The Bare Necessities for Doing Undergraduate Multivariable Calculus

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ABSTRACT

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Hadas Brandes

Students in two mathematics streams at Concordia University start their programs on similar footing in terms of pre-requisite courses; their paths soon split in the two directions set by the Pure and Applied Mathematics (MATH) courses and the Major in Mathematics and Statistics (MAST) courses. In particular, likely during their first year of studies, the students set out to take a two-term arrangement of Multivariable Calculus in the form of MAST 218 – 219 and MATH 264 – 265, respectively. There is an ongoing discussion about the distinction between the MAST and MATH courses, and how it is justified. This thesis seeks to address the matter by identifying the mathematics that is *essential* for students to learn in order to succeed in each of these courses. We apply the Anthropological Theory of the Didactic (ATD) in order to model the knowledge to be taught and to be learned in MAST 218 and MATH 264, as decreed by the curricular documents and course assessments. The ATD describes units of mathematical knowledge in terms of a practical block (*tasks* to be done and *techniques* to accomplish them) and a theoretical block that frames and justifies the practical block. We use these notions to model the knowledge to be taught and learned in each course and reflect on the implications of the inclusion and exclusion of certain units of knowledge in the minimal core of what students need to learn. Based on these models, we infer that the learning of Multivariable Calculus in both courses follows in a tradition observed in single-variable calculus courses, whereby students develop compartmentalized units of knowledge. That is, we find that it is *necessary* for students in MAST 218 and MATH 264 to specialize in techniques that apply to certain routine tasks, and to this end, it suffices to learn bits and pieces of theoretical knowledge that are not unified in a mathematically-informed way. We briefly consider potential implications of such learning in the wider context of the MATH and MAST programs.

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In honor of the cats who bravely and honorably fought to keep me from working on this thesis, who sat on my books for a midday nap and chewed on my paper for a midnight snack, three last words before we tuck in; Meshi, Arge, and Twink.

List of Figures

List of Tables

Chapter I: Introduction

The idea that sparked this thesis was an ongoing discussion about two courses in Concordia University's Mathematics and Statistics Department: MAST 218 and MATH 264, or 'Multivariable Calculus I' for students in the Major in Mathematics and Statistics program (MAST) and students in the Pure and Applied Mathematics Specialization (MATH). Historically, the courses were created to cater to the two different programs. Today, there's discussion as to whether the distinction is still relevant. One of the goals at the onset of my research was to look at the "distance" between the two courses. A second goal came about in the application of the Anthropological Theory of the Didactic (ATD) to the study of undergraduate mathematics, which in itself was an important theoretical and operationalization exercise. The first and second goals happened to fit together well, as it turned out useful in the exercise of applying the ATD to have to analyze two courses, as opposed to just one. As I explain below, through these two initial aims I ultimately set to identify and describe the minimal knowledge essential for MAST 218 and MATH 264 students to learn in order to succeed in their course.

The ATD is a research programme developed by Chevallard (1999) that focuses on didactic practices as they occur in an *institution*. I don't go into the details of discussing the meaning of "institution" in the context of ATD – it suffices to say that it refers to a group of persons with common goals, with a common set of rules and norms that organize their behavior and interactions, and with a set of strategies to achieve those goals. The institution's features are somewhat independent of the individuals who participate in it; the institution and its features persist over time – before and after individuals join and leave; and there are mechanisms to share the rules, norms and strategies with new comers. The university, the mathematics and statistics department, MAST 218, MATH 264, the classroom, are examples of institutions (for a further discussion on the meaning of "institution" in the context of ATD, see Hardy, 2009a). Chevallard contends that every institutional didactic practice is forged by a variety of internal and external conditions that stem from all levels of society – from governmental policies to those of a particular teaching institution, social and cultural norms, all the way down to the experiences of the instructors and students directly involved in a given didactic practice. In simple terms, the ultimate goal of any (mathematics) educational institution is to share/transmit/teach (mathematical) knowledge. However, from the perspective of ATD, knowledge does not exist in the vacuum, rather, it is bound to the institution in which it is shared and somehow connected to the knowledge shared in other related institutions; such connection is called *transposition* and is of didactic nature in the context of educational institutions. Didactic transpositions take place along a spectrum of knowledge in which *scholarly* *mathematics* (the mathematical knowledge developed, shared and used by the experts – the mathematicians) is transposed into the knowledge to be taught in a given institution, up to a transposition into knowledge actually learned by the students. Chevallard (1999) distinguishes several stages in this transformation of mathematical knowledge: scholarly knowledge, knowledge to be taught, knowledge actually taught, knowledge to be learned, and knowledge actually learned. An essential feature of the ATD is an epistemological model called *praxeology* – this feature allows the researcher to model the mathematical knowledge in any one of the stages of the didactic transposition.

The notion of *praxeology* serves to model a unit of knowledge. Chevallard (1999) defines it by a *theoretical block* and a *practical block*. The theoretical block of a praxeology consists of a theory Θ, a set of concepts and assumptions, and a technology θ, a set of notions and arguments justified by the theory; Θ and θ come about through a discourse that joins the elements of each set in some rational way. The practical block contains the tasks T to be achieved in a praxeology and the techniques τ with which to complete these tasks. The technology from the theoretical block produces, explains, and justifies the techniques, thereby connecting the two blocks into a single unit of knowledge, a praxeology, denoted $\Pi = (T, \tau; \Theta, \theta)$. I expand on this notion and provide examples in Chapter III.

The language and concepts in the ATD helped sharpen the initial aim of this thesis of *measuring the distance* between MAST 218 and MATH 264. The goal became to identify the minimal knowledge that students of 218 and 264 need to learn to succeed in these courses; thus, I set out to characterize the knowledge that is *essential* for students to learn in order to provide acceptable solutions in their final exams. To this end, I treated three instances of didactic transposition: I created a reference model based on the scholarly multivariable calculus knowledge that is to be transposed, a model of the knowledge to be taught in both courses, as indicated by the curricular documents, and a model of the knowledge to be learned, as determined by the final examinations in each course as these largely determine students' success rate in the courses. The main purpose of the model of the knowledge to be taught is to model the knowledge to be learned.

In addition to the goal of identifying and describing (in terms of praxeologies) the minimal knowledge required for MAST 218 and MATH 264 students to learn to succeed in the course, I aim to examine what MAST and MATH students *gain* and what they may be *missing* given the core of the knowledge to be learned. That is, I hope to identify discrepancies between the knowledge to be taught and the knowledge to be learned, and to conjecture the effects of the exclusions.

This thesis begins with a two-part literature review in Chapter II. The first section considers research on the learning of calculus, with specific attention to studies on *tasks* that students have to do. The second section discusses the relevance and affordances of the ATD to the evaluation of university mathematics education. I introduce concepts and constructs of the ATD to allow for this discussion, though I expand on them more fully in Chapter III, which I preview next. Section II.ii and Chapter III together fund a fuller depiction of the theoretical framework for this thesis.

Chapter III gives an overview of ATD as a theoretical framework and an explanation of my application of the ATD to this study. I provide here the operational definition of *knowledge to be taught* and *knowledge to be learned* in this thesis.

I describe the institutional context of MAST 218 and MATH 264 in Chapter IV; as per the ATD, no mathematical praxeology can be discussed in the void, and the didactic transposition of a unit of mathematical knowledge exists in a given institution defined by its norms and organizations. I open with the stated missions of the two programs that house these courses: the Major in Mathematics in Statistics (MAST) and the Pure and Applied Mathematics Specialization (MATH). I proceed with the entry requirements and degree requirements for each; I situate both courses in their respective programs; and finally describe the heavily-coordinated, multi-section quality of MAST 218 and MATH 264.

In Chapter V, I discuss my reference model of the scholarly knowledge from which starts the transposition of multivariable calculus that is ultimately to be taught and learned in 218/264.

Chapter VI centers on the knowledge to be taught (KT) in the two courses. In a first section, I explain my methodology for constructing a model of the knowledge to be taught: how I identified the praxeologies, how I coded and recorded their theoretical and practical blocks, and how and why I cross-referenced among elements of theoretical blocks and among elements of the practical blocks. I present the models that resulted from this work in the second section of the chapter.

In Chapter VII, I restate the aim of having a model of the knowledge to be learned (KL) in the context of this thesis; I present my methodology for identifying the praxeologies of the KL; finally, I present a model of the knowledge to be learned and append to each praxeology a discussion of the units of KT that are present and absent in the KL.

Chapter VIII is a two-part discussion. It opens with a discussion of the *ideal student* in these courses: the student who possesses, at the minimum, the knowledge *required* to provide suitable solutions in the final exam. I describe the ideal MAST 218 student and the ideal MATH 264 student in terms of the structure $(T, \tau; \Theta, \theta)$ of the praxeologies they need to learn to succeed in their course; we will see that the theoretical blocks (Θ, θ) of the praxeologies of the KT are distorted in the transposition into knowledge to be learned. I conclude with some comments on the affordances and limitations of ATD in this study.

I close in Chapter IX by recapitulating the core components of this thesis and reviewing certain aspects of my methodology. In a second section, I briefly conjecture the implications of the absence of certain components of the KT in the KL. I conclude with a brief proposal of a follow-up study aim at capturing the knowledge at each step of the didactic transposition in MAST 218 and MATH 264.

Chapter II: Literature Review

This work is set in undergraduate mathematics education; specifically, in two courses that, together with their sequels, set the tone for MATH and MAST[1] students' first year of undergraduate mathematics education as they move from MAST 218/MATH 264 onto MAST 219/MATH 265, the second halves of a two-term approach to multivariable calculus. The following two-part literature review gives context to this study. I first go over some of the research on the learning of calculus; in particular, I consider studies that have probed into the tasks that students have to do. I then review literature on the affordances of the Anthropological Theory of the Didactic (ATD) to the study of university mathematics education. This sets the stage for the use of the ATD in this study, which I discuss in Chapter III.

Section II.i: Teaching and Learning of Calculus

Research on the learning of calculus often probes at the nature of students' learning of calculus; from their concept image (in the sense of Tall & Vinner, 1981) and use of core notions such as limits (Cottrill et al., 1996; Hardy, 2009b; Swinyard & Larsen, 2012), continuity (Vinner, 1987), and rate of change (Thompson, 1994), down to the hand played by algebraic dexterity (Gray, Loud, & Sokolowski, 2009; White & Mitchelmore, 1996). Important concepts in mathematics education have been developed and initially illustrated against students' learning of calculus; e.g., cognitive obstacles (Tall & Vinner, 1981) and epistemological obstacles (Sierpinska, 1990; Sierpinska, 1994). There's a pattern that indicates calculus students mostly engage in procedural work that requires no more than a superficial image of these concepts; for instance, Tall & Vinner (1981) identify various distinctions between students' personal concept image of limit and continuity and the formal concepts as decreed by the professional mathematical community. Some studies further suggest that the images students form in calculus courses can become epistemological obstacles; for instance, the image students form of limit in traditional calculus courses as something that a function *approaches* might impede their understanding and use of the formal definition of limit (Cornu, 2002).

My study is situated within this body of research; my goal is to describe the knowledge that *multivariable* calculus students need to learn to succeed in the first half of a year-long sequence of courses. In using the lens of the Anthropological Theory of the Didactic (see the following section and Chapter III), I aim to characterize this knowledge in terms of a theoretical block (a discourse unified by a

[1] Recall from the introduction that MATH and MAST are the two programs in which the courses MATH 264 and MAST 218 occur; MATH refers to the Pure and Applied Specialization and MAST to the Major in Mathematics and Statistics.

set of axioms, definitions, and theorems) and practical block (a set of tasks and techniques supported and explained by the theoretical block). In a later stage, I hope to conjecture potential effects of the type of knowledge that is to be learned on students' images of multivariate calculus concepts, and, on a macro-level, on the perceptions or misconceptions they may develop about mathematics. Given that the multivariable calculus courses (MAST 218 and MATH 264) that I consider follow a format similar to that of their earlier counterparts (see section IV.v), I expand on a few studies that have characterized the tasks that calculus students typically do in these earlier counterparts and how these tasks may relate to their conceptualization (or lack thereof) of key calculus notions.

In conjunction with students' particular formulations of calculus concepts, studies also focus on the role played by the routine problems that typify students' single-variable calculus studies (Hardy, 2009b; Lithner, 2004; Selden, Selden, Hauk, & Mason, 1999). These studies emphasize the exercise-driven quality of the course assessments, in the sense of Selden et al.'s (1999) *very* or *moderately routine* problems, which, respectively, "mimic sample problems found in the text or lectures, except for minor changes in wording, notation, coefficients, constants, or functions that students view as incidental to the way the problems are solved" and "can be solved by well-practiced methods" such as "change of variable integration problems in calculus" (p.18).

The exercise-driven quality of the course assessments extends to elements of the course curricula (Lithner, 2004). Calculus textbooks traditionally adhere to a definition-theorem-example-exercise format, wherein the exercises repeat the problematics of the examples and algorithms outlined in the text. Lithner (2004) measured the extent to which intrinsic mathematical properties play a role in the minimal reasoning required to solve routine tasks in traditional Calculus textbooks. His metrics were three types of reasoning, each of which is a step up from the previous in its use of intrinsic mathematical properties:

- identification by similarities (IS), whereby a strategy for tackling a problem is chosen based on the similarities of certain surface properties between the new problem and a known problem (e.g. given a limit-finding problem, note whether the limit is taken at a numerical value of infinity and identify the type of function involved); the implemented strategy is identical to that of the known problem;
- local plausible reasoning (LPR) is required if components in the exercise are similar to a known problem but differ in a few local parts (e.g. if the known exercise involves a whole number exponent but the new exercise involves a non-integer rational exponent); some reasoning

rooted in mathematical properties must be made to ensure the strategy is suitable in the new exercise; the implemented strategy is identical to that of the known problem;

- global plausible reasoning (GPR) is required if the "strategy choice is founded on analyzing and considering the intrinsic mathematical properties of the components in the exercise" and the solution is formed similarly (Lithner, 2004, p. 419).

Lithner's classification of reasoning types runs along a scale of how big a role is played by the mathematical properties intrinsic to the problem versus the reapplication of known algorithms; this scale runs parallel to Selden & Selden's (1999) spectrum of problems from *very routine* to *very non-routine*, which vary based on how familiar the solver is with the given problem. Schoenfeld (1985) similarly distinguishes between the very and the non-routine as simply *exercises* and *problems*. The more routine the problem, the less interaction is required of the solver with the mathematics specific to that problem.

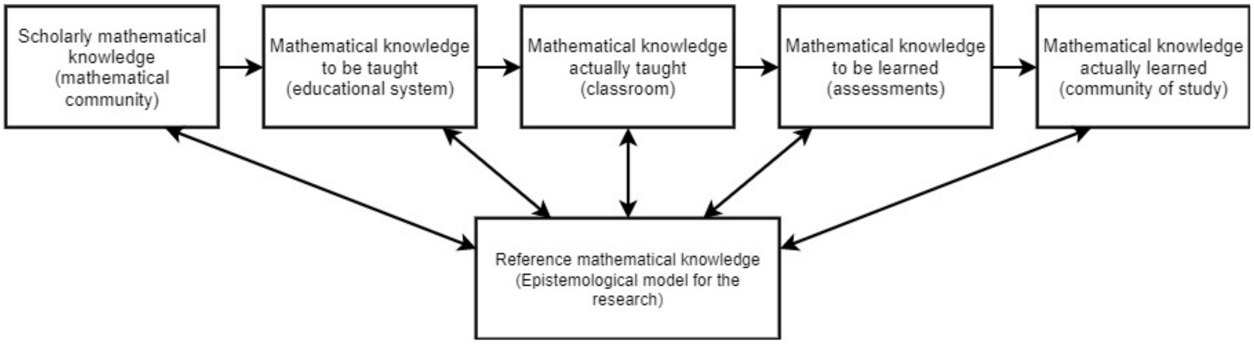
The assessments in North-American calculus courses are largely drawn from the course textbook, which Lithner (2004) demonstrated to be steeped routine problems. Accordingly, he found students' strategies to be anchored in what they recalled superficially rather than in the mathematics specific to the given problems (Lithner, 2000). This correlates with calculus students' failure to complete non-routine problems (Selden et al., 1989; Selden et al., 1999; Hardy, 2009b). Lithner's finding that the great majority of calculus textbook exercises require no more than IS reasoning suggests that students' superficial reliance on intrinsic mathematical properties in favor of the recall of algorithms may have roots in their learning environment – or, in the language of ATD, in their institutional context.

Section II.ii: Using ATD to study UME[2]

Chevallard's *Theory of Didactic Transposition* (1985) lights on the anthropological backdrop of the mathematics found in educational settings. First, it recognizes that this mathematics occurs *in a setting* – that is, an *institution.* From there, the *Theory of Didactic Transposition* underlines the trail whence that contextualized mathematics originated; Chevallard details a transposition that starts with mathematics in its originally produced or scholarly form and phases into mathematics to be taught as prescribed by a curriculum, actually taught by course instructors, and eventually learned by students. The mathematics cannot be separated from its contextual elements. The following is a diagram typically used to represent

[2] To make sense of this brief literature review on the affordances of Anthropological Theory of the Didactic (ATD) to the study of undergraduate mathematics education (UME), I introduce briefly some key concepts that are further expanded and discussed in the chapter addressing the theoretical framework of this thesis (Chapter III).

didactic transposition from scholarly knowledge to knowledge actually learned by students (adapted from Barbé et al. (2005) and Bosch et al. (2005)):



This is a commonly used representation, but its linearity doesn't account for the complexity of didactic transposition as knowledge is likely transposed in more than one direction (e.g. knowledge to be learned may influence knowledge actually taught, in the sense that teaching in the classroom may be informed by any upcoming assessments). The processes involved in the transpositions are complex and most research has focused mainly on the 'final' products of each stage.

The Anthropological Theory of the Didactic (Chevallard, 2002) provides an epistemological model to describe mathematical knowledge as a human activity (thus the anthropology word in the title) as it occurs in a given institution (research mathematics; applied mathematics; engineering; school mathematics at different educational levels; mathematics teacher training institutes, etc.). ATD holds that mathematical knowledge can be described in terms of an organization of mathematical nature. This "organization of mathematical knowledge," called in ATD a (mathematical) praxeology[3], is a special case of a praxeology, a system of four components that together model an activity[4]:

- a set T of types of tasks which indicate the nature and goals of the activity;
- a set τ of techniques available to accomplish each type of task;
- a technology[5] θ that justifies these techniques (a logos about the techniques); and

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[3] Chevallard chose the word "praxeology" to reflect that the model of knowledge refers to a discourse (logos) about a practice (praxis). In some texts, Chevallard and his colleagues refer to praxeologies as organization of knowledge, and more specifically, as mathematical organizations. We stick to the word praxeology to emphasize its nature: the logos about a praxis (in the same sense that the words biology, anthropology, sociology, etc., are used).
[4] Although I expand on the concepts brought forward by the theory of ATD used in this thesis in Chapter III, it is necessary for the discussion here to present these concepts.
[5] The word "technology" is a literal translation form the word "technologie," the French term used by the proponents of ATD. We stick to the word "technology" in this thesis, as it has been so far the preferred English translation, although we agree with colleagues who have recently proposed that "methodology" was perhaps a more accurate

• a theory Θ that justifies the technology.

The ATD models each of the steps in the didactic transposition in the form of these units of knowledge called *praxeologies*, each formed of a theoretical and practical block (Chevallard, 1999). I expand on these in an overview of the theoretical assumptions of ATD in section III.i. For now, it suffices to note that the theoretical block of a praxeology consists of the *technology*, that is, notions and arguments bound by some rational discourse and justified by a *theory*, its conceptual basis. The theoretical block explains and warrants the tasks and corresponding techniques that form the practical block of a praxeology. Furthermore, it is the theoretical block that "makes it possible to preserve the practice and communicate it to others, so that they, too, can participate in it" (this suggests a didactic intention in any (cultural) institutional practice; an activity cannot become part of a practice if there are no means with which to teach and maintain it – hence 'Anthropological Theory of the *Didactic*') (Hardy, 2009b, p.5).

Further, the Anthropological Theory of the Didactic (ATD) maintains that that "the study of any didactic problem needs to adopt a particular standpoint (model) of the involved mathematical practices" (Bosch, Chevallard, & Gascón, 2005, p.4). Such a *reference model* sets the platform from which the researcher observes the didactic transposition in a study of a given didactic problem (see Chapter V).

The ATD lends itself to the study of university mathematics education (UME) and has been used as such on some occasions (Bergé, 2008; Barquero, Bosch, & Gascón, 2008; Hardy, 2009b; De Vleeschouwer, 2010). Winsløw, Barquero, de Vleeschouwer, & Hardy (2014) discuss the affordances of the ATD to the study of UME in the form of its theoretical assumptions and the models it provides. I outline below some of the main themes in the application of ATD to UME.

Section II.ii.a: Conditions and constraints

The conditions and constraints that shape mathematics in UME stem mainly from internal and external sources: e.g., university policies, professors, and students; ministerial policies, society, and cultural context. A university's rules and principles can trickle down to the mathematics classroom; Winsløw et al. mention external funding and admission conditions (p.99) as factors that can ultimately fashion didactic practices.

University professors both teach and research. Since teaching is not their only duty, it can be difficult for teachers of university mathematics to expand on their didactic processes, in terms of choosing

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translation (Giovanniello, 2017; Pelczer, 2017; personal communication between my supervisor and Anna Sierpinska).

questions to work on, techniques to work with, and the theory on which the work is based (Winsløw et al., 2014). This results in a restricted 'action space' (Chevallard, 2002a) as the course mathematics is tightly wound to the textbook and time and curricular constraints lead to "heavily transposed" theory and practice (Winsløw et al., 2014, p.99).

On the other end, external conditions also appear in the shape of students' past and current mathematical and personal experiences, expectations, needs, and aspirations (Winsløw et al., 2014). Students from various schools, cities, states or provinces, and countries arrive with a trove of peculiarities. Their experience in mathematics is colored by their primary and high-school studies and now heads a slew of transition problems throughout their university mathematics studies (discussed further below). Winsløw et al. (2014) attribute students' transition problems to "discontinuities between the mathematical praxeologies that appear in school, at the beginning of university studies, and in the more advanced parts of such studies" (p.99). Students' specific mathematical constraints are tinted with their academic needs and professional expectations (and those imposed by the culture and society they live in); the didactic situation of UME is therefore defined by more than an aim to prepare students for university mathematics and its professional applications/uses.

Section II.ii.b: Transitions in students' praxeologies

Winsløw et al. (2014) explain that an institution's praxeological model is well-fitted with teachers' praxeologies but ill-reflected in students' praxeologies. Teachers include in their view the theoretical and practical blocks as well as the ways in which the theory justifies and explains the practice; students, at the pre-university level and in some cases at the university level, tend to have a praxeology defined mostly by practice. This is especially observed in differential and integral calculus courses where assessment is concerned mostly with the practical block and, while at times it may pick at the theoretical, it generally does not address the ways in which the theoretical maintains the practical. This may have a precedent in the way the knowledge is taught in the classroom, as teachers don't necessarily have time to justify the tasks and techniques, given often-hefty curricula to deliver. Students, for their part, tacitly accept the existence of a theoretical discourse supporting the practical (the techniques) without concerning themselves with it (Hardy, 2009b; Winsløw et al., 2014). Their work focuses mainly in recognizing types of tasks and identifying a suitable technique (Hardy; 2009b; Winsløw et al., 2014), much as in Lithner's identification of similarities reasoning (2004), Schoenfeld's *exercises* (1985), or Selden et al.'s very and moderately routine problems (1999).

As students progress in their undergraduate mathematics studies they undergo two transitions. Where once they might have tacitly ignored theoretical blocks and worked exclusively within the practical block of a praxeology, they increasingly have to engage with theory and technology in their completion of tasks. Winsløw et al. (2014) call the transition from praxeologies that are purely practical to praxeologies that include a theoretical *and* a practical block a *first transition* to university mathematical praxeologies (p.101). For example, prior to the first transition, students complete tasks such as using the derivative rules to find the derivative of a function. Here, differentiability is an always-met *condition* of the functions upon which students act in the tasks they do. A first transition would occur once tasks require students to explicitly *acknowledge* differentiability before applying derivative rules (e.g. '*if* f is differentiable, then …'; or 'f does not have a derivative there because it's not differentiable at that point'). Prior to the *first transition*, it is sufficient for students to attend only to the practical block of the mathematical knowledge; at the other end of the transition, students are required to explicitly acknowledge the theoretical block as the justification for the techniques they choose to use for the completion of a task.

A *second transition* follows when students reach courses whose curricula and assessment prioritize what once may have been the theory and technology of a practical block; as students transition into proof-making and validating, the theoretical blocks of the past become the practical blocks of their present. Students' tasks lie in proving and their techniques directly draw from elements of a theoretical block. For instance, the second transition will have occurred in a student who knows to use the definition or theorems about continuity to prove that, *if a function is continuous*, then some property of that function is true. The characteristics of a second transition are that students both explicitly acknowledge and use the theoretical block to *generate* a technique for achieving a task.

Chapter III: Anthropological Theory of the Didactic

In this chapter, I present an overview of the Anthropological Theory of the Didactic (ATD) by outlining three of its main theoretical assumptions and explaining its affordances to research methodology in the form of *praxeologies*. I conclude with a description of my application of the ATD to this study of two undergraduate multivariable calculus courses.

Section III.i: An overview

This study is set in the framework of the Anthropological Theory of the Didactic (ATD). As discussed just previously in Section II.ii, this *scientific research programme* (Lakatos, 1974) can be used to good effect in research concerning university mathematics education (UME) (Winsløw et al., 2014).

The ATD is driven by a core of three assumptions. First, that the "interactions of students, teachers and mathematical knowledge in [a given teaching context]" are marked by "the transformation of *scholarly knowledge* to *knowledge to be taught* to *knowledge actually taught and learned* (Chevallard 1985; Bosch & Gascón, 2006)" (Winsløw et al., 2014, p.96). This is termed a *didactic transposition*; its study plots the transformation of a unit of knowledge from conception to conversion into knowledge to be taught and then learned in a certain context – that is, in an institution, which must also be taken into account for its role in the transposition. To study a didactic transposition, the researcher must create an epistemological model of the knowledge; this *reference model* frames the research questions, data collection, and data analysis (Bosch et al., 2005; Winsløw et al., 2014).

A second key assumption of the ATD supplies an approach to the study of a didactic transposition: that "human activity, and in particular mathematical activity, can be modelled with the notion of *praxeologies*" (Winsløw et al., 2014, p.97). The ATD therefore calls for "explicit models of mathematical activity" that constitute praxeologies and give stage for the description and analysis of an activity.

Chevallard (1999) explains a praxeology (henceforth, Π) in terms of its theoretical and practical blocks. The latter consists of types of tasks (T) and the techniques (τ) used to perform them; the former couples the theory (Θ) and technologies (θ) that justify the practical block; $\Pi = (T, \tau; \Theta, \theta)$. Winsløw et al. (2014) describe technology[6] as "a set of notions and arguments arranged into a more or less rational discourse to provide a first description, explanation and justification of the techniques, and also to organize the different types of tasks and techniques" and theory as an "abstract set of concepts and arguments, which functions as a basis and support of the technology" (p.97). For instance, the

[6] See note 4 in Section II.ii.

completeness of R (theory) frames the definitions of supremum and infimum (technologies) that can then be used to produce techniques for tasks that do with the bounds of subsets of R. The language of theory-technology-task-technique accommodates praxeologies of different ranges in precision. Depending on whether a praxeology describes a single type of task (e.g. finding the limit of rational functions at points of indetermination), a set of tasks joined by some technology (e.g. finding the limit of rational functions), or a wider range united by a theory (e.g. determining whether the limit of a rational function exists at a point and finding it if it does), it is termed either a *point, local,* or *regional* praxeology (Winsløw et al., 2014).

The environment in which a praxeology occurs is the focus of a third basic assumption. The ATD stresses the importance of "the conditions that enable or favour the development of [a] praxeology in the institution, and the constraints that tend to impede that development" ((Winsløw et al., 2014, p.98). These may occur along several levels of what Chevallard (2002b) presents as a "hierarchy of levels of didactic codetermination": civilization, society, teaching institution, pedagogy, discipline, domain, sector, theme, question/subject (as presented in Winsløw et al., 2014, p. 98).

Section III.ii: Applying the ATD to the study of MAST 218 and MATH 264

The goal of this study is to describe and analyze the minimal units of knowledge that a student needs in order to achieve academic success in MAST 218 or MATH 264. The framework's assumption of the role of didactic transposition in the interactions between teachers, students, and mathematical knowledge suggests the importance of considering the chain that links scholarly knowledge, knowledge to be taught, knowledge actually taught, knowledge to be learned, and knowledge actually learned. To the end of describing the minimal knowledge that is *required* of students to learn in MAST 218 and MATH 264, I construct three models; each corresponding to a moment of didactic transposition:

- A reference model, that is, a model of the scholarly knowledge;
- a model of the knowledge to be taught in MAST 218/MATH 264; more specifically, I model the knowledge that *could* be taught, as decreed by the course outline and textbook. This is distinct from the knowledge *actually* taught, which stems from students' and instructors' combined experiences; and
- a model of the knowledge to be learned, as indicated by the course assessments.

In what follows I describe the operational definitions of the *knowledge to be taught* and *knowledge to be learned*.

Section III.ii.a: Knowledge to be Taught

This section outlines the operational definition of *knowledge to be taught* (KT) proposed in this study. In Section VI.i, I explain at length how I constructed my model of the KT in MAST 218/MATH 264. In short, the source materials are the syllabus (see Appendix C) and textbook, James Stewart's *Multivariable Calculus*, which are the same for both courses. (Potential variations of the textbook edition from year to year do not manifest themselves as a difference in content and approach.) The textbook is formed of chapters, each split into sections. The course outline sequences the sections to be taught and lists end-of-section exercises that form the assignments for the course. By "knowledge to be taught," I mean the praxeologies that constitute the mathematical knowledge in the sections and exercises of the textbook indicated on the course outline.

Before I specify how praxeologies are manifested in the textbook, a note on the notions of theory and technology and how they occur in this text. As discussed in the overview above, the ATD defines theory as the conceptual basis for a technology, which in turn consists of the "rational discourse" and "notions and arguments" within it that together allow for and support the practical block of a praxeology (Winsløw et al., 2014, p.97). I argue that in the case of the mathematical knowledge delivered in this course, technology and theory can be taken as one. There is no clear distinction between the two in the textbook. The technologies throughout are set in the geometry and algebra of three-dimensional space organized in the Cartesian system, and at times in the wider theory of Euclidean metric spaces. However, the concepts and assumptions (the *theory* in the words of ATD) are not made explicit and tend to be weaved into the notions and arguments (the technology). For example, the notions of closed and bounded sets are defined in R^2 and the terms are then used in the statement of the Extreme Value Theorem; there is no further discourse on the concepts of "closed," "bounded," or "sets," nor of their value to the validity of this theorem. Further, the focus of the mathematical knowledge to be taught in this multivariable calculus course is mainly in the practical blocks, as will be seen later on. Thus, for the purpose of this study, it was sufficient to compile the set of items that form the theoretical blocks of the praxeologies of the KT without making any distinctions between theory and technology.

Mathematical knowledge is manifested in the textbook through point, local, and regional praxeologies in the different layers of the structure of Stewart's *Multivariable Calculus*. The syllabus lists sections from four chapters (and a couple of sections from a fifth); the sections from each chapter are unified by a set of technologies and tasks, and it is in this sense that the textbook's chapters form regional praxeologies. For instance, Chapter 14, Partial Derivatives, unifies by the theory of (mostly three-dimensional) Euclidian metric space the praxeologies $\Pi 16$ Functions of Several Variables, $\Pi 17$ Limits and Continuity, $\Pi 18$ Partial Derivatives, $\Pi 19$ Tangent Planes and Linear Approximations, $\Pi 20$ Chain Rule, $\Pi 21$ Directional Derivatives and the Gradient Vector, $\Pi 22$ Maximum and Minimum Values, and $\Pi 23$ Lagrange Multipliers (described in Section VI.ii.b).

Each section in the textbook is centered on a given technology. For instance, section 12.5, Equations of Lines and Planes, consists of a set of tasks and techniques unified by the technology of linear equations (e.g. finding equations for lines, planes, or distances between objects in space). Section 12.4, Cross Product, expands on applications and properties of the technology of cross product. Section 14.4, Tangent Planes and Linear Approximation, is driven by the linearization of functions and includes various tasks produced by this technology: finding the tangent plane to a surface, to approximate a function by its linearization, to find the differential of a function, to show that a function is differentiable at a point… Each section brings with it tasks and associated techniques that are produced and explained by that section's particular technology. The sections of the textbook have the quality of *local praxeologies*.

I consulted the examples and end-of-section exercises listed on the course outline to determine the tasks to be taught in accordance with each section's theoretical block. This is where point praxeologies may arise. For instance, in the section Equations of Lines and Planes, I identified the tasks $T_{10.3.1}$ (to find a vector equation of a plane), $T_{10.3.2}$ (to find a scalar equation of a plane), and $T_{10.3.3}$ (to find a linear equation of a plane). (More on the notation $T_{i.j.k}$ in Section VI.i.) These three are the same type of task $T_{10.3}$, to find the equation of a plane. They all rely on the same technology of vectors and normal vectors and consist of different techniques that accomplish the same task of finding an equation for a plane. Thus, $T_{10.3} = \{T_{10.3.1}, T_{10.3.2,} T_{10.3.3}\}$ constitutes an example of a point praxeology in the knowledge to be taught.

Section III.ii.b: Knowledge to be Learned

In an operational sense, I define the *knowledge to be learned* (KL) as the subset of the knowledge to be taught which students need to know in order to provide solutions to the questions on final exams. This operationalization, although useful to describe and characterize the KL, does not properly reflect the fact that a transposition takes place and that some of the praxeological elements (likely, elements of the theoretical block) are more likely ill-defined than well-defined subsets of the praxeological elements of the KT. While the KL may borrow *elements* of the KT praxeologies, the discourse that unifies the two blocks of a praxeology might be distorted in the transposition.

In Chapter VII, I elaborate on why final exams are suitable for determining the knowledge students must learn, as well as the methodology I adopted to map the KL as a subset of the KT.

Section III.ii.c: Institutional context of the didactic transposition

The ATD contents that no element of the didactic transposition can be talked about in the void; this transposition occurs within a given institution. For instance, the completeness property of R takes on different meanings and roles in the distinct institutions of Calculus and Analysis (Bergé, 2008). Another example: the knowledge learned about limits in Calculus differs from that learned in Analysis. The notion of function occurs in a variety of domains of mathematics, but there's a distinction between its existence in high-school mathematics and a university Set Theory course. The mathematics that exists in different institutions differs according to their respective goals and norms, even if the mathematical knowledge at the source is the same. The didactic transposition of any mathematical knowledge is therefore determined by the institution in which it occurs.

In concert with the importance attributed by the ATD to the institutional context of a didactic transposition, I describe in Chapter IV the institutional contexts of MAST 218 and 264, base my methodology on the institutions formed by these courses (by using their respective curricular and assessment documents as a source of data), and later consider the potential effects of the institutional context on the knowledge to be learned in these courses.

Chapter IV: Institutional Context of MAST 218 and MATH 264

MAST 218 and MATH 264 are housed in two programs in Concordia University's Department of Mathematics and Statistics. MAST 218 is the Multivariable Calculus I course for students in the Major in Mathematics and Statistics (MAST); MATH 264 is the sister course for those in the Pure and Applied Mathematics specialization (MATH). The institutional context of the courses will serve to understand who the students are, their background, what their degree entails, and how these multivariable calculus course fit into their paths of study.

Section IV.i: Mission

The Mathematics and Statistics Department explains the mission of its programs on the departmental website as follows:

> [The Major in Mathematics and Statistics] is designed for students who wish to enter the job market right after graduation. As a Mathematics and Statistics student, you'll uncover the mathematical structure of random systems such as the economy and the stock market, health and survival, and weather forecasting. You will build a solid foundation in linear algebra, calculus, probability and number theory, and learn to use professional software tools for mathematics and data analysis applications.

More precisely,

> The Major in Mathematics & Statistics is a 42-credit program with a common core of 36 credits. It is aimed at students who would like to have a good background in the mathematical sciences, but whose goals are to enter the job market upon graduation, rather than to pursue graduate studies. The focus of the Major is on the applicable nature of the mathematical sciences as tools for solving, and as ways of thinking about, a wide range of problems. Certain selected topics will be covered in each course accompanied by the use of appropriate software applications.[7]

Those who opt for the 60-credit specialization in Pure and Applied Mathematics are promised to

> enter a field that has both a rich history and many future career possibilities. As a mathematician, you'll design and analyze mathematical models and develop systems for testing and evaluation.

[7] http://www.concordia.ca/artsci/math-stats/programs/undergraduate/mathematics-statistics-ba-bsc.html

In essence, you will use mathematics to find creative solutions for systems such as communications, software development, encryption technologies, banking and drug testing.

Through labs and lectures, you will learn to use professional software tools for mathematics and data analysis applications. You'll also learn to think in the abstract and fine-tune your analytical skills.

After graduation, students have the knowledge and skill to design and analyze mathematical systems in any number of scientific or business fields. This program is designed to prepare students for graduate studies.[8]

In sum, the major stream aims for the applications of mathematics and a professional career, while the pure and applied program is a preparation for graduate studies in mathematics.

Section IV.ii: Entry requirements

The cut-off average for those applying to the pure and applied program is a C+ for university transfers, a 75% average for Canadian high-school students, or a 24 CRC[9] for Cegep students; for the MAST, the cut-off is a C for university transfers, 70% for Canadian high-school students, or holding a Cegep diploma. That being said, "students with less than a 70% average in Cegep Mathematics courses [are] required to take a six-credit "transition" Calculus and Linear Algebra course (MAST 214) upon entry into the MATH/STAT Major" and this course does "not count for credits in the major."[10] In any case, the cut-off average for admission is more of an indicator and "may change depending on the applicant pool."[11]

Other than the slight difference in cut-off averages, both programs have the same entry requirements – differential and integral single-variable calculus and introductory linear algebra in the form of courses equivalent to Concordia University's MATH 203, 205, and 204, respectively. In Quebec, those who have a Cegep degree in programs such as the Pure and Applied or Health Sciences will have

[8] http://www.concordia.ca/artsci/math-stats/programs/undergraduate/pure-applied-mathematics-ba-bsc.html
[9] The CRC, cote du rangement collégial in French, is a score that ranks a student's academic performance in Quebec colleges. It is based on statistical methods that consider students' individual performance along with the strength of various groups (e.g. their high school, their classmates in Cegep courses…) Universities use this score for admission purposes.
[10] http://www.concordia.ca/academics/undergraduate/calendar/current/sec31/31-200.html#programs
[11] http://www.concordia.ca/artsci/math-stats/programs/undergraduate/mathematics-statistics-ba-bsc.html
http://www.concordia.ca/artsci/math-stats/programs/undergraduate/pure-applied-mathematics-ba-bsc.html

completed courses equivalent to these. Students may enter the programs without having completed all three courses, as long as they do within the first 30 credits of their degree.

Differential single-variable calculus, often titled "Calculus I," is typically held in a one-term course that covers functional notation, differentiation of polynomials, the power, product, quotient, and chain rules, differentiation of elementary functions, implicit differentiation, higher derivatives, maxima and minima, as well as applications to tangents to plane curves, graphing, related rates, and approximations using the differential; the course concludes with antiderivatives, definite integrals, and area[12], in anticipation of integral calculus. Students learn to apply the differential techniques from Calculus I and integration techniques from the following Calculus II to a set of routine problems, the solutions to which are steeped in algebraic methods.

"Calculus II" acquaints students with techniques of integration (substitutions, integration by parts, partial fractions), improper integrals, physical applications of the definite integral, infinite series and tests for convergence, power series, and Taylor's theorem[13].

Lastly, the introductory linear algebra course includes the "[a]lgebra and geometry of vectors, dot and cross products, lines and planes," and "[s]ystem of equations, operations on matrices, rank, inverse, quadratic form, and rotation of axes."[14] Students are folded into a practice of linear algebra methods and techniques specific to vectors and matrices as well as their applications to the solution spaces of systems of equations.

Section IV.iii: Degree Requirements

The following are the degree requirements for the two programs; the left column indicates the number of credits to be completed from the list of courses on the right, each of which is worth 3 credits.

60 BA or BSc Specialization in Pure and Applied Mathematics

30 MATH 251 Linear Algebra I, 252 Linear Algebra II, 264 Advanced Calculus I, 265 Advanced Calculus II, 354 Numerical Analysis, 361 Operations Research, 364 Analysis I, 365 Analysis II; STAT 249 Probability I, 250 Statistics

[12] Description extracted from Concordia University's Undergraduate Calendar http://www.concordia.ca/academics/undergraduate/calendar/current/sec31/31-200.html
[13] Description extracted from Concordia University's Undergraduate Calendar http://www.concordia.ca/academics/undergraduate/calendar/current/sec31/31-200.htmls
[14] Description extracted from Concordia University's Undergraduate Calendar http://www.concordia.ca/academics/undergraduate/calendar/current/sec31/31-200.html

12 MATH 366 Complex Analysis I, 369 Abstract Algebra I, 370 Ordinary Differential Equations, 464 Real Analysis

3 Chosen from MAST 217 Introduction to Mathematical Thinking, 232 Mathematics with Computer Algebra

9 Chosen from any other 400-level MATH/STAT courses

6 MATH/STAT chosen with prior departmental approval

42 BA or BSc Major in Mathematics and Statistics

33 COMP 218 or 248; MAST 217 Introduction to Mathematical Thinking or COMP 232; MAST 218 Multivariable Calculus I, 219 Multivariable Calculus II, 221 Applied Probability, 232 Mathematics with Computer Algebra, 234 Linear Algebra and Applications I, 235 Linear Algebra and Applications II, 324 Introduction to Optimization, 331 Mathematical Modelling, 333 Applied Statistics

3 Chosen from MAST 330 Differential Equations, 332 Techniques in Symbolic Computations

3 Chosen from MAST 223 Introduction to Stochastic Methods of Operations Research, 334 Numerical Analysis, 335 Investment Mathematics, 397 Topics in Mathematics and Statistics, 398 Reading Course in Mathematics and Statistics

3 Chosen with prior departmental approval[15]

Section IV.iv: Multivariable Calculus in the MAST and MATH Programs

Multivariable Calculus is taught in a sequence of two courses: MAST 218 and 219 in the major program and MATH 264 and 265 in the pure and applied specialization. MAST 218 and MATH 264 are equivalent courses that share the same course outline but have different final examinations. They are requisite for registration to MAST 219 and MATH 265, where again the course outline is identical and the exams different.

The prerequisites for MAST 218/MATH 264 are the same as the mathematics courses in the entry requirements: differential and integral calculus and introductory linear algebra. These courses can therefore be taken early in a student's mathematical studies. 218 and 219 are mandatory courses in the MAST program and are requisite for one more core course: MAST 221, Applied Probability. The list of core

[15] http://www.concordia.ca/academics/undergraduate/calendar/current/sec31/31-200.html

courses common to MATH students and to which 264/265 are a gateway is far longer: STAT 249 (Probability I), STAT 250 (Statistics), MATH 354 (Numerical Analysis), MATH 366 (Complex Analysis I), MATH 370 (Ordinary Differential Equations), and MATH 464 (Real Analysis). This adds up to a third of a MATH student's program credits. Thus, on one hand this study deals with MAST 218, a course which MAST students must pass to obtain their degree but which is not an entry point for the entirety of the program and therefore holds less power over the course of an undergraduate career. On the other hand, there's MATH 264: a gateway to a vast swath of its program's courses.

Section IV.v: Multi-Section Quality of MAST 218/MATH 264

The three prerequisite courses for entry into the MAST and MATH programs (and registration in MAST 218/MATH 264) are actually entry points into all STEM programs. Because of this, they attract an unusually high number of students and are administered accordingly. The hundreds of registrants for each prerequisite course are split into 4-6 (or more) crowded classrooms every semester; these sections are heavily coordinated by a strict curriculum, a course examiner, and common assessments (Sierpinska, Bobos, & Knipping, 2008). A course outline specifies what to teach every week along with recommended exercises from the textbook. A course examiner – a department professor not teaching any of that term's sections – writes the assessments so students in all sections write the same assignments, midterms, and exams. The identical assignments are completed in an online platform which accepts only students' final answer and evaluates it immediately upon submission of a response. Final exams are consistent from term to term in both format and content; the course examiner conducts a common grading system by assigning each instructor the same 3-4 questions to correct on *all* students' exams, thereby implementing some standard in the marking of all students registered in the course. Past final exams are readily available to students at the university print shop as well as online; students' expectations are also given considerable weight, as concern with students' reactions gains traction and prevents changes being made to the final exams. Finally, instructors (both long-standing staff and graduate students teaching the course for the first time) may change from section to section or term to term; but they need to exercise their freedom within the restrictions set by the coordination across sections and terms.

The highly-coordinated flavor of the prerequisite mathematics courses extends to MAST 218 and MATH 264. Even in terms where only a single section is offered for one of these courses, the multi-section quality of the prerequisites persists. MAST 218 and MATH 264 share the same course outline, a document copied and pasted from term to term and which specifies what to teach every week. The weight attributed to the various assessments remains the same as the grade distribution in the mathematics prerequisite courses. The assignments for 218 and 264 are often the same exercises from the book, the difference being that MAST students are allocated the questions whose answers appear at the end of the textbook and MATH students are allocated questions of the same type but whose answers are not available. One exam is given for all MAST sections and another for the MATH sections, and these exams are consistent across terms. On the whole, the previous comments on the features of stemming from the multisection quality of the prerequisite courses are all applicable to the case of 218/264.

Chapter V: Reference Model

I present my reference model (see Sections III.i, III.ii) of the scholarly knowledge at the backbone of the MAST 218/MATH 264 curricula in the discussion below. I considered the scholarly mathematical knowledge upon which rests the multivariable calculus in our courses before and after handling the knowledge to be taught; before, in order to obtain a perspective unimpeded by the specifics of the curriculum, and after, to reflect on how the knowledge to be taught in 218/264 fits into my epistemological model. As discussed in the introduction, one of my main goals is to examine what students are gaining or missing given the minimal core of the knowledge they must learn to succeed in this first multivariable calculus course. The following account of the relevant scholarly mathematics abetted my concluding conjectures on the effects of the knowledge to be learned on students' mathematics learning.

The course outline points at the topics which are to be transposed into knowledge taught and learned: in the first half of the course are parametric curves, polar curves, conic sections, vector algebra, equations of lines and planes, cylinders and quadric surfaces; in the second half, vector functions and vector calculus, multivariable functions and their limits, partial derivatives, tangent planes and linear approximations, the chain rule, directional derivatives and the gradient vector, and a study of maxima and minima of multivariable functions. Upon consulting Marsden, Tromba, and Weinstein's *Basic Multivariable Calculus* (1993), I surmised that the mathematics involved is rooted in the study of curves and surfaces in the context of two regional praxeologies: $\Pi 1$ Curves in Euclidean space and $\Pi 2$ Surfaces in Euclidean space, each unified by the theory of Euclidean metric space. The main tasks in $\Pi 1$ are $T_{1.1}$, to represent plane, space, and n-dimensional curves in parametric form, and the former two in algebraic and graphical form as well; and $T_{1.2}$, to describe these curves at given points. Similarly, the main tasks in the practical block of $\Pi 2$ Surfaces in Euclidean metric space are to represent ($T_{2.1}$) and describe ($T_{2.2}$) surfaces particularly in R^3 but also generally in R^n. $\Pi 1$ and $\Pi 2$ rest on the following concepts taken in context of Euclidean metric space: dimension, coordinate systems, real and vector-valued functions (along with domain, target, range, and graph), injectivity and surjectivity of functions, neighborhood of a point, limits, continuity, and differentiability of multivariable functions; paths in the plane and in space, parametrization of curves, level curves; parametrized and geometric surfaces, level surfaces; and vector algebra.

The theory and tasks outlined above are the extent of the model I formed prior to tackling the knowledge to be taught; there is a wide spectrum of technologies and associated techniques available to support the tasks of representing and describing curves and surfaces. The first few chapters of Luther

Pfahler Eisenhart's *Treatise on the Differential Geometry of Curves and Surfaces* (1909) provide a list of technologies that complete the theoretical blocks of $\Pi 1$ and $\Pi 2$. For instance, he expands on a score of technologies that pertain to space curves, among them the notions of parametric equations, arc length, tangent to a curve, order of contact, normal plane, curvature and radius of first curvature, osculating plane, principal normal and binormal, osculating circle, center of first curvature, torsion, form of a curve in the neighborhood of a point, etc. On the whole, in light of these technologies, the tasks $T_{1.2}$ and $T_{2.2}$ of describing curves and surfaces can be seen to aim for these objects' local geometric properties and invariant quantities.

My reference model of the multivariable calculus scholarly knowledge that serves as a starting point of the transposition into knowledge to be taught to students in the MATH and MAST programs rests much in the application of multivariable calculus to the study of curves and surfaces in the neighborhood of a point. The 218/264 course outline does include sections on particular examples of curves and surfaces – conic sections and quadric surfaces, to be precise (it remains to be seen whether these are approached with the lens of differential calculus). Beyond the affordances of calculus to differential geometry, I include in my reference model knowledge specific to multivariable calculus in the theory of completeness, limits, continuity, differentiability, Riemann integration, etc. in R^n .

Chapter VI: Knowledge to be taught

As discussed in Section III.ii.a, the knowledge to be taught in MAST 218 and MATH 264 can be captured in the form of praxeologies. Based on the assumptions of the ATD, mathematical activity consists of theoretical and practical components and can be modelled accordingly (see Section III.i). This chapter opens with a description of my methodology in creating such a two-part model of the knowledge to be taught: a theoretical block formed of theory and technology; and a practical block formed of tasks and techniques sustained by the theoretical block. This starts with identifying the praxeologies and their theoretical and practical blocks, continues with the coding and recording of the components of these blocks, and ends with the cross-referencing I undertook to bring out the dynamics between the praxeologies throughout the course.

The first steps of the methodology – identifying, recording, and coding the praxeologies – allowed the creation of a model of the knowledge to be taught. Having this model is one of the stated aims of my research, in and of itself, but also serves the further goal of constructing a model of the knowledge to be learned and exploring the inclusion and exclusion of certain units of knowledge in the minimal core of what students need to learn, and what the implications might be for students' calculus and general mathematics studies. The cross-referencing I did also serves this goal; by having a map of connections between the theoretical blocks of the KT (knowledge to be taught) praxeologies and within and across their practical blocks, it was possible to get a more granular image of what students do or do not need to learn in the detail of the ways in which they must learn to use components of one praxeology in the context of another.

Section VI.ii opens with an above-ground view of the knowledge to be taught; that is, a view of the grid of connections between the theoretical blocks of the praxeologies, powered by the cross-referencing discussed in the first part of this chapter. This is followed by a ground-view of the knowledge to be taught, where the contents of the praxeologies are presented: the theoretical blocks first and practical blocks second.

Section VI.i: Methodology

The following is a description of how I constructed the model of the knowledge to be taught in MAST 218/ MATH 264: how I identified the praxeologies; how I recorded and coded the items in their theoretical and practical blocks; and how and why I identified cross-references – making explicit the connections across theoretical blocks and across and within practical blocks.

Section VI.i.a: Identifying the praxeologies

As discussed in Section III.ii.a, the knowledge to be taught in MAST 218 and MATH 264 is defined in this thesis as the praxeologies that constitute the mathematical knowledge in the textbook sections that are listed on the syllabus. This knowledge can be modelled in one fell swoop: both courses have the same course outline, which lists sections along with end-of-section exercises from the courses' textbook, James Stewart's *Multivariable Calculus*. I used the outlines from the years 2014-2017 for both courses. But for two sections on power series (leading to a section on Taylor series, which appears in all outlines) that are included in the Fall 2014 outline for MATH 264 but not in the others, the course outlines are identical.

Thus, my model of the knowledge to be taught is based on those course outlines and on the 8th edition of Stewart's *Multivariable Calculus* – the courses' textbook. The different editions of this textbook bear little to no difference; at most, the functions in the exercises are shuffled or constants are changed. In Section III.ii.a, I explained that each section in the text can be treated as a (local) praxeology; accordingly, I extracted from the textbook the theoretical and practical blocks that constitute each praxeology (Π) and borrowed the section titles to name them. The theoretical blocks consist of the definitions, theorems, corollaries, or properties present in the text. To identify the tasks to be taught in a given Π, I considered the examples as well as the end-of section exercises indicated in the course outline. In most cases, I associated tasks with techniques that are showcased either in the examples or in the discussion portions of the text.

Section VI.i.b: Coding and recording the theoretical and practical blocks

The course outline, which is shared in common by 218 and 264, sequences 24 sections in Stewart's *Multivariable Calculus*. The corresponding praxeologies, denoted by $\Pi1, \Pi2, \dots, \Pi24$, are numbered according to their order of appearance on this outline, that is, the order in which they are to be taught – with the exception of the section on Taylor Series. I coded this review section (indicated on the outline as "review") of material previously taught in MATH 205 (integral calculus) as $\Pi24$ (despite it being the 8th section in line)[16].

As discussed in the previous section, I found the theoretical block of each praxeology in the definitions, theorems, and explanations in the textbook section to which the praxeology corresponds. I

[16] In all honesty, I worked on this section last with the expectation that none of the later sections in the course outline would build on Taylor Series – and they did not, indeed – and forgot to skip a digit in numbering the sections following this one.

recorded the components of the theoretical blocks as items and listed them and the textbook passages that describe them in their order of appearance in the text so as to reflect the discourse that pieces these items together. In particular, items that are presented as theorems in the textbook are indicated as such here as well. The lists of items that form the theoretical block for each praxeology are presented in the next section and the corresponding document with passages extracted from the book in Appendix D. The following is an example of the components of a theoretical block of a praxeology:

> Terms that are defined: Differentiable, Differential, Increment of a function, Linear approximation (or tangent plane approximation), Linear approximation, Linearization, Tangent plane and Equations to describe it
>
> Theorem: Differentiability of a function

This is a list of terms that are defined in the context of a *discourse* (e.g. “equation to describe a tangent plane” does not refer only to the equation but also to the explanation – in the textbook – of how it’s derived from the definition of tangent plane); it is in that way that these form the theoretical block of $\Pi 19$, Tangent Planes and Linear Approximations. These items produce, explain, and justify the techniques used to perform the tasks in the same praxeology.

The practical block of each praxeology Π consists of the tasks and the techniques with which the tasks are to be accomplished in the context of this course. Each task is denoted $\mathrm{T}_{\mathrm{i.j}}$, where i indicates the Π in which the task is to be taught and j serves to differentiate between types of task (see Section III.ii.a for a discussion of the meaning of a ‘task type’). In some cases, a third digit is used, $\mathrm{T}_{\mathrm{i.j.k}}$, when several tasks $\mathrm{T}_{\mathrm{i.j.1}}, \mathrm{T}_{\mathrm{i.j.2}}, \dots, \mathrm{T}_{\mathrm{i.j.n}}$ differ in some sense but have the common goal of some task $\mathrm{T}_{\mathrm{i.j}}$, as in the previously discussed (Section III.ii.a) case of $T_{10.3.1}, T_{10.3.2,}$ and $T_{10.3.3}$, which all have the same goal as $T_{10.3}$, to find an equation for a plane, but each of which aims for a particular equation (vector, scalar, and linear, respectively) . The technique for performing a task $\mathrm{T}_{\mathrm{i.j}}$ is coded $\tau_{\mathrm{i.j}}$, or $\tau_{\mathrm{i.j.1}}, \tau_{\mathrm{i.j.2}}, \dots, \tau_{\mathrm{i.j.n}}$ when several techniques are taught for dealing with a given task. For instance, task $T_{3.2}$, to sketch/graph a curve given in polar coordinates, can be tackled with four techniques: $\tau_{3.2.1}$ (plot points and join them to sketch the curve), $\tau_{3.2.2}$ (use the graph of the curve in Cartesian coordinates to read "the values of r that correspond to increasing values of theta" (Stewart, 2015, p.702), $\tau_{3.2.3}$ (use the symmetries of a curve, if any: symmetry about the polar axis; symmetry about the pole; symmetry about the line pi/2...), and $\tau_{3.2.4}$ (use a graphing device; if it doesn't have a "built-in polar graphing command" (p.705), convert to parametric equations ($T_{3.1}, \tau_{3.1}$)). Finally, I denote particular combinations of tasks and techniques, that is, the practical block of a praxeology, or subsets of the practical block, as $\left(\mathrm{T}_{i,j}, \tau_{i,j}\right)$.

Section VI.i.c: Cross-referencing the praxeologies

I present in this section the cross-referencing I did between the theoretical blocks of the praxeologies and within and across their practical blocks. By "cross-references" in theoretical blocks, I mean instances in which the textbook's definition or explanation involves a term introduced in another section of the text. In the context of the practical blocks, I consider two cases of "cross-references": when similar types of tasks appear in different local praxeologies and when a technique for a given task involves steps that correspond to another task or technique.

In general, references in the textbook are not made explicit; it is up to the reader to recoup the meaning of concepts as they appear throughout the book. This may be challenging for students at this level. An implicit reference makes one of two presumptions: that students have internalized the concept, or that they at least recognize that is has been previously introduced. As will be seen later in Chapter VII, however, students' learning may not be driven by the same principles as those that run the construction of mathematical knowledge as material to be taught. For that matter, an explicit outline of the cross-references that connect the praxeologies in the knowledge to be taught may help determine the missing links in what students need to learn (recall from Section III.ii.b the working definition of knowledge to be learned as a subset of knowledge to be taught).

The main goal of making explicit the connections between the theoretical blocks of the local praxeologies is to explain the role of these praxeologies in the course as a whole. For example, I found that the regional praxeologies of Partial Derivatives (local praxeologies $\Pi 16 - \Pi 23$) and Vector Functions (local praxeologies $\Pi 12 - \Pi 15$) rely greatly on the local praxeology $\Pi 10$, Equations of Lines and Planes. This, along with $\Pi 10$ belonging to the domain of linear algebra and not calculus, suggests that this local praxeology is included in the knowledge to be taught to support the technologies that appear in the praxeologies of Partial Derivatives and Vector Functions: equations of planes and lines are necessary for construction of technologies that appear in the praxeologies of Partial Derivatives (e.g., the definition and equation of a normal line to a level surface at a point) and Vector Functions (e.g., the definition and equation of an osculating plane to a curve at a point). A map of the dynamics between the praxeologies whose knowledge is to be taught may therefore help provide reasons for which knowledge is or isn't to be learned; in particular, a map might elucidate why some praxeologies or certain portions of praxeologies have wider representation that others in the final examinations.

The discussion on theoretical blocks extends to the practical blocks; cross-references help underline tasks and techniques that are particularly useful in a variety of praxeologies, in the knowledge both to be taught and learned. Cross-referencing the task-technique pairs that are steps in other tasks ensured the inclusion in the model of knowledge to be learned of task-technique pairs that might be too basic to be specifically tested for on a final exam. For example, students are never specifically tasked with finding the value of a multivariable function ($T_{16.1}, \tau_{16.1}$); this is hardly the making of a meaty exam problem. But it's certainly essential for students to be able to execute this task, as is indicated by the frequency with which ($T_{16.1}, \tau_{16.1}$) shows up in the description of solutions to exam problems (see the models of knowledge to be learned in Section VII.iii). On the whole, then, cross-referencing subsets of the practical blocks contributes to the end-game of identifying the knowledge that students must learn to succeed in the course and the knowledge that isn't quite as essential for that.

On another but related note, I found an operational advantage to having cross-references. This has to do with the KL model thought as a subset of the KT model. For instance, consider technique $\tau_{21.5}$ for performing task $T_{21.5}$, "to find the equation of the normal line to a level surface at a point," from the local praxeology $\Pi 21$ Directional Derivatives and the Gradient Vector:

$\boldsymbol{\tau_{21.5}}$

find the gradient vector at that point ($T_{21.2}, \tau_{21.2}$) and use that as the line's directional numbers to form its symmetric equations ($T_{10.1.3}, \tau_{10.1.3.1}$).

The two steps in $\tau_{21.5}$ correspond to task-technique pairs identified elsewhere; one in this praxeology, $\Pi 21$, and one in another, $\Pi 10$. These cross-references proved useful in modelling the knowledge to be learned to solve exam problems that involved normal lines to level surfaces. Rather than retrace all the steps that may be involved in the solutions to these questions, I was able to use these cross-references instead.

In sum, the overarching purpose of the model of the knowledge to be taught is to model the knowledge to be learned. Mapping connections in the form of cross-references between and within the praxeologies of the knowledge to be taught adds to the description of knowledge to be learned. The contribution of cross-references manifests itself most in the analysis of the knowledge to be learned; they can explain the presence or absence of praxeologies (or parts thereof) and suggest the effects of inclusions and exclusions.

Section VI.i.c.i: Cross-references across theoretical blocks

In each theoretical block Πi, I tagged items with the theoretical block components to which they make reference to and that belong to Πj, when $i \neq j$. Consider the terms that are defined in the theoretical block of $\Pi 10$, Equations of Lines and Planes:

> Direction numbers of a line [Tages: $\Pi 6, \Pi 7$], Distance from a point to a plane in R^3 [Tags: $\Pi 6$], Linear equations in R^3 [Tags: $\Pi 6$], Normal vector of a plane [Tags: $\Pi 7, \Pi 8$], Parametric equations of a line [Tags: $\Pi 1, \Pi 6, \Pi 7$], Point-slope form of a line [Tags: $\Pi 6$], Scalar equation of the plane [Tags: $\Pi 6$], Skew lines, Symmetric equations of a line [Tags: $\Pi 6, \Pi 7$], Vector equation of a line [Tags: $\Pi 6, \Pi 7, (\Pi 1)$], Vector equation of a line segment [Tags: $\Pi 7$], Vector equation of the plane [Tags: $\Pi 6, \Pi 7, \Pi 8$]

For instance, item **"**Normal vector of a plane" is tagged with "$\Pi 7$" and "$\Pi 8$" because the notions of vectors and parallel vector (defined in $\Pi 7$) and of orthogonal vectors (defined in $\Pi 8$) occur in the textbook's definition of normal vector of a plane (see Appendix D for the textbook excerpts defining the items in the theoretical blocks). This work is summarized in a map (presented soon in Section VI.ii) of the cross-references made in theoretical blocks of different praxeologies.

Section VI.i.c.ii: Cross-references within and across practical blocks

Cross-references in practical blocks make explicit the interconnections between tasks and techniques across and within praxeologies. In my model, where each type of task is given by a code $T_{i.j}$ and a description of the task, I tagged a task with the code of another task if they have a *similar* goal and involve similar technology; by 'tagging' I mean that in the description of a task, I reference in brackets the task to which it is similar. I illustrate with an example: consider the two tasks

> $\boldsymbol{T_{22.4}}$ (from $\Pi 22$ Maximum and minimum values)
>
> To use extreme values in conjunction with techniques from other praxeologies to solve various problems in 2 or 3 dimensions
>
> $\boldsymbol{T_{23.2}}$ (from $\Pi 23$ Lagrange multipliers)
>
> To solve extreme value problems that can be modeled by a function f(x,y,z) subject to a constraint g(x,y,z) = k

These tasks are *similar* in that they both have the goal of solving extreme value problems that can be modelled by multivariable functions and completed with techniques based in the technology of partial derivatives. In my model, I therefore 'tagged' $T_{22.4}$ by writing $T_{23.2}$ in brackets next to it (i.e. "$T_{22.4}$ $(T_{23.2})$") and vice-versa to indicate the similarity.

Techniques $\tau_{\mathrm{i.j}}$ are tagged with task-technique pairs that correspond to steps in the process required by $\tau_{\mathrm{i.j}}$; 'tagging' here means that in the description of a technique, I referenced in brackets task-technique pairs that corresponded to each step. For example, consider task $T_{13.2.1}$ and its associated technique $\tau_{13.2.1}$:

$\boldsymbol{T_{13.2.1}}$

To find the unit tangent vector at a point

$\boldsymbol{\tau_{13.2.1}}$

- find the tangent vector at the give point, which is the task-technique pair $(T_{13.1}, \tau_{13.1})$

- then find the unit vector in that direction, which is $(T_{7.4}, \tau_{7.4.1})$.

Just as with the map of the connections between theoretical blocks, identifying the use of tasks in the completion of other tasks helps uncover the dynamics of the knowledge to be taught throughout the course. Additionally, these references were especially helpful in creating a model of the knowledge to be learned; as discussed later in section VII.ii, I identified praxeologies of knowledge to be learned from the knowledge required to solve final exam questions; to this end, I constructed solutions to exam questions by identifying tasks from the praxeologies of the knowledge to be taught. Thus, it was useful to have cross-references describing steps in the technique for a given task, as these traced out all the steps that might be required for the solution to an exam question.

Section VI.ii: The model

I present in this section the model of the knowledge to be taught in MAST 218/MATH 264. I do this in two stages. First, we consider the praxeologies from afar, focusing on the connections forged by the cross-referencing between them rather than the mathematical content specific to teach. I call this a sky-view of the model, as it allows us to see the praxeologies mapped out in the course as a whole, but not the features that make up each praxeology. We zoom in on the praxeologies in the next stage: the ground-view of each praxeology consists of the row of items forming its theoretical block and a tree of the tasks and techniques that make up its practical block. Thus, the praxeologies that comprise the knowledge to be taught are mapped along their theoretical and practical axes in section VI.ii.b; but first, in section VI.ii.a, I acknowledge that these praxeologies are to be taught (in some way or another) in the context of a course. Taken together, the ground- and sky-view constitute a multi-dimensional model of the knowledge to be taught.

Section VI.ii.a: Praxeologies, as seen from the sky

The first figure I present is a bird's eye view of the praxeologies that together make up the knowledge to be taught (see Figure 2). More precisely, this figure is a map that features the cross-references I identified across the theoretical blocks (see Section VI.i.c, Cross-referencing the praxeologies). Each local praxeology appears in a box. Recall that the praxeologies in my model each correspond to a section from the course textbook; there, these sections are subdivisions of chapters, and so I reflect this structure by color coding the praxeology boxes by chapter. As explained in the language of the ATD (section III.ii.a), each chapter constitutes a regional praxeology. Thus, color-coding by chapter helps trace the connections between local praxeologies both within the regional praxeology to which they belong and across the totality of knowledge to be taught. For instance, the regional praxeology of Partial Derivatives, previously identified as the sum total of the local praxeologies $\Pi 16$ through $\Pi 23$ (see section III.ii.a), builds a great deal on theory and technology from the local praxeologies $\Pi 6$ Three-Dimensional Coordinate Systems, $\Pi 7$ Vectors, $\Pi 8$ Dot Product, $\Pi 10$ Equations of Lines and Planes, so these become "pre-requisite" praxeologies ("pre-requisite" knowledge). The local praxeologies that do with gradients require technology pertaining to $\Pi 12$ Vector Functions and Space Curves and $\Pi 13$ Derivatives and Integrals of Vector Functions. Therefore, full grasp on the theoretical blocks of the regional praxeology of Partial Derivatives includes not only the theory and technology specific to that praxeology but also components of local praxeologies belonging to a different regional praxeology to which it refers or on which it builds – "pre-requisite" praxeologies. In the next section, I present the theoretical block for each local praxeology in the form of a list of the components specific to that praxeology; the map of the cross-referencing between them, shown below, makes explicit the 'prerequisite' relations.

A few words on how to read the map of cross-references between the KT theoretical blocks (Figure 2): two boxes are bound by an arrow if the theoretical block of the praxeology in the box at the endpoint of the arrow has items that build on concepts introduced in the praxeology at the initial point of the arrow. Arrows are either full or dotted. A full arrow indicates that the endpoint praxeology uses a term that was defined in the initial-point praxeology; in this case, the reader would have to know the concept from that 'initial' praxeology to understand it in the new context. For instance, the definition of the arc length function of a curve in $\Pi 14$ builds on the concept of vector function from $\Pi 12$; such a reference would warrant a full arrow. A dotted arrow indicates that the endpoint praxeology involves a concept that had previously appeared in the initial-point praxeology, but that the discourse in the endpoint praxeology provides enough information for this concept to be understood without reference to its initial occurrence in the book. For instance, while the notion of trace of a surface first appears in $\Pi 11$ (Cylinders and Quadric Surfaces), it comes up again in $\Pi 16$ (Functions of Several Variables), $\Pi 18$ (Partial Derivatives), and $\Pi 19$ (Tangent Planes and Linear Approximations); in none of these cases is it necessary to have first read the definition of trace in $\Pi 11$. The distinction between the two types of cross-references highlights the extent to which a given local praxeology provides mathematical knowledge that is needed in other local praxeologies. In the grand scheme of things, this might help explain the presence or absence of tasks from some KT praxeologies in the knowledge to be learned (see sections VIII.i.b and VIII.i.c).

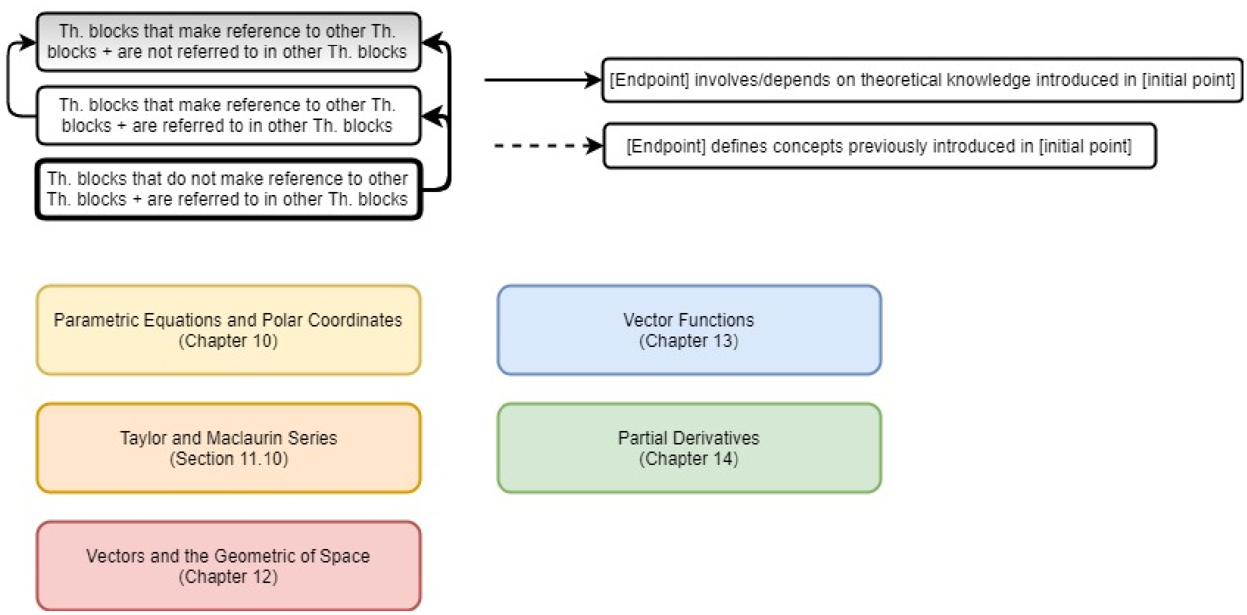


Figure 1. Description of elements in the map (*Figure 2*) of cross-references between theoretical blocks of Π of knowledge to be taught.

In Figure 2, the map of cross-references between theoretical blocks of praxeologies of the knowledge to be taught, the local praxeologies are represented in color-coded boxes, as in Figure 1. The color reflects the textbook chapter in which the local praxeology occurs as a section. The black-and-white boxes in Figure 1 indicate the kinds of connections that the theoretical block of a local praxeology may have to the theoretical blocks of local praxeologies. Figure 2 follows below:

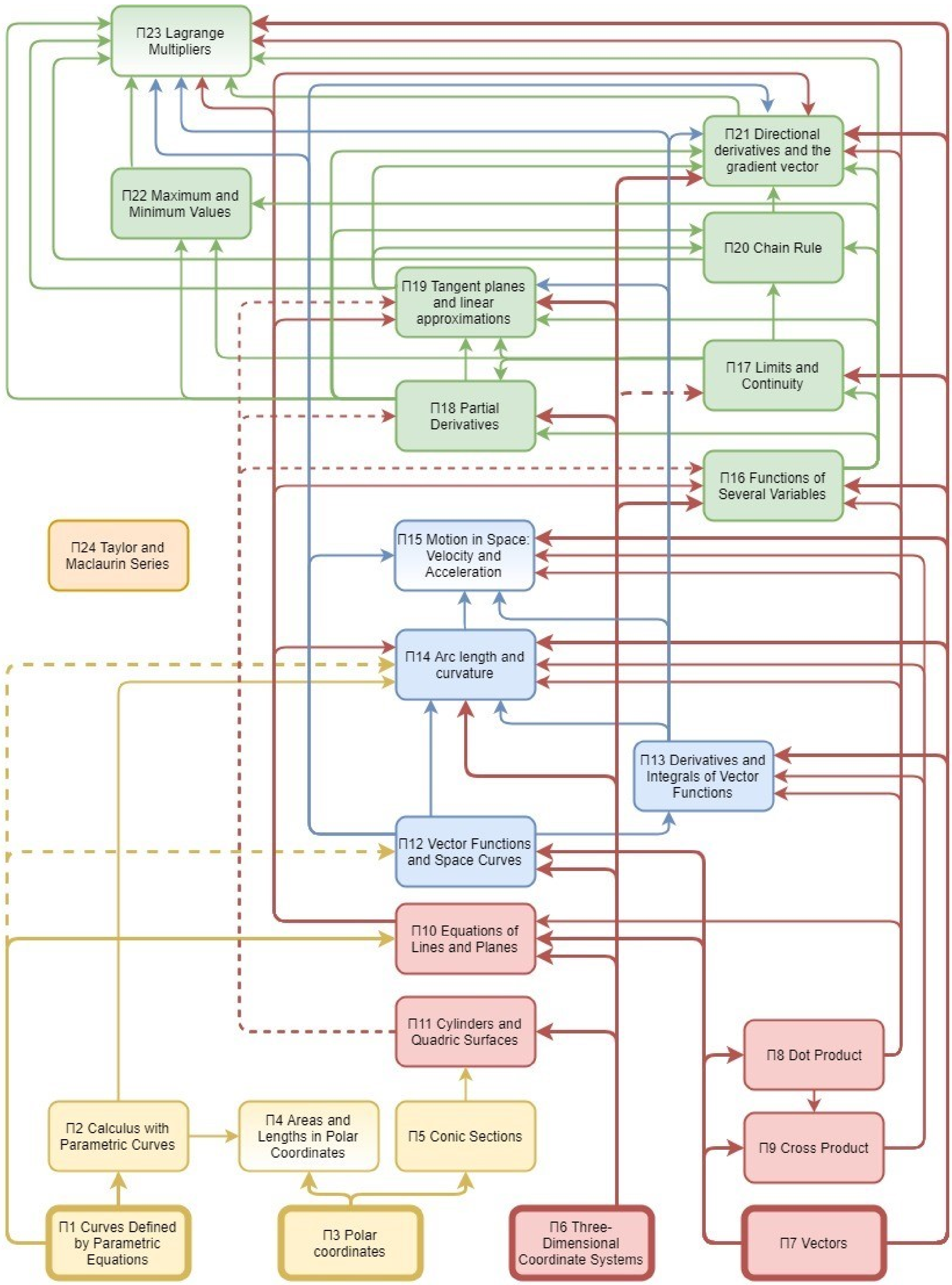


Figure 2. Map of the cross-references between theoretical blocks of Π of the knowledge to be taught in MAST 218/MATH 264.

The map of the praxeologies of 218/264 suggests that they are taught in a way such that the earlier half of the course supplies technologies that can be used to support theory and practice in the later praxeologies. Students are first introduced to parametric plane curves; they are taught polar coordinates and how to find areas and lengths of polar curves; this first chapter ends with the equations of conic sections. The next chapter introduces the Cartesian three-dimensional coordinate system, direction vectors, defines vectors in terms of components, and goes over vector operations and applications; this chapter concludes with the equations of cylinders, quadric surfaces, lines, and planes. From here on, the third and fourth chapters covered in the course build on the praxeologies from the first two. In particular, the chapter on vector functions recycles the notions of parameter and parametric curves seen at the start of the course. All praxeologies related to vector functions build on vector theory and technology introduced in $\Pi 7$ Vectors, and many also make use of theory from $\Pi 8$ Dot product and $\Pi 9$ Cross product. The last chapter, Partial Derivatives, dips into the three-dimensional coordinate system presented in $\Pi 6$ and uses the technology of vectors and the dot product to do so. Vector functions and their derivatives come up in the teaching of $\Pi 19$ Tangent planes and linear approximations, $\Pi 21$ Directional derivatives and the gradient vector, and $\Pi 23$ Lagrange multipliers.

Some of the local praxeologies in the former half of the course are not built on to the same extent as the notions of parameter and vectors: $\Pi 2$, the calculus of parametric curves, is not referred to in the later teachings of the calculus of general space curves (barring the exception of the generalization of the arc length formula, where the authors remind the reader of the more specific version seen earlier on). Polar coordinates ($\Pi 3$), areas and lengths in polar coordinates ($\Pi 4$), conic sections ($\Pi 5$), and cylinders and quadric surfaces ($\Pi 11$) are an independent bunch as well. Knowledge of the equations and definitions of the curves and surfaces covered in $\Pi 5$ and $\Pi 11$ is not necessary for handling of the knowledge to be taught in later chapters. The notion of traces first introduced in $\Pi 11$ does reappear in $\Pi 16$ Functions of several variables, $\Pi 18$ Partial derivatives, and $\Pi 19$ Tangent planes and linear approximations, but its use in these later praxeologies is in no way dependent on its initial appearance in $\Pi 11$.

On the whole, the pairings of vector functions with space curves and multivariable functions with surfaces are the crown jewels of our multivariable course... On the fringes of this dominion, there lurk Taylor and Maclaurin Series. The power to approximate that these series afford to those who yield them is left unspoken; the knowledge to be taught is rather a regimen of finding the Taylor series and radii of convergence of classes of functions familiar from high-school algebra courses (neatly tabled and summarized at the end of the chapter).

Section VI.ii.b: Praxeologies, as seen from the ground

Each praxeology is presented in this section as the combination of its theoretical block – the list of items (definitions, theorems, etc.) of which it consists – and its practical block – the tasks and techniques justified and explained by the theoretical block. The sky-view of the praxeology gave us a perspective on how the praxeologies build on one another; this, in turn, allowed us to surmise that some local praxeologies may be taught for the prerequisite quality of their mathematical content, that is, for their use in praxeologies that are taught later in the term. The view from above also highlighted some 'islands' in the knowledge to be taught – that is, local praxeologies that are either faintly or not at all connected to the majority of the praxeologies. Now we'll walk the grounds of the praxeologies: the theoretical and practical blocks formed by the principles discussed in section III.ii.a and the methodology described in section VI.i.

Section VI.ii.b.i: The Theoretical Blocks

The following is the list of items (terms that are defined, theorems) that, together with the discourse that binds them, form the theoretical block for each praxeology. If an item builds on an item from a different Π, this Π is tagged in brackets. The excerpts from the textbook explaining each item are in Appendix D.

$\Pi 1$ Curves Defined by Parametric Equations: Initial point of a parametric curve, Parameter, Parametric curve, Parametric equations, Terminal point of a parametric curve

$\Pi 2$ Calculus with Parametric Curves: Arc Length of a parametric curve [Tags: $\Pi 1$], Area under a parametric curve [Tags: $\Pi 1$], Concavity of a parametric curve [Tags: $\Pi 1$], Tangent of a parametric curve representing a differentiable function [Tags: $\Pi 1$].

$\Pi 3$ Polar Coordinates: Transformations between the polar and coordinate systems, Polar axis, Polar coordinates, Polar coordinate system, Pole, Product Rule, Tangent to a polar curve [Tags: $\Pi 2$].

$\Pi 4$ Areas and Lengths in Polar Coordinates: Arc length of a polar curve [Tags: $\Pi 2$], Area of a region bounded by a polar curve, Area of a sector of a circle, Product Rule, Riemann sums.

$\Pi 5$ Conic Sections: Axis - Major axis of an ellipse, Minor axis of an ellipse, Of a parabola; Cartesian equation of an ellipse, Cartesian equation of a hyperbola, Cartesian equation of a parabola, Directrix of a parabola, of a conic section; Eccentricity, Ellipse – equation of, focus of; of a hyperbola, of a parabola, of a conic section; Hyperbola, Parabola, Rotation in polar coordinates [Tags: $\Pi 3$], Theorem 5.1 – Classification of conic sections by eccentricity [Tags: $\Pi 3$], Theorem 5.2 – Polar equation of conic section [Tags: $\Pi 3$], Vertex of an ellipse, of a hyperbola, of a parabola.

Π6 Three-Dimensional Coordinate Systems: Coordinate axes, Coordinates in space, Coordinate planes, Distance formula in three dimensions, Equation of a sphere, Equation in R^3s, Origin, Projection onto the coordinate planes, Three-dimensional rectangular coordinate system.

Π7 Vectors: Algebraic scalar multiplication, Algebraic vector addition, Components of a vector, Definition of scalar multiplication, Definition of vector addition, Difference of vectors, Equivalent/Equal vectors, Initial point of a vector, Length of a vector in R^2 and in R^3, Magnitude/length of a vector, Negative of a vector, Parallel vectors, Position vector, Properties of vectors, Relation between the representation of a vector and the vector, Representations of a vector, Standard basis vectors, Terminal point of a vector, Triangle law for vector addition, Unit vector, Unit vector in the direction of a given vector, Vector - Displacement vector, n-dimensional vector; Zero vector

Π8 Dot Product: Angle between two vectors [Tags: $\Pi7$], Components of the unit vector in the direction of a given vector [Tags: $\Pi7$], Direction angles [Tags: $\Pi7$], Direction cosines [Tags: $\Pi7$], Displacement vector of an object [Tags: $\Pi7$], Dot (/scalar/inner) product [Tags: $\Pi7$], Orthogonal (perpendicular) vectors [Tags: $\Pi7$], Orthogonality and dot product [Tags: $\Pi7$], Projection - Scalar projection [Tags: $\Pi7$], Vector projection [Tags: $\Pi7$]; Properties of the Dot Product [Tags: $\Pi7$], Theorem: dot product of two vectors and the angle between them [Tags: $\Pi7$], Work done by a constant force [Tags: $\Pi7$]

Π9 Cross Product: Coplanar vectors [Tags: $\Pi7$], Corollary of Theorem 9.2: parallel vectors [Tags: $\Pi7$], Cross product [Tags: $\Pi7, \Pi8$], Determinant of order 2, of order 3; Geometric interpretation of cross product [Tags: $\Pi7$], Geometric interpretation of triple scalar product [Tags: $\Pi7$], Properties of the cross product [Tags: $\Pi7$], Scalar triple product [Tags: $\Pi7, \Pi8$], Theorem 9.1: Orthogonality of cross product [Tags: $\Pi7, \Pi8$], Theorem 9.2: Magnitude of the cross product [Tags: $\Pi7, \Pi8$], Torque [Tags: $\Pi7, \Pi8$], Vector triple product

Π10 Equations of Lines and Planes: Direction numbers of a line [Tages: $\Pi6, \Pi7$], Distance from a point to a plane in R^3 [Tags: $\Pi6$], Linear equations in R^3 [Tags: $\Pi6$], Normal vector of a plane [Tags: $\Pi7, \Pi8$], Parametric equations of a line [Tags: $\Pi1, \Pi6, \Pi7$], Point-slope form of a line [Tags: $\Pi6$], Scalar equation of the plane [Tags: $\Pi6$], Skew lines, Symmetric equations of a line [Tags: $\Pi6, \Pi7$], Vector equation of a line [Tags: $\Pi6, \Pi7, (\Pi1)$], Vector equation of a line segment [Tags: $\Pi7$], Vector equation of the plane [Tags: $\Pi6, \Pi7, \Pi8$]

Π11 Cylinders and Quadric Surfaces: Cone [Tags: $\Pi5, \Pi6$], Cylinder [Tags: $\Pi6$], Ellipsoid [Tags: $\Pi5, \Pi6$], Elliptic paraboloid [Tags: $\Pi5, \Pi6$], Hyperboloid of one sheet [Tags: $\Pi5, \Pi6$], Hyperboloid of two sheets

[Tags: $\Pi5, \Pi6$],Hyperbolic paraboloid [Tags: $\Pi5, \Pi6$], Quadric surface [Tags: $\Pi5, \Pi6$], Rulings of a cylinder [Tags: $\Pi6$], Traces (or cross-sections) of a surface [Tags: $\Pi6$]

$\Pi12$ Vector Functions and Space Curves: Component functions of a vector function [Tags: $\Pi7$], Continuous vector function [Tags: $\Pi7$], Limit of a vector function [Tags: $\Pi7$], Parameter [Tags: $\Pi6, \Pi7, (\Pi1)$], Parametric equations of a space curve [Tags: $\Pi6, \Pi7, (\Pi1)$], Space curve [Tags: $\Pi6, \Pi7, (\Pi1)$], Vector-valued function [Tags: $\Pi7$]

$\Pi13$ Derivatives and Integrals of Vector Functions: Derivative of a vector function [Tags: $\Pi7, \Pi12$], Integral of a continuous vector function [Tags: $\Pi7, \Pi12$], Second derivative of a vector function [Tags: $\Pi12$], Tangent line [Tags: $\Pi7, \Pi12$], Tangent vector [Tags: $\Pi7, \Pi12$], Theorem 13.1: derivative of a vector function [Tags: $\Pi7, \Pi12$], Theorem 13.2: differentiation rules [Tags: $\Pi7, \Pi8, \Pi9, \Pi12$], Theorem 13.3: Fundamental theorem of calculus [Tags: $\Pi7, \Pi12$], Unit tangent vector [Tags: $\Pi7, \Pi12$]

$\Pi14$ Arc Length and Curvature: Arc length function [Tags: $\Pi7, \Pi12, \Pi13, (\Pi1)$], Binomial vector [Tags: $\Pi7, \Pi8, \Pi9, \Pi12, \Pi13$], Corollary of Theorem 14.1: Curvature of a plane curve, Curvature [Tags: $\Pi7, \Pi12, \Pi13$], Length of a curve [Tags: $\Pi2, \Pi6, \Pi7, \Pi12, \Pi13$], Normal plane [Tags: $\Pi8, \Pi10, \Pi13$], Normal vector [Tags: $\Pi7, \Pi8, \Pi9, \Pi12, \Pi13$], Osculating circle (or circle of curvature) [Tags: $\Pi8, \Pi10, \Pi13$], Osculating plane [Tags: $\Pi8, \Pi10, \Pi13$], Parametrizing a curve with respect to arc length [Tags: $\Pi7, \Pi12, \Pi13, (\Pi1)$], Smooth curve [Tags: $\Pi7, \Pi12, \Pi13$], Smooth parametrization [Tags: $\Pi7, \Pi12, \Pi13$], Theorem 14.1: Curvature of a curve using only its vector function [Tags: $\Pi7, \Pi9, \Pi12, \Pi13$]

$\Pi15$ Motion in Space – Velocity and Acceleration: Acceleration [Tags: $\Pi12, \Pi13$], Newton's Second Law of Motion [Tags: $\Pi12$], Normal component of acceleration [Tags: $\Pi7, \Pi8, \Pi9, \Pi12, \Pi13, \Pi14$], Speed [Tags: $\Pi7, \Pi12, \Pi13$], Tangential component of acceleration [Tags: $\Pi7, \Pi8, \Pi9, \Pi12, \Pi13, \Pi14$], Velocity vector [Tags: $\Pi7, \Pi12, \Pi13$]

$\Pi16$ Functions of Several Variables: Function of two variables [Tags: $\Pi6$], Domain of a two-variable function [Tags: $\Pi6$], Graph of a two-variable function [Tags: $\Pi6$], Linear function of two variables [Tags: $\Pi6$, $\Pi10$], Range of a two-variable function [Tags: $\Pi6$], Level curves of a two-variable function [Tags: $(\Pi11)$]; Function of three variables, Domain of a three-variable function, Level surfaces of a three-variable function; Function of n variables [Tags: $\Pi7, \Pi8$], Domain of an n-variable function [Tags: $\Pi7, \Pi8$]

$\Pi17$ Limits and Continuity of Multivariable Functions: Continuous [Tags: $\Pi16$], Limit laws [Tags: $\Pi16$], Limit of a two-variable function [Tags: $\Pi16$], Polynomial function of two variables [Tags: $\Pi16$], Continuity of polynomial functions [Tags: $\Pi16$], Rational function of two variables [Tags: $\Pi16$], Continuity of rational functions [Tags: $\Pi16$]

$\Pi18$ Partial Derivatives: Clairaut's Theorem [Tags: $\Pi16$, $\Pi17$], Interpretation of partial derivatives [Tags: $\Pi6, \Pi11, \Pi16$], Partial derivatives of a two-variable function [Tags: $\Pi16$], Partial derivatives of functions of three or more variables [Tags: $\Pi16$], Partial derivatives of order 3 or higher [Tags: $\Pi16$, $\Pi17$], Partial differential equations, Second partial derivatives [Tags: $\Pi16$]

$\Pi19$ Tangent Planes and Linear Approximations: Differentiable [Tags: $\Pi16, \Pi18$], Differential [Tags: $\Pi16, \Pi18$], Increment of a function [Tags: $\Pi16, \Pi18$], Linear approximation (or tangent plane approximation) [Tags: $\Pi16, \Pi18$], Linear approximation of a two-variable function in the notation of differentials [Tags: $\Pi16, \Pi18$], Linearization [Tags: $\Pi16, \Pi18$], Tangent plane - Definition [Tags: $\Pi7, \Pi13, \Pi16, \Pi17, \Pi18, (\Pi11)$], Tangent plane - Equation [Tags: $\Pi7, \Pi10, \Pi13, \Pi16, \Pi17, \Pi18, (\Pi11)$], Theorem: Differentiability of a function [Tags: $\Pi16, \Pi17, \Pi18$]

$\Pi20$ Chain rule: Chain rule (case 1) [Tags: $\Pi16, \Pi18, \Pi19$], Chain rule (case 2) [Tags: $\Pi16, \Pi18, \Pi19$], Chain rule (general version) [Tags: $\Pi16, \Pi18, \Pi19$, Differentiation of a single-variable function defined implicitly [Tags: $\Pi16, \Pi17, \Pi18, \Pi19$], Differentiation of a two-variable function defined implicitly [Tags: $\Pi16, \Pi17, \Pi18, \Pi19$]

$\Pi21$ Directional Derivatives and the Gradient Vector: Directional derivative [Tags: $\Pi6, \Pi7, \Pi10, \Pi13, \Pi16, \Pi18$], Directional derivative in the notation of the gradient vector [Tags: $\Pi8, \Pi19$], Gradient vector [Tags: $\Pi7, \Pi8, \Pi12, \Pi16, \Pi18, \Pi19$], Normal line to a surface at a point [Tags: $\Pi8, \Pi10, \Pi12, \Pi13, \Pi16, \Pi19, \Pi20$], Tangent plane to a level surface at a point [Tags: $\Pi8, \Pi10, \Pi12, \Pi13, \Pi16, \Pi19, \Pi20$], Theorem 21.1: Existence of the directional derivatives of a function, formula in terms of partial derivatives [Tags:$\Pi7, \Pi16, \Pi18, \Pi19, \Pi20$], Theorem 21.2: Maximizing the directional derivative [Tags: $\Pi7, \Pi8, \Pi19$]

$\Pi22$ Maximum and Minimum Values: Absolute maximum [Tags: T $\Pi16$], Absolute minimum [Tags: $\Pi16$], Bounded set, Closed set, Critical point [Tags: $\Pi16$, $\Pi18$], Extreme value theorem for functions of two variables [Tags: $\Pi16$, $\Pi17$], Local maximum [Tags: $\Pi16$], Local maximum value [Tags: $\Pi16$], Local minimum [Tags: $\Pi16$], Local minimum value [Tags: $\Pi16$], Theorem 22.1: Partial derivatives at local maxima and minima [Tags: $\Pi16, \Pi18$], Second derivatives test [Tags: $\Pi16, \Pi17, \Pi18$]

$\Pi23$ Lagrange Multipliers: Lagrange multiplier [Tags: $\Pi7, \Pi8, \Pi12, \Pi13, \Pi16, \Pi19, \Pi20, \Pi21, \Pi22$], Method of Lagrange Multipliers (one constraint) [Tags: $\Pi7, \Pi16, \Pi21, \Pi22$], Method of Lagrange

Multipliers (two constraints) [Tags: $\Pi7, \Pi16, \Pi18, \Pi21, \Pi22$], The geometric basis of Lagrange's method for functions of two variables [Tags: $\Pi7, \Pi8, \Pi12, \Pi13, \Pi16, \Pi19, \Pi20, \Pi21, \Pi22$], Two constraints [Tags: $\Pi7, \Pi8, \Pi10, \Pi16$, $\Pi21$, $\Pi22$]

$\Pi24$ Taylor and Maclaurin Polynomials: Binomial series, Maclaurin series, nth-degree Taylor polynomial, Remainder of a Taylor series, Taylor series, Taylor's inequality, Theorem 24.1: Coefficients in the power series expansion of a function, Theorem 24.2

Section VI.ii.b.ii: The Practical Blocks

To make practical blocks more accessible, I constructed each of them in the form of a graph with tasks and techniques in the place of vertices; full arrows connect types of tasks with specific tasks and dotted arrows connect these to corresponding techniques. This representation makes for easier differentiation of the different types of tasks, especially in praxeologies that involve a multitude of tasks; while the coding of the tasks is designed to reflect this (i.e., $T_{2.3}$ and $T_{2.5}$ are two different task types in the practical block of $\Pi2$), a visual representation reinforces this difference. Alternatively, a graph can help highlight the similarities between tasks in cases such as the following: $T_{10.1.1}$ (to find a vector equation of a line), $T_{10.1.2}$ (to find parametric equations of a line), and $T_{10.1.3}$ (to find symmetric equations of a line) are three tasks taught in the textbook, and are really just variations on the task type $T_{10.1}$ (to find an equation of a line). A graphical representation also lends itself well to cases of task types that are taught to be performed via a slew of different techniques. Following the first of these graphs below, I present a guide explaining the anatomy of these trees.

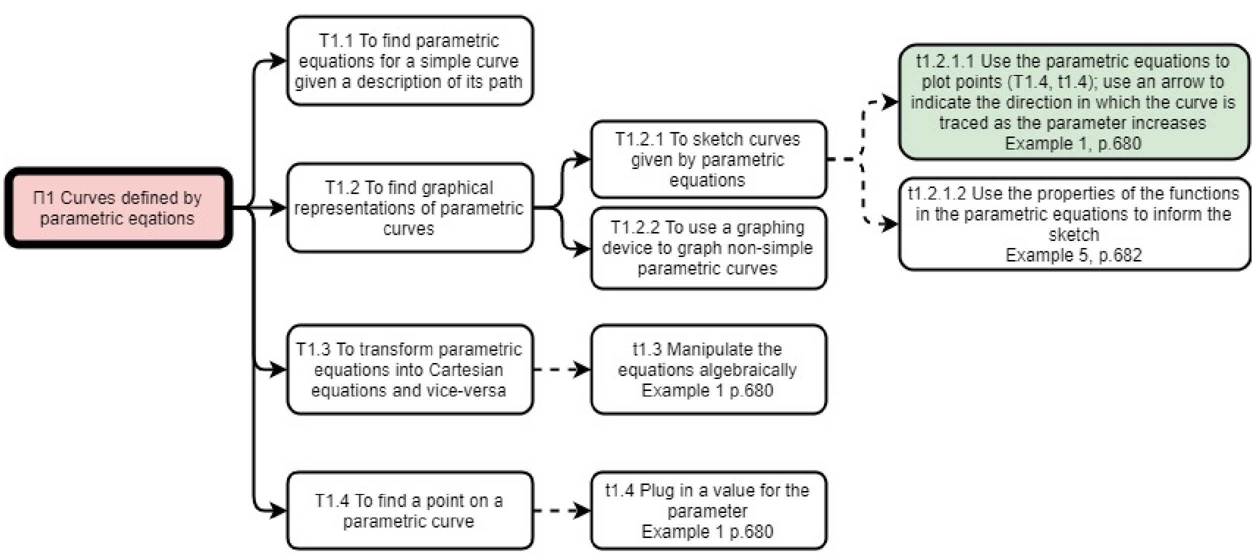


Figure 3. Practical block of KT $\Pi1$.

The following is a guide for reading the figures that depict the practical blocks of praxeologies Πi of the knowledge to be taught (KT):

Praxeology (fat-bordered pink boxes) – Each tree branches out the tasks and techniques that together make up the practical block of a praxeology; the name of the praxeology (borrowed from the title of the textbook section to which it corresponds) is indicated in each tree by a central pink box.

Tasks (full arrows) – From the central pink box flow out 'full' arrows that indicate the main tasks to be taught within this praxeology. In some cases, a task is shown to split into a set of other tasks. For example, in $\Pi 1$, two (full) arrows branch out from task $T_{1.2}$ into tasks $T_{1.2.1}$ and $T_{1.2.2}$. This means that $T_{1.2.1}$ and $T_{1.2.2}$ appear as tasks in the textbook; I found them to share a common purpose (in the sense discussed in section III.ii.a), and therefore piled them under the umbrella $T_{1.2}$.

Techniques (dotted arrows) – Apart from a handful of tasks which could be tackled with too wild an array of techniques, most in this course are linked with at most a handful of techniques. Whenever possible, these boxes include a reference to an example in the textbook (Stewart, 2015) that demonstrates the technique.

Cross-references – **Task boxes are shaded yellow** if the task is a variation of a task from another praxeology (e.g. "To find the length of a polar curve" in $\Pi 4$ is akin to the task of finding "the length of a parametric curve" in $\Pi 2$), the difference being in the object to which the technology is applied. **Technique boxes are shaded green** if the procedure outlined within involve steps that are themselves task-technique pairs coded in another praxeology.

I proceed with the maps of the remaining 23 praxeologies of the knowledge to be taught in MAST 218 and MATH 264:

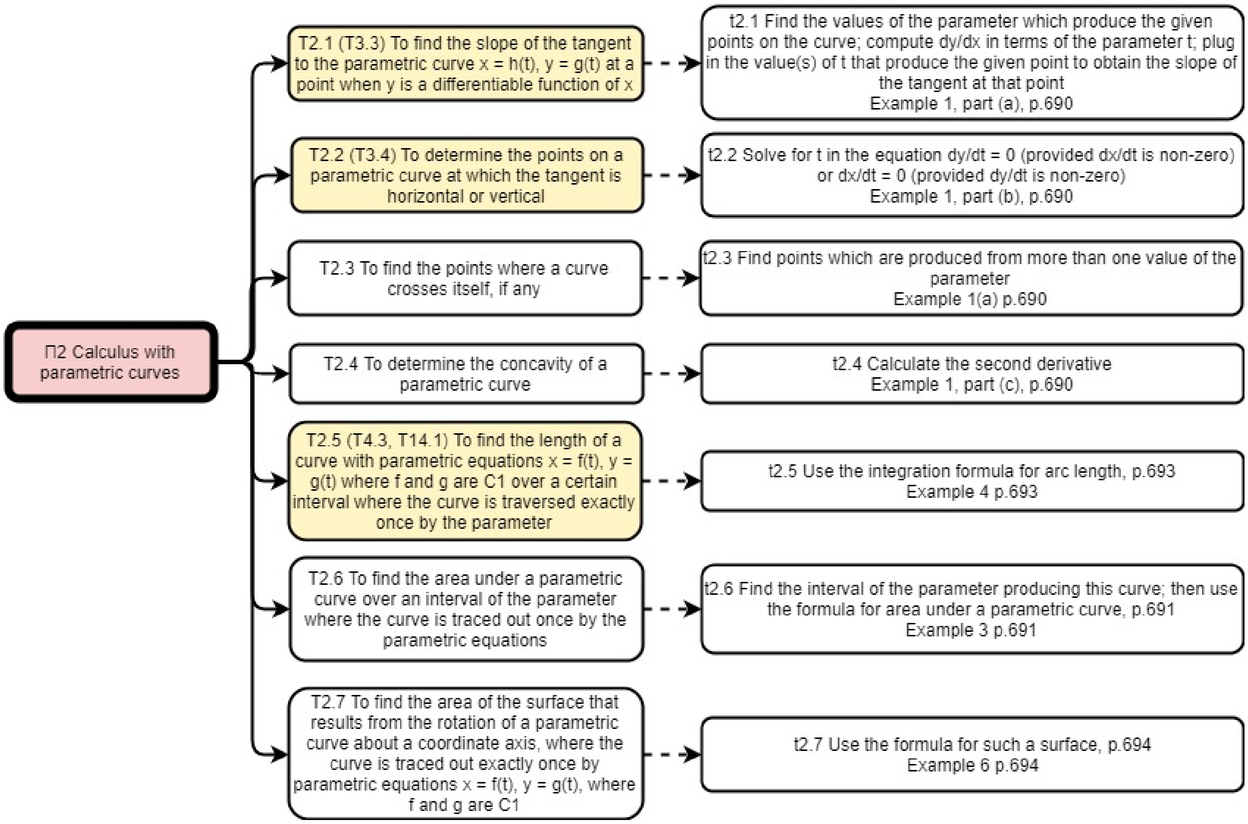


Figure 4. Practical block of KT Π2.

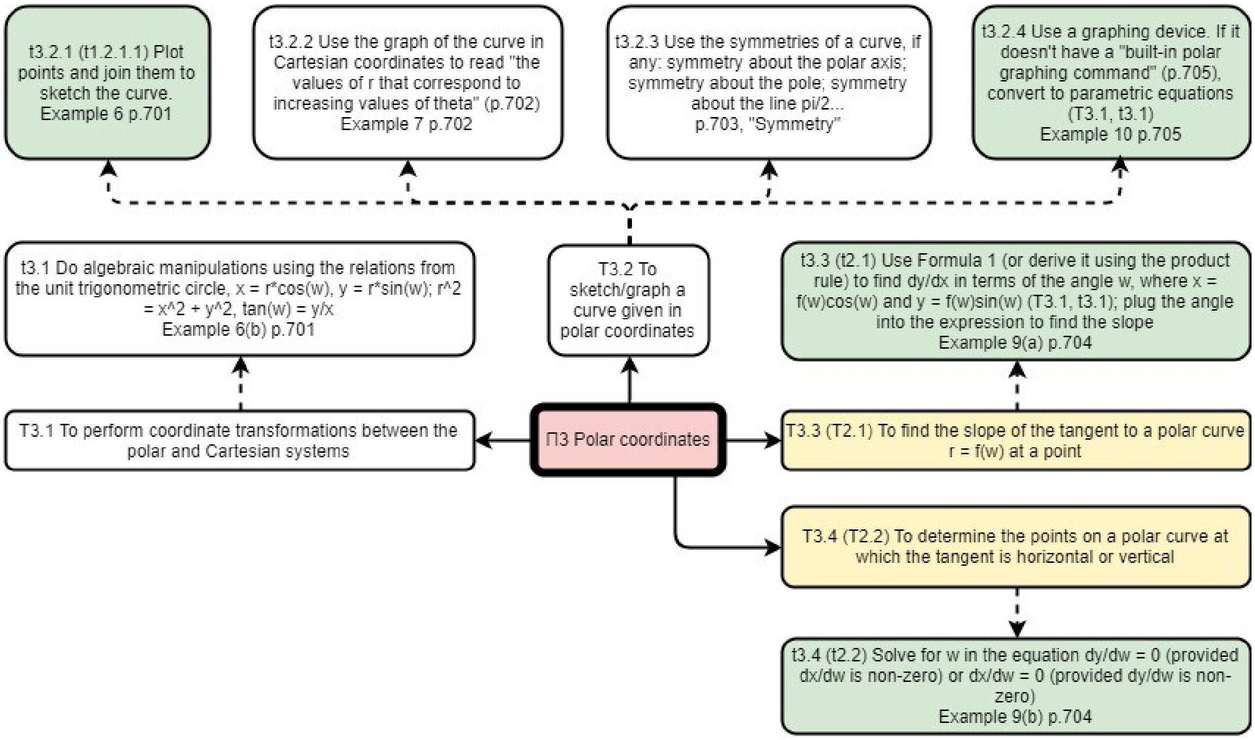


Figure 5. Practical block of KT Π3.

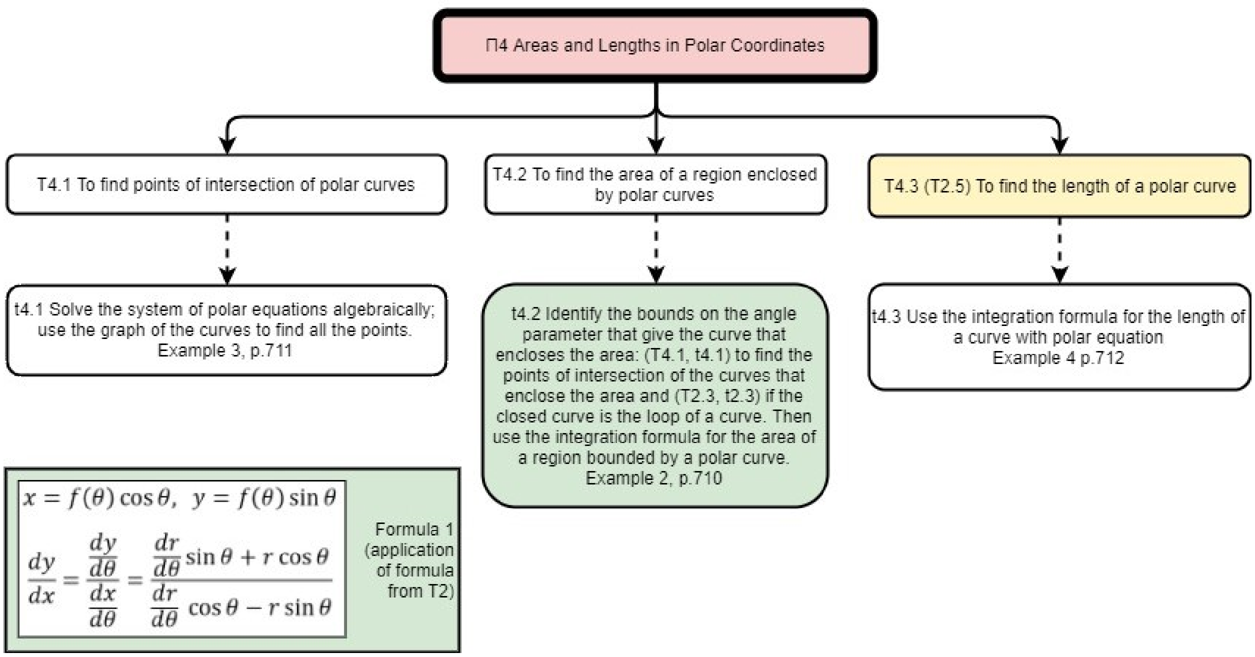


Figure 6. Practical block of KT Π4.

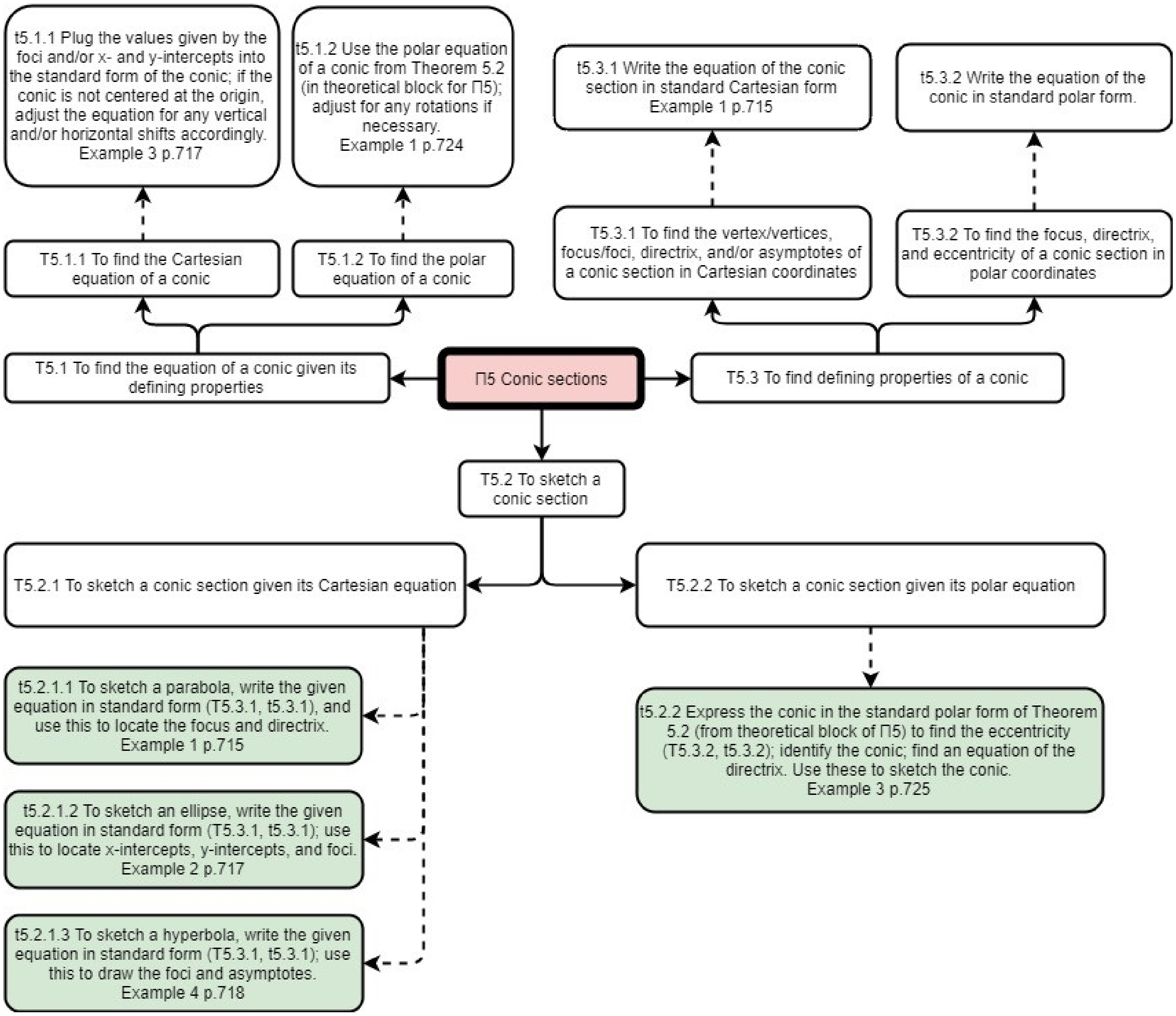


Figure 7. Practical block of KT П5.

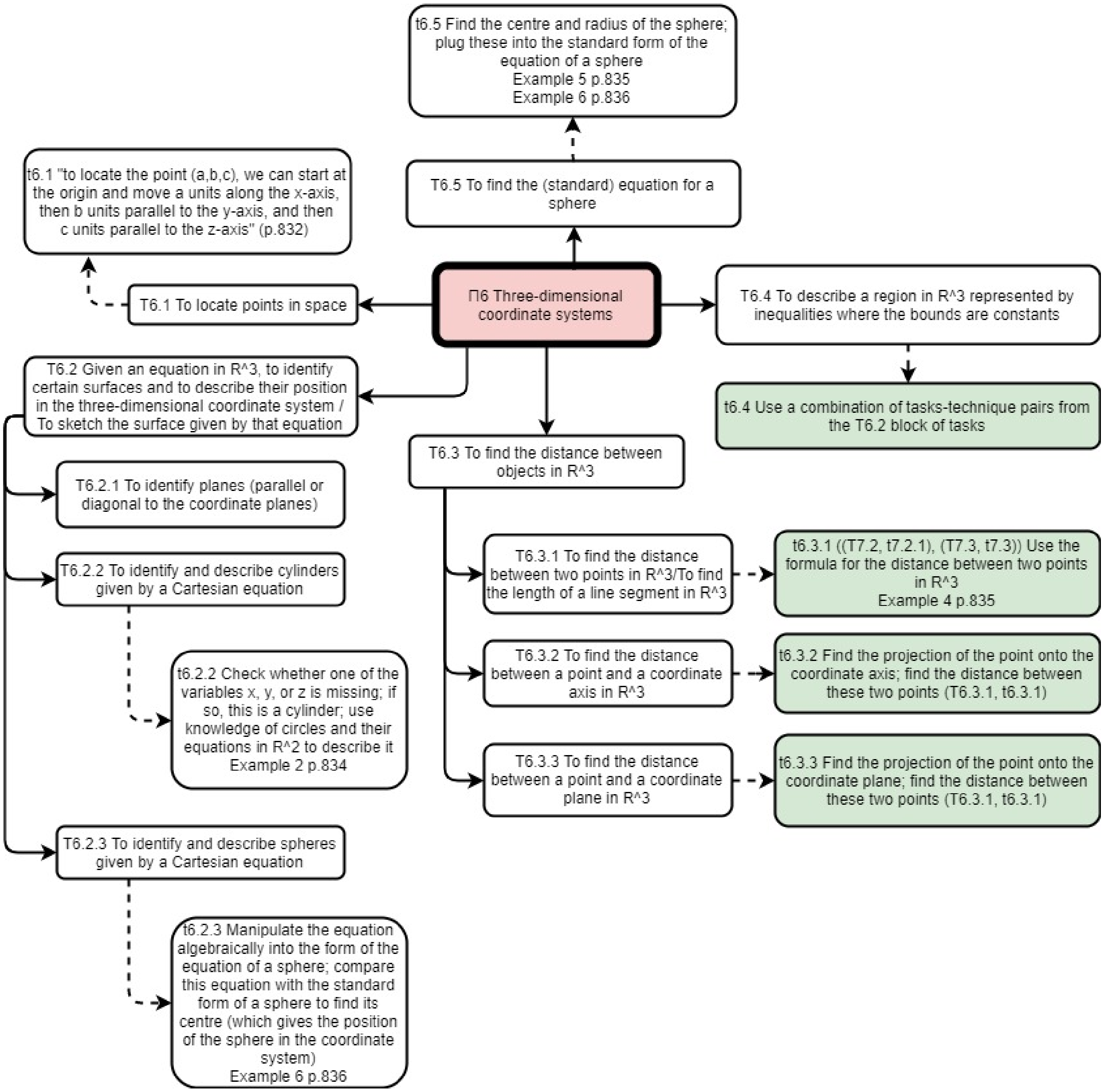


Figure 8. Practical block of KT Π6.

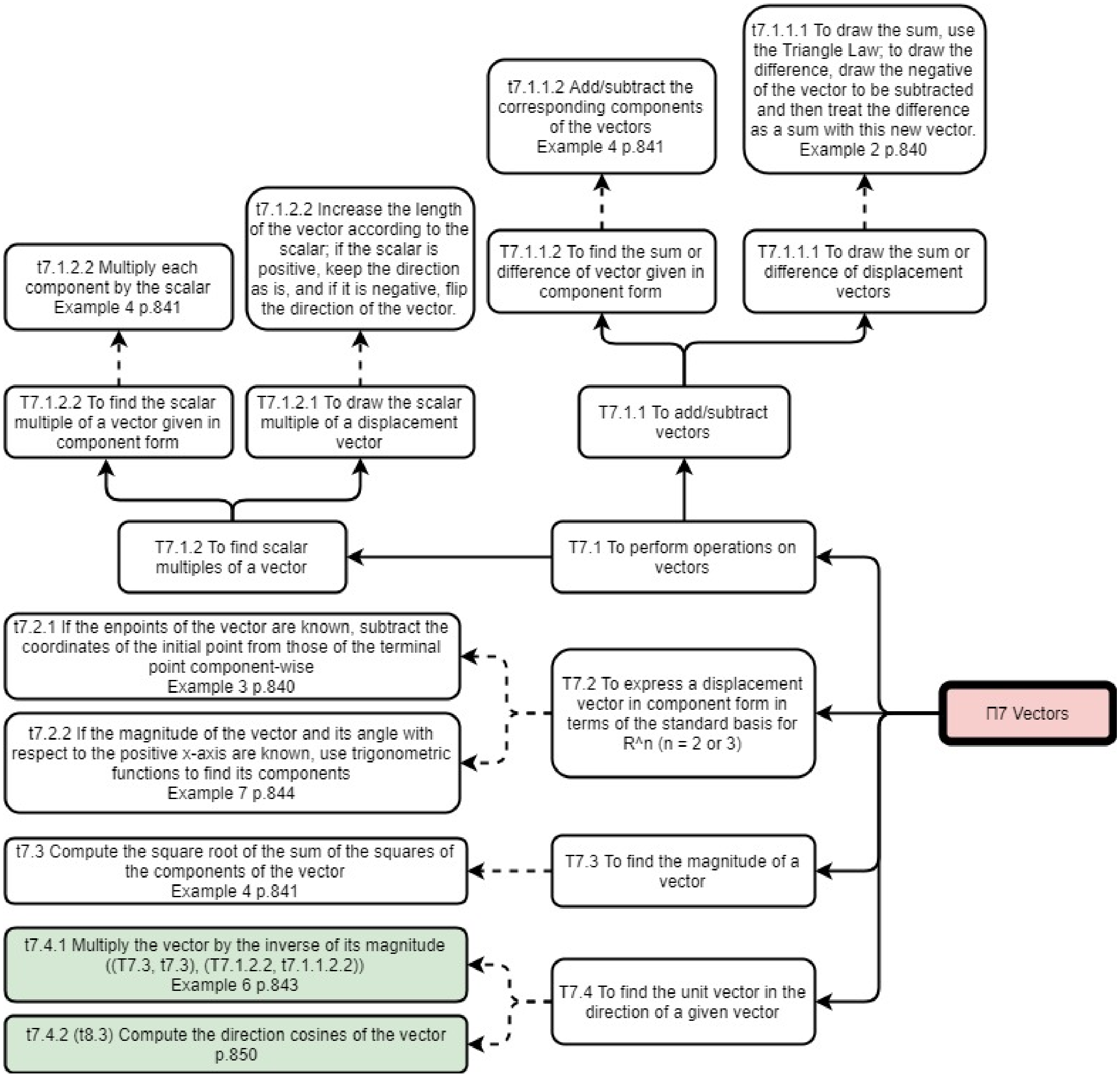


Figure 9. Practical block of KT Π7.

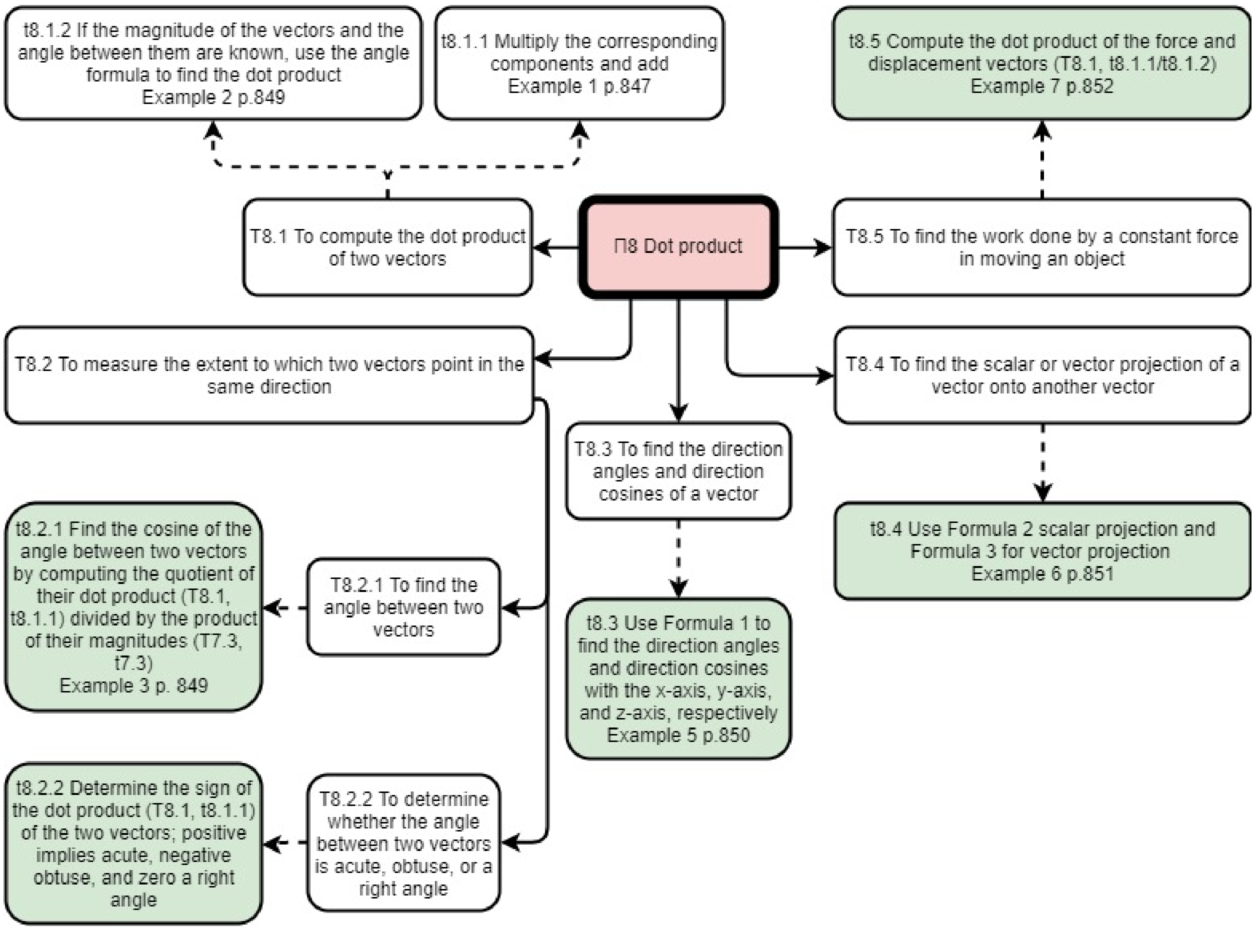


Figure 10. Practical block of KT Π8.

$$\cos\alpha = \frac{a_1}{|\mathbf{a}|} \qquad \cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|} \qquad \text{where } \mathbf{a} = (a_1, a_2, a_3)$$

Formula 1
(T7.3, t7.3)

$$\operatorname{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

Formula 2
((T8.1, t8.1.1),(T7.3, t7.3))

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Formula 3
((T8.1, t8.1.1),(T7.3, t7.3),(T7.1.2.2., t7.1.2.2))

Figure 11. Formulas referred to in the practical block of KT Π8.

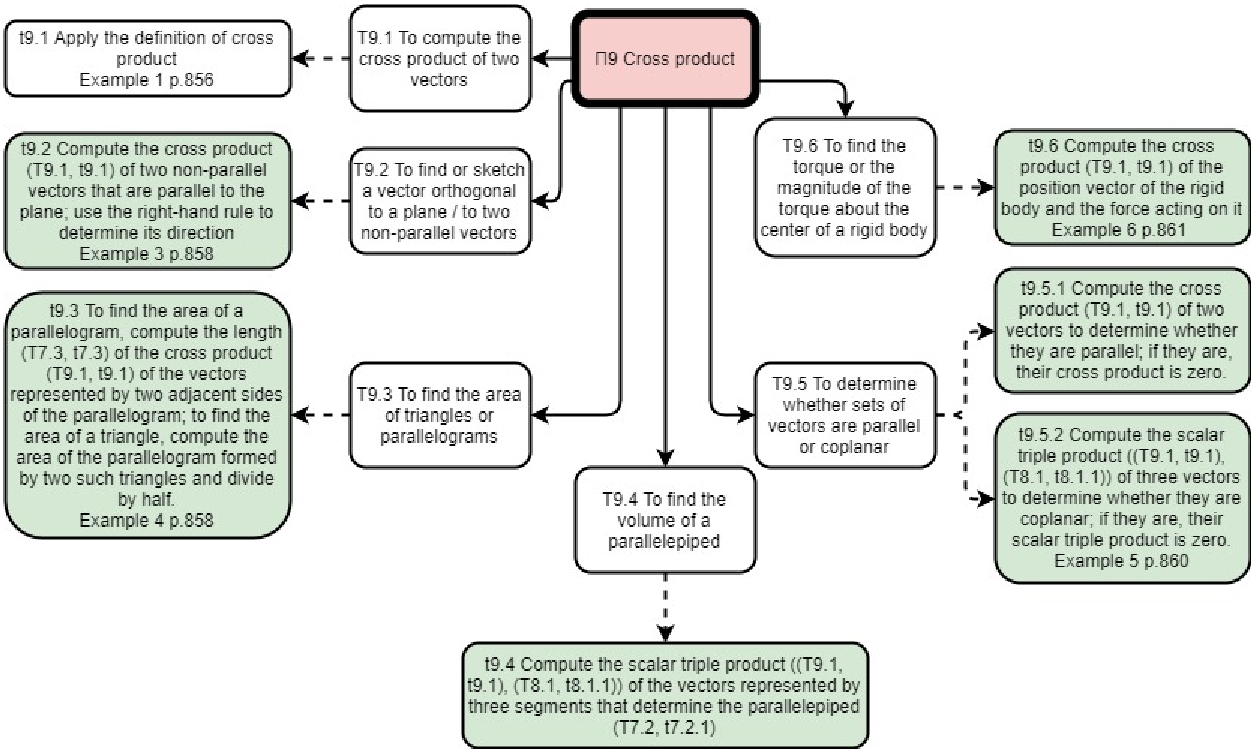


Figure 12. Practical block of KT Π9.

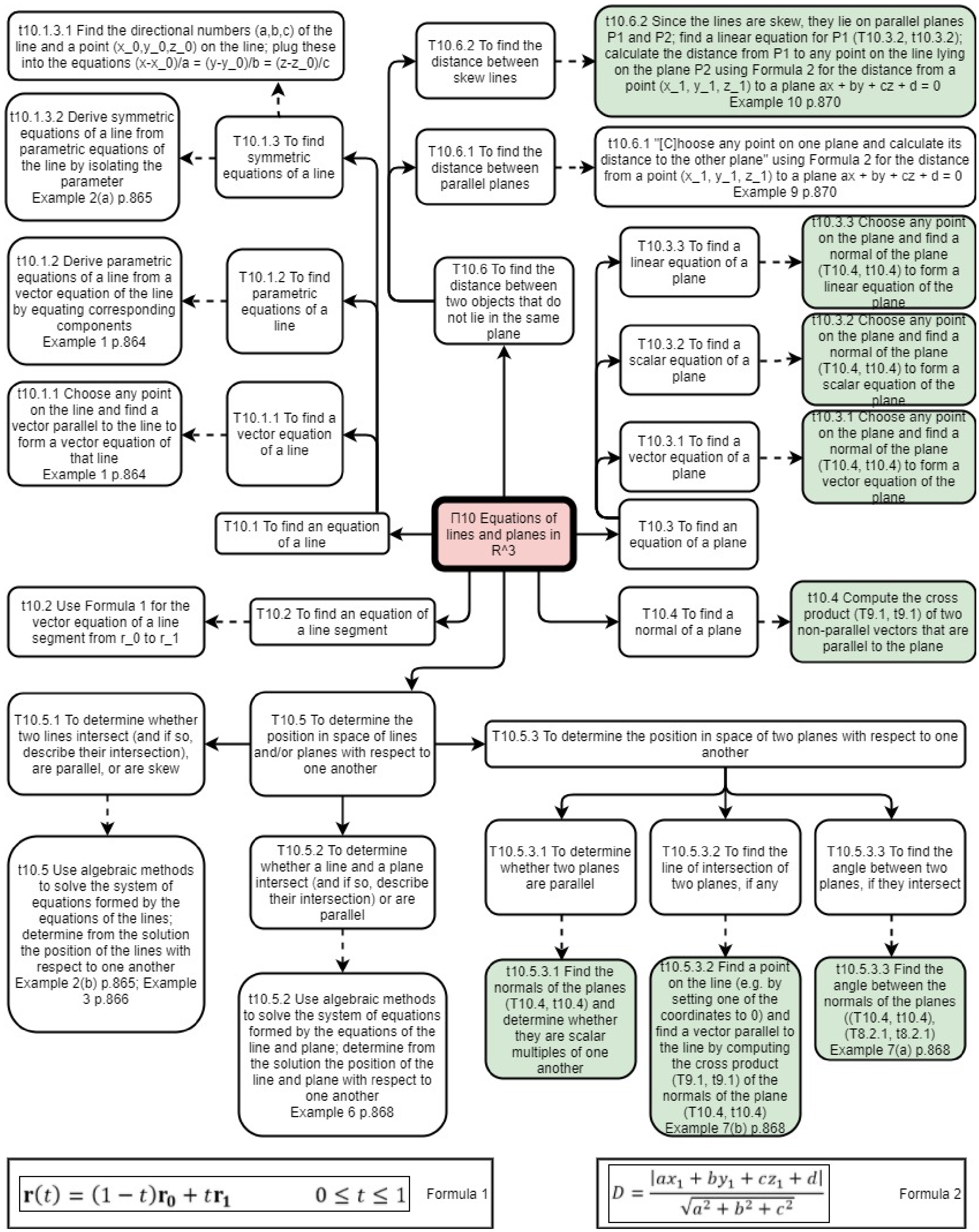


Figure 13. Practical block of KT Π10.

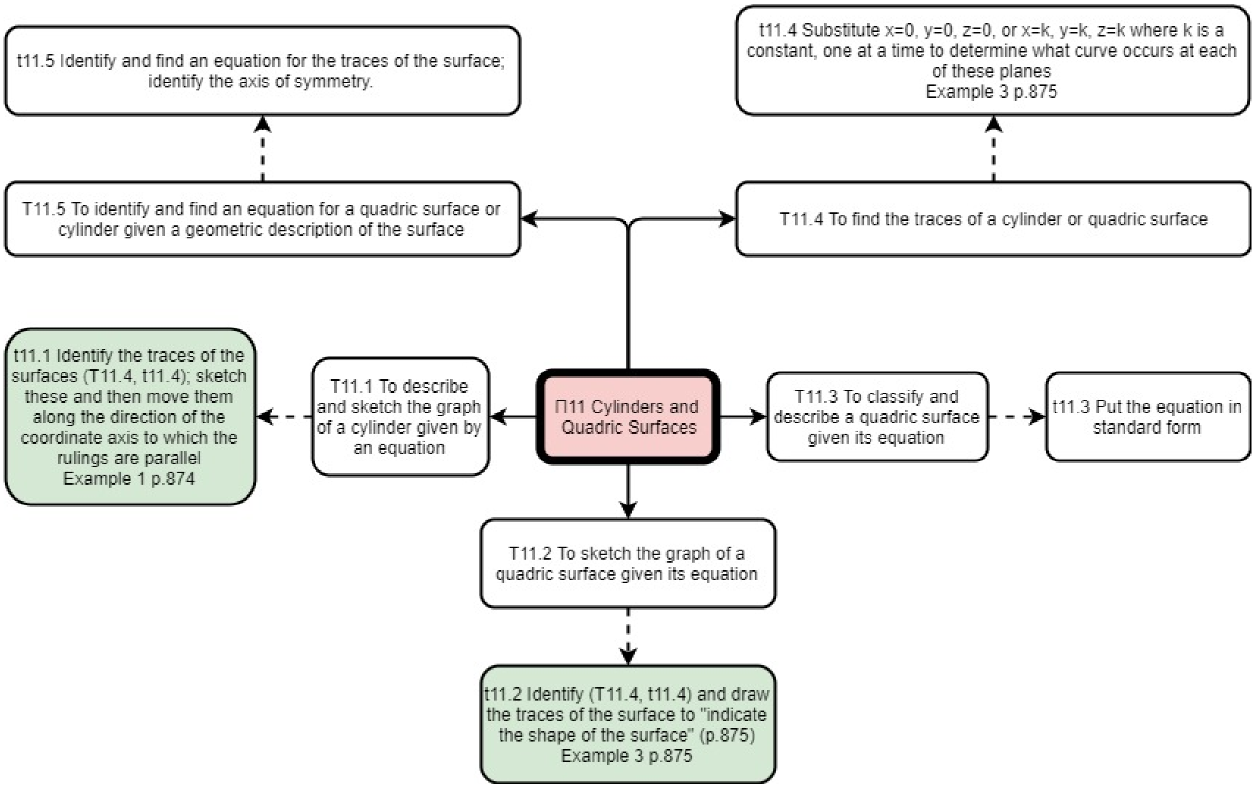


Figure 14. Practical block of KT П11.

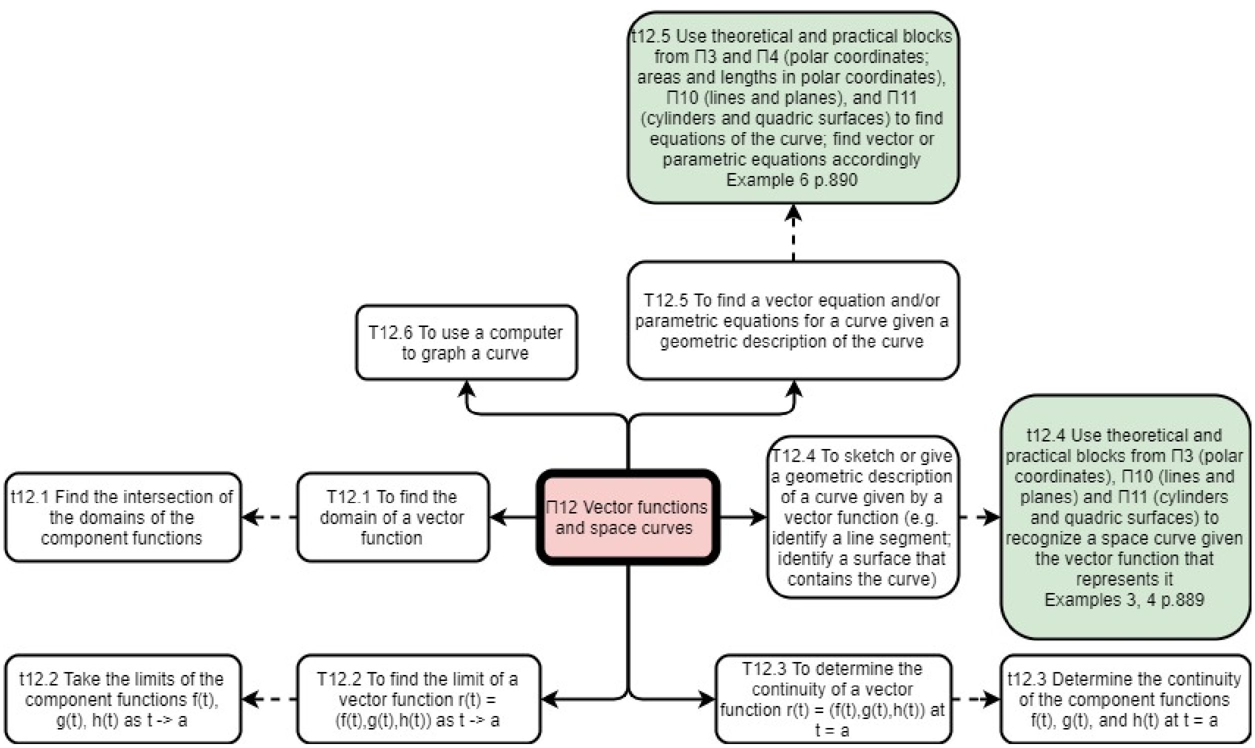


Figure 15. Practical block of KT П12.

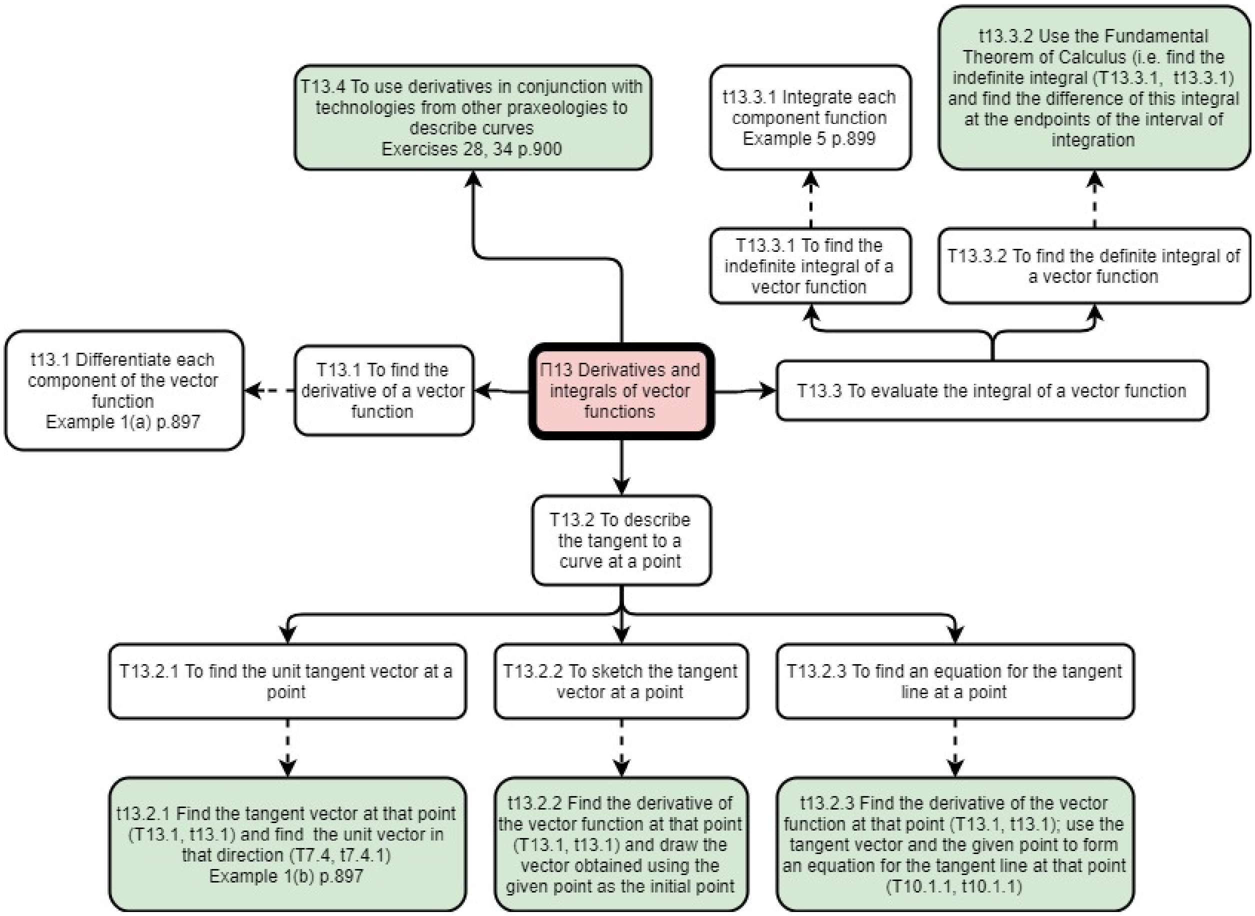


Figure 16. Practical block of KT Π13.

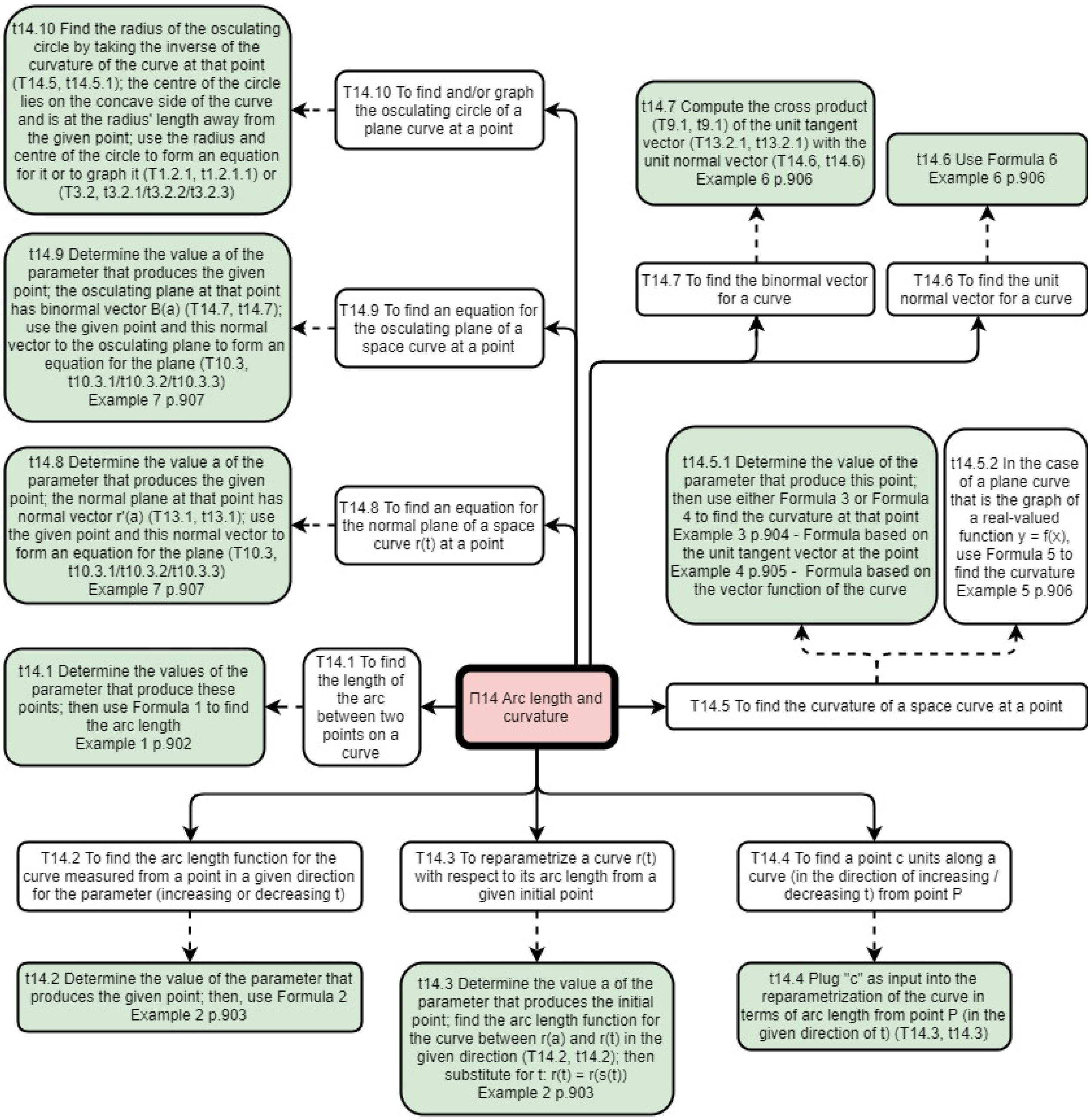


Figure 17. Practical block of KT Π14.

$L = \int_a^b \lvert \mathbf{r}'(t) \rvert dt$	Formula 1 ((T13.1, t13.1),(T7.3, t7.3))
$s(t) = \int_a^t \lvert \mathbf{r}'(u) \rvert du$	Formula 2 ((T13.1, t13.1),(T7.3, t7.3))
$\kappa(t) = \dfrac{\lvert \mathbf{T}'(t) \rvert}{\lvert \mathbf{r}'(t) \rvert}$	Formula 3 ((T13.2.1, t13.2.1),(T13.1, t13.1),(T7.3, t7.3),(T7.1.2.2, t7.1.2.2))
$\kappa(t) = \dfrac{\lvert \mathbf{r}'(t) \times \mathbf{r}''(t) \rvert}{\lvert \mathbf{r}'(t) \rvert^3}$	Formula 4 ((T13.1, t13.1),(T9.1, t9.1),(T7.3, t7.3))
$\kappa(x) = \dfrac{\lvert f''(x) \rvert}{\left[1 + (f'(x))^2\right]^{3/2}}$	Formula 5
$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\lvert \mathbf{T}'(t) \rvert}$	Formula 6 ((T13.2.1, t13.2.1),(T13.1, t13.1),(T7.3, t7.3),(T7.1.2.2, t7.1.2.2))

Figure 18. Formulas referred to in the practical block of KT Π14.

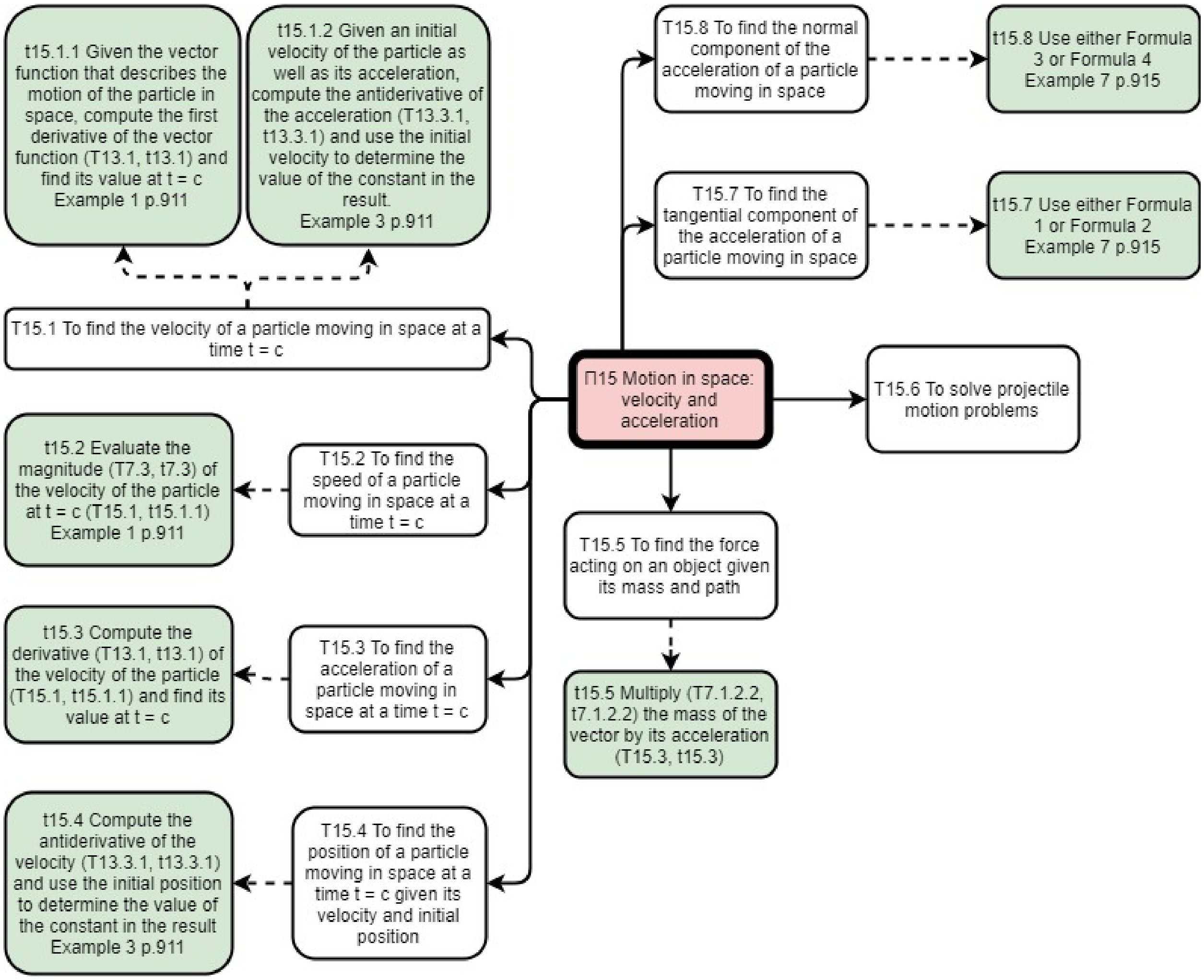


Figure 19. Practical block of KT Π15.

$a_T = v'$ Formula 1 (T15.2, t15.2)	$a_N = \kappa v^2$ Formula 3 ((T14.5, t14.5.1),(T15.2, t15.2))
$a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{\lvert\mathbf{r}'(t)\rvert}$ Formula 2 ((T13.1,t13.1),(T8.1, t8.1), (T7.3, t7.3))	$a_N = \dfrac{\lvert\mathbf{r}'(t)\times\mathbf{r}''(t)\rvert}{\lvert\mathbf{r}'(t)\rvert}$ Formula 4 ((T13.1,t13.1),(T9.1, t9.1), (T7.3, t7.3))

where $v = |\mathbf{v}|$ is the speed of the particle

Figure 20. Formulas referred to in the practical block of KT Π15.

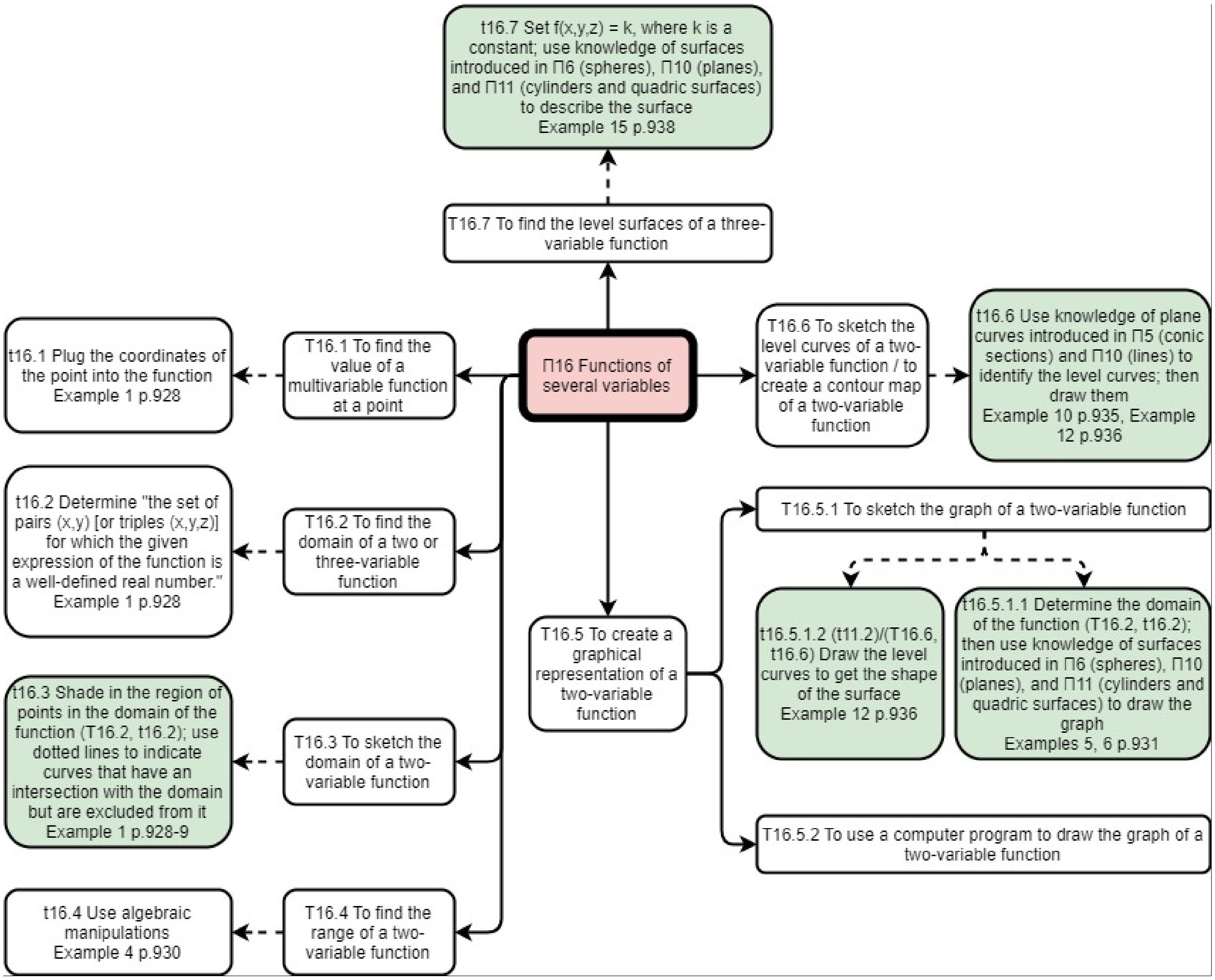


Figure 21. Practical block of KT Π16.

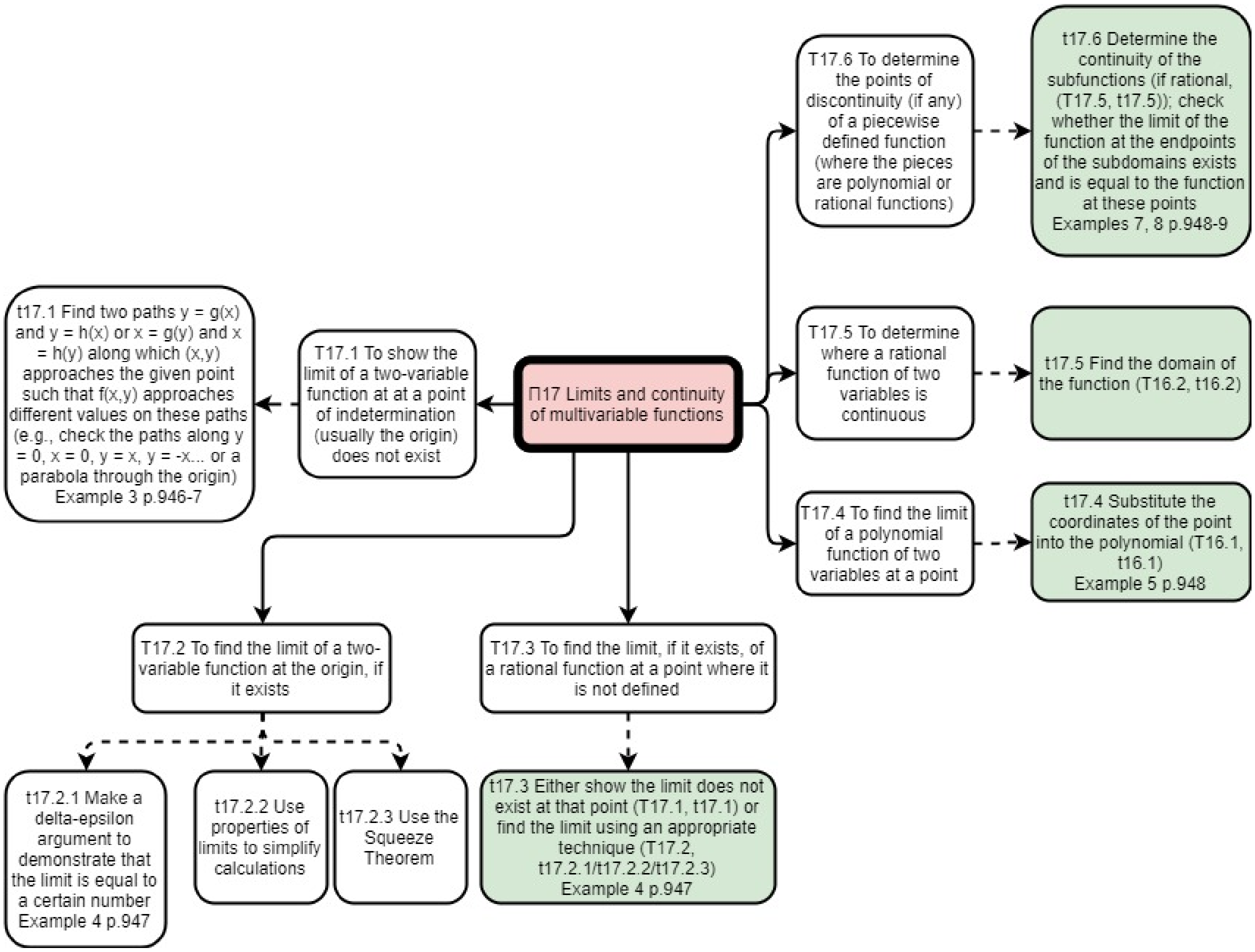


Figure 22. Practical block of KT Π17.

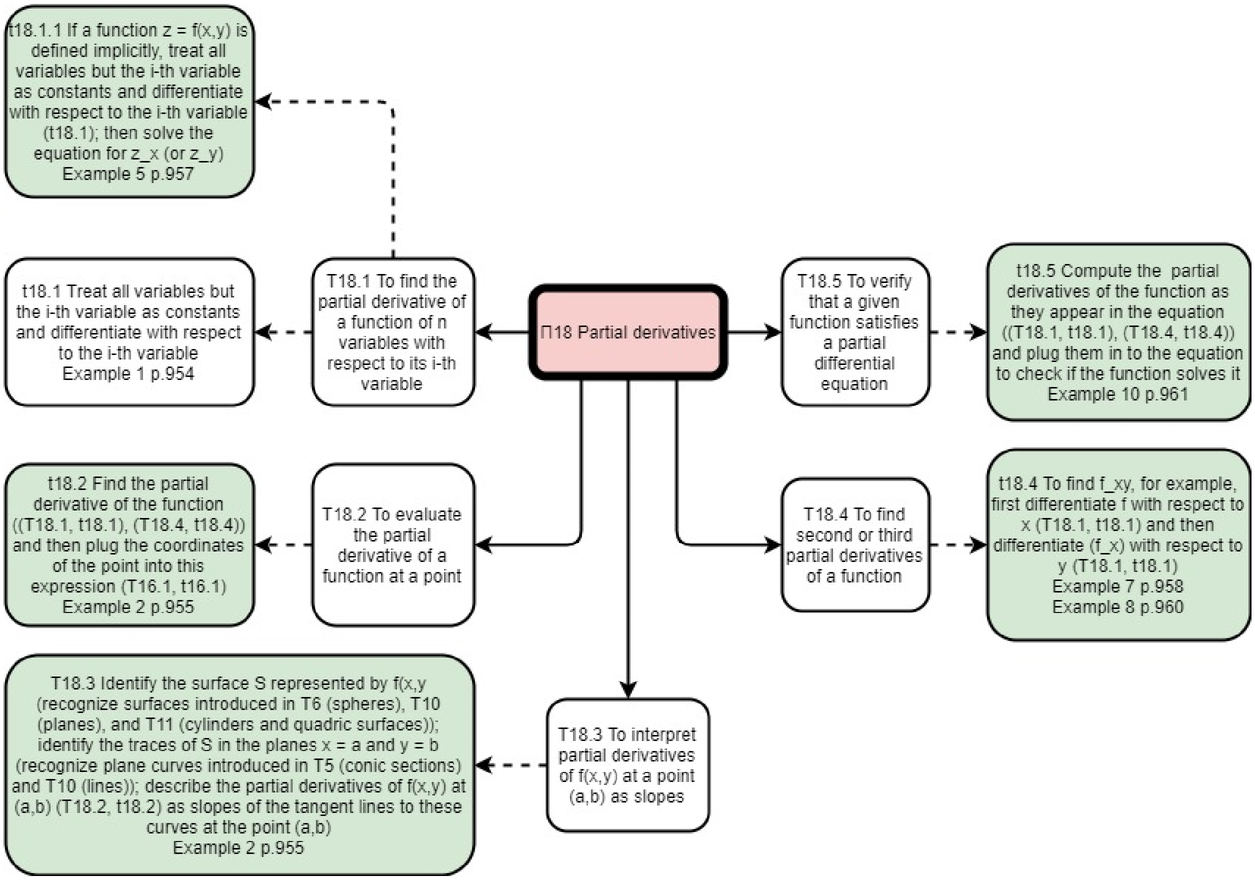


Figure 23. Practical block of KT Π18.

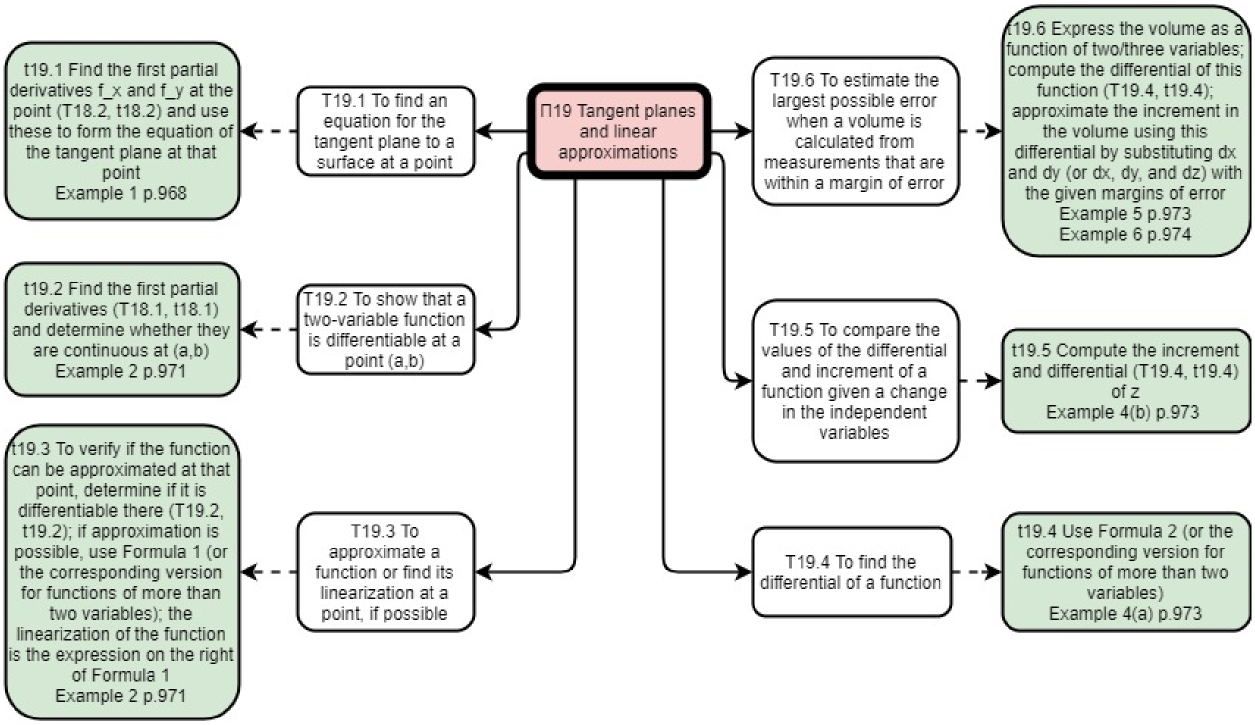


Figure 24. Practical block of KT Π19.

$$\boxed{f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

Formula 1
((T18.1, t18.1),(T16.1, t16.1)

$$\boxed{dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

Formula 2
(T18.1, t18.1)

Figure 25. Formulas referred to in the practical block of KT Π19.

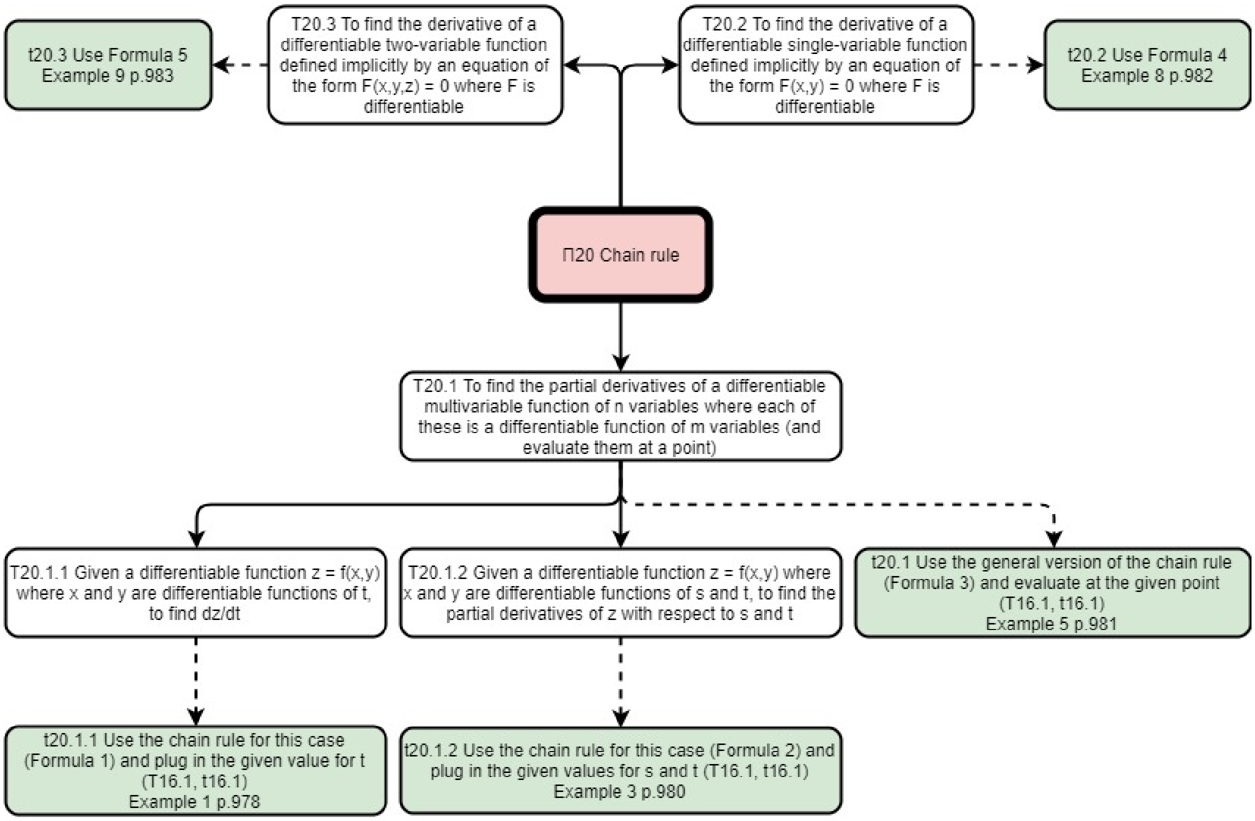


Figure 26. Practical block of KT Π20.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Formula 1 (T18.1, t18.1)

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Formula 2 (T18.1, t18.1)

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Formula 3 (T18.1, t18.1)

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Formula 4 (T18.1, t18.1)

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Formula 5 (T18.1, t18.1)

Figure 27. Formulas referred to in the practical block of KT Π20.

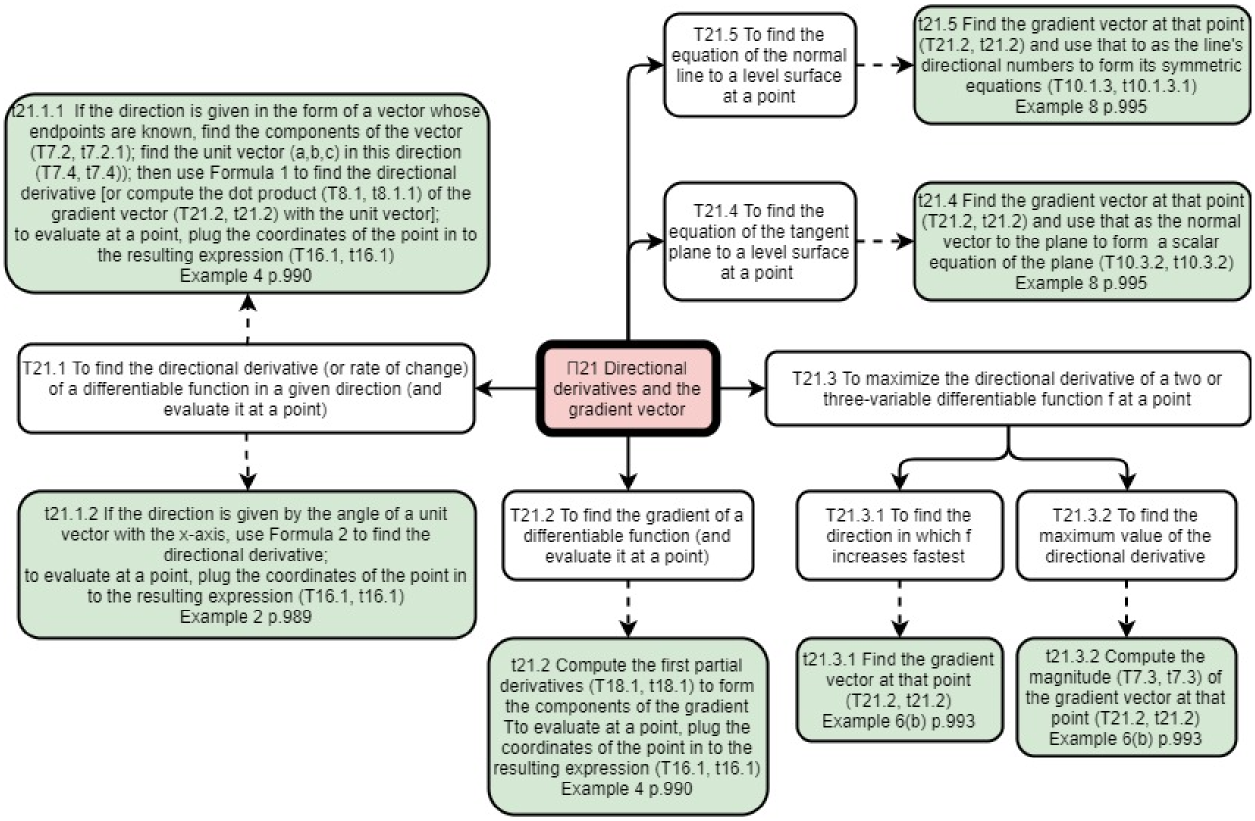


Figure 28. Practical block of KT П21.

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$$
$$D_{\mathbf{u}}f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c$$

Formula 1
(T18.1, t18.1)

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

Formula 2
(T18.1, t18.1)

Figure 29. Formulas referred to in the practical block of KT П21.

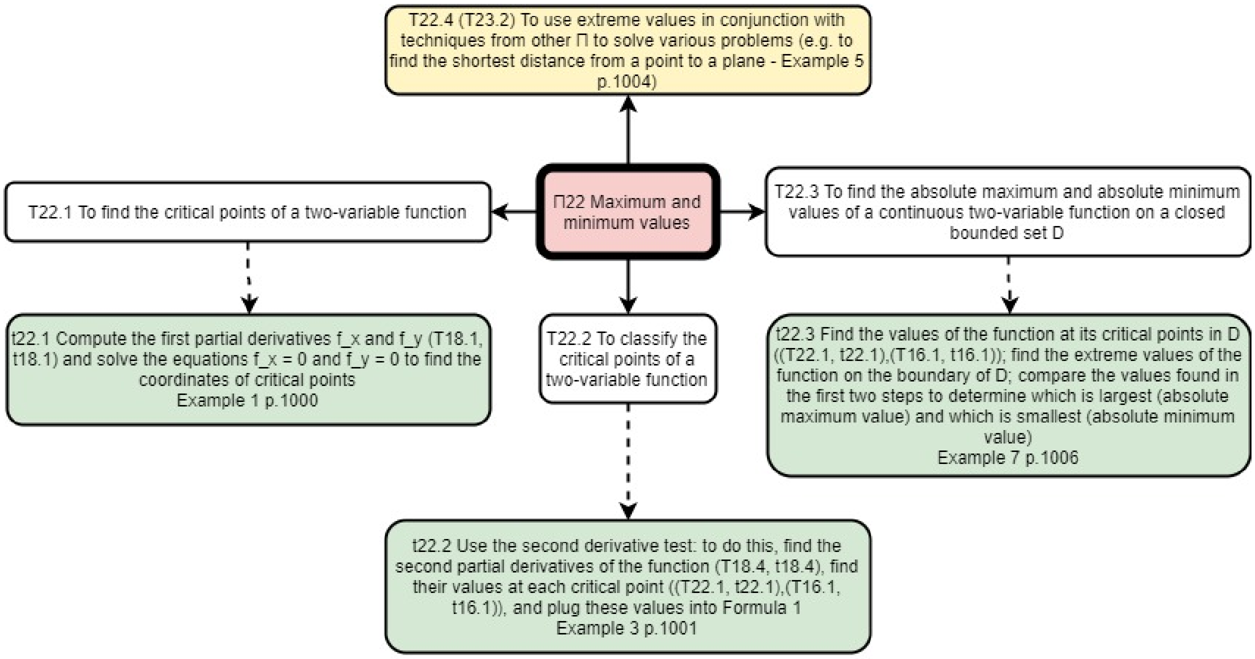


Figure 30. Practical block of KT Π22.

$$\boxed{D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2}$$

Formula 1 (T18.4, t18.4)

Figure 31. Formulas referred to in the practical block of KT Π22.

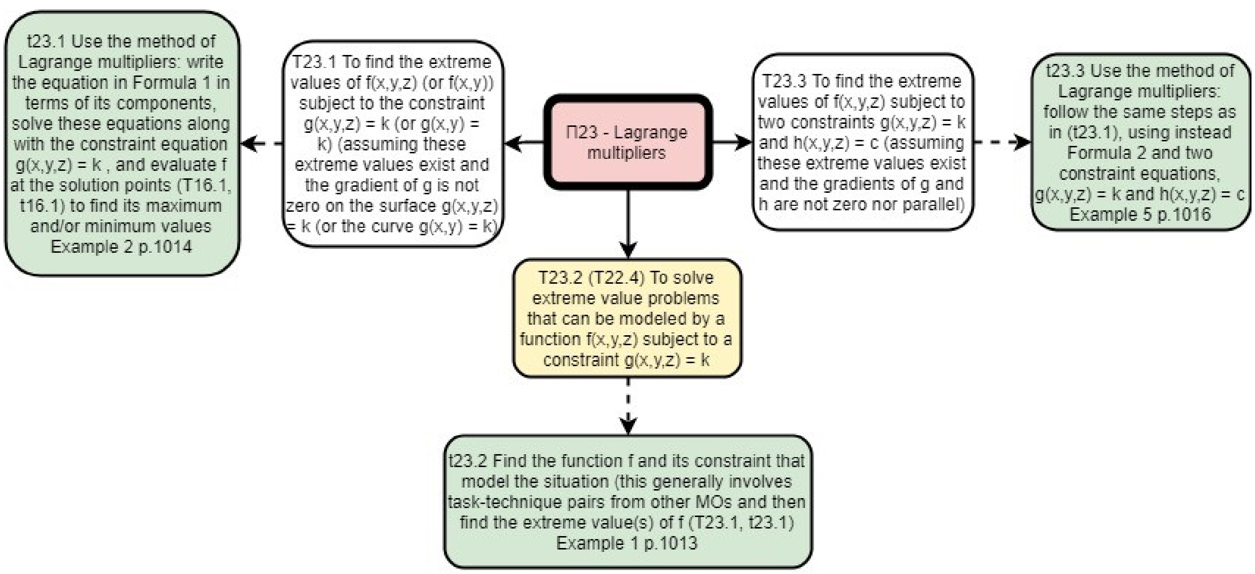


Figure 32. Practical block of KT Π23.

$$\boxed{\nabla f(x, y, z) = \lambda \nabla g(x, y, z)}$$

Formula 1
((T21.2, t21.2), (T7.1.2.2, t7.1.2.2))

$$\boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0,) + \mu \nabla h(x_0, y_0, z_0)}$$

Formula 2
((T21.2, t21.2), (T7.1.2.2, t7.1.2.2), (T7.1.1.2, t7.1.1.2))

Figure 33. Formulas referred to in the practical block of KT Π23.

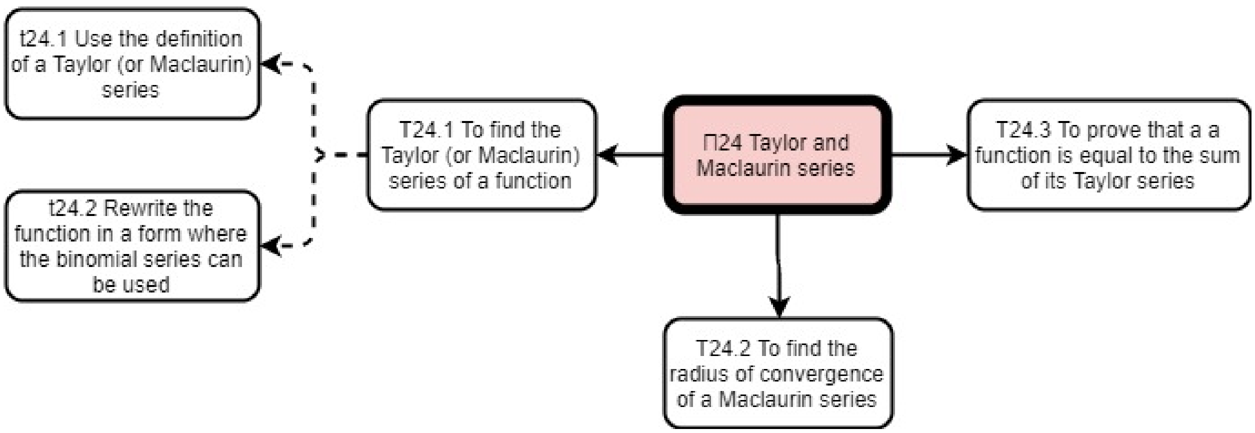


Figure 34. Practical block of KT Π24.

Chapter VII: Knowledge to be learned

This chapter opens with a recapitulation of the main goal of having a model of the knowledge to be learned. The methodology I used to create it is explained in a second section: I outline how I identified the praxeologies forming the knowledge to be learned and then how I created the model from there. I finally present the model in a third section organized by praxeology, wherein the model for each praxeology is followed by a discussion of the knowledge to be learned within.

Section VII.i: Aim

A cursory examination of the 218/264 finals reveals that the solutions to the exam problems are richly apt to be sequenced as task-technique pairs from the model of knowledge to be taught. Accordingly, I model the knowledge to be learned in each course as a subset of the knowledge to be taught. As discussed in the introduction (p.8), the purpose is to measure the distance between the two models: that is, to identify the knowledge to be taught that's also to be learned, to propose explanations of the inclusions, and to conjecture the effects of the exclusions. I begin by describing my approach to identifying the practical blocks of the knowledge to be learned; I then present and explore the model for each mathematical organization in the knowledge to be learned. The case-by-case analysis where I discuss the knowledge that is and isn't to be learned in each mathematical organization delivers a lay of the land. In a second stage, I take a more global approach, taking account of the institutional context, reference model, and model of the knowledge to be taught to identify the undercurrents that characterize the knowledge to be learned in 218 and 264.

Section VII.ii: Methodology

Section VII.ii.a: Identifying the praxeologies

My model is based on final examinations from the recent past. Altogether, I considered six MAST 218 and six MATH 264 exams from the years 2012-2015 (Appendices A and B). These exams are typically 8 – 10 questions long, with some of these split into 2-4 sub-questions themselves. In both courses, students are graded according to the scheme that yields the better mark: 90% final exam, 10% assignments; or 60% final exam, 30% midterm, and 10% assignments. It is suitable to draw out the knowledge to be learned solely from the final examinations as the grading scheme burdens them with the bulk of determining a student's academic performance in the course. Further, in most of the semesters I considered for this study, the assignments came straight out of the exercise pages of the textbook. Since the particular exercises that were chosen were mirror images of examples from the text, it was more appropriate to treat them as knowledge to be taught than knowledge to be learned; in any case, whatever the grading scheme these exercises shape only 10% of students' final grades.

I began my study by describing the solution to each[17] exam question in terms of KT task-technique pairs that occur in the solution. I did not record steps in the solution that do with single-variable calculus from the prerequisite courses; the purpose of this model was purely to identify the subset of the KT model which is also to be learned. I reflect on the role of algebraic and single-variable differentiation techniques in the discussion that follows the model. Apart from a handful of cases in the MATH 264 exams, these sequences of task-technique pairs constitute complete solutions to the exam questions.

This formed the first step in the creation of a KL model: for each of MAST 218 and MATH 264, a table listed the problems in each final exam. I enumerated the problems and appended a description of the task. I omitted certain details in favor of focusing on task *types* rather than specificity.

Here is an instance of this work. Consider problem 3b from the MATH 264 Fall 2012 exam,

Find the tangent plane T that touches S at $(x,y) = (2,1)$

where the surface S is given by $z = f(x,y) = 1 - e^{-\left(\frac{1}{4}x^2+y^2\right)}$. I recorded this as "to find the tangent plane to a surface at a point." This task corresponds identically to task $T_{19.1}$ from the KT model; in turn, the technique for this task requires the completion of $T_{18.2}$: to find the value of the partial derivative of a function at a point. Thus, problem 3b is recorded as the task "to find the tangent plane to a surface at a point" and associated with the KT sequence $[(T_{18.2}, \tau_{18.2}), (T_{19.1}, \tau_{19.1})]$. In this example, the exam task happened to correspond to a KT task; this is not always the case. Nonetheless, the approach remains: identify the sequence of KT task-technique pairs that occur in the solution.

The next step in the creation of a KL model was to group together tasks that are of the same type and identify the praxeologies that form the 218/264 exams. In many cases, sets of task types corresponded to KT local praxeologies; I combined local praxeologies to form regional ones that themselves occur in the knowledge to be taught. The following are the regional praxeologies of the knowledge to be learned:

[17] I did not consider Bonus questions as these, by definition, are not *required*, and my end-goal is to determine the minimal knowledge *essential* for students to learn to succeed on their exam.

MAST 218	MATH 264
Partial derivatives and surfaces	Partial derivatives and surfaces
Space curves and vector functions	Space curves and vector functions
Equations of lines and planes and distances in R^3	Distances in R^3
Limits of rational functions	Limits of rational functions
Polar curves	Polar curves
Taylor series	Taylor series
Conic sections	Conic sections
Spheres	
Quadric surfaces	
Parametric plane curves	

Table 1. Praxeologies of knowledge to be learned in MAST 218 and MATH 264.

Section VII.ii.b: Creating the model

The final model is a display of the practical block of each Π that makes up the knowledge to be learned in MAST 218 and MATH 264. This display combines the models for each course and uses colors to distinguish between the two; this readily allows for a comparison of the knowledge to be learned in both courses. Further, a simultaneous look at the two courses allows observations about one to highlight aspects of the second. My model takes the form of the figure below (Figure 33). The praxeologies that I identified in the knowledge to be learned (denoted KL Π from now on) are at times local and other times regional; the local KL Π correspond to a single KT Π and the regional KL Π to a set of local KT Π. In general:

$$\text{KL } \Pi = \{\text{KT } \Pi \mid \text{tasks based on practical block of KT } \Pi \text{ occur as exam questions }\}$$

For instance, upon considering the exam tasks that do with the regional praxeology Partial Derivatives and Surfaces, I noted that a cluster of tasks deals with the chain rule – as in the KT $\Pi 20$, the Chain Rule – and another with directional derivatives and the gradient vector – as in the KT $\Pi 21$. Thus, the KL Π Partial Derivatives and Surfaces is partitioned into sub-clusters of tasks that match up with KT $\Pi 16$ and $\Pi 18 - \Pi 23$. Each such sub-cluster lists tasks that occur as questions in the 218/264 exams.

The figure below displays the structure of the models of knowledge to be learned: the yellow box at the bottom indicates the KL Π; the black and white box at the top indicates the corresponding KT Π

whose tasks occur in exams and combine to form the KL Π; as noted above, some KL Π are associated with only one KT Π, others with several. The tasks that occur in MAST 218 exams are indicated below the KT Π they belong to in red boxes, MATH 264 in blue; those that occur in both – in gradiented red-and-blue.

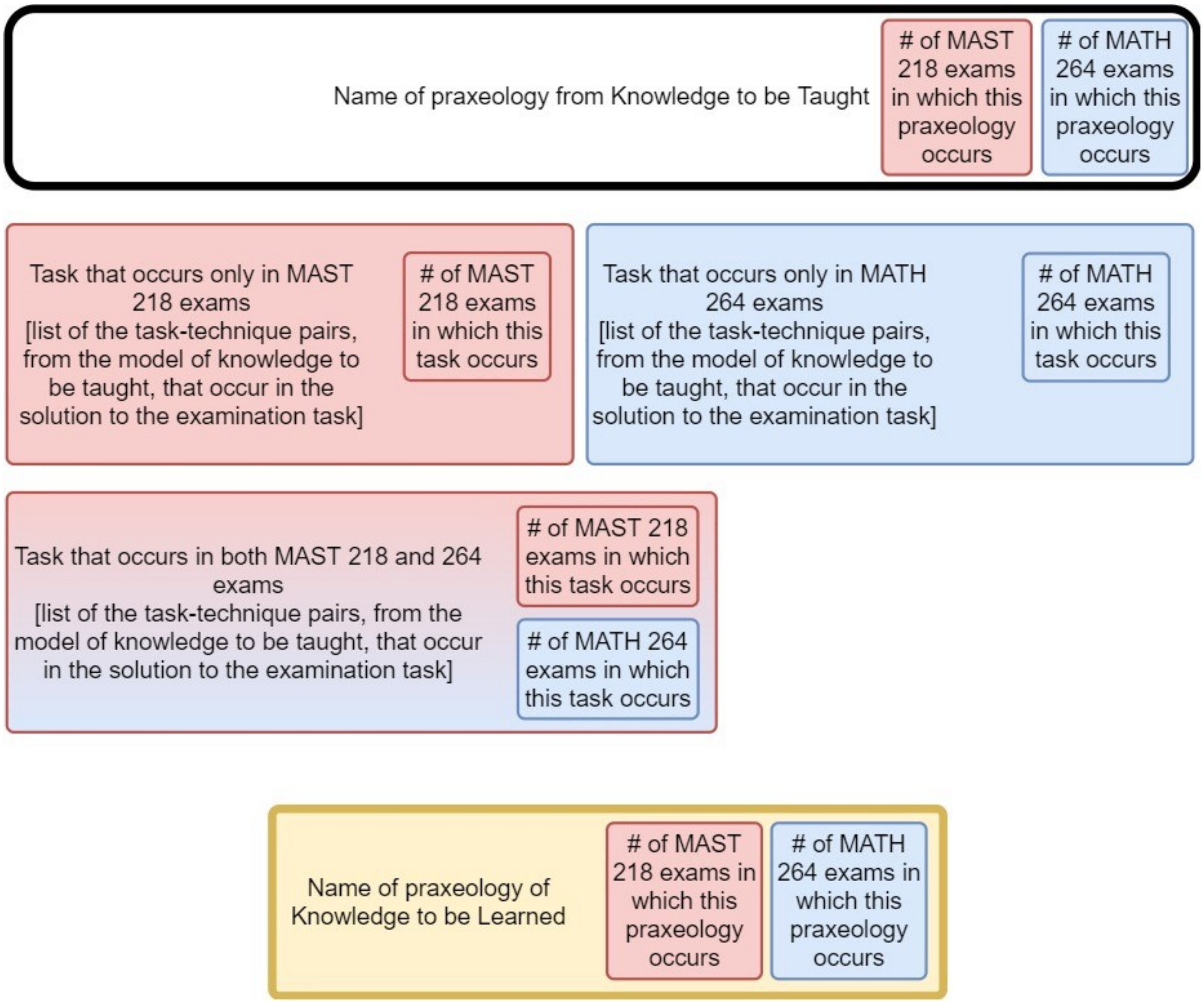


Figure 35. Structure of the display of a praxeology of knowledge to be learned.

To more faithfully represent the weight attributed to various praxeologies and specific tasks, the model of each KL Π is girded with three layers of frequency data:

1. the number of MAST 218 exams in which a task occurs is indicated in a red box and the number of MATH 264 exams in which it occurs in a blue box.

These tasks are clustered under the KT Π to which they belong; this Π is indicated in a black-and white box and

2. the number of exams in which it occurs is marked therein.

Without this additional layer of information, it might be easy to misinterpret the weight given to certain tasks in the exams. For instance, three tasks from KT $\Pi 11$ Cylinders and Quadric Surfaces appear in MAST 218 exams, once each. Thus, the box for each of them is marked "1" in red, meaning that each task occurs in one MAST 218 examination. For all the reader knows, these three tasks may occur in three different exams (out of six) – this would mean that this Π bears bulk in the knowledge to be learned in MAST 218. However, all three of these particular tasks occurred in the same examination. The box "$\Pi 11$ Cylinders and Quadric Surfaces" is therefore marked with "1" in red (i.e. 1 (one) MAST 218 exam). The reader now knows that the tasks in the cluster $\Pi 11$ occur in only one exam – and so they occur as a single problem and are absent in 5 out of 6 exams. Thus, by considering both the number of exams in which a KT Π appears and the number of exams in which its tasks occur, it's possible to gain a more accurate account of the knowledge to be learned. The last layer of frequency indicated in each KL Π is

3. the number of exams in which the regional Π occurs. To obtain this number, I counted the number of exams in which the totality of the (local) tasks within the Π occur.

Section VII.iii: The model

The following is an updated version of Table 1, appended with the number of occurrences of each praxeology in the knowledge to be learned.

KL Praxeology	Number of MAST 218 exams (/6) in which KL Π occurs	Number of MATH 264 exams (/6) in which KL Π occurs
Partial derivatives and surfaces	6	6
Space curves and vector functions	6	4
Equations of lines and planes & distances in R^3	6	1
Polar curves	5	4
Limits of rational functions	4	3
Taylor series	2	4
Conic sections	1	2
Spheres	1	0
Quadric surfaces	1	0
Parametric plane curves	4	0

Table 2. Number of MAST 218 and MATH 264 exams in which KL praxeologies occur.

Before I embark on a general discussion of the landscape of these multivariate calculus courses, I present and examine each of the regions into which the knowledge to be learned is divvied up – the praxeologies tabled up top.

Section VII.iii.a: Partial Derivatives and Surfaces ($\Pi 16, \Pi 18$ to $\Pi 23$)

Π16 Functions of several variables | 1 | 1

- To find the domain of a two-variable function (T16.2, t16.2) | 1 | 1
- To sketch the domain of a two-variable function (T16.3, t16.3) | 1 | 1

Π18 Partial derivatives | 3 | 5

- To find the first partial derivatives of a function (T18.1, t18.1) | 1
- To find the value of the first partial derivatives of a function at a point (T16.1, t16.1) | 1
- To find the first partial derivatives of a two-variable function defined implicitly [(T18.1, t18.1),(T18.1, t18.1.1)] | 3 | 2
- To verify that a two-variable function satisfies a partial differential equation [(T18.1, t18.1), (T18.5, t18.5)] | 2

Π20 Chain Rule | 2 | 3

- To use the chain rule to find the partial derivative of a differentiable function of two variables, each of which is a differentiable function of one variable [(T18.1, t18.1), (T20.1.1, t20.1.1)] | 2
- To use the chain rule to find the partial derivative of a differentiable function (of two/three variables, each of which is a differentiable function of two variables) [(T18.1, t18.1), (T16.1, t16.1), (T20.1, t20.1) / (T20.1.2, t20.1.2)] | 3

Π19 Tangent planes and linear approximations | 3 | 2

- To find the equation of the tangent plane to a surface at a point [(T16.1, t16.1),(T18.2, t18.2), (T19.1, t19.1)] | 1 | 2
- To find a point on a surface such that the tangent plane at this point is parallel to a given plane [(T18.1, t18.1.1), (T19.1, t19.1)] | 2
- To use the tangent plane equation at a point (a,b) to approximate the value of the function at a point close to (a,b) (T19.3, t19.3) | 1

Partial derivatives and surfaces | 6 | 6

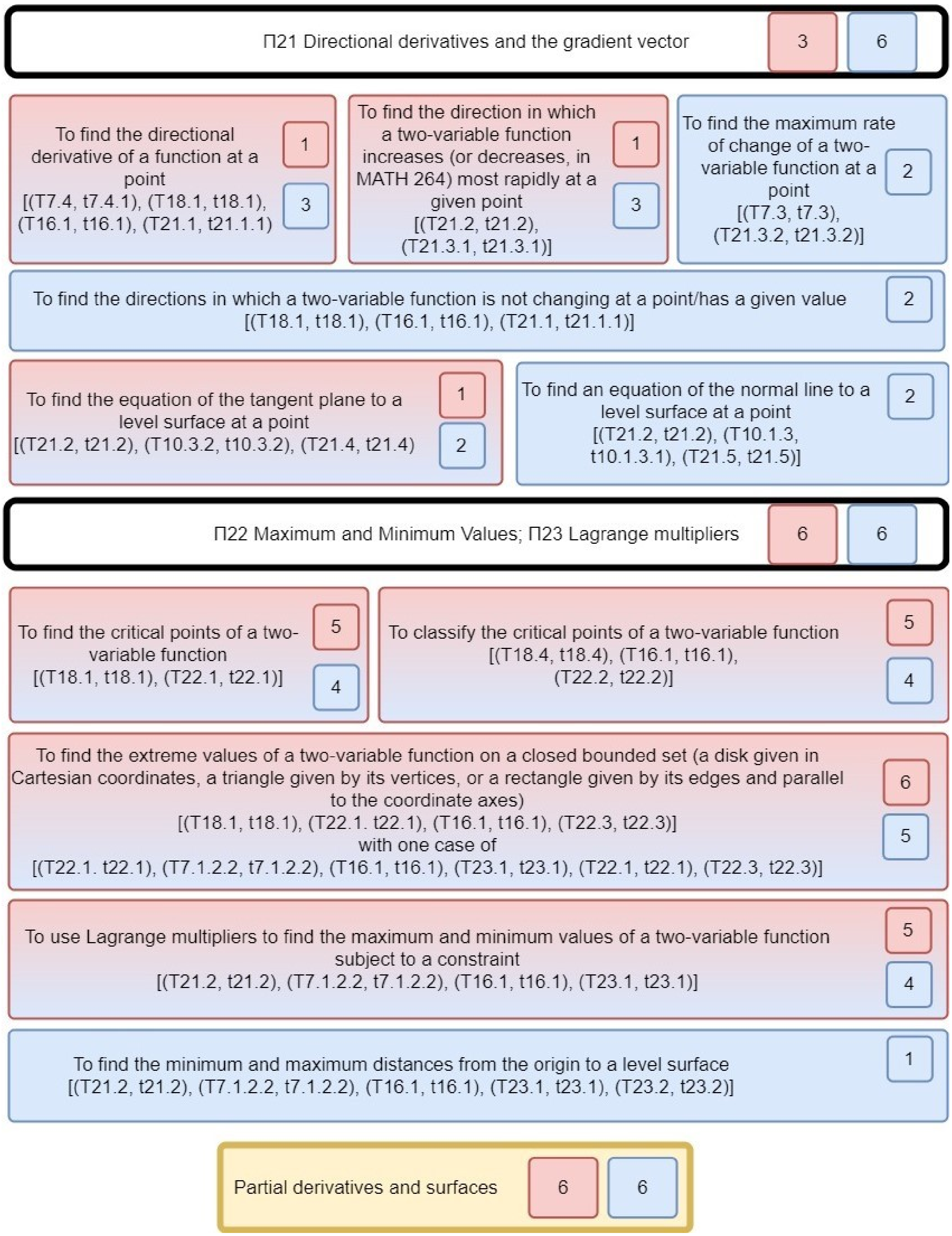


Figure 36. Model of the KL Π Partial derivatives and surfaces.

By far the most tested praxeology of the lot, Partial Derivatives and Surfaces is the hallmark of every MAST 218 and MATH 264 exam: it stars in at least 2-4 questions for major students and 4-5 in pure and applied students' exams. The tasks here all rely on students' capacity to use the technique of partial derivatives. Students need not know the definition of partial derivatives in terms of limits, but they do need to know *how to find* the partial derivatives of a function: differentiate only with respect to the variable in question and treat the others as constants.

The tasks in this KL Π can be partitioned among 7 KT Π:

- $\Pi 16$ Functions of Several variables (MAST 218 – 1; MATH 264 – 1)

  In both MAST 218 and MATH 264, students are asked to find and sketch the domain of a two-variable function: the former are treated to the function $f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$ and the latter to $f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$. To provide an appropriate solution to this exercise, students need to know that the square root is only defined for nonnegative (real) numbers ('real' in parentheses, as the mathematics at this level of study is tacitly assumed to occur in real space). From here it's a matter of determining the values of x and y for which the radicands are nonnegative. In both cases, students need to recognize the inequality $x^2 + y^2 \leq c^2$ (where c is a constant) represents a disk of radius c. Finally, students need to know that the domain of a function $f(x) = g(x) = h(x)$ is the intersection of the domains of g and h.

  This exercise is akin to those students are tasked with in high-school algebra courses – the difference is in the extra independent variable. This does not translate to a difference in technique or the need for new knowledge, however. Equations of disks in R^2 are a core component in high-school algebra, as is the task of finding the domain of a square-root function. What's new is in that the function is in two variables; but the task students need to do projects them back onto the plane of high-school algebra. They don't need to make any explicit links between the domain they find and the two-variable function. In the theoretical block of the knowledge to be taught, the graph of a two-variable function is defined as a surface "lying directly above or below its domain in the xy-plane" (Stewart, 2015, p.930). Students don't need to make the link between the significance of a two-variable function and the procedures they apply to it in the practical block.

- $\Pi 18$ Partial derivatives (MAST 218 – 3; MATH 264 – 5)

This set of tasks is constituted of those where students are to find partial derivatives of various functions but not yet use them for much further purpose. We see here that MAST 218 students specifically need to learn how to find the first partial derivatives of implicitly-defined functions (i.e. to differentiate both sides of the equation with respect to a single variable and then isolate the derivative). MAST 218 students needed to be able to do this in half of the six exams I studied; in the same number of MATH 264 exams, students were also tasked with finding the first partial derivative of a function, though not always an implicitly-defined one. On two other occasions MATH students needed to verify whether a function satisfies a partial differential equation, in what is perhaps a nod to the Ordinary Differential Equation course students in the Pure and Applied stream are to take later on. (The Differential Equations course available to students in the major stream is optional.)

- $\Pi 20$ Chain Rule (MAST 218 – 2; MATH 264 – 3)

The questions that lead students to apply the chain rule are straightforward requests to find the first partial derivatives of functions of variables that are themselves functions as well; in some cases, they are also instructed to use the chain rule. In all cases, students aren't burdened with the task of determining that the chain rule is indeed applicable; differentiability is not a property so much as an assumption for the functions in the exams. The only difference between how MAST 218 and MATH 264 students are tested here is in the number of variables involved. The reason for this 'dissimilarity' is hard to discern – it simply requires a repetition of the same computations.

- $\Pi 19$ Tangent planes and linear approximations (MAST 218 – 3; MATH 264 – 2)

The main task here is to find an equation of the tangent plane to a surface at a point. To do this, students need to find the first partial derivatives of the given function and use these to produce the equation of the plane. This task is routine in the KT $\Pi 19$ – indeed, it corresponds to task ($\mathrm{T}_{19.1}$, $\tau_{19.1}$), so the equation of a tangent plane to a surface at a point belongs to the trove of items that students are familiar with from the textbook examples and (assignment) exercises. The alternative is to be able to derive the equation of the tangent plane – a task which is not expected of students, and requires control of items from various theoretical blocks of the knowledge to be taught: the geometric interpretation of the partial derivatives of a two-variable function at a point $P(a, b, c)$ as "the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$" (Stewart, 2015, p.955); the definition of tangent plane at a point as the plane that contains these two tangent lines; and the derivation of the equation of this tangent plane, which dabbles in a range of algebraic manipulations.

MAST 218 students sometimes need to combine the routine task $(T_{19.1}, \tau_{19.1})$ of finding the equation of a tangent plane with their theoretical knowledge of parallel planes in order "to find a point on a surface such that the tangent plane at this point is parallel to a given plane" (this occurs in two exams). In terms of the relevant theoretical knowledge, students must know that the normal of a plane determines its direction and be able to extract from the equation of a plane its normal. This is knowledge from linear algebra which is taught in this course in the KT $\Pi 10$ Equations of lines and planes.

Finally, the matter of linear approximations comes up only once, and this in a MATH 264 exam when students are tasked with finding the tangent plane to a surface at a point and then using the equation they've formed to approximate the value of the function at a nearby point. Given the absence of such an exercise from the rest of the exams, it seems that linear approximations are not given much value as knowledge to be learned. This, combined with the routine nature of the task that *does* regularly appear (to find the equation of the tangent plane to a surface at a point) as knowledge to be learned from the KT $\Pi 19$, Tangent Planes and Linear Approximations, has far-reaching implications: students are tested for their ability to form an equation of a tangent plane, but not on the theoretical meaning of these planes and their descriptive value in approximating surfaces. Students need to be able to find the partial derivatives of the given function; they need to be able to recall the standard equation of a tangent plane; they need to plug the values of the partial derivatives at the given point into the equation. They don't need to explain the relation between a surface that is the graph of the function and its tangent planes.

- $\Pi 21$ Directional derivatives and the gradient vector (MAST 218 – 3; MATH 264 – 6)

This Π from the knowledge to be taught is a staple of MATH 264 exams and appears in half of the MAST 218 exams. Some of the tasks from this KT Π that occur in the exams have to do with the rate of change of a function at a point: to find the direction in which a function increases the most, to find its maximum rate of change at a point, and to find the directions in which it has a given rate of change. The rest of the problems do with finding the equations of tangent planes and normal lines to level surfaces.

Before I discuss the knowledge to be learned from the practical and theoretical blocks of $\Pi21$, I lay a few words on some of the 'prerequisite' technologies that support the techniques used to solve the tasks at hand but aren't specific to $\Pi21$. From a practical standpoint, students must be fluent in the calculation of magnitudes of vectors and, apart from this, know how to find the unit vector in the direction of a given vector (this in particular requires knowing scalar multiplication of vectors). Depending on how a student learns to find the directional derivative of a point, they may or may not need to know how to compute a dot product. Finally, the definition of gradient vector rests on a knowledge of what vectors are. These snippets are from the practical and theoretical blocks of the KT $\Pi7$, Vectors, and in some cases, the component definition of dot product from $\Pi8$, Dot Product.

The exam tasks that relate to $\Pi21$ all require students to find either the directional derivative or the gradient vector of a two of three-variable function. This is one more exercise in computing partial derivatives. Students don't have to do much with directional derivatives further than this. On one occasion, a MAST 218 exam had students compute the directional derivative of a function at a point in a given direction. On two occasions, MATH 264 students had to find the direction in which the directional derivative of a function has a given value. These students had to go one step further than finding an expression for the directional derivative of a function – they had to set it up in an equation to determine the point that yields a particular value; an algebraic matter. Grosso modo, there isn't much ado about directional derivatives beyond computing them.

Computing gradient vectors, on the other hand, is a task that students must learn to execute in describing certain aspects of surfaces that are the graphs of functions and level surfaces. While this isn't of much import in MAST 218 exams, appearing as it does in only two of them, it is a significant technique for MATH 264 students to learn and know how to use in the context of surfaces. In terms of the theory they need to learn, they need to know that the direction in which a two-variable function increases the most is given by its gradient vector; concordantly, they also need to know that the maximum rate of change of a function is the magnitude of its gradient. This is a theorem in the theoretical block for the KT $\Pi21$. Students must also view the gradient vector in its capacity as a normal to the tangent plane to a level surface at a point. This knowledge, coupled with knowing how to construct the equation of a plane given its normal or the equation of a line given a vector parallel to it (the stuff of KT $\Pi10$, Equations of lines and planes), enables students to achieve the tasks of finding the equations of tangent planes and normal lines to level surfaces. Hence, the theoretical knowledge to be learned about gradient vectors is their role in determining the direction in which a function increases the most at a point and in fixing the tangent plane to level surfaces at a point.

- $\Pi22$ Maximum and minimum values and $\Pi23$ Lagrange multipliers (MAST 218 – 6; MATH 264 – 6)

  Students in 218 and 264 alike must be adept at the procedures for finding the local and global maxima and minima of functions. Students are invariably assigned the routine tasks that drive the theoretical blocks of $\Pi22$ and $\Pi23$: to find the critical points of a function and to classify them; to find the extreme values of a function over a closed bounded set (always a disk, rectangle, or triangle); and to use Lagrange multipliers to find the extreme values of a function subject to a single constraint.

  For the first task, students must know that critical points are those where all partial derivatives are zero – they may even have an intuitive understanding of why this is so. Thus, they find the function's partial derivatives, set them equal to zero, and solve. For the second task, students must apply the second derivative test – of which the underlying technology, properties of the Hessian matrix, is absent from the knowledge to be taught, but which is fully within reach of 218/264 students' computational skill. In fact, beyond having an intuitive hold of the relation between critical points and derivatives, students are held responsible purely for procedural handiwork. To find the extreme values of functions over closed bounded sets, they may again visualize why these values can be found on a closed bounded set, even if they have yet to form a rigorous conceptualization of 'closed' 'bounded' 'sets' – as individual concepts and as a conglomerate which 264 students are set to encounter in their later Analysis courses. Thus, students unwittingly apply the Extreme Value Theorem; its assumptions are always satisfied in the questions on the exam. In bona fide fashion, functions are always continuous, and the rectangles, disks, and triangles they're defined on are always closed and bounded. Finally, students receive explicit instructions to use Lagrange multipliers to find the extreme values of two-variable functions subject to a single condition. Yet again, this amounts to a sequence of routine tasks involving computation of derivatives and systems of equations – the "Method of Lagrange Multipliers," outlined as such in the textbook – whose execution requires zero recall of the theoretical discussion that funds it.

On the whole, the mathematical organization of Partial Derivatives and Surfaces in the knowledge to be learned belies the theoretical block at its basis. For one, the functions dealt to students are differentiable by default. Students needn't spare a thought as to why there is even a directional derivative or gradient vector to be found in the given cases. Since differentiability is consistently a non-issue, it follows that this bit of theoretical knowledge is not to be learned. I expand on the treatment of differentiability, continuity, and limit in this multivariable calculus course later, following the discussion of the remaining praxeologies in the knowledge to be learned. On the basis of the current mathematical organization, I suggest only that technologies in the knowledge to be taught about surfaces are trimmed down in the knowledge to be learned; students are to learn them as *techniques* useful in the execution of tasks that do with two or three-variable functions. Directional derivatives and gradient vectors are no longer technologies contingent upon certain conditions; they are exercises to be done, rather than quantities that help study certain classes of functions; and in this metamorphosis from technology to task to technique, directional derivatives and gradients lose their descriptive prowess.

The practical blocks of the KT Π fare similarly. The practical block of the knowledge to be learned in the praxeology Partial derivatives and Surfaces is a selection of discrete tasks from the practical block of the KT Π. More precisely, students don't generally have to do a multitude of tasks that together would give a portrait of a function at a point. They alternately have to find a tangent plane at a point, or a directional derivative, or simply a partial derivative. Only once do students have to use the tangent plane at a point on a surface to approximate the value of the function at a nearby point in the domain. The matter of the domain of functions comes up once. The various technologies students have to learn to use are virtually distinct from the theory of surfaces that they support. The absence in all examinations of the task of sketching a surface is most indicative of a rupture between the computations students must do and the surfaces that their results are meant to describe. Yet again, it seems that the technologies of this mathematical organization – continuity, differentiability, differentials (which, along with increments of a function, are no-shows), gradient vectors, directional derivatives – are not to be learned in any theoretical capacity. The first of these are properties which are both assumed and forgotten; the latter occur in the computation of gradients and derivatives in a set of routine tasks and techniques that deal with functions given in their algebraic form. In sum, the knowledge to be learned in the mathematical organization of Partial Derivatives and Surfaces is bereft of a cohesive account of partial derivatives in their relation to surfaces.

Section VII.iii.b: Space Curves and Vector Functions ($\Pi12$ to $\Pi15$)

Π12 Vector functions and space curves | 1 | 0

- To show the derivative of the magnitude of a vector function is equal to the dot product of the derivative vector and the vector function divided by the magnitude of the vector function [(T7.3, t7.3), (T8.1, t8.1.1)] | 1
- To show that if a vector function has constant length over an interval, then the derivative vector is perpendicular to the vector function at all points of this interval (T8.2.2, t8.2.2) | 1

Π13 Derivatives and integrals of vector functions | 3 | 2

- To find the unit tangent vector of a space curve [(T13.1, t13.1), (T7.4, t7.4.1), (T13.2.1, t13.2.1)] | 2 | 1
- To find an equation of the tangent line to a space curve [(T13.1, t13.1), (T1.4, t1.4), (T13.2.3, t13.2.3)] | 1
- To find the angle between two curves at their intersection points (MAST 218) (T1.3, (T13.1, t13.1), (T7.3, t7.3), (T8.2.1, t8.2.1), T13.4) (MATH 264) [(T2.3, t2.3), (T13.1, t13.1), (T7.3, t7.3), (T8.2.1, t8.2.1), T13.4] | 1 | 1

Π14 Arc length and curvature | 5 | 4

- To find the curvature of a curve [(T13.2.1, t13.2.1), (T13.1, t13.1), (T7.3, t7.3), (T7.1.2.2, t7.1.2.2), (T14.5, t14.5.1)] | 2
- To find the length of a space curve [(T13.1, t13.1), (T7.3, t7.3), (T14.1, t14.1)] | 4 | 1
- To find a point on a curve where the curvature is zero [(T13.1, t13.1), (T7.3, 7t.3), (T7.1.2.2, t7.1.2.2), (T14.5, t14.5)] | 1
- To find the principal normal vector of a curve [(T13.1, t13.1), (T7.3, t7.3), (T7.1.2.2, t7.1.2.2), (T14.6, t14.6)] | 1
- To show the curvature of a given curve approaches 0 as t approaches 0, ∞ | 1
- To find an expression for the normal plane to a curve [(T13.1, t13.1), (T10.3, t10.3.1/t10.3.2/t10.3.3), (T14.8, t14.8)] | 1
- To find a point on a curve where the normal plane is parallel to a given plane [(T13.1, t13.1), T13.4] | 1 | 2
- To find an equation of the osculating plane of a curve at a point [(T13.2.1, t13.2.1), (T14.6, t14.6), (T9.1, t9.1), (T14.7, t14.7), (T10.3, t10.3.1/t10.3.2/t10.3.3), (T14.9, t14.9)] | 1

Space curves and vector functions | 6 | 4

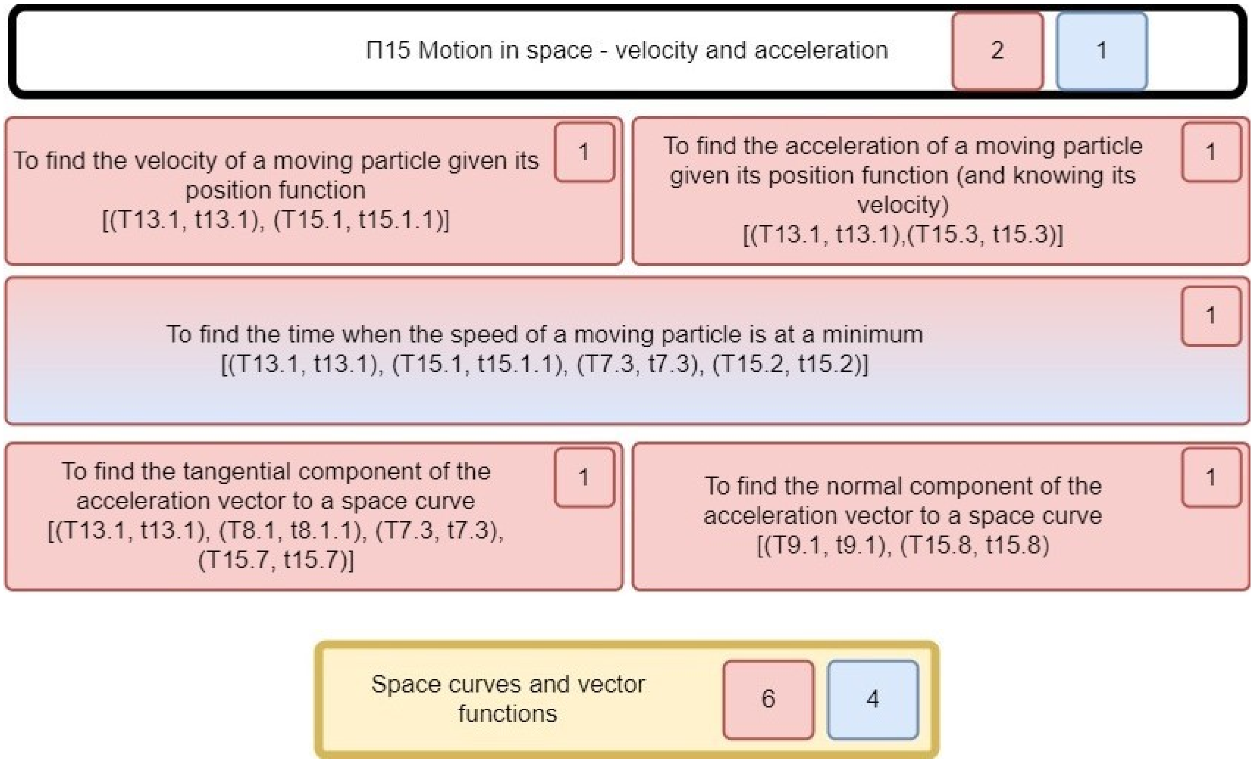


Figure 37. Model of the KL Π Space curves and vector functions.

I explore the knowledge to be learned in the context of each KT Π in decreasing order of their number of occurrences in my set of final exams.

- $\Pi 14$ Arc length and curvature (MAST 218 – 5; MATH 264 – 4)

  It's hardly a curve-ball that arc length and curvature form a praxeology that's among the most frequently tested in 218 and 264 assessments. For one, they are cornerstone quantities in the description of space curves. Additionally, recall the map of cross-references between the theoretical blocks of the knowledge to be taught: $\Pi 14$ scaffolds upon theory and technology that relate to parametric curves, three-dimensional coordinate systems, equations of lines and planes, and the triad of vectors, dot product, and cross product. This seems an opportunity to test a great scope of students' knowledge in one fell swoop. One of the aims of the discussion that follows is to determine the minimal scope of theoretical and practical knowledge to be learned for a student to execute the tasks in this mathematical organization.

  MAST 218 students primarily need to find the length of a curve. They have a formula: the arc length of a curve given by vector function $\mathbf{r}(t)$, with $a \leq t \leq b$, is given by

$$\int_a^b |\mathbf{r}'(t)|\,dt.$$

Finding the arc length of a curve is therefore an exercise in computing the derivative of a vector function, finding the magnitude of the vector function, and finally integrating over the interval $[a, b]$. The conditions necessary for the application of this formula are always met - the components of the vector functions in the exam are continuously differentiable and the vector function injective.

It is unnecessary for students to learn the importance of arc length as an invariant property of a curve; none of the tasks highlight in any way the independence of arc length of the parametrization of a curve. None of the tasks highlight the *utility* of this property, either: absent are the tasks (to be taught) of reparametrizing a curve using its arc length function or of finding a point on a curve a number of units away from a given point on the curve. As far as the arc length of a space curve is concerned, students need only know a formula. What they're missing out on is the purpose of studying arc length to begin with.

The curvature of space curves receives a similar sentence. MATH 264 students must find it, and MAST 218 students must identify a point on the curve where it is zero. Both tasks whittle down to a formula (from a choice of two) for finding the curvature κ of a curve $\boldsymbol{r}(t)$ *at* a point:

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \quad \text{or} \quad \kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

where $\mathbf{T}$ is the unit tangent vector of $\mathbf{r}$.

This task, which at face-value seems to be about curvature, is more accurately an exercise in computing derivatives and magnitudes of vector functions. On the one exam (from the twelve considered) where students are relieved of the duty to learn by heart the library of formulas relating to properties of space curves, students have to turn their focus elsewhere: for a given vector function, find its curvature, and then show that $\kappa(t) \to 0$ as $t \to 0, \infty$, and to find the maximum value of $\kappa(t)$. Students are not asked to explain nor observe how the first task they complete in effect ensures that the second task can be achieved. This is reminiscent of two routine tasks from the North-American standard Differential Calculus course: given a single-variable function, to find its limit at zero and at infinity and to find its maximum value. Altogether, the nature of the exam tasks is such that students don't need to learn the relation between a space curve and its curvature; their job rests with the vector calculus techniques that relate to it. They can turn a blind eye to the theoretical underpinnings and definition of curvature and successfully answer exam questions where the term ‘curvature’ shows up.

The story repeats in the remaining tasks. On separate occasions, students have to find the equations for the normal and osculating planes of a curve. In all, the tasks that students must learn to undertake any task in this mathematical organization are to find the derivative of vector functions; to find the unit tangent vector to a curve at a point; to find the magnitude of a vector; to do scalar multiplication; and to form an equation for a plane, given its normal. These tasks form the underbelly of the mathematical organization of Arc Length and Curvature.

What is at the forefront of this mathematical organization? Altogether, the various geometric properties of curves that are to be taught – tangent, normal, binormal vectors; tangent, normal, osculating planes; the osculating circle – can be learned as entities unrelated to one another. (On that note: the osculating circle is exempt from the exams I studied.) Students are never required to use these geometric properties cohesively to give a local description of a curve. They have to find equations for a different property on each exam; and in most cases, they don’t need to have an intuitive understanding of these properties. For instance, students needn’t know that the osculating plane at a point P on a curve is “the plane that comes closest to containing the part of the curve near P” (Stewart, 2015, p.907). Thus, it’s not necessary for students to form a full-fledged local conception of a curve. The task $\Pi 14.10$ from the practical block of $\Pi 14$ might have required students to do so: to graph the osculating circle of a plane curve at a point. Just as in the case of Partial Derivatives and Surfaces, the absence of a graphing task is telling of the discrete nature of the knowledge to be learned in the Π Arc Length and Curvature – discrete both in the disconnectedness of the tasks, with different exams poking at points of curves once with an osculating plane, once with a normal vector, another time with curvature, and in how students must learn differential and integral calculus techniques to find these geometric properties but not what these properties actually signify.

- $\Pi 13$ Derivatives and integrals of vector functions (MAST 218 – 3; MATH 264 – 2)

Most of the questions that do tasks specific to the practical block of $\Pi 13$ have students find either the tangent vector or the tangent line to a curve at a point. Thus, as in their first differential calculus course, students need to know that the tangent to a curve at a point is given by the derivative of the function it represents. To find the tangent line, students need to combine this knowledge with technique taught in $\Pi 10$, Equations of lines and planes.

In one exam in each of 218 and 264, students are to find the angle of intersection between two curves. For this, they must know that the tangent line to a curve at a point approximates it locally; upon finding equations for the tangent vectors to the two curves at the point of intersection, students must know how to find the angle between two vectors (task $T_{8.2.1}$ from the KT Π, Dot product). This task requires students to engage with the theoretical block of a KT Π. Students must settle on an appropriate model for the situation (tangent lines to model the curve locally at the given point) and select a task from the practical block accordingly.

- $\Pi 15$ Motion in space - velocity and acceleration (MAST 218 – 2; MATH 264 – 1)

In this little-represented mathematical organization, students must know how to obtain the velocity, acceleration, and speed of a curve at a point. To this end, they need to learn that the velocity of a curve at a point is given by the derivative at that point, that acceleration is the derivative of velocity, and speed the magnitude of the velocity. In one MAST 218 exam, students are tasked with finding these three quantities given the position function of a particle. Additionally, they're asked to find the particle's minimum speed. To this end, they must compute the magnitude of the velocity and find its minimum using knowledge learned in single-variable calculus and relearned, in a sense, in $\Pi 22$, Maximum and minimum values of multivariable functions.

A second MAST 218 exam asked of students to find the tangential and normal components of the acceleration vector of a space curve. This amounts to an application of formulas provided in the knowledge to be taught. As in the case of most technologies referred to in this KL Π of vector functions and space curves, students don't need to provide a geometric interpretation of these concepts and formulas.

In sum, the exam tasks drawn from the KT $\Pi 15$ Motion in space are firmly rooted in the task of finding the derivative of a vector function. The tasks don't delve much further than this. For instance, $T_{15.6}$, to solve projectile motion problems, is absent from the knowledge to be learned. This Π in the knowledge to be learned seems a missed opportunity to hand MAST 218 students problems of a more applicative nature (in line with the MAST program's stated mission of focusing on this aspect of mathematics).

- $\Pi 12$ Vector functions and space curves (MAST 218 – 1; MATH 264 – 0)

In a turn of events, students are not asked to *find* anything, but to validate a property of vector functions: they must show that

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t)$$

This task is non-routine in the sense that students aren't given a particular vector function to work with. This means they must produce an appropriate representation for $\boldsymbol{r}(t)$ that will enable them to work with the technologies involved. To this end, students should write $\mathbf{r}(t)$ in terms of its component functions:

$$\mathbf{r}(t) = (f_1(t), f_2(t), \dots, f_n(t))$$

It's not clear if students are expected to provide as general a proof as possible (i.e. n component functions) or if they may assume the vector function to have, say, three components. In any case, the first task is to represent $\mathbf{r}(\mathrm{t})$ component-wise. This is an unusual requirement that calls for a theoretical approach to vector functions. Students don't generally need to explicitly think about vector functions having the form of a vector; they receive concrete vector functions along with some task that requires them to do calculations. Students who do choose to express $\mathbf{r}(t)$ in the appropriate form can move on to the more computational job of applying the definitions of magnitude, dot product, and derivative of vector functions.

The second part of the exam question is the following: "If the vector $\mathbf{r}(t)$ has constant length in the interval (a, b), show that the derivative vector is perpendicular to $\mathbf{r}(t)$ at all points of this interval." To show this, students need to use the first part of the problem. They need to know that the derivative of a constant is zero; further, they need to know that two vectors are orthogonal if their dot product is zero.

This problem requires students to work on a more abstract level than usual: they need to have some awareness of symbolic representations and their affordances in choosing to represent $\mathbf{r}(t)$ component-wise. It's difficult to assess just how much reflection was required of students taking this examination to determine an approach to the question. Tasks specific to proving properties of vector functions and their derivatives make a brief appearance in the textbook; the extent to which this type of task was emphasized by the instructor is beyond my reach. Additionally, since this question occurred in one of the *earlier* final exams that I studied (2013), I don't know if prior finals (accessible to students at the university print shop) bandied about such tasks more frequently than in recent years' exams.

It is essential for 218 and 264 students alike to be fluent in differentiation and vector algebra techniques to engage in the Π of Vector functions and space curves. They need to know derivative rules for single-variable functions to execute virtually all the tasks in this Π, as their respective techniques all rest on the differentiation of vector functions. They need also employ the following subset of vector algebra tasks and techniques taught in the course: $(T_{7.1.2.2}, \tau_{7.1.2.2})$ (scalar multiplication of vectors given in component form), $(T_{7.3}, \tau_{7.3})$ (to find the magnitude of a vector), $(T_{7.4}, \tau_{7.4})$ (to find the unit vector in the direction of a given vector), and $(T_{8.1}, \tau_{8.1.1})$ (to find the dot product of two vectors given in component form). This summarizes the minimal technical skills students need to have to undertake any of the tasks from $\Pi 12$ through $\Pi 15$ (the constituent KT Π of the KL Π Vector functions and space curves).

The knowledge to be taught in the chapter Vector Functions mostly relates to the differential geometry of space curves. The minimal scope of knowledge to be *learned*, however, does not bridge the link between curves and the calculus of vector functions. The routine nature of the tasks, mostly of the types “to find an invariant quantity of a curve” or “to find the equation of a geometric property of curves,” enables students to learn formulas and does not require them to develop neither an intuitive nor a rigorous definition of these invariants and properties of curves.

Section VII.iii.c: Equations of Lines and Planes and Distances in R^3 ($\Pi 10$)

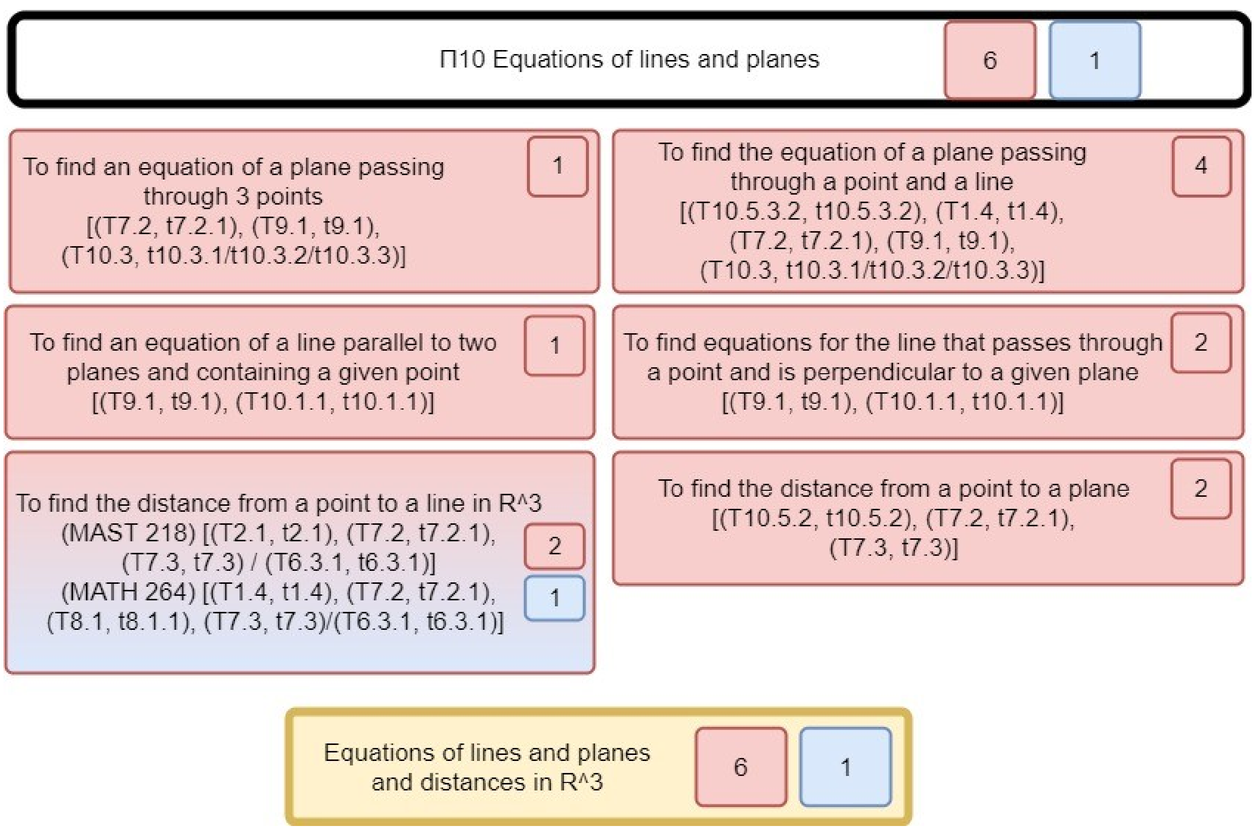


Figure 38. Model of the KL Π Equations of lines and planes and distances in R^3.

Tasks whose goal is to find the equation of a line or a plane (*not* for a higher purpose) occur in all six MAST exams but only in one MATH exam. This KL Π corresponds to the KT $\Pi 10$ Equation of lines and planes.

The knowledge to be learned to answer the exam questions specific to this Π forms a quite specific praxeology. Equations of planes are to be found in two setups:

1. Given three points in the plane;
2. Given a point and a line that lie in the plane.

The second case reduces to the first. From this point, the procedure is use the three points to find two vectors parallel to the plane (but not to each other), to find their cross product and therefore a normal to the plane, and to use this normal to form an equation for the plane. The theoretical knowledge at the base of this task: that a plane is determined by a normal and a point on the plane; that the cross product of two vectors produces a vector orthogonal to both; that the displacement from one point to another in space can be represented by a vector; and an understanding of the notation used for the component form of points and vectors alike.

Students tasked with finding the equations of lines are given a variety of scenarios:

1. A line that is the intersection of two planes (this occurs as a sub-step in the task "to find the equation of a plane passing through a point and a line);
2. A line parallel to two (intersecting) planes and containing a given point;
3. A line perpendicular to a plane and containing a given point.

The first of these corresponds to ($T_{10.5.3.2}, \tau_{10.5.3.2}$), where the technique is to find a point on the line (e.g. by setting one of the coordinates to 0) and find a vector parallel to the line by computing the cross product of the normal vectors of the planes. The second case is in fact a re-wording of the first. The third case requires student to find two vectors on the plane and compute their cross product to find a vector parallel to the line. In all, the theoretical knowledge to be learned is that a line is determined by a point and a vector parallel to it; to recognize from the equation of a plane its normal; and to know that the cross product of two vectors produces a vector orthogonal to both.

Beyond knowing the form of the equations of a lines and planes, MAST 218 students need to know how these objects are determined by vectors parallel or perpendicular to them. They also need to know an important theorem about cross product and how to compute it. This does not need to be learned by MATH students.

In one MATH 264 and four MAST 218 exams, students need to find the distance between a point P and either a line or a plane. To this end, they must find the point Q on the line (or plane) whose distance from P is minimal. It's essential for students to realize this point is such that $\overrightarrow{PQ}$ is orthogonal to the line (or plane). The rest of the task is in modelling this theoretical knowledge: choose an arbitrary point $Q\ (x, y, z)$ on the line, express the vector $\overrightarrow{PQ}$ in component form, find a vector $\mathbf{u}$ in the direction of the line, and solve the equation

$$\overrightarrow{PQ} \cdot \mathbf{u} = 0$$

This produces the point Q and allows students to compute the distance between P and the line: the magnitude of $\overrightarrow{PQ}$.

Equations of lines and planes and distances in R^3 constitutes a major mathematical organization for which MAST 218 students must be prepared. They must be adept at representing vectors, computing dot and cross products, and finding the magnitude of vectors; they also need to know how to relate vectors and vector operations to lines and planes. It is curious that, on each of the multivariable calculus exams in the major stream, at least one question is barren of any calculus. This is not so for those in the pure and applied stream; but that's not to say that they needn't learn some of the linear algebra techniques that form the core of this mathematical organization. After all, their learning of line and plane equations supports the tasks in the earlier-discussed Π of Partial Derivatives and surfaces; and their learning of the definition of orthogonality based on dot product, the computation of dot product and vector magnitude is also key to performing tasks in both the praxeologies of Partial Derivatives and Vector Functions. The role of $\Pi 10$, Equations of Lines and Planes, in the context of the knowledge to be taught seems to be to support both theory and practice of praxeologies of concepts belonging to calculus. Why is it that MAST 218 students need to learn linear algebra techniques for the endgame of solving linear algebra problems – in a multivariable calculus course? I conjecture an answer in the conclusion.

Section VII.iii.d: Parametric Plane Curves ($\Pi 2$)

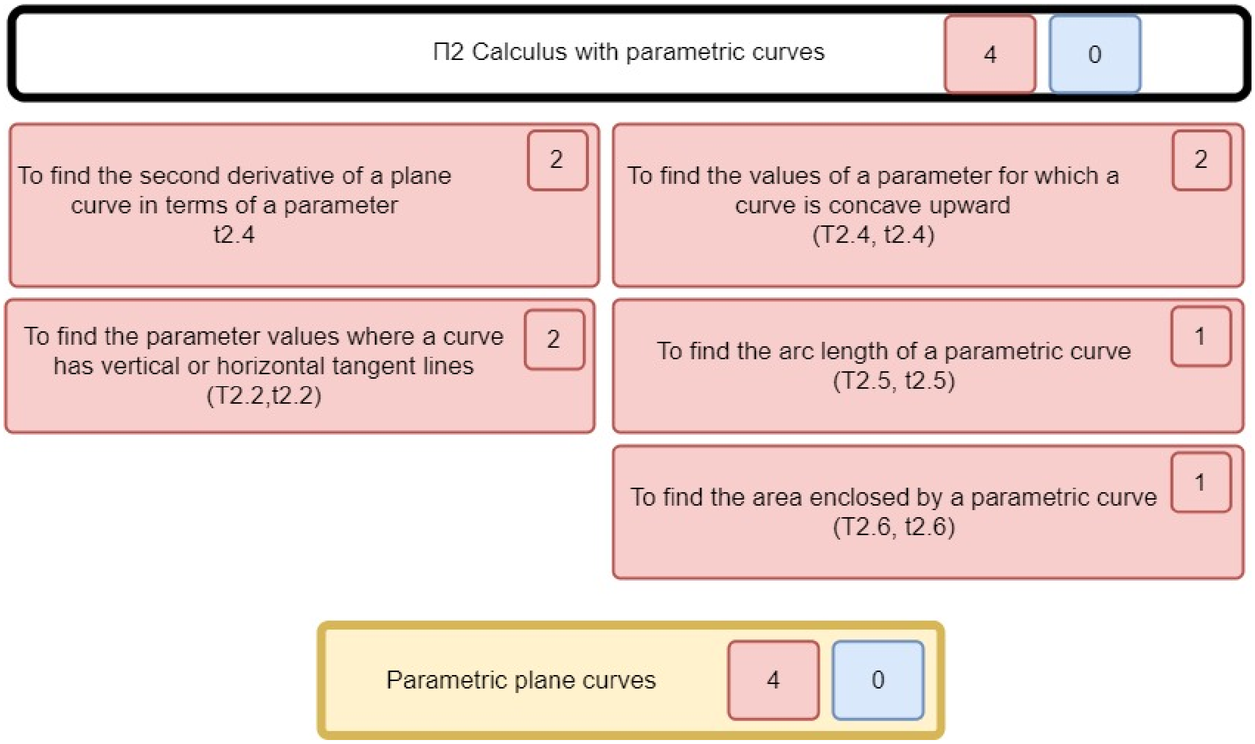


Figure 39. Model of the KL Π Parametric plane curves.

Only MAST 218 students are tested on the techniques taught in the first chapters of the course on plane curves and parametric equations. The tasks they are assigned all do with the calculus of parametric curves, as taught in the KT $\Pi 2$, Calculus with parametric curves. Almost all the tasks from the practical block of this KT Π occur throughout the four exams that exhibit problems specific to plane curves.

To find the parameter values where a curve has vertical or horizontal tangent lines $(T_{2.2}, \tau_{2.2})$ is a task that occurs in two exams. Knowing the properties of a vertical or horizontal tangent allows students to determine the equations to solve: $\frac{dx}{dt} = 0$ or $\frac{dy}{dt} = 0$. This is essentially an exercise of single-variable calculus, but could be considered as a review for students of the notion of tangent, a recurring concept throughout the knowledge to be learned in this course.

MAST 218 are also tasked with finding the second derivative of a parametric curve $(\tau_{2.4})$ – sometimes purely for the sake of computing it and sometimes to determine the value of a parameter for which a curve is concave upward $(T_{2.4}, \tau_{2.4})$. This boils down to the following: given parametric equations $x = f(t)$ and $y = g(t)$, where x and y are differentiable functions of t and y a differential function of x, to differentiate y with respect to x. The technique is an instantiation of the chain rule. Since y is a differentiable function of x and x of t,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Thus, as long as $\frac{dx}{dt}$ is nonzero,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad (*)$$

The textbook notes that students could remember this equation "by thinking of canceling the dt's" (Stewart, 2015, p.689). The alternative to this trick – deriving the equation anew – requires students to use the technique encapsulated in the three inches of paper preceding this sentence. (The chain rule for single-variable functions.) Finding the second derivative of a parametric curve rests on this same technique. Since $\frac{dy}{dx}$ here is in fact a differentiable function of x and t, Formula $(*)$ can be applied to find $\frac{d^2y}{dx^2}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Beyond knowing of the existence of this formula (along with the formula itself or how to derive it), students need to know that the concavity of a plane curve is determined by the sign of its second derivative. Yet again, both the technique and bit of theory to be learned here are matters of single-variable calculus.

Integration pops up in the wake of two tasks: to find the arc length of a parametric curve and to find the area enclosed by a parametric curve. Both tasks are to be executed with the flick of a formula. Here, as with the cases of differentiation with parametric curves, students don't need to have a rigorous or even intuitive understanding of parametric curves; but they do need to understand the symbolic representation that encapsulates these curves and recognize the presence of a parameter as a marker of parametric equations. They need to be able to differentiate and integrate using the given equations appropriately, as indicated by the formulas. They don't quite need to learn the relation between parametric equations (that define a parametric curve) and the curve traced out by the parameter therein. Students may learn this relation, but they don't *need* to in order to complete the tasks in this mathematical organization.

I will venture an explanation as to why MATH 264 exams exclude questions that call for the techniques specific to this mathematical organization and what the role of this Π may be in that course. Recall that parametric curves are taught in the first week. The exercises in differentiation and integration with parametric curves are opportunities for students to freshen up on the techniques they learned in their single-variable calculus course. The theoretical block of this Π also allows students to brush up on the concept of tangent, a key technology in much of the theory to be taught later on in the course. Introducing parametric curves, finally, familiarizes students with the notion of parameter, major in the later teaching of vector functions and space curves. These last concepts – vector functions and space curves – seem to occupy much space in the knowledge to be learned by MATH 264 students. The techniques learned in the context of vector functions are more general than the very specific ones to be learned for parametric curves. Perhaps the specificity of the parametric curve techniques makes them less attractive, if the aim is to arm students with techniques with greater potential for application.

In sum, the practical block of $\Pi 2$, Calculus with parametric curves, is not knowledge to be learned in MATH 264. Upon considering the map of cross-references between the theoretical blocks in the knowledge to be taught, however, the knowledge to be learned in this Π seems to rest mainly in its introduction of parameters. This technology comes in handy in $\Pi 11$ Equations of Lines and Planes, $\Pi 12$ Vector Functions and Space Curves, and $\Pi 14$ Arc Length and Curvature – three Π central to the knowledge to be learned in MAST 218 and MATH 264 alike. If we view $\Pi 2$ in this light, the question may switch from “why is its practical block not to be learned by MATH 264 students?” to “why is its practical block to be learned by MAST 218 students?” I believe the answer may lie in the institutional context of the course and will conjecture in the conclusion (chapter IX).

Section VII.iii.e: Limits of Rational Functions ($\Pi 17$)

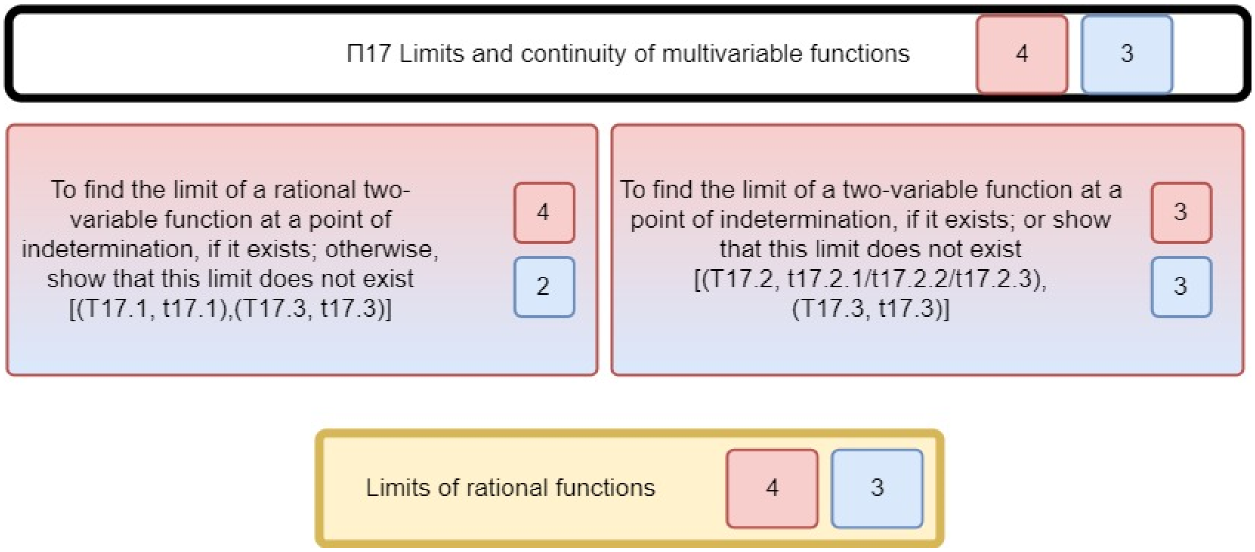


Figure 40. Model of the KL Π Limits of rational functions.

Tasks that involve the notion of limit are a core component of the 218/264 final exams. The following are the two types of tasks that students encounter; they are both preceded with the statement

Find the limit, if it exists, or show that the limit does not exist.

In both tasks, students are given an expression $\lim_{(x,y)\to(0,0)} f(x,y)$ where f takes on an indeterminate form at the origin. To show the limit does not exist, it suffices to find two paths along which the function approaches different values in a neighborhood of the origin; if the limit exists, it's necessary to conjecture what this limit might be and then use an $\varepsilon - \delta$ argument to show that it's indeed the limit. I outline the details of each of these tasks as they occur on the final exams.

Type 1

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + 2y^2}$$

This rational function is similar to those that occur in the set of exercises assigned to students from the textbook (see exercises 5-22 in (Stewart, 2015, p.950)). In the knowledge to be taught in $\Pi 17$, three paths in particular are used in examples to show that the limit of a rational function does not exist: the lines $y = x, y = -x$, and the parabola $y = x^2$. As a general technique for the task of determining whether a function does not have a limit at a given point, students learn to substitute any two of these expressions for y into the function:

[Along the line $y = x$, the function takes the value]

$$\frac{2xx}{x^2 + 2x^2} = \frac{2x^2}{3x^2} = \frac{2}{3}$$

[Using the properties of limits, it follows that the limit of the function as it approaches the origin along the line $y = x$ is $\frac{2}{3}$.]

Along the line $y = -x$, the function takes the value

$$\frac{2x(-x)}{x^2 + 2(-x)^2} = -\frac{2x^2}{3x^2} = -\frac{2}{3}$$

[Using the properties of limits, it follows that the limit of the function as it approaches the origin along the line $y = -x$ is $-\frac{2}{3}$.]

The statements I include in brackets reflect the knowledge to be taught, but it is not clear whether it is expected of students to demonstrate this reasoning or if it is sufficient to apply the algorithm of "substitute x and $-x$ for y; if the result is different, the limit does not exist." Since the function approaches different values along two different paths nearing the origin, the limit does not exist at that point. Students may be under tight time constraints in an exam situation, and therefore don't have much time, will, or even the habit to try out different approaches. They might automatically reach for the $y = x$, $y = -x$ test to determine if the function has a limit at the origin upon recognizing the type of task. Students who choose to consider the mathematical properties of the given rational function may note (intuitively) that the numerator is an odd function while the denominator is even, thereby implying the function may take on different values in the different quadrants. In either case, the most readily available strategy is to plug in x and $-x$ in place of y and see what happens. Depending on the expectations stated in class by their instructors, it may or may not be necessary for students to demonstrate (e.g. by using the term 'path' appropriately) a somewhat intuitive understanding that the existence of the limit of a multivariable function at a point is contingent upon the function approaching the same value along any path near that point. Thus, I surmise that one of the minimal units of knowledge to be learned from $\Pi 17$, Limits, is the algorithm described above that aims to compare the value of a function $f(x, y)$ for different values of y.

To complete the task described above, it is not essential for students to learn the institutional definition of limit is, as found in the textbook:

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $\quad (x, y) \in D$ and $\quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $\quad |f(x, y) - L| < \varepsilon$

(Stewart, 2015, p.944))

A particular version of the $\varepsilon - \delta$ argument *does* need to be learned for students to use an appropriate technique for the second type of task:

Type 2

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}$$

The function to be evaluated here is also similar to the ones students are tasked with in assignments (see exercises 5-22 in (Stewart, 2015, p.950)). In all of the MAST 218 and MATH 264 exams I considered, **Type 2** tasks all had four things in common:

1. The function is rational in all cases but one (the exception involves a trigonometric function);
2. The denominator in the expression is $x^2 + y^2$;
3. The limit is taken at the origin;
4. The limit is zero.

To accomplish this task, it is sufficient on some occasions to use properties of limits ($\tau_{17.2.2}$) or the Squeeze Theorem ($\tau_{17.2.3}$) to find this limit, both of which require an appropriate set of algebraic manipulations but not an understanding of the limit; students may also use the $\varepsilon - \delta$ technique ($\tau_{17.2.1}$). The standardized form of the functions in this type of task allows for an algorithmic application of the $\varepsilon - \delta$ technique that – due to its algorithmic nature – may not require a full understanding of the concepts. I explain: to begin this task, students need to first assess whether the limit is likely to exist or not. Upon plugging in a couple of values for x or y, they find that the limit may be zero; unless the function can be simplified, this requires the use of δ and ε. The solution (following the steps as taught in the text, as in p.947 of (Stewart, 2015)) would then go like this, as per the technique $\tau_{17.2.1}$:

Step 1: Apply the definition of limit in terms of ε and δ:

Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$$\text{if } 0 < \sqrt{x^2 + y^2} < \delta \text{ then } \left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$$

Step 2: Make appropriate algebraic manipulations of the expression $\frac{|3x^2y|}{x^2+y^2}$ to obtain an inequality of the sort $\frac{|3x^2y|}{x^2+y^2} \leq c\sqrt{x^2 + y^2}$ where c is a constant.

Step 3: Choose $\delta = \varepsilon/c$, so that

$$\left|\frac{3x^2y}{x^2 + y^2} - 0\right| \leq c\sqrt{x^2 + y^2} < c\delta = c\left(\frac{\varepsilon}{c}\right) = \varepsilon$$

The steps in $\tau_{17.2.1}$ that pertain to ε and δ *could* be replicated without an understanding of the concepts; it is not necessary for students to have a definition of limit that corresponds in any way to the formal definition, which includes a variety of concepts that are known to be of great difficulty for students to grasp (neighborhood of a point, the meaning of the quantifiers 'for all' and 'there exists', etc.). Step 2 is the only one where students are required to *produce* technique, but here what they require is algebraic dexterity that is quite unrelated to the notion of limit.

In all, I conclude from the given tasks and the techniques available from $\Pi 17$ that the knowledge that is *essential* for MATH 264 and MAST 218 students to learn from this praxeology is of an algorithmic nature and requires algebraic manipulations. In tasks where students find the limit of a function, it is *not* essential to have an understanding of what it means for a limit to exist; in tasks where students show that the limit of a function does not exist at a point, they may benefit from learning that the limit of a function fails to exist at a point if the function has different values as it approaches the point along a certain path. In the latter case, though, algebraic manipulations are *sufficient* to determine that the limit of a function does not exist.

Section VII.iii.f: Polar Curves ($\Pi 3 - \Pi 4$)

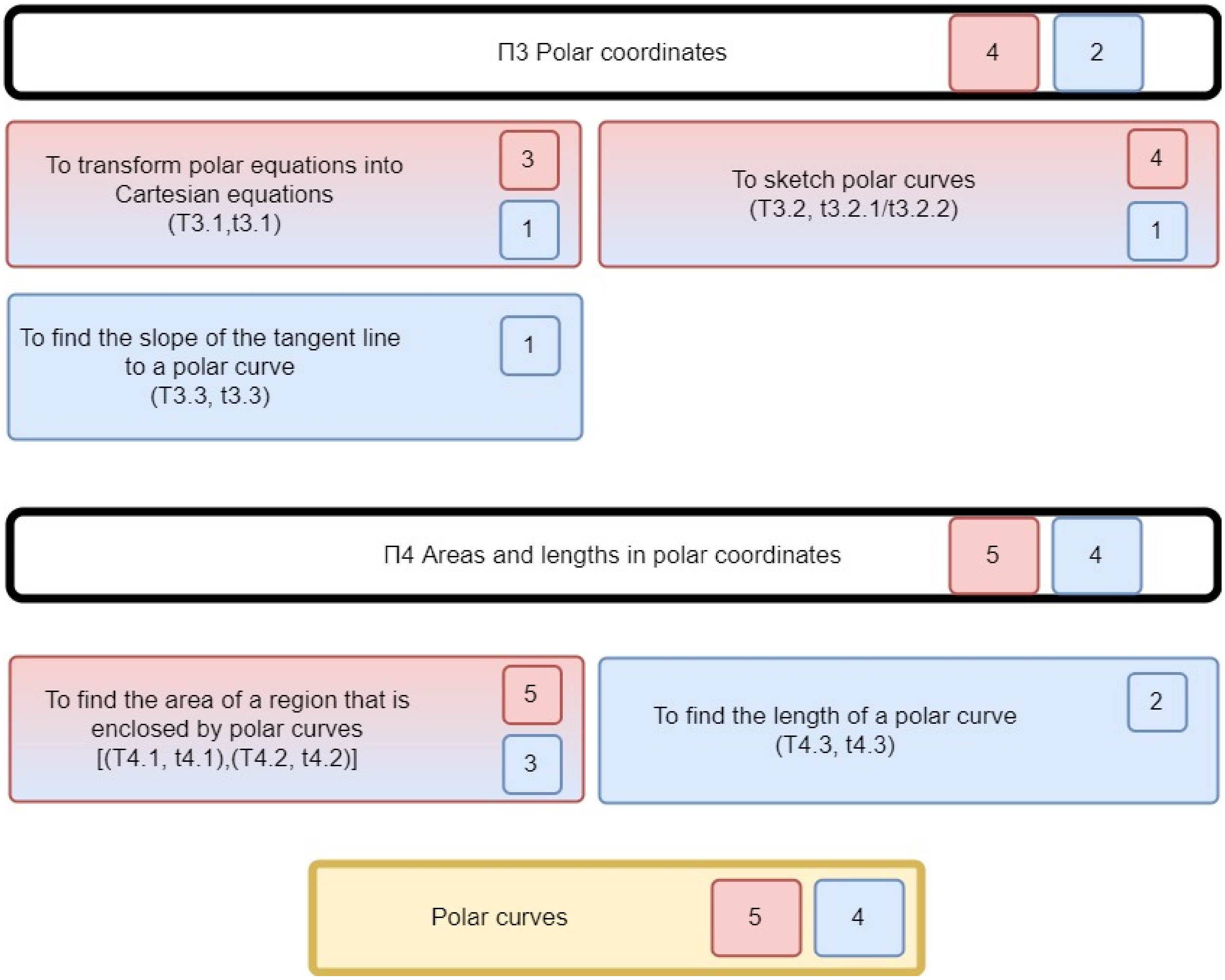


Figure 41. Model of the KL Π Polar curves.

One of the more frequently occurring praxeologies in 218/264 exams, Polar curves are to be learned from two KT Π: Π3 Polar coordinates and Π4 Areas and lengths in polar coordinates.

- Π3 Polar coordinates (MAST 218 – 4; MATH 264 – 2)

  A task that students are often given in 218/264 exams is to identify a polar curve by finding its Cartesian equations and then sketch that curves. Students must therefore be adept at the algebraic manipulations involved in the transformation from polar to Cartesian equations. The polar equations in the exams are invariably of the following forms:

$$\gamma_1\colon\ r = 2 + \sin\theta, \quad 0 \le \theta \le 2\pi$$

$$\gamma_2\colon\ r = 3\sin\theta, \quad 0 \le \theta \le \pi$$

up to a change in the constants and where the trigonometric functions are either sine or cosine. Students learn the specific algebraic manipulations necessary to convert these into Cartesian equations in examples from the textbook. For instance, Example 6 (Stewart, 2015, p.701) outlines the procedure for converting curves of the form of γ_2:

1. Use the equations established from relations in the trigonometric unit circle, choosing equations based on whether the trigonometric function is sine or cosine:

   Use the equation $y = r\sin\theta$ to write $\sin\theta = y/r$, so the equation $r = 3\sin\theta$ becomes $r = 3y/r$.

2. Express the Cartesian variable in terms of the polar variable, and again use the equations established from relations in the trigonometric unit circle:

   It follows that $3y = r^2 = x^2 + y^2$, which gives $x^2 + y^2 - 3y = 0$.

3. Complete the square to obtain the Cartesian equation for the curve:

   Completing the square, we obtain $x^2 + \left(y - \frac{3}{2}\right)^2 = \frac{9}{4}$

Students must recognize the standard equation of circles to identify the curve given by this equation as well as the center and radius of the circle obtained. A recipe of similar taste produces the Cartesian equation of curves of the form of γ_1. First, it's critical for students to be familiar with the basic relations established in the trigonometric circle. Students must also be fluent in the application of these very specific procedures lest they lose valuable exam time; this particular procedure also comes with the perk of yielding a recognizable Cartesian equation (students are always asked to identify the curve in these questions).

To sketch these curves, students must know how to plot points in polar coordinates. To plot a polar curve of the form γ_1: $r = 2 + \sin\theta, 0 \leq \theta \leq 2\pi$, students are taught to first sketch it in Cartesian coordinates by shifting the sine curve up by 2 units. A similar approach can be taken for curves of the form of γ_2. Students need to learn to use the Cartesian graph of the curve to determine "the values of r that correspond to increasing values of θ" (Stewart, 2015, p.702). From this data they plot points and approximate the parts of the curve between them. For instance, if, when θ increases from α to β, r decreases from a to b, then identify in a polar coordinate system the points with these polar coordinates, and connect them using rounded curves.

The tasks to be learned in the $\Pi 3$, Polar coordinates, require highly specific techniques that dabble in algebraic manipulations and the relations established between polar and Cartesian coordinates by the trigonometric unit circle. On one occasion, MATH 264 students were asked to find the slope of a tangent to a polar curve. The technique for this task couples knowledge of the unit circle relations with a formula forged by a combination of the chain and product rules in order to find the derivative of y with respect to x (the slope of the tangent). All in all, students are tasked with learning an algebraically-specific set of techniques to tackle a slim number of questions on polar curves that may occur in MATH 264 exams but are almost sure to be present in MAST 218 finals.

- $\Pi 4$ Areas and lengths in polar coordinates (MAST 218 – 5; MATH 264 – 4)

MAST 218 students exclusively have to find the area of a region that is enclosed by polar curves while MATH 264 students are either given the same task or have to find the length of a polar curve. Both cases are tackled by a formula that students presumably need to learn by heart. The area problem is often paired with the above task of converting polar equations to Cartesian equations, where students are given two curves to convert, identify, and sketch; in these cases, students must find the intersection of the two curves before applying the integration formulas. In all, these tasks especially test students' technique in integrating single-variable functions.

Section VII.iii.g: Conic Sections ($\Pi 5$)

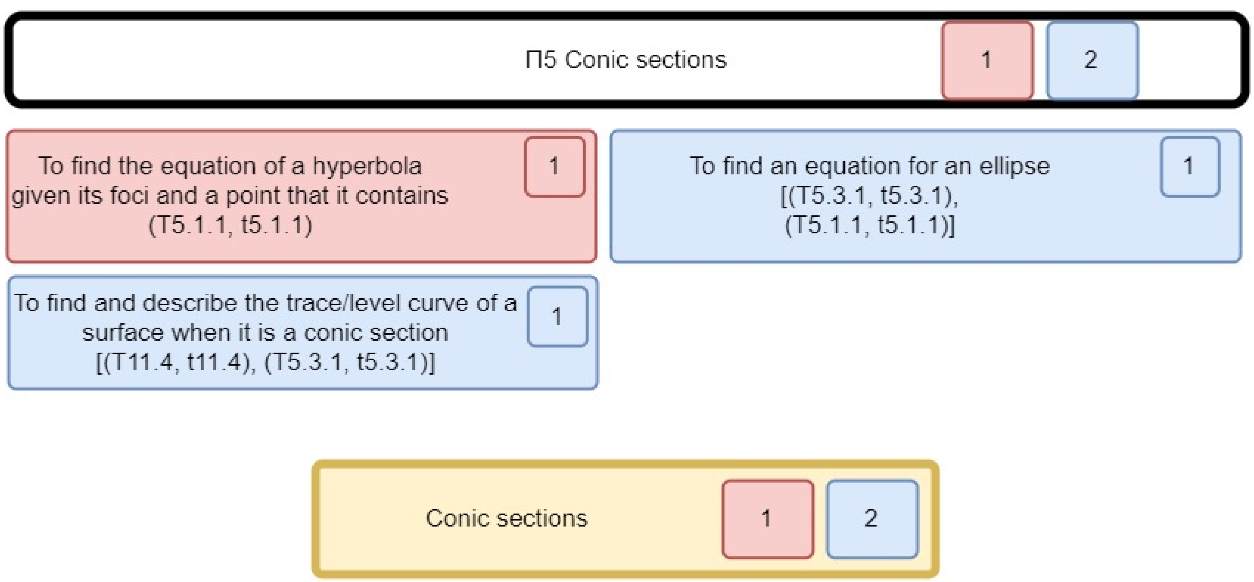


Figure 42. Model of the KL Π Conic sections.

Conic sections make brief appearances in 218 and 264 exams alike. In one instance, students have to find the Cartesian equation of a hyperbola; in another, the Cartesian equation of an ellipse. A third instance tacks on the task of describing a conic section in the context of surfaces – here students are asked to describe the level curve of a surface ("given by $z = \frac{1}{2}$"); upon substituting $\frac{1}{2}$ for z in the equation for the surface and manipulating the equation appropriately, students find that the level curve is an ellipse. Altogether, the three questions indicate that students are expected to recognize the equations of conic sections and, conversely, to use their geometric properties (e.g. foci) to form their standard Cartesian equation.

Section VII.iii.h: Quadric Surfaces and Spheres $(\Pi 6, \Pi 11)$

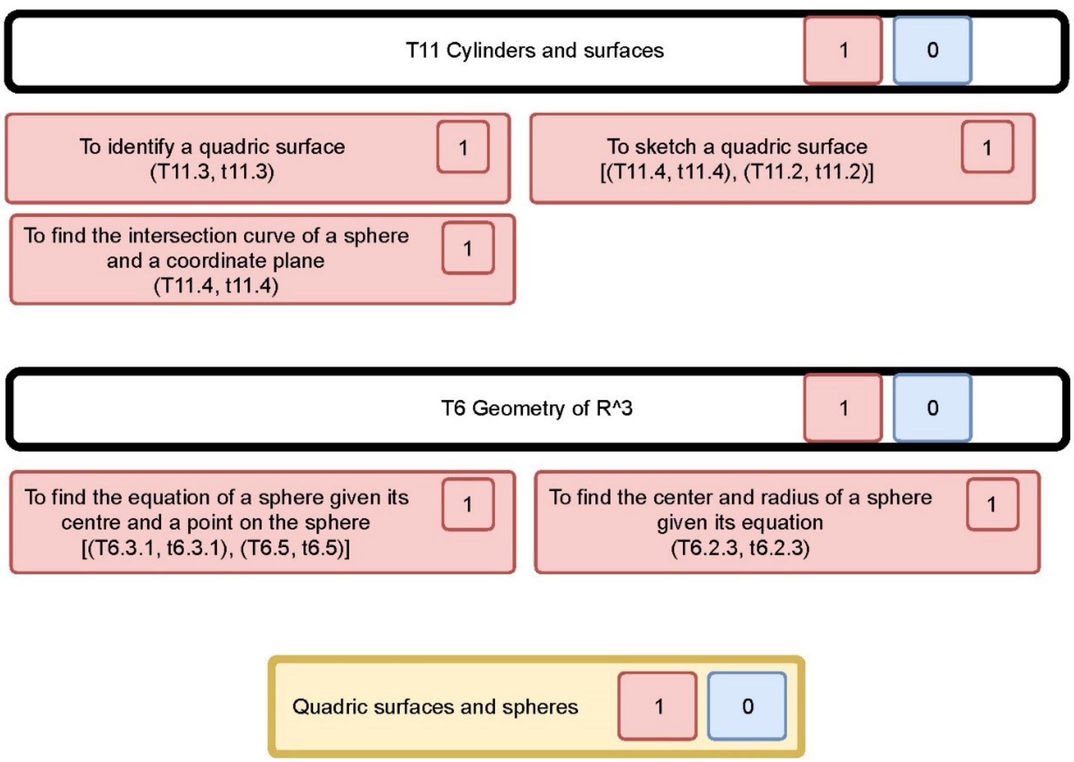


Figure 43. Model of the KL Π Quadric surfaces and spheres.

Notice that the tasks on quadric surfaces and spheres occur all occur in a single MAST 218 exam.

The spherically-concerned problem has students find the equation of a sphere given its center and a point on the sphere; to find the intersection curve of this sphere and one of the coordinate planes (the yz-plane, specifically); and to find the center and radius of an altogether different sphere given by the following equation:

$$x^2 + y^2 + z^2 - 8x + 2y + 6z + 1 = 0$$

The first of these tasks is an action of recall to fetch the standard equation of spheres. The second task requires students to know that coordinate planes can be described by the coordinate which remains constant along that plane. That is, they need to connect "the yz-plane" with the equation $x = 0$. The remainder of this task is an algebraic matter, as is the technique necessary for finding the sphere given by the above non-standard equation. Note that the wording of the third question informs students that this equation gives a sphere; thus, students can decide which manipulations to enact based on the final shape of they need the equation to be in. The minimum knowledge to be learned in relation to spheres is therefore their standard equation and an agility in the algebraic manipulations that can reshape quadratic equations (that represent spheres) to take on the standard form of spheres.

Three problems later, students are asked to identify and sketch the graph of the surface given by

$$4x^2 + 4y^2 - 8y + z^2 = 0$$

To identify the surface given by this equations, students need to know that the graph of a second-degree equation in three variables is a quadric surface. To sketch the graph of this surface, students need to find some of its traces, sketch them, and from this glean the shape of the entire surface.

Section VII.iii.i: Taylor Series ($\Pi 24$)

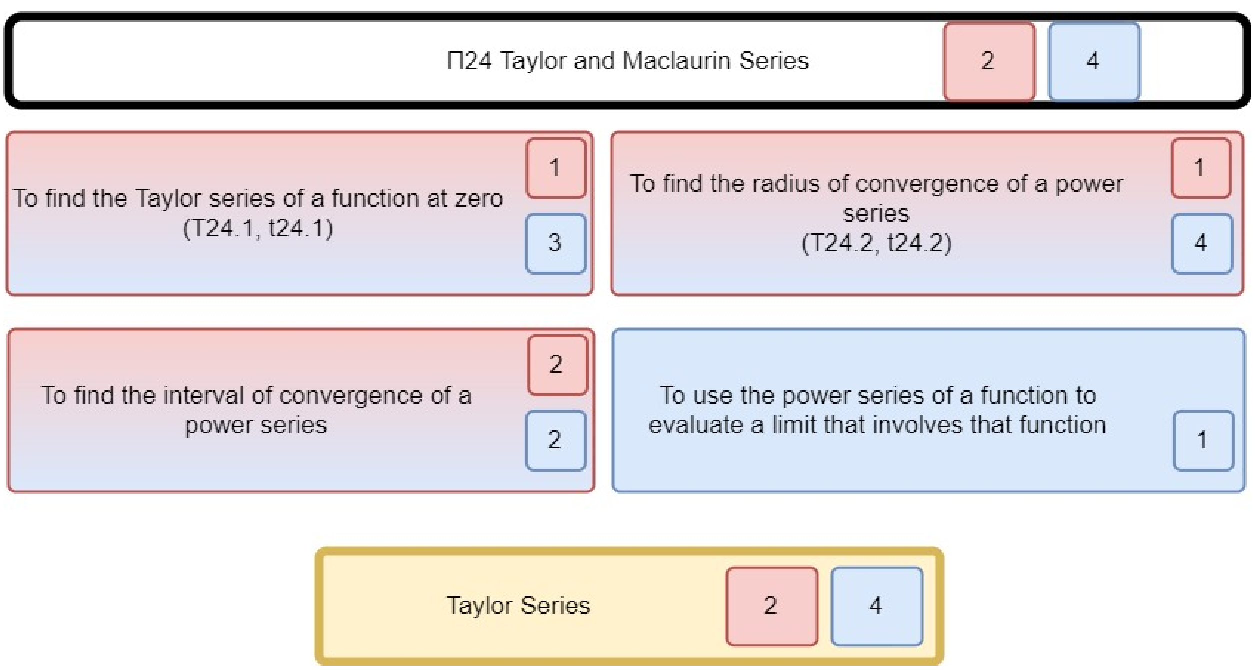


Figure 44. Model of the KL Π Taylor Series.

The knowledge to be learned in $\Pi 24$ is similar to what's taught and learned in the prerequisite MATH 205 course, where Taylor series are covered in the last week of the term. Here, the topic is indicated as a "review" on the course outline. There is no apparent link between the task and techniques for finding Taylor series of single-variable functions, their radii or interval of convergence, and the rest knowledge to be taught and learned in this multivariable calculus course.

Chapter VIII: Discussion

Based upon the praxeology-specific discussions in Chapter VII on the knowledge to be learned, I portray an ideal student for each of MAST 218 and MATH 264. By 'ideal student,' I mean a student who has learned all that is necessary to succeed in the course; by 'succeed,' I mean to get acceptable solutions in the final exam. I follow this discussion with some comments on the affordances and limitations of ATD in this study.

Section VIII.i: Ideal students

I propose a portrait of the ideal students in MAST 218 and MATH 264 in three parts: the shared attributes of these ideal students, who follow a coordinated curriculum after all; the knowledge to be learned specifically by the ideal MAST 218 student; and that which is specific to the MATH 264 student. By addressing the similarities and peculiarities of the ideal students in these courses, I hope to address one of the questions that had spurred this study and shed some light on the distance between MAST 218 and MATH 264.

Section VIII.i.a: The shared attributes of the ideal MAST 218 and MATH 264 students

MAST 218 and MATH 264 students dabble in the themes of partial derivatives and surfaces; space curves and vector functions; equations of lines and planes and distance in R^3; limits of rational functions; polar curves; Taylor Series; 218 students also fiddle with parametric plane curves; conic sections; and quadratic surfaces and spheres. The MAST 218 spread includes in it tasks drawn from almost all the praxeologies in the knowledge to be taught; what it shares in common with MATH 264 is mostly taught in the second half of the course, with the exception of polar curves, distances in R^3, and Taylor series. Let's call 'ideal student' one who has the requisite knowledge to write acceptable solutions in a final exam. What, then, are the praxeologies of an ideal student in MAST 218 or MATH 264? How might we characterize these praxeologies?

I argue that the shared attributes of the ideal MAST 218 and MATH 264 students, beyond any particular mathematical themes, are in the characteristics of these students' praxeologies. Below, I consider which parts of the KT praxeologies are to be learned and characterize them in the language of of Lithner (2004) and Selden et al. (1999). I conclude by situating the knowledge to be learned by the ideal MAST 218 and MATH 264 students in the path of university mathematics transitions outlined by Winsløw et al. (2014) (see section II.ii.b).

Of the 24 KT praxeologies, 19 explicitly occur in questions in the MAST 218 final exams and 17 in MATH 264 exams. That being said, four of the praxeologies that do not occur explicitly do so implicitly in both 218/264 in the shape of subtasks or because they introduce concepts built on by other praxeologies; and one more praxeology occurs implicitly in 218. This means that all 24 of the KT praxeologies occur in some shape or form as KL praxeologies in MAST 218, and 21 of them as KL praxeologies in MATH 264. (Of course, this is over the span of six final exams, so a student in a given semester might not enjoy the delight of such a thorough final; but I take the sum of the six exams as representative of what's fair game in any given semester, thereby constituting the knowledge to be learned in either course.) Thus, the knowledge to be learned in both courses is not necessarily a subset of the knowledge to be taught in the sense that some praxeologies are to be learned while others aren't. Rather, as I aim to show below, it is a subset in the sense of what's *left* of these praxeologies after the didactic transposition of knowledge to be taught into knowledge to be learned.

One of the most obvious ways in which the MAST 218/MATH 264 KT praxeologies change in this transposition is in the down-sizing of certain practical blocks. For instance, consider the praxeology of Polar Coordinates, which occurs in identical form in the 218/264 exams (see section VII.iii.f). The ideal student in either course can convert polar equations into Cartesian equations and sketch the polar curve given by the equations – *given the same two particular polar expressions*. The ideal student's *topos* ('action space') (Chevallard, 2002a) does not need to extend beyond the point praxeology specific to each of these particular functions. The KT local praxeology of polar coordinates is concentrated into one KL point praxeology. Practical blocks are also downsized in the more obvious sense – for instance, from an array of 6 types of tasks in $\Pi 19$, only two appear in the final exams. The practical block of $\Pi 12$ vanishes completely from the ideal student's topos.

Regardless of whether a praxeology's KT practical block remains relatively intact or is stripped down to a point praxeology or two, most of the praxeologies seem to undergo a surgical removal of their theoretical backing in the transposition from KT to KL. For instance, consider $\Pi 18, \Pi 20, \Pi 21, \Pi 22$, and $\Pi 23$, local KT praxeologies that constitute much of the regional praxeology of Partial Derivatives and whose KT and KL practical blocks are nearly identical. These constitute a number of questions out of the 8-10 in each MAST 218 and MATH 264 exam (see section VII.iii.a). The practical blocks of $\Pi 18$ and $\Pi 20$, Partial Derivatives and Chain Rule respectively, are reduced to computational tasks whereby the ideal student needs to apply the appropriate differentiation algorithm; the geometric interpretation of partial derivatives as slopes is unneeded and the ideal student does not need to know any of the theory or technology at the backbone of the procedures. The ideal student does not need to learn the limit-based definition of partial derivative nor the definitions and roles of limits, continuity, and differentiability in the concepts of derivative, gradient, minima and maxima of a function; if anything, the ideal student may benefit from conceptualizing directional derivatives as the derivatives of a function in given directions, but students are not asked to express any such conceptualizations. Thus, the theoretical blocks of $\Pi 18$ and $\Pi 20$ vanish in the transposition of the praxeologies from KT to KL. This is similarly the case for $\Pi 22$ and $\Pi 23$; the ideal student is fluent in the algorithms prescribed by these praxeologies but doesn't need to justify or explain them. The exception is in $\Pi 21$, where the student first needs to know that the components of the gradient vector of a function are the function's partial derivatives, but must also associate with gradient some meaning beyond its symbolic representation (e.g. that the maximum rate of change of a function is the magnitude of its gradient). The absence of theoretical blocks in the ideal student's praxeology is manifested in several ways, then: first, in that the student does not need to justify the validity or choice of technique (e.g. by stating that the chain rule is applicable since the functions involved are all differentiable); second, in that the final exam questions do not ask of students to interpret their results (e.g. by making a sketch of a surface near a point where some geometric properties of the surface were computed); and finally, in that it is not necessary to have more than a superficial grasp of the concepts in the theoretical blocks to undertake the types of tasks in the final exams. I expand on this last point.

In general, the ideal student can recognize task types and identify the appropriate technique, in reasoning similar to Lithner's *identification of similarities* (IS) (2004) (see section II.i). For instance, consider the limit-related tasks in the final exams. As previously discussed (section VI.iii.e), the question is always the same: find the limit of a function $f(x,y)$ at the origin, if it exists, or show that it does not exist. There is little variation in the given function f from one exam to the next: either it is an odd rational function with no limit or it involves a trigonometric component which could be rid of to reduce f to a rational function in the process of a $\varepsilon - \delta$ argument (in these cases, the exam functions invariably have limit 0). Such functions did occur as exercises in these students' assignments; students who accessed previous years' exams will have also encountered them there. Further, this type of task (beyond the particular function) is identical to the type of task taught in $\Pi 17$. Thus, IS describes the strategy undertaken by the ideal student as the tasks in the KL praxeologies are mostly identical to their counterparts in the KT; further, the types of KT of tasks that occur as KL are mostly associated with only one technique. Consider also $\Pi 15$, Motion in Space – Velocity and Acceleration. Exam tasks taken from $\Pi 15$ require students to find the velocity, acceleration, and speed of a particle given its position function.

However, projectile problems – arguably the endgame of $\Pi 15$ – are absent from the KL. These are the types of problems that do not have one single applicable technique. Hence, the ideal student needs to know and use task-technique pairs that occur in the KT praxeologies in order to use reasoning similar to Lithner's IS (2004). I gave only the two examples of $\Pi 17$ and $\Pi 15$, but they are representative of most of the types of exercises students are dealt in MAST 218 examinations. Therefore, there seems to be no discontinuity between this course and students' prior mathematics schooling, where their responsibility was also mainly in the recognition of types of task and the choice of an appropriate known technique (Winsløw et al., 2004).

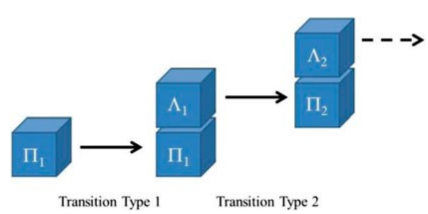
IS reasoning is defined by a lack of reflection required on the intrinsic mathematical properties particular to the problem at hand (Lithner, 2004). Indeed, in the case just discussed of velocity, acceleration, and speed, the ideal student can view these superficially as first derivative, second derivative, and magnitude of the velocity, respectively; it is not necessary for the student to base velocity and acceleration in the concept of limit. Consider also $\Pi 14$, Arc Length and Curvature. There is little to no difference between the KT and KL tasks from $\Pi 14$; the variations require only some extra use of theoretical knowledge from $\Pi 10$. For instance, instead of the task of finding the equation of a normal plane to a curve at a point, students might be asked to find a point on a curve where the normal plane is parallel to a given plane. Apart from this difference, the $\Pi 14$ tasks and associated techniques are the same in the KT and KL. To identify and make use of these similarities, the ideal student needs to be familiar with the terms in the question statements (arc length, curvature, normal plane, unit normal vector, binormal vector, osculating plane…) and to know the formulas for deriving them. This is surface knowledge; the ideal student is not tested on the definitions of these quantities and geometric properties as they relate to a curve at a point (e.g. a student might need to find the equation for an osculating plane, but does not need to explain what the osculating plane describes). This neglect of the intrinsic mathematical properties (Lithner, 2004) of the curves which students (unwittingly) describe suggests that the theoretical block of $\Pi 14$ (and many other praxeologies, for that matter) need not be present in the ideal MAST 218 student.

At the beginning of this discussion, I set out to show that theoretical blocks are missing from the ideal student's topos in a few senses: the student is not required to use theory or technology to justify or explain the techniques chosen to complete a task, and at times is even told which technique to use (e.g. students are instructed to 'use Lagrange Multipliers' or 'use the chain rule'); the student is not required to interpret the numerical or algebraic results of their calculus in any way; and it suffices for the student to learn the components of the theoretical blocks only superficially. In all, it seems that MAST 218 and

MATH 264 follow in the pre-university mathematics tradition whereby students need not link the practical and theoretical blocks of a praxeology (Winsløw et al., 2014). Further, the components of the practical blocks themselves are discrete, as the ideal student does not need to combine different tasks in any way – for instance, the ideal student must know how to find the invariant quantities of a curve, but needn't provide a local description of a curve based on its invariant quantities (see section VII.iii.b). This may be called the "*compartmentalization* of knowledge in calculus courses" described by Winsløw et al. (2014, p.104).

I have considered which subsets of the praxeologies students are tested for and in particular how their practical and theoretical blocks are manifested in the ideal student's praxeologies. On the whole, it appears that the ideal student in MAST 218 and MATH 264 needs to know a surface version of the KT theoretical blocks: they need to know terms and associated formulas, in some cases have some intuitive image of certain concepts, and be fluent in the algorithms described by the technologies (e.g. the method of Lagrange multipliers). This surface acquisition of the theoretical block aids the ideal student in recognizing types of task that are either very or moderately routine (Selden et al., 1999) in the KT practical blocks, as well as in identifying the suitable technique for completing the task. I use the terms *very* and *moderately routine* based on the evidence that most of the final exam questions corresponded identically or nearly identically to a task type from a KT praxeology; as previously mentioned, the variations generally occurred in the form of some additional algebraic formulation that is not specific to calculus.

In light of the absence of theoretical blocks in the ideal student's praxeologies, I conclude that these praxeologies are actually practical blocks. This places MAST 218 and MATH 264 in the stage prior to the first transition in university mathematics education:



Taken from (Winsløw et al., 2014, p.101)

Where Π refers to the practical block of a praxeology and Λ to its theoretical block. As discussed in section II.ii.b, the first transition in undergraduate mathematics studies occurs when students no longer work strictly within the practical block of a praxeology and begin to incorporate a theoretical block; a second transition occurs when students' past theoretical blocks become their current practical blocks (e.g. when they start making and validating proofs).

I conjecture that the sequel (and mandatory) courses MAST 219 and MATH 265 follow in the same steps as those set by 218 and 264; I base this on the continued institutional context of the courses, both in terms of the students' backgrounds and in their equally heavily-coordinated and controlled quality with detailed course outlines, course examiner, and common assessments. Most importantly, the curriculum follows in the same textbook, implying that the knowledge to be taught is of a similar flavor to that of 218/264. Assuming that the knowledge to be learned continues similarly, this would imply that MATH and MAST students' first year of undergraduate mathematics includes a sequence of courses that sets the *ideal student's* topos prior to the first of two transitions in the learning of undergraduate mathematics.

Section VIII.i.b: Some specifics on the ideal MAST 218 student

The MAST 218 students are responsible for snippets of a few KT praxeologies that MATH 264 students do not need to learn: the standard equations of spheres and quadratic surfaces (from $\Pi 6$ and $\Pi 11$, respectively) and much of the practical block of $\Pi 2$, Calculus with Parametric Curves (to be specific, the techniques to the tasks involve single-variable differentiation and the recall of some formulae).

More substantially, the main peculiarity of the ideal MAST 218 student is her proficiency in the practical and at times theoretical blocks of $\Pi 10$, Equations of Lines and Planes. This is interesting as $\Pi 10$ is a unit of knowledge that MAST 218 need to learn to succeed in the prerequisite course MATH 204 (introductory linear algebra; see section IV.ii); but $\Pi 10$ is not *specific* to the domain of multivariable calculus, and indeed its role in the knowledge to be taught seems to be mostly to support knowledge developed in the KT regional praxeologies Vector Functions and Partial Derivatives. Nevertheless, the MAST 218 student can flexibly identify and combine different task types from $\Pi 10$ as necessary in order to find the equations of a planes and lines and planes (in the process, using $(T_{7.1.2.2}, \tau_{7.1.2.2})$, $(T_{7.2}, \tau_{7.2.1})$, and $(T_{7.3}, \tau_{7.3})$ from the practical block of $\Pi 7$, Vectors, and the notions of parallel and orthogonal vectors along with their relation to scalar multiplication and the dot product). Further, because the task of finding the equation either of a plane or line (or both) required a variation of subtasks from one exam to the next, I surmise that perhaps the ideal student can use the mathematical properties intrinsic to each of these linear algebraic situations to determine which strategy to implement (Lithner, 2004).

Section VIII.i.c: Some specifics on the ideal MATH 264 student

The ideal MATH 264 student can complete any of the task types in the practical blocks of the following KT praxeologies: $\Pi 24$ (Taylor Polynomials), $\Pi 14$ (Arc Length and Curvature), and the Partial Derivatives-related praxeologies $\Pi 16, \Pi 18, \Pi 20, \Pi 21, \Pi 22$, and $\Pi 23$. (The ideal MAST 218 is responsible for only certain task types in each of these praxeologies.) This ideal student can also complete a few tasks from the other KT praxeologies (as discussed in section VI.iii), but does not need to accomplish tasks *specific* to the KT $\Pi 7$ (Vectors), $\Pi 8$ (Dot Product), $\Pi 9$ (Cross Product), nor $\Pi 10$ (Equations of Lines and Planes). As discussed above in the case of the ideal MAST 218 student, the knowledge in these praxeologies is *prerequisite* to MATH 264 both mathematically and officially by departmental policies (see section IV.ii). Given the prominent role of lines and planes in the theoretical blocks of the KT regional praxeologies Vector Functions ($\Pi 12 - \Pi 15$) and Partial Derivatives ($\Pi 16 - \Pi 23$) (see section VI.ii.a to view the map of cross-references between the theoretical blocks of the KT praxeologies), I surmise that the ideal MATH 264 student needs to know the minimal subsets of $\Pi 7 - \Pi 10$ that are necessary to access knowledge in $\Pi 12 - \Pi 15$ and $\Pi 16 - \Pi 23$: from $\Pi 7$, the notion and component representation of vectors, the notion and formula of length of a vector, the relation between scalar multiplication and parallel vectors, and the definition of unit vector; to do scalar multiplication $(T_{7.1.2.2}, \tau_{7.1.2.2})$, to express a vector in component form given its endpoints $(T_{7.2}, \tau_{7.2.1})$, to find the length of a vector $(T_{7.3}, \tau_{7.3})$, and to find the unit vector in the direction of a given vector $(T_{7.4}, \tau_{7.4})$; from $\Pi 8$, the definition of orthogonal vectors; and from $\Pi 10$, the equations of lines and planes.

I noted in section IV.iv that MATH 264 is a gateway to a significant portion of the MATH program: STAT 249 (Probability I), STAT 250 (Statistics), MATH 354 (Numerical Analysis), MATH 366 (Complex Analysis I), MATH 370 (Ordinary Differential Equations), and MATH 464 (Real Analysis). Briefly, I note that the notion of *partial differential equations* does occur as knowledge to be learned (see section VII.iii.a; the ideal student can verify that a function satisfies a partial differential equation); more concretely, the ideal MATH 264 student certainly needs to be able to find partial derivatives of functions, a technique that can be handy in MATH 370. On the whole, it might be interesting to 'compare' the knowledge to be learned by the *ideal student* in MATH 264 with the knowledge she has to learn in these other courses, to investigate in what sense this course is *prerequisite* or prepares this student for the mathematical knowledge that is planned to be taught in these courses.

Section VIII.ii: Affordances and limitations of ATD

The ATD turns to the institutional components that can shape a student's learning of mathematics, rather than the cognitive phenomena that tend to be the focus of mathematics education research (Winsløw et al., 2014). By using praxeologies to model the stages in a didactic transposition and with the aid of a reference model (see section III.i), the researcher can situate and gain insight into students' ways of knowing. For instance, Hardy (2009b) explains that while calculus students operate within the practical blocks of a praxeology, they don't just tacitly accept (if at all) the mathematical theoretical block but rather replace it with a model idiosyncratic to their beliefs. The difficulties students may incur later in Analysis courses can be viewed in terms of how students need to restructure the blocks of their praxeologies by first dispensing of the social, cognitive, or didactic justifications they've adopted (Winsløw et al., 2014). On the whole, the study of students' difficulties in university mathematics can benefit from an ATD approach as it provides a framework for addressing and interpreting the discrepancies between students' praxeologies, their teachers', the institutional, and the scholarly.

The ATD advises to study the discrepancies among layers of a transposition in its institutional context. In the setting of university mathematics education, the researcher needs to account for the internal and external conditions and constraints determined by university and departmental policies, professors' responsibilities, the structure of a course, and students' mathematical and personal experiences. The external conditions brought in by the students can be exceptionally varied as universities increasingly recruit on the international level (Winsløw et al., 2014); variations may be especially felt in courses taken in the first halves of a university degree, such as calculus and analysis courses.

In the absence of the holistic take required by ATD for describing a didactic transposition, the affordances of the theory may be limited in some ways. For this study, I considered only the knowledge to be taught and learned in a course by examining curricular documents and course assessments. I used my model of KT to model KL by tracing tasks in the exam questions back to similar tasks in the KT praxeologies (see section VII.ii on my methodology). In the absence of a model of the knowledge *actually* taught, however, this approach proved tenuous in treating unusual exam questions. These occurred chiefly in a few questions on one of the earlier MATH 264 exams and in the form of (MATH 264) bonus questions. I took the examiner's choice to position certain questions as bonuses as an indication that the knowledge (in both theoretical and practical senses) required to solve them is not *essential* for MATH 264 students to learn. I therefore chose to exclude these from my model. As for the non-bonus unusual questions, the techniques required to solve them proved unfit to be described as a subset of KT. Access to the knowledge actually taught would have indicated the extent to which these questions were indeed unusual, and perhaps pointed at the strategy expected of students. Thus, my methodology of describing KL tasks in terms of KT tasks was insufficient to capture instances of KL that deviated from the norm. By and large, the exam questions lent themselves well to this methodology and the fringe cases were scarce, so this was not a significant problem in this study. But perhaps for courses that have a less heavily-coordinated flavor and with more variation in the assessments, an introductory study that considers only fragments of a didactic transposition might not be as appropriate or might require some changes in the methodology.

Chapter IX: Conclusions

I open my concluding chapter with a review of the core components of this study: this consists of a recapitulation of my work with some intermittent remarks on the advantages and pitfalls of the methodology. I follow with some conjectures about the implications of what MAST 218 and MATH 264 students need or don't necessarily need to learn; and I finish with a suggestion for follow-up research.

Section IX.i: Review

The goal of this study was to determine the minimal units of knowledge that are *essential* for MAST 218 and MATH 264 students to learn in order to provide acceptable solutions in their final exams. To identify these 'bare necessities,' I used an approach based on the ATD. I extracted a model of the knowledge to be taught from the curricular documents; I did this by identifying praxeologies and their practical and theoretical blocks in the textbook sections listed on the course outline. I then modelled the knowledge to be learned as a subset of the KT by identifying sequences of tasks and techniques from the model of KT that form a solution for each final exam question. I did this for all (non-bonus) questions in six past final exams for each of MAST 218 and MATH 264 from the recent past (2012-2015); given the heavily-coordinated quality of these courses from one term to the next, I presume the sum of six finals for each course is representative of the knowledge to be learned by students in any particular term.

This work resulted in the creation of a model of the KT in the shape of 24 local praxeologies that reflect the 24 sections listed on the course outline. I represented the practical block of each praxeology in the form of a tree; in the place of vertices I wrote a more or less direct description of the tasks and techniques, and used arrows to indicate whether the endpoint is a task or technique. Given that most of the practical blocks included a substantial number of different types of tasks and associated techniques, I found it helpful to represent each practical block as a graph. I also listed the components (definitions, theorems…) of the theoretical block that frames each of these practical blocks; the pitfall is that these lists do not reflect the *discourse* that combines the items therein, which is key to the makings of a theoretical block as the components of theory and technology are not discrete. To make up for this shortcoming, I attached the relevant excerpts from the textbook as a reference in Appendix D. Finally, I created a grid mapping the cross-references (in the sense that a term or concept from one section is used later on in another section) made between the praxeologies' theoretical blocks. This grid reflects cross-references that are generally *implicit* in the text, meaning that it is up to the reader to recognize previously-defined terms or concepts and recoup their meaning. I also cross-referenced tasks and techniques in the practical blocks; in particular, in the description of a given technique, I identified steps that correspond to tasks that occur elsewhere (be it in the same or a different practical block). While my models of the practical and theoretical block of each praxeology depict the mathematical knowledge and activity *specific* to the praxeology, the cross-referencing between the blocks identified the mathematical knowledge and activity that are 'prerequisite' for each praxeology. The cross-referencing uncovered further dynamics of the knowledge to be taught. For instance, it pointed at some praxeologies which are heavily referenced by other praxeologies, suggesting that they may be taught for their prerequisite capacity (e.g. the praxeologies that belong to the domain of linear algebra).

In a first step toward a model of the KL, I used the practical blocks of the KT Π to identify the sequences $(T_i, \tau_i)_i$ that describe a solution. A potential dent in the methodology here is that I did not account for flexibility in the creation of a solution; while solutions to a particular exam question may vary to some extent, I described the solution of each exam question using only *one* sequence of task-technique pairs from the KT ; I based the choice of $(T_i, \tau_i)_i$ on the operational definition of the KL as a subset of the KT, meaning that I described solutions to exam questions in terms of pairs $(T_i, \tau_i)_i$ that solve the 'problem' in the quickest possible way – thereby circumventing other, perhaps longer, sequences $(T_i, \tau_i)_i$ that would function as solutions just as well but perhaps require different units of knowledge.

The model of the KL consists of regional praxeologies unified by a mostly absent theoretical block, in the sense that students can do tasks and techniques in these praxeologies without knowing the theory or technology that explain and justify them. I partition each regional praxeology of the KL among the local praxeologies of the KT and describe the tasks within in a more or less direct way; the techniques are given by the set of $(T_i, \tau_i)_i$ (from the practical blocks of the KT) required for the completion of the task. The representation of the KL in terms of the local praxeologies of the KT as well as in terms of the tasks and techniques of the KT has the advantage that it directly relates the KL to the KT. The readily-available link between the KT and the KL can suggest the transposition that occurs between the two.

I capped my findings on the knowledge to be learned in MAST 218 and MATH 264 with a discussion of what the ideal student in each course might look like. That is, I described the praxeologies that would enable a student in each course to provide acceptable solutions in the final exam. I inferred from the models of the KL that the praxeologies of the ideal student – in either course – are subsets of the practical blocks in the KT; the theoretical blocks of the KT are not *required* to be learned in more than a superficial and compartmentalized sense. Students need to recognize terms and have a surface definition for them. For instance, it's necessary for the ideal student to know that the normal to a curve at a point is a line (or vector) orthogonal to the curve. It's also necessary to know the associated formulas. It's not necessary to explain the intrinsic mathematical properties or uses of components of the theoretical blocks, however. For instance, the ideal student needn't know the utility of an equation for the tangent plane to a surface; the ideal student need only be able to *find* such an equation in certain circumstances (e.g. given the equation for a level surface and a point that it contains).

On the whole, it seems that the ideal student can get by with a compartmentalized subset of the KT praxeologies. A model of the knowledge *actually* learned by students in these courses would indicate more concretely the nature of students' theoretical blocks. For instance, in a study of students' learning of single-variable calculus, Hardy (2009b) used elements of the ATD and found that students use social, cognitive, and didactic reasoning to justify and explain the tasks and techniques with which they engage. In any case, the evidence from this study seems sufficient to suggest that the KL in MAST 218 and MATH 264 exists almost strictly within the practical blocks of the praxeologies of the KT. This situates the ideal student for both these courses in a stage prior to the *first transition* of university mathematics education (see section II.ii.b). Given the indications that MAST 219 and MATH 265, the second halves of the two-term approach to multivariable calculus, are similar in content and quality to MAST 218 and MATH 264 (the textbook, the format of course outlines, grading schemes, and multi-section quality remain unchanged), this suggests that the first year of MAST and MATH students' mathematical studies includes a sequence of courses where it is not essential to make theoretical considerations.

Section IX.ii: Conjectures

Based on what was just discussed at the end of section IX.i and on the collected evidence and analysis of the praxeologies of the KL (section VI.iii, where I present the model of the KL), I conjecture implications of certain gaps in the knowledge that's *necessary* for MAST 218 and MATH 264 students to learn. I focus on two matters in particular given their prominent role in my reference model: the absence of the concepts of limit, continuity, and differentiation in the knowledge that's essential for students to learn; and the role (or lack thereof) of curves and surfaces in the knowledge to be learned.

Section IX.ii.a: Limits, continuity, and differentiation matters

A subset of the theoretical blocks of KT Π that vanishes in the transposition from KT to KL includes the concepts of limit, continuity, and differentiability. Both MAST and MATH students can be successful in their first multivariable calculus courses and not have an image of limits, continuity, and differentiability that is rooted in the mathematical definitions of the terms. For that matter, the ideal student in both courses can get by without even knowing the *terms* 'continuous' and 'differentiable'; these are part of the theory in the KT that justifies and explains when certain technologies (e.g. partial derivatives) are defined, but in final exams the functions given are *always* continuous and differentiable. Students are not tasked with determining nor acknowledging that these conditions are satisfied. The situation is similar with limits. It is enough for students' knowledge of limits to be confined to the point praxeologies built around the tasks of 'finding the limit of a two-variable function at a point of indetermination (specifically when this point is the origin)' and 'showing the limit of a rational two-variable function does not exist at the origin.' The requisite theoretical knowledge for the former includes properties of limits, the Squeeze Theorem, and an application of a particular $\varepsilon - \delta$ argument that, in this institution, comes down to algebraic manipulation. To show the limit of a function does not exist, in the second case, the requisite theoretical knowledge is that the limit of a two-variable function at a point does not exist if the function approaches different values along two paths near that point – although, as discussed in section VI.iii.e, this may not be necessary either. At no point is it necessary for the ideal student to learn a definition of the limit of a function at a point that is based in any way in the formal definition.

I conjecture the implications of the absence of the concepts of limits, continuity, and differentiability being in the knowledge to be learned. As far as the ideal student is concerned, these are terms that happened to occur in the knowledge to be taught (and perhaps the knowledge actually taught as well); but as they are not an issue in the tasks students are responsible for completing, the conditions of limits, continuity, and differentiability are assumptions (that students can ignore) rather than properties of certain classes of functions. Indeed, students are not confronted with functions that fail to be differentiable, and only at times (in the knowledge to be taught) with a small set of functions (i.e. rational) that have points of discontinuity. For MATH 264 students, this might contribute to difficulties in Analysis where limits and continuity become the objects at the center of students' mathematical activity and are no longer concepts whose significance and definition are not even requisite knowledge.

Beyond the mathematical knowledge specific to these concepts, I wonder whether the treatment of properties as *assumptions* may contribute to a misconception about the structure of mathematics. MATH 264 students spend the first two terms of their mathematics degree in courses that do not require them to reflect or acknowledge differences between assumptions and things that need proving; or on the arbitrary (man-made) nature of definitions. This runs in contrast to the axiomatic nature of mathematics, and, I surmise, might contribute to difficulties as students transition to courses where their tasks lie in proving, and therefore on an understanding of the role of assumptions and conditions.

Section IX.ii.b: The absence of curves and surfaces

In my reference model, I considered the multivariable calculus at the start of the transition from scholarly knowledge to knowledge to be taught and learned in MAST 218 and MATH 264; I viewed it as calculus that pertains in particular to the description of curves and surfaces. On the surface, this seems to be the case in the final examinations: students are asked to find geometric properties of curves and surfaces. However, it seems to me that students needn't acknowledge that this is what they are doing. Their tasks are largely of a computational nature; the question statement includes the terms 'curve' and 'surface,' but the meaning of these is in their algebraic or parametric representation. As previously discussed (Section VIII.i.a), tangent and normal lines and planes, osculating planes and circles, etc. are terms that the ideal student needs to know, and be able to represent in algebraic notation; but it's not essential for the ideal student to have a geometric image of these concepts. The implication of this, I think, is that students don't have to be aware of the mathematical properties intrinsic to any particular task. It seems that the ideal student needs to learn multivariable calculus *techniques* but does not need to connect the results of these techniques to any geometric object, even though the work is largely set in R^2 and R^3. This may contribute to a perception that mathematics resides purely in symbolic representation and manipulations. These are undeniably crucial to mathematics, but making a habit of detaching symbols from their context is unsustainable in the mathematical activity that MATH 264 students are headed for.

Section IX.iii: Follow-Up Research

My application of the ATD to MAST 218 and MATH 264 focused on two instances in the didactic transposition outlined by the ATD: the knowledge to be taught and the knowledge to be learned. Combined, these provide a snapshot of the minimal praxeologies that students need to have in order to provide acceptable solutions in their final exams. The models and analyses I suggest of the KL do not describe what students actually learn and are not informed by what is actually taught.

The tight time constraints to which university courses are subjected, and in particular MAST 218 and MATH 264 given their laden curricula, contribute to a significant discrepancy between the knowledge to be taught and that which is actually taught. Professors have to make decisions, perhaps based on the established standards of final exams, regarding what to teach during class time and how to do so. In courses typified by procedure-driven assessments, such as calculus, it is likely for instructors to spend less time expanding on theory and technology (perhaps limiting themselves to the presentation of key definitions and theorems) and even less on the justifications and explanations that link the theoretical and practical blocks (Winsløw et al., 2014). In this case, only a skeleton of the theoretical block of the KT praxeology remains in the knowledge actually taught as the teacher accepts the theoretical justifications but perhaps does not make explicit the links between theory and practice. The gap between knowledge to be taught and knowledge actually learned is significant, especially at this level of undergraduate mathematics studies where students' learning is driven less by the textbook than by the instructor's teachings.

I therefore propose to follow-up on this study with a research project that includes all steps in the didactic transposition in each of MAST 218 and MATH 264: the scholarly knowledge, the knowledge to be taught, the knowledge actually taught, the knowledge to be learned, and the knowledge actually learned. A full-blown approach would be necessary to make any conclusive inferences that could, perhaps, be the basis for recommendations for teaching and assessments.

To capture the entire scope of a didactic transposition, and in recognition of the role of an institutional context in this transposition, I suggest a research project that monitors a few sections of a multivariable calculus course in a given term as follows. In what follows I suggest an approach to modelling the knowledge *actually* taught and *actually* learned, the two instances that were not addressed in the current study. To construct a model of the knowledge actually taught, I propose a combination of two approaches. A first approach is to regularly interview the instructors throughout the term to discern their praxeologies of the mathematics, what they plan to teach in the classroom, and to obtain instructors' reflections on what they ended up teaching; this could provide a model of the knowledge actually taught as perceived by the instructor. A second approach might be to collect the class notes for each lesson taken by a number of the students (the same ones throughout the semester) and model the praxeologies found therein; this would provide a model of the knowledge actually taught as perceived by the students. A combination of the two might be helpful in tracing the transition from knowledge to be taught to knowledge *actually* learned, and situating the knowledge *actually* taught along the spectrum between the two. To model the last stage in the transposition, I propose collecting students' final examinations and using their solutions as basis for a model of the knowledge actually learned.

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Appendix A – MAST 218 Final Examinations

Dec. 2012

Instructions: Only approved calculators permitted. Answer all eight numbered questions. The value for each part is indicated in square brackets in the margin (out of a possible total of 100). Show all your steps. Write the complete solution on the right hand pages of your examination booklet only.

MARKS: marks for each problem are shown in front of the problems.

↓MARKS

12 **Problem 1 :** Consider the plane curve defined by the parametric equations:

$$x = t^3 - 3t, \quad y = t^3 - 3t^2 \; .$$

(a). Find d^2y/dx^2 in terms of t.

(b). Find the points on the curve where the tangent is horizontal or vertical.

12 **Problem 2 :** Consider the curves γ_1 and γ_2 defined by polar equations:

$$\gamma_1 : \quad r = 2 + \sin\theta, \quad 0 \le \theta \le 2\pi; \qquad \gamma_2 : \quad r = 3\sin\theta, \quad 0 \le \theta \le \pi$$

(a). Identify the curve γ_2 by finding its **Cartesian equations** in (x, y)−coordinates, and **sketch the polar curves** γ_1 and γ_2.

(b). Find the area A of the region that lies inside the first curve γ_1 and outside the second curve γ_2.

12 **Problem 3 :** Consider the space curve $\mathbf{r}(t) = \langle\, 2t^{3/2},\ \cos(2t),\ \sin(2t)\,\rangle$.
Find the length of the curve for $t \in [1, 3]$.

12 **Problem 4 :** (a). Find an equation of the plane PL passing through the three points: $A(2,1,1)$, $B(-1,-1,10)$ and $C(1,3,-4)$.

(b). Find equations, expressed in implicit form, for the line passing through the point B that is perpendicular to the plane PL obtained in (a).

14 **Problem 5 :** Let the surface $z = f(x,y)$ be defined implicitly by the equation:

$$\sin(xyz) = x^2y^2 + z^2 - 1 .$$

(a). Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(b). Find the equation of the tangent plane at the point $(1,1,0)$.

(c). In which direction does f increase most rapidly at this point?

12 **Problem 6 :** Find the limit, if it exists, or show that the limit does not exist:

$$(a).\quad \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}, \qquad (b).\quad \lim_{(x,y)\to(0,0)} \frac{x^4y^4\sin^2(x^2y^2)}{x^2+y^2}.$$

14 **Problem 7 :** Consider the function $f(x,y) = (y^2 - x^2)e^y$.

(a). Find all critical points of $f(x,y)$.

(b). Classify the critical points obtained in (a) as local minimum, local maximum, or saddle points.

(c). Find the absolute maximum and minimum values of f on the set

$$D = \{(x,y) : x^2 + y^2 \leq 4\}.$$

12 **Problem 8 :** Use **Lagrange Multipliers** to find the maximum and minimum values of $f(x,y) = e^{xy}$ subject to the constraint: $x^3 + y^3 = 16$.

GOOD LUCK !!!

Dec. 2013

Instructions:

Only approved calculators permitted.

Problems of equal value. Do any 8 problems.

Show all your steps. Write complete solutions on the right hand pages of your examination booklet only.

Problem 1 : (a) Find an equation of the sphere that passes through the point $(6,-2,3)$ and has center $(-1,2,1)$.

(b) Find the curve in which this sphere intersects the $yz-$plane.

(c) Find the center and radius of the sphere:

$$x^2+y^2+z^2-8x+2y+6z+1=0$$

Problem 2 : Consider the curves γ_1 and γ_2 defined by polar equations:

$r(t)=3\cos\theta$ and $r(t)=1+\cos\theta$

(a) Sketch both of them. Find the Cartesian equation of any one of them in $(x,y)-$coordinates.

(b) Find the area A of the region that lies inside γ_1 and outside γ_2.

Problem 3 : A curve is defined by the parametric equations:

$$x=\int_1^t \frac{\cos u}{u}du, \qquad y=\int_1^t \frac{\sin u}{u}du, \qquad 1\le t\le 2\pi$$

(a) Find the t values where the curve has vertical tangent lines.

(b) Find the length of the curve from $t=1$ to $t=\pi/2$.

Problem 4 : Identify and sketch the graph of the surface:

$$4x^2+4y^2-8y+z^2=0.$$

Problem 5 : Let $P = (1, 2, -2)$, and let L be the line given parametrically

$$L: \ (0, 3, 1) + t\ \langle 2, -1, 3\rangle, \quad -\infty < t < +\infty.$$

(a) Find the distance of the point P from the line L;

(b) Find the equation of the plane Q passing through point P and the line L;

Problem 6 : Let $\mathbf{r}(t)$, $t \in (a, b) \in \mathbb{R}$, be a differentiable vector valued function of t.

(a) Show that
$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

(b) If the vector $\mathbf{r}(t)$ has constant length in the interval (a, b), show that the derivative vector is perpendicular to $\mathbf{r}(t)$ at all points of this interval.

Problem 7 : At what point on the curve $x = t^3$, $y = 3t$, $z = t^4$ is the normal plane parallel to the plane $6x + 6y - 8z - 1 = 0$?

Problem 8 : Let the position function of a particle be given by
$$\mathbf{r}(t) = t^2\ \mathbf{i} + 5t\ \mathbf{j} + +(t^2 - 16t)\ \mathbf{k}\ .$$
Compute the velocity and acceleration of the particle at any time t.
When is the speed a minimum?

Problem 9 : (a) Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the function $x^2 + 2y^2 + 3z^2 = 1$.

(b) Compute the directional derivative of the function $f(x, y, z) = xe^y + ye^z + ze^x$ at the point $(0, 0, 0)$ in the direction of the vector $\mathbf{v} = 5\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Problem 10 : Find the absolute maximum and minimum values of the function $f(x, y) = x + y - xy$ over the closed triangular region with vertices $(0, 0)$, $(0, 2)$ and $(4, 0)$.

GOOD LUCK !!!

Dec. 2014

Special Instructions: Calculators permitted. Lined paper booklets.

READ THE QUESTIONS CAREFULLY !!!

SHOW ALL WORK !!! JUSTIFY ALL STEPS !!!

GOOD LUCK !!!

MARKS: marks for each problem are shown in front of the problems.

↓MARKS

10 **Problem 1 :** Consider the plane curve defined by the parametric equations: $x = \sin^2 t$, $y = \cos t$, $0 \le t \le 2\pi$.

(a). Find d^2y/dx^2 in terms of t.

(b). Find the values of t at which the plane curve is concave upward.

10 **Problem 2 :** Consider the curves γ_1 and γ_2 defined by polar equations:

$$\gamma_1 : \quad r = 1 - \cos\theta, \qquad \gamma_2 : \quad r = 1 + \cos\theta.$$

(a). Sketch the polar curves γ_1 and γ_2.

(b). Find the area A of the region which lies inside γ_1 and outside γ_2.

10 **Problem 3 :** Consider the function

$$f(x) = ln\left(\frac{1+x}{1-x}\right)$$

(a). Find the Taylor series of $f(x)$ at $x = 0$.

(b). Find the interval of convergence for the Taylor series obtained in (a).

10 **Problem 4 :** (a). Find an equation of the plane passing through the point $A(1,2,3)$ and the line $(3t, 2t, t)$, $-\infty < t < +\infty$.

(b). Find an equation of the line passing through the point $(2,2,0)$ and perpendicular to the plane obtained in (a).

(c). Find the distance from the point $(2,2,0)$ to the plane obtained in (a).

10 **Problem 5 :** Consider the space curve $\mathbf{r}(t) = \langle 3t\cos t,\ 4t,\ 3t\sin t\rangle$.

(a). Find an equation of the tangent line to the curve at $\mathbf{r}(1)$.

(b). Find the length of the curve for $t \in [0,4]$. You can use a formula

$$\int \sqrt{1+u^2}du = \frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln(u+\sqrt{1+u^2}) + C\,.$$

10 **Problem 6 :** Consider the space curve $\mathbf{r}(t) = \langle t,\ 2t,\ t^2\rangle$.

(a). Find the unit tangent vector $\mathbf{T}(1)$ and the principal normal vector $\mathbf{N}(1)$ of the curve at $t=1$.

(b). **Use the Chain Rule** to find the partial derivatives $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$, where

$$u = xe^{yz^2}, \quad x = \ln(st), \quad y = t^3, \quad z = s^2+t^2.$$

10 **Problem 7 :** Find the limit, if it exists, or show that the limit does not exist:

$$(a).\quad \lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2}\,, \qquad (b).\quad \lim_{(x,y)\to(0,0)} \frac{\sin(x^2y^2)}{x^4+y^4}\,.$$

10 **Problem 8 :** Consider the function $f(x,y) = (y^2+x^2)e^{x^2-y^2}$

(a). Find all critical points of $f(x,y)$.

(b). Classify those critical points obtained in (a) as points of local minimum, local maximum, or saddle points.

10 **Problem 9 :** For the function $f(x,y) = 3-2x+y+xy$, find the absolute maximum and minimum values of $f(x,y)$ in the region enclosed by the curves $y=x^2$ and $y=4$.

10 **Problem 10 :** Use the **Lagrange Multipliers** to find the maximum and minimum values of $f(x,y) = xy^2$ subject to the constraint: $4x^2+9y^2=36$.

Apr. 2013

Instructions: Only approved calculators permitted. The value for each problem is indicated in square brackets in the margin (out of a possible total of 100). Show all your steps. Write the complete solution on the right hand pages of your examination booklet only.

MARKS: marks for each problem are shown in front of the problems.

↓MARKS

[4] **Problem 1 :** Find and sketch the domain of the function

$$f(x,y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}\ .$$

[14] **Problem 2 :** Consider the curves γ_1 and γ_2 defined by polar equations:

$$\gamma_1 : \quad r = 3 + \sin\theta, \quad 0 \le \theta \le 2\pi; \qquad \gamma_2 : \quad r = 4\sin\theta, \quad 0 \le \theta \le \pi$$

(a) Identify the curve γ_2 by finding its **Cartesian equations** in $(x, y)-$coordinates, and **sketch the polar curves** γ_1 and γ_2.

(b) Find the area A of the region that lies inside the first curve γ_1 and outside the second curve γ_2.

[12] **Problem 3 :** Consider the curve $r(t) = \langle\ \sqrt{3}\ t,\ \sin(t),\ \cos(-t)\ \rangle, \quad 0 \le t \le 1.$

(a) Find the unit tangent vector $\vec{T}(t)$ of the curve.

(b) Find the length of the curve.

[12] **Problem 4 :** Let $P = (1,1,1)$, and let L be the line given parametrically

$$L: \ (0,1,0) + t\,\langle 1,2,2\rangle, \qquad -\infty < t < +\infty.$$

(a) Find the distance of point P from line L;

(b) Find the equation of the plane Q passing through point P and the line L;

[12] **Problem 5 :** Let the surface $z = f(x,y)$ be defined implicitly by the equation:

$$x^2 + 2y^2 - 3z^2 = 3\ .$$

(a) Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(b) Find the equation of the tangent plane at the point $(2,-1,1)$.

[14] **Problem 6 :** Find the limit, if it exists, or show that the limit does not exist:

$$(a) \quad \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}\,, \qquad\qquad (b) \quad \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}.$$

[18] **Problem 7 :** Consider the function $f(x,y) = x^3 - 6xy + 8y^3$.

(a) Find all critical points of $f(x,y)$.

(b) Classify the critical points obtained in (a) as local minimum, local maximum, or saddle points.

(c) Find the absolute maximum and minimum of $f(x,y)$ on the square bounded by the lines: $x=0$, $y=0$, $x=1$ and $y=1$.

[14] **Problem 8 :** Use **Lagrange Multipliers** to find the maximum and minimum values of $f(x,y) = y^2 - x^2$ subject to the constraint: $\frac{1}{4}x^2 + y^2 = 1$.

GOOD LUCK !!!

Apr. 2014

Solve as many problems as you can; each problem is 10% worth. No books and notes are permitted, only calculators if necessary.

Problem 1. Find the length of the curve $x = e^{2t}\cos 3t$, $y = e^{2t}\sin 3t$, $z = e^{2t}$, $0 \le t \le \pi$.

Problem 2. Find equation of a line which is parallel to the planes $x + 2y - 3z = 1$ and $-2x + 3y + z = -1$ and goes through the point $P(3,2,1)$.

Problem 3. Find the angle between the line $y = x + 1$ and the hyperbola $xy = 2$ at their points of intersection.

Problem 4. Find the area inside the closed curve $x = t - t^2$, $y = t - 3t^2 + 2t^3$ $(0 \le t \le 1)$.

Problem 5. Find a point on the curve $x = t^3 - 6t^2 + 13t$, $y = t^3 - 6t^2 + 11t$ where its curvature is zero.

Problem 6. Find a point on the surface $x^3 + \frac{y^3}{2} + \frac{z^3}{3} = 1$ such that the tangent plane at this point is parallel to the plane $x + y + z = 0$.

Problem 7. Let $f(x, y)=\tan(x+y)$. Let $x=r\cos\theta$, $y=r\sin\theta$. Find $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial \theta}$.

Problem 8. Find the equation of a hyperbola with the foci (-1, 0) and (3, 0) containing the point (0,0).

Problem 9. Find all the critical points of the function $f(x, y)=\sin x \sin y$ and classify them (i.e. find which of them are points of local minimum, local maximum, or saddle points).

Problem 10. Find the absolute minimum of the function $f(x, y)=x^2-4xy+y^2$ in the rectangle $0\le x\le 1$, $0\le y\le 5$.

GOOD LUCK!

Apr. 2015

FINAL EXAM MAST218, Winter 2015

Evaluation out of 90. Each question worth 10 points.

1. Find the area enclosed by the loop of the curve $r = \cos\theta - \sec\theta$ where (r, θ) are polar coordinates.

2. Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$

3. Find equation of the plane that passes through the point $(-1, 2, 1)$ and contains the line of intersection of the planes $x + y - z = 2$ and $2x - y + 3z = 1$.

4. Find the tangential and normal components of the acceleration vector to the curve

$$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k}$$

at $t = 1$.

5. Find the limit if it exists, or show that it does not exist:

$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$$

6. Are there any point on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?

7. Find local and global maxima and minima, and saddle points of the function $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

8. Find the absolute maximum and minimum values of the function $f(x, y) = x + y - xy$ on the closed triangular region with vertices $(2, 0)$, $(0, 2)$ and $(0, -2)$.

9. Find the extreme values of $f(x, y) = x^2 + y^2 + 4x - 4y$ on the region $x^2 + y^2 \leq 9$. Use the method of Lagrange multipliers for analysis of the boundary.

10. Bonus (10 points): A curve is defined by the parametric equation $x(t) = \int_1^t \frac{\cos u}{u}$; $y(t) = \int_1^t \frac{\sin u}{u}$. Find the length of the arc of the curve from the origin to the nearest point where there is a vertical tangent line.

1

Appendix B – MATH 264 Final Examinations

Dec. 2012

Instructions: *Please answer all 5 questions which carry equal marks.*
Explain your work carefully.
Approved calculators are permitted.

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1. Consider the integral $J(x) = \int_0^x \frac{1-\cos(t)}{t^2}\,dt$.
   (a) Assuming that the integrand $f(t) = (1-\cos(t))/t^2$ is continuous at $t = 0$, find $f(0)$ and sketch the graph of $f(t)$ for $-4\pi \le t \le 4\pi$.
   (b) Does $J(x)$ have local maxima or minima? If yes, find them. If no, justify why there are none.
   (c) Find a Taylor series about $x = 0$ for $J(x)$.
   (d) What is the interval of convergence of this Taylor series?
   (e) Find a Taylor polynomial $T(x)$ to approximate $J(x)$ with error $< 10^{-5}$ for $0 \le x \le 1$. Explain!
   (f) Sketch the graph of $J(x)$ for $0 \le x \le 1$.

2. Consider the curve in $\Re^2$ which has the polar equation
$$r(\theta) = \frac{3}{1-\frac{1}{2}\sin(\theta)}, \quad \theta \in [0, 2\pi].$$
   (a) Find the largest and smallest $r(\theta)$, and the corresponding values of θ.
   (b) Sketch the curve.
   (c) Find the enclosed area.
   (d) Find the slope of the tangent line to the curve when $\theta = 0$.
   (e) Give an integral expression for the length of the part of the curve in the first quadrant $x \ge 0, y \ge 0$. Do not evaluate the integral.

3. Consider the surface $S:\ z = f(x,y) = 1 - e^{-(\frac{1}{4}x^2+y^2)}$.
   (a) Find and describe precisely the level curve given by $z = \frac{1}{2}$.
   (b) Find the tangent plane T that touches S at $(x,y) = (2,1)$.
   (c) Use T to approximate the value $f(2.1, 0.9)$.
   (d) Find the directional derivative $D_{\mathbf{u}}f$ at $(x,y) = (2,1)$, in the direction defined by $\mathbf{v} = \langle -1, 2\rangle$.

4.

(a) If $g(x,y,z) = c$, a constant

(i) explain what is meant by $\partial x/\partial y$ and

(ii) show that

$$\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) = -1. \qquad (*)$$

(b) Verify $(*)$ for the special case $g(x,y,z) = x^2 + 3y^2 + z^2 - 2z = 7$.

(c) If $Q = \ln(1 + x^2 + y^2 + z^2)$, where $x = se^t$, $y = te^s$, and $z = e^s e^t$, find

(i) expressions for $\partial Q/\partial s$ and $\partial Q/\partial t$, and

(ii) their explicit values for $(s,t) = (0,1)$.

5. Consider the function $z = f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9x$.

(a) Find the critical points of $f(x,y)$ for $(x,y) \in \Re^2$.

(b) Classify these critical points as loc min, loc max, or saddle points.

(c) Find the extrema of $f(x,y)$ when x and y are restricted to the curve $x^2 + y^2 = 1$.

HINT: let $x = \cos(t)$ and $y = \sin(t)$, with $t \in [0, 2\pi]$.

Instructions: Answer all numbered questions 1-10. The value for each part is indicated in square brackets in the margin (out of a possible total of 60). Only try the **Bonus Question** if you find that you have enough time left over *after* having replied to all ten regular questions. Lined examination booklets will be provided. Write all relevant calculations and results on the right hand pages. Left hand pages are for rough work only, and will not be read or graded by the examiners. Only calculators of the type authorized by the Department of Mathematics and Statistics may be used. Any books, notes, or other recorded materials may not be used.

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[6] **1.** Find the length of the planar curve defined in polar co-ordinaes by

$$r = \sin^3(\theta/3), \quad 0 \leq \theta \leq \pi.$$

[6] **2.** Find an equation for the ellipse that shares a vertex and a focus with the parabola

$$x^2 + y = 100.$$

and that has its other focus at the origin.

[6] **3.** Find the Maclaurin series for the function

$$f(x) = \ln(4 - x)$$

and its interval of convergence.

[6] **4.** Find the curvature of the ellipse

$$x = 3\cos t, \quad y = 4\sin t$$

at the points $(3, 0)$ and $(0, 4)$.

[6] **5.** Find an equation of the osculating plane of the curve

$$x = \sin 2t, \quad y = t, \quad z = \cos 2t$$

at the point $(0, \pi, 1)$.

[6] **6.** Find equations of the *tangent plane* and *the normal line* to the surface defined by

$$z = e^x \cos y$$

at the point $(0, 0, 1)$.

[3] **7.** If

$$v = x^2 \sin y + ye^{xy}$$

where

$$x = s + 2t \quad \text{and} \quad y = st,$$

use the chain rule to find $\frac{\partial v}{\partial s}$ and $\frac{\partial v}{\partial t}$ when $(s, t) = (0, 1)$.

[6] **8.** Find the directional derivative of

$$f(x, y, z) = x^2 y + x\sqrt{1 + z}$$

at the point $(1, 2, 3)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

[6] **9.** Identify the local maxima and minima and saddle points of

$$f(x, y) = x^3 - 6xy + 8y^3$$

and determined the values of the function at these points.

[6] **10.** Find the points where the absolute maximum and minimum values of the function

$$f(x, y) = e^{-x^2 - y^2}(x^2 + 2y^2)$$

attains its absolute maximum and minimum values within the disk defined by

$$x^2 + y^2 \leq 4.$$

What are these values?

[3] **BONUS QUESTION**

An airplane has flying speed 180 km/hr in still air. The pilot takes off from an airfield and heads due north according to the plane's compass. After 30 minutes of flight time the pilot notices that, due to the wind, the plane has actually travelled only 80 km at an angle 5^0 east of north.

(a) What is the wind velocity?

(b) In which direction should the pilot have headed in order to reach the intended destination?

(Ignore the correction due to the finite time required to reach full air speed. If your calculator doesn't have sines and cosines, approximate these for 5^0 to three significant figures using a Taylor series approximation, with π taken as 2.1415.)

Dec. 2014

Special Instructions

▷ Only approved calculators are allowed. Justify all your answers. All questions have equal value.

Formulaire

For a curve $\mathbf{r}(t)$ in $\mathbb{R}^3$, Arclength $s(t) = \int_a^t |\mathbf{r}'(u)|du$, $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, $\kappa = |\dfrac{d\mathbf{T}}{ds}| = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

---

1. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ with $x(t) = t^3, y(t) = 3t, z(t) = t^4$. Give an expression for the normal plane to $\mathbf{r}(t)$ of the form

$$A(t)x + B(t)y + C(t)z = D(t).$$

At what point is the normal plane parallel to the plane $6x + 6y - 8z = 1$?

2. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ with $x(t) = t, y(t) = 1/t, z(t) = 0$ for $t > 0$. Compute $\kappa(t)$. Show that $\kappa(t) \to 0$ as $t \to 0, \infty$. Find the maximum value of $\kappa(t)$.

3. Write a power series expansion around $x = 0$ of $f(x) = x - \ln(1 + x)$. Use it to evaluate the

$$\lim_{x \to 0} f(x)/x^2.$$

Give a power series for $\int f(x)/x^2$ around $x = 0$. What is its radius of convergence?

4. Suppose that temperature at every point (x, y) is given by

$$T(x, y) = 10e^{-x^2-y^2}.$$

(a) Find the direction in which the temperature is *decreasing* most rapidly at the point $(1, -2)$, and give the rate of change in this direction.

(b) Find all directions in which the temperature is *not changing* at the point $(1, -2)$. answer.

5. Let C be the curve defined by the parametric equations

$$\begin{aligned} x &= t^2 \\ y &= t^3 - t, \end{aligned}$$

where $t \in \mathbb{R}$. Find a point at which C intersects itself, and find the angle of this intersection.

6. Find and classify the critical points of the function

$$f(x, y) = 2x^3 - 6xy + 3y^2.$$

7. Find the minimum and maximum values of the function

$$f(x, y) = x^2 + y^2 - 2x - 5$$

on the region defined by the inequality

$$x^2 + 2y^2 \leq 16.$$

8. Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{3^n(x-2)^n}{\sqrt{n}}.$$

9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$yz = \ln(x + z).$$

10. Find the minimum and maximum distances from the origin $O = (0, 0, 0)$ to the surface of the ellipsoid given by

$$x^2 + y^2/4 + z^2/9 = 1.$$

Be sure to justify your answer.

Apr. 2012

Answer all questions. Each problem is worth 10 marks. No Calculators allowed.

1. Find and sketch the domain of the function

$$f(x, y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$$

2. Evaluate the limit or show that it does not exist:

(a) $\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+y^2}$ (b) $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$

3. If $z = xy + xe^{\frac{y}{x}}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + z$.

4. Find the equations of the tangent plane and the normal line to the surface

$$x^2 + y^2 - 3z^2 = 3 \quad \text{at the point } (2, -1, 1)$$

5. If $v = x^2 \sin y + ye^{xy}$, where $x = s + 2t$ and $y = st$, use the Chain Rule to find $\frac{\partial v}{\partial s}$ and $\frac{\partial v}{\partial t}$ when $s = 0$ and $t = 1$.

6. Find the directional derivative of $f(x, y, z) = x^2y + x\sqrt{1+z}$ at the point $(1, 2, 3)$ in the direction of $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

7. Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{y}$ at the point (2,1). In which direction does it occur?

8. Find parametric equations of the tangent line at the point (-2,2,4) to the curve of interesection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

9. Find the absolute maximim and minimum values of $f(x, y) = 4xy^2 - x^2y^2 - xy^3$ on the closed triangular region in the xy-plane defined by the vertices $(0, 0), (0, 6), (6, 0)$.

10. Use a Lagrange multiplier to find the maximum and minimum of $f(x, y, z) = xyz$ subject to the constraint $x^2 + y^2 + z^2 = 3$.

1

Apr. 2013

Instructions: Only approved calculators permitted. The value for each problem is indicated in square brackets in the margin (out of a possible total of 100). Show all your steps. Write the complete solution on the right hand pages of your examination booklet only.

MARKS: marks for each problem are shown in front of the problems.

↓MARKS

[6] **Problem 1 :** Find the Taylor series for $f(x) = e^x + e^{2x}$ at the point 0 and its radius of convergence.

[14] **Problem 2 :** Consider the curves γ_1 and γ_2 defined by polar equations:

$$\gamma_1: \quad r = 3 + \sin\theta, \quad 0 \le \theta \le 2\pi; \qquad \gamma_2: \quad r = 4\sin\theta, \quad 0 \le \theta \le \pi$$

(a). Identify the curve γ_2 by finding its **Cartesian equations** in (x, y)−coordinates, and **sketch the polar curves** γ_1 and γ_2.

(b). Find the area A of the region that lies inside the first curve γ_1 and outside the second curve γ_2.

[12] **Problem 3 :** Consider the curve $\gamma = (\sqrt{3}\,t,\ \sin(t),\ \cos(-t)), \quad 0 \le t \le 1.$

(a) Find the unit tangent $\vec{T}$ and unit normal $\vec{N}$ vectors of γ;

(b) Find the length of the curve γ.

[12] **Problem 4 :** Let $P = (1, 1, 1)$, and let L be the line given parametrically

$$L: \ (0,1,0) + t\langle 1,2,2\rangle, \quad -\infty < t < +\infty.$$

(a) Find the distance of point P from line L;

(b) Find the equation of the plane Q passing through point P and the line L;

(c) Find the symmetric equations of the line L_2 passing through point P and perpendicular to the plane Q;

[14] **Problem 5 :** Let the surface $z = f(x, y)$ be defined implicitly by the equation:

$$x^2 + 2y^2 - 3z^2 = 3 \ .$$

(a). Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(b). Find the equation of the tangent plane at the point $(2, -1, 1)$.

(c). In which direction does f increase most rapidly at this point?

[10] **Problem 6 :** Find the limit, if it exists, or show that the limit does not exist:

$$(a). \quad \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}\ , \qquad (b). \quad \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}.$$

[18] **Problem 7 :** Consider the function $f(x, y) = x^3 - 6xy + 8y^3$.

(a). Find all critical points of $f(x, y)$.

(b). Classify the critical points obtained in (a) as local minimum, local maximum, or saddle points.

(c). Find the absolute maximum and minimum of $f(x, y)$ on the square bounded by the lines: $x = 0$, $y = 0$, $x = 1$ and $y = 1$.

[14] **Problem 8 :** Use **Lagrange Multipliers** to find the maximum and minimum values of $f(x, y) = y^2 - x^2$ subject to the constraint: $\frac{1}{4}x^2 + y^2 = 1$.

GOOD LUCK !!!

Apr. 2015

FINAL EXAM MATH 264, Winter 2015

Evaluation out of 90. Each question worth 10 points.

1. Find the area of the region enclosed by one loop of the curve $r = \sin 4\theta$ where (r, θ) are polar coordinates.

2. Find the radius and interval of convergence of the power series $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$.

3. At what point of the curve $\mathbf{r}(t) = (t^3, 3t, t^4)$ is the normal plane parallel to the plane $6x+6y-8z = 1$?

4. The position function of a particle is given by $\mathbf{r}(t) = (t^2, 5t, t^2 - 16t)$. When is the speed a minimum?

5. Find the limit if it exists, or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

6. Let $z = y + f(x^2 - y^2)$ where f is a differentiable function. Compute $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$.

7. Find the directions in which the directional derivative of $f(x, y) = ye^{-xy}$ at the point $(0, 2)$ has the value 1.

8. Find the absolute minimum and absolute maximum of the function $f(x, y) = x^3 - 3x - y^3 + 12y$ in the quadrilateral whose vertices are $(-2, 3)$, $(2, 3)$, $(2, 2)$ and $(-2, -2)$.

9. Find maximum and minimum values of the function $f(x, y, z) = 2x + 2y + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

10. Bonus (10 points). Among all triangles with perimeter 2 find the triangle with maximal area. Hint: use the Heron's formula for the area $A = \sqrt{(1-x)(1-y)(1-z)}$, where x, y, z are sides of the triangle.

Appendix C – MATH 218/MATH 264 Course Outline

Department of Mathematics & Statistics
Concordia University

MATH 264 (MAST 218)
Multivariable Calculus I
Winter 2015

Instructor:	
Prerequisites:	Math 205 or an equivalent Calculus II course.
Text:	*Multivariable Calculus*, 7th Edition by J. Stewart, Brooks/Cole.
Assignments:	Assignments are ***very important*** as they indicate the level of difficulty of the problems that the students are expected to solve. Therefore, every effort should be made to **do and understand the assignment problems.** The assignments will be corrected and graded.
Web Resources:	Many excellent animated illustrations to the text of the book are collected at the site www.stewartcalculus.com, see TEC (Tools for Enriching Calculus) for the edition 6. Regular use of this resource is much recommended.
Use of Computer Algebra System:	It is optional but much recommended to install and use Maple or Mathematica. These computer tools can be used to verify and illustrate any analytical results you get while doing your assignment problems.
Calculators:	Electronic communication devices (including cell phones) are not allowed in the examination rooms. Only calculators approved by the Department (with a sticker attached as proof of approval) are permitted in the class test(s) and final examination. The preferred calculators are the **Sharp EL 531** and the **Casio FX 300MS**, available at the Concordia Bookstore.
Test:	Midterm exam covering the first six weeks will be given in week 7 or 8.
Final Grade:	The highest of the following: • 90% final exam, 10% assignments. • 30% midterm, 10% assignments, and 60% final exam.

Departmental website: http://www.mathstat.concordia.ca

Week	Sections	Topics	Assignments
1	10.1, 10.2	Parametric equations of curves.	p.665: 7,11,24,37,46; p.675: 3,13,19,43,51;
2	10.3, 10.4, 10.5	Areas and lengths in polar coordinates. Conic sections.	p.686: 9,23,35,55; p.692: 5,11,25,45; p.700: 7,13,23,29,45;
3	10.6, 11.10, 12.1	Conic sections in polar coordinates. Taylor series: review. Three-dimensional coordinate systems.	p.708: 3,11,15; p.789: 9,11,19, 49; p.814: 3,9,11, 19;
4	12.2, 12.3, 12.4	Vectors. Dot product. Cross product.	p.822: 5,8,21,33; p.830: 7,13,21,43,51; p.838: 5,11,29,43;
5	12.5, 12.6	Equations of lines and planes. Cylinders and quadric surfaces.	p.848: 9,25,53,65,73; p.856: 5,7,17,21-28,41,43;
6	13.1, 13.2	Vector functions and space curves. Derivatives and integrals of vector functions.	p.869: 5,19,21-26,29,47; p.876: 9,21,23,33,37;
7	13.3, 13.4	Arc length and curvature of space curve. Velocity and acceleration.	p.884: 1,13,25,31,43,49; p.894: 7,13,19,35;
8	14.1, 14.2	Functions of several variables, their limits and continuity.	p.912: 8,13,29,32,59-64; p.923: 7,11,21,41;
9	14.3, 14.4	Partial derivatives. Tangent planes and linear approximation.	p.935: 5-8,23,35,53,61,75; p.946: 5,15,35;
10	14.5, 14.6	Chain rule. Directional derivatives and gradient vector.	p.954: 3,9,21,33,41; p.967: 5,9,21,27,31,45,53;
11	14.7	Maximum and minimum values.	p.977: 4,11,17,29,35,39,45;
12	14.8	Lagrange multipliers.	p.987: 5,9,13,15,19,33,43;
13		**Review**	

Appendix D – Excerpts from the Textbook for the Theoretical Blocks of the KT Praxeologies

Π1 Curves Defined by Parametric Equations

Parameter; parametric equations; parametric curve

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**.

p.680

Initial and terminal points of a parametric curve

In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leq t \leq b$$

has **initial point** $\big(f(a), g(a)\big)$ and **terminal point** $\big(f(b), g(b)\big)$.

p.681

Π2 Calculus with Parametric Curves

Tangent of a parametric curve representing a differentiable function; concavity of a parametric curve

Tags - T1 Parametric equation for a curve

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t), y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$, we can solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if } \frac{dx}{dt} \neq 0$$

[This equation] (which you can remember by thinking of canceling the dt's) enables us to find the slope dy/dx of the tangent of a parametric curve without having to eliminate the parameter t.

p.689

[I]t is also useful to consider d^2y/dx^2. This can be found by replacing y by dy/dx in [the equation for the slope of the tangent to a parametric curve]:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

p.689

This second derivative is used to determine the concavity of a parametric curve.

Area under a parametric curve

Tags - T1 Parametric equation for a curve

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)dx$, where $F(x) \geq 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t), \alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

$$A = \int_a^b y\, dx = \int_\alpha^\beta g(t)f'(t)dt \qquad [\text{or } \int_\beta^\alpha g(t)f'(t)dt$$

p.691

Arc Length of a parametric curve

Tags - T1 Parametric equation for a curve

If a curve C is described by the parametric equation $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

p.693

Π3 Polar coordinates

Coordinate system, Cartesian coordinates, polar coordinate system, pole, polar axis, polar coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents that pole for any value of θ.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies on the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

p.698

Transformations between the polar and Cartesian coordinate systems

The following relations between Cartesian and polar coordinates are based on the definition of the trigonometric functions and the relations established within the trigonometric unit circle. See Figure 5 on p.699.

$$x = r\cos\theta\,, y = r\sin\theta \quad \text{and} \qquad r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}$$

Tangent to a polar curve

Tags – T2 Tangent to a parametric curve

The textbook derives a formula for the slope of the tangent to a polar curve $r = f(\theta)$ from the formula for the tangent to parametric curves (T2) and an application of the product rule to the parametric equations

$$x = f(\theta)\cos\theta, \;\; y = f(\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

p.703

Π4 Areas and Lengths in Polar Coordinates

Area of a sector of a circle

$$A = \frac{1}{2}r^2\theta$$

Where r is the radius and θ is the radian measure of the central angle of the sector of a circle.

This formula is used along with Riemann sums to "develop the formula for the area of a region whose boundary is given by a polar equation."

p.709

Riemann sums

Riemann sums are used to develop a formula for the area bounded by a polar curve. See p.709-710.

Area of a region bounded by a polar curve (T4)

The area A of a region bounded by a polar curve $r = f(\theta)$ is given by

$$A = \int_a^b \frac{1}{2}r^2 d\theta$$

p.710

Arc length of a polar curve

Tags – T2 Arc length of a parametric curve

The length of a curve with polar equation $r = f(\theta), a \le \theta \le b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

p.712

This formula is derived from the formula for the arc length of a parametric curve (T2) and an application of the product rule.

Π5 Conic Sections

Parabola; focus, directrix, vertex, and axis of a parabola

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 2. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

p.714

Ellipse and foci of an ellipse

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is constant (See Figure 6). These two fixed points are called the **foci** (plural of **focus**).

p.716

Cartesian equation of an ellipse; vertices, major axis, and minor axis of an ellipse; symmetry of an ellipse

[Consider an ellipse that has its] foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 7 so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$.

Then the equation of the ellipse [...] [is]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (*)$$

[where $b^2 = a^2 - c^2$; since $a^2 - c^2 \leq a^2$, it follows that $0 < b \leq a$.]

The points $(a, 0)$ and $(-a, 0)$ are called the **vertices** of the ellipse and the line segment joining the vertices is called the **major axis**.

The line segment joining $(0, b)$ and $(0, -b)$ is the **minor axis**.

[Note that the equation of the ellipse] is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

p.716

If the foci of an ellipse are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in $(*)$. The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geq b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

p.717

Hyperbola and foci of a hyperbola

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the **foci**) is a constant.

p.717

Cartesian equation of a hyperbola; vertices of a hyperbola; symmetry of a hyperbola

[W]hen the foci are on the x-axis at $(\pm c, 0)$and the difference of distances is

$|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (*)$$

where $c^2 = a^2 + b^2$. Notice that the x-intercepts are again $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola. But if we put $x = 0$ in Equation $(*)$ we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation $(*)$ and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$$

This shows that $x^2 \geq a^2$, so $|x| = \sqrt{x^2} \geq a$. Therefore we have $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its *branches*.

When we draw a hyperbola it is useful to first draw its **asymptotes**, which are the dashed lines $y = \left(\frac{b}{a}\right)x$ and $y = -\left(\frac{b}{a}\right)x$ shown in Figure 12 (p.718). Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes.

If the foci of a hyperbola are on the y-axis, then by reversing the roles of x and y we obtain the following information:

The hyp erbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$,vertices $(0, \pm a)$, and asymptotes $y = \pm\left(\frac{a}{b}\right)x$.

p.718

Cartesian equation of a parabola

An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

p.715

Rotation in polar coordinates

Tags – T3 Polar coordinates

[T]he graph of $r = f(\theta - \alpha)$ is the graph of $r = f(\theta)$ rotated counterclockwise about the origin through an angle α.

p.725

Theorem 5.1 – Classification of conic sections by eccentricity

Tags – T3 Polar coordinates, Transformation between Cartesian and polar coordinates

Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is

(a) An ellipse if $e < 1$
(b) A parabola if $e = 1$
(c) A hyperbola if $e > 1$

p.722

Theorem 5.2 – Polar equation of conic section

Tags – T3 Polar coordinates

A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

p.724

Π6 Three-Dimensional Coordinate Systems

Origin, coordinate axes, coordinate planes

In order to represent points in space, we first choose a fixed point O (the origin) and three directed lines through O that are perpendicular to each other, called the **coordinate axes** and labeled the x-axis, y-axis, and z-axis. Usually we think of the x- and y-axes as being horizontal and the z-axis as being vertical, and we draw the orientation of the axes as in Figure 1. The direction of the z-axis is determined by the **right-hand rule** as illustrated in Figure 2: If you curl the fingers of your right hand around the z-axis in the direction of a 90° counterclockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis.

The three coordinate axes determine the three **coordinate planes** illustrated in Figure 3(a). the xy-plane is the plane that contains the x- and y-axes; the yz-plane contains the y- and z-axes; the xz-plane contains the x- and z- axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant**, in the foreground, is determined by the positive axes.

p.832

Coordinates in space

[I]f P is any point in space, let a be the (directed) distance from the yz-plane to P, let b be the distance from the xz-plane to P, and let c be the distance from the xy-plane to P. We represent the point P by the ordered triple (a, b, c) of real numbers and we call a, b, and c the **coordinates** of P; a is the x-coordinate, b is the y-coordinate, and c is the z-coordinate. Thus, to locate the point (a, b, c), we can start at the origin and move a units along the x-axis, then b units parallel to the y-axis, and then c units parallel to the z-axis as in Figure 4.

p.832

Projection onto the coordinate planes

If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates $(a, b, 0)$ called the **projection** of P onto the xy-plane. Similarly $R(0, b, c)$ and $S(a, 0, c)$ are the projections of P onto the yz-plane and xz-plane, respectively.

p.833

Three-dimensional rectangular coordinate system

The Cartesian product $R \times R \times R = \{(x, y, z) | x, y, z \in R\}$ is the set of all ordered triples of real numbers and is denoted by R^3. We have given a one-to-one correspondence between points P in space and ordered triples (a, b, c) in R^3. It is called a **three-dimensional rectangular coordinate system**.

p.833

Equation in R^3

In three-dimensional analytic geometry, an equation in x, y, and z represents a *surface* in R^3.

p.833

[Note that w]hen an equation is given, we must understand from the context whether it represents a curve in R^2 or a surface in R^3.

p.834

An example (Example 1, p.833) is given to clarify that an equation that contains less than three variables (e.g. $y = 5$), if given in R^3, should be understood in that context.

Distance formula in three dimensions

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

p.835

The proof in the textbook (p.835) constructs a rectangular box with $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ at opposite ends. The distance between P_1 and P_2 is found upon two applications of the Pythagorean Theorem and a bit of algebra (combining the two equations that result from the Pythagorean Theorem).

Equation of a sphere

An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

p.835

Π7 Vectors

Vector; displacement vector; initial and terminal points of a vector; equivalent/equal vectors; and the zero vector

The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both magnitude and direction.

[…]

For instance, suppose a particle moves along a line segment from point A to point B. The corresponding **displacement vector $\mathbf{v}$**, shown in Figure 1, has **initial point** A (the tail) and **terminal point** B (the tip) and we indicate this by writing $\mathbf{v} = \overrightarrow{AB}$.

"We say that $\mathbf{u}$ and $\mathbf{v}$ are **equivalent** (or **equal**) and we write $\mathbf{u} = \mathbf{v}$" if "the vector $\mathbf{u} = \overrightarrow{CD}$ has the same length and the same direction as $\mathbf{v}$ even [if] it is in a different position."

The **zero vector**, denoted by $\mathbf{0}$, has length 0. It is the only vector with no specific direction.

p.838

Definition of vector addition; triangle law for vector addition

If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the sum $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

p.838

This is called the **triangle law** for vector addition.

Definition of scalar multiplication; parallel vectors; negative of a vector

If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

p.839

Notice that two nonzero vectors are **parallel** if they are scalar multiples of one another. In particular, the vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but points in the opposite direction. We call it the **negative** of v.

p.839

Difference of vectors

By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

p.839

Components of a vector; representations of a vector; position vector

If we place the initial point of a vector $\mathbf{a}$ at the origin of a rectangular coordinate system, then the terminal point of $\mathbf{a}$ has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (See Figure 11). These coordinates are called the **components** of $\mathbf{a}$ and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \qquad \text{or} \qquad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

p.840

[T]he vectors shown in Figure 12 are all equivalent to the vector $\overrightarrow{OP} = \langle 3,2 \rangle$ whose terminal point is $P(3,2)$. What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward. We can think of all these geometric vectors as **representations** of the algebraic vector $\mathbf{a} = \langle 3,2 \rangle$. The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3,2)$ is called the **position vector** of the point P.

In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$.

p.840

Relation between the representation of a vector and the vector

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

p.840

Magnitude/length of a vector

The **magnitude** or **length** of the vector $\mathbf{v}$ is the length of any of its representations and is denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

p.841

Length of a vector in R^2 and in R^3

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

p.841

Algebraic vector addition and scalar multiplication

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = c$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

p.841

n-dimensional vector

An n-dimensional vector is an ordered n-tuple:

$$\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$$

Where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of **a.** Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

p.842

We denote by V_n the set of all n-dimensional vectors.

Properties of vectors

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

p.842

Standard basis vectors

Let

$$\mathbf{i} = \langle 1,0,0 \rangle \qquad \mathbf{j} = \langle 0,1,0 \rangle \qquad \mathbf{k} = \langle 0,0,1 \rangle$$

These vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are called the **standard basis vectors**.

Similarly, in two dimensions we define $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$.

p.842

Unit vector

A **unit vector** is a vector whose length is 1.

p.843

Unit vector in the direction of a given vector

In general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

p.843

Π8 Dot Product

Dot/scalar/inner product

Tags – T7 Components of a vector

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** [also called **scalar product** or **inner product**] of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

p.847

Properties of the Dot Product

Tags – T7 Vectors, V_3, magnitude of a vector, algebraic vector addition and scalar multiplication, zero vector

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

p.847

The proof of these properties (p.848) relies mainly on the definition of dot product and some algebraic manipulations.

Angle between two vectors

Tags – T7 Representations of a vector, parallel vectors

[T]he **angle $\boldsymbol{\theta}$ between a and b** … is defined to be the angle between the representations of $\mathbf{a}$ and $\mathbf{b}$ that start at the origin, where $0 \leq \theta \leq \pi$.

Note that if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

p.848

Theorem 8.1: the dot product of two vectors and the angle between them[18]

Tags – T7 Vectors, magnitude of a vector

If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

p.848

The proof in the textbook (p.848) uses the Law of Cosines to establish an equation involving the vectors $\mathbf{a}$ and $\mathbf{b}$; properties of the dot product are then applied to manipulate this equation.

Orthogonal (perpendicular) vectors

Tags – T7 Vectors

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \frac{\pi}{2}$.

The zero vector $\mathbf{0}$ is considered to be perpendicular to all vectors.

p.849

Orthogonality and dot product

Tags – T7 Vectors, magnitude of a vector

[Theorem 8.1] gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \frac{\pi}{2} = 0$$

and conversely if $\mathbf{a} \cdot \mathbf{b} = 0$, then $\cos \theta = 0$, so $\theta = \frac{\pi}{2}$.

[Therefore, t]wo vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

p.849

Direction angles and direction cosines of a vector

Tags – T7 Vectors, magnitude of a vector, standard basis vectors

The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ (in the interval $[0, \pi]$) that $\mathbf{a}$ makes with the positive x-, y-, and z-axes, respectively. (See Figure 3.)

The cosines of these direction angles, $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, are called the **direction cosines** of the vector $\mathbf{a}$. Using [Theorem 8.1,] we obtain

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}$$

Similarly, we also have

$$\cos \beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos \gamma = \frac{a_3}{|\mathbf{a}|}$$

p.850

[18] The textbook acknowledges that this may be taken as the definition of dot product (p.848)

Components of the unit vector in the direction of a given vector

Tags – T7 Component form of vectors, magnitude of a vector, algebraic scalar multiplication of a vector, unit vector in the direction of a given vector

Recall that the direction cosines of a vector $\mathbf{a} = (a_1, a_2, a_3)$ are

$$\cos\alpha = \frac{a_1}{|\mathbf{a}|} \qquad \cos\beta = \frac{a_2}{|\mathbf{a}|} \qquad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

By squaring the expressions in [these equations] and adding, we see that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

We can also use [these equations] to write

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle = \langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma\rangle$$
$$= |\mathbf{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$

Which says that the direction cosines of **a** are the components of the unit vector in the direction of **a.**

p.850

Vector projection, scalar projections

Tags – T7 Representations of vectors, initial point of a vector, magnitude of a vector

Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\mathrm{proj}_{\mathbf{a}}\mathbf{b}$**.**

The **scalar projection** of b onto a (also called the **component of b** along **a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b**. (See Figure 5.) This is denoted by $\mathrm{comp}_{\mathbf{a}}\mathbf{b}$. Observe that it is negative if $\frac{\pi}{2} \le \theta \le \pi$. The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\qquad \mathrm{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\qquad \mathrm{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

p.851

Displacement vector of an object; work done by a constant force

Tags – T7 Vectors, magnitude of a vector

In Section 5.4 we defined the work done by a constant force F in moving an object through a distance d as $W = Fd$, but this applies only when the force is directed along the line of motion of the object. Suppose, however, that the constant force is a vector $\mathbf{F} = \overrightarrow{PR}$ pointing in some other direction, as in Figure 6. If the force moves the object from P to Q, then the **displacement vector** is $\mathbf{D} = \overrightarrow{PQ}$. The **work** done by this force along $\mathbf{D}$ and the distance moved:

$$W = (|\mathbf{F}|\cos\theta)|\mathbf{D}|$$

But then, from Theorem 8.1, we have

$$W = |\mathbf{F}||\mathbf{D}|\cos\theta = \mathbf{F}\cdot\mathbf{D}$$

Thus the work done by a constant force $\mathbf{F}$ is the dot product $\mathbf{F}\cdot\mathbf{D}$, where $\mathbf{D}$ is the displacement vector.

p.851-2

Π9 Cross Product

Cross product (vector product); determinants of order 2 and 3

Tags – T7 Component form of a vector

T8 Orthogonal vectors

If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

[The cross product] is also called the **vector product**. Note that $\mathbf{a}\times\mathbf{b}$ is defined only when $\mathbf{a}$ and $\mathbf{b}$ are *three-dimensional* vectors.

p.855

This definition is preceded by some computations with dot products to demonstrate the link between the definition of cross product and the search for a vector perpendicular to two given vectors.

In order to make [this definition] easier to remember, we use the notation of determinants. A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & b \end{vmatrix} = ad - bc$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

If we now rewrite [the definition of cross product] using second-order determinants and the standard basis vectors $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$, we see that the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

[W]e often write

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

p.855

Theorem 9.1: Orthogonality of the cross product

Tags – T7 vectors

T8 Orthogonality, orthogonality and dot product

The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

p.856

The proof established orthogonality by computing the dot product of $\mathbf{a} \times \mathbf{b}$ with $\mathbf{a}$ and showing it is 0.

Theorem 9.2: Magnitude of the cross product[19]

Tags – T7 Vectors, magnitude of a vector

T8 Angle between vectors

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \le \theta \le \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

p.857

The proof follows from the definitions of cross product and length of a vector and some algebraic manipulations.

Corollary of Theorem 9.2: parallel vectors

Tags – T7 vectors, parallel vectors

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

p.857

Geometric interpretation of cross product

Tags – T7 Magnitude of a vector

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

p.858

Properties of the cross product

Tags – T7 vectors, component form of a vector, algebraic vector addition and scalar multiplication

If $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

p.859

The textbook proves property 5 and notes that “these properties can be proved by writing the vectors in terms of their components and using the definition of a cross product” (p.859).

Scalar triple product

Tags – T7 Vectors

T8 Dot product

[19] At this point, the textbook acknowledges that its definition of cross product is equivalent to defining it as a “vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, whose orientation is determined by the right-hand rule, and whose length is $|\mathbf{a}||\mathbf{b}| \sin \theta$” (p.857)

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 [from Properties of the cross product] is called the **scalar triple product** of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Notice from Equation 12 [in the proof of property 5 (p.859)] that we can write the scalar triple product as a determinant:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

p.859

Coplanar vectors

Tags - T7 Vectors

Three vectors are said to be **coplanar** if they lie in the same plane (p.859).

Geometric interpretation of triple scalar product

Tags – T7 Vectors, magnitude of a vector

T8 Angle between vectors

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. (See Figure 3.) The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \frac{\pi}{2}$.) Therefore the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{a}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

p.859

Therefore, to summarize:

The volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

p.860

Vector triple product

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ that occurs in Property 6 [from Properties of the cross product] is called the **vector triple product** of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

p.860

Torque

Tags – T7 Vectors, magnitude of a vector

T8 Angle between vectors

[C]onsider a force $\mathbf{F}$ acting on a rigid body at a point given by a position vector $\mathbf{r}$. (For instance, if we tighten a bolt b applying a force to a wrench as in Figure 4, we product a turning effect.) The **torque** $\boldsymbol{\tau}$ (relative to the origin) is defined to be the cross product of the position and force vectors

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

and measures the tendency of the body to rotate about the origin. The direction of the torque vector indicates the axis of rotation. According to [Theorem 2: Magnitude of the cross product], the magnitude of the torque vector is

$$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{f}| \sin\theta$$

where θ is the angle between the position and force vectors.

p.860

$\Pi 10$ Equations of Lines and Planes

Point-slope form of a line

Tags – T6 Three-dimensional coordinate systems

A line in the xy-plane is determined when a point on the line and the direction of the line (its slope or angle of inclination) are given. The equation of the line can then be written using the point-slope form.

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L.

p.863

Vector equation of a line

Tags – T6 Three-dimensional coordinate systems

T7 Vectors, position vectors, representation of a vector, Triangle Law for vector addition, parallel vectors

(T1 Parameters)

In three dimensions the direction of a line is conveniently described by a vector, so we let $\mathbf{v}$ be a vector parallel to L. Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r_0}$ and $\mathbf{r}$ be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If **a** is the vector with representation $\overrightarrow{PP_0}$, as in Figure 1, then the Triangle Law for vector addition gives $\mathbf{r} = \mathbf{r_0} + \mathbf{a}$. But, since $\mathbf{a}$ and $\mathbf{v}$ are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$. Thus

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$$

which is a **vector equation** of L. Each value of the **parameter** t gives the position vector $\mathbf{r}$ of a point on L. In other words, as t varies, the line is traced out by the tip of the vector $\mathbf{r}$.

p.864

Parametric equations of a line

Tags – T1 Parametric equations, parameter

T6 Three-dimensional coordinate systems

T7 Component form of a vector

Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c\rangle$ are

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

p.864

The textbook derives the parametric equations for a line from the vector equation of a line by expressing each vector in component form and equating the components (p.864).

Symmetric equations and direction numbers of a line

Tags – T6 Three-dimensional coordinate systems

T7 Component form of a vector, parallel vectors

[I]f a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L. Since any vector parallel to $\mathbf{v}$ could also be used, we see that any three numbers proportional to a, b, and c could also be used as a set of direction numbers for L.

Another way of describing a line L is to eliminate the parameter t from [its parametric equations: $x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$].

If none of a, b, or c is 0, we can solve each of these equations for t:

$$t = \frac{x-x_0}{a} t = \frac{y-y_0}{b} t = \frac{z-z_0}{c}$$

Equating these results, we obtain

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

These equations are called **symmetric equations** of L.

If one of a, b, or c is 0, we can still eliminate t. For instance, if $a = 0$, we could write the equations of L as

$$x = x_0 \qquad \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This means that L lies in the vertical plane $x = x_0$.

p.865

Vector equation of a line segment

Tags – T7, Vectors, properties of vectors

The line segment from $\mathbf{r_0}$ to $\mathbf{r_1}$ is given by the vector equation

$$\mathbf{r}(t) = (1-t)\mathbf{r_0} + t\mathbf{r_1} \qquad 0 \le t \le 1$$

p.866

Normal vector of a plane

Tags – T7 Vectors, parallel vector

T8 Orthogonal vector

A single vector parallel to a plane is not enough to convey the "direction" of the plane, but a vector perpendicular to the plane does completely specify its direction. Thus a plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector**.

p.867

Vector equation of the plane

Tags – T6 Three-dimensional coordinate systems

T7 Position vectors, representation of a vector, difference of vectors

T8 Dot product and orthogonality, properties of the dot product

Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r_0}$ and $\mathbf{r}$ be the position vectors of P_0 and P. Then the vector $\mathbf{r} - \mathbf{r_0}$ is represented by $\overrightarrow{P_0P}$. (See Figure 6.) The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane. In particular, $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r_0}$ and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$$

Either [of these equations] is called a **vector equation of the plane**.

p.867

Scalar equation of the plane

Tags – T6 Three-dimensional coordinate systems

A **scalar equation of the plane** through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

p.867

Linear equations in R^3

Tags – T6 Three-dimensional coordinate systems

By collecting the terms in [the scalar equation of a plane, $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$], we can rewrite the equation of a plane as

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$. [This equation] is called a **linear equation** in x, y, and z. Conversely, it can be shown that if a, b, and c are not all 0, then [this linear equation] represents a plane with normal vector $\langle a, b, c \rangle$.

p.867

Since a linear equation in x, y, and z represents a plane and two nonparallel planes intersect in a line, it follows that two linear equations can represent a line. The points (x, y, z) that satisfy both $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ lie on both of these planes, and so the pair of linear equations represents the line of intersection of the planes (if they are not parallel).

p.869

Distance from a point to a plane in R^3

Tags – T6 Three-dimensional coordinate systems

The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

p.870

Skew lines

Two lines in R^3 are said to be **skew** if they have no intersection and are not parallel (that is, if they do not lie in the same plane) (p.866). Skew lines "can be viewed as lying on two parallel planes" (p.870).

Π11 Cylinders and Quadric Sections

Traces of a surface

Tags – T6 Three-dimensional coordinate systems

[T]he curves of intersection of [a] surface with planes parallel to the coordinate planes […] are called **traces** (or **cross-sections**) of the surface.

p.874

Cylinder; rulings of a cylinder

Tags – T6 Three-dimensional coordinate systems

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve.

p.874

If one of the variables x, y, or z is missing from the equation of a surface, then the surface is a cylinder.

p.874

Quadric surface

Tags – T6 Three-dimensional coordinate systems

T5 Conic sections

A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z**.** The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, B, C, \dots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane.

p.875

Ellipsoid

Tags – T6 Three-dimensional coordinate systems

T5 Conic sections

An **ellipsoid** is a quadric surface all of whose traces are ellipses:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

If $a = b = c$, the ellipsoid is a sphere.

p.877

Elliptic paraboloid

Tags – T6 Three-dimensional coordinate systems

T5 Conic sections

An **elliptic paraboloid** is a quadric surface whose horizontal traces are ellipses and vertical traces parabolas; the "variable raised to the first power indicates the axis of the paraboloid":

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

p.877

Hyperbolic paraboloid

Tags – T6 Three-dimensional coordinate systems

T5 Conic sections

A **hyperbolic paraboloid** is a quadric surface whose horizontal traces are hyperbolas and vertical traces parabolas:

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Cone

Tags – T6 Three-dimensional coordinate systems

T5 Conic sections

A **cone** is a quadric surface whose horizontal traces are ellipses and "vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$":

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

p.877

Hyperboloid of one sheet

Tags – T6 Three-dimensional coordinate systems

T5 Conic sections

A **hyperboloid of one sheet** is a quadric surface whose horizontal traces are ellipses and vertical traces hyperbolas; "the axis of symmetry corresponds to the variable whose coefficient is negative":

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

p.877

Hyperboloid of two sheets

Tags – T6 Three-dimensional coordinate systems

T5 Conic sections

A **hyperboloid of two sheets** is a quadric surface whose horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$ and vertical traces are hyperbolas; the two minus signs indicate two sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

p.877

Π12 Vector Functions and Space Curves

Vector-valued function; component functions of a vector function

Tags – T7 Vectors, V_3, component form of a vector/components of a vector

In general, a function if a rule that assigns to each element in the domain an element in the range. A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions $\mathbf{r}$ whose values are three-dimensional vectors. This means that for every number t in the domain of $\mathbf{r}$ there is a unique vector in V_3 denoted by $\mathbf{r}(t)$. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f, g, and h are real-valued functions called the **component functions** of $\mathbf{r}$ and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

p.888

Limit of a vector function

Tags – T7 Vectors, component form of a vector/components of a vector

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

provided the limits of the component functions exist.

p.888

Continuous vector function

Tags – T7 Vectors, component form of a vector/components of a vector

A vector function $\mathbf{r}$ is **continuous at $\boldsymbol{a}$** if

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$$

In view of [the definition of the limit of a vector function], we see that $\mathbf{r}$ is continuous at a if and only if its component functions f, g, and h are continuous at a.

p.889

Space curve; parametric equations of a space curve; parameter

Tags – T6 Three-dimensional coordinate systems

(T1 – Parametric equations of a curve, parameter, plane curves)

T7 Vector, component form of a vector, position vector of a point, standard basis vectors

There is a close connection between continuous vector functions and space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t) \qquad y = g(t) \qquad z = h(t) \qquad (*)$$

and t varies throughout the interval I, is called a **space curve.** The equations in $(*)$ are called **parametric equations of $\boldsymbol{C}$** and t is called a **parameter**. We can think of C as being traced out by a moving particle whose position at time t is $\big(f(t), g(t), h(t)\big)$. If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C. Thus any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in Figure 1.

p.889

Plane curves can also be represented in vector notation. For instance, the curve given by the parametric equations $x = t^2 - 2t$ and $y = t + 1$ could also be described by the vector equation

$$r(t) = \langle t^2 - 2t, t+1 \rangle = (t^2 - 2t)\mathbf{i} + (t+1)\mathbf{j}$$

where $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$.

p.889

Π13 Derivatives and Integrals of Vector Functions

Derivative of a vector function; tangent vector; tangent line; unit tangent vector

Tags – T12 Vector functions

T7 Position vectors, difference of vectors, scalar multiple of a vector, magnitude of a vector, unit vector

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1. If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t+h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector. If $h > 0$, the scalar multiple $\left(\frac{1}{h}\right)\left(\mathbf{r}(t+h) - \mathbf{r}(t)\right)$ has the same direction as $\mathbf{r}(t+h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. We will also have occasion to consider the **unit tangent vector**, which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

p.896

Theorem 13.1: derivative of a vector function

Tags – T12 Vector functions

T7 Components of a vector, standard basis vectors, properties of vectors

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(\mathrm{t}) = \langle f'(\mathrm{t}), g'(\mathrm{t}), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

p.896

The proof in the textbook (p.896) relies on the definition of derivative of a vector function, properties of vector addition and scalar multiplication (T7), and the definition of the derivative of real-valued functions.

Theorem 13.2: differentiation rules[20]

Tags – T12 Vector functions

T7 Vector addition and scalar multiplication

[20] There are virtually no tasks to be taught that require these rules; there's just one example in the textbook that demonstrates their utility, but nothing more.

T8 Dot product

T9 Cross product

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \mathbf{u}'(t)+\mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$
6. $\frac{d}{dt}\big[\mathbf{u}\big(f(t)\big)\big] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

p.898

The textbook notes that "[t]his theorem can be proved either directly from [the definition of the derivative of a vector function] or by using [Theorem 13.1] and the corresponding differentiation formulas for real-valued functions" (p.898). The proof for Formula 4 is provided (p.898).

Second derivative of a vector function

Tags – T12 Vector functions

Just as for real-valued functions, the **second derivative** of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

p.898

Integral of a continuous vector function

Tags – T12 Vector functions

T7 Vectors, component form of a vector/components of a vector, standard basis vectors

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of $\mathbf{r}$ in terms of the integrals of its components f, g, and h as follows.

$$\begin{aligned}\int_a^b \mathbf{r}(t)\,dt &= \lim_{n\to\infty}\textstyle\sum_{i=1}^{n}\mathbf{r}(t_i^*)\Delta t\\ &= \lim_{n\to\infty}[(\textstyle\sum_{i=1}^{n} f(t_i^*)\Delta t)\mathbf{i}+(\textstyle\sum_{i=1}^{n} g(t_i^*)\Delta t)\mathbf{j}+(\textstyle\sum_{i=1}^{n} h(t_i^*)\Delta t)\mathbf{k}]\end{aligned}$$

and so

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)dt\right)\mathbf{i}+\left(\int_a^b g(t)dt\right)\mathbf{j}+\left(\int_a^b h(t)dt\right)\mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

p.899

We use the notation $\int \mathbf{r}(t)dt$ for **indefinite integrals** (**antiderivatives**).

p.899

Theorem 13.3: Fundamental theorem of calculus

Tags – T12 Vector functions

T7 Difference of vectors

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

p.899

Π14 Arc Length and Curvature

Length of a curve

Tags – T13 Derivative of a vector function

T12 Vector equations, space curves

T7 Magnitude of a vector

T6 Three-dimensional coordinate system

T2 Length of a parametric (plane) curve

Suppose that [a space curve] has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle, a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t), y = g(t), z = h(t)$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Notice that both of the arc length formulas [here for space curves and in the MO T2 for plane curves] can be put into the more compact form

$$L = \int_a^b |\mathbf{r}'(t)|dt$$

p.902

Arc length function; parametrizing a curve with respect to arc length

Tags – T13 Derivative of a vector function

T12 Vector functions, space curve

T7 Standard basis vectors, magnitude of a vector

(T1 Parametrization of a curve)

[S]uppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad\qquad a \leq t \leq b$$

Where $\mathbf{r}'$ is continuous and C is traversed exactly once as t increases from a to b. We define its **arc length function** s by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \qquad (*)$$

Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. (See Figure 3.) If we differentiate both sides of [equation $(*)$] using Part 1 of the Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $\mathbf{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by [equation $(*)$], then we may be able to solve for t as a function of s: $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t: $\mathbf{r} = \mathbf{r}\big(t(s)\big)$.

p.903

Curvature; smooth parametrization; smooth curve

Tags – T13 Tangent vector, derivative of a vector function

T12 Parametric equations of a curve

T7 Magnitude of a vector, unit tangent vector

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I. A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

If C is a smooth curve defined by the vector function $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicates the direction of the curve. […]

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.) Because the unit tangent vector has constant length, only changes in the direction contribute to the rate of change of $\mathbf{T}$.

The **curvature** of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector.

The curvature is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule (Theorem 13.2.3, Formula 6) to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \text{and} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|$$

But $\frac{ds}{dt} = |\mathbf{r}'(t)|$ from [the Fundamental Theorem of Calculus applied to the arc length function], so

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

p.904

Theorem 1: Curvature of a curve using only its vector function

Tags – T13 Derivative of a vector function, second derivative of a vector function

T12 Vector functions, space curve

T9 Cross product

T7 Magnitude of a vector

The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

p.904

Corollary of Theorem 14.1: Curvature of a plane curve

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $r'(x) = \mathbf{i} + f'(x)\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by [Theorem 14.1],

$$\kappa(x) = \frac{|f''(x)|}{\left[1 + \left(f'(x)\right)^2\right]^{3/2}}$$

p.905

Normal and binomial vectors

Tags – T13 Tangent vector

T12 Vector function

T9 Cross product

T8 Orthogonal vectors, dot product

T7 Unit vector, magnitude of a vector

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t, we have $T(t) \cdot T'(t) = 0$ by Example 13.2.4 [if $|\mathbf{r}(t)|$ is constant, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t], so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that, typically, $\mathbf{T}'(t)$ is itself not a unit vector. But at any point where [the curvature] $\kappa \neq 0$ we can define the **principal unit normal vector** $\mathbf{N}(t)$ (or simply **unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector**. It is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ and is also a unit vector. (See Figure 6.)

p.906

Normal and osculating planes; osculating circle (or circle of curvature)

Tags - T13 Tangent vector

T10 Vectors which determine a plane containing a point

T8 Orthogonal vectors

The plane determined by the normal and binomial vectors $\mathbf{N}$ and $\mathbf{B}$ at a point P on a curve C is called the **normal plane** of C at P. It consists of all lines that are orthogonal to the tangent vector $\mathbf{T}$. The plane determined by the vectors T and N is called the **osculating plane** of C at P. [...] it is the plane that comes closest to containing the part of the curve near P. (For a plane curve, the osculating plane is simply the plane that contains the curve.)

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which N points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the **osculating circle** (or the **circle of curvature**) of C at P.

p.907

Π15 Motion in Space – Velocity and Acceleration

Velocity vector

Tags – T13 Tangent vector, tangent line

T12 Vector functions, space curves

T7 Difference of vectors

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. Notice from Figure 1 that, for small values of h, the vector

$$\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} \qquad (*)$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$. Its magnitude measures the size of the displacement vector per time. The vector $(*)$ gives the average velocity over a time interval of length h and its limit is the **velocity vector** $\mathbf{v}(t)$ at time t:

$$\mathbf{v}(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

Thus the velocity vector is also the tangent vector and points in the direction of the tangent line.

p.910

Speed

Tags – T13 Derivative of a vector function

T12 Vector functions, space curves

T7 Magnitude of a vector

The **speed** of [a] particle [moving through space] at time t is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$. This is appropriate because, from [the definition of the velocity vector] and from Equation 13.3.7 $\left[\frac{ds}{dt} = |\mathbf{r}'(t)|\right]$, we have

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt} = \text{rate of change of distance with respect to time}$$

p.910

Acceleration

Tags – T13 First and second derivatives of a vector function

T12 Vector functions, space curve

[T]he **acceleration** of [a] particle [moving through space] is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

p.911

Newton's Second Law of Motion

Tags – T12 Vector functions

[I]f, at a time t, a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

$$\mathbf{F}(t) = m\mathbf{a}(t)$$

p.912

Tangential and normal components of acceleration

Tags – T14 Curvature, tangent vector, normal vector, osculating plane of a space curve

T13 First and second derivatives of a vector function

T12 Vector function, space curve

T9 Cross product

T8 Dot product

T7 Vector addition, scalar multiplication, magnitude of a vector

When we study the motion of a particle, it is often useful to resolve the acceleration into two components, one in the direction of the tangent and the other in the direction of the normal. If we write $v = |\mathbf{v}|$ for the speed of the particle, then […]

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \boldsymbol{N}$$

[where $\mathbf{a}$ is the acceleration of the particle, $\mathbf{T}$ the unit tangent vector, κ the curvature, and $\mathbf{N}$ the unit normal vector.]

Writing a_T and a_N for the tangential and normal components of acceleration, we have

$$\mathbf{a} = a_T\mathbf{T} + a_N\boldsymbol{N} \quad (*)$$

where

$$a_T = v' \qquad \text{and} \qquad a_N = \kappa v^2$$

This resolution is illustrated in Figure 7.

Let's look at what formula $(*)$ says. The first thing to notice is that the binormal vector $\mathbf{B}$ is absent. No matter how an object moves through space, its acceleration always lies in the plane of $\mathbf{T}$ and $\mathbf{N}$ (the osculating plane). (Recall that $\mathbf{T}$ gives the direction of motion and $\mathbf{N}$ points in the direction the curve is turning.) Next we notice that the tangential component of acceleration is v', the rate of change of speed, and the normal component of acceleration is κv^2, the curvature times the square of the speed. […]

Although we have expressions for the tangential and normal components of acceleration [above], it's desirable to have expressions that depend only on $\mathbf{r}, \mathbf{r}'$, and $\mathbf{r}''$. To this end we take the dot product of $\mathbf{v} = v\mathbf{T}$ with $\mathbf{a}$ as given by Equation $(*)$: [...]

Therefore

$$a_T = v' = \frac{(\mathbf{v} \cdot \mathbf{a})}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

Using the formula for curvature given by Theorem 13.3.10, we have

$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} |\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

p.914-5

Π16 Functions of Several Variables

Function of two variables; domain; range

Tags – T6 Three-dimensional rectangular coordinate system

A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) | (x, y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

[...]

If a function f is given by a formula and no domain is specified, then the domain of f is understood to be the set of all pairs (x, y) for which the given expression is a well-defined real number.

p.928

Graph of a two-variable function

Tags – T6 Three-dimensional rectangular coordinate system, equation in R^3

If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in R^3 such that $z = f(x, y)$ and (x, y) is in D.

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane (see Figure 5).

p.930

Linear function of two variables

Tags - T6 Three-dimensional rectangular coordinate system

T10 Linear equation of a plane

[The] function

$$f(x, y) = ax + by + c$$

[...] is called a **linear function**. The graph of such a function has the equation

$$z = ax + by + c \text{ or } ax + by - z + c = 0$$

so it is a plane.

p.931

Level curves

(Tags – T11 Traces of a surface)

The **level curves** of a function f of two variables are the curves with equations $f(x,y)=k$, where k is a constant (in the range of f).

A level curve $f(x,y)=k$ is the set of all points in the domain of f at which f takes on a given value k. In other words, it shows where the graph of f has height k.

You can see from Figure 11 the relation between level curves and horizontal traces. The level curves $f(x,y)=k$ are just the traces of the graph of f in the horizontal plane $z=k$ projected down to the xy-plane.

p.933

Function of three variables; domain

A **function of three variables**, f, is a rule that assigns to each ordered triple (x,y,z) in a domain $D\subset R^3$ a unique real number denoted by $f(x,y,z)$.

p.937

Level surfaces of a three-variable function

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x,y,z)=k$, where k is a constant. If the point (x,y,z) moves along a level surface, the value of $f(x,y,z)$ remains fixed.

p.939

Function of n variables; domain

Tags – T8 Dot product

T7 n-dimensional vectors

A **function of n variables** is a rule that assigns a number $x=f(x_1,x_2,\dots,x_n)$ to an n-tuple $(x_1,x_2,\dots,x_n)$ of real numbers. We denote by R^n the set of all such n-tuples. […]

The function f is a real-valued function whose domain is a subset of R^n. Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x}=\langle x_1,x_2,\dots,x_n\rangle$, we often write $f(\mathbf{x})$ in place of $f\langle x_1,x_2,\dots,x_n\rangle$. With this notation we can rewrite the function defined in [an equation $C=f(x_1,x_2,\dots,x_n)=c_1x_1+c_2x_2+\cdots+c_nx_n$] as

$$f(\mathbf{x})=\mathbf{c}\cdot\mathbf{x}$$

where $\mathbf{c}=\langle c_1,c_2,\dots,c_n\rangle$ and $\mathbf{c}\cdot\mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

In view of the one-to-one correspondence between points $(x_1,x_2,\dots,x_n)$ in R^n and their position vectors $\mathbf{x}=\langle x_1,x_2,\dots,x_n\rangle$ in V_n, we have three ways of looking at a function f defined on a subset of R^n:

1. As a function of n real variables $x_1,x_2,\dots x_n$
2. As a function of a single point variable $(x_1,x_2,\dots,x_n)$
3. As a function of a single vector variable $\mathbf{x}=\langle x_1,x_2,\dots,x_n\rangle$

We will see that all three points of view are useful.

p.938

Π17 Limits and Continuity of Multivariable Functions

Limit of a two-variable function

Tags – T16 Function of two variables, domain of a two-variable function

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $\quad (x, y) \in D$ and $\quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad$ then $\quad |f(x, y) - L| < \varepsilon$

Other notations for the limit in [this definition] are

$$\lim_{\substack{x\to a\\ y\to b}} f(x, y) = L \quad \text{and} \quad f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

p.944

[This definition] says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0). The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction n of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b). Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

[To summarize:]

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

p.945

Limit laws

Tags – T16 Function of two variables

The Limit Laws listed in Section 1.6 can be extended to functions of two variables: the limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on. In particular, the following equations are true.

$$\lim_{(x,y)\to(a,b)} x = a \qquad \lim_{(x,y)\to(a,b)} y = b \qquad \lim_{(x,y)\to(a,b)} c = c$$

The Squeeze Theorem also holds.

p.947

Continuous

Tags – T16 Function of two variables

A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

[…]

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains.

p.948

Polynomial function of two variables; rational function of two variables; continuity of polynomial and rational functions on their domain

Tags – T16 Function of two variables, domain of a two-variable function

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. A **rational function** is a ratio of polynomials.

The limits in [Limit laws] show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous. Since any polynomial can be built up out of the simple functions f, g, and h by multiplication and addition, it follows that *all polynomials are continuous on* R^2. Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

p.948

Limits and continuity of functions of three or more variables

Tags – T16 Function of three or more variables

(T6 Distance between points in R^3)

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f. Because the distance between two points (x, y, z) and (a, b, c) in $R^3$is given by $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, we can write the precise definition as follows: for ever number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y, z) \text{ is in the domain of } f \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$$

$$\text{then } |f(x, y, z) - L| < \varepsilon$$

The function f is **continuous** at (a, b, c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c)$$

p.949

Vector notation for limits of multivariable functions

Tags – T16 Function of n variables

T7 n-dimensional vectors, magnitude of a vector

If f is defined on a subset D of R^n, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } \mathbf{x} \in D \text{ and } 0 < |\mathbf{x} - \mathbf{a}| < \delta \text{ then } |f(\mathbf{x}) - L| < \varepsilon$$

p.950

Π18 Partial Derivatives

Partial derivatives of a two-variable functions

Tags – T16 Functions of several variables

[I]f f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of f with respect fo x at (a, b)** and denote It by $f_x(a, b)$. Thus

$$f_x(a,b) = g'(a) \text{ where } g(x) = f(x,b) \qquad (*)$$

By the definition of the derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

and so Equation $(*)$ becomes

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Similarly, the **partial derivative of f with respect to y at (a, b)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed $(x = a)$ and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

p.953

In summary:

If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

p.954

Interpretation of partial derivatives

Tags – T16 Functions of several variables

T11 Traces of a surface

T6 Three-dimensional coordinate systems

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S. By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.). Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

Note that the curve $C_1$is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$. The curve $C_2$is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.

Thus the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a,b,c)$ to the traces C_1 and C_2 of S in the planes $y=b$ and $x=a$.

[…] [P]artial derivatives can also be interpreted as *rates of change*. If $z=f(x,y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed.

p.955

Partial derivatives of functions of three or more variables

Tags – T16 Functions of several variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is defined as

$$f_x(x,y,z)=\lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x,y,z)$ with respect to x. If $w=f(x,y,z)$, then $f_x=\partial w/\partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In u is a function of n variables, $u=f(x_1,x_2,\dots,x_n)$, its partial derivatives with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i}=\lim_{h\to 0}\frac{f(x_1,\dots,x_{i-1},x_i+h,x_{i+1},\dots,x_n)-f(x_1,\dots,x_i,\dots,x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i}=\frac{\partial f}{\partial x_i}=f_{x_i}=D_i f$$

p.957-8

Second partial derivatives

Tags – T16 Functions of several variables

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x, (f_x)_y, (f_y)_x, (f_y)_y$, which are called the **second partial derivatives** of f. If $z=f(x,y)$ we use the following notation:

$$(f_x)_x=f_{xx}=f_{11}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=\frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y=f_{xy}=f_{12}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 z}{\partial y\partial x}$$

$$(f_y)_x=f_{yx}=f_{21}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 z}{\partial x\partial y}$$

$$(f_y)_y=f_{yy}=f_{22}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 z}{\partial y}$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y\partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx}, the order is reversed.

p.958

Clairaut's Theorem

Tags – T17 Continuous multivariable functions

T16 Multivariable functions

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

p.959

Partial derivatives of order 3 or higher

Tags – T17 Continuous multivariable functions

T16 Multivariable functions

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

p.960

Partial differential equations

Partial derivatives occur in *partial differential equations* that express certain physical laws. For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** Pierre Laplace (1749-1827). Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

p.960

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Describes the motion of a waveform.

p.960

Π19 Tangent Planes and Linear Approximations

Definition of tangent plane

Tags – T18 Partial derivatives

T17 Continuity of a function

T16 Multivariable function

T13 Tangent line to a curve

(T11 Traces of a surface)

T6 Geometry of R^3

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on s. As in the preceding section, let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)

We will see in Section 14.6 that if C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane. Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

p.968

Equation of tangent plane

Tags – T18 Partial derivatives of two-variable functions as slopes

T17 Continuity of a function

T16 Multivariable function

T13 Tangent line to a curve

(T11 Traces of a surface)

T10 Equations of lines and planes

T6 Geometry of R^3

We know from Equation 12.5.7 that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$z - z_0 = a(x - x_0) + b(y - y_0) \quad (*)$$

If Equation $(*)$ represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in Equation $(*)$ gives

$$z - z_0 = a(x - x_0) \qquad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a. But from Section 14.3 we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$. Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation $(*)$, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2, so $b = f_y(x_0, y_0)$.

p.968

In summary:

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

p.968

Linearization; linear approximation (or tangent plane approximation)

Tags – T18 Partial derivatives

T16 Linear function of two variables and its graph, functions of two variables

The linear function whose graph is [the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$], namely

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is called the **linearization** of f at (a, b) and the approximation

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

p.970

For […] functions [of more than two variables] the **linear approximation** is

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

and the linearization $L(x, y, z)$ is the right side of this expression.

p.973

Differentiable; increment of a function

Tags – T18 Partial derivatives

T16 Multivariable functions

[C]onsider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b)$$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$. By analogy with [the definition of differentiability of a one-variable function $y = f(x)$ in terms of the increment of y if x changes from a to $a + \Delta x$] we define the differentiability of a function of two variables as follows.

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

p.970

A differentiable function [of more than two variables] is defined by an expression similar to the one in [the definition above].

p.973

If $w = f(x, y, z)$, then the **increment** of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x,y,z)$$

p.973

Theorem: Differentiability of a function

Tags – T18 Partial derivatives

T17 Continuity of a function

T16 Multivariable functions

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

p.971

Differential

Tags – T18 Partial derivatives

T16 Multivariable functions

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

(Compare with [the definition of the differential of $y = f(x)$: $dy = f'(x)dx$].) Sometimes the notation df is used in place of dz.

p.972

The **differential** dw [of a function of three variables] is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial z}{\partial z}dz$$

p.974

Linear approximation of a two-variable function in the notation of differentials

Tags – T18 Partial derivatives

T16 Multivariable functions

If we take $dx = \Delta x = x - a$ and $dy = \Delta y = y - b$ in [the definition of a differential of a function $z = f(x, y)$], then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

So, in the notation of differentials, the linear approximation [$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ of f at (a, b)] can be written as

$$f(x, y) \approx f(a, b) + dz$$

p.972

П20 Chain Rule

Chain rule (case 1)

Tags – T19 Differentiable function

T18 Partial derivative

T16 Multivariable function

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

p.978

Since we often write $\partial z/\partial x$ in place of $\partial f/\partial x$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

p.978

The proof is provided in the textbook (p.978).

Chain rule (case 2)

Tags – T19 Differentiable function

T18 Partial derivative

T16 Multivariable function

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

p.979

The proof is omitted in the textbook.

Chain rule (general version)

Tags – T19 Differentiable function

T18 Partial derivative

T16 Multivariable function

Suppose that u is a differentiable function of the n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \dots, t_m$. Then u is a function of $t_1, t_2, \dots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \dots, m$.

p.980

The proof is omitted in the textbook.

Differentiation of a single-variable function defined implicitly

Tags – T19 Differentiable function

T18 Partial derivative

T17 Continuity of a function

T16 Multivariable function

The Chain Rule can be used to give a more complete description of the process of implicit differentiation that was introduced in Sections 2.6 and 14.3 [T18 Partial derivatives]. We suppose that an equation of the form $F(x,y)=0$ defines y implicitly as a differentiable function of x, that is, $y=f(x)$, where $F(x,f(x))=0$ for all x in the domain of f. If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x,y)=0$ with respect to x. Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

But $dx/dx=1$, so if $\partial F/\partial y\neq 0$, we solve for dy/dx and obtain

$$\frac{dy}{dx}=\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=-\frac{F_x}{F_y} \qquad (*)$$

To derive this equation we assumed that $F(x,y)=0$ defines y implicitly as a function of x. The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid: it states that if F is defined on a disk containing (a,b), where $F(a,b)=0, F_y(a,b)\neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x,y)=0$ defines y as a function of x near the point (a,b) and the derivative of this function is given by Equation $(*)$.

p.982

Differentiation of a two-variable function defined implicitly

Tags – T19 Differentiable function

T18 Partial derivative

T17 Continuity of a function

T16 Multivariable function

[S]uppose that z is given implicitly as a function $z=f(x,y)$ by an equation of the form $F(x,y,z)=0$. This means that $F(x,y,f(x,y))=0$ for all (x,y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x,y,z)=0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

But

$$\frac{\partial}{\partial x}(x)=1 \qquad \text{and} \qquad \frac{\partial}{\partial x}(y)=0$$

so this equation becomes

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

If $\partial F/\partial x\neq 0$, we solve for $\partial z/\partial x$ and we obtain the first formula in [the following equations]. The formula for $\partial z/\partial y$ is obtained in a similar manner.

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid: if F is defined within a sphere containing (a,b,c), where $F(a,b,c)=0, F_z(a,b,c)\neq 0$, and F_x, F_y, and F_z

are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by [the equations above].

p.983

Π21 Directional Derivatives and the Gradient Vector

Directional derivative

Tags – T18 Partial derivatives, rate of change

T16 Multivariable function, graph of a two-variable function

T13 Tangent line to a curve

T10 Plane determined by a point and two vectors parallel to it

T7 Unit vector

T6 Three-dimensional coordinate systems

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors $\mathbf{i}$ and $\mathbf{j}$.

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.) To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of $\mathbf{u}$ intersects S in a curve C. (See Figure 3.) The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\mathbf{u}$.

If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\mathbf{u}$ and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h. Therefore $x - x_0 = ha, y - y_o = hb$, so $x = x_0 + ha, y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of u, which is called the directional derivative of f in the direction of $\mathbf{u}$.

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

By comparing [the definition of directional derivative] with [that of the partial derivatives f_x and f_y of a function$f(x, y)$], we see that if $\mathbf{u} = \mathbf{i} = \langle 1,0 \rangle$, then $D_{\mathbf{i}} f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0,1 \rangle$, then $D_{\mathbf{j}} f = f_y$. In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

p.986-7

For functions of three variables we can define directional derivatives in a similar manner. Again $D_{\mathbf{u}}f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector $\mathbf{u}$.

The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

If we use vector notation, then we can write both definitions [...] of the directional derivative in the compact form

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$. This is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + t\mathbf{u}$ (Equation 12.5.1 [in T10] and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

p.991

Theorem 1: Existence of the directional derivatives of a function, formula in terms of partial derivatives

Tags – T20 Chain rule

T19 Differentiable function

T18 Partial derivative

T16 Multivariable function

T7 Unit vector

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

p.988

The proof is provided in the textbook (p.988-9).

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in the theorem above becomes

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

p.989

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then the same method that was used to prove [the above theorem in the case of two-variable functions] can be used to show that

$$D_{\mathbf{u}}f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

p.991

Gradient vector

Tags – T19 Differentiable function

T18 Partial derivatives

T16 Multivariable function

T12 Vector function

T8 Dot product

T7 Vectors, standard basis vectors

Notice from [Theorem 1] that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\begin{aligned} D_{\mathbf{u}}f(x,y) &= f_x(x,y)a + f_y(x,y)b \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \mathbf{u} \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So we give it a special name (the *gradient*) of f and a special notation ($\mathbf{grad}\ f$ or ∇f, which is read "del f").

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

p.990

For a function f of three variables, the **gradient vector**, denoted by ∇f or $\mathbf{grad}\ f$, is

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$

or, for short,

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

p.991

Directional derivative in the notation of the gradient vector

Tags – T19 Differentiable function

T8 Dot product

With [the] notation for the gradient vector, we can rewrite [the equation

$D_{\mathbf{u}}f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle \cdot \mathbf{u}$] for the directional derivative of a differentiable function as

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

p.990

Similar notation for the directional derivative of functions of more than two variables (p.991).

Theorem 2: Maximizing the directional derivative

Tags - **T19 Differentiable function**

T8 Dot product

T7 Magnitude of a vector

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

p.992

The proof is based on the expression of the directional derivative in the notation of the gradient vector and uses the version of dot product that involves the angle between two vectors (p.992).

Tangent plane to a level surface at a point; normal line to a surface at a point

Tags – T20 Chain rule

T19 Differentiable function

T16 Function of three variables, level surface of a function of three variables

T13 Tangent vector to a curve at a point

T12 Vector functions and space curves

T10 Scalar equation of a plane, normal to a plane, symmetric equations of a line

T8 Dot product, dot product and orthogonality

Suppose S is a surface with equation $F(x,y,z)=k$, that is, it is a level surface of a function F of three variables, and let $P(x_0,y_0,z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through the point P. Recall from Section 13.1 [T12 Vector functions and space curves] that the curve C is described by a continuous vector function $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$. Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0)=\langle x(t),y(t),z(t)\rangle$. Since C lies on S, any point $(x(t),y(t),z(t))$ must satisfy the equation of S, that is,

$$F\big(x(t),y(t),z(t)\big)=k \qquad (*)$$

If x,y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation [$(*)$] as follows:

$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}=0 \qquad (**)$$

But, since $\nabla F=\langle F_x,F_y,F_z\rangle$ and $\mathbf{r}'(t)=\langle x'(t),y'(t),z'(t)\rangle$, Equation [$(**)$] can be written in terms of a dot product as

$$\nabla F\cdot\mathbf{r}'(t)=0$$

In particular, when $t=t_0$ we have $\mathbf{r}(t_0)=\langle x_0,y_0,z_0\rangle$

$$\nabla F(x_0,y_0,z_0)\cdot\mathbf{r}'(t_0)=0\,(***)$$

Equation [$(***)$] says that *the gradient vector at P, $\nabla F(x_0,y_0,z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P*. (See Figure 9.) If $\nabla F(x_0,y_0,z_0)\neq\mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x,y,z)=k$ **at** $P(x_0,y_0,z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0,y_0,z_0)$. Using the standard equation of a plane (Equation 12.5.7), we can write the equation of this tangent plane as

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, by Equation 12.5.3, its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

p.994

Π22 Maximum and Minimum Values

Local maximum, local maximum value; local minimum, local minimum value; absolute maximum, absolute minimum

Tags – T16 Function of two variables, domain of a two-variable function

A function of two variables has a **local maximum** at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \le f(a, b)$ for all points (x, y) in the disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \ge f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum *value***.

If the inequalities in [the above definition] fold for *all* points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

p.1000

Theorem 1: Partial derivatives at local maxima and minima

Tags – T18 Partial derivatives

T16 Function of two variables

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

p.1000

Critical point

Tags – T18 Partial derivatives

T16 Function of two variables

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. [Theorem 1] says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f. However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

p.1000

Second derivatives test

Tags – T18 First and second partial derivatives

T17 Continuous multivariable functions

T16 Function of two variables

((T9 Determinant of order 2))

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Note 1 In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

Note 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

Note 3 To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

p.1001

Closed set

Just as a closed interval contains its endpoints, a **closed set** in R^2 is one that contains all its boundary points. [A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.]

p.1005

Bounded set

A **bounded set** in R^2 is one that is contained within some disk.

p.1005

Extreme value theorem for functions of two variables

Tags – T17 Continuous multivariable functions

T16 Function of two variables

If f is continuous on a closed, bounded set D in R^2, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

p.1005

Π23 Lagrange Multipliers

The geometric basis of Lagrange's method for functions of two variables; Lagrange multiplier

Tags – T22 Extreme values

T21 Normal line, gradient vector

T20 Chain rule

T19 Differentiable

T16 Multivariable functions, level curve of a two-variable function, level surface of a three-variable function

T13 Tangent vector

T12 Vector functions, space curves

T8 Orthogonal

T7 Parallel vectors/scalar multiple of a vector, zero vector

[W]e start by trying to find the extreme values of $f(x,y)$ subject to a constraint of the form $g(x,y)=k$. In other words, we seek the extreme values of $f(x,y)$ when the point (x,y) is restricted to lie on the level curve $g(x,y)=k$. Figure 1 shows this curve together with several level curves of f. These have the equations $f(x,y)=c$, where $c=7,8,9,10,11$. To maximize $f(x,y)$ subject to $g(x,y)=k$ is to find the largest value of c such that the level curve $f(x,y)=c$ intersects $g(x,y)=k$. It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.) This means that the normal lines at the point (x_0,y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0,y_0)=\lambda\nabla g(x_0,y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$. Thus the point (x,y,z) is restricted to lie on the level surface S with equation $g(x,y,z)=k$. Instead of the level curves in Figure 1, we consider the level surfaces $f(x,y,z)=c$ and argue that if he maximum value of f is $f(x_0,y_0,z_0)=c$, then the level surface $f(x,y,z)=c$ is tangent to the level surface $g(x,y,z)=k$ and so the corresponding gradient vectors are parallel.

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0,y_0,z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$ that lies on S and passes through P. If t_0 is the parameter value corresponding to the point P, then $\mathbf{r}(t_0)=\langle x_0,y_0,z_0\rangle$. The composite function $h(t)=f(x(t),y(t),z(t))$ represents the values that f takes on the curve C. Since f has an extreme value at (x_0,y_0,z_0), it follows that h has an extreme value at t_0, so $h'(t_0)=0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0,y_0,z_0)x'(t_0)+f_y(x_0,y_0,z_0)y'(t_0)+f_z(x_0,y_0,z_0)z'(t_0) \\ &= \nabla f(x_0,y_0,z_0)\cdot\mathbf{r}'(t_0) \end{aligned}$$

This shows that the gradient vector $\nabla f(x_0,y_0,z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C. But we already know from Section 14.6 [T21 Directional derivatives and the gradient vector] that the gradient vector of $g, \nabla g(x_0,y_0,z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve. (See Equation 14.6.18.) this means that the gradient vectors $\nabla f(x_0,y_0,z_0)$ and $\nabla \mathrm{g}(x_0,y_0,z_0)$ must be parallel. Therefore, if $\nabla g(x_0,y_0,z_0)\neq\mathbf{0}$, there is a number λ such that

$$\nabla f(x_0,y_0,z_0)=\lambda\nabla \mathrm{g}(x_0,y_0,z_0)$$

The number λ in [this equation] is called a **Lagrange multiplier**.

p.1011-1012

Method of Lagrange Multipliers (one constraint)

Tags - T22 Maximum and minimum values

T21 Gradient vector

T16 Three-variable functions, level surface of a three-variable function

T7 Zero vector, scalar multiplication of a vector

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

p.1012

Two constraints

Tags - T22 Maximum and minimum values

T21 Gradient vector

T16 Three-variable functions, level surface of a three-variable function

T10 Plane determined by two vectors

T8 Orthogonal

T7 Zero vector, sum and scalar multiplication of vectors

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.) Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to C at P. But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C. This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are not zero and not parallel.) So there are numbers λ and μ (called Lagrange multipliers) such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0,) + \mu \nabla h(x_0, y_0, z_0)$$

p.1016

Method of Lagrange Multipliers (two constraints)

Tags - T22 Maximum and minimum values

T21 Gradient vector

T18 Partial derivatives

T16 Three-variable functions, level surface of a three-variable function

T7 Scalar multiplication of a vector

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ, and μ. These equations are obtained by writing [the equation $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0,) + \mu \nabla h(x_0, y_0, z_0)$] in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

p.1016

Π24 Taylor Series

Theorem 1: Power series expansion of a function

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

p.800

Taylor series; Maclaurin series

Substituting [the] formula for c_n [in Theorem 1] back into the series, we see that *if* f has a power series expansion at a, then it must be of the following form.

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

The series in [this equation] is called the **Taylor series of the function f at a** (or **about a** or **centered at a**). For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}x^2 + \cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

p.800

nth-degree Taylor polynomial

[T]he partial sums [of the Taylor series of a function f at a] are

$$\begin{aligned} T_n(x) &= \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Notice that T_n is a polynomial of degree n called the **nth-degree Taylor polynomial of f at a.**

p.801

Remainder of a Taylor series

In general, $f(x)$ is the sum of its Taylor series if

$$\lim_{n \to \infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}[f(x) - R_n(x) = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

We have therefore proved the following theorem [Theorem 2].

p.801

Theorem 2

If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

p.801

Taylor's inequality

If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \leq d$$

p.802

Binomial series

[T]he Maclaurin series of $f(x) = (a+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n$$

This series is called the **binomial series**.

[…]

The traditional notation for the coefficients in the binomial series is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

and these numbers are called the **binomial coefficients**.

p.806

In summary:

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Although the binomial series always converges when $|x| < 1$, the question of whether or not it converges at the endpoints, ± 1, depends on the value of k. It turns out that the series converges at 1 if $-1 < k \leq 0$

and at both endpoints if $k \geq 0$. Notice that if k is a positive integer and $n > k$, then the expression for $\binom{k}{n}$ contains a factor $(k - k)$, so $\binom{k}{n} = 0$ for $n > k$. This means that the series terminates and reduces to the ordinary Binomial Theorem when k is a positive integer.

p.807