On some refinements of the embedding of critical Sobolev spaces into BMO, and a study of stability for parabolic equations with time delay

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ABSTRACT

On some refinements of the embedding of critical Sobolev spaces into BMO, and a study of stability for parabolic equations with time delay

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Van Schaftinen [77] showed that the inequalities of Bourgain and Brezis [11], [12] give rise to new function spaces that refine the classical embedding $W^{1,n}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$. It was suggested by Van Schaftingen [77] that similar results should hold in the setting of bounded domains $\Omega \subset \mathbb{R}^n$ for $bmo_r(\Omega)$ and $bmo_z(\Omega)$ classes.

The first part of this thesis contains the proofs of these conjectures as well as the development of a non-homogeneous theory of Van Schaftingen spaces on $\mathbb{R}^n$. Based on the results in the non-homogeneous setting, we are able to show that the refined embeddings can also be established for *bmo* spaces on Riemannian manifolds with bounded geometry, introduced by Taylor [68].

The stability of parabolic equations with time delay plays an important role in the study of non-linear reaction-diffusion equations with time delay. While the stability regions for such equations without convection on bounded time intervals were described by Travis and Webb [70], the problem remained unaddressed for the equations with convection. The need to determine exact regions of stability for such equations appeared in the context of the work of Mei and Wang on the Nicholson equation with delay [50].

In the second part of this thesis, we study the parabolic equations with and without convection on $\mathbb{R}$. It has been shown that the presence of convection terms can change the regions of stability. The implications for the stability problems for non-linear equations are also discussed.

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Notation List

- BMO, the space of functions of bounded mean oscillation
- *bmo*, the non-homogeneous space of functions of bounded mean oscillation
- VMO, the space of functions of vanishing mean oscillation
- *vmo*, the non-homogeneous space of functions of vanishing mean oscillation
- $W^{s,p}$, Sobolev space
- $B_q^{s,p}$, Besov space
- $F_q^{s,p}$, Triebel-Lizorkin space
- div, divergence operator
- H^1, Hardy space
- L_k^1, Lebesgue integrable differentiable forms of order k
- Υ_k^1, differentiable forms of order k which are Lebesgue integrable with their exterior derivatives
- $\otimes$, tensor product
- $\wedge$, wedge product
- $\approx$, equality up to a constant factor

Part I

Function spaces between critical Sobolev spaces and *bmo*

Chapter 1

Introduction

1.1 The John-Nirenberg space $\mathrm{BMO}(\mathbb{R}^n)$

Let f be a locally integrable function on $\mathbb{R}^n$. Given a cube $Q \subset \mathbb{R}^n$ (henceforth by a cube we will understand a cube with sides parallel to the axes), we denote the average of f over Q by f_Q, i.e.

$$f_Q = \frac{1}{|Q|} \int_Q f(x)dx,$$

where $|Q|$ is the Lebesgue measure of Q.

In 1961 John and Nirenberg introduced the space of functions of bounded mean oscillation (BMO).

Definition 1.1.1. We say that $f \in \mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|_{\mathrm{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty.$$

Note that $\|\cdot\|_{\mathrm{BMO}}$ is a norm on the quotient space of functions modulo constants.

Functions of bounded mean oscillations turned out to be the right substitute for L^∞ functions in a number of questions in analysis. Let us consider three examples that illustrate that.

Example 1.1.2. Let T be a convolution operator bounded on $L^2(\mathbb{R}^n)$ such that its kernel K satisfies the following cancellation condition: there exists $A > 0$ such that

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \le A, \text{ for all } y \neq 0.$$

Then for any $p \in (1, \infty)$ there exists $C_p > 0$ so that

$$\|Tf\|_{L^p} \leq C_p \|f\|_{L^p}, \text{ for any } f \in L^2 \cap L^p.$$

In other words T can be extended to a bounded linear operator on L^p, for any $p \in (1, \infty)$ (for the proof see e.g. [65], Chapter II).

The result does not hold for $f \in L^\infty(\mathbb{R}^n)$. However, as was shown by Peetre [55], T can be extended to a continuous linear operator on $\mathrm{BMO}(\mathbb{R}^n)$.

Example 1.1.3. Let K_s be the Riesz potential of order $s \in (0, n)$, i.e. the distribution with Fourier transform

$$\hat{K}_s(\xi) = |\xi|^{-s},$$

and I_s be the convolution operator with kernel K_s. Then for any p, q such that $1 < p < q < \infty$ and $1/q = 1/p - s/n$, there exists $C_{p,q}$ such that

$$\|I_s f\|_{L^q} \leq C_{p,q} \|f\|_p, \forall f \in L^p.$$

The inequality fails for the limiting case $p = 1, q = \infty$ and $s = n$. However, substituting the L^∞ norm by the one of BMO, we have

$$\|I_n f\|_{\mathrm{BMO}} \leq C \|f\|_{L^1}, \forall f \in L^1$$

(see [64], Section 6.3 in Chapter IV).

Example 1.1.4. Let $W^{1,p}(\mathbb{R}^n)$ be the Sobolev space of functions $f \in L^p$ such that $\nabla f \in L^p$. The embedding theorem of Gagliardo-Nirenberg-Sobolev (see e.g. [65], Chapter V) asserts that for any $p \in [1, n)$ there exists C_p such that

$$\|f\|_{L^{np/(n-p)}} \leq C_p \|f\|_{W^{1,p}}, \forall f \in W^{1,p}.$$

The inequality fails for $p = n$, so we do not have the embedding $W^{1,n}$ into L^∞. However, it follows from the Poincare inequality that for some constant C

$$\|f\|_{\mathrm{BMO}} \leq C \|f\|_{W^{1,n}}, \forall f \in W^{1,n}$$

(see e.g. [27], Section 5.8).

1.2 Refinements of the embedding $W^{1,n} \subset \mathrm{BMO}$

In the last example we showed that a good substitute for L^∞ in the Sobolev embedding theorem is BMO. The question of how optimal this embedding is and whether it can be improved, has been an active research topic. It was proven by John and Nirenberg [39] that if $f \in \mathrm{BMO}(\mathbb{R}^n)$, then for each $C > 0$,

$$e^{C|f|} \in L^1_{loc}(\mathbb{R}^n).$$

This fact is precise in the sense that there exists $f \in \mathrm{BMO}(\mathbb{R}^n)$ such that for any $\epsilon > 0$ and $C > 0$,

$$e^{C|f|^{1+\epsilon}} \notin L^1_{loc}(\mathbb{R}^n).$$

However, Pokhozhaev [57], Trudinger [74] and Yudovich [82] independently showed that for any $f \in W^{1,n}(\mathbb{R}^n)$, we have

$$e^{C|f|^{n/n-1}} \in L^1_{loc}(\mathbb{R}^n), \text{ for any } C > 0,$$

which says that the embedding $W^{1,n} \subset \mathrm{BMO}$ is not optimal on the scale of Orlicz spaces.

Later Brezis and Wainger [15] generalized the Pokhozhaev-Trudinger-Yudovich result to a larger class of functions. Namely, they showed that if f and ∇f both belong to the Lorentz space $L^{n,q}$, $1 < q < \infty$ (see e.g. [7] for the definition of Lorentz spaces), then for any $C > 0$ and $q' = \frac{q}{q-1}$

$$e^{C|f|^{q'}} \in L^1_{loc}(\mathbb{R}^n).$$

More detailed exposition of the progress in this direction can be found in [56].

1.3 New inequalities for critical Sobolev spaces

The starting point of the research described in this work was the following estimate of Bourgain, Brezis and Mironescu [13], Proposition 4:

Theorem 1.3.1. *Let Γ be a closed rectifiable curve in $\mathbb{R}^n$ with unit tangent vector τ and let $u \in C_0^\infty(\mathbb{R}^n)$. Then*

$$\left| \int_\Gamma u(x)\tau(x)dx \right| \leq C_n |\Gamma| \|\nabla u\|_{L^n},$$

where C_n depends on the dimension only.

We note that $\int_\Gamma u(x)\tau(x)dx$ is an element of $\mathbb{R}^n$ in the above theorem as well as in the next one. The proof of Theorem 1.3.1 was given using Littlewood-Paley theory; an elementary proof based only on the Morrey-Sobolev embedding theorem was found by Van Schaftingen [76].

A more general form of this result was given in [11]

Theorem 1.3.2. *For every $u \in C_0^\infty(\mathbb{R}^n)$ and $F \in L^1(\mathbb{R}^n;\mathbb{R}^n)$ such that $divF = 0$,*

$$\left|\int_{\mathbb{R}^n} u(x)F(x)dx\right| \leq C_n\|\nabla u\|_{L^n}\|F\|_{L^1}.$$

Bourgain and Brezis showed that Theorem 1.3.2 follows from Theorem 1.3.1 and Smirnov's theorem on the integral representation of divergence-free vector fields [63]. A direct and elementary proof of Theorem 1.3.2 was soon given by Van Schaftingen [75]. It has been shown in [78] that $\|\nabla u\|_{L^n}$ can be relaxed to $\|u\|_{\dot{F}^s_{p,q}}$, where $\dot{F}^s_{p,q}$ is the homogeneous Triebel-Lizorkin space with $p > 1$, $s = n/p$ and $q > 0$.

1.4 Van Schaftingen's classes

1.4.1 Divergence-free case

It was noted in [8] and [77] that $\|\nabla u\|_{L^n}$ in Theorem 1.3.2 cannot be replaced by $\|u\|_{\mathrm{BMO}}$. In order to understand the relationship between Theorem 1.3.2 and the classical embedding $\dot{W}^{1,n} \subset \mathrm{BMO}$ (here $\dot{W}^{1,n}$ is the homogeneous Sobolev space), Van Schaftingen in [77] considered the class of distributions

$$D_{n-1}(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n) : \|u\|_{D_{n-1}} < \infty\},$$

where

$$\|u\|_{D_{n-1}} := \sup\{|u(\phi_i)| : \Phi = (\phi_1, \ldots, \phi_n) \in \mathcal{D}(\mathbb{R}^n;\mathbb{R}^n), div\ \Phi = 0, \|\Phi\|_{L^1} \leq 1\}.$$

It was shown that for $1 < p < \infty$, $\dot{W}^{n/p,p} \subset D_{n-1} \subset \mathrm{BMO}$, where both inclusions are proper, thus establishing a new refinement of the classic embedding $W^{1,n} \subset \mathrm{BMO}$.

The definition of D_{n-1} was motivated by Theorem 1.3.2 and it is natural to ask what is the relation between D_{n-1} and Theorem 1.3.1. Using Smirnov's theorem, Van Schaftingen showed the following result

Theorem 1.4.1 ([77]). *Let $u \in D_{n-1}$ be a continuous function. Then*

$$\|u\|_{D_{n-1}} = \sup_{\Gamma} \frac{1}{|\Gamma|} \left| \int_{\Gamma} u(x)\tau(x)dx \right|,$$

where the supremum is taken over all closed C^1 smooth curves Γ with unit tangent vectors τ.

1.4.2 General case

It is natural to consider a different scenario, replacing the divergence operator $\nabla \cdot \Phi$ in the above by the curl, $\nabla \times \Phi$, or more generally exterior differentiation d. Van Schaftingen defined a scale of spaces D_k using the k-differential forms

$$\Phi(x) = \sum_{1 \le i_1 < \cdots < i_k \le n} \phi_{i_1,\dots i_k}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and introduced the following

Definition 1.4.2. For $1 \le k \le n$, D_k is defined as

$$D_k(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n) : \|u\|_{D_k} < \infty\},$$

where

$$\|u\|_{D_k} := \sup\{|u(\phi_{i_1,\dots,i_k})| : \Phi \in \mathcal{D}(\mathbb{R}^n; \Lambda^k(\mathbb{R}^n)),\ d\Phi = 0,\ \|\Phi\|_{L^1} \le 1\}.$$

It was shown in [77] that the D_k classes lie strictly between the critical Sobolev spaces and BMO. Namely, the following proper inclusions are continuous

$$\dot{W}^{1,n} \subset D_{n-1} \subset \cdots \subset D_1 \subset \text{BMO}.$$

The analogue of Theorem 1.4.1 for D_1 functions is the following

Theorem 1.4.3 ([77]). *Let $u \in D_1$ be continuous. Then*

$$\|u\|_{D_1} = \sup_{\Sigma} \frac{1}{|\Sigma|} \left| \int_{\Sigma} u(y)\nu(y)d\sigma(y) \right|,$$

where the supremum is taken over all closed smooth connected $n-1$ dimensional surfaces Σ with unit normal vectors ν.

1.5 Applications to linear elliptic PDEs

Let us consider the system of equations in $\mathbb{R}^n, n \geq 2$

$$\Delta U = F, \tag{1.5.1}$$

where $F \in L^p(\mathbb{R}^n; \mathbb{R}^n)$. For $p \in (1, \infty)$, the regularity of the solution U comes from the Calderon-Zygmund theory and the Sobolev embedding theorem. For example, if $1 < p < n/2$, the Calderon-Zygmund theory tells us that

$$\|D^2 U\|_p \leq C\|F\|_p$$

and by the Sobolev embedding theorem, one has

$$\|DU\|_{p'} \leq C_p \|F\|_p,$$

and

$$\|U\|_{p''} \leq C'_p \|F\|_p,$$

where $1/p' = 1/p - 1/n$ and $1/p'' = 1/p - 2/n$.

None of these estimates holds for $p = 1$. However, using Theorem 1.3.2, Bourgain and Brezis showed that it is possible to obtain the regularity of U with $F \in L^1$ under the restriction: $divF = 0$. More precisely, they proved

Theorem 1.5.1 ([11]). *If U is a solution of (1.5.1), where $F \in L^1$ and $divF = 0$, then for some constant $C > 0$,*

$$\max(\|DU\|_{L^{n/(n-1)}(\mathbb{R}^n)}, \|U\|_{L^{n/(n-2)}(\mathbb{R}^n)}) \leq C\|F\|_1, \ \textit{for } n \geq 3$$

and

$$\max(\|DU\|_{L^n(\mathbb{R}^n)}, \|U\|_{L^\infty(\mathbb{R}^n)}) \leq C\|F\|_{L^1(\mathbb{R}^n)}, \ \textit{for } n = 2.$$

1.6 Results and structure of Part 1

The structure of the rest of this part is as follows. In Chapter 2 we recall the basic notions used in this work: distributions, differential forms and currents. We also review the basic theory of Hardy spaces on $\mathbb{R}^n$ and bounded Lipschitz domains in $\mathbb{R}^n$.

In Chapter 3 we develop the non-homogeneous theory on $\mathbb{R}^n$. In Section 3.1 we define the class of special differential forms similar to the Sobolev space $W^{1,1}$. In Section 3.2 we use these forms in order to define a new type of function spaces $d^k(\mathbb{R}^n)$, which are the non-homogeneous analogs of Van Schaftingen's classes $D_k(\mathbb{R}^n)$. In the same section we show that $d^k(\mathbb{R}^n)$ form a monotone family of spaces (Lemma 3.2.4) and prove the following result

Theorem (3.2.6)**.** *$d^1(\mathbb{R}^n)$ is continuously embedded into the space $bmo(\mathbb{R}^n)$ and $\exists C > 0$ so that for any $u \in d^k(\mathbb{R}^n)$, $1 \le k \le n$*

$$\|u\|_{bmo} \le C\|u\|_{d^k}.$$

In Section 3.3.1 we define an even finer scale of function spaces v^k and show the chain of continuous embeddings

$$W^{1,n}(\mathbb{R}^n) \subset v^{n-1}(\mathbb{R}^n) \subset \cdots \subset v^1(\mathbb{R}^n) \subset vmo(\mathbb{R}^n).$$

In Section 3.3.2, we prove that the space $v^{n-1}(\mathbb{R}^n)$ can be characterized by the following theorem.

Theorem (3.3.7)**.** *Let $u \in C(\mathbb{R}^n)$. Then $u \in v^{n-1}(\mathbb{R}^n)$ if and only if*

$$\sup_{\partial\gamma=\emptyset} \frac{1}{|\gamma|} \left| \int_\gamma u(t)\tau(t)dt \right| + \sup_{|\gamma|\ge 1} \frac{1}{|\gamma|} \left| \int_\gamma u(t)\tau(t)dt \right| < \infty,$$

where the suprema are taken over smooth curves γ with finite lengths $|\gamma|$, boundaries $\partial\gamma$ and unit tangent vectors τ.

In Section 3.4 we prove that d^k classes are invariant under the tensor multiplication by smooth function (Theorem 3.4.1). Using this result we give explicit examples of d^k functions in Section 3.5. In Section 3.6 we give an application of our d^k spaces for one elliptic system by proving the following fact

Theorem (3.6.1)**.** *Let $F \in L^1(\mathbb{R}^2)$ and $divF \in L^1(\mathbb{R}^2)$. Then the system $(I - \Delta)U = F$ admits a unique solution U such that*

$$\|U\|_\infty + \|\nabla U\|_2 \le C(\|F\|_1 + \|divF\|_1).$$

In Sections 3.7-3.8 we define d^k spaces on bounded Lipschitz domains Ω: $d^k(\Omega)$ and $d^k_z(\Omega)$. The main result of Section 3.8 is the following theorem, which gives an affirmative answer to the question posed by Van Schaftingen

Theorem (3.8.11)**.** *Any $u \in d^1(\Omega)$ is a $bmo_r(\Omega)$ function as there exists $C > 0$ such that*

$$\|u\|_{bmo_r(\Omega)} \leq C\|u\|_{d^1(\Omega)} \; \forall u \in d^1(\Omega).$$

In Chapter 4, we recall some basics of Riemannian manifolds and define the notion of bounded geometry. Further, building on our own results of Chapter 3 we prove the refined embeddings between critical Sobolev space and *bmo* on Riemannian manifolds with bounded geometry.

Theorem (4.4.2)**.** *Let M be the Riemannian manifold with bounded geometry. Then the following continuous embeddings are true*

$$W^{1,n}(M) \subset d^{n-1}(M) \subset \cdots \subset d^1(M) \subset bmo(M).$$

Chapter 2

Preliminaries

2.1 Distributions

Let Ω be an open subset of $\mathbb{R}^n$. By multi-index α we denote an n-tuple of non-negative integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. By $|\alpha|$ we will understand the sum of the components, $\alpha_1 + \cdots + \alpha_n$. By the partial derivative ∂_x^α we will understand the mixed derivative with respect to the variable $x \in \mathbb{R}^n$,

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}.$$

We use the Schwartz notation

$$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) |\ \mathit{supp}\ f \subset \Omega \text{ is compact}\}$$

and say that $\{f_n\} \subset \mathcal{D}(\Omega)$ converges to $f \in \mathcal{D}(\Omega)$ if there exists a compact $K \subset \Omega$ such that $\mathit{supp}\ f_n \subset K$ for all n and for any fixed multi-index α

$$\lim_{n\to\infty} \sup_{x\in\Omega} |\partial^\alpha f_n(x) - \partial^\alpha f(x)| = 0.$$

The space of continuous linear functionals on $\mathcal{D}(\Omega)$ is denoted by $\mathcal{D}'(\Omega)$ and is called the space of distributions.

For the class of test functions

$$\mathcal{E}(\Omega) = \{f \in C^\infty(\Omega)\},$$

we say that $\{f_n\} \subset \mathcal{E}(\Omega)$ converges to $f \in \mathcal{E}(\Omega)$ if for any fixed multi-index α and any fixed compact $K \subset \Omega$

$$\lim_{n\to\infty} \sup_{x\in K} |\partial^\alpha f_n(x) - \partial^\alpha f(x)| = 0.$$

The space of continuous linear functionals on $\mathcal{E}(\Omega)$, denoted by $\mathcal{E}'(\Omega)$ coincides with the space of compactly supported distributions (see e.g. Theorem 2.3.1 in [37]).

Finally, we will say that $f : \mathbb{R}^n \to \mathbb{R}$ is a Schwartz function and write $f \in S(\mathbb{R}^n)$ if $f \in C^\infty(\mathbb{R}^n)$ and $\sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha f(x)| < \infty$, for any $N > 0$ and any multi-index α. The topology on $\mathcal{S}(\mathbb{R}^n)$ is defined as follows: we say that $f_n \to f$ in $\mathcal{S}(\mathbb{R}^n)$, if for any fixed α and $N > 0$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha f_n(x) - \partial^\alpha f(x)| = 0.$$

The space of continuous linear functionals on $S(\mathbb{R}^n)$ is denoted by $S'(\mathbb{R}^n)$ and is called the space of tempered distributions.

2.2 Differential forms

Let $k \geq 1$. We denote the class of k-asymmetric tensors on $\mathbb{R}^n$ by $\Lambda^k(\mathbb{R}^n)$. In other words, each element $T \in \Lambda^k(\mathbb{R}^n)$ is a real-valued k-linear form on $\mathbb{R}^n$ such that for any vectors $v_1, \ldots, v_k$ ($v_i \in \mathbb{R}^n$) and any permutation $\sigma \in S_k$

$$T(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = sgn(\sigma) T(v_1, \ldots, v_n).$$

Note that Λ^1 coincides with the space of linear functionals on $\mathbb{R}^n$.

For $T \in \Lambda^k(\mathbb{R}^n)$ and $P \in \Lambda^l(\mathbb{R}^n)$, the wedge product of P and Q is an asymmetric tensor defined by

$$T \wedge P(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in S_{k,l}} sgn(\sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) P(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

where $S(k,l)$ is the set of permutations σ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$. The basic algebraic properties of the wedge products and alternating tensors can be found e.g. in [42]. In particular, it is well known that any $T \in \Lambda^k(\mathbb{R}^n)$ can be written as

$$T = \sum_{1\leq i_1 < i_2 < \cdots < i_k \leq n} T_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $T_{i_1,\dots,i_k}$ are scalars and $\{dx^j\}_{j=1}^n$ is the standard dual basis in $\mathbb{R}^n$, i.e. for any $a = (a_1,\dots,a_n) \in \mathbb{R}^n$, $dx^j(a) = a_j$.

In order to alleviate the notation, we will adopt the following convention: let $I = (i_1,\dots,i_k)$ be a set of k indices $1 \le i_1 < i_2 \cdots < i_k \le n$; then we put by definition $|I| = k$ and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

Definition 2.2.1. Let $O \subset \mathbb{R}^n$ be an open set and $1 \le k \le n$. A differential k-form ω is a map assigning to each $y \in O$ a k-asymmetric tensor

$$y \mapsto \sum_{|I|=k} \omega_I(y) dx^I,$$

where each ω_I is a smooth function on O. The space of differential k-forms will be denoted by $\mathcal{E}^k(O)$.

Moreover, we put $\mathcal{E}^0(O) = C^\infty(O)$.

Definition 2.2.2. Let $O \subset \mathbb{R}^n$ be an open set. The exterior derivative $d : \mathcal{E}^k(O) \mapsto \mathcal{E}^{k+1}(O)$ is defined for $k = 0$ by

$$d\omega(y) = \sum_{i=1}^n \partial_i \omega(y) dx^i$$

and for $1 \le k < n$ by

$$d\omega(y) = d \sum_{|I|=k} \omega_I(y) dx^I = \sum_I \sum_{j=1}^n \partial_j \omega_I(y) dx^j \wedge dx^I.$$

We put $d\omega = 0$ for any $\omega \in \mathcal{E}^n(O)$.

Definition 2.2.3. The Hodge operator $\star : \mathcal{E}^k(O) \mapsto \mathcal{E}^{n-k}(O)$ is defined by

$$\star\omega = \star \sum_{|I|=k} \omega_I(y) dx^I = \sum_{|I|=k} \omega_I(y) dx^{I^c},$$

where $I^c = (j_1,\dots,j_{n-k})$, $1 \le j_1 < j_2 < \dots j_{n-k} \le n$ and $j_s \ne i_l$ for any $s \in [1, n-k]$ and $l \in [1,k]$.

Definition 2.2.4. The co-differential operator $\delta : \mathcal{E}^k(O) \mapsto \mathcal{E}^{k-1}(O)$ is defined as the following composition of d and $\star$:

$$\delta\omega = (-1)^{nk+n+1} \star d(\star\omega)$$

2.3 Currents

Let $\Omega \subset \mathbb{R}^n$ be open. We consider the space of k-differential forms with $\mathcal{D}(\Omega)$ components and denote it by $\mathcal{D}^k(\Omega)$. In other words

$$\mathcal{D}^k(\Omega) = \{\Psi(y) = \sum_{|I|=k} \psi_I(y) dx^I : \psi_I \in \mathcal{D}(\Omega)\}.$$

The convergence in this space is understood in the component-wise sense: $\{\Psi^j = \sum_{|I|=k} \psi_I^j dx^I\}_{j=1}^{\infty}$ converges to $\Psi = \sum_{|I|=k} \psi_I dx^I$ in $\mathcal{D}^k(\Omega)$, if each $\psi_I^j \to \psi_I$, as $j \to \infty$ in $\mathcal{D}(\Omega)$.

Definition 2.3.1. The space of continuous linear functionals on $\mathcal{D}^k(\Omega)$ is called the space of currents of degree k and denoted by $\mathcal{D}_k(\Omega)$.

It is not difficult to show (see e.g. Proposition 3.2.1 in [24]) that any current $\Phi \in \mathcal{D}_k(\Omega)$ can be written as

$$\Phi = \sum_{|I|=k} \phi_I dx^I,$$

where ϕ_I are 0-currents, i.e. distributions in $\mathcal{D}'(\Omega)$. The action of $\Phi \in \mathcal{D}_k$ on $\Psi \in \mathcal{D}^k$ becomes

$$\Phi(\Psi) = \sum_I \phi_I(\psi_I).$$

For any $\Phi \in \mathcal{D}_k$, we define its exterior derivative $d\Phi$ as an element in $\mathcal{D}_{k+1}$ defined by

$$d\Phi(\Psi) = -\Phi(\delta\Psi),$$

where δ is the exterior co-derivative.

We also denote the boundary of a current $\Phi \in \mathcal{D}_k(\Omega)$ by $\partial\Phi \in \mathcal{D}_{k-1}(\Omega)$, defined by

$$\partial\Phi(\Psi) = \Phi(d\Psi), \text{ for any } \Psi \in \mathcal{D}^{k-1}(\mathbb{R}^n).$$

Similarly, we can introduce a component-wise topology on $\mathcal{E}^k(\Omega)$.

Definition 2.3.2. The space of continuous linear functionals on $\mathcal{E}^k(\Omega)$ is called the space of compactly supported currents of degree k and denoted by $\mathcal{E}_k(\Omega)$.

2.4 Hardy spaces on $\mathbb{R}^n$

2.4.1 Definitions

In this section, we recall the definition and basic properties of the local Hardy space $h^1(\mathbb{R}^n)$ introduced by Goldberg [30].

Let us fix $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int \phi \neq 0$. For $f \in L^1(\mathbb{R}^n)$, we define the local maximal function $m_\phi f(x)$ by

$$m_\phi f(x) = \sup_{0<t<1} |\phi_t * f(x)|.$$

Definition 2.4.1. We say that f belongs to the local Hardy space $h^1(\mathbb{R}^n)$ if $m_\phi f \in L^1(\mathbb{R}^n)$ and we put

$$\|f\|_{h^1} := \|m_\phi f\|_{L^1}.$$

It is useful to compare h^1 with the classic real Hardy space $H^1(\mathbb{R}^n)$, which can be defined using the global maximal function M_ϕ,

$$M_\phi f(x) := \sup_{t>0} |\phi_t * f(x)|,\ \ f \in L^1(\mathbb{R}^n).$$

Definition 2.4.2. We say that f belongs to the Hardy space $H^1(\mathbb{R}^n)$ if $M_\phi f \in L^1(\mathbb{R}^n)$, and we put

$$\|f\|_{H^1} := \|M_\phi f\|_{L^1}.$$

It follows from the definitions of the maximal functions that $m_\phi f(x) \leq M_\phi f(x)$ for any $f \in L^1$ and $x \in \mathbb{R}^n$. Therefore $H^1 \subset h^1$. One of the reasons why it is often more convenient to deal with a larger space h^1 instead of H^1 is that $\mathcal{S}(\mathbb{R}^n) \subset h^1(\mathbb{R}^n)$, while any $f \in H^1(\mathbb{R}^n)$ has to satisfy $\int_{\mathbb{R}^n} f = 0$. It is important to note that $f \in h^1(\mathbb{R}^n)$ and $\int\limits_{\mathbb{R}^n} f = 0$ do not imply that $f \in H^1(\mathbb{R}^n)$ (see Theorem 3 in [30]). However, the following is true

Lemma 2.4.3. *If $f \in h^1(\mathbb{R}^n)$, $\int\limits_{\mathbb{R}^n} f(x)dx = 0$ and supp $f \subset B$, where B is a bounded subset of $\mathbb{R}^n$, then there exists $C_B > 0$ such that*

$$\|f\|_{H^1} \leq C_B \|f\|_{h^1}.$$

Proof. We will follow the argument of Goldberg (see the proof of Theorem 3 in [30]). Without loss of generality we can assume that B is a ball containing the origin, say $B = B(0, R)$ for some $R > 0$. We need to show that

$$\|M_\phi f(x)\|_{L^1} \leq C\|f\|_{h^1}.$$

Since $f \in h^1(\mathbb{R}^n)$, and $\|m_\phi f\|_{L^1} = \|f\|_{h^1}$ by definition, it remains to estimate $\sup_{t\geq 1} |\phi_t * f|$.

On the one hand, for any $x \in \mathbb{R}^n$,

$$\sup_{t\geq 1} |\phi_t * f(x)| \leq \|\phi_t\|_{L^\infty}\|f\|_{L^1} \leq \|\phi\|_{L^\infty}\|f\|_{h^1}.$$

Therefore, if we put $\tilde{B} = B(0, 2R)$

$$\int_{\tilde{B}} \sup_{t\geq 1} |\phi_t * f(x)|dx \leq C_\phi R^n \|f\|_{h^1}. \tag{2.4.1}$$

On the other hand, due to the cancellation

$$\int f(x)dx = 0,$$

one has

$$|\phi_t * f(x)| = \left|\int_B [\phi_t(x - y) - \phi_t(x)]f(y)dy\right| \leq$$

$$\leq CR \sup_{y\in B} |\nabla\phi_t(x - y)| \cdot \|f\|_{L^1} \leq CRt^{-n-1} \sup_{y\in B} \left|\nabla\phi\left(\frac{x - y}{t}\right)\right| \cdot \|f\|_{h^1}.$$

Due to the smoothness and rapid decay of ϕ, there exists $K > 0$ such that $|\nabla\phi(y)| \leq K|y|^{-n-1}$. Hence, for $x \in \mathbb{R}^n \setminus \tilde{B}$

$$|\phi_t * f(x)| \leq CR\|f\|_{h^1} \sup_{|y|\leq R} |x - y|^{-n-1} \leq CR\|f\|_{h^1}(|x| - R)^{-n-1}$$

Therefore

$$\int_{\mathbb{R}^n\setminus\tilde{B}} \sup_{t\geq 1} |\phi_t * f(x)|dx \leq C\|f\|_{h^1} R \int_{|x|\geq 2R} \frac{dx}{(|x| - R)^{n+1}} \leq C'\|f\|_{h^1}. \tag{2.4.2}$$

Estimates (2.4.1-2.4.2) yield the lemma. $\square$

2.4.2 Density of $\mathcal{D}(\mathbb{R}^n)$ in $h^1(\mathbb{R}^n)$

It was shown by Goldberg ([30], p. 35) that $\mathcal{S}(\mathbb{R}^n)$ is dense in $h^1(\mathbb{R}^n)$. Later, we will need to use a slightly stronger fact

Lemma 2.4.4. *The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $h^1(\mathbb{R}^n)$.*

Proof. We employ the fact that $h^1(\mathbb{R}^n)$ is a space of the Triebel-Lizorkin category: $h^1(\mathbb{R}^n) = F_2^{0,1}(\mathbb{R}^n)$ (see e.g. [71] for the definition of $F_q^{s,p}$ spaces). By the embedding theorem ([71], p. 47), there is a continuous embedding

$$W^{1,1}(\mathbb{R}^n) \subset F_2^{0,1}(\mathbb{R}^n) = h^1(\mathbb{R}^n).$$

Since $\mathcal{S}$ is dense in h^1 and any Schwartz function can be approximated by $\mathcal{D}$ functions in the $W^{1,1}$ norm, the result follows. □

2.4.3 Dual space of $h^1(\mathbb{R}^n)$

Definition 2.4.5 ([30])**.** We say that $f \in L^1_{loc}(\mathbb{R}^n)$ belongs to $bmo(\mathbb{R}^n)$ if

$$\|f\|_{bmo} := \sup_{l(Q)\le 1} \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx + \sup_{l(Q)\ge 1} \frac{1}{|Q|}\int_Q |f(x)|dx < \infty,$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$ and Q are cubes with sides parallel to the axes, of side-length $l(Q)$.

Notice that $\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le \frac{2}{|Q|}\int_Q |f(x)|dx$ and therefore $bmo(\mathbb{R}^n)$ is a subspace of John-Nirenberg space $\mathrm{BMO}(\mathbb{R}^n)$. Moreover, if $\|f\|_{bmo} = 0$ then $f = 0$ a.e. on $\mathbb{R}^n$, unlike in $\mathrm{BMO}(\mathbb{R}^n)$, where constant functions are identified with $f \equiv 0$.

Later we will need the following theorem of Goldberg:

Theorem 2.4.6 ([30])**.** *The space $bmo(\mathbb{R}^n)$ is isomorphic to the space of continuous linear functionals on $h^1(\mathbb{R}^n)$.*

2.4.4 Pseudo-differential operators on h^1

In order to formulate another useful result of Goldberg, we recall the definition of a pseudo-differential operator.

Definition 2.4.7. We say that $\sigma(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is a symbol of order $m \in \mathbb{Z}$ and write $\sigma \in S^m$ if for any pair of multi-indices α and β there exists $C_{\alpha,\beta}$ such that

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \text{ for all } x, \xi.$$

Definition 2.4.8. We say that a linear map $P(x,D) : \mathcal{S}(\mathbb{R}^n) \mapsto C^\infty(\mathbb{R}^n)$ is a pseudo-differential operator of order m and write $P(x,D) \in \mathrm{OPS^m}$, if P acts on $u \in \mathcal{S}(\mathbb{R}^n)$ as follows:

$$\begin{aligned} P(x,D)u(x) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i(x-y)\xi}\sigma(x,\xi)u(y)dyd\xi = \\ &= \int_{\mathbb{R}^n} e^{2\pi i x\xi}\sigma(x,\xi)\hat{u}(\xi)d\xi, \end{aligned}$$

for some $\sigma \in S^m$.

It follows from the definition that differential operators are pseudo-differential operators with $\sigma(x,\xi) = p(\xi)$ for some polynomial p. Here is an example of a pseudo-differential operator of a negative order:

Example 2.4.9. Let $m > 0$ and G_m be the Bessel potential of order m, defined by

$$G_m f(x) = \int (1+|\xi|^2)^{-m/2}\hat{f}(\xi)e^{2\pi i x\xi}d\xi.$$

Then by the Leibniz rule

$$\partial_\xi^\alpha(1+|\xi|^2)^{-m/2} = \sum_{0\leq|\beta|\leq|\alpha|} C_{\alpha,\beta,m}\xi^\beta(1+|\xi|^2)^{-(m+|\alpha|+|\beta|)/2},$$

which implies that $G_m \in \mathrm{OPS^{-m}}(\mathbb{R}^n)$.

It is known (see e.g. 3.3 in [69]) that for $P \in \mathrm{OPS^m}$ and $Q \in \mathrm{OPS^k}$, the product $PQ \in \mathrm{OPS^{m+k}}$. This gives another example:

Example 2.4.10. Let G_m be the Bessel potential and $P(D) = \sum_{|\alpha|=k} c_\alpha \partial^\alpha$, then $P(D)G_m \in \mathrm{OPS^{k-m}}$.

Theorem 2.4.11 ([30])**.** *If $T \in \mathrm{OPS^0}$, then there exists a constant $C > 0$ such that*

$$\|Tf\|_{h^1} \leq C\|f\|_{h^1} \text{ for any } f \in \mathcal{S}(\mathbb{R}^n).$$

Therefore, any $T \in \mathrm{OPS^0}$ can be extended to a continuous linear operator on $h^1(\mathbb{R}^n)$.

2.4.5 Atomic decomposition

Definition 2.4.12. An $H^1(\mathbb{R}^n)$ atom is a Lebesgue measurable function a, supported on a cube Q, such that

$$\|a\|_{L^2(Q)} \leq |Q|^{-1/2}$$

and

$$\int_Q a(x)dx = 0.$$

Definition 2.4.13. [19] An $h^1(\mathbb{R}^n)$ atom is a Lebesgue measurable function a, supported on a cube Q such that

$$\|a\|_{L^2(Q)} \leq |Q|^{-1/2}$$

and

$$\left|\int_Q a(x)dx\right| \leq |Q|^{1/n}$$

Remark 2.4.14. The original definition of $h^1(\mathbb{R}^n)$ atoms introduced by Goldberg required a stronger moment condition $\int_Q a(x)dx = 0$ for $|Q| \leq 1$. However, it can be shown (see e.g [21]) that any h^1 atom in the above sense is an $h^1(\mathbb{R}^n)$ function.

Theorem 2.4.15 ([28])**.** *Let $f \in L^1(\mathbb{R}^n)$. Then $f \in H^1(\mathbb{R}^n)$ if and only if there exist sequences of $H^1(\mathbb{R}^n)$ atoms $\{a_k\}$ and real numbers $\{\lambda_k\} \subset \mathbb{R}$ such that $\sum |\lambda_k| < \infty$ and*

$$\sum_k \lambda_k a_k \to f \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Furthermore,

$$\|f\|_{H^1} \approx \inf\{\sum_k |\lambda_k| : f = \sum_k \lambda_k a_k\},$$

where the infimum is taken over all atomic decompositions of f.

Theorem 2.4.16 ([21], [30])**.** *Let $f \in L^1(\mathbb{R}^n)$. Then $f \in h^1(\mathbb{R}^n)$ if and only if there exist a sequence of $h^1(\mathbb{R}^n)$ atoms $\{a_k\}$ and real numbers $\{\lambda_k\} \subset \mathbb{R}$ such that $\sum |\lambda_k| < \infty$ and*

$$\sum_k \lambda_k a_k \to f, \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Furthermore,

$$\|f\|_{h^1} \approx \inf\{\sum_k |\lambda_k| : f = \sum_k \lambda_k a_k\},$$

where the infimum is taken over all atomic decompositions of f.

2.5 Hardy spaces on domains

2.5.1 Definitions

Definition 2.5.1. [20], [52]

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. The space $h^1_r(\Omega)$ consists of elements of $L^1(\Omega)$ which are the restrictions to Ω of elements of $h^1(\mathbb{R}^n)$, i.e.

$$h^1_r(\Omega) = \{f \in L^1(\Omega) : \exists F \in h^1(\mathbb{R}^n) : F = f \text{ on } \Omega\}.$$

We can consider this as a quotient space equipped with the quotient norm

$$\|f\|_{h^1_r(\Omega)} := \inf\{\|F\|_{h^1(\mathbb{R}^n)} : F = f \text{ on } \Omega\}.$$

Definition 2.5.2. [19] The space $h^1_z(\Omega)$ is defined to be the subspace of $h^1(\mathbb{R}^n)$ consisting of those elements which are supported on $\overline{\Omega}$.

Definition 2.5.3. [20] The space $H^1_z(\Omega)$ is defined to be the subspace of $H^1(\mathbb{R}^n)$ consisting of those elements which are supported on $\overline{\Omega}$.

It follows directly from the introduced definitions that as sets of functions in $L^1(\Omega)$

$$H^1_z(\Omega) \subset h^1_z(\Omega) \subset h^1_r(\Omega).$$

Both of these embeddings are strict: any $f \in \mathcal{D}(\Omega)$, such that $\int f \neq 0$ belongs to $h^1_z(\Omega) \setminus H^1_z(\Omega)$; the existence of $f \in h^1_r(\Omega) \setminus h^1_z(\Omega)$ is shown in [19], Proposition 6.4.

2.5.2 Atomic decomposition

Local versions of atoms are defined as follows

Definition 2.5.4. An $H^1(\mathbb{R}^n)$ atom supported in an open set $\Omega \subset \mathbb{R}^n$ is called an $H^1_z(\Omega)$ atom. An $h^1(\mathbb{R}^n)$ atom supported in an open set $\Omega \subset \mathbb{R}^n$ is called an $h^1_z(\Omega)$ atom.

Theorem 2.5.5 (Theorem 3.3 in [20])**.** *Let Ω be a bounded Lipschitz domain and $f \in L^1(\Omega)$. Then $f \in H^1_z(\Omega)$ if and only if there exist a sequence of $H^1_z(\Omega)$ atoms $\{a_k\}$ and real numbers $\{\lambda_k\} \subset \mathbb{R}$ such that $\sum |\lambda_k| < \infty$ and*

$$\sum_k \lambda_k a_k \to f \text{ in } \mathcal{D}'(\Omega).$$

Furthermore,

$$\|f\|_{H^1} \approx \inf\{\sum_k |\lambda_k| : f = \sum_k \lambda_k a_k\},$$

where the infimum is taken over all atomic decompositions of f.

A similar decomposition holds for $h^1_z(\Omega)$ functions

Theorem 2.5.6 (Theorem 3.2 in [20])**.** *Let Ω be a bounded Lipschitz domain and $f \in L^1(\Omega)$. Then $f \in h^1_z(\Omega)$ if and only if there exist a sequence of $h^1_z(\Omega)$ atoms $\{a_k\}$ and real numbers $\{\lambda_k\} \subset \mathbb{R}$ such that $\sum |\lambda_k| < \infty$ and*

$$\sum_k \lambda_k a_k \to f, \text{ in } \mathcal{D}'(\Omega).$$

Furthermore,

$$\|f\|_{h^1} \approx \inf\{\sum_k |\lambda_k| : f = \sum_k \lambda_k a_k\},$$

where the infimum is taken over all atomic decompositions of f.

2.5.3 Density of $\mathcal{D}(\Omega)$

The following lemma is an analogue of Lemma 2.4.4. It is a special case of the result established by Triebel for smooth $\partial\Omega$ (see e.g. [73], p. 46) and in a more general setting, including Lipschitz domains, obtained in [17].

Lemma 2.5.7. *[17] Let Ω be a bounded Lipschitz domain. Then the space $\mathcal{D}(\Omega)$ is dense in $h^1_r(\Omega)$.*

Since, $h^1_z(\Omega) \subset h^1_r(\Omega)$ as a subset, the lemma implies that any $f \in h^1_z(\Omega)$ can be approximated by a $\mathcal{D}(\Omega)$ function in the $h^1_r(\Omega)$ norm. Later we will use the fact that $\mathcal{D}(\Omega)$ is dense in $f \in h^1_z(\Omega)$ in the stronger h^1-norm.

Lemma 2.5.8. *Let Ω be a domain of $\mathbb{R}^n$. Then the set of $\mathcal{D}(\Omega)$ functions is dense in $h^1_z(\Omega)$.*

Proof. Given $\epsilon > 0$ and $f \in h^1_z(\Omega)$, we need to find $\tilde{f} \in \mathcal{D}(\Omega)$ such that $\|f - \tilde{f}\|_{h^1} \leq \epsilon$.

By Theorem 2.5.6,

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where $\{\lambda_j\} \in l^1$, a_j are supported in closed cubes $Q_j \subset \Omega$,

$$\|a_j\|_{\text{atom}} := |Q_j|^{1/2}\|a_j\|_{L^2(Q_j)} \leq 1,$$

$$\left|\int_{Q_j} a_j(x)dx\right| \leq |Q_j|^{1/n}$$

and for some $C > 0$ depending only on the dimension n

$$\|a_j\|_{h^1(\mathbb{R}^n)} \leq C. \tag{2.5.1}$$

Therefore, choosing $N \geq 1$ large enough and putting $g \in h^1_z(\Omega)$ as

$$g = \sum_{j=1}^{N} \lambda_j a_j,$$

we can ensure that

$$\|f - g\|_{h^1} \leq C \sum_{j=N+1}^{\infty} |\lambda_j| < \epsilon/2. \tag{2.5.2}$$

In order to approximate g, we will use the method of convolutions. Let $\delta \in (0, 1)$ and ϕ be a smooth non-negative function supported in the unit ball centred at the origin such that $\|\phi\|_{L^1} = 1$. As usual, we put $\phi_\delta(x) = \delta^{-n}\phi(x/\delta)$ and define

$$g_\delta = g * \phi_\delta = \sum_{j=1}^{N} \lambda_j a_j * \phi_\delta =: \sum_{j=1}^{N} \lambda_j a_j^\delta.$$

We can always assume that δ is small so that *supp* $a_j^\delta \subset Q_j^\delta \subset \Omega$, where Q_j^δ is a cube with the same center as Q_j and $|Q_j^\delta| \leq 2|Q_j|$, for $1 \leq j \leq N$. Then $g_\delta \in \mathcal{D}(\Omega)$,

$$\left|\int a_j^\delta(x)dx\right| \leq \int \left|\int a_j(x-y)dx\right| |\phi_\delta(y)|dy \leq |Q_j|^{1/n} \leq |Q_j^\delta|^{1/n},$$

and

$$\|a_j - a_j^\delta\|_{L^2(Q_j^\delta)} \leq \|a_j\|_{L^2} + \|a_j^\delta\|_{L^2} \leq 2\|a_j\|_{L^2} \leq$$

$$\leq 2|Q_j|^{-1/2} \leq 2\sqrt{2}|Q_j^\delta|^{-1/2}.$$

In other words $a_j - a_j^\delta$ are $h^1_z(\Omega)$ atoms supported in Q_j^δ. Hence, with $C > 0$ as in (2.5.1)

$$\|g - g_\delta\|_{h^1} \leq \sum_{j=1}^{N} |\lambda| \|a_j - a_j^\delta\|_{h^1} \leq C \sum_{j=1}^{N} |\lambda_j| \|a_j - a_j^\delta\|_{\text{atom}} \leq$$

$$\leq C\sqrt{2}\max_{1\leq j\leq N}|Q_j|^{1/2}\sum_{j=1}^{N}|\lambda_j|\|a_j-a_j^\delta\|_{L^2}\leq C'\|f\|_{h^1}\max_{1\leq j\leq N}\|a_j-a_j*\phi_\delta\|_{L^2}.$$

Since, $\|h-h*\phi_\delta\|_{L^2}\to 0$ for any $h\in L^2$, we can choose $\delta>0$ small enough to have

$$\|a_j-a_j*\phi_\delta\|_{L^2}\leq \epsilon/(2C'\|f\|_{h^1}),\ j=1,\ldots,N$$

and therefore have

$$\|g-g_\delta\|_{h^1}\leq\epsilon/2. \tag{2.5.3}$$

Putting $\tilde{f}=g_\delta\in\mathcal{D}(\Omega)$ and using (2.5.2) and (2.5.3), we obtain the desired

$$\|f-\tilde{f}\|_{h^1}\leq\|f-g\|_{h^1}+\|g-\tilde{f}\|_{h^1}<\epsilon.$$

□

2.5.4 Dual spaces

Definition 2.5.9. The space $bmo_z(\Omega)$ is defined to be a subspace of $bmo(\mathbb{R}^n)$ consisting of those elements which are supported on $\bar{\Omega}$, i.e.

$$bmo_z(\Omega)=\{g\in bmo(\mathbb{R}^n): g=0 \text{ on } \mathbb{R}^n\setminus\bar{\Omega}\}$$

with

$$\|g\|_{bmo_z(\Omega)}=\|g\|_{bmo(\mathbb{R}^n)}.$$

Definition 2.5.10. [19]

Let Ω be a bounded Lipschitz domain. A function $g\in L^1_{loc}(\Omega)$ is said to belong to $bmo_r(\Omega)$ if

$$\|g\|_{bmo_r(\Omega)}=\sup_{|Q|\leq 1}\frac{1}{|Q|}\int_Q|g(x)-g_Q|dx+\sup_{|Q|>1}\frac{1}{|Q|}\int_Q|g(x)|dx<\infty,$$

where suprema are taken over all cubes $Q\subset\Omega$. The space of such functions equipped with norm $\|\cdot\|_{bmo_r(\Omega)}$ is called $bmo_r(\Omega)$.

Theorem 2.5.11 ([18], [52]). *The space $bmo_z(\Omega)$ is isomorphic to the dual of $h^1_r(\Omega)$.*

Theorem 2.5.12 ([18], [40]). *The space $bmo_r(\Omega)$ is isomorphic to the dual of $h^1_z(\Omega)$.*

Chapter 3

d^k spaces and continuous embeddings into bmo spaces

3.1 Differential k-forms of Sobolev type on $\mathbb{R}^n$

In this section we are interested in the currents associated with $L^1(\mathbb{R}^n)$ functions.

Definition 3.1.1. For $1 \leq k \leq n$, we define $L^1_k(\mathbb{R}^n)$ as subspace of $\mathcal{D}_k(\mathbb{R}^n)$ consisting of elements $\Phi = \sum_{|I|=k} \phi_I dx^I$ such that all components ϕ_I are $L^1(\mathbb{R}^n)$ functions. This space is equipped with the norm

$$\|\Phi\|_{L^1_k} = \sum_{|I|=k} \|\phi_I\|_{L^1(\mathbb{R}^n)}.$$

An analogue of the $W^{1,1}$ Sobolev space for differential forms is defined as follows

Definition 3.1.2. Let $1 \leq k \leq n-1$. We say that $\Phi \in \mathcal{D}_k(\mathbb{R}^n)$ belongs to $\Upsilon^1_k(\mathbb{R}^n)$ if $\Phi \in L^1_k(\mathbb{R}^n)$ and $d\Phi \in L^1_{k+1}(\mathbb{R}^n)$. We equip $\Upsilon^1_k(\mathbb{R}^n)$ with the norm

$$\|\Phi\|_{\Upsilon^1_k} = \|\Phi\|_{L^1_k} + \|d\Phi\|_{L^1_{k+1}}.$$

Proposition 3.1.3. *The class of compactly supported Υ^1_k differential forms is dense in Υ^1_k.*

Proof. Given $\epsilon > 0$, we need to show that there exists a compactly supported $\tilde{\Phi} \in \Upsilon^1_k(\mathbb{R}^n)$ such that

$$\|\Phi - \tilde{\Phi}\|_{\Upsilon^1_k} \leq \epsilon. \tag{3.1.1}$$

Let us first of all notice that L^1_k differential forms can be $\wedge$-multiplied by smooth differential forms: if $k, l \geq 0$ such that $k + l \leq n$, $\Phi = \sum_{|I|=k} \phi_I dx^I \in L^1_k(\mathbb{R}^n)$ and $\Psi = \sum_{|J|=l} \psi_J dx^J \in \mathcal{D}^l(\mathbb{R}^n)$, then for some constant $C_{k,l}$

$$\|\Phi \wedge \Psi\|_{L^1_{k+l}} \leq C_{k,l} \|\Phi\|_{L^1_k} \max_J \|\psi_J\|_{L^\infty(\mathbb{R}^n)}. \tag{3.1.2}$$

Let us choose a smooth function η supported in $|x| < 2$ such that $\eta(x) = 1$, for $|x| \leq 1$ and $0 \leq \eta(x) \leq 1$ for any x. Put, for $r > 0$,

$$\tilde{\Phi}_r(y) = \sum_{|I|=k} \phi_I(y)\eta(y/r)dx^I.$$

Then $\|\Phi - \tilde{\Phi}_r\|_{\Upsilon^1_k} = \|\Phi - \tilde{\Phi}_r\|_{L^1_k} + \|d(\Phi - \tilde{\Phi}_r)\|_{L^1_{k+1}}$.

Since all ϕ_I are $L^1(\mathbb{R}^n)$ functions, there exists $R_1 > 0$ such that

$$\sum_{|I|=k} \|\phi_I(y)\|_{L^1(|y|>R)} < \epsilon/3.$$

Choosing $r > R_1$. we have

$$\|\Phi - \tilde{\Phi}_r\|_{L^1_k} = \sum_{|I|=k} \|\phi_I(y)(1 - \eta(y/r))\|_{L^1(\mathbb{R}^n)} < \epsilon/3. \tag{3.1.3}$$

Let us denote the components of $d\Phi$ by ω_J; then by the product rule one has

$$d\tilde{\Phi}_r = \sum_{|J|=k+1} \omega_J(y)\eta(y/r)dx^J + \frac{(-1)^k}{r} \sum_{|I|=k} \phi_I(y)dx^I \wedge \sum_{j=1}^n \partial_j\eta(y/r)dx^j$$

and

$$\|d\Phi - d\tilde{\Phi}_r\|_{L^1_{k+1}} \leq \sum_{|J|=k} \|\omega_J(y)(1 - \eta(y/r))\|_{L^1(\mathbb{R}^n)} +$$

$$+ r^{-1}\|\Phi \wedge (\sum_{j=1}^n \partial_j\eta(y/r)dx^j)\|_{L^1_{k+1}}.$$

Again, since $\omega_J \in L^1(\mathbb{R}^n)$, we can find a large enough $R_2 \geq R_1$ such that

$$\sum_{|J|=k} \|\omega_J(y)\|_{L^1(|y|>R_2)} < \epsilon/3.$$

Hence, recalling (3.1.2) and choosing r large enough we can ensure that

$$\|d\Phi - d\tilde{\Phi}_r\|_{L^1_{k+1}} \leq 2\epsilon/3. \tag{3.1.4}$$

Now (3.1.3) and (3.1.4) imply (3.1.1). $\square$

Proposition 3.1.4. *The space of $\mathcal{D}^k(\mathbb{R}^n)$ differential forms is dense in $\Upsilon^1_k(\mathbb{R}^n)$.*

Proof. Due to the previous result, it is enough to show that for any compactly supported $\tilde{\Phi} \in \Upsilon^1_k(\mathbb{R}^n)$ and $\epsilon > 0$, there exists $\Psi \in \mathcal{D}^k(\mathbb{R}^n)$ such that

$$\|\tilde{\Phi} - \Psi\|_{\Upsilon^1_k} \leq \epsilon.$$

Let η be as in the proof above and normalized in L^1-norm. Then we put $\eta_r(x) = r^{-n}\eta(x/r)$ and

$$\Psi_r(y) = \tilde{\Phi} * \eta_r(y) := \sum_{|I|=k} \tilde{\phi}_I * \eta_r(y) dx^I.$$

Since the functions $\tilde{\phi}_I$ are compactly supported, so are $\tilde{\phi}_I * \eta_r$. Moreover, $\tilde{\phi}_I * \eta_r$ are smooth and $\partial^\alpha(\tilde{\phi}_I * \eta_r) = \tilde{\phi}_I * \partial^\alpha \eta_r$, so $\tilde{\phi}_I * \eta_r \in \mathcal{D}(\mathbb{R}^n)$. It is well known (see e.g. Chapter I in [66]) that $\|\tilde{\phi}_I * \eta_r - \tilde{\phi}_I\|_{L^1} \to 0$, as $r \to 0$. Thus $\Psi_r \to \Phi$ in $L^1_k(\mathbb{R}^n)$.

Moreover, the definitions of d and $\star$ show that for any $F \in \mathcal{D}^{k+1}(\mathbb{R}^n)$, $d(F * \eta_r) = d(F) * \eta_r$ and $\star(F * \eta_r) = \star(F) * \eta_r$. Hence $\delta(F * \eta_r) = \delta(F) * \eta_r$ and

$$d\Psi_r(F) = -\Psi_r(\delta F) = -\tilde{\Phi}(\delta(F) * \eta_r) = -\tilde{\Phi}(\delta(F * \eta_r)) = d(\tilde{\Phi}) * \eta_r(F).$$

Therefore, $d\Psi_r = d(\tilde{\Phi}) * \eta_r \to d\Psi$ in L^1_{k+1} as $r \to 0$ and, for sufficiently small $r > 0$,

$$\|\tilde{\Phi} - \Psi_r\|_{\Upsilon^1_k} \leq \epsilon.$$

□

3.2 d^k spaces on $\mathbb{R}^n$

In [75], Van Schaftingen showed that a slightly more general version of Theorem 1.3.2 is true

Theorem 3.2.1. *For some constant $C > 0$, the inequality*

$$\left| \int u(x) F_i(x) dx \right| \leq C \|u\|_{W^{1,n}} (\sum_{i=1}^n \|F_i\|_{L^1} + \|divF\|_{L^1})$$

holds for all $u \in W^{1,n}$ and $F = (F_1, \ldots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$, where $F_i \in \mathcal{D}(\mathbb{R}^n)$.

This result suggests the introduction of non-homogeneous versions of the Van Schaftingen classes D_k, as follows.

Definition 3.2.2. Let $1 \le k \le n$. We say that $u \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $d^k(\mathbb{R}^n)$ if

$$\sup_{\|\Phi\|_{\Upsilon^k(\mathbb{R}^n)} \le 1} \max_{|I|=k} |u(\phi_I)| < \infty, \tag{3.2.1}$$

where the suprema are taken over all $\Phi = \sum_{|I|=k} \phi_I dx^I \in \mathcal{D}^k(\mathbb{R}^n)$. We will denote this supremum by $\|u\|_{d^k}$.

Remark 3.2.3. Proposition 3.1.3 suggests that the domain of $u \in d^k(\mathbb{R}^n)$ can be extended to include all components of $\Upsilon^1_k(\mathbb{R}^n)$ forms. Let $u \in \mathcal{D}'(\Omega)$ and $\tilde{u}$ be a linear map from $\mathcal{D}^k(\Omega)$ to $\left(\mathbb{R}^{\binom{n}{k}}, \|\cdot\|_{max}\right)$, associated to u by

$$\tilde{u}\left(\sum_{|I|=k} \phi_I dx^I\right) = (u(\phi_I)).$$

Then $u \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $d^k(\mathbb{R}^n)$, if and only if $\tilde{u}$ can be extended to a bounded linear map from $\Upsilon^1_k(\mathbb{R}^n)$ to $\binom{n}{k}$ dimensional Euclidean space equipped with the *max* norm.

Note that $\Upsilon^1_n(\mathbb{R}^n) = L^1(\mathbb{R}^n)$, so $d^n(\mathbb{R}^n)$ is isomorphic to $L^\infty(\mathbb{R}^n)$.

Lemma 3.2.4. *Let $1 \le k < l \le n$ and $u \in d^l(\mathbb{R}^n)$. Then $u \in d^k(\mathbb{R}^n)$ and $\|u\|_{d^k(\mathbb{R}^n)} \le \|u\|_{d^l(\mathbb{R}^n)}$. In other words, the following embeddings are continuous*

$$d^n(\mathbb{R}^n) \subset d^{n-1}(\mathbb{R}^n) \subset \cdots \subset d^1(\mathbb{R}^n)$$

Proof. It is enough to consider the case $k = l - 1$, because the general case will follow from it by induction. Let $1 \le l \le n$, $u \in d^l(\mathbb{R}^n)$ and

$$\Phi(x) = \sum_{|I|=l-1} \phi_I(x) dx^I \in \mathcal{D}^{l-1}(\mathbb{R}^n).$$

We need to show for any component ϕ_I,

$$|u(\phi_I)| \le \|u\|_{d^l} \|\Phi\|_{\Upsilon^1_{l-1}}.$$

Fix any such I. Since $|I| = l - 1 < n$, there exists $j \in [1, n]$ such that $dx^I \wedge dx^j \ne 0$. Put $\tilde{\Phi}(x) = \Phi(x) \wedge dx^j$. Then $\tilde{\Phi} \in \mathcal{D}^l$ and $\|\tilde{\Phi}\|_{\Upsilon^1_l} \le \|\Phi\|_{\Upsilon^1_{l-1}}$. Moreover, by construction, one of the components of $\tilde{\Phi}$ equals to $\pm\phi_I dx^I \wedge dx^j$. Since $u \in d^l(\mathbb{R}^n)$, we have

$$|u(\phi_I)| \le \|u\|_{d^l} \|\tilde{\Phi}\|_{\Upsilon^1_l} \le \|u\|_{d^l} \|\Phi\|_{\Upsilon^1_{l-1}}.$$

□

In terms of d^k spaces, Theorem 3.2.1 can be formulated as follows.

Theorem 3.2.5. *$W^{1,n}(\mathbb{R}^n)$ is continuously embedded into $d^{n-1}(\mathbb{R}^n)$ and $\exists C > 0$ so that for any $u \in W^{1,n}$*

$$\|u\|_{d^{n-1}} \leq C\|u\|_{W^{1,n}}.$$

Our main result in this section is the following

Theorem 3.2.6. *$d^1(\mathbb{R}^n)$ is continuously embedded into the space $bmo(\mathbb{R}^n)$ and $\exists C > 0$ so that for any $u \in d^k(\mathbb{R}^n)$, $1 \leq k \leq n$*

$$\|u\|_{bmo} \leq C\|u\|_{d^k}.$$

Remark 3.2.7. This result is a non-homogeneous analogue of the main theorem in [77]. We adapt the proof of that theorem to the non-homogeneous setting.

Proof. By Lemma 3.2.4, it is enough to prove the case $k = 1$. The argument is based on the fact that $bmo(\mathbb{R}^n)$ is the dual space of $h^1(\mathbb{R}^n)$. We claim that given $f \in \mathcal{D}(\mathbb{R}^n)$, there exist n differential forms $\{\Phi^j\}_{j=1}^n \subset \Upsilon_1^1(\mathbb{R}^n)$ such that for some C independent of f,

$$\|\Phi^j\|_{\Upsilon_1^1} \leq C\|f\|_{h^1}, \tag{3.2.2}$$

$$f = \sum_{i=1}^n \phi_i^i, \tag{3.2.3}$$

where

$$\Phi^j = \sum_{i=1}^n \phi_i^j dx^i.$$

Assuming the claim the proof is easy. Let $u \in d^1(\mathbb{R}^n)$. For arbitrary $f \in \mathcal{D}(\mathbb{R}^n)$, let Φ^j be such that (3.2.2) and (3.2.3) are true. Then by the Remark 3.2.3 we can apply u to ϕ_i^i to have

$$|u(f)| \leq \sum_{i=1}^n |u(\phi_i^i)| \leq \sum_{i=1}^n \|u\|_{d^1}\|\Phi^i\|_{\Upsilon_1^1} \leq Cn\|u\|_{d^1}\|f\|_{h^1}. \tag{3.2.4}$$

By the density of $\mathcal{D}$ in h^1 and the duality $bmo = (h^1)'$, we conclude that $u \in bmo(\mathbb{R}^n)$.

In order to prove the claim, let $f \in \mathcal{D}$ be arbitrary and consider the equation

$$(I - \Delta)v = f \text{ in } \mathbb{R}^n.$$

Then $v = \mathcal{J}(f)$, where $\mathcal{J}$ is a convolution operator whose kernel is the Bessel potential of order 2, G_2 (see Example 2.4.9). For $j \in [1, n]$, let

$$\Phi^j = \sum_{i=1}^{n} \left(\frac{\mathcal{J}}{n} - \partial_i \partial_j \mathcal{J} \right) (f) dx^i.$$

Since $f \in \mathcal{D} \subset \mathcal{S}$, all components of Φ^j are $\mathcal{S}$ functions and

$$d\Phi^j = \sum_{1 \le i < k \le n} \left(\frac{\partial_i \mathcal{J} - \partial_k \mathcal{J}}{n} \right) (f) \; dx^i \wedge dx^k.$$

By Examples 2.4.9 and 2.4.10,

$$\frac{\mathcal{J}}{n} - \partial_i \partial_j \mathcal{J} \in OPS^{-2}(\mathbb{R}^n) + OPS^0(\mathbb{R}^n) \subset OPS^0(\mathbb{R}^n)$$

and

$$\left(\frac{\partial_i \mathcal{J} - \partial_k \mathcal{J}}{n} \right) \in OPS^{-1}(\mathbb{R}^n) \subset OPS^0(\mathbb{R}^n).$$

Recalling Theorem 2.4.11, we see that the components of Φ^j and $d\Phi^j$ are h^1 functions and for some C independent of f,

$$\|\Phi^j\|_{L^1_1} + \|d\Phi^j\|_{L^1_2} \le C\|f\|_{h^1},$$

which proves (3.2.2). Finally, $\{\Phi^j\}$ satisfy (3.2.3) for

$$\sum_{i=1}^{n} \left(\frac{\mathcal{J}}{n} - \partial_i \partial_i \mathcal{J} \right) f = \mathcal{J}(f) - \Delta \mathcal{J}(f) = (I - \Delta)\mathcal{J}(f) = f.$$

□

Corollary 3.2.8. *For $1 \le k \le n$, the space $d^k(\mathbb{R}^n)$ equipped with the norm $\| \cdot \|_{d^k}$ is a Banach space.*

Proof. Let $\{u_m\}_{m=0}^{\infty}$ be a Cauchy sequence in d^k. The above theorem shows that u_m is a Cauchy sequence in $bmo(\mathbb{R}^n)$. Since bmo is a complete Banach space, there exists $u \in bmo(\mathbb{R}^n)$, such that $u_m \to u$ in $\| \cdot \|_{bmo}$. Moreover, for any $\Phi = \sum_{|I|=k} \phi_I dx^I \in \mathcal{D}^k(\mathbb{R}^n)$ and $j \ge 0$, using duality of bmo and h^1 and the fact that each $\phi_I \in \mathcal{D} \subset h^1$,

$$\left| \int (u_j - u)\phi_I \right| = \lim_{m \to \infty} \left| \int (u_j - u_m)\phi_I \right| \le$$

$$\le \lim_{m \to \infty} \|u_j - u_m\|_{d^k} \|\Phi\|_{\Upsilon^1_k},$$

which shows that $u \in d^k(\mathbb{R}^n)$, and $\|u_j - u\|_{d^k} \to 0$, as $j \to \infty$. □

Summing up the results of this section, we can now say that for $1 \leq k \leq n$,

$$W^{1,n}(\mathbb{R}^n) \subset d^{n-1}(\mathbb{R}^n) \subset \cdots \subset d^1(\mathbb{R}^n) \subset bmo(\mathbb{R}^n).$$

3.3 v^k classes on $\mathbb{R}^n$

3.3.1 Definition and embeddings

Definition 3.3.1. Let $1 \leq k \leq n$. We define the class $v^k(\mathbb{R}^n)$ as the closure of $C_0(\mathbb{R}^n)$ functions in the norm $\|\cdot\|_{d^k}$. Here

$$C_0(\mathbb{R}^n) = \{u :\in C(\mathbb{R}^n) : \lim_{|x|\to\infty} u(x) = 0\}.$$

First of all we notice that by Proposition 3.2.4, $v^k(\mathbb{R}^n)$ form a monotone family of spaces

$$v^n(\mathbb{R}^n) \subset v^{n-1}(\mathbb{R}^n) \subset \cdots \subset v^1(\mathbb{R}^n).$$

The appropriate subspace that will contain all v^k functions was studied by Dafni [22] and Bourdaud [10].

Definition 3.3.2. [22] $vmo(\mathbb{R}^n)$ is the subspace of $bmo(\mathbb{R}^n)$ functions satisfying

$$\lim_{\delta\to 0} \sup_{l(Q)\leq\delta} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx = 0 \tag{3.3.1}$$

and

$$\lim_{R\to\infty} \sup_{l(Q)>1, Q\cap B(0,R)=\emptyset} \frac{1}{|Q|} \int_Q |f(x)| dx = 0. \tag{3.3.2}$$

Theorem 3.3.3 ([22])**.** *$vmo(\mathbb{R}^n)$ is the closure of $C_0(\mathbb{R}^n)$ in $bmo(\mathbb{R}^n)$.*

An immediate consequence of this result and Theorem 3.2.6 is

Theorem 3.3.4. *For $1 \leq k \leq n$, the space $v^k(\mathbb{R}^n)$ is embedded into $vmo(\mathbb{R}^n)$.*

Corollary 3.3.5. *$v^1(\mathbb{R}^n)$ does not contain $d^n(\mathbb{R}^n)$ as a subspace. In particular, $v^k(\mathbb{R}^n)$ are proper subspaces of $d^k(\mathbb{R}^n)$ for $k = 1, \ldots, n$.*

Proof. Recall that $d^n(\mathbb{R}^n)$ coincides with $L^\infty(\mathbb{R}^n)$. If L^∞ was a subspace of $v^1(\mathbb{R}^n)$, then by the last theorem we would have $L^\infty \subset vmo(\mathbb{R}^n)$. However, choosing f as a characteristic function of the quadrant $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0\}$, we have an example of an L^∞ function that does not satisfy (3.3.1). So $L^\infty \not\subset vmo(\mathbb{R}^n)$. □

Finally, we recall that $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$ for any $p \in [1, \infty)$. Therefore by Van Schaftingen's Theorem 3.2.1, we have $W^{1,n} \subset v^{n-1}(\mathbb{R}^n)$.

All in all, we conclude that the following embeddings hold

$$W^{1,n}(\mathbb{R}^n) \subset v^{n-1}(\mathbb{R}^n) \subset \cdots \subset v^1(\mathbb{R}^n) \subset vmo(\mathbb{R}^n).$$

3.3.2 Intrinsic definition of the space v^{n-1}

Definition 3.3.6. For $u \in d^{n-1}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, we will use the following notation

$$\|u\|_* = \sup_{\partial\gamma=\emptyset} \frac{1}{|\gamma|} \left| \int_\gamma u(t)\tau(t)dt \right| + \sup_{|\gamma|\geq 1} \frac{1}{|\gamma|} \left| \int_\gamma u(t)\tau(t)dt \right|,$$

where the suprema are taken over smooth curves γ with finite lengths $|\gamma|$, boundaries $\partial\gamma$ and unit tangent vectors τ.

Our goal is to prove the following result which plays the role of Theorem 1.4.1 in the non-homogeneous setting.

Theorem 3.3.7. *There are constants $c_1, c_2 > 0$ such that for every $u \in d^{n-1}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$,*

$$c_1\|u\|_* \leq \|u\|_{d^{n-1}} \leq c_2\|u\|_*.$$

The proof is based on the following three lemmas

Lemma 3.3.8. *There exists $C > 0$ such that for any γ with $\partial\gamma = \emptyset$ or $|\gamma| \geq 1$,*

$$\frac{1}{|\gamma|} \left| \int_\gamma u(y)\tau(y)dy \right| \leq C\|u\|_{d^{n-1}}.$$

Proof. The proof is based on the argument of Bourgain and Brezis [11].

Let $\eta \geq 0$ be a smooth radial function on $\mathbb{R}^n$, compactly supported in $|x| \leq 1$, such that $\|\eta\|_{L^1} = 1$. As usual we put $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$. Let us define the $(n-1)$-form

$$\Phi^\epsilon(x) = \sum_{j=1}^n \left(\int_\gamma \eta_\epsilon(t - x)\tau_j(t)dt \right) dx^{I_j}, x \in \mathbb{R}^n,$$

where $I_j = (i_1, \dots, i_{n-1}), i_k \neq j$.

The reason to introduce this differential form is the following equality

$$\left|\int_\gamma u(t)\tau(t)dt\right| = \lim_{\epsilon\to 0}\left|\int_\gamma \tau(t)\int_{\mathbb{R}^n} u(x)\eta_\epsilon(x-t)dxdt\right| =$$

$$= \lim_{\epsilon\to 0}\left|\int u(x)\phi_I^\epsilon dx\right|,$$

where ϕ_I^ϵ are components of Φ^ϵ. By the Remark 3.2.3, we need to estimate $\|\Phi^\epsilon\|_{\Upsilon^1_{n-1}}$. It is clear that $\|\Phi^\epsilon\|_{L^1_{n-1}} \leq n\|\eta_\epsilon\|_{L^1}|\gamma| = n|\gamma|$. Moreover,

$$d\Phi^\epsilon(x) = -\left(\int_\gamma \nabla\eta_\epsilon(y-x)\cdot\tau(y)dy\right)dx^1\wedge\cdots\wedge dx^n =$$

$$= [\eta_\epsilon(a-x) - \eta_\epsilon(b-x)]dx^1\wedge\cdots\wedge dx^n.$$

Therefore $\|d\Phi^\epsilon\|_{L^1_n}$ is 0 if γ is closed or ≤ 2 if γ is not closed. Finally,

$$\frac{1}{|\gamma|}\left|\int_\gamma u(s)\tau(s)ds\right| \leq \frac{1}{|\gamma|}\limsup_{\epsilon\to 0}\left|\int u(x)\phi_I^\epsilon dx\right| \leq \|u\|_{d^k}(2+n),$$

because, for non-closed γ, $|\gamma| \geq 1$. So we proved the lemma with $C = n+2$. □

In order to prove the converse estimate, Bourgain and Brezis evoked the decomposition theorem of Smirnov.

Theorem 3.3.9 ([63]). *For any compactly supported $\Phi \in L^1_{n-1}(\mathbb{R}^n)$, with $d\Phi = 0$, there exists a sequence of positive numbers $\{\mu_j^m\}$ and closed smooth curves $\{\gamma_j^m\}$ such that for all $m \geq 1$,*

$$\sum_{j=1}^{\infty}|\mu_j^m||\gamma_j^m| \leq \|\Phi\|_{L^1_{n-1}}$$

and for every $u \in C(\mathbb{R}^n)$ and $1 \leq i \leq n$

$$\sum_{j=1}^{\infty}\mu_j^m\int_{\gamma_j^m} u(s)\tau_i(s)ds \to \int u(x)\phi_i(x)dx, \text{ as } m\to\infty,$$

where ϕ_i are the components of Φ.

In our case $d\Phi \in L^1_{n-1}(\mathbb{R}^n)$ does not necessarily vanish and we need a more general version of Smirnov's theorem, which we formulate in the following form

Theorem 3.3.10 (Theorem C in [63]). *Let $\Phi \in \Upsilon^1_{n-1}(\mathbb{R}^n)$. Then there exist $P \in \Upsilon^1_{n-1}(\mathbb{R}^n)$ and $Q \in \Upsilon^1_{n-1}(\mathbb{R}^n)$ such that*

- $\|\Phi\|_{L^1_{n-1}} = \|P\|_{L^1_{n-1}} + \|Q\|_{L^1_{n-1}}$,
- $dP = 0$ *and we can apply the previous theorem to* P
- $dQ = d\Phi$.

Moreover, there exist $\{\lambda^l_j\}$ *and smooth curves* $\tilde{\gamma}^l_j$ *(not necessarily closed) such that for all* $l \geq 1$

$$\sum_{j=1}^{\infty} |\lambda^l_j||\tilde{\gamma}^l_j| \leq \|Q\|_{L^1_{n-1}},$$

$$\sum_{j=1}^{\infty} |\lambda^l_j| \leq \|dQ\|_{L^1_n}$$

and for $1 \leq i \leq n$

$$\sum_{j=1}^{\infty} \lambda^l_j \int_{\tilde{\gamma}^l_j} u(s)\tau_i(s)ds \to \int u(x)q_i(x)dx, \text{ as } l \to \infty.$$

where q_i *are the components of* Q.

Let us introduce an auxiliary norm for $u \in C(\mathbb{R}^n)$:

$$\|u\|_{**} = \sup_{\partial\gamma=\emptyset} \frac{1}{|\gamma|} \left| \int_{\gamma} u(s)\tau(s)ds \right| + \sup_{|\gamma|<1} \left| \int_{\gamma} u(s)\tau(s)ds \right| + \sup_{|\gamma|\geq 1} \frac{1}{|\gamma|} \left| \int_{\gamma} u(s)\tau(s)ds \right|.$$

Lemma 3.3.11. *For any $u \in d^{n-1}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$,*

$$\|u\|_{d^{n-1}(\mathbb{R}^n)} \leq 2\|u\|_{**}.$$

Proof. By the definition of $d^{n-1}(\mathbb{R}^n)$, there exists

$$\Phi = \sum_{i=1}^{n} \phi_i dx^1 \wedge \dots \widehat{dx^i} \wedge \dots dx^n \in \mathcal{D}^{n-1}(\mathbb{R}^n)$$

such that

$$\|\Phi\|_{L^1_{n-1}} + \|d\Phi\|_{L^1} \leq 1$$

and

$$\|u\|_{d^{n-1}} \le 2 \max_I |u(\phi_I)|. \tag{3.3.3}$$

Let us apply Theorem 3.3.10 to Φ. Then Φ can be decomposed into the sum of P and Q such that $d\Phi = dQ$, $\|\Phi\|_{L^1_{n-1}} = \|P\|_{L^1_{n-1}} + \|Q\|_{L^1_{n-1}}$ and Q is a weak limit of the linear combination of the curves $\tilde{\gamma}^l_j$ in the sense that

$$\sum_{j=1}^{\infty} \tilde{\lambda}^l_j \int_{\tilde{\gamma}^l_j} u(s)\tau_i(s)ds \to \int u(x)q_i(x)dx, \text{ as } l \to \infty,$$

where

$$\sum_{j=1}^{\infty} |\tilde{\lambda}^l_j|(1 + |\tilde{\gamma}^l_j|) \le \|Q\|_{L^1_{n-1}} + \|dQ\|_{L^1} \le 1, \text{ for all } l \ge 1.$$

Moreover, applying Theorem 3.3.9 to P, we get a sequence of closed curves γ^l_j and numbers λ^l_j such that

$$\sum_{j=1}^{\infty} \lambda^l_j \int_{\gamma^l_j} u(s)\tau_i(s)ds \to \int u(x)p_i(x)dx, \text{ as } l \to \infty$$

and

$$\sum_{j=1}^{\infty} |\lambda^l_j||\gamma^l_j| \le \|P\|_{L^1_{n-1}} \le 1 \text{ for all } l \ge 1.$$

All in all,

$$\int u(x)\phi_i(x)dx = \lim_{l\to\infty} \sum_{j=1}^{\infty} \lambda^l_j \int_{\gamma^l_j} u(s)\tau_i(s)ds + \sum_{j=1}^{\infty} \tilde{\lambda}^l_j \int_{\tilde{\gamma}^l_j} u(s)\tau_i(s)ds$$

and

$$\left|\int u(x)\phi_i(x)dx\right| \le \sup_{l,j} \left|\frac{1}{|\gamma^l_j|}\int_{\gamma^l_j} u(s)\tau_i(s)ds\right| + \tag{3.3.4}$$

$$+ \sup_{l,|\tilde{\gamma}^l_j|<1} \left|\int_{\tilde{\gamma}^l_j} u(s)\tau_i(s)ds\right| + \sup_{l,|\tilde{\gamma}^l_j|\ge 1} \left|\frac{1}{|\tilde{\gamma}^l_j|}\int_{\tilde{\gamma}^l_j} u(s)\tau_i(s)ds\right| \le \|u\|_{**}.$$

The result follows from (3.3.3) and (3.3.4). □

Lemma 3.3.12. *For any $u \in C(\mathbb{R}^n)$*

$$\|u\|_* \le \|u\|_{**} \le 4\|u\|_*.$$

Proof. The first inequality follows from the definitions of the norms. In order to see the second one, we need to show that

$$\sup_{|\gamma|<1} \left| \int_\gamma u(s)\tau(s)ds \right| \leq \sup_{\partial\gamma=\emptyset} \frac{3}{|\gamma|} \left| \int_\gamma u(s)\tau(s)ds \right| + \sup_{|\gamma|\geq 1} \frac{3}{|\gamma|} \left| \int_\gamma u(s)\tau(s)ds \right|.$$

Let us consider any γ with $|\gamma| < 1$ and $\partial\gamma = \{a, b\}$. We can always find γ' such that $1 < |\gamma'| < 2$ and $\gamma'' := \gamma + \gamma'$ is a closed curve.

Then

$$\begin{aligned} \left| \int_\gamma u(s)\tau(s)ds \right| &\leq \left| \int_{\gamma''} u(s)\tau(s)ds \right| + \left| \int_{\gamma'} u(s)\tau(s)ds \right| \\ &\leq \left| \frac{3}{|\gamma''|} \int_{\gamma''} u(s)\tau(s)ds \right| + \left| \frac{3}{|\gamma'|} \int_{\gamma'} u(s)\tau(s)ds \right| \end{aligned}$$

□

3.4 Tensor product of d^k functions

Let $u \in C(\mathbb{R}^n)$ and $v \in C(\mathbb{R}^m)$, then we can define the tensor product of u and v as an element of $C(\mathbb{R}^{n+m})$ such that

$$u \otimes v(x, y) = u(x)v(y) \text{ for all } x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

This notion can extended to the case $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^m)$ (see e.g. Theorem 5.1.1 in [37]) as follows: there exists a unique $w \in \mathcal{D}'(\mathbb{R}^{n+m})$ such that

$$w(\phi_1 \otimes \phi_2) = u(\phi_1)v(\phi_2),$$

for any $\phi_1 \in \mathcal{D}(\mathbb{R}^n)$ and $\phi_2 \in \mathcal{D}(\mathbb{R}^m)$ and

$$w(\phi) = u(v(\phi(x, y))) = v(u(\phi(x, y))).$$

Notice that if $u \in d^n(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$ and $v \in d^m(\mathbb{R}^m) = L^\infty(\mathbb{R}^m)$, then $u \otimes v \in d^{n+m}(\mathbb{R}^{n+m}) = L^\infty(\mathbb{R}^{n+m})$. This can be partially extended to $u \in d^k$, $k < n$ as follows

Theorem 3.4.1. *Let $v(y) = \Pi_{j=1}^m v_j(y_j)$, where $v_j \in \mathcal{D}(\mathbb{R})$ and $u \in \mathcal{D}'(\mathbb{R}^n)$.*

- *If $u \in d^k(\mathbb{R}^n)$, then $u \otimes v \in d^k(\mathbb{R}^{n+m})$ and there exists $C > 0$ such that*

$$\|u \otimes v\|_{d^k(\mathbb{R}^{n+m})} \leq C\|v\|_{C^1(\mathbb{R}^m)}\|u\|_{d^k(\mathbb{R}^n)}.$$

- *If* $u \otimes v \in d^k(\mathbb{R}^{n+m})$ *and* $v \neq 0$, *then* $u \in d^k(\mathbb{R}^n)$ *and there exists* $C_v > 0$ *such that*

$$\|u\|_{d^k(\mathbb{R}^n)} \leq C_v \|u \otimes v\|_{d^k(\mathbb{R}^{n+m})}.$$

We will prove two lemmas below, from which the theorem will immediately follow by induction in m. Before doing this, let us note a useful corollary of Theorem 3.4.1.

Corollary 3.4.2. *The embeddings* $d^n(\mathbb{R}^n) \subset d^{n-1}(\mathbb{R}^n) \subset \cdots \subset d^1(\mathbb{R}^n)$ *are proper.*

Proof. It is well known (see e.g. 5.6.1 in [27]) that there exists $u \in W^{1,n} \setminus L^\infty(\mathbb{R}^n)$, $n \geq 2$. Thus by Theorem 3.2.5, $u \in d^{n-1} \setminus d^n$ and the embedding $d^n(\mathbb{R}^n) \subset d^{n-1}(\mathbb{R}^n)$ is proper. This is all we need when $n = 2$.

For $n \geq 3$, we can argue by induction on n. Suppose the embeddings

$$d^{n-1}(\mathbb{R}^{n-1}) \subset d^{n-2}(\mathbb{R}^{n-1}) \subset \cdots \subset d^1(\mathbb{R}^{n-1})$$

are all proper. In order to find $w \in d^k(\mathbb{R}^n) \setminus d^{k+1}(\mathbb{R}^n)$, $2 \leq k \leq n-2$, consider $u \in d^k(\mathbb{R}^{n-1}) \setminus d^{k+1}(\mathbb{R}^{n-1})$ and put $w(x,y) = u(x) \otimes 1(y) = u(x)$. Then by the first part of Theorem 3.4.1, $w(x,y) \in d^k(\mathbb{R}^n)$. On the other hand the second part of Theorem 3.4.1 asserts that if $w(x,y) \in d^{k+1}(\mathbb{R}^n)$ then we would have $u \in d^{k+1}(\mathbb{R}^{n-1})$, which would contradict to the choice of u. So $w \in d^k(\mathbb{R}^n) \setminus d^{k+1}(\mathbb{R}^n)$. □

Lemma 3.4.3. *Let* $v \in L^\infty(\mathbb{R})$ *be fixed and* $\|v\|_{L^\infty} > 0$. *If* $u \otimes v \in d^k(\mathbb{R}^{n+1})$ *for some* $u \in \mathcal{D}'(\mathbb{R}^n)$, *then* $u \in d^k(\mathbb{R}^n)$ *and*

$$\|u\|_{d^k(\mathbb{R}^n)} \leq C_v \|u \otimes v\|_{d^k(\mathbb{R}^{n+1})},$$

for some $C_v > 0$ *independent of* u.

Proof. Given $\Phi \in \mathcal{D}^k(\mathbb{R}^n)$ such that

$$\Phi(x) = \sum_{|I|=k} \phi_I(x) dx^I,$$

we need to show that for some C_v

$$|u(\phi_I)| \leq C_v \|u \otimes v\|_{d^k(\mathbb{R}^{n+1})} \|\Phi\|_{\Upsilon^1_k(\mathbb{R}^n)}.$$

We can always find $\theta \in \mathcal{D}(\mathbb{R})$ such that $|\int_{\mathbb{R}} v(y)\theta(y)dy| > \frac{1}{2}\|v\|_{L^\infty} > 0$. Consider $\tilde{\Phi}(x,y) \in \mathcal{D}^k(\mathbb{R}^{n+1})$ defined by

$$\tilde{\Phi}(x,y) = \Phi(x)\theta(y) = \sum_{|I|=k} \phi_I(x)\theta(y)dx^I.$$

Then $d\tilde{\Phi}(x,y) = d\Phi(x)\theta(y) + \Phi(x) \wedge \theta'(y)dy$ and

$$\|\tilde{\Phi}\|_{\Upsilon^k(\mathbb{R}^{n+1})} \leq C\|\theta\|_{W^{1,1}}\|\Phi\|_{\Upsilon^k(\mathbb{R}^n)}.$$

Therefore, for any component ϕ_I

$$\begin{aligned}
|u(\phi_I)| = \frac{|(u \otimes v)(\phi_I(x)\theta(y))|}{|v(\theta)|} &< \frac{2}{\|v\|_{L^\infty}}|(u \otimes v)(\phi_I(x)\theta(y))| \\
&\leq \frac{2}{\|v\|_{L^\infty}}\|u \otimes v\|_{d^k(\mathbb{R}^{n+1})}\|\tilde{\Phi}\|_{\Upsilon^1_k(\mathbb{R}^{n+1})} \\
&\leq \left(\frac{C\|\theta\|_{W^{1,1}}}{\|v\|_{L^\infty}}\right)\|u \otimes v\|_{d^k(\mathbb{R}^{n+1})}\|\Phi\|_{\Upsilon^1_k(\mathbb{R}^n)}.
\end{aligned}$$

Finally, putting $C_v = \frac{C\|\theta\|_{W^{1,1}}}{\|v\|_{L^\infty}}$, we complete the proof. □

The proof of the converse fact is more technical. It is convenient first to adopt some notations that will simplify the argument.

Let $\Phi(x) = \sum_{|I|=k} \phi_I(x)dx^I \in \mathcal{D}^k(\mathbb{R}^{n+1})$. For $x \in \mathbb{R}^{n+1}$, we denote by $y \in \mathbb{R}$ its last coordinate and by $\tilde{x} \in \mathbb{R}^n$ the first n coordinates. We divide the set of indices I in the decomposition of Φ into two groups. Let R be the set of $I = (i_1, \ldots, i_k)$ in the decomposition of Φ with $i_k = n+1$, and L be the rest. Then

$$\Phi = \sum_{I \in L} \phi_I(\tilde{x},y)d\tilde{x}^I + \sum_{I=(\tilde{I},n+1)\in R} \phi_I(\tilde{x},y)d\tilde{x}^{\tilde{I}} \wedge dy. \tag{3.4.1}$$

We will denote these sums by Φ_L and Φ_R, respectively. Finally, we introduce the integration over the last coordinate as $\int_{\mathbb{R}_y} \Phi(\tilde{x},y) \in \mathcal{D}^{k-1}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}_y} \Phi(\tilde{x},y) = \int_{\mathbb{R}_y} \Phi_R(\tilde{x},y) := \sum_{(\tilde{I},n+1)\in R} \left(\int_{\mathbb{R}} \phi_I(\tilde{x},y)dy\right) d\tilde{x}^{\tilde{I}}$$

(if $\Phi = \Phi_L$, we agree that $\int_{\mathbb{R}_y} \Phi(\tilde{x},y) = 0$). Note that by the Fubini theorem,

$$\left\|\int_{\mathbb{R}_y} \Phi(\tilde{x},y)\right\|_{L^1_{k-1}(\mathbb{R}^n)} \leq \|\Phi\|_{L^1_k(\mathbb{R}^{n+1})}.$$

Lemma 3.4.4. *Let* $u \in d^k(\mathbb{R}^n)$ *and* $v \in \mathcal{D}(\mathbb{R})$. *Then* $u \otimes v \in d^k(\mathbb{R}^{n+1})$ *and*

$$\|u \otimes v\|_{d^k(\mathbb{R}^{n+1})} \leq C\|v\|_{C^1(\mathbb{R})}\|u\|_{d^k(\mathbb{R}^n)},$$

where $C > 0$ *is independent of* u, v.

Proof. We need to consider an arbitrary

$$\Phi = \sum_{|I|=k} \phi_I dx^I \in \mathcal{D}^k(\mathbb{R}^{n+1})$$

and show that for some $C > 0$

$$\left| u \left(\int_{\mathbb{R}} v(y)\phi_I(\tilde{x}, y)dy \right) \right| \leq C\|v\|_{C^1}\|u\|_{d^k}\|\Phi\|_{\Upsilon_k^1(\mathbb{R}^n)}. \tag{3.4.2}$$

Let Φ be decomposed as in (3.4.1)

$$\Phi = \Phi_L + \Phi_R := \sum_{I \in L} \phi_I(\tilde{x}, y)d\tilde{x}^I + \sum_{I=(\tilde{I}, n+1) \in R} \phi_I(\tilde{x}, y)d\tilde{x}^{\tilde{I}} \wedge dy.$$

We claim that the differential forms

$$\Phi^1(\tilde{x}) = \sum_{I \in L} \left(\int_{\mathbb{R}} v(y)\phi_I(\tilde{x}, y)dy \right) d\tilde{x}^I$$

$$\Phi^2(\tilde{x}) = \sum_{I=(\tilde{I}, n+1) \in R} \left(\int_{\mathbb{R}} v(y)\phi_I(\tilde{x}, y)dy \right) d\tilde{x}^{\tilde{I}}$$

satisfy the estimates

$$\|\Phi^1\|_{\Upsilon_k^1(\mathbb{R}^n)} \leq C\|v\|_{L^\infty}\|\Phi\|_{\Upsilon_k^1(\mathbb{R}^{n+1})} \tag{3.4.3}$$

and

$$\|\Phi^2\|_{\Upsilon_{k-1}^1(\mathbb{R}^n)} \leq C\|v\|_{C^1}\|\Phi\|_{\Upsilon_k^1(\mathbb{R}^{n+1})}. \tag{3.4.4}$$

Notice that (3.4.3) and the definition of $d^k(\mathbb{R}^n)$ imply (3.4.2) for $I \in L$. For $I \in R$, (3.4.2) follows from (3.4.4), Lemma 3.2.4 and the definition of $d^k(\mathbb{R}^n)$. So it is enough to establish (3.4.3) and (3.4.4).

Clearly,

$$\left\| \int_{\mathbb{R}} v(y)\phi_I(\tilde{x}, y)dy \right\|_{L^1(\mathbb{R}^n)} \leq \|v\|_{L^\infty}\|\phi_I\|_{L^1(\mathbb{R}^{n+1})}.$$

So both $\|\Phi^1\|_{L_k^1}$ and $\|\Phi^2\|_{L_{k-1}^1}$ are controlled by $\|v\|_{C^1}\|\Phi\|_{L_k^1(\mathbb{R}^{n+1})}$.

In order to estimate $\|d\Phi^1\|_{L^1_{k+1}}$ and $\|d\Phi^2\|_{L^1_k}$, we make use of our notation.

First of all we note that by our choice of Φ^1

$$\Phi^1(\tilde{x}) = \int_{\mathbb{R}_y} \Phi_L(\tilde{x}, y) \wedge v(y)dy.$$

Moreover, from the definitions of the exterior derivative and $\int_{\mathbb{R}_y}$, one has

$$d\Phi^1(\tilde{x}) = d\int_{\mathbb{R}_y} \Phi_L \wedge v(y)dy = \int_{\mathbb{R}_y} d[\Phi_L \wedge v(y)dy], \tag{3.4.5}$$

where the first d is the differentiation with respect to $\tilde{x}$ and the last one is with respect to $\tilde{x}$ and y. Since $\Phi_R \wedge v(y)dy = 0$, we have

$$d[\Phi \wedge v(y)dy] = d[\Phi_L \wedge v(y)dy]$$

and therefore

$$\int_{\mathbb{R}_y} d[\Phi_L \wedge v(y)dy] = \int_{\mathbb{R}_y} d[\Phi \wedge v(y)dy]. \tag{3.4.6}$$

Noticing that $v(y)dy$ is a closed 1-form we combine (3.4.5) and (3.4.6) to obtain

$$d\Phi^1(\tilde{x}) = \int_{\mathbb{R}_y} d\Phi \wedge v(y)dy.$$

Applying Fubini's theorem we have

$$\|d\Phi^1\|_{L^1_{k+1}} = \left\| \int_{\mathbb{R}_y} d\Phi \wedge v(y)dy \right\|_{L^1_{k+1}(\mathbb{R}^n)} \le$$

$$\le \|d\Phi \wedge v(y)dy\|_{L^1_{k+2}(\mathbb{R}^{n+1})} \le \|v\|_{L^\infty} \|d\Phi\|_{L^1_{k+1}(\mathbb{R}^{n+1})}$$

(the last inequality is justified by (3.1.2)). This proves (3.4.3).

Further, notice that

$$\Phi^2(\tilde{x}) = \int_{\mathbb{R}_y} v(y)\Phi_R(\tilde{x}, y)$$

and

$$\int_{\mathbb{R}_y} d[v(y)\Phi_L(\tilde{x}, y)] = 0,$$

because for each $I \in L$, $\int_{\mathbb{R}_y} d\tilde{x}^I = 0$ and $\int_{\mathbb{R}} \partial_y(v(y)\phi_I(\tilde{x}, y))dy = 0$. Hence

$$d\Phi^2(\tilde{x}) = d\int_{\mathbb{R}_y} v(y)\Phi_R(\tilde{x}, y) = \int_{\mathbb{R}_y} d[v(y)\Phi_R(\tilde{x}, y)] = \int_{\mathbb{R}_y} d[v(y)\Phi(\tilde{x}, y)].$$

Therefore, by Fubini's theorem and (3.1.2) we obtain

$$\|d\Phi^2\|_{L^1_k} \leq \left\| \int_{\mathbb{R}_y} d[v(y)\Phi(\tilde{x},y)] \right\|_{L^1_k} \leq$$

$$\left\| \int_{\mathbb{R}_y} v'(y)dy \wedge \Phi(\tilde{x},y)] \right\|_{L^1_k} + \left\| \int_{\mathbb{R}_y} v(y)d\Phi(\tilde{x},y)] \right\|_{L^1_k} \leq \|v\|_{C^1} \|\Phi\|_{\Upsilon^1_k(\mathbb{R}^{n+1})},$$

which proves (3.4.4).

□

3.5 Examples of $d^k(\mathbb{R}^n)$ functions

In this section, we want to show that there are more functions in $d^k(\mathbb{R}^n)$ besides those in $W^{1,n}(\mathbb{R}^n)$.

3.5.1 Triebel-Lizorkin and Besov functions

We recall that Sobolev space $W^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$ is a special case of more general classes of functions

$$W^{s,p}(\mathbb{R}^n) = F^{s,p}_p(\mathbb{R}^n) = B^{s,p}_p(\mathbb{R}^n),$$

here $F^{s,p}_q$, $s \in \mathbb{R}$, $0 < p, q < \infty$ is the space of Triebel-Lizorkin and $B^{s,p}_q(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, is the Besov space (see e.g. [32] or [72] for definitions).

It was shown in [78] (see Proposition 2.1 there), that $\mathring{F}^{s,p}_q \subset D_{n-1}$ for all $sp = n$, $1 < p < \infty$, $0 < q < \infty$ (here $\mathring{F}^{s,p}_q$ is a homogeneous Triebel-Lizorkin space). Recalling the embedding theorems (see e.g. Ex 6.5.2 in [32])

$$\mathring{B}^{s,p}_{\min(p,q)} \subset \mathring{F}^{s,p}_q \subset \mathring{B}^{s,p}_{\max(p,q)},$$

one can obtain the embedding $B^{s,p}_q \subset D_{n-1}$ for $0 < q < \infty$. The case $q = \infty$ remains open (see Open problem 1 in [79]).

One can notice that the proof of Proposition 2.1 in [78] is exactly the same as the proof of Theorem 1.5 in [76]. In fact it can be extended to the non-homogeneous setting as

Theorem 3.5.1. *Let $1 < p < \infty$, $1 < q < \infty$. Then there exists constants C_1 and C_2 such that*

$$\|u\|_{d^{n-1}} \le C_1 \|u\|_{F_q^{n/p,p}}$$

and

$$\|u\|_{d^{n-1}} \le C_2 \|u\|_{B_q^{n/p,p}}.$$

3.5.2 Locally Lipschitz functions

The following proposition provides a simple sufficient condition to ensure that $u \in d^{n-1}(\mathbb{R}^n)$.

Proposition 3.5.2. *Let $u \in W^{1,1}_{loc}(\mathbb{R}^n \setminus \{0\})$. If $|x|(u(x) + \nabla u(x)) \in L^\infty(\mathbb{R}^n)$, then $u \in d^{n-1}(\mathbb{R}^n)$ and*

$$\|u\|_{d^{n-1}} \le C\||x|(|u| + |\nabla u|)\|_{L^\infty}.$$

Proof. The proof follows from integration by parts as in the proof of Proposition 4.3 in [77].

We need to show that for any $\Phi = \sum_{j=1}^{n} \phi_j(x) dx^1 \wedge \dots \widehat{dx}^j \wedge \dots dx^n \in \mathcal{D}^{n-1}(\mathbb{R}^n)$, we have

$$\left| \int u(x)\phi_j(x)dx \right| \le C\||x|(u(x) + \nabla u(x))\|_{L^\infty} \|\Phi\|_{\Upsilon^1_{n-1}}.$$

Note that

$$\int x_j (\sum_i \phi_i \partial_i u) dx = -\int \phi_j u dx - \int x_j u \cdot (\sum_i \partial_i \phi_i) dx.$$

So

$$\left| \int u(x)\phi_j(x)dx \right| \le n\||x|\nabla u\|_{L^\infty} \|\Phi\|_{L^1_{n-1}} + \||x|u\|_{L^\infty} \|d\Phi\|_{L^1_n}.$$

□

The proposition allows us to give an example of $u \in d^{n-1}$ which is not covered by the previous classes of functions, the Bessel potential G_n.

Remark 3.5.3. A typical example of $u \in D^{n-1} \setminus W^{1,n}$ in [77] is the function $u(x) = \log|x|$. However, this function does not belong to $bmo(\mathbb{R}^n)$ and therefore is not in any d^k, $1 \le k \le n$ as

$$\sup_{|Q|>1} \frac{1}{|Q|} \int_Q |\log|y||dy = \infty.$$

Example 3.5.4. Let $G_n(x)$ be the Bessel potential of order n, i.e. the function whose Fourier transforms is given by $\hat{G}_n(\xi) = (1 + |\xi|^2)^{-n/2}$.

The fact that G_n satisfies the conditions of the last proposition follows from the fact that G_n is a continuously differentiable function on $\mathbb{R}^n \setminus \{0\}$ and the asymptotic formulas for the Bessel potentials (see e.g. [3], pp. 415-417):

$$G_n(x) \sim C_1 \log |x|, \text{ as } x \to 0,$$

$$G_n(x) \sim C_2 |x|^{-1/2} e^{-|x|}, \text{ as } x \to \infty.$$

Moreover,

$$\frac{\partial}{\partial x_i} G_n(x) = C_s' \cdot \frac{x_i}{|x|} K_1(|x|),$$

where K_1 is the Bessel-Macdonald function of order 1, with the asymptotics

$$K_1(r) \sim C_3 r^{-1}, \text{ as } r \to 0+$$

$$K_1(r) \sim C_4 r^{-1/2} e^{-r}, \text{ as } r \to \infty.$$

Combining this example and Theorem 3.4.1, we obtain more examples

Proposition 3.5.5. *Let $v_1, \ldots, v_l$ be $\mathcal{D}(\mathbb{R})$ functions such that $v_i \not\equiv 0$. Then*

$$G_n(x_1, \ldots, x_{n-l}) \otimes v_1(x_{n-l+1}) \cdots \otimes v_l(x_n) \in d^k(\mathbb{R}^n)$$

if and only if $1 \leq k < n - l \leq n$.

3.6 Application to PDE

We will illustrate how non-homogeneous d^k spaces can be used in the analysis of classic PDE.

The following result was shown in [11]: if $\Delta U = F$ in $\mathbb{R}^2$ and $divF = 0$, then

$$\|U\|_\infty + \|\nabla U\|_2 \leq C\|F\|_1.$$

As it has been noted in [14] (see Remark 2.1 there), one can relax the condition $divF = 0$ to $divF \in L^1$ to obtain

$$\|\nabla U\|_2 \leq C(\|F\|_1 + \|divF\|_1).$$

However (as is also noted in [14] without explanation), U may be no longer be an L^∞ vector field.

Let us explain this using Theorem 3.2.6. Let $g(x) = \log|x|$. Then $g * F$ is continuous for any $F \in L^1_1$ and if

$$\|U\|_\infty = (2\pi)^{-1}\|g * F\|_\infty \le C(\|F\|_1 + \|divF\|_1)$$

were true for any $F \in \mathcal{D}^k(\mathbb{R}^2)$, then we would have

$$|g * F(0)| = |\int g(x)F(x)dx| \le C\|F\|_{\Upsilon^1_1},$$

and $g(x) = \log|x|$ would be an d^1 function and by Theorem 3.2.6, $\log|x| \in bmo(\mathbb{R}^2)$. However, this is false by Remark 3.5.3.

So the solution of equation $\Delta U = F \in \mathbb{R}^2$ can be essentially unbounded even if $divF \in L^1$, because the fundamental solution of Δ in $\mathbb{R}^2$ is not an element of $d^1(\mathbb{R}^2)$.

Based on the examples of $d^{n-1}(\mathbb{R}^n)$ functions, one can guess that the situation should be better in the case of the Helmholtz equation.

Indeed, the following proposition shows that solutions to the Helmholtz equation can be fully controlled even under relaxed conditions.

Theorem 3.6.1. *Let $F \in L^1(\mathbb{R}^2)$ and $divF \in L^1(\mathbb{R}^2)$. Then the system $(I - \Delta)U = F$ admits a unique solution U such that*

$$\|U\|_\infty + \|\nabla U\|_2 \le C(\|F\|_1 + \|divF\|_1).$$

Proof. Without loss of generality we can assume that $F \in \mathcal{S}(\mathbb{R}^2;\mathbb{R}^2)$ The solution U has the form $U(x) = G_2 * F(x)$, where $G_2(x)$ is the Bessel potential of order 2. By Example 3.5.4, $G_2 \in d^1(\mathbb{R}^2)$. Thus for any $x \in \mathbb{R}^2$,

$$|U(x)| = |G_n * F(x)| \le \|G_2\|_{d^1}\|\tau_x F\|_{\Upsilon^1_1(\mathbb{R}^2)} = \|J_2\|_{d^1}\|F\|_{\Upsilon^1_1(\mathbb{R}^2)},$$

where τ_x is the translation operator defined by $(\tau_x f)(y) = f(y - x)$. In other words

$$\|U\|_\infty \le C(\|F\|_1 + \|divF\|_1). \tag{3.6.1}$$

In order to control ∇U notice that the decay of F and G_2 implies

$$\int |\nabla U_i(x)|^2 dx = -\int U_i(x)\Delta U_i(x)dx =$$

$$= \int U_i(x)F_i(x)dx - \int U_i^2(x)dx.$$

Hence, recalling that U is a convolution of the L^1 functions G_2 and F,

$$\|\nabla U\|_2 \leq C\|U\|_\infty^{1/2}(\|F\|_1 + \|U\|_1)^{1/2} \leq C\|U\|_\infty^{1/2}\|F\|_1^{1/2}.$$

Using (3.6.1) we complete the proof. □

3.7 L^1 forms on domains

Let Ω be a domain in $\mathbb{R}^n$. We start by defining local versions of the spaces L_k^1 and Υ_k^1 associated with Ω.

Definition 3.7.1. For $1 \leq k \leq n$, we define $L_k^1(\Omega)$ as subspace of $\mathcal{D}_k(\Omega)$ consisting of elements $\Phi = \sum_{|I|=k} \phi_I dx^I$ such that all components ϕ_I are $L^1(\Omega)$ functions. This space is equipped with the norm

$$\|\Phi\|_{L_k^1} = \sum_{|I|=k} \|\phi_I\|_{L^1(\Omega)}.$$

Definition 3.7.2. Let $1 \leq k \leq n-1$. We say that $\Phi \in \mathcal{D}_k(\Omega)$ belongs to $\Upsilon_k^1(\Omega)$, if $\Phi \in L_k^1(\Omega)$ and $d\Phi \in L_{k+1}^1(\Omega)$. We equip $\Upsilon_k^1(\Omega)$ with the norm

$$\|\Phi\|_{\Upsilon_k^1} = \|\Phi\|_{L_k^1} + \|d\Phi\|_{L_{k+1}^1}.$$

Finally, we define

$$\Upsilon_{k,0}^1(\Omega) = \overline{\mathcal{D}^k(\Omega)},$$

where the closure is taken with respect to the $\Upsilon_k^1(\Omega)$ norm.

Unlike in the case of $\mathbb{R}^n$, $\mathcal{D}^k(\Omega)$ are not dense in $\Upsilon_k^1(\Omega)$ (hence the definition of $\Upsilon_{k,0}^1$). In order to define the appropriate density result, we need the following definitions.

Definition 3.7.3. Let Ω be a bounded domain in $\mathbb{R}^n$. By $\mathcal{D}^k(\overline{\Omega})$, $1 \leq k \leq n$, we denote the space of restrictions of $\mathcal{D}^k(\mathbb{R}^n)$ differential forms to Ω.

Definition 3.7.4. Let Ω be a bounded domain. We say that its boundary $\partial\Omega$ satisfies the segment condition if for any $z \in \partial\Omega$ there exists $R_z > 0$, an orthogonal transformation $A \in O(n)$ and a continuous function F defined in a neighborhood of the hyper-plane $\mathbb{R}^{n-1}$ such that

$$\Omega \cap B_{R_z}(z) = \{Ay : y = (y', y_n) \in \mathbb{R}^n, |y - A^{-1}(z)| < R_z,\ y_n > F(y')\}. \tag{3.7.1}$$

If the function F can be chosen as a Lipschitz function, the domain Ω is called a (strongly) Lipschitz domain.

Proposition 3.7.5. *Let $1 \leq k \leq n$.*

- *If Ω is any open subset of $\mathbb{R}^n$, then $\Upsilon_k^1(\Omega) \cap \mathcal{E}^k(\Omega)$ is dense in $\Upsilon_k^1(\Omega)$.*
- *If Ω is a bounded domain with $\partial\Omega$ satisfying the segment condition, then $\mathcal{D}^k(\bar{\Omega})$ is dense in $\Upsilon_k^1(\Omega)$.*

Proof. Both of these facts are well known for Sobolev spaces (see e.g. [1] or [44]). In particular, the first part of the proposition is an analogue of the famous Meyers-Serrin theorem [51], and can be proved in exactly the same way. We will show the proof of the second part following the argument in [67] for Sobolev spaces.

Since Ω is bounded, $\partial\Omega$ is compact and we can find a finite number, say $m < \infty$, of $z_j \in \partial\Omega$, $R_j > 0$, $A_j \in O(n)$ and $F_j \in C(\mathbb{R}^{n-1})$ satisfying (3.7.1) with $\partial\Omega \subset \bigcup_{j=1}^m B_{R_j}(z_j)$. Since the number of balls B_{R_j} is finite, there exists $\delta > 0$ such that

$$\bar{\Omega} \subset \bigcup_j B_{R_j}(z_j) \cup \{x \in \Omega | dist(x, \partial\Omega) > \delta\}. \tag{3.7.2}$$

Let $\theta_1, \ldots, \theta_m, \eta$ be the partition of unity associated to the covering (3.7.2) of $\bar{\Omega}$, i.e. $\theta_j \in \mathcal{D}(B_{R_j}(z_j))$, $j = 1, \ldots m$, $\eta \in \mathcal{D}(\{x \in \Omega | dist(x, \partial\Omega) > \delta\})$ and

$$\sum_{j=1}^m \theta_j(x) + \eta(x) = 1 \text{ for any } x \in \bar{\Omega}.$$

Let $\Phi \in \Upsilon_k^1(\Omega)$ be arbitrary. We need to show that for any small $\epsilon > 0$ there exist $\Phi_\epsilon \in \mathcal{D}^k(\bar{\Omega})$ such that

$$\|\Phi - \Phi_\epsilon\|_{\Upsilon_k^1} \to 0, \text{ as } \epsilon \to 0.$$

By the first part of the theorem we can assume without loss of generality that $\Phi \in \mathcal{E}^k(\Omega) \cap \Upsilon^1_k(\Omega)$.

Let us decompose Φ using the partition of unity we introduced

$$\Phi = \sum_{j=1}^{m} \theta_j \Phi + \eta(x)\Phi = \sum_{j=1}^{m} \Phi^j + \Phi^0.$$

Each $\Phi^j \in \Upsilon^1_k(\Omega)$. Moreover Φ^0, being supported in Ω, is an element of $\mathcal{D}^k(\Omega)$. Therefore we need to approximate each Φ^j, $j \in [1, m]$ by forms $\Phi^j_\epsilon \in \mathcal{D}^k(\bar{\Omega}_j)$. Once it is done, the form $\Phi_\epsilon = \Phi^0 + \sum\limits_{j=1}^{m} \Phi^\epsilon \in \mathcal{D}(\bar{\Omega})$ will be the approximation of Φ with

$$\|\Phi - \Phi_\epsilon\|_{\Upsilon^1_k} \to 0 \text{ as } \epsilon \to 0.$$

We make use of the segment condition. Fix $j \in [1, m]$, put $\Omega_j = \Omega \cap B_{R_j}$ and

$$\Omega'_j = A_j^{-1}(\Omega_j) = \{y = (y', y_n) : |y - A^{-1}z_j| < R_j, y_n > F(y')\}.$$

Let A^*_j be the pull back associated with A_j, i.e. $A^*_j : \Upsilon^1_k(\Omega_j) \to \Upsilon^1_k(\Omega'_j)$ is a linear map defined by the action $A^*_j\Phi(y) = \sum\limits_{|I|=k} \phi_I(Ay)dy^I$, for any $\Phi(x) = \sum\limits_{|I|=k} \phi_I(x)dx^I \in \Upsilon^1_k(\Omega_j)$. It is known that the pull-back commutes with exterior differentiation and with the wedge product, and preserves the norm $\|A^*_j\Phi\|_{\Upsilon^1_k(\Omega'_j)} = |\mathrm{detA_j}|\|\Phi\|_{\Upsilon^1_\mathrm{k}(\Omega_\mathrm{j})} = \|\Phi\|_{\Upsilon^1_\mathrm{k}(\Omega_\mathrm{j})}$ (see e.g. [61], pp 22-23).

So if for arbitrarily small $\epsilon > 0$ and each $\Psi^j = A^*_j\Phi^j$ we can find $\Psi^j_\epsilon \in \mathcal{D}^k(\bar{\Omega}'_j)$ such that

$$\|\Psi^j - \Psi^j_\epsilon\|_{\Upsilon^1_k(\Omega'_j)} \to 0 \text{ as } \epsilon \to 0,$$

then $(A^{-1})^*\Psi^j_\epsilon \in \mathcal{D}^k(\bar{\Omega}_j)$ will be an approximation of Φ^j:

$$\|\Phi^j - (A^{-1})^*\Psi^j_\epsilon\|_{\Upsilon^1_k(\Omega_j)} = \|(A^{-1})^*\Psi^j - (A^{-1})^*\Psi^j_\epsilon\|_{\Upsilon^1_k(\Omega_j)} \to 0 \text{ as } \epsilon \to 0.$$

Notice that Ψ^j vanishes on $\{y \in B_{R_j}(A^{-1}z_j) : y_n > F(y')\}$ and we can extend Ψ^j by 0 to an element of $\Upsilon^1_k(\{y = (y', y_n) : y_n > F(y')\})$. Finally, for sufficiently small $\epsilon > 0$, we put

$$\Psi^j_\epsilon(y', y_n) = \Psi^j(y', y_n + \epsilon).$$

Such Ψ^j_ϵ is an element of $\mathcal{D}^k(\bar{\Omega}'_j)$ and as $\epsilon \to 0$

$$\|\Psi^j - \Psi^j_\epsilon\|_{L^1_k(\Omega'_j)} \to 0$$

and

$$\|d\Psi^j - d\Psi^j_\epsilon\|_{L^1_{k+1}(\Omega'_j)} \to 0,$$

due to the facts that d commutes with translations and $\|f(\cdot + y) - f(\cdot)\|_{L^1} \to 0$ as $|y| \to 0$ for any $f \in L^1(\mathbb{R}^n)$. □

3.8 d^k classes on Lipschitz domains Ω

In this section we define d^k classes on domains. Everywhere in this section we assume Ω to be a bounded Lipschitz domain in $\mathbb{R}^n$.

Definition 3.8.1. Let $1 \le k \le n$. A distribution $u \in \mathcal{D}'(\Omega)$ is said to belong to $d^k(\Omega)$ if there exists $C > 0$ such that $|u(\phi_I)| \le C\|\Phi\|_{\Upsilon^1_k(\Omega)}$ for any

$$\Phi = \sum_{|I|=k} \phi_I dx^I \in \mathcal{D}^k(\Omega).$$

We denote the space of such distributions by $d^k(\Omega)$ and equip it with the norm

$$\|u\|_{d^k(\Omega)} := \sup\{|u(\phi_I)| : \Phi \in \mathcal{D}^k(\Omega); \|\Phi\|_{\Upsilon^1_k(\Omega)} \le 1\}.$$

Remark 3.8.2. Let $1 \le k \le n$. We want to consider distributions $u \in \mathcal{E}'(\Omega)$ such that $|u(\phi_I)| \le C\|\Phi\|_{\Upsilon^1_k(\Omega)}$ for some finite $C > 0$ and any

$$\Phi = \sum_{|I|=k} \phi_I dx^I \in \mathcal{D}^k(\mathbb{R}^n).$$

The class of $\mathcal{E}'(\Omega) \cap d^k(\mathbb{R}^n)$, equipped with the norm $\|\cdot\|_{d^k(\mathbb{R}^n)}$ forms an incomplete normed space. Therefore we define $d^k_z(\Omega)$ as follows.

Remark 3.8.3. The definitions we use were suggested by Van Schaftingen in [77]. It is also possible to define $d^k(\Omega)$ as we did in Remark 3.2.3. Any $u \in \mathcal{D}'(\Omega)$ defines a linear map $\tilde{u} : \mathcal{D}^k(\Omega) \to \mathbb{R}^{\binom{n}{k}}$ by

$$\tilde{u}\left(\sum_{|I|=k} \phi_I dx^I\right) = (u(\phi_I))_I.$$

By Proposition 3.7.5, $u \in d^k(\Omega)$ if and only if $\tilde{u}$ can be extended to a bounded linear map from $\Upsilon^1_{k,0}(\Omega)$ to $(\mathbb{R}^{\binom{n}{k}}, \|\cdot\|_{max})$.

3.8.1 $d_z^k(\Omega)$ spaces

All properties of $d_z^k(\Omega)$ spaces can be deduced from the previous results and the following definition

Definition 3.8.4. Let $1 \le k \le n$. Then

$$d_z^k(\Omega) = \{u \in d^k(\mathbb{R}^n) : supp\ u \in \overline{\Omega}\}.$$

Remark 3.8.5. It is clear that $d_z^k(\Omega)$ is a closed subspace of $d^k(\mathbb{R}^n)$, hence complete, and $\mathcal{E}'(\Omega) \cap d^k(\mathbb{R}^n) \subset d_z^k(\Omega)$. Conversely, any $u \in d_z^k$ is the weak limit of $\mathcal{E}'(\Omega) \cap d^k(\mathbb{R}^n)$. Indeed, consider any $u \in d^k(\mathbb{R}^n)$ supported in $\bar{\Omega}$. By Theorem 3.2.6 and the definition of $bmo_z(\bar{\Omega})$, $u \in bmo_z(\Omega)$. In particular $u \in L^1(\Omega)$. Let η_j be a sequence of $\mathcal{D}(\Omega)$ functions such that $\lim_{j\to\infty} \eta_j = \chi_\Omega$, the characteristic function of Ω. Then by Lebesgue's dominated convergence theorem, for any $\Phi \in \mathcal{D}^k(\bar{\Omega})$ and I,

$$\int_\Omega u(x)\phi_I(x)dx = \lim_{j\to\infty} \int_\Omega (\eta_j u)(x)\phi_I(x)dx.$$

This shows that $u = \lim_{j\to\infty}(\eta_j u)$ is a weak limit.

Combining this definition with Lemma 3.2.4 we obtain

Proposition 3.8.6. *The spaces $d_z^k(\Omega)$ form a monotone family, i.e. the following embeddings hold*

$$d_z^n(\Omega) \subset d_z^{n-1}(\Omega) \subset \cdots \subset d_z^1(\Omega).$$

Proposition 3.8.7. *Let Ω be a bounded Lipschitz domain and $W_0^{1,n}(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ functions in the norm $\|\cdot\|_{W^{1,n}(\Omega)}$. Then $W_0^{1,n}(\Omega)$ is continuously embedded into $d_z^{n-1}(\Omega)$.*

Proof. The space $W_0^{1,n}(\Omega)$ can be characterized (see e.g. Theorem 5.29 in [1]) as follows: let $f \in W^{1,n}(\Omega)$, then $f \in W_0^{1,n}(\Omega)$ if and only if the extension of f by zero to $\mathbb{R}^n \setminus \bar{\Omega}$ belongs to $W^{1,n}(\mathbb{R}^n)$. Using this characterization, we can identify any $u \in W_0^{1,n}(\Omega)$ with $\tilde{u} \in W^{1,n}(\mathbb{R}^n)$ supported in $\bar{\Omega}$. By Van Schaftingen's theorem such $\tilde{u}$ is an element of $d^{n-1}(\mathbb{R}^n)$ and is supported in $\bar{\Omega}$. Therefore by the last proposition $\tilde{u} \in d_z^{n-1}(\Omega)$. □

Proposition 3.8.8. *The space $d_z^1(\Omega)$ is a proper subspace of $bmo_z(\Omega)$.*

Proof. It follows immediately from Theorem 3.2.6, Proposition 3.8.4 and the definition of $bmo_z(\Omega)$. □

All in all, we can see that the spaces $d_z^k(\Omega)$ form a family of intermediate spaces between $W_0^{1,n}(\Omega)$ and $bmo_z(\Omega)$.

3.8.2 $d^k(\Omega)$ spaces

It follows directly from the definitions of $d^k(\mathbb{R}^n)$ and $d^k(\Omega)$, that $u \to u|_\Omega$ maps $d^k(\mathbb{R}^n)$ to $d^k(\Omega)$ and

$$\|u|_\Omega\|_{d^k(\Omega)} \leq \|u\|_{d^k(\mathbb{R}^n)}, \tag{3.8.1}$$

where $u|_\Omega$ stands for the restriction of u to Ω.

Repeating verbatim the proof of Proposition 3.2.4, one obtains

Proposition 3.8.9. *Let $1 \leq k < l \leq n$ and $u \in d^l(\Omega)$. Then $u \in d^k(\Omega)$ and $\|u\|_{d^k(\Omega)} \leq \|u\|_{d^l(\Omega)}$. In other words*

$$d^n(\Omega) \subset d^{n-1}(\Omega) \subset \cdots \subset d^1(\Omega).$$

In order to show that $W^{1,n}(\Omega) \subset d^{n-1}(\Omega)$, we recall the extension property of Sobolev spaces. It is well-known (see e.g. Theorem 5.24 in [1]) that if Ω is a Lipschitz domain then there exists a bounded linear operator $E : W^{l,p}(\Omega) \to W^{l,p}(\mathbb{R}^n)$ such that $Eu = u$ almost everywhere in Ω for all $u \in W^{l,p}(\Omega)$. If we consider such an extension E on $W^{1,n}(\Omega)$ and recall (3.8.1) and Theorem 3.2.5, then

$$\|u\|_{d^{n-1}(\Omega)} = \|Eu|_\Omega\|_{d^{n-1}(\Omega)} \leq \|Eu\|_{d^{n-1}(\mathbb{R}^n)} \leq$$

$$\leq \|Eu\|_{W^{1,n}(\mathbb{R}^n)} \leq \|E\|\|u\|_{W^{1,n}(\Omega)}.$$

In other words,

Proposition 3.8.10. *If Ω is a bounded Lipschitz domain, then $W^{1,n}(\Omega)$ is continuously embedded into $d^{n-1}(\Omega)$.*

The following result is the analogue of Theorem 3.2.6 on Lipschitz domains.

Theorem 3.8.11. *Any $u \in d^1(\Omega)$ is a $bmo_r(\Omega)$ function and*

$$\|u\|_{bmo_r(\Omega)} \leq C\|u\|_{d^1(\Omega)}.$$

The proof is more technical than the one of Theorem 3.2.6 because of the presence of $\partial\Omega$. Firstly, we state a corollary of the Nečas inequality:

$$\|f\|_{L^2(\Omega)} \leq C(\|f\|_{W^{-1,2}(\Omega)} + \|\nabla f\|_{W^{-1,2}(\Omega)}) \forall f \in L^2(\Omega).$$

Lemma 3.8.12 ([5], Lemma 10)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. If $g \in L^2(\Omega)$ and $\int g = 0$, then there exists a vector-valued function $F \in W_0^{1,p}(\Omega, \mathbb{R}^n)$ such that*

$$\begin{cases} \mathrm{divF} = \mathrm{g}, \ \textit{in } \Omega \\ \|DF\|_{L^2} \leq C\|g\|_2. \end{cases}$$

Here DF is a matrix $\partial_j F_i$ and $C > 0$ depends only on the Lipschitz constant of Ω.

Using this lemma we prove the following

Lemma 3.8.13. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. If $g \in H^1_z(\Omega)$, then there exists a vector-valued function $F \in W_0^{1,1}(\Omega, \mathbb{R}^n)$ such that*

$$\begin{cases} \mathrm{divF} = \mathrm{g}, \ \textit{in } \Omega \\ \|DF\|_{L^1} \leq C\|g\|_{H^1}. \end{cases}$$

Proof. Let $g \in H^1_z(\Omega)$. Then by Theorem 2.5.5, it can be decomposed into $H^1_z(\Omega)$ atoms $a_i \in L^2(\mathbb{R}^n)$ as

$$g = \sum_{i=1}^{\infty} \lambda_i a_i$$

and

$$\sum_{i=1}^{\infty} |\lambda_i| \leq 2\|g\|_{H^1}.$$

For each $i \geq 1$, by means of Lemma 3.8.12, we can find $V^i \in W_0^{1,2}(Q_i, \mathbb{R}^n)$, such that

$$\begin{cases} \mathrm{divV^i = a_i \ in \ Q_i} \\ \|DV^i\|_{L^2} \leq C\|a_i\|_{L^2}. \end{cases}$$

As $W_0^{1,2}(B_i)$ fields, V^i can be continuously extended by 0 to $W^{1,2}(\Omega)$. We denote these extensions by the same V^i. We claim that $F = \sum_{i=1}^{\infty} \lambda_i V^i$ is the solution we seek.

Indeed, since a_i are atoms, we have

$$\|DV^i\|_{L^1} \le |Q_i|^{1/2}\|DV^i\|_{L^2} \le C|Q_i|^{1/2}\|a_i\|_{L^2} \le C_1 \text{ for all } i \ge 1.$$

Therefore, the partial sums $\sum_{i=1}^{N} \lambda_i DV^i$, supported in Ω, converge to an element of $W_0^{1,1}(\Omega, \mathbb{R}^{n\times n})$ and

$$\|DF\|_{L^1} \le C_1 \sum_i |\lambda_i| \le C\|g\|_{H^1}.$$

Finally, by the construction of F,

$$\mathrm{div}F = \sum_i \lambda_i \cdot \mathrm{div}V^i = \sum_i \lambda_i a_i = g.$$

□

Now we can prove the last theorem of this section

Proof of Theorem 3.8.11. We will use the duality between $h_z^1(\Omega)$ and $bmo_r(\Omega)$ asserted by Theorem 2.5.12. By Lemma 2.5.8, it is enough to show that for any $f \in \mathcal{D}(\Omega)$ and $u \in d^1(\Omega)$

$$|u(f)| \le C\|u\|_{d^1}\|f\|_{h^1}. \tag{3.8.2}$$

Given $f \in \mathcal{D}(\Omega)$, we write f as the sum $f = g + \theta$, where

$$g = f - \int f(x)dx \cdot \psi,$$

$$\theta = \int f(x)dx \cdot \psi,$$

where $\psi \in \mathcal{D}(\Omega)$ is any function with $\int \psi = 1$.

Note that $\theta \in \mathcal{D}(\Omega)$ with $\|\theta\|_{h^1} \le \|\psi\|_{L^\infty}\|f\|_{h^1}$ and $\|\theta\|_{W^{1,1}} \le \|f\|_{h^1}\|\psi\|_{W^{1,1}}$. Moreover if we define $\Theta = \sum_{i=1}^{n} \theta dx^i \in \mathcal{D}^1(\Omega)$, then $\|\Theta\|_{\Upsilon_1^1(\Omega)} \le C\|\psi\|_{W^{1,1}}\|f\|_{h^1}$. Therefore

$$|u(\theta)| \le \|u\|_{d^1(\Omega)}\|\Theta\|_{\Upsilon_1^1(\Omega)} \le C_\psi\|u\|_{d^1(\Omega)}\|f\|_{h^1}. \tag{3.8.3}$$

On the other hand, for $g \in \mathcal{D}(\Omega)$, we recall Lemma 2.4.3 to see that $g \in H_z^1(\Omega)$ and

$$\|g\|_{H^1} \le C_\Omega\|g\|_{h^1} \le C'_\psi\|f\|_{h^1}. \tag{3.8.4}$$

Hence, Lemma 3.8.13 is applicable and there exists $F \in W_0^{1,1}(\Omega;\mathbb{R}^n)$ such that

$$\begin{cases} \mathrm{divF} = \mathrm{g}, \text{ in } \Omega \\ \|DF\|_{L^1(\Omega;\mathbb{R}^{n\times n})} \le C\|g\|_{H^1}. \end{cases}$$

Using this F, we introduce n differential forms

$$\Phi^j = \sum_{i=1}^n \partial_i F_j dx^i$$

and claim that all $\Phi^j \in \Upsilon^1_{1,0}(\Omega)$ and $\|\Phi^j\|_{\Upsilon^1_1(\Omega)} \le C'_\psi \|f\|_{h^1}$. Assuming the claim and recalling that u is well defined on components of $\Upsilon^1_{1,0}(\Omega)$ forms (see Remark 3.8.3), one has

$$|u(g)| = |u(\sum_{i=1}^n \partial_i F_i)| \le \sum_{i,j=1}^n |u(\partial_i F_j)| \le \tag{3.8.5}$$

$$\le n\|u\|_{d^1(\Omega)} \max_{1\le j\le n} \|\Phi^j\|_{\Upsilon^1_1(\Omega)} \le C\|u\|_{d^1(\Omega)}\|f\|_{h^1}.$$

We complete the proof by deducing (3.8.2) from (3.8.3), (3.8.5) and the triangle inequality.

In order to prove the claim, we note that $d\Phi^j = 0$ by construction and all components of Φ^j are $L^1(\Omega)$ functions, bounded in the L^1-norm by a multiple of $\|g\|_{H^1}$. Recalling (3.8.4), we may conclude that

$$\|\Phi^j\|_{\Upsilon^1_1(\Omega)} = \|\Phi^j\|_{L^1_1(\Omega)} \le C\|f\|_{h^1}.$$

Furthermore, $F_j \in W_0^{1,1}(\Omega)$ for $j = 1, \ldots, n$, which means that there exist sequences $\{F_j^m\}_{m=1}^\infty \subset \mathcal{D}(\Omega)$ such that $\|\partial_i F_j^m - \partial_i F_j\|_{L^1(\Omega)} \to 0$, as $m \to \infty$. Hence, by forming closed $\mathcal{D}^1(\Omega)$-forms

$$\Phi^{j,m} = \sum_{i=1}^n \partial_i F_j^m dx^i,$$

we can construct $\mathcal{D}^1(\Omega)$ approximations of Φ^j, such that as $m \to \infty$,

$$\|\Phi^{j,m} - \Phi^j\|_{\Upsilon^1_1(\Omega)} = \|\Phi^{j,m} - \Phi^j\|_{L^1_1(\Omega)} \to 0,$$

which shows that $\Phi^j \in \Upsilon^1_{1,0}(\Omega)$ for $j = 1, \ldots, n$. □

Chapter 4

Functions on Riemannian manifolds with bounded geometry

4.1 Preliminaries

4.1.1 Riemannian manifolds

Let M be a connected smooth manifold, and TM and T^*M be its tangent and cotangent bundles respectively (see e.g. [42] for the basic theory of manifolds). The space of smooth maps $X : M \to TM$, we will denote by $\Gamma(TM)$.

A metric g on M is a smooth function $g : TM \times TM \to \mathbb{R}$ such that

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is symmetric bi-linear and positive definite for all $p \in M$. The manifold M equipped with a metric g is called a Riemannian manifold and is denoted by (M, g).

4.1.2 Connection and co-variant derivative

A connection ∇ is defined as a smooth map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ which is linear in both variables, i.e. for each $\alpha, \beta \in \mathbb{R}$ and $X, Y, Z \in \Gamma(TM)$

$$\nabla(\alpha X + \beta Z, Y) = \alpha\nabla(X, Y) + \beta\nabla(Z, Y),$$

$$\nabla(X, \alpha Y + \beta Z) = \alpha\nabla(X, Y) + \beta\nabla(X, Z)$$

and satisfying the following relations for any smooth $f : M \to \mathbb{R}$

$$\nabla(fX, Y) = f\nabla(X, Y),$$

$$\nabla(X, fY) = f\nabla(X, Y) + X(Df)Y.$$

For a fixed $X \in \Gamma(TM)$, the map $\nabla_X := \nabla(X, \cdot) : \Gamma(TM) \to \Gamma(TM)$ is called the (covariant) derivative in the direction X.

Among infinitely many connections on M, we will only be interested in the Levi-Civita connection ∇, which satisfies two additional conditions: it is consistent with the Riemannian metric

$$Zg(X, Y) = g(\nabla(Z, X)) + g(X, \nabla(Z, Y)), \forall X, Y, Z \in \Gamma(TM)$$

and is torsion-free in the sense

$$\nabla(X, Y) - \nabla(Y, X) = [X, Y],$$

where the commutator $[X, Y] \in \Gamma(TM)$ is defined as a vector field satisfying $[X, Y](\theta) = X(D(Y(\theta))) - Y(D(X(\theta)))$ for any $\theta \in T^*M$.

4.1.3 Geodesics and exponential maps

If $\gamma : (a, b) \subset \mathbb{R} \to M$ is smooth, then $\gamma'(t) \in T_{g(t)}M$ for each $t \in (a, b)$. We say that γ is a geodesic if $\nabla_{\gamma'(t)}\gamma'(t) = 0$ for all $t \in (a, b)$.

One of the basic facts about geodesics is the following result (the proof can be found on p. 65 in [25])

Proposition 4.1.1. *Let (M, g) be a Riemannian manifold. For every point $p \in M$, there exists a neighborhood of p, $O_p \subset M$, $\epsilon > 0$,*

$$U_p = \{(q, X) : q \in O_p, X \in T_qM, g(X, X) < \epsilon^2\}$$

and a smooth map $\gamma_{q,X}(t) : U_p \times (-2, 2) \to M$ such that for every fixed $(q, X) \in U_p$, $\gamma_{q,X}$ is the unique geodesic satisfying $\gamma_{q,X}(0) = q$ and $\gamma'_{q,X}(0) = X$.

The uniqueness of $\gamma_{q,X}$ in the proposition should be understood as follows. If there exists $\tilde{\gamma} : (a,b) \to M$, $a < 0 < b$, such that $\tilde{\gamma}(0) = q$ and $\tilde{\gamma}'(0) = X$, then $\tilde{\gamma} = \gamma_{q,X}$ on $(a,b) \cap (-2,2)$.

This allows us to introduce the exponential maps.

Definition 4.1.2. Let $p \in M$ and $B_\epsilon(0) = \{X \in T_pM : g(X,X) < \epsilon^2\}$ as in the last proposition, then the map $\exp_p : B_\epsilon(0) \to M$ defined by

$$\exp_p(X) = \gamma_{p,X}(1), \text{ for } X \in B_\epsilon(0)$$

is called the exponential map at p.

It follows from the inverse function theorem (see e.g. Proposition 2.9 in [25]) that for each $p \in M$, $\exp_p$ is a diffeomorphism of some neighborhood of the origin in T_pM onto its image in M, which contains $p \in M$ for $\exp_p(0) = p$. It shows that $\exp_p$ defines a coordinate chart on M. Such coordinates are called the normal geodesic coordinates of M.

Let us note an important property of homogeneity of geodesics: $\gamma : (-a,a) \to M$ is a geodesic with $\gamma(0) = p$ and $\gamma'(0) = X \in T_pM$, if and only if $\gamma_c(t) := \gamma(ct)$, $c > 0$ is a geodesic defined on $(-a/c, a/c)$ and $\gamma_c(0) = p$, $\gamma_c'(0) = cX$. This implies that if given $p \in M$, the domain of the map $\gamma_{q,X}(t)$ in the proposition above can be extended to $U_p \times (-2c, 2c)$ for some $c > 1$, then $\exp_p$ admits an extension to $B_{c\epsilon}(0) \subset T_pM$. The case $c = \infty$ is of special importance.

Definition 4.1.3. A Riemannian manifold (M,g) is called geodesically complete if the exponential map $\exp_p$ admits an extension to the entire tangent space T_pM or equivalently, if any geodesic γ with $\gamma(0) = p$ can be extended to a geodesic $\gamma : \mathbb{R} \to M$.

4.1.4 Riemannian manifold as a metric space

Given any smooth curve $\gamma : [a,b] \to M$, the derivative $\gamma'(t) \in T_{g(t)}M$ and one can define the length of γ by

$$|\gamma| = \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}dt.$$

The distance between any two points $p, q \in M$ then can be defined as

$$d(p,q) = \inf_{\gamma:\gamma(a)=p,\gamma(b)=q} |\gamma|.$$

It is known (see e.g. [25], p. 146) that $d(\cdot,\cdot)$ is a metric. The completeness of such metric space is characterized by the Hopf-Rinow theorem: (M,g) equipped with the above metric d is a complete metric space if and only if (M,g) is geodesically complete.

4.1.5 Integration on (M,g)

We recall that on an oriented smooth manifold M of dimension n, there exist nowhere vanishing n-differential forms, called volume elements. By fixing one of volume elements $dx^1 \wedge \cdots \wedge dx^n$, we define the integral of an n-differential form $\omega : M \to \Lambda^n(M)$ via an arbitrary covering of M by coordinate charts $\{(U_i,\phi_i)\}$ compatible with orientation and a subordinate partition of unity $\{\psi_i\}$

$$\int_M \omega = \sum_i \int_{\phi_i(U_i)} (\psi_i f_i) \circ \phi^{-1}(x) dx,$$

where f_i are representations of ω in the coordinates $\phi_i = (x^1,\ldots,x^n)$

$$\omega = f_i dx^1 \wedge \cdots \wedge dx^n.$$

On an oriented Riemannian manifold, there exists the canonical volume element is defined as a n-differential form

$$d\text{vol} = \sqrt{det(g_{i,j})}\, dx^1 \wedge \cdots \wedge dx^n$$

in any local coordinates $(x^1,\ldots,x^n)$ such that $(\partial_{x^1},\ldots,\partial_{x^n})$ is a positively oriented basis of the tangent space and $g_{i,j}$ is a matrix representation of g in these coordinate systems.

4.2 Riemannian manifolds with bounded geometry

Let (M,g) be a complete Riemannian manifold. Then $\exp_p$ is defined on T_pM and, as mentioned earlier, for sufficiently small $r_p > 0$, maps $B_{r_p}(0) \in T_pM$ diffeomorphically onto an open subset of M. Let us denote by $\text{inj}_M(p)$, the supremum of all such $r_p > 0$ and define the injectivity radius of M as

$$\text{inj}_M := \inf\{\text{inj}_M(p) : p \in M\}.$$

Definition 4.2.1. A Riemannian manifold (M, g) is called a manifold with bounded geometry if

1. M is complete and connected;

2. $\mathrm{inj}_M > 0$;

3. For every multi-index α, there exists $C_\alpha > 0$ such that $|D^\alpha g_{i,j}| \leq C_\alpha$ in the normal geodesic coordinates $(\Omega_p(r_p), \exp_p^{-1})$.

Examples of manifolds with bounded geometry include compact Riemannian manifold, $\mathbb{R}^n$ and $\mathbb{H}^n$ (see e.g. [26]).

4.2.1 Tame partition of unity

Let (M, g) be a Riemannian manifold with bounded geometry. For $\delta \in (0, \mathrm{inj}_M)$, we denote by $\Omega_\delta(p)$, the image $B_\delta(0)$ by the map $\exp_p$ which is called a geodesic ball with radius δ centered at p.

Proposition 4.2.2 ([72] p. 284)**.** *For sufficiently small $\delta > 0$ there exists a uniformly locally finite covering of M by a sequence of geodesic balls $\{\Omega_\delta(p_j)\}_{j\in\mathbb{Z}_+}$ and a corresponding smooth partition of unity $\{\psi_j\}_{j\in\mathbb{Z}_+}$ subordinate to $\{\Omega_\delta(p_j)\}_{j\in\mathbb{Z}_+}$.*

Such covering and partition of unity we will call following Taylor [68], a tame covering and a tame partition of unity.

4.3 $W^{s,p}(M)$, $h^1(M)$ and bmo(M)

Definition 4.3.1 ([72], Chapter 7)**.** Let (M, g) be a Riemannian manifold with bounded geometry and let $\{\psi_j\}$ be a tame partition of unity subordinate to a tame cover by geodesic balls $\{\Omega_\delta(p_j)\}$. The Sobolev space $W^{s,p}(M)$, $1 < p < \infty$, $s > 0$ is defined as

$$W^{s,p}(M) = \{f \in \mathcal{D}'(M) : \sum_{j\in\mathbb{Z}_+} \|\psi_j f \circ \exp_{p_j}\|^p_{W^{s,p}(\mathbb{R}^n)} < \infty\}$$

Taylor in [68], introduced versions of Hardy spaces and *bmo* on manifolds with bounded geometry. One way to define $h^1(M)$ is as follows:

Definition 4.3.2 ([68] Corollary 2.4)**.** Let $f \in \mathcal{D}'(M)$ and $\{\psi_j\}$ a tame partition of unity subordinate to a tame cover by geodesic balls $\{\Omega_\delta(p_j)\}$. We say that $f \in h^1(M)$ if $\sum_j \|(\psi_j f)\circ \exp_{p_j}\|_{h^1(\mathbb{R}^n)} < \infty$. We equip the space $h^1(M)$ with the norm

$$\|f\|_{h^1(M)} = \sum_j \|(\psi_j f) \circ \exp_{p_j}\|_{h^1(\mathbb{R}^n)}.$$

The space $bmo(M)$ is defined similarly

Definition 4.3.3 ([68] Corollary 3.4)**.** Let $f \in L^1_{loc}(M)$ and $\{\psi_j\}$ a tame partition of unity subordinate to a tame cover by geodesic balls $\{\Omega_\delta(p_j)\}$. We say that $f \in bmo(M)$ if $\sum_j \|(\psi_j f) \circ \exp_{p_j}\|_{bmo(\mathbb{R}^n)} < \infty$. We equip the space $bmo(M)$ with the norm

$$\|f\|_{bmo(M)} = \sum_j \|(\psi_j f) \circ \exp_{p_j}\|_{bmo(\mathbb{R}^n)}.$$

Remark 4.3.4. All these classes of functions have equivalent global definitions. However, for our purposes it is more convenient to use the introduced versions. We refer to [68], [4] and [72] for alternative definitions and the proofs of their equivalence.

4.4 $d^k(M)$ spaces and the embedding into $bmo(M)$

Definition 4.4.1. Let $\{\psi_j\}$ be a tame partition of unity subordinate to a tame cover by geodesic balls $\{\Omega_\delta(p_j)\}$. We say that $u \in \mathcal{D}'(M) \in d^k(M)$ if for each j, $(\psi_j u)\circ\exp_{p_j} \in d^k(\mathbb{R}^n)$ and

$$\|u\|_{d^k(M)} := \sum_j \|(\psi_j u) \circ \exp_{p_j}\|_{d^k(\mathbb{R}^n)} < \infty.$$

We complete this part with the result which immediately follows from the definitions of the spaces $W^{1,n}(M)$, $d^k(M)$, $bmo(M)$ and the results of Section 3.2: Lemma 3.2.4 and Theorems 3.2.5, 3.2.6.

Theorem 4.4.2. *Let M be the Riemannian manifold with bounded geometry. Then the following continuous embeddings are true*

$$W^{1,n}(M) \subset d^{n-1}(M) \subset \cdots \subset d^1(M) \subset bmo(M)$$

Part II

Delay parabolic equations

Chapter 5

Introduction

5.1 Non-spatial population models with time delay

The simplest mathematical model of population dynamics is the so-called Malthusian growth model

$$\frac{dx}{dt} = \lambda x,$$

where $x(t)$ is the population size at moment t and $\lambda > 0$ is the reproductive rate or difference between birth and death rate. This model can be used to model population for short periods but fails to predict the long-term future, because $x(t)$ grows exponentially fast as $t \to \infty$.

A more realistic population model is known as the Verhulst growth model is derived from the principle that the population change $\frac{dx}{dt}$ is proportional to the size of population x and the remaining resources $K - x$. In other words,

$$\frac{dx}{dt} = \lambda x \left(1 - \frac{x}{K}\right), \tag{5.1.1}$$

where $K > 0$ is the so-called carrying capacity of the population. The factor $1 - x/K$ on the right hand side of (5.1.1), which makes it different from the exponential model, can be viewed as a self-regulatory mechanism of the system or a feedback to the depleting resources. Given initial condition $x(0) = x_0 < K$, one can easily solve the equation to obtain

$$x(t) = \frac{x_0 K}{x_0 - (x_0 - K)e^{-\lambda t}}.$$

This function is monotonically increasing from x_0 to K.

However this monotonicity does not always agree with the long-term observations of some populations. Certain species showed the presence of regular cycles in the population profiles (see e.g. [34], [59]). One of the explanations of this phenomenon was suggested by a renown ecologists Hutchinson [38]. He suggested that the feedback of the system occurs with some time delay and suggested a modified equation, which is known as the Hutchinson equation

$$\frac{dx(t)}{dt} = \lambda x(t)\left(1 - \frac{x(t-\tau)}{K}\right),$$

where $\tau > 0$ is the time delay parameter. It is far more difficult to study the solutions of this equation than of (5.1.1). In order to illustrate this, we note that the change of variables $y(t) = -1 + x(t)/K$, $s = \tau t$ transforms the equation into

$$\frac{dy(s)}{ds} = -\lambda\tau y(s-1)\left(1 + y(s)\right), \tag{5.1.2}$$

which was studied by for the first time by Wright [80]. It was shown by Wright that the trivial solution $y(t) = 0$ is asymptotically stable if $\lambda\tau < \pi/2$ and unstable if $\lambda\tau > \pi/2$, which implies the corresponding results for the equilibrium state $x(t) = K$ of the Hutchinson's equation (in [38] this result is attributed to Lars Onsager, p 237 in [38]). However, the question whether $y = 0$ is a globally stable solution of (5.1.2), posed by Wright in 1955 remains open.

5.2 Spatial population models with time delay

The spatial population models are used to simulate the process of how the population disperse in space and grows in time.

5.2.1 Models without delay

The fundamental spatial model, based on the extension of the logistic equation (5.1.1) is known as the Fisher-KPP equation

$$\frac{\partial u(t,x)}{\partial t} - D\Delta_x u(t,x) = (\alpha - \beta u(t,x))u(t,x). \tag{5.2.1}$$

Here $u(t,x)$ is a population size at location x at time t, Δ_x is the Laplacian in variable x, D is a diffusion coefficient, and $\alpha > 0$ and $\beta > 0$ are the birth and death rates.

It was proposed for $x \in \mathbb{R}$ in 1937 by Fisher [29]. At the same time it was studied in two dimensional case and with a more general reaction term on the right by Kolmogorov, Petrovskiy and Piskunov (KPP) [41].

A linearized version of Fisher-KPP model can be obtain from (5.2.1) by choosing $\beta = 0$ (thus assuming the Malthusian growth)

$$\frac{\partial u(t,x)}{\partial t} - D\Delta_x u(t,x) = \alpha u(t,x). \tag{5.2.2}$$

This equation is sometimes called the Skellam model after J.G. Skellam who first proposed to consider it in [62].

A more general approach to the modelling of population dynamics in space and time is to consider a reaction-diffusion equation

$$\frac{\partial u(t,x)}{\partial t} - D\Delta_x u(t,x) = F(u(t,x)), \tag{5.2.3}$$

where F is the so-called growth function, which is assumed to be smooth and satisfy $F(0) = F(u_+) = 0$, for some $u_+ > 0$.

Clearly, a linear function F corresponds to (5.2.2), while the choice $F(u) = \alpha u - \beta u^2$ gives (5.2.1). Other examples often used in the biological models are the Ricker function $F(u) = Cue^{-ku}$ [58] and the Beverton-Holt function $F(u) = \frac{Cu}{1+ku}$ [9]. Another example of the growth function is $F(u) = \frac{3u^2}{1+2u^2} - u$, which is interesting for biologists for it exhibits the so-called strong Allee effect ($F(u) < 0$ for $u \in (0, 1/2)$ and $F(u) > 0$ for $u \in (1/2, 1)$). A detailed discussion of these models and their generalizations in the non-spatial setting can be found in [54].

5.2.2 Models with time delay

The incorporation of the time lag into (5.2.3) gives a more general model

$$\frac{\partial u(t,x)}{\partial t} - D\Delta_x u(t,x) = F(u(t,x), u(t-\tau)). \tag{5.2.4}$$

There have been numerous studies of this type of equations over the last 25 years. In [16], Britton considered the case $F(u,v) = u(1 + \alpha u + v)$. In particular, his equation

$$\frac{\partial u(t,x)}{\partial t} - D\Delta_x u(t,x) = u(t,x)(1 + \alpha u(t,x) + u(t-\tau,x))$$

generalizes the Hutchinston equation (corresponds to $D = 0$) and the Fisher-KPP equation (corresponds to $\tau = 0$).

The time-delay version of the Ricker function $F(u, v) = pve^{-\alpha v} - \beta u$ became known as the Nicholson function after the work [33], where it was shown that the model based on such growth function explains the experimental observations of blowflies by Nicholson [53] . The properties of spatial models with the Nicholson growth function were studied in [2], [31] [43],[49] and other works.

5.2.3 The structure of Part 2

The structure of the rest of this part is as follows. In Section 6 we discuss the stability of travelling wave solutions of non-linear equation and the corresponding linearized problem. We formulate our main results regarding some linear parabolic equations with time delay, which completely characterizes the stability region for these type of equations. In Section 7 we prove these results.

Chapter 6

Main results

6.1 Travelling wave solutions and their stability

Following [47] and [50], we consider the initial value problem

$$\frac{\partial u(t,x)}{\partial t} - D\Delta_x u(t,x) = b(u(t-\tau,x)) - d(u(t,x)) \tag{6.1.1}$$

$$u(t,x) = u_0(t,x), \text{ where } (t,x) \in [-\tau,0]\times\mathbb{R}^n,$$

where b, d are non-linear functions that we will call the birth and death rate functions. We assume the functions satisfy the following conditions

(H.1) $b(0) = d(0) = 0$ and there exists $u_+ > 0$ such that $b, d \in C^2[0, u_+]$ are non-decreasing functions and $b(u_+) = d(u_+)$

(H.2) $b'(0) > d'(0) \geq 0$, $0 \leq b'(u_+) < d'(u_+)$

(H.3) For $u \in (0, u_+)$, $d''(u) \geq 0$ and $b''(u) \leq 0$

This model is quite general and includes the Fisher-KPP model with time delay and the spatial model of Nicholson equation as special cases.

The conditions imply that the equation

$$\frac{du(t)}{dt} = b(u(t-\tau)) - d(u(t))$$

has two equilibrium solutions $u = u_- := 0$ and $u = u_+$, the latter of which is stable while the former is not. The last condition prevents the existence of other equilibrium points between 0 and u_+.

One the fundamental question in the theory of nonlinear PDE is the existence and stability of the travelling wave solution between these two equilibrium states.

Definition 6.1.1. A special solution of (6.1.1) of the form $u(t,x) = \phi(x \cdot \theta + ct)$ with $\theta \in \mathbb{R}^n : |\theta| = 1$ and $\phi(s) \to u_\pm$ as $s \to \pm\infty$ is called a travelling wave solution with velocity c .

It is possible to show (see Proposition 1.1 in [50]) that for any $\tau > 0$ there exists $c_* > 0$ such that there exist travelling waves with any velocity $c \geq c_*$ and there is no travelling wave with velocity $c < c_*$.

The progress in the study of the stability of travelling wave solutions can be summarized as follows. For sufficiently large $c > c_*$ the local stability of travelling wave solutions was shown in [49] (see also [48] for a more general result). The global stability for all $c > c_*$ was obtained later in [45] (see also [46] for a more general version). The global stability of the travelling waves including the critical case $c = c_*$ was established in [47] and [50].

It was shown in Section 3 of [50] that using the conditions imposed on the birth and death functions, the question of stability of travelling wave solutions can be reduced to the study of the linear parabolic equation with time delay

$$\partial_t u(t,x) - \Delta_x u(t,x) + w \cdot \nabla_x u + \alpha u(t,x) = \beta u(t-\tau,x),\ t > 0,\ x \in \mathbb{R}^n, \qquad (6.1.2)$$

where $\alpha > 0, \beta > 0$ are specific contants depending on the wave speed c.

6.2 Parabolic equations with time delay

The equation (5.2.4) with $F(u,v) = \alpha u - \beta v$, where α and β are called birth and mortality rates, generalizes the Skellam model (5.2.2). The asymptotic behavior of their solutions was completely understood by Travis and Webb in [70]. More precisely, Travis and Webb, in their famous work [70] (see also [81]), considered the question of stability of the following

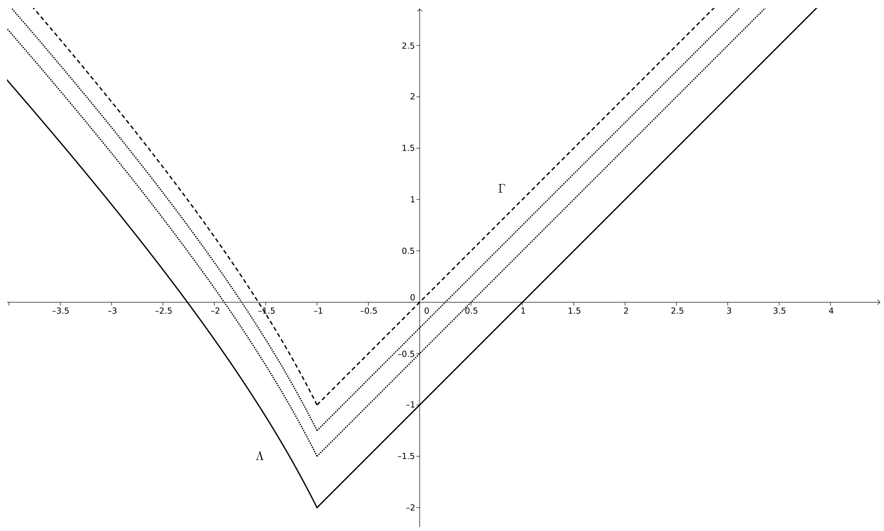

Figure 6.1: Stability regions for equation (6.2.1) with $\tau = 1$ on intervals of length $\pi, 2\pi, 4\pi$

boundary value problem:

$$(\partial_t - \Delta_x + \alpha I)u(t,x) - \beta u(t-\tau,x) = 0,\ t>0,\ x \in [0,\pi], \tag{6.2.1}$$

$$u(t,0) = u(t,\pi) = 0, t > 0$$

$$u(t,x) = \phi(t,x) \in C([-\tau,0], L^2[0,\pi]).$$

It was shown that the exact stability region is an open set bounded below by the solid graph Λ shown in Figure 6.1. More precisely, if $(\tau\beta, \tau\alpha)$ lies strictly above the curve, then there exist $K, \epsilon > 0$ such that $|u(t,x)| \leq K\|\phi\|e^{-\epsilon t}$, uniformly in $x \in \mathbb{R}^n$. Otherwise, there exists a solution $u(t,x)$ and x_0 such that

$$\limsup_{t\to\infty} |u(t,x_0)| > 0.$$

It is not difficult to see that Travis-Webb's reasoning implies that by amplifying the period from π to $R\pi$, the stability region shifts upward by $1/R$ (dotted curves on Figure 7.3.1). So it is natural to expect that the epigraph of the limiting curve (the dashed curve Γ on Figure 6.1) should be the stability region of the equation on the real line.

The stability problem of travelling wave described above suggests to consider a more general setting and include first order derivatives in equation and consider the following initial value problem:

$$(\partial_t - \Delta_x + w \cdot \nabla_x + \alpha I)u(t,x) - \beta u(t-\tau,x) = 0, \tag{6.2.2}$$

$$u(t,x) = \phi(t,x) \in C([-\tau,0], L^1(\mathbb{R}^n)), \tag{6.2.3}$$

where $\alpha, \beta \in \mathbb{R}$, $\tau > 0$, $w \in \mathbb{R}^n$, $u(t,x) \in C(\mathbb{R}, L^1(\mathbb{R}^n))$.

It turns out that the following theorems are true for the limiting graph Γ, which will be precisely defined by below in Definition 7.1.1.

Theorem 6.2.1 (A)**.** *Let $\tau > 0$ and $\alpha, \beta \in \mathbb{R}$ be such that $\Gamma(\tau\beta) \leq \alpha\tau$ and $\alpha\tau > -1$. Then there exists $K > 0$ such that, for any $u(x,t) \in C([-\tau,\infty), L^1(\mathbb{R}^n))$ satisfying (6.2.2) with $w = 0$ and (6.2.3),*

$$|u(x,t)| \leq K\|\phi\| t^{-n/2}.$$

Moreover, if $\tau\alpha > \Gamma(\tau\beta)$, then there exists $\epsilon > 0$ such that

$$|u(x,t)| \leq K\|\phi\| e^{-\epsilon t}.$$

Theorem 6.2.2 (B)**.** *Let $\tau > 0$, $w \in \mathbb{R}^n \setminus \{0\}$ $\alpha \in \mathbb{R}$, and $\beta > 0$ be such that $\Gamma(\tau\beta) \leq \alpha\tau$. Then there exists $K > 0$ such that, for any $u(x,t) \in C([-\tau,\infty), L^1(\mathbb{R}^n))$ satisfying (6.2.2) and (6.2.3),*

$$|u(x,t)| \leq K\|\phi\| t^{-n/2}.$$

Moreover, if $\tau\alpha > \Gamma(\tau\beta)$, then there exists $\epsilon > 0$ such that

$$|u(x,t)| \leq K\|\phi\| e^{-\epsilon t}.$$

Theorem 6.2.3 (C)**.** *Given $\tau > 0$, $\beta < 0$, $w \in \mathbb{R}^n \setminus \{0\}$, there exists $\alpha \in \mathbb{R}$ such that $\tau\alpha > \Gamma(\tau\beta)$ and there exists $u(x,t) \in C([-\tau,\infty), L^1(\mathbb{R}^n))$ satisfying (6.2.2) such that*

$$\limsup_{t\to\infty} |u(0,t)| = \infty.$$

So if w in (6.2.2) equals zero, then indeed, every solution inside the stability region decays exponentially as $t \to \infty$. Moreover, for critical α, β, all solutions decay at least algebraically as $t \to \infty$. The same is true for the general $w \in \mathbb{R}^n$ when $\beta > 0$. However, in the left half plane, the boundary of the stability region lies strictly above the graph Γ.

Chapter 7

The Proofs of the main theorems

7.1 Preliminaries

We start with the following simple delay ODE

$$v'(t) + av(t) - be^{i\theta}v(t-1) = 0, \tag{7.1.1}$$

where $a, b, \theta \in \mathbb{R}$ are some constants and $v(t)$ is a continuous function specified on $t \in [-1, 0]$.

Definition 7.1.1. For $x \in \mathbb{R}$ and $\theta \in (0, \pi)$, we define

$$\Gamma(x, \theta) = F(t(x, \theta), \theta) = (\theta - t)\cot(t),$$

where $t(x, \theta) \in (0, \pi)$ is implicitly defined by $x = \frac{\theta - t}{\sin(t)}$.

It is not difficult to check that for each $x \in \mathbb{R}$ the following limits exist

$$\lim_{\theta \to 0+} \Gamma(x, \theta), \ \lim_{\theta \to \pi-} \Gamma(x, \theta).$$

We extend the definition $\Gamma(x, \theta)$ for $\theta \in [0, \pi]$, by continuity and for $[\pi, 2\pi]$, by putting $\Gamma(x, \pi + \omega) = \Gamma(x, \pi - \omega)$, $\omega \in (0, \pi)$. This makes $\Gamma(x, \cdot)$ a continuous function on $[0, 2\pi]$, which we can further extend periodically to $\mathbb{R}$. The graphs of $\Gamma(x, \theta)$ are shown in Figure 7.1.

In order to simplify notation we put

$$\Gamma(x) := \Gamma(x, 0) = \lim_{\theta \to 0+} \Gamma(x, \theta).$$

The following theorem shows how the function $\Gamma(x, \theta)$ plays the role of the stability criteria for equation (7.1.1).

Theorem 7.1.2 (Hayes-Sakata)**.** *Let $a, b, \theta \in \mathbb{R}$.*

If $a > \Gamma(b, \theta)$ then there exists $K(a, b, \theta), \epsilon(a, b, \theta) > 0$ such that

$$|v(t)| \leq K\|v\|_{C[-1,0]}e^{-\epsilon t}, \tag{7.1.2}$$

for any continuous solution of (7.1.1).

If $a \leq \Gamma(b, \theta)$, then there exists a solution of (7.1.1), $v(t)$, such that

$$\limsup_{t\to\infty} |v(t)| > 0.$$

In the case $\theta = 0$ this result was proved by Hayes in [36], and for arbitrary θ it was considered by Sakata in [60].

How is it relevant to our initial value problem (6.2.2)? Let $u(t, x)$ be a solution of (6.2.2). Then applying the Fourier transform in x, we obtain a delay ODE

$$\left(\frac{d}{dt} + (|\xi|^2 + iw \cdot \xi + \alpha)I\right)\hat{u}(\xi, t) - \beta\hat{u}(\xi, t - \tau) = 0. \tag{7.1.3}$$

Changing variables by putting $\gamma = \frac{w\cdot\xi}{|\xi|}$, $r = |\xi|$, and $u_r(t) = \hat{u}(t, \xi)$, we have

$$u_r'(t) + (\alpha + r^2 + i\gamma r)u_r(t) - \beta u_r(t - \tau) = 0,\ t > 0. \tag{7.1.4}$$

Lemma 7.1.3. *Let $\alpha, \beta, \gamma \in \mathbb{R}$, $\tau, r \geq 0$.*

If $u_r(t)$ is a solution of (7.1.4) then

$$u_r(t) = e^{-i\gamma tr}v_r(t/\tau), \tag{7.1.5}$$

where

$$v_r'(s) + (a + \tau r^2)v_r(s) - be^{icr}v_r(s - 1) = 0, s > 0 \tag{7.1.6}$$

with $a = \tau\alpha$, $b = \tau\beta$, $c = \tau\gamma$.

Proof. Change variables: $s = t/\tau$, $\tilde{u}_r(\cdot) = u_r(\tau\cdot)$. Then

$$\tilde{u}_r'(s) + \tau(\alpha + r^2 + i\gamma r)\tilde{u}_r(s) - \tau\beta\tilde{u}_r(s - 1) = 0$$

Now put $v_r(s) = e^{i\gamma\tau rs}\tilde{u}_r(s) = e^{i\gamma tr}u_r(t)$. □

So, if $u(t,x)$ is a solution of (6.2.2), then for each $\xi \in \mathbb{R}^n$,

$$|\hat{u}(t,\xi)| = |v_\xi(t/\tau)|,$$

where $v_\xi(t)$ is a solution of equation (7.1.1), whose asymptotic behavior as $t \to \infty$ is described by estimate (7.1.2). The problem is that, estimate (7.1.2) is not directly applicable for our purposes. In order to get back to $u(x,t)$, we need to invert the Fourier transform and deal with

$$|u(t,x)| \leq \int |\hat{u}(\xi,t)| d\xi.$$

So we need a more quantitative estimate than (7.1.2), with a more explicit dependence on ξ.

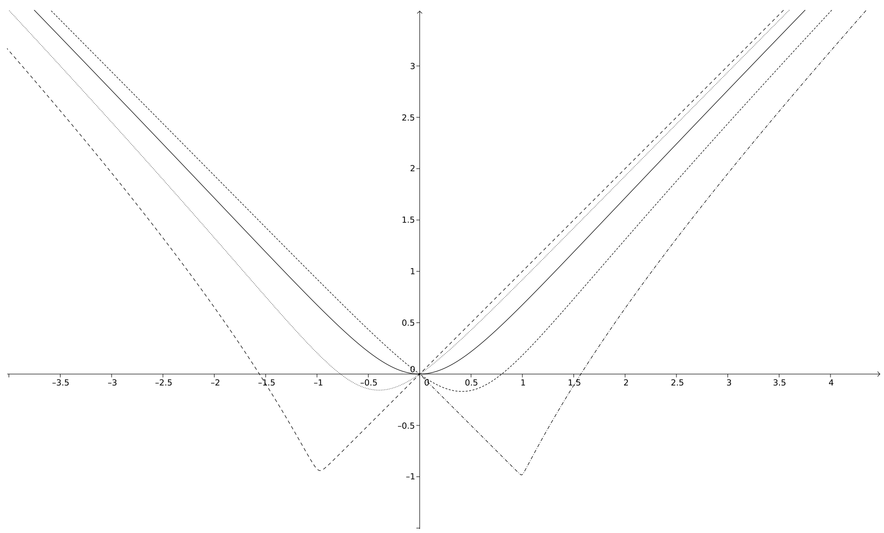

Figure 7.1: Graphs of $\Gamma(x,\theta)$: for values $\theta = 0$, $\theta = \pi/4$, $\theta = \pi/2$, $\theta = 3\pi/2$ and $\theta = \pi$ from the left to the right

7.2 Proofs of theorems A-C

Our main goal is to obtain estimates for solutions of (7.1.4) similar to the theorem of Hayes-Sakata but also exhibiting the role of the parameter r.

In the next section we will prove the following results

Theorem 7.2.1. *Let $u_r(t)$ be a solution of (7.1.4). If $\alpha \geq |\beta|$, then*

$$|u_r(t)| \leq \sqrt{2}\|u_r\|_{C[-\tau,0]} \left(\frac{1+\tau|\beta|}{1+\tau\alpha+\tau r^2}\right)^{t/\tau}$$

Theorem 7.2.2. *If $\beta < 0$ and $\alpha\tau \geq \Gamma(\tau\beta) > -1$, then for any $R > 0$ there exist $C(a,b,\tau,R), \epsilon(a,b,\tau,R) > 0$ such that*

$$|u_r(t)| \leq C\|u_r\|_{C[-\tau,0]} e^{-\epsilon r^2 t} e^{-\mu_* t/(2\tau)}, \ \forall t > 0, \ 0 \leq r \leq R$$

for all $u_r \in C[-1,\infty)$, solving (7.1.4) with $\gamma = 0$. Here

$$\mu_* = \sup\{\mu \geq 0 : \alpha\tau - \mu \geq \Gamma(\tau\beta e^{\mu})\}.$$

Theorem 7.2.3. *Let $\beta < 0$, $\tau > 0$, $\gamma \neq 0$. Then there exist α such that $\tau\alpha > \Gamma(\tau\beta)$, $R > 0$ and $u_r(t)$, solutions of (7.1.4), such that*

$$\lim_{t>0} |u_r(t)| = \infty, \ r \in [R/2, R].$$

We start by showing how Theorems 7.2.1-7.2.3 imply Theorems A-C.

7.2.1 Proof of Theorem A

Let $u(x,t)$ be a solution of (6.2.2), with $w = 0$. Then for each $\xi \in \mathbb{R}^n$, the function $u_r(t) = \hat{u}(t,\xi)$ satisfies (7.1.4) with $r = |\xi|$, $\gamma = 0$. Let $\alpha\tau \geq \Gamma(\tau\beta)$. There are two possible cases

$\beta > 0$ **case**

If $\alpha\tau \geq \Gamma(\tau\beta)$ and $\beta > 0$, then $\alpha \geq \beta > 0$. Applying Theorem 7.2.1,

$$|\hat{u}(\xi,t)| \leq \sqrt{2}\|\hat{u}(\xi,t)\|_{C[-\tau,0]} \left(\frac{1+\tau\beta}{1+\tau\alpha+\tau|\xi|^2}\right)^{t/\tau},$$

for each fixed ξ. This estimate shows that for large t, $\hat{u}(\xi,t) \in L^1$ in the variable ξ. Therefore, by the Fourier inversion,

$$|u(x,t)| = \left|\int \hat{u}(\xi,t)e^{-i\xi x} d\xi\right| \leq \sqrt{2}\int \sup_{-\tau \leq t \leq 0} |\hat{u}(\xi,t)| \left(\frac{1+\tau\beta}{1+\tau\alpha+\tau|\xi|^2}\right)^{t/\tau} d\xi \leq$$

$$\leq \sqrt{2} \sup_{-\tau \leq t \leq 0} \|u(\cdot,t)\|_{L^1} \int \left(\frac{1+\tau\beta}{1+\tau\alpha+\tau|\xi|^2} \right)^{t/\tau} d\xi$$

Observe that:

if $\alpha = \beta$ then

$$\int \left(\frac{1+\tau\beta}{1+\tau\beta+\tau|\xi|^2} \right)^{t/\tau} d\xi = O(t^{-n/2});$$

if $\alpha > \beta$ then

$$\int \left(\frac{1+\tau\beta}{1+\tau\alpha+\tau|\xi|^2} \right)^{t/\tau} d\xi = O(e^{-\epsilon t})$$

for some $\epsilon > 0$.

So, we conclude that if $\beta > 0$ and $\alpha = \beta$ then any solution of (6.2.2) with initial data (6.2.3) $u(x,t)$ satisfies

$$|u(x,t)| \leq K\|\phi\|_{C([-\tau,0],L^1)} t^{-n/2}$$

and if $\alpha > \beta$, then $\exists \epsilon > 0$ such that

$$|u(x,t)| \leq K\|\phi\|_{C([-\tau,0],L^1)} e^{-\epsilon t}.$$

$\beta < 0$ **case**

Let $\alpha\tau \geq \Gamma(\beta\tau) > -1$ and $\beta < 0$, and take $R > 0$ be large enough to ensure $\alpha + R^2 > |\beta|$. By Fourier inversion,

$$|u(x,t)| \leq \int_{|\xi| \leq R} |\hat{u}(\xi,t)| d\xi + \int_{|\xi| > R} |\hat{u}(\xi,t)| d\xi.$$

The first integral can be estimated by Theorem 7.2.2,

$$\int_{|\xi| \leq R} |\hat{u}(\xi,t)| d\xi \leq K_R \|\phi\| e^{-\mu_* t} \int e^{-\epsilon|\xi|^2 t} d\xi \leq K'_R \|\phi\| e^{-\mu_* t} t^{-n/2},$$

where $\mu_* > 0$ if $\alpha\tau > \Gamma(\beta\tau)$ and $\mu_* = 0$ if $\alpha\tau = \Gamma(\beta\tau)$. The second integral is handled by Theorem 7.2.1 as above,

$$\int_{|\xi| > R} |\hat{u}(\xi,t)| d\xi \leq L\|\phi\| e^{-\delta t},$$

with some $\delta > 0$.

So we conclude that if $\beta < 0$ and $u(x,t)$ is a solution of (6.2.2) and :

if $\alpha\tau \geq \Gamma(\beta\tau)$ then

$$|u(x,t)| \leq K\|\phi\| t^{-n/2},$$

if $\alpha\tau > \Gamma(\beta\tau)$ then

$$|u(x,t)| \leq K\|\phi\|e^{-\epsilon t}$$

for some $\epsilon > 0$.

7.2.2 Proof of Theorem B

Let $\alpha \geq |\beta|$. Note that in the proof of Theorem A, for $\beta > 0$, we did not use the fact that $w = 0$. Therefore using the same argument as in the case $\beta > 0$ above, one can see that any solution $u(x,t)$ of (6.2.2) satisfies

$$|u(x,t)| \leq K\|\phi\|t^{-n/2}.$$

Moreover if $\alpha > \beta$, then $\exists\epsilon > 0$ such that

$$|u(x,t)| \leq K\|\phi\|e^{-\epsilon t}.$$

7.2.3 Proof of Theorem C

Let $\beta < 0$, $w \neq 0$, $\eta \in [1/2, 1]$ and $\gamma = \eta|w|$. Let α and $R > 0$ be as in Theorem 7.2.3. It follows from the proof of that theorem that such α and $R > 0$ can be chosen uniformly in $\eta \in [1/2, 1]$.

Then there exists complex valued function $\lambda_\eta(r)$ on $[R/2, R]$ with $\Re(\lambda_\eta(r)) \geq \epsilon_0$ on $r \in [R/2, R]$ and such that

$$\lambda_\eta(r) + (\alpha + r^2 + i\eta|w|r) - \beta e^{-\tau\lambda_\eta(r)} = 0,\ r \in [R/2, R],\ \eta \in [1/2, 1]$$

Let ψ be a positive Schwartz function such that $\psi(\xi) = 0$ if

$$\xi \cdot w \leq 1/2|w||\xi|$$

or

$$|\xi^2| \notin [R/2, R].$$

Denote

$$\rho(t,\xi) = \psi(\xi)\exp(\lambda_{\eta(\xi)}(|\xi|)t).$$

Finally, put

$$u(x,t) = \int \psi(\xi) e^{\lambda_{\eta(\xi)}(|\xi|)t} e^{-i\xi x} d\xi,$$

where $\eta(\xi) = \cos(\xi, w)$. By construction such function is a solution of (6.2.2) and

$$|u(0,t)| = \left| \int \psi(\xi) e^{\lambda_{\eta(\xi)}(|\xi|)t} d\xi \right| \geq C e^{\epsilon_0 t} \left| \int e^{i\Im(\lambda_{\eta(\xi)}(|\xi|)t)} \right|$$

Therefore, there exists α such that $\Gamma(\beta) < \alpha < |\beta|$ and a solution $u(x,t)$ of (6.2.2) so that

$$\limsup_{t\to\infty} |u(0,t)| = \infty.$$

7.3 Proofs of theorems 7.2.1-7.2.3

We will need the following proposition which is a corollary of a more general principle (e.g. Theorem 5.4 of [23] or Theorem 4.3 in [6])

Proposition 7.3.1. *Let $a, b, \theta \in \mathbb{R}$ and*

$$h_{a,b,\theta}(\lambda) = \lambda + a - be^{-\lambda + i\theta}.$$

1. *If all zeros of $h_{a,b}(\lambda)$ lie in the half plane $\Re(\lambda) \leq 0$ and all purely imaginary zeros $\lambda = iy$ are simple, then there exists $K(a,b) > 0$ such that for every $v(t) \in C[-1,\infty]$, satisfying (7.1.1), we have*

$$|v(t)| \leq K(a,b) \|v\|_{C[-1,0]}, \ \forall t > 0.$$

2. *If all zeros of $h_{a,b}(\lambda)$ lie in the half-plane $\Re(\lambda) < -\epsilon < 0$, then there is $K(a,b) > 0$ so that*

$$|v(t)| \leq K(a,b) \|v\|_{C[-1,0]} e^{-\epsilon t}, \ t > 0,$$

for all continuous solutions of (7.1.1).

The following lemma is self-evident.

Lemma 7.3.2. *Let $a, b, \theta \in \mathbb{R}$, and $\mu \in \mathbb{R}$. Let $v(t)$ be continuous on $[-1, 0]$ and*

$$v'(t) + av(t) - be^{i\theta} v(t-1) = 0.$$

If we let $w(t) = e^{\mu t}v(t)$, then $w(t)$ is a continuous solution of

$$w'(t) + a_* w(t) - b_* e^{i\theta} w(t-1) = 0, \tag{7.3.1}$$

with

$$a_* = a - \mu, \quad b_* = be^{\mu}. \tag{7.3.2}$$

Let us outline our strategy, before going into details.

By Lemma 7.1.3, it is enough to estimate solutions of equation (7.1.6). So we start with it.

When r in (7.1.6) equals 0, the Hayes-Sakata theorem gives a necessary and sufficient condition to ensure the stability of all solutions: if $a > \Gamma(b)$ then there exist $K(a,b)$ and $\epsilon(a,b)$ so that

$$|v_0(t)| \leq K(a,b)\|v_0\|_{C[-1,0]} e^{-\epsilon(a,b)t}.$$

If we assume that $c = 0$ in (7.1.6), then as $r > 0$ increases, the corresponding point $(b, a+\tau r)$ is moving more into the region of stability. In this case we can expect a better order of decay, i.e. $\epsilon(a + r^2, b)$ should be an increasing function of r. Though, potentially, the function $K(a + r^2, b)$ could also start growing, it turns out to be uniformly bounded in the region $a > |b|$. In this case we obtain the most accurate estimate - Theorem 7.2.1. We cannot say that $K(a,b)$ is uniformly bounded in $a > \Gamma(b)$, because it blows up to ∞ near the point $(-1,-1)$, so we will need to be careful near that point. Finally, controlling the growth of ϵ as a function of r, we will prove Theorem 7.2.2.

If $c \neq 0$, the situation is different. Figure 7.1 shows that as θ increases the graphs are "moving"to the right. If the point (a,b), of the parameters in (7.1.6), lies in the left half plane, then as r grows, the point $(b, a + \tau r^2)$ moves upward as before, but now the graph shifts to the right at the rate proportional to r. As a result, for small $r > 0$, the point $(b, a + \tau r^2)$ will end up below the stability region, which is the epigraph of $\Gamma(x, cr)$. This will prove Theorem 7.2.3.

7.3.1 Proof of Theorem 7.2.1

Lemma 7.3.3. *Let $a > 0$ and $u \in C[-1, \infty)$ satisfy*

$$u'(t) + au(t) - ae^{i\theta}u(t-1) = 0. \tag{7.3.3}$$

If for some $m \in \mathbb{N}$

$$|\Re(e^{ik\theta}u)(t)|, |\Im(e^{ik\theta}u)(t)| \leq 1, \ k = 0, 1, \ldots, m; \ t \in [-1, 0],$$

then for non-negative integer l so that $l \leq m$,

$$|\Re(e^{ik\theta}u)(t)|, |\Im(e^{ik\theta}u)(t)| \leq 1, \ k = 0, 1, \ldots m - l; \ t \in [l-1, l]. \tag{7.3.4}$$

Proof. We will prove only the real-part case using induction in l (imaginary parts can be considered exactly in the same way).

For $l = 0$, there is nothing to prove. Assume that the claim is true for $l = l_0 < m$. Put $u(t) = e^{-at}C(t)$. Then $C(l_0) = e^{al_0}u(l_0)$ and

$$C'(t) = e^{at}ae^{i\theta}u(t-1), \ t \in [l_0, l_0 + 1].$$

Clearly, $C(t) = C(l_0) + \int_{l_0}^{t} C'(y)dy$. Multiply the function C by exponentials $e^{ik\theta}$ ($k \leq m - l_0 - 1$) and use (7.3.3) to get

$$e^{ik\theta}C(t) = e^{al_0}e^{ik\theta}u(l_0) + \int_{l_0}^{t} ae^{ay}e^{i(k+1)\theta}u(y-1)dy.$$

Then using our assumption,

$$\Re(e^{ik\theta}C(t)) \leq e^{al_0} + \int_{l_0}^{t} ae^{by}dy = e^{at}$$

and

$$\Re(e^{ik\theta}C(t)) \geq -e^{al_0} - \int_{l_0}^{t} ae^{ay}dy = -e^{at}$$

for $t \in [l_0, l_0 + 1], k \in [0, m - l_0 - 1]$. Hence, $|\Re(e^{ik\theta}u(t))| \leq 1$. □

Corollary 7.3.4. *Let $a > 0, \theta \in \mathbb{R}$. If $u \in C[-1, \infty)$ is a solution of*

$$u'(t) + au(t) - ae^{i\theta}u(t - \tau) = 0,$$

then

$$|u(t)| \leq \sqrt{2}\|u\|_{C[-\tau, 0]}, \ t \geq 0.$$

Lemma 7.3.5. *Let $a \geq |b|$, $c \in \mathbb{R}$ and $r, \tau > 0$.*

If $v_r(t) \in C[-1, \infty)$ is a solution of

$$v_r'(t) + (a + \tau r^2)v_r(t) - be^{icr}v_r(t-1) = 0, t > 0,$$

then

$$|v_r(t)| \leq \sqrt{2}\|v_r\|_{C[-1,0]} \left(\frac{1 + |b|}{1 + a + \tau r^2}\right)^t,$$

all $t, r > 0$.

Proof. Fix r and let $z = \sup\{\mu : \mu + |b|e^{\mu} \leq a + \tau r^2\}$. If $v(r)$ is a continuous solution of

$$v'(t) + (a + \tau r^2)v(t) = be^{icr}v(t-1), t > 0,$$

then by Lemma 7.3.2, $v(t) = w(t)e^{-zt}$, where

$$w'(t) + dw(t) - de^{i(cr + \arg(b))}w(t-1) = 0,$$

where $d = a - z = |b|e^z$. By the above corollary,

$$|w(t)| \leq \sqrt{2}\|w(t)\| \leq \sqrt{2}\|v(t)\|_{C[-1,0]}.$$

Therefore

$$|v(t)| \leq \sqrt{2}\|v(t)\|_{C[-1,0]}e^{-zt}$$

As it follows from the definition of z

$$a + \tau r^2 - z = |b|e^z$$

$$a + \tau r^2 = z + |b|e^z \leq (1 + |b|)e^z - 1$$

$$z \geq \ln\left(\frac{1 + a + \tau r^2}{1 + |b|}\right).$$

Finally,

$$|v(t)| \leq \sqrt{2}\|v(t)\|_{C[-1,0]} \left(\frac{1 + |b|}{1 + a + \tau r^2}\right)^t.$$

$\square$

Now applying Lemmas 7.1.3 and 7.3.5, we obtain Theorem 7.2.1.

7.3.2 Proofs of Theorems 7.2.2 and 7.2.3

Proof of Theorem 7.2.2

Put
$$K(\alpha,\beta)=\sup_{v}\sup_{t>0}\frac{|v(t)|e^{\mu_* t/2}}{\|v\|_{C[-1,0]}},$$
where the outer supremum is taken over all continuous solutions of
$$v'(t)+\alpha v(t)-\beta v(t-1)=0 \tag{7.3.5}$$
and
$$\mu_*=\sup\{\mu\geq 0:\Gamma(\beta e^{\mu})+\mu\leq\alpha\}. \tag{7.3.6}$$

By this choice, every solution of (7.3.5) satisfies
$$|v(t)|\leq K(\alpha,\beta)\|v\|_C e^{-\mu_* t/2}.$$

It follows from the proof of Theorem 5.2 in [35] and Lemma 7.3.2, that $K(\alpha,\beta)$ is a continuous function on $S=\{(\beta,\alpha):\alpha>\Gamma(b)\}$. The function $K(\alpha,\beta)$ roughly speaking plays the role of $\sqrt{2}$ that we had in the last lemma. Note that $K(\alpha,\beta)$ can be continuously extended to $\bar{S}\setminus(-1,-1)$, because if $\alpha=\Gamma(\beta)>-1$, then all imaginary solutions of $\lambda+\alpha-\beta e^{-\lambda}=0$ are simple. However, for $\alpha=\Gamma(-1)=-1$, $\lambda=0$ is a double root of $\lambda+\alpha-\beta e^{-\lambda}=0$ and by Proposition 7.3.1, $K(-1,-1)=\infty$.

Lemma 7.3.6. *Let $b\in(-\infty,0)$, $a\geq\Gamma(b)>-1$ and $r,\tau>0$. Let μ_* be defined by (7.3.6).*

For any $R>0$ there exists $C(a,b,\tau,R),\epsilon(a,b,\tau,R)>0$ such that
$$|v_r(t)|\leq C\|v_r\|_{C[-1,0]}e^{-\epsilon\tau r^2 t}e^{-\mu_* t/2},\ \forall t>0,\ 0\leq r\leq R$$
for all $v_r\in C[-1,\infty)$ satisfying
$$v_r'(t)+(a+\tau r^2)v_r(t)=bv_r(t-1),t>0.$$

Proof. As we showed above
$$|v_r(t)|\leq K(\alpha+\tau r^2,\beta)\|v_r\|_{C[-1,0]}e^{-z(r)t/2},$$

where

$$z(r) = \sup\{\mu \geq 0 : a + \tau r^2 - \mu \geq \Gamma(\beta e^{\mu})\}$$

Since $K(a,b)$ is continuous and can be extended to $\partial S \setminus (-1,-1)$ and $a > -1$, there exists $C > 0$ such that $K(a + \tau r^2, b) < C$. So all we need to do is to show that for some ϵ,

$$z(r) \geq z(0) + \epsilon\tau r^2,\ 0 \leq r \leq R$$

or

$$z'(r) \geq \tau\epsilon r^2, \quad 0 \leq r \leq R.$$

By the definition of $z(r)$

$$a + \tau r^2 - z(r) = \Gamma(be^{z(r)})$$

and for $x = be^z \neq -1$

$$z'(r) = \frac{2\tau r}{1 + \Gamma'(x)x}.$$

We complete the proof by showing that for $|x| < L$, $x \neq -1$

$$-1 < \Gamma'(x)x < 2L. \tag{7.3.7}$$

If $x \geq -1$, then $\Gamma(x) = x$ and $\Gamma'(x) = 1$. For $x = -t/\sin(t) < -1$, $\Gamma(x(t)) = -t\cot(t)$. Then

$$\frac{d}{dt}\Gamma(x(t)) = \frac{t - \cos(t)\sin(t)}{\sin^2(t)}$$

$$\frac{dx}{dt} = \frac{t\cos(t) - \sin(t)}{\sin^2(t)}$$

$$\Gamma'(x) = \frac{t - \cos(t)\sin(t)}{t\cos(t) - \sin(t)},$$

where $x = -\frac{t}{\sin(t)}$, $t \in (0,\pi)$. It is not difficult to show that

$$-2 \leq \frac{t - \cos(t)\sin(t)}{t\cos(t) - \sin(t)} \leq -1,\ t \in (0,\pi).$$

So for $-L < x < -1$.

$$0 \leq \Gamma'(x)x \leq 2L.$$

□

Combining Lemma 7.1.3 and the last lemma, we obtain Theorem 7.2.2.

Proof of Theorem 7.2.3

The final goal is to show that if $\gamma \neq 0$ then it is impossible to obtain results similar to the ones established above. More precisely, for any $\beta < 0$, there exists $R > 0$ and continuous functions $u_r(t)$, solutions of (7.1.4) with $\tau\alpha > \Gamma(\tau\beta)$, such that

$$\sup_{t>0} |u_r(t)| = \infty, \ \forall r \in (0, R).$$

Lemma 7.3.7. *Let $a, b \in \mathbb{R}$, $\tau, r > 0$ and*

$$z(r) = \sup\{\mu \in \mathbb{R} : a + \tau r^2 - \mu \geq \Gamma(be^{\mu}, cr)\}.$$

If $c \neq 0$, then for any $b < 0$, there exists $a > \Gamma(b)$ and $R > 0$ such that

$$z(r) < 0, \forall r \in [R/2, R].$$

Proof. Since $\Gamma(x, \theta)$ is even with respect to θ we can assume that $c > 0$. We will show that given $c > 0$ and $b < 0$, there exists $\epsilon = \epsilon(b, c) > 0$ so that

$$\lim_{r \to 0+} z'(r) \leq -\epsilon < 0.$$

This will be enough to prove the theorem because then for some small $R > 0$

$$z(r) = z(0) + \int_0^r z'(s)ds \leq z(0) - \epsilon r/2, \ r \in [R/2, R].$$

Hence, choosing $a - \Gamma(b) > 0$ so small that $z(0) < \epsilon R/8$, we will have

$$z(r) \leq -\epsilon R/8 < 0, \ r \in [R/2, R].$$

Recall that for $\theta \neq 0 \mod \pi$,

$$\Gamma(x, \theta) = F(t(x, \theta), \theta),$$

where

$$F(t, \theta) = (\theta - t)\cot(t)$$

and $t(x, \theta) \in (0, \pi)$, so that

$$x = \frac{\theta - t}{\sin(t)}.$$

Let us fix $t_0 \in (0, \pi)$ so that $b = -t_0/\sin(t_0)$.

By the definition of $z(r)$,

$$a + \tau r^2 - z(r) = \Gamma(be^{z(r)}, cr) = F(t(be^{z(r)}, cr), cr)$$

$$2\tau r - z' = F'_t(t, cr)(t'_x(be^z, cr)[be^z z'] + ct'_\theta(be^z, cr)) + cF'_\theta(t, cr),$$

and

$$z' = \frac{2\tau r - cF'_\theta(t, cr) - cF'_t(t, cr)t'_\theta(be^z, cr)}{1 + F'_t(t, cr)t'_x(be^z, cr)be^z}.$$

Furthermore,

$$F'_\theta(t, \theta) = \cot(t)$$

$$F'_t(t, \theta) = -\frac{\sin(t)\cos(t) + (\theta - t)}{\sin^2(t)}$$

and

$$t'_\theta = \frac{\sin(t)}{\sin(t) + (\theta - t)\cos(t)}.$$

So

$$F'_\theta(t, \theta) + F'_t(t, \theta)t'_\theta(x, \theta) = \frac{\cos(t)}{\sin(t)} - \frac{\sin(t)\cos(t) + (\theta - t)}{\sin(t)(\sin(t) + (\theta - t)\cos(t))}$$

$$= \frac{-(\theta - t)\sin^2(t)}{\sin(t)(\sin(t) + (\theta - t)\cos(t))} = \frac{t - \theta}{1 + (\theta - t)\cot(t)} = \frac{t - \theta}{1 + \Gamma(x, \theta)}$$

Therefore, denoting $x = be^{z(r)}$,

$$z'(r) = \frac{2\tau r - c(t(x, cr) - cr)/(1 + \Gamma(x, \theta))}{1 + \Gamma'_x(x, cr)x}$$

and recalling (7.3.7) one can see that

$$\lim_{r \to 0+} z'(r) = \frac{-ct_0}{(1 + \Gamma(b))(1 + \Gamma'_x(b)b)} \leq -\epsilon < 0,$$

for some $\epsilon > 0$. □

Lemma 7.3.8. *Given $b < 0$, $\tau > 0$, $c \neq 0$, there exists $a > \Gamma(b, cr)$, $R > 0$ and $v_r(t) \in C[-1, \infty)$, solutions of (7.1.6) such that*

$$\sup_{t > 0} |v_r(t)| = \infty, \ \ r \in [R/2, R].$$

Proof. By Lemma 7.3.2,

$$v_r(t) = e^{-z(r)} w_r(t)$$

where

$$w_r'(t) + \Gamma(be^{z(r)}, cr) w_r(t) - be^{z(r)} e^{icr} w_r(t-1) = 0.$$

By part (b) of Theorem 7.1.2, there are always $w_r(t)$ such that

$$\limsup_{t \to \infty} |w_r(t)| > 0.$$

□

Combining the last result with Lemma 7.1.3, we obtain Theorem 7.2.3.

7.4 Implications to the stability of travelling waves

One of the motivations to study the Skellam model with time delay was to give a more rigorous proof of Theorem 2.3. in [50]. Since there we deal with positive coefficients α, β, this goal was achieved by proving Theorem B.

In its current form, the negative result (Theorem C) is not applicable to population models, because a negative death rate β does not have a biological meaning. However, if both α and β are negative, then we can swap the roles of the variables in (6.1.2) and have a positive death rate $-\alpha$ and positive birth rate $-\beta$. Such growth rates cannot be obtained from the linearization of the model (6.1.1) satisfying conditions (H.1) - (H.3). It seems plausible that the right setting in which the negative result can be used is the growth model with the Allee effect, where $b(u) - d(u)$ has three zeros $0, u_1, u_2$ and $b - d$ is negative and concave up on $(0, u_1)$ and positive and concave down on (u_1, u_2).

Bibliography

[1] R. Adams and J. Fournier, *Sobolev Spaces*, Academic Press, 2003.

[2] M. Aguerrea, C. Gomez, and S. Trofimchuk, *On uniqueness of semi-wavefronts*, Mathematische Annalen **354** (2012), no. 1, 73–109.

[3] N. Aronszajn and K. Smith, *Theory of Bessel potentials I*, Annales de l'institut Fourier **11** (1961), 385–475.

[4] T. Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampere Equations*, Springer-Verlag, New York, 1982.

[5] P. Auscher, E. Russ, and P. Tchamitchian, *Hardy Sobolev spaces on strongly Lipschitz domains of* $\mathbb{R}^n$, Journal of Functional Analysis **218** (2005), no. 1, 54–109.

[6] R. Bellman and K. Cooke, *Differential-difference Equations*, Academic Press, New York-London, 1963.

[7] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic press, 1988.

[8] F. Bethuel, G Orlandi, and D. Smets, *Approximations with vorticity bounds for the Ginzburg–Landau functional*, Communications in Contemporary Mathematics **6** (2004), no. 05, 803–832.

[9] R. Beverton and S. Holt, *On the Dynamics of Exploited Fish Populations*, Springer Science & Business Media, 2012.

[10] G. Bourdaud, *De* $\mathrm{BMO}(\mathbb{R}^n)$ *et de bmo*$(\mathbb{R}^n)$, Ann. Inst. Fourier, Grenoble **52** (2002), no. 4, 1187–1218.

[11] J. Bourgain and H. Brezis, *New estimates for the Laplacian, the div-curl, and related Hodge systems*, C. R. Math. Acad. Sci. Paris **7** (2004), no. 338, 539–543.

[12] ______, *New estimates for elliptic equations and hodge type systems*, Journal of the European Mathematical Society **9** (2007), no. 2, 277–315.

[13] J. Bourgain, H. Brezis, and P. Mironescu, *$H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg–Landau equation*, Publications mathématiques de l'IHES **99** (2004), 1–115.

[14] H. Brezis and J. Van Schaftingen, *Boundary estimates for elliptic systems with L^1-data*, Calculus of Variations **30** (2007), 369–388.

[15] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Communications in Partial Differential Equations **5** (1980), no. 7, 773–789.

[16] N. Britton, *Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model*, SIAM Journal on Applied Mathematics **50** (1990), no. 6, 1663–1688.

[17] A. Caetano, *Approximation by functions of compact support in Besov-Triebel-Lizorkin spaces on irregular domains*, Studia Math **142** (2000), no. 1, 47–63.

[18] D-C. Chang, *The dual of Hardy spaces on a bounded domain in $\mathbb{R}^n$*, Forum Math **6** (1994), no. 1, 65–81.

[19] D-C. Chang, G. Dafni, and E. Stein, *Hardy spaces, BMO, and boundary value problems for the Laplacian on a smooth domain in $\mathbb{R}^n$*, Transactions of the American Mathematical Society **351** (1999), no. 4, 1605–1661.

[20] D-C. Chang, S. Krantz, and E. Stein, *H^p theory on a smooth domain in $\mathbb{R}^N$ and elliptic boundary value problems*, Journal of functional analysis **114** (1993), no. 2, 286–347.

[21] G. Dafni, *Hardy spaces on strongly pseudoconvex domains in $\mathbb{C}^n$ and domains of finite type in $\mathbb{C}^2$*, ProQuest LLC, Ann Arbor, MI, 1993, Thesis (Ph.D.)–Princeton University.

[22] ———, *Local VMO and weak convergence in h^1*, Canadian mathematical bulletin **45** (2002), no. 1, 46–59.

[23] O. Diekmann, S. Van Gils, S. Lunel, and H. Walther, *Delay Equations: Functional-, Complex-, and Nonlinear analysis*, Springer Science & Business Media, 2012.

[24] T-C. Dinh and N. Sibony, *Introduction to the theory of currents*, Lecture notes (2005).

[25] M-P. Do Carmo, *Riemannian Geometry*, Mathematics: Theory and Applications, vol. 115, Birkhäuser Boston, 1992.

[26] J. Eldering, *Normally Hyperbolic Invariant Manifolds – the Noncompact Case*, Berlin:Springer, 2013.

[27] L. Evans, *Partial Differential Equations*, Graduate Series in Mathematics, Vol. 19, American Mathematical Society, 2010.

[28] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Mathematica **129** (1972), no. 1, 137–193.

[29] R. Fisher, *The wave of advance of advantageous genes*, Annals of Eugenics **7** (1937), no. 4, 355–369.

[30] D. Goldberg, *A local version of real Hardy spaces*, Duke mathematical journal **46** (1979), no. 1, 27–42.

[31] A. Gomez and S. Trofimchuk, *Global continuation of monotone wavefronts*, Journal of the London Mathematical Society **89** (2014), no. 1, 47–68.

[32] L. Grafakos, *Modern Fourier analysis*, Graduate Texts in Mathematics, vol. 250, Springer, 2009.

[33] W. Gurney, S. Blythe, and R. Nisbet, *Nicholson's blowflies revisited*, Nature **287** (1980), 17–21.

[34] K. Hadeler and G Bocharov, *Where to put delays in population models, in particular in the neutral case*, Canadian Applied Mathematics Quarterly **11** (2003), 159–173.

[35] J. Hale, *Theory of Functional Differential Equations*, Applied Mathematical Sciences, vol. 3, Springer-Verlag, New York-Heidelberg, 1977.

[36] N. Hayes, *Roots of the transcendental equation associated with a certain difference-differential equation*, J. London Math. Soc. **25** (1950), 226–232.

[37] L. Hormander, *The Analysis of Linear Partial Differential operators I*, Springer-Verlag, 1990.

[38] E. Hutchinson, *Circular causal systems in ecology*, Annals of the New York Academy of Sciences **50** (1948), no. 4, 221–246.

[39] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Communications on pure and applied Mathematics **14** (1961), no. 3, 415–426.

[40] A. Jonsson, P. Sjögren, and H. Wallin, *Hardy and Lipschitz spaces on subsets of* $\mathbb{R}^n$, Studia Math **80** (1984), no. 2, 141–166.

[41] A Kolmogorov, I. Petrovskii, and N. Piskunov, *A study of the equation of diffusion with increase in the quantity of matter, and its application to a biological problem*, Bul. Moskovskogo Gos. Univ **1** (1937), no. 7, 1–26.

[42] J. Lee, *Introduction to Smooth Manifolds*, Springer, 2003.

[43] C-K. Lin, C-T. Lin, Y. Lin, and M Mei, *Exponential stability of nonmonotone traveling waves for Nicholson's blowflies equation*, SIAM Journal on Mathematical Analysis **46** (2014), no. 2, 1053–1084.

[44] V. Maz'ya, *Sobolev Spaces: with Applications to Elliptic Partial Differential Equations*, Springer, 2011.

[45] M. Mei, C-K. Lin, C-T Lin, and J. So, *Traveling wavefronts for time-delayed reaction–diffusion equation I: local nonlinearity*, Journal of Differential Equations **247** (2009), no. 2, 495–510.

[46] M. Mei, C-K Lin, C-T. Lin, and J. So, *Traveling wavefronts for time-delayed reaction–diffusion equation II: nonlocal nonlinearity*, Journal of Differential Equations **247** (2009), no. 2, 511–529.

[47] M. Mei, C. Ou, and X-Q. Zhao, *Global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations*, SIAM Journal on Mathematical Analysis **42** (2010), no. 6, 2762–2790.

[48] M. Mei and J. So, *Stability of strong travelling waves for a non-local time-delayed reaction–diffusion equation*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **138** (2008), no. 03, 551–568.

[49] M. Mei, J. So, M. Li, and S. Shen, *Asymptotic stability of travelling waves for Nicholson's blowflies equation with diffusion*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **134** (2004), no. 03, 579–594.

[50] M. Mei and Y. Wang, *Remark on stability of traveling waves for nonlocal Fisher-KPP equations*, Int. J. Numer. Anal. Model. Ser. B **2** (2011), 379–401.

[51] N. Meyers and J. Serrin, *H=W*, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 1055–1056.

[52] A. Miyachi, *H^p spaces over open subsets of* $\mathbf{R}^n$, Studia Math. **95** (1990), no. 3, 205–228.

[53] A. Nicholson, *An outline of the dynamics of animal populations.*, Australian journal of Zoology **2** (1954), no. 1, 9–65.

[54] M. Panik, *Growth Curve Modeling: Theory and Applications*, John Wiley & Sons, 2014.

[55] J. Peetre, *On convolution operators leaving $L^{p,\lambda}$ spaces invariant*, Annali di Matematica Pura ed Applicata **72** (1966), no. 1, 295–304.

[56] L. Pick, *Optimal Sobolev embeddings*, Nonlinear analysis, function spaces and applications **6** (1999), 156–199.

[57] S. Pohozhaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR **165** (1965), 36–39.

[58] W Ricker, *Stock and recruitment*, Journal of the Fisheries Research Board of Canada **11** (1954), no. 5, 559–623.

[59] S. Ruan, *Delay differential equations in single species dynamics*, Delay Differential Equations and Applications (2006), 477–517.

[60] S. Sakata, *Asymptotic stability for a linear system of differential-difference equations*, Funkcial. Ekvac. **41** (1998), no. 3, 435–449.

[61] G. Schwarz, *Hodge Decomposition – a Method for Solving Boundary Value Problems*, Springer, 2006.

[62] J. Skellam, *Random dispersal in theoretical populations*, Biometrika **38** (1951), no. 1, 196–218.

[63] S. Smirnov, *Decomposition of solenoidal vector charges into elementary solenoids, and the structure of normal one-dimensional flows*, Algebra i Analiz **5** (1993), no. 4, 206–238.

[64] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 2016.

[65] ———, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 2016.

[66] E. Stein and G. Weiss, *Introduction to Harmonic Analysis on Euclidean spaces*, Princeton, Princeton University Press, 1971.

[67] L. Tartar, *An Introduction to Sobolev Spaces and Interpolation Spaces*, Springer Science & Business Media, 2007.

[68] M. Taylor, *Hardy spaces and bmo on manifolds with bounded geometry*, Journal of Geometric Analysis **19** (2009), no. 1, 137–190.

[69] ———, *Partial differential equations II: Qualitative studies of linear equations*, vol. 116, Springer Science & Business Media, 2013.

[70] C. Travis and G. Webb, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc. **200** (1974), 395–418.

[71] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics, Birkhäuser Verlag, 1983.

[72] ———, *Theory of Function Spaces II*, Monographs in Mathematics, Birkhäuser Verlag, Basel, 1992.

[73] ———, *The Structure of Functions*, Springer Science & Business Media, 2012.

[74] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech **17** (1967), no. 5, 473–483.

[75] J. Van Schaftingen, *Estimates for L^1-vector fields*, Comptes Rendus Mathematique **339** (2004), no. 3, 181 – 186.

[76] ———, *A simple proof of an inequality of Bourgain, Brezis and Mironescu*, Comptes Rendus Mathematique **338** (2004), no. 1, 23–26.

[77] ———, *Function spaces between BMO and critical Sobolev spaces*, Journal of Functional Analysis (2006), no. 236, 490–516.

[78] ———, *Limiting fractional and lorentz space estimates of differential forms*, Proceedings of the American Mathematical Society (2010), 235–240.

[79] ———, *Limiting Bourgain-Brezis estimates for systems of linear differential equations: Theme and variations*, Journal of Fixed Point Theory and Applications **15** (2014), no. 2, 273–297.

[80] E. Wright, *The stability of solutions of non-linear difference-differential equations*, Proceedings of the Royal Society of Edinburgh. Section A. Mathematical and Physical Sciences **63** (1950), no. 1, 18–26.

[81] J. Wu, *Theory and Applications of Partial Functional-differential Equations*, Applied Mathematical Sciences, vol. 119, Springer-Verlag, New York, 1996.

[82] V. Yudovich, *Some estimates connected with integral operators and with solutions of elliptic equations*, Dokladi Akademii Nauk SSSR **138** (1961), no. 4, 805.