Three Essays in Theoretical and Empirical Derivative Pricing

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Abstract

Three Essays in Theoretical and Empirical Derivative Pricing

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The first essay investigates the option-implied investor preferences by comparing equilibrium option pricing models under jump-diffusion to option bounds extracted from discrete-time stochastic dominance (SD). We show that the bounds converge to two prices that define an interval comparable to the observed option bid-ask spreads for S&P 500 index options. Further, the bounds' implied distributions exhibit tail risk comparable to that of the return data and thus shed light on the dark matter of the divergence between option-implied and underlying tail risks. Moreover, the bounds can better accommodate reasonable values of the ex-dividend expected excess return than the equilibrium models' prices. We examine the relative risk aversion coefficients compatible with the boundary distributions extracted from index return data. We find that the SD-restricted range of admissible RRA values is consistent with the macro-finance studies of the equity premium puzzle and with several anomalous results that have appeared in earlier option market studies.

The second essay examines theoretically and empirically a two-factor stochastic volatility model. We adopt an affine two-factor stochastic volatility model, where aggregate market volatility is decomposed into two independent factors; a persistent factor and a transient factor. We introduce a pricing kernel that links the physical and risk neutral distributions, where investor's equity risk preference is distinguished from her variance risk preference. Using simultaneous data from the S&P 500 index and options markets, we find a consistent set of parameters that characterizes the index dynamics under physical and risk-neutral distributions. We show that the proposed decomposition of variance factors can be characterized by a different persistence and different sensitivity of the variance factors to the volatility shocks. We obtain negative prices for both variance factors, implying that investors are willing to pay for insurance against increases in volatility risk, even if those increases have little persistence. We also obtain negative correlations between shocks to the market returns and each volatility factor, where correlation is less significant in transient factor and therefore has a less significant effect on the index skewness. Our empirical results indicate that unlike stochastic volatility model, join restrictions do not lead to the poor performance of two-factor SV model, measured by Vega-weighted root mean squared errors.

In the third essay, we develop a closed-form equity option valuation model where equity returns are related to market returns with two distinct systematic components; one of which captures transient variations in returns and the other one captures persistent variations in returns. Our proposed factor structure and closed-form option pricing equations yield separate expressions for the exposure of equity options to both volatility components and overall market returns. These expressions allow a portfolio manager to hedge her portfolio's exposure to the underlying risk factors. In cross-sectional analysis our model predicts that firms with higher transient beta have a steeper term structure of implied volatility and a steeper implied volatility moneyness slope. Our model also predicts that variances risk premiums have more significant effect on the equity option skew when the transient beta is higher. On the empirical front, for the firms listed on the Dow Jones index, our model provides a good fit to the observed equity option prices.

To my parents, Zahra and Khosro.

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Chapter 1

Shedding Light on a Dark Matter: Jump Diffusion and Option-Implied Investor Preferences

1.1 Introduction

The "dark matter" in finance was recently defined by Ross [2015, Page 616] as the "very low probability of a catastrophic event and the impact that changes in that perceived probability can have on asset prices". More specifically, it is the inability of virtually the entire empirical research dealing with the pricing of rare events in the option markets to achieve a "reasonable" reconciliation of the implied rare event probabilities extracted from options to the observed historical frequency of such events in the underlying market. Note that the study's proposed recovery method of the option-implied distribution did not solve the problem, since the recovered probability of extreme drops in underlying returns of the S&P 500 index was more than 10 times the one extracted from the historical record of the index.[1] Although the option-implied return distribution is supposedly forward-looking and need not be the same as the one extracted from historical returns, it should not imply implausible statistical behavior that has never been observed in the real world.

In this paper we address these issues by presenting a model of the derivation of the risk neutral or Q-distribution for an asset whose returns follow jump diffusion asset dynamics based on stochastic dominance (SD). Unlike previous studies that relied on general equilibrium considerations and data from both the underlying and the option market, our model relies *only* on underlying market data and uses the much weaker assumption of a monotone decreasing pricing kernel to derive its results; this assumption has not been contradicted empirically by direct tests.[2] Using model parameter values derived from several empirical jump diffusion studies in the S&P 500 market and reverse engineering of the equilibrium models implied by our derived option prices, we show that SD produces "reasonable" estimates of the key implied risk aversion and equity premium for most cases, thus shedding light into the dark matter. We also show that the SD-implied Q-distribution, unlike the ones recovered from observed option prices, has comparable left tail risk as the one extracted from observed index returns, with the option-implied probability of extreme events virtually indistinguishable from the one estimated from the underlying market.

The main motivation for abandoning the general equilibrium approach is the fact that it refers to the frictionless economy, which is by definition unobservable. While frictions in the form of a bid-ask spread in observed prices are immaterial for the underlying market, where they are very low, their magnitude in the option market makes the extraction of the *true*

[1] See Ross [2015, Pages 642-643].

[2] See Barone-Adesi et al. [2012], Barone-Adesi et al. [2008] and Beare and Schmidt [2016].

equilibrium option price a highly uncertain prospect. For instance on January 19, 2017, at around 13.30 EST, the S&P 500 index was trading at 2267 and the close to ATM February puts and calls with strikes at 2265 and 2270 were trading at spreads of 6.8% and 6.15% of their respective midpoints. These spreads escalated dramatically, especially for calls, for OTM options in the same cross section: for a strike of 2100 the corresponding put had an 18.2% spread, and the spread for the 2400 strike call was a stunning 60.87%.[3] With this kind of uncertainty over the proper data to use in fitting the model it becomes virtually impossible to assess the source of the dark matter.

Equilibrium models are established either based on the production economy or on the exchange economy. In a production setting a representative investor, almost always assumed to be of the Constant Relative Risk Aversion (CRRA) type, chooses her optimal level of consumption in each period and invests the rest in the production for future consumption, where the production technology grows stochastically and the initial endowment is constant. The large literature on this model includes Brock [1982], Cox et al. [1985], Cochrane [1991], and Cochrane [1996]. Studies that consider jumps in the production process are Ahn and Thompson [1988] and Bates [1988, 1991]. Pan [2002] and Liu and Pan [2003] also include jumps in a production economy but in a partial equilibrium setting as they only study the price of derivatives and disregard the price of assets. In addition, there are several equilibrium studies in an exchange economy based on consumption asset pricing, where aggregate endowment is stochastic such as, among others, Lucas Jr [1978], Breeden [1979], and more recently Bates [2008], and Santa-Clara and Yan [2010].

By contrast, the stochastic dominance literature is slimmer, even though it appeared more than 30 years ago. It was first introduced by Perrakis and Ryan [1984], Levy [1985], Ritchken [1985], and subsequently extended by Perrakis [1986] and Ritchken and Kuo [1988, 1989]. More recently Constantinides and Perrakis [2002, 2007] extended it to incorporate proportional transaction costs, an extension that was tested empirically in Constantinides et al. [2009] and Constantinides et al. [2011]. Jump diffusion valuation elements for a specific type of insurance derivatives were applied in the SD context in Perrakis and Boloorforoosh [2013], while Oancea and Perrakis [2014](OP, 2014) established the formal equivalence of SD to the Black and Scholes [1973] model under simple diffusion asset dynamics for both index and equity options.

[3] Note that these numbers are probably underestimates: the VIX volatility on that particular date was more than 7% below its historical average of around 19%. It is well-known that the bid-ask spreads rise when volatility is high, which in turn is associated with low returns and tail risk.

This paper presents the SD theory for index options in a general jump diffusion context and examines the equilibrium models' results within the framework of SD. We derive upper and lower bounds on option prices based on the parameters of the physical distribution of the underlying return process, whose width is comparable to the observed bid-ask spread in the option market. We then compare these bounds to equilibrium models' predicted option values and the associated risk neutral volatility and mean return of the underlying asset as functions of the relative risk aversion (RRA) parameter.

We use jump diffusion asset dynamics parameters extracted from available econometric studies in the S&P 500 underlying index market. We rely on the fact that the SD bounds are independent of RRA but rely, on the other hand, on the ex-dividend mean return of the underlying asset; as we point out, for the most frequently used underlying, the S&P 500 index, that range is widely assumed to lie within known limits. Further, we observe that the derived bounds are relatively insensitive to the parameters of the jump component provided the total volatility is kept constant; this is important because there is a large variability in the estimates of these parameters depending on the time span of the data.[4]

By contrast, the econometric literature has presented widely divergent values of the RRA coefficient. Even within the option pricing models and associated empirical research the RRA coefficient varies widely between studies and even within the same study.[5] As we show in this paper, many of these RRA values yield economically meaningless results within any equilibrium model, since either the option price or the implied mean return are beyond any reasonable values. This is true even a fortiori for RRA estimates extracted out of the equity premium puzzle literature, which can be more than five times as large.[6]

We provide expressions for the pricing kernels implied by the SD bounds in equilibrium analysis and for the implied RRA values, when they exist. We show that the SD lower bound implies a monotone decreasing pricing kernel and a risk neutral distribution that do not have a representative investor counterpart but are consistent with empirical evidence that shows a negative RRA. We also show that the SD upper bounds extracted from several econometric estimations of jump diffusion parameters for the S&P 500 index are consistent with RRA estimates as high as the equity premium ones, the only option price-implied estimates to

[4] See, for instance, Andersen et al. [2002, Tables 3 and 6] and Tauchen and Zhou [2011, Table 4].

[5] See, for instance, Rosenberg and Engle [2002] and Bliss and Panigirtzoglou [2004], who find coefficients ranging from 2 to 12 and 1.97 to 7.91 respectively.

[6] The reported RRA estimates are 41 for Mehra and Prescott [1985], 40 to 50 for Cochrane and Hansen [1992], and more than 35 for Campbell and Cochrane [1999].

possess this property; in fact, the puzzle disappears in SD-implied RRA parameters and associated mean returns. We show that the SD upper bounds' implied RRA is also consistent with the more recent stylized models in the equity premium studies that include rare events in a representative investor's consumption growth in an attempt to reconcile the estimates with observed quantities and solve the puzzle.[7] Hence, the SD approach allows us to include option market considerations in these macro-finance studies, an inclusion that is not feasible with the traditional option market equilibrium models.

We also note that, although in principle the equilibrium analysis is consistent with a general set of assumptions, its application to option pricing models with jump diffusion asset dynamics follows a very stylized framework that makes its results a subset of the SD analysis. Indeed, the representative investor assumption with time-additive preferences implies a monotone decreasing pricing kernel in the underlying asset return which, when combined with a given RRA value, yields endogenously the option price, the underlying mean return and the riskless rate. Assuming that the latter variable is exogenously given,[8] the derived SD bounds define a range of admissible values of the option price whose width is a function of the mean return and otherwise relies only on the kernel monotonicity; they should, therefore, contain all "reasonably valued" option-mean return pairs and the RRA values that produce them. Our results show that this happens only for a very narrow range of results that is extremely sensitive to the return distribution parameters, a failure of the equilibrium analysis consistent with Ross [2015] "dark matter" remark.

This chapter proceeds as follows. Section (1.2) presents the jump diffusion stochastic dominance bounds as the limits of the discrete time SD bounds following a modified version of the approach in Oancea and Perrakis [2014]. Section (1.3) presents a summary of the dominant equilibrium approach and extracts the implied bounds on the RRA parameter from the SD option bounds. Section (1.4) applies these results in several empirically important cases and shows that the SD bounds can reconcile several of the apparently puzzling results derived by earlier studies. Section (1.5) concludes. In the Appendix (1.E) we discuss the implications of combining jump processes with stochastic volatility (SV) diffusion, which can be handled

[7] See, for instance, Barro [2009], Wachter [2013], Backus et al. [2011] and Martin [2013].

[8] As discussed in Oancea and Perrakis [2014], although the constant riskless rate may not be justified in practice, its effect on option values is generally recognized as minor in short- and medium-lived options. It has been adopted without any exception in all equilibrium based jump-diffusion option valuation models that have appeared in the literature. See the comments in Bates [1991, note 30] and Amin and Ng [1993, page 891]. Bakshi et al. [1997] found that stochastic interest rates do not improve the goodness of fit in a model featuring stochastic volatility and jumps.

with SD as long as the pricing of the systematic risk of SV is done independently.[9]

1.2 Jump Diffusion Option Pricing Under Stochastic Dominance

The SD approach derives upper and lower bounds on the option prices in a multiperiod discrete time context and then finds the limits of these bounds as the time partition tends to zero. The derivation of the bounds was done in earlier studies, most recently in Oancea and Perrakis [2014] and will not be repeated here. We summarize the results and assumptions of the SD model before applying them to jump diffusion.

In a discrete time model trading occurs at a finite number of trading dates $t = 0, 1, ..., T$ of length Δt. We consider an index as the underlying asset with current price S_t and return $\left(S_{t+\Delta t} - S_t\right)/S_t \equiv z_{t+\Delta t}$ in each time interval. We also consider a riskless asset with return equal to R in each time period with r as a continuous time counterpart of return, where $(1+R) = e^{r\Delta t} = 1 + r\Delta t + o(\Delta t)$. The SD bounds are derived under the following set of assumptions.

1. There exists at least one utility-maximizing risk averse investor (the trader) in the economy who holds only the index and the riskless asset;[10]

2. This particular investor is marginal in the option market;

3. The riskless rate is non-random.[11]

These market equilibrium assumptions are quite general, insofar as they do not require that all agents have the properties that we assign to traders, thus allowing a market with heterogeneous agents and the existence of other investors with different portfolio holdings than the trader.

Let $P(z_{j,t+\Delta t})$ denotes the physical return distribution, assumed continuous without loss of

[9]In this respect the SD approach is no different from the alternative equilibrium approach. See, for instance, Liu et al. [2005, footnote 9].

[10]This assumption implies that the pricing kernel in any multiperiod equilibrium model is a monotone decreasing function of the return.

[11]See footnote 8.

generality. By assumption, $E\left[z_{t+\Delta t}\left|S_t\right.\right] > R$.[12] Similarly, let $z_{\min,t+\Delta t}$ denotes the lowest possible return, which is initially assumed to be strictly greater than -1. In our equilibrium, we also define the upper (lower) bounds, $\overline{C}_t(S_t)$ ($\underline{C}_t(S_t)$), on the admissible call option prices as the reservation write (purchase) prices of the option under market equilibrium that excludes the presence of stochastically dominant strategies. Violations of the bounds trigger investment strategies that increase the expected utility of any trader by introducing a corresponding short (long) option in her portfolio.

To derive the bounds $\overline{C}_t(S_t)$ and $\underline{C}_t(S_t)$ we recursively apply the Lemma 1 and Proposition 1 in Oancea and Perrakis [2014]. Note that the derivation of the bounds depends on the convexity of the call option prices and payoff, a property which clearly holds for the jump-diffusion dynamics as well.[13]

Lemma 1.1. *If the option price $C_t(S_t)$ is convex for any t then it lies within the following bounds:*

$$\frac{1}{1+R}E^{L_t}\left[C_{t+\Delta t}\left(S_t(1+z_{t+\Delta t})\right)\right] \leq C_t(S_t) \leq \frac{1}{1+R}E^{U_t}\left[C_{t+\Delta t}\left(S_t(1+z_{t+\Delta t})\right)\right], \qquad (1.2.1)$$

where E^{U_t} and E^{L_t} denote respectively expectations taken with respect to the distributions

$$U(z_{t+\Delta t}) = \begin{cases} P\left(z_{t+\Delta t}\left|S_t\right.\right) & \textit{with probability } \frac{R-z_{\min,t+\Delta t}}{E(z_{t+\Delta t})-z_{\min,t+\Delta t}} \\ 1_{z_{\min,t+\Delta t}} & \textit{with probability } \frac{E(z_{t+\Delta t})-R}{E(z_{t+\Delta t})-z_{\min,t+\Delta t}} \equiv Q \end{cases} \qquad (1.2.2)$$

$$L(z_{t+\Delta t}) = P\left(z_{t+\Delta t}|S_t, z_{t+\Delta t} \leq z_t^*\right)$$

such that $E\left(z_{t+\Delta t}|S_t, z_{t+\Delta t} \leq z_t^*\right) = R$.

---

[12] When the underlying asset is the index, as Section (1.2), this assumption is always true. However, when we have stocks with negative beta and non-decreasing pricing kernel, the equivalent assumption would be $\hat{z}_{n,t+\Delta t} < R$.

[13] The convexity of the option with respect to the underlying stock price holds in all cases in which the return distribution has *i.i.d.* time increments, in all univariate state-dependent diffusion processes, and in bi-variate (stochastic volatility) diffusions under most assumed conditions; see Merton [1973] and Bergman et al. [1996]. The SD approach also applies to non-convex option payoffs but no closed form exists for the limiting distributions.

Proof. See Lemma 1 in Oancea and Perrakis [2014]. □

Remark 1.1. *Note that U_t and L_t are risk neutral as $E^{U_t}(1+z_{t+\Delta t}) = E^{L_t}(1+z_{t+\Delta t}) = R$, that is the distributions U_t and L_t are the incomplete market counterparts of the risk neutral probabilities of the binomial model, in the sense that when the underlying asset tends to diffusion at the continuous time limit both distributions tend to the same risk neutral diffusion.* □

Remark 1.2. *Note that the upper bound pricing kernel, which is related to U_t, spikes at $z_{\min,t+\Delta t}$ and is constant thereafter while the lower bound pricing kernel, which is related to L_t, is zero for $z_{t+\Delta t} > z_t^*$ and constant positive elsewhere. We will discuss more about these two pricing kernels in Section ((1.3)).* □

Proposition 1.1. *Under the monotonicity of the pricing kernel assumption and for a discrete distribution of the stock return z_t, all admissible option prices lie between the upper and lower bounds $\overline{C}_t(S_t)$ and $\underline{C}_t(S_t)$, evaluated by the following recursive expressions*

$$\begin{aligned}
\overline{C}_T(S_T) &= C_T(S_T) = (S_T - K)^+ \\
\overline{C}_t(S_t) &= \frac{1}{1+R} E^{U_t}\left[\overline{C}_{t+\Delta t}\left(S_t(1+z_{t+\Delta t})\right)|S_t\right] \\
\underline{C}_t(S_t) &= \frac{1}{1+R} E^{L_t}\left[\underline{C}_{t+\Delta t}\left(S_t(1+z_{t+\Delta t})\right)|S_t\right]
\end{aligned} \tag{1.2.3}$$

where E^{U_t} and E^{L_t} denote expectations taken with respect to the distributions given in (1.2.2).

Proof. See Proposition 1 in Oancea and Perrakis [2014]. □

Remark 1.3. *Note that in the special case where a stock can become worthless within a single elementary time period $(t, t+\Delta t)$ we have $z_{\min,t+\Delta t} = -1$, irrespective of the underlying index dynamics. In such a case the upper bound distribution is no longer risk neutral and can be extracted by (1.2.4) where the expectation is taken with respect to the actual distribution $P(z_{t+\Delta t}|S_t)$ rather than the upper bound distribution in (1.2.2).* □

$$\begin{aligned}
\overline{C}_T(S_T) &= (S_T - K)^+ \\
\overline{C}_t(S_t) &= \frac{E^P\left[\overline{C}_{t+\Delta t}\left(S_t(1+z_{t+\Delta t})\right)|S_t\right]}{E\left[1+z_{t+\Delta t}|S_t\right]}
\end{aligned} \tag{1.2.4}$$

This important special case yields a looser upper bound on the call option prices but also a convenient closed form solution when the underlying return follows jump-diffusion dynamics. It also holds when multiplied by the round-trip transaction cost in an economy with frictions in the form of proportional transaction costs in the underlying asset; see Constantinides and Perrakis [2002, Proposition 1].

Now, we turn our attention to the case where the underlying asset returns follow a jump-diffusion dynamic. We model the returns as a sum of two components, one of which will tend to a diffusion with a probability of $1-\lambda_t\Delta t$, and the other to a jump process. Therefore, the return dynamic has the following form.[14]

$$z_{t+\Delta t} = \begin{cases} \left[\mu(S_t,t)-\lambda_t k\right]\Delta t + \sigma(S_t,t)\epsilon\sqrt{\Delta t} & \text{with probability} \quad (1-\lambda_t\Delta t) \\ \left[\mu(S_t,t)-\lambda_t k\right]\Delta t + \sigma(S_t,t)\epsilon\sqrt{\Delta t} + (j_t-1) & \text{with probability} \quad (\lambda_t\Delta t) \end{cases} \tag{1.2.5}$$

In this expression ϵ has a bounded distribution of mean zero and variance one, $\epsilon \sim D(0,1)$ and $\epsilon_{\min} \leq \epsilon \leq \epsilon_{\max}$, but otherwise unrestricted. With probability $\lambda_t\Delta t$ there is a jump with amplitude equal to j_t. This amplitude is a random variable with distribution $j_t \sim D_{jt}(\mu_{jt},\sigma_{jt})$. Although our results may be extended to allow for dependence of both jump intensity and jump amplitude distributions on S_t, we shall adopt the common assumption in the literature that the jump process is state- and time-independent, with $\lambda_t = \lambda$, $j_t = j$. Similarly, it is commonly assumed that jump amplitude is log-normally distributed,[15] implying that $J = \ln(j) \sim N(\mu_j - \frac{1}{2}\sigma_j^2, \sigma_j^2)$ with $\mu_j = \ln\left(E\left[j\right]\right)$ and $k = e^{\mu_j} - 1$. In our case we adopt more general assumptions, with the distribution D_j restricted to a non-negative support, so that the variable j takes values with $0 \leq j_{\min}$ but otherwise unrestricted; it can, therefore, accommodate all types of continuous or discrete distributions, as in Fu et al. [2016]. With this specification we set in (1.2.5), $\mu(S_t,t) \equiv \mu_t$, $\sigma(S_t,t) \equiv \sigma_t$, $\lambda_t = \lambda$, $j_t = j$, and represent the discrete time by

$$z_{t+\Delta t} = (\mu_t - \lambda k)\Delta t + \sigma_t \epsilon\sqrt{\Delta t} + (j-1)\Delta N, \tag{1.2.6}$$

[14]For simplicity dividends are ignored throughout this paper. All results can be easily extended to the case where the stock has a known and constant dividend yield, as in index options. In the latter case the instantaneous mean in (1.2.6) and (1.2.7) is net of the dividend yield.

[15]Kou [2002] and Kou and Wang [2004] use a double exponential jump size distribution, which is analytically convenient in computing the first passage time to an option exercise barrier.

where N is a Poisson counting process with intensity λ.

In the remainder of this section we first present conditions that establish the convergence of the return process described in (1.2.5) and (1.2.6) to a mixed jump-diffusion process. We then extract the two option bound distributions from (1.2.1) and (1.2.2) and find their convergence to continuous time expressions following the approach in Oancea and Perrakis [2014] for convergence of (1.2.6) to a diffusion process in the absence of jumps. That approach defines a sequence of stock prices and associated probability measures and proves that the proposed sequence converges[16] in distribution to a diffusion and its probability converges weakly to the respected probability measure. Therefore, the mean and the variance of the discrete process converge weakly to the equivalent parameters of the diffusion process. Then we close this section by numerical analysis regarding the proposed upper and lower bounds on the option prices. In the case of jump diffusion we may prove the following lemma.

Lemma 1.2. *The discrete process described by (1.2.6) converges weakly to the jump-diffusion process (1.2.7) as the time interval approaches to zero.*

$$dS_t/S_t = (\mu_t - \lambda k)\,dt + \sigma_t dW + (j-1)\,dN \tag{1.2.7}$$

Proof. See Appendix (1.A). □

For the discrete time process (1.2.6), which tends to a jump-diffusion (1.2.7), a unique option price can be derived by arbitrage methods alone only if we have zero volatility and the jump amplitude takes exactly one value when a jump occurs. In such a case (1.2.6) is binomial and it can be readily verified that the upper bound distributions, U_t, and the lower bound distribution, L_t, coincide and the stochastic dominance approach yields the same unique option price as the binomial jump process in Cox et al. [1979]. Otherwise, we must examine the two bounds separately.

For the option upper bound we apply the transformation (1.2.2) to the discretization (1.2.6), assuming first that $j_{\min} > 0$. For such a process we note that as Δt decreases, there exists h, such that for any $\Delta t \leq h$, the minimum outcome of the jump component is less than the minimum outcome of the diffusion component, $(j_{\min} - 1) < (\mu_t \Delta t + \sigma_t \epsilon_{\min} \sqrt{\Delta t})$. Consequently, for any $\Delta t \leq h$, the minimum outcome of the return distribution is $(j_{\min} - 1)$,

[16] More details on the weak convergence and its properties for Markov processes can be found at Ethier and Kurtz [2009], or Stroock and Varadhan [2007].

which is the value that we substitute for the minimum return, $z_{\min,t+\Delta t}$, in the transformation (1.2.2). With such a substitution we have now the following result for the jump diffusion upper bound on the call option price.

Proposition 1.2. *When the underlying asset follows a jump-diffusion process described by (1.2.7) the upper option bound is the expected payoff discounted by the riskless rate of an option on an asset whose dynamics are described by the jump-diffusion process*

$$dS_t/S_t = \left(r - (\lambda + \lambda_{Ut})\, k^U\right) dt + \sigma_t dW_t^Q + \left(j_t^U - 1\right) dN_t^Q, \tag{1.2.8}$$

where the upper bound risk-neutral jump intensity is $\lambda^U = \lambda + \lambda_{Ut}$ *and*

$$\lambda_{Ut} = -\frac{\mu_t - r}{j_{\min} - 1}, \tag{1.2.9}$$

and j_t^U *is a mixture of jumps with intensity* $\lambda + \lambda_{Ut}$ *and distribution and mean*

$$\begin{aligned} j_t^U &= \begin{cases} j & \text{with probability } \frac{\lambda}{\lambda+\lambda_{Ut}} \\ j_{\min} & \text{with probability } \frac{\lambda_{Ut}}{\lambda+\lambda_{Ut}} \end{cases} \\ E\left[j_t^U - 1\right] &= k^U = \left(\frac{\lambda}{\lambda + \lambda_{Ut}}\right)k + \left(\frac{\lambda_{Ut}}{\lambda + \lambda_{Ut}}\right)(j_{\min} - 1) \end{aligned} \tag{1.2.10}$$

Proof. See Appendix (1.B). ◻

By definition of the convergence of the discrete time process, Proposition (1.2) states that the call upper bound is the discounted expectation of the call payoff under the risk neutral jump-diffusion process given by (1.2.8). We may, therefore, use the results derived by Merton [1976] for options on assets following jump-diffusion processes with the jump risk fully diversifiable.[17] Applying Merton's approach to the jump-diffusion process given by (1.2.8), we find that the upper bound on call option prices for the jump-diffusion process (1.2.7) must satisfy the following partial differential equation (PDE), with terminal condition $C(S_T, T) = \max\{S_T - K, 0\}$:

[17] Note that we do not assume here that the jump risk is diversifiable.

$$\frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + \left[r - (\lambda + \lambda_{Ut})k^U\right] S\frac{\partial \overline{C}}{\partial S} - \frac{\partial \overline{C}}{\partial T} + (\lambda + \lambda_{Ut})E^U\left[\overline{C}(Sj_t^U) - \overline{C}(S)\right] = r\overline{C} \tag{1.2.11}$$

An important special case of the upper bound is when the lower limit of the jump amplitude is equal to 0, in which case $j_{\min} = 0$ and the return distribution has an absorbing state in which the stock becomes worthless and so the lowest possible return would be $z_{1,t+\Delta t} = z_{\min,t+\Delta t} = -1$; this is the case described in Remark (1.3) and equation (1.2.4), in which as we saw the option upper bound is the expected payoff with the actual distribution, discounted by the expected return on the stock. Hence, this is identical to the Merton [1976, Equation 14] case with r replaced by μ, yielding

$$\frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + \left[\mu_t - \lambda k\right] S\frac{\partial \overline{C}}{\partial S} - \frac{\partial \overline{C}}{\partial T} + \lambda E^U\left[\overline{C}(Sj_t^U) - \overline{C}(S)\right] = \mu_t\overline{C}. \tag{1.2.12}$$

If (1.2.12) holds and as in Bates [1991] we assume, in addition, that the diffusion parameters are constant and the jump amplitude has a lognormal distribution with $\ln(j) \sim N(\mu_j - \frac{1}{2}\sigma_j^2, \sigma_j^2)$ where $k = E[j-1] = e^{\mu_j} - 1$, then the distribution of the asset prices given that n jumps occurred is conditionally normal, with the following mean and variance.

$$\begin{aligned} \mu_n &= \mu - \lambda k + \frac{n}{T}\mu_j \\ \sigma_n^2 &= \sigma^2 + \frac{n}{T}\sigma_j^2 \end{aligned} \tag{1.2.13}$$

Hence, if n jumps occurred, the option price would be a Black-Scholes expression with μ_n replacing the riskless rate r, or $BS\left(S, X, T, \mu_n, \sigma_n\right)$. Integrating (1.2.12) would then yield the following upper bound, which can be obtained directly from Merton [1976] by replacing r by μ.

$$\begin{aligned} &\bar{C}\left(S, X, T, \mu_n, \sigma_n\right) = \sum_{n=0}^{\infty} \frac{e^{\lambda' T}\left(\lambda' T\right)^n}{n!}\left[SN\left(d_{1n}\right) - Xe^{-\mu_n T}N(d_{2n})\right] \\ &d_{1n} = \frac{\ln\left(S/X\right) + \left(\mu_n + 0.5\sigma_n^2\right)T}{\sqrt{\sigma_n^2 T}},\ d_{2n} = d_{1n} - \sqrt{\sigma_n^2 T} \\ &\ln\left(j\right) \sim N\left[\mu_j - 0.5\sigma_j^2, \sigma_j^2\right],\ j \sim \text{lognormal}\left[e^{\mu_j}, e^{2\mu_j}\left(e^{\sigma_j^2} - 1\right)\right] \\ &k = E\left[j-1\right],\ k = e^{\mu_j} - 1,\ \ln\left(1+k\right) = \mu_j,\ \lambda' = \lambda\left(1+k\right) \end{aligned} \tag{1.2.14}$$

When the jump distribution is not normal, the conditional asset distribution given n jumps is the convolution of a normal and n jumps distribution. The upper bound cannot be obtained in closed form, but it is possible to obtain the characteristic function of the bounds distribution. We will extract the boundary distribution's characteristic function, its pricing kernel, and the respected properties in the Appendix (1.D). Similar approaches can be applied to the integration of equation (1.2.12), which holds whenever $-1 < (j_{\min} - 1) < 0$. Closed form solutions can also be found whenever the amplitude of the jumps is fixed as, for instance, when there is only an up or a down jump of a fixed size. A PDE similar to (1.2.12) also holds if the process has only "up" jumps, in which case $(j_{\min} - 1) = 0$ and the lowest return $z_{\min}$ in (1.2.2) comes from the diffusion component. In such a case the key probability Q of (1.2.2) is the same as in the case of diffusion, discussed in the proof of Proposition 2 of Oancea and Perrakis [2014]. In that situation, equation (1.2.11) still holds with $\lambda_{Ut} = 0$, implying that the option upper bound is the Merton [1976] bound, with the jump risk fully diversifiable.

The option lower bound for the jump-diffusion process given by (1.2.7) and its discretization (1.2.6) is found by a similar procedure. We apply $L(z_{t+\Delta t})$ from (1.2.2) to the process (1.2.6) and we prove in the Appendix ((1.C)) the following result.

Proposition 1.3. *When the underlying asset follows a jump-diffusion process described by (1.2.7), the lower option bound is the expected payoff discounted by the riskless rate of an option on an asset whose dynamics is described by the jump-diffusion process*

$$dS_t/S_t = \left[r - \lambda k^L\right] dt + \sigma_t dW_t^Q + \left(j_t^L - 1\right) dN_t^Q \tag{1.2.15}$$

where the lower bound's jump intensity remains the same, $\lambda^L = \lambda$, and j_t^L is absolute jump size with the truncated distribution $j|j \leq \bar{j}_t$.

The mean of the relative jump size, k^L, and the value of truncation boundary, $\bar{j}_t$, can be obtained by solving the following equations.

$$\begin{aligned} \mu_t - r &= \lambda k - \lambda k^L \\ k^L &= E\left(j - 1|j \leq \bar{j}_t\right) \end{aligned} \tag{1.2.16}$$

Proof. See Appendix (1.C). □

Observe that (1.2.16) always has a solution since $\mu_t > r$ by assumption. The limiting distribution includes the whole diffusion component and a truncated jump component. Unlike simple diffusion, the truncation does not disappear as $\Delta t \to 0$. As with the upper bound, we can apply the Merton [1976] approach to derive the PDE satisfied by the option lower bound, which is given by

$$\frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \underline{C}}{\partial S^2} + \left[r - \lambda k^L\right] S \frac{\partial \underline{C}}{\partial S} - \frac{\partial \underline{C}}{\partial T} + \lambda E^L \left[\underline{C}(S j_t^L) - \underline{C}(S)\right] = r\underline{C} \tag{1.2.17}$$

with terminal condition $\underline{C}_T = C(S_T, T) = \max\{S_T - K, 0\}$. The solution of (1.2.17) can be obtained in closed form only when the jump amplitudes are fixed, since even when the jumps are normally distributed, the lower bound jump distribution is truncated.

Observe that the jump components in both $\overline{C_t}(S_t)$ and $\underline{C}_t(S_t)$ are now state-dependent if μ_t, the diffusion component of the instantaneous expected return on the stock, is state-dependent, even though the actual jump process is independent of the diffusion. In many empirical applications of jump-diffusion processes, which were on the S&P 500 index options, the unconditional estimates are considered unreliable. On the other hand there is consensus that the unconditional mean is in the $4-6\%$ range;[18] this is reflected in the numerical results. Observe also that for normally distributed jumps the only parameters that enter into the computation of the bounds are the mean of the process, the volatility of the diffusion and the parameters of the jump component. Hence, the information requirements are the same as in the more traditional approaches, with the important difference that the mean of the process replaces the risk aversion parameter. This difference favors the SD approach, as the consensus that exists for the values of the mean of the process does not extend to the risk aversion parameter, as we shall see in the next section.

We illustrate in Table (1.1) and Figure (1.1), the convergence of the bounds under a jump-diffusion process for an ATM option with $X = 100$, time to maturity $T = 0.25$ years, and the annual parameters: $r = 2\%, \mu = 4\%, \sigma = 20\%, \lambda = 0.6, \mu_j = -0.05, \sigma_j = 7\%$. In our numerical analysis, the diffusion process was approximated by a sequence of trinomial trees constructed according to the algorithm of Kamrad and Ritchken [1991]. The jump process was approximated by a sequence of multinomial trees with up to 1000 time periods, which is based on the algorithm of Amin [1993], where the jump amplitude distribution is lognormal. For each tree, the upper and lower bound risk-neutral probability distributions were computed by applying equation (1.2.2) respectively to the single-period distribution.

[18] See Fama and French [2002], Constantinides [2002] and Dimson et al. [2006].

The two option bounds were evaluated as discounted expectations of the option payoff under the two risk neutral distributions described in Propositions (1.2) and (1.3). In order to evaluate the bounded jump amplitudes discussed in the case where $j_{min} > 0$, the distribution was truncated to a worst-case jump return of -20%. The truncation limit is chosen to meet the observed jump amplitude in econometric studies of jump diffusion. We also computed the upper bound under the assumption that the return distribution is unbounded. As a reference point and for ease of comparisons, we report the Merton [1976] price, the jump-diffusion dynamic with diversified jump risk.

[Table (1.1) about here]

[Figure (1.1) about here]

The results presented in Table (1.1) show the jump-diffusion upper and lower bounds on the call options price. The maximum spread between the bounds is about 4.6% of the midpoint, lower than the observed bid-ask spread for at-the-money call options on the S&P 500 index noted in the Introduction. As expected from Proposition (1.2), equations (1.2.8)-(1.2.10), the upper bound is directly related to the diffusion risk premium and therefore the spread is an increasing function of $\mu - r$ while the lower bound is almost constant: unreported results show that the upper bound rises from 4.59 to 4.75 and to 4.91 for a risk premium equal to 4% and 6% respectively, while the lower bound stays approximately constant around 4.38. Unreported results also show that the bounds are much tighter for in-the-money options and the spread decreases to less than 2% for the base case. Similar unreported results show that the spread rises to 9.1% for the base case parameters when the options are 10% out-of-the-money, much lower than the observed spread noted in the introduction. Note that the range of values of μ implies an ex-dividend risk premium range from 2% to 6%; a range that covers what most people would consider the appropriate value of such a premium in many important indexes. For the most commonly chosen risk premium of 4%, corresponding to $\mu = 6\%$, the spread of at-the-money options is about 8.1%, a tight bound if we consider the average bid-ask spread for at-the-money call options on the S&P 500 index.

The range of allowable option prices in the stochastic dominance approach is the exact counterpart of the inability of the "traditional" arbitrage-based approaches to produce a single option price for jump diffusion processes without an arbitrarily chosen risk aversion parameter, even when the models have been augmented in that case by general equilibrium considerations. Recall also that for any partition of the time to expiration, and by extension at the continuous time limit the SD bounds behave like no arbitrage bounds, in the sense that if the option prices fail to lie between the bounds any risk-averse investor can increase her expected utility by choosing a dominant portfolio containing the underlying, the riskless asset and a long or short option position.[19] We further address this issue in the next section.

A major advantage of the stochastic dominance bounds in the jump-diffusion case is their relative insensitivity in the jump parameters, provided the total volatility is kept constant. Table (1.2) shows the value of the bounds for the ATM options for various values of the intensity parameter λ ranging from 0 to 1.9, with the total volatility $\sigma^2 + \lambda \left[\left(\mu_j - 0.5\sigma_j^2 \right)^2 + \sigma_j^2 \right]$ kept constant to the base case value of 0.04444 by adjusting σ_j and the remaining parameters are kept constant as in the base case.[20] As we can see, the bounds are tight and relatively insensitive to λ, while the spread decreases in λ from 5.24% to 4.1%. This weak dependence of the bounds on λ is particularly important, given the difficulty of estimating the parameters and the impossibility of estimating meaningful option prices by the "traditional" method for all but the lowest values of the ranges of λ and the admissible risk aversion parameters.[21]

[Table (1.2) about here]

1.3 Equilibrium Analysis

In this section we consider the traditional approach to the extraction of the risk neutral distribution based on general equilibrium in the production or exchange economy, in which the underlying returns follow a jump-diffusion process, and compare its results to the stochastic dominance bounds of the previous section. Since the pricing kernel links the physical and risk-neutral densities in a general equilibrium setup, we derive the upper and lower bounds' pricing kernels in the SD approach, which are independent of investor's preferences. We then use these kernels to restrict the preferences of the representative investor in the general equilibrium approach and extract appropriate bounds on the preference of the representative investor, which depend on option moneyness and time to maturity. Finally, we compare the

[19] See Oancea and Perrakis [2014].

[20] We discuss the choice of the range of intensity values in the next section.

[21] For instance, for a risk aversion coefficient of 7, the mid-range of the Rosenberg and Engle [2002] estimates, and for $\lambda = 10$ the total volatility of the option rises from 26.3% to 93% and becoming explosive on higher values.

SD implied bounds on the relative risk aversion (RRA) coefficient with those commonly used in the option pricing literature and those extracted from macroeconomic data, mostly based on consumptions and market return data.

We follow the general equilibrium analysis with asset return dynamics given by (1.2.7) and a representative investor of the CRRA-type, with γ denoting the RRA coefficient. Of particular interest for our purposes are the expressions for the equilibrium pricing kernel π_t under the CRRA assumption, and the corresponding parameter mapping from the physical or P-distribution to the risk neutral Q-distribution. These mappings satisfy the requirements that $E_t\left[d\left(\pi_t S_t\right)\right] = 0$ and $E_t\left[d\pi_t/\pi_t\right] = -rdt$. The derivation of the following expressions can be found in several studies.[22]

The general expression for the pricing kernel process in a general equilibrium model with a CRRA representative investor with RRA equal to γ follows the dynamics in equations (1.3.1) and (1.3.2).

$$d\pi_t/\pi_t = \left(-r - \lambda E\left[j_t^{\pi} - 1\right]\right) dt - \eta dW_t + \left(j_t^{\pi} - 1\right) dN_t, \tag{1.3.1}$$

$$\eta = \gamma\sigma, \quad j_t^{\pi} = j_t^{-\gamma}. \tag{1.3.2}$$

where η, the market price of diffusive risk, is proportional to volatility and $j_t^{\pi} - 1$ is the relative jump amplitude of the pricing kernel process.

The correspondence between the physical and risk neutral jump distribution parameters for the CRRA investor is given by

$$\lambda^Q = \lambda E\left[j_t^{-\gamma}\right], \quad k^Q = \frac{E\left[\left(j_t - 1\right) j_t^{-\gamma}\right]}{E\left[j_t^{-\gamma}\right]}. \tag{1.3.3}$$

Note also that in this model the total equilibrium risk premium must be equal to the sum of the diffusive risk premium and the jump risk premium, $\mu_t - r = \gamma\sigma^2 + \lambda k - \lambda^Q k^Q$.

For a lognormal jump amplitude $\ln\left(j_t\right) \sim N\left[\mu_j - 0.5\sigma_j^2, \sigma_j^2\right]$, we have the following transformations.

[22] See, for instance, Bates [1991, 2006], Pan [2002], and Liu et al. [2005].

$$\lambda^Q = \lambda \exp\Big[-\gamma\mu_j + \frac{1}{2}\gamma\left(\gamma+1\right)\sigma_j^2\Big] \tag{1.3.4}$$

$$k^Q = E^Q\Big[j_t^Q - 1\Big] = \exp\left[\mu_j - \gamma\sigma_j^2\right] - 1 = \exp\left[\mu_j^Q\right] - 1 \tag{1.3.5}$$

With these relations the risk neutral jump diffusion dynamics become now

$$dS_t/S_t = \Big[r - \lambda^Q E^Q\left[j_t^Q - 1\right]\Big]dt + \sigma_t dW_t^Q + \left[j_t^Q - 1\right]dN_t^Q \tag{1.3.6}$$

Equations (1.3.1)-(1.3.6) summarize and describe completely the mapping from the P- to the risk neutral Q-distribution for a general equilibrium analysis of jump diffusion derivatives pricing given the existence of a representative CRRA investor, the only case that has appeared so far in the literature. It can be seen easily from these expressions that the equilibrium pricing kernel is monotone decreasing in the underlying asset return and that the total risk premium is endogenously given as a function of the RRA parameter. Since the stochastic dominance bounds include all option prices consistent with a decreasing pricing kernel and with expected risk premium smaller than or equal to the one used in deriving the bounds, we may now derive the limits on γ implied by the SD bounds of the previous section.

To embed the SD bounds in an equilibrium model we first note that the pricing kernel equation (1.3.1) should still hold but that in the absence of a representative CRRA investor (1.3.2) no longer holds. On the other hand, Propositions (1.2) and (1.3) introduce two risk-neutral distributions that yield the upper and lower option bounds when the underlying asset follows the jump-diffusion process. The violation of any of these two bounds implies that any trader can improve her utility by introducing the corresponding short or long option positions in her portfolio. Since utility maximization given the P-distribution (including the total risk premium) is a first step in the equilibrium approach, the SD bounds should be satisfied by the option price derived in an equilibrium model. The next result, part of which is obvious from (1.2.8)-(1.2.10) and (1.2.15)-(1.2.16) and the rest is proven in the Appendix (1.D), helps establish bounds on the admissible equilibrium model values of γ given the P-distribution.

Proposition 1.4. *When the underlying asset follows a jump-diffusion process described by (1.3.6) the option bounds' corresponding risk neutral parameters are:*

For the upper bound:

$$\begin{aligned}
&\lambda^Q = \lambda^U \Rightarrow \lambda E\left[j_t^\pi\right] = \lambda + \lambda_{Ut} = \lambda - \frac{\mu_t - r}{j_{\min} - 1}, \\
&k^Q = k^U \Rightarrow E^Q\left[j_t^Q - 1\right] = \frac{1}{E\left[j_t^\pi\right]} \times E\left[(j_t - 1) \times j_t^\pi\right] = \frac{\lambda}{\lambda + \lambda_{Ut}} k + \frac{\lambda_{Ut}}{\lambda + \lambda_{Ut}} \left(j_{\min} - 1\right).
\end{aligned} \tag{1.3.7}$$

For the lower bound:

$$\begin{aligned}
&\lambda^Q = \lambda^L = \lambda, \\
&k^Q = k^L \Rightarrow E^Q\left[j_t^Q - 1\right] = \frac{1}{E\left[j_t^\pi\right]} \times E\left[(j_t - 1) \times j_t^\pi\right] = E\left(j_t - 1 | j \leq \bar{j}_t\right).
\end{aligned} \tag{1.3.8}$$

If the jump amplitude is a truncated lognormal, the characteristic function of the jump component's distribution is $e^{\lambda T(f_j(\varphi)-1)}$, where $f_j(\varphi) \equiv E(j^{i\varphi})$ is the characteristic function of the jump amplitude. In such a case the means and variances of the return distributions under the upper and lower bounds' Q-distributions are given by expressions (1.D.9)-(1.D.10) and (1.D.17)-(1.D.18) of the Appendix (1.D) and their truncated counterparts are given by (1.D.11)-(1.D.12) and (1.D.19)-(1.D.20). □

In the next section we use the equilibrium expressions summarized and/or derived in this section in order to find implicit bounds on the admissible values of the RRA parameter γ given the SD bounds estimated from the P-distribution parameters. These estimates are from option pricing studies containing jump diffusion and studies associated with the equity premium puzzle initially identified by Mehra and Prescott [1985].

1.4 RRA Values Implied by Stochastic Dominance Option Bounds

An exact expression giving the limits of the RRA compatible with the boundary risk neutral distributions of Propositions (refprop2) and (refprop3) is not available in closed form, especially in view of the fact that the transformed jump amplitudes are not lognormal. Such limits can only be defined numerically for a given set of parameters. In what follows we first find these limits for our base case and then examine several parameter values extracted from existing econometric studies of the S&P 500 returns' P-distribution. Figure (1.2) shows the admissible range of values of γ in the case of ATM options for our base case parameter values and for two alternative upper bounds, one based on the entire lognormal distribution $j_{min} = 0$ and the other on a lognormal distribution truncated at a worst-case return of -20%, i.e. $j_{min} = 0.8$. We find the implied RRA using the Bates [1991] jump-diffusion model to derive the equilibrium call option prices for a continuum of relative risk aversion coefficients up to 10.

[Figure (1.2) about here]

With respect to the SD lower bound, the only admissible value of a relative risk aversion coefficient for CRRA investor in Figure (1.2) is negative and equal to -1.72 for our base case, violating the risk aversion principle for the representative investor. This is not a surprising SD result, given that the bound lies below the Merton value, where the jump risk is unsystematic, which is also the Bates [1991] jump-diffusion price if the representative investor's RRA is zero. More to the point, several econometric studies of S&P 500 index options based on the equilibrium approach and CRRA utilities have persistently documented negative values of γ, starting with Jackwerth [2000] and including Aït-Sahalia and Lo [2000] and especially Ziegler [2007]. The latter study examined various potential explanations of this perverse result without reaching any definitive conclusion.[23] The SD lower bound results are possible explanations of these negative γ findings, even though the implied pricing kernel is increasing. What they imply is that the equilibrium model cannot account for several risk neutral jump diffusion distributions compatible with the underlying P-distribution and the much weaker SD assumption of a declining pricing kernel. Since our purpose is the analysis of the admissible equilibrium model solutions within the SD framework, we shall ignore hereafter the SD lower bound and assume that the lowest SD-compatible value of γ is 0.

From the SD upper bound, in our base case the maximum SD-admissible γ is 5.49 for the truncated lognormal, rising to almost 7 for the case of $j_{min} = 0$, as illustrated in Figure (1.2). Note that, unlike the equilibrium model, the SD upper bound does not imply the same γ for all degrees of moneyness, as shown in Figure (1.3). Nonetheless, the range of upper bound-implied γ is relatively narrow, starting from 7.1 for 2% OTM up to 7.7 for

[23] Ziegler [2007] considers several potential explanations for the U-shaped and negative implied risk aversion patterns: (I) preference aggregation, both with and without stochastic volatility and jumps in returns; (II) misestimation of investors' beliefs caused by stochastic volatility, jumps, or a Peso problem; (III) heterogeneous beliefs.

2% ITM. Unreported results show a similar narrow range of relative risk aversion also holds when the moneyness is kept constant but the time to expiration is varied from 0.083 to 1 year for the base case parameters.

[Figure (1.3) about here]

[Table (1.3) about here]

Table (1.3) shows the equilibrium option prices for our base case parameters and for the RRA range of Figures Figure (1.2) and Figure (1.3). The table also shows the corresponding implied mean μ and the risk neutral parameters of the jump component λ^Q and k^Q from (1.3.4)-(1.3.5) for the continuum of RRA.[24] It is clear that the SD-restricted range of admissible RRA values needs to be tightened even further in order to accommodate reasonable values of the ex-dividend expected excess return, which is taken equal to 2% in our SD base case but rises to unreasonably high values when the RRA exceeds 2.

[Table (1.4) about here]

[Table (1.5) about here]

The SD implied upper bound on the relative risk aversion varies relatively slowly for a wide range of moneyness and time to maturity, as shown in Figure (1.3) and Tables (1.4) and (1.5). Following the base case scenario, the 2% OTM call option reduces the SD implied upper bound on the RRA from 7.37 to 7.11 and the 2% ITM call option increases the upper bound on the RRA from 7.37 to 7.72. Similarly, as we increase time to maturity from one month to six months, the SD upper bounds on the call option prices increase by a factor of more than 2, from 2.56 to 6.97, but the implied upper bound on the risk aversion increases only from 6 to 8.56. On the other hand, the sensitivity of the implied RRA to changes in risk premium is significantly higher than that of the option bound, as can be seen in Tables (1.4) and (1.5) when the ex-dividend risk-premium increases from 2% to 4%

Since the SD-implied RRA is parameter dependent, we examine it for the parameter values that were estimated in earlier studies. Such studies fall into two categories, option market-based and macro-finance studies attempting to explain the equity premium puzzle.

[24]Implied mean return is calculated based on the jump diffusion equilibrium risk premium, $\mu_t - r = \gamma\sigma^2 + \lambda k - \lambda^Q k^Q$, in Section (1.3).

In empirical tests of the former category, a jump diffusion model is often included in a nested model that also includes stochastic volatility;[25] only a few of these studies are reviewed here. Bates [1991] applied the nested models to Deutsche mark currency options, and in a subsequent study Bates [2000] to S&P 500 futures options, while Pan [2002] and Rosenberg and Engle [2002] examined S&P 500 index options, and Bliss and Panigirtzoglou [2004] FTSE 100 and S&P 500 index futures options. In these tests the parameters of the implied risk neutral distribution are extracted from cross sections of observed option prices and attempts are made to reconcile these option-based distributions with data from the market of the underlying asset. All studies stress the importance of jump risk premia in these reconciliation attempts.

Such reconciliations have not always been crowned with success, with the result that reported estimates of γ vary widely between studies. They range from an arbitrarily chosen value of 2 for Bates [1991] to 3.94 estimated by the same author in Bates [2006] using both return and option data, to a value up to 10 by Liu and Pan [2003], where they quantify the gain of including derivatives in portfolio optimization in the presence of jumps. Bliss and Panigirtzoglou [2004] choose the risk aversion parameter between 3.37 and 9.52 to produce subjective densities that best fit the distributions of realized values. In a bootstrap estimate of the RRA based on observed 5-week S&P 500 options they report a minimum of -1.34 and a maximum of 8.17 for the relative risk aversion; note the approximate consistency of these varying estimates with the γ limits implied from our SD bounds shown in Figure (1.2) and Table (1.5) for the same maturity and a 4% risk premium. Liu et al. [2005] adjust the risk aversion coefficient to 3.49 to match an observed total equity premium when the underlying process follow jump-diffusion dynamic while the representative agent is averse not only to diffusive and jump risk but also to uncertainty aversion. However, as they point out, the data implied RRA coefficient has to be considerably larger than 3.49 if they only incorporate diffusive risk and jump risk in justifying the pronounced smirk pattern.

Although the risk aversion values in these studies are mostly consistent with the SD implied bounds on RRA, the SD results are extracted uniquely from estimates of the underlying returns P-distribution. Compared to the equilibrium approach's estimates, they require an additional parameter, the total risk premium $\mu_t - r$, but do not require knowledge of

[25] The equilibrium model does not allow stochastic volatility and jumps in linking the $P-$ and Q-distributions. Although Duffie et al. [2000] have presented option prices under general Q-distributions containing both stochastic volatility and jumps, to our knowledge the only stochastic volatility pricing kernel was derived by Christoffersen et al. [2013] in the context of the Heston [1993]. For stochastic volatility in the SD context see the Appendix (1.E).

γ. Unlike γ, there are reliable historical estimates of μ_t, even the largest of which defines tighter bounds on γ than those available from empirical studies that rely on the option market. They can, therefore, verify the consistency of the two markets in a more reliable manner than the equilibrium approach. Note that the inconsistencies and inability of the equilibrium approach to reconcile the evidence of the underlying and option markets has already been mentioned in several earlier studies.[26]

A key issue in all the jump diffusion option pricing models is the accurate estimation of the parameters, since the Q-distributions for the option market fluctuate widely even for small differences in the parameter estimates. Further, the total risk premium does not appear explicitly and must be estimated from γ and the P-parameters, equal to $\gamma\sigma^2 + \lambda k - \lambda^Q k^Q$ as in (1.3.4) and (1.3.5). Since this premium is also a byproduct of the P-estimation, a successful reconciliation of the two markets must also verify the consistency of the premium with the value of γ used in the option market valuation. This is generally not done in most studies.

[Table (1.6) about here]

[Figure (1.4) about here]

We carry out this exercise for several econometric estimations of jump diffusion parameters shown in Table (1.6), whose results differ substantially not only between studies but also between different data series within the same study. From the parameter estimates, we extract the appropriate RRA coefficient to match the reported P-distribution excess return in Column 3 of Table (1.6). We find that γ should be below 2 in Andersen et al. [2002], and Eraker et al. [2003], and below 2.5 in Ramezani and Zeng [2007] and Honore [1998]. Therefore, none of the extracted underlying jump diffusion parameters can accommodate relative risk aversion coefficient above 2.5. Figure (1.4) shows the relationship between γ and the corresponding jump diffusion equilibrium risk premium $\gamma\sigma^2 + \lambda k - \lambda^Q k^Q$.

The SD implied bounds on the relative risk aversion can also provide information on the RRA coefficient extracted in macro finance studies. The RRA coefficients used in the option pricing literature are much lower than those of the equity premium puzzle studies, where Mehra and Prescott [1985] report a coefficient of 41, Cochrane and Hansen [1992] report RRA in the range of 40-50, and Campbell and Cochrane [1999] expects a value more than

[26]See Eraker et al. [2003, P. 1294], Broadie et al. [2007], Broadie et al. [2009], and Ross [2015].

35,[27] although some argue that risk aversion this large implies implausible behavior along other dimensions;[28] note that these studies relied on pure diffusion dynamics of consumption growth. Table (1.7) provides a partial explanation for this discrepancy.

[Table (1.7) about here]

In Table (1.7) we use the jump diffusion parameters of Table (1.6) to estimate the SD upper bound option prices (Column 3) and then extract the implied upper bound relative risk aversion (Column 4) by equating these option upper bounds with the equilibrium option prices from the Bates [1991] model. Observe that for all of these parameter estimates the upper bound RRA values are similar to the ones found in the equity premium puzzle literature, the only option pricing model that can achieve such high γ values. Since the SD upper bound gives the highest admissible option price implied by the P-distribution parameters, this price is equivalent to, ceteris paribus, the largest possible RRA coefficient compatible with the preferences of the representative option trader in an equilibrium model.

From the upper bound γ in Column 4 we estimate the implied equity premium (Column 5), using the Mehra and Prescott [1985] estimates, which were derived in the absence of rare events affecting consumption for CRRA investors. We have $\ln\left(E_t\left[R_{e,t+1}\right]\right) - \ln R_f = \gamma\sigma^2_{\Delta \ln C}$,[29] where the implied riskless rate R_f is found from the equation $\ln R_f = -\ln\beta + \gamma\mu_{\Delta \ln C} - 0.5\gamma^2\sigma^2_{\Delta \ln C}$, and where $\beta = 0.99$, $\mu_{\Delta \ln C} = 0.01919$ and $\sigma^2_{\Delta \ln C} = 0.0011767$.[30] As we see, such a risk premium estimate is significantly lower than the observed risk premium (Column 6) in four out of the six cases. Thus we still observe the equity premium puzzle as extensively addressed by different authors,[31] as the corresponding equilibrium premium is lower than the observed one in most cases. Since the SD implied upper bound is like a no arbitrage bound, one possible explanation for the above puzzle may be that index options are overpriced from the option trader's perspective, as claimed in several empirical studies.

[27] See also the survey article by Kocherlakota [1996]

[28] See Campanale et al. [2010]

[29] These estimates remained essentially unchanged when the data was extended to 2005 and then to 2009. See Barro [2006, Section 1.F] and Backus et al. [2011]. This dataset has been used widely in most recent studies of the equity premium.

[30] This is equivalent to the distribution of real consumer expenditure with mean of 0.02 and standard deviation of 0.035.

[31] A good summary of the puzzle and its possible resolutions is in Mehra and Prescott [2003]. The expressions are in Mehra and Prescott [2003].

Alternatively, the equity premium estimated from the extracted value of the RRA and an equation applicable only to diffusion dynamics is incorrect and needs to be adjusted for rare event risk.

We examine this alternative explanation by exploring the consistency of the upper bound-implied RRA with the results of more recent equity premium puzzle studies that go beyond simple diffusion and consider the presence of fat tails in the consumption distribution.[32] In particular Barro [2006] has shown that rare disasters may account for high equity risk premia by using the international consumption dataset while maintaining a tractable framework of a representative agent with time-additive isoelastic preferences.[33] In his model the equity premium is given by $\ln\left(E_t\left[R_{e,t+1}\right]\right) - \ln R_f = \phi\gamma\sigma^2_{\Delta \ln C} + \lambda E_t\left[\left(e^{-\gamma J} - 1\right)\left(1 - e^{\phi J}\right)\right]$,[34] where $\phi = 1$ is the leverage effect that used to model dividends as a levered consumption and J is again the amplitude of the consumption disaster risk, assumed lognormal, $\ln J \sim N(\mu_j, \sigma_j^2)$.

We apply the above equity premium equation using the upper bound RRA for the jump diffusion parameter estimates of Eraker et al. [2003], reported in Table (1.6). The implied equity premium in the presence of consumption disaster is 8.95, a level of premium that is above the observed 7.5% premium evaluated under the assumption that the risk free rate is 5%, but is close to the observed 8.5% premium if that rate is assumed to be a more realistic 4%.[35] More to the point, we verify the tail risk of the upper bound-implied bootstrapped Q-distribution and compare it to that of the corresponding P-distribution with the study's parameter estimates. The two distributions are shown in Figure (1.5).

[Figure (1.5) about here]

As we see, the two distributions have virtually identical tails: the probability of a three-month decline in return in excess of 20% is equal to 0.00065 and 0.00099 on the basis of the P and Q distributions respectively. By comparison, in the reported results of Ross [2015, Pages 642-643] the latter has 10 times higher tail risk as the former. The SD upper bound distribution corresponding to the lognormal distribution truncated at 80% yields a

[32]See, for instance, Barro [2006], Wachter [2013] and Martin [2013].

[33]See Barro and Ursua [2008], Gabaix [2008], and Wachter [2013] to name a few.

[34]This is the continuous-time counterpart of Barro [2006] reproduced in Wachter [2013, Section I.G and Appendix C].

[35]We assume that the consumption disaster has the mean, volatility, and intensity equal to 0.3, 0.15, and 0.01 respectively, following Backus et al. [2011].

virtually identical result. These demonstrations are perhaps the most powerful evidence of the advantage of the SD method vis-à-vis the equilibrium approach in equity premium studies if we want to extract the appropriate RRA value from the option market, insofar as it can illuminate the "dark matter" that motivates this paper.

1.5 Extensions and Conclusions

The results presented in Section (1.2) yield bounds for jump-diffusion index option prices that are relatively simple to compute and reasonably tight for most empirically important cases. In addition, the bounds can also accommodate state-dependent diffusion parameters, even though their computation would be difficult. Last but not least, the SD approach does not assume simultaneous equilibrium in the options and the underlying asset markets, an equilibrium that is not realistic if the options do not trade in an organized or a liquid market, as with catastrophe derivative instruments, where the instruments trade over the counter and the underlying process follows rare-event dynamics.[36]

The discrete time approach of the bounds estimation allows several significant extensions to jump-diffusion option pricing. Thus, the valuation of American options is obvious, due to the discrete nature of the bounds. Further, the incorporation of proportional transaction costs is available for some (but not all) European or American option cases following the general results of Constantinides and Perrakis [2002, 2007]. Most important, a comparison of the SD jump diffusion bounds with the dominant equilibrium model's results that are nested by it showed that several empirical puzzles of that model are consistent with the more general SD context.

[36]See Perrakis and Boloorforoosh [2013].

1.A Proof of Lemma (1.2)

We prove the convergence of the discretization (1.2.6) in the i.i.d. case[37] where $\mu_t - \lambda k = \mu$, $\sigma_t = \sigma$, $j_t = j$. Convergence in the non-i.i.d. case follows from the convergence criteria for stochastic integrals, presented in Duffie and Protter [1992]. It is shown in an appendix, available from the authors on request.

The characteristic function of the terminal stock price at time T for a \$1 initial price under the jump-diffusion process (1.2.7) is

$$\begin{aligned}\varphi_{jD}(\omega) &= \exp\left[i\omega\mu T - \frac{\omega^2\sigma^2 T}{2}\right]\exp\left(-\lambda T\right)\sum_{N=0}^{\infty}\frac{(\lambda T)^N}{N!}\left[\varphi_j(\omega)\right]^N \\ &= \exp\left[i\omega\mu T - \frac{\omega^2\sigma^2 T}{2}\right]\exp\left[\lambda T(\varphi_j(\omega)-1)\right],\end{aligned} \tag{1.A.1}$$

where $\varphi_j(\omega)$ is the characteristic function of the jump distribution. The first exponential corresponds to the diffusion component and the second to the jump component.

The characteristic function of the discretization (1.2.6) is

$$\varphi(\omega) = (\lambda\Delta t\varphi_j(\omega) + 1 - \lambda\Delta t)\left[\exp(i\omega\mu\Delta t)\varphi_\epsilon(\omega\sigma\sqrt{\Delta t})\right], \tag{1.A.2}$$

where $\varphi_\epsilon(\omega)$ is the characteristic function of ϵ.[38] Since the distribution of ϵ has mean 0 and variance 1, we have

$$\begin{aligned}E\left[\epsilon\right] &= 0 = i\varphi_\epsilon'(0), \\ E\left[\epsilon^2\right] &= 1 = -\varphi_\epsilon''(0).\end{aligned}$$

By the Taylor expansion of $\varphi_\epsilon(\omega)$, we get

[37] The proof is similar to that of Theorem 21.1 in Jacod and Protter [2003]. An alternative proofs is in Fu et al. [2016].

[38] If instead of (1.2.6) we have a mixture of the diffusion and jump components then the characteristic function becomes $\varphi(\omega) = \lambda\Delta t\varphi_j(\omega) + (1-\lambda\Delta t)\left[\exp(i\omega\mu\Delta t)\varphi_\epsilon(\omega\sigma\sqrt{\Delta t})\right]$. The multiperiod convolution, however, still converges to (1.A.3).

$$\varphi(\omega) = (\lambda \Delta t \varphi_j(\omega) + 1 - \lambda \Delta t) \left[\exp(i\omega\mu\Delta t) \Big[1 - \frac{\omega^2\sigma^2\Delta t}{2} + \omega^2\sigma^2\Delta t\, h\big(\omega\sigma\sqrt{\Delta t}\big) \Big] \right],$$

where $h(\omega) \to 0$ as $\omega \to 0$. The multi-period convolution has the characteristic function $\varphi(\omega)^{(T/\Delta t)}$. Taking the limit, we have

$$\begin{aligned}
\lim_{\Delta t \to 0} [\varphi(\omega)]^{T/\Delta t} &= \lim_{\Delta t \to 0} \exp \left[\frac{T}{\Delta t} \ln \left(\lambda \Delta t \varphi_j(\omega) + 1 - \lambda \Delta t \right) \right. \\
&\quad \left. + \frac{T}{\Delta t} \ln \left[\exp(i\omega\mu\Delta t) \Big[1 - \frac{\omega^2\sigma^2\Delta t}{2} + \omega^2\sigma^2\Delta t\; h\big(\omega\sigma\sqrt{\Delta t}\big) \Big] \right] \right] \qquad (1.A.3) \\
&= \exp \left[\lambda T \left(\varphi_j(\omega) - 1 \right) + i\omega\mu T - \frac{\omega^2\sigma^2 T}{2} \right]
\end{aligned}$$

after applying $l'H\hat{o}pital's$ rule. Equation (1.A.3) is, however, the same as equation (1.A.1), the characteristic function of (1.2.7). So, Levy's continuity theorem [39] proves the weak convergence of (1.2.6) to (1.2.7), QED. □

Another way to characterize the limit process is its generator. Denote by $Z_{D,t}$ the diffusion component and by $Z_{j,t}$ the jump component of the return process. Therefore, we have

$$\begin{aligned}
&\lim_{\Delta t \to 0} \frac{E\left[f(S_{t+\Delta\, t},\, t + \Delta t) \right] - f(S_t,\, t)}{\Delta t} = \\
&= \lim_{\Delta t \to 0} \tfrac{E\left[f\left(S_t(1+Z_{D,\,t+\Delta t}),\, t+\Delta t \right) \right] - f(S_t,\, t)}{\Delta t} + \lambda \Delta t \tfrac{E[f\left(S_t(1+Z_{j,\,t+\Delta t}),\, t+\Delta t \right)] - f(S_t,\, t)}{\Delta t} \qquad (1.A.4) \\
&= (\mu_t - \lambda k) S \tfrac{\partial f}{\partial S} + \tfrac{1}{2} \sigma_t^2 S^2 \tfrac{\partial^2 f}{\partial S^2} + \lambda E\left[f(Sj) - f(S) \right],
\end{aligned}$$

which gives us the generator of the price process described by (1.2.7), QED. □

[39] See for instance Jacod and Protter [2003, Theorem 19.1].

1.B Proof of Proposition (1.2)

We follow the proof of Proposition 2 in Oancea and Perrakis [2014] and consider the same multiperiod discrete time option bounds, obtained by successive expectations under the risk-neutral upper bound distribution. We then seek the limit of this distribution as $\Delta t \to 0$. The multiperiod upper bound distribution is given by

$$U(z_{t+\Delta t}) = \begin{cases} P\left(z_{t+\Delta t}\,|S_t\right) & \text{with probability} \quad \frac{R - z_{\min,t+\Delta t}}{E(z_{t+\Delta t}) - z_{\min,t+\Delta t}} \\ 1_{z_{\min,t+\Delta t}} & \text{with probability} \quad \frac{E(z_{t+\Delta t}) - R}{E(z_{t+\Delta t}) - z_{\min,t+\Delta t}} \equiv Q \end{cases}, \tag{1.B.1}$$

where $P(z_{t+\Delta t}\,|S_t)$ is the physical probability of return at each state at time $t+\Delta t$ and $1_{z_{\min,t+\Delta t}}$ is the physical probability for the lowest possible return. Assuming jump-diffusion dynamics as (1.2.7), the minimum outcome of the returns distribution is $j_{\min} - 1$, as discussed in Section (1.2). Since $z_{\min,t+\Delta t} = j_{\min} - 1$ the martingale transformation for the U-distribution clearly does not involve the diffusion component, which stays the same. The U-distribution is now a convolution of the diffusion component and a jump component with amplitude equal to $j_{\min} - 1$ and $j - 1$ with corresponding probabilities of Q and $1 - Q$ respectively, where Q is defined by the following equation.

$$\begin{aligned} Q &\equiv \frac{E(z_{t+\Delta t}) - R}{E(z_{t+\Delta t}) - z_{\min,t+\Delta t}} = \frac{E(z_{t+\Delta t}) - r\Delta t}{E(z_{t+\Delta t}) - (j_{\min} - 1)} \\ &= \frac{\mu_t \Delta t - r\Delta t}{\mu_t \Delta t - \sigma \max(|\epsilon|)\sqrt{\Delta t} - (j_{\min} - 1)} = -\frac{\mu_t - r}{(j_{\min} - 1)}\Delta t = \lambda_{Ut}\Delta t, \end{aligned} \tag{1.B.2}$$

where λ_{Ut} is defined in Proposition (1.2).

Observe that λ_{Ut} is always positive since $(j_{\min} - 1) < 0$ and $E(z_{t+\Delta t}) > r\Delta t$. Hence, considering the multiperiod upper bound distribution (1.B.1) and equation (1.2.6), the discrete time upper bound process is as follows:

$$z_{t+\Delta t} = \begin{cases} z_{D,t+\Delta t} + (j-1)\Delta N & \text{with probability} \quad 1 - \lambda_{Ut}\Delta t \\ z_{D,t+\Delta t} + (j_{\min} - 1)\Delta N & \text{with probability} \quad \lambda_{Ut}\Delta t \end{cases}. \tag{1.B.3}$$

The outcomes of this process and their probabilities are as follows:

$$z_{t+\Delta t} = \begin{cases} z_{D,t+\Delta t} & \text{with probability} \quad (1-\lambda\Delta t)(1-\lambda_{Ut}\Delta t) \\ z_{D,t+\Delta t} + (j-1) & \text{with probability} \quad \lambda\Delta t(1-\lambda_{Ut}\Delta t) \\ z_{D,t+\Delta t} + (j_{\min}-1) & \text{with probability} \quad \lambda_{Ut}\Delta t \end{cases} . \tag{1.B.4}$$

By removing the terms in $o(\Delta t)$, the upper bound process outcomes become

$$z_{t+\Delta t} = \begin{cases} z_{D,t+\Delta t} & \text{with probability} \quad 1-(\lambda+\lambda_{Ut})\Delta t \\ z_{D,t+\Delta t} + (j_t^U-1) & \text{with probability} \quad (\lambda+\lambda_{Ut})\Delta t \end{cases} . \tag{1.B.5}$$

where j_t^U is given by (1.2.10). This process, however, corresponds to (1.2.8), QED. $\square$

The generator of the price process, which is also reflected in equation (1.2.11), is

$$A^U f = \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 f}{\partial S^2} + \left[r - (\lambda+\lambda_{Ut})k^U\right] S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial T} + (\lambda+\lambda_{Ut})E^U\left[f(Sj_t^U) - f(S)\right] \tag{1.B.6}$$

,QED. $\square$

1.C Proof of Proposition (1.3)

The proof is very similar to those of Lemma (1.2) and Proposition (1.2). Assuming, for simplicity, that both ϵ and j have continuous distributions, we may apply the multiperiod lower bound distribution, given by

$$L(z_{t+\Delta t}) = P\left(z_{t+\Delta t}\,|S_t\,, z_{t+\Delta t} \le z_t^*\right) \quad \text{such that} \quad E\left(z_{t+\Delta t}\,|S_t\,, z_{t+\Delta t} \le z_t^*\right) = R. \tag{1.C.1}$$

From the convergence of the return process without the jump component to the diffusion process as in (1.2.6) and (1.2.7),[40] it is clear that as $\Delta t \to 0$ all the outcomes of the diffusion component will be lower in absolute value than $|j_t|$. Therefore, the limiting distribution will include the whole diffusion component and a truncated jump component. The maximum jump outcome in this truncated distribution is obtained from the condition that the distribution is risk neutral, which is expressed in (1.2.16). We observe that the lower bound distribution over $(t,\, t+\Delta t)$ is the sum of the diffusion component and a jump of intensity λ and log-amplitude distribution j_t^L, the truncated distribution $\{j|j \leq \bar{j}_t\}$.

$$z_{t+\Delta t} = \begin{cases} z_{D,t+\Delta t} & \text{with probability} \quad 1-\lambda\Delta t \\ z_{D,t+\Delta t} + (j_t^L - 1)\Delta N & \text{with probability} \quad \lambda\Delta t \end{cases} \tag{1.C.2}$$

By Lemma (1.2) this process converges weakly for to the jump-diffusion process (1.2.15), QED. □

The generator of the price process is

$$A^L f = \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 f}{\partial S^2} + \left[r - \lambda k^L\right] S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial T} + \lambda E^L \left[f(S j_t^L) - f(S)\right] \tag{1.C.3}$$

which appears in equation (1.2.17), QED. □

1.D Characteristic Function and Moments of Return Dynamics

When the underlying process under P is defined by equation (1.2.7), then the log return process is

$$\begin{aligned} Ln\left(S_t/S_0\right) &= \left[\left(\mu - \frac{1}{2}\sigma^2 - \lambda k\right)t + \sigma W_t + \sum_{i=1}^{N_t} J_i\right] \\ &= \left[\left(\mu - \frac{1}{2}\sigma^2 - \lambda k\right)t + \sigma W_t + \sum_{i=1}^{N_t} \ln(j_i)\right] \end{aligned} \tag{1.D.1}$$

[40]More detail can be find in Oancea and Perrakis [2014].

The characteristic function of the log return process can be defined as the following expectation of the log-return density function.

$$
\begin{aligned}
f_\varphi\left(\ln(S_t/S_0)\right) &= E\left[\exp\left(i\varphi\ln(S_t/S_0)\right)\right] \\
&= E\Big[\exp\big[i\varphi\big(\mu-\frac{1}{2}\sigma^2-\lambda k\big)t\big]\Big]\ E\left[\exp\left(i\varphi\sigma W_t\right)\right]\ E\Big[\exp\big(\sum_{i=1}^{N_t} i\varphi J_i\big)\Big] \\
&= \exp\Big[i\varphi\big(\mu-\frac{1}{2}\sigma^2-\lambda k\big)t\Big]\ \exp\big[\frac{1}{2}\left(i\varphi\sigma\right)^2 t\big]\ E\Big[\exp\big(\sum_{i=1}^{N_t} i\varphi\ln(j_i)\big)\Big] \\
&= \exp\Big[i\varphi\big(\mu-\frac{1}{2}\sigma^2-\lambda k\big)t-\frac{1}{2}\varphi\sigma^2 t\Big]\ \big[\exp\big(\lambda t E(j^{i\varphi}-1)\big)\big] \\
&= \exp\Big[i\varphi\mu t-\frac{1}{2}i\varphi\sigma^2 t-i\varphi\lambda t k-\frac{1}{2}\varphi^2\sigma^2 t+\lambda t E\left(j^{i\varphi}-1\right)\Big]
\end{aligned}
$$

$$
f_\varphi\left(\ln(S_t/S_0)\right) = \exp\Big[i\varphi\mu t-\frac{1}{2}i\varphi\left(1-i\varphi\right)\sigma^2 t+\lambda\big[E\big(j^{i\varphi}-1\big)-i\varphi k\big]t\Big] \tag{1.D.2}
$$

Using the above characteristic function, the mean and the volatility of the log return process can be defined with the derivatives of the characteristic function.

$$
\begin{aligned}
E\Big[\ln(S_t/S_0)\Big] &= (-i)\frac{\partial f}{\partial\varphi}\,|_{\varphi=0}\ = \big(\mu-\frac{1}{2}\sigma^2+\lambda E\left[\ln j\right]-\lambda k\big)t \\
Var\Big[\ln(S_t/S_0)\Big] &= (-i)^2\frac{\partial^2 f}{\partial\varphi^2}\,|_{\varphi=0}\ = \big(\sigma^2+\lambda\left(E\left[\ln j\right]\right)^2+\lambda\left(Var\left[\ln j\right]\right)\big)\,t
\end{aligned}
$$

When the jump size is log normal, $Ln(j)\sim N\left(\mu_j-\frac{1}{2}\sigma_j^2,\sigma_j^2\right)$ or $j\sim LogN\left(e^{\mu_j},e^{2\mu_j}\left(e^{\sigma_j}-1\right)\right)$,

$$
E\left[\ln(S_t/S_0)\right] = \mu t-\frac{1}{2}\sigma^2 t+\lambda\big(\mu_j-\frac{1}{2}\sigma_j^2\big)t-\lambda k t \tag{1.D.3}
$$

$$
Var\left[\ln(S_t/S_0)\right] = \sigma^2 t+\lambda\Big[\big(\mu_j-\frac{1}{2}\sigma_j^2\big)^2+\sigma_j^2\Big]t \tag{1.D.4}
$$

For the risk-neutral process $J^Q=\ln(j^Q)\sim N\left(\mu_j-\gamma\sigma_j^2-\frac{1}{2}\sigma_j^2,\ \sigma_j^2\right)$

$$
f_\varphi^Q\Big[\ln\frac{S_t}{S_0}\Big] = \exp\Big[i\varphi r t-\frac{1}{2}i\varphi\left(1-i\varphi\right)\sigma^2 t+\lambda^Q t\left[E^Q\left(j^{i\varphi}-1\right)-i\varphi k^Q\right]\Big] \tag{1.D.5}
$$

$$E^Q\left[\ln(S_t/S_0)\right] = rt - \frac{1}{2}\sigma^2 t + \lambda^Q\big[\mu_j - \gamma\sigma_j^2 - \frac{1}{2}\sigma_j^2\big]t - \lambda^Q k^Q t \tag{1.D.6}$$

$$Var^Q\left[\ln(S_t/S_0)\right] = \sigma^2 t + \lambda^Q\Big[\big(\mu_j - \gamma\sigma_j^2 - \frac{1}{2}\sigma_j^2\big)^2 + \sigma_j^2\Big]t \tag{1.D.7}$$

Following Proposition (1.2), when the underlying asset follows the dynamic of (1.2.8), the upper bound characteristic function and its first two central moments can be defined similarly by equations (1.D.8), (1.D.9), and (1.D.10) respectively.

$$f_\varphi^U\Big[\ln\frac{S_t}{S_0}\Big] = \exp\Big[i\varphi rt - \frac{1}{2}i\varphi\left(1 - i\varphi\right)\sigma^2 t + \left(\lambda + \lambda_{Ut}\right)\big[E^U\left(j^{i\varphi} - 1\right) - i\varphi k^U\big]\,t\Big] \tag{1.D.8}$$

$$\begin{aligned} E^U\Big[\ln\frac{S_t}{S_0}\Big] &= rt - \frac{1}{2}\sigma^2 t + \left(\lambda + \lambda_{Ut}\right)E^U\left[\ln(j^U)\right]t - \left(\lambda + \lambda_{Ut}\right)k^U t \\ &= rt - \frac{1}{2}\sigma^2 t + \lambda\big(\mu_j - \frac{1}{2}\sigma_j^2\big)t + \lambda_{Ut}\left(\ln j_{\min}\right)t - \left(\lambda + \lambda_{Ut}\right)k^U t \end{aligned} \tag{1.D.9}$$

$$\begin{aligned} Var^U\Big[\ln\frac{S_t}{S_0}\Big] &= \sigma^2 t + \left(\lambda + \lambda_{Ut}\right)\left(E^U\left[\ln(j^U)\right]\right)^2 t + \left(\lambda + \lambda_{Ut}\right)\left(Var^U\left[\ln(j^U)\right]\right)t \\ &= \sigma^2 t + \frac{1}{\lambda + \lambda_{Ut}}\Big[\lambda\big(\mu_j - \frac{1}{2}\sigma_j^2\big) + \lambda_{Ut}\left(\ln\left(j_{\min}\right)\right)\Big]^2 t + \lambda\sigma_j^2 t \end{aligned} \tag{1.D.10}$$

In our analysis of the upper bound, we discuss the limiting distribution that includes the diffusion component and a truncated jump component with truncation limit chosen to meet the observed jump amplitude in econometric studies of jump diffusion.[41] In this case the first and second central moments can be defined by equations (1.D.11) and (1.D.12) where Φ is the normal cumulative function and ϕ is the normal probability function.

[41]See Lien [1985].

$$E^{U}\left[\ln\frac{S_t}{S_0}\,|j>j_{\min}\right]=rt-\frac{1}{2}\sigma^2 t$$
$$+\left[\lambda\big(\mu_j-\frac{1}{2}\sigma_j^2\big)+\lambda_{Ut}\left(\ln j_{\min}\right)+\sqrt{\lambda\left(\lambda+\lambda_{Ut}\right)}\sigma_j\frac{\phi\left(a_0\right)}{1-\Phi\left(a_0\right)}\right]t$$
$$-\left(\lambda+\lambda_{Ut}\right)k^{U}\times\frac{\Phi\left(\sigma_j-a_0\right)}{\Phi\left(-a_0\right)}t \tag{1.D.11}$$

$$Var^{U}\left[\ln\frac{S_t}{S_0}\,|j>j_{\min}\right]=\sigma^2 t+\left(\lambda+\lambda_{Ut}\right)\times$$
$$\left[\frac{\lambda}{\lambda+\lambda_{Ut}}\big(\mu_j-\frac{1}{2}\sigma_j^2\big)+\frac{\lambda_{Ut}}{\lambda+\lambda_{Ut}}\left(\ln j_{\min}\right)+\sqrt{\frac{\lambda}{\lambda+\lambda_{Ut}}}\sigma_j\frac{\phi\left(a_0\right)}{1-\Phi\left(a_0\right)}\right]^2 t$$
$$+\lambda\sigma_j^2\left[1+\frac{a_0\phi\left(a_0\right)}{1-\Phi\left(a_0\right)}-\big(\frac{\phi\left(a_0\right)}{1-\Phi\left(a_0\right)}\big)^2\right] \tag{1.D.12}$$

where $a_0=\left[\ln\left(j_{\min}\right)-\left(\mu_j-0.5\times\sigma_j^2\right)\right]/\sigma_j$.

Another important special case discussed in Remark (1.3) and equation (1.2.12) where the lower limit of the jump amplitude is equal to 0. Therefore, $j_{\min}=0$ and the return distribution has an absorbing state in which the stock becomes worthless. In this case the upper bound characteristic function and its central moments are as follow.

$$f_{\varphi}^{j_{\min}=-1}\left[\ln\frac{S_t}{S_0}\right]=\exp\left[i\varphi\mu t-\frac{1}{2}i\varphi\left(1-i\varphi\right)\sigma^2 t+\lambda\left[E\left(j^{i\varphi}-1\right)-i\varphi k\right]t\right] \tag{1.D.13}$$

$$E^{j_{\min}=-1}\left[\ln(S_t/S_0)\right]=\mu t-\frac{1}{2}\sigma^2 t+\lambda\big(\mu_j-\frac{1}{2}\sigma_j^2\big)t-\lambda kt \tag{1.D.14}$$

$$Var^{j_{\min}=-1}\left[\ln(S_t/S_0)\right]=\sigma^2 t+\lambda\left[\big(\mu_j-\frac{1}{2}\sigma_j^2\big)^2+\sigma_j^2\right]t \tag{1.D.15}$$

Similarly, we introduce the lower bound characteristic function and its central moments when the underlying asset follows the dynamic of (1.2.15), as in Proposition (1.3).

$$f_{\varphi}^{L}\left[\ln\frac{S_t}{S_0}\right]=\exp\left[i\varphi rt-\frac{1}{2}i\varphi\left(1-i\varphi\right)\sigma^2 t+\lambda\left[E^{L}\left(j^{i\varphi}-1\right)-i\varphi k^{L}\right]t\right] \tag{1.D.16}$$

$$E^L\left[\ln(S_t/S_0)\right] = rt - \frac{1}{2}\sigma^2 t + \lambda E^L\left[\ln(j^L)\right]t - \lambda k^L t \tag{1.D.17}$$

$$Var^L\left[\ln(S_t/S_0)\right] = \sigma^2 t + \lambda\left(E^L\left[\ln(j^L)\right]\right)^2 t + \lambda\left(Var^L\left[\ln(j^L)\right]\right)t \tag{1.D.18}$$

Accordingly, if the distribution of $J = \ln(j)$ is normal and truncated at the upper bound $\ln(\bar{j})$ then the central moments are given by (1.D.19) and (1.D.20) where $b_0 = \left[\ln(\bar{j}) - \left(\mu_j - 0.5\sigma_j^2\right)\right]/\sigma_j$.

$$\begin{aligned} E^L\Big[\ln\frac{S_t}{S_0}\,|j<\bar{j}\,\Big] &= rt - \frac{1}{2}\sigma^2 t + \lambda E^L\left[\ln(j^L)\,|j<\bar{j}\,\right] - \lambda E^L\left[j^L - 1\,|j<\bar{j}\,\right]t \\ &= rt - \frac{1}{2}\sigma^2 t + \lambda\Big[\big(\mu_j - \frac{1}{2}\sigma_j^2\big) + \sigma_j\frac{\phi(b_0)}{\Phi(b_0)}\Big] \\ &- \lambda k \times \frac{\Phi(-\sigma_j + b_0)}{\Phi(b_0)}t \end{aligned} \tag{1.D.19}$$

$$\begin{aligned} Var^L\Big[\ln\frac{S_t}{S_0}\,|j<\bar{j}\,\Big] &= \sigma^2 t + \lambda\Big[E^L\left[\ln(j^L)\,|j<\bar{j}\,\right]\Big]^2 t + \lambda\Big[Var^L\left[\ln(j^L)\,|j<\bar{j}\,\right]\Big]t \\ &= \sigma^2 t + \lambda\Big[\mu_j - \frac{1}{2}\sigma_j^2 + \sigma_j\frac{\phi(b_0)}{\Phi(b_0)}\Big]^2 t \\ &+ \lambda\sigma_j^2\Big[1 - \frac{b_0\phi(b_0)}{\Phi(b_0)} - \big(\frac{\phi(b_0)}{\Phi(b_0)}\big)^2\Big]t \end{aligned} \tag{1.D.20}$$

1.E Stochastic Volatility and Jumps Under Stochastic Dominance

Here we discuss how the incorporation of stochastic volatility (SV) will affect the jump diffusion SD bounds on index options. SV introduces an additional source of systematic risk, which can be handled either by arbitrage or by equilibrium considerations. We sketch below an extension of our approach to the pricing of jump risk that can incorporate SV, provided its systematic risk implications are handled outside our model.

In a combined SV and jump-diffusion process, the stock returns are still given by (1.2.7) but the volatility σ_t is random and follows a general diffusion, often a mean-reverting Ornstein-

Uhlenbeck process.[42] In our case we use a general form with an unspecified instantaneous mean $m(\sigma_t^2)$ and volatility $s(\sigma_t^2)$. The asset dynamics then become

$$\begin{aligned} dS_t/S_t &= (\mu_t - \lambda k)dt + \sigma_t dW_1 + (j-1)dN \\ d\sigma_t^2 &= m(\sigma_t^2)dt + s(\sigma_t^2)dW_2, \end{aligned} \tag{1.E.1}$$

where the two Brownian motions are correlated as $dW_1.dW_2 = \rho\sigma_t^2 dt$. The following discrete representation (1.E.2) can be easily shown by applying Lemma (1.2) to converge to (1.E.1):[43]

$$\begin{aligned} \left(S_{t+\Delta t} - S_t\right)/S_t \equiv z_{t+\Delta t} &= \mu(S_t)\Delta t + \sigma_t \epsilon\sqrt{\Delta t} + (j-1)\Delta N \\ \sigma_{t+\Delta t}^2 - \sigma_t^2 &= m(\sigma_t^2)\Delta t + s(\sigma_t^2)\varsigma\sqrt{\Delta t} \end{aligned} \tag{1.E.2}$$

Where ς is an error term of mean 0 and variance 1, and with correlation $\rho(\sigma_t^2)$ between ϵ and ς. In what follows we shall assume that this correlation is constant.

Under reasonable regularity conditions the pricing kernel at time t conditional on the state variable vector (S_t, σ_t) is monotone decreasing. Similarly, for any given σ_t the option price is convex in the stock price.[44] Hence, for any given volatility path over the interval $[0, T]$ to option expiration the option prices at any time t are bound by the expressions $\overline{C}_t(S_t, \sigma_t)$ and $\underline{C}_t(S_t, \sigma_t)$ given in (1.2.3). Since both of these expressions are expected option payoffs under risk neutral distributions, we can apply arbitrage methods as in Merton [1976] to price the options given a price $\xi(S_t, \sigma_t, t)$ for the volatility risk. Propositions (1.2) and (1.3), therefore, hold and the admissible option's upper bound satisfies the PDE in (1.E.3) and its lower counterpart satisfies the PDE in equation (1.E.4).

[42] See Heston [1993].

[43] We also use the proof of the convergence of the diffusion process discussed in Oancea and Perrakis [2014]. In the extension of the proof to stochastic volatility, the only difference is related to the vector ϕ_t in applying the Lindeberg condition, which is now a two-dimensional (S_t, σ_t^2) vector.

[44] The pricing kernel monotonicity holds if the kernel does not include a separate variance preference parameter; see Christoffersen et al. [2013, Pages 1966-1967]. For the convexity see the results of Bergman et al. [1996].

$$
\begin{aligned}
&\frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + \left[r - (\lambda + \lambda_{Ut})k^U\right] S \frac{\partial \overline{C}}{\partial S} + \rho \sigma_t s(\sigma_t^2) \frac{\partial^2 \overline{C}}{\partial S \partial \sigma_t^2} + \frac{1}{2} s^2(\sigma_t^2) \frac{\partial^2 \overline{C}}{\partial^2 \sigma_t^2} + \\
&+ \left[m(\sigma_t^2) - \xi(S_t, \sigma_t, t)\right] \frac{\partial \overline{C}}{\partial \sigma_t^2} - \frac{\partial \overline{C}}{\partial T} + (\lambda + \lambda_{Ut}) E^U \left[\overline{C}(S j_t^U) - \overline{C}(S)\right] = r\overline{C}
\end{aligned}
\tag{1.E.3}
$$

$$
\begin{aligned}
&\frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \underline{C}}{\partial S^2} + \left[r - \lambda k^L\right] S \frac{\partial \underline{C}}{\partial S} + \rho \sigma_t s(\sigma_t^2) \frac{\partial^2 \underline{C}}{\partial S \partial \sigma_t^2} + \frac{1}{2} s^2(\sigma_t^2) \frac{\partial^2 \underline{C}}{\partial^2 \sigma_t^2} + \\
&+ \left[m(\sigma_t^2) - \xi(S_t, \sigma_t, t)\right] \frac{\partial \underline{C}}{\partial \sigma_t^2} - \frac{\partial \underline{C}}{\partial T} + \lambda E^L \left[\underline{C}(S j_t^L) - \underline{C}(S)\right] = r\underline{C}
\end{aligned}
\tag{1.E.4}
$$

The estimation of (1.E.3)-(1.E.4) under general conditions presents computational challenges that lie outside the scope of this paper and remains a topic for future research.

References

Ahn, C. M. and H. E. Thompson (1988). Jump-diffusion processes and the term structure of interest rates. *Journal of Finance 43*(1), 155–174.

Aït-Sahalia, Y. and A. W. Lo (2000). Nonparametric risk management and implied risk aversion. *Journal of Econometrics 94*(1), 9–51.

Amin, K. I. (1993). Jump diffusion option valuation in discrete time. *Journal of Finance 48*(5), 1833–1863.

Amin, K. I. and V. K. Ng (1993). Option valuation with systematic stochastic volatility. *Journal of Finance 48*(3), 881–910.

Andersen, T. G., L. Benzoni, and J. Lund (2002). An empirical investigation of continuous-time equity return models. *Journal of Finance 57*(3), 1239–1284.

Backus, D., M. Chernov, and I. Martin (2011). Disasters implied by equity index options. *Journal of Finance 66*(6), 1969–2012.

Bakshi, G., C. Cao, and Z. Chen (1997). Empirical performance of alternative option pricing models. *The Journal of Finance 52*(5), 2003–2049.

Barone-Adesi, G., H. Dall'O, and V. Vovchak (2012). Is the price kernel monotone. *ACRN Journal of Finance and Risk Perspectives 1*(2), 43–68.

Barone-Adesi, G., R. F. Engle, and L. Mancini (2008). A garch option pricing model with filtered historical simulation. *Review of Financial Studies 21*(3), 1223–1258.

Barro, R. J. (2006). Rare disasters and asset markets in the twentieth century. *Quarterly Journal of Economics 121*(3), 823–866.

Barro, R. J. (2009). Rare disasters, asset prices, and welfare costs. *American Economic Review 99*(1), 243–264.

Barro, R. J. and J. F. Ursua (2008). Consumption disasters in the twentieth century. *American Economic Review 98*(2), 58–63.

Bates, D. S. (1988). Pricing options under jump-diffusion processes. *Working Paper, The Wharton School, University of Pennsylvania.*

Bates, D. S. (1991). The crash of'87: Was it expected? the evidence from options markets. *Journal of Finance 46*(3), 1009–1044.

Bates, D. S. (2000). Post-'87 crash fears in the s&p 500 futures option market. *Journal of Econometrics 94*(1), 181–238.

Bates, D. S. (2006). Maximum likelihood estimation of latent affine processes. *Review of Financial Studies 19*(3), 909–965.

Bates, D. S. (2008). The market for crash risk. *Journal of Economic Dynamics and Control 32*(7), 2291–2321.

Beare, B. K. and L. D. W. Schmidt (2016). An empirical test of pricing kernel monotonicity. *Journal of Applied Econometrics 31*(2), 338–356.

Bergman, Y. Z., B. D. Grundy, and Z. Wiener (1996). General properties of option prices. *Journal of Finance 51*(5), 1573–1610.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy 81*(3), 637–654.

Bliss, R. R. and N. Panigirtzoglou (2004). Option-implied risk aversion estimates. *Journal of Finance 59*(1), 407–446.

Breeden, D. T. (1979). An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics 7*(3), 265–296.

Broadie, M., M. Chernov, and M. Johannes (2007). Model specification and risk premia: Evidence from futures options. *The Journal of Finance 62*(3), 1453–1490.

Broadie, M., M. Chernov, and M. Johannes (2009). Understanding index option returns. *Review of Financial Studies 22*(11), 4493–4529.

Brock, W. A. (1982). Asset prices in a production economy. In *The Economics of Information and Uncertainty*, pp. 1–46. University of Chicago Press.

Campanale, C., R. Castro, and G. L. Clementi (2010). Asset pricing in a production economy with chew–dekel preferences. *Review of Economic Dynamics 13*(2), 379–402.

Campbell, J. Y. and J. H. Cochrane (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy 107*(2), 205–251.

Christoffersen, P., S. Heston, and K. Jacobs (2013). Capturing option anomalies with a variance-dependent pricing kernel. *Review of Financial Studies 26*(8), 1963–2006.

Cochrane, J. H. (1991). Production-based asset pricing and the link between stock returns and economic fluctuations. *Journal of Finance 46*(1), 209–237.

Cochrane, J. H. (1996). A cross-sectional test of an investment-based asset pricing model. *Journal of Political Economy 104*(3), 572–621.

Cochrane, J. H. and L. P. Hansen (1992). Asset pricing explorations for macroeconomics. In *NBER Macroeconomics Annual 1992, Volume 7*, pp. 115–182. MIT Press.

Constantinides, G. M. (2002). Rational asset prices. *Journal of Finance 57*(4), 1567–1591.

Constantinides, G. M., M. Czerwonko, J. Carsten Jackwerth, and S. Perrakis (2011). Are options on index futures profitable for risk-averse investors? empirical evidence. *Journal of Finance 66*(4), 1407–1437.

Constantinides, G. M., J. C. Jackwerth, and S. Perrakis (2009). Mispricing of s&p 500 index options. *Review of Financial Studies 22*(3), 1247–1277.

Constantinides, G. M. and S. Perrakis (2002). Stochastic dominance bounds on derivatives prices in a multiperiod economy with proportional transaction costs. *Journal of Economic Dynamics and Control 26*(7), 1323–1352.

Constantinides, G. M. and S. Perrakis (2007). Stochastic dominance bounds on american option prices in markets with frictions. *Review of Finance 11*(1), 71–115.

Cox, J. C., J. E. Ingersoll, and S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica 53*(2), 385–407.

Cox, J. C., S. A. Ross, and M. Rubinstein (1979). Option pricing: A simplified approach. *Journal of Financial Economics 7*(3), 229–263.

Dimson, E., P. Marsh, and M. Staunton (2006). The worldwide equity premium: a smaller puzzle. In *EFA 2006 Zurich Meetings Paper*.

Duffie, D., J. Pan, and K. Singleton (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica 68*(6), 1343–1376.

Duffie, D. and P. Protter (1992). From discrete-to continuous-time finance: Weak convergence of the financial gain process1. *Mathematical Finance 2*(1), 1–15.

Eraker, B., M. Johannes, and N. Polson (2003). The impact of jumps in volatility and returns. *Journal of Finance 58*(3), 1269–1300.

Ethier, S. N. and T. G. Kurtz (2009). *Markov processes: characterization and convergence*, Volume 282. John Wiley & Sons.

Fama, E. F. and K. R. French (2002). The equity premium. *Journal of Finance 57*(2), 637–659.

Fu, M. C., B. Li, G. Li, and R. Wu (2016). Option pricing for a jump-diffusion model with general discrete jump-size distributions. *Management Science*.

Gabaix, X. (2008). Variable rare disasters: A tractable theory of ten puzzles in macro-finance. *American Economic Review 98*(2), 64–67.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies 6*(2), 327–343.

Honore, P. (1998). Pitfalls in estimating jump-diffusion models. *Available at SSRN 61998*.

Jackwerth, J. C. (2000). Recovering risk aversion from option prices and realized returns. *Review of Financial Studies 13*(2), 433–451.

Jacod, J. and P. E. Protter (2003). *Probability essentials*. Springer Science & Business Media.

Kamrad, B. and P. Ritchken (1991). Multinomial approximating models for options with k state variables. *Management Science 37*(12), 1640–1652.

Kocherlakota, N. R. (1996). The equity premium: It's still a puzzle. *Journal of Economic literature 34*(1), 42–71.

Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science 48*(8), 1086–1101.

Kou, S. G. and H. Wang (2004). Option pricing under a double exponential jump diffusion model. *Management Science 50*(9), 1178–1192.

Levy, H. (1985). Upper and lower bounds of put and call option value: Stochastic dominance approach. *Journal of Finance 40*(4), 1197–1217.

Lien, D.-H. D. (1985). Moments of truncated bivariate log-normal distributions. *Economics Letters 19*(3), 243–247.

Liu, J. and J. Pan (2003). Dynamic derivative strategies. *Journal of Financial Economics 69*(3), 401–430.

Liu, J., J. Pan, and T. Wang (2005). An equilibrium model of rare-event premia and its implication for option smirks. *Review of Financial Studies 18*(1), 131–164.

Lucas Jr, R. E. (1978). Asset prices in an exchange economy. *Econometrica 46*(6), 1429–1445.

Martin, I. W. (2013). Consumption-based asset pricing with higher cumulants. *Review of Economic Studies 80*(2), 745–773.

Mehra, R. and E. C. Prescott (1985). The equity premium: A puzzle. *Journal of Monetary Economics 15*(2), 145–161.

Mehra, R. and E. C. Prescott (2003). The equity premium in retrospect. *Handbook of the Economics of Finance 1*, 889–938.

Merton, R. (1973). Theory of rational option pricing. *Bell Journal of Economics 4*(1), 141–183.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics 3*(1), 125–144.

Oancea, I. M. and S. Perrakis (2014). From stochastic dominance to black–scholes: An alternative option pricing paradigm. *Risk and Decision Analysis 5*(2), 99–112.

Pan, J. (2002). The jump-risk premia implicit in options: Evidence from an integrated time-series study. *Journal of Financial Economics 63*(1), 3–50.

Perrakis, S. (1986). Option bounds in discrete time: Extensions and the pricing of the american put. *Journal of Business 59*(1), 119–141.

Perrakis, S. and A. Boloorforoosh (2013). Valuing catastrophe derivatives under limited diversification: A stochastic dominance approach. *Journal of Banking & Finance 37*(8), 3157–3168.

Perrakis, S. and P. J. Ryan (1984). Option pricing bounds in discrete time. *Journal of Finance 39*(2), 519–525.

Ramezani, C. A. and Y. Zeng (2007). Maximum likelihood estimation of the double exponential jump-diffusion process. *Annals of Finance 3*(4), 487–507.

Ritchken, P. and S. Kuo (1989). On stochastic dominance and decreasing absolute risk averse option pricing bounds. *Management Science 35*(1), 51–59.

Ritchken, P. H. (1985). On option pricing bounds. *Journal of Finance 40*(4), 1219–1233.

Ritchken, P. H. and S. Kuo (1988). Option bounds with finite revision opportunities. *Journal of Finance 43*(2), 301–308.

Rosenberg, J. V. and R. F. Engle (2002). Empirical pricing kernels. *Journal of Financial Economics 64*(3), 341–372.

Ross, S. (2015). The recovery theorem. *Journal of Finance 70*(2), 615–648.

Santa-Clara, P. and S. Yan (2010). Crashes, volatility, and the equity premium: Lessons from s&p 500 options. *The Review of Economics and Statistics 92*(2), 435–451.

Stroock, D. W. and S. S. Varadhan (2007). *Multidimensional diffusion processes*. Springer.

Tauchen, G. and H. Zhou (2011). Realized jumps on financial markets and predicting credit spreads. *Journal of Econometrics 160*(1), 102–118.

Wachter, J. A. (2013). Can time-varying risk of rare disasters explain aggregate stock market volatility? *Journal of Finance 68*(3), 987–1035.

Ziegler, A. (2007). Why does implied risk aversion smile? *Review of Financial Studies 20*(3), 859–904.

Table 1.1: The Convergence of the Stochastic Dominance Upper and Lower Option Bounds

Periods	**Lower Bound**	**Merton Price**	**Upper Bound** ($j_{min}-1 > -1$)	**Upper Bound** ($j_{min}-1 = -1$)
5	4.3443	4.4455	4.5521	4.6920
10	4.3764	4.4455	4.5972	4.6964
15	4.3606	4.4455	4.5671	4.6983
20	4.3694	4.4455	4.5784	4.6990
25	4.3757	4.4455	4.5878	4.6994
30	4.3800	4.4455	4.5955	4.6996
35	4.3752	4.4455	4.5851	4.7000
40	4.3802	4.4455	4.5938	4.7001
45	4.3763	4.4455	4.5858	4.7003
50	4.3811	4.4455	4.5943	4.7003
60	4.3743	4.4455	4.5815	4.7006
70	4.3772	4.4455	4.5854	4.7007
80	4.3797	4.4455	4.5892	4.7007
90	4.3820	4.4455	4.5929	4.7008
100	4.3781	4.4455	4.5856	4.7009
150	4.3804	4.4455	4.5881	4.7010
200	4.3838	4.4455	4.5931	4.7011
250	4.3837	4.4455	4.5922	4.7011
300	4.3842	4.4455	4.5927	4.7011
350	4.3851	4.4455	4.5939	4.7012
400	4.3834	4.4455	4.5904	4.7012
450	4.3849	4.4455	4.5928	4.7012
500	4.3838	4.4455	4.5906	4.7012
600	4.3848	4.4455	4.5920	4.7012
700	4.3860	4.4455	4.5940	4.7012
800	4.3855	4.4455	4.5926	4.7013
900	4.3852	4.4455	4.5920	4.7013
1,000	4.3852	4.4455	4.5918	4.7013

The table shows the convergence of the jump-diffusion bounds for an ATM option with $X = 100$ and time to maturity $T = 0.25$ years with $r = 2\%$, $\mu = 4\%$, $\sigma = 20\%$, $\lambda = 0.6$, $\mu_j = -0.05$, $\sigma_j = 7\%$, annual parameters. The jump amplitude distribution is lognormal. In the case $j_{min} - 1 > -1$, the distribution was truncated to a worst-case jump return of -20%. In the last column we present the case when the lower limit of the jump amplitude is equal to 0, in which $j_{min} - 1 = -1$, that is the return distribution has an absorbing state where the stock becomes worthless.

Table 1.2: Sensitivity of the SD Option Bounds to the Jump Parameters

Lambda	**Jump Vol.** (σ_j)	**Lower Bound**	**Merton Price**	**Upper Bound** ($j_{min}-1 > -1$)	**Upper Bound** ($j_{min}-1 = -1$)
0.0	0.0000	4.2275	4.2312	4.2348	4.4842
0.1	0.1996	4.2601	4.3417	4.4832	4.5966
0.2	0.1377	4.2927	4.3991	4.5332	4.6548
0.3	0.1093	4.3226	4.4253	4.5603	4.6813
0.4	0.0918	4.3482	4.4364	4.5755	4.6924
0.5	0.0794	4.3689	4.4417	4.5849	4.6976
0.6	0.0700	4.3852	4.4455	4.5918	4.7013
0.7	0.0624	4.3977	4.4491	4.5973	4.7048
0.8	0.0560	4.4074	4.4529	4.6020	4.7086
0.9	0.0505	4.4151	4.4568	4.6061	4.7124
1.0	0.0456	4.4214	4.4606	4.6099	4.7161
1.1	0.0412	4.4267	4.4643	4.6134	4.7197
1.2	0.0371	4.4314	4.4679	4.6166	4.7231
1.3	0.0332	4.4356	4.4713	4.6198	4.7264
1.4	0.0295	4.4393	4.4745	4.6228	4.7296
1.5	0.0258	4.4427	4.4777	4.6257	4.7326
1.6	0.0221	4.4457	4.4808	4.6285	4.7356
1.7	0.0183	4.4486	4.4838	4.6313	4.7385
1.8	0.0140	4.4514	4.4868	4.6340	4.7413
1.9	0.0085	4.4541	4.4897	4.6366	4.7441

The table shows the jump-diffusion bounds for an ATM option with $X = 100$ and time to maturity $T = 0.25$ years, and annual parameters $r = 2\%$, $\mu = 4\%$, $\sigma = 20\%$, $\mu_j = -0.05$, for various values of the intensity parameter and the jump amplitude volatility σ_j. We vary the jump volatility and intensity, keeping the overall volatility of the jump-diffusion constant equal to 0.04444. The jump amplitude distribution is lognormal. In the case, $j_{min} - 1 > -1$ the distribution was truncated to a worst-case jump return of -20%.

Table 1.3: Sensitivity of the SD Option Bounds to the Jump Parameters

Risk Aversion	**Call Price**	**Implied Mean**	**Risk Neutral Jump Intensity**	**Risk Neutral Jump Size**
−1.00	4.4007	−0.0240	0.5721	−0.0418
0.00	4.4198	0.0200	0.6000	−0.0464
0.50	4.4307	0.0422	0.6156	−0.0488
1.00	4.4425	0.0644	0.6323	−0.0511
1.50	4.4554	0.0869	0.6503	−0.0534
2.00	4.4694	0.1095	0.6696	−0.0557
2.50	4.4847	0.1322	0.6904	−0.0580
3.00	4.5012	0.1551	0.7126	−0.0604
3.25	4.5101	0.1667	0.7244	−0.0615
4.00	4.5388	0.2016	0.7621	−0.0649
6.00	4.6359	0.2977	0.8846	−0.0741
8.00	4.7723	0.3991	1.0470	−0.0831
10.00	4.9648	0.5085	1.2639	−0.0920
20.00	7.9741	1.3808	4.3456	−0.1355
40.00	65.6746	49.8924	223.4470	−0.2162

This table shows the sensitivity of the equilibrium jump-diffusion call option prices to the coefficient of relative risk aversion γ for a continuum of coefficients up to 40. The base case parameters are $S = 100$, $X = 100$, $T = 0.25$, $r = 2\%$, $\mu = 4\%$, $\sigma = 20\%$, $\lambda = 0.6$, $\mu_j = -0.05$, $\sigma_j = 7\%$. The call option prices are based on the Bates [1991] jump-diffusion model. Implied mean return is calculated based on the jump diffusion equilibrium risk premium in Section (1.3) as $\mu_t - r = \gamma\sigma^2 + \lambda k - \lambda^Q k^Q$. Risk neutral jump intensity and risk neutral jump size are based on the equilibrium transformation in equations (1.3.4) and (1.3.5).

Table 1.4: Sensitivity of the SD-implied RRA Bounds to the Market Risk Premium

	2% Risk Premium			4% Risk Premium		
Money ness (X/S)	**Implied Relative Risk Aversion**	**Upper Bound (Truncated)**	**Upper Bound**	**Implied Relative Risk Aversion**	**Upper Bound (Truncated)**	**Upper Bound**
0.95	8.36	7.5917	7.7691	11.46	7.7681	8.1059
0.96	8.13	6.9206	7.0831	11.20	7.0968	7.4072
0.97	7.92	6.2842	6.4325	10.97	6.4588	6.7427
0.98	7.72	5.6835	5.8182	10.74	5.8551	6.1137
0.99	7.54	5.1193	5.2410	10.54	5.2867	5.5210
1.00	7.37	4.5918	4.7013	10.34	4.7538	4.9653
1.01	7.24	4.1030	4.2011	10.18	4.2587	4.4487
1.02	7.11	3.6508	3.7384	10.03	3.7994	3.9693
1.03	6.99	3.2345	3.3122	9.88	3.3752	3.5265
1.04	6.87	2.8530	2.9216	9.74	2.9854	3.1194
1.05	6.77	2.5060	2.5664	9.62	2.6297	2.7480

This table shows the sensitivity of the SD implied upper bound relative risk aversion to the moneyness and unconditional mean return. SD upper bound on call option prices are calculated for $j_{min} - 1 = -0.8$ (columns 3 and 6) and for the full support of jump distribution (columns 4 and 8). The parameters are $S = 100$, $T = 0.25$, $r = 2\%$, $\mu = 4\% - 6\%$, $\sigma = 20\%$, $\lambda = 0.6$, $\mu_j = -0.05$, $\sigma_j = 7\%$.

Table 1.5: The Sensitivity of the SD-implied Upper Bound RRA to the Time-to-Maturity

Time to Maturity	**Implied Relative Risk Aversion**	**Upper Bound**	**Implied Relative Risk Aversion**	**Upper Bound**
0.08	6.00	2.5685	8.72	2.6554
0.10	6.17	2.8358	8.93	2.9402
0.15	6.64	3.5404	9.48	3.6978
0.20	7.03	4.1509	9.94	4.3616
0.25	7.37	4.7013	10.34	4.9653
0.30	7.67	5.2081	10.70	5.5255
0.35	7.95	5.6830	11.02	6.0540
0.40	8.19	6.1315	11.31	6.5560
0.45	8.43	6.5594	11.58	7.0376
0.50	8.65	6.9705	11.83	7.5025
0.55	8.84	7.3643	12.05	7.9499
0.60	9.02	7.7453	12.26	8.3844
0.65	9.20	8.1151	12.46	8.8078
0.70	9.36	8.4752	12.65	9.2215
0.75	9.52	8.8267	12.83	9.6267
0.80	9.68	9.1708	13.01	10.0245
0.85	9.80	9.5005	13.15	10.4072
0.90	9.95	9.8310	13.31	10.7913
0.95	10.10	10.1568	13.47	11.1708
1.00	10.20	10.4667	13.60	11.5336

This table shows the sensitivity of the SD implied upper bound relative risk aversion to the time to maturity of the options from one-month expiration until one year to expiration and unconditional mean return. The SD upper bound on the call option prices is for the whole support of jump distribution. The base case parameters are $S = 100$, $X = 100$, $r = 2\%$, $\mu = 4\% - 6\%$, $\sigma = 20\%$, $\lambda = 0.6$, $\mu_j = -0.05$, $\sigma_j = 7\%$.

Table 1.6: The Empirical Jump Diffusion Parameter Estimates of the S&P 500 Index

Paper	**Dates**	**Equity Premium**	**Vol.(σ)**	**Jump Intensity**	**Mean Jump**	**Jump Vol.**	**Risk Free**
Honore [1998]	1928-1988	7.94%	10.04%	62.15	-0.13%	1.9%	5.0%
Andersen et al. [2002]	1953-1996	3.22%	9.91%	12.63	0.00%	2.6%	5.1%
Andersen et al. [2002]	1980-1996	10.80%	11.38%	14.89	0.00%	3.4%	5.1%
Ramezani and Zeng [2007][a]	1926-2003	2.56%	13.49%	10.63	0.08%	2.4%	5.0%*
Ramezani and Zeng [2007][b]	1926-2003	5.08%	12.70%	18.57	0.05%	2.0%	5.0%*
Eraker et al. [2003]	1980-1999	7.50%	12.91%	1.51	-2.59%	4.1%	5.0%*

[a] Based on raw returns.
[b] Based on dividend-adjusted returns.

This table shows the empirical jump diffusion parameters for the S&P 500 Index as measured in the corresponding econometric studies that assume that the underlying process is jump diffusion. All the reported parameters are annual. * indicates cases where the reported studies did not estimate the risk-free rate, arbitrarily set at 5%. The differences in jump parameters between Eraker et al. [2003] and the other studies stems from the fact that Eraker et al. [2003] captures small jumps with stochastic volatility, which leads to a lower jump intensity and higher mean and volatility of the jumps.

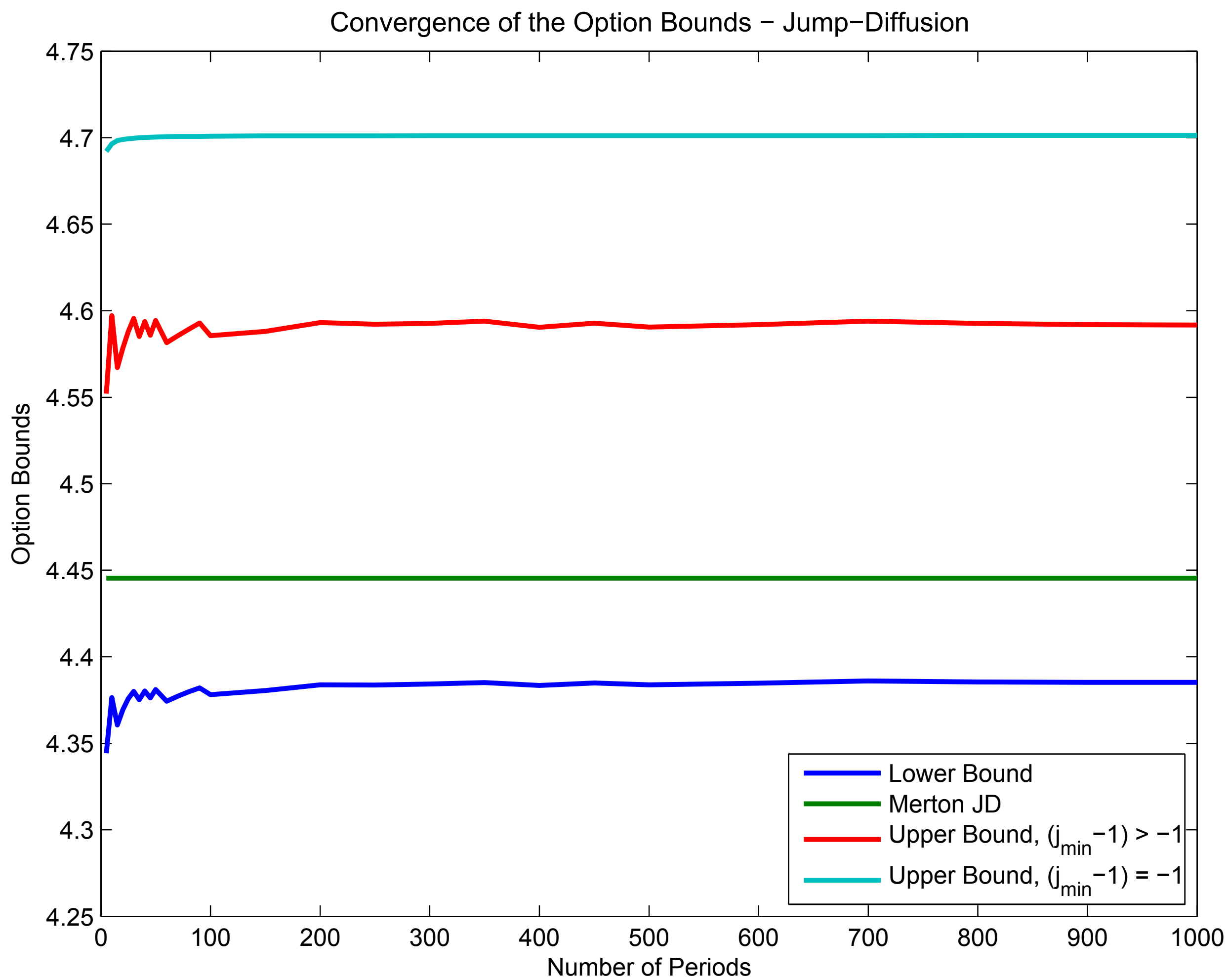
Table 1.7: The SD-implied Bounds on the Relative Risk Aversion Coefficient

Underlying Parameters	Dates	JD Upper Bound Option Prices	Implied Upper Bound Relative Risk Aversion	Implied Equity Premium	Observed Equity Premium
Base Case Parameters		4.70	7.0	0.93%	2.00%
Honore [1998]	1928-1988	5.49	37.5	4.10%	7.94%
Andersen et al. [2002]	1953-1996	3.89	26.5	3.52%	3.32%
Andersen et al. [2002]	1980-1996	5.82	28.5	3.69%	10.80%
Ramezani and Zeng [2007][a]	1926-2003	4.13	33.5	3.99%	2.56%
Ramezani and Zeng [2007][b]	1926-2003	4.47	47.5	3.84%	5.08%
Eraker et al. [2003]	1980-1999	4.62	22.5	3.10%	7.50%

[a] Based on raw returns.
[b] Based on dividend-adjusted returns.

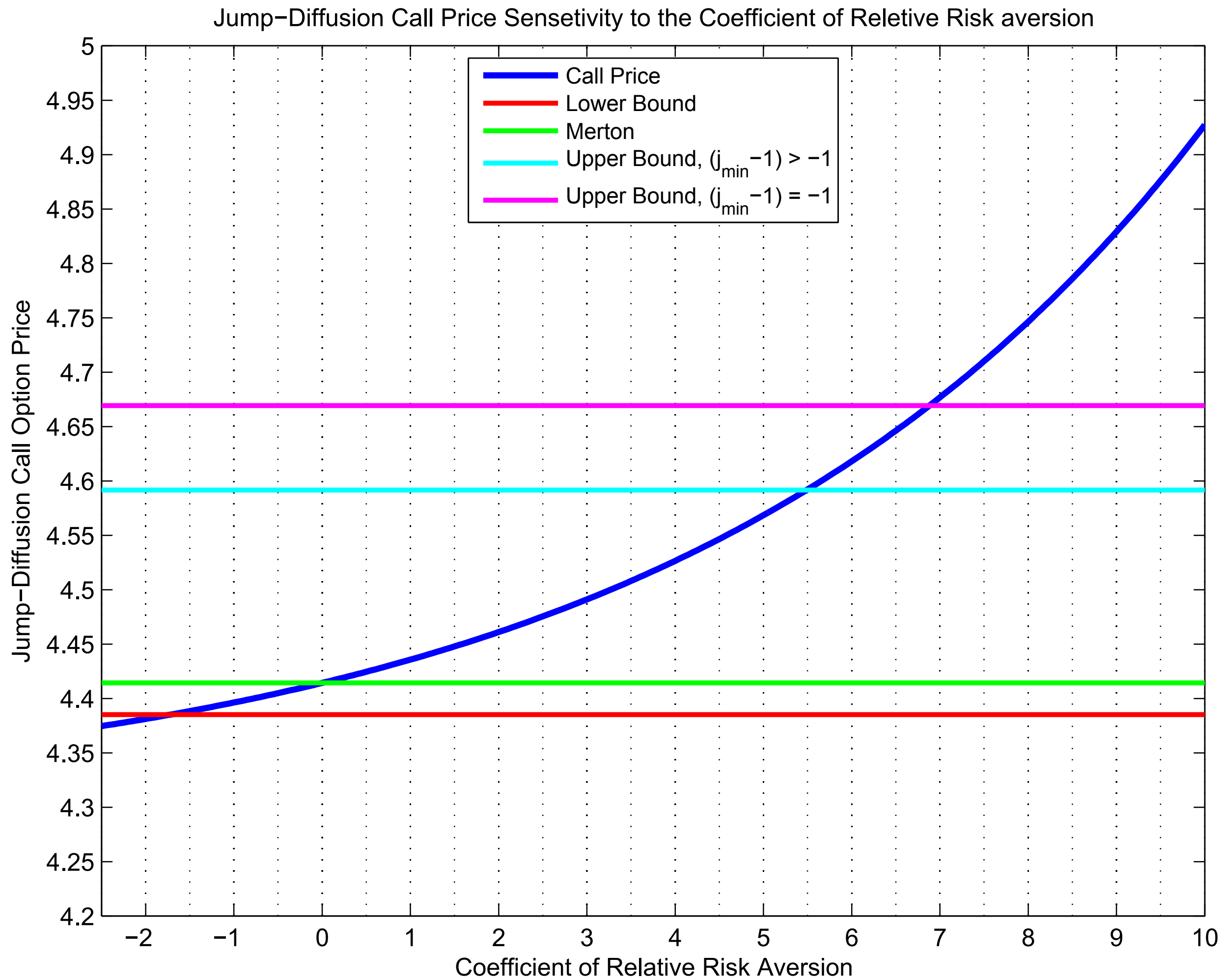
This table shows the implied upper bound RRA and corresponding implied equity premium for the studies reported in Table (1.6). Implied upper bound relative risk aversion (column 4) is defined by using the upper bound option prices (column 3) together with the equilibrium option prices from Bates [1991]. Implied equity premium is calculated following Mehra and Prescott [1985]. The consumption data is annual U.S. data from 1890 to 2004 from Barro [2006], where the growth rate of real consumer expenditure per person has a mean of 0.020 and its standard deviation is 0.035.

Figure 1.1: The Convergence of Jump-Diffusion Stochastic Dominance Option Bounds



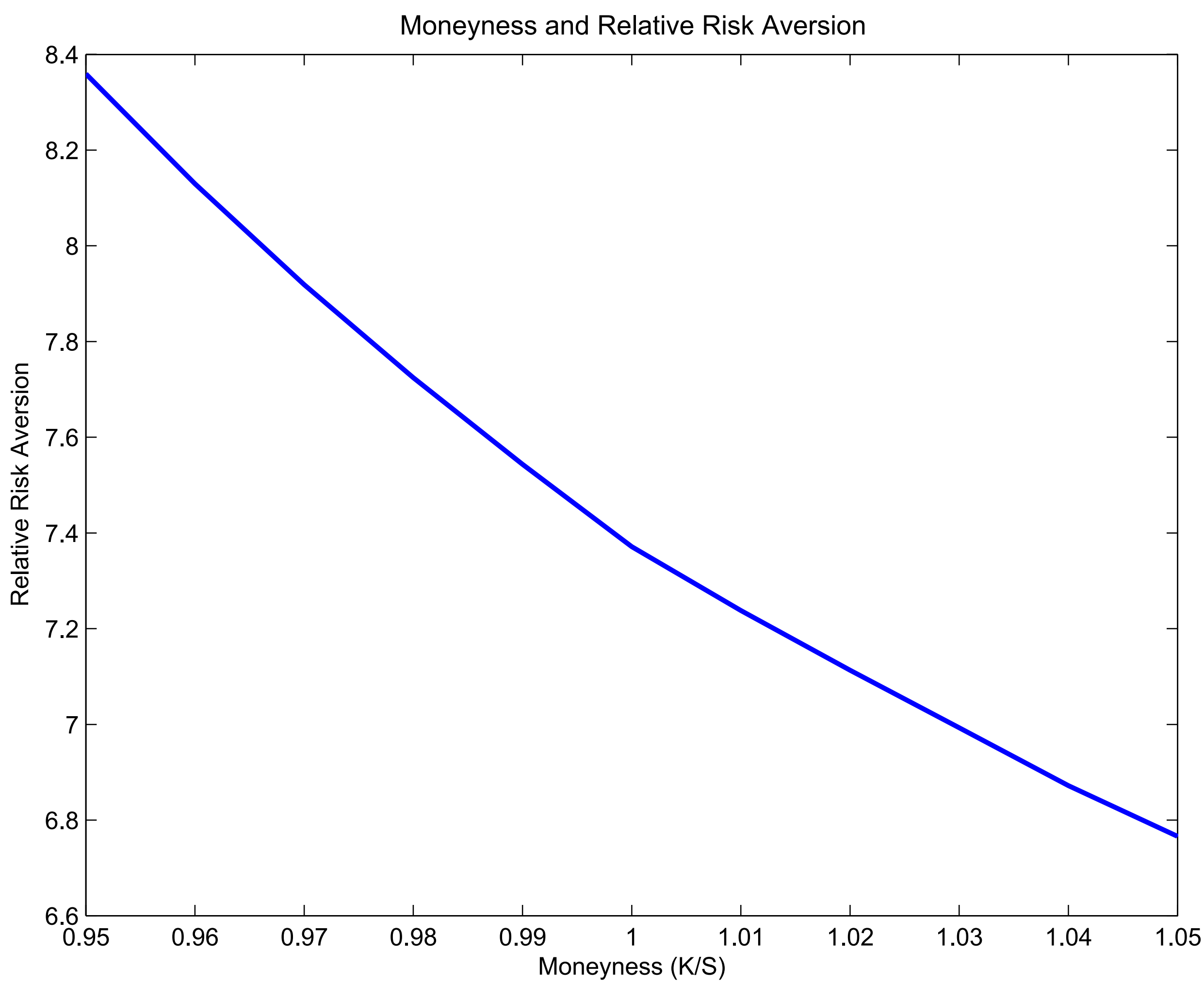
This figure illustrates the convergence of the option bounds under a jump-diffusion process for an ATM option with $X = 100$, time to maturity $T = 0.25$ years, and with the following annual parameters: $r = 2\%$, $\mu = 4\%$, $\sigma = 20\%$, $\lambda = 0.6$, $\mu_j = -0.05$, $\sigma_j = 7\%$. The jump size distribution is lognormal. In the case $j_{min} - 1 > -1$, the distribution was truncated to a worst case jump return of -20%. When $j_{min} - 1 = -1$, the return distribution has an absorbing state where the stock becomes worthless.

Figure 1.2: The Sensitivity of JD Call Option Prices to the Coefficient of RRA



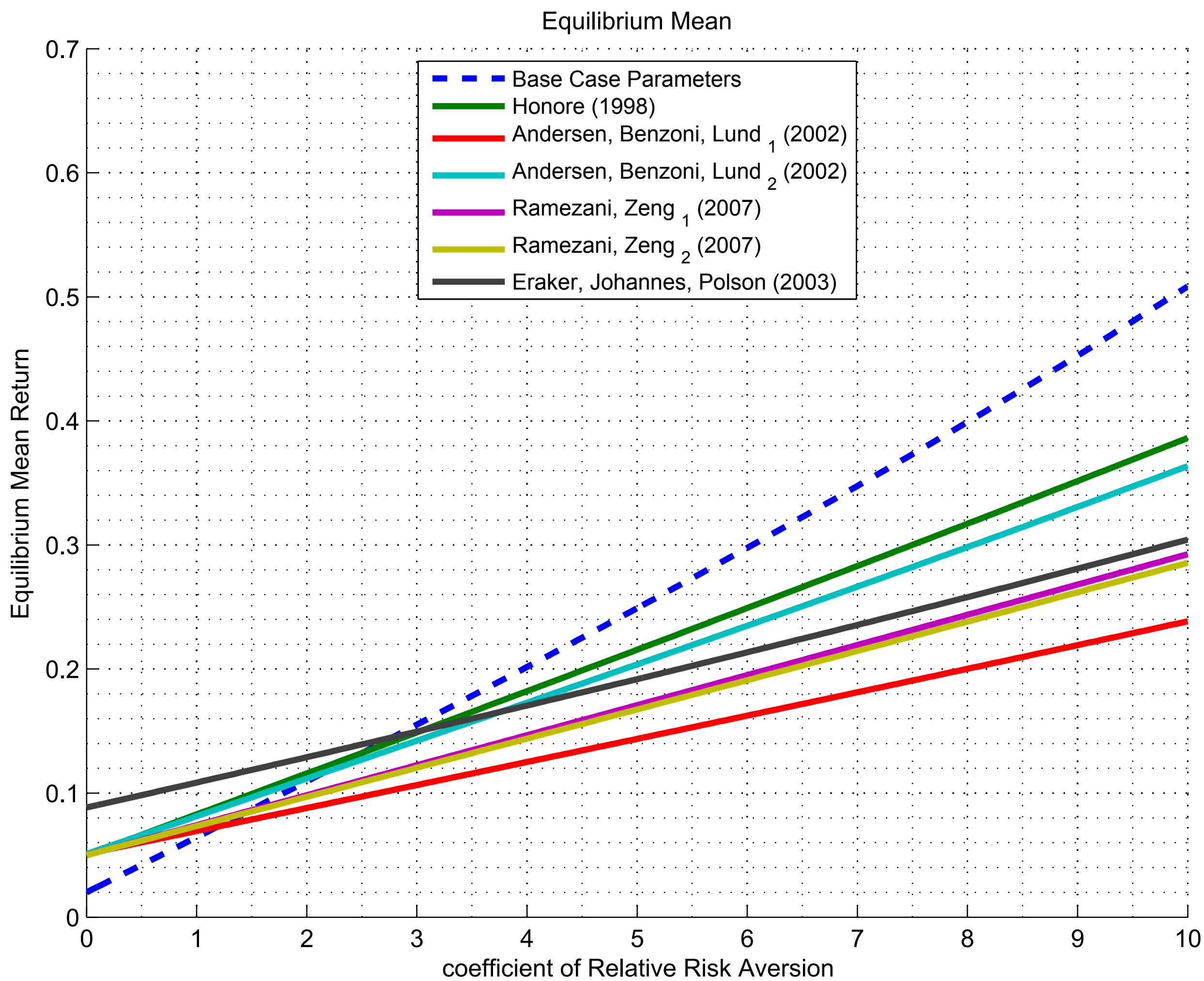
This figure shows the sensitivity of the equilibrium jump-diffusion call option prices to the coefficient of relative risk aversion for a continuum of coefficients up to 10. The parameters are $S = 100$, $X = 100$, $T = 0.25$, $r = 2\%$, $\mu = 4\%$, $\sigma = 20\%$, $\lambda = 0.6$, $\mu_j = -0.05$, $\sigma_j = 7\%$. The price of call option is based on the Bates [1991] jump-diffusion model. In case $j_{min} - 1 > -1$, the upper bounds distribution is truncated to a worst-case jump return of -20%. When $j_{min} - 1 = -1$, the lower limit of the jump amplitude is set to 0 and the return distribution has an absorbing state where the stock becomes worthless.

Figure 1.3: The SD-implied Upper Bound on the RRA Coefficient Versus Option Moneyness



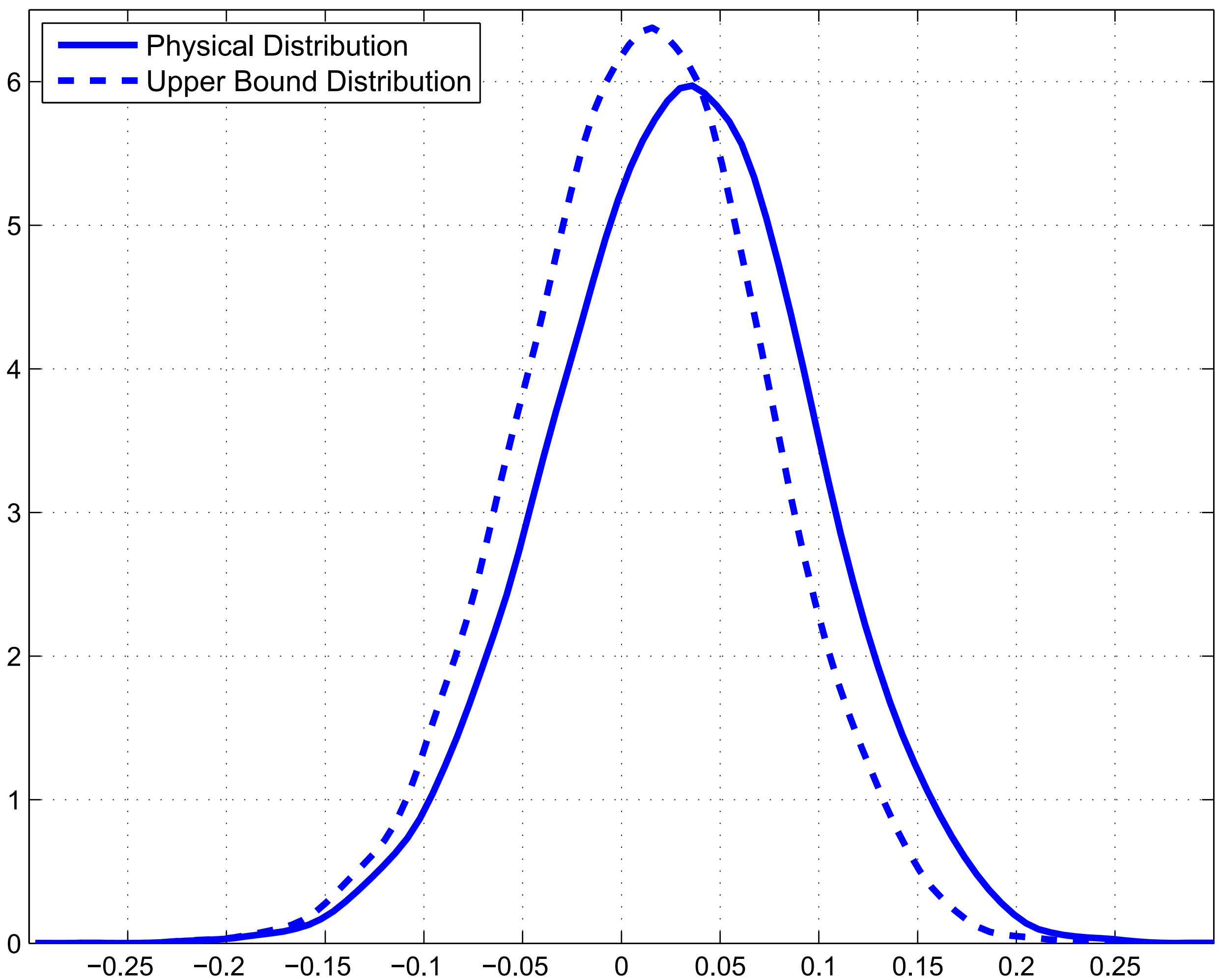
This figure describes the sensitivity of the SD-implied upper bound on the relative risk aversion to the moneyness. SD bound is defined based on the base case parameters $S = 100$, $T = 0.25$, $r = 2\%$, $\mu = 4\%$, $\sigma = 20\%$, $\lambda = 0.6$, $\mu_j = -0.05$, $\sigma_j = 7\%$ on the entire support of the jump distribution.

Figure 1.4: The Equilibrium Mean Return Versus the Coefficient of Relative Risk Aversion



This figure shows the sensitivity of the equilibrium mean of the jump-diffusion return process to the coefficient of relative risk aversion for a continuum of coefficients up to 10. The base case parameters are $r = 2\%$, $\mu = 4\%$, $\sigma = 20\%$, $\lambda = 0.6$, $\mu_j = -0.05$, $\sigma_j = 7\%$. This relation is based on the well-known equilibrium risk premium where $\mu_t - r = \varphi_\sigma + \varphi_j$ and $\varphi_\sigma = \gamma\sigma^2$ and $\varphi_j = \lambda k - \lambda^Q k^Q$. We draw the equilibrium mean return based on the parameters estimated from underlying S&P 500 index returns in Honore [1998], Andersen et al. [2002], Ramezani and Zeng [2007], and Eraker et al. [2003]. More details regarding the underlying parameters are given in Table (4.4).

Figure 1.5: The Bootstrapped Densities - SD Upper Bound Versus Physical Distributions



This figure shows the bootstrapped density for the physical distribution and upper bound SD distribution based on the parameters estimated from underlying S&P 500 index returns in Eraker et al. [2003].

Chapter 2

The Transient and The Persistent Variance Risk Premium

2.1 Introduction

The dynamics of index return volatility and their role in pricing options have had a long history following the classic early works by Wiggins [1987] and Heston [1993], that recognized the volatility's stochastic nature and managed to derive closed form expressions for the resulting European options. Related early contributions were also by Duan [1995], Duan et al. [1999], and Heston and Nandi [2000] under GARCH return dynamics, with option prices derived either by numerical methods or with closed form expressions. More recent studies, however, have pointed out that a single factor stochastic volatility (SV) or GARCH is not sufficient to represent both the underlying (P) and the risk neutral (Q) measures of the joint dynamics of returns and variances for the key S&P 500 index and its options.[1] In particular, these studies show that one-factor models are incapable of simultaneously fitting the persistence of volatility and the volatility of volatility, and that two volatility factors (one with persistent dynamics and one with transient dynamics) are needed to explain return volatility dynamics; similar considerations apply also to option-based risk neutral returns.

This paper examines index option pricing under two SV factors, where aggregate market volatility is decomposed into a more persistent volatility component, which has nearly a unit root, and a transitory volatility component, which has more rapid time decay. Building up on Christoffersen et al. [2009] model, we adopt an affine two-factor SV process for the underlying index returns and introduce an admissible pricing kernel to find the risk-neutral returns dynamics and to price European options.[2] As in the one-factor volatility of Christoffersen et al. [2013], we also introduce an associated component volatility model (bivariate GARCH model) and derive the corresponding pricing kernel linking the P- and Q-distributions.

We investigate empirically the pricing performance of our two-factor SV model in S&P 500 options by estimating the joint dynamics of returns and variances under the P and Q measures.[3] First, we filter two vectors of daily spot variances using the Particle Filter (PF)

[1] See, for instance, Bollerslev and Zhou [2002], Alizadeh et al. [2002], and Chernov et al. [2003] for the P-returns and Bates [2000], Christoffersen et al. [2008], and Christoffersen et al. [2009] for the option-based Q-distribution.

[2] Note that the extracted risk-neutral dynamics are not restricted to the introduced admissible pricing kernel, where investor's variance risk preference is distinguished from her equity risk preference. In other words, we can obtain the risk-neutral dynamics without completely characterizing the equilibrium in economy. To do so, we specify a class of Radon-Nikodym derivatives and derive restrictions that ensure the existence of equivalent martingale measure, which makes the discounted stock price process a martingale.

[3] Joint estimation appropriately weights returns and option data and simultaneously address the model's

method.[4,5] We follow the conventional filtration procedure of similar studies but provide a novel and methodologically important solution for the challenging issue of how to separate the two variance components' paths. We then use a likelihood-based loss function that combines a return-based and an option-based likelihood functions to obtain a consistent set of structural parameters for the two-factor SV model; a parameter set that simultaneously captures the information contents embedded in the time-series of index returns and cross-sections of options prices.[6,7] In other words, the resulting estimates are therefore consistent with the return data and option data. Further, joint estimation allow us to obtain two separate variance risk premiums; a transient variance risk premium and a persistent variance risk premium. To the best of our knowledge, this is the first study that estimates consistent P- and Q-parameters from underlying index return and option data and reports variance risk premium for a persistent and a transient component.

In empirical analysis, using the data from index and option market, we find that one of the volatility factors is highly persistent (persistent component) while the immediate impact of volatility shocks on the other volatility factor is bigger but short-lived (transient component). We also find the same level of persistence in the transient and persistent variance components when we only use option data, which is consistent with previous studies in option market. The unconditional transient and persistent variances are consistent with the average filtered spot transient and persistent variance components. Consistent with our intuition, we observe that the transient volatility component is much more volatile than the persistent volatility component. The same result holds when we use only option data.

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ability to fit the time-series of returns and cross-section of option prices. The importance of joint estimation of the structural parameters of the underlying returns and volatility dynamics has been addressed in Bates [1996], Chernov and Ghysels [2000], Pan [2002], Eraker [2004], and Broadie et al. [2007] among others.

[4]For the application of PF in estimating the model parameters see Gordon et al. [1993], Johannes et al. [2009], Johannes and Polson [2009], Christoffersen et al. [2010], and Boloorforoosh [2014].

[5]Note that unlike discrete-time GARCH model, filtering spot volatility in continuous-time stochastic volatility is a cumbersome task because volatility is a latent variable.

[6]According to Christoffersen et al. [2009, Section 6], "an integrated analysis of multifactor models using option data as well as underlying returns out to be done."

[7]The main challenge in such an efficient joint estimation procedure is its heavy computational burden. To overcome this challenge, previous studies mostly focused on a very short time-series and/or weekly/monthly option dataset, See Pan [2002] and Eraker [2004]. However, we managed to keep a large time-series of returns and the entire cross-section of daily option prices over the same time span using high performance computing techniques as well as parallel computing techniques.

We find negative prices for both variance components, namely $\lambda_1 = -1.0798$ and $\lambda_2 = -1.0355$. Our finding implies that investors are willing to pay for an insurance against an increase in volatility risk, even if that increase has little persistence. To the best of our knowledge none of the previous studies of two-factor stochastic volatility models reports the price of the variance risk factors as they either focused on the options market data or the underlying returns data. The negative variance risk premium for both transient and persistent variance components are consistent with the findings in Adrian and Rosenberg [2008]. Using a large cross-section of stock returns data, they find negative and significant prices for both short-run and long-run volatility components.[8]

We obtain negative correlation between shocks to the market returns and each variance component, implying that both components are important in capturing the so-called leverage effect. We find that the point estimate of the transient correlation parameter is less negative ($\rho_2 = -0.2173$) compared to the persistent correlation parameter ($\rho_1 = -0.6918$) and therefore, it has a less significant effect on the skewness and kurtosis of the return dynamics and thus on the volatility smirk. In other words, the persistent correlation factor has more significant effect on the return skewness and on the price of out-of-money put options. We observe the same pattern between correlation parameters when we estimate the model only with option data.

Extensive empirical evidence supports the presence of two volatility components in the dynamics of the market returns. In the P-distribution domain the relative performance of the two-factor SV structure compared to its one-factor counterpart in capturing the dynamics of the exchange rate and equity returns has been examined in several studies.[9,10] These studies document that two volatility factors, one with a persistent dynamic and one with a transient dynamic, are needed to characterize volatility dynamics, since one-factor models are incapable of simultaneously fitting the persistence of volatility and the volatility of volatility. For instance, Chernov et al. [2003] suggest that the addition of a second volatility factor breaks the link between tail thickness and volatility persistence. They show that the second SV factor in affine models leads to a significant improvement relative to a single SV models in capturing the return dynamics. They also find that when the second volatility factor

[8] Note that Adrian and Rosenberg [2008] introduce a discrete-time model where short-run and long-run volatility components are distinguished by construction whereas in our models we do not impose any restrictions on the variance dynamics other than variance shocks are independent.

[9] See, for instance, Bollerslev and Zhou [2002], Alizadeh et al. [2002], and Chernov et al. [2003].

[10] There is also widespread evidence that multifactor volatility model is needed to capture the term structure of the interest rate. See Dai and Singleton [2000, 2002] among others.

is allowed to have its own correlation with returns, the correlation parameters can take on both positive and negative values, contrary to the findings in single factor volatility models, where the correlation parameter is always found to be negative.

Similar considerations also hold for the Q-distribution. Previous studies in the option markets document that SV models are helpful in the modeling of the volatility smirk by incorporating a leverage effect[11] and in the modeling of the volatility term structure effect by incorporating mean reversion in variance dynamics.[12] Empirical studies observe that the shape of the volatility smirk can be either flat or steep at a given volatility level, however stochastic volatility models cannot accommodate both at the same time for a given parametrization.[13] This so called structural problem in one factor SV models is more restraining especially when estimating the model parameters using multiple cross-sections of options data. Such a restriction is mostly related to the fact that in one-factor SV models the correlation between stock returns and variance is constant across all cross-sections of option contracts regardless of the level and shape of the volatility. Multiple SV models, on the other hand, can better capture the time-varying nature of the smirk as the correlation between stock returns and total volatility is stochastic.[14] Such models, therefore, have more flexibility to fit the term structure of the volatility and to control the level and the slope of volatility smirk in cross-sections of option prices.[15] Moreover, the conditional skewness and kurtosis are more flexible for given levels of conditional variance. Our own empirical results confirm that these important characteristics lead to superior performance of multifactor SV model compared to its single factor counterpart.

Similar inconsistencies in the joint estimation of the SV model are illustrated by Broadie et al. [2007]. They note the failure of SV model to reconcile the P- and Q-estimates of certain structural parameters of the SV model, namely the correlation coefficient and volatility of volatility, and conclude that the SV model is basically misspecified. They also show that the joint restrictions on the returns and volatility dynamics under the P and Q measures

[11] See, among others, Bakshi et al. [1997], Bates [2000], and Jones [2003].

[12] Egloff et al. [2010, Page 1289] show that the upward sloping autocorrelation term structure of variance swap rate quotes points to the existence of multiple variance risk factors. They also find that upward sloping mean term structure of variance swap rate quotes is evidence for non-zero market prices for variance risk factors.

[13] See Derman [1999].

[14] Christoffersen et al. [2009, Equation 15] show that the correlation between returns and total volatility in a two-factor SV model is stochastic.

[15] See, for instance, Egloff et al. [2010] and Mencía and Sentana [2013].

leads to the poor performance of the SV model, measured by the high level of IVRMSE. They indicate that due to the joint-restriction, SV model cannot generate sufficient amounts of conditional skewness and kurtosis. However, in our empirical analysis, we find that joint restrictions on the P and Q dynamics does not lead to the poor performance of our two-factor SV model.

Although our study is not the first one to examine multifactor SV and GARCH models, it is the only one to present consistent P- and Q-parameter estimates both theoretically and empirically. For instance Bates [2000] examined a multifactor specification in option pricing by relying on the Q-distribution only. Christoffersen et al. [2008] introduced a two-component GARCH model, which can generate more flexible skewness and volatility of volatility dynamics in capturing the dynamics of the S&P 500 index returns and in pricing European S&P 500 call options. They document that the empirical performance of the volatility component model is significantly better than that of the benchmark GARCH(1,1) model, both in-sample and out-of-sample. They also find that the proposed volatility component specification could better capture the volatility term structure. Nonetheless, the absence of an explicit pricing kernel linking the P- and Q-distributions in that study necessitated either the use of an arbitrary price of volatility risk or the estimation of the risk neutral parameters by relying on the Q-distribution only. Christoffersen et al. [2009] further explore multiple variance factors model Q-distribution only and find that it can generate stochastic correlation between total instantaneous volatility and stock returns. They also illustrate the importance of multiple variance factors by analyzing the principal components of Black-Scholes implied volatility of of S&P 500 index options.

This chapter proceeds as follows. Section (2.2) presents the theoretical model for pricing index options. Section (2.3) contains the description of the data sets. In Section (2.4) we discuss the methodology for estimation of the structural parameters that characterize the dynamics of index return and variance components under both P and Q distributions. Section (2.5) presents the estimation results. In section (2.6) we investigate the performance of the model and report in-sample goodness-of-fit statistics. Section (2.7) examines the stability of the model and measures the out-of-sample performance of the model. Section (2.8) concludes. The appendix provides the proofs of the theoretical results.

2.2 Model Setup

We start by a multiple-factor stochastic volatility dynamics that governs the market index returns under the P-distributions and then introduce a pricing kernel that links the P-dynamics to their risk-neutral counterparts by imposing appropriate martingale's restrictions on pricing kernel. We complete the the index model by deriving a closed-form pricing equation for index options. We then introduce a GARCH model under physical distribution which is similar to our multiple-factor stochastic volatility model with two independent volatility dynamics. The risk neutral GARCH dynamics is also defined using a discrete-time analog of our continuous-time pricing kernel.

2.2.1 The Multifactor Stochastic Volatility Model

We assume the following two-factor stochastic volatility process governing the dynamics of the market index returns and variance under the physical distributions.

$$\begin{aligned}
dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t} \\
dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t}
\end{aligned} \tag{2.2.1}$$

where, as in Christoffersen et al. [2009] we assume the stochastic structure (2.2.2).

$$\begin{aligned}
&dw_{1,t} \cdot dz_{1,t} = \rho_1 dt, \ -1 \leq \rho_1 \leq +1 \\
&dw_{2,t} \cdot dz_{2,t} = \rho_2 dt, \ -1 \leq \rho_2 \leq +1 \\
&dw_{1,t} \cdot dw_{2,t} = 0 \\
&\rho_1^2 + \rho_2^2 \leq +1
\end{aligned} \tag{2.2.2}$$

As in Heston [1993], θ_1 and θ_2 are unconditional average variance components, κ_1 and κ_2 capture the speed of mean reversion in each variance components, and σ_1 and σ_2 measure the volatility of variance components. The market equity risk premiums are denoted by $\mu_1 v_{1,t}$ and $\mu_2 v_{2,t}$. Following Bollerslev and Zhou [2006] we expect that μ_1 and μ_2 measure the persistent and transient "continuous-time" volatility feedback effects or risk-return trade-offs. The instantaneous correlation between shocks to the market returns and shocks to the persistent variance component is measured by ρ_1 and the instantaneous correlations between market returns and the transient variance component is given by ρ_2. As in Bollerslev and Zhou [2006], we expect that ρ_1 and ρ_2 account for persistent and transient "continuous-time" leverage (asymmetry) effect.

Note that (2.2.2) implies that the total return variance and the correlation between return and total variance are as follows.

$$\begin{aligned} \mathrm{Var}_t[dS_t/S_t] &= v_{1,t}dt + v_{2,t}dt = v_t dt \\ \mathrm{Corr}_t[dS_t/S_t, dV_t] &= \frac{\rho_1\sigma_1 v_{1,t} + \rho_2\sigma_2 v_{2,t}}{\sqrt{\sigma_1^2 v_{1,t} + \sigma_2^2 v_{2,t}}\,\sqrt{v_{1,t}+v_{2,t}}}dt \end{aligned} \tag{2.2.3}$$

We may then prove the following result.

Proposition 2.1. *The market index has the following dynamics under the risk-neutral measure:*

$$\begin{aligned} dS_t/S_t &= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \ , \\ dv_{1,t} &= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t} \ , \\ dv_{2,t} &= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t} \ , \end{aligned} \tag{2.2.4}$$

where, $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$. *The market price of risk factors are*

$$\begin{aligned} \psi_{1,t} &= \frac{\sigma_1\mu_1 - \rho_1\lambda_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}} \ , \ \psi_{2,t} = \frac{\sigma_2\mu_2 - \rho_2\lambda_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}} \ , \\ \psi_{3,t} &= \frac{\lambda_1 - \rho_1\sigma_1\mu_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}} \ , \ \psi_{4,t} = \frac{\lambda_2 - \rho_2\sigma_2\mu_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}} \ . \end{aligned} \tag{2.2.5}$$

One admissible pricing kernel that links the physical dynamics in (2.2.1) to the risk-neutral dynamics in (2.2.4) takes the following exponential affine form.

$$\frac{M_t}{M_0} = \left(\frac{S_t}{S_0}\right)^{\phi} \exp\left[\delta t + \eta_1\int_0^t v_{1,s}ds + \eta_2\int_0^t v_{2,s}ds + \zeta_1(v_{1,t}-v_{1,0}) + \zeta_2(v_{2,t}-v_{2,0})\right] \tag{2.2.6}$$

As in Christoffersen et al. [2013], $\{\delta,\ \eta_1,\ \eta_2\}$ governs the time-preferences, while $\{\phi,\ \zeta_1,\ \zeta_2\}$ governs the respected risk aversion to the index and variance risk factors, all of which are defined in the appendix.

Proof. See Appendix (2.A). □

We note that the introduced nonlinear log pricing kernel in (2.2.6) is one way of "completing the market" and linking P- to Q- dynamics, where ζ_1, ζ_2 capture the nonlinearity of the log pricing kernel.[16] Transforming the physical dynamics in (2.2.1) into the risk neutral dynamics in (2.2.4) can also be done by assuming the following standard stochastic discount factor and without explicit assumptions about the investor's variance preferences. The proof of such a transformation can be found in Appendix (2.B).

$$\frac{dM_t}{M_t} = -rdt - \psi_t' dW_t \ , \tag{2.2.7}$$

where $\psi_t \equiv [\psi_{1,t}, \psi_{2,t}, \psi_{3,t}, \psi_{4,t}]$ is the vector of market price of risk factors and $W_t \equiv [z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}]$ is the vector of innovations in return and variance.

To embed the options market data into the estimation of structural parameters, we determine a closed-from expression for the price of the European call options, with strike price K and time to maturity τ, by inverting the conditional characteristic function of the log spot index prices, $x_t = \ln(S_t)$.

$$C_t(S_t, K, v_{1,t}, v_{2,t}, \tau) = S_t P_1 - Ke^{-r\tau} P_2 \ , \tag{2.2.8}$$

where,

$$\begin{aligned} P_1 &= \frac{1}{2} + \frac{1}{\pi}\frac{1}{S_t e^{r\tau}} \int_0^\infty \Re\Big[\frac{e^{-i\phi \ln K} \tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi - i)}{i\phi}\Big] d\phi \ , \\ P_2 &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\Big[\frac{e^{-i\phi \ln K} \tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi)}{i\phi}\Big] d\phi \ , \end{aligned} \tag{2.2.9}$$

[16] Note also that ζ_1, ζ_2 affect a wedge between physical and risk neutral structural parameters of volatility dynamics.

and where the risk-neutral conditional characteristic function of the natural logarithm of the index price at expiration, $x_{t+\tau}$, is

$$\tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi) \equiv \mathrm{E}_t^Q \left[\exp(i\phi x_{t+\tau}) \mid x_t\right] . \tag{2.2.10}$$

Since the two-factor SV model in (2.2.4) is an affine process, following Heston [1993], the conditional risk-neutral characteristic function in (2.2.10) has the following affine exponential form.[17]

$$\tilde{f}(v_{1,t}, v_{2,t}, \tau, \phi) = \exp\left[i\phi x_t + i\phi r\tau + A_1(\tau,\phi) + A_2(\tau,\phi) + B_1(\tau,\phi)v_{1,t} + B_2(\tau,\phi)v_{2,t}\right] , \tag{2.2.11}$$

where[18] for every $j = \{1, 2\}$

$$\begin{aligned}
A_j(\tau,\phi) &= \frac{\tilde{\kappa}_j\tilde{\theta}_j}{\sigma_j^2}\Big[(\tilde{\kappa}_j - \rho_j\sigma_j i\phi - d_j)\tau - 2\ln\big[\frac{1 - c_j e^{-d_j\tau}}{1 - c_j}\big]\Big] \\
B_j &= \frac{\tilde{\kappa}_j - \rho_j\sigma_j i\phi - d_j}{\sigma_j^2}\big[\frac{1 - e^{-d_j\tau}}{1 - c_j e^{-d_j\tau}}\big] \\
c_j &= \frac{\tilde{\kappa}_j - \rho_j\sigma_j i\phi - d_j}{\tilde{\kappa}_j - \rho_j\sigma_j i\phi + d_j} \\
d_j &= \sqrt{(\tilde{\kappa}_j - \rho_j\sigma_j i\phi)^2 + \sigma_j^2\phi(\phi + i)} \, .
\end{aligned} \tag{2.2.12}$$

2.2.2 The Component Volatility Model (Bivariate GARCH)

Since the seminal papers of Engle [1982] and Bollerslev [1986] several ARCH-type models have been proposed where the main difference is in parametrization of the conditional

[17] Note that the conditional risk-neutral characteristic function of the natural logarithm of return, $x_{t+\tau} - x_t = \ln(S_{t+\tau}/S_t)$, can be defined with the same expression as (2.2.11) but without the first component, $i\phi x_t$.

[18] Following Duffie et al. [2000], the coefficients A_1, A_2, B_1, and B_2 are the solutions of a system of Riccati equations subject to appropriate boundary conditions. For the ease of computation we modify these solutions based on the little Heston trap formulation of Albrecher et al. [2006].

variance and asymmetry effect. Extensive empirical evidence examine the importance of conditional heteroskedasticity and variance mean reversion in modeling index returns and index options.

Note that in ARCH-type models volatility is considered as a deterministic process, whereas in case of SV models volatility has a fully stochastic nature.

Engle and Lee [1999] introduce a component extension to the simple GARCH(1,1) model where the unconditional mean of the conditional variance process is time-varying and provide empirical evidence that the component model provides a very good fit to return data. Christoffersen et al. [2008] consider an affine version of component volatility model of Engle and Lee [1999] by generalizing the affine Gaussian GARCH(1,1) Heston and Nandi [2000] as follows.

$$\begin{aligned}
R_t &\equiv \ln\left(\frac{S_t}{S_{t-1}}\right) = r + \left(\mu - \frac{1}{2}\right)h_t + \sqrt{h_t}z_t \\
h_t &= q_t + \beta_h(h_{t-1} - q_{t-1}) + \alpha_h\big((z_{t-1} - \gamma_h\sqrt{h_{t-1}})^2 - (1 + \gamma_h^2 q_{t-1})\big) \\
q_t &= w_q + \beta_q q_{t-1} + \alpha_q\big((z_{t-1} - 1)^2 - 2\gamma_q\sqrt{h_{t-1}}z_{t-1})^2\big) \ ,
\end{aligned} \tag{2.2.13}$$

where h_t is referred to as the total conditional variance, q_t as the long-run component of conditional variance, and therefore $h_t - q_t$ as the short-run component conditional variance with zero unconditional mean. This volatility component model is relatively simple since both of the volatility components, h_t and q_t, are characterized by nonlinear functions of a single innovation z_{t-1}. A richer model of return volatility includes multiple innovations.[19]

We introduce a component volatility model (bivariate GARCH model) which is similar to our two-factor stochastic volatility model in the sense that volatility components are independent. We extend the Heston and Nandi [2000] affine Gaussian GARCH(1,1) model that yields a closed-form option valuation formula similar to our SV model. Note that several studies investigate the limits of GARCH models as the time intervals become small and find that for a given GARCH model, there could be a several continuous-time limits and several GARCH models could converge to a continuous-time stochastic volatility model.[20] A discrete time analog of our SV model under the physical measure can be defined as follows.

[19]See for instance Feunou and Tédongap [2012], Christoffersen et al. [2010], and Khrapov and Renault [2016].

[20]See Corradi [2000].

$$
\begin{aligned}
R_t \equiv \ln(\frac{S_t}{S_{t-1}}) &= r + (\mu_1 - \frac{1}{2})h_{1,t} + (\mu_2 - \frac{1}{2})h_{2,t} + \varepsilon_{1,t} + \varepsilon_{2,t} \\
h_{1,t} &= w_1 + \beta_1 h_{1,t-1} + \alpha_1 (z_{1,t-1} - \gamma_1 \sqrt{h_{1,t-1}})^2 \\
h_{2,t} &= w_2 + \beta_2 h_{2,t-1} + \alpha_2 (z_{2,t-1} - \gamma_2 \sqrt{h_{2,t-1}})^2
\end{aligned}
\tag{2.2.14}
$$

where r is the daily continuously compounded interest rate, $\varepsilon_{1,t} = \sqrt{h_{1,t}} z_{1,t}$, $\varepsilon_{2,t} = \sqrt{h_{2,t}} z_{2,t}$, and $z_{1,t}$ and $z_{2,t}$ are standard normal distributions. $h_{1,t}+h_{2,t}$ is the conditional variance of the log return in period t. The autoregressive parameters β_1 and β_2 determine the persistence of the each variance component and the innovation parameters α_1 and α_2 determine the variance of variance and thus kurtosis in each variance component. γ_1 and γ_2 capture the so-called leverage effect, asymmetry in the response of each volatility component to positive versus negative return shocks. Note that in our specification, the conditional mean return is

$$
\mathrm{E}_{t-1}[S_t/S_{t-1}] = \mathrm{E}_{t-1}[\exp(R_t)] = \exp(r + \mu_1 h_{1,t} + \mu_2 h_{2,t}) \ . \tag{2.2.15}
$$

The expected future variance is a linear function of current variance and long-run average (unconditional) variance.

$$
\begin{aligned}
\mathrm{E}_{t-1}[h_{t+1}] &= \mathrm{E}_{t-1}[h_{1,t+1} + h_{2,t+1}] \\
&= (\beta_1 + \alpha_1\gamma_1^2)h_{1,t} + (1 - \beta_1 - \alpha_1\gamma_1^2)\,\mathrm{E}[h_{1,t}] \\
&+ (\beta_2 + \alpha_2\gamma_2^2)h_{2,t} + (1 - \beta_2 - \alpha_2\gamma_2^2)\,\mathrm{E}[h_{2,t}]
\end{aligned}
\tag{2.2.16}
$$

where $\mathrm{E}[h_{1,t}] \equiv \sigma_1^2 = (w_1 + \alpha_1)/(1 - \beta_1 - \alpha_1\gamma_1^2)$ and $\mathrm{E}[h_{2,t}] \equiv \sigma_2^2 = (w_2 + \alpha_2)/(1 - \beta_2 - \alpha_2\gamma_2^2)$ are long-run average (unconditional) component variance. We refer to $(\beta_1 + \alpha_1\gamma_1^2)$ and $(\beta_2 + \alpha_2\gamma_2^2)$ as the persistence of the variance component. A high level of persistence (close to one) implies that shocks that push variance away from its long-run average will persist for a long time. The conditional variance of h_{t+1} is also linear in past variance.

$$
\mathrm{Var}_{t-1}[h_{t+1}] = \mathrm{Var}_{t-1}[h_{1,t+1} + h_{2,t+1}] = 2\alpha_1^2 + 4\alpha_1^2\gamma_1^2 h_{1,t} + 2\alpha_2^2 + 4\alpha_2^2\gamma_2^2 h_{2,t} \tag{2.2.17}
$$

The conditional covariance between stock returns and variance is

$$\mathrm{Cov}_{t-1}(R_t, h_{t+1}) = \mathrm{Cov}_{t-1}(R_t, h_{1,t+1} + h_{2,t+1}) = -2\alpha_1\gamma_1 h_{1,t} - 2\alpha_2\gamma_2 h_{2,t}\ . \tag{2.2.18}$$

We transform the physical stock price process (2.2.14) to the corresponding risk neutral process using a discrete-time analog of the continuous-time pricing kernel (2.2.6).

$$\frac{M_t}{M_0} = (\frac{S_t}{S_0})^{\phi} \exp\left[\delta t + \eta_1 \sum_{s=1}^{t} h_{1,s} + \eta_2 \sum_{s=1}^{t} h_{2,s} + \zeta_1(h_{1,t+1} - h_{1,1}) + \zeta_2(h_{2,t+1} - h_{2,1})\right] \tag{2.2.19}$$

where parameters $\{\delta,\ \eta_1,\ \eta_2\}$ govern the time-preference, and parameters $\{\phi,\ \zeta_1,\ \zeta_2\}$ govern the respected risk aversion to equity risk and to variance risk factors. Note that ζ_1 and ζ_2 capture the non-linearity of the log pricing kernel.

Proposition 2.2. *Given the physical GARCH process (2.2.14) and the pricing kernel (2.2.19), the risk neutral innovations may be characterized by the following transformations.*

$$\begin{aligned} z^*_{1,t} &= \sqrt{1-2\alpha_1\zeta_1}\big(z_{1,t} + (\mu_1 + \frac{\alpha_1\zeta_1}{1-2\alpha_1\zeta_1})\sqrt{h_{1,t}}\big) \\ z^*_{2,t} &= \sqrt{1-2\alpha_2\zeta_2}\big(z_{2,t} + (\mu_2 + \frac{\alpha_2\zeta_2}{1-2\alpha_2\zeta_2})\sqrt{h_{2,t}}\big) \end{aligned} \tag{2.2.20}$$

Hence, the corresponding risk-neutral GARCH process may be characterized as follows

$$\begin{aligned} R_t \equiv \ln(\frac{S_t}{S_{t-1}}) &= r - \frac{1}{2}h^*_{1,t} - \frac{1}{2}h^*_{2,t} + \sqrt{h^*_{1,t}}z^*_{1,t} + \sqrt{h^*_{2,t}}z^*_{2,t} \\ h^*_{1,t} &= w^*_1 + \beta_1 h^*_{1,t-1} + \alpha^*_1(z^*_{1,t-1} - \gamma^*_1\sqrt{h^*_{1,t-1}})^2 \\ h^*_{2,t} &= w^*_2 + \beta_2 h^*_{2,t-1} + \alpha^*_2(z^*_{2,t-1} - \gamma^*_2\sqrt{h^*_{2,t-1}})^2 \end{aligned} \tag{2.2.21}$$

where conditional variance under physical and risk-neutral distributions are linked as

$$h^*_{1,t} = \frac{h_{1,t}}{1-2\alpha_1\zeta_1}\ ,\ h^*_{2,t} = \frac{h_{2,t}}{1-2\alpha_2\zeta_2} \tag{2.2.22}$$

and for every $j = \{1, 2\}$ the parameters mapping may be given by

$$\begin{aligned} \alpha_j^* &= \frac{\alpha_j}{(1-2\alpha_j\zeta_j)^2} \\ w_j^* &= \frac{w_j}{1-2\alpha_j\zeta_j} \\ \gamma_j^* &= (\mu_j - \frac{1}{2} + \gamma_j)(1-2\alpha_j\zeta_j) + \frac{1}{2} \end{aligned} \tag{2.2.23}$$

Proof. The proof of this proposition is very similar to its continuous-time counterpart. We show that the GARCH model under physical measure (2.2.14) is linked to the GARCH model under risk-neutral measure (2.2.21) with the proposed pricing kernel (2.2.19) by specifying a set of sufficient conditions (2.2.20), (2.2.22), and (2.2.23). We first impose Euler equation for the risk-free asset and subsequently impose Euler equation for the underlying asset to find this parameters mapping. See Appendix (2.C). □

Note that linking P- to Q- dynamics can also be done through a log-linear pricing kernel. But, log-linear pricing kernel within GARCH models does not incorporate directly the effect of variance premium on risk neutralization. However, variance dependent pricing kernel allows to directly incorporate the effect of variance premium as $-2\alpha\zeta$ in risk neutralization. A negative variance premium yields higher level of risk-neutral variances compared to the physical variances as $h^*_{1,t}$ exceeds $h_{1,t}$ and $h^*_{2,t}$ exceeds $h_{2,t}$. Negative variance premium also yields higher level of risk neutral innovation parameters α^*_1 and α^*_2 and hence increases the risk neutral variance persistence, $(\beta_1 + \alpha^*_1\gamma^{*2}_1)$ and $(\beta_2 + \alpha^*_2\gamma^{*2}_2)$.

2.3 Data

We obtain daily prices of S&P 500 index options from the OptionMetrics volatility surface data set, which is based on the midpoint of bid-ask quotes. Our sample of S&P 500 index options is from January 4, 1996 through December 29, 2011. We follow the data cleaning routine commonly used in the empirical option pricing literature: we remove options with implied volatility less than 5% and greater than 150%; we also follow the filtering rules in Bakshi et al. [1997] to remove options that violate various no-arbitrage conditions. We focus on out-of-the-money (OTM) options with maturity up to and including one-year and with

10% moneyness (spot price over strike price).[21,22] Our option-based optimization function minimizes the squared deviations between model and market option prices and therefore may put greater weight on expensive in-the-money (ITM) and long-maturity options.[23] Moreover, ITM S&P 500 call options are less liquid than OTM call options. To prevent such biases in our optimization, we discard all ITM options and use OTM S&P 500 put options and convert them into ITM call options. After cleaning, we have 345,710 S&P 500 index option quotes together with daily underlying returns. This is the dataset that we use to filter daily spot variances and to estimate a set of structural parameters.

Table (2.1) presents the descriptive statistics of the call option contracts in our sample sorted by moneyness (stock price over strike price) and day-to-maturity (DTM). Note that we focus on OTM option contracts, which means S/K is below 1 for OTM call contracts. After cleaning, we have 208,098 out-of-the-money call option contracts with an average day-to-maturity of 143 days, an average price of \$35.59, an average implied volatility of 20.64%, and an average delta of 0.37. Table (2.2) reports the descriptive statistics of the put option contracts in our sample sorted by moneyness and day-to-maturity. After cleaning, we use 137,612 out-of-the-money (S/K is above 1) put option contracts with an average day-to-maturity of 136 days, an average price of \$32.11, an average implied volatility of 24.34%, and an average delta of -0.29. Note that Panel C in Tables (2.1) and (2.2) reflect the well-known volatility smirk in index options, as implied volatility is larger for OTM put options (Table (2.2), Panel C) compared to the OTM call options (Table (2.1), Panel C).

[Table (2.1) about here]

[Table (2.2) about here]

---

[21] This range of moneyness implies that we keep OTM call options with moneyness less than 1.1 and OTM put options with moneyness greater than 0.9.

[22] As discussed in previous section, multiple-factor SV models could better capture the slope and the level of smirk compare to single-factor SV models. Therefore, unlike similar analysis, we undertake a more extensive calibration exercise by incorporating the information content of options on longer maturity horizons and wider moneyness ranges. For instance, Ait-Sahalia and Kimmel [2007, Section 7] only include short-maturity at-the-money S&P 500 Index Options; Eraker [2004] use 3,270 call options contracts recorded over 1,006 trading days; Jones [2003] models are estimated using a sample of 3537 S&P 100 index options from January 1986 to June 2000.

[23] See Huang and Wu [2004].

The data for daily index level, index return, and the dividend yields are from CRSP. In our analysis we first adjust daily index level with dividend yields and then compute the option prices using the dividends adjusted returns. Risk-free interest rates for all maturities are estimated by linear interpolation between the closest zero-coupon rates using the Zero Coupon Yield Curve data from OptionMetrics.

2.4 Estimation Methodology

To estimate the parameters of two-factor stochastic volatility model of the index we follow the literature on the estimation of stochastic volatility models, where the main challenge is the estimation of unobserved latent volatilities. There are several approaches to estimate stochastic volatility model. Our own approach combines the information from underlying index and option markets to impose consistency between structural parameters under P and Q distributions, known as joint estimation. Therefore, we use a likelihood function that contains a return-based component and an option-based component, as in Santa-Clara and Yan [2010] and Christoffersen et al. [2013].[24] Here we do a joint-estimation by filtering the two vectors of daily spot variances, $\{v_{1,t}, v_{2,t}\}$, and simultaneously estimating a set of structural parameters of the dynamics of index returns and variances, including the market price of each variance component, $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_2, \rho_1, \rho_2, \lambda_1, \lambda_2\}$. Note that joint estimation allow us to have reliable prices of variance risk factors, as we can get a consistent set of structural parameters between the P and Q distributions.

Since the market variances are unobserved state variables, we first extract daily instantaneous persistent and transient variance components using the Particle Filter (PF) method. This optimal filtering methodology provides a tool for learning about unobserved shocks and states from discretely observed prices generated by continuous-time models.[25] Although we generally follow the conventional filtration procedure in the literature, we provide a novel approach to the challenge of filtering the two separate variance paths. Our proposed solution is not trivial and to the best of our knowledge is novel and constitutes a methodological

[24] Consistency can also be imposed through moment-based and simulation-based methods; see Ait-Sahalia and Kimmel [2007], Eraker [2004], Jones [2003], Chernov and Ghysels [2000], and Pan [2002]. Other approaches use only option-based data to estimate only the Q distribution; Bakshi et al. [1997], Bates [2000], Huang and Wu [2004], and Christoffersen et al. [2009].

[25] For the application of PF in estimating the model parameters see Gordon et al. [1993], Johannes et al. [2009], Johannes and Polson [2009], Christoffersen et al. [2010], and Boloorforoosh [2014].

contribution to the option pricing literature.

2.4.1 The Return Based Likelihood Function

To define the return-based likelihood function and filter spot variances, we start by discretizing the returns dynamics (2.2.1). Applying Ito's lemma to equation (2.2.1), gives the dynamics of logarithm of stock prices as follows.

$$\begin{aligned} d\ln(S_t) &= (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \ , \\ dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t} \ , \\ dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t} \ , \end{aligned} \tag{2.4.1}$$

where, $\mu \equiv r + \mu_1 v_{1,t} + \mu_2 v_{2,t}$. We discretize (2.4.1) using the Euler scheme.[26] Equation (2.4.2) models the relation between observed index prices and unobserved variances at time $t + \Delta t$ conditional on the time t variances.

$$\begin{aligned} \ln(S_{t+\Delta t}) - \ln(S_t) &= (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))\Delta t + \sqrt{v_{1,t}\Delta t}\ z_{1,t+\Delta t} + \sqrt{v_{2,t}\Delta t}\ z_{2,t+\Delta t} \ , \\ v_{1,t+\Delta t} &= v_{1,t} + \kappa_1(\theta_1 - v_{1,t})\Delta t + \sigma_1\sqrt{v_{1,t}\Delta t}\ w_{1,t+\Delta t} \ , \\ v_{2,t+\Delta t} &= v_{2,t} + \kappa_2(\theta_2 - v_{2,t})\Delta t + \sigma_2\sqrt{v_{2,t}\Delta t}\ w_{2,t+\Delta t} \ . \end{aligned} \tag{2.4.2}$$

Brownian shocks $z_{1,t+\Delta t}$, $z_{2,t+\Delta t}$, $w_{1,t+\Delta t}$, and $w_{2,t+\Delta t}$ are normal random variables with mean zero and variance one. From the first equation in (2.4.2) we use the observed daily index log-prices ($\ln(S_t)$, $\ln(S_{t+\Delta t})$) to first filter the daily return's shocks ($z_{1,t+\Delta t}$, $z_{2,t+\Delta t}$) and then, using the filtered shocks in returns and the last two equation in (2.4.2), we filter daily spot variances ($v_{1,t+\Delta t}$, $v_{2,t+\Delta t}$). Note that we filter filter the summation of return shocks $z_{1,t+\Delta t} + z_{2,t+\Delta t}$ as we cannot separate the daily observed shocks into two components, $z_{1,t+\Delta t}$ and $z_{2,t+\Delta t}$. Therefore, we rewrite the underlying dynamics as (2.4.3), given that the return shocks are uncorrelated and then discretize this dynamics.

[26] According to Eraker [2004] and Li et al. [2008] the discretization bias of the Euler scheme is negligible for daily data.

$$
\begin{aligned}
&d\ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))dt + \sqrt{v_{1,t} + v_{2,t}}dz_t \ , \\
&dv_{1,t} = \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t} \ , \\
&dv_{2,t} = \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t} \ ,
\end{aligned}
\tag{2.4.3}
$$

with the correlation structure:

$$
\begin{aligned}
&dw_{1,t} \cdot dz_t = \rho_1 dt, \ -1 \leq \rho_1 \leq +1 \ , \\
&dw_{2,t} \cdot dz_t = \rho_2 dt, \ -1 \leq \rho_2 \leq +1 \ , \\
&dw_{1,t} \cdot dw_{2,t} = 0 \ .
\end{aligned}
\tag{2.4.4}
$$

We decompose the variance shocks into orthogonal components as in (2.4.5) and then discretize the return dynamics (2.4.3) using the Euler scheme and shock's decomposition (2.4.5).[27]

$$
\begin{aligned}
&dw_{1,t} = \rho_1 dz_t + \sqrt{1 - \rho_1^2}\ dB_{1,t} \\
&dw_{2,t} = \rho_2 dz_t - \frac{\rho_1 \rho_2}{\sqrt{1 - \rho_1^2}}\ dB_{1,t} + \sqrt{\frac{1 - \rho_1^2 - \rho_2^2}{1 - \rho_1^2}}\ dB_{2,t} \\
&\langle dB_{1,t}\ , dB_{2,t} \rangle = 0
\end{aligned}
\tag{2.4.5}
$$

$$
\begin{aligned}
&\ln(S_{t+\Delta t}) - \ln(S_t) = (\mu - \frac{1}{2}(v_{1,t} + v_{2,t}))\Delta t + \sqrt{(v_{1,t} + v_{2,t})\Delta t}\ z_{t+\Delta t} \ , \\
&v_{1,t+\Delta t} = v_{1,t} + \kappa_1(\theta_1 - v_{1,t})\Delta t + \sigma_1\sqrt{v_{1,t}\Delta t}\ w_{1,t+\Delta t} \ , \\
&v_{2,t+\Delta t} = v_{1,t} + \kappa_2(\theta_2 - v_{2,t})\Delta t + \sigma_2\sqrt{v_{2,t}\Delta t}\ w_{2,t+\Delta t} \ ,
\end{aligned}
\tag{2.4.6}
$$

where, $z_{t+\Delta t}$, $w_{1,t+\Delta t}$, and $w_{2,t+\Delta t}$ are all $N(0,1)$. Now, using daily index log-returns, we proceed to filter the spot variances from the discretized model in (2.4.6) given the correlation structure in (2.4.5).

We follow Pitt [2002][28] and adopt a particular implementation of the PF, which is referred

[27] Note that the quadratic variations of the transformed using the proposed shocks decomposition (2.4.5) should remain the same as $\sqrt{dt}$.

[28] See Pitt [2002], Christoffersen et al. [2010], and Boloorforoosh [2014] for a detailed description of the PF algorithm.

to as the sampling-importance-resampling (SIR) PF. This implementation of PF method allow us to approximate the true density of the persistent variance component ($v_{1,t}$) and the transient variance component ($v_{2,t}$) using two sets of particles that are updated recursively through equations (2.4.6). In other words, we recursively simulate next period particles of each variance component until we have the empirical distributions of each variance factor over the entire sample. That is, given N particles of $\{v^j_{1,t}\}^N_{j=1}$, N particles of $\{v^j_{2,t}\}^N_{j=1}$, simulated return shocks, and $w_{1,t+\Delta t}$ and $w_{2,t+\Delta t}$ we generate the next period particles, N particles $\{v^j_{1,t+\Delta t}\}^N_{j=1}$ and another N particles $\{v^j_{2,t+\Delta t}\}^N_{j=1}$ at any time $t + \Delta t$.

We start by simulating return's shocks $z^j_{t+\Delta t}$ given the initial value of structural parameters Θ_0 and current variance particles $\{v^j_{1,t}, v^j_{2,t}\}$, on every day t and for every particle $j = 1, 2, ..., N$, according to (2.4.7). Then using (2.4.8) we simulate volatility shocks $w^j_{1,t+\Delta t}$ and $w^j_{1,t+\Delta t}$. Note that $\epsilon^j_{1,t+\Delta t}$ and $\epsilon^j_{2,t+\Delta t}$ are independent standard normal random variables.

$$z^j_{t+\Delta t} = \Big[\ln(S_{t+\Delta t}/S_t) - (\mu - \frac{1}{2}(v^j_{1,t} + v^j_{2,t}))\Delta t\Big] / \sqrt{(v^j_{1,t} + v^j_{2,t})\Delta t} \tag{2.4.7}$$

$$\begin{aligned} w^j_{1,t+\Delta t} &= \rho_1 z^j_{t+\Delta t} + \sqrt{1-\rho_1^2}\ \epsilon^j_{1,t+\Delta t} \\ w^j_{2,t+\Delta t} &= \rho_2 z^j_{t+\Delta t} - \frac{\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\ \epsilon^j_{1,t+\Delta t} + \sqrt{\frac{1-\rho_1^2-\rho_2^2}{1-\rho_1^2}}\epsilon^j_{2,t+\Delta t} \end{aligned} \tag{2.4.8}$$

Then, given the simulated return's shocks $\{z^j_{t+\Delta t}\}^N_{j=1}$ and simulated shocks to the persistent and transient variance components $\{w^j_{1,t+\Delta t}\}^N_{j=1}$ and $\{w^j_{2,t+\Delta t}\}^N_{j=1}$, we simulate next period variance particles $\{\tilde{v}^j_{1,t+\Delta t}\}$ and $\{\tilde{v}^j_{2,t+\Delta t}\}$, for every day t according to (2.4.9).

$$\begin{aligned} \tilde{v}^j_{1,t+\Delta t} &= v^j_{1,t} + \kappa_1(\theta_1 - v_{1,t})\Delta t + \sigma_1\sqrt{v_{1,t}\Delta t}\ w_{1,t+\Delta t} \\ \tilde{v}^j_{2,t+\Delta t} &= v^j_{2,t} + \kappa_2(\theta_2 - v_{2,t})\Delta t + \sigma_2\sqrt{v_{2,t}\Delta t}\ w_{2,t+\Delta t} \end{aligned} \tag{2.4.9}$$

This is the "Sampling Step," at the end of which we generate N possible daily values for the persistent variance component $v_{1,t+\Delta t}$ and another N possible daily values for the transient variance component $v_{2,t+\Delta t}$ over the entire sample. In the next step, "Importance Step," we evaluate importance of the sampled daily particles by assigning appropriate weights $\tilde{W}^j_{t+\Delta t}$ to the simulated daily particles using a multivariate normal distribution. Intuitively, these weights, $\tilde{W}^j_{t+\Delta t}$, are likelihood that the next day return at $t + 2\Delta t$ is generated by this set of particles. Then, the probability of each daily particle can be defined by normalizing the weights within each day according to (2.4.12). Note that these weights are the basis of our likelihood function under the P distribution.

$$(r_{t+2\Delta t}|\{\tilde{v}_{1,t+\Delta t}\ ,\tilde{v}_{2,t+\Delta t}\}) \sim N\big[(\mu - \frac{1}{2}(\tilde{v}_{1,t+\Delta t} + \tilde{v}_{2,t+\Delta t}))\Delta t\ ,(\tilde{v}_{1,t+\Delta t} + \tilde{v}_{2,t+\Delta t})\Delta t\big] \tag{2.4.10}$$

$$\tilde{W}^{j}_{t+\Delta t} = \frac{1}{\sqrt{2\pi(\tilde{v}^{j}_{1,t+\Delta t} + \tilde{v}^{j}_{2,t+\Delta t})\Delta t}} \cdot \exp\Big(- \frac{1}{2}\frac{\big(\ln(\frac{S_{t+2\Delta t}}{S_{t+\Delta t}}) - (\mu - \frac{1}{2}(\tilde{v}^{j}_{1,t+\Delta t} + \tilde{v}^{j}_{2,t+\Delta t}))\Delta t\big)^2}{(\tilde{v}^{j}_{1,t+\Delta t} + \tilde{v}^{j}_{2,t+\Delta t})\Delta t}\Big) \tag{2.4.11}$$

$$\breve{W}^{j}_{t+\Delta t} = \frac{\tilde{W}^{j}_{t+\Delta t}}{\sum_{j=1}^{N}\tilde{W}^{j}_{t+\Delta t}} \tag{2.4.12}$$

Note that combining independent shocks $z_{1,t}$ and $z_{2,t}$ in (2.4.3) imposes a restriction on the weights of daily variance particles. Therefore, the importance probability is assigned to the summation of return's shocks. However, estimation results show that the path of filtered spot persistent variance component and transient variance component in our two-factor SV model are not sensitive to this assumption. We investigate the sensitivity of our result to this weighting assumption by estimating daily spot variances using the two-step iterative approach, following Huang and Wu [2004]. We do not observe significant difference between filtered spot variances in two-step iterative approach and those filtered with particle filter method.

In the last step, "Resampling Step," we find the empirical distribution of smoothly resampled daily particles. Following the Pitt [2002] algorithm, we draw smoothed daily particles by assigning uniform distributions to the raw daily particles for persistent and transient variance components. As in the sampling step, we start from the beginning of the sample period and recursively simulate the next period daily particles using the smoothly resampled daily particles. The procedure continues until we have the empirical distributions of the persistent and transient variance components over the entire sample.

Given the appropriate weights (2.4.12), we define the return-based likelihood function as follows.

$$LLR \propto \sum_{t=1}^{T} \ln\left(\frac{1}{N}\sum_{j=1}^{N} \breve{W}_t^j(\Theta)\right) \tag{2.4.13}$$

Our implementation uses the maximum likelihood importance sampling (MLIS) methodology to maximize LLR criterion. Note that return-based likelihood function (2.4.13) is a function of the structural parameters of the market model under P measure, $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_1, \rho_1, \rho_2\}$. Note also that the filtered daily spot persistent variance component $v_{1,t}^P$ and transient variance component $v_{2,t}^P$ can be defined as the average of the smoothly resampled particles.

$$\hat{v}_{1,t}^P = \frac{1}{N}\sum_{j=1}^{N} v_{1,t}^j\ , \qquad \hat{v}_{2,t}^P = \frac{1}{N}\sum_{j=1}^{N} v_{2,t}^j \tag{2.4.14}$$

2.4.2 The Option Based Likelihood Function

In order to fully specify the market dynamics under the Q measure, we need to estimate a set of structural parameters for the market model under Q measure $\tilde{\Theta} \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_1, \rho_1, \rho_2, \lambda_1, \lambda_2\}$, a vector of daily spot persistent variance component $\hat{v}_{1,t}^Q$, and a vector of daily spot transient variance component $\hat{v}_{2,t}^Q$. Unobserved daily spot persistent and transient variance components under the Q measure can be filtered using the PF method. We follow the same procedure as described in (2.4.7)-(2.4.12) for the market variances under P measure while using structural parameters under Q measure, $\{\tilde{\kappa}_1, \tilde{\kappa}_2, \tilde{\theta}_1, \tilde{\theta}_2, \sigma_1, \sigma_1, \rho_1, \rho_2\}$. Note that $\tilde{\kappa}^i = \kappa^i + \lambda^i$ and $\tilde{\theta}^i = \frac{\kappa^i\theta^i}{\kappa^i+\lambda^i}$ for $i = \{1, 2\}$ according to the Proposition (2.1). We may obtain daily spot persistent and transient variance components under Q measure as the average of the smoothly resampled daily particles for each component of market variance.

$$\hat{v}_{1,t}^Q = \frac{1}{N}\sum_{j=1,Q}^{N} v_{1,t}^j\ , \qquad \hat{v}_{2,t}^Q = \frac{1}{N}\sum_{j=1,Q}^{N} v_{2,t}^j \tag{2.4.15}$$

Define the option-based likelihood function using a Vega-weighted loss function for the index options, where Vega is the Black-Scholes sensitivity of the option price with respect to volatility.[29] The Vega- weighted option pricing errors serves as an approximation to the implied volatility root mean squared errors,[30] which is a very popular loss function. This Vega-weighted loss function does not require a numerical inversion of the Black and Scholes [1973] model price and thus is helpful in large scale optimization problems such as ours.

Define normalized option pricing errors as follows.

$$\eta_n = (C_n^O - C_n^M(\tilde{\Theta}, \hat{v}_1^Q, \hat{v}_2^Q, S_t, K, \tau))/Vega_n \ , \qquad n = 1, \dots, M \tag{2.4.16}$$

where C_n^O is the observed daily option prices and $C_n^M(\tilde{\Theta}, \hat{v}_1^Q, \hat{v}_2^Q, S_t, K, \tau)$ is the model price of index option n, according to pricing equation (2.2.8), given the filtered spot persistent and transient variance component and structural parameters under Q measure. M is the total number of index option contracts and $Vega_n$ is the Black and Scholes [1973] option Vega for the option n. Then we may obtain the option-based likelihood as follows.[31]

$$LLO \propto -\frac{1}{2}\Big(M\ln(2\pi) + \sum_{n=1}^{M}\big(\ln(s^2) + \eta_n^2/s^2\big)\Big) \ , \tag{2.4.17}$$

Combining the returns-based likelihood function (2.4.13) and the options-based likelihood function (2.4.17), we have the total likelihood function. Our implementation usses the nonlinear least squares importance sampling (NLSIS) estimation mythology to solve the following optimization and to estimate the structural parameters of the market model $\hat{\Theta}$ and $\hat{\tilde{\Theta}}$ and daily spot persistent and transient variance components.

$$\max_{\Theta, \tilde{\Theta}}\big(LLR + LLO\big) \ . \tag{2.4.18}$$

It is important to note that our optimization algorithm is iterative. Each iteration starts with an initial set of structural parameters, which then will be used to filter daily spot volatilities using the information content of index returns. Then, given spot volatilities and

[29] Note that while several loss functions have been used in option pricing literature, option theory does not suggest an specific loss function as pricing equations do not contain an error term. Therefore, the appropriate loss functions are defined according to econometric considerations as well as convenience.

[30] See for example Carr and Wu [2007] and Christoffersen et al. [2009].

[31] Note that we replace s^2 by its sample analog $\hat{s}^2 = \frac{1}{M}\sum_{n=1}^{N}\eta_n^2$.

observed option prices, next set of optimal parameters can be reached by minimizing the option pricing errors over the entire sample. The procedure iterates until an optimal set of structural parameters is reached and thereby we obtain final vectors of transient and variance spot variance components.

2.5 Parameter Estimation Results

This section reports the filtered daily spot variance components together with the structural parameter estimates for the two-factor SV model. As described in the Data Section, we use a long time-series of daily S&P 500 index returns and the entire cross-section of S&P 500 option prices over the period from January 4, 1996 to December 29, 2011. Given the slow mean-reversion in the dynamic of market volatility, it is important to let the data set span a long time series. This is in particular important in our analysis as we decompose the overall market volatility into two independent components and would like to characterize the dynamics of transient and persistent variance components.

In what follows we set the market risk premium μ equal to the sample average daily index returns. We use 10% OTM index options and then put-call-parity to convert OTM puts into ITM calls. Table (2.3) reports structural parameter estimates (under P measure) that characterize the dynamics of index returns and its persistent and transient variance components. Panel A provides result of the joint estimation; a consistent set of parameters under P and Q measures. Therefore, the speeds of mean reversion and the unconditional mean of the persistent and transient variance components under Q-measure are linked to their P-measure equivalents through the market prices of the volatility risk factors ($\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$).[32] To provide a basis for further comparison and to examine the goodness of fit of the two-factor SV model under the joint-estimation, we also estimate structural parameters using only option data. This result is provided in Panel C.

[Table (2.3) about here]

As discussed, the purpose of two-factor stochastic volatility model is to capture independent movements in the underlying returns and option prices over time. Consistent with previous studies in both discrete time GARCH models and continuous time stochastic volatility mod-

[32]see Proposition (2.1).

els, we find that one of the volatility factors is highly persistent and the other one is highly mean-reverting. In joint-estimation, we find that the first variance component is slowly mean-reverting with $\kappa_1 = 1.4271$ under physical measure while the rate of mean reversion in the second variance component is much higher with $\kappa_2 = 3.5874$ under the physical measure.[33] The point estimate of mean reversion parameters from option-based estimation is similar to those from joint estimation. Using options data only, we find that $\tilde{\kappa}_1 = 0.2267$ and $\tilde{\kappa}_2 = 2.9137$, which is consistent with the speed of mean reversion from joint estimation where under Q-measure $\tilde{\kappa}_1 = 0.3473$ and $\tilde{\kappa}_2 = 2.5520$.

To gain a better intuition about persistent and transient variance components we define the half-life ($T_{1/2}$) of a variance component as the number of weeks that it takes for autocorrelation of each variance component to decay to half of its weekly autocorrelation level. Half-life can be computed as $T_{1/2} = \ln(\phi/2)/\ln(\phi)$ where $\Delta t = 7/365$ and $\phi = \exp(-\kappa\Delta t)$, denoting weekly autocorrelation of time-series each variance component. The risk neutral point estimate of mean reversion speed in transient variance component implies a half-life around 15 weeks while it is 105 weeks in the persistent variance component, almost 7 times larger than its transient counterpart. These values confirm that first variance component is highly persistent while the second one is highly auto-correlated and thus the immediate impact of variance shocks on this component is larger but short-lived.

We observe that the unconditional persistent variance under P-measure is $\theta_1 = 0.0026$, which is much less than the unconditional transient variance $\theta_2 = 0.0171$. The unconditional risk neutral persistent and transient variance components are $\tilde{\theta}_1 = 0.0106$ and $\tilde{\theta}_2 = 0.0240$ which correspond to 10.30% and 15.49% volatility per year. Note that the unconditional variance of both components are consistent with the average filtered daily spot persistent variance and daily spot transient variance over the entire sample.

Consistent with our intuition, we observe a wide spread between the volatility of variance in the persistent and transient variance components. As a result of joint estimation we find that $\sigma_1 = 0.0855$ and $\sigma_2 = 0.3496$. This result is consistent with the option-based estimation where we find that transient variance component is much more volatile with $\sigma_2 = 0.5678$ compared to the persistent variance component with $\sigma_1 = 0.0958$. Higher level of volatility of variance in option-based estimation compared to the joint estimation is consistent with previous studies[34]

[33] These value correspond to a daily variance persistence of $1 - 1.4271/365 = 0.9961$ for the first component and $1 - 3.5874/365 = 0.9901$ for the second component.

[34] For instance, Bates [2000] reports that option-based estimates of volatility of variance is larger than the

We find negative prices for both variance components where $\lambda_1 = -1.0798$ and $\lambda_2 = -1.0355$. These negative prices imply that investors are willing to pay for an insurance against an increase in volatility risk, even if that increase has little persistence. To the best of our knowledge none of the previous studies of two-factor stochastic volatility models in option market reports the prices of the variance risk factors as they either focused on the options market data or the underlying index returns data. Our negative prices for both variance components is consistent with asset pricing studies where the short-run and the long-run volatility components are priced cross-sectional asset pricing factors. Adrian and Rosenberg [2008]use a large cross-section of individual stocks over a very long period and find that prices of both short-run and long-run variance components are negative and highly significant. Therefore, our join estimation result confirm that there is a consensus of opinions about the price of transient and persistent variance components among option traders and equity traders.

Our joint estimation results show that correlation between shocks to the index returns and shocks to the persistent variance component is $\rho_1 = -0.6918$. The correlation between shocks to the index returns and shocks to the transient variance component is $\rho_2 = -0.2173$. ρ_1 and ρ_2 captures asymmetry in the response of persistent and transient variance components to positive versus negative return shocks and can be considered as the persistent and transient continuous time leverage (asymmetry) effect. The leverage effect induces negative skewness in index returns and thus yields a volatility smirk. Our results show that that leverage effect is more significant in the persistent variance component compared to the transient variance component. Therefore, persistent variance component has more significant effect on the dynamic of index skewness. Using the data from option market only, we find that $\rho_1 = -0.91$ and $\rho_1 = -0.49$. The higher absolute level of option implied correlation coefficients compared to those of joint estimation is partly related to the well documented fact that risk neutral distribution is more negatively skewed.

Our persistent and transient correlation coefficients are almost consistent with those of previous studies in option market. The average correlation coefficients in Christoffersen et al. [2009, Table 3] are $\rho_1 = -0.96$ for their first variance component and $\rho_2 = -0.83$ for their second variance component.[35] Bates [2000] also reports the structural parameter estimates of a two-factor SV model using 1988-1993 S&P 500 futures option prices. He obtains one

one obtained from time-series-based estimates.

[35] Christoffersen et al. [2009] use data on European S&P 500 call option quotes over the period 1990-2004. Note that they estimate a separate set of structural parameters for every year in their sample.

set of structural parameters over the entire sample where $\rho_1 = -0.78$ and $\rho_2 = -0.38$. To provide a basis for comparison, we also estimate structural parameters using options data only over the same sample period and find $\rho_1 = -0.91$ and $\rho_2 = -0.49$. There are potential explanations for differences between the reported estimates of the correlation coefficients in these studies, not in the least, the very different data set and the very different time span. Despite differences in the magnitude of the coefficients, the point estimates for the correlation coefficients are negative for both persistent and transient variance components across all these studies. Further, the transient variance component has lower (in absolute value) level of correlation compared to the persistent variance components in all these studies.

To provide some empirical evidence on the difference between persistent and transient variance components over time, we plot the paths of filtered variance components. Figure (2.1) plots filtered time series of risk-neutral spot variance components of S&P 500 index based on our two-factor stochastic volatility model. Panel A shows time series of persistent variance component and Panel B shows time series of transient variance component. The blue plots are based on the Particle Filter method using data from both S&P 500 index and option markets (joint estimation) and the red plots are filtered spot variances using only S&P 500 options data.

[**Figure (2.1) about here**]

Naturally, the overall patterns of persistent and transient variance components filtered from joint estimation are consistent with those filtered from options data only. However, option implied variance components are more volatile in the sense that when variance increases, it tends to do more sharply compared to the one filtered based on joint estimation and thus exhibit more spikes. In particular, this pattern in more pronounced in the transient variance component (Panel B). The observed sharper spikes in option-based filtered variance in the two-factor SV model is consistent with previous studies of one-factor SV model. The smoother variance paths in joint-estimation is partly due to smooth resampling procedure in SIR PF method and partly due to imposed consistency between parameter estimates under P and Q measures.

To provide more intuition about the total risk neutral variance in our two-factor SV model, Figure (2.2) combines persistent and transient variance components and plots time series of total spot variance versus model-free option-implied VIX volatility index. As we expect, the time series of option implied total spot variance is closely related to the VIX volatility index. Further, the time series of total spot variance from joint estimation follow the same pattern as the VIX volatility index. However, due to joint restrictions, the total spot variance from joint estimation do not exhibits volatility spikes as large as those observed in the VIX volatility index.

[Figure (2.2) about here]

2.6 Model Performance and In-Sample Fit

We measure the goodness of fit using the following Vega-weighted root mean squared option pricing errors (Vega RMSE) as it is consistent with the loss function that we used in the the optimization routine.

$$\text{Vega RMSE} \equiv \sqrt{\frac{1}{N}\sum_{n,t}^{M}\Big(\frac{C_{n,t}^{O} - C_{n,t}^{M}(\hat{\hat{\Theta}}, \hat{v}_{1,t}^{Q}, \hat{v}_{2,t}^{Q})}{Vega_{n,t}}\Big)^2}\,, \tag{2.6.1}$$

where, $C_{n,t}^{O}$ is the observed price of index option n on day t, $C_{n,t}^{M}$ is the model price for the same index option on the same day, and $Vega_{n,t}$ is the Black-Scholes option Vega for the same option contract on the same day. To provide a reference for comparison, we also report the implied volatility root mean squared error (IVRMSE).

$$\text{IVRMSE} \equiv \sqrt{\frac{1}{N}\sum_{n,t}^{M}\big(IV_{n,t}^{O} - IV(C_{n,t}^{M}(\hat{\hat{\Theta}}, \hat{v}_{1,t}^{Q}, \hat{v}_{2,t}^{Q}))\big)^2}\,, \tag{2.6.2}$$

where, $IV_{n,t}^{O}$ is the Black-Scholes implied volatility of observed option n on day t and $IV(C_{n,t}^{M}(\hat{\hat{\Theta}}, \hat{v}_{1,t}^{Q}, \hat{v}_{2,t}^{Q}))$ is the Black-Scholes implied volatility of the model option price for the same index option on the same day.

Table (2.4) reports in-sample goodness-of-fit for the two-factor stochastic volatility model over the entire sample, 1996 through 2011 for various maturities. Panels A and B report in-sample fit for calls and puts separately. The right panel reports model fit based on the joint estimation while the left panel gives reports option-based fit. We find that the overall

Vega-weighted RMSE of joint estimation and option-based estimation are 2.56% and 0.98% respectively. Note that the overall IVRMSE are 2.59% and 0.99% respectively, which means that Vega-weighted RMSE could be used as an approximation of IVRMSE. Overall, our two-factor SV model provides a better fit to call option contracts compared to put option contracts, which is consistent with the findings in one-factor stochastic volatility model.

Note that joint estimation imposes a consistency between physical and risk neutral parameters which are otherwise not identical. Such a restriction is not required in option-based estimation which could partly explain the better in-sample fit of option-based estimation compared to joint estimation. However, the reported RMSEs confirms that unlike stochastic volatility model, joint restrictions on return and variance dynamics under P and Q measures does not lead to the poor performance of the two-factor SV model.

Broadie et al. [2007] refer to the inconsistency between the option-based estimates of certain structural parameters in SV model and the parameter estimates from underlying time-series of returns and indicate that the SV model is basically misspecified. In particular, they state that the point estimates of the correlation coefficient and volatility of volatility are incompatible under the P and Q measures. They also show that the joint restrictions on the returns and volatility dynamics under the P and Q measures lead to the poor performance of the stochastic volatility model, measured by high level of RMSE. Using S&P 500 returns and futures options data over the period of 1987 through 2003, they find IVRMSE of 1.1% for the option-based estimation and 8.73% while imposing time-series consistency.

They note that this poor performance of SV model indicates the inability of the SV models to generate sufficient amounts of conditional skewness and kurtosis. This drawback in standard SV models is mainly attributed to the fact that the estimated conditional higher moments are highly correlated with the estimated conditional variance. By contrast, in-sample fit of our two-factor SV model is significantly improved relative to the Heston SV model. Further, the spread between Vega-weighted RMSE of joint estimation and option-based estimation is reduced significantly in the two-factor SV model versus the Heston SV model. The better performance of two-factor SV model is due to the fact that it can generate stochastic correlation between volatility and stock returns. This feature enables the two-factor SV model to better capture the conditional skewness and kurtosis.[36]

[36]Previous studies show that using the option data only two factor SV model improves on the benchmark SV model both in-sample and out-of-sample, see Christoffersen et al. [2009, Section 3.1].

2.7 Model Stability and Out-of-Sample Performance

In order to examine the stability of the two-factor SV model of index and its out-of-sample performance, we divide the dataset into two subsample periods. The first subsample is from January 1996 through December 2003 and contains 169,800 daily option contracts. The second one is from January 2004 to December 2011 which contains 175,910 daily option contracts. Using both daily returns and option data we filter spot daily persistent variance path and transient variance path and repeat the joint estimation routine within each subsample. Table (2.5) reports the parameter estimates within each subsample (Panels A and B). For the sake of comparison, Panels C and D also report the parameter estimates from option-based estimation. The main results of the subsample tests are as follows.

First, we find that PF is a reliable filtering technique even within shorter sample period of 8 years. We observe that the time series of total spot daily variances under risk neutral measure is largely consistent with the time series of the VIX option implied volatility index within each subsample period.

Second, the parameter estimates within each subsample period is largely inline with those obtained from whole-sample estimates. Moreover, within each subsample period, the joint estimation results is also consistent with option-based parameter estimates. We find that point estimate for the transient mean reversion parameter is higher in the second subsample period while the opposite is true for the persistent mean reversion speed. Overall, the level and order of parameter estimates are almost consistent within both subsample periods and also across both estimation methods (joint estimation and option-based estimation).[37]

Third, the correlation coefficients between transient and persistent variance shocks and return shocks within subsample periods remain consistent with the ones estimated over the entire sample period and those reported in previous studies[38] in the sense that the magnitude of persistent correlation coefficient is higher than its transient counterpart. Further, the transient and persistent remain negative with the same order within two subsample pe-

[37] Christoffersen et al. [2009, Table 3] report annual risk neutral parameter estimates for the two-factor SV model over the period 1990 through 2004 using data from S&P 500 index option data. Our option-based subsample parameter estimates are mostly consistent with their average annual result except for the volatility of volatility parameter. Apart from differences in the size of sample, this difference in point estimates may partly be explained by the fact that the annual parameter estimates in Christoffersen et al. [2009] does not satisfy the Feller condition. Feller [1951] shows that a square root process is strictly positive if $2\kappa\theta > \sigma^2$.

[38] See Section 6.

riods, confirming our previous findings that investors are willing to pay to avoid transient and highly mean reverting volatility shocks.

Fourth, we evaluate our model fit within both subsample periods and report Vega RMSEs and IVRMSEs separately for calls and puts and for different maturities. Entries in Table (2.6) and Table (2.7) are inline with model fit over the entire sample period, reported in Table (2.4). Our joint estimation result show a better in-sample fit over the second subsample period as Vega RMSEs and IVRMSEs are reduced.

Last, in order to measure the out-of-sample performance of the two-factor SV model in capturing the behaviour of S&P 500 index options, we use the parameter estimates form the first subsample (1996-2003). Given the parameter estimates from the first subsample period, we use Particle Filter methods to filter risk neutral spot daily persistent and transient variance components over the second subsample period and then compute the IVRMSEs and Vega RMSE over the second subsample (2004-2011). Table (2.8) reports the summary statistics of the out-of-sample performance for different maturities and for calls and puts separately. Comparing out-of-sample entries in (2.8) with those of in-sample in (2.7) over the same period supports the stable performance of the two-factor SV model either in joint-estimation or in option-based estimation.

2.8 Concluding Remarks

In this paper we investigate a two-factor stochastic volatility model where the aggregate market volatility is decomposed into a persistent and a transient volatility components. We extend the pricing kernel in Christoffersen et al. [2013], where investor's equity preference is distinguished from her variance preference, and introduce an admissible pricing kernel that links the proposed market dynamics under P and Q measures. We also discuss alternative pricing kernel for risk neutralization without separating equity and variance preferences. As the proposed two-factor specification is affine, we obtain a closed-from pricing expression for European call options. We use a long time-series of daily S&P 500 index returns and the entire cross-section of S&P 500 option prices over the same time span. We filter time series of persistent and transient spot variance components and simultaneously estimate a set of structural parameters that characterizes the dynamics of index return and variance components.

In empirical analysis, we show that the proposed decomposition of volatility can be characterized by different sensitivity of the variance components to the volatility shocks and different persistence in variance components. Consistent with the previous studies in both discrete time GARCH models and continuous time stochastic volatility models, we find that one of the volatility component is highly persistent and the other one is highly mean-reverting, where immediate impact of volatility shocks on the transient volatility component is bigger but short-lived. We obtain negative risk premium for both variance components, implying that investors are willing to pay for insurance against increases in volatility risk, even if such increases have little persistence. The negative risk premiums of both variance components are consistent with the findings in equity market where Adrian and Rosenberg [2008] find that short-run and long-run variance components are priced factors with negative risk premium. We also obtain negative correlations between shocks to the index returns and shocks to the transient and persistent variance components. In particular, we observe that the persistent correlation coefficient has more significant effect on the dynamics of index skewness.

Our model provides good fit to observed option prices both in- and out-of-sample, measured by Vega-weighted root mean squared option pricing errors and implied volatility root mean squared errors. More to the point, we find that unlike stochastic volatility model, joint restrictions on return and variance dynamics under P and Q measures does not lead to the poor performance of our two-factor SV model.

2.A Proof of Proposition 2.1

We impose the condition that the product of the price of any traded asset and the pricing kernel under physical measure is a martingale. We also impose the condition that the discounted price of any traded asset under risk neutral measure is also a martingale. We show that the two-factor stochastic volatility process under physical measure in (2.2.1) are linked to its risk-neutral counterpart in (2.2.4) by the unique arbitrage free pricing kernel introduced in (2.2.6) and deduce restrictions on the time-preference parameters, $\{\delta, \eta_1, \eta_2\}$, risk-aversion (equity aversion) parameter, ϕ, and variance preference parameters (variance aversion), $\{\zeta_1, \zeta_2\}$. We close this proof by showing how physical Wiener processes $\{z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}\}$ are linked to risk neutral Wiener processes $\{\tilde{z}_{1,t}, \tilde{z}_{2,t}, \tilde{w}_{1,t}, \tilde{w}_{2,t}\}$ by equity premium $\{\mu_1, \mu_2\}$ and variance premium $\{\lambda_1, \lambda_2\}$ parameters.

Consider that index return under physical and risk-neutral measures follows the dynamics (2.A.1) and (2.A.2).

$$
\begin{aligned}
dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(\rho_1 dz_{1,t} + \sqrt{1-\rho_1^2}dB_{1,t}) \\
dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(\rho_2 dz_{2,t} + \sqrt{1-\rho_2^2}dB_{2,t})
\end{aligned}
\tag{2.A.1}
$$

$$
\begin{aligned}
dS_t/S_t &= r dt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \\
dv_{1,t} &= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(\rho_1 d\tilde{z}_{1,t} + \sqrt{1-\rho_1^2}d\tilde{B}_{1,t}) \\
dv_{2,t} &= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(\rho_2 d\tilde{z}_{2,t} + \sqrt{1-\rho_2^2}d\tilde{B}_{2,t})
\end{aligned}
\tag{2.A.2}
$$

Then, following Christoffersen et al. [2013], we show that the pricing kernel links the physical and risk neutral measures has the following exponential affine form.

$$
\frac{M_t}{M_0} = \left(\frac{S_t}{S_0}\right)^{\phi} \exp\left[\delta t + \eta_1 \int_0^t v_{1,s}ds + \eta_2 \int_0^t v_{2,s}ds + \zeta_1(v_{1,t} - v_{1,0}) + \zeta_2(v_{2,t} - v_{2,0})\right]
\tag{2.A.3}
$$

Note that in the sprite of Cox et al. [1985] and Heston [1993] we assume that the market price of each variance risk factor is proportional to spot variance. Therefore, the risk neutral process in (2.A.2) can be defined as follows.

$$
\begin{aligned}
dS_t/S_t &= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \\
dv_{1,t} &= (\kappa_1(\theta_1 - v_{1,t}) - \lambda_1 v_1)dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t} \\
dv_{2,t} &= (\kappa_2(\theta_2 - v_{2,t}) - \lambda_2 v_2)dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t}
\end{aligned}
\tag{2.A.4}
$$

The log stock price process under physical measure and log pricing kernel process have the following dynamics respectively.

$$
d(\log(S_t)) = (r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \tag{2.A.5}
$$

$$
d(\log(M_t)) = \phi \cdot d(\log(S_t)) + (\delta + \eta_1 v_{1,t} + \eta_2 v_{2,t})dt + \zeta_1 dv_{1,t} + \zeta_2 dv_{2,t} \tag{2.A.6}
$$

Replacing (2.A.5) and (2.A.1) into (2.A.6) we have:

$$
\begin{aligned}
d(\log(M_t)) =& \big[\phi(r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t}) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} \\
&+ \zeta_1\kappa_1(\theta_1 - v_{1,t}) + \zeta_2\kappa_2(\theta_2 - v_{2,t})\big]dt \\
&+ \big[\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sigma_1\sqrt{v_{1,t}}\big]dz_{1,t} + \big[\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sigma_2\sqrt{v_{2,t}}\big]dz_{2,t} \\
&+ \big[\zeta_1\sigma_1\sqrt{v_{1,t}}\sqrt{1-\rho_1^2}\big]dB_{1,t} + \big[\zeta_2\sigma_2\sqrt{v_{2,t}}\sqrt{1-\rho_2^2}\big]dB_{2,t}.
\end{aligned}
\tag{2.A.7}
$$

As $dM_t/M_t = d(\log(M_t)) + \frac{1}{2}[d(\log(M_t))]^2$ we have

$$
\begin{aligned}
dM_t/M_t =& \Big[\phi(r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t}) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} \\
&+ \zeta_1\kappa_1(\theta_1 - v_{1,t}) + \zeta_2\kappa_2(\theta_2 - v_{2,t}) + \frac{1}{2}\phi^2(v_{1,t} + v_{2,t}) \\
&+ \phi(\zeta_1\rho_1\sigma_1 v_{1,t} + \zeta_2\rho_2\sigma_2 v_{2,t}) + \frac{1}{2}\zeta_1^2\sigma_1^2 v_{1,t}^2 + \frac{1}{2}\zeta_2^2\sigma_2^2 v_{2,t}^2\Big]dt \\
&+ \Big[\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sigma_1\sqrt{v_{1,t}}\Big]dz_{1,t} + \Big[\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sigma_2\sqrt{v_{2,t}}\Big]dz_{2,t} \\
&+ \Big[\zeta_1\sigma_1\sqrt{v_{1,t}}\sqrt{1-\rho_1^2}\Big]dB_{1,t} + \Big[\zeta_2\sigma_2\sqrt{v_{2,t}}\sqrt{1-\rho_2^2}\Big]dB_{2,t}.
\end{aligned} \tag{2.A.8}
$$

The first restriction on the pricing kernel is that the product of the money market account, $B_t = B_0\exp(rt)$, and the pricing kernel, M_t, should be a martingale under physical measure. Therefore, $E[d(B_t \cdot M_t)] = 0$ or $E[dM_t/M_t] = -rdt$.

$$
\begin{aligned}
&\Big[\phi(r + \mu_1 v_{1,t} + \mu_2 v_{2,t} - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t}) + \delta + \eta_1 v_{1,t} + \eta_2 v_{2,t} + \zeta_1\kappa_1(\theta_1 - v_{1,t}) + \zeta_2\kappa_2(\theta_2 - v_{2,t}) \\
&+ \frac{1}{2}\phi^2(v_{1,t} + v_{2,t}) + \phi(\zeta_1\rho_1\sigma_1 v_{1,t} + \zeta_2\rho_2\sigma_2 v_{2,t}) + \frac{1}{2}\zeta_1^2\sigma_1^2 v_{1,t}^2 + \frac{1}{2}\zeta_2^2\sigma_2^2 v_{2,t}^2\Big]dt = -rdt
\end{aligned} \tag{2.A.9}
$$

As (2.A.9) holds for $v_{1,t} = v_{2,t} = 0$,

$$
\delta = -r(\phi + 1) - \zeta_1\kappa_1\theta_1 - \zeta_2\kappa_2\theta_2. \tag{2.A.10}
$$

(2.A.9) also holds for $v_{1,t} = v_{2,t} = \infty$.

$$
\begin{aligned}
\eta_1 &= -\phi\mu_1 + 1/2\phi + \zeta_1\kappa_1 - 1/2(\phi^2 + \zeta_1^2\sigma_1^2 + 2\phi\zeta_1\sigma_1\rho_1) \\
\eta_2 &= -\phi\mu_2 + 1/2\phi + \zeta_2\kappa_2 - 1/2(\phi^2 + \zeta_2^2\sigma_2^2 + 2\phi\zeta_2\sigma_2\rho_2)
\end{aligned} \tag{2.A.11}
$$

The second restriction on the pricing kernel is based on the fact that $[S_t.M_t]$ is also a martingale under physical measure. Therefore, $E[d(S_t \cdot M_t)] = 0$. As a result of this restriction we have

$$v_{1,t}(\mu_1 + \phi + \zeta_1\sigma_1\rho_1) + v_{2,t}(\mu_2 + \phi + \zeta_2\sigma_2\rho_2) = 0,$$
$$\phi = \frac{-1}{v_{1,t} + v_{2,t}}\big[(\mu_1 + \zeta_1\sigma_1\rho_1)v_{1,t} + (\mu_2 + \zeta_2\sigma_2\rho_2)v_{2,t}\big]. \tag{2.A.12}$$

If we impose the restriction that $\mu_1 + \zeta_1\sigma_1\rho_1 \equiv \mu_2 + \zeta_2\sigma_2\rho_2$, then (2.A.12) can be simplified as follows.

$$\phi = -(\mu_1 + \zeta_1\sigma_1\rho_1) = -(\mu_2 + \zeta_2\sigma_2\rho_2) \tag{2.A.13}$$

We impose the third restriction on pricing kernel so that for any asset $U \equiv U(S, v_1, v_2, t)$, $[U(t).M_t]$ is also a martingale under P-distribution. Therefore, $E[d(U \cdot M_t)] = E[dU.M_t + U.dM_t + dU.dM_t] = 0$. Replacing M_t and dM_t into this equation we have the following restriction where $U_S = \partial U(S, v_1, v_2, t)/\partial S$, $U_{v_1} = \partial U(S, v_1, v_2, t)/\partial v_1$, and $U_{v_2} = \partial U(S, v_1, v_2, t)/\partial v_2$.

$$\begin{aligned}
&- rU + U_t + U_S(r + \mu_1 v_{1,t} + \mu_2 v_{2,t})S + U_{v_{1,t}}\kappa_1(\theta_1 - v_{1,t}) + U_{v_{2,t}}\kappa_2(\theta_2 - v_{2,t}) \\
&+ \frac{1}{2}U_{SS}(v_{1,t} + v_{2,t}) + \frac{1}{2}U_{v_{1,t}v_{1,t}}\sigma_1^2 v_{1,t} + \frac{1}{2}U_{v_{2,t}v_{2,t}}\sigma_2^2 v_{2,t} + U_{Sv_{1,t}}\rho_1\sigma_1 v_{1,t} + U_{Sv_{2,t}}\rho_2\sigma_2 v_{2,t} \\
&+ (U_S S\sqrt{v_{1,t}} + U_{v_{1,t}}\rho_1\sigma_1\sqrt{v_{1,t}})(\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sigma_1\sqrt{v_{1,t}}) \\
&+ (U_S S\sqrt{v_{2,t}} + U_{v_{2,t}}\rho_2\sigma_2\sqrt{v_{2,t}})(\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sigma_2\sqrt{v_{2,t}}) \\
&+ U_{v_{1,t}}\zeta_1\sigma_1^2 v_{1,t}(1 - \rho_1^2) + U_{v_{2,t}}\zeta_2\sigma_2^2 v_{2,t}(1 - \rho_2^2) = 0
\end{aligned} \tag{2.A.14}$$

The last restriction is based on the fact that discounted price process should be a martingale under risk neutral measure. Therefore, for any asset, $U(S, v_1, v_2, t)$, whose payoff depends on the state variables $\{S, v_1, v_2\}$, U/B_t is a Q-martingale. This restriction implies that $E^Q[d(U/B_t)] = 0$ or equivalently $E^Q[d(U(S, v_1, v_2, t))] = rU(S, v_1, v_2, t)$.

$$\begin{aligned}
&U_t + rSU_S + U_{v_{1,t}}(\kappa_1(\theta_1 - v_{1,t}) - \lambda_1 v_{1,t}) + U_{v_{2,t}}(\kappa_1(\theta_1 - v_{1,t}) - \lambda_2 v_{2,t}) + \frac{1}{2}U_{SS}(v_{1,t} + v_{1,t}) \\
&+ \frac{1}{2}U_{v_{1,t}v_{1,t}}\sigma_1^2 v_{1,t} + \frac{1}{2}U_{v_{2,t}v_{2,t}}\sigma_2^2 v_{2,t} + U_{Sv_{1,t}}\rho_1\sigma_1 v_{1,t} + U_{Sv_{2,t}}\rho_2\sigma_2 v_{2,t} = rU.
\end{aligned} \tag{2.A.15}$$

Replace (2.A.15) from the last restriction into (2.A.14) from the third restriction.

$$
\begin{aligned}
&U_S(\mu_1 v_{1,t} + \mu_2 v_{2,t})S + U_{v_{1,t}}\lambda_1 v_{1,t} + U_{v_{2,t}}\lambda_2 v_{2,t} \\
&+ (U_S S\sqrt{v_{1,t}} + U_{v_{1,t}}\rho_1\sigma_1\sqrt{v_{1,t}})(\phi\sqrt{v_{1,t}} + \zeta_1\rho_1\sigma_1\sqrt{v_{1,t}}) \\
&+ (U_S S\sqrt{v_{2,t}} + U_{v_{2,t}}\rho_2\sigma_2\sqrt{v_{2,t}})(\phi\sqrt{v_{2,t}} + \zeta_2\rho_2\sigma_2\sqrt{v_{2,t}}) \\
&+ U_{v_{1,t}}\zeta_1\sigma_1^2 v_{1,t}(1-\rho_1^2) + U_{v_{2,t}}\zeta_2\sigma_2^2 v_{2,t}(1-\rho_2^2) = 0
\end{aligned}
$$

$$
\begin{aligned}
&U_S(\mu_1 v_{1,t} + \mu_2 v_{2,t})S + U_{v_{1,t}}\lambda_1 v_{1,t} + U_{v_{2,t}}\lambda_2 v_{2,t} \\
&+ U_S S\phi v_{1,t} + U_S S\zeta_1\rho_1\sigma_1 v_{1,t} + U_{v_{1,t}}\rho_1\sigma_1\phi v_{1,t} + U_{v_{1,t}}\zeta_1\sigma_1^2 v_{1,t} \\
&+ U_S S\phi v_{2,t} + U_S S\zeta_2\rho_2\sigma_2 v_{2,t} + U_{v_{2,t}}\rho_2\sigma_2\phi v_{2,t} + U_{v_{2,t}}\zeta_2\sigma_2^2 v_{2,t} = 0
\end{aligned}
\tag{2.A.16}
$$

From the second restriction in (2.A.12) we know that $\mu_1 v_{1,t} + \mu_2 v_{2,t} = -\phi v_{1,t} - \zeta_1\rho_1\sigma_1 v_{1,t} - \phi v_{2,t} - \zeta_2\rho_2\sigma_2 v_{2,t}$. Therefore, we can further simplify (2.A.16).

$$
U_{v_{1,t}}\big(\rho_1\sigma_1\phi + \lambda_1 + \zeta_1\sigma_1^2\big)v_{1,t} + U_{v_{2,t}}\big(\rho_2\sigma_2\phi + \lambda_2 + \zeta_2\sigma_2^2\big)v_{2,t} = 0 \tag{2.A.17}
$$

One admissible solution for (2.A.17) would be:

$$
\begin{aligned}
\rho_1\sigma_1\phi + \lambda_1 + \zeta_1\sigma_1^2 = 0 \\
\rho_2\sigma_2\phi + \lambda_2 + \zeta_2\sigma_2^2 = 0
\end{aligned}
\tag{2.A.18}
$$

If we combine restrictions in (2.A.18) with those introduced in (2.A.13) and replace them back into (2.A.13) we have ϕ, ζ_1, and ζ_2.

$$
\begin{aligned}
\zeta_1 = \frac{\rho_1\sigma_1\mu_1 - \lambda_1}{\sigma_1^2(1-\rho_1^2)} \\
\zeta_2 = \frac{\rho_2\sigma_2\mu_2 - \lambda_2}{\sigma_2^2(1-\rho_2^2)}
\end{aligned}
\tag{2.A.19}
$$

$$\phi = -\mu_1 - \frac{\rho_1^2\sigma_1^2\mu_1 - \lambda_1\rho_1\sigma_1}{\sigma_1^2(1-\rho_1^2)} = -\mu_2 - \frac{\rho_2^2\sigma_2^2\mu_2 - \lambda_2\rho_2\sigma_2}{\sigma_2^2(1-\rho_2^2)} \tag{2.A.20}$$

Therefore, an admissible pricing kernel linking the P and Q dynamics in (2.A.1) and (2.A.2) is as follows.

$$\frac{dM_t}{M_t} = -rdt - \mu_1\sqrt{v_{1,t}}dz_{1,t} - \mu_2\sqrt{v_{2,t}}dz_{2,t} + \frac{\rho_1\sigma_1\mu_1 - \lambda_1}{\sigma_1^2(1-\rho_1^2)}dB_{1,t} + \frac{\rho_2\sigma_2\mu_2 - \lambda_2}{\sigma_2^2(1-\rho_2^2)}dB_{2,t} \tag{2.A.21}$$

This is the pricing kernel introduced in (2.1).

Now, we show that how physical shocks are linked to risk neutral shocks through equity premium $\{\mu_1, \mu_2\}$ and variance premium $\{\lambda_1, \lambda_2\}$ parameters.

$$\begin{aligned} d\tilde{z}_{1,t} &= dz_{1,t} + (\psi_{1,t} + \rho_1\psi_{3,t})dt \\ d\tilde{z}_{2,t} &= dz_{2,t} + (\psi_{2,t} + \rho_2\psi_{4,t})dt \\ d\tilde{w}_{1,t} &= dw_{1,t} + (\psi_{3,t} + \rho_1\psi_{1,t})dt \\ d\tilde{w}_{2,t} &= dw_{2,t} + (\psi_{4,t} + \rho_2\psi_{2,t})dt \end{aligned} \tag{2.A.22}$$

Replace physical shocks in return dynamics (2.2.1) by risk neutral shocks introduced in (2.A.22).

$$\begin{aligned} dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt \\ &\quad + \sqrt{v_{1,t}}d\tilde{z}_{1,t} - (\psi_{1,t} + \rho_1\psi_{3,t})\sqrt{v_{1,t}}dt + \sqrt{v_{2,t}}d\tilde{z}_{2,t} - (\psi_{2,t} + \rho_2\psi_{4,t})\sqrt{v_{2,t}}dt \end{aligned} \tag{2.A.23}$$

As a result of risk neutralization in (2.A.23), the expected stock returns in (2.A.23) should be equal to the risk free rate of returns. Therefore, we have the following restriction.

$$(\mu_1 v_{1,t} + \mu_2 v_{2,t})dt = (\psi_{1,t} + \rho_1 \psi_{3,t})\sqrt{v_{1,t}}dt + (\psi_{2,t} + \rho_2 \psi_{4,t})\sqrt{v_{2,t}}dt \tag{2.A.24}$$

One possible solution of (2.A.24) is as follows.

$$\begin{aligned} \mu_1 \sqrt{v_{1,t}} &= \psi_{1,t} + \rho_1 \psi_{3,t} \\ \mu_2 \sqrt{v_{2,t}} &= \psi_{2,t} + \rho_2 \psi_{4,t} \end{aligned} \tag{2.A.25}$$

Similarly, we replace the proposed transformation in (2.A.22) into the dynamics of volatilities in (2.2.1).

$$\begin{aligned} dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1 \sqrt{v_{1,t}} d\tilde{w}_{1,t} - \sigma_1 \sqrt{v_{1,t}}(\psi_{3,t} + \rho_1 \psi_{1,t})dt \\ dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2 \sqrt{v_{2,t}} d\tilde{w}_{2,t} - \sigma_2 \sqrt{v_{2,t}}(\psi_{4,t} + \rho_2 \psi_{2,t})dt \end{aligned} \tag{2.A.26}$$

The risk-neutral variance dynamics in (2.A.26) should be equivalent to those in (2.A.4), where the market price of variance risk factors is proportional to spot variance. Therefore, we have following restrictions:

$$\begin{aligned} \sigma_1 \sqrt{v_{1,t}}(\psi_{3,t} + \rho_1 \psi_{1,t}) &= \lambda_1 v_{1,t} \\ \sigma_2 \sqrt{v_{2,t}}(\psi_{4,t} + \rho_2 \psi_{2,t}) &= \lambda_2 v_{2,t} \end{aligned} \tag{2.A.27}$$

Combining the restrictions in (2.A.25) and (2.A.27), we have the following results, which link the physical distribution (2.2.1) to the risk neutral distribution (2.2.4).

$$
\begin{aligned}
\psi_{1,t} &= \frac{\sigma_1\mu_1 - \rho_1\lambda_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}} \\
\psi_{2,t} &= \frac{\sigma_2\mu_2 - \rho_2\lambda_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}} \\
\psi_{3,t} &= \frac{\lambda_1 - \rho_1\sigma_1\mu_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}} \\
\psi_{4,t} &= \frac{\lambda_2 - \rho_2\sigma_2\mu_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}}
\end{aligned}
\tag{2.A.28}
$$

2.B Risk Neutral Distribution

Risk neutral distribution in (2.2.4) can also be extracted by assuming the following standard stochastic discount factor, without explicit assumptions about the investor's variance preferences.

$$
\frac{dM_t}{M_t} = -rdt - \psi_t' dW_t \ , \tag{2.B.1}
$$

where $\psi_t \equiv [\psi_{1,t}, \psi_{2,t}, \psi_{3,t}, \psi_{4,t}]$ is the vector of market price of risk factors and $W_t \equiv [z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}]$ is the vector of innovations in market index return and variance components. Given the SDF in (2.B.1), the change-of-measure from P to Q distribution has the following exponential form.

$$
\frac{dQ}{dP}(t) \equiv M_t \exp(rt) = \exp\Big[-\int_0^t \psi_u' dW_u - \frac{1}{2}\int_0^t \psi_u' d\langle W, W'\rangle_u \psi_u \Big] \tag{2.B.2}
$$

where $\langle W, W'\rangle$ is the covariance operator.

We follow the notion of Doléans-Dade exponential (stochastic exponential) and define the stochastic exponential $\varepsilon(\cdot)$ as follow.

$$
\varepsilon\Big(\int_0^t \vartheta_u' dW_u\Big) \equiv \exp\Big[\int_0^t \vartheta_u' dW_u - \frac{1}{2}\int_0^t \vartheta_u' d\langle W, W'\rangle_u \vartheta_u\Big] \tag{2.B.3}
$$

Therefore, the change-of-measure (2.B.2) can be expressed in term of stochastic exponential as

$$\frac{dQ}{dP}(t) = \varepsilon\Big(\int_0^t -\psi_u' dW_u\Big) \tag{2.B.4}$$

Applying Ito's lemma, we get the following dynamic for the log stock price process under physical measure.

$$\log\big(\frac{S_t}{S_0}\big) = (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})t - \frac{1}{2}v_{1,t}t + \int_0^t \sqrt{v_{1,u}}dz_{1,u} - \frac{1}{2}v_{2,t} + \int_0^t \sqrt{v_{2,u}}dz_{2,u} \tag{2.B.5}$$

Given (2.B.5) and definition of stochastic exponential (2.B.3) we have

$$\frac{S_t}{S_0} = \exp\big[(r + \mu_1 v_{1,t} + \mu_2 v_{2,t})t\big]\varepsilon\Big(\int_0^t \sqrt{v_{1,u}}dz_{1,u}\Big)\varepsilon\Big(\int_0^t \sqrt{v_{2,u}}dz_{2,u}\Big) \tag{2.B.6}$$

To find the market prices of risk we impose the restriction that the product of the price of any traded asset and the pricing kernel under physical measure is a P-martingale. Given the change-of-measure (2.B.2), the following process, $N(t)$, should be a P-martingale.

$$N(t) \equiv \frac{S_t}{S_0}\frac{dQ}{dP}(t)\exp(-rt) \tag{2.B.7}$$

where

$$\begin{aligned} N(t) =& \exp\big[(\mu_1 v_{1,t} + \mu_2 v_{2,t})t\big] \\ & \varepsilon\Big(\int_0^t \sqrt{v_{1,u}}dz_{1,u}\Big)\varepsilon\Big(-\int_0^t \psi_{1,u}dz_{1,u} - \int_0^t \psi_{3,u}dw_{1,u}\Big) \\ & \varepsilon\Big(\int_0^t \sqrt{v_{2,u}}dz_{2,u}\Big)\varepsilon\Big(-\int_0^t \psi_{2,u}dz_{2,u} - \int_0^t \psi_{4,u}dw_{2,u}\Big) \end{aligned} \tag{2.B.8}$$

Using the properties of a stochastic exponential $\varepsilon(\cdot)$, $\varepsilon(X_t)\varepsilon(Y_t) = \varepsilon(X_t + Y_t)\exp(\langle X, Y\rangle_t)$ we can rewrite the process of $N(t)$ as follows.

$$
\begin{aligned}
N(t) = &\exp\left[(\mu_1 v_{1,t} + \mu_2 v_{2,t})t\right] \\
&\varepsilon\Big(\int_0^t \big(\sqrt{v_{1,u}} - \psi_{1,u}\big)dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u}\Big) \exp\Big[-\int_0^t \sqrt{v_{1,u}}(\psi_{1,u} + \rho_1 \psi_{3,u})du\Big] \\
&\varepsilon\Big(\int_0^t \big(\sqrt{v_{2,u}} - \psi_{2,u}\big)dz_{2,u} - \int_0^t \psi_{4,u} dw_{2,u}\Big) \exp\Big[-\int_0^t \sqrt{v_{2,u}}(\psi_{2,u} + \rho_2 \psi_{4,u})du\Big]
\end{aligned}
\tag{2.B.9}
$$

From the definition of a stochastic exponential we know that $\varepsilon(\cdot)$ are P-martingales. Thus, the process $N(t)$ is a P-martingale when the following restriction holds.

$$
\exp\left[(\mu_1 v_{1,t} + \mu_2 v_{2,t})t\right] \exp\big[-\int_0^t \sqrt{v_{1,u}}(\psi_{1,u} + \rho_1 \psi_{3,u})du\big] \exp\Big[-\int_0^t \sqrt{v_{2,u}}(\psi_{2,u} + \rho_2 \psi_{4,u})du\Big] = 1
\tag{2.B.10}
$$

The restriction in (2.B.10) can be satisfied if

$$
\begin{aligned}
\mu_1 v_{1,t} t - \sqrt{v_{1,t}}(\psi_{1,t} + \rho_1 \psi_{3,t})t = 0 \\
\mu_2 v_{2,t} t - \sqrt{v_{2,t}}(\psi_{3,t} + \rho_2 \psi_{4,t})t = 0
\end{aligned}
\tag{2.B.11}
$$

To fully specify the market prices of risk we assume that market price of variance risk factors are proportional to spot volatilites, following Heston [1993].

$$
\begin{aligned}
(\psi_{3,t} + \rho_1 \psi_{1,t}) &= \frac{v_{1,t}}{\sigma_1 \sqrt{v_{1,t}}} \lambda_1 \\
(\psi_{4,t} + \rho_2 \psi_{2,t}) &= \frac{v_{2,t}}{\sigma_2 \sqrt{v_{2,t}}} \lambda_2
\end{aligned}
\tag{2.B.12}
$$

Combining the restrictions in (2.B.11) and (2.B.12), we have the following market price of risk factors. Note that these prices are the same as those we find in Proposition (2.1).

$$
\begin{aligned}
\psi_{1,t} &= \frac{\sigma_1\mu_1 - \rho_1\lambda_1}{(1-\rho_1^2)}\frac{\sqrt{v_{1,t}}}{\sigma_1} \\
\psi_{2,t} &= \frac{\sigma_2\mu_2 - \rho_2\lambda_2}{(1-\rho_2^2)}\frac{\sqrt{v_{2,t}}}{\sigma_2} \\
\psi_{3,t} &= \frac{\lambda_1 - \rho_1\sigma_1\mu_1}{(1-\rho_1^2)}\frac{\sqrt{v_{1,t}}}{\sigma_1} \\
\psi_{4,t} &= \frac{\lambda_2 - \rho_2\sigma_2\mu_2}{(1-\rho_2^2)}\frac{\sqrt{v_{2,t}}}{\sigma_2}
\end{aligned} \tag{2.B.13}
$$

Given the market price of risk factors (2.B.13), we can apply Girsanov's theorem to find transform physical innovations in (2.2.1) to its risk neutral counterpart in (2.2.4).

$$
\begin{aligned}
d\tilde{z}_{1,t} &= dz_{1,t} + \psi_{1,t}dt + \rho_1\psi_{3,t}dt \\
d\tilde{z}_{2,t} &= dz_{2,t} + \psi_{2,t}dt + \rho_2\psi_{4,t}dt \\
d\tilde{w}_{1,t} &= dw_{1,t} + \psi_{3,t}dt + \rho_1\psi_{1,t}dt \\
d\tilde{w}_{2,t} &= dw_{2,t} + \psi_{4,t}dt + \rho_2\psi_{2,t}dt
\end{aligned} \tag{2.B.14}
$$

With some algebra we have the following transformations.

$$
\begin{aligned}
d\tilde{z}_{1,t} &= dz_{1,t} + \mu_1\sqrt{v_{1,t}}dt \\
d\tilde{z}_{2,t} &= dz_{2,t} + \mu_2\sqrt{v_{2,t}}dt \\
d\tilde{w}_{1,t} &= dw_{1,t} + (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt \\
d\tilde{w}_{2,t} &= dw_{2,t} + (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt
\end{aligned} \tag{2.B.15}
$$

Replacing $dz_{1,t}, dz_{2,t}, dw_{1,t}, dw_{2,t}$ from (2.B.15) into the physical dynamics in (2.2.1) and knowing that $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$ we obtain risk neutral return and variance dynamics.

$$
\begin{aligned}
dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
&= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}(d\tilde{z}_{1,t} - \mu_1\sqrt{v_{1,t}}dt) + \sqrt{v_{2,t}}(d\tilde{z}_{2,t} - \mu_2\sqrt{v_{2,t}}dt) \\
&= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t}
\end{aligned} \tag{2.B.16}
$$

$$
\begin{aligned}
dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(d\tilde{w}_{1,t} - (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt) \\
&= (\kappa_1\theta_1 - (\kappa_1 + \lambda_1)v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t} \\
&= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t}
\end{aligned}
\tag{2.B.17}
$$

$$
\begin{aligned}
dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(d\tilde{w}_{2,t} - (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt) \\
&= (\kappa_2\theta_2 - (\kappa_2 + \lambda_2)v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t} \\
&= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t}
\end{aligned}
\tag{2.B.18}
$$

2.C Proof of Proposition 2.2

We show that the GARCH model under physical measure (2.2.14) is linked to the GARCH model under risk-neutral measure (2.2.21) with the proposed pricing kernel (2.2.19) by specifying a set of sufficient conditions (2.2.20), (2.2.22), and (2.2.23). We first impose Euler equation for the risk-free asset and subsequently impose Euler equation for the underlying asset to find this parameters mapping.

Given the pricing kernel (2.2.19), we have

$$
\frac{M_t}{M_{t-1}} = \big(\frac{S_t}{S_{t-1}}\big)^{\phi} \exp\Big[\delta + \eta_1 h_{1,t} + \eta_2 h_{2,t} + \zeta_1(h_{1,t+1} - h_{1,t}) + \zeta_2(h_{2,t+1} - h_{2,t})\Big] \tag{2.C.1}
$$

Rewrite the physical GRACH dynamics (2.2.14) as follows.

$$
\begin{aligned}
S_t/S_{t-1} &= \exp\big[r + (\mu_1 - \frac{1}{2})h_{1,t} + (\mu_2 - \frac{1}{2})h_{2,t} + \sqrt{h_{1,t}}z_{1,t} + \sqrt{h_{2,t}}z_{2,t}\big] \\
h_{1,t+1} - h_{1,t} &= w_1 + (\beta_1 - 1)h_{1,t} + \alpha_1(z_{1,t} - \gamma_1\sqrt{h_{1,t}})^2 \\
h_{2,t+1} - h_{2,t} &= w_2 + (\beta_2 - 1)h_{2,t} + \alpha_2(z_{2,t} - \gamma_2\sqrt{h_{2,t}})^2
\end{aligned}
\tag{2.C.2}
$$

Substitute the dynamics (2.C.2) into (2.C.1)

$$
\begin{aligned}
\frac{M_t}{M_{t-1}} = \exp\Big[& r\phi + (\mu_1 - \frac{1}{2})\phi h_{1,t} + (\mu_2 - \frac{1}{2})\phi h_{2,t} + \sqrt{h_{1,t}}\phi z_{1,t} + \sqrt{h_{2,t}}\phi z_{2,t} \\
& + \delta + \eta_1 h_{1,t} + \eta_2 h_{2,t} \\
& + w_1\zeta_1 + (\beta_1 - 1)\zeta_1 h_{1,t} + \alpha_1\zeta_1(z_{1,t} - \gamma_1\sqrt{h_{1,t}})^2 \\
& + w_2\zeta_2 + (\beta_2 - 1)\zeta_2 h_{2,t} + \alpha_2\zeta_2(z_{2,t} - \gamma_2\sqrt{h_{2,t}})^2\Big] \,.
\end{aligned} \tag{2.C.3}
$$

Expanding squares and collecting some terms yield the following expression for a one-day pricing kernel.

$$
\begin{aligned}
\frac{M_t}{M_{t-1}} = \exp\Big[& r\phi + \delta + w_1\zeta_1 + w_2\zeta_2 \\
& + \big((\mu_1 - \frac{1}{2})\phi + \eta_1 + (\beta_1 - 1)\zeta_1 + \alpha_1\gamma_1^2\zeta_1\big)\ h_{1,t} \\
& + \big((\mu_2 - \frac{1}{2})\phi + \eta_2 + (\beta_2 - 1)\zeta_2 + \alpha_2\gamma_2^2\zeta_2\big)\ h_{2,t} \\
& + (\phi - 2\alpha_1\gamma_1\zeta_1)\sqrt{h_{1,t}}z_{1,t} + (\alpha_1\zeta_1)z_{1,t}^2 \\
& + (\phi - 2\alpha_2\gamma_2\zeta_2)\sqrt{h_{2,t}}z_{2,t} + (\alpha_2\zeta_2)z_{2,t}^2\Big]
\end{aligned} \tag{2.C.4}
$$

Before imposing the Euler equation, we introduce the expectations (2.C.5), where $z_{1,t}$ and $z_{2,t}$ follow a standard normal distribution.

$$
\begin{aligned}
\mathrm{E}\left[\exp(2a_1b_1z_{1,t} + a_1z_{1,t}^2)\right] &= \exp\left[-\frac{1}{2}\ln(1-2a_1) + \frac{2a_1^2b_1^2}{1-2a_1}\right] \\
\mathrm{E}\left[\exp(2a_2b_2z_{2,t} + a_2z_{2,t}^2)\right] &= \exp\left[-\frac{1}{2}\ln(1-2a_2) + \frac{2a_2^2b_2^2}{1-2a_2}\right]
\end{aligned} \tag{2.C.5}
$$

where in our case

$$
\begin{aligned}
a_1 = \alpha_1\zeta_1 \ , \ b_1 &= \frac{\phi - 2\alpha_1\gamma_1\zeta_1}{2\alpha_1\zeta_1}\sqrt{h_{1,t}} \\
a_2 = \alpha_2\zeta_2 \ , \ b_2 &= \frac{\phi - 2\alpha_2\gamma_2\zeta_2}{2\alpha_2\zeta_2}\sqrt{h_{2,t}}
\end{aligned} \tag{2.C.6}
$$

and thus

$$
\begin{aligned}
2a_1^2b_1^2 &= 2\alpha_1^2\zeta_1^2\big(\frac{\phi-2\alpha_1\gamma_1\zeta_1}{2\alpha_1\zeta_1}\big)^2h_{1,t} = \frac{1}{2}\big(\phi-2\alpha_1\gamma_1\zeta_1\big)^2h_{1,t} \\
2a_2^2b_2^2 &= 2\alpha_2^2\zeta_2^2\big(\frac{\phi-2\alpha_2\gamma_2\zeta_2}{2\alpha_2\zeta_2}\big)^2h_{2,t} = \frac{1}{2}\big(\phi-2\alpha_2\gamma_2\zeta_2\big)^2h_{2,t}
\end{aligned} \tag{2.C.7}
$$

Therefore, conditional expectations of the last two lines of pricing kernel (2.C.4) may be simplified as follows.

$$
\begin{aligned}
\mathrm{E}_{t-1}\Big[\exp\big[(\phi-2\alpha_1\gamma_1\zeta_1)\sqrt{h_{1,t}}z_{1,t}+\alpha_1\zeta_1z_{1,t}^2\big]\Big] &= \exp\big[-\frac{1}{2}\ln(1-2\alpha_1\zeta_1)+\frac{\phi-2\alpha_1\gamma_1\zeta_1}{2(1-2\alpha_1\zeta_1)}h_{1,t}\big] \\
\mathrm{E}_{t-1}\Big[\exp\big[(\phi-2\alpha_2\gamma_2\zeta_2)\sqrt{h_{2,t}}z_{2,t}+\alpha_2\zeta_2z_{2,t}^2\big]\Big] &= \exp\big[-\frac{1}{2}\ln(1-2\alpha_2\zeta_2)+\frac{\phi-2\alpha_2\gamma_2\zeta_2}{2(1-2\alpha_2\zeta_2)}h_{2,t}\big]
\end{aligned} \tag{2.C.8}
$$

We begin the proof by imposing the Euler equation for the risk-free asset.

$$
\mathrm{E}_{t-1}\big[\frac{M_t}{M_{t-1}}\big] = \exp(-r) \tag{2.C.9}
$$

Substituting (2.C.4) into (2.C.9), taking conditional expectation, and using the results (2.C.8) yield

$$
\begin{aligned}
\mathrm{E}_{t-1}\big[\frac{M_t}{M_{t-1}}\big] = \exp\Big[& r\phi+\delta+w_1\zeta_1+w_2\zeta_2 \\
&+\big((\mu_1-\frac{1}{2})\phi+\eta_1+(\beta_1-1)\zeta_1+\alpha_1\gamma_1^2\zeta_1\big)\ h_{1,t} \\
&+\big((\mu_2-\frac{1}{2})\phi+\eta_2+(\beta_2-1)\zeta_2+\alpha_2\gamma_2^2\zeta_2\big)\ h_{2,t} \\
&-\frac{1}{2}\ln(1-2\alpha_1\zeta_1)+\frac{\phi-2\alpha_1\gamma_1\zeta_1}{2(1-2\alpha_1\zeta_1)}h_{1,t} \\
&-\frac{1}{2}\ln(1-2\alpha_2\zeta_2)+\frac{\phi-2\alpha_2\gamma_2\zeta_2}{2(1-2\alpha_2\zeta_2)}h_{2,t}\Big] = \exp(-r)
\end{aligned} \tag{2.C.10}
$$

Taking logs requires

$$
\begin{aligned}
&(1+\phi)r + \delta + w_1\zeta_1 + w_2\zeta_2 - \frac{1}{2}\ln(1-2\alpha_1\zeta_1) - \frac{1}{2}\ln(1-2\alpha_2\zeta_2) \\
&+\Big[\big((\mu_1 - \frac{1}{2})\phi + \eta_1 + (\beta_1 - 1)\zeta_1 + \alpha_1\gamma_1^2\zeta_1\big) + \frac{\phi - 2\alpha_1\gamma_1\zeta_1}{2(1-2\alpha_1\zeta_1)}\Big] h_{1,t} \\
&+\Big[\big((\mu_2 - \frac{1}{2})\phi + \eta_2 + (\beta_2 - 1)\zeta_2 + \alpha_2\gamma_2^2\zeta_2\big) + \frac{\phi - 2\alpha_2\gamma_2\zeta_2}{2(1-2\alpha_2\zeta_2)}\Big] h_{2,t} = 0
\end{aligned}
\tag{2.C.11}
$$

Therefore, one possible solution of (2.C.11) can be defined as follows.

$$
\begin{aligned}
&\delta = -(\phi+1)r - \zeta_1 w_1 - \zeta_2 w_2 + \frac{1}{2}\ln(1-2\zeta_1\alpha_1) + \frac{1}{2}(1-2\zeta_2\alpha_2) \\
&\eta_1 = -(\mu_1 - \frac{1}{2})\phi - \zeta_1\alpha_1\gamma_1^2 + (1-\beta_1)\zeta_1 - \frac{(\phi - 2\zeta_1\alpha_1\gamma_1)^2}{2(1-2\zeta_1\alpha_1)} \\
&\eta_2 = -(\mu_2 - \frac{1}{2})\phi - \zeta_1\alpha_2\gamma_2^2 + (1-\beta_2)\zeta_2 - \frac{(\phi - 2\zeta_2\alpha_2\gamma_2)^2}{2(1-2\zeta_2\alpha_2)}
\end{aligned}
\tag{2.C.12}
$$

Then, we impose the Euler equation for the underlying index.

$$
\mathrm{E}_{t-1}\big[\frac{S_t}{S_{t-1}} \times \frac{M_t}{M_{t-1}}\big] = 1 \tag{2.C.13}
$$

where

$$
\frac{M_t}{M_{t-1}} \times \frac{S_t}{S_{t-1}} = \big(\frac{S_t}{S_{t-1}}\big)^{(\phi+1)} \exp\Big[\delta + \eta_1 h_{1,t} + \eta_2 h_{2,t} + \zeta_1(h_{1,t+1} - h_{1,t}) + \zeta_2(h_{2,t+1} - h_{2,t})\Big]. \tag{2.C.14}
$$

Following the results in (2.C.10), we replace ϕ by $\phi + 1$ and we have

$$
\begin{aligned}
\mathrm{E}_{t-1}\big[\frac{M_t}{M_{t-1}} \times \frac{S_t}{S_{t-1}}\big] = \exp\Big[& r(\phi+1) + \delta + w_1\zeta_1 + w_2\zeta_2 \\
& + \big((\mu_1 - \frac{1}{2})(\phi+1) + \eta_1 + (\beta_1 - 1)\zeta_1 + \alpha_1\gamma_1^2\zeta_1\big)\; h_{1,t} \\
& + \big((\mu_2 - \frac{1}{2})(\phi+1) + \eta_2 + (\beta_2 - 1)\zeta_2 + \alpha_2\gamma_2^2\zeta_2\big)\; h_{2,t} \\
& - \frac{1}{2}\ln(1 - 2\alpha_1\zeta_1) + \frac{(\phi+1) - 2\alpha_1\gamma_1\zeta_1}{2(1-2\alpha_1\zeta_1)} h_{1,t} \\
& - \frac{1}{2}\ln(1 - 2\alpha_2\zeta_2) + \frac{(\phi+1) - 2\alpha_2\gamma_2\zeta_2}{2(1-2\alpha_2\zeta_2)} h_{2,t}\Big] = \exp(-r)
\end{aligned}
\tag{2.C.15}
$$

Taking logs and substituting δ, η_1 and η_2 from (2.C.12) yield the following restriction.

$$
(\mu_1 - \frac{1}{2}) + (\mu_2 - \frac{1}{2}) + \frac{1 + 2\phi - 4\alpha_1\gamma_1\zeta_1}{2(1-2\alpha_1\zeta_1)}\; h_{1,t} + \frac{1 - 2\phi - 4\alpha_2\gamma_2\zeta_2}{2(1-2\alpha_2\zeta_2)}\; h_{2,t} = 0 \tag{2.C.16}
$$

Therefore, one admissible solution for the risk aversion parameter would be

$$
\phi = -(\mu_1 - \frac{1}{2} + \gamma_1)(1 - 2\alpha_1\zeta_1) + \gamma_1 - \frac{1}{2} = -(\mu_2 - \frac{1}{2} + \gamma_2)(1 - 2\alpha_2\zeta_2) + \gamma_2 - \frac{1}{2} \tag{2.C.17}
$$

To complete the proof, we need to specify how physical shocks $z_{1,t}$ and $z_{2,t}$ are transformed to risk-neutral shocks $z^*_{1,t}$ and $z^*_{2,t}$. We use the fact that the risk-neutral distribution is proportional to the physical distribution times pricing kernel. We also use the fact that $z_{1,t}$ and $z_{2,t}$ are independent.

$$
f^*_{t-1}(S_t) = \frac{M_t}{\mathrm{E}_{t-1}[M_t]} \times f_{t-1}(S_t) \tag{2.C.18}
$$

Using the proposed pricing kernel and physical dynamics and after some algebra, we find that the mean and variance may shift according to the following transformations.

$$
\begin{aligned}
z^*_{1,t} &= \sqrt{1-2\alpha_1\zeta_1}\big(z_{1,t} + (\mu_1 + \frac{\alpha_1\zeta_1}{1-2\alpha_1\zeta_1})\sqrt{h_{1,t}}\big) \\
z^*_{2,t} &= \sqrt{1-2\alpha_2\zeta_2}\big(z_{2,t} + (\mu_2 + \frac{\alpha_2\zeta_2}{1-2\alpha_2\zeta_2})\sqrt{h_{2,t}}\big)
\end{aligned}
\qquad (2.C.19)
$$

Note that the risk-neutral (2.2.21) dynamics can be derived by replacing the risk-neutral shocks (2.C.19) into the physical dynamics (2.2.14).

References

Adrian, T. and J. Rosenberg (2008). Stock returns and volatility: Pricing the short-run and long-run components of market risk. *The Journal of Finance 63*(6), 2997–3030.

Ait-Sahalia, Y. and R. Kimmel (2007). Maximum likelihood estimation of stochastic volatility models. *Journal of Financial Economics 83*(2), 413–452.

Albrecher, H., P. Mayer, W. Schoutens, and J. Tistaert (2006). The little heston trap. *status: published 6*(05).

Alizadeh, S., M. W. Brandt, and F. X. Diebold (2002). Range-based estimation of stochastic volatility models. *The Journal of Finance 57*(3), 1047–1091.

Bakshi, G., C. Cao, and Z. Chen (1997). Empirical performance of alternative option pricing models. *The Journal of Finance 52*(5), 2003–2049.

Bates, D. S. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *Review of Financial Studies 9*(1), 69–107.

Bates, D. S. (2000). Post-'87 crash fears in the s&p 500 futures option market. *Journal of Econometrics 94*(1), 181–238.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy 81*(3), 637–654.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics 31*(3), 307–327.

Bollerslev, T. and H. Zhou (2002). Estimating stochastic volatility diffusion using conditional moments of integrated volatility. *Journal of Econometrics 109*(1), 33–65.

Bollerslev, T. and H. Zhou (2006). Volatility puzzles: a simple framework for gauging return-volatility regressions. *Journal of Econometrics 131*(1), 123–150.

Boloorforoosh, A. (2014). Is idiosyncratic volatility risk priced? evidence from the physical and risk-neutral distributions. *Concordia University Working Paper*.

Broadie, M., M. Chernov, and M. Johannes (2007). Model specification and risk premia: Evidence from futures options. *The Journal of Finance 62*(3), 1453–1490.

Carr, P. and L. Wu (2007). Stochastic skew in currency options. *Journal of Financial Economics 86*(1), 213–247.

Chernov, M., A. R. Gallant, E. Ghysels, and G. Tauchen (2003). Alternative models for stock price dynamics. *Journal of Econometrics 116*(1), 225–257.

Chernov, M. and E. Ghysels (2000). A study towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of options valuation. *Journal of Financial Economics 56*(3), 407–458.

Christoffersen, P., R. Elkamhi, B. Feunou, and K. Jacobs (2010). Option valuation with conditional heteroskedasticity and nonnormality. *Review of Financial Studies 23*(5), 2139–2183.

Christoffersen, P., S. Heston, and K. Jacobs (2009). The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well. *Management Science 55*(12), 1914–1932.

Christoffersen, P., S. Heston, and K. Jacobs (2013). Capturing option anomalies with a variance-dependent pricing kernel. *Review of Financial Studies 26*(8), 1963–2006.

Christoffersen, P., K. Jacobs, and K. Mimouni (2010). Volatility dynamics for the s&p 500: Evidence from realized volatility, daily returns, and option prices. *Review of Financial Studies 23*(8), 3141–3189.

Christoffersen, P., K. Jacobs, C. Ornthanalai, and Y. Wang (2008). Option valuation with long-run and short-run volatility components. *Journal of Financial Economics 90*(3), 272–297.

Corradi, V. (2000). Reconsidering the continuous time limit of the garch (1,1) process. *Journal of Econometrics 96*(1), 145–153.

Cox, J. C., J. E. Ingersoll, and S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica 53*(2), 385–407.

Dai, Q. and K. J. Singleton (2000). Specification analysis of affine term structure models. *The Journal of Finance 55*(5), 1943–1978.

Dai, Q. and K. J. Singleton (2002). Expectation puzzles, time-varying risk premia, and affine models of the term structure. *Journal of Financial Economics 63*(3), 415–441.

Derman, E. (1999). Regimes of volatility. *Risk 4*, 55–59.

Duan, J.-C. (1995). The garch option pricing model. *Mathematical Finance 5*(1), 13–32.

Duan, J.-C., G. Gauthier, and J.-G. Simonato (1999). An analytical approximation for the garch option pricing model. *Journal of Computational Finance 2*(4), 75–116.

Duffie, D., J. Pan, and K. Singleton (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica 68*(6), 1343–1376.

Egloff, D., M. Leippold, and L. Wu (2010). The term structure of variance swap rates and optimal variance swap investments. *Journal of Financial and Quantitative Analysis 45*(5), 1279–1310.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica 50*(4), 987–1007.

Engle, R. F. and G. Lee (1999). *A permanent and transitory component model of stock return volatility, in Robert F. Engle, and Halbert L. White, eds.: Cointegration, Causality, and Forecasting: A Festschrift in Honor of CliveW. J. Granger*. Oxford University Press.

Eraker, B. (2004). Do stock prices and volatility jump? reconciling evidence from spot and option prices. *The Journal of Finance 59*(3), 1367–1404.

Feller, W. (1951). Two singular diffusion problems. *Annals of Mathematics 54*(1), 173–182.

Feunou, B. and R. Tédongap (2012). A stochastic volatility model with conditional skewness. *Journal of Business & Economic Statistics 30*(4), 576–591.

Gordon, N. J., D. J. Salmond, and A. F. Smith (1993). Novel approach to nonlinear/non-gaussian bayesian state estimation. In *IEE Proceedings F (Radar and Signal Processing)*, Volume 140, pp. 107–113. IET.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies 6*(2), 327–343.

Heston, S. L. and S. Nandi (2000). A closed-form garch option valuation model. *Review of Financial Studies 13*(3), 585–625.

Huang, J.-z. and L. Wu (2004). Specification analysis of option pricing models based on time-changed lévy processes. *The Journal of Finance 59*(3), 1405–1440.

Johannes, M. and N. Polson (2009). Particle filtering. *Handbook of Financial Time Series*, 1015–1029.

Johannes, M. S., N. G. Polson, and J. R. Stroud (2009). Optimal filtering of jump diffusions: Extracting latent states from asset prices. *Review of Financial Studies 22*(7), 2759–2799.

Jones, C. S. (2003). The dynamics of stochastic volatility: evidence from underlying and options markets. *Journal of Econometrics 116*(1), 181–224.

Khrapov, E. R. S. and E. Renault (2016). Affine option pricing model in discrete time. *Working Paper*.

Li, H., M. T. Wells, and L. Y. Cindy (2008). A bayesian analysis of return dynamics with lévy jumps. *Review of Financial Studies 21*(5), 2345–2378.

Mencía, J. and E. Sentana (2013). Valuation of vix derivatives. *Journal of Financial Economics 108*(2), 367–391.

Pan, J. (2002). The jump-risk premia implicit in options: Evidence from an integrated time-series study. *Journal of Financial Economics 63*(1), 3–50.

Pitt, M. K. (2002). Smooth particle filters for likelihood evaluation and maximisation. the warwick economics research paper series. *University of Warwick, Department of Economics.*

Santa-Clara, P. and S. Yan (2010). Crashes, volatility, and the equity premium: Lessons from s&p 500 options. *The Review of Economics and Statistics 92*(2), 435–451.

Wiggins, J. B. (1987). Option values under stochastic volatility: Theory and empirical estimates. *Journal of Financial Economics 19*(2), 351–372.

Table 2.1: S&P 500 Index Call Option Data Characteristics by Moneyness and Maturity

Panel A: Number of call option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤0.92	152	3,371	12,690	8,782	24,995
0.92<S/K≤0.94	642	8,220	17,345	8,342	34,549
0.94<S/K≤0.96	4,033	14,436	18,557	8,096	45,122
0.96<S/K≤0.98	10,761	17,202	17,000	7,167	52,130
S/K>0.98	13,052	16,137	15,628	6,485	51,302
All	28,640	59,366	81,220	38,872	208,098

Panel B: Average price of call option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤0.92	13.6200	15.5478	23.0998	47.0797	24.8368
0.92<S/K≤0.94	11.7434	16.1440	26.2574	56.2993	27.6110
0.94<S/K≤0.96	9.9935	18.0151	34.2459	69.4400	32.9236
0.96<S/K≤0.98	11.5532	24.4015	44.6126	82.1867	40.6885
S/K>0.98	18.5235	35.5330	57.9296	95.6642	51.9126
All	13.0867	21.9283	37.2290	70.1340	35.5945

Panel C: Average implied volatility of call option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤0.92	0.4071	0.2299	0.1894	0.1791	0.2514
0.92<S/K≤0.94	0.3163	0.2034	0.1760	0.1831	0.2197
0.94<S/K≤0.96	0.2213	0.1792	0.1770	0.1881	0.1914
0.96<S/K≤0.98	0.1784	0.1741	0.1833	0.1958	0.1829
S/K>0.98	0.1715	0.1829	0.1900	0.2028	0.1868
All	0.2589	0.1939	0.1831	0.1898	0.2064

Panel D: Average delta of call option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤0.92	0.2316	0.2302	0.2724	0.3726	0.2767
0.92<S/K≤0.94	0.2329	0.2549	0.3121	0.4268	0.3067
0.94<S/K≤0.96	0.2381	0.2984	0.3832	0.4827	0.3506
0.96<S/K≤0.98	0.2996	0.3843	0.4608	0.5319	0.4191
S/K>0.98	0.4422	0.4976	0.5377	0.5771	0.5136
All	0.2889	0.3331	0.3932	0.4782	0.3733

Note to Table: This table reports the summary statistics of out-of-the-money S&P 500 call option contracts in our sample, from January 1, 1996 to December 31, 2011. The implied volatilities and the deltas are from the OptionMetrics volatility surface data set. S denotes the price of the S&P 500 index, K the option strike price, and DTM denotes the number of calandar days to maturity.

Table 2.2: S&P 500 Index Put Option Data Characteristics by Moneyness and Maturity

Panel A: Number of put option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤1.02	10,776	13,499	13,463	5,904	43,642
1.02<S/K≤1.04	7,163	10,951	12,018	5,008	35,140
1.04<S/K≤1.06	3,699	8,083	10,399	5,317	27,498
1.06<S/K≤1.08	1,248	5,334	8,105	3,908	18,595
S/K>1.08	385	3,173	5,591	3,588	12,737
All	23,271	41,040	49,576	23,725	137,612

Panel B: Average price of put option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤1.02	18.7121	30.3521	44.9423	63.5550	39.3904
1.02<S/K≤1.04	13.9689	25.4113	40.1731	59.5418	34.7738
1.04<S/K≤1.06	12.7334	21.7862	34.1231	55.3294	30.9930
1.06<S/K≤1.08	14.0224	20.8254	30.5229	44.3883	27.4397
S/K>1.08	16.1005	20.9994	30.9259	43.7921	27.9545
All	15.1075	23.8749	36.1375	53.3213	32.1103

Panel C: Average implied volatility of put option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤1.02	0.1929	0.1933	0.1992	0.2121	0.1994
1.02<S/K≤1.04	0.2194	0.2134	0.2158	0.2127	0.2153
1.04<S/K≤1.06	0.2646	0.2314	0.2233	0.2313	0.2376
1.06<S/K≤1.08	0.3342	0.2599	0.2367	0.2200	0.2627
S/K>1.08	0.4255	0.2904	0.2583	0.2343	0.3021
All	0.2873	0.2377	0.2266	0.2221	0.2434

Panel D: Average delta of put option contracts

	DTM≤30	30<DTM≤91	91<DTM≤182	DTM>182	All
S/K≤1.02	-0.3931	-0.3988	-0.3931	-0.3631	-0.3870
1.02<S/K≤1.04	-0.2860	-0.3221	-0.3403	-0.3334	-0.3204
1.04<S/K≤1.06	-0.2348	-0.2699	-0.2932	-0.3060	-0.2760
1.06<S/K≤1.08	-0.2194	-0.2395	-0.2579	-0.2612	-0.2445
S/K>1.08	-0.2175	-0.2209	-0.2431	-0.2547	-0.2341
All	-0.2702	-0.2902	-0.3055	-0.3037	-0.2924

Note to Table: This table reports the summary statistics of out-of-the-money S&P 500 put option contracts in our sample, from January 1, 1996 to December 31, 2011. The implied volatilities and delta are from the OptionMetrics volatility surface data set. S denotes the price of the S&P 500 index, K the option strike price, and DTM denotes the number of calandar days to maturity.

Table 2.3: Market Parameter Estimates

Panel A: Parameter Estimates (Physical) - Joint Estimation

κ_1	κ_2	θ_1	θ_2	σ_1	σ_2	ρ_1	ρ_2	λ_1	λ_2
1.4271	3.5874	0.0026	0.0171	0.0855	0.3496	-0.6918	-0.2173	-1.0798	-1.0355

Panel B: Parameter Estimates (Risk Neutral) - Options-based Estimation

$\tilde{\kappa}_1$	$\tilde{\kappa}_2$	$\tilde{\theta}_1$	$\tilde{\theta}_2$	σ_1	σ_2	ρ_1	ρ_2
0.2267	2.9137	0.0590	0.0100	0.0958	0.5678	-0.9135	-0.4934

Note to Table: This table reports the structural parameter estimates of the S&P 500 Index for the two-factor stochastic volatility model. The reported results in Panel A are from the joint estimation using the daily S&P 500 index returns and options data. Structural parameters in Panel B are estimated using only options data. In both panels, we use 10% OTM call and put options over the period 1996-2011. As in Proposition (2.1), $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$. Therefore, risk neutral parameters from joint estimation are $\tilde{\kappa}_1 = 0.3473$, $\tilde{\kappa}_2 = 2.5520$, $\tilde{\theta}_1 = 0.0106$, $\tilde{\theta}_2 = 0.0240$.

Table 2.4: Goodness of Fit

	Option Based Estimation				Joint Estimation		
	Number of Obs.	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**
	Panel A: Goodness of Fit - Call Option Contracts						
DTM≤30	28,640	1.2956			2.7171		
30<DTM≤91	59,366	0.8695			2.5104		
91<DTM≤182	81,220	0.6913			2.3505		
DTM>182	38,872	0.8943			2.6032		
All	208,098	0.8846	0.9132	4.4244	2.5299	2.5637	12.4210
	Panel B: Goodness of Fit - Put Option Contracts						
DTM≤30	23,271	1.6193			2.8857		
30<DTM≤91	41,040	1.0712			2.4509		
91<DTM≤182	49,576	0.8342			2.4941		
DTM>182	23,725	1.0440			2.5256		
All	137,612	1.1064	1.1167	4.5879	2.5877	2.6389	10.8418
	Panel C: Goodness of Fit - All Option Contracts						
DTM≤30	51,911	1.4497			2.7946		
30<DTM≤91	100,406	0.9571			2.4835		
91<DTM≤182	130,796	0.7486			2.4180		
DTM>182	62,597	0.9538			2.5665		
All	345,710	0.9790	0.9992	4.4428	2.5566	2.5939	11.5335

Note to Table: This table reports in-sample goodness-of-fit for our two-factor stochastic volatility model over the entire sample, 1996 through 2011 for various maturities. We also report in-sample fit for calls and puts separately. All numbers are in percentage points. We compute the Vega-weighted root mean squared error (Vega RMSE) along with the implied volatility root mean squared error (IVRMSE). We also report the ratio of IVRMSE over the average implied volatility. To provide a basis for caparison the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.

Table 2.5: Subsample Parameter Estimates

κ_1	κ_2	θ_1	θ_2	σ_1	σ_2	ρ_1	ρ_2	λ_1	λ_2
Panel A: Joint Estimation:1996 - 2003									
1.2138	3.2780	0.0033	0.0195	0.0855	0.3220	-0.6514	-0.2985	-1.1008	-0.9755
Panel B: Joint Estimation (2003 - 2011)									
1.1274	4.2337	0.0069	0.0289	0.0793	0.4675	-0.5102	-0.3086	-1.0684	-1.0351
Panel C: Options-based Estimation (1996-2003)									
0.1794	2.6176	0.0437	0.0104	0.0912	0.3732	-0.8891	-0.4434		
Panel D: Options-based Estimation (2003-2011)									
0.1117	3.4731	0.0623	0.0247	0.0837	0.6692	-0.7550	-0.6497		

Note to Table: This table reports the structural parameter estimates of the S&P 500 Index for the two-factor stochastic volatility model over two subsample period. The first subsample is from January 1996 to December 2003 and the second one is from January 2004 to December 2011. The point estimates in Panel A and Panel B are from the joint estimation using the daily S&P 500 index returns and options data. Entries in Panel C and Panel D are estimated using only options data. In both panels, we use 10% OTM call and put options over the period 1996-2011.

Table 2.6: Subsample Goodness of Fit (1996-2003)

	Option Based Estimation				Joint Estimation		
	Number of Obs.	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**
Panel A: Subsample Goodness of Fit (1996-2003) - Call Option Contracts							
DTM≤30	14,267	1.2355			2.9061		
30<DTM≤91	30,414	0.8397			2.8784		
91<DTM≤182	39,160	0.7194			2.7826		
DTM>182	18,237	0.7593			3.0274		
All	102,078	0.8514	0.8846	4.5041	2.8787	2.9137	12.8697
Panel B: Subsample Goodness of Fit (1996-2003) - Put Option Contracts							
DTM≤30	11,775	1.5167			3.3108		
30<DTM≤91	20,282	1.1038			2.9729		
91<DTM≤182	24,137	0.8742			2.9596		
DTM>182	11,528	1.0111			2.9025		
All	67,722	1.1006	1.1067	4.7416	3.0462	3.1389	11.9169
Panel C: Subsample Goodness of Fit (1996-2003) - All Option Contracts							
DTM≤30	26,042	1.3698			3.1091		
30<DTM≤91	50,696	0.9542			2.9218		
91<DTM≤182	63,297	0.7820			2.8691		
DTM>182	29,765	0.8655			2.9682		
All	169,800	0.9586	0.9792	4.5567	2.9592	3.0055	12.2725

Note to Table: This table reports in-sample goodness-of-fit for our two-factor stochastic volatility model over the entire sample, 1996 through 2011 for various maturities. We also report in-sample fit for calls and puts separately. All numbers are in percentage points. We compute the Vega-weighted root mean squared error (Vega RMSE) along with the implied volatility root mean squared error (IVRMSE). We also report the ration of IVRMSE over the average implied volatility. To provide a basis for caparison the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.

Table 2.7: Subsample Goodness of Fit (2004-2011)

	Option Based Estimation				**Joint Estimation**		
	Number of Obs.	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**
Panel A: Subsample Goodness of Fit (2004-2011) - Call Option Contracts							
DTM≤30	14,373	1.3526			2.5715		
30<DTM≤91	28,952	0.8998			2.1570		
91<DTM≤182	42,060	0.6640			1.9298		
DTM>182	20,635	0.9985			2.0532		
All	106,020	0.9155	0.9471	4.1833	2.2014	2.3017	10.1665
Panel B: Subsample Goodness of Fit (2004-2011) - Put Option Contracts							
DTM≤30	11,496	1.7181			2.4266		
30<DTM≤91	20,758	1.0383			1.9112		
91<DTM≤182	25,439	0.7944			1.9656		
DTM>182	12,197	1.0741			2.0348		
All	69,890	1.1121	1.1437	4.3421	2.0802	2.1294	8.0843
Panel C: Subsample Goodness of Fit (2004-2011) - All Option Contracts							
DTM≤30	25,869	1.5259			2.5109		
30<DTM≤91	49,710	0.9601			2.0487		
91<DTM≤182	67,499	0.7159			1.9459		
DTM>182	32,832	1.0273			2.0445		
All	175,910	0.9982	1.0297	4.2046	2.1480	2.2348	9.1255

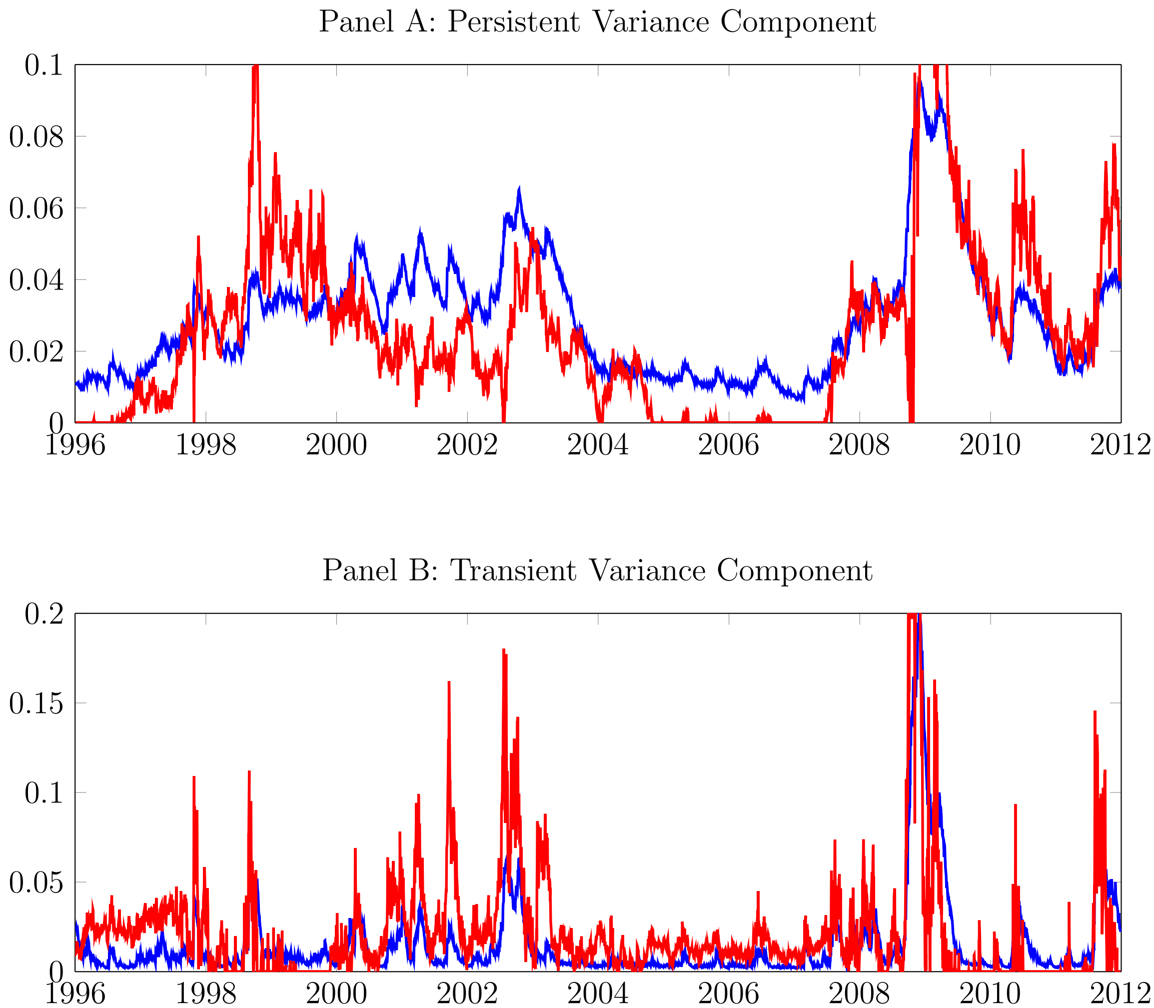
Note to Table: This table reports goodness-of-fit for our two-factor stochastic volatility model over the subsample from January 2004 through December 2011 for various maturities. We also report in-sample fit for calls and puts separately. All numbers are in percentage points. We compute vega-weighted root mean squared error (Vega RMSE) along with implied volatility root mean squared error (IVRMSE). We also report the ration of IVRMSE over the average implied volatility. To provide a basis for caparison the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.

Table 2.8: Out of Sample Goodness of Fit (2004-2011)

	Option Based Estimation				**Joint Estimation**		
	Number of Obs.	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**
Panel A: Out of Sample Goodness of Fit (2004-2011) - Call Option Contracts							
DTM≤30	14,373	1.4764			2.7853		
30<DTM≤91	28,952	0.9372			2.2801		
91<DTM≤182	42,060	0.6902			1.9978		
DTM>182	20,635	1.0797			2.1189		
All	106,020	0.9753	0.9985	4.4103	2.2201	2.3907	10.5596
Panel B: Out of Sample Goodness of Fit (2004-2011) - Put Option Contracts							
DTM≤30	11,496	1.8064			2.5780		
30<DTM≤91	20,758	1.1048			1.9984		
91<DTM≤182	25,439	0.8359			1.9856		
DTM>182	12,197	1.1153			2.1478		
All	69,890	1.1708	1.2142	4.6097	2.1259	2.2087	8.3853
Panel C: Out of Sample Goodness of Fit (2004-2011) - All Option Contracts							
DTM≤30	25,869	1.6313			2.6952		
30<DTM≤91	49,710	1.0105			2.1670		
91<DTM≤182	67,499	0.7485			1.9932		
DTM>182	32,832	1.0931			2.1297		
All	175,910	1.0573	1.0893	4.4480	2.1831	2.3201	9.4737

Note to Table: This table reports out-of-sample goodness-of-fit for our two-factor stochastic volatility model over the period from January 2004 through December 2011 for various maturities. We also report out-of-sample fit for calls and puts separately. All numbers are in percentage points. Out-of-sample daily spot persistent and transient variance components are filtered with Particle Filter method given the in-sample structural parameter estimates over the period January 1996 through December 2003. The Vega RMSE along with the IVRMSE are computed given in-sample structural parameters and filtered variance components. We also report the ratio of IVRMSE over the average implied volatility. To provide a basis for capari-son the left panel reports pricing errors based on the option data and the right panel reports those of joint estimation.

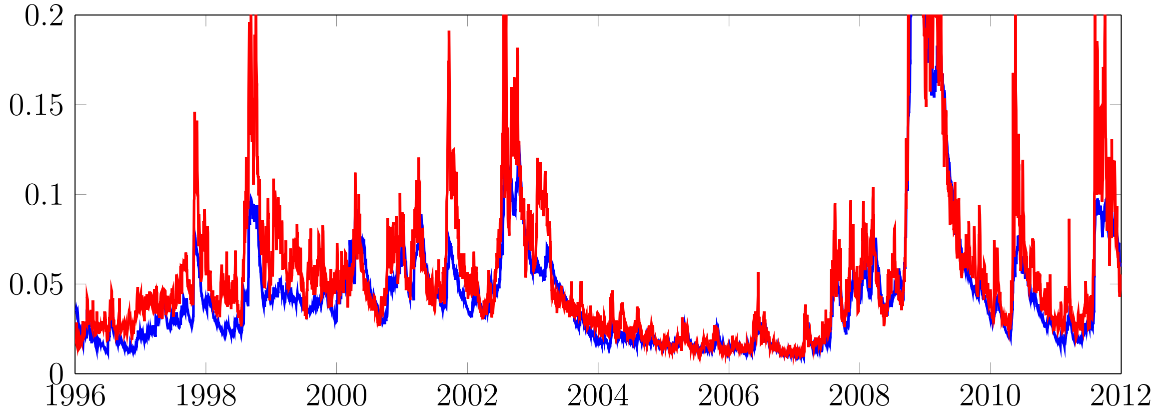
Figure 2.1: The S&P 500 Index Spot Variance Components Paths



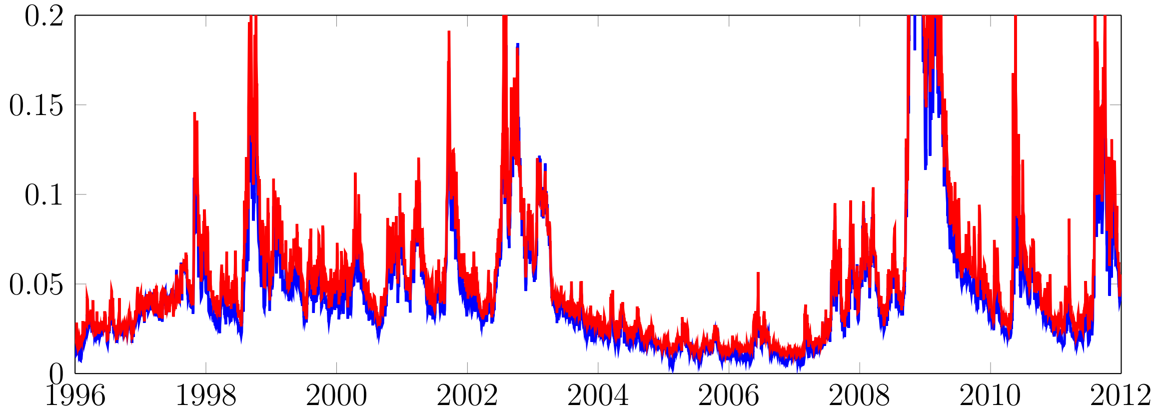
Note to Figure: We plot time series of risk-neutral spot variances for the S&P 500 index in the two-factor stochastic volatility model. Panel A shows time series of persistent variance component and Panel B shows time series of transient variance component. The blue plots are based on the Particle Filter method using data from both S&P 500 index and option markets (joint estimation). The red plots are filtered spot variances using data from S&P 500 option market only.

Figure 2.2: The S&P 500 Index Total Spot Variance Path Versus VIX

Panel A: Joint Estimation Versus VIX



Panel B: Option-Based Estimation Versus VIX



Note to Figure: We plot time series of risk-neutral total spot variance for the S&P 500 index by combining persistent and transient variance components of the two-factor stochastic volatility model. The blue plots in Panel A is based on the Particle Filter method using data from both S&P 500 index and option markets (joint estimation). The blue plot in Panel B is based on data from S&P 500 option market only. Red plots in both panels are time series of the VIX option implied volatility index.

Chapter 3

The Transient and The Persistent Factor Structure in Equity Options

3.1 Introduction

This paper extend two factor stochastic volatility models to the equilibrium pricing of equity options and find that the existence of multiple volatility components in the dynamics of index has significant implications for equity option prices. Extensive empirical evidence supports the presence of two volatility components in the dynamics of the market index. These studies document that a single factor stochastic volatility (SV) is not sufficient to represent both the underlying (P) and the risk neutral (Q) measures of the joint dynamics of returns and variances for the key S&P 500 index and its options. In the P-distribution domain two volatility components are required to simultaneously capture the persistence of volatility and the volatility of volatility.[1] In the Q-distribution domain, multiple SV models have more flexibility to fit the term structure of the volatility and to control the level and the slope of the volatility smirk in cross-sections of options.[2] Given the performance of two volatility components in capturing the stylized facts relative to one-factor SV models, we examine how equity options respond to the existence of two volatility components in the dynamics of the market index.

We extend the one-volatility-factor model in Christoffersen et al. [2015] and assume that individual equity returns are related to the market index with two distinct systematic components (two constant factor loadings), as well as an idiosyncratic component which is stochastic and follows the standard square root process. Hence, equity returns are related to the market index with two distinct betas, one of which captures the transient variations in market returns and the other one captures its persistent counterpart. We obtain a closed-form option pricing equation for individual equity options as the proposed model belongs to the affine class of models. We show that instantaneous expected returns of equity options depend on both transient and persistent betas.

In empirical analysis, we estimate the structural parameters and filter spot idiosyncratic variance for the firms listed in the Dow Jones index. We find that proposed option pricing model provides a good fit to the observed equity option prices across all of the 27 firms, both in-sample and out-of-sample. Further, the in-sample performance of our model over the one-factor structure of Christoffersen et al. [2015] together with its cross-sectional im-

[1] See, for instance, Chernov et al. [2003].

[2] See Christoffersen et al. [2009], Egloff et al. [2010], and Mencía and Sentana [2013] among others. Egloff et al. [2010, Page 1289] show that the upward slope of autocorrelation term structure of variance swap rate quotes points to the existence of multiple variance risk factors.

plications regarding IV term-structure, moneyness slope, and equity option skew support the importance of transient and persistent factor loadings in pricing equity options. Our estimation results show that the transient and persistent betas have quite different values across all the firms: in our sample of 27 firms, the transient beta has values ranging from 1.01 to 1.35, while the persistent beta is about half the value, range from 0.34 to 0.68. Our empirical investigation of this model using individual equity option prices finds support for the proposed factor structure in equity options.

Our models' framework is especially important for a portfolio manager who hedges her portfolio's exposure to the systematic risk factors in the portfolio of stocks and options.[3] Our proposed factor structure and closed-form option pricing equation make this analysis readily available and yields similar closed-form expressions for the exposure of equity options to the transient and persistent market variance components in addition to its exposure to the overall market returns. We also obtain a closed-form expression for the expected equity option returns and show that exposures to the level of market index and market variance components affect the expected equity option returns. In other words, we are able to disentangle the effect of market risk premium from those of persistent and transient variance risk premiums on the expected equity option returns.

The proposed factor structure has a number of important cross-sectional implications for equity options. Our model predicts that firms with higher transient betas have higher implied volatilities. It also predicts that firms with higher transient betas have steeper term structures of implied volatility while the persistent betas have a marginal effect on the implied volatility term structures. It also predicts that the implied volatility moneyness slopes are steeper for the firms with the higher transient betas while the persistent betas have a much less significant effect on the moneyness slopes. Consistent with previous studies, we find that the variance risk premium has a significant effect on the equity option skew. More to the point, our model predicts that it is the transient variance risk premium that mainly drives the slope of equity implied volatility smile for individual equities.

Our proposed factor structure in equity options is motivated by the extensive empirical evidence that supports the presence of two variance components in the dynamics of the market index.[4] In the P-distribution domain, they document that two volatility factors are needed to explain the volatility dynamics, since one-factor models are incapable of simultaneously

[3]The proposed framework is equally important for risk managers and dispersion traders.

[4]The aggregate market volatility is decomposed into two independent components, one with persistent dynamics and the other one with transient dynamics.

fitting the persistence of volatility and the volatility of volatility in the dynamics of the market index. Chernov et al. [2003] suggest that the addition of a second volatility factor breaks the link between tail thickness and volatility persistence and leads to a significant improvement relative to a single SV models in capturing the return dynamics. Bollerslev and Zhou [2002] and Alizadeh et al. [2002] documents the importance of two volatility components in capturing the dynamics of exchange rates. According to Dai and Singleton [2000, 2002] multifcator volatility models are needed to model the term structure of the interest rate.

Extensive empirical evidence in the Q-distribution domain also point toward the existence of two variance components. Egloff et al. [2010] and Mencía and Sentana [2013] find that two-factor SV models have more flexibility to fit the term structure of the volatility and to control the level and the slope of the volatility smirk in cross-sections of option prices. Christoffersen et al. [2009] show that multiple SV models can better capture the time-varying nature of the smirk as the correlation between stock returns and total volatility is stochastic and thus can generate sufficient amounts of conditional skewness and kurtosis. Egloff et al. [2010, Page 1289] show that the upward sloping autocorrelation term structure of variance swap rate quotes points to the existence of multiple variance risk factors. In a model free framework, Christoffersen et al. [2009] find that the first two principal components of the Black-Scholes implied variances on a sample of S&P 500 index options together explain more than 95% of the variation in the implied variances.

Within the asset pricing models, when market volatility is stochastic, the classical Intertemporal CAPM (ICAPM) model of Merton [1973] and Merton [1980] implies that in the presence of two state variables, namely return and volatility, the excess returns on the market portfolio should also be related to the volatility of the market. In other words, the asset risk premiums are not only determined by the covariation of asset returns with the market returns, but also by its covariation with the state variables that govern the market volatility.[5] More recently, Adrian and Rosenberg [2008] show that the equilibrium pricing kernel depends on both the short- and long-run volatility components as well as the excess market returns. Using a large cross section of data from individual equity returns, they find negative and highly significant risk premiums for both volatility components. Our paper extends the insights of these earlier studies into the pricing of equity options, formulates the simultaneous equilibrium of both equity underlying and option markets, and tests empirically the derived

[5]See also Ang et al. [2006], who show that the show that the aggregate market volatility is a significant cross-sectional asset pricing factor.

results.

This chapter proceeds as follows. Section (3.2) presents the theoretical model for pricing individual equity options. In Section (3.3) we discuss the properties and implications of the model. Section (3.4) contains the description of the data sets. In Section (3.5) we discuss the estimation methodologies and then present the estimation results in Section (3.6). Section (3.7) investigate the performance of the model and its goodness-of-fit. Section (3.8) concludes. The appendix provides the proofs of the theoretical results.

3.2 Model Setup

We start by a multiple-factor stochastic volatility dynamics that governs the market index returns under the P-distributions and then introduce its risk neutral counterparts as in Ghanbari [2016]. We then describe the dynamics of individual equity returns under P-distribution and introduce an appropriate stochastic discount factor (SDF) to find the equity dynamics under Q-distribution. Last, we derive a closed-form equation that gives the price of individual equity options.

We assume the following two-factor stochastic volatility process governing the dynamics of the market index returns and variance under the physical distributions.

$$\begin{aligned}
dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t} \\
dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t} \ ,
\end{aligned} \tag{3.2.1}$$

with two independent variance components as described in the following stochastic structure (3.2.2).

$$\begin{aligned}
&\langle dw_{1,t}, dz_{1,t}\rangle = \rho_1 dt, \ -1 \leq \rho_1 \leq +1 \\
&\langle dw_{2,t}, dz_{2,t}\rangle = \rho_2 dt, \ -1 \leq \rho_2 \leq +1 \\
&\langle dw_{1,t}, dw_{2,t}\rangle = 0 \\
&\rho_1^2 + \rho_2^2 \leq +1
\end{aligned} \tag{3.2.2}$$

The model parameters have the conventional definition as in the Heston [1993] SV model:

κ_1 and κ_2 capture the speed of mean reversion of each variance component, θ_1 and θ_2 are the unconditional average variances of persistent and transient variance components, and σ_1 and σ_2 measure the volatility of variance components. The instantaneous correlation between shocks to market returns and shocks to the persistent variance component is described by ρ_1 and the instantaneous correlation between shocks to market returns and shocks to the transient variance component is given by ρ_2, known as "continuous-time" leverage effect. Note that $\mu_1 v_{1,t} + \mu_2 v_{2,t}$ is the index equity risk premium.[6]

Following Ghanbari [2016], the market index has the following dynamics under the risk-neutral measure.[7]

$$\begin{aligned} dS_t/S_t &= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \ , \\ dv_{1,t} &= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t} \ , \\ dv_{2,t} &= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t} \ , \end{aligned} \tag{3.2.3}$$

where, $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, and $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$, and where λ_1 and λ_2 are the prices of the variances risk factors as in the single factor SV model.

For individual equities, we assume that equity returns are related to the market returns with two distinct systematic risk factors and two constant factor loadings β_1^i and β_2^i. Following Bakshi et al. [2003] we assume that idiosyncratic shocks to equity returns ξ_t^i follows a standard square-root process. This assumption allows us to characterize the differences in the moments' dynamics of individual equity and index options.[8]

$$\begin{aligned} dS_t^i/S_t^i &= \mu^i dt + \beta_1^i(\mu_1 v_{1,t}dt + \sqrt{v_{1,t}}dz_{1,t}) + \beta_2^i(\mu_2 v_{2,t}dt + \sqrt{v_{2,t}}dz_{2,t}) + \sqrt{\xi_t^i}dz_t^i \\ d\xi_t^i &= \kappa^i(\theta^i - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^i \end{aligned} \tag{3.2.4}$$

where κ^i, θ^i, and σ^i can be defined as for their market counterparts. ρ^i is the correlation

[6]Note that (3.2.2) implies that the total return variance $\text{Var}_t[dS_t/S_t] = v_{1,t}dt + v_{2,t}dt \equiv v_t dt$.

[7]A complete derivations of risk neutral dynamics of two-factor SV model together with appropriate pricing kernel that links P- and Q-dynamics are in Ghanbari [2016].

[8]Our model can be extended to examine the idiosyncratic variance risk premium while incorporating two-factor structure in the dynamics of equity returns. We discuss the implications of priced idiosyncratic variance in the following section.

coefficient between idiosyncratic return innovations and idiosyncratic variance innovations for every individual equity i. This parameter captures an asymmetry in the relation between idiosyncratic volatility and individual equity returns.[9] Given the specification (3.2.4) the total instantaneous variance for stock i at time t under physical measure is given by

$$v_t^i \equiv (\beta_1^i)^2 v_{1,t} + (\beta_2^i)^2 v_{2,t} + \xi_t^i \tag{3.2.5}$$

In order to price options on individual equities, Proposition (3.1) gives the risk neutral dynamics of an individual equity i by assuming a conventional stochastic discount factor, given the physical dynamics (3.2.2) and (3.2.4). We also assume that the prices of market variance components are proportional to the spot volatility components.[10]

Proposition 3.1. *Using a conventional stochastic discount factor and given the dynamics of the individual equity returns under P-measure (3.2.4), the following dynamics govern its Q-measure counterparts.*

$$\begin{aligned} dS_t^i/S_t^i &= rdt + \beta_1^i\sqrt{v_{1,t}}d\tilde{z}_{1,t} + \beta_2^i\sqrt{v_{2,t}}d\tilde{z}_{2,t} + \sqrt{\xi_t^i}d\tilde{z}_t^i \\ d\xi_t^i &= \kappa^i(\theta^i - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^i \end{aligned} \tag{3.2.6}$$

The market prices of risk factors are

$$\begin{aligned} \psi_{1,t} &= \frac{\sigma_1\mu_1 - \rho_1\lambda_1}{\sigma_1(1-\rho_1^2)}\sqrt{v_{1,t}}\ ,\ \psi_{2,t} = \frac{\sigma_2\mu_2 - \rho_2\lambda_2}{\sigma_2(1-\rho_2^2)}\sqrt{v_{2,t}}\ , \\ \psi_{1,t}^i &= \frac{\mu^i - r}{\sqrt{\xi_t^i}(1-(\rho^i)^2)}\ ,\ \psi_{2,t}^i = -\frac{\mu^i - r}{\sqrt{\xi_t^i}}\frac{\rho^i}{1-(\rho^i)^2}\ . \end{aligned} \tag{3.2.7}$$

Proof. See Appendix (3.A). □

As the dynamics of individual equities are affine, the conditional risk-neutral characteristic function of the natural logarithm of the equity price i is derived analytically in the following

[9] Following Andersen et al. [2001] we expect that the observed asymmetry should be weaker but still present for individual equities.

[10] We can simply extend our model and consider the priced idiosyncratic variance risk by assuming that idiosyncratic variance risk is also proportional to the spot idiosyncratic volatility. In this case, $\tilde{\kappa}^i = \kappa^i + \lambda^i$, $\tilde{\theta}^i = \frac{k^i\theta^i}{k^i+\lambda^i}$. Further details are provided in the proof of the Proposition (3.1).

proposition. We may then compute the closed-from pricing equation for European equity call options with strike price K and time to maturity τ. See also Appendix B.

Proposition 3.2. *Given the dynamics of the individual equity returns under the Q-measure (3.2.6), the risk-neutral conditional characteristic function of the natural logarithm of individual equity price i, $x^i_{t+\tau} = \ln(S^i_{t+\tau})$, is:*

$$
\begin{aligned}
\tilde{f}^i(x^i_t, v_{1,t}, v_{2,t}, \xi^i, \beta^i_1, \beta^i_2, \tau, \phi) &\equiv \mathrm{E}^Q_t\left[\exp(i\phi x^i_{t+\tau}) \mid x^i_t\right] \\
&= \exp\big[i\phi x^i_t + i\phi r\tau - A_1(\tau,\phi) - A_2(\tau,\phi) - B(\tau,\phi) \\
&\quad + C_1(\tau,\phi)v_{1,t} + C_2(\tau,\phi)v_{2,t} + D(\tau,\phi)\xi^i_t\big] \ ,
\end{aligned} \tag{3.2.8}
$$

where, the expressions for $A_1(\tau,\phi)$, $A_2(\tau,\phi)$, $B(\tau,\phi)$, $C_1(\tau,\phi)$, $C_2(\tau,\phi)$, and $D(\tau,\phi)$ are provided within the proof. Then, individual equity option prices may be found as follows.

$$
C^i_t(S^i_t, K, \tau) = S^i_t P^i_1 - K e^{-r\tau} P^i_2 \ , \tag{3.2.9}
$$

where,

$$
\begin{aligned}
P^i_1 &= \frac{1}{2} + \frac{1}{\pi}\frac{1}{S^i_t e^{r\tau}}\int_0^\infty \Re\Big[\frac{e^{-i\phi\ln K}\tilde{f}^i(v_{1,t}, v_{2,t}, \xi^i_t, \tau, \phi - i)}{i\phi}\Big]d\phi \ , \\
P^i_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\Big[\frac{e^{-i\phi\ln K}\tilde{f}^i(v_{1,t}, v_{2,t}, \xi^i_t, \tau, \phi)}{i\phi}\Big]d\phi \ .
\end{aligned} \tag{3.2.10}
$$

Proof. See Appendix (3.B). □

3.3 Model Properties and Implications

This section explores, both theoretically and numerically, some of the implications of the proposed two-factor structure in the dynamics of equity returns. In particular, we examine the relative importance of the transient and persistent volatility components on the sensitivity of the equity option prices with respect to the level of the market index and with respect to each variance component. We also investigate the effects of factor loadings β^i_1 and β^i_2 and their importance on the instantaneous expected returns of individual equity options. We close this section by exploring a number of important cross-sectional implications of two-factor structure in equity options, some of which shed some lights on the relations between the systematic risk factors and moments of the conditional distribution of equity returns.

In the numerical analysis, we fix parameters as follows; structural parameters for the market index model are from Christoffersen et al. [2009], for individual equities the parameters are set to replicate the observed patterns in the one-factor model of Christoffersen et al. [2015]. Further, these parameter values highlight the importance of two-factor structure relative to one-factor structure in examining the properties and cross-sectional implications of factor structure in equity options. Since we are interested in the role of the persistent beta, β_1^i, and the transient beta, β_2^i, we explore the model properties for different sets of betas while keeping the total unconditional risk-neutral equity variance constant.

The total unconditional risk-neutral equity variance is evaluated at its mean reverting value equal to $\tilde{v}^i \equiv (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$. Note that we fix the total unconditional risk-neutral market variance to 0.05, with its persistent component $\tilde{\theta}_1 = 0.006$ and transient component $\tilde{\theta}_2 = 0.044$. Therefore, for every set of betas, the unconditional idiosyncratic equity variance can be defined by $\theta^i = \tilde{v}^i - (\beta_1^i)^2\tilde{\theta}_1 - (\beta_2^i)^2\tilde{\theta}_2$. The spot market persistent and transient variance components are set to $v_{1,t} = 0.012$ and $v_{2,t} = 0.048$ respectively and the total spot equity variance is set to $v_t^i = 0.05$. Consequently, for different sets of betas, the spot idiosyncratic variance under the physical measure can be defined as $\xi_t^i = v_t^i - (\beta_1^i)^2 v_{1,t} - (\beta_2^i)^2 v_{2,t}$. We choose the remaining structural parameters of the market and equity dynamics as follows: $\{\tilde{\kappa}_1 = 0.18, \tilde{\kappa}_2 = 2.8, \sigma_1 = 3.6, \sigma_2 = 0.29, \rho_1 = -0.96, \rho_2 = -0.83\}$ and $\{\tilde{\kappa}^i = 0.8, \sigma^i = 0.2, \rho^i = 0\}$. We fix the risk-free rate at 4% per year and examine at-the-money equity options with 3 months to maturity. We explore the model properties and their cross-sectional implications by assuming the ratio of spot index price over spot equity price as $S_t^i/S_t = 0.1$.

The proposed two-factor structure explicitly shows how changes in the level of the spot market index are translated into the equivalent changes in the equity option prices. It also allow us to examine how equity option prices respond to variations in the persistent and transient market variance components. The following proposition establishes these relations and creates a basis for further sensitivity analysis.

Proposition 3.3. *Given the closed-form equity option pricing expression in Proposition (3.2), the sensitivity of the individual equity call option prices C_t^i with respect to the level of* *the market index S_t may be given by:*

$$\frac{\partial C_t^i}{\partial S_t} = \frac{\partial C_t^i}{\partial S_t^i}\frac{S_t^i}{S_t}(\beta_1^i + \beta_2^i) \ . \tag{3.3.1}$$

Further, the sensitivity of the individual equity call option prices C_t^i with respect to the market variance components $v_{1,t}$ and $v_{2,t}$ are:

$$\begin{aligned} \frac{\partial C_t^i}{\partial v_{1,t}} &= \frac{\partial C_t^i}{\partial v_t^i}(\beta_1^i)^2 \ , \\ \frac{\partial C_t^i}{\partial v_{2,t}} &= \frac{\partial C_t^i}{\partial v_t^i}(\beta_2^i)^2 \ . \end{aligned} \tag{3.3.2}$$

where the total spot variance for equity i is $v_t^i = (\beta_1^i)^2 v_{1,t} + (\beta_2^i)^2 v_{2,t} + \xi_t^i$.

Proof. See Appendix (3.C). □

We interpret the expression (3.3.1) as the "market delta" and the expressions (3.3.2) as the "persistent market vega" and "transient market vega" for call options on equity i. Figure (3.1) shows the market sensitivity of the model-implied equity call option prices given the structural parameter values defined above. We plot the market delta for different sets of betas to examine the relative importance of transient and persistent factors. Consistent with Christoffersen et al. [2015], we find that firms with different sets of betas have different sensitivities to changes in the level of the market index. Consistent with Proposition (3.3), we observe that firm's with higher transient (persistent) beta are more sensitive to the changes in the level of the market index when we keep persistent (transient) beta constant. The same is also true for firms with higher average beta. Although, we cannot distinguish between the effect of transient and persistent betas on market delta per se, we observe that at-the-money equity call option prices are relatively more sensitive to the transient beta. Note that the top panel of figure (3.1) replicates the market delta following the calibration in the one-factor model of Christoffersen et al. [2015].

[Figure (3.1) about here]

Figures (3.2) and (3.3) plot the sensitivity of the model-implied equity call option prices with respect to the persistent and transient market variance components using the parameter values described above. Christoffersen et al. [2015] find that firms with higher betas are more sensitive to changes in the market volatility. Our model predicts the same pattern with respect to the total market volatility. More to the point, we find that firms with higher persistent betas are more sensitive to changes in the persistent variance component while the effect of the transient beta on the persistent market vega is marginal but reverse. Further, firms with higher transient betas are more sensitive to changes in the transient variance factor while the effect of the persistent beta on the transient market vega is reverse but significant. In other words, persistent beta has an important effect on the transient market vega across different level of moneyness (See Figure (3.3)). This distinctive property of our model allows a portfolio manager to better examine the exposure of her portfolio to the variations in market returns,[11] a feature that is absent in the single factor structure of Christoffersen et al. [2015]. Comparing the level of transient market vega and persistent market vega, our model predicts that equity call option prices are more sensitive to the transient volatility component compared to the persistent volatility component.

[Figure (3.2) about here]

[Figure (3.3) about here]

Our two-factor structure and closed-form equity option pricing formula allow us to shed some light on the relation between the expected returns of individual equity options and the characteristics of market returns and variance components as expressed in Proposition (3.4) below. This result allows us to disentangle the effect of the market risk premium from those of variance component risk premiums on the equity call option returns. It also shows how equity betas play a direct role on the equity call option returns. In particular, the second component in the right-hand-side (RHS) of equation (3.3.3), which is related to the market risk premium, affects the equity call option returns through the market delta by an adjustment factor which includes the persistent and transient betas. Moreover, the third component in the RHS of (3.3.3), which is related to variances risk premiums, shows how equity betas affect the equity call option returns through the total market vega of equity call

[11] Remember that market vega is the amount of money per underlying share that the option value will gain or loose as market volatilities rise or fall by 1%. It is also important as value of some option strategies are partially sensitive to changes in volatility.

options. Note that $\partial C_t^i / \partial v_t$ measures the total market vega of equity call options.

Proposition 3.4. *Given the closed-form equity option pricing expression (3.B.12)-(3.B.13), the dynamics of the market index (3.2.1) and individual equity returns (3.2.4), the instantaneous expected excess returns on individual equity call options under the physical measure can be characterized as follows.*

$$\begin{aligned} \frac{1}{dt} E_t^P \left[\frac{dC_t^i}{C_t^i} - rdt \right] &= \left[(\mu^i - r) \frac{S_t^i}{C_t^i} \right] \frac{\partial C_t^i}{\partial S_t^i} \\ &+ \left[\frac{\beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{1,t}}{\beta_1^i + \beta_2^i} \frac{S_t}{C_t^i} \right] \frac{\partial C_t^i}{\partial S_t} \\ &+ \left[\frac{(\beta_1^i)^2 \lambda_1 v_{1,t} + (\beta_2^i)^2 \lambda_2 v_{2,t}}{(\beta_1^i)^2 + (\beta_2^i)^2} \frac{1}{C_t^i} \right] \frac{\partial C_t^i}{\partial v_t} \end{aligned} \tag{3.3.3}$$

Proof. See Appendix (3.D). □

Our proposed two-factor structure has also important cross-sectional implications for equity options. Christoffersen et al. [2015] document that firms with higher betas have a steeper term structure of implied volatility. However, our model moves further and provides a novel term structure effect. In particular, we show how the term structure of implied volatility responds differently to the transient and persistent variations in market returns. Using the parameter values introduced at the beginning of this section, we show how β_1^i and β_2^i have different and non-trivial effects on the implied volatility term structures of individual equity options. Figure (3.4) plots the model implied volatility for at-the-money equity call options with respect to time-to-maturity for different sets of betas. Consistent with the finding in Christoffersen et al. [2015] (the top LHS panel), the higher the average betas the steeper the term structures of the implied volatility of equity options (the top RHS panel). In particular, our model predicts that the term structures of implied volatility of equity options is more sensitive to the transient beta (the bottom LHS panel) while the impact of the persistent beta on the term structures of implied volatility of equity options is marginal (the bottom RHS panel).[12] In other words, firms with higher transient betas have a term structure of implied volatility that co-moves more with the market term structure of IV.

[12]Note that in all the graphs the total unconditional equity variance under the risk neutral measure is fixed at $\tilde{v}^i = (\beta_1^i)^2 \tilde{\theta}_1 + (\beta_2^i)^2 \tilde{\theta}_2 + \theta^i = 0.11$.

[Figure (3.4) about here]

Figure (3.5) plots the model implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. Consistent with Christoffersen et al. [2015], reported in the top left-hand-side panel, our model predicts that the higher the beta of a firm, the steeper the implied volatility moneyness slope of its equity options (reported in the top panel). More to the point, our factor structure separates the effect of transient and persistent betas and predicts that the persistent beta has a marginal effect on the slope of implied volatility of equity options across moneyness (the bottom right panel). We note that the observed moneyness slope in Christoffersen et al. [2015] is mainly driven by the transient beta (the bottom LHS panel). This important result links our findings to those of Bakshi et al. [2003] who show that the market index distribution is more negatively skewed than the idiosyncratic equity return distribution. Given our proposed factor structure, our model predicts that firms with higher transient beta exhibit more negative skewness, which is consistent with previous studies.

[Figure (3.5) about here]

We close this section by discussing the implications of two-factor structure on the relation between the market variance risk premiums and the equity option skew. Figure (3.6) plots the difference between the model implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. The implied volatility difference is computed as the difference between equity call option IV when we increase variance component risk premiums from $\lambda_1 = \lambda_2 = -0.5$ to $\lambda_1 = \lambda_2 = 0$. As expected, the variance risk premiums have a more significant effect on the implied volatility of equity call options when the beta is higher (the top RHS panel). In particular, we observe that the transient beta has a more significant effect on the slope of equity implied volatility smile (the bottom LHS panel) compared to the persistent beta (the bottom RHS panel). In other words, in-the-money equity call options are getting relatively more expensive for firms with higher transient betas when we increase variance risk premiums. Note that for all the graphs the total unconditional equity variance is fixed $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$. Note also that the top LHS panel replicates the same pattern following the calibration in the one-factor model of Christoffersen et al. [2015].

[Figure (3.6) about here]

3.4 Data

For individual equities, we choose all the firms listed in the Dow Jones Industrial Average index and collect equity options data from OptionMetrics.[13] We keep all options up to 10% moneyness and with maturity up to and including 1 year. Note that options on individual equities are American, the price of which could be affected by early exercise premium. To prevent any bias in the estimation of the structural parameters of equities and daily spot idiosyncratic variance, the loss function needs to be defined based on the implied volatility as implied volatilities and deltas for the equity options reported in OptionMetrics are computed by the Cox et al. [1979] binomial tree model. Otherwise, if the loss function is based on mean-squared option pricing errors, we either need to restrict our sample to out-of-the-money equity options that are less sensitive to early exercise premium or have to covert the American-style equity options into European-style equity options by taking into account the early exercise premium. Due to the computational burden of such adjustments and considering the closed-from European option pricing equation in Proposition (3.2), we focus on OTM equity options.[14]

To filter daily spot market transient and persistent variance components, we use data from S&P 500 index and option markets. We obtain S&P 500 index option prices from the OptionMetrics volatility surface data set from January 4, 1996 through December 29, 2011. We follow the data cleaning routine commonly used in the empirical option pricing literature: we remove options with implied volatility less than 5% and greater than 150%; we also follow the filtering rules in Bakshi et al. [1997] to remove options that violate various no-arbitrage conditions. We focus on out-of-the-money (OTM) option contract with maturity up to and including one-year and with moneyness (spot over strike price) up to 10%.[15] After cleaning, our sample contains 345,710 S&P 500 index option contracts.

The data for daily equity prices, equity returns, daily index level, index returns, and the dividend yields are from CRSP. In the empirical analysis, we first adjust daily equity prices and index level with dividend yields and then compute option prices using the dividend-adjusted returns. Risk-free interest rates for all maturities are estimated by linear interpolation be-

[13]Note that we drop the Bank of America, the Kraft Foods Incorporation, and the Travelers Companies Incorporation.

[14]See Bakshi et al. [2003] and Christoffersen et al. [2015].

[15]See Ghanbari [2016] for detailed description of the S&P 500 index options data set and its summary statistics.

tween the closest zero-coupon rates of the Zero Coupon Yield Curve from OptionMetrics.

Table (3.1) presents the descriptive statistics of the option contracts that are used to filter daily spot market variances and daily spot idiosyncratic variance, and to estimate the structural parameters for individual equities and market index. This table reports the number of available call and put option contracts for each firm after data cleaning. For every firm, we also report the average number of days-to-maturity and average implied volatility of option contracts in our sample. Overall, we have 4,241,990 equity call options and 3,209,990 equity put options with an average days-to-maturity of 135 days. On average, for every firm we have 275,999 option contracts with an average implied volatility of 28.52%.

[**Table (3.1) about here**]

Tables (3.2) and (3.3) provide further details regarding equity call options and put options. On average we observe that equity call options in our sample are more expensive (2.688 for calls versus 2.344 for puts), more sensitive to underlying equity prices and volatilties, have lower implied volatility (27.32% for calls and 29.73% for puts), and have a greater number of days-to-maturity (137 days for calls and 134 days for puts.)

[**Table (3.2) about here**]

[**Table (3.3) about here**]

3.5 Estimation Methodology

Our estimation methodology is twofold. At the market index level, we do a joint-estimation to filter the vectors of daily spot variance components and to estimate a set of structural parameters. Then, for every individual equity i, we filter daily spot idiosyncratic variance and structural parameters, given the filtered transient and persistent spot variance components of the market index.

3.5.1 Estimation of the Index Model

To estimate the structural parameters and filter daily spot idiosyncratic variance for the firms listed in our sample, we first filter the time-series of the transient and persistent spot market variance components. We follow the approach in Ghanbari [2016] for the estimation of the two-factor stochastic volatility model of the market index which combines information from underlying index and option markets. We use a two-component likelihood function, a return-based component and an option-based component, to impose consistency between structural parameters under P and Q distributions. To filter unobserved transient and persistent spot variance components, we use the sampling-importance-resampling (SIR) implementation of the Particle Filter (PF) methods.[16]

Our optimization function is as follows.

$$\max_{\Theta,\tilde{\Theta}} \big(LLR + LLO \big) \ , \tag{3.5.1}$$

where LLR is the return-based and LLO is the option-based likelihood functions and Θ is the set of structural parameters of the market index model under P-measure and $\tilde{\Theta}$ is the equivalent set under Q-measure.

$$LLR \propto \sum_{t=1}^{T} \ln \left(\frac{1}{N} \sum_{j=1}^{N} \breve{W}_t^j(\Theta) \right) \prime \tag{3.5.2}$$

where $\breve{W}_t^j$ is the normalized weight of particle j at time t, N is the number of daily particles, and $\Theta \equiv \{\kappa_1, \kappa_2, \theta_1, \theta_2, \sigma_1, \sigma_2, \rho_1, \rho_2, \lambda_1, \lambda_2\}$.

$$LLO \propto -\frac{1}{2} \Big(M \ln(2\pi) + \sum_{n=1}^{M} \big(\ln(s^2) + \eta_n^2 / s^2 \big) \Big) \ , \tag{3.5.3}$$

where M is the total number of index option contracts and η_n is the Vega-weighted loss function for index option n.

[16] See Ghanbari [2016] for implantation of PF in the context of two-factor stochastic volatility model. See Pitt [2002] for a detailed description of the PF algorithm.

$$\eta_n = (C_n^O - C_n^M(\tilde{\Theta}, \hat{v}_1^Q, \hat{v}_2^Q, S_t, K, \tau))/Vega_n\ , \qquad n = 1, \ldots, M\ , \tag{3.5.4}$$

where C_n^O is the observed price of call option n and $C_n^M(\tilde{\Theta}, \hat{v}_1^Q, \hat{v}_2^Q, S_t, K, \tau)$ is the model price of call option n.[17] $Vega_n$ is the Black and Scholes [1973] option Vega for the same option contract. Note that we obtain daily persistent ($\hat{v}_{1,t}^Q$) and transient ($\hat{v}_{2,t}^Q$) spot variance components under Q measure as the average of smoothly re-sampled particles of daily variance components.

$$\hat{v}_{1,t}^Q = \frac{1}{N}\sum_{j=1}^{N} v_{1,t}^j\ , \qquad \hat{v}_{2,t}^Q = \frac{1}{N}\sum_{j=1}^{N} v_{2,t}^j \tag{3.5.5}$$

Our index optimization algorithm is iterative. Each iteration starts with an initial set of structural parameters, which then will be used to filter transient and persistent daily spot variance components using the information content of index returns. Then, given spot variance components, structural parameters of the index, and observed option prices, the next set of optimal parameters can be reached by minimizing the option pricing errors over the entire sample. The procedure iterates until an optimal set of structural parameters is reached and thereby we obtain the final vectors of transient and persistent spot variance components.

3.5.2 Estimation of the Individual Equity Model

We estimate a set of structural parameters $\tilde{\Theta}^i \equiv \{\kappa^i, \theta^i, \sigma^i, \rho^i, \beta_1^i, \beta_2^i\}$ and a vector of daily spot idiosyncratic variances $\{\xi_t^i\}$ for each individual equity in our sample following the two-step iterative approach of Bates [2000] and Huang and Wu [2004]. In the first step, given a set of initial structural parameters for each equity, $\tilde{\Theta}_0^i$, we estimate a vector of daily spot idiosyncratic variance conditional on a set of risk-neutral structural parameters of the market model, $\hat{\tilde{\Theta}}$, and filtered daily risk-neutral spot variance components, $\{\hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q\}$. Using a Vega-weighted loss function, the set of daily spot idiosyncratic variance $\hat{\xi}_t^i$ for every firm i can be obtained as the solution to the following optimization problem, which minimizes the Vega-weighted daily mean-squared option pricing errors.

[17] See Ghanbari [2016] for option pricing equation under two-factor stochastic volatility model.

$$\hat{\xi}_t^i = \arg\min_{\xi_t^i} \sum_{n=1}^{M_t^i} (C_{n,t}^{i,O} - C_{n,t}^{i,M_t^i}(\tilde{\Theta}_0^i, \hat{\tilde{\Theta}}, \hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q, \xi_t^i))^2 / (Vega_{n,t}^i)^2 \ , \qquad t = 1, \ldots, T \ , \tag{3.5.6}$$

where M_t^i is the total number of available option contracts for the equity i on day t, $C_{n,t}^{i,O}$ is the observed price of equity option n for stock i on day t, $C_{n,t}^{i,M_t^i}$ is the model price for the same option obtained from equity pricing equation (3.2.9), and $Vega_{n,t}^i$ is the Black-Scholes option Vega for the same equity option contract. Note that we repeat the optimization in (3.5.6) every day and for every equity to estimate a vector of spot idiosyncratic variances over the entire sample.

The second step estimates the structural parameters $\tilde{\Theta}^i$ for firm i, by minimizing sum of daily Vega-weighted mean-squared option pricing errors over the entire sample, given filtered daily spot idiosyncratic variance obtained in the first step, the dynamics of the market index and the filtered daily spot variance components. We may then solve the the following optimization problem.

$$\hat{\tilde{\Theta}}^i = \arg\min_{\xi_t^i} \sum_{n=1}^{M^i} (C_n^{i,O} - C_n^{i,M^i}(\tilde{\Theta}^i, \hat{\tilde{\Theta}}, \hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q, \hat{\xi}_t^i))^2 / (Vega_n^i)^2 \ , \tag{3.5.7}$$

where $M^i \equiv \sum_{t=1}^T M_t^i$ is the total number of available option contracts for equity i. For every equity, the procedure iterates between the optimizations in (3.5.6) and (3.5.7) to minimize the pricing error until the change in the RMSE of the estimation in the second step is no longer significant. Note that every new iteration starts based on the structural parameters of the previous iteration, $\tilde{\Theta}_0^i = \hat{\tilde{\Theta}}^i$.

3.6 Parameter Estimation Results

This section first reports the filtered daily spot variance components together with the structural parameter estimates for the S&P 500 Index. We use a long time-series of daily S&P 500 index returns and the entire cross-section of S&P 500 option prices that span the period from January 4, 1996 to December 29, 2011. The market risk premium is set to the sample average daily index returns. We use OTM index options with up to 10% moneyness and then convert the OTM puts into ITM calls through put-call parity.[18] To provide a basis for further comparison and to examine the model fit under the joint-estimation, we also report the structural parameters of the market model, estimated only from option data.

Table (3.4) reports the parameter estimates (under P measure) that characterize the dynamics of the S&P 500 index returns and its variance components from the joint estimation. Therefore, we obtain the same value for correlation coefficients ρ and volatility of variance components σ under P and Q measures while the speed of mean reversion and the unconditional mean of the variance components under P and Q measures are linked through the market prices of the transient and persistent variance components risk factors ($\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{\kappa_1\theta_1}{\kappa_1+\lambda_1}$, and $\tilde{\theta}_2 = \frac{\kappa_2\theta_2}{\kappa_2+\lambda_2}$).[19] Note that we assume that the transient and persistent beta coefficients are the same under P and Q measures following Serban et al. [2008].

[Table (3.4) about here]

We also report structural parameters and daily spot idiosyncratic variance for 27 firms listed in the Dow Jones Industrial Average Index. The parameter estimates and latent idiosyncratic variance are conditional on the transient and persistent spot variance components $\hat{v}_{1,t}^{Q}$ and $\hat{v}_{2,t}^{Q}$ and structural parameters $\hat{\hat{\Theta}}$ reported in Table (3.4). The data for individual equities starts from June 1, 1996 rather than January 1, 1996. Note that we drop the first 5 months of each equity's data set to prevent any estimation bias, as the filtered spot market variance components are noisy in the first months of the estimation period. Note also that S&P 500 Index options are European style while the individual equity options are American style, the price of which might be affected by early exercise premium. To reduce the bias in the calculation of equity option prices using the closed-form pricing equation in Proposition (3.2)

[18] See Ghanbari [2016] for the descriptive statistics of the index data set.

[19] See Ghanbari [2016] for further discussion of the parameter estimates from joint estimation. The study also reports parameter values from option-based estimation.

we focus on OTM options.[20,21]

Table (3.5) reports the structural parameter estimates that characterize the dynamics of the individual equity returns and idiosyncratic variance under the Q measure. The table also contains the point estimates of the persistent and transient betas for 27 firms in our sample.

[Table (3.5) about here]

The speed of mean reversion for risk-neutral idiosyncratic variance ranges from $\tilde{\kappa}^i = 0.3920$ for Coca Cola to $\tilde{\kappa}^i = 1.7078$ for 3M. This range of $\tilde{\kappa}^i$ is implies that most of the firms in our sample have highly persistent idiosyncratic variance with average speed of mean reversion 0.8055. In other words, the average half-life of idiosyncratic variance for the firms in our sample is almost 46 weeks, implying that it takes 46 weeks for the idiosyncratic variance autocorrelation to decay to half of its weekly autocorrelation. We also find that most of the firms in our sample have an idiosyncratic variance that is more persistent than the overall market variance.

The unconditional risk neutral idiosyncratic variance of the firms in our sample starts from $\tilde{\theta}^i = 0.0093$ for General Electric and increases up to $\tilde{\theta}^i = 0.0756$ for Hewlett-Packard. The point estimates for the volatility of the idiosyncratic variance range from $\sigma^i = 0.0670$ for General Electric to $\sigma^i = 0.3967$ for Hewlett-Packard. For all the firms in our sample, the average point estimates for the volatility of the idiosyncratic variance is 0.1823. The correlation between shocks to equity returns and shocks to idiosyncratic variance is negative for all the equities (except for Verizon) and ranges from $\rho^i = -0.99$ for JP Morgan to $\rho^i = 0.512$ for Verizon.

[20] Bakshi et al. [2003] show that for OTM S&P 100 American options the early exercise premium is negligible. They estimate two separate implied volatilities: the implied volatility that equates the option price to the American option price from binomial tree model, and the implied volatility that equates the option price to the Black-Scholes price where the discounted dividends are subtracted from the spot price. They find that although American option implied volatility is smaller than its Black-Scholes counterparts, the difference is negligible and within the bid-ask spread.

[21] Using the data of the firms listed on Dow Jones Index, Christoffersen et al. [2015] show that the early exercise premium is negligible for equity call options. As a robustness test, we also estimate the equity model by using only the equity call options rather than OTM calls and puts. We find that the point estimates of structural parameters are quite similar to our base case estimation where we use OTM put and call option contracts. This result is available from the author upon request.

The betas estimates are novel and to the best of our knowledge this is the first study that reports the option-implied persistent beta and transient beta for individual equities and thus there is no benchmark for further comparisons. However, we find that firms respond differently to transient and persistent variations in market index returns. The persistent beta ranges from $\beta_1^i = 0.3430$ for American Express to $\beta_1^i = 0.6798$ for IBM. The transient beta starts from $\beta_2^i = 1.0125$ for Procter & Gamble and increases to $\beta_2^i = 1.3466$ for JP Morgan. The average persistent beta is 0.4899 and the average transient beta is 1.2284. Across all 27 firms in our sample the transient beta is always greater than the persistent beta, implying that for the large capitalization firms listed in the Dow Jones index, transient and larger variations in the market tend to be related to the proportionally larger systematic price reactions across equities than persistent and smaller variations in the market index.

Our point estimates of the transient and persistent option-implied betas are similar to the continuous beta and jump beta of Todorov and Bollerslev [2010] who introduce a framework to separate and identify continuous and discontinuous systematic risks. Using high frequency data from a large cross-section of forty large-capitalized individual stocks, they find that the average jump betas are larger than the continuous betas with few exceptions. Although we only use option data and estimate ad-hoc constant beta over the entire sample, we observe a similar pattern as theirs between our transient and persistent betas.[22]

As discussed in Section 3, the proposed two-factor structure has important implications for equity option market deltas, market Vegas, and instantaneous expected returns of equity options. We also show how this two-factor structure affects the slope of the term structure and moneyness of implied volatility of individual equity options. Along these lines, our findings of different sensitivities to the systematic transient and persistent risk factors may corroborate the theoretical implications of our model. The beta estimates have further implications for portfolio management, suggesting the importance of different strategies for hedging transient versus persistent systematic market variations.

We close this section by providing more intuition about the idiosyncratic variance across the firms in our sample by presenting the distributional properties of the filtered spot idiosyn-

[22]The assumption of constant transient and persistent betas allow us to keep the affine specification of the dynamics of individual equity and derive a closed-form equity option pricing equation. We can, however, estimate time-varying betas by modifying our estimation procedure. We can fix the structural parameters of the market and individual equities and estimate conditional betas and spot idiosyncratic variance on a daily basis, given the transient and persistent spot variance components using a loss function very similar to 3.5.7.

cratic variance. Table (3.6) reports the mean, median, standard deviation, and the maximum of the filtered spot idiosyncratic variances for every firm conditional on the structural parameters of the two-factor SV model of index and the filtered market spot variance components. We observe that for all the firms the median is significantly lower than the mean, implying that the mean estimates of the filtered spot idiosyncratic volatilities are driven by outliers that may be common to all firms.

[Table (3.6) about here]

3.7 In-Sample Fit and Out-of-Sample Performance

The Goodness of fit of the proposed tow-factor structure in equity options is measured by the Vega-weighted root mean squared option pricing errors (Vega RMSE) as it is consistent with the loss function that was used in the model estimation.

$$\text{Vega RMSE}^i \equiv \sqrt{\frac{1}{N}\sum_{n,t}^{M^i}\big(\frac{C_{n,t}^{O,i} - C_{n,t}^{i,M^i}(\tilde{\Theta}^i, \hat{\tilde{\Theta}}, \hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q, \hat{\xi}_t^i)}{Vega_{n,t}^i}\big)^2}\,, \tag{3.7.1}$$

where, $C_{n,t}^{O,i}$ is the observed price of option n on day t written on individual equity i, $C_{n,t}^{i,M^i}$ is the model price of the same option written on individual equity i on the same day, and $Vega_{n,t}^i$ is the Black-Scholes option Vega for the same equity option contract on the same day. We also report the implied volatility root mean squared error (IVRMSE) measured as follows.

$$\text{IVRMSE}^i \equiv \sqrt{\frac{1}{N}\sum_{n,t}^{M}\left(IV_{n,t}^{O,i} - IV(C_{n,t}^{i,M^i}(\tilde{\Theta}^i, \hat{\tilde{\Theta}}, \hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q, \hat{\xi}_t^i))\right)^2}\,, \tag{3.7.2}$$

where, $IV_{n,t}^{O,i}$ is the Black-Scholes implied volatility of observed option n written on individual equity i on day t and $IV(C_{n,t}^{i,M^i}(\tilde{\Theta}^i, \hat{\tilde{\Theta}}, \hat{v}_{1,t}^Q, \hat{v}_{2,t}^Q, \hat{\xi}_t^i))$ is the Black-Scholes implied volatility of the model price for the same equity option on the same day.

Table (3.7) provides goodness-of-fit statistics for 27 the firms in our sample, both in-sample and out-of-sample. Using option data over the period 1996-2011, we find that all the firms in our sample has a Vega RMSE below 2 except for Cisco and Chevron. We find similar in-sample performance when the goodness-of-fit is measure by IVRMSE. The average Vega RMSEs and IVRMSEs across all the firms are 1.61% and 1.59% respectively. The average relative IVRMSE, measured as the ratio of IVRMSE over the average Black-Scholes IV, is 5.66%. We find that Boeing has the best fit with IVRMSE of 1.35% and Cisco has the worst fit with IVRMSE of 2.12%; however, the fit is quite similar across the firms. Overall we conclude that the model provides a reasonably good fit for all 27 firms.

We find that our model has a relatively better in-sample fit compared to the one-factor structure model. For the firms listed on Dow Jones index, Christoffersen et al. [2015, Table 4] find that the average IVRMSE is 1.66%.[23] Further, comparing goodness-of-fits in our model with those of Heston model for the same firms, reported in Christoffersen et al. [2015, Table A.2], also supports the performance of our model. Overall, the in-sample performance of our model over the one-factor structure together with its cross-sectional implications regarding IV term-structure, moneyness slope, and equity option skew support the importance of transient and persistent factor loadings in pricing equity options.

Entries in the last column of Table (3.7) reports out-of-sample performance of the equity model. We divide the data set into two subsample periods. using data from 1996 to 2003 we estimate structural parameters for the index model, for every individual equity, and filter persistent and transient daily spot index variance components, and spot idiosyncratic variance for all the firms. In the next step we filter spot idiosyncratic variance for all the firms over the period 2004 to 2011, given spot variance components and structural parameters in the first subsample period. Note that we use an optimization function similar to (3.5.6). We find that the model provides good out-of-sample fit. For most of the firms, the out-of-sample Vega RMSEs are consistent with their in-sample Vega RMSEs. Overall, the average Vega RMSE is 1.81% across all 27 firms.[24]

[23]Note that their sample span the period 1996 to 2010.

[24]The out-of-sample performance can also be examined with spot idiosyncratic variance obtained from one-day ahead $(t+1)$ forecast of idiosyncratic variance for individual equity i given the in-sample structural parameter estimates and time t spot idiosyncratic variance. One-day ahead $(t+1)$ forecast of idiosyncratic variance may be computed as $\hat{\xi}^i_{t+1|t} \equiv \mathrm{E}_t[\xi^i_{t+1}] = \theta^i + (\xi^i_t - \theta^i)(1 - \exp(-\frac{\kappa^i}{252}))$. However, this approach may be more suitable for instance if in-sample fit is based on a Wednesday options and then out-of-sample fit can be examined based on the Thursday options.

3.8 Concluding Remarks

Motivated by the extensive empirical evidence that supports the existence of two volatility components in the dynamics of index, we examine how individual equity option prices respond to transient and persistent factor loadings. We adopt a two-factor stochastic volatility model as in Ghanbari [2016] where aggregate market volatility is decomposed into two independent volatility components, a transient component and a persistent component. Then we extend the model in Christoffersen et al. [2015] and assume that individual equity returns are related to market index returns with two distinct systematic components and an idiosyncratic component, which is stochastic and follows a standard square root process. We derive a closed form pricing equation for individual equity call options where equity option prices depend on two constant factor loadings, a transient beta and a persistent beta.

For the firms listed on Dow Jones Index, we estimate structural parameters and filter spot idiosyncratic variances, which together characterize the dynamics of the individual equity under the risk-neutral measure. Given the level of IVRMSEs, we find that our model provides a good-fit both in-sample and out-of-sample. We also report the point estimates of transient and persistent betas for 27 firms. We find that for all the firms, the transient beta is always greater than the persistent beta, implying that for large capitalization firms listed in the Dow Jones index, transient and larger variations in the market tends to be related to the proportionally larger systematic price reactions across equities than persistent and smaller variations in the market index. It also supports the presence of a two-factor structure in our model. Along this line, the different sensitivities to the systematic transient and persistent risks may corroborate the theoretical implication of our model. The beta estimates have further implications for portfolio management, suggesting the importance of different strategies for hedging transient versus persistent systematic market variations.

Our equity option pricing model sheds some lights on the impact of systematic price changes on the equity option prices. We find closed-form expressions for the sensitivity of the equity option prices to the changes in the index level (market delta) and changes in the persistent and transient variance components (persistent and transient market vega) and show how transient and persistent betas may affect the expected returns of individual equity options through market delta and vegas. Our closed-form pricing equation and proposed factor structure allow a portfolio manager to hedge her portfolio exposure to the level of the market index, and to the persistent and transient variations in the market index.

We show that the proposed two-factor structure has important cross-sectional implications for equity options. Consistent with the findings of Duan and Wei [2009], our model predicts that firms with a higher beta have a higher implied volatility. More to the point, we find that firms with a higher transient beta have a steeper term structure of implied volatility and a steeper implied volatility moneyness slope. We also observe that the variance risk premium has a more significant effect on the implied volatility smile of equity options (equity option skew) when the transient beta is higher. Overall, the in-sample performance of our model over the one-factor structure, its out-of-sample performance, together with its cross-sectional implications regarding IV term structure, moneyness slope, and equity option skew support the importance of transient and persistent factor loadings in pricing equity options.

3.A Proof of Proposition 3.1

We transform the physical dynamics of individual equity returns (3.A.1) to its risk neutral counterparts (3.A.2) by assuming an appropriate stochastic discount factor (SDF).

$$\begin{aligned} &dS_t^i/S_t^i = \mu^i dt + \beta_1^i(\mu_1 v_{1,t}dt + \sqrt{v_{1,t}}dz_{1,t}) + \beta_2^i(\mu_2 v_{2,t}dt + \sqrt{v_{2,t}}dz_{2,t}) + \sqrt{\xi_t^i}dz_t^i \\ &d\xi_t^i = \kappa^i(\theta^i - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^i \end{aligned} \tag{3.A.1}$$

$$\begin{aligned} &dS_t^i/S_t^i = rdt + \beta_1^i\sqrt{v_{1,t}}d\tilde{z}_{1,t} + \beta_2^i\sqrt{v_{2,t}}d\tilde{z}_{2,t} + \sqrt{\xi_t^i}d\tilde{z}_t^i \\ &d\xi_t^i = \kappa^i(\theta^i - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^i \end{aligned} \tag{3.A.2}$$

where

$$\begin{aligned} &\langle dz_t^i, dw_t^i\rangle = \rho^i dt \\ &\langle dz_t^i, dw_t^j\rangle = 0 \ \forall(i \neq j) \end{aligned} \tag{3.A.3}$$

As individual equity returns are linked to the market index returns with a two-factor model and two constant factor loadings β_1 and β_2, the proposed SDF should jointly specify the risk neutral distributions of the market index and individual equity returns. Remember that the dynamics of market index returns under the P- and Q-measure are as follows.

$$\begin{aligned} dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\ dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(\rho_1 dz_{1,t} + \sqrt{1-\rho_1^2}dB_{1,t}) \\ dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(\rho_2 dz_{2,t} + \sqrt{1-\rho_2^2}dB_{2,t}) \end{aligned} \tag{3.A.4}$$

$$\begin{aligned} dS_t/S_t &= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t} \\ dv_{1,t} &= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(\rho_1 d\tilde{z}_{1,t} + \sqrt{1-\rho_1^2}d\tilde{B}_{1,t}) \\ dv_{2,t} &= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(\rho_2 d\tilde{z}_{2,t} + \sqrt{1-\rho_2^2}d\tilde{B}_{2,t}) \end{aligned} \tag{3.A.5}$$

where

$$
\begin{aligned}
&\langle dw_{1,t}, dz_{1,t}\rangle = \rho_1 dt,\ -1 \le \rho_1 \le +1 \\
&\langle dw_{2,t}, dz_{2,t}\rangle = \rho_2 dt,\ -1 \le \rho_2 \le +1 \\
&\langle dw_{1,t}, dw_{2,t}\rangle = 0 \\
&\rho_1^2 + \rho_2^2 \le +1
\end{aligned} \tag{3.A.6}
$$

We assume the following standard SDF.

$$
\frac{dM_t}{M_t} = -rdt - \psi_t' dW_t \ , \tag{3.A.7}
$$

where $\psi_t \equiv [\psi_{1,t}, \psi_{2,t}, \psi_{3,t}, \psi_{4,t}, \psi_{1,t}^i, \psi_{2,t}^i]$ $i = \{1, 2, \cdots, n\}$ is the vector of market price of risk factors and $W_t \equiv [z_{1,t}, z_{2,t}, w_{1,t}, w_{2,t}, z_t^i, w_t^i]$ $i = \{1, 2, \cdots, n\}$ is the vector of innovations in market return, market variance components, equity i return, and equity i idiosyncratic variance. Given the SDF in (3.A.7), the change-of-measure from P- to Q-distribution has the following exponential form.

$$
\frac{dQ}{dP}(t) \equiv M_t \exp(rt) = \exp\Big[- \int_0^t \psi_u' dW_u - \frac{1}{2}\int_0^t \psi_u' d\langle W, W'\rangle_u \psi_u \Big] \tag{3.A.8}
$$

where $\langle W, W'\rangle$ is the covariance operator.

We follow the notion of Doléans-Dade exponential (stochastic exponential) and define the stochastic exponential $\varepsilon(\cdot)$ as follow.

$$
\varepsilon\Big(\int_0^t \vartheta_u' dW_u \Big) \equiv \exp\Big[\int_0^t \vartheta_u' dW_u - \frac{1}{2}\int_0^t \vartheta_u' d\langle W, W'\rangle_u \vartheta_u \Big] \tag{3.A.9}
$$

Therefore, the change-of-measure (3.A.8) can be expressed in term of stochastic exponential as

$$
\frac{dQ}{dP}(t) = \varepsilon\Big(\int_0^t -\psi_u' dW_u \Big) \tag{3.A.10}
$$

Applying Ito's lemma, for every individual equity i, we have the following dynamic under the physical measure.

$$\log\big(\frac{S_t^i}{S_0^i}\big) = \big[\mu^i + \beta_1^i\mu_1 v_{1,t} + \beta_2^i\mu_2 v_{2,t} - \frac{1}{2}(\beta_1^i)^2 v_{1,t} - \frac{1}{2}(\beta_2^i)^2 v_{2,t} - \frac{1}{2}\xi_t^i\big]t \\ + \beta_1^i\int_0^t \sqrt{v_{1,u}}dz_{1,u} + \beta_2^i\int_0^t \sqrt{v_{2,u}}dz_{2,u} + \int_0^t \sqrt{\xi_u^i}dz_u^i \tag{3.A.11}$$

Given (3.A.11) and definition of stochastic exponential (3.A.9) we have

$$\frac{S_t^i}{S_0^i} = \exp\big[(\mu^i + \beta_1^i\mu_1 v_{1,t} + \beta_2^i\mu_2 v_{2,t})t\big]\varepsilon\Big(\int_0^t \beta_1^i\sqrt{v_{1,u}}dz_{1,u} + \int_0^t \beta_2^i\sqrt{v_{2,u}}dz_{2,u} + \int_0^t \sqrt{\xi_u^i}dz_u^i\Big) \tag{3.A.12}$$

Note that

$$\varepsilon\Big(\int_0^t \beta_1^i\sqrt{v_{1,u}}dz_{1,u}\Big) = \exp\Big[\int_0^t \beta_1^i\sqrt{v_{1,u}}dz_{1,u} - \frac{1}{2}\int_0^t (\beta_1^i)^2 v_{1,u}du\Big] \tag{3.A.13}$$

To find the market prices of risk we impose the restriction that the product of the price of any individual equity and the pricing kernel under physical measure is a P-martingale. Given the change-of-measure (3.A.10), for every individual equity i, the following process $N(t)$ should be a P-martingale.

$$N(t) \equiv \frac{S_t^i}{S_0^i}\frac{dQ}{dP}(t)\exp(-rt) \tag{3.A.14}$$

where

$$\begin{aligned} N(t) = {} & \exp\big[(-r + \mu^i + \beta_1^i\mu_1 v_{1,t} + \beta_2^i\mu_2 v_{2,t})t\big] \\ & \varepsilon\Big(\int_0^t \beta_1^i\sqrt{v_{1,u}}dz_{1,u}\Big)\varepsilon\Big(-\int_0^t \psi_{1,u}dz_{1,u} - \int_0^t \psi_{3,u}dw_{1,u}\Big) \\ & \varepsilon\Big(\int_0^t \beta_2^i\sqrt{v_{2,u}}dz_{2,u}\Big)\varepsilon\Big(-\int_0^t \psi_{2,u}dz_{2,u} - \int_0^t \psi_{4,u}dw_{2,u}\Big) \\ & \varepsilon\Big(\int_0^t \sqrt{\xi_u^i}dz_u^i\Big)\varepsilon\Big(-\int_0^t \psi_{1,u}^i dz_u^i - \int_0^t \psi_{2,u}^i dw_u^i\Big) \\ & \varepsilon\Big(-\sum_{j\notin i}\int_0^t \psi_{1,u}^j dz_u^j - \sum_{j\notin i}\int_0^t \psi_{2,u}^j dw_u^j\Big) \end{aligned} \tag{3.A.15}$$

We decompose $N(t)$ into two orthogonal components $N(t) \equiv I(t)L(t)$ and then make sure that $I(t)$ and $L(t)$ are a P-martingale.

$$\begin{aligned}I(t) =& \exp\left[(-r + \mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t\right] \\ & \varepsilon\Big(\int_0^t \beta_1^i \sqrt{v_{1,u}} dz_{1,u}\Big) \varepsilon\Big(-\int_0^t \psi_{1,u} dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u}\Big) \\ & \varepsilon\Big(\int_0^t \beta_2^i \sqrt{v_{2,u}} dz_{2,u}\Big) \varepsilon\Big(-\int_0^t \psi_{2,u} dz_{2,u} - \int_0^t \psi_{4,u} dw_{2,u}\Big) \\ & \varepsilon\Big(\int_0^t \sqrt{\xi_u^i} dz_u^i\Big) \varepsilon\Big(-\int_0^t \psi_{1,u}^i dz_u^i - \int_0^t \psi_{2,u}^i dw_u^i\Big)\end{aligned} \tag{3.A.16}$$

$$L(t) = \varepsilon\Big(-\sum_{j \notin i} \int_0^t \psi_{1,u}^j dz_u^j - \sum_{j \notin i} \int_0^t \psi_{2,u}^j dw_u^j\Big) \tag{3.A.17}$$

From the definition of a stochastic exponential we know that $\varepsilon(\cdot)$ are P-martingales and so does $L(t)$. Therefore, we only need to make sure that $I(t)$ is also a P-martingale. Using the properties of a stochastic exponential $\varepsilon(\cdot)$, $\varepsilon(X_t)\varepsilon(Y_t) = \varepsilon(X_t + Y_t)\exp(\langle X, Y\rangle_t)$ and the correlation structure (3.A.3) and (3.A.6) we can rewrite the process of $I(t)$ as follows.

$$\begin{aligned}I(t) =& \exp\left[(-r + \mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t\right] \\ & \varepsilon\Big(\int_0^t (\beta_1^i \sqrt{v_{1,u}} - \psi_{1,u}) dz_{1,u} - \int_0^t \psi_{3,u} dw_{1,u}\Big) \exp\Big[-\int_0^t \beta_1^i \sqrt{v_{1,u}} (\psi_{1,u} + \rho_1 \psi_{3,u}) du\Big] \\ & \varepsilon\Big(\int_0^t (\beta_2^i \sqrt{v_{2,u}} - \psi_{2,u}) dz_{2,u} - \int_0^t \psi_{4,u} dw_{2,u}\Big) \exp\Big[-\int_0^t \beta_2^i \sqrt{v_{2,u}} (\psi_{2,u} + \rho_2 \psi_{4,u}) du\Big] \\ & \varepsilon\Big(\int_0^t (\sqrt{\xi_u^i} - \psi_{1,u}^i) dz_u^i - \int_0^t \psi_{2,u}^i dw_u^i\Big) \exp\Big[-\int_0^t \sqrt{\xi_u^i} (\psi_{1,u}^i + \rho^i \psi_{2,u}^i) du\Big]\end{aligned} \tag{3.A.18}$$

Thus, given $\varepsilon(\cdot)$ are P-martingales, the process $I(t)$ is a P-martingale when the following restriction holds.

$$\begin{aligned}
&\exp\left[(-r + \mu^i + \beta_1^i \mu_1 v_{1,t} + \beta_2^i \mu_2 v_{2,t})t\right] \\
&\exp\left[-\int_0^t \beta_1^i \sqrt{v_{1,u}}(\psi_{1,u} + \rho_1 \psi_{3,u})du\right] \exp\left[-\int_0^t \beta_2^i \sqrt{v_{2,u}}(\psi_{2,u} + \rho_2 \psi_{4,u})du\right] \\
&\exp\left[-\int_0^t \sqrt{\xi_u^i}(\psi_{1,u}^i + \rho^i \psi_{2,u}^i)du\right] = 1
\end{aligned} \tag{3.A.19}$$

The restriction (3.A.19) holds if the following conditions for the market index, (3.A.20), and for every individual equity i, (3.A.21), hold.

$$\begin{aligned}
\mu_1 v_{1,t} t - \sqrt{v_{1,t}}(\psi_{1,t} + \rho_1 \psi_{3,t})t = 0 \\
\mu_2 v_{2,t} t - \sqrt{v_{2,t}}(\psi_{3,t} + \rho_2 \psi_{4,t})t = 0
\end{aligned} \tag{3.A.20}$$

$$-rt + \mu^i t - \sqrt{\xi_t^i}(\psi_{1,t}^i + \rho^i \psi_{2,t}^i)t = 0 \tag{3.A.21}$$

To fully specify the market prices of risk we assume that market price of variance risk factors are proportional to spot volatility components, following Heston [1993].

$$\begin{aligned}
(\psi_{3,t} + \rho_1 \psi_{1,t}) = \frac{v_{1,t}}{\sigma_1 \sqrt{v_{1,t}}} \lambda_1 \\
(\psi_{4,t} + \rho_2 \psi_{2,t}) = \frac{v_{2,t}}{\sigma_2 \sqrt{v_{2,t}}} \lambda_2
\end{aligned} \tag{3.A.22}$$

If we assume that the idiosyncratic variance is also a priced risk factor, then its price is also proportional to the spot idiosyncratic volatility for every individual equity i. Otherwise, $\lambda^i = 0$.

$$(\psi_{2,t}^i + \rho^i \psi_{1,t}^i) = \frac{\xi_t^i}{\sigma^i \sqrt{\xi_t^i}} \lambda^i \tag{3.A.23}$$

Combining the restrictions in (3.A.20) and (3.A.22), we have the following market price of risk factors.

$$\begin{aligned}
\psi_{1,t} &= \frac{\sigma_1\mu_1 - \rho_1\lambda_1}{(1-\rho_1^2)}\frac{\sqrt{v_{1,t}}}{\sigma_1} \\
\psi_{2,t} &= \frac{\sigma_2\mu_2 - \rho_2\lambda_2}{(1-\rho_2^2)}\frac{\sqrt{v_{2,t}}}{\sigma_2} \\
\psi_{3,t} &= \frac{\lambda_1 - \rho_1\sigma_1\mu_1}{(1-\rho_1^2)}\frac{\sqrt{v_{1,t}}}{\sigma_1} \\
\psi_{4,t} &= \frac{\lambda_2 - \rho_2\sigma_2\mu_2}{(1-\rho_2^2)}\frac{\sqrt{v_{2,t}}}{\sigma_2}
\end{aligned} \tag{3.A.24}$$

Combining the restrictions in (3.A.21) and (3.A.23) and given that idiosyncratic variance is not priced, we have the following results for every individual equity.

$$\begin{aligned}
\psi^i_{1,t} &= \frac{\mu^i - r}{\sqrt{\xi^i_t}(1-(\rho^i)^2)} \\
\psi^i_{2,t} &= (-\frac{\mu^i - r}{\sqrt{\xi^i_t}} + \frac{\xi^i\lambda^i}{\sigma^i})\frac{\rho^i}{1-(\rho^i)^2}
\end{aligned} \tag{3.A.25}$$

Given the market prices of risk factors (3.A.24) (3.A.25), we apply the Girsanov's theorem to transform physical innovations of the market index dynamics (3.A.4) and individual equity dynamics (3.A.1) to their risk neutral counterparts in (3.A.5) and (3.A.2). Note that we assume idiosyncratic variance is not priced and thus $\lambda^i = 0$.

$$\begin{aligned}
d\tilde{z}^i_t &= dz^i_t + \psi^i_{1,t}dt + \rho^i\psi^i_{2,t}dt \\
d\tilde{z}_{1,t} &= dz_{1,t} + \psi_{1,t}dt + \rho_1\psi_{3,t}dt \\
d\tilde{z}_{2,t} &= dz_{2,t} + \psi_{2,t}dt + \rho_2\psi_{4,t}dt \\
d\tilde{w}_{1,t} &= dw_{1,t} + \psi_{3,t}dt + \rho_1\psi_{1,t}dt \\
d\tilde{w}_{2,t} &= dw_{2,t} + \psi_{4,t}dt + \rho_2\psi_{2,t}dt
\end{aligned} \tag{3.A.26}$$

With some algebra we have the following transformations.

$$
\begin{aligned}
d\tilde{z}_t^i &= dz_t^i + (\mu^i - r)dt/\sqrt{\xi_t^i} \\
d\tilde{z}_{1,t} &= dz_{1,t} + \mu_1\sqrt{v_{1,t}}dt \\
d\tilde{z}_{2,t} &= dz_{2,t} + \mu_2\sqrt{v_{2,t}}dt \\
d\tilde{w}_{1,t} &= dw_{1,t} + (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt \\
d\tilde{w}_{2,t} &= dw_{2,t} + (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt
\end{aligned} \tag{3.A.27}
$$

Replacing $dz_t^i, dw_t^i, dz_{1,t}, dz_{2,t}, dw_{1,t}, dw_{2,t}$ from (3.A.27) into the physical dynamics in (3.A.1) and (3.A.4) and knowing that $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$ we obtain risk neutral return and variance components dynamics.

$$
\begin{aligned}
dS_t^i/S_t^i &= \mu^i dt + \beta_1^i(\mu_1 v_{1,t}dt + \sqrt{v_{1,t}}dz_{1,t}) + \beta_2^i(\mu_2 v_{2,t}dt + \sqrt{v_{2,t}}dz_{2,t}) + \sqrt{\xi_t^i}dz_t^i \\
&= \mu^i dt + \beta_1^i(\mu_1 v_{1,t}dt + \sqrt{v_{1,t}}(d\tilde{z}_{1,t} - \mu_1\sqrt{v_{1,t}}dt)) \\
&\quad + \beta_2^i(\mu_2 v_{2,t}dt + \sqrt{v_{2,t}}(d\tilde{z}_{2,t} - \mu_2\sqrt{v_{2,t}}dt)) + \sqrt{\xi_t^i}(d\tilde{z}_t^i - (\mu^i - r)dt/\sqrt{\xi_t^i}) \\
&= rdt + \beta_1^i\sqrt{v_{1,t}}d\tilde{z}_{1,t} + \beta_2^i\sqrt{v_{2,t}}d\tilde{z}_{2,t} + \sqrt{\xi_t^i}d\tilde{z}_t^i
\end{aligned} \tag{3.A.28}
$$

$$
\begin{aligned}
dS_t/S_t &= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}dz_{1,t} + \sqrt{v_{2,t}}dz_{2,t} \\
&= (r + \mu_1 v_{1,t} + \mu_2 v_{2,t})dt + \sqrt{v_{1,t}}(d\tilde{z}_{1,t} - \mu_1\sqrt{v_{1,t}}dt) + \sqrt{v_{2,t}}(d\tilde{z}_{2,t} - \mu_2\sqrt{v_{2,t}}dt) \\
&= rdt + \sqrt{v_{1,t}}d\tilde{z}_{1,t} + \sqrt{v_{2,t}}d\tilde{z}_{2,t}
\end{aligned} \tag{3.A.29}
$$

$$
\begin{aligned}
dv_{1,t} &= \kappa_1(\theta_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}(d\tilde{w}_{1,t} - (\lambda_1/\sigma_1)\sqrt{v_{1,t}}dt) \\
&= (\kappa_1\theta_1 - (\kappa_1 + \lambda_1)v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t} \\
&= \tilde{\kappa}_1(\tilde{\theta}_1 - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}d\tilde{w}_{1,t}
\end{aligned} \tag{3.A.30}
$$

$$
\begin{aligned}
dv_{2,t} &= \kappa_2(\theta_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}(d\tilde{w}_{2,t} - (\lambda_2/\sigma_2)\sqrt{v_{2,t}}dt) \\
&= (\kappa_2\theta_2 - (\kappa_2 + \lambda_2)v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t} \\
&= \tilde{\kappa}_2(\tilde{\theta}_2 - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}d\tilde{w}_{2,t}
\end{aligned} \tag{3.A.31}
$$

3.B Proof of Proposition 3.2

Given the Q dynamics of index returns and individual equities returns in (3.2.3) and (3.2.6), applying Ito's lemma on x_t^i, delivers the following expression.

$$\begin{aligned} x_{t+\tau}^i - x_t^i = r\tau - \frac{1}{2}\big[{\beta_1^i}^2 v_{1,t:t+\tau} + {\beta_2^i}^2 v_{2,t:t+\tau} + \xi_{t:t+\tau}^i\big]\tau \\ + \beta_1^i \int_t^{t+\tau} \sqrt{v_{1,u}} d\tilde{z}_{1,u} + \beta_2^i \int_t^{t+\tau} \sqrt{v_{2,u}} d\tilde{z}_{2,u} + \int_t^{t+\tau} \sqrt{\xi_t^i} d\tilde{z}_u^i \end{aligned} \tag{3.B.1}$$

For the ease of notations we define:

$$\begin{aligned} \tilde{z}_{v_{1,\tau}} &\equiv \int_t^{t+\tau} \sqrt{v_{1,u}} d\tilde{z}_{1,u} \ , \\ \tilde{z}_{v_{2,\tau}} &\equiv \int_t^{t+\tau} \sqrt{v_{2,u}} d\tilde{z}_{2,u} \ , \\ \tilde{z}_{\xi_\tau^i} &\equiv \int_t^{t+\tau} \sqrt{\xi_u^i} d\tilde{z}_u^i \ . \end{aligned}$$

By the definition of risk-neutral conditional characteristic function of log-returns in (3.2.8) we have:[25]

$$\tilde{f}^i(\tau,\phi) = E_t^Q\Big[\exp\big[i\phi(r\tau - \frac{1}{2}({\beta_1^i}^2 v_{1,t:t+\tau} + {\beta_2^i}^2 v_{2,t:t+\tau} + \xi_{t:t+\tau}^i)\tau + \beta_1^i \tilde{z}_{v_{1,\tau}} + \beta_2^i \tilde{z}_{v_{2,\tau}} + \tilde{z}_{\xi_\tau^i})\big]\Big]. \tag{3.B.2}$$

Define the stochastic exponential $\zeta(\cdot)$ as follows.

$$\zeta\big(\int_0^t w_u' dW_u\big) \equiv \exp\big[\int_0^t w_u' dW_u - \frac{1}{2}\int_0^t w_u' d\langle W, W'\rangle w_u\big] \tag{3.B.3}$$

Therefore,

[25]For compactness, the dependence of risk-neutral conditional characteristic function to x_t^i, $v_{1,t}$, $v_{2,t}$, ξ_t^i, β_1^i, and β_2^i is suppressed in (3.B.2).

$$\zeta\big(i\phi\beta_1^i\ \tilde{z}_{v_{1,\tau}}\big) = \exp\big[i\phi\beta_1^i\ \tilde{z}_{v_{1,\tau}} - \frac{1}{2}(i\phi\beta_1^i)^2\langle\tilde{z}_{v_{1,\tau}}, \tilde{z}_{v_{1,\tau}}\rangle\big]$$
$$= \exp\big[i\phi\beta_1^i\ \tilde{z}_{v_{1,\tau}} + \frac{1}{2}\phi^2{\beta_1^i}^2 v_{1,t:1+\tau}\big]. \tag{3.B.4}$$

Similar to (3.B.4), define $\zeta\big(i\phi\beta_2^i\ \tilde{z}_{v_{2,\tau}}\big)$ and $\zeta\big(i\phi\ \tilde{z}_{\xi_\tau^i}\big)$ and then combine these three stochastic exponential with (3.B.2) to get the following risk-neutral conditional characteristic function.

$$\tilde{f}^i(\tau,\phi) = e^{i\phi r\tau}E_t^Q\Big[\zeta\big(i\phi\beta_1^i\ \tilde{z}_{v_{1,\tau}}\big)\zeta\big(i\phi\beta_2^i\ \tilde{z}_{v_{2,\tau}}\big)\zeta\big(i\phi\ \tilde{z}_{\xi_\tau^i}\big)\exp\big[-g_1 v_{1,t:t+\tau} - g_2 v_{2,t:t+\tau} - g_3\xi_{t:t+\tau}^i\big]\Big] \tag{3.B.5}$$

where, $g_1 = \frac{1}{2}i\phi{\beta_1^i}^2(1-i\phi)$, $g_2 = \frac{1}{2}i\phi{\beta_2^i}^2(1-i\phi)$, and $g_3 = \frac{1}{2}i\phi(1-i\phi)$. Following Carr and Wu [2004], we define a new change-of-measure from Q-measure to C-measure as follows.[26]

$$\frac{dC}{dQ}(t) \equiv \zeta\big(i\phi\beta_1^i\ \tilde{z}_{v_{1,\tau}}\big)\zeta\big(i\phi\beta_2^i\ \tilde{z}_{v_{2,\tau}}\big)\zeta\big(i\phi\ \tilde{z}_{\xi_\tau^i}\big) \tag{3.B.6}$$

The Radon-Nikodym derivatives of C with respect to Q in (3.B.6) allows to write (3.B.5) as

$$\tilde{f}^i(\tau,\phi) = e^{i\phi r\tau}E_t^Q\Big[\frac{\frac{dC}{dQ}(T)}{\frac{dC}{dQ}(t)}\exp\big[-g_1 v_{1,t:t+\tau} - g_2 v_{2,t:t+\tau} - g_3\xi_{t:t+\tau}^i\big]\Big]$$
$$= e^{i\phi r\tau}E_t^C\Big[\exp\big[-g_1 v_{1,t:t+\tau} - g_2 v_{2,t:t+\tau} - g_3\xi_{t:t+\tau}^i\big]\Big]. \tag{3.B.7}$$

Accordingly, we transform the risk-neutral shocks to index returns volatlities and to the idiosyncratic returns volatility to their C-measure counterparts by applying the extension of Grisanov's theorem within the complex plane.

$$\begin{aligned} d\tilde{w}_{1,t} &= dw_{1,t}^C + (i\phi\rho_1\beta_1^i\sqrt{v_{1,t}})dt \\ d\tilde{w}_{2,t} &= dw_{2,t}^C + (i\phi\rho_2\beta_2^i\sqrt{v_{2,t}})dt \\ d\tilde{w}_t^i &= dw_t^{i,C} + (i\phi\rho^i\sqrt{\xi_t^i})dt \end{aligned} \tag{3.B.8}$$

As a results, the index volatilities dynamics and idiosyncratic volatility dynamics of individ-

[26] As the Radon-Nikodym derivatives in(3.B.6) is defined based on the stochastic exponential $\zeta(\cdot)$, it is Martingale by definition.

ual equity under the C-measure are

$$\begin{aligned}
dv_{1,t} &= \kappa_1^C(\theta_1^C - v_{1,t})dt + \sigma_1\sqrt{v_{1,t}}dw_{1,t}^C \ , \\
dv_{2,t} &= \kappa_2^C(\theta_2^C - v_{2,t})dt + \sigma_2\sqrt{v_{2,t}}dw_{2,t}^C \ , \\
d\xi_t^i &= \kappa^{i,C}(\theta^{i,C} - \xi_t^i)dt + \sigma^i\sqrt{\xi_t^i}dw_t^{i,C},
\end{aligned} \tag{3.B.9}$$

where,

$$\begin{aligned}
\kappa_1^C &= \tilde{\kappa}_1 - i\phi\rho_1\beta_1^i\sigma_1 \quad & \theta_1^C &= \tilde{\kappa}_1\tilde{\theta}_1/\kappa_1^C \ , \\
\kappa_2^C &= \tilde{\kappa}_2 - i\phi\rho_2\beta_2^i\sigma_2 \quad & \theta_2^C &= \tilde{\kappa}_2\tilde{\theta}_2/\kappa_2^C \ , \\
\kappa^{i,C} &= \kappa^i - i\phi\rho^i\sigma^i \quad & \theta^{i,C} &= \kappa^i\theta^i/\kappa^{i,C} \ .
\end{aligned}$$

Using the closed-form solution of the moment generating functions of $E_t^C[\exp(-g_1 v_{1,t:t+\tau})]$, and $E_t^C[\exp(-g_2 v_{2,t:t+\tau})]$, and $E_t^C[\exp(-g_3 \xi_{t:t+\tau}^i)]$, the risk-neutral conditional characteristic function of log individual equity prices has the following affine form.

$$\begin{aligned}
\tilde{f}^i(v_{1,t}, v_{2,t}, \xi_t^i, \tau, \phi) = \exp\big[&i\phi x_t^i + i\phi r\tau - A_1(\tau,\phi) - A_2(\tau,\phi) - B(\tau,\phi) \\
&- C_1(\tau,\phi)v_{1,t} - C_2(\tau,\phi)v_{2,t} - D(\tau,\phi)\xi_t^i\big],
\end{aligned} \tag{3.B.10}$$

$$
\begin{aligned}
A_1(\tau,\phi) &= \frac{\tilde{\kappa}_1\tilde{\theta}_1}{\sigma_1^2}\Big[2\ln\big[1-\frac{d_1-\kappa_1^C}{2d_1}(1-e^{-d_1\tau})\big]+(d_1-\kappa_1^C)\tau\Big],\\
A_2(\tau,\phi) &= \frac{\tilde{\kappa}_2\tilde{\theta}_2}{\sigma_2^2}\Big[2\ln\big[1-\frac{d_2-\kappa_2^C}{2d_2}(1-e^{-d_2\tau})\big]+(d_2-\kappa_2^C)\tau\Big],\\
B(\tau,\phi) &= \frac{\tilde{\kappa}^i\tilde{\theta}^i}{{\sigma^i}^2}\Big[2\ln\big[1-\frac{d^i-\kappa^{i,C}}{2d^i}(1-e^{-d^i\tau})\big]+(d^i-\kappa^{i,C})\tau\Big],\\
C_1(\tau,\phi) &= \frac{2g_1(1-e^{-d_1\tau})}{2d_1-(d_1-\kappa_1^C)(1-e^{-d_1\tau})},\\
C_2(\tau,\phi) &= \frac{2g_2(1-e^{-d_2\tau})}{2d_2-(d_2-\kappa_2^C)(1-e^{-d_2\tau})},\\
D(\tau,\phi) &= \frac{2g^i(1-e^{-d^i\tau})}{2d^i-(d^i-\kappa^{i,C})(1-e^{-d^i\tau})},\\
d_1 &= \sqrt{(\kappa_1^C)^2+2\sigma_1^2 g_1}\ ,\\
d_2 &= \sqrt{(\kappa_2^C)^2+2\sigma_2^2 g_2}\ ,\\
d^i &= \sqrt{(\kappa^{i,C})^2+2{\sigma^i}^2 g^i}\ ,\\
g_1 &= \frac{1}{2}i\phi{\beta_1^i}^2(1-i\phi)\ ,\\
g_2 &= \frac{1}{2}i\phi{\beta_2^i}^2(1-i\phi)\ ,\\
g^i &= \frac{1}{2}i\phi(1-i\phi)\ .
\end{aligned}
\tag{3.B.11}
$$

We determine the price of a European call option on an individual equity with the strike price K and the time-to-maturity τ by inverting the risk-neutral conditional characteristic function of log-returns.[27]

$$
C_t^i(S_t^i,K,\tau) = S_t^i P_1^i - Ke^{-r\tau}P_2^i\ , \tag{3.B.12}
$$

[27]Note that the risk-neutral conditional characteristic function of the logarithm of individual equity returns, $x_{t+\tau}^i - x_t^i = \ln(S_{t+\tau}^i/S_t^i)$, can be defined with the same expression as (3.B.10) but without the first component, $i\phi x_t^i$.

where,

$$
\begin{aligned}
P_1^i &= \frac{1}{2} + \frac{1}{\pi}\frac{1}{S_t^i e^{r\tau}} \int_0^\infty \Re\Big[\frac{e^{-i\phi \ln K}\tilde{f}^i(v_{1,t}, v_{2,t}, \xi_t^i, \tau, \phi - i)}{i\phi}\Big] d\phi \ , \\
P_2^i &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\Big[\frac{e^{-i\phi \ln K}\tilde{f}^i(v_{1,t}, v_{2,t}, \xi_t^i, \tau, \phi)}{i\phi}\Big] d\phi \ .
\end{aligned}
\tag{3.B.13}
$$

3.C Proof of Proposition 3.3

Proof of the Proposition (3.3) is available upon request.

3.D Proof of Proposition 3.4

Proof of the Proposition (3.4) is available upon request.

References

Adrian, T. and J. Rosenberg (2008). Stock returns and volatility: Pricing the short-run and long-run components of market risk. *The Journal of Finance 63*(6), 2997–3030.

Alizadeh, S., M. W. Brandt, and F. X. Diebold (2002). Range-based estimation of stochastic volatility models. *The Journal of Finance 57*(3), 1047–1091.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and H. Ebens (2001). The distribution of realized stock return volatility. *Journal of Financial Economics 61*(1), 43–76.

Ang, A., R. J. Hodrick, Y. Xing, and X. Zhang (2006). The cross-section of volatility and expected returns. *The Journal of Finance 61*(1), 259–299.

Bakshi, G., C. Cao, and Z. Chen (1997). Empirical performance of alternative option pricing models. *The Journal of Finance 52*(5), 2003–2049.

Bakshi, G., N. Kapadia, and D. Madan (2003). Stock return characteristics, skew laws, and the differential pricing of individual equity options. *Review of Financial Studies 16*(1), 101–143.

Bates, D. S. (2000). Post-'87 crash fears in the s&p 500 futures option market. *Journal of Econometrics 94*(1), 181–238.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy 81*(3), 637–654.

Bollerslev, T. and H. Zhou (2002). Estimating stochastic volatility diffusion using conditional moments of integrated volatility. *Journal of Econometrics 109*(1), 33–65.

Carr, P. and L. Wu (2004). Time-changed lévy processes and option pricing. *Journal of Financial Economics 71*(1), 113–141.

Chernov, M., A. R. Gallant, E. Ghysels, and G. Tauchen (2003). Alternative models for stock price dynamics. *Journal of Econometrics 116*(1), 225–257.

Christoffersen, P., M. Fournier, and K. Jacobs (2015). The factor structure in equity options. *Rotman School of Management Working Paper* (2224270).

Christoffersen, P., S. Heston, and K. Jacobs (2009). The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well. *Management Science 55*(12), 1914–1932.

Cox, J. C., S. A. Ross, and M. Rubinstein (1979). Option pricing: A simplified approach. *Journal of Financial Economics 7*(3), 229–263.

Dai, Q. and K. J. Singleton (2000). Specification analysis of affine term structure models. *The Journal of Finance 55*(5), 1943–1978.

Dai, Q. and K. J. Singleton (2002). Expectation puzzles, time-varying risk premia, and affine models of the term structure. *Journal of Financial Economics 63*(3), 415–441.

Duan, J.-C. and J. Wei (2009). Systematic risk and the price structure of individual equity options. *Review of Financial Studies 22*(5), 1981–2006.

Egloff, D., M. Leippold, and L. Wu (2010). The term structure of variance swap rates and optimal variance swap investments. *Journal of Financial and Quantitative Analysis 45*(5), 1279–1310.

Ghanbari, H. (2016). Transient and persistent variance risk premium. *Working Paper*.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies 6*(2), 327–343.

Huang, J.-z. and L. Wu (2004). Specification analysis of option pricing models based on time-changed lévy processes. *The Journal of Finance 59*(3), 1405–1440.

Mencía, J. and E. Sentana (2013). Valuation of vix derivatives. *Journal of Financial Economics 108*(2), 367–391.

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica 41*(5), 867–887.

Merton, R. C. (1980). On estimating the expected return on the market: An exploratory investigation. *Journal of Financial Economics 8*(4), 323–361.

Pitt, M. K. (2002). Smooth particle filters for likelihood evaluation and maximisation. the warwick economics research paper series. *University of Warwick, Department of Economics*.

Serban, M., J. P. Lehoczky, and D. J. Seppi (2008). Cross-sectional stock option pricing and factor models of returns. *Working Paper*.

Todorov, V. and T. Bollerslev (2010). Jumps and betas: A new framework for disentangling and estimating systematic risks. *Journal of Econometrics 157*(2), 220–235.

Table 3.1: Data Sample Summary

Company	Ticker	Call	Put	All Options	Avg DTM	Avg IV
S&P 500 Index	**SPX**	208,098	137,612	345,710	141	22.49%
Alcoa	**AA**	134,112	106,732	240,844	130	35.16%
American Express	**AXP**	143,880	109,422	253,302	132	31.62%
Boeing	**BA**	149,949	116,967	266,916	131	30.52%
Caterpillar	**CAT**	145,951	113,189	259,140	130	32.04%
Cisco	**CSCO**	127,223	100,605	227,828	128	36.92%
Chevron	**CVX**	178,737	132,901	311,638	135	24.56%
Dupont	**DD**	162,592	122,417	285,009	135	27.43%
Disney	**DIS**	145,656	114,062	259,718	138	29.84%
General Electric	**GE**	151,825	112,771	264,596	141	27.74%
Home Depot	**HD**	145,260	113,691	258,951	134	30.92%
Hewlett-Packard	**HPQ**	127,524	101,302	228,826	131	35.36%
IBM	**IBM**	164,543	125,043	289,586	135	27.09%
Intel	**INTC**	123,444	98,783	222,227	135	36.09%
Johnson & Johnson	**JNJ**	189,496	137,546	327,042	140	21.83%
JP Morgan	**JPM**	149,895	110,342	260,237	132	31.60%
Coca Cola	**KO**	178,611	131,747	310,358	141	23.03%
McDonald's	**MCD**	163,946	126,156	290,102	138	26.05%
3M	**MMM**	176,339	131,127	307,466	135	24.82%
Merck	**MRK**	160,622	120,662	281,284	134	27.68%
Microsoft	**MSFT**	138,523	106,266	244,789	140	30.69%
Pfizer	**PFE**	145,288	112,830	258,118	141	28.63%
Procter & Gamble	**PG**	186,969	137,111	324,080	139	22.12%
AT&T	**T**	174,932	123,359	298,291	135	25.85%
United Technologies	**UTX**	166,534	126,111	292,645	134	26.64%
Verizon	**VZ**	167,457	117,498	284,955	138	26.02%
Walmart	**WMT**	165,015	127,833	292,848	138	25.74%
Exxon Mobil	**XOM**	177,667	133,517	311,184	137	24.07%
Average		157,111	118,889	275,999	135	28.52%
Minimum		123,444	98,783	222,227	128	21.83%
Maximum		189,496	137,546	327,042	141	36.92%

Note to Table: This table reports the number of available call and put options for index and for each firm in our sample. Our sample contains options with moneyness up to 10% and maturity up to and including 1 year over the period 1996-2011. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM) and the average Black-Scholes implied volatility (Avg IV) of available contracts.

Table 3.2: Data Sample Summary - Call Options

Ticker	Avg Price	Min Price	Max Price	Avg IV	Min. IV	Max IV	Avg Delta	Avg Vega	Avg DTM
SPX	35.59	1.876	195.53	20.64%	7.03%	74.98%	0.373	251.02	143
AA	2.256	0.110	14.121	34.20%	16.93%	153.65%	0.442	8.385	130
AXP	3.218	0.375	27.372	30.28%	12.72%	148.17%	0.436	12.612	133
BA	3.022	0.375	14.928	29.57%	16.06%	89.57%	0.429	13.062	131
CAT	3.351	0.376	15.375	30.98%	16.01%	103.28%	0.432	13.882	131
CSCO	2.364	0.093	32.268	35.87%	15.93%	107.08%	0.441	7.251	129
CVX	3.196	0.375	15.509	23.45%	12.79%	94.43%	0.416	16.718	137
DD	2.319	0.375	13.407	26.25%	12.29%	92.26%	0.427	10.961	136
DIS	1.899	0.375	17.498	28.56%	6.95%	95.86%	0.441	8.422	139
GE	2.385	0.375	27.865	26.38%	6.90%	148.93%	0.438	10.855	143
HD	2.215	0.375	15.933	29.72%	14.84%	100.91%	0.435	9.111	136
HPQ	2.869	0.375	46.162	34.47%	15.32%	97.89%	0.445	9.303	132
IBM	4.976	0.361	36.790	25.83%	11.93%	86.82%	0.416	23.901	136
INTC	2.946	0.375	28.764	35.20%	17.34%	90.86%	0.455	9.389	136
JNJ	2.391	0.375	14.911	20.44%	9.66%	70.84%	0.409	14.260	142
JPM	2.759	0.131	19.016	30.02%	11.19%	160.94%	0.431	11.158	133
KO	2.080	0.375	10.651	21.73%	8.27%	69.30%	0.416	11.767	143
MCD	2.008	0.375	13.560	24.80%	11.58%	78.87%	0.429	10.308	139
MMM	3.608	0.375	17.730	23.66%	12.51%	79.62%	0.413	18.890	136
MRK	2.797	0.375	23.758	26.56%	14.29%	85.20%	0.432	12.354	136
MSFT	3.143	0.375	29.554	29.44%	12.22%	87.86%	0.450	11.448	141
PFE	2.175	0.375	22.262	27.57%	14.20%	100.98%	0.441	8.982	143
PG	2.770	0.375	19.779	20.77%	9.28%	64.34%	0.409	16.262	142
T	1.611	0.075	9.373	24.41%	10.04%	82.25%	0.432	7.657	137
UTX	3.247	0.375	22.284	25.34%	13.16%	82.34%	0.417	16.273	135
VZ	2.078	0.375	12.448	24.58%	9.22%	86.98%	0.444	9.779	141
WMT	2.199	0.375	17.836	24.52%	11.16%	67.26%	0.418	11.103	140
XOM	2.688	0.375	15.079	22.92%	12.58%	84.79%	0.414	14.474	139
Avg.	2.688	0.334	20.527	27.32%	12.42%	96.71%	0.430	12.169	137

Note to Table: This table reports the number of available call option contracts for the index and for each firm in our sample. Our sample contains call options with moneyness up to 10% and maturity up to and including 1 year over the period 1996-2011. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM), the average Black-Scholes implied volatility (Avg IV), the average Black-Scholes delta (Avg Delta), and the average Black-Scholes vega (Avg Vega) of available contracts.

Table 3.3: Data Sample Summary - Put Options

Ticker	Avg Price	Min Price	Max Price	Avg IV	Min IV	Max IV	Avg Delta	Avg Vega	Avg DTM
SPX	32.11	2.640	195.53	24.34%	8.90%	82.74%	-0.292	227.67	136
AA	1.908	0.110	14.121	36.13%	17.39%	159.25%	-0.342	7.840	129
AXP	2.821	0.375	27.372	32.95%	12.20%	149.37%	-0.340	11.851	130
BA	2.604	0.375	14.928	31.47%	17.43%	93.33%	-0.339	12.114	130
CAT	2.981	0.376	15.375	33.11%	17.86%	104.41%	-0.340	12.959	130
CSCO	2.120	0.093	32.268	37.97%	16.34%	112.08%	-0.351	6.862	128
CVX	2.754	0.375	15.509	25.67%	11.68%	98.59%	-0.327	15.499	134
DD	1.978	0.375	13.407	28.61%	13.70%	94.19%	-0.333	10.133	133
DIS	1.618	0.375	17.498	31.11%	14.31%	99.48%	-0.343	7.738	137
GE	2.018	0.375	27.865	29.09%	7.10%	149.59%	-0.337	10.048	140
HD	1.946	0.375	15.933	32.12%	14.03%	103.50%	-0.343	8.508	133
HPQ	2.368	0.375	46.162	36.25%	16.45%	94.06%	-0.350	8.721	129
IBM	4.535	0.361	36.790	28.35%	12.38%	90.96%	-0.336	22.422	134
INTC	2.596	0.375	28.764	36.97%	16.35%	92.03%	-0.353	9.103	134
JNJ	2.081	0.375	14.911	23.22%	9.61%	77.42%	-0.327	13.112	137
JPM	2.471	0.131	19.016	33.19%	11.99%	169.06%	-0.337	10.568	131
KO	1.827	0.375	10.651	24.34%	9.52%	67.51%	-0.330	10.878	139
MCD	1.727	0.375	13.560	27.30%	12.47%	74.29%	-0.336	9.455	136
MMM	3.175	0.375	17.730	25.99%	13.82%	86.39%	-0.329	17.609	134
MRK	2.316	0.375	23.758	28.80%	9.07%	88.64%	-0.334	11.504	132
MSFT	2.821	0.375	29.554	31.94%	11.20%	94.44%	-0.349	11.241	139
PFE	1.864	0.375	22.262	29.68%	13.95%	75.78%	-0.343	8.501	140
PG	2.435	0.375	19.779	23.47%	9.58%	74.12%	-0.327	15.103	137
T	1.400	0.075	9.373	27.30%	10.25%	86.45%	-0.334	7.206	134
UTX	2.904	0.375	22.284	27.94%	13.62%	87.87%	-0.333	15.167	133
VZ	1.728	0.375	12.448	27.45%	10.94%	89.81%	-0.330	9.118	135
WMT	1.979	0.375	17.836	26.97%	11.44%	72.69%	-0.335	10.324	136
XOM	2.309	0.375	15.079	25.22%	12.79%	97.18%	-0.329	13.299	136
Avg.	2.344	0.334	20.527	29.73%	12.87%	99.35%	-0.337	11.366	134

Note to Table: This table reports the number of available put option contracts for the index and for each firm in our sample. Our sample contains put options with moneyness up to 10% and maturity up to and including 1 year over the period 1996-2011. We rely on the implied volatility surface data set provided by OptionMetrics. For each firm, we also report the average number of days-to-maturity (Avg DTM), the average Black-Scholes implied volatility (Avg IV), the average Black-Scholes delta (Avg Delta), and the average Black-Scholes vega (Avg Vega) of available contracts.

Table 3.4: Market Parameter Estimates

Panel A: Parameter Estimates (Physical) - Joint Estimation

κ_1	κ_2	θ_1	θ_2	σ_1	σ_2	ρ_1	ρ_2	λ_1	λ_2
1.4271	3.5874	0.0026	0.0171	0.0855	0.3496	-0.6918	-0.2173	-1.0798	-1.0355

Panel B: Parameter Estimates (Risk Neutral) - Options-based Estimation

$\tilde{\kappa}_1$	$\tilde{\kappa}_2$	$\tilde{\theta}_1$	$\tilde{\theta}_2$	σ_1	σ_2	ρ_1	ρ_2		
0.2267	2.9137	0.0590	0.0100	0.0958	0.5678	-0.9135	-0.4934		

Note to Table: This table reports the structural parameter estimates of the S&P 500 Index for the two-factor stochastic volatility model. The reported results in Panel A are from the joint estimation using the daily S&P 500 index returns and options data. Structural parameters in Panel B are estimated using only options data. In both panels, we use OTM call and put options with moneyness up to 10% over the period 1996-2011. As in Proposition (3.1), $\tilde{\kappa}_1 = \kappa_1 + \lambda_1$, $\tilde{\kappa}_2 = \kappa_2 + \lambda_2$, $\tilde{\theta}_1 = \frac{k_1\theta_1}{k_1+\lambda_1}$, $\tilde{\theta}_2 = \frac{k_2\theta_2}{k_2+\lambda_2}$. Therefore, risk neutral parameters from the joint estimation are $\tilde{\kappa}_1 = 0.3473$, $\tilde{\kappa}_2 = 2.5520$, $\tilde{\theta}_1 = 0.0106$, $\tilde{\theta}_2 = 0.0240$.

Table 3.5: Individual Equity Parameter Estimates

Company	Ticker	$\tilde{\kappa}$	$\tilde{\theta}$	σ	ρ	β_1	β_2
Alcoa	**AA**	0.7253	0.0202	0.1612	-0.87	0.3850	1.3159
American Express	**AXP**	0.7663	0.0128	0.1009	-0.91	0.3430	1.3203
Boeing	**BA**	0.7692	0.0235	0.1757	-0.97	0.4108	1.3046
Caterpillar	**CAT**	0.6354	0.0291	0.1984	-0.84	0.3608	1.3215
Cisco	**CSCO**	0.6804	0.0653	0.3599	-0.81	0.4420	1.2508
Chevron	**CVX**	0.9390	0.0097	0.0913	-0.88	0.5816	1.1538
Dupont	**DD**	0.8702	0.0137	0.1310	-0.92	0.4949	1.2888
Disney	**DIS**	0.6995	0.0247	0.1841	-0.89	0.4462	1.2854
General Electric	**GE**	0.5694	0.0093	0.0670	-0.85	0.4968	1.3111
Home Depot	**HD**	0.6912	0.0340	0.2379	-0.83	0.4278	1.3097
Hewlett-Packard	**HPQ**	0.6159	0.0756	0.3967	-0.64	0.4432	1.2458
IBM	**IBM**	0.7717	0.0186	0.1676	-0.78	0.6798	1.2853
Intel	**INTC**	0.8160	0.0295	0.2123	-0.84	0.4322	1.2652
Johnson & Johnson	**JNJ**	0.6492	0.0238	0.2015	-0.95	0.5574	1.0197
JP Morgan	**JPM**	0.8606	0.0193	0.1836	-0.99	0.4483	1.3466
Coca Cola	**KO**	0.3920	0.0291	0.1895	-0.87	0.6077	1.0897
McDonald's	**MCD**	0.9305	0.0262	0.2109	-0.97	0.4754	1.1359
3M	**MMM**	1.7078	0.0107	0.1569	-0.86	0.5886	1.1752
Merck	**MRK**	1.2259	0.0105	0.1073	-0.89	0.5018	1.2276
Microsoft	**MSFT**	0.7777	0.0108	0.0710	-0.81	0.4513	1.2739
Pfizer	**PFE**	0.8957	0.0210	0.1724	-0.88	0.5067	1.2166
Procter & Gamble	**PG**	0.5107	0.0470	0.3056	-0.85	0.5782	1.0125
AT&T	**T**	0.6972	0.0098	0.0830	-0.93	0.5116	1.2126
United Technologies	**UTX**	0.9778	0.0271	0.2606	-0.83	0.5221	1.2668
Verizon	**VZ**	0.8423	0.0102	0.0970	0.51	0.4719	1.1838
Walmart	**WMT**	0.6533	0.0314	0.2136	-0.86	0.4695	1.1724
Exxon Mobil	**XOM**	1.0785	0.0148	0.1849	-0.94	0.5925	1.1764
Average		0.8055	0.0244	0.1823	-0.820	0.4899	1.2284
Min		0.3920	0.0093	0.0670	-0.990	0.3430	1.0125
Max		1.7078	0.0756	0.3967	0.512	0.6798	1.3466

Note to Table: This table reports the risk-neutral structural parameter estimates for individual equities conditional on the structural parameters of the S&P 500 index and the vectors of filtered spot market variance components. This table also reports the persistent beta β_1^i and the transient beta β_2^i for individual equity i. The market parameters and spot variance components are estimated using OTM call and put options over the period 1996-2011 with moneyness up to 10%. For individual equities, we use OTM call and put options with moneyness up to 10% over the period 1996-2011, where we drop the first five months.

Table 3.6: Distributional Properties of Spot Idiosyncratic Volatility

Company	**Ticker**	**Mean**	**Std dev**	**Max**	**Median**
Alcoa	**AA**	0.1259	0.1387	0.6879	0.0900
American Express	**AXP**	0.1068	0.1489	0.7138	0.0692
Boeing	**BA**	0.0633	0.0442	0.2484	0.0521
Caterpillar	**CAT**	0.0783	0.0628	0.4395	0.0587
Cisco	**CSCO**	0.1497	0.1328	0.8274	0.0987
Chevron	**CVX**	0.0293	0.0267	0.2126	0.0260
Dupont	**DD**	0.0460	0.0476	0.2526	0.0292
Disney	**DIS**	0.0636	0.0515	0.2661	0.0460
General Electric	**GE**	0.0618	0.0938	0.6134	0.0413
Home Depot	**HD**	0.0741	0.0600	0.3230	0.0510
Hewlett-Packard	**HPQ**	0.1250	0.1231	0.4893	0.0903
IBM	**IBM**	0.0439	0.0482	0.2620	0.0260
Intel	**INTC**	0.1206	0.0882	0.6408	0.0927
Johnson & Johnson	**JNJ**	0.0225	0.0257	0.2340	0.0116
JP Morgan	**JPM**	0.1070	0.1325	0.9138	0.0786
Coca Cola	**KO**	0.0268	0.0308	0.1729	0.0133
McDonald's	**MCD**	0.0389	0.0345	0.1638	0.0277
3M	**MMM**	0.0297	0.0304	0.1645	0.0180
Merck	**MRK**	0.0438	0.0367	0.2189	0.0358
Microsoft	**MSFT**	0.0749	0.0614	0.4605	0.0647
Pfizer	**PFE**	0.0490	0.0425	0.2021	0.0356
Procter & Gamble	**PG**	0.0256	0.0326	0.2411	0.0103
AT&T	**T**	0.0522	0.0532	0.5365	0.0359
United Technologies	**UTX**	0.0399	0.0374	0.2126	0.0258
Verizon	**VZ**	0.0428	0.0438	0.3520	0.0280
Walmart	**WMT**	0.0436	0.0550	0.2870	0.0193
Exxon Mobil	**XOM**	0.0234	0.0210	0.1556	0.0204
Average		0.0633	0.0631	0.3812	0.0443
Minimum		0.0225	0.0210	0.1556	0.0103
Maximum		0.1497	0.1489	0.9138	0.0987

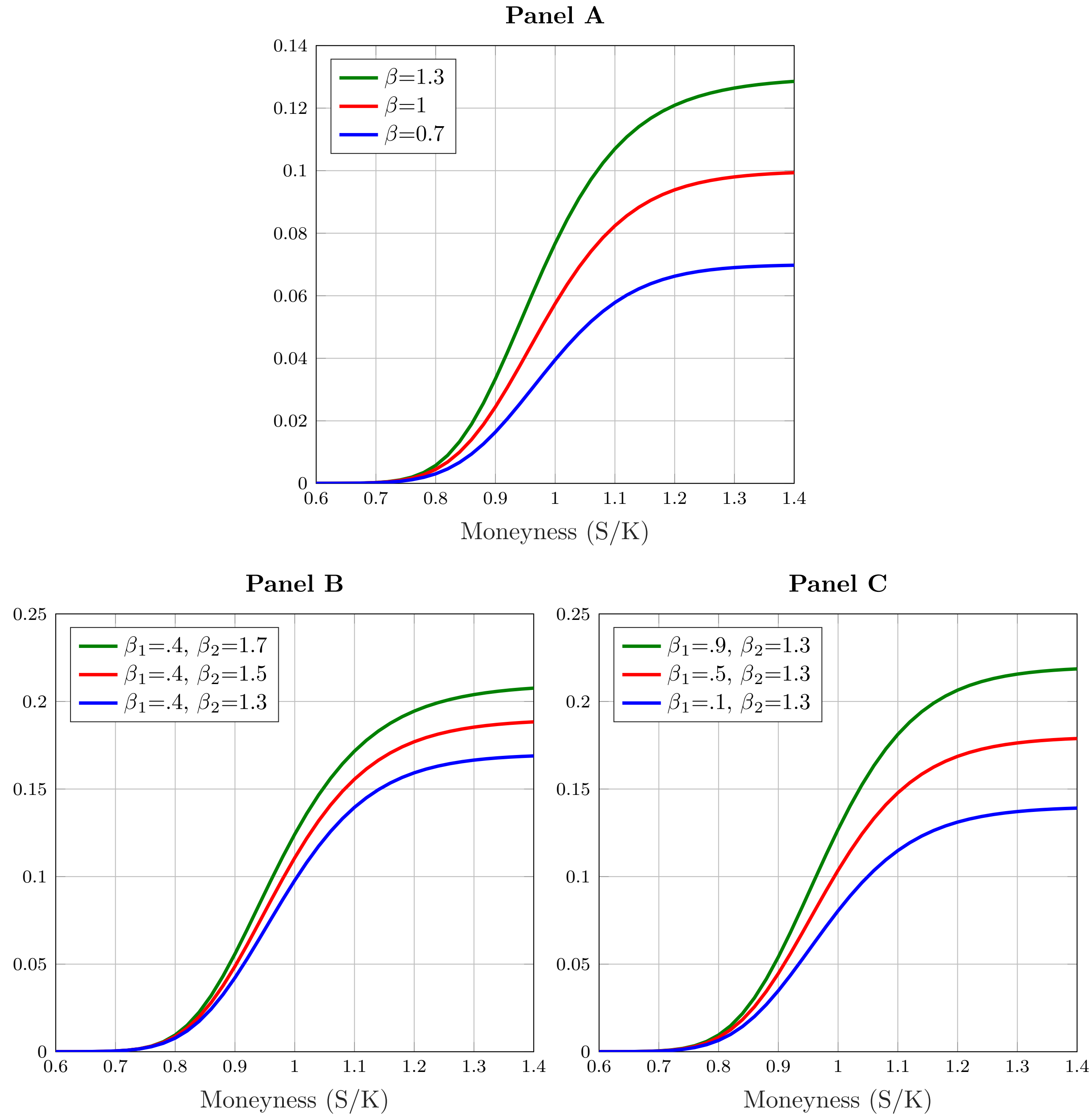
Note to Table: This table reports the mean, median, standard deviation, and maximum of spot idiosyncratic variance for every firm i conditional on the structural parameters of the S&P 500 index and filtered spot market variance components. The reported results are based on OTM call and put index option and individual equity option contracts with moneyness up to 10% over the period 1996-2011.

Table 3.7: Goodness of Fit

	In-Sample			Out-of-Sample
Ticker	**Vega RMSE**	**IV RMSE**	**IVRMSE/ Avg. IV**	**Vega RMSE**
AA	1.84	1.87	5.32	2.24
AXP	1.82	1.79	5.66	2.14
BA	1.41	1.35	4.42	1.97
CAT	1.50	1.47	4.59	1.68
CSCO	2.14	2.12	5.74	2.23
CVX	2.02	1.95	7.94	2.24
DD	1.42	1.41	5.14	1.53
DIS	1.75	1.69	5.66	1.97
GE	1.84	1.86	6.71	1.93
HD	1.58	1.54	4.98	1.72
HPQ	1.53	1.53	4.33	1.87
IBM	1.46	1.42	5.24	1.61
INTC	1.56	1.58	4.38	1.68
JNJ	1.42	1.40	6.41	1.65
JPM	1.85	1.82	5.76	2.08
KO	1.54	1.46	6.34	1.62
MCD	1.34	1.33	5.11	1.59
MMM	1.41	1.39	5.60	1.74
MRK	1.36	1.41	5.09	1.46
MSFT	1.67	1.64	5.34	1.75
PFE	1.49	1.46	5.10	1.73
PG	1.39	1.37	6.19	1.39
T	1.98	1.96	7.58	2.21
UTX	1.48	1.44	5.41	1.54
VZ	1.56	1.55	5.96	1.59
WMT	1.57	1.55	6.02	1.76
XOM	1.66	1.63	6.77	1.82
Average	1.61	1.59	5.66	1.81

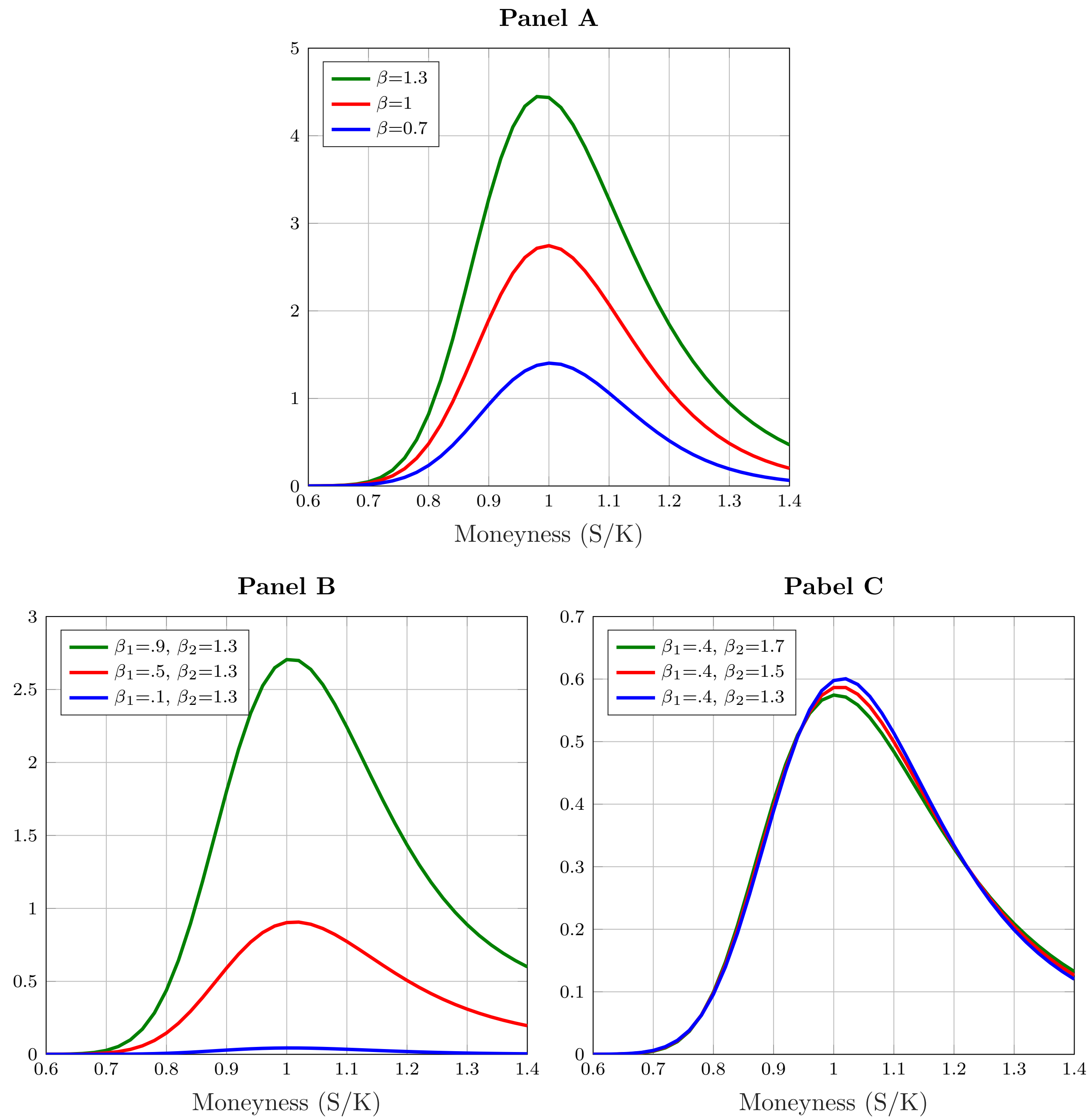
Note to Table: This table reports goodness-of-fit statistics for individual equity options. In-sample statistics are computed using options over the entire sample, 1996-2011. All numbers are in percentage points. We compute the Vega-weighted root mean squared error (Vega RMSE) along with the implied volatility root mean squared error (IVRMSE). We also report the ratio of IVRMSE over the average Black-Scholes implied volatility. We also report out-of-sample Vega RMSE over the period 2004-2011, given the in-sample parameter estimates, market spot variance components, and spot idiosyncratic variance over the period 1996-2003.

Figure 3.1: Market Delta of Equity Call Options



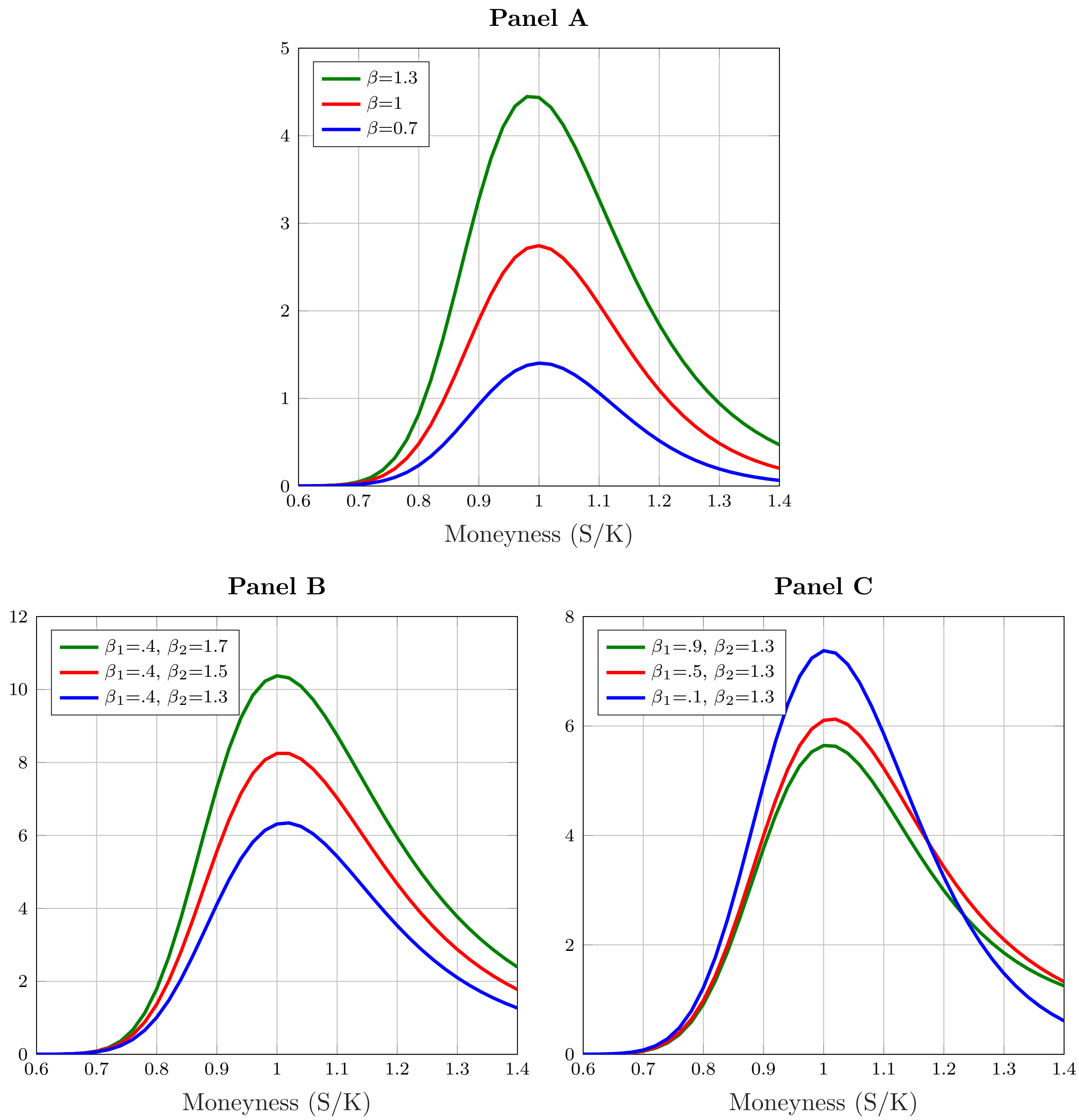
Note to Figure: This figure plots the sensitivity of the model-implied equity call option prices with respect to the level of market index for different sets of betas. Panel A shows this sensitivity following the calibration in in one-factor structure model of Christoffersen et al. [2015] while Panels B and C are the sensitivity in our two-factor structure model. Panel B, shows market delta when persistent beta is constant and Panel C is market delta when transient beta is constant.

Figure 3.2: Persistent Market Vega of Equity Call Options



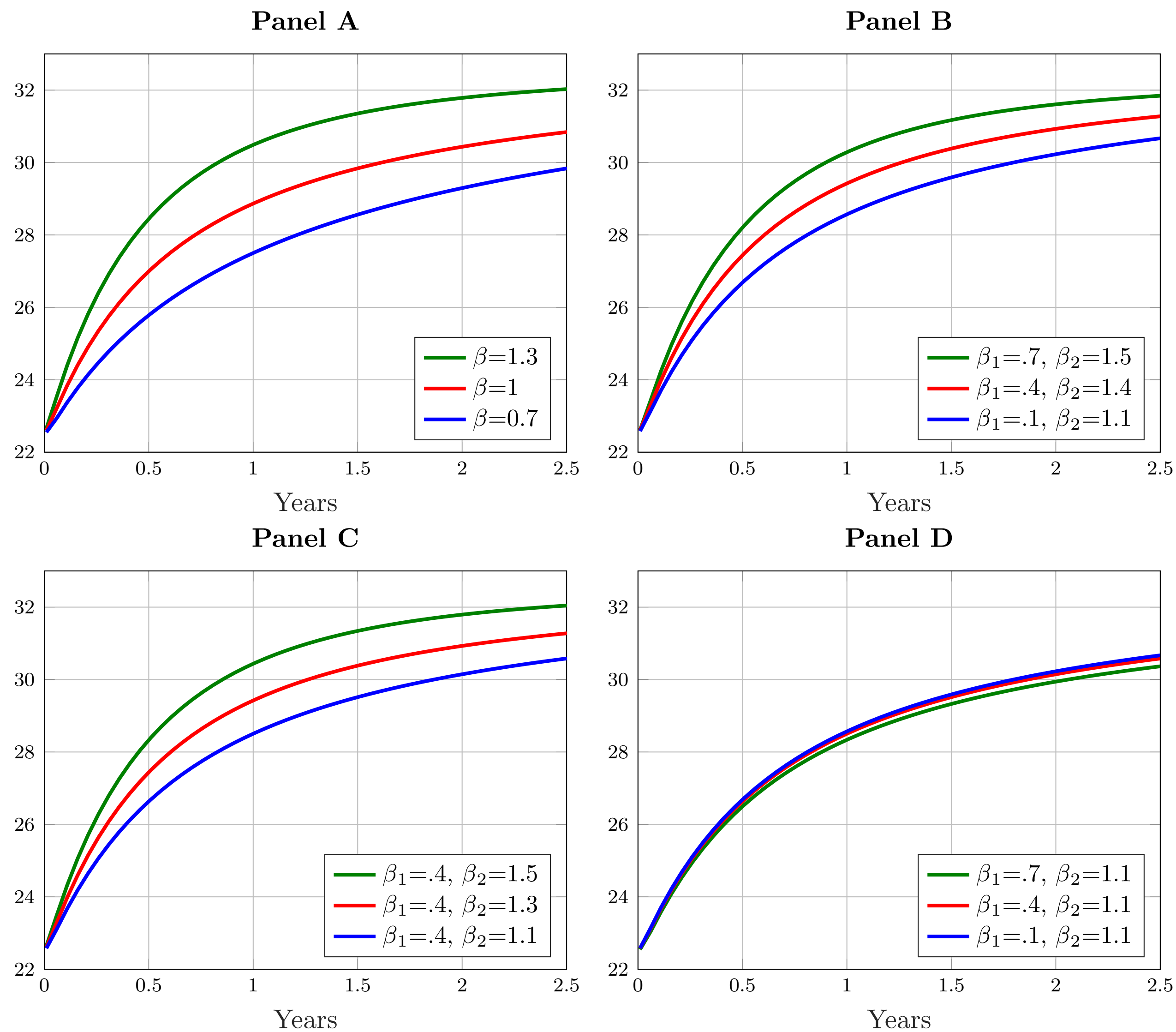
Note to Figure: This figure plots the sensitivity of the model-implied equity call option prices with respect to the persistent variance component for different sets of betas. Panel A shows this sensitivity following the calibration in in one-factor structure model while Panels B and C are the sensitivity in our two-factor structure model. Panel B, shows the persistent market vega when transient beta is constant and Panel C is the persistent market vega when persistent beta is constant. Note also that for all the graphs the total unconditional equity variance is fixed, $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$.

Figure 3.3: Transient Market Vega of Equity Call Options



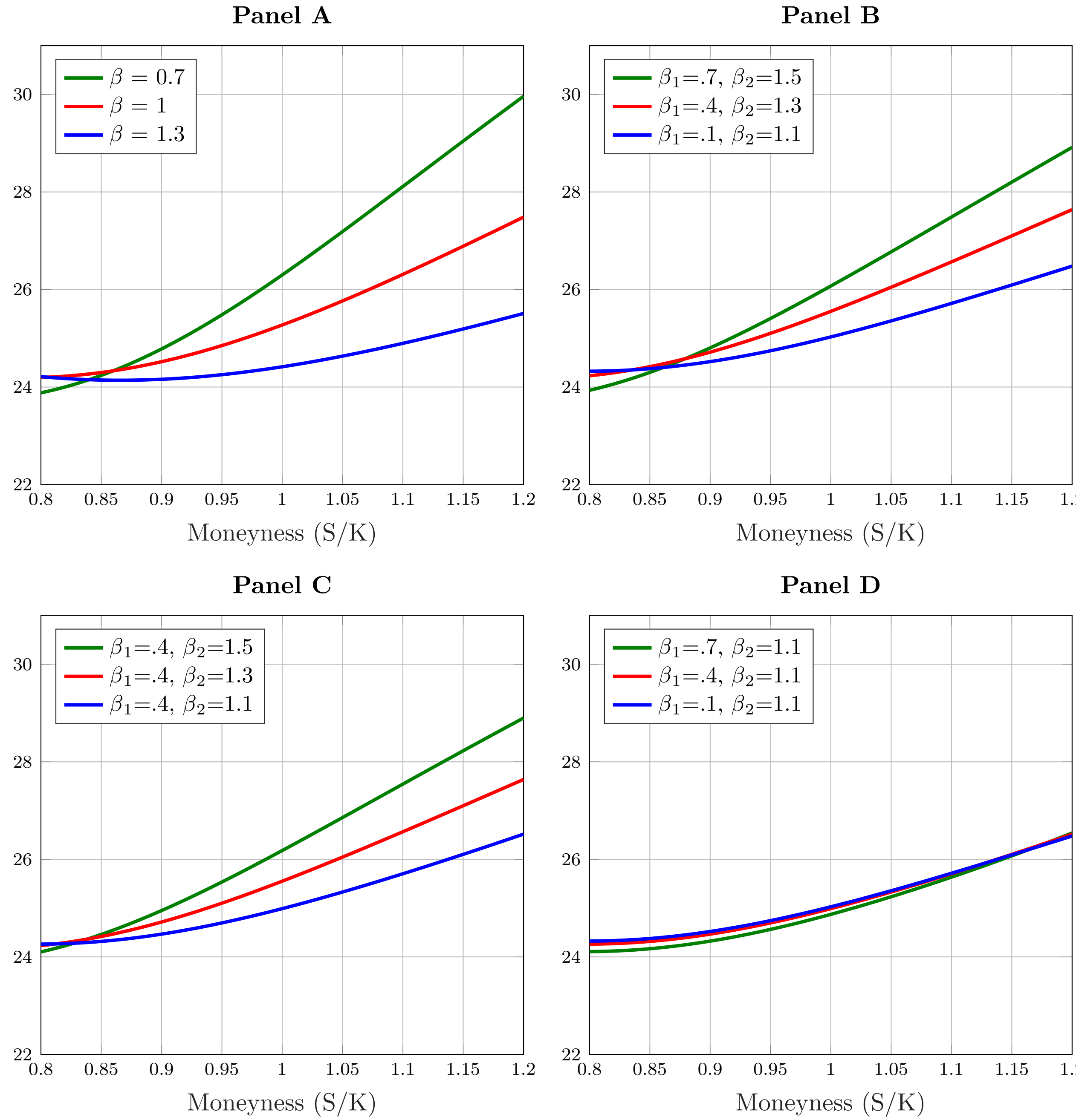
Note to Figure: This figure plots the sensitivity of the model-implied equity call option prices with respect to the transient variance component for different sets of betas. Panel A shows this sensitivity following the calibration in in one-factor structure model while Panels B and C are the sensitivity in our two-factor structure model. Panel B, shows the transient market vega when persistent beta is constant and Panel C is the transient market vega when transient beta is constant. Note also that for all the graphs the total unconditional equity variance is fixed, $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$.

Figure 3.4: Persistent and Transient Betas and Implied Volatility Term Structure



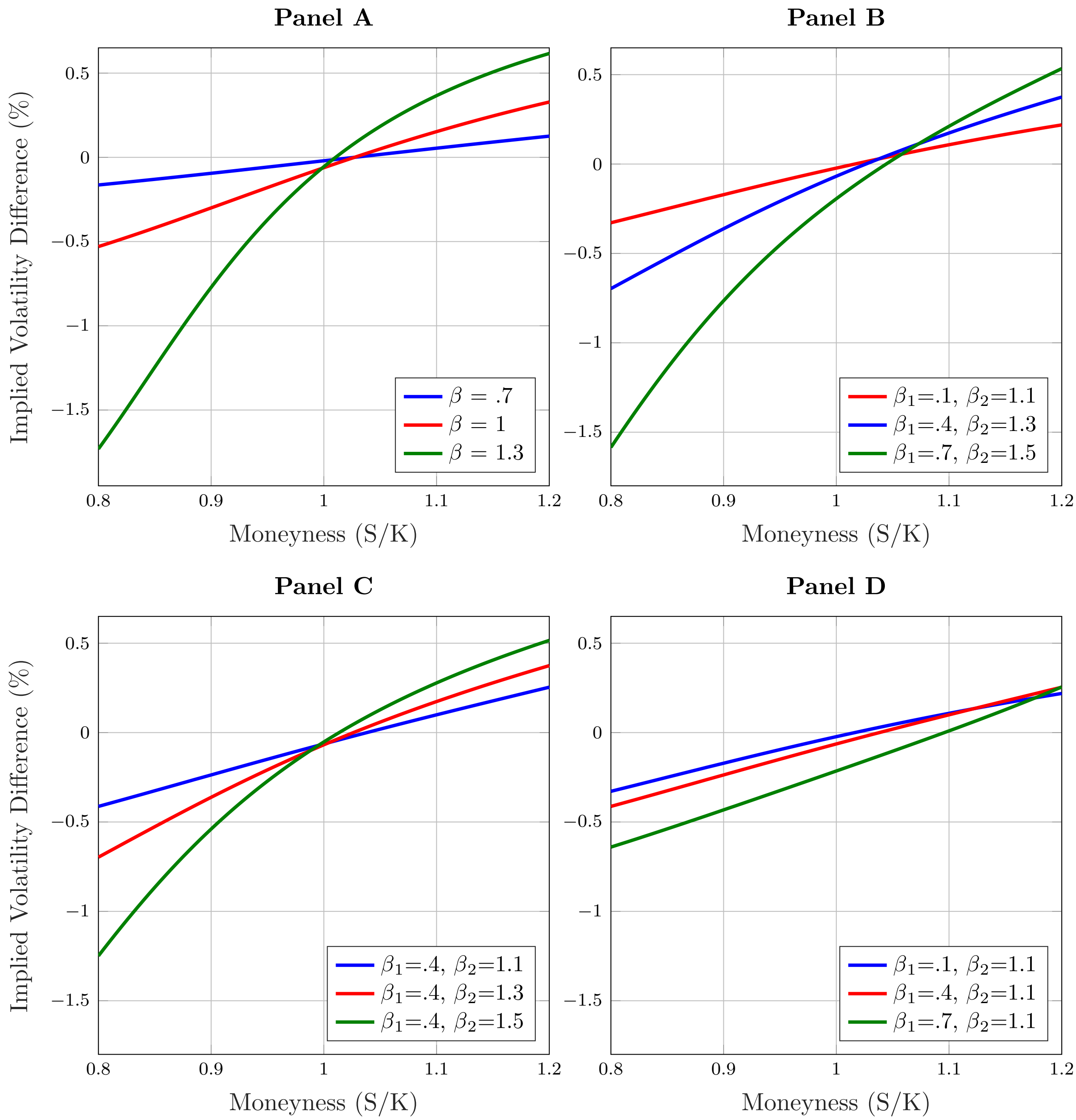
Note to Figure: This figure plots the model-implied volatility for at-the-money equity call options with respect to the time-to-maturity for different sets of betas. Panel A shows the term-structure effect following the one-factor structure model and Panel B replicates the same IV structure with our two-factor structure model. Panels C shows IV term structure when persistent beta β_1^i is constant and Panel D shows IV term structure when transient beta β_2^i is constant. Note that for all the graphs the total unconditional equity variance is fixed, $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$. We also fix the total unconditional risk-neutral market variances to 0.05, with $\tilde{\theta}_1 = 0.006$ and $\tilde{\theta}_2 = 0.044$. Therefore, the unconditional idiosyncratic equity variance for every set of betas can be defined by $\theta^i = \tilde{v}^i - (\beta_1^i)^2\tilde{\theta}_1 - (\beta_2^i)^2\tilde{\theta}_2$. The spot market variance components are set equal to $v_{1,t} = 0.012$ and $v_{2,t} = 0.048$ and the total spot equity variance is $v_t^i = 0.05$. Consequently, we define the spot idiosyncratic variance for different sets of betas as $\xi_t^i = v_t^i - (\beta_1^i)^2 v_{1,t} - (\beta_2^i)^2 v_{2,t}$. We choose the remaining structural parameters of the market and equity dynamics as follows: $\{\tilde{\kappa}_1 = 0.18, \tilde{\kappa}_2 = 2.8, \sigma_1 = 3.6, \sigma_2 = 0.29, \rho_1 = -0.96, \rho_2 = -0.83\}$ and $\{\tilde{\kappa}^i = 0.8, \sigma^i = 0.2, \rho^i = 0\}$. We keep the risk-free rate at 4% per year and the ratio of spot index price over spot equity price is equal to $S_t^i/S_t = 0.1$. Note that the Y axis is Implied Volatility

Figure 3.5: Persistent and Transient Betas and Implied Volatility Across Moneyness



Note to Figure: This figure plots the model-implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. Panel A shows the IV moneyness slope following the one-factor structure model and Panel B replicates the same IV moneyness slope with our two-factor structure model. Panels C shows IV moneyness slope when persistent beta β_1^i is constant and Panel D shows IV moneyness slope when transient beta β_2^i is constant. Note that for all the graphs the total unconditional equity variance is fixed at $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$. Note also that the Y axis is Implied Volatility.

Figure 3.6: Persistent and Transient Variances Risk Premiums and Implied Volatility Smile



Note to Figure: This figure plots the difference between model-implied volatility for three-month equity call options with respect to the moneyness (S/K) for different sets of betas. The implied volatility difference is the difference between IV when $\lambda_1 = \lambda_2 = -0.5$ and when $\lambda_1 = \lambda_2 = 0$. Panel A shows the effect of market variance risk premium on equity option skew (slope of IV curve) following the calibration in one-factor structure model while Panel B replicates the same effect in our two-factor structure model. Panels C shows IV difference when persistent beta β_1^i is constant and Panel D shows IV difference when transient beta β_2^i is constant. Note that for all the graphs the total unconditional equity variance is fixed, $\tilde{v}^i = (\beta_1^i)^2\tilde{\theta}_1 + (\beta_2^i)^2\tilde{\theta}_2 + \theta^i = 0.11$.