. Finally one student obtained $2e$ as if applying the Power Rule.

The remaining three of the 63 students did not attempt this problem, and one student's work could not be categorized.

Note: The remaining problems will be analyzed in less detail.

4.4.4 Problems with part d

Reminder: $f(x) = sin\left(tan\sqrt{1 + x^3}\right)$

Part (d) was a composition of four elementary functions, requiring the use of Chain Rule. Problems identifying the structure of this function (D-Struc) were apparent in 13 (~21%) student solutions, and was shown in errors with the Chain Rule, applying the Product Rule instead, evaluating a derivative at the derivative of an inner function, and treating a trigonometric function as a variable (Mis-Notation2). A good portion of the students, 49 (~78%) of 63 multiplied four derivatives. As in the other problems requiring the Chain Rule, these numbers include those who made errors with the elementary derivatives, but still knew to multiply four derivatives. Other common errors were with the derivative of the trigonometric functions, as well as the Power Rule. Lastly, one student did not attempt this problem.

An example of Mis-Notation2 is the following: $(\tan(f(g(x))))' = \tan \cdot (f'(g(x)) \cdot g'(x)) + f(g(x)) \sec^2$, whereby the student applies the Product Rule, treating tan as a variable. Further, for this student $(\tan)' = \sec^2$.

4.4.5 Problems with part e

Reminder: $f(x) = x^{cosx}$ (use logarithmic differentiation)

In this problem, students were specifically asked to use the logarithmic differentiation procedure. Fifty-seven (~90%) students attempted to use logarithmic differentiation, where 39 (~62%) did so correctly, and 18 (~29%) did so incorrectly. Two students attempted to solve this problem by writing $x^{cosx} =$

$\left(e^{lnx}\right)^{cosx}$, whereas three students attempted some other incorrect method. Lastly, one student did not attempt this problem.

Common errors in the logarithmic differentiation procedure include five students who did not take the natural logarithm of both sides of the equation, and 8 (~13%) students did not implicitly derive the left-hand side, or made errors with this implicit differentiation. Further, 9 (~14%) students made an error with, or "did not bring down the $\cos x$ exponent", thus not properly recalling the logarithmic laws. Twelve (~19%) students made an error with, or did not use the Product Rule, and lastly, 12 (~19%) students made an error with, or did not replace y with x^{cosx}, thus leaving the derivative with respect to x, as a function of x and y.

In summary, the following false rules appeared to be implicitly used by the students:

FR-Eq4 $\frac{1}{a}x = b \rightarrow x = b \cdot \frac{1}{a}$

FR-Ln2 $\ln x = x \ln$

FR-Log5 $\ln\left(a^b\right) = a \cdot \ln(b)$

FR-Log6 $\ln\left(a^b\right) = \ln(b \cdot a)$

Note: FR-Eq4 is yet another example of the illusion of terms "moving" from one side of an equation to another. FR-Ln2 demonstrates treating a logarithmic function as a variable. One student wrote $\ln\left(x^{\cos x}\right) = \ln(\cos x) \cdot x$ (FR-Log5), while another wrote $\ln\left(x^{\cos x}\right) = \ln(\cos x \cdot x)$ (FR-Log6), which are examples of the false rules about logarithms.

4.5 Students' responses to Problem 5

Problem 5 was comprised of three parts. In part (a), students were asked to verify that a point belonged to a curve given by an equation, and to find the equation of the tangent line at that point. Part (b) was a related rates question in which students were asked to find the rate of change of one variable when the rate of change of the other related quantity was provided. Lastly, in part (c), students were asked to apply l'Hospital's Rule to evaluate the limit of a function. For part (a), we did not expect any arithmetic errors in plugging in the coordinates of the point into the variables in the equation, as students were asked to show that the point belonged to the curve. Thus, their work would reflect equality, satisfying the condition of the equation. We did however, expect errors in the implicit differentiation, as the second term in the expression contained a product of x and y. For part (b), we expected errors in the implicit differentiation with respect to time. Also, since it was a word problem we expected students to miss that the given numerical value was negative, since the rate was decreasing. For part (c), we expected errors in the differentiation of $-e^{-x}$, due to the double negative signs, as well as errors in recognizing indeterminate forms, and indeterminacy.

As a reminder, Problem 5 was given as follows:

Part a) *Verify that the point* $(2,1)$ *belongs to the curve defined by the equation*

$x^2 + 2xy + 4y^2 = 12$*, and find the equation of the tangent line to the curve at this point.*

Part b) *A particle is moving along a hyperbola* $xy = 8$*. As it reaches the point* $(4,2)$*, the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?*

Part c) *Use the l'Hôpital's rule to evaluate the* $lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

In Table 5 we present the distribution of correct and incorrect answers to this problem.

Table 5. Distribution of correct and incorrect answers to Problem 5.

n=63	correct	incorrect	other	total
P5				
Part a - Point belonging to curve	54 (86%)	0	9 (14%)	63
Part a - Equation of tangent line	29 (46%)	23 (37%)	11 (17%)	63
Part b - Related rates	20 (32%)	34 (54%)	9 (14%)	63
Part c - L'Hôpital's Rule	39 (62%)	21 (33%)	3 (5%)	63

Remarks: the "other" column includes all students who omitted the respective portion of the problem. The common mistakes amongst students are described in the following sections.

4.5.1 Problems with part a

Most of the students who obtained an incorrect equation of the tangent line, did so because they had an incorrect slope. They either made an error with the implicit differentiation of the function, or they made arithmetic errors in plugging in the coordinate of the point, after differentiating correctly. Ten (~16%) students used the standard form $y = mx + b$, and 43 (~68%) students used the point-slope form $y - y_1 = m(x - x_1)$ to find the equation of the tangent line.

The common errors with implicit differentiation included 21 (~33%) students who made some error with the middle term, either an incorrect Product Rule, not applying it, not distributing the coefficient of 2 to both terms, or not treating y as a function of x. Other errors include a non-zero derivative of the constant, and arithmetic errors in calculating the slope at the given point

In summary, the following false rule appeared to be implicitly used by the students:

FR-Dist2 $\qquad a(b + c) = ab + c$

For example, in implicitly differentiating the middle term $2xy$, a common solution was a variation of the form $2y + xy'$, in which the coefficient 2 was not properly distributed. Further, students had difficulties achieving a covariational understanding of functions (D-Covar). For example, in differentiating $2xy$ with respect to x, some students did not take into account that y was a function of x. Answers in which y was not a dependent variable were of the form $2x + 2y$, whereby some included errors with the distributive law.

4.5.2 Problems with part b

The students who obtained an incorrect rate of change either did so because they made an error in the implicit differentiation with respect to t, did not differentiate with respect to t (D-Covar), or they missed that the given quantity was negative. A total of 21 (~33%) students made an error with implicit differentiation, while 21 (~33%) students indicated that $\frac{dy}{dt}$ equaled 3 instead of -3. Another common error was the non-zero derivative of the constant.

An example of a solution in which covariational thinking of functions was defective is the following: $\frac{d}{dt}(xy) = x \cdot \frac{dy}{dx} \cdot y \cdot \frac{dx}{dy}$.

4.5.3 Problems with part c

Seventeen (~27%) students obtained the wrong limit because they incorrectly derived the numerator or the denominator. Note that no one confused l'Hospital's Rule with the Quotient Rule. Five students did not recognize indeterminacy, either claiming that a division by zero was infinity, or that it was zero. Lastly, five students made arithmetic errors.

Most of the errors occurred with the derivative of $-e^{-x}$, where answers included: e^{x}, the derivative of an exponential is *the* exponential function; $-e^{-x}$, the derivative of an exponential is the *same* exponential function; and xe^{-x}, where a possible false rule could be $(e^{ax})' = axe^{ax}$.

4.6 STUDENTS' RESPONSES TO PROBLEM 6

Problem 6 was comprised of two parts. In part (a), students were asked to find the slope m, of the secant line joining two points. In part (b), students were asked to find all the points $x = c$ on an interval such that $f'(c) = m$. For part (a), we did not expect many errors in finding the y-coordinates, however students must have been able to understand different function notation. Namely, that $f(k)$ in $(k, f(k))$ requires plugging in k for x in the equation. We also did not expect many errors in finding the slope, as students are used to this numerical activity since secondary school. For part (b), the existence of a solution to this problem is guaranteed by the Mean Value Theorem, however students could have successfully completed this problem procedurally, without even knowing the theorem. We expected some errors in differentiating the polynomial, in setting the derivative equal to m, and in solving for c in $f'(c) = m$, as $f'(c)$ was a quadratic equation. The equation was a very simple quadratic equation and errors in solving such equations were included in false rules.

As a reminder, Problem 6 was given as follows:

> *Let* $f(x) = x^3 - 3x + 2$
>
> ***Part a)*** *Find the slope* m *of the secant line joining the points* $(-2, f(-2))$ *and* $(2, f(2))$.
>
> ***Part b)*** *Find all points* $x = c$ *(if any) on the interval* $[-2, 2]$ *such that* $f'(c) = m$.

In Table 6 we present the distribution of correct and incorrect answers to this problem.

Table 6. Distribution of correct and incorrect answers to Problem 6.

n=63	correct	incorrect	other	total
P6				
Part a - Slope of secant line	53 (84%)	7 (11%)	3 (5%)	63
Part b - Mean Value Theorem	42 (67%)	18 (28%)	3 (5%)	63

Remarks: The "other" column includes all students who omitted the respective portion of the problem. The common mistakes amongst students are described in the following sections.

4.6.1 Problems with part a

Three students obtained an incorrect slope due to arithmetic errors. Two students incorrectly calculated $f(-2)$ and/or $f(2)$, and one student made an error with the slope formula (D-App). Four students incorrectly copied $f(x)$, thus obtaining incorrect points. Lastly, three students did not attempt this portion of the problem.

The error made with the slope formula was as follows: $m = \frac{y_2 - y_1}{x_1 - x_2}$

The error above may have been due to inattention, as both x-coordinates were coefficients of 2, whereby one was positive, and one was negative.

4.6.2 Problems with part b

Fifty-nine (~94%) students correctly found $f'(c)$, and one student did not find $f'(c)$. Rather, this student calculated the slope again as the slope of the secant line, and obtained $f'(c)$ to be 1. This student did not continue the problem. Four students correctly found $f'(c)$ however made it equal to 0, instead of m. Lastly, three students did not attempt this portion of the problem.

The main errors that occurred solving for c include five students who made algebraic errors, two students who determined c to be only the positive root, and one student who claimed that no such points existed, without providing justification.

In summary, the following false rules appeared to be used implicitly by students:

FR-Sqrt1	$x^2 = k \rightarrow x = \sqrt{k}$
FR-Sqrt2	$x^2 = k \rightarrow x = \pm k$
FR-Eq2	$x - a = b \rightarrow x = -a - b$
FR-Eq6	$ax = b \rightarrow x = \frac{b-a}{a}$
FR-Dist10	$ac^2 - b^2 = (c + a)(c - b)$

Note: FR-Sqrt1 & 2 demonstrate misunderstandings with square root expressions. FR-Eq2 & 6 demonstrate misunderstandings about the rules required to process equations. FR-Dist10 demonstrates an incorrect use of the distributive law.

4.7 Students' responses to Problem 7

Problem 7 was comprised of three parts. In part (a), students were asked to find the derivative of a function using the definition of the derivative. In part (b), students were asked to find the linearization formula for the function at a point. In part (c), students were asked to find the differential dy and to evaluate it for specific values of x, and dx. For part (a), we expected a number of errors with the distributive law, for example in expanding the cubic term. For part (b), it was expected that students would make errors recalling the linearization formula, confusing the variable and parameters, x, a, $f(a)$, and $f'(a)$ (D-App). For part (c), we expected that students would make mistakes in recalling the differential equation. We did not expect many errors in the substitution of the numerical values for x and dx.

As a reminder, Problem 7 was given as follows:

> *Consider the function* $f(x) = x^3 - 2x^2 + 1$
>
> ***Part a)*** *Use the definition of the derivative to find the formula for* $f'(x)$.
>
> ***Part b)*** *Write the linearization formula for* f *at* $a = 2$.
>
> ***Part c)*** *Find the differential* dy *and evaluate it for the values* $x = 2$ *and* $dx = 0.2$.

In Table 7 we present the distribution of correct and incorrect answers to this problem.

Table 7. Distribution of correct and incorrect answers to Problem 7.

n=63	correct	incorrect	other	total
P7				
Part a - Definition of derivative	44 (70%)	18 (28%)	1 (2%)	63
Part b - Linearization	41 (65%)	19 (30%)	3 (5%)	63
Part c - Differential	31 (49%)	20 (32%)	12 (19%)	63

Remarks: The "other" column includes all students who omitted the respective portion of the problem. The common mistakes amongst students are described in the following sections.

4.7.1 Problems with part a

Seven (~11%) students either did not include the limit operator, or made some error in recalling the definition of the derivative. Three students used the differentiation rules, rather than the definition of the derivative. One student's work could not be categorized, and three students did not attempt this portion of the problem.

Twelve (~19%) students made algebraic errors, which were mainly violations of the distributive law. The main errors occurred in expanding the cubic term $(x+h)^3$.

In summary, the following false rules appeared to be used implicitly by students:

FR-Dist1	$-(a+b)=-a+b$
FR-Dist3	$(a+b)^n=a^n+b^n$
FR-Dist4	$(a+b)^n=a+b^n$
FR-Dist5	$(a+b)^2=a^2+ab+b^2$
FR-Dist6	$(a+b)^3=(a+b)^2+(a+b)^3$ (right-hand side properly expanded)
FR-Dist7	$(a+b)^3=a^3+ba^2+2a^2+3ab+b^3$
FR-Dist8	$(a+b)^2=a^2+4ab+b^2$
FR-Dist9	$(a+b)^3=a^3+a^2b+b^3$
FR-Frac3	$\frac{a+c}{b+c}=\frac{a}{b}$

Note: All the false rules listed above demonstrate errors with the distributive law. An example of FR-Dist1 was $-(x^3-2x^2+1)=-x^3-2x^2+1$. An example of FR-Dist3 was $(x+h)^3=x^3+h^3$. Further, FR-Frac3 also includes a misunderstanding of the following algebraic property: If $b,c\neq 0$, then $\frac{a}{b}=\frac{ac}{bc}$, whereby the student also applies this property to addition: $\frac{a+c}{b+c}=\frac{a}{b}$.

4.7.2 Problems with parts b and c

For part (b), 17 (~33%) students provided an incorrect equation for the linearization of the function, and/or confused the variable and parameters, x, a, $f(a)$, and $f'(a)$. This demonstrates their inability to recognize the linearization of f at a, as the equation of a tangent line at the point $(a,f(a))$ (D-App).

Some distributive and arithmetic errors were also apparent. Three students did not attempt this portion of the problem.

For part (c), the majority of the errors that occurred included using a wrong equation for the differential, i.e. using $f(x)$ instead of $f'(x)$, using $L(x)$ and evaluating it at $x = 2$, and $x = 0.2$, or not solving for the differential dy. Students made minor errors with the derivative, and some students' work could not be categorized. Lastly, 12 (~19%) students did not attempt this portion of the problem.

In summary, the following incorrect linearization equations were provided by students:

a) $L(a) = L(a) + L'(a) + (x - f'(a))$

b) $L(x) = f'(a)(x - a)f(a)$

c) $L(x) = f(a) - f'(a)(x - a)$

d) $L(x) = f(x) + f'(x)(x - a)$

e) $L(x) = f(a) + f'(a) \cdot f(x + a)$

f) $L(x) = f(a) - f'(x)(x - a)$

g) $L(x) = f'(a) + f(a)(a - 1)$

h) $y - f(a) = f'(a)(a - x)$

Note: The incorrect linearization equations above demonstrate the students' confusion with the variables and parameters. This difficulty in applying the linearization formula is coded as D-App. This also demonstrates an incorrect understanding of the structure of linear equations (D-Struc).

In summary, the following false rules appeared to be implicitly used by students:

FR-Dist14	$a + b(x + c) = (a + b)(x + c)$
FR-Eq4	$\frac{1}{a} x = b \rightarrow x = b \cdot \frac{1}{a}$

Note: An example of FR-Dist14 is student #13 who wrote $1 + 4(x - 2) = 5(x - 2)$. FR-Eq4 is the belief in terms "moving" from one side of an equation to another.

4.8 STUDENTS' RESPONSES TO PROBLEM 8

Problem 8 was comprised of two different parts. In part (a), students were asked to find the absolute extrema of a function on a closed interval. In part (b), students were asked to find the dimensions of a box with a minimal surface area. In part (a), given that the function was continuous and in a closed interval, it was expected that students would use the Closed Interval Method to find the absolute extrema. We did not expect many algebraic errors in finding the derivative of the polynomial as this is quite routine. However, we did expect errors in the students' attempts at finding the critical points, as the derivative was a polynomial of degree 3. Also, in using the Closed Interval Method, we expected that some students would forget substituting the numerical values of the endpoints in $f(x)$. As well, we expected some minor arithmetic errors in the substitution of the critical points, as well as for the endpoints. For part (b), it was expected that students would incorrectly set up the given information, i.e. make errors with the equations for volume and surface area (D-Mod). Even though the object was a rectangular box, and students have seen this object time and time again in calculus problems, students still made errors with the equations. We expected some algebraic errors in finding the critical points, as the derivative of the surface area contained a subtraction and a ratio. Finally, we expected that students would not verify if the critical point was the absolute extreme point minimizing the surface area. This expectation stems from the fact that in all routine optimization problems, the critical point found is always the absolute extreme point sought.

As a reminder, Problem 8 was given as follows:

> ***Part a)*** *Find the absolute maximum and minimum values of* $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ *on the interval* $[-2, 3]$.
>
> ***Part b)*** *A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions of the box that minimize the surface area.*

In Table 8 we present the distribution of correct and incorrect answers to this problem.

Table 8. Distribution of correct and incorrect answers to Problem 8.

n=63	correct	incorrect	other	total
P8				
Part a - Closed Interval Method	25 (40%)	34 (54%)	4 (6%)	63
Part b - Optimization	18 (29%)	36 (57%)	9 (14%)	63

Remarks: For part (a), a solution was categorized as "incorrect", if one or both of the absolute extreme points were incorrect. For part (b), a solution was categorized as "incorrect", if both dimensions were not provided, or one of the two was incorrect. Further, this category includes 20 (~32%) students who were unable to complete the optimization problem. The "other" column includes all students who omitted the respective portion of the problem. The common mistakes amongst students are described in the following sections.

4.8.1 Problems with part a

Fifty-seven (~90%) students correctly obtained the derivative of the function. Only two students made errors with the Power Rule. Thirty-seven (~59%) students correctly identified all three critical points, while 19 (~30%) students made algebraic errors. The majority of the errors were with the distributive law, and forgetting to include $x = 0$ as a critical point. Lastly, three students did not attempt finding the critical points.

In evaluating $f(x)$ at the critical and endpoint points, 7 (~11%) and 9 (~14%) students made arithmetic errors, respectively. Three students incorrectly evaluated the points at $f'(x)$ instead of $f(x)$. Further, 9 (~14%) students did not use the Closed Interval Method as they did not evaluate $f(x)$ at the endpoints.

In summary, the following false rules appeared to be implicitly used by the students:

FR-Eq1 $\quad x + a = 0 \ \rightarrow x = a$

FR-Ops3 $\quad abx = bx(ab)$

FR-Ops4 $\quad abx = bx(b)$

FR-Dist12 $\quad x^2 - (m - n)x - mn = (x - m)(x - n)$

Note: FR-Eq1 demonstrates the illusion that terms "move" from one side of an equation to another without the use of any operations. FR-Dist12 is an error with the distributive law.

For the students that correctly "factored", the following are examples in which some students omitted some of the solutions for the critical points.

a) $kx\ (x + m)(x - n) = 0 \rightarrow x = -m, x = n$

b) $x(kx^2 - kx - 2k) = 0 \rightarrow x = 0$

c) $kx^2 - kx - 2k = 0 \rightarrow x(kx - k) - 2k = 0 \rightarrow kx - k = 0 \rightarrow x = 1$

Note: Students' omission of $x = 0$ as a critical point can be seen in (a); students' omission of the factors of the quadratic equation as critical points can be seen in (b); and lastly, in (c), the student only appeared to be concerned with a special portion of the equation.

4.8.2 Problems with part b

Representing variables with letters and creating formulas is a nontrivial algebraic activity (D-Mod). A total of 32 (~51%) students correctly set up both the volume and surface area equations. Even though the given object was a rectangular box, 10 (~16%), and 15 (~24%) students made errors with the volume, and surface area equations, respectively. For those that set up the volume equation properly, most students correctly found the expression for height, $h = \frac{32000}{x^2}$. Some made errors in substituting this expression for h in the surface area equation, while others did not substitute the variable, rather they incorrectly differentiated the product with respect to x. Others made minor mistakes with the derivative (one reversed the Quotient Rule), thus obtaining an incorrect critical point. Further, of the students that successfully found the critical point, six (~10%) students either did not find the numerical value of the height of the box, or made errors in solving for the height, which are simple numerical tasks. Lastly, for those students that found a critical point (whether correct or not), only 12 (~19%) students verified whether the critical point obtained was the absolute minimum.

In summary, the following are consequences of conceptual difficulties with covariational thinking about variables (D-Covar), as well as difficulties in modeling a relationship among variables (D-Mod).

a) $V = x^3$

b) $V = x^2$

c) $V = 2xh$

d) $V = 4xh$

e) $V = 5x^2$

f) $V = 4\pi r^2$

g) $S = 2x$

h) $S = xh$

i) $S = 2x + 4y + 4h$

j) $S = 8x$

k) $S = x^2 + 5xh$

l) $S = 5x^2h$

m) $S = 5x^2$

n) $S = x^2 + xy$

o) $S = 2x + 4xh$

p) $S = 4x^2 + h$

q) $S = x^2$

Note: An incorrect understanding and representation of the volume of a rectangular prism can be seen in (a)-(f). Note that (a) can be explained by the student assuming it was a cube and not a rectangular prism. An incorrect understanding and representation of the surface area of a rectangular box can be seen in (g)-(q).

The following false rules appeared to be implicitly used by the students:

FR-Ops1	$a(-bx) = -ba$, ignoring the variable
FR-Eq5	$ax = -b \rightarrow x = \frac{b}{a}$, ignoring the sign
FR-Ops2	$ax\frac{b}{x^2} = \frac{b}{x}$, ignoring a constant

4.9 Summary of difficulties, misconceptions and false rules

4.9.1 Algebraic difficulties

D-Struc: Difficulty in correctly decoding the structure of an algebraic expression

D-App: Difficulty in applying a formula to a given situation

D-Mod: Difficulty in modeling a relationship among variables by an equation

D-Covar: Difficulty in achieving a covariational understanding of functions: difficulty discerning which quantity is a function of which quantity and how

4.9.2 Algebraic misconceptions

4.9.2.1 Misconceptions related to the concept of function

4.9.2.1.1 Misconceptions about the inverse of a function

Mis-InvF1: The inverse of a function is a step-by-step procedure to follow

Mis-InvF2: The inverse of a function is obtained by changing the sign of each term

4.9.2.1.2 Misconceptions about notation

Mis-Notation1: A letter without a sign, e.g., "x" denotes a non-negative number; a letter with a negative sign, e.g., "$-x$" denotes a negative number.

Mis-Notation2: Abbreviations of names of transcendental functions or numbers such as "ln", "e", "sin", "cos", etc. are processed as variables.

4.9.2.1.3 Misconceptions about the domain of a function

Mis-Dom1: The domain of a function equals the domain of its algebraic expression

Mis-Dom2: The domain of a composition of functions is the domain of the algebraic expression of the composite function

Mis-Dom3: The domain of the inverse function is the domain of the algebraic expression of the inverse function

Mis-Dom4: To find the domain of a function f solve $g(x) = 0$ or $g(x) \geq 0$ or $g(x) > 0$, where $g(x)$ is the whole algebraic expression for f or some part of it

4.9.3 False algebraic rules

4.9.3.1 Assumption of linearity of non-linear functions

FR-Lin-Log	$\log(a \pm b) = \log(a) \pm \log(b)$	
FR-Lin-Sqrt	$\sqrt{a \pm b} = \sqrt{a} \pm \sqrt{b}$	

4.9.3.2 False rule about the absolute value function

FR-Abs	$\lvert x\rvert = x$ and $\lvert -x\rvert = x$	

4.9.3.3 False rules about the square root function

FR-Sqrt1	$x^2 = k \rightarrow x = \sqrt{k}$	
FR-Sqrt2	$x^2 = k \rightarrow x = \pm k$	
FR-Sqrt3	$\sqrt{x^2} = x$	(possible consequence of FR-abs1)

FR-Sqrt4 $\sqrt{x^2} = \pm x$

4.9.3.4 False rules about logarithms (other than the assumption of linearity)

FR-Log1 $\log(a+b) = \log(a)\log(b)$ (variation of FR-Log2 applied to sum)

FR-Log2 $\log(a-b) = \frac{\log(a)}{\log(b)}$

FR-Log3 $\log(a-b) = \log\left(\frac{a}{b}\right)$ (variation of FR-Log4)

FR-Log4 $\log(a-b) = \log\left(\frac{a}{-b}\right)$

FR-Log5 $\log\left(a^b\right) = a\log(b)$

FR-Log6 $\log\left(a^b\right) = \log(b \cdot a)$

FR-Log7 $y = \log_a x \Leftrightarrow y^a = x$

Note: FR-Log1 & FR-Log3 were not found in the student solutions to the final examination, however were apparent during in class and during one-to-one discussions.

4.9.3.5 False rules about the natural logarithm

FR-Ln1 $\ln(x) = x$

FR-Ln2 $\ln x = x\ln$ (consequence of Mis-Notation2)

4.9.3.6 False rules about the exponential function

FR-Exp1 $e^x = x$

FR-Exp2 $e^{ax} = ax$ (generalization of FR-Exp1)

FR-Exp3 $e^{\ln x} = 1$

4.9.3.7 False rules about expressions with radicals

FR-Rad1 $\sqrt[a]{x^b} = ax^{\frac{b}{2}}$

FR-Rad2 $\sqrt{ax^b} = \sqrt{a} \cdot x^b$

FR-Rad3 $\sqrt{ax^b} = a \cdot x^{\frac{b}{2}}$

FR-Rad4	$\sqrt{a+b} \cdot \sqrt{a+b} = a^2 + b^2$
FR-Rad5	$\sqrt{a+b} \cdot \sqrt{a-b} = (a-b)^2$

4.9.3.8 False rules related to algebraic operations

Rules violating the distributive property

FR-Dist1	$-(a+b) = -a+b$
FR-Dist2	$a(b+c) = ab+c$
FR-Dist3	$(a+b)^n = a^n + b^n$
FR-Dist4	$(a+b)^n = a + b^n$
FR-Dist5	$(a+b)^2 = a^2 + ab + b^2$
FR-Dist6	$(a+b)^3 = (a+b)^2 + (a+b)^3$
FR-Dist7	$(a+b)^3 = a^3 + ba^2 + 2a^2 + 3ab + b^3$
FR-Dist8	$(a+b)^2 = a^2 + 4ab + b^2$
FR-Dist9	$(a+b)^3 = a^3 + a^2b + b^3$
FR-Dist10	$ac^2 - b^2 = (c+a)(c-b)$
FR-Dist11	$-x^2 + x + ab = (-x-a)(x-b)$
FR-Dist12	$x^2 - (m-n)x - mn = (x-m)(x-n)$
FR-Dist13	$(ab+cd) = (a+c)(b+d)$
FR-Dist14	$a + b(x+c) = (a+b)(x+c)$

False commutativity

FR-Com	$a + bc = ca + b$

False rule about adding 0

FR-Zero	$a + 0 = 0$

Other false rules about operations

FR-Ops1	$a(-bx) = -ba$	
FR-Ops2	$ax \cdot \frac{b}{x^2} = \frac{b}{x}$	
FR-Ops3	$abx = bx(ab)$	
FR-Ops4	$abx = bx(b)$	
FR-Ops5	$(ax)^n = ax^n$	(generalization of FR-Rad3, if $n \in \mathbb{R}$)

4.9.3.9 False rules about processing equations

FR-Eq1	$x + a = 0 \to x = a$	(illusion of moving terms to other side)
FR-Eq2	$x - a = b \to x = -a - b$	
FR-Eq3	$ax = b \to x = \frac{a}{b}$	
FR-Eq4	$\frac{1}{a}x = b \to x = b \cdot \frac{1}{a}$	
FR-Eq5	$ax = -b \to x = \frac{b}{a}$	
FR-Eq6	$ax = b \to x = \frac{b-a}{a}$	

4.9.3.10 False rules about algebraic fractions

FR-Frac1	$\frac{ab+c}{bd} = \frac{a+c}{d}$
FR-Frac2	$\frac{ab+cd}{a+c} = b + d$
FR-Frac3	$\frac{a+c}{b+c} = \frac{a}{b}$

4.10 SOME CONSEQUENCES OF THE DIFFICULTIES, MISCONCEPTIONS AND FALSE RULES FOR CALCULATION OF DERIVATIVES

In general, the false rules about differentiation that students invented could be attributed to the difficulties coded in section 4.9 as D-Struc, D-App, D-Mod and D-Covar. We will give a few (2-3) examples of such rules for each of these difficulties, as mentioned in analyses of problems from 4 to 8.

More examples can be found in the Supplemental Documentation file, posted on the web (https://drive.google.com/open?id=0B2kYIbyY4SGRbERhcWtDXzVMMzQ).

One consequence of the D-Struc difficulty was the false rule saying that the derivative of a function was the derivative of each elementary function in their respective position. Thus the Power Rule, the Product Rule, the Quotient Rule, and the Chain Rule were not correctly applied in some student solutions. Here are a few examples:

$(u^v)' = (u')^{v'}$

$(uv)' = u'v'$

$\left(\frac{u}{v}\right)' = \frac{u'}{v'}$

Some students' mistakes appeared to be a consequence of the D-App difficulty. For example,

$\left(\frac{u}{v}\right)' = \frac{uv'-u'v}{v^2}$

$\left(\frac{u}{v}\right)' = \frac{u'v'-u}{v^2}$

Other mistakes appeared to be a consequence of D-Mod and D-Covar difficulties. Here are two examples: We assume that y is a function of x, and we differentiate with respect to x. Also note that a is a constant.

$(axy)' = ax + ay$

$(axy)' = a$, treating xy as one single independent variable

5 CONSTRUCTION OF A PLACEMENT TEST FOR AN ALGEBRA BASED CALCULUS I COURSE

Students who take a Calculus I course - a pre-university level course – at the university come with highly varied background knowledge in mathematics. Some have good algebraic skills, some have barely passed the prerequisite algebra courses or passed them a long time ago. As we have seen in chapters 3 and 4, passing the final examination in the Calculus I course at Concordia University is algebraically quite demanding. There exist approaches to Calculus that do not require so much algebra (e.g., the Harvard Consortium Calculus) but this is not the case for the MATH 203 Calculus I course at Concordia. Therefore to reduce the rate of failure in the course (and unnecessary frustration in students) it would be good to offer students a Placement Test to assess whether they have the algebraic skills and knowledge necessary to succeed in this particular Calculus I course or rather would benefit from reviewing the pre-calculus material.

The analyses presented in chapters 3 and 4 can help in constructing such a test. Students' difficulties, misconceptions and false rules, and their manifestations in solving the final examination problems identified in the analyses can serve to structure the test and provide ideas for the choices in multiple choice items and for true-or-false questions.

In this chapter, we present some examples of placement test items constructed based on our analyses and data.

5.1 ITEMS ADDRESSING ALGEBRAIC DIFFICULTIES

5.1.1 Items addressing difficulties in decoding the structure of algebraic expressions (D-Struc)

The example below uses the manifestations of the difficulty in finding the formula of a composition of functions in students' solutions to Problem 1a of the final examination, involving composition of a square root function with a linear function. A similar test item can be constructed using the data from students' solutions of Problem 1b, involving logarithmic functions.

Example: TI-D-Struc-Ex1

Let $f(x) = \sqrt{x+1}$ and $g(x) = 4x - 3$.

The formula for the function $g \circ f$ is:

A. $g(f(x)) = 4\sqrt{x+1} - 3$

B. $g \circ f = 4(\sqrt{x+1} - 3)$

C. $g \circ f = 4\,x\,\sqrt{x+1} - 3$

D. $g(f(x)) = 4(\sqrt{x+1} - 3)$

The formula for the function $f \circ g$ is:

A. $f(g(x)) = \sqrt{4x-2}$

B. $f \circ g = \sqrt{(4x-3)} + 1$

C. $f \circ g = \sqrt{4x-2}$

D. $f(g(x)) = \sqrt{4x-3} + 1$

5.1.2 Items addressing difficulties related to the application of a formula to a given situation (D-App)

This difficulty was most apparent in students' application of differentiation rules, such as the Chain Rule or the Product Rule, or the Quotient Rule. In a placement test for a Calculus course we cannot ask students to calculate a derivative, because they do not know this concept yet. But we can give them a formula and ask them to apply it, without getting into the meaning of the formula. Here is an example of such item.

Example: TI-D-App-Ex1

With some functions in mathematics, we can associate a function that we denote by writing the letter representing the function with an apostrophe; for example if the name of the function is f then the associated function's name is f'. Suppose we read this symbol as "f prime", and name the operation of calculating the f prime for a function f, "priming". There are rules for priming different types of functions. For example, if $f(x) = ax^n$ where a and n are any constant real numbers, then $f'(x) = nax^{n-1}$; and if $f(x) = \sin x$ then $f'(x) = \cos x$. There are also more general rules for "priming" functions made of other functions. For example, the rule for priming the quotient of two functions is as follows:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)\cdot f'(x) - g'(x)\cdot f(x)}{g(x)^2}.$$

Let $(x) = \frac{\sin x}{-6x^{\frac{1}{2}}}$. Use the information given above to decide which, if any, of the expressions below are correct responses to the question: Calculate $H'(x)$:

A. $H'(x) = \frac{\cos x}{-3x^{-\frac{1}{2}}}$

B. $H'(x) = \frac{\left(-3x^{-\frac{1}{2}}\right)\cdot\sin x+\left(6x^{\frac{1}{2}}\right)\cdot\cos x}{36x}$

C. $H'(x) = \frac{\left(-6x^{\frac{1}{2}}\right)\cdot\cos x-\left(-3x^{-\frac{1}{2}}\right)\cdot\sin x}{36x}$

D. $H'(x) = \frac{\left(-6x^{\frac{1}{2}}\right)\cdot\cos x-\left(-3x^{-\frac{1}{2}}\right)\cdot\sin x}{-36x}$

E. None of the above

Another example can be the following, if essay type items are allowed on the test. Such items would have to be manually graded.

Example: TI-D-App-Ex2

Given exp (x, n) means the same as x^n, write $x^n \cdot x^m = x^{n+m}$ in this new notation.

5.1.3 Items addressing difficulties related to modeling relationships among variables by equations (D-Mod)

An example of an item addressing this difficulty could be as follows.

Example: TI-D-Mod-Ex1

A tourist agency rents 42-seat buses for transportation of tourists on sightseeing tours. The number 42 includes the driver's seat and the seat for the tour guide. The agency pays $125 per bus per tour. If x is the number of tourists that signed up for a sightseeing tour, let $P(x)$ denote the price the agency will pay for renting the buses.

What is the rule for the function $P(x)$?

A. $f(x) = \frac{x}{40} \cdot 125$

B. $f(x) = \frac{40}{x} \cdot 125$

C. $f(x) = 125 \cdot \left(\textit{the smallest integer greater than} \frac{x}{40}\right)$

D. $f(x) = 125 \cdot \left(\textit{the smallest integer greater than} \frac{40}{x}\right)$

Another example, can be taken from Carlson's (1998) test items regarding functions, and could be modified as follows.

Example: TI-D-Mod-Ex2

Which of the following expresses the diameter of a circle as a function of its area? Let d represent the diameter, and A represent the area of the circle.

A.	$A = \pi r^2$	B.	$r = \pm\sqrt{\frac{A}{\pi}}$
C.	$A = \frac{\pi}{4} d^2$	D.	$d = 2\sqrt{\frac{A}{\pi}}$
E.	$d = \pm 2\sqrt{\frac{A}{\pi}}$	F.	$r = \sqrt{\frac{A}{\pi}}$
G.	$A = \frac{\pi}{4} r^2$	H.	$d = \frac{1}{2}\sqrt{\frac{\pi}{A}}$
I.	$d = \frac{1}{2}\sqrt{\frac{A}{\pi}}$	J.	$r = \sqrt{\frac{\pi}{A}}$

K. the diameter of a circle is not a function of its area

5.1.4 Items addressing the difficulty in achieving a covariational understanding of functions (D-Covar)

An example of an item addressing this difficulty could be as follows.

Example: TI-D-Covar-Ex1

Given $f(a) = xa^2 + ya + c$

a) What is $f(2)$?

A. $f(2) = 2a^2 + 2a + c$

B. $f(2) = 2a^2 + ya + c$

C. $f(2) = 4x + 2y + c$

D. $f(2) = 14$

E. The value of the function at 2 cannot be calculated.

b) f is a function of which variable?

A. a

B. x

C. y

D. c

Errors observed in student solutions to problems with implicit differentiation not only demonstrated their inability to apply the appropriate differentiation rules, but also their difficulties in covariational understanding of functions. A related rates problem taken from Stewart (2016, p. 245), and modified as follows is the next example addressing this difficulty.

Example: TI-D-Covar-Ex2

Air is being pumped into a spherical balloon so that its volume increases at a rate of $100cm^3/s$. Let r represent the radius of the sphere, V represent the volume of the sphere, and t represent time. Note that the volume of a sphere is given by the following equation $V = \frac{4}{3}\pi r^3$. Which of the following statements are true? (Mark all that apply).

A.	r is a function of V	B.	V is a function of r
C.	r is a function of t	D.	V is a function of t
E.	t is a function of r	F.	t is a function of V
G.	None of the above are true		

5.2 ITEMS ADDRESSING ALGEBRAIC MISCONCEPTIONS

5.2.1 Items about the inverse of a function

The next test item examples address the misconception Mis-InvF1. The inverse of a function is a step-by-step procedure to follow.

Example: TI-Mis-InvF1-Ex1

Given $f(x) = \ln(e^x - 3)$ and $g(x) = \ln(e^x + 3)$, John and Mary are asked whether the two functions are inverses of the each other.

John starts by renaming x and y in $f(x)$ and then by working to isolate y as such:

$x = \ln(e^y - 3)$ (John switches x and y)

$e^x = e^{\ln(e^y-3)}$ (John says that since we have a ln we need to put an e in front)

$e^x = e^y - 3$ (John also says that since we had $e^{\ln(e^y-3)}$ we could bring down the $e^y - 3$)

$e^x + 3 = e^y$ (John says to move the -3 to other side)

$\ln(e^x + 3) = y$ (John concludes that they are inverses)

Mary verifies if $f\big(g(x)\big) = g\big(f(x)\big) = x$.

$$f\big(g(x)\big) = f\big(\ln(e^x + 3)\big) = \ln\big(e^{\ln(e^x+3)} - 3\big) = \ln(e^x + 3 - 3) = \ln(e^x) = x$$

$$g\big(f(x)\big) = g\big(\ln(e^x - 3)\big) = \ln\big(e^{\ln(e^x-3)} + 3\big) = \ln(e^x - 3 + 3) = \ln(e^x) = x$$

Whose solution is correct?

a) John

b) Mary

c) Both

d) None

The next example asks to assess the validity of a solution of a problem to find the formula of the inverse of a given function, if possible.

Example: TI-Mis-InvF1-Ex2

Given the function: $f(x) = \sqrt{x^2 - 1}$

John works on finding the inverse of this function. He follows the procedure from the textbook and writes the following solution. Is John's solution correct? YES/NO

STEP 1: Write the equation: $y = \sqrt{x^2 - 1}$

STEP 2: Solve the equation for x:

$y^2 = x^2 - 1$

$x^2 = y^2 + 1$

$x = \pm\sqrt{y^2 + 1}$

STEP 3: Interchange x and y

$y = \pm\sqrt{x^2 + 1}$

Answer: $f^{-1}(x) = \pm\sqrt{x^2 + 1}$

The next example addresses, among other erroneous ideas about inverse functions found in student's solutions to Problem 1b, the misconception Mis-InvF2: The inverse of a function is obtained by changing the sign of each term. It asks to recognize the algebraic expression of the inverse of a given function.

Example: TI-Mis-InvF2-Ex1

Given the function $f(x) = \ln(e^x - 3)$. Which, if any, of the following expressions is a correct formula for the inverse of this function?

A. $f^{-1}(x) = -\ln\,(-e^x + 3)$

B. $x = \ln(e^y - 3)$

C. $f^{-1}\colon\ x = \frac{\ln e^y}{\ln 3}$

D. $f^{-1}(x) = \ln(e^x + 3)$

E. $x = \ln(e^y + 3)$

F. $y = \ln(e^x + 3)$

G. $f^{-1}(x) = \ln\left(\frac{e^x}{3}\right)$

H. $f^{-1}(x) = \ln\left(\frac{e^x}{-3}\right)$

I. $y = \ln(e^x)\ln(3)$

J. The function has no inverse.

5.2.2 Items addressing misconceptions about notation

We give two examples here, one for each identified misconception related to notation.

The first example is about the value of a letter variable without a sign being always positive and the value of a letter variable with a negative sign being always negative.

Example: TI-Mis-Notation1-Ex1

True or false?

A. For any number x, the number $-5x$ is a negative number.

B. For any number a, the number a is positive.

C. For any number x, the number $5x$ is a positive number.

D. For any number a, the number $-a$ is negative.

The second example addresses Mis-Notation2: Abbreviations of names of transcendental functions or numbers such as “ln”, “e”, “sin”, “cos”, etc. are processed as variables.

Example: TI-Mis-Notation2-Ex1

True or false?

1. If $\ln x + 3 = \ln y$, then $\frac{\ln x+3}{\ln} = y$
2. $\ln(e^x - 3) = e^{x(\ln)} e^{x(e^x-3)}$
3. If $\tan x = \frac{\sin x}{\cos x}$ then $\sin = \cos(\tan)$
4. $2\sin(2 + x) = 4\sin + 2x\sin$

5.2.3 Items addressing misconceptions about the domain of a function

We start with an example of an item addressing Mis-Dom1: The domain of a function equals the domain of its algebraic expression.

Example: TI-Mis-Dom1-Ex1

A tourist agency rents 42-seat buses for transportation of tourists on sightseeing tours. The number 42 includes the driver's seat and the seat for the tour guide. The agency pays $125 per bus per tour. If x is the number of tourists that signed up for a sightseeing tour, let $f(x)$ denote the price the agency will pay for renting the buses.

What is the domain of the function $f(x)$?

A.	$\mathbb{R}$	B.	$\mathbb{N}$
C.	$\mathbb{Z}$	D.	$x \neq 0$
E.	$x \geq 0$	F.	$x > 0$

G. $\{x \in \mathbb{N}: x \text{ } is \text{ } divisible \text{ } by \text{ } 40\}$

H. the function $f(x)$ has no domain

The next examples address misconceptions about the domain of a composition of functions, in particular Mis-Dom2: The domain of a composition of functions is the domain of the algebraic expression of the composite function, and the analogous Mis-Dom3, about the domain of the inverse function.

Example: TI-Mis-Dom2+3-Ex1

Let $f(x) = \sqrt{x}$, $x \in \mathbb{R}_0^+$ and $g(x) = x^2$, $x \in \mathbb{R}$

(a) The domain of the function $f \circ g$ is:

A.	$\mathbb{R}$	B.	$\mathbb{R}_0^+$
C.	$x \in \mathbb{R}$	D.	$x \in \mathbb{R}_0^+$
E.	$x = 0$	F.	$x \geq 0$
G.	$x > 0$	H.	$x \neq 0$
I.	$(-\infty, +\infty)$	J.	$[0, +\infty)$

(b) The domain of the function $g \circ f$ is:

A.	$\mathbb{R}$	B.	$\mathbb{R}_0^+$
C.	$x \in \mathbb{R}$	D.	$x \in \mathbb{R}_0^+$
E.	$x = 0$	F.	$x \geq 0$
G.	$x > 0$	H.	$x \neq 0$
I.	$(-\infty, +\infty)$	J.	$[0, +\infty)$

(c) The value of the function $f \circ g$ at $x = -5$ is:

A.	5	B.	-5
C.	25	D.	$\sqrt{25}$
E.	$\sqrt{(-5)^2}$	F.	$\sqrt{-25}$
G.	$\|-5\|$		
H.	There is no value. The function is not defined at $x = -5$.		

Example TI-Mis-Dom2-Ex2

Given $f(x) = \sqrt{x}$, $g(x) = \sqrt{2-x}$, $f \circ g = \sqrt{\sqrt{2-x}}$, and $g \circ f = \sqrt{2-\sqrt{x}}$,

(a) the domain of the function $f \circ g$ is:

A.	$0 \leq x \leq 4$	B.	$x \geq 0$
C.	$x > 0$	D.	$x < 2$
E.	$x \leq 2$	F.	$x < 4$
G.	$x \leq 4$	H.	$(0, 4)$
I.	$(-\infty, 2) \cup (2, \infty)$		

(b) the domain of the function $g \circ f$ is:

A.	$0 \leq x \leq 4$	B.	$x \geq 0$
C.	$x > 0$	D.	$x < 2$

E. $x \leq 2$ F. $x < 4$

G. $x \leq 4$ H. $(0, 4)$

I. $(-\infty, 4) \cup (4, \infty)$

Below are examples addressing Mis-Dom4: To find the domain of a function f solve $g(x) = 0$ or $g(x) \geq 0$ or $g(x) > 0$, where $g(x)$ is the whole algebraic expression for f or some part of it.

Example: TI-Mis-Dom4-Ex1

Given $f(x) = \frac{\sqrt{2x+3}}{e^x - 1}$, four students describe how they found the domain of the function.

John said, "I made it equal to zero, solved for x, and got $x = -\frac{3}{2}$, so my domain is $\left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$."

Bob said, "I made $2x + 3$ greater than zero and obtained $x > -\frac{3}{2}$ as my domain".

Peter said, "Since the expression under the root cannot be negative, I made $2x + 3 \geq 0$, and got $x \geq -\frac{3}{2}$".

Mary said, "I checked when the denominator equaled to zero and got $x = 0$, so the domain is $x \neq 0$".

Jane said, "I did both what Peter did and Mary did, and got $\left[-\frac{3}{2}, 0\right) \cup (0, \infty)$."

Whose solution is correct?

A. John

B. Bob

C. Peter

D. Mary

E. Jane

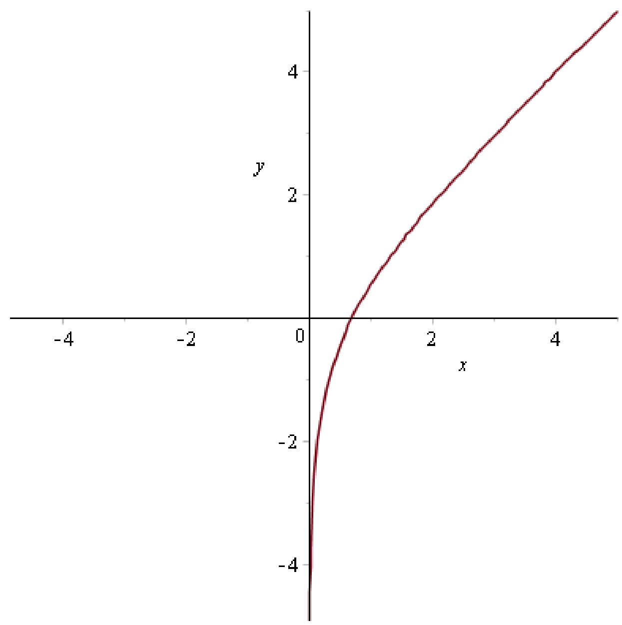
F. All are correct

G. None are correct

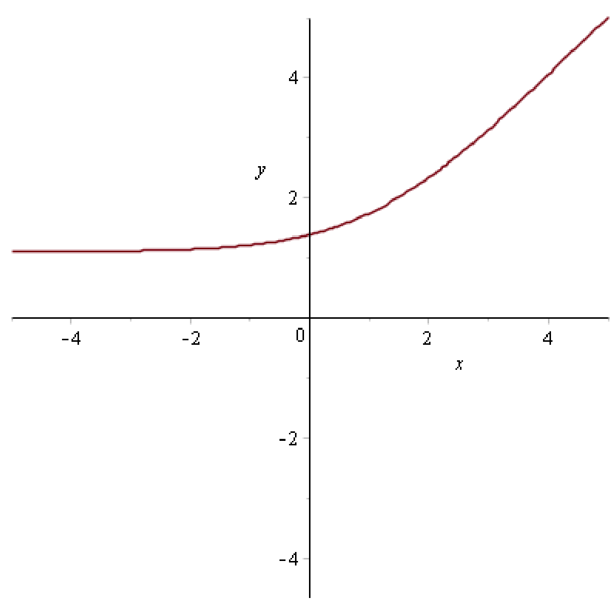
Example: TI-Mis-Dom4-Ex2

Which of the following curves is the most likely to be the graph of the function $f(x) = \ln\left(e^{x} - 3\right)$?

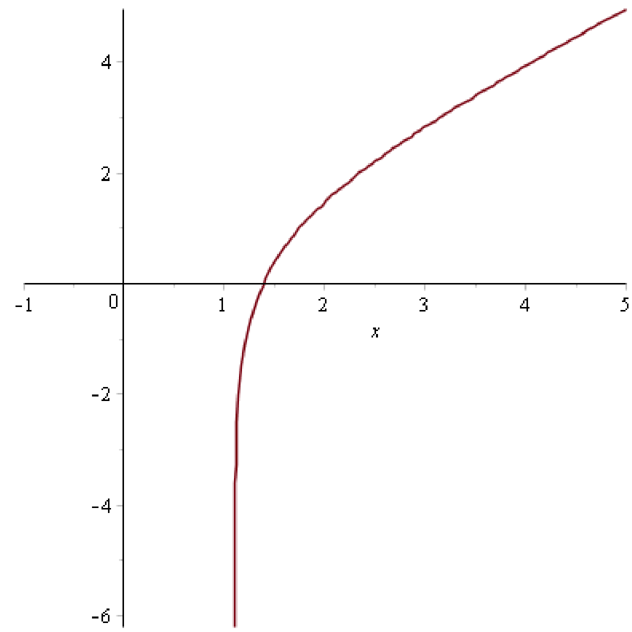
A)



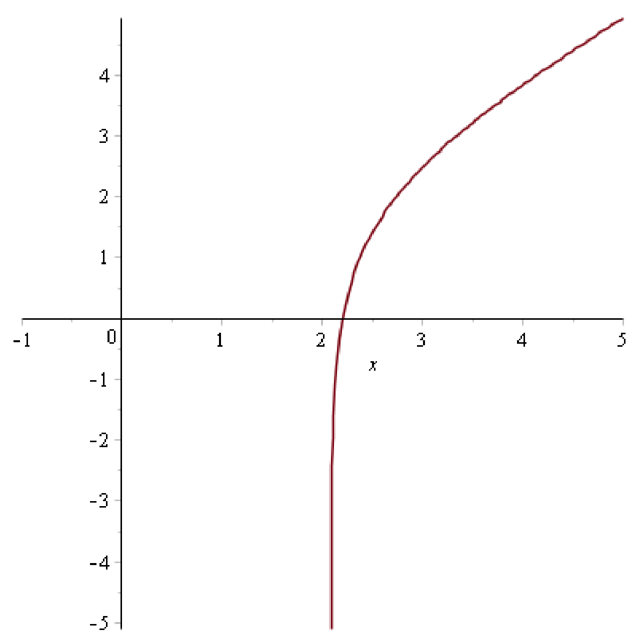
B)



C)



D)



5.3 ITEMS ADDRESSING ALGEBRAIC FALSE RULES

5.3.1 Items addressing false rules about linearity of non-linear functions

Example: TI-FR-Lin-Ex1

True or false?

1. $\sqrt{4x-2} = 2x - \sqrt{2}$, (FR-Lin-Sqrt & FR-Rad2)
   where x is any real number greater or equal to $\frac{1}{2}$
2. $4(\sqrt{x+1}) - 3 = 4\sqrt{x} + 4 - 3$, (FR-Lin-Sqrt)
   where x is any real number greater or equal to -1
3. $\ln(e^y - 3) = \ln(e^y) - \ln(3)$ (FR-Lin-Log)

5.3.2 Items addressing false rules about logarithms

Example: TI-FR-Log+Ln-Ex1

True or false?

1. $\ln(e^x - 3) = \frac{\ln e^x}{\ln 3}$ (FR-Log2)
2. $\ln(e^x - 3) = \ln\left(\frac{e^x}{3}\right)$ (FR-Log3)
3. $\ln\left(\frac{e^x}{3}\right) = x - \ln(3)$
4. If $y = \log_{\mathrm{e}}(e^x - 3)$ then $y^e = e^x - 3$ (FR-Log7)

5.3.3 Items addressing false rules about the natural logarithmic and exponential functions

Example: TI-FR-Ln+Exp-Ex1

Which of these is the correct answer to the question: Solve for x in $e^x + 3 > 0$.

A. $x > \ln(-3)$
B. $x > -3$ (FR-Exp1 or FR-Ln1)
C. $x > -3\ln$ (FR-Ln2)
D. $x > -\ln(3)$ (FR-Ln2)
E. x is $\mathbb{R}$
F. None of the above

5.3.4 Items addressing false rules about expressions with radicals

Example: TI-FR-Sqrt+Rad-Ex1

True or false?

1. If $x^2 = 4$ then $x = 2$ (FR-Sqrt1)
2. $\sqrt{(-4)^2} = -4$ (FR-Sqrt3)
3. $\sqrt[3]{(x+4)^2} = (x+4)^{\frac{3}{2}}$
4. $\sqrt[4]{2x+3} = (2x+3)^{\frac{1}{4}}$
5. $\sqrt{x+\pi} \cdot \sqrt{x+\pi} = x^2 + \pi^2$ (FR-Rad4)

5.3.5 Items addressing false rules about algebraic operations

Example: TI-FR-Dist+Ops-Ex1

True or false?

1. $(2x + \pi y) = (2 + \pi)(x + y)$ (FR-Dist13)
2. $(9 + x)^2 = 81 + x^2$ (FR-Dist3)
3. $x^2 + (2 + \sqrt{2})x + 2\sqrt{2} = (x + 2)(x + \sqrt{2})$
4. $ex\left(\frac{3}{x^2}\right) = \frac{3}{x}$ (FR-Ops2)

Example: TI-FR-Dist-Ex2

Which of the following is the correct expansion of $(2 - x)^3$? (Check all that apply).

A. $2^3 - x^3$ (FR-Dist3)

B. $(2 - x)^2 + (2 - x)^3$ (FR-Dist6)

C. $2^3 - 4x + 8 - 6x - x^3$ (FR-Dist7)

D. $2^3 - 12x + 6x^2 - x^3$

E. $2^3 - 4x + 2x^2 - x^3$ (combination of several rules)

F. $2^3 - 8x + 4x^2 - x^3$ (combination of several rules)

5.3.6 Items addressing false rules about processing equations

Example: TI-FR-Eq-Ex1

True or false?

1. if $x - 4 = -6$ then $x = -4 + 6$ (FR-Eq2)
2. if $-\frac{e^4}{2}x = -5$ then $x = \frac{10}{e^4}$
3. if $-\pi x = \sqrt{3}$ then $x = -\frac{\pi}{\sqrt{3}}$ (FR-Eq3)
4. if $\frac{2}{3}x = \sqrt{5}$ then $x = \frac{2\sqrt{5}}{3}$ (FR-Eq4)

5.3.7 Items addressing false rules about algebraic fractions

Example: TI-FR-Frac-Ex1

True or false?

1. $\frac{2x^2+4}{3x} = \frac{2x+4}{3}$ (FR-Fraq1)
2. $\frac{3x+2y}{3+y} = x + 2$ (FR-Frac2)
3. $\frac{7x-49}{7\pi} = \frac{x-7}{\pi}$
4. $\frac{x+e}{y+e} = \frac{x}{y}$ (FR-Frac3)

6 CONCLUSIONS AND RECOMMENDATIONS

An introductory one-variable Calculus course is a prerequisite for many STEM programs. Candidates aspiring to enter these programs have not always taken this course in secondary school or in college, or they have, but did not do well in it. Our university offers these candidates a condensed, fast paced version of the course, where the instructor's role is not so much to teach as to guide students in learning the material on their own; help them prepare to pass the final examination. Many of these candidates, having been out of school for a while, have forgotten much of the more elementary mathematics they learned before; some never knew it well. Fast pace of the course and weak background knowledge of more elementary mathematics are among the reasons that researchers give for the high failure rates in this type of course, and for the attrition of students in STEM related programs (Hagman, Johnson, & Fosdick, 2017).

Our research brought about the idea that this course, the way it has been conceived in our institution, along with the traditional Stewart textbook, necessitates a large amount of algebraic manipulations to solve various problems. In multiple instances, we have quoted the author using algebraic techniques to solve problems, while other methods were possible, but not always provided. For example, there is a whole chapter that is devoted to finding limits, whereby different limit laws are used, and different algebraic techniques are demonstrated, and expected to be used as the correct solutions. An example of flawed algebraic knowledge coupled with a belief that Calculus is algebra is provided in the solution of the work by student #60, in section 4.3.4. The student appears to "algebraize" the theorem that if $g(x) \to \infty$ then $\lim_{x\to\infty} \frac{1}{g(x)} = 0$, in the form of the rule "$\frac{1}{\infty} = 0$". It is this phenomenon that pushed some mathematics educators to teach Calculus in a qualitative, and conceptual way, with minimal use of algebra. One example is the "Harvard Calculus" approach, which can be seen in the Calculus textbook by Gleason and Hughes-Hallet (1998). This approach to Calculus proposes a number of changes to the traditional one, such as represented in Stewart's textbook, including reducing the focus on algebra. As an example, in finding limits of functions, the numerical approach with a decimal answer is sufficient for these authors (Gleason & Hughes-Hallet, 1998). Further, their belief is that every topic should be presented in four ways: verbally, numerically, graphically, and algebraically, and that solutions to problems are often given in at least two of these ways. Algebraic techniques are not used unless necessary. Stewart tends to use the 4-way approach to represent functions, however does not necessarily provide two types of solutions for every topic in the textbook. At Concordia University, we are using the traditional Stewart textbook, thus it is evident that students will be exposed to, and will use a lot of algebraic knowledge to succeed in this course.

Students taking MATH 203 at Concordia University tend to have various algebraic skill levels, as some will have marginally passed, while others will have obtained very high grades in the prerequisite algebra, and function courses. For example, many students do not know why the following would be true: $x - a = b \rightarrow x = b + a$. Some believe that when a term moves from one side of an equation to another, the sign in front of the term must change. Not understanding why this is true, while extending this flawed belief to other operations can explain some of the false rules we have identified. When such questions arose during class or during office hours, the correct method, or simplified theory behind why a property, or law was true was always provided. Understanding the *why* of something to be true is a better way to retain the information than simple memorization. Unfortunately, there were students who did not attend classes, and/or those who did not ask questions. Algebraic misconceptions have been shown to be quite difficult to unlearn (Booth, Barbieri, Eyer, & Paré-Blagoev, 2014), thus without addressing students' prior algebraic difficulties, misconceptions, and false rules, we anticipated that these errors would still be apparent in their solutions to the final examination. This was one of the reasons for our analysis of the students' solutions to the final examination, as well as rich data availability. If students possessed these algebraic manifestations at the beginning of the course, and at the end, we hoped on being able to capture them with the final examination.

This idea of "moving" a term from one side to another of equality can be traced back to students' prior experience regarding arithmetic, and early algebra, where they had been used to interpreting the equals sign as an instruction to compute ("2 + 3 = ■", or "2 + 3 = ?"), and to writing the solution of an equation always on the right side of the equals sign. A study with elementary school children, using non-traditional forms such as $\blacksquare = 2 + 5$, rather than the traditional form $2 + 5 = \blacksquare$, yielded students with a better understanding of mathematical equivalence (McNeil, Fyfe, Petersen, Dunwiddie, & Brletic-Shipley, 2011). It would be interesting to know, if possible, if students taking MATH 200, MATH 201, and MATH203 had prior experience with non-traditional forms of arithmetic expressions?

The inability to understand mathematical equivalence, among many other skills obtained in early experiences with arithmetic and algebra can shape students' algebraic knowledge, resulting in difficulties, misconceptions, and false rules such as the ones identified in this thesis. Those algebraic difficulties, misconceptions and false rules, as well as examples of their manifestation in students' solutions to one typical final examination in the Calculus I course for science students are the main result of the thesis.

The category of **algebraic difficulties** refers to students' lack of knowledge regarding certain algebraic skills, necessary in solving calculus problems in this particular course. Difficulties apparent in the students' solutions included: 1) difficulties with the structure of the algebraic expressions of functions (D-Struc), which for example affected the students' ability to discern which differentiation rule to apply and how; 2) difficulties in applying a formula to a given situation (D-App), revealed, in particular, in students' errors in applying the differentiation rules; 3) difficulties in modeling relationships among variables (D-Mod), affecting students' ability to solve related rates and optimization problems; and 4) difficulties with covariational thinking (D-Covar) with consequences in implicit differentiation. Thus the algebraic difficulties identified have an important impact on one's ability to perform well in Calculus.

Algebraic misconceptions refer to false or limited beliefs about algebraic concepts, processes and notation. Some of those identified in the research were related to the concept of the domain of a function. The misconception that the domain of a function is the maximal set of numbers for which an algebraic expression has a numerical value (Mis-Dom1) was, in fact, implicitly conveyed in the course, and assumed in the formulation of the final examination where students were asked to find the domains of functions given only by their algebraic expressions. There is nothing wrong with the concept of the maximal set of numbers for which an algebraic expression has a numerical value. But calling it "the domain" and not distinguishing it from the concept of domain introduced when defining the general concept of function causes much confusion in the students' minds as to the meaning of this word in the context of the Calculus course. We found a great variety of interpretations of this word in students' solutions. Some students resorted to procedurally applying some condition to the whole expression, or to some special element of the algebraic expression. Further, students' errors in solving the problem about the domain were also consequences of their misconceptions about elementary functions and their properties, such as the natural logarithmic, and square root functions. Other misconceptions identified were related to the notions of the inverse of a function, and to algebraic notation.

Algebraic false rules included errors with some false belief regarding an algebraic procedure, law, or property. The most common algebraic false rules identified include errors in logarithmic laws, in expressions with radicals, and in the distributive law, to name a few.

After examining all the students' solutions to the final examination, compiling the information, and creating a coding scheme, we created sample items for a placement test to determine whether students would be ready for a Calculus course such as the one provided at Concordia University. Calculus placement tests already exist in the literature, such as the ones by Carlson, Oehrtman, and Engelke (2010). We can surely create test items that are inspired by existing instruments, however the items that we created include answers that are based on Concordia University students' difficulties, misconceptions, and false rules regarding algebraic expressions, as demonstrated in their solutions. Although false rules have been known to be infrequent and unstable, we observed an overlap of our false rules with some of the 99 as proposed by Payne & Squibb (1990). Common false rules included errors with the distributive law. One idea is that if students are being taught to use for example the "FOIL method", and not that to apply the distributive law, if this method is recalled incorrectly, it can produce errors that students are unable to detect or correct. If math is but a random set of steps, based on unconnected ideas and procedures, then students can do as they please with an algebraic expression. They do not understand why they cannot for example "cancel" a term from a fraction in which the common term is included in only one term of a sum.

This research was done and reported with a practical utility in mind. The way we coded the sources of students' algebraic errors and cross-referenced them in the thesis was intended to facilitate searching the text for ideas and examples for the construction of test items. For example, if one would like to test students' knowledge of functions and domain (perhaps in a final examination of prerequisite courses), one needs to search "Mis-Dom" in this thesis to find all instances of misconceptions related to domain.

Unfortunately, various algebraic difficulties, misconceptions and false rules can be ingrained in students' mind from early on, and are known to be difficult to overcome. It is difficult to determine how early, or at what point during formal education a student can form a misconception. Further, we cannot control what the elementary and secondary school teachers are saying or doing in the classroom. In Quebec, elementary school teachers obtain their teaching license after having completed a bachelor's degree of 4 years in an education program. Throughout their program, they are prepared to teach all subjects, whereby they are only required to take a small number of one-term courses[10], which are solely focused on the teaching of mathematics (Sierpinska & Osana, 2012). If elementary school teachers have their own algebraic difficulties, misconceptions, and false rules, these can unfortunately be passed on to their

[10] The number of courses in mathematics teaching varies from university to university, but it is always a very small percentage of the number of all courses in the program.

students. Further, these elementary school teachers are not knowledgeable enough to recognize students' difficulties, nor that perhaps what they are saying can lead to misconceptions and false-rules. Aspiring high school mathematics teachers in Quebec can obtain their teaching certificate, by completing a 60-credit program after having completed an undergraduate degree in mathematics, or by completing a 120-credit program in education, in which 36-51 credits are university mathematics courses. Further, in Quebec, graduates of undergraduate or graduate mathematics programs cannot teach below the CEGEP level, unless they have an education degree; thus we arrive at an impasse. Mathematicians cannot teach the young, when these difficulties, misconceptions and false rules start their manifestations, nor are the licensed teachers themselves necessarily knowledgeable enough to do so. Sierpinska and Osana (2012) have proposed developing a mathematical knowledge base for aspiring elementary school teachers, however proposing this knowledge base does not guarantee that it will be implemented.

We can however propose some formation for our teachers at Concordia University, for the MATH 200 level courses. These teachers can not only be made aware of the false rules identified in this thesis, but also be instructed themselves on what to say (and more importantly on what not to say). For example, I would strongly recommend that teachers refer to the associative, commutative, and distributive laws solely, rather than use the creative acronyms of names created for different methods, such as the "FOIL method", or the "reverse FOIL method". Also, we should take the extra time to indicate why a term appears to move from one side of an equation to the other. As simple as this idea may seem, it was a source of a number of false rules identified, applied to different operations. When discussing functions and their domains, it is proposed that we provide the definitions in the formal sense, and then inform students that functions in the textbook are referred to by their algebraic expressions, even though an expression is not sufficient to determine a function. This would at the very least provide students with a distinction between the formal definition of the domain of functions, and the natural domain as is implicitly used in the textbook.

Although we cannot change the contents of the textbook, it is proposed that the solutions offered in class lean away from algebraic solutions. We may not be prepared to change our view of Calculus to the "Harvard Calculus" approach, not now, maybe not ever, however we can use examples from textbooks such as the one by Gleason and Hughes-Hallet (1998) to move away from algebraic-based solutions. For example, in finding the limit of a ratio of functions, in which one of the expressions contains a radical expression, it is proposed to use reasoning rather than algebra to help students understand the notion of limits, rather than limits being yet another algebraic procedure. As was discussed throughout this thesis, routine questions regarding various topics in this course involved many algebraic manipulations. Our categorization of algebraic errors has great implications for the teaching and learning of Calculus. In its current state, we have shown that students lacking algebraic knowledge consequently make errors in all topics of this Calculus course. For example, in this course curve sketching as well as all the steps involved are very important, and generally constitute a large proportion of the final exam grade. We can all agree that understanding the relationship between functions and their derivatives is at the heart of such a problem, however if given a function in which a student cannot find the critical points – not because they do not understand what critical points are, nor because they do not know how to go about doing so, but rather because they become stuck at the simple algebra required to do so post differentiating, such a problem does not accurately assess their knowledge of the content. Questions assessing the relationship between a function and its derivative would be more beneficial, such as perhaps matching the graphs of functions with their derivatives. Or perhaps one could provide a list of possible critical points, whereby the students would need to verify which of the ones provided were critical points, in order to proceed with the steps involved in the sketch. This would isolate and test one's understanding of the important material, not blurring the solution with algebraic skills.

Further, in preparing the final examination, I was strongly encouraged to follow the topics listed in the course outline, to structure the layout and content of the final like that of previous final examinations, and to choose problems from the textbook. Thus the structure, topics, and problems were highly routine. Due to various algebraic skill levels, I tried to choose problems that were not algebraically demanding. My reasons for doing so included that I wanted to assess Calculus related knowledge and not have this blurred with deficiencies in algebra. For example, the derivative of one of the functions in the final was a quadratic polynomial, which contained a quadratic term and a constant. Thus, distributivity was not exactly an issue here, rather errors with the square root function became apparent (FR-Sqrt). Factoring quadratic polynomials is known to be very difficult for students. If the final examination contained more complicated quadratic polynomials, a student lacking the algebraic skill required to use the distributive law, would be unable to correctly solve a number of problems, giving them a poor grade or even causing them to fail the course. Since the final exam is strongly weighted in the course, deciding to change the content of the final exam will need to be done very carefully. For example, with the current limit problems, are we testing their knowledge of limits, or only their knowledge of algebra, as most of the problems are solved with algebraic techniques? Questions of the form, "what occurs to the numerator when x gets closer and closer to a?", and similarly for the denominator, would provide use with insight into the students' thought processes. Thus, in MATH 203's current state, using a traditional textbook, and keeping the same final examination format, algebraic knowledge is still very important for student success in this course, which leaves us with some thoughts about our goals for this course. Should we try to delineate algebra and calculus within the problems chosen, so as to only test knowledge of Calculus related content, or do we believe that algebra is an important foundation, and necessary for all mathematics courses?

Another idea that intrigues me is if having for example the basic algebra course offered at Concordia University (MATH 200 - Fundamental Concepts of Algebra) with algebraic expressions that are solely with letters, without numerical coefficients would improve students' algebraic skills? Would students develop less false rules, if they learn that any letter can represent a variable or a coefficient, if they do not work with coefficients in the arithmetic sense? For example, problems would be of the form $-c = -bx + ayb$, and students would be asked to solve for any of the possible letters. The reason for avoiding numerical coefficients altogether is to try to eliminate misconceptions that students have such as variables being placeholders, or that a letter without a sign such as x denotes a non-negative number. Further, this research did not study classroom interactions, teacher lesson preparation, etc. However, one idea that could avoid Mis-Notation1 is that the phrase "negative b" should be omitted from ones vocabulary altogether, when referring to variables. One should say "the opposite of b", whatever b may be. Slips, in which teachers refer to the opposite of b as negative b, could lead to this misconception.

With regards to the introductory class on elementary functions offered at Concordia University (MATH 201 - Elementary Functions), if all functions were provided in their formal sense, including their domain, along with their graphs (when possible), would the act of finding the domain of an algebraic expression be all that important, as this act is an algebraic activity?

Another interesting topic for discussion is the way in which mathematics courses are compartmentalized in North America, as opposed to having yearlong courses in which different mathematical domains are integrated. In North America, university mathematics instruction divides mathematics into separate subjects, e.g., algebra, functions, calculus, linear algebra, etc. taught as separate courses, and offered in a 13-week semester. Having these separate courses, students appear to have difficulties in connecting the ideas learnt, as they feel that they learn new disconnected ideas with every new topic. As mentioned in an article by Krussel (1998), "... mathematics is a confusing array of disconnected facts, rules, and definitions." As example, the linearization formula is an equation of the tangent line at a point, yet students had difficulties in seeing this link, and made errors in recognizing which letters were variables, and parameters, besides making errors with the structure of the linear equation.

Although we have analyzed the algebra involved in a Calculus course and typical final examination, determining that students can do poorly if assessments are largely algebraic, we are not advocating to further separate the two subjects. Rather we are only stating that algebra is quite important for student success in this course, and the need for a good placement test would lessen the failure rate, and student frustration. Perhaps the need for a crash course, addressing the algebraic misconceptions and false rules is required prior to taking the Calculus course.

Another thought is whether students who are not continuing in a mathematics related field would benefit from taking the traditional based course, or whether the “Harvard Calculus” type of course would be sufficient as a prerequisite? Future research warrants studying whether STEM related programs, and further STEM related careers truly benefit from all the content taught in the introductory Calculus course. As Hagman, Johnson, and Fosdick (2017) pointed out, there is a high attrition in STEM related programs, due to characteristics of courses such as the introductory Calculus course described here. Although these introductory courses are intended to be a weeding of sorts, perhaps we are weeding out students that can excel and be very talented in STEM related careers. Such as someone who mistook the arugula plant for a weed, and removed it from my family’s garden, after we had taken care of it, and were waiting for it to grow a little more to reap its benefits.

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