Multivariate Robust Vector-Valued Range Value-at-Risk

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Abstract

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In a multivariate setting, the dependence between random variables has to be accounted for modeling purposes. Various of multivariate risk measures have been developed, including bivariate lower and upper orthant Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR). The robustness of their estimators has to be discussed with the help of sensitivity functions, since risk measures are estimated from data.

In this thesis, several univariate risk measures and their multivariate extensions are presented. In particular, we are interested in developing the bivariate version of a robust risk measure called Range Value-at-Risk (RVaR). Examples with different copulas, such as the Archimedean copula, are provided. Also, properties such as translation invariance, positive homogeneity and monotonicity are examined. Consistent empirical estimators are also presented along with the simulation. Moreover, the sensitivity functions of the bivariate VaR, TVaR and RVaR are obtained, which confirms the robustness of bivariate VaR and RVaR as expected.

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CONTENTS

LIST OF FIGURES

LIST OF TABLES

1. INTRODUCTION

Volatility and risks of financial markets have recently increased significantly with the globalization of economy and financial innovation. For companies, risk management is crucial to their success. Entities are interested in risk measures in order to allocate capital and maintain solvency.

Different univariate risk measures have been proposed in the literature. Consider a random loss variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with its cumulative distribution function (cdf) F_X. The term Value-at-Risk (VaR) found its way through the G-30 report published in July 1993 see [14] for details. It is the loss in market value that can only be exceeded with a probability of at most $1-\alpha$ where α often takes value 0.95 or 0.99. However, *Artzner et al.* (1999) show that VaR is not a coherent risk measure and it does not provide any information about the tail of the distribution, suggesting two specific risk measures called Tail Conditional Expectation (TCE) and Worst Conditional Expectation (WCE). TCE evaluates the average value of VaR over all confidence levels greater than α while WCE is the expected loss under the condition that the set of worst events occurs. An alternative risk measure called Conditional Value-at-Risk (CVaR) is used to optimize portfolios by *Rockafellar and Uryasev* (2000). Another remedy for the deficiencies of VaR is Expected Shortfall (ES) proposed by *Acerbi and Tasche* (2002). WCE is closely related to TCE, but in general does not coincide with it. For discrete random variables, the WCE could be greater than the TCE. WCE is only useful in a theoretical setting since it requires the knowledge of the whole underlying probability space while TCE is easy to compute but not coherent for discrete random variables. ES and CVaR are two different interpretations of the weighted average between the VaR and losses exceeding VaR. ES is more precise than CVaR and easier to compute, since it considers the effect of jump points. Specially, ES is continuous and monotonic with respect to the significant level α. Although all these measures have been widely studied, univariate risk measures are not enough for current financial markets since financial risks are strongly interconnected and cannot only be managed individually or by aggregation.

In reality, companies have to consider the dependence between risks so that they can get the accurate capital allocation, and systemic and global risk evaluation. Systemic risk refers to the risks imposed by interdependencies in a system. Univariate risk measures are unable to be used for heterogeneous classes of homogeneous risks. Therefore, multivariate risk measures have been developed in the last decade. An extension of the Worst Conditional Expectation (WCE) in *Artzner et al.* (1999) is called the Multivariate Worst Conditional Expectation (MWCE), which is proposed by *Jouini et al.* (2004). In the same framework, *Bentahar* (2006) introduces a quantile-based risk measure called vector-valued Tail Conditional Expectation (TCE). Furthermore, *Tahar and Lépinette* (2012, 2014) propose the Generalized Worst Conditional Expectation (GWCE).

Embrechts and Puccetti (2006), *Nappo and Spizzichino* (2009) and *Prékopa* (2012) use the notion of quantile curves to define a multivariate risk measure called upper and lower orthant VaR. Based on the same idea, *Cossette et al.* (2013) redefine the upper and lower orthant VaR and propose the upper and lower orthant TVaR in *Cossette et al.* (2015). At the same time, *Cousin and Di Bernardino* (2013) develop a vectorized version of the upper and lower orthant VaR. Moreover, multivariate extensions of CTE and CoVaR are developed in *Cousin and Di Bernardino* (2014, 2015). CoVaR represents VaR for a financial institution, conditional on the boundary of the α-level set, and measures a financial institution's contribution to the system's risk. A drawback of multivariate VaR is that it represents the boundary of the α-level set and no information above is provided, similarly to the univariate VaR. Furthermore, relationships holding for univariate risk measures can be totally different in a multivariate setting. Thus, in this thesis, we will compare and summarize the relationships between these multivariate risk measures.

Most risk measures are defined as functions of the loss distribution which should be estimated from data in applications. *Cont et al.* (2010) define risk measurement procedure and analyze the robustness of different risk measures. They point out the conflict between the subadditivity and robustness and propose a robust risk measure called weighted VaR (WVaR). *Bignozzi and Tsanakas* (2016) also suggest to use the truncated version of TVaR (or Range-Value-at-Risk (RVaR)) which is same as WVaR when the mean of the loss distribution is infinite. We will develop the multivariate RVaR in this thesis, in order to provide a new robust multivariate risk measure.

2. UNIVARIATE RISK MEASURES

2.1 Preliminaries

A risk measure $\rho(X)$ for a univariate risk X corresponds to the required assets that have to be maintained such that the financial position $\rho(X) - X$ is acceptable for regulators. Since there are several ways to define risk measures, an appropriate choice becomes crucial for both regulators and entrepreneurs. Properties of coherent risk measures proposed by *Artzner et al.* (1999) can be significant criteria.

Definition 2.1.1. *For random variables X and Y, a risk measure ρ is a coherent risk measure if it satisfies the following four axioms,*

1. *(Translation invariance) For all $c \in \mathbb{R}$, $\rho(X + c) = \rho(X) + c$.*
2. *(Positive homogeneity) If $c \geq 0$, then $\rho(cX) = c\rho(X)$.*
3. *(Monotonicity) For $X \leq Y$, then $\rho(X) \leq \rho(Y)$.*
4. *(Subadditivity) $\rho(X + Y) \leq \rho(X) + \rho(Y)$.*

The interpretations of these axioms have been well documented in the literature (see, e.g., [2] for details). Translation invariance indicates that the addition of a certain amount of losses increases the risk measure by the same amount. Positive homogeneity indicates that the risk measure is proportional to the size of risk. For example, if we measure the losses in different currency units, the results will follow the same scale. Monotonicity indicates that the portfolio which always has higher losses should have a higher risk measure. Subadditivity indicates that the risk can be diversified by combining portfolios. Risk measures satisfying all these axioms are more reasonable and acceptable.

Moreover, since risk measures are estimated from historical data in practice, the robustness of their estimators is a relevant question. Robust statistics can be defined as statistics that are not unduly affected by outliers. To clarify this definition, consider a sample generated from a log-normal distribution which is heavy-tailed. The occurrence of huge losses will significantly shift up the sample average. Using sample average as the estimator of mean would lead to large mean squared error (MSE). Therefore, it is not a robust statistic.

Now, consider a continuous random variable X with cdf $F \in \mathbb{D}$ where $\mathbb{D}$ is the convex set of cdfs. Notice that a risk measure is distribution-based if $\rho(X_1) = \rho(X_2)$ when $F_{X_1} = F_{X_2}$. Hence, we use $\rho(F) \triangleq \rho(X)$ to represent the distribution-based risk measures. To quantify the sensitivity of a risk measure to the change in the distribution, the sensitivity function is used in this thesis. This method is used in *Cont et al.* (2010) and can be explained as the one-sided directional derivative of the effective risk measure at F in the direction δ_z.

Definition 2.1.2. *Consider ρ, a distribution-based risk measure of a continuous random variable X with distribution function $F \in \mathbb{D}$. For $\varepsilon \in [0,1)$, set $F_\varepsilon = \varepsilon\delta_z + (1-\varepsilon)F$ such that $F_\varepsilon \in \mathbb{D}$. $\delta_z \in \mathbb{D}$ is the probability measure which gives mass 1 to $\{z\}$. The distribution F_ε is differentiable at any $x \neq z$ and has a jump point at the point $x = z$. The sensitivity function is defined by*

$$S(z) = S(z; F) \triangleq \lim_{\varepsilon \to 0^+} \frac{\rho(F_\varepsilon) - \rho(F)}{\varepsilon},$$

for any $z \in \mathbb{R}$ such that the limit exist.

Furthermore, for a robust statistic, the value of sensitivity function will not go to infinity when z becomes arbitrarily large. In other word, the bounded sensitivity function makes sure that the risk measure will not blow up when a small change happens.

2.2 Definitions and Properties

Value-at-Risk (VaR) introduced in the G-30 report published in July 1993 provides the lower bound which covers the $100\alpha\%$ of the possible losses. In other words, it gives us the probable maximum loss under a given significance level.

Definition 2.2.1. *For a random variable X with cumulative distribution function (cdf) F_X, the Value-at-Risk at significance level $\alpha \in (0,1)$ is given by*

$$VaR_\alpha(X) = \inf\{x \in \mathbb{R} : F_X(x) \geq \alpha\}.$$

Note that for a continuous random variable X with strictly increasing cdf, $VaR_\alpha(X) = F_X^{-1}(\alpha)$, is also called the α-quantile, where F_X^{-1} is the inverse function of cdf.

It is well known that VaR is not a coherent risk measure since it is not subadditive. Moreover, VaR fails to give any information beyond the level α. Therefore, risk measures which quantify the magnitude of loss of the worst $100(1-\alpha)\%$ cases are developed, such as the Tail Conditional Expectation (TCE), the Worst Conditional Expectation (WCE), the Conditional Value-at-Risk (CVaR) and the Expected Shortfall (ES).

Definition 2.2.2. *For a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with cdf F_X and $VaR_\alpha(X)$ defined in Definition 2.2.1, the Tail Conditional Expectation at significance level*

$\alpha \in [0,1]$ *is given by*

$$TCE_\alpha(X) = E\left[X | X \geq VaR_\alpha(X)\right].$$

If X *is a continuous random variable,* $TCE_\alpha(X)$ *could be wrote into the following form,*

$$TCE_\alpha(X) = TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(X)du.$$

TCE proposed by *Artzner et al.* (1999) measures the average loss given that the loss is no less than the $100\alpha\%$ of all possible cases. However, like VaR, it is not coherent since the subadditivity can only be satisfied when the random variable is continuous. To figure out this problem, the Worst Conditional Expectation is proposed in the same article.

Definition 2.2.3. *For a random variable* X *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with cdf* F_X*, the Worst Conditional Expectation at significance level* $\alpha \in [0,1]$ *is defined by*

$$WCE_\alpha(X) = \sup\left\{E[X|A] | P(A) \geq 1-\alpha, A \in \mathcal{F}\right\}.$$

WCE is the maximum expected loss of at least $100(1-\alpha)\%$ cases. Hence, it depends not only on the distribution of X but also on the structure of the underlying probability space. Thus, it seems hopeless to compute the value of it in practice, when the probability space is infinite. Therefore, to find a coherent risk measure which is computable, *Rockafellar and Uryasev* (2000) propose the CVaR.

Definition 2.2.4. *For a random variable* X *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with cdf* F_X*, the Conditional Value-at-Risk at significance level* $\alpha \in [0,1)$ *is defined by*

$$CVaR_\alpha(X) = \inf\left\{a + \frac{E[X-a]^+}{1-\alpha} : a \in \mathbb{R}\right\}.$$

Note the number a is selected to minimize the value of CVaR. CVaR is defined without VaR and it is subadditive, which makes this measure used to optimize portfolios. In addition, ES is proposed by *Acerbi and Tasche* (2002), making some modifications on the definition of TCE such that ES is subadditive when the distribution is discrete.

Definition 2.2.5. *For a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with cdf F_X and $VaR_\alpha(X)$ defined as in Definition 2.2.1, the Expected Shortfall at significance level $\alpha \in [0,1)$ is defined by*

$$ES_\alpha(X) = \frac{1}{1-\alpha}\left\{E[X1_{\{X \geq VaR_\alpha(X)\}}] + VaR_\alpha(X)[1-\alpha - P(X \geq VaR_\alpha(X))]\right\}.$$

Acerbi and Tasche (2002) show that the ES is a coherent risk measure (see, e.g., [1] for details) and discuss the relationships between the TCE, WCE, CVaR and ES, considering that all of these risk measures are used to evaluate the same thing i.e. the expected losses of the worst $100(1-\alpha)\%$ cases. The differences in definitions lead to the differences in the numerical results (see, section 2.4 for details) and properties.

Proposition 2.2.1. *For a discrete random variable X, $WCE_\alpha(X)$, $TCE_\alpha(X)$, $CVaR_\alpha(X)$ and $ES_\alpha(X)$ have the following relationships,*

$$TCE_\alpha(X) \leq WCE_\alpha(X) \leq ES_\alpha(X) \leq CVaR_\alpha(X).$$

Proposition 2.2.2. *For a continuous random variable X, $WCE_\alpha(X)$, $TCE_\alpha(X)$, $CVaR_\alpha(X)$ and $ES_\alpha(X)$ have the following relationships,*

$$TCE_\alpha(X) = WCE_\alpha(X) = ES_\alpha(X) = CVaR_\alpha(X).$$

Note that Proposition 2.2.1 and Proposition 2.2.2 are proved by *Acerbi and Tasche* (2002) (see, e.g., [1] for details).

In next section, we will discuss the robustness of the risk measures presented so far.

2.3 Robustness

Proposition 2.3.1. *For a continuous random variable X with cdf F, the sensitivity function of $VaR_\alpha(X)$ is given by*

$$S(z) = \begin{cases} -\frac{1-\alpha}{f(VaR_\alpha(F))}, & z < VaR_\alpha(X) \\ \frac{\alpha}{f(VaR_\alpha(F))}, & z > VaR_\alpha(X) \\ 0, & z = VaR_\alpha(X) \end{cases}$$

which is a bounded function. Thus, VaR is a robust risk measure.

Note, $VaR_\alpha(F) \triangleq VaR_\alpha(X)$ since VaR is a distribution-based risk measure. The way to prove the Proposition 2.3.1 is presented by *Cont et al.* (2010). The basic idea is to measure the effect of the small change at a point on the risk measures using the sensitivity function. A bounded sensitivity function can be obtained for the robust risk measure.

Proof. Fix $z \in \mathbb{R}$ and set $F_\varepsilon = \varepsilon\delta_z + (1-\varepsilon)F$ such that $F_0 \equiv F$, where $F \in \mathbb{D}$ and the direction of change $\delta_z \in \mathbb{D}$. The distribution F_ε is differentiable at any $x \neq z$ with $F_\varepsilon^{'}(x) = (1-\varepsilon)f(x) > 0$ and has a jump (of size $\varepsilon \in [0,1)$) at the point $x = z$. Hence,

$$VaR_\alpha(F_\varepsilon) = F_\varepsilon^{-1}(\alpha) = \begin{cases} F^{-1}\left(\frac{\alpha}{1-\varepsilon}\right), & \alpha < (1-\varepsilon)F(z) \\ F^{-1}\left(\frac{\alpha-\varepsilon}{1-\varepsilon}\right), & \alpha \geq (1-\varepsilon)F(z) + \varepsilon \\ \text{z}, & \text{otherwise.} \end{cases}$$

Thus, the sensitivity function of $VaR_\alpha(X)$ can be evaluated by

$$
\begin{aligned}
S(z) &= \lim_{\varepsilon \to 0^+} \frac{VaR_\alpha(F_\varepsilon) - VaR_\alpha(F)}{\varepsilon} = \left[\frac{d}{d\varepsilon} VaR_\alpha(F_\varepsilon)\right]_{\varepsilon=0} \\
&= \begin{cases} -\frac{1-\alpha}{f(VaR_\alpha(F))}, & z < VaR_\alpha(X) \\ \frac{\alpha}{f(VaR_\alpha(F))}, & z > VaR_\alpha(X) \\ 0, & z = VaR_\alpha(X). \end{cases}
\end{aligned}
$$

Note that the sensitivity function of $VaR_\alpha(X)$ is bounded by two horizontal lines and has a jump point at $z = VaR_\alpha(X)$. Consider a dataset with $VaR_\alpha(X)$. Then, adding a value smaller than $VaR_\alpha(X)$ would decrease $VaR_\alpha(X)$ and vice versa.

□

Proposition 2.3.2. *For a continuous random variable X, let $\rho(X) = TCE_\alpha(X) = WCE_\alpha(X) = CVaR_\alpha(X) = ES_\alpha(X)$. Then, the sensitivity function of $\rho(X)$ is given by*

$$
S(z) = \begin{cases} VaR_\alpha(X) - \rho(X), & z < VaR_\alpha(X) \\ \frac{z - \alpha VaR_\alpha(X)}{1-\alpha} - \rho(X), & z \geq VaR_\alpha(X). \end{cases}
$$

Note, $TCE_\alpha(X)$, $WCE_\alpha(X)$, $CVaR_\alpha(X)$ and $ES_\alpha(X)$ have unbounded sensitivity functions, which means they are not robust.

Proof. Let $\rho(X) = TCE_\alpha(X) = WCE_\alpha(X) = ES_\alpha(X) = CVaR_\alpha(X)$, then the sensitivity function of $\rho(X)$ is given by

$$S(z) = \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{1-\alpha} \int_\alpha^1 \frac{VaR_u(F_\varepsilon) - VaR_u(F)}{\varepsilon} du \right\}$$

$$= \frac{1}{1-\alpha} \int_\alpha^1 \lim_{\varepsilon \to 0^+} \left\{ \frac{VaR_u(F_\varepsilon) - VaR_u(F)}{\varepsilon} \right\} du$$

$$= \frac{1}{1-\alpha} \int_\alpha^1 \left[\frac{d}{d\varepsilon} VaR_u(F_\varepsilon) \right]_{\varepsilon=0} du$$

$$= \begin{cases} \frac{1}{1-\alpha} \int_\alpha^1 \left[\frac{d}{d\varepsilon} F^{-1}\left(\frac{u-\varepsilon}{1-\varepsilon}\right) \right]_{\varepsilon=0} du, & F(z) < \alpha \\ \frac{1}{1-\alpha} \left\{ \int_\alpha^{F(z)} \left[\frac{d}{d\varepsilon} F^{-1}\left(\frac{u}{1-\varepsilon}\right) \right]_{\varepsilon=0} du + \int_{F(z)}^1 \left[\frac{d}{d\varepsilon} F^{-1}\left(\frac{u-\varepsilon}{1-\varepsilon}\right) \right]_{\varepsilon=0} du \right\}, & F(z) \geq \alpha \end{cases}$$

$$= \begin{cases} VaR_\alpha(X) - \rho(X), & z < VaR_\alpha(X) \\ \frac{z - \alpha VaR_\alpha(X)}{1-\alpha} - \rho(X), & z \geq VaR_\alpha(X). \end{cases}$$

Note that the sensitivity function of $\rho(X)$ is a linear function of z. When the jump happens on the right tail of the distribution, it goes to infinity. Therefore, $TCE_\alpha(X)$, $WCE_\alpha(X)$, $ES_\alpha(X)$ and $CVaR_\alpha(X)$ are not robust. □

Since risk measures providing information on tails of distribution are not robust, *Cont et al.* (2010) present the Range Value-at-Risk.

Definition 2.3.1. *For a continuous random variable X with cdf F_X, the univariate Range*

Value-at-Risk at level range $[\alpha_1, \alpha_2] \subseteq [0,1]$ *is defined by*

$$RVaR_{\alpha_1,\alpha_2}(X) = E\left[X | VaR_{\alpha_1}(X) \le X \le VaR_{\alpha_2}(X)\right] = \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} VaR_u(X)du.$$

Note that $RVaR_{\alpha_1,\alpha_2}(X)$ is not a coherent risk measure since it does not satisfy the subadditivity. For $\alpha_2 < 1$, $RVaR_{\alpha_1,\alpha_2}(X)$ is always well bounded,

$$\begin{aligned} RVaR_{\alpha_1,\alpha_2}(X) &= \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} VaR_u(X)du \\ &\le \frac{1}{\alpha_2 - \alpha_1}\int_{\alpha_1}^{\alpha_2} VaR_{\alpha_2}(X)du = VaR_{\alpha_2}(X). \end{aligned}$$

Moreover, the robustness of RVaR can be proved in the same way as the one we used before.

Proposition 2.3.3. *For a continuous random variable* X *with cdf* F*, RVaR is not sensitive to the small change at any point* x*. In other word, its sensitivity function is bounded and given by*

$$S(z) = \begin{cases} \frac{(1-\alpha_1)VaR_{\alpha_1}(X) - (1-\alpha_2)VaR_{\alpha_2}(X)}{\alpha_2 - \alpha_1} - RVaR_{\alpha_1,\alpha_2}(X), & z < VaR_{\alpha_1}(X) \\ \frac{z - \alpha_1 VaR_{\alpha_1}(X) - (1-\alpha_2)VaR_{\alpha_2}(X)}{\alpha_2 - \alpha_1} - RVaR_{\alpha_1,\alpha_2}(X), & VaR_{\alpha_1}(X) \le z \le VaR_{\alpha_2}(X) \\ \frac{\alpha_2 VaR_{\alpha_2}(X) - \alpha_1 VaR_{\alpha_1}(X)}{\alpha_2 - \alpha_1} - RVaR_{\alpha_1,\alpha_2}(X), & z > VaR_{\alpha_2}(X) \end{cases}$$

which means RVaR is a robust risk measure.

Proof. The sensitivity function of $RVaR_{\alpha_1,\alpha_2}(X)$ can be obtained as follows.

$$S(z) = \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \frac{VaR_u(F_\varepsilon) - VaR_u(F)}{\varepsilon} du \right\}$$

$$= \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \lim_{\varepsilon \to 0^+} \left\{ \frac{VaR_u(F_\varepsilon) - VaR_u(F)}{\varepsilon} \right\} du$$

$$= \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \left[\frac{d}{d\varepsilon} VaR_u(F_\varepsilon) \right]_{\varepsilon=0} du$$

$$= \begin{cases} \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \left[\frac{d}{d\varepsilon} F^{-1}\left(\frac{u-\varepsilon}{1-\varepsilon}\right)\right]_{\varepsilon=0} du, & F(z) < \alpha_1 \\ \frac{1}{\alpha_2-\alpha_1} \left\{ \int_{\alpha_1}^{F(z)} \left[\frac{d}{d\varepsilon} F^{-1}\left(\frac{u}{1-\varepsilon}\right)\right]_{\varepsilon=0} du + \int_{F(z)}^{\alpha_2} \left[\frac{d}{d\varepsilon} F^{-1}\left(\frac{u-\varepsilon}{1-\varepsilon}\right)\right]_{\varepsilon=0} du \right\}, & \alpha_1 \le F(z) \le \alpha_2 \\ \frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \left[\frac{d}{d\varepsilon} F^{-1}\left(\frac{u}{1-\varepsilon}\right)\right]_{\varepsilon=0} du, & F(z) > \alpha_2 \end{cases}$$

$$= \begin{cases} \frac{(1-\alpha_1)VaR_{\alpha_1}(X) - (1-\alpha_2)VaR_{\alpha_2}(X)}{\alpha_2-\alpha_1} - RVaR_{\alpha_1,\alpha_2}(X), & z < VaR_{\alpha_1}(X) \\ \frac{z - \alpha_1 VaR_{\alpha_1}(X) - (1-\alpha_2)VaR_{\alpha_2}(X)}{\alpha_2-\alpha_1} - RVaR_{\alpha_1,\alpha_2}(X), & VaR_{\alpha_1}(X) \le z \le VaR_{\alpha_2}(X) \\ \frac{\alpha_2 VaR_{\alpha_2}(X) - \alpha_1 VaR_{\alpha_1}(X)}{\alpha_2-\alpha_1} - RVaR_{\alpha_1,\alpha_2}(X), & z > VaR_{\alpha_2}(X). \end{cases}$$

□

This result shows that the sensitivity function of $RVaR_{\alpha_1,\alpha_2}(X)$ is linear in z over the interval $[VaR_{\alpha_1}(X), VaR_{\alpha_2}(X)]$ and constant over other intervals. It is bounded, which means RVaR is a robust risk measure.

2.4 Numerical Examples

In this section, the examples for discrete and continuous random variables are presented, respectively. The results illustrate relationships between the risk measures we discussed in the previous sections.

Example 2.4.1. *Let X, which is the loss of a policy, be a discrete random variable with the probability distribution as list in the following Table.*

Tab. 2.1: Probability distribution of the discrete variable X

Ω	ω_1	ω_2	ω_3	ω_4	ω_5
X	0	$\frac{1}{3}$	1	$\frac{3}{2}$	3
$P(X = x_i)$	20%	30%	10%	30%	10%

Let $\alpha = 80\%$, then

$$VaR_{0.8}(X) = \frac{3}{2} = 1.5$$

can make sure that 80% of losses can be covered. Then,

$$TCE_{0.8}(X) = E[X|X \geq VaR_{0.8}] = \frac{\frac{3}{2} * 0.3 + 3 * 0.1}{0.3 + 0.1} = \frac{7.5}{4} = 1.875$$

can be evaluated based on the value of $VaR_{0.8}(X)$.

$WCE_{0.8}(X)$ can be calculated by maximizing the expectation of the events set which happens with probability larger than 0.2. Therefore, the events set $A = \{\omega_3, \omega_5\}$ will be selected after comparing the results of different combinations. Then,

$$WCE_{0.8}(X) = \frac{1 * 0.1 + 3 * 0.1}{0.1 + 0.1} = 2.$$

$CVaR_\alpha(X) = \inf\left\{a + \frac{E[X-a]^+}{1-\alpha} : a \in \mathbb{R}\right\}$, in this case we choose $a = VaR_{0.8}(X) = \frac{3}{2}$, then

$$CVaR_{0.8}(X) = \frac{3}{2} + \frac{(3-\frac{3}{2})*0.1}{1-0.8} = \frac{9}{4} = 2.25$$

is the minima for any possible value of a.

$ES_\alpha(X) = \frac{1}{1-\alpha}\left\{E[X1_{\{X \geq VaR_\alpha(X)\}}] + VaR_\alpha(X)[1-\alpha-P(X \geq VaR_\alpha(X))]\right\}$, therefore

$$ES_{0.8}(X) = \frac{1}{1-0.8}\left\{\frac{3}{2}*0.3 + 3*0.1 + \frac{3}{2}*(1-0.8-0.4)\right\} = 2.25.$$

The results show that $TCE_\alpha(X) \leq WCE_\alpha(X) \leq ES_\alpha(X) \leq CVaR_\alpha(X)$, which coincides with the Proposition 2.2.1.

Example 2.4.2. *Consider a continuous random variable X with cdf $F(x)$.*

According to the Definition 2.2.2,

$$TCE_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(X)du$$

Furthermore, let $A = \{X \geq VaR_\alpha(X)\}$ such that $WCE_\alpha(X)$ can be maximized. Then,

$$\begin{aligned} WCE_\alpha(X) &= \sup\{E[X|A]|P(A) \geq 1-\alpha, A \in \mathcal{F}\} \\ &= E[X|X \geq VaR_\alpha(X)] = TCE_\alpha(X). \end{aligned}$$

Moreover, when $a = VaR_\alpha(X)$,

$$\begin{aligned} CVaR_\alpha(X) &= a + \frac{E[X-a]^+}{1-\alpha} \\ &= VaR_\alpha(X) + \frac{E[X-VaR_\alpha(X)]^+}{1-\alpha} \\ &= VaR_\alpha(X) + \frac{\int_{VaR_\alpha(X)}^\infty [x-VaR_\alpha(X)]dF(x)}{1-\alpha} \\ &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(X)du = TCE_\alpha(X). \end{aligned}$$

Finally,

$$\begin{aligned}
ES_\alpha(X) &= \frac{1}{1-\alpha}\left\{E[X1_{\{X\geq VaR_\alpha(X)\}}] + VaR_\alpha(X)\left[1-\alpha-P(X\geq VaR_\alpha(X))\right]\right\} \\
&= \frac{1}{1-\alpha}\left\{\int_{VaR_\alpha(X)}^{\infty} xdF(x) + VaR_\alpha(X)\left[1-\alpha-(1-\alpha)\right]\right\} \\
&= \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(X)du = TCE_\alpha(X).
\end{aligned}$$

In conclusion, for a continuous random variable, $TCE_\alpha(X) = WCE_\alpha(X) = ES_\alpha(X) = CVaR_\alpha(X)$. Let $Y \sim N(\mu, \sigma^2)$ such that $X = e^Y$ is a log-normal distributed random variable. Then the cumulative distribution function of X is given by

$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right) \Rightarrow VaR_\alpha(X) = e^{\sigma\Phi^{-1}(\alpha)+\mu},$$

where Φ refers to the standard normal distribution.

$$\begin{aligned}
TCE_\alpha(X) = WCE_\alpha(X) = ES_\alpha(X) &= CVaR_\alpha(X) \\
&= \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(X)du \\
&= \frac{1}{1-\alpha}\int_\alpha^1 e^{\sigma\Phi^{-1}(u)+\mu}du \\
&= \frac{e^{\mu+\frac{1}{2}\sigma^2}}{1-\alpha}\left[\Phi\left(\sigma-\Phi^{-1}(\alpha)\right)\right].
\end{aligned}$$

The result is obtained by using a change of variable of the form $a = \Phi^{-1}(u)$.

In the next example, simulation results are obtained to show the robustness of risk measures.

Example 2.4.3. *To estimate risk measures mentioned in the Example 2.4.2, we generate two data sets with sample size* $n = 100$ *from a log-normal distribution with the same mean* $E(X) = 100$ *and coefficients of variation* $CV(X) = \sqrt{Var(X)}/E(X)$ *taking values 1 and 2, respectively. Then the parameters* (μ, σ) *for each groups are (4.2586, 0.8326) and (3.8005, 1.2686).*

Let $X_{(1)}, X_{(2)}, ..., X_{([n\alpha])}, X_{([n\alpha]+1)}, ..., X_{(n)}$ denote the data in the sample arranged in increasing order. Then, for n large enough, the empirical estimator of $VaR_\alpha(X)$ will be the statistics $X_{([n\alpha]+1)}$. The empirical estimator of $\rho(X) = TCE_\alpha(X) = WCE_\alpha(X) = ES_\alpha(X) = CVaR_\alpha(X)$ is given by

$$\widehat{\rho}(X) = \frac{X_{([n\alpha]+1)} + ... + X_{(n)}}{n - [n\alpha]}.$$

Moreover, $RVaR_{\alpha_1,\alpha_2}(X)$ can be calculated as

$$\begin{aligned} RVaR_{\alpha_1,\alpha_2}(X) &= \frac{1}{\alpha_2 - \alpha_1} \int_{VaR_{\alpha_1}(X)}^{VaR_{\alpha_2}(X)} \frac{1}{t\sqrt{2\pi\sigma^2}} t e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}} dt \\ &= \frac{e^{\mu + \frac{1}{2}\sigma^2}}{\alpha_2 - \alpha_1} \left[\Phi\left(\sigma - \Phi^{-1}(\alpha_1)\right) - \Phi\left(\sigma - \Phi^{-1}(\alpha_2)\right)\right], \end{aligned}$$

where Φ refers to the standard normal cumulative distribution function.

The empirical estimator of $RVaR_{\alpha_1,\alpha_2}(X)$ is defined by

$$\widehat{RVaR}_{\alpha_1,\alpha_2}(X) = \frac{X_{([n\alpha_1]+1)} + ... + X_{([n\alpha_2]+1)}}{[n\alpha_2] - [n\alpha_1] + 1}.$$

The simulation results are presented in the Table 2.2, with $\alpha_2 = 0.99$.

Tab. 2.2: $\widehat{RVaR}_{\alpha_1,\alpha_2}(X)$ and $\widehat{\rho}(X)$

	$RVaR_{\alpha_1,\alpha_2}(X)$		$\widehat{RVaR}_{\alpha_1,\alpha_2}(X)$		$\rho(X)$		$\widehat{\rho}(X)$	
CV	1	2	1	2	1	2	1	2
$\alpha_1 = 0.90$	287.91	388.61	242.03	314.68	326.75	494.83	269.87	396.46
$\alpha_1 = 0.95$	351.77	520.70	301.45	498.97	416.66	706.73	331.65	592.96

The dependence of these risk measures on different significance levels is observed, risk measures are increasing with α_1. Furthermore, Table 2.2 shows that $RVaR_{\alpha_1,\alpha_2}(X)$ is more robust than $TCE_\alpha(X)$, $WCE_\alpha(X)$, $ES_\alpha(X)$ and $CVaR_\alpha(X)$ since large variance has a smaller impact on this statistic.

3. MULTIVARIATE RISK MEASURES

We have reviewed several popular univariate risk measures and examined their properties and robustness. Before discussing properties and robustness of some established multivariate risk measures, we review copulas which are frequently used to model dependent random variables.

3.1 Copulas

Definition 3.1.1. *Let $\boldsymbol{X} = \{X_1, X_2, \dots, X_d\}$ be a random vector with cdf F. Set $U_i = F_i(x_i) \sim U[0,1]$, $i = 1, \dots, d$. Then, the copula $C : [0,1]^d \to [0,1]$ of F is given by*

$$C(u_1, \dots, u_d) = P(U_1 \leq u_1, \dots, U_d \leq u_d), \qquad u_i \in [0,1], \quad i = 1, \dots, n.$$

This definition is proposed by *Nelsen* (1999). Moreover, for a random vector $(U_1, \dots, U_d)$ with cdf C, $P(a_1 \leq U_1 \leq b_1, \dots, a_d \leq U_d \leq b_d)$ is non-negative. Furthermore, Sklar's theorem (see *Sklar* (1959)) states that any multivariate cumulative distibution function can be expressed in terms of its marginal distributions and a copula.

Theorem 3.1.1. *(Sklar's Theorem) Let F be a d-dimensional distribution function with marginal distributions $F_1, \dots, F_d$, then there exists a copula C such that*

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)).$$

Note, if $F_1, \ldots, F_d$ are continuous, then C is unique.

In this thesis we mainly discuss bivariate risk measures. Consider X_1 and X_2 with marginal cdf's F_1 and F_2, respectively. Then, we have

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)).$$

For what follows, the survival copula is defined by

$$\begin{aligned}\overline{C}(u_1, u_2) &= P(U_1 > u_1, U_2 > u_2) \\ &= 1 - P(U_1 \leq u_1) - P(U_2 \leq u_2) + P(U_1 \leq u_1, U_2 \leq u_2) \\ &= 1 - u_1 - u_2 + C(u_1, u_2), \quad (u_1, u_2) \in [0,1]^2.\end{aligned}$$

Finally, we will evaluate the empirical estimators of bivariate risk measures in Chapter 4 with the empirical copula C_n. For a random sample (X_{l1}, X_{l2}), $l = 1, \ldots, n$, C_n is defined by

$$C_n(u_1, u_2) = \frac{1}{n} \sum_{l=1}^{n} \mathbf{1}_{\{F_{n,1}(X_{l1}) \leq u_1, F_{n,2}(X_{l2}) \leq u_2\}},$$

where $F_{n,i}$ represents the empirical cdf of $\mathbf{X}_i = \{X_{1i}, \ldots, X_{ni}\}$, $i = 1, 2$. Well known copulas and families of copulas are defined as follow.

Fréchet Family

First, we consider a pair of independent random variables. If X_1 is independent of X_2, then

$$F(x_1, x_2) = F_1(x_1) F_2(x_2), \quad (x_1, x_2) \in \mathbb{R}^2,$$

or

$$\Pi(u_1, u_2) = u_1 u_2, \quad (u_1, u_2) \in [0,1]^2.$$

Also, one has the upper and lower Fréchet-Hoeffding bounds proposed in *Fréchet* (1951) and *Hoeffding* (1940), respectively

$$M(u_1, u_2) = \min(u_1, u_2), \qquad W(u_1, u_2) = \max(0, u_1 + u_2 - 1),$$

for $(u_1, u_2) \in [0, 1]^2$.

Theorem 3.1.2. *For an arbitrary bivariate copula* $C : [0, 1]^2 \to [0, 1]$ *and any* $(u_1, u_2) \in [0, 1]^2$,

$$W(u_1, u_2) \le C(u_1, u_2) \le M(u_1, u_2).$$

Note $M(u_1, u_2)$ indicates that random variables are comonotonic (perfect positive dependence) whereas $W(u_1, u_2)$ indicates that random variables are countermonotonic (perfect negative dependence).

Archimedean Family

The Archimedean Family introduced by *Nelsen* (2006) is of the form

$$C(u_1, u_2; \theta) = \psi^{-1}(\psi(u_1; \theta) + \psi(u_2; \theta); \theta), \quad (u_1, u_2) \in [0, 1]^2, \quad \theta \in \Theta,$$

where ψ is the generator function and satisfies following properties,

(1) $\psi(0) = \infty$ and $\psi(1) = 0$,

(2) $\psi'(t) < 0$,

(3) $\psi''(t) > 0$.

Moreover, the parameter θ dictates the dependence between the random variables. Below are presented some Archimedean copulas, most of which will be used throughout this thesis.

(1) Gumbel Copula

For $\theta \in [1, \infty)$, the Gumbel copula is given by

$$C(u_1, u_2; \theta) = e^{-\left\{[-\ln(u_1)]^\theta + [-\ln(u_2)]^\theta\right\}^{\frac{1}{\theta}}},$$

with the generator $\psi(t; \theta) = (-\ln(t))^\theta$ and the inverse generator $\psi^{-1}(t; \theta) = e^{-t^{\frac{1}{\theta}}}$. Specially, for $\theta = 1$, we have

$$C(u_1, u_2) = e^{-\{[-\ln(u_1)] + [-\ln(u_2)]\}} = u_1 u_2 = \Pi(u_1, u_2),$$

is the independent copula.

(2) Frank Copula

For $\theta \in (-\infty, 0) \cup (0, \infty)$, the Frank copula is defined by

$$C(u_1, u_2; \theta) = -\frac{1}{\theta} \ln\left[1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}\right],$$

with the generator $\psi(t; \theta) = -\ln(\frac{e^{-\theta t} - 1}{e^{-\theta} - 1})$ and the inverse generator $\psi^{-1}(t; \theta) = -\frac{1}{\theta} \ln[1 + e^{-t}(e^{-\theta} - 1)]$.

(2) Clayton Copula

For $\theta \in [-1, 0) \cup (0, \infty)$, we have the Clayton copula defined by

$$C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}},$$

with the generator $\psi(t; \theta) = \frac{1}{\theta}(t^{-\theta} - 1)$ and the inverse generator $\psi^{-1}(t; \theta) = (1 + \theta t)^{-\frac{1}{\theta}}$. Note, $\Pi(u_1, u_2)$ is obtained if $\theta = 0$. Moreover, $W(u_1, u_2)$ and $M(u_1, u_2)$ are attained by setting $\theta = -1$ and $\theta \to \infty$, respectively.

Figure 3.1 and Figure 3.2 show the dependence structure of variables with Gumbel, Frank and Clayton copulas, respectively. Frank copula is relatively symmetric, whereas Gumbel and Clayton copulas have stronger dependence in the right and left tails, respectively.

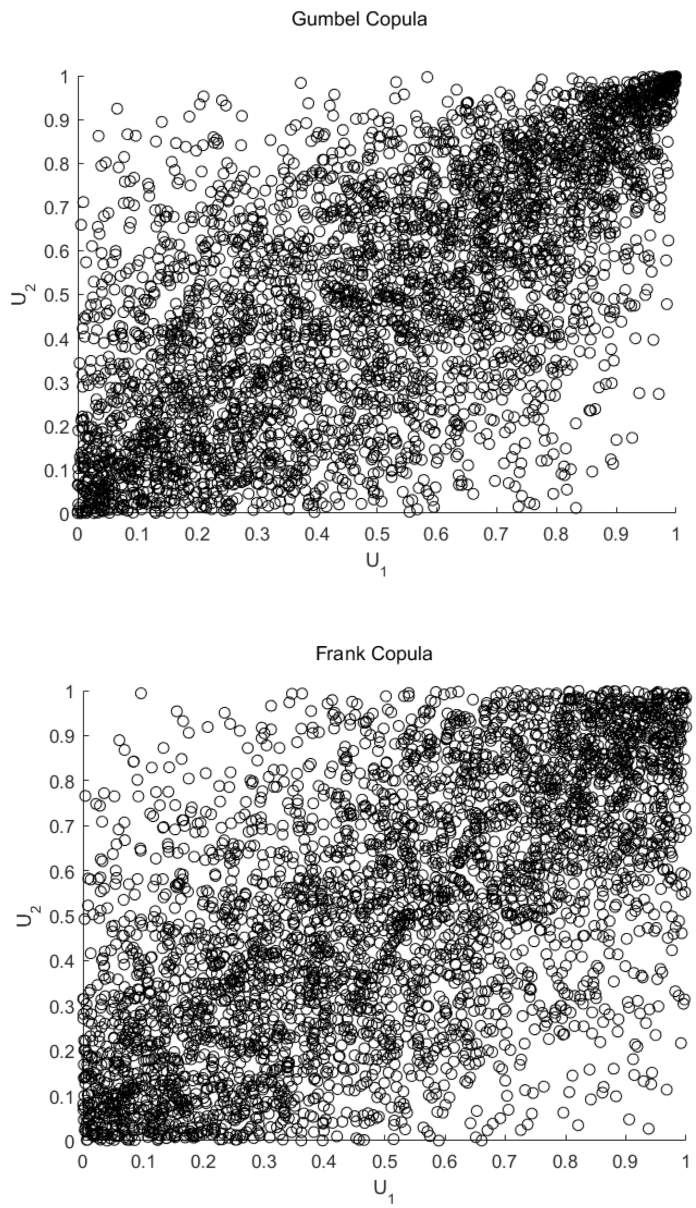

Fig. 3.1: Gumbel and Frank Copulas with dependent parameters $\theta = 2$ and $\theta = 5$

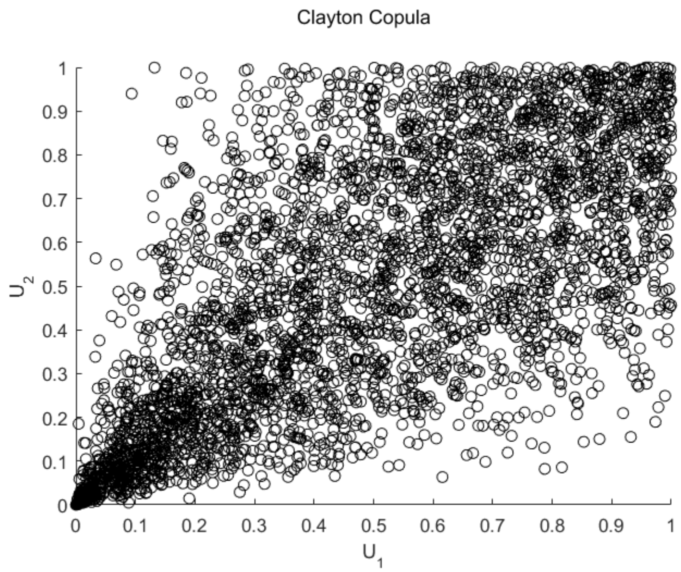

Fig. 3.2: Clayton Copula with dependent parameters $\theta = 2$

3.2 Bivariate VaR and TVaR

First, let us introduce the bivariate orthant based VaR proposed by *Embrechts and Puccetti* (2006).

Definition 3.2.1. *Let $\boldsymbol{X} = (X_1, X_2)$ be a random vector with joint cdf $F_{\boldsymbol{X}}$ and joint suvivial function (sf) $\bar{F}_{\boldsymbol{X}}$. At significance level $\alpha \in [0,1]$, the bivariate lower orthant VaR is defined by*

$$\underline{VaR}_\alpha(\boldsymbol{X}) = \partial\{\boldsymbol{x} \in \mathbb{R}^2 : F_{\boldsymbol{X}}(\boldsymbol{x}) \geq \alpha\}, \qquad (eq.\ 3.1)$$

and bivariate upper orthant VaR is defined by

$$\overline{VaR}_\alpha(\boldsymbol{X}) = \partial\{\boldsymbol{x} \in \mathbb{R}^2 : \bar{F}_{\boldsymbol{X}}(\boldsymbol{x}) \leq 1 - \alpha\}. \qquad (eq.\ 3.2)$$

Note, in a bivariate setting, the relationship between the cumulative distribution function and the survival function, namely $F(x) = 1 - \bar{F}(x)$, does not hold. For this reason, bivariate VaR need to be defined as the lower and upper orthant VaR separately using either the cdf or sf. In addition, ∂ denotes the boundary of the set. To study the behavior of bivariate orthant based VaR and get the bivariate extension of TVaR, *Cossette et al.* (2013, 2015) rewrite Eq. 3.1 and Eq. 3.2.

Definition 3.2.2. *Let $\boldsymbol{X} = (X_1, X_2)$ be a random vector with joint cdf $F_{\boldsymbol{X}}$ and joint sf $\bar{F}_{\boldsymbol{X}}$. At significance level $\alpha \in [0,1]$, the bivariate lower orthant VaR is defined by*

$$\underline{VaR}_\alpha(\boldsymbol{X}) = \{(x_1, \underline{VaR}_{\alpha,x_1}(\boldsymbol{X})), x_1 \geq VaR_\alpha(X_1)\},$$

or

$$\underline{VaR}_\alpha(\boldsymbol{X}) = \{(\underline{VaR}_{\alpha,x_2}(\boldsymbol{X}), x_2), x_2 \geq VaR_\alpha(X_2)\},$$

and the bivariate upper orthant VaR is defined by

$$\overline{VaR}_\alpha(\boldsymbol{X}) = \{(x_1, \overline{VaR}_{\alpha,x_1}(\boldsymbol{X})), x_1 \leq VaR_\alpha(X_1)\},$$

or

$$\overline{VaR}_\alpha(\boldsymbol{X}) = \{(\overline{VaR}_{\alpha,x_2}(\boldsymbol{X}), x_2), x_2 \leq VaR_\alpha(X_2)\}.$$

For $i, j = 1, 2$ $(i \neq j)$,

$$\underline{VaR}_{\alpha,x_i}(\boldsymbol{X}) = VaR_{\frac{\alpha}{F_{X_i}(x_i)}}\left(X_j | X_i \leq x_i\right), \quad x_i \geq VaR_\alpha(X_i)$$

and

$$\overline{VaR}_{\alpha,x_i}(\boldsymbol{X}) = VaR_{\frac{\alpha - F_{X_i}(x_i)}{1 - F_{X_i}(x_i)}}\left(X_j | X_i \geq x_i\right), \quad x_i \leq VaR_\alpha(X_i).$$

Example 3.2.1. *Consider the random vector (X_1, X_2) with joint cdf is defined with a Gumbel copula with dependent parameter $\theta = 1.5$ and marginals $X_1 \sim$ Weibull (2, 50) and $X_2 \sim$*

Weibull (2, 150). Then, we get $\underline{VaR}_{0.95,x_1}(\boldsymbol{X})$ *on Figure 3.3 and* $\overline{VaR}_{0.99,x_1}(\boldsymbol{X})$ *on Figure 3.4. Let* u_{x_1} *(respectively* u_{x_2}*) and* l_{x_1} *(respectively* l_{x_2}*) denote the essential upper and lower support of* X_1 *(respectively* X_2*). Note,*

$$\underline{VaR}_{0.95,x_1}(\boldsymbol{X}) \subset [VaR_{0.95}(X_1), u_{x_1}] \times [VaR_{0.95}(X_2), u_{x_2}],$$

and

$$\overline{VaR}_{0.99,x_1}(\boldsymbol{X}) \subset [l_{x_1}, VaR_{0.99}(X_1)] \times [l_{x_2}, VaR_{0.99}(X_2)].$$

Also, Figure 3.3 and Figure 3.4 show the convexity of the curve which has been studied in Cossette et al. (2013).

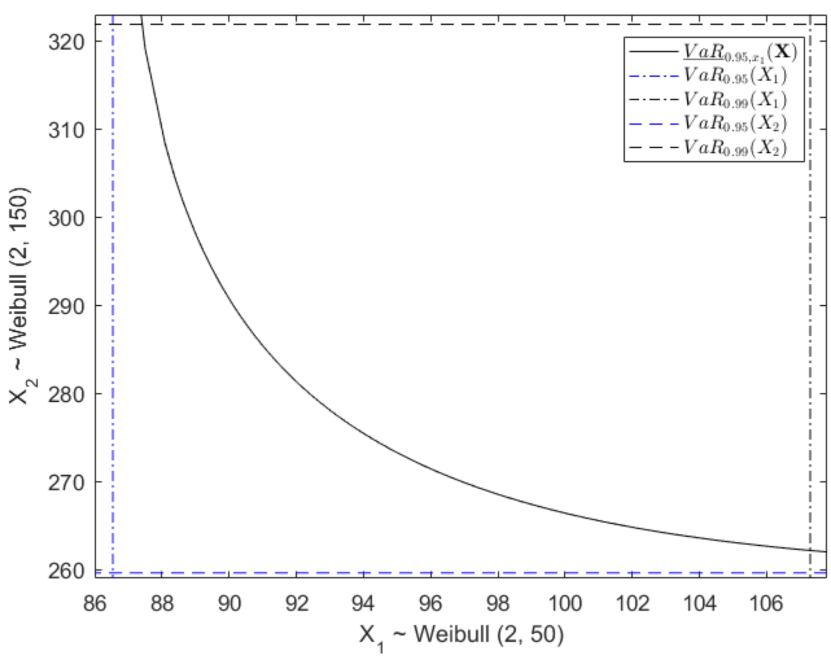

Fig. 3.3: Lower orthant VaR at level 0.95

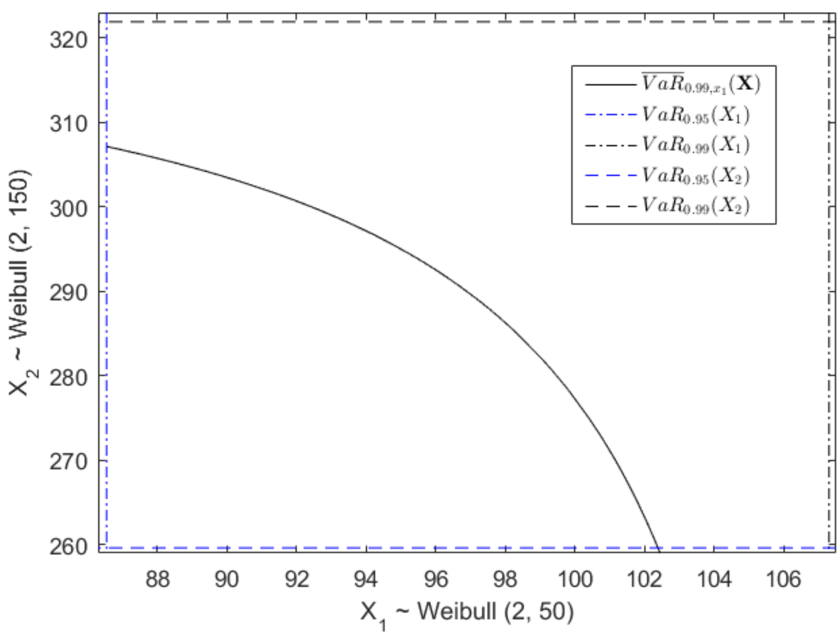

Fig. 3.4: Upper orthant VaR at level 0.99

Proposition 3.2.1. *Let $\boldsymbol{X} = (X_1, X_2)$ be a continuous random vector. Let ϕ_1 and ϕ_2 be real functions defined on the supports of X_1 and X_2, respectively.*

1. *(Translation invariance) For all $\boldsymbol{c} = (c_1, c_2) \in \mathbb{R}^2$ and $i, j = 1, 2$, $i \neq j$, then*

$$\underline{VaR}_{\alpha, x_j + c_j}(\boldsymbol{X} + \boldsymbol{c}) = \underline{VaR}_{\alpha, x_j}(\boldsymbol{X}) + c_i,$$

$$\overline{VaR}_{\alpha, x_j + c_j}(\boldsymbol{X} + \boldsymbol{c}) = \overline{VaR}_{\alpha, x_j}(\boldsymbol{X}) + c_i.$$

2. *(Positive homogeneity) For all $\boldsymbol{c} = (c_1, c_2) \in \mathbb{R}^2_+$ and $i, j = 1, 2$, $i \neq j$, then*

$$\underline{VaR}_{\alpha, c_j x_j}(\boldsymbol{c}\boldsymbol{X}) = c_i \underline{VaR}_{\alpha, x_j}(\boldsymbol{X}), \qquad \overline{VaR}_{\alpha, c_j x_j}(\boldsymbol{c}\boldsymbol{X}) = c_i \overline{VaR}_{\alpha, x_j}(\boldsymbol{X}).$$

3. *(Negative transformations) For all $\boldsymbol{c} = (c_1, c_2) \in \mathbb{R}^2_-$ and $i, j = 1, 2$, $i \neq j$, then*

$$\underline{VaR}_{\alpha, c_j x_j}(\boldsymbol{c}\boldsymbol{X}) = c_i \overline{VaR}_{1-\alpha, x_j}(\boldsymbol{X}), \qquad \overline{VaR}_{\alpha, c_j x_j}(\boldsymbol{c}\boldsymbol{X}) = c_i \underline{VaR}_{1-\alpha, x_j}(\boldsymbol{X}).$$

In general,

(1) *For increasing functions ϕ_1 and ϕ_2, $i, j = 1, 2$, $i \neq j$,*

$$\underline{VaR}_{\alpha, \phi_j(x_j)}(\phi(\boldsymbol{X})) = \phi_i(\underline{VaR}_{\alpha, x_j}(\boldsymbol{X})),$$

$$\overline{VaR}_{\alpha, \phi_j(x_j)}(\phi(\boldsymbol{X})) = \phi_i(\overline{VaR}_{\alpha, x_j}(\boldsymbol{X})).$$

(2) *For decreasing functions ϕ_1 and ϕ_2, $i, j = 1, 2$, $i \neq j$,*

$$\underline{VaR}_{\alpha, \phi_j(x_j)}(\phi(\boldsymbol{X})) = \phi_i(\overline{VaR}_{1-\alpha, x_j}(\boldsymbol{X})),$$

$$\overline{VaR}_{\alpha, \phi_j(x_j)}(\phi(\boldsymbol{X})) = \phi_i(\underline{VaR}_{1-\alpha, x_j}(\boldsymbol{X})).$$

4. *(Monotonicity) Let* $\boldsymbol{X} = (X_1, X_2)$ *and* $\boldsymbol{X}' = (X_1', X_2')$ *be two pairs of risks with joint cdf's* $F_{\boldsymbol{X}}$ *and* $F_{\boldsymbol{X}'}$ *respectively.* $\boldsymbol{X}$ *is said to be more concordant than* $\boldsymbol{X}'$, *denoted* $\boldsymbol{X} \prec_{co} \boldsymbol{X}'$, *if* $F_{\boldsymbol{X}}(\boldsymbol{x}) \leq F_{\boldsymbol{X}'}(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \mathbb{R}^2$. *Then,*

$$\underline{VaR}_\alpha(\boldsymbol{X}') \prec \underline{VaR}_\alpha(\boldsymbol{X}), \qquad \overline{VaR}_\alpha(\boldsymbol{X}) \prec \overline{VaR}_\alpha(\boldsymbol{X}').$$

Proof. The proof of Proposition 3.2.1 is presented in *Cossette et al.* (2013). □

Definition 3.2.3. *Let* $\boldsymbol{X} = (X_1, X_2)$ *be a random vector with joint cdf* $F_{\boldsymbol{X}}$ *and joint suvivial function (sf)* $\bar{F}_{\boldsymbol{X}}$. *At significance level* $\alpha \in [0, 1]$, *the bivariate lower orthant TVaR is given by*

$$\underline{TVaR}_\alpha(\boldsymbol{X}) = \left\{ \left(x_i, \underline{TVaR}_{\alpha, x_i}(\boldsymbol{X})\right), x_i \geq VaR_\alpha(X_i), i = 1, 2 \right\},$$

and bivariate upper orthant TVaR is given by

$$\overline{TVaR}_\alpha(\boldsymbol{X}) = \{ \left(x_i, \overline{TVaR}_{\alpha, x_i}(\boldsymbol{X})\right), x_i \leq VaR_\alpha(X_i), i = 1, 2 \}.$$

Note that for $i, j = 1, 2$ $(i \neq j)$,

$$\begin{aligned} \underline{TVaR}_{\alpha, x_i}(\mathbf{X}) &= E[X_j | X_j > \underline{VaR}_{\alpha, x_i}(\mathbf{X}), X_i \leq x_i] \\ &= \frac{1}{F_{X_i}(x_i) - \alpha} \int_\alpha^{F_{X_i}(x_i)} \underline{VaR}_{u, x_i}(\mathbf{X}) du, \end{aligned}$$

and

$$\begin{aligned} \overline{TVaR}_{\alpha, x_i}(\mathbf{X}) &= E[X_j | X_j > \overline{VaR}_{\alpha, x_i}(\mathbf{X}), X_i \geq x_i] \\ &= \frac{1}{1 - \alpha} \int_\alpha^1 \overline{VaR}_{v, x_i}(\mathbf{X}) dv. \end{aligned}$$

Example 3.2.2. *Consider the same random vector defined in Example 3.2.1 with significance level* $\alpha = 0.99$. $\underline{TVaR}_{0.99, x_1}(\boldsymbol{X})$ *(respectively* $\overline{TVaR}_{0.99, x_1}(\boldsymbol{X})$*) is obtained based on the Definition 3.2.3. For comparison, we also plot* $\underline{VaR}_{0.99, x_1}(\boldsymbol{X})$ *(respectively* $\overline{VaR}_{0.99, x_1}(\boldsymbol{X})$*) on*

Figure 3.5 and Figure 3.6. Note that $\underline{TVaR}_{0.99,x_1}(\boldsymbol{X})$ *(respectively* $\overline{TVaR}_{0.99,x_1}(\boldsymbol{X})$*) goes to* $TVaR_{0.99}(X_2)$ *when* X_1 *approaches to* u_{x_1} *(respectively* l_{x_1}*). These results are obtained using the numerical integration tools in Matlab.*

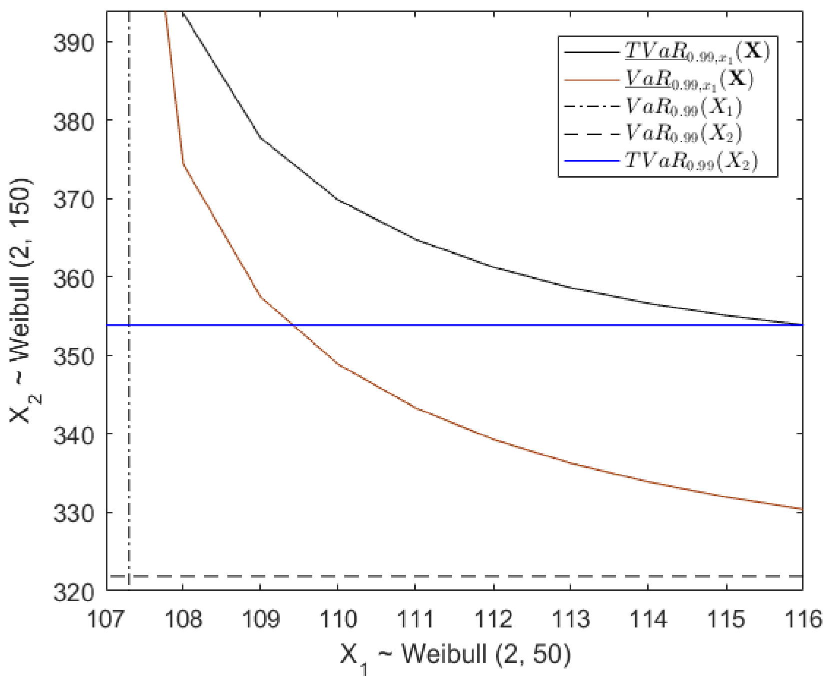

Fig. 3.5: Lower orthant VaR and TVaR at level 0.99

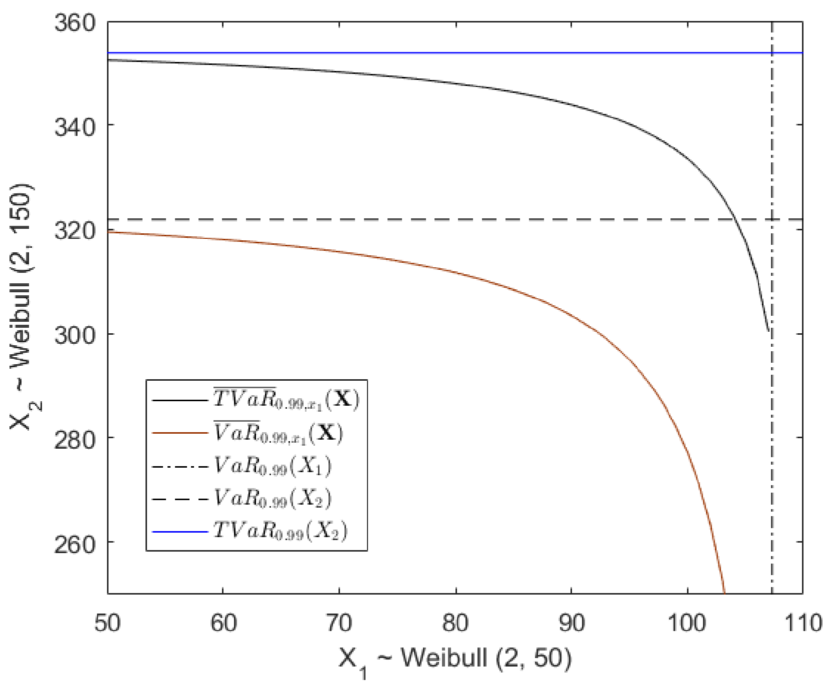

Fig. 3.6: Upper orthant VaR and TVaR at level 0.99

Proposition 3.2.2. *Let* $\boldsymbol{X} = (X_1, X_2)$ *be a continuous random vector.*

1. *(Translation invariance) For all* $\boldsymbol{c} = (c_1, c_2) \in \mathbb{R}^2$ *and* $i, j = 1, 2$, $i \neq j$, *then*

$$\underline{TVaR}_{\alpha, x_j + c_j}(\boldsymbol{X} + \boldsymbol{c}) = \underline{TVaR}_{\alpha, x_j}(\boldsymbol{X}) + c_i,$$

$$\overline{TVaR}_{\alpha, x_j + c_j}(\boldsymbol{X} + \boldsymbol{c}) = \overline{TVaR}_{\alpha, x_j}(\boldsymbol{X}) + c_i.$$

2. *(Positive homogeneity) For all* $\boldsymbol{c} = (c_1, c_2) \in \mathbb{R}^2_+$ *and* $i, j = 1, 2$, $i \neq j$, *then*

$$\underline{TVaR}_{\alpha, c_j x_j}(\boldsymbol{cX}) = c_i \underline{TVaR}_{\alpha, x_j}(\boldsymbol{X}), \qquad \overline{TVaR}_{\alpha, c_j x_j}(\boldsymbol{cX}) = c_i \overline{TVaR}_{\alpha, x_j}(\boldsymbol{X}).$$

3. *(Monotonicity) Let* $\boldsymbol{X} = (X_1, X_2)$ *and* $\boldsymbol{X}' = (X_1', X_2')$ *be two pairs of risks with joint cdf* $F_{\boldsymbol{X}}$ *and* $F_{\boldsymbol{X}'}$ *respectively. If* $\boldsymbol{X} \prec_{co} \boldsymbol{X}'$, *then*

$$\underline{TVaR}_{\alpha}(\boldsymbol{X}') \prec \underline{TVaR}_{\alpha}(\boldsymbol{X}), \qquad \overline{TVaR}_{\alpha}(\boldsymbol{X}) \prec \overline{TVaR}_{\alpha}(\boldsymbol{X}').$$

Proof. The proof of Proposition 3.2.2 can be found in *Cossette et al.* (2015). □

Jouini et al. (2004), *Bentahar* (2006) and *Tahar and Lépinette* (2012, 2014) extend the multivariate WCE and TCE. Furthermore, *Cousin and Di Bernardino* (2013, 2014, 2015) propose a series of multivariate risk measures developed from a different aspect.

Definition 3.2.4. *Let* $\boldsymbol{X} = (X_1, X_2)$ *denote a bivariate random vector on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *For* $\alpha \in [0, 1]$, *the vector-valued Worst Conditional Expectation at level* α *is defined by*

$$WCE_{\alpha}(\boldsymbol{X}) = \left\{ \boldsymbol{x} \in \mathbb{R}^2 : E[\boldsymbol{x} - \boldsymbol{X} | A] \succeq 0, \forall A \in \mathcal{F} \text{ such that } P(A) \geq 1 - \alpha \right\}.$$

Definition 3.2.5. *Let* $\boldsymbol{X} = (X_1, X_2)$ *denote a bivariate random vector on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *For* $\alpha \in [0, 1]$, *the vector-valued Tail Conditional Expectation at level* α *is* *defined by*

$$TCE_\alpha(\boldsymbol{X}) = \{\boldsymbol{x} \in \mathbb{R}^2 : E[\boldsymbol{x} - \boldsymbol{X} | \boldsymbol{X} \in A] \succeq 0, \forall A \in \mathcal{Q}_\alpha(\boldsymbol{X})\},$$

where $\mathcal{Q}_\alpha(\boldsymbol{X}) = \{A \subseteq \mathbb{R}^2 : P(\boldsymbol{X} \in A) \geq 1 - \alpha\}$.

Note that the vector-valued $WCE_\alpha(\mathbf{X})$ and $TCE_\alpha(\mathbf{X})$ are the natural extension of their real-valued versions. Therefore, they share similar properties.

Definition 3.2.6. *Let* $\boldsymbol{X} = (X_1, X_2)$ *denote a bivariate random vector on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *For* $\alpha \in [0, 1]$, *the Generalized Worst Conditional Expectation at level* α *is defined by*

$$GWCE_\alpha(\boldsymbol{X}) = \bigcup_{\tilde{\boldsymbol{X}}} WCE_\alpha(\tilde{\boldsymbol{X}}),$$

where random variables $\tilde{\boldsymbol{X}}$ *on* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *having the same distribution as* $\boldsymbol{X}$.

Proposition 3.2.3. *Let* $\boldsymbol{X} = (X_1, X_2)$ *denote a bivariate random vector on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *having a continuous probability density. Then*

$$WCE_\alpha(\boldsymbol{X}) = TCE_\alpha(\boldsymbol{X}) = GWCE_\alpha(\boldsymbol{X}).$$

The proof of this Proposition is presented in [4].

Definition 3.2.7. *Let* $\boldsymbol{X} = (X_1, X_2)$ *denote a bivariate random vector on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with joint cdf* $\boldsymbol{F}_{\boldsymbol{X}}$ *and joint sf* $\bar{\boldsymbol{F}}_{\boldsymbol{X}}$. *The multivariate lower-orthant* VaR^* *at level* $\alpha \in [0, 1]$ *is defined by*

$$\underline{VaR}^*_\alpha(\boldsymbol{X}) = E[\boldsymbol{X} | F_{\boldsymbol{X}}(\boldsymbol{X}) = \alpha],$$

and the multivariate upper-orthant VaR^* *is defined by*

$$\overline{VaR}^*_\alpha(\boldsymbol{X}) = E[\boldsymbol{X} | \bar{F}_{\boldsymbol{X}}(\boldsymbol{X}) = 1 - \alpha].$$

Definition 3.2.8. *Let $\boldsymbol{X} = (X_1, X_2)$ denote a bivariate random vector on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with joint cdf $\boldsymbol{F_X}$ and joint sf $\bar{\boldsymbol{F}}_{\boldsymbol{X}}$. The multivariate lower-orthant Conditional Tail Expectation at level $\alpha \in [0,1]$ is defined by*

$$\underline{CTE}_\alpha(\boldsymbol{X}) = E[\boldsymbol{X} | F_{\boldsymbol{X}}(\boldsymbol{X}) \geq \alpha],$$

and the multivariate upper-orthant Conditional Tail Expectation is defined by

$$\overline{CTE}_\alpha(\boldsymbol{X}) = E[\boldsymbol{X} | \bar{F}_{\boldsymbol{X}}(\boldsymbol{X}) \leq 1 - \alpha].$$

Definition 3.2.9. *Let $\boldsymbol{X} = (X_1, X_2)$ denote a bivariate random vector on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with joint cdf $\boldsymbol{F_X}$ and joint sf $\bar{\boldsymbol{F}}_{\boldsymbol{X}}$. For a significant level $\alpha \in [0,1]$, the multivariate lower-orthant CoVaR is defined by*

$$\underline{CoVaR}_{\alpha,\omega}(\boldsymbol{X}) = VaR_\omega(\boldsymbol{X} | F_{\boldsymbol{X}}(\boldsymbol{X}) = \alpha),$$

and the multivariate upper-orthant CoVaR is defined by

$$\overline{CoVaR}_{\alpha,\omega}(\boldsymbol{X}) = VaR_\omega(\boldsymbol{X} | \bar{F}_{\boldsymbol{X}}(\boldsymbol{X}) = 1 - \alpha),$$

where $\omega = \{\omega_1, \omega_2\}$ is the significance level vector of marginal risk with $\omega_i \in [0,1]$.

Note that similar properties mentioned in the Proposition 3.2.1 also hold for the multivariate upper and lower orthant VaR, CTE and CoVaR (see, e.g., [9], [10], [11] for details).

3.3 Bivariate Lower Orthant RVaR

In the following part of this chapter, we will propose a new multivariate risk measure called bivariate lower and upper orthant RVaR based on the results in *Cossette et al.* (2013, 2015). Its properties, such as translation invariance, positive homogeneity and monotonicity, will be discussed.

Definition 3.3.1. *Consider a continuous random vector* $\boldsymbol{X} = (X_1, X_2)$ *on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. Bivariate lower orthant RVaR at level range* $[\alpha_1, \alpha_2] \subseteq [0, 1]$ *is given by*

$$\underline{RVaR}_{\alpha_1,\alpha_2}(\boldsymbol{X}) = ((x_1, \underline{RVaR}_{\alpha_1,\alpha_2,x_1}(\boldsymbol{X})), (\underline{RVaR}_{\alpha_1,\alpha_2,x_2}(\boldsymbol{X}), x_2)),$$

where

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = E[X_j | \underline{VaR}_{\alpha_1,x_i}(\boldsymbol{X}) \le X_j \le VaR_{\alpha_2}(X_j), X_i \le x_i],$$

for

$$\underline{VaR}_{\alpha_1,VaR_{\alpha_2}(X_j)}(\boldsymbol{X}) \le x_i \le VaR_{\alpha_2}(X_i), \quad i, j = 1, 2 (i \ne j).$$

This definition is based on the univariate RVaR and bivariate lower orthant VaR. As we can see, $RVaR_{\alpha_1,\alpha_2}(X)$ is the expectation of a random variable X given that it belongs to the interval $[VaR_{\alpha_1}(X), VaR_{\alpha_2}(X)]$. Hence, we start from bounding the $\mathbf{X}$ in the rectangle area

$$[VaR_{\alpha_1}(X_i), VaR_{\alpha_2}(X_i)] \times [VaR_{\alpha_1}(X_j), VaR_{\alpha_2}(X_j)].$$

Considering the effect of the random variable X_i on X_j, $\underline{VaR}_{\alpha_1,x_i}(\mathbf{X})$ and $\underline{VaR}_{\alpha_2,x_i}(\mathbf{X})$ are applied. However, only $\underline{VaR}_{\alpha_1,x_i}(\mathbf{X})$ could lie in the above area, which has been shown in Example 3.2.1. Therefore, we require $\underline{VaR}_{\alpha_1,x_i}(\mathbf{X}) \le X_j \le VaR_{\alpha_2}(X_j)$ to make sure that the lower bound at level α_1 can be achieved and the upper bound at level α_2 will not be exceeded.

The next result shows that $\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X})$ has a similar expression as $\underline{TVaR}_{\alpha,x_i}(\mathbf{X})$ which can be expressed as the integration of $\underline{VaR}_{\alpha,x_i}(\mathbf{X})$.

Proposition 3.3.1. *For a continuous random vector* $\boldsymbol{X} = (X_1, X_2)$ *with joint cdf* $F(x_1, x_2)$ *and marginal cdf's* $F_{X_1}(x_1)$ *and* $F_{X_2}(x_2)$*,* $\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X})$ *can be restated as*

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = \frac{1}{F(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} \int_{\alpha_1}^{F(x_i, VaR_{\alpha_2}(X_j))} \underline{VaR}_{u,x_i}(\boldsymbol{X}) du,$$

for

$$\underline{VaR}_{\alpha_1, VaR_{\alpha_2}(X_j)}(\boldsymbol{X}) \leq x_i \leq VaR_{\alpha_2}(X_i), \quad i,j = 1,2(i \neq j).$$

Proof.

$$\begin{aligned}
\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}) &= E[X_j | \underline{VaR}_{\alpha_1,x_i}(\mathbf{X}) \leq X_j \leq VaR_{\alpha_2}(X_j), X_i \leq x_i] \\
&= \int_{\underline{VaR}_{\alpha_1,x_i}(\mathbf{X})}^{VaR_{\alpha_2}(X_j)} \frac{x_j dF_{X_i}(x_j)}{\frac{F(x_i, VaR_{\alpha_2}(X_j))}{F_{X_i}(x_i)} - \frac{\alpha_1}{F_{X_i}(x_i)}} \\
&= \frac{F_{X_i}(x_i)}{F(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} \int_{\underline{VaR}_{\alpha_1,x_i}(\mathbf{X})}^{VaR_{\alpha_2}(X_j)} x_j dF_{X_i}(x_j).
\end{aligned}$$

Note, we have

$$\underline{VaR}_{\alpha,x_i}(\mathbf{X}) = VaR_{\frac{\alpha}{F_{X_i}(x_i)}}(X_j | X_i \leq x_i).$$

Then, using $u = F_{X_i}(x_j) = P(X_j \leq x_j | X_i \leq x_i)$,

$$\begin{aligned}
\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}) =& \frac{F_{X_i}(x_i)}{F(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} \int_{\alpha_1/F_{X_i}(x_i)}^{F_{X_i}(VaR_{\alpha_2}(X_j))} F_{X_i}^{-1}(u) du \\
=& \frac{1}{F(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} \int_{\alpha_1}^{F_{X_i}(VaR_{\alpha_2}(X_j) F_{X_i}(x_i))} F_{X_i}^{-1}\left(\frac{u}{F_{X_i}(x_i)}\right) du \\
=& \frac{1}{F(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} \int_{\alpha_1}^{F_{X_i}(VaR_{\alpha_2}(X_j) F_{X_i}(x_i))} \underline{VaR}_{u,x_i}(\mathbf{X}) du \\
=& \frac{1}{F(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} \int_{\alpha_1}^{F(x_i, VaR_{\alpha_2}(X_j))} \underline{VaR}_{u,x_i}(\mathbf{X}) du.
\end{aligned}$$

□

Example 3.3.1. *Consider the random vector* $\boldsymbol{X} = (X_1, X_2)$ *defined in Example 3.2.1. Let the confidence level range be* $\alpha_1 = 0.95$ *and* $\alpha_2 = 0.99$*. Then, we get bivariate lower orthant RVaR in Figure 3.7 and Figure 3.8. For comparison, we plot* $\underline{VaR}_{0.95,x_i}(\boldsymbol{X})$ *on the same graph.*

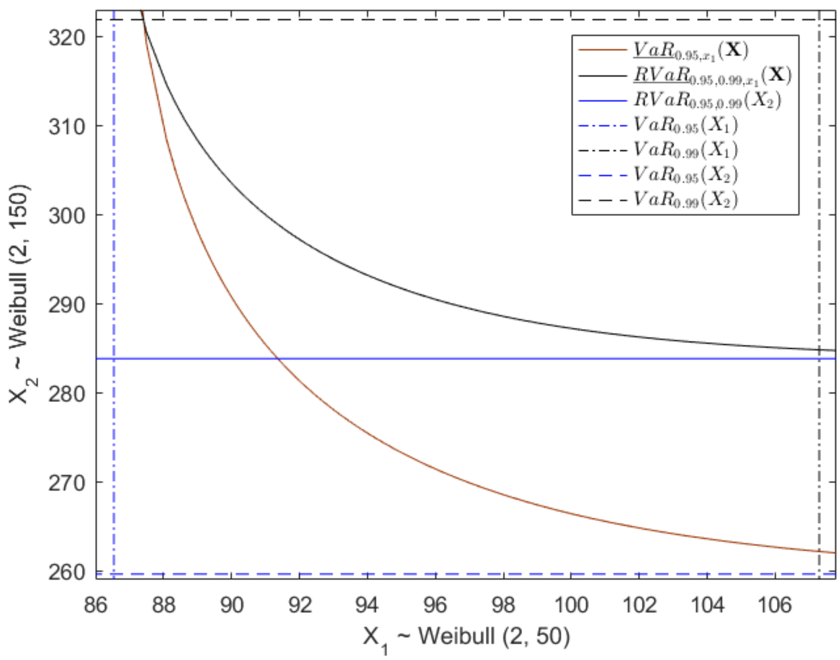

Fig. 3.7: Lower orthant VaR at level 0.95 and RVaR at level range $[0.95, 0.99]$ for fixed values of X_1

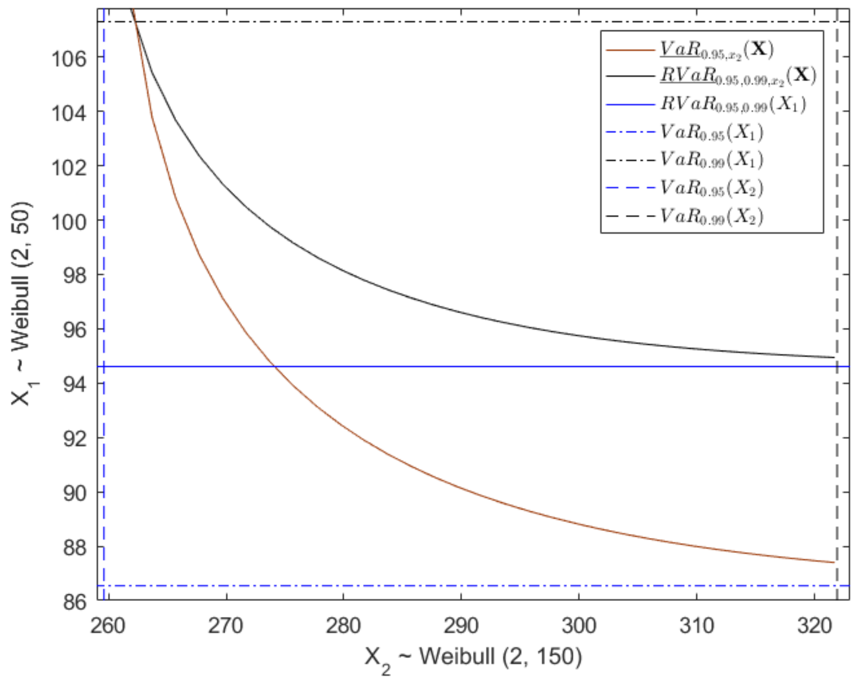

Fig. 3.8: Lower orthant VaR at level 0.95 and RVaR at level range $[0.95, 0.99]$ for fixed values of X_2

The shape of the lower orthant RVaR curve is shown in Figure 3.7 and Figure 3.8. One can observe that $\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X})$ converges to the univariate RVaR when x_i $(i = 1, 2)$ approaches infinity. Also, when x_i gets close to $VaR_{\alpha_1}(X_i)$, $\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X})$ approaches $VaR_{\alpha_2}(X_j)$.

3.4 Bivariate Upper Orthant RVaR

Definition 3.4.1. *Consider a continuous random vector* $\boldsymbol{X} = (X_1, X_2)$ *on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. Bivariate upper orthant RVaR at level range* $[\alpha_1, \alpha_2] \subseteq [0, 1]$ *is given by*

$$\overline{RVaR}_{\alpha_1,\alpha_2}(\boldsymbol{X}) = ((x_1, \overline{RVaR}_{\alpha_1,\alpha_2,x_1}(\boldsymbol{X})), (\overline{RVaR}_{\alpha_1,\alpha_2,x_2}(\boldsymbol{X}), x_2)),$$

where

$$\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = E[X_j | VaR_{\alpha_1}(X_j) \leq X_j \leq \overline{VaR}_{\alpha_2,x_i}(\boldsymbol{X}), X_i \geq x_i],$$

for

$$VaR_{\alpha_1}(X_i) \leq x_i \leq \overline{VaR}_{\alpha_2, VaR_{\alpha_1}(X_j)}(\boldsymbol{X}), \quad i, j = 1, 2 (i \neq j).$$

Similarly as for the lower orthant RVaR we can define the bivariate upper orthant RVaR. We consider the impact of the random variable X_i on X_j and require that $VaR_{\alpha_1}(X_j) \leq X_j \leq \overline{VaR}_{\alpha_2,x_i}(\mathbf{X})$ to make sure that the upper level α_2 can be achieved and the lower level α_1 will not be exceeded. Note, here we only modify the upper bound of the interval since only the curve $\overline{VaR}_{\alpha_2,x_i}(\mathbf{X})$ could lie in the bounded area

$$[VaR_{\alpha_1}(X_i), VaR_{\alpha_2}(X_i)] \times [VaR_{\alpha_1}(X_j), VaR_{\alpha_2}(X_j)].$$

The following result provides the expression of $\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X})$ in the form of the integration of $\overline{VaR}_{\alpha,x_i}(\mathbf{X})$.

Proposition 3.4.1. *Consider* $\boldsymbol{X} = (X_1, X_2)$ *with joint sf* $\bar{F}(x_1, x_2)$ *and marginal sf's* $\bar{F}_{X_2}(x_2)$ *and* $\bar{F}_{X_2}(x_2)$*, respectively. Then,* $\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X})$ *can be expressed by*

$$\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = \frac{1}{\alpha_2 - (1 - \bar{F}(x_i, VaR_{\alpha_1}(X_j)))} \int_{1-\bar{F}(x_i, VaR_{\alpha_1}(X_j))}^{\alpha_2} \overline{VaR}_{v,x_i}(\boldsymbol{X}) dv,$$

for

$$VaR_{\alpha_1}(X_i) \leq x_i \leq \overline{VaR}_{\alpha_2, VaR_{\alpha_1}(X_j)}(\boldsymbol{X}), \quad i, j = 1, 2 (i \neq j).$$

Proof. We have that

$$\begin{aligned}
\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}) &= E[X_j|VaR_{\alpha_1}(X_j) \le X_j \le \overline{VaR}_{\alpha_2,x_i}(\mathbf{X}), X_i \ge x_i] \\
&= \int_{VaR_{\alpha_1}(X_j)}^{\overline{VaR}_{\alpha_2,x_i}(\mathbf{X})} \frac{x_j dF_{\bar{X}_i}(x_j)}{\frac{\bar{F}(x_i,VaR_{\alpha_1}(X_j))}{1-F_{X_i}(x_i)} - \frac{1-\alpha_2}{1-F_{X_i}(x_i)}} \\
&= \frac{1-F_{X_i}(x_i)}{\bar{F}(x_i,VaR_{\alpha_1}(X_j)) - 1 + \alpha_2} \int_{VaR_{\alpha_1}(X_j)}^{\overline{VaR}_{\alpha_2,x_i}(\mathbf{X})} x_j dF_{\bar{X}_i}(x_j).
\end{aligned}$$

Note, one has

$$\overline{VaR}_{\alpha,x_i}(\mathbf{X}) = VaR_{\frac{\alpha-F_{X_i}(x_i)}{1-F_{X_i}(x_i)}}(X_j|X_i \ge x_i).$$

Then, using $v = F_{\bar{X}_i}(x_j) = P(X_j \le x_j|X_i \ge x_i)$,

$$\begin{aligned}
\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}) =& \frac{1-F_{X_i}(x_i)}{\bar{F}(x_i,VaR_{\alpha_1}(X_j)) - 1 + \alpha_2} \int_{1-\frac{\bar{F}_{X_i}(VaR_{\alpha_1}(X_j))}{1-F_{X_i}(x_i)}}^{\frac{\alpha_2-F_{X_i}(x_i)}{1-F_{X_i}(x_i)}} F_{\bar{X}_i}^{-1}(v)dv \\
=& \frac{1}{\bar{F}(x_i,VaR_{\alpha_1}(X_j)) - 1 + \alpha_2} \int_{1-\bar{F}_{X_i}(VaR_{\alpha_1}(X_j))}^{\alpha_2} F_{\bar{X}_i}^{-1}\left(\frac{v-F_{X_i}(x_i)}{1-F_{X_i}(x_i)}\right)dv \\
=& \frac{1}{\alpha_2 - (1-\bar{F}(x_i,VaR_{\alpha_1}(X_j)))} \int_{1-\bar{F}(x_i,VaR_{\alpha_1}(X_j))}^{\alpha_2} \overline{VaR}_{v,x_i}(\mathbf{X})dv.
\end{aligned}$$

□

Example 3.4.1. *According Proposition 3.4.1, one gets the curve of* $\overline{RVaR}_{\alpha_1,\alpha_2,x_1}(\boldsymbol{X})$ *and* $\overline{RVaR}_{\alpha_1,\alpha_2,x_2}(\boldsymbol{X})$ *in Figure 3.9 and Figure 3.10, respectively. For comparison, we plot the curve of the upper orthant* VaR *in the same graph.*

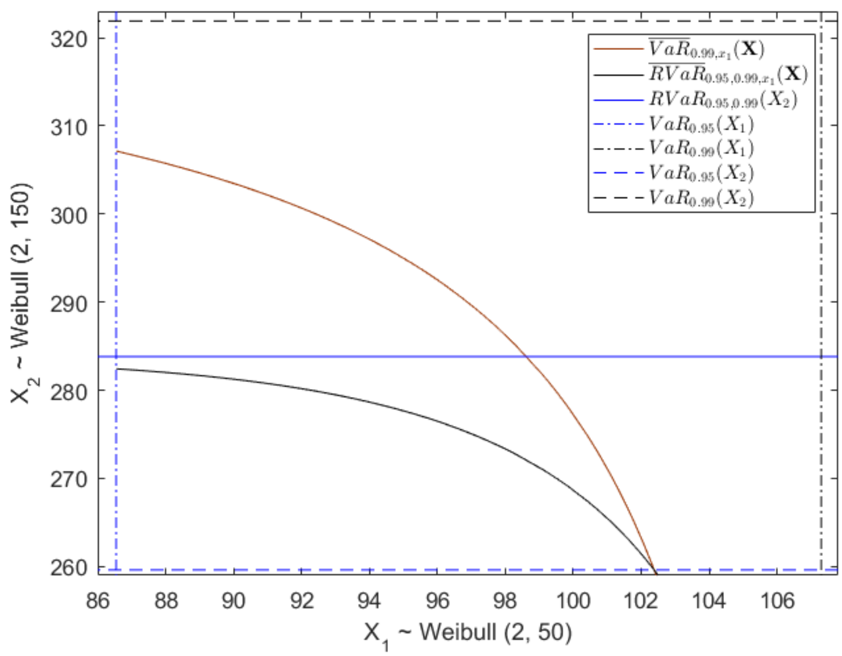

Fig. 3.9: Upper orthant VaR at level 0.99 and RVaR at level range $[0.95, 0.99]$ for fixed values of X_1

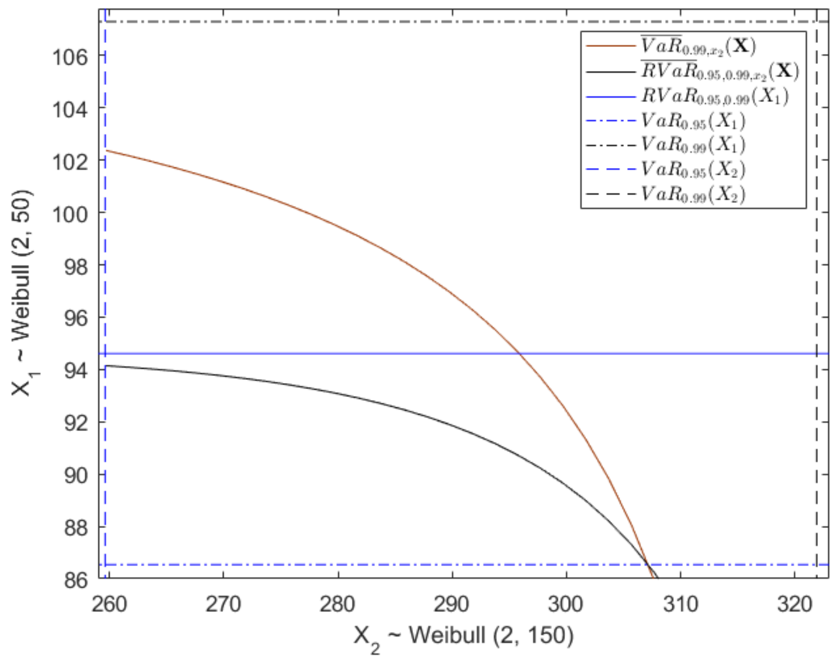

Fig. 3.10: Upper orthant VaR at level 0.99 and RVaR at level range $[0.95, 0.99]$ for fixed values of X_2

Figure 3.9 and Figure 3.10 show the shape of the upper orthant RVaR curve. $\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X})$ converges to $RVaR_{0.95,0.99}(X_j)$ when x_i $(i = 1, 2)$ gets close to lower support of X_i. Also, when x_i approaches $VaR_{\alpha_2}(X_i)$, $\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X})$ approaches $VaR_{\alpha_1}(X_j)$. As a result, the curves of bivariate RVaR are bounded by the curves of univariate VaR, which is similar to the univariate RVaR.

3.5 Properties of Bivariate RVaR

Proposition 3.5.1. *Let $\boldsymbol{X} = (X_1, X_2)$ be a continuous random vector.*

1. *(Translation invariance) For all $\boldsymbol{c} = (c_1, c_2) \in \mathbb{R}^2$ and $i, j = 1, 2$, $i \neq j$, then*

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_j+c_j}(\boldsymbol{X}+\boldsymbol{c}) = \underline{RVaR}_{\alpha_1,\alpha_2,x_j}(\boldsymbol{X}) + c_i,$$

$$\overline{RVaR}_{\alpha_1,\alpha_2,x_j+c_j}(\boldsymbol{X}+\boldsymbol{c}) = \overline{RVaR}_{\alpha_1,\alpha_2,x_j}(\boldsymbol{X}) + c_i.$$

2. *(Positive homogeneity) For all $\boldsymbol{c} = (c_1, c_2) \in \mathbb{R}^2_+$ and $i, j = 1, 2$, $i \neq j$, then*

$$\underline{RVaR}_{\alpha_1,\alpha_2,c_jx_j}(\boldsymbol{cX}) = c_i\underline{RVaR}_{\alpha_1,\alpha_2,x_j}(\boldsymbol{X}),$$

$$\overline{RVaR}_{\alpha_1,\alpha_2,c_jx_j}(\boldsymbol{cX}) = c_i\overline{RVaR}_{\alpha_1,\alpha_2,x_j}(\boldsymbol{X}).$$

3. *(Monotonicity) Let $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{X}' = (X_1', X_2')$ be two pairs of risks with joint cdf's $F_{\mathbf{X}}$ and $F_{\mathbf{X}'}$ respectively. If $\boldsymbol{X} \prec_{co} \boldsymbol{X}'$, then*

$$\underline{RVaR}_{\alpha_1,\alpha_2}(\boldsymbol{X}') \prec \underline{RVaR}_{\alpha_1,\alpha_2}(\boldsymbol{X}),$$

$$\overline{RVaR}_{\alpha_1,\alpha_2}(\boldsymbol{X}) \prec \overline{RVaR}_{\alpha_1,\alpha_2}(\boldsymbol{X}').$$

Proof. Here we need to use the properties of bivariate VaR in Proposition 3.2.1 to proof the above results.

For Translation invariance,

$$\begin{aligned}
\underline{RVaR}_{\alpha_1,\alpha_2,x_j+c_j}(\mathbf{X}+\mathbf{c}) =& \frac{\int_{\alpha_1}^{F(x_j,VaR_{\alpha_2}(X_i))} \underline{VaR}_{u,x_j+c_j}(\mathbf{X}+\mathbf{c})du}{F(x_j, VaR_{\alpha_2}(X_i)) - \alpha_1} \\
=& \frac{\int_{\alpha_1}^{F(x_j,VaR_{\alpha_2}(X_i))} \underline{VaR}_{u,x_j}(\mathbf{X}) + c_i du}{F(x_j, VaR_{\alpha_2}(X_i)) - \alpha_1} \\
=& \frac{\int_{\alpha_1}^{F(x_j,VaR_{\alpha_2}(X_i))} \underline{VaR}_{u,x_j}(\mathbf{X})du}{F(x_j, VaR_{\alpha_2}(X_i)) - \alpha_1} + c_i \\
=& \underline{RVaR}_{\alpha_1,\alpha_2,x_j}(\mathbf{X}) + c_i.
\end{aligned}$$

For Positive Homogeneity,

$$\begin{aligned}\underline{RVaR}_{\alpha_1,\alpha_2,c_jx_j}(\mathbf{cX}) =& \frac{\int_{\alpha_1}^{F(x_j,VaR_{\alpha_2}(X_i))} \underline{VaR}_{u,c_jx_j}(\mathbf{cX})du}{F(x_j,VaR_{\alpha_2}(X_i))-\alpha_1}\\ =& \frac{\int_{\alpha_1}^{F(x_j,VaR_{\alpha_2}(X_i))} c_i\underline{VaR}_{u,x_j}(\mathbf{X})du}{F(x_j,VaR_{\alpha_2}(X_i))-\alpha_1}\\ =& \frac{c_i\int_{\alpha_1}^{F(x_j,VaR_{\alpha_2}(X_i))} \underline{VaR}_{u,x_j}(\mathbf{X})du}{F(x_j,VaR_{\alpha_2}(X_i))-\alpha_1}\\ =& c_i\underline{RVaR}_{\alpha_1,\alpha_2,x_j}(\mathbf{X}).\end{aligned}$$

Using the same way, we could get similar results for upper orthant RVaR.

Moreover, if $\mathbf{X} \prec_{co} \mathbf{X}'$, then $\underline{VaR}_{\alpha,x_i}(\mathbf{X}') \prec \underline{VaR}_{\alpha,x_i}(\mathbf{X})$. According to the Definition 3.3.1, we have

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}) = E[X_j|\underline{VaR}_{\alpha_1,x_i}(\mathbf{X}) \le X_j \le VaR_{\alpha_2}(X_j), X_i \le x_i],$$

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i'}(\mathbf{X}) = E[X_j'|\underline{VaR}_{\alpha_1,x_i'}(\mathbf{X}') \le X_j' \le VaR_{\alpha_2}(X_j'), X_i' \le x_i'],$$

and $\mathbf{X}$, $\mathbf{X}'$ have same marginal cdfs. Hence, the expectation over the interval $[\alpha_1, \alpha_2]$ will following the same pattern, say

$$\underline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}') \prec \underline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}).$$

Also, since

$$\bar{F}_{\mathbf{x}}(\mathbf{X}) = 1 - F(X_1) - F(X_2) + F_{\mathbf{x}}(\mathbf{X}),$$

$$\bar{F}_{\mathbf{x}'}(\mathbf{X}') = 1 - F(X_1) - F(X_2) + F_{\mathbf{x}'}(\mathbf{X}'),$$

then $\bar{F}_{\mathbf{x}}(\mathbf{X}) \le \bar{F}_{\mathbf{x}'}(\mathbf{X}')$. Thus, $\overline{VaR}_{\alpha,x_i}(\mathbf{X}) \prec \overline{VaR}_{\alpha,x_i}(\mathbf{X}')$. Again, for the same reason, we can conclude that

$$\overline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}) \prec \overline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}').$$

□

Consider the random vectors $\mathbf{X}_M$, $\mathbf{X}_W$ and $\mathbf{X}_\Pi$ which denote the monotonic, inverse monotonic and independent vector, respectively. They have following relationship

$$\mathbf{X}_W \prec_{co} \mathbf{X}_\Pi \prec_{co} \mathbf{X}_M,$$

which means according to the Proposition 3.5.1, we have

$$\underline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}_M) \prec \underline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}_\Pi) \prec \underline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}_W),$$

and

$$\overline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}_W) \prec \overline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}_\Pi) \prec \overline{RVaR}_{\alpha_1,\alpha_2}(\mathbf{X}_M).$$

Example 3.5.1. *Consider a bivariate random vector (X_1, X_2) which is either comonotonic, counter-conomotonic or independent. We obtain the lower orthant RVaR based on the Proposition 3.3.1. For $i, j = 1, 2$ $(i \neq j)$,*

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}_\Pi) = \frac{1}{\alpha_2 F_{X_i}(x_i) - \alpha_1} \int_{\alpha_1}^{\alpha_2 F_{X_i}(x_i)} VaR_{\frac{u}{F_{X_i}(x_i)}}(X_j)du,$$

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}_M) = \frac{1}{F_{X_i}(x_i) - \alpha_1} \int_{\alpha_1}^{F_{X_i}(x_i)} VaR_u(X_j)du,$$

and

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}_W) = \frac{1}{F_{X_i}(x_i) + \alpha_2 - 1 - \alpha_1} \int_{\alpha_1}^{F_{X_i}(x_i)+\alpha_2-1} VaR_{u-F_{X_i}(x_i)+1}(X_j)du.$$

If the random vector above is defined with exponential marginal cdfs, i.e. $X_i \sim Exp(\lambda_i)$, then we get the following results.

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}_\Pi) = \frac{1}{\alpha_2 F_{X_i}(x_i) - \alpha_1}\left(\frac{1}{\lambda_i}\left[(F_{X_i}(x_i) - \alpha_2 F_{X_i}(x_i))\ln(1-\alpha_2)\right.\right.$$
$$\left.\left. -(F_{X_i}(x_i) - \alpha_1)\ln\left(\frac{F_{X_i}(x_i) - \alpha_1}{F_{X_i}(x_i)}\right) + (\alpha_2 F_{X_i}(x_i) - \alpha_1)\right]\right),$$
$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}_M) = \frac{1}{F_{X_i}(x_i) - \alpha_1}\left(\frac{1}{\lambda_i}\left[(1 - F_{X_i}(x_i))\ln(1 - F_{X_i}(x_i))\right.\right.$$
$$\left.\left. - (1-\alpha_1)\ln(1-\alpha_1) + (F_{X_i}(x_i) - \alpha_1)\right]\right),$$
$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}_W) = \frac{1}{(F_{X_i}(x_i) + \alpha_2 - 1) - \alpha_1}\left(\frac{1}{\lambda_i}\left[(1-\alpha_2)\ln(1-\alpha_2)\right.\right.$$
$$\left.\left. - (F_{X_i}(x_i) - \alpha_1)\ln(F_{X_i}(x_i) - \alpha_1) + (F_{X_i}(x_i) + \alpha_2 - 1 - \alpha_1)\right]\right).$$

Proposition 3.5.2. *Let $\boldsymbol{X} = (X_1, X_2)$ be a pair of random variables with cdf $F_{\mathbf{X}}$ and marginal distributions F_{X_1} and F_{X_2}. Assume that $F_{\mathbf{X}}$ is continuous and strictly increasing. Then, for $i, j = 1, 2$ and $i \neq j$,*

$$\lim_{x_i \to \underline{VaR}_{\alpha_1, VaR_{\alpha_2}(X_j)}(\boldsymbol{X})} \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = VaR_{\alpha_2}(X_j),$$
$$\lim_{x_i \to \overline{VaR}_{\alpha_2, VaR_{\alpha_1}(X_j)}(\boldsymbol{X})} \overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = VaR_{\alpha_1}(X_j).$$

Moreover,

$$\lim_{x_i \to u_{x_i}} \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = RVaR_{\alpha_1,\alpha_2}(X_j),$$
$$\lim_{x_i \to l_{x_i}} \overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = RVaR_{\alpha_1,\alpha_2}(X_j),$$

where u_{x_i} (or l_{x_i}) represents the upper (or lower) support of the rv. X_i.

Proof. One has that

$$\lim_{x_i \to \underline{VaR}_{\alpha_1, VaR_{\alpha_2}(X_j)}(\mathbf{X})} \underline{VaR}_{\alpha_1,x_i}(\mathbf{X}) = VaR_{\alpha_2}(X_j).$$

Thus, integrating this constant on the interval $[\alpha_1, F(x_i, VaR_{\alpha_2}(X_j))]$ results in $VaR_{\alpha_2}(X_j)$. Similarly, we can prove that when x_i approaches the upper bound $\overline{VaR}_{\alpha_2, VaR_{\alpha_1}(X_j)}(\mathbf{X})$, $\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X})$ will approaches $VaR_{\alpha_1}(X_j)$. Furthermore, we have that

$$\lim_{x_i \to u_{x_i}} \underline{VaR}_{u,x_i}(\mathbf{X}) = VaR_u(X_j).$$

Combined with $F(u_{x_i}, VaR_{\alpha_2}(X_j)) = \alpha_2$, we get the result that

$$\lim_{x_i \to u_{x_i}} \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}) = \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} VaR_u(X_j) du = RVaR_{\alpha_1,\alpha_2}(X_j).$$

Similarly, we can prove the limit of $\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X})$ is also $RVaR_{\alpha_1,\alpha_2}(X_j)$. □

Now, we consider the behavior of aggregate risks defined as follows:

$$\mathbf{S} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} X_i \\ Y_i \end{pmatrix},$$

where S_1 and S_2 denote the aggregate amount of claims for two different business class respectively. X_i and Y_i represent the risks within each class, where $i = 1, \ldots, n$, such that $S_1 = \sum_{i=1}^{n} X_i$ and $S_2 = \sum_{i=1}^{n} Y_i$.

Unlike univariate TVaR, the univariate RVaR does not satisfy the subadditivity. Hence, it seems impossible to prove that the bivariate RVaR is subadditive. However, if we suppose that $(X_1, \ldots, X_n)$ (respectively $(Y_1, \ldots, Y_n)$) is comonotonic, the following results can be obtained.

Proposition 3.5.3. *Let $(X_1, \ldots, X_n)$ (respectively $(Y_1, \ldots, Y_n)$) be comonotonic with cdf's $F_{X_1}, \ldots, F_{X_n}$ (respectively $G_{Y_1}, \ldots, G_{Y_n}$). The dependence structure between $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ is unknown. Then,*

$$\underline{RVaR}_{\alpha_1,\alpha_2,S_1}(\boldsymbol{S}) = \sum_{i=1}^{n} \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(X_i, Y_i),$$

$$\underline{RVaR}_{\alpha_1,\alpha_2,S_2}(\boldsymbol{S}) = \sum_{i=1}^{n} \underline{RVaR}_{\alpha_1,\alpha_2,y_i}(X_i, Y_i),$$

and

$$\overline{RVaR}_{\alpha_1,\alpha_2,S_1}(\boldsymbol{S}) = \sum_{i=1}^{n} \overline{RVaR}_{\alpha_1,\alpha_2,x_i}(X_i, Y_i),$$

$$\overline{RVaR}_{\alpha_1,\alpha_2,S_2}(\boldsymbol{S}) = \sum_{i=1}^{n} \overline{RVaR}_{\alpha_1,\alpha_2,y_i}(X_i, Y_i).$$

Proof. Define $F_{S_1}^{-1}(u) = \sum_{i=1}^{n} F_{X_i}^{-1}(u)$ and $F_{S_2}^{-1}(u) = \sum_{i=1}^{n} G_{Y_i}^{-1}(u)$. If $(X_1, \ldots, X_n)$ (respectively $(Y_1, \ldots, Y_n)$) is comonotonic, then there exists a uniform random variable U_1 (respectively U_2) such that $S_1 = F_{S_1}^{-1}(U_1)$ (respectively $S_2 = F_{S_2}^{-1}(U_2)$). Hence,

$$\begin{aligned}
\underline{RVaR}_{\alpha_1,\alpha_2,S_2}(\mathbf{S}) &= \frac{\int_{\alpha_1}^{F(s_1, VaR_{\alpha_2}(S_2))} \underline{VaR}_{u,S_2}\left(F_{S_1}^{-1}(U_1), F_{S_2}^{-1}(U_2)\right) du}{F(s_1, VaR_{\alpha_2}(S_2)) - \alpha_1} \\
&= \frac{\int_{\alpha_1}^{F(s_1, VaR_{\alpha_2}(S_2))} F_{S_1}^{-1}\left(\underline{VaR}_{u,F_{S_2}(s_2)}(U_1, U_2)\right) du}{F(s_1, VaR_{\alpha_2}(S_2)) - \alpha_1} \\
&= \sum_{i=1}^{n} \frac{\int_{\alpha_1}^{F(x_i, VaR_{\alpha_2}(Y_i))} \underline{VaR}_{u,y_i}\left(F_{X_i}^{-1}(U_1), G_{Y_i}^{-1}(U_2)\right) du}{F(x_i, VaR_{\alpha_2}(Y_i)) - \alpha_1} \\
&= \sum_{i=1}^{n} \frac{\int_{\alpha_1}^{F(x_i, VaR_{\alpha_2}(Y_i))} \underline{VaR}_{u,y_i}(X_i, Y_i)\, du}{F(x_i, VaR_{\alpha_2}(Y_i)) - \alpha_1} \\
&= \sum_{i=1}^{n} \underline{RVaR}_{\alpha_1,\alpha_2,y_i}(X_i, Y_i).
\end{aligned}$$

Other equations in the Proposition 3.5.3 can be proved in the same way. □

In conclusion, the bivariate RVaR has similar properties to the bivariate VaR and TVaR, such as translation invariance, positive homogeneity and monotonicity. Furthermore, it has an advantage over bivariate VaR and TVaR. Compared to bivariate VaR, bivariate TVaR and RVaR provide essential information about the tail of the distribution. Moreover, $\underline{TVaR}_{\alpha,x_i}(\mathbf{X})$ and $\overline{TVaR}_{\alpha,x_i}(\mathbf{X})$ will go to infinity when X_i approaches $VaR_\alpha(X_i)$ whereas the bivariate RVaR is bounded in the area $[VaR_{\alpha_1}(X_i), VaR_{\alpha_2}(X_i)] \times [VaR_{\alpha_1}(X_j), VaR_{\alpha_2}(X_j)]$. This measure could be useful for insurance companies that must set aside capital for risks that are sent to a reinsurer after having reached a certain level. Assume that the insurance company transfers the risks to the reinsurer when the total losses exceed the VaR at level α_2. To be considered solvent, the insurance company need to measure the risks with truncated data. In this case, multivariate RVaR could be helpful.

Next, we will propose the empirical estimator of bivariate RVaR with numerical examples. The robustness of multivariate risk measures will be checked.

4. EMPIRICAL ESTIMATORS AND ROBUSTNESS OF RVAR

4.1 Empirical Estimator for Bivariate RVaR

Definition 4.1.1. *Consider a series of observations* $\boldsymbol{X} = (X_1, X_2)$ *with* $X_i = (x_{1i}, \ldots, x_{ni})$, $i = 1, 2$. *Additionally, we have* $\boldsymbol{x}_l = (x_{l1}, x_{l2}) \in \mathbb{R}_+^2$, $l = 1, \ldots, n$. *Denote* F_n *and* $F_{n,i}$, *the empirical cdf's (ecdf) for* $\boldsymbol{X}$ *and* X_i, *respectively. We define the estimator for the lower orthant RVaR, for a fixed* X_i, *by*

$$\underline{RVaR}^n_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = \frac{\int_{\alpha_1}^{F_n(x_i, VaR_{\alpha_2}(X_j))} \underline{VaR}^n_{u,x_i}(\boldsymbol{X}) du}{F_n(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1}.$$

For $m \in \mathbb{N}$ *large enough, let* $s = \frac{F_n(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1}{m}$ *and* $u_k = \alpha_1 + ks$, *then the above expression can be simplified into*

$$\begin{aligned} \underline{RVaR}^n_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) &= \sum_{k=1}^{m} \frac{\underline{VaR}^n_{u_k,x_i}(\boldsymbol{X}) \cdot s}{F_n(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} \\ &= \sum_{k=1}^{m} \frac{\underline{VaR}^n_{u_k,x_i}(\boldsymbol{X})}{m}, \end{aligned}$$

where $\underline{VaR}^n_{u,x_i}(\boldsymbol{X}) = \inf\{x_j \in \mathbb{R}_+ : F_{n,x_i}(x_j) \geq u\}$ *is the empirical lower orthant* VaR *given* X_i *and* F_{n,x_i} *the ecdf of* $\boldsymbol{X}$ *given the same* X_i.

Note, $\underline{VaR}^n_{u,x_i}(\mathbf{X})$ is the smallest value of X_j given X_i such that F_n is larger than u. Similarly, we define the empirical estimator of upper orthant $RVaR$ as follows.

Definition 4.1.2. *Consider a series of observations* $\boldsymbol{X} = (X_1, X_2)$ *with* $X_i = (x_{1i}, \ldots, x_{ni})$, $i = 1, 2$. *Additionally, we have* $\boldsymbol{x}_l = (x_{l1}, x_{l2}) \in \mathbb{R}^2_+$, $l = 1, \ldots, n$. *Denote* $\bar{F}_n$ *and* $\bar{F}_{n,i}$, *the empirical sf's (esf) for* $\boldsymbol{X}$ *and* X_i, *respectively. For a fixed* X_i, *the estimator for the lower orthant RVaR is defined by*

$$\overline{RVaR}^n_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) = \frac{\int_{1-\bar{F}_n(x_i, VaR_{\alpha_1}(X_j))}^{\alpha_2} \overline{VaR}^n_{v,x_i}(\boldsymbol{X}) du}{\alpha_2 - (1 - \bar{F}_n(x_i, VaR_{\alpha_1}(X_j)))}.$$

For $m \in \mathbb{N}$ *large enough. Let* $s = \frac{\alpha_2 - (1 - \bar{F}_n(x_i, VaR_{\alpha_1}(X_j)))}{m}$ *and* $v_k = 1 - \bar{F}_n(x_i, VaR_{\alpha_1}(X_j)) + ks$, *then the above expression can be simplified into*

$$\begin{aligned} \overline{RVaR}^n_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) &= \sum_{k=1}^{m} \frac{\overline{VaR}^n_{v_k,x_i}(\boldsymbol{X}) \cdot s}{\alpha_2 - (1 - \bar{F}_n(x_i, VaR_{\alpha_1}(X_j)))} \\ &= \sum_{k=1}^{m} \frac{\overline{VaR}^n_{v_k,x_i}(\boldsymbol{X})}{m}, \end{aligned}$$

where $\overline{VaR}^n_{v,x_i}(\boldsymbol{X}) = \inf\{x_j \in \mathbb{R}_+ : \bar{F}_{n,x_i}(x_j) \leq 1 - v\}$ *is the empirical upper orthant VaR given* X_i *and* $\bar{F}_{n,x_i}$ *the esf of* $\boldsymbol{X}$ *given the same* X_i.

The following proposition, based on the proof of the consistency of bivariate VaR in *Cousin and Di Bernardino* (2013), shows the consistency of the bivariate RVaR in Hausdorff distance. For $\alpha \in (0, 1)$ and $r, \zeta > 0$, consider the ball

$$E = B(\{\mathbf{x} \in \mathbb{R}^2_+ : |F(x) - \alpha| \leq r\}, \zeta).$$

Denote $m^{\nabla} = \inf_{\mathbf{x} \in E} \parallel (\nabla F)_{\mathbf{x}} \parallel$ as the infimum of the Euclidean norm of the gradient vector and $M_H = \sup_{\mathbf{x} \in E} \parallel (HF)_{\mathbf{x}} \parallel$ as the matrix norm of the Hessian matrix evaluated at $\mathbf{x}$ for a twice differentiable $F(x_1, x_2)$.

Proposition 4.1.1. *Let $[\alpha_1, \alpha_2] \subset (0,1)$ and $F(x_1, x_2)$ be twice differentiable on $\mathbb{R}^2$. Assume there exists $r, \zeta > 0$ such that $m^{\nabla} > 0$ and $M_H < \infty$. Assume for each n, F_n is continuous with probability one (wp1) and*

$$\| F - F_n \| \xrightarrow[n\to\infty]{wp1} 0.$$

And let $F_{n,i}$ be the consistent estimator of F_i, we have

$$\underline{RVaR}^n_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}) \xrightarrow[n\longrightarrow\infty]{wp1} \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}).$$

Proof. According to the theorem 2.1 in Beck (2015), we have

$$\underline{VaR}^n_{u,x_i}(\mathbf{X}) \xrightarrow[n\longrightarrow\infty]{wp1} \underline{VaR}_{u,x_i}(\mathbf{X}).$$

for any $u \in (0,1)$. From the assumption, one has

$$F_{n,i} \xrightarrow[n\longrightarrow\infty]{wp1} F_i.$$

Then,

$$\begin{aligned}\mathbf{1}_{\left[\alpha_1, F_n(x_i, VaR_{\alpha_2}(X_j))\right]}(u) &= \begin{cases} 1, & u \in [\alpha_1, F_n(x_i, VaR_{\alpha_2}(X_j))] \\ 0, & otherwise \end{cases} \\ &\xrightarrow[n\longrightarrow\infty]{wp1} \begin{cases} 1, & u \in [\alpha_1, F(x_i, VaR_{\alpha_2}(X_j))] \\ 0, & otherwise \end{cases} \\ &= \mathbf{1}_{\left[\alpha_1, F(x_i, VaR_{\alpha_2}(X_j))\right]}(u).\end{aligned}$$

As a result, it can be seen that

$$\frac{\underline{VaR}^n_{u,x_i}(\mathbf{X})\mathbf{1}_{\left[\alpha_1, F_n(x_i, VaR_{\alpha_2}(X_j))\right]}(u)}{F_n(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} \xrightarrow[n\longrightarrow\infty]{wp1} \frac{\underline{VaR}_{u,x_i}(\mathbf{X})\mathbf{1}_{\left[\alpha_1, F(x_i, VaR_{\alpha_2}(X_j))\right]}(u)}{F(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1}.$$

Therefore, by the dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \underline{RVaR}^n_{\alpha_1,\alpha_2,x_i}(\mathbf{X}) &= \lim_{n\to\infty} \int \frac{\underline{VaR}^n_{u,x_i}(\mathbf{X})\mathbf{1}_{\left[\alpha_1, F_n(x_i, VaR_{\alpha_2}(X_j))\right]}(u)}{F_n(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} du \\
&= \int \frac{\underline{VaR}_{u,x_i}(\mathbf{X})\mathbf{1}_{\left[\alpha_1, F(x_i, VaR_{\alpha_2}(X_j))\right]}(u)}{F(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} du \\
&= \frac{\int_{\alpha_1}^{F(x_i, VaR_{\alpha_2}(X_j))} \underline{VaR}_{u,x_i}(\mathbf{X}) du}{F(x_i, VaR_{\alpha_2}(X_j)) - \alpha_1} \\
&= \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}).
\end{aligned}$$

Note, the consistency of upper orthant RVaR could be proved in the same way. □

Next, we will check the robustness of the estimator of bivariate VaR, TVaR and RVaR. Since all of them are distribution-based risk measures, the sensitivity function can be used to quantify the robustness.

4.2 Robustness of Multivariate Risk Measures

Proposition 4.2.1. *For a pair of continuous random variables* $\boldsymbol{X}$ *with joint cdf* $F(x_1, x_2)$ *and marginals* $F_{X_1}(x_1)$ *and* $F_{X_2}(x_2)$*, the sensitivity function of* $\underline{VaR}_{\alpha,x_i}(\boldsymbol{X})$ *is given by*

$$S(z) = \begin{cases} -\frac{F_{X_i}(x_i)-\alpha}{f_{x_i}\left[\underline{VaR}_{\alpha,x_i}(\boldsymbol{X})\right]F_{X_i}(x_i)}, & z < \underline{VaR}_{\alpha,x_i}(\boldsymbol{X}) \\ \frac{\alpha}{f_{x_i}\left[\underline{VaR}_{\alpha,x_i}(\boldsymbol{X})\right]F_{X_i}(x_i)}, & z > \underline{VaR}_{\alpha,x_i}(\boldsymbol{X}) \\ 0, & \text{otherwise} \end{cases}$$

which is bounded. Thus, $\underline{VaR}_{\alpha,x_i}(\boldsymbol{X})$ *is a robust risk measure.*

Proof. Let $F_{x_i}(x_j) = P(X_j \leq x_j | X_i \leq x_i)$ be the conditional distribution of X_j knowing X_i, $i, j = 1, 2$ $(i \neq j)$. For any fixed x_i and $\varepsilon \in [0, 1)$, set $F_{\varepsilon,x_i}(x_j) = \varepsilon\delta_z + (1-\varepsilon)F_{x_i}(x_j)$. The distribution F_{ε,x_i} is differentiable at any $x_j \neq z$ with $F'_{\varepsilon,x_i}(x_j) = (1-\varepsilon)f_{x_i}(x_j) > 0$ and has a jump at the point $x_j = z$.

We have that

$$\underline{VaR}_{\alpha,x_i}(\mathbf{X}) = VaR_{\frac{\alpha}{F_{X_i}(x_i)}}(X_j | X_i \leq x_i).$$

Then,

$$\begin{aligned}
\underline{VaR}_{\alpha,x_i}(F_{\varepsilon,x_i}) &= F^{-1}_{\varepsilon,x_i}\left(\frac{\alpha}{F_{X_i}(x_i)}\right) \\
&= \begin{cases} F^{-1}_{x_i}\left(\frac{\alpha}{(1-\varepsilon)F_{X_i}(x_i)}\right), & \frac{\alpha}{F_{X_i}(x_i)} < (1-\varepsilon)F_{x_i}(z) \\ F^{-1}_{x_i}\left(\frac{\alpha/F_{X_i}(x_i)-\varepsilon}{1-\varepsilon}\right), & \frac{\alpha}{F_{X_i}(x_i)} \geq (1-\varepsilon)F_{x_i}(z)+\varepsilon; \\ \text{z}, & \text{otherwise.} \end{cases}
\end{aligned}$$

As a consquence, the sensitivity function of $\underline{VaR}_{\alpha,x_i}(\mathbf{X})$ can be evaluated by

$$\begin{aligned}
S(z) &= \lim_{\varepsilon \to 0^+} \frac{\underline{VaR}_{\alpha,x_i}(F_{\varepsilon,x_i}) - \underline{VaR}_{\alpha,x_i}(F_{x_i})}{\varepsilon} \\
&= \left[\frac{d}{d\varepsilon}\underline{VaR}_{\alpha,x_i}(F_{\varepsilon,x_i})\right]_{\varepsilon=0} \\
&= \begin{cases} -\frac{F_{X_i}(x_i)-\alpha}{f_{x_i}\left[\underline{VaR}_{\alpha,x_i}(\mathbf{X})\right]F_{X_i}(x_i)}, & z < \underline{VaR}_{\alpha,x_i}(\mathbf{X}) \\ \frac{\alpha}{f_{x_i}\left[\underline{VaR}_{\alpha,x_i}(\mathbf{X})\right]F_{X_i}(x_i)}, & z > \underline{VaR}_{\alpha,x_i}(\mathbf{X}) \\ 0, & z = \underline{VaR}_{\alpha,x_i}(\mathbf{X}). \end{cases}
\end{aligned}$$

The result shows that $\underline{VaR}_{\alpha,x_i}(\mathbf{X})$ has a bounded sensitivity function for any fixed x_i, which means it is a robust statistic. Note that this conclusion coincides with the one associated with the univariate VaR. □

Proposition 4.2.2. *For a pair of continuous random variables* $\boldsymbol{X}$ *with joint cdf* $F(x_1, x_2)$ *and marginals* $F_{X_1}(x_1)$ *and* $F_{X_2}(x_2)$*, the sensitivity function of* $\underline{RVaR}_{0.95,0.99,x_i}(\boldsymbol{X})$ *is given by*

$$S(z) = \begin{cases} \frac{(F_{X_i}(x_i)-\alpha_1)\underline{VaR}_{\alpha_1,x_i}(\boldsymbol{X})-(F_{X_i}(x_i)-\beta)VaR_{\alpha_2}(X_j)}{\beta-\alpha_1} - \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}), & z < \underline{VaR}_{\alpha_1,x_i}(\boldsymbol{X}) \\ \frac{zF_{X_i}(x_i)-\alpha_1\underline{VaR}_{\alpha_1,x_i}(\boldsymbol{X})-(F_{X_i}(x_i)-\beta)VaR_{\alpha_2}(X_j)}{\beta-\alpha_1} - \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}), & \underline{VaR}_{\alpha_1,x_i}(\boldsymbol{X}) \le z \le VaR_{\alpha_2}(X_j) \\ \frac{\beta VaR_{\alpha_2}(X_j)-\alpha_1\underline{VaR}_{\alpha_1,x_i}(\boldsymbol{X})}{\beta-\alpha_1} - \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}), & z > VaR_{\alpha_2}(X_j) \end{cases}$$

which is a bounded function. Thus, $\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X})$ *is robust.*

Proof. The sensitivity function of

$$\underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}) = \frac{1}{\beta-\alpha_1}\int_{\alpha_1}^{\beta} \underline{VaR}_{u,x_i}(\mathbf{X})du,$$

where $\beta = F(x_i, VaR_{\alpha_2}(X_j))$ is given by

$$S(z) = \frac{1}{\beta-\alpha_1}\int_{\alpha_1}^{\beta}\left[\lim_{\varepsilon\to 0^+}\frac{\underline{VaR}_{u,x_i}(F_{\varepsilon,x_i})-\underline{VaR}_{u,x_i}(F_{x_i})}{\varepsilon}\right]du$$

$$= \frac{1}{\beta-\alpha_1}\int_{\alpha_1}^{\beta}\left[\frac{d}{d\varepsilon}\underline{VaR}_{u,x_i}(F_{\varepsilon,x_i})\right]_{\varepsilon=0} du$$

$$= \begin{cases} \frac{1}{\beta-\alpha_1}\int_{\alpha_1}^{\beta} -\frac{F_{X_i}(x_i)-u}{f_{x_i}\left[\underline{VaR}_{u,x_i}(\mathbf{X})\right]F_{X_i}(x_i)}du, & z < \underline{VaR}_{\alpha,x_i}(\mathbf{X}) \\ \frac{1}{\beta-\alpha_1}\left\{\int_{\alpha_1}^{F(x_i,z)}\frac{u}{f_{x_i}\left[\underline{VaR}_{u,x_i}(\mathbf{X})\right]F_{X_i}(x_i)}du \right. & \\ \left. + \int_{F(x_i,z)}^{\beta} -\frac{F_{X_i}(x_i)-u}{f_{x_i}\left[\underline{VaR}_{u,x_i}(\mathbf{X})\right]F_{X_i}(x_i)}du\right\}, & \underline{VaR}_{\alpha,x_i}(\mathbf{X}) \le z \le VaR_{\alpha_2}(X_j) \\ \frac{1}{\beta-\alpha_1}\int_{\alpha_1}^{\beta}\frac{u}{f_{x_i}\left[\underline{VaR}_{u,x_i}(\mathbf{X})\right]F_{X_i}(x_i)}du, & z > VaR_{\alpha_2}(X_j) \end{cases}$$

$$= \begin{cases} \frac{(F_{X_i}(x_i)-\alpha_1)\underline{VaR}_{\alpha_1,x_i}(\mathbf{X})-(F_{X_i}(x_i)-\beta)VaR_{\alpha_2}(X_j)}{\beta-\alpha_1} - \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}), & z < \underline{VaR}_{\alpha_1,x_i}(\mathbf{X}) \\ \frac{zF_{X_i}(x_i)-\alpha_1\underline{VaR}_{\alpha_1,x_i}(\mathbf{X})-(F_{X_i}(x_i)-\beta)VaR_{\alpha_2}(X_j)}{\beta-\alpha_1} - \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}), & \underline{VaR}_{\alpha_1,x_i}(\mathbf{X}) \leq z \leq VaR_{\alpha_2}(X_j) \\ \frac{\beta VaR_{\alpha_2}(X_j)-\alpha_1\underline{VaR}_{\alpha_1,x_i}(\mathbf{X})}{\beta-\alpha_1} - \underline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}), & z > VaR_{\alpha_2}(X_j) \end{cases}$$

Furthermore, the sensitivity function of $\underline{TVaR}_{\alpha,x_i}(\mathbf{X})$ can be obtained, when $\beta = F_{X_i}(x_i)$. Then,

$$S(z) = \begin{cases} \underline{VaR}_{\alpha,x_i}(\mathbf{X}) - \underline{TVaR}_{\alpha,x_i}(\mathbf{X}), z < \underline{VaR}_{\alpha,x_i}(\mathbf{X}) \\ \frac{zF_{X_i}(x_i)-\alpha\underline{VaR}_{\alpha,x_i}(\mathbf{X})}{F_{X_i}(x_i)-\alpha} - \underline{TVaR}_{\alpha,x_i}(\mathbf{X}), z \geq \underline{VaR}_{\alpha,x_i}(\mathbf{X}). \end{cases}$$

Obviously, it is linear in z, which implies that $\underline{TVaR}_{\alpha,x_i}(\mathbf{X})$ is not a robust statistic, which also coincides with univariate TVaR. □

Proposition 4.2.3. *For a pair of continuous random variables $\boldsymbol{X}$ with joint sf $\bar{F}(x_1, x_2)$ and marginals $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, the sensitivity function of $\overline{VaR}_{\alpha,x_i}(\boldsymbol{X})$ is given by*

$$S(z) = \begin{cases} -\frac{1-\alpha}{f_{\bar{x}_i}\left[\overline{VaR}_{\alpha,x_i}(\boldsymbol{X})\right](1-F_{X_i}(x_i))}, & z < \overline{VaR}_{\alpha,x_i}(\boldsymbol{X}) \\ \frac{\alpha-F_{X_i}(x_i)}{f_{\bar{x}_i}\left[\overline{VaR}_{\alpha,x_i}(\boldsymbol{X})\right](1-F_{X_i}(x_i))}, & z > \overline{VaR}_{\alpha,x_i}(\boldsymbol{X}) \\ 0, & z = \overline{VaR}_{\alpha,x_i}(\boldsymbol{X}). \end{cases}$$

The bounded sensitivity function implies $\overline{VaR}_{\alpha,x_i}(\boldsymbol{X})$ is a robust risk measure.

Proof. Let $F_{\bar{x}_i}(x_j) = P(X_j \leq x_j | X_i \geq x_i)$ be the conditional distribution of X_j given $X_i \geq x_i$, $i, j = 1, 2$. For any fixed x_i and $\varepsilon \in [0, 1)$, set $F_{\varepsilon,\bar{x}_i}(x_j) = \varepsilon\delta_z + (1-\varepsilon)F_{\bar{x}_i}(x_j)$. $F_{\varepsilon,\bar{x}_i}$ is differentiable at any $x_j \neq z$ with $F'_{\varepsilon,\bar{x}_i}(x_j) = (1-\varepsilon)f_{\bar{x}_i}(x_j) > 0$ and has a jump at the point $x_j = z$.

Then, given that $\overline{VaR}_{\alpha,x_i}(\mathbf{X}) = VaR_{\frac{\alpha-F_{X_i}(x_i)}{1-F_{X_i}(x_i)}}(X_j|X_i \geq x_i)$, we have

$$\overline{VaR}_{\alpha,x_i}(F_{\varepsilon,\bar{x}_i}) = F^{-1}_{\varepsilon,\bar{x}_i}\left(\frac{\alpha - F_{X_i}(x_i)}{1 - F_{X_i}(x_i)}\right)$$
$$= \begin{cases} F^{-1}_{\bar{x}_i}\left(\frac{\alpha-F_{X_i}(x_i)}{(1-\varepsilon)(1-F_{X_i}(x_i))}\right), & \frac{\alpha-F_{X_i}(x_i)}{1-F_{X_i}(x_i)} < (1-\varepsilon)F_{\bar{x}_i}(z) \\ F^{-1}_{\bar{x}_i}\left(\frac{\frac{\alpha-F_{X_i}(x_i)}{1-F_{X_i}(x_i)}-\varepsilon}{1-\varepsilon}\right), & \frac{\alpha-F_{X_i}(x_i)}{1-F_{X_i}(x_i)} \geq (1-\varepsilon)F_{\bar{x}_i}(z) + \varepsilon; \\ \text{z}, & \text{otherwise.} \end{cases}$$

Hence, the sensitivity function of $\overline{VaR}_{\alpha,x_i}(\mathbf{X})$ can be obtained by

$$S(z) = \lim_{\varepsilon \to 0^+} \frac{\overline{VaR}_{\alpha,x_i}(F_{\varepsilon,\bar{x}_i}) - \overline{VaR}_{\alpha,x_i}(F_{\bar{x}_i})}{\varepsilon}$$
$$= \left[\frac{d}{d\varepsilon}\overline{VaR}_{\alpha,x_i}(F_{\varepsilon,\bar{x}_i})\right]_{\varepsilon=0}$$
$$= \begin{cases} -\frac{1-\alpha}{f_{\bar{x}_i}\left[\overline{VaR}_{\alpha,x_i}(\mathbf{X})\right](1-F_{X_i}(x_i))}, & z < \overline{VaR}_{\alpha,x_i}(\mathbf{X}) \\ \frac{\alpha-F_{X_i}(x_i)}{f_{\bar{x}_i}\left[\overline{VaR}_{\alpha,x_i}(\mathbf{X})\right](1-F_{X_i}(x_i))}, & z > \overline{VaR}_{\alpha,x_i}(\mathbf{X}) \\ 0, & z = \overline{VaR}_{\alpha,x_i}(\mathbf{X}). \end{cases}$$

Like $\underline{VaR}_{\alpha,x_i}(\mathbf{X})$, $\overline{VaR}_{\alpha,x_i}(\mathbf{X})$ also has a bounded sensitivity function, which means it is robust. And differences in results is because that bivariate lower and upper orthat RVaR are evaluated using cdf and sf, respectively.

□

Proposition 4.2.4. *For a pair of continuous random variables $\boldsymbol{X}$ with joint sf $\bar{F}(x_1, x_2)$ and marginals $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, the sensitivity function of $\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X})$ is given by*

$$S(z)=\begin{cases}\frac{(1-\beta)VaR_{\alpha_1}(X_j)-(1-\alpha_2)\overline{VaR}_{\alpha_2,x_i}(\boldsymbol{X})}{\alpha_2-\beta}-\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}), & z<VaR_{\alpha_1}(X_j)\\ \frac{z(1-F_{X_i}(x_i))-(\beta-F_{X_i}(x_i))VaR_{\alpha_1}(X_j)-(1-\alpha_2)\overline{VaR}_{\alpha_2,x_i}(\boldsymbol{X})}{\alpha_2-\beta}-\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}), & VaR_{\alpha_1}(X_j)\le z\le \overline{VaR}_{\alpha_2,x_i}(\boldsymbol{X})\\ \frac{(\alpha_2-F_{X_i}(x_i))\overline{VaR}_{\alpha_2,x_i}(\boldsymbol{X})-(\beta-F_{X_i}(x_i))VaR_{\alpha_1}(X_j)}{\alpha_2-\beta}-\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X}), & z>\overline{VaR}_{\alpha_2,x_i}(\boldsymbol{X}).\end{cases}$$

The bounded function provides that $\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\boldsymbol{X})$ is robust.

Proof. The sensitivity function of

$$\overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X})=\frac{1}{\alpha_2-\beta}\int_{\beta}^{\alpha_2}\overline{VaR}_{v,x_i}(\mathbf{X})dv,$$

where $\beta=1-\bar{F}(x_i,VaR_{\alpha_1}(X_j))$ is given by

$$\begin{aligned}S(z)=&\frac{1}{\alpha_2-\beta}\int_{\beta}^{\alpha_2}\left[\lim_{\varepsilon\to 0^+}\frac{\overline{VaR}_{v,x_i}(F_{\varepsilon,\bar{x}_i})-\overline{VaR}_{v,x_i}(F_{\bar{x}_i})}{\varepsilon}\right]dv\\ =&\frac{1}{\alpha_2-\beta}\int_{\beta}^{\alpha_2}\left[\frac{d}{d\varepsilon}\overline{VaR}_{v,x_i}(F_{\varepsilon,\bar{x}_i})\right]_{\varepsilon=0}dv\\ =&\begin{cases}\frac{1}{\alpha_2-\beta}\int_{\beta}^{\alpha_2}-\frac{1-v}{f_{\bar{x}_i}\left[\overline{VaR}_{v,x_i}(\mathbf{X})\right](1-F_{X_i}(x_i))}dv, & z<VaR_{\alpha_1}(X_j)\\ \frac{1}{\alpha_2-\beta}\left\{\int_{\beta}^{1-\bar{F}(x_i,z)}\frac{v-F_{X_i}(x_i)}{f_{\bar{x}_i}\left[\overline{VaR}_{v,x_i}(\mathbf{X})\right](1-F_{X_i}(x_i))}dv\right.\\ \left.+\int_{1-\bar{F}(x_i,z)}^{\alpha_2}-\frac{1-v}{f_{\bar{x}_i}\left[\overline{VaR}_{v,x_i}(\mathbf{X})\right](1-F_{X_i}(x_i))}dv\right\}, & VaR_{\alpha_1}(X_j)\le z\le\overline{VaR}_{\alpha_2,x_i}(\mathbf{X})\\ \frac{1}{\alpha_2-\beta}\int_{\beta}^{\alpha_2}\frac{v-F_{X_i}(x_i)}{f_{\bar{x}_i}\left[\overline{VaR}_{v,x_i}(\mathbf{X})\right](1-F_{X_i}(x_i))}dv, & z>\overline{VaR}_{\alpha_2,x_i}(\mathbf{X})\end{cases}\end{aligned}$$

$$= \begin{cases} \frac{(1-\beta)VaR_{\alpha_1}(X_j)-(1-\alpha_2)\overline{VaR}_{\alpha_2,x_i}(\mathbf{X})}{\alpha_2-\beta} - \overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}), & z < VaR_{\alpha_1}(X_j) \\ \frac{z(1-F_{X_i}(x_i))-(\beta-F_{X_i}(x_i))VaR_{\alpha_1}(X_j)-(1-\alpha_2)\overline{VaR}_{\alpha_2,x_i}(\mathbf{X})}{\alpha_2-\beta} - \overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}), & VaR_{\alpha_1}(X_j) \leq z \leq \overline{VaR}_{\alpha_2,x_i}(\mathbf{X}) \\ \frac{(\alpha_2-F_{X_i}(x_i))\overline{VaR}_{\alpha_2,x_i}(\mathbf{X})-(\beta-F_{X_i}(x_i))VaR_{\alpha_1}(X_j)}{\alpha_2-\beta} - \overline{RVaR}_{\alpha_1,\alpha_2,x_i}(\mathbf{X}), & z > \overline{VaR}_{\alpha_2,x_i}(\mathbf{X}). \end{cases}$$

Furthermore, the sensitivity function of $\overline{TVaR}_{\alpha,x_i}(\mathbf{X})$ can be obtained, when $\beta = \alpha$ and $\alpha_2 = 1$. Then,

$$S(z) = \begin{cases} \overline{VaR}_{\alpha,x_i}(\mathbf{X}) - \overline{TVaR}_{\alpha,x_i}(\mathbf{X}), z < \overline{VaR}_{\alpha,x_i}(\mathbf{X}) \\ \frac{z(1-F_{X_i}(x_i))-(\alpha-F_{X_i}(x_i))\overline{VaR}_{\alpha,x_i}(\mathbf{X})}{1-\alpha} - \overline{TVaR}_{\alpha,x_i}(\mathbf{X}), z \geq \overline{VaR}_{\alpha,x_i}(\mathbf{X}). \end{cases}$$

The sensitivity function of $\overline{TVaR}_{\alpha,x_i}(\mathbf{X})$ is similar to the one of $\underline{TVaR}_{\alpha,x_i}(\mathbf{X})$ because they have similar definitions. Therefore, $\overline{TVaR}_{\alpha,x_i}(\mathbf{X})$ is not robust. □

4.3 Simulation

To estimate $\underline{RVaR}_{0.95,0.99}(\mathbf{X})$ and $\overline{RVaR}_{0.95,0.99}(\mathbf{X})$, we begin with a study of $\underline{VaR}_{0.95}(\mathbf{X})$ and $\overline{VaR}_{0.99}(\mathbf{X})$. In particular, simulations are ran each with the sample size of $n = 1000$ and $n = 4000$, respectively. Marginally, the random variables are distributed with $X_1 \sim$ Weibull (2, 50) and $X_2 \sim$ Weibull (2, 150). The dependence is represented by a Gumbel copula with $\theta = 1.5$. The results of the simulation are presented in Figure 4.1 and Figure 4.2.

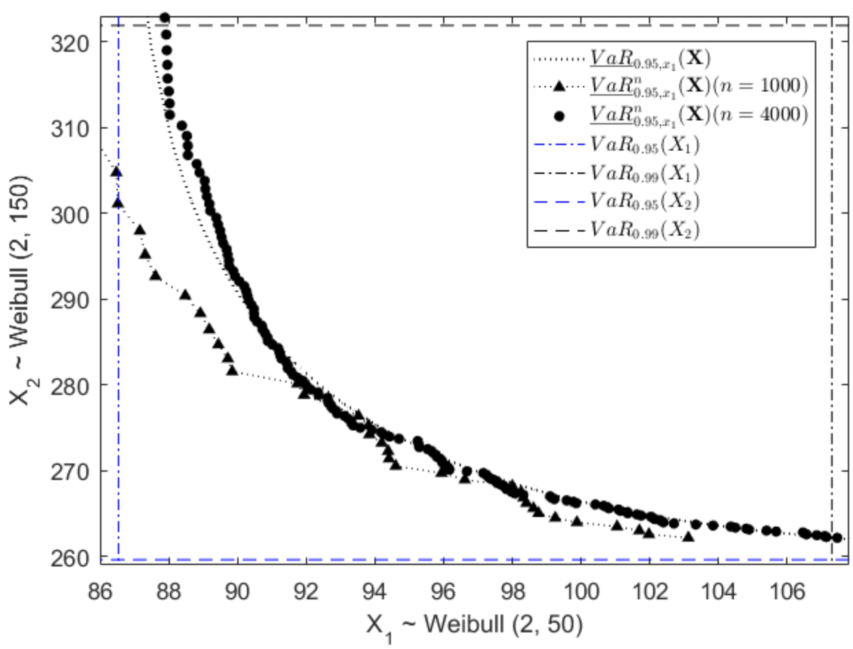

Fig. 4.1: Empirical estimator of the lower orthant VaR at level 0.95

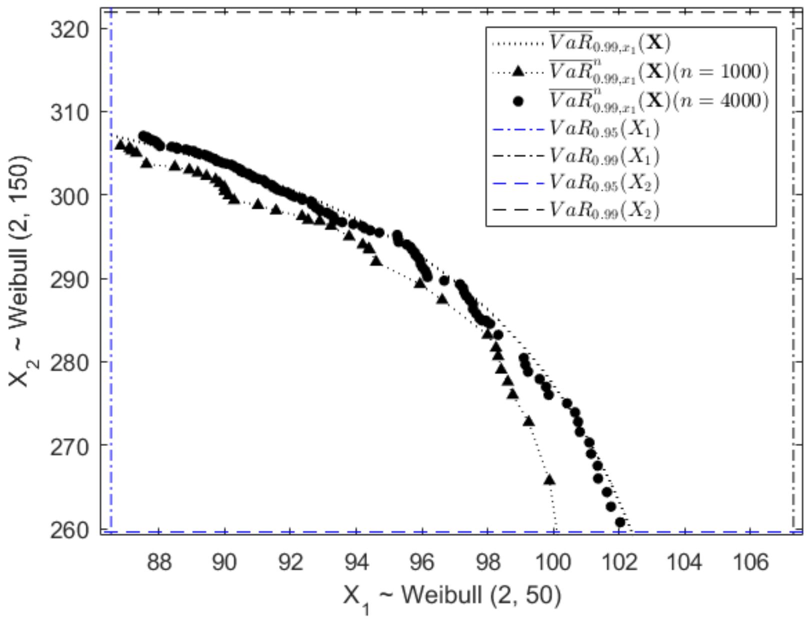

Fig. 4.2: Empirical estimator of the upper orthant VaR at level 0.99

Notice that the choice of the sample size n is useful in order to find the estimation results of the bivariate lower and upper orthant VaR. Higher value of n is more likely that we can get the closer points to the theoretical inversion of cdf. The estimates of $\underline{RVaR}_{0.95,0.99}(\mathbf{X})$ and $\overline{RVaR}_{0.95,0.99}(\mathbf{X})$ with $n = 4000$ are presented in Figure 4.3 and Figure 4.4, respectively. For comparison, the curve with solid triangle represents the result with $m = 100$ and the dark circle represents the result with $m = 250$.

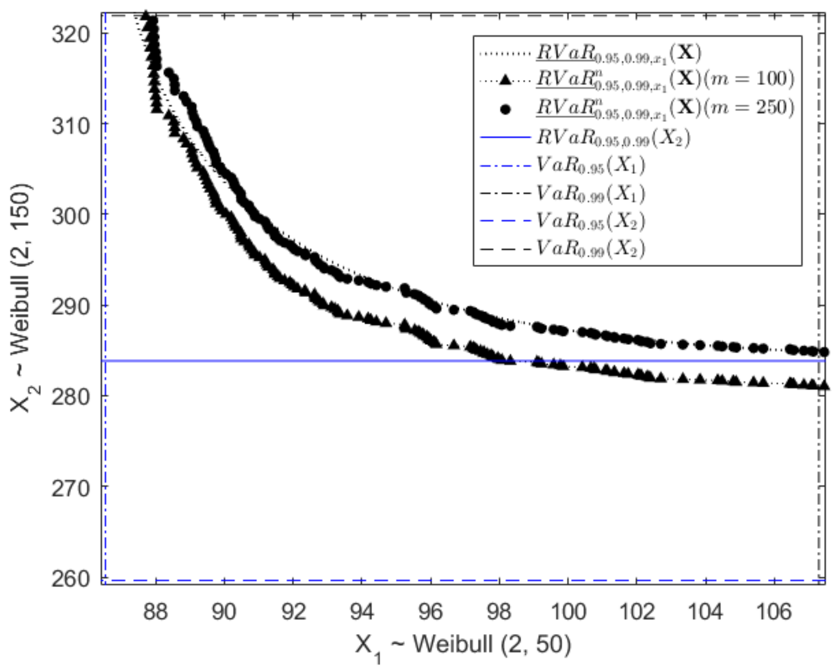

Fig. 4.3: Empirical estimator of the lower orthant RVaR at level range $[0.95, 0.99]$

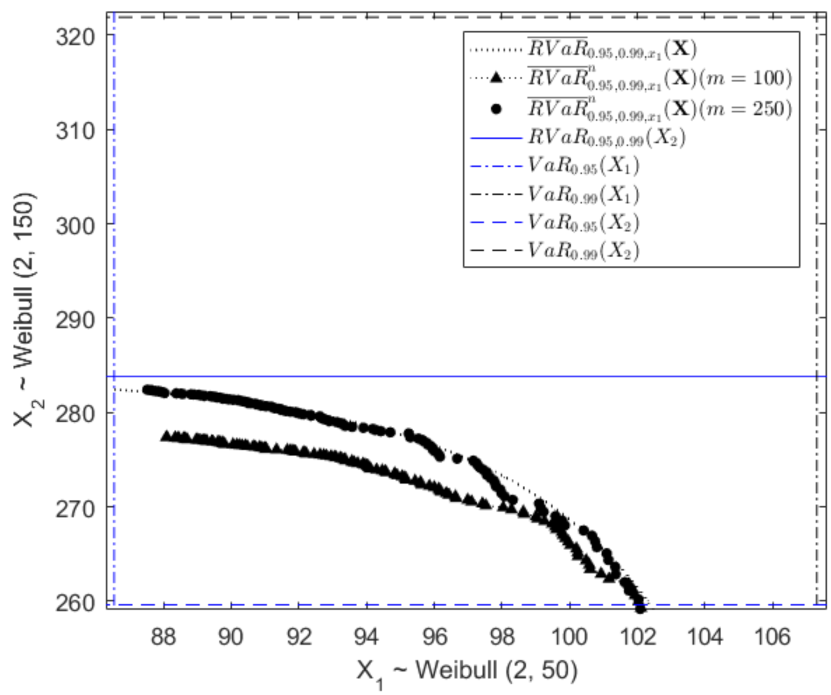

Fig. 4.4: Empirical estimator of the upper orthant RVaR at level range $[0.95, 0.99]$

As can be seen, the differences between the theoretical values and their empirical estimations are extremely small. This could be explained by the robustness and the consistency of the empirical estimators of bivariate VaR and RVaR. The choice of m is important for the average part of the equation. Higher value of m leads to the better estimator. Also, the sample size is large enough, since 4% (of $n = 4000$) data will be used. In addition, RVaR is estimated with a truncated dataset, which could reduce the impact of huge values. As a result, the estimations of lower and upper orthant VaR and RVaR are quite accurate.

To end this chapter, we use an example to illustrate the effect of the dependence between the random variables on the shape of the bivariate RVaR curves. Consider the random vector (X_1, X_2) jointed by a Gumbel copula with dependent parameters $\theta = 1.4$, $\theta = 1.5$ and $\theta = 1.6$, respectively. With marginals $X_1 \sim$ Gamma (2, 0.1) and $X_2 \sim$ Gamma (1, 0.05), we get the theoretical values and estimators of $\underline{RVaR}_{0.95,0.99,x_1}(\mathbf{X})$ on Figure 4.5 and $\overline{RVaR}_{0.95,0.99,x_1}(\mathbf{X})$ on Figure 4.6.

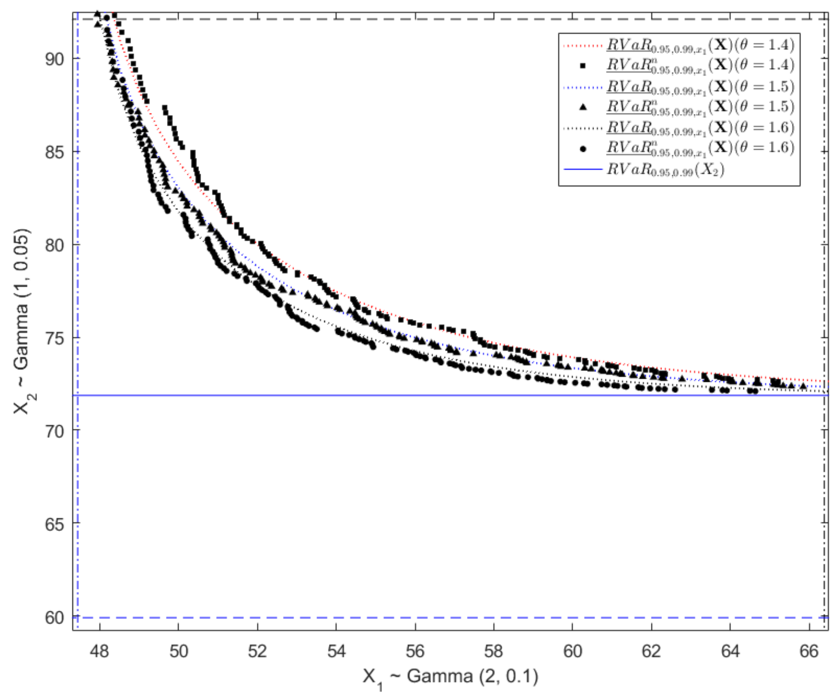

Fig. 4.5: Empirical estimator of the lower orthant RVaR at level range $[0.95, 0.99]$ with dependent parameters $\theta = 1.4$, $\theta = 1.5$ and $\theta = 1.6$

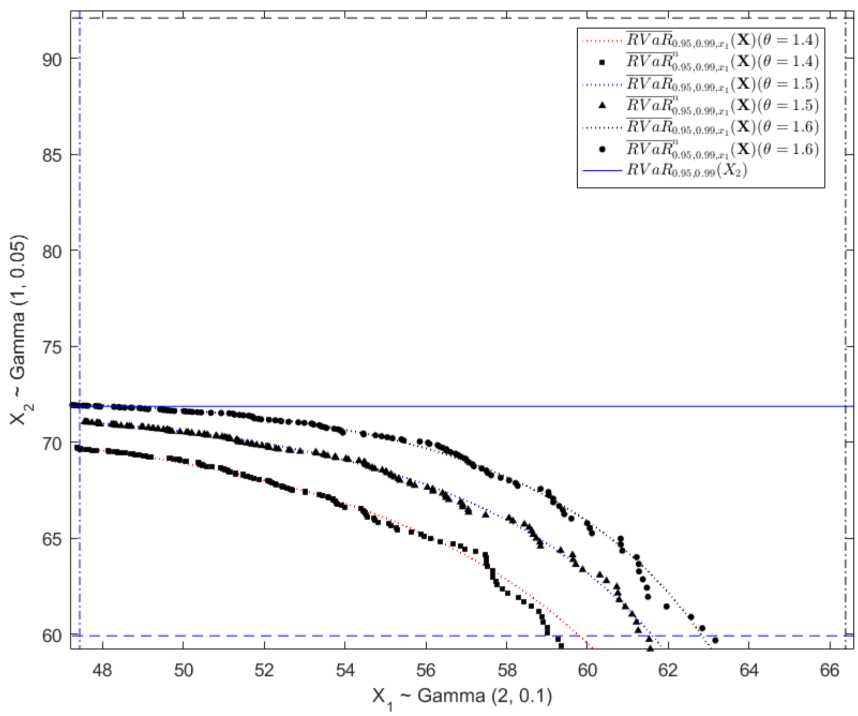

Fig. 4.6: Empirical estimator of the upper orthant RVaR at level range $[0.95, 0.99]$ with dependent parameters $\theta = 1.4$, $\theta = 1.5$ and $\theta = 1.6$

Both Figure 4.5 and Figure 4.6 illustrate that the theoretical curves of bivariate RVaR with different marginals have similar characters. They are bounded by values of the univariate VaR and converge to the univariate RVaR. Moreover, the convexity of the curves are related to the dependent parameter of the copula. θ has a larger effect on $\overline{RVaR}_{0.95,0.99,x_1}(\mathbf{X})$ than on $\underline{RVaR}_{0.95,0.99,x_1}(\mathbf{X})$. The curve with larger θ is more convex. Also, larger θ results in a lower curve of $\underline{RVaR}_{0.95,0.99,x_1}(\mathbf{X})$ and a higher curve of $\overline{RVaR}_{0.95,0.99,x_1}(\mathbf{X})$.

5. CONCLUSION

We review various types of univariate risk measures, including the VaR, TCE, WCE, CVaR, ES and RVaR, and discuss the relationship between them in the discrete and continuous settings, respectively. Robustness of risk measures is accounted since risk measures are usually estimated from historical data. Under the assumption that the random variable has a continuous cumulative distribution function, the sensitivity functions of univariate risk measures which are used to quantify the impact of a small perturbation of the cdf are evaluated. The results show that univariate VaR and RVaR are robust risk measures.

While our focus here is on multivariate risk measures, the method of sensitivity functions can be extended for distribution-based multivariate risk measures. Sensitivity functions are obtained for the bivariate lower and upper orthant VaR and TVaR proposed by *Cossette et al.* (2013, 2015). Bivariate lower and upper orthant VaR is robust, whereas TVaR is not. This result is similar to the one for univariate VaR and TVaR, which motivates the investigation of the bivariate RVaR. Moreover, bivariate lower and upper orthant RVaR have nice properties such as translation invariance, positive homogeneity and monotonicity. Specially, subadditivity can be satisfied for aggregated risks if each risk class is monotonic. Numerical examples with different copula families are provided with graphical representations.

Finally, the empirical estimators of the bivariate lower and upper orthant RVaR are proposed. The robustness and consistency of such estimators are confirmed. Furthermore, the simulations illustrate that empirical estimators yield accurate results. Also, bivariate RVaR might be applied into capital allocation. For example, lower orthant RVaR can be used when the loss distribution has a heavy tail. For example, if a random vector $\mathbf{X} = (X_1, X_2)$ is distributed with marginal distributions which have infinite means, then $TVaR_\alpha(X_1)$ and $TVaR_\alpha(X_2)$ are infinite. As a result, the allocation couple of bivariate TVaR based on approaches provided in *Cossette et al.* (2015) cannot always be calculated whereas bivariate RVaR always admits a solution, since univariate and bivariate RVaR are always bounded. Further studies can be developed from these aspects.

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