On the Upper Bound of Petty's Conjecture in 3 Dimensions

Emilie Cyrenne

A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements

for the Degree of Master of Science (Mathematics) at

Concordia University

Montreal, Quebec, Canada

March 2018

CONCORDIA UNIVERSITY

School of Graduate Studies

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By: Emilie Cyrenne

Entitled: On the Upper Bound of Petty's Conjecture in 3 Dimensions

and submitted in partial fulfillment of the requirements for the degree of

Master of Science (Mathematics)

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Dr. Alina Stancu Supervisor

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ABSTRACT

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Emilie Cyrenne

Among the various important aspects within the theory of convex geometry is that of the field of affine isoperimetric inequalities. Our focus deals with validating the upper bound of Petty's conjecture relating the volume of a convex body and that of its associated projection body. We begin our study by providing some background properties pertaining to convexity as seen through the lens of Minkowski theory. We then show that Petty's conjecture holds true in a certain class of 3-dimensional non-affine deformations of simplices. More precisely, we prove that any simplex in $\mathbb{R}^3$ attains the upper bound in comparison to any deformation of a simplex by a Minkowski sum with a small line segment. As part of our theoretical analysis, we make use of mixed volumes and Maclaurin series expansion in order to simplify the targeted functionals. Finally, we provide an example validating what is known in the literature as the reverse and direct Petty projection inequality. In all cases, Mathematica is used extensively as our means of visualizing the plots of our selected convex bodies and corresponding projection bodies.

Acknowledgments

Dr. Stancu has played an indispensable role in my journey as a graduate student, and more particularly, in the preparation of my thesis. Her keen interest in analytic geometry, coupled with her devotion to thorough work, has served both as a source of inspiration and motivation. Aspiring to achieve the same goal, Dr. Stancu and I worked as a team. Our many interactions - both face-to-face meetings and email correspondence - have played an invaluable contribution to my work. Her commitment to excellence has fostered a challenging yet gratifying work dynamic. Ultimately, the strong bond we have developed has categorically enriched my academic experience. I will forever be thankful to her.

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

Among the various branches of convex geometry, the study of inequalities is prominent. As well as being of independent interest, many of these inequalities have been applied to various mathematical contexts such as ordinary and partial differential equations, functional analysis, linear programming, etc. Often, convexity is naturally necessary when seeking the existence and uniqueness of extremal values. The isoperimetric problem, whose aim is to determine a geometric figure having maximal area for a given perimeter, is one such example.

In this paper, we examine more closely the field of affine isoperimetric inequalities, in which functionals associated to convex bodies remain invariant (unchanged) under affine transformations [10]. Of particular interest is the relationship relating the volume of a convex body (compact, convex set with non-empty interior) and that of its projection body. A ratio of these two volumes, raised at appropriate powers such that the ratio is invariant under scaling, is the subject of two outstanding conjectures in convex geometry. The study of projection bodies came about in the early 1900s by Minkowski, who showed that for every convex body $K \subset X$, there exists a corresponding unique centrally symmetric (symmetric with respect to the origin) convex body ΠK, denoted as the projection body of K [10]. A first conjecture was due to Petty who claimed that:

$$\frac{\mathrm{Vol}_n(\Pi K)}{\mathrm{Vol}_n(K)^{n-1}} \geq \frac{\omega_{n-1}^n}{\omega_n^{n-2}} \tag{1.1}$$

with equality if and only if K is an ellipsoid [11]. Here, K denotes an arbitrary convex body in $\mathbb{R}^n$, $\mathrm{Vol}_n(\cdot)$ the n-dimensional volume and ω_n the n-dimensional volume of the n-dimensional unit ball [3].

In other words, Petty conjectured that the functional

$$P_n(K) = \frac{\mathrm{Vol}_n(\Pi K)}{\mathrm{Vol}_n^{n-1}(K)} \tag{1.2}$$

is minimal for ellipsoids. There has been much research done on proposing an upper bound for P_n. Schneider conjectured that, for centrally symmetric convex bodies, 2^n is an upper bound [13]. Brannen disproved Schneider's claim by establishing counterexamples for $n \geq 3$ and instead, proposed that for all n-dimensional convex bodies K, the value $\frac{(n+1)n^n}{n!}$ is an upper bound, which is the value that the above functional reaches for simplices [3]. Both conjectured extreme values for this Petty functional are referred to as Petty's conjecture. We focus on calculating the value of P_n for 3-dimensional affine images of simplices and specific non-affine deformations of simplices defined using a Minkowski sum of segments. We validate that the upper bound of Petty's conjecture holds true in the class of deformations of this type.

On a more theoretical basis, we want to show in general that the upper bound of Petty's conjecture is true for any deformation of a simplex by a Minkoswki sum with a segment. This involves some background on mixed volumes and their related properties, among which is the linearity of mixed volumes.

Finally, we state what is known in the literature as Petty's projection inequality involving the volume of a convex body and that of the polar of its associated projection body. Petty's projection inequality states that:

$$\mathrm{Vol}_n^{n-1}(K)\mathrm{Vol}_n(\Pi K)^\star \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n \tag{1.3}$$

with equality if and only if K is an ellipsoid. The inequality shows that the minimal value of this new functional, $Q_n(K) = \mathrm{Vol}_n^{n-1}(K)\mathrm{Vol}_n(\Pi K)^\star$, is reached for simplices - this is known as the Zhang projection inequality [14] or the reverse Petty inequality. Petty himself proved that the maximum of Q_n is attained only for ellipsoids. We provide an example validating the reverse and the direct Petty projection inequality for the union of a simplex and a line segment of arbitrary length. In addition, we comment on the duality of the two functionals, $\mathrm{Vol}_n(\Pi K)$ and $\mathrm{Vol}_n(\Pi K)^\star$. Petty's projection inequality for polar bodies was first introduced because the original problem, that pertaining to the conjectured bounds of P_n, was too hard to prove.

The present paper is divided as follows. Chapter 1 is an introduction to the theory of convexity and projection bodies. In Chapter 2, we present our results, validating Petty's conjecture, and elaborate on our calculations. Chapter 3 provides a theoretical approach to how certain deformations of simplices do not hinder Petty's conjecture. We conclude with some insight on Petty's projection inequality for polar bodies.

1.2 Convexity and Convex Sets

Before embarking on our journey of projection bodies, we must first elaborate on what is meant by convexity and convex sets. Convex geometry is a specific branch in geometry pertaining to the study of convex sets. Among numerous other advantages, convexity facilitates optimization problems by efficiently identifying the feasible region and ensuring an optimal solution. Furthermore, convexity allows for results obtained in lower-dimensional space to have equal reasoning and application in infinite-dimensional theory.

A subset C of a vector space X over $\mathbb{R}$ is called convex if the line segment joining any two points in C also lies in C, i.e.,

$C \subseteq X$, C is convex if $C \neq \emptyset$ and $\forall x, y \in C$ and $\forall \lambda \in [0,1], \lambda \mathrm{x} + (1-\lambda)y \in C$.

The set $\{\lambda \mathrm{x} + (1-\lambda)y : 0 \leq \lambda \leq 1\}$ is a closed line segment connecting the endpoints x and y. In this way, a convex combination is defined as a linear combination of points having non-negative coefficients that sum to one. The above definition applies to any Euclidean space $\mathbb{R}^n$. The simplest examples of nonempty convex sets are singletons, intervals and the entire space $\mathbb{R}^n$. Interestingly, convexity intertwines with the notion of means. For example, any convex combination satisfying $\lambda \mathrm{x} + (1-\lambda)y$ with $\lambda \in$ (0,1) is the weighted arithmetic mean of x and y. Likewise, the weighted geometric mean of x and y pertains to the concavity (negative convexity) of functions.

1.2.1 Some Properties of Convex Sets

a. Arithmetic summation and multiplication by reals preserve convexity: if C is a convex set in X and $\lambda_1, ..., \lambda_k \in [0,1]$ such that $\sum_{i=1}^{k} \lambda_i = 1$, then the set $\lambda_1 C_1 + ... + \lambda_k C_k = \left\{\sum_{i=1}^{k} \lambda_i x_i : x_i \in C, i = 1, ..., k, k \geq 2, k \in \mathbb{N}\right\}$ is convex.

   Proof by induction: For $k = 2$, the set $\{\lambda_1 x_1 + \lambda_2 x_2 : x_1, x_2 \in C\}$ is convex since C is convex. Assume that the set $\left\{\sum_{i=1}^{k} \lambda_i x_i : x_i \in C, i = 1, ..., k, k \geq 2, k \in \mathbb{N}\right\}$ is convex. We will show it is convex for $k = k+1$. Let $x_1, ..., x_{k+1}$ be arbitrary points in C, and let $\lambda_1, \ldots, \lambda_{k+1}$ be real non-negative numbers with $\sum_{i=1}^{k+1} \lambda_i = 1$. Then $\lambda_1 + ... + \lambda_k = 1 - \lambda_{k+1}$. If $\lambda_{k+1} = 1$, then $\lambda_1 x_1 + ... + \lambda_{k+1} x_{k+1} = x_{k+1} \in C$ (all $\lambda_i' s = 0, i = 1, ..., k$). If $\lambda_{k+1} \neq 1$, then $\frac{\lambda_1}{1-\lambda_{k+1}} + \frac{\lambda_2}{1-\lambda_{k+1}} + ... + \frac{\lambda_k}{1-\lambda_{k+1}} = 1$. By the hypothesis of induction , $\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} x_i \in C$. Since $x_{k+1} \in C$ (and by definition of convexity), we have $(1-\lambda_{k+1}) \sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} x_i + \lambda_{k+1} x_{k+1} \in C$. Then $(1-\lambda_{k+1}) \left(\frac{\lambda_1 x_1}{1-\lambda_{k+1}} + \frac{\lambda_2 x_2}{1-\lambda_{k+1}} + ... + \frac{\lambda_k x_k}{1-\lambda_{k+1}}\right) + \lambda_{k+1} x_{k+1} \in C$. Cancelling the necessary terms, we get $\sum_{i=1}^{k+1} \lambda_i x_i \in C$.

b. Images of convex sets under affine maps are convex: if $\phi : X \to X$ is an affine mapping and $C \subset X$ is convex, then $\phi(C) = \{\phi(x) = \mathrm{Lx} + v : \mathrm{L}$ is a linear operator, $\mathrm{x} \in X, v \in X$ is fixed$\} \subset X$ is also convex.

Proof: Let $x, y \in C$. Then $\phi(x), \phi(y) \in \phi(C)$. The convexity of C implies that $\lambda x + (1-\lambda)y \in C \ \forall \lambda \in [0,1]$. Since ϕ is affine, then $\phi(\lambda x + (1-\lambda)y) = \lambda\phi(x) + (1-\lambda)\phi(y) \in \phi(C)$. Thus, $\phi(C)$ is convex.

c. Convex sets are closed under arbitrary intersections: if $\{K_\alpha\}_{\alpha \in A}$ is an arbitrary collection of convex sets, then their intersection $K := \bigcap_{\alpha \in A} K_\alpha$ is also convex.

Proof: Let $\{K_\alpha\}_{\alpha \in A}$ be a family of convex sets, and let $K = \cap_{\alpha \in A} K_\alpha$. Then $\forall x, y \in K$, we have $x, y \in K_\alpha, \forall \alpha \in A$ (and all y's are convex by assumption). Hence, $\forall \alpha \in A$ and $\lambda \in [0,1], \lambda x + (1-\lambda)y \in K_\alpha$. Thus $\lambda x + (1-\lambda)y \in K$, implying that K is convex.

Definition 1.2.1. *Let A be an arbitrary set of X. Then the convex hull of A, denoted by conv(A), is the set of all convex combinations of A. In other words,*

$$conv(A) = \left\{ \sum_{i=1}^{m} \lambda_i x_i : \forall m \geq 2, \forall x_i \in A \ , \lambda_i \in [0,1] \text{ with } \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Equivalently, conv(A) is the intersection of all convex sets containing A. According to our third property of convex sets, the convex hull of a set is necessarily convex - indeed, conv(A) is the smallest convex set containing A. For example, the convex hull of a set of finite vertices in $\mathbb{R}^n$ is called a polytope. If $\{x_1 \ , ..., \ x_{n+1}\}$ is a set of $(n+1)$-points in $\mathbb{R}^n$ such that x_i - x_j , $\forall$ i > j, is a linearly independent set, then conv $(\{x_1 \ , ..., \ x_{n+1}\})$ is called a simplex. Simplices in $\mathbb{R}$ are simply line segments. In $\mathbb{R}^2$, simplices are triangles. In $\mathbb{R}^3$, simplices are tetrahedra. The trend continues.

Definition 1.2.2. *Let A, B be convex sets of $\mathbb{R}^n$. The Minkowski sum of A and B, denoted as $A + B = \{z \in \mathbb{R}^n : z = x + y, \ \text{for some } x \in A \text{ and } y \in B\}$.*

1.3 Convex Bodies

At the heart of the geometric theory of convex sets lies the space of convex bodies, which are defined in n-dimensional Euclidean space as compact, convex sets with non-empty interior. The class of convex bodies is closed under Minkowski addition. Convex bodies are uniquely characterized by their support functions.

Definition 1.3.1. *We call* $h_K : \mathbb{S}^{n-1} \to \mathbb{R},\ h_K(x) = \sup\{x \cdot y : y \in K\}$ *the support function of* K*, where* $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ *is the* $(n-1)$*-dimensional unit sphere in* $\mathbb{R}^n$*.*

The support function of a convex body K, containing the origin in its interior, in a direction u is the orthogonal distance from the supporting hyperplanes of K of normal u to the origin. At every point on the boundary of a convex body, there exists at least one hyperplane such that the convex body lies in one of the two closed halfspaces defined by this supporting hyperplane. From this definition, any convex body is the intersection of closed halfspaces containing K that have supporting hyperplanes as boundaries. At any given point on the boundary, the supporting hyperplane is not unique as there exists points where the boundary is not smooth, i.e., supporting hyperplanes passing through vertices (corner points) are not unique. It is known that every convex body has a unique support function with respect to the origin and that any convex body K is completely determined by its support function.

Within the theory of convexity lies an important subtopic known as duality, a notion interchangeable with that of polarity. In broad terms, duality is mostly used in functional analysis, whereas polarity is commonly applied by geometers. A direct interplay between functional analysis and geometry is established when studying norms. For any norm $\|\cdot\|$, the dual norm is the norm $\|\cdot\|^\star = \sup_{y\neq 0} \frac{x\cdot y}{\|y\|} = \sup_{||y||=1}(x \cdot y)$.

Definition 1.3.2. *Let* K *be a convex body in* $\mathbb{R}^n$ *with* $0 \in int(K)$*. We define the polar of* K *the set* $K^\star = \{y \in \mathbb{R}^n : x \cdot y \leq 1, \forall x \in K\}$*.*

We remark that the polar of the unit ball with respect to any norm is the unit ball with respect to the corresponding dual normed space. We come back to duality in the last section - for now, we focus on polarity and some of its properties.

The properties of K are equally reflected in those of $K^\star$. To start with, we note that if K is a convex body containing the origin in its interior, then $K^\star$ is also a convex body with $0 \in Int\,(K^\star)$. If P is a polytope in $\mathbb{R}^n$ such that $P = \bigcap_{i=1}^{k} \{x \in \mathbb{R}^n : x \cdot n_i \leq 1\}$, as any polytope may be expressed as an intersection of a finite number of halfspaces, then $P^\star = \text{conv}\,\{n_i\}$, $i = 1, \ldots, k$, where each n_i is a unit vector in $\mathbb{R}^n$. The size and shape of the polar body tend to be *inverted* to that of the original set. For example, the polar of a 3-dimensional cube is an octahedron - 6 faces and 8 vertices for the original set correspond to 8 faces and 6 vertices for the polar. In $\mathbb{R}^2$, a long rectangle extended over the x-axis with vertices at $(500, \frac{1}{2}), (-100, \frac{1}{2}), (500, -\frac{1}{2}), (-100, -\frac{1}{2})$ has a taller, compressed diamond as its polar, with corners at $(-\frac{1}{100}, 0), (\frac{1}{500}\ , 0)$ and $(0, \pm 2)$. This shows that the polar of a polytope is highly dependent on the choice of the origin. We also note that polars of simplices are simplices.

A key aspect pertaining to polarity is that $K^\star$ is always convex, regardless of whether or not K is itself convex.

1.3.1 Some Properties of Polarity

a. If K is a convex set, then $K^{\star\star} = K$.

   Proof: We need to prove $K \subseteq K^{\star\star}$ and $K^{\star\star} \subseteq K$

   Recall that 0 (the zero vector in $\mathbb{R}^n$) $\in K \subset \mathbb{R}^n$, while

   $K^\star = \{y \in \mathbb{R}^n : x \cdot y \leq 1,\ \forall x \in K\}$ and

   $K^{\star\star} = \{z \in \mathbb{R}^n : z \cdot y \leq 1,\ \forall y \in K^\star\}$.

   Case 1: $K \subseteq K^{\star\star}$

Let $x_0 \in K$, x_0 is an arbitrary point in K. Then $x_0 \cdot y \leq 1$, $\forall y \in K^\star$, thus $x_0 \in K^{\star\star}$.

Case 2: $K^{\star\star} \subseteq K$

Assume there exists $z_0 \in K^{\star\star} \setminus K$, ($z_0 \in K^{\star\star}$, but $z_0 \notin K$.) Since $z_0 \notin K$, then by the Hahn-Banach Separation Theorem, there exists a separating hyperplane for z_0 and K. By definition, there is a vector $n \in \mathbb{R}^n$ such that $z_0 \cdot n > 1$ and $z \cdot n \leq 1$, $\forall z \in K$. This means $n \in K^\star$. However, this contradicts that $z_0 \cdot n > 1$ (because $z_0 \in K^{\star\star}$). Therefore, our assumption was wrong and $K^{\star\star} \subseteq K$.

Note: K need not be convex; $K^\star$ is always convex. Thus, the first case always holds.

b. Polarity reverses set inclusion: if $K_1 \subseteq K_2 \Rightarrow {K_2}^\star \subseteq {K_1}^\star$

Proof: Let $y \in {K_2}^\star$. Then, by definition, $x \cdot y \leq 1$, $\forall x \in K_2$. This implies that $x \cdot y \leq 1$, $\forall x \in K_1$ (since $K_1 \subset K_2$.) Thus $y \in {K_1}^\star$.

c. If P is symmetric with respect to the origin, then so is $P^\star$.

Proof: Let P be a convex body symmetric with respect to the origin. Then $x \in P \Rightarrow x \in -P$. Now $P^\star = \{y \in \mathbb{R}^n : x \cdot y \leq 1, \ \forall x \in P\}$. If $y \in P^\star$, then $x \cdot y \leq 1$, $\forall x \in P$. Since $x \in -P$, $x \cdot y \leq 1$, $\forall x \in -P \Rightarrow y \in -P^\star$.

1.4 Projection Bodies

The study of projection bodies and their polars is of rather recent investigation. The reason for their emergence is mainly due to their connectedness to several areas of mathematics, the most common being geometric tomography, a field gathering information pertaining to a geometric object based on data obtained from its sections or projections [5]. We are concerned with projections of convex bodies and the significant role they play in the branch of geometric inequalities. Knowledge extracted from the projections allows for the determination of the original body. For example, if the convex body is centrally symmetric, then the size of its projections, up to translation, suffices in tracing back the body [7]. In lay terms, the word projection refers to a shadow projected orthogonally onto a line or planar surface.

For every convex body K, there exists a corresponding centered convex body called the projection body of K, denoted as ΠK. As for all convex bodies, projection bodies are defined explicitly by their support functions. The latter is defined as follows:

Definition 1.4.1. *[3] If K is a convex body in $\mathbb{R}^n$, then the support function of its projection body ΠK is $h_{\Pi K}(u) = Vol_{n-1}(K, u)$, where $Vol_{n-1}(K, u)$ is the $(n-1)$-dimensional volume of the projection of K onto a hyperplane passing through the origin orthogonal to the unit vector $u \in \mathbb{S}^{n-1}$.*

In other words, the support function of ΠK in the direction of u is the $(n-1)$-dimensional volume (area if $K \in \mathbb{R}^3$) of the projection of K onto a hyperplane of normal u. By the Cauchy projection formula, we have that $\mathrm{Vol}_{n-1}(K, u) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |u \cdot v| \, d\mu(K, v)$, where $d\mu(K, v)$ is the surface area measure of K at the point on the surface of K with outward-unit normal vector v in $\mathbb{R}^n$ [4]. Thus, a symmetric convex body ΠK is the projection body of K if its support function is defined as above.

To our great advantage, if K is a polytope, its surface area is concentrated on a finite number of unitary directions (the outward-normals to the $(n-1)$-dimensional faces), therefore, as we will see, the above formula simplifies.

Put simply, following Brannen's reasoning for calculating the projection body ΠK of a polytope [3], we analyze the area and outward-unit normal vector corresponding to each top dimensional face of the convex body. Then, according to Brannen, projection bodies are finite sums of segments, and generally, for any convex body K, they are limits of finite sums of segments. Each outward-unit normal vector has an associated opposite vector - the line segment connecting the endpoints, multiplied by the area of each face, is called an "area segment". For a polytope K, the pairwise Minkowski sum of all area segments of K gives us the resulting projection body. To obtain the volume of the projection body, we calculate the sum of the absolute value of the determinant whose column vectors are precisely twice one of the endpoints (positive or negative) pertaining to each area segment. We provide our exact calculations in the following chapter, starting with the explicit calculations concerning the volume of projection bodies for non-affine transformations of the right tetrahedron in $\mathbb{R}^3 = \text{conv}\left((0,0,0),(1,0,0),(0,1,0),(0,0,1)\right)$.

Finally, it is important to note that the optimization problem we consider, namely Petty's conjecture, is affine invariant in the sense that $P_n(K) = P_n(AK)$, where $A = L + x$ is any affine transformation of $\mathbb{R}^n$ onto itself with linear part L whose determinant is non-zero. Thus, the value of the functional P_n for any simplex is equivalent to its value on the right tetrahedron, which is why we may consider it as our reference simplex. Later in our analysis, we will also use this invariance to consider the simplex regular.

A first step towards understanding the affine invariance lies upon the property that $\Pi(K + x) = \Pi K,\ \forall x \in \mathbb{R}^n$. In other words, as the projection body of K is formed by the size of the projections of K, no translation of K will change the size of the projections of K, hence the shape of ΠK remains unchanged. The second ingredient is the invariance under scaling of the functional $P_n(K) = \frac{\text{Vol}_n(\Pi K)}{\text{Vol}_n(K)^{n-1}}$. If K is multiplied by any scaling factor, then the resulting projection body of K will expand/contract by the same scaling factor at the power $(n-1)$ as the surface area of K determines ΠK. Thus, the latter ratio defining $P_n(K)$ does not change in the presence of a scaling factor.

For more details pertaining to the previous sections, we refer the reader to [6] and [12].

Chapter 2

Calculations/Analysis of Calculations

In this chapter, we present the projection body of the right tetrahedron in $\mathbb{R}^3$, its volume, as well as the projection bodies and corresponding volumes of two unit non-affine transformations of the right tetrahedron. Finally, we draw conclusions as to the validation of Petty's projection inequality in $\mathbb{R}^3$.

The following essential fact will be used for the construction of each projection body:

Lemma 2.0.1. *[4] If K is a polytope in $\mathbb{R}^3$ such that $u_1, \ldots, u_m$, $m \geq 4$, are the unit outer normals to the faces of K whose areas are, correspondingly, $a_1, \ldots, a_m$, then the projection body of K is the Minkowski sum of the segments of direction u_i and length equal to area a_i,*

$$\Pi K = \frac{a_1}{2}[-u_1, u_1] + \ldots + \frac{a_m}{2}[-u_m, u_m]. \tag{2.1}$$

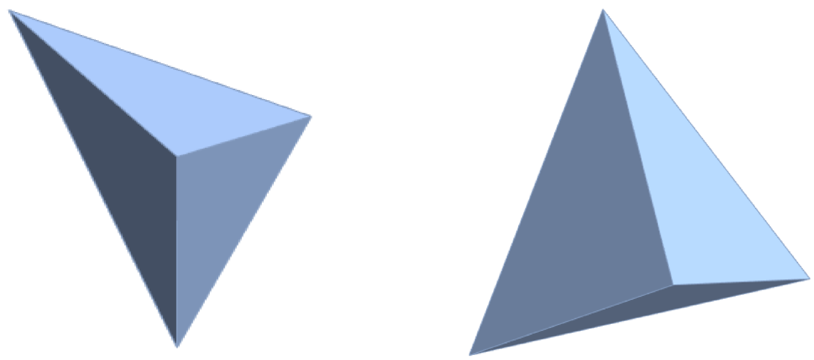
2.1 Calculations for $K = T$, where $T =$ right tetrahedron in $\mathbb{R}^3$

Let $K = \text{conv}(\{0,0,0\},\{1,0,0\},\{0,1,0\},\{0,0,1\})$.

Two different views for the plot of K

Using *Mathematica*, we present two views of K, using the command

$K =$ ConvexHullMesh[{0,0,0},{1,0,0},{0,1,0},{0,0,1}]



(a) View 1 (b) View 2

Figure 2.1: Two different views for the right tetrahedron in $\mathbb{R}^3$

Furthermore, we compute the areas of the faces of K in order to apply the previous lemma. These calculations were initially done using *Mathematica* - we reproduce here the syntax.

Faces of K

a. Right triangle 1

b. Right triangle 2

c. Right triangle 3

d. Equilateral triangle

We denote by: u_i the outward-unit normal vector with respect to face i, a_i the area of face i, $a_i u_i$ the area segment corresponding to face i.

- Right triangle 1: vertices are $(0,0,0),(0,1,0),(0,0,1)$

$(0,0,0)-(0,0,1)=\langle 0,0,-1\rangle$

$(0,0,0)-(0,1,0)=\langle 0,-1,0\rangle$

$\langle 0,0,-1\rangle \times \langle 0,-1,0\rangle = \langle -1,0,0\rangle$

$||\langle -1,0,0\rangle|| = 1$

$a_1 = \frac{1}{2}||\langle -1,0,0\rangle|| = \frac{1}{2}$

$u_1 = \langle -1,0,0\rangle$

$-\,u_1 = \langle 1,0,0\rangle$

$a_1\left(u_1\right) = \left\langle \frac{-1}{2},0,0\right\rangle; a_1\left(-u_1\right) = \left\langle \frac{1}{2},0,0\right\rangle$

- Right triangle 2: vertices are $(0,0,0),(1,0,0),(0,0,1)$

$(0,0,0)-(0,0,1)=\langle 0,0,-1\rangle$

$(0,0,0)-(1,0,0)=\langle -1,0,0\rangle$

$\langle 0,0,-1\rangle \times \langle -1,0,0\rangle = \langle 0,1,0\rangle$

$||\langle 0,1,0\rangle|| = 1$

$a_2 = \frac{1}{2}||\langle 0,1,0\rangle|| = \frac{1}{2}$

$u_2 = \langle 0,1,0\rangle$

$-\,u_2 = \langle 0,-1,0\rangle$

$a_2\left(u_2\right) = \left\langle 0,\frac{1}{2},0\right\rangle; a_2\left(-u_2\right) = \left\langle 0,\frac{-1}{2},0\right\rangle$

- Right triangle 3: vertices are $(0,0,0),(1,0,0),(0,1,0)$

$(0,0,0)-(1,0,0)=\langle -1,0,0\rangle$

$(0,0,0)-(0,1,0)=\langle 0,-1,0\rangle$

$\langle -1,0,0\rangle \times \langle 0,-1,0\rangle = \langle 0,0,1\rangle$

$||\langle 0,0,1\rangle|| = 1$

$a_3 = \frac{1}{2}||\langle 0,0,1\rangle|| = \frac{1}{2}$

$u_3 = \langle 0,0,1\rangle$

$-u_3 = \langle 0,0,-1\rangle$

$a_3\,(u_3) = \left\langle 0,0,\frac{1}{2}\right\rangle; a_3\,(-u_3) = \left\langle 0,0,\frac{-1}{2}\right\rangle$

- Equilateral triangle: vertices are $(1,0,0),(0,1,0),(0,0,1)$

$(1,0,0)-(0,0,1) = \langle 1,0,-1\rangle$

$(1,0,0)-(0,1,0) = \langle 1,-1,0\rangle$

$\langle 1,0,-1\rangle \times \langle 1,-1,0\rangle = \langle -1,-1,-1\rangle$

$||\langle -1,-1,-1\rangle|| = \sqrt{3}$

$a_4 = \frac{1}{2}||\langle -1,-1,-1\rangle|| = \frac{\sqrt{3}}{2}$

$u_4 = \langle \frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\rangle$

$-u_4 = \langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\rangle$

$a_4\,(u_4) = \left\langle \frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right\rangle; a_4\,(-u_4) = \left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$

For the simplicity of the calculations, we consider

$$2\Pi K = a_1[-u_1,u_1] + \ldots + a_m[-u_m,u_m], \tag{2.2}$$

thus

$2\Pi K = [-a_1u_1, a_1u_1] + [-a_2u_2, a_2u_2] + [-a_3u_3, a_3u_3] + [-a_4u_4, a_4u_4]\,.$

$2\Pi K = \left[\left\langle \frac{1}{2},0,0\right\rangle, \left\langle \frac{-1}{2},0,0\right\rangle\right] + \left[\left\langle 0,\frac{-1}{2},0\right\rangle, \left\langle 0,\frac{1}{2},0\right\rangle\right] + \left[\left\langle 0,0,\frac{-1}{2}\right\rangle, \left\langle 0,0,\frac{1}{2}\right\rangle\right]$

$+ \left[\left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle, \left\langle \frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right\rangle\right].$

$\Rightarrow 2\Pi K$, as well as its rescaling ΠK, is the convex hull of, at most, $2^4 = 16$ possible vectors.

$2\Pi K = \text{conv}(\{1,0,0\}, \{0,-1,-1\}, \{1,0,1\}, \{0,-1,0\}, \{1,1,0\}, \{0,0,-1\}, \{0,0,0\},$
$\{1,1,1\}, \{-1,-1,-1\}, \{0,0,1\}, \{-1,-1,0\}, \{0,1,0\}, \{-1,0,-1\}, \{0,1,1\}, \{-1,0,0\}).$

Plot of 2ΠK

ConvexHullMesh$[\{1,0,0\}, \{0,-1,-1\}, \{1,0,1\}, \{0,-1,0\}, \{1,1,0\}, \{0,0,-1\}, \{0,0,0\},$
$\{1,1,1\}, \{-1,-1,-1\}, \{0,0,1\}, \{-1,-1,0\}, \{0,1,0\}, \{-1,0,-1\}, \{0,1,1\}, \{-1,0,0\}]$

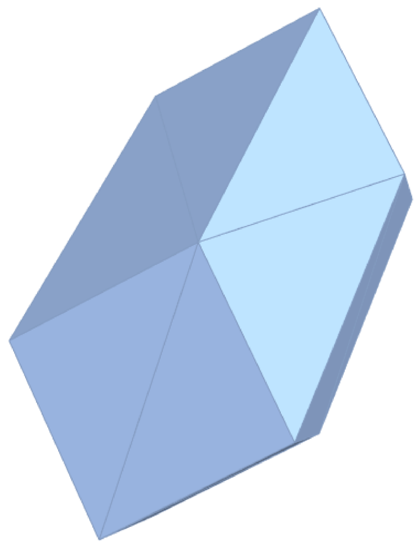


Figure 2.2: Plot of $2\Pi K$ for K = right tetrahedron in $\mathbb{R}^3$

We calculate the volume of $2\Pi K$ using Brannen's formula [4] as follows:

$\text{Vol}(2\Pi K) = |w_1, w_2, w_3| + |w_1, w_2, w_4| + |w_1, w_3, w_4| + |w_2, w_3, w_4|$

$w_1 = 2 \cdot a_1\,(u_1) = \langle -1, 0, 0\rangle$

$w_2 = 2 \cdot a_2\,(u_2) = \langle 0, 1, 0\rangle$

$w_3 = 2 \cdot a_3\,(u_3) = \langle 0, 0, 1\rangle$

$w_4 = 2 \cdot a_4\,(u_4) = \langle -1, -1, -1\rangle$

- $|w_1, w_2, w_3|$

$$\text{Det}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1$$

- $|w_1, w_2, w_4|$

$$\text{Det}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{pmatrix} = 1$$

- $|w_1, w_3, w_4|$

$$\text{Det}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} = -1$$

- $|w_2, w_3, w_4|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} = -1$$

Therefore, we get that $\text{Vol}(2\Pi K) = 1 + 1 + 1 + 1 = 4$ and $\text{Vol}(\Pi K) = 4/2^3 = \frac{1}{2}$. Now, $\text{Vol}(K) = \frac{a_b \cdot h}{3} = \frac{\frac{1}{2} \cdot 1}{3} = \frac{1}{6}$.

Thus, Petty's functional for the right tetrahedron in $\mathbb{R}^3$, and consequently for any tetrahedron in $\mathbb{R}^3$, due to its invariance under linear transformations, is: $\frac{\text{Vol}(\Pi K)}{\text{Vol}(K)^2} = \frac{\frac{1}{2}}{\left(\frac{1}{6}\right)^2} = 18.$

2.2 Calculations for $K = T + I_{[0,1,0]}\epsilon$, where T = right tetrahedron in $\mathbb{R}^3$

In this subsection, we consider a deformation of the right simplex T by taking its Minkowski sum with a segment of length ϵ in the direction of the vector $(0, 1, 0)$. We reproduce the calculations performed with *Mathematica* to illustrate the convex polytope, to calculate its volume and, further, to analyze the effect of this transformation on the projection body of the deformed simplex.

$$K = T + I_{[0,1,0]}\epsilon = \text{conv}(\{1,0,0\}, \{0,0,1\}, \{1,\epsilon,0\}, \{0,1+\epsilon,0\}, \{0,\epsilon,1\}, \{0,\epsilon,0\}, \{0,0,0\}).$$

Plot of K if $\epsilon = 1$

$K = \text{ConvexHullMesh}[\{1,0,0\}, \{0,0,1\}, \{1,1,0\}, \{0,2,0\}, \{0,1,1\}, \{0,1,0\}, \{0,0,0\}]$

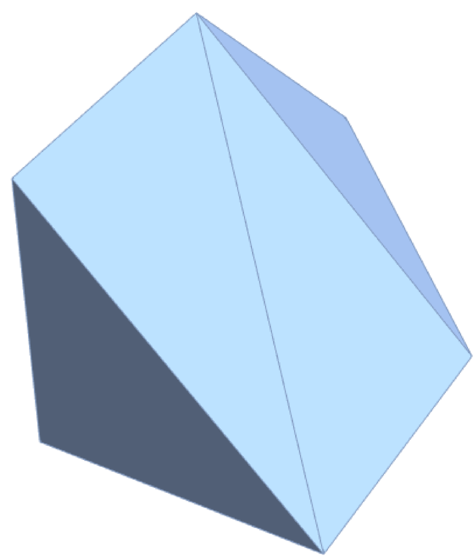


Figure 2.3: Plot of $K = T + I_{[0,1,0]}$ in $\mathbb{R}^3$

Plot of K if $\epsilon = \frac{1}{2}$

$K = \text{ConvexHullMesh}[\{1,0,0\},\{0,0,1\},\{1,\frac{1}{2},0\},\{0,\frac{3}{2},0\},\{0,\frac{1}{2},1\},\{0,\frac{1}{2},0\},\{0,0,0\}]$

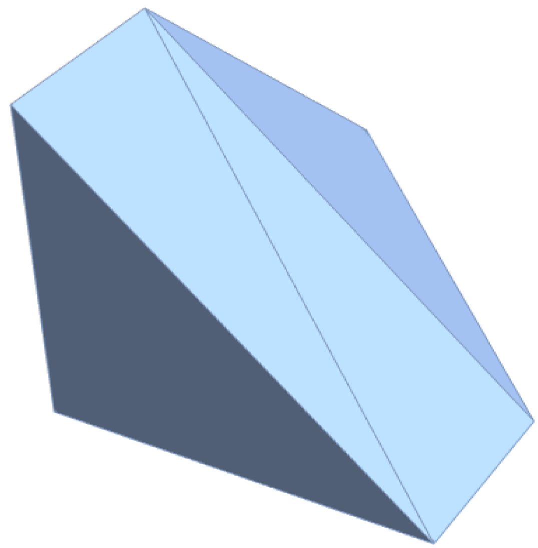


Figure 2.4: Plot of $K = T + I_{[0,\frac{1}{2},0]}$ in $\mathbb{R}^3$

Faces of K

a. Right triangle

b. Equilateral triangle

c. Rectangle

d. Trapezoid 1

e. Trapezoid 2

We denote by: u_i the outward-unit normal vector with respect to face i, a_i the area of face i, $a_i u_i$ the area segment corresponding to face i.

- Right triangle: vertices are $(0,0,0),(1,0,0),(0,0,1)$

$(0,0,0)-(0,0,1)=\langle 0,0,-1\rangle$

$(0,0,0)-(1,0,0)=\langle -1,0,0\rangle$

$\langle 0,0,-1\rangle \times \langle -1,0,0\rangle = \langle 0,1,0\rangle$

$||\langle 0,1,0\rangle|| = 1$

$a_1 = \frac{1}{2}||\langle 0,1,0\rangle|| = \frac{1}{2}$

$u_1 = \langle 0,1,0\rangle$

$-u_1 = \langle 0,-1,0\rangle$

$a_1\left(u_1\right) = \left\langle 0,\frac{1}{2},0\right\rangle; a_1\left(-u_1\right) = \left\langle 0,\frac{-1}{2},0\right\rangle$

- Equilateral triangle: vertices are $(0,\epsilon,1),(0,1+\epsilon,0),(1,\epsilon,0)$

$(1,\epsilon,0)-(0,\epsilon,1)=\langle 1,0,-1\rangle$

$(1,\epsilon,0)-(0,1+\epsilon,0)=\langle 1,-1,0\rangle$

$\langle 1,0,-1\rangle \times \langle 1,-1,0\rangle = \langle -1,-1,-1\rangle$

$||\langle -1,-1,-1\rangle|| = \sqrt{3}$

$a_2 = \frac{1}{2}||\langle -1,-1,-1\rangle|| = \frac{\sqrt{3}}{2}$

$u_2 = \langle \frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\rangle$

$-u_2 = \langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\rangle$

$a_2\left(u_2\right) = \left\langle \frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right\rangle; a_2\left(-u_2\right) = \left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$

- Rectangle: vertices are $(0,0,1),(0,\epsilon,1),(1,\epsilon,0),(1,0,0)$

$(1,0,0)-(0,0,1)=\langle 1,0,-1\rangle$

$(1,0,0)-(1,\epsilon,0)=\langle 0,-\epsilon,0\rangle$

$\langle 1,0,-1\rangle \times \langle 0,-\epsilon,0\rangle = \langle -\epsilon,0,-\epsilon\rangle$

$||\langle -\epsilon,0,-\epsilon\rangle|| = \sqrt{2}\epsilon$

$a_3 = \text{length} \cdot \text{width} = \sqrt{2}\epsilon$

$u_3 = \langle \frac{-\sqrt{2}}{2},0,\frac{-\sqrt{2}}{2}\rangle$

$-u_3 = \langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \rangle$

$a_3\left(u_3\right) = \langle -\epsilon, 0, -\epsilon \rangle \, ; a_3\left(-u_3\right) = \langle \epsilon, 0, \epsilon \rangle$

- Trapezoid 1: vertices are $(0,0,0), (0,1+\epsilon,0), (1,\epsilon,0), (1,0,0)$

$(0,\epsilon,0) - (1,\epsilon,0) = \langle -1,0,0 \rangle$

$(0,\epsilon,0) - (0,1+\epsilon,0) = \langle 0,-1,0 \rangle$

$\langle -1,0,0 \rangle \times \langle 0,-1,0 \rangle = \langle 0,0,1 \rangle$

$||\langle 0,0,1 \rangle|| = 1$

$a_4 = \frac{(B+b)h}{2} = \frac{((1+\epsilon)+\epsilon)\cdot 1}{2} = \frac{1}{2} + \epsilon$

$u_4 = \langle 0,0,1 \rangle$

$-u_4 = \langle 0,0,-1 \rangle$

$a_4\left(u_4\right) = \left\langle 0, 0, \frac{1}{2} + \epsilon \right\rangle ; a_4\left(-u_4\right) = \left\langle 0, 0, \frac{-1}{2} - \epsilon \right\rangle$

- Trapezoid 2: vertices are $(0,0,1), (0,\epsilon,1), (0,1+\epsilon,0), (0,0,0)$

$(0,\epsilon,0) - (0,\epsilon,1) = \langle 0,0,-1 \rangle$

$(0,\epsilon,0) - (0,1+\epsilon,0) = \langle 0,-1,0 \rangle$

$\langle 0,0,-1 \rangle \times \langle 0,-1,0 \rangle = \langle -1,0,0 \rangle$

$||\langle -1,0,0 \rangle|| = 1$

$a_5 = \frac{(B+b)h}{2} = \frac{((1+\epsilon)+\epsilon)\cdot 1}{2} = \frac{1}{2} + \epsilon$

$u_5 = \langle -1,0,0 \rangle$

$-u_5 = \langle 1,0,0 \rangle$

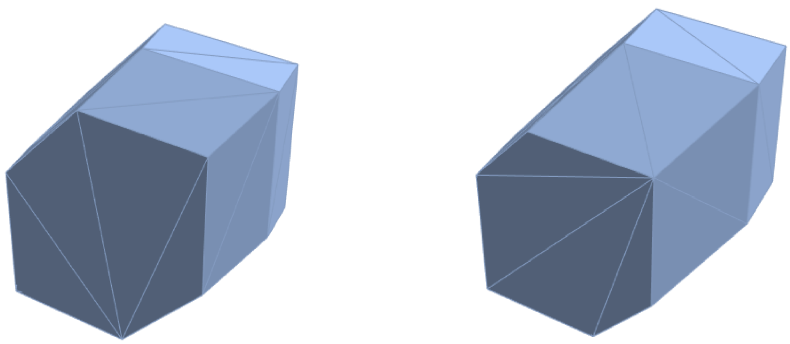
$a_5\left(u_5\right) = \left\langle \frac{-1}{2} - \epsilon, 0, 0 \right\rangle ; a_5\left(-u_5\right) = \left\langle \frac{1}{2} + \epsilon, 0, 0 \right\rangle$

$$2\Pi K = [-a_1u_1, a_1u_1] + [-a_2u_2, a_2u_2] + [-a_3u_3, a_3u_3] + [-a_4u_4, a_4u_4] + [-a_5u_5, a_5u_5] \, .$$

$$2\Pi K = \left[\left\langle 0, \frac{-1}{2}, 0 \right\rangle, \left\langle 0, \frac{1}{2}, 0 \right\rangle\right] + \left[\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle, \left\langle \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2} \right\rangle\right] + \left[\langle \epsilon, 0, \epsilon \rangle, \langle -\epsilon, 0, -\epsilon \rangle\right]$$
$$+ \left[\left\langle 0, 0, \frac{-1}{2} - \epsilon \right\rangle, \left\langle 0, 0, \frac{1}{2} + \epsilon \right\rangle\right] + \left[\left\langle \frac{1}{2} + \epsilon, 0, 0 \right\rangle, \left\langle \frac{-1}{2} - \epsilon, 0, 0 \right\rangle\right] .$$

$\Rightarrow 2\Pi K$ (and ΠK) is the convex hull of, at most, $2^5 = 32$ possible vectors.

$2\Pi K = \text{conv}(\{1{+}2\epsilon, 0, 0\}, \{0, 0, 0\}, \{1{+}2\epsilon, 0, 1{+}2\epsilon\}, \{0, 0, 1{+}2\epsilon\}, \{1, 0, -2\epsilon\}, \{-2\epsilon, 0, -2\epsilon\},$
$\{-2\epsilon, 0, 1\}, \{1, 0, 1\}, \{2\epsilon, -1, -1\}, \{-1, -1, -1\}, \{2\epsilon, -1, 2\epsilon\}, \{-1, -1, 2\epsilon\},$
$\{0, -1, -1{-}2\epsilon\}, \{-1{-}2\epsilon, -1, -1{-}2\epsilon\}, \{0, -1, 0\}, \{-1{-}2\epsilon, -1, 0\}, \{1{+}2\epsilon, 1, 0\}, \{0, 1, 0\},$
$\{1{+}2\epsilon, 1, 1{+}2\epsilon\}, \{0, 1, 1{+}2\epsilon\}, \{1, 1, -2\epsilon\}, \{-2\epsilon, 1, -2\epsilon\}, \{-2\epsilon, 1, 1\}, \{1, 1, 1\}, \{2\epsilon, 0, -1\},$
$\{-1, 0, -1\}, \{2\epsilon, 0, 2\epsilon\}, \{-1, 0, 2\epsilon\}, \{0, 0, -1{-}2\epsilon\}, \{-1{-}2\epsilon, 0, -1{-}2\epsilon\}, \{-1{-}2\epsilon, 0, 0\})$



(a) $2\Pi K$ for $K = T + I_{[0,1,0]}$ (b) $2\Pi K$ for $K = T + I_{[0,\frac{1}{2},0]}$

Figure 2.5: Two different plots of $2\Pi K$ in $\mathbb{R}^3$

We calculate the volume of $2\Pi K$ using again Brannen's formula [4] as follows:

$\text{Vol}(2\Pi K) = \sum_{1\le i<j<k\le n} |w_i, w_j, w_k| = |w_1, w_2, w_3| + |w_1, w_2, w_4| + |w_1, w_2, w_5| + |w_1, w_3, w_4| + |w_1, w_3, w_5| + |w_1, w_4, w_5| + |w_2, w_3, w_4| + |w_2, w_4, w_5| + |w_2, w_3, w_5| + |w_3, w_4, w_5|$

$w_1 = 2 \cdot a_1(u_1) = \langle 0, 1, 0\rangle$

$w_2 = 2 \cdot a_2(u_2) = \langle -1, -1, -1\rangle$

$w_3 = 2 \cdot a_3(u_3) = \langle -2\epsilon, 0, -2\epsilon\rangle$

$w_4 = 2 \cdot a_4(u_4) = \langle 0, 0, 1 + 2\epsilon\rangle$

$w_5 = 2 \cdot a_5(u_5) = \langle -1 - 2\epsilon, 0, 0\rangle$

- $|w_1, w_2, w_3|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ -2\epsilon & 0 & -2\epsilon \end{pmatrix} = 0$$

- $|w_1, w_2, w_4|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 1+2\epsilon \end{pmatrix} = 1+2\epsilon$$

- $|w_1, w_2, w_5|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ -1-2\epsilon & 0 & 0 \end{pmatrix} = 1+2\epsilon$$

- $|w_1, w_3, w_4|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -2\epsilon & 0 & -2\epsilon \\ 0 & 0 & 1+2\epsilon \end{pmatrix} = 2\epsilon(1+2\epsilon)$$

- $|w_1, w_3, w_5|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -2\epsilon & 0 & -2\epsilon \\ -1-2\epsilon & 0 & 0 \end{pmatrix} = 2\epsilon(1+2\epsilon)$$

- $|w_1, w_4, w_5|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1+2\epsilon \\ -1-2\epsilon & 0 & 0 \end{pmatrix} = -(1+2\epsilon)^2$$

- $|w_2, w_3, w_4|$

$$\text{Det}\begin{pmatrix} -1 & -1 & -1 \\ -2\epsilon & 0 & -2\epsilon \\ 0 & 0 & 1+2\epsilon \end{pmatrix} = -2\epsilon(1+2\epsilon)$$

- $|w_2, w_3, w_5|$

$$\text{Det}\begin{pmatrix} -1 & -1 & -1 \\ -2\epsilon & 0 & -2\epsilon \\ -1-2\epsilon & 0 & 0 \end{pmatrix} = -2\epsilon(1+2\epsilon)$$

- $|w_2, w_4, w_5|$

$$\text{Det}\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1+2\epsilon \\ -1-2\epsilon & 0 & 0 \end{pmatrix} = (1+2\epsilon)^2$$

- $|w_3, w_4, w_5|$

$$\text{Det}\begin{pmatrix} -2\epsilon & 0 & -2\epsilon \\ 0 & 0 & 1+2\epsilon \\ -1-2\epsilon & 0 & 0 \end{pmatrix} = 0$$

Thus, $\text{Vol}(2\Pi K)$ is expressed as a function of ϵ in the following way :

$\text{Simplify}[0 + (1+2\epsilon) + (1+2\epsilon) + 2\epsilon(1+2\epsilon) + 2\epsilon(1+2\epsilon) + (1+2\epsilon)^2 + 2\epsilon(1+2\epsilon) + (1+2\epsilon)^2 + 2\epsilon(1+2\epsilon) + 0] = 4(1+5\epsilon+6\epsilon^2)$

Now the convex body K may be decomposed as a triangular prism at the base of a right tetrahedron. We can therefore express the volume of K as the sum of the volumes of the triangular prism and the right tetrahedron.

$$\text{Vol}(K) = a_b \cdot h + \frac{a_b \cdot h}{3} = \frac{1}{2}\epsilon + \frac{\frac{1}{2}\cdot 1}{3} = \frac{\epsilon}{2} + \frac{1}{6}.$$

Consequently, for $K = T + I_{[0,1,0]}\epsilon$, we have the following expression as Petty's functional:

$$\frac{\text{Vol}(\Pi K)}{\text{Vol}(K)^2} = \frac{\frac{1}{2}\left(1 + 5\epsilon + 6\epsilon^2\right)}{\left(\frac{\epsilon}{2} + \frac{1}{6}\right)^2} = \frac{18(1+2\epsilon)}{(1+3\epsilon)}. \tag{2.3}$$

.

2.3 Calculations for $K = T + I_{[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}]}\epsilon$, where $T =$ right tetrahedron in $\mathbb{R}^3$

The deformation of the right simplex is taken here in the direction of the vector $(1,1,1)$ (normalized) by a Minkowski sum with a segment of length ϵ. We repeat the corresponding calculations of the projection body and its volume to see the effect of this deformation on the value of Petty's functional.

$K = T + I_{[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}]}\epsilon = \text{conv}\Big(\{0,0,0\},\{1,0,0\},\{0,1,0\},\{0,0,1\},\{\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\},$
$\{1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\},\{\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\},\{\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}}\}\Big).$

Plot of K if $\epsilon = 1$

$K = \text{ConvexHullMesh}[\{0,0,0\},\{1,0,0\},\{0,1,0\},\{0,0,1\},\{\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\},\{1+\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\},$
$\{\frac{1}{\sqrt{3}},1+\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\},\{\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},1+\frac{1}{\sqrt{3}}\}]$

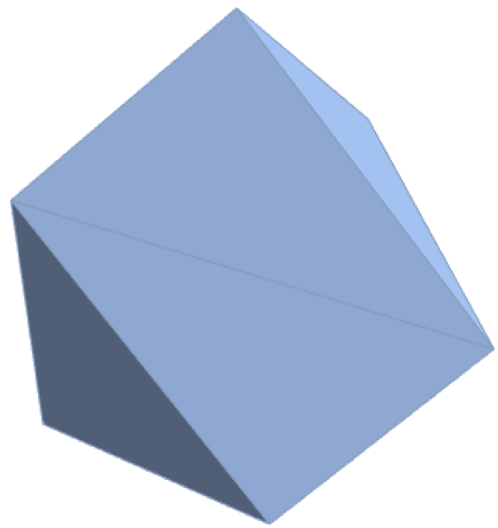


Figure 2.6: Plot of $K = T + I_{[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}]}$ in $\mathbb{R}^3$

Plot of K if $\epsilon = \frac{1}{2}$

$K =$ ConvexHullMesh$[\{0,0,0\},\{1,0,0\},\{0,1,0\},\{0,0,1\},\{\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}}\},\{1+\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}}\},$
$\{\frac{1}{2\sqrt{3}},1+\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}}\},\{\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}},1+\frac{1}{2\sqrt{3}}\}]$

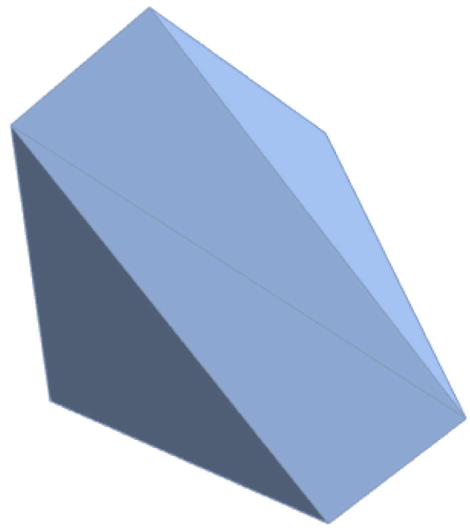


Figure 2.7: Plot of $K = T + I_{[\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}}]}$ in $\mathbb{R}^3$

Faces of K

a. Right triangle 1

b. Right triangle 2

c. Right triangle 3

d. Equilateral triangle

e. Rectangle 1

f. Rectangle 2

g. Rectangle 3

We denote by: u_i the outward-unit normal vector with respect to face i, a_i the area of face i, $a_i u_i$ the area segment corresponding to face i.

- Right triangle 1: vertices are $(0,0,1),(1,0,0),(0,0,0)$

$(0,0,0)-(0,0,1)=\langle 0,0,-1\rangle$

$(0,0,0)-(1,0,0)=\langle -1,0,0\rangle$

$\langle 0,0,-1\rangle \times \langle -1,0,0\rangle = \langle 0,1,0\rangle$

$||\langle 0,1,0\rangle|| = 1$

$a_1 = \frac{1}{2}||\langle 0,1,0\rangle|| = \frac{1}{2}$

$u_1 = \langle 0,1,0\rangle$

$-\,u_1 = \langle 0,-1,0\rangle$

$a_1\left(u_1\right) = \left\langle 0, \frac{1}{2}, 0\right\rangle ; a_1\left(-u_1\right) = \left\langle 0, \frac{-1}{2}, 0\right\rangle$

- Right triangle 2: vertices are $(1,0,0),(0,1,0),(0,0,0)$

$(0,0,0)-(1,0,0)=\langle -1,0,0\rangle$

$(0,0,0)-(0,1,0)=\langle 0,-1,0\rangle$

$\langle -1,0,0\rangle \times \langle 0,-1,0\rangle = \langle 0,0,1\rangle$

$||\langle 0,0,1\rangle|| = 1$

$a_2 = \frac{1}{2}||\langle 0,0,1\rangle|| = \frac{1}{2}$

$u_2 = \langle 0,0,1\rangle$

$-\,u_2 = \langle 0,0,-1\rangle$

$a_2\left(u_2\right) = \left\langle 0, 0, \frac{1}{2}\right\rangle ; a_2\left(-u_2\right) = \left\langle 0, 0, \frac{-1}{2}\right\rangle$

- Right triangle 3: vertices are $(0,0,1),(0,1,0),(0,0,0)$

$(0,0,0)-(0,0,1)=\langle 0,0,-1\rangle$

$(0,0,0)-(0,1,0)=\langle 0,-1,0\rangle$

$\langle 0,0,-1\rangle \times \langle 0,-1,0\rangle = \langle -1,0,0\rangle$

$||\langle -1,0,0\rangle|| = 1$

$a_3 = \frac{1}{2}||\langle -1,0,0\rangle|| = \frac{1}{2}$

$u_3 = \langle -1,0,0\rangle$

$-u_3 = \langle 1,0,0\rangle$

$a_3\,(u_3) = \left\langle \frac{-1}{2},0,0\right\rangle; a_3\,(-u_3) = \left\langle \frac{1}{2},0,0\right\rangle$

- Equilateral triangle: vertices are $(1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}),(\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}),(\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}})$

$(\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}) - (1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}) = \langle -1,1,0\rangle$

$(\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}}) - (1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}) = \langle -1,0,1\rangle$

$\langle -1,1,0\rangle \times \langle -1,0,1\rangle = \langle 1,1,1\rangle$

$||\langle 1,1,1\rangle|| = \sqrt{3}$

$a_4 = \frac{1}{2}||\langle 1,1,1\rangle|| = \frac{\sqrt{3}}{2}$

$u_4 = \langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\rangle$

$-u_4 = \langle \frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\rangle$

$a_4\,(u_4) = \left\langle \frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle; a_4\,(-u_4) = \left\langle \frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right\rangle$

- Rectangle 1: vertices are $(0,0,1),(1,0,0),(\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}}),(1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}})$

$(1,0,0) - (0,0,1) = \langle 1,0,-1\rangle$

$(\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}}) - (0,0,1) = \langle \frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle$

$\langle 1,0,-1\rangle \times \langle \frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle = \langle \frac{\epsilon}{\sqrt{3}},\frac{-2\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle$

$||\langle \frac{\epsilon}{\sqrt{3}},\frac{-2\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle|| = \sqrt{\left(\frac{\epsilon}{\sqrt{3}}\right)^2 + \left(\frac{-2\epsilon}{\sqrt{3}}\right)^2 + \left(\frac{\epsilon}{\sqrt{3}}\right)^2} = \sqrt{\frac{6\epsilon^2}{3}} = \sqrt{2}\epsilon$

$a_5 = \text{length} \cdot \text{width} = \sqrt{2}\epsilon$

$u_5 = \langle \frac{\sqrt{6}}{6},\frac{-\sqrt{6}}{3},\frac{\sqrt{6}}{6}\rangle$

$-u_5 = \langle \frac{-\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{-\sqrt{6}}{6}\rangle$

$a_5\,(u_5) = \left\langle \frac{\epsilon}{\sqrt{3}},\frac{-2\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\right\rangle; a_5\,(-u_5) = \left\langle \frac{-\epsilon}{\sqrt{3}},\frac{2\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}}\right\rangle$

- Rectangle 2: vertices are $(1,0,0),(0,1,0),(1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}),(\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}})$

$(\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}})-(0,1,0)=\langle\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle$

$(0,1,0)-(1,0,0)=\langle -1,1,0\rangle$

$\langle\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle\times\langle -1,1,0\rangle=\langle\frac{-\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}},\frac{2\epsilon}{\sqrt{3}}\rangle$

$||\langle\frac{-\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}},\frac{2\epsilon}{\sqrt{3}}\rangle||=\sqrt{\left(\frac{-\epsilon}{\sqrt{3}}\right)^2+\left(\frac{-\epsilon}{\sqrt{3}}\right)^2+\left(\frac{2\epsilon}{\sqrt{3}}\right)^2}=\sqrt{\frac{6\epsilon^2}{3}}=\sqrt{2}\epsilon$

$a_6=\text{length}\cdot\text{width}=\sqrt{2}\epsilon$

$u_6=\langle\frac{-\sqrt{6}}{6},\frac{-\sqrt{6}}{6},\frac{\sqrt{6}}{3}\rangle$

$-u_6=\langle\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6},\frac{-\sqrt{6}}{3}\rangle$

$a_6\left(u_6\right)=\left\langle\frac{-\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}},\frac{2\epsilon}{\sqrt{3}}\right\rangle;a_6\left(-u_6\right)=\left\langle\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{-2\epsilon}{\sqrt{3}}\right\rangle$

- Rectangle 3: vertices are $(0,0,1),(0,1,0),(\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}),(\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}})$

$(\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},1+\frac{\epsilon}{\sqrt{3}})-(0,0,1)=\langle\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle$

$(0,1,0)-(0,0,1)=\langle 0,1,-1\rangle$

$\langle\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle\times\langle 0,1,-1\rangle=\langle\frac{-2\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle$

$||\langle\frac{-2\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\rangle||=\sqrt{\left(\frac{-2\epsilon}{\sqrt{3}}\right)^2+\left(\frac{\epsilon}{\sqrt{3}}\right)^2+\left(\frac{\epsilon}{\sqrt{3}}\right)^2}=\sqrt{\frac{6\epsilon^2}{3}}=\sqrt{2}\epsilon$

$a_7=\text{length}\cdot\text{width}=\sqrt{2}\epsilon$

$u_7=\langle\frac{-\sqrt{6}}{3},\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{6}\rangle$

$-u_7=\langle\frac{\sqrt{6}}{3},\frac{-\sqrt{6}}{6},\frac{-\sqrt{6}}{6}\rangle$

$a_7\left(u_7\right)=\left\langle\frac{-2\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\right\rangle;a_7\left(-u_7\right)=\left\langle\frac{2\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}}\right\rangle$

$2\Pi K=[-a_1u_1,a_1u_1]+[-a_2u_2,a_2u_2]+[-a_3u_3,a_3u_3]+[-a_4u_4,a_4u_4]+[-a_5u_5,a_5u_5]+$
$[-a_6u_6,a_6u_6]+[-a_7u_7,a_7u_7]$

$2\Pi K=\left[\left\langle 0,\frac{-1}{2},0\right\rangle,\left\langle 0,\frac{1}{2},0\right\rangle\right]+\left[\left\langle 0,0,\frac{-1}{2}\right\rangle,\left\langle 0,0,\frac{1}{2}\right\rangle\right]+\left[\left\langle\frac{1}{2},0,0\right\rangle,\left\langle\frac{-1}{2},0,0\right\rangle\right]$
$+\left[\left\langle\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}\right\rangle,\left\langle\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle\right]+\left[\left\langle\frac{-\epsilon}{\sqrt{3}},\frac{2\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}}\right\rangle,\left\langle\frac{\epsilon}{\sqrt{3}},\frac{-2\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\right\rangle\right]+\left[\left\langle\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{-2\epsilon}{\sqrt{3}}\right\rangle,\left\langle\frac{-\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}},\frac{2\epsilon}{\sqrt{3}}\right\rangle\right]+$
$\left[\left\langle\frac{2\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}},\frac{-\epsilon}{\sqrt{3}}\right\rangle,\left\langle\frac{-2\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}},\frac{\epsilon}{\sqrt{3}}\right\rangle\right]$

$\Rightarrow 2\Pi K$ (and ΠK) is the convex hull of, at most, $2^7 = 128$ possible vectors.

Due to the lengthy *Mathematica* code used to illustrate the plot of $2\Pi K$, we omit the code and instead, simply provide the following respective plots:

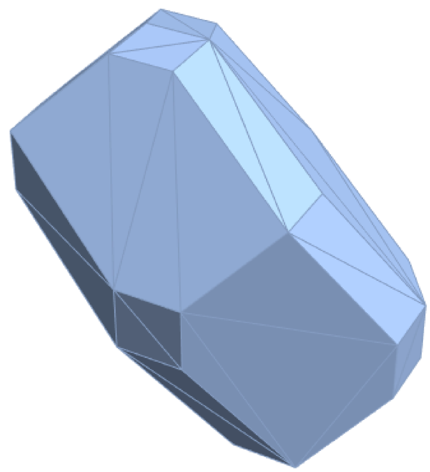


Figure 2.8: Plot of $2\Pi K$ for $K = T + I_{\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]}$ in $\mathbb{R}^3$

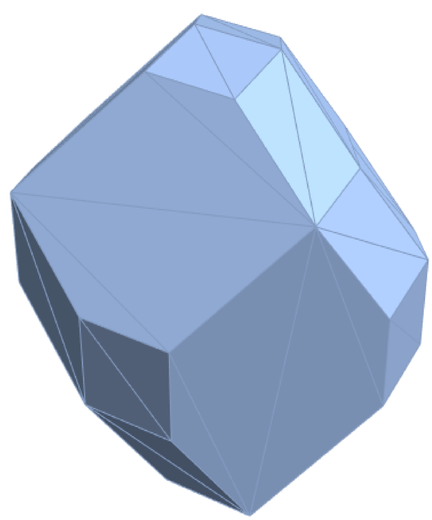


Figure 2.9: Plot of $2\Pi K$ for $K = T + I_{\left[\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}}\right]}$ in $\mathbb{R}^3$

Once more, we calculate the volume of $2\Pi K$ using Brannen's formula [4] as follows:

$$\mathrm{Vol}(\Pi K) = \sum_{1\leq i<j<k\leq n} |w_i, w_j, w_k|$$

$$w_1 = 2 \cdot a_1\,(u_1) = \langle 0, 1, 0\rangle$$

$$w_2 = 2 \cdot a_2\,(u_2) = \langle 0, 0, 1\rangle$$

$$w_3 = 2 \cdot a_3\,(u_3) = \langle -1, 0, 0\rangle$$

$$w_4 = 2 \cdot a_4\,(u_4) = \langle -1, -1, -1\rangle$$

$$w_5 = 2 \cdot a_5\,(u_5) = \langle \tfrac{2\epsilon}{\sqrt{3}}, \tfrac{-4\epsilon}{\sqrt{3}}, \tfrac{2\epsilon}{\sqrt{3}}\rangle$$

$$w_6 = 2 \cdot a_6\,(u_6) = \langle \tfrac{-2\epsilon}{\sqrt{3}}, \tfrac{-2\epsilon}{\sqrt{3}}, \tfrac{4\epsilon}{\sqrt{3}}\rangle$$

$$w_7 = 2 \cdot a_7\,(u_7) = \langle \tfrac{-4\epsilon}{\sqrt{3}}, \tfrac{2\epsilon}{\sqrt{3}}, \tfrac{2\epsilon}{\sqrt{3}}\rangle$$

- $|w_1, w_2, w_3|$

  $$\mathrm{Det}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = -1$$

- $|w_1, w_2, w_4|$

  $$\mathrm{Det}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} = -1$$

- $|w_1, w_2, w_5|$

  $$\mathrm{Det}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = \frac{2\epsilon}{\sqrt{3}}$$

- $|w_1, w_2, w_6|$

  $$\mathrm{Det}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = \frac{-2\epsilon}{\sqrt{3}}$$

- $|w_1, w_2, w_7|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = \frac{-4\epsilon}{\sqrt{3}}$$

- $|w_1, w_3, w_4|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} = -1$$

- $|w_1, w_3, w_5|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = \frac{2\epsilon}{\sqrt{3}}$$

- $|w_1, w_3, w_6|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = \frac{4\epsilon}{\sqrt{3}}$$

- $|w_1, w_3, w_7|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = \frac{2\epsilon}{\sqrt{3}}$$

- $|w_1, w_4, w_5|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 0$$

- $|w_1, w_4, w_6|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = 2\sqrt{3}\epsilon$$

- $|w_1, w_4, w_7|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 2\sqrt{3}\epsilon$$

- $|w_1, w_5, w_6|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = -4\epsilon^2$$

- $|w_1, w_5, w_7|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = -4\epsilon^2$$

- $|w_1, w_6, w_7|$

$$\text{Det}\begin{pmatrix} 0 & 1 & 0 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = -4\epsilon^2$$

- $|w_2, w_3, w_4|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} = 1$$

- $|w_2, w_3, w_5|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = \frac{4\epsilon}{\sqrt{3}}$$

- $|w_2, w_3, w_6|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = \frac{2\epsilon}{\sqrt{3}}$$

- $|w_2, w_3, w_7|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = \frac{-2\epsilon}{\sqrt{3}}$$

- $|w_2, w_4, w_5|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 2\sqrt{3}\epsilon$$

- $|w_2, w_4, w_6|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = 0$$

- $|w_2, w_4, w_7|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = -2\sqrt{3}\epsilon$$

- $|w_2, w_5, w_6|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = -4\epsilon^2$$

- $|w_2, w_5, w_7|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = -4\epsilon^2$$

- $|w_2, w_6, w_7|$

$$\text{Det}\begin{pmatrix} 0 & 0 & 1 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = -4\epsilon^2$$

- $|w_3, w_4, w_5|$

$$\text{Det}\begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 2\sqrt{3}\epsilon$$

- $|w_3, w_4, w_6|$

$$\text{Det}\begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = 2\sqrt{3}\epsilon$$

- $|w_3, w_4, w_7|$

$$\text{Det}\begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 0$$

- $|w_3, w_5, w_6|$

$$\text{Det}\begin{pmatrix} -1 & 0 & 0 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = 4\epsilon^2$$

- $|w_3, w_5, w_7|$

$$\text{Det}\begin{pmatrix} -1 & 0 & 0 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 4\epsilon^2$$

- $|w_3, w_6, w_7|$

$$\text{Det}\begin{pmatrix} -1 & 0 & 0 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 4\epsilon^2$$

- $|w_4, w_5, w_6|$

$$\text{Det}\begin{pmatrix} -1 & -1 & -1 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \end{pmatrix} = 12\epsilon^2$$

- $|w_4, w_5, w_7|$

$$\text{Det}\begin{pmatrix} -1 & -1 & 1 \\ \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 12\epsilon^2$$

- $|w_4, w_6, w_7|$

$$\text{Det}\begin{pmatrix} -1 & -1 & -1 \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 12\epsilon^2$$

- $|w_5, w_6, w_7|$

$$\text{Det}\begin{pmatrix} \frac{2\epsilon}{\sqrt{3}} & \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \\ \frac{-2\epsilon}{\sqrt{3}} & \frac{-2\epsilon}{\sqrt{3}} & \frac{4\epsilon}{\sqrt{3}} \\ \frac{-4\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} & \frac{2\epsilon}{\sqrt{3}} \end{pmatrix} = 0$$

Thus, Vol$(2\Pi K)$ is expressed as a function of ϵ in the following way : Simplify$[1+1+\frac{2\epsilon}{\sqrt{3}}+\frac{2\epsilon}{\sqrt{3}}+\frac{4\epsilon}{\sqrt{3}}+1+\frac{2\epsilon}{\sqrt{3}}+\frac{4\epsilon}{\sqrt{3}}+\frac{2\epsilon}{\sqrt{3}}+0+2\sqrt{3}\epsilon+2\sqrt{3}\epsilon+4\epsilon^2+4\epsilon^2+4\epsilon^2+1+\frac{4\epsilon}{\sqrt{3}}+\frac{2\epsilon}{\sqrt{3}}+\frac{2\epsilon}{\sqrt{3}}+2\sqrt{3}\epsilon+0+2\sqrt{3}\epsilon+4\epsilon^2+4\epsilon^2+4\epsilon^2+2\sqrt{3}\epsilon+2\sqrt{3}\epsilon+0+4\epsilon^2+4\epsilon^2+4\epsilon^2+12\epsilon^2+12\epsilon^2+12\epsilon^2+0] = 4+20\sqrt{3}\epsilon+72\epsilon^2$

Now, K may be expressed as the union of a triangular prism and the right tetrahedron. Thus, $\text{Vol}\,(K) = a_b \cdot h + \frac{1}{6}$.

$\text{Vol}\,(K) = \frac{\sqrt{3}}{2} \cdot \epsilon + \frac{1}{6}$.

We conclude that, for $K = T + I_{\left[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right]}\epsilon$, we have the following expression as Petty's functional:

$$\frac{\text{Vol}\,(\Pi K)}{(\text{Vol}\,(K))^2} = \frac{\frac{1}{2}+\frac{5}{2}\sqrt{3}\epsilon+9\epsilon^2}{\left(\frac{1}{6}+\frac{\sqrt{3}}{2}\epsilon\right)^2} = \frac{18(\sqrt{3}+6\epsilon)}{(\sqrt{3}+9\epsilon)}.$$

2.4 Analysis of Calculations

Using the results shown above, we validate that Petty's functional $P_n(K) \leq \frac{\text{Vol}_n(\Pi T)}{(\text{Vol}_n(T))^{n-1}}$, where $n = 3$, T is the right tetrahedron and K is a non-affine transformation of T of the form $K = T + \sigma\epsilon, \epsilon$ in $(0, 1]$, and σ is a segment of unit length.

2.4.1 Analysis of $P_3(K)$ for $K = T + I_{[0,1,0]}\epsilon$ with various values of $\epsilon \in (0, 1]$

Let

$$P_3(K) = \frac{\mathrm{Vol}_3(\Pi K)}{\mathrm{Vol}_3(K)^2} = 18\frac{(1+2\epsilon)}{(1+3\epsilon)} \ . \tag{2.4}$$

ϵ	0.1	0.25	0.5	0.75	1
$P_3(K)$	16.62	16.55	14.40	13.85	13.50

Table 2.1: $P_3(K)$ for various values of $\epsilon \in (0, 1]$

We can write: $P_3(\epsilon) = 18\frac{(1+2\epsilon)}{(1+3\epsilon)} = 18\left[\frac{1+3\epsilon-\epsilon}{1+3\epsilon}\right] = 18\left[1 - \left(\frac{1}{1+3\epsilon}\right)\epsilon\right]$.

Letting $f(\epsilon) = \frac{1}{1+3\epsilon}$, and $\epsilon \to 0$, we approximate $P_3(K)$ using the second degree MacLaurin series expansion as follows: $f(\epsilon) \approx f(0) + f'(0)\epsilon + \frac{f''(0)}{2!}\epsilon^2$,

$f'(\epsilon) = -\frac{3}{(1+3\epsilon)^2}$,

$f''(\epsilon) = \frac{18}{(1+3\epsilon)^3}$,

$\Rightarrow f(\epsilon) \approx 1 - 3\epsilon + 18\epsilon^2$.

Thus, $P_3(\epsilon) \approx 18\left[1 - (1 - 3\epsilon + 18\epsilon^2)\,\epsilon\right] = 18\left[1 - \epsilon + 3\epsilon^2 - 18\epsilon^3\right]$.

2.4.2 Analysis of $P_3(K)$ for $K = T + I_{[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}]}\epsilon$ with various values of $\epsilon \in (0, 1]$

Let

$$P_3(K) = \frac{\mathrm{Vol}_3(\Pi K)}{\mathrm{Vol}_3(K)^2} = 18\frac{(\sqrt{3}+6\epsilon)}{(\sqrt{3}+9\epsilon)} \ . \tag{2.5}$$

ϵ	0.1	0.25	0.5	0.75	1
$P_3(K)$	15.95	14.61	13.67	13.23	12.97

Table 2.2: $P_3(K)$ for various values of $\epsilon \in (0,1]$

We can write: $P_3(\epsilon) = 18\left[\frac{\sqrt{3}+6\epsilon}{\sqrt{3}+9\epsilon}\right] = 18\left[\frac{\sqrt{3}+9\epsilon-3\epsilon}{\sqrt{3}+9\epsilon}\right] = 18\left[1-\left(\frac{3}{\sqrt{3}+9\epsilon}\right)\epsilon\right] = 18\left[1-\left(\frac{3}{\sqrt{3}(1+3\sqrt{3}\epsilon)}\right)\epsilon\right] = 18\left[1-\left(\frac{\sqrt{3}}{1+3\sqrt{3}\epsilon}\right)\epsilon\right]$.

Letting $g(\epsilon) = \frac{\sqrt{3}}{1+3\sqrt{3}\epsilon}$, and $\epsilon \to 0$, we approximate $P_3(K)$ using the second degree MacLaurin series expansion as follows: $g(\epsilon) \approx g(0) + g'(0)\epsilon + \frac{g''(0)}{2!}\epsilon^2$,

$g'(\epsilon) = -\frac{9}{(1+3\sqrt{3}\epsilon)^2}$,

$g''(\epsilon) = \frac{54\sqrt{3}}{(1+3\sqrt{3}\epsilon)^3}$,

$\Rightarrow g(\epsilon) \approx \sqrt{3} - 9\epsilon + 27\sqrt{3}\epsilon^2$.

Thus, $P_3(\epsilon) \approx 18\left[1-\left(\sqrt{3}-9\epsilon+27\sqrt{3}\epsilon^2\right)\epsilon\right] = 18\left[1-\sqrt{3}\epsilon+9\epsilon^2-27\sqrt{3}\epsilon^3\right]$.

Letting $f(\epsilon) = 18\,\frac{(1+2\epsilon)}{(1+3\epsilon)}$ and $g(\epsilon) = 18\,\frac{(\sqrt{3}+6\epsilon)}{(\sqrt{3}+9\epsilon)}$, we see that $f(\epsilon) > g(\epsilon),\ \forall\epsilon \in (0,1]$.

Hence, we conclude with the following proposition:

Proposition 2.4.1. *The right tetrahedron T is a local maximizer of the functional $P_n(K)$ along the 1-parameter family of convex bodies $K = T + \epsilon\sigma$, with small $\epsilon > 0$.*

Chapter 3

Theoretical Approach to Simplices

We know that for any polyhedra K other than a simplex, or any convex body for that matter, there exists a simplex T, $K \neq T$, and a direction σ such that $T + \epsilon\sigma \subset K$, and likewise, $\Pi(T + \epsilon\sigma) \subset \Pi K$. We want to show theoretically that $\mathrm{Vol}_n(\Pi K)\mathrm{Vol}_n(K)^{1-n} \leq \mathrm{Vol}_n(\Pi T)\mathrm{Vol}_n(T)^{1-n}$, where K is a deformation of a simplex in a direction of a vector (thus not an affine transformation) and T is any polyhedra, as a first step toward showing the latter inequality for any convex body K. Knowledge on mixed volumes is required in providing the underlying theory behind the above set inclusions.

3.1 Mixed Volumes

The theory of mixed volumes owes much of its development to A.D. Aleksandrov, Minkowski, Hadwiger and many other well-known mathematicians [1]. Mixed volumes reflect a mutual measure of size associated to various convex bodies, dependent upon the shape of the bodies, and the relative orientation they have with one another [8]. Key inequalities emanating from mixed volume theory reflect only partly their importance in the general scheme of convexity. Among other applications, mixed volumes bridge the gap between algebraic and convex geometry, and are essential ingredients to various topics, namely combinatorics and probability theory [1].

Much of the underlying framework behind the theory of mixed volumes dates back to Minkowski, who stated that the volume of a linear combination $\sum_{i=1}^{m} \alpha_i P_i$ of convex bodies $P_i \subset \mathbb{R}^n$ is a homogeneous polynomial of degree n, where $\alpha_i \geq 0$ and summation here refers to the Minkowski sum, as defined earlier. The coefficients of this polynomial expansion are precisely the mixed volumes. More explicitly, if $P_1, ..., P_m \subset \mathbb{R}^n$ are convex bodies (for example, polytopes in $\mathbb{R}^n$), and $\alpha_1, ..., \alpha_m \geq 0$ are real numbers, then $\mathrm{Vol}_n\left(\alpha_1 P_1 + ... + \alpha_m P_m\right) = \sum_{i_1,...,i_n=1}^{m} \alpha_{i_1} \cdots \alpha_{i_n} V^{(n)}\left(P_{i_1}, ..., P_{i_n}\right)$. The coefficient $V^{(n)}\left(P_{i_1}, ..., P_{i_n}\right)$ of the monomial $\alpha_{i_1} \cdots \alpha_{i_n}$ is called the mixed volume of $P_{i_1}, ..., P_{i_n}$ [1].

In $\mathbb{R}^2$ for example, $\mathrm{Vol}_2\left(\alpha_1 P_1 + \alpha_2 P_2\right) = {\alpha_1}^2 V^{(2)}\left(P_1, P_1\right) + \alpha_1\alpha_2 V^{(2)}\left(P_1, P_2\right) + \alpha_2\alpha_1 V^{(2)}\left(P_2, P_1\right) + {\alpha_2}^2 V^{(2)}\left(P_2, P_2\right)$. Now, $V^{(2)}\left(P_2, P_1\right) = V^{(2)}\left(P_1, P_2\right)$ since mixed volumes are symmetric in any of their entries, and $V^{(2)}\left(P_1, P_1\right) = \mathrm{Vol}_2\left(P_1\right)$. Therefore, $\mathrm{Vol}_2\left(\alpha_1 P_1 + \alpha_2 P_2\right) = {\alpha_1}^2 \mathrm{Vol}_2\left(P_1\right) + 2\alpha_1\alpha_2 V^{(2)}\left(P_1, P_2\right) + {\alpha_2}^2 \mathrm{Vol}_2\left(P_2\right)$. For simplicity, we have kept above the notation of volume $\mathrm{Vol}_2(\,.\,)$ for the area of compact sets in $\mathbb{R}^2$.

Definition 3.1.1. *Let $P_1, ..., P_n \subset \mathbb{R}^n$ be compact polytopes. The n-mixed volume of $P_1, ..., P_n$ is defined as the following:*

$$V^{(n)}\left(P_1, ..., P_n\right) = \frac{1}{n} \sum_{u \in Norm(P_1,...,P_n)} h_{P_n}(u) V^{(n-1)}\left(P_1(u), ..., P_{n-1}(u)\right) \tag{3.1}$$

where $h_{P_i}(u)$ is the support function with respect to face P_i having outer normal vector u, $Norm(P_n)$ denotes the set of outer normals to the $(n-1)$-dimensional faces of $P_1, \ldots, P_n$, and $P_i(u)$ is the top-dimensional face of P_i of outer unit normal vector u.

For example, in $\mathbb{R}$, we define $V^{(1)}(P) = V_1(P) = b - a$ (equal to the length of the interval P). In $\mathbb{R}^2$, the mixed volume is simply the "mixed area". We have $V^{(2)}\left(P_1, P_2\right) = \frac{1}{2} \sum_{u \in \mathrm{Norm}(P_1)} h_{P_2} V^{(1)}\left(P_1(u)\right)$. The support function h_{P_2} is precisely the perpendicular distance from the origin to the line passing through a vertex of P_2, parallel to the appropriate side of P_1.

3.1.1 Properties of Mixed Volumes [1]

a. Symmetric in any of its entries:

$V^{(n)}\left(P_1, ..., P_{n-1}, P_n\right) = V^{(n)}\left(P_1, ..., P_n, P_{n-1}\right);$

b. Translation invariant:

$V^{(n)}\left(P_1, ..., P_n\right) = V^{(n)}\left(P_1 + x, P_2, ..., P_n\right),\ \forall x \in \mathbb{R}^n;$

c. Monotonic with respect to set inclusion:

If $P_1 \subseteq \tilde{P}_1$, then $V^{(n)}\left(P_1, P_2, ..., P_n\right) \leq V^{(n)}\left(\tilde{P}_1, P_2, ..., P_n\right);$

d. Non-negative:

$V^{(n)}\left(P_1, ..., P_n\right) \geq 0;$

e. Positively homogeneous in each argument:

$\forall \alpha \geq 0:\ V^{(n)}\left(\alpha P_1, ..., P_n\right) = \alpha V^{(n)}\left(P_1, ..., P_n\right);$

f. Additive in each argument with respect to Minkowski addition:

$V^{(n)}\left(\alpha \mathrm{P}_1 + \beta \overline{P_1}, P_2, ..., P_n\right) = \alpha \mathrm{V}^{(n)}\left(P_1, P_2, ..., P_n\right) + \beta \mathrm{V}^{(n)}\left(\overline{P_1}, P_2, ..., P_n\right);$

g. $V^{(n)}(P, ..., P) = \mathrm{Vol}_n(P)$.

3.2 Theoretical Breakdown

Our objective in this section is to show that Petty's conjecture holds in 3 dimensions for any deformation of a simplex by the Minkowski sum with a segment, that is, not only for the directions validated in Chapter 2. Concretely, we want to show the following for $n = 3$:

$$F(K) = \mathrm{Vol}_n(\Pi K)\mathrm{Vol}_n(K)^{1-n} \leq F(T) = \mathrm{Vol}_n(\Pi T)\mathrm{Vol}_n(T)^{1-n} \tag{3.2}$$

where K is the Minkowski sum of T, the right tetrahedron in $\mathbb{R}^3$, and a line segment of arbitrary small length and direction.

Let us denote $K = T + \epsilon \cdot I$, where $I = \frac{1}{2}[-u_I, u_I]$ such that u_I is the direction of I and $\epsilon > 0$ is small.

Based on our previous results, we already have $\mathrm{Vol}_3(\Pi T)\mathrm{Vol}_3(T)^{-2} = 18$. Thus, it remains to show:

$$\mathrm{Vol}_3(\Pi(T + \epsilon \cdot I))\mathrm{Vol}_3(T + \epsilon \cdot I)^{-2} \leq 18. \tag{3.3}$$

3.2.1 The linear approximation of $\mathrm{Vol}_3(T + \epsilon \cdot I)$

We begin by expressing $\mathrm{Vol}_3(T + \epsilon \cdot I)$ using mixed volumes.

By definition, we have: $\mathrm{Vol}_3(\alpha_1 P_1 + \alpha_2 P_2) = \sum_{i_1,i_2,i_3=1}^{2} \alpha_{i_1}\alpha_{i_2}\alpha_{i_3} V^{(3)}(P_{i_1}, P_{i_2}, P_{i_3}) = {\alpha_1}^3 V^{(3)}(P_1, P_1, P_1) + \alpha_1\alpha_2\alpha_1 V^{(3)}(P_1, P_2, P_1) + {\alpha_1}^2\alpha_2 V^{(3)}(P_1, P_1, P_2) + \alpha_1{\alpha_2}^2 V^{(3)}(P_1, P_2, P_2) + \alpha_2{\alpha_1}^2 V^{(3)}(P_2, P_1, P_1) + {\alpha_2}^2\alpha_1 V^{(3)}(P_2, P_2, P_1) + \alpha_2\alpha_1\alpha_2 V^{(3)}(P_2, P_1, P_2) + {\alpha_2}^3 V^{(3)}(P_2, P_2, P_2)$.

Letting $\alpha_1 = 1, \alpha_2 = \epsilon, P_1 = T$, and $P_2 = I$, and using properties of mixed volumes, we get:

$\mathrm{Vol}_3(T + \epsilon \cdot I) = V^{(3)}(T, T, T) + \epsilon \mathrm{V}^{(3)}(T, I, T) + \epsilon \mathrm{V}^{(3)}(T, T, I) + \epsilon^2 V^{(3)}(T, I, I) + \epsilon \mathrm{V}^{(3)}(I, T, T) + \epsilon^2 V^{(3)}(I, I, T) + \epsilon^2 V^{(3)}(I, T, I) + \epsilon^3 V^{(3)}(I, I, I) = \mathrm{Vol}_3(T) + 3\epsilon \mathrm{V}^{(3)}(T, T, I) + 3\epsilon^2 V^{(3)}(T, I, I) + \epsilon^3 \mathrm{Vol}_3(I)$.

Since I is an arbitrary segment in $\mathbb{R}^3$, $\mathrm{Vol}_3(I) = 0$. Furthermore, the term $V^{(3)}(T, I, I)$ is negligible (when $\epsilon \to 0, \epsilon^2$ becomes significantly small). Thus, what we want to focus on is : $\mathrm{Vol}_3(T + \epsilon \cdot I) \approx \mathrm{Vol}_3(T) + 3\epsilon \mathrm{V}^{(3)}(T, T, I) + O(\epsilon^2) = \frac{1}{6} + 3\epsilon \mathrm{V}^{(3)}(T, T, I) + O(\epsilon^2)$.

Use of the second degree MacLaurin series expansion

We use the second degree MacLaurin series expansion to approximate
$\mathrm{Vol}_3(T+\epsilon\cdot I)^{-2} = \frac{1}{\mathrm{Vol}_3(T+\epsilon\cdot I)^2} \approx \frac{1}{\left[\frac{1}{6}+3\epsilon \mathrm{V}^{(3)}(T,T,I)\right]^2}$.
Letting $f(\epsilon) = \frac{1}{\left[\frac{1}{6}+3\epsilon \mathrm{V}^{(3)}(T,T,I)\right]^2}$, and ϵ close to 0, we have:

$$f(\epsilon) \approx f(0) + f'(0)\epsilon + \frac{f''(0)}{2!}\epsilon^2$$
$$f(0) = \frac{1}{\left(\frac{1}{6}\right)^2} = 36$$
$$f'(\epsilon) = -\frac{6V^{(3)}(T,T,I)}{\left(\frac{1}{6}+3\epsilon \mathrm{V}^{(3)}(T,T,I)\right)^3}; f'(0) = -\frac{6V^{(3)}(T,T,I)}{\left(\frac{1}{6}\right)^3} = -1296V^{(3)}(T,T,I)$$
$$f''(\epsilon) = \frac{54\left(V^{(3)}(T,T,I)\right)^2}{\left(\frac{1}{6}+3\epsilon \mathrm{V}^{(3)}(T,T,I)\right)^4}; f''(0) = \frac{54\left(V^{(3)}(T,T,I)\right)^2}{\left(\frac{1}{6}\right)^4} = 69984\left(V^{(3)}(T,T,I)\right)^2.$$

Thus, $\mathrm{Vol}_3(T+\epsilon\cdot I)^{-2} \approx 36 - 1296\epsilon \mathrm{V}^{(3)}(T,T,I) + 34992\epsilon^2\left(V^{(3)}(T,T,I)\right)^2 =$
$216\mathrm{Vol}_3(T) - 1296\epsilon V^{(3)}(T,T,I) + 34992\epsilon^2\left(V^{(3)}(T,T,I)\right)^2$.

3.2.2 The linear approximation of Vol$_3(\Pi(T+\epsilon\cdot I))$

The Projection Body of a Polytope

Let us start with the definition of the projection body of a polytope in $\mathbb{R}^3$ [4]. While the general definition of the projection body of a convex body in $\mathbb{R}^3$ applies, we can deduce a simplified form of the definition in the case of a polytope. To present it here, we need to establish first some notation:

a. P : polytope in $\mathbb{R}^3$;

b. F_i : faces of P, $i = 1, ..., n$;

c. u_i : outward unit normal vector to the face F_i;

d. a_i : area of face F_i;

e. $[-a_iu_i, a_iu_i]$: area segment of P corresponding to the i-th face of length a_i parallel to u_i with midpoint at the origin.

Then, by definition, $2\Pi K = \sum_{i=1}^{n} [-a_iu_i, a_iu_i]$ is the Minkowski sum of n area segments and thus the convex hull of, at most, 2^n vectors.

Example 1: $K = T$, the right tetrahedron in $\mathbb{R}^3$.

In this case, T has 4 faces and, thus,

$\Rightarrow 2\Pi T = \sum_{i=1}^{4} [-a_iu_i, a_iu_i]$ is the Minkowski sum of 4 area segments and thus the convex hull of, at most, $2^4 = 16$ vectors. More precisely,

$\Rightarrow 2\Pi T = \left[\left\langle \frac{1}{2}, 0, 0\right\rangle, \left\langle \frac{-1}{2}, 0, 0\right\rangle\right] + \left[\left\langle 0, \frac{-1}{2}, 0\right\rangle, \left\langle 0, \frac{1}{2}, 0\right\rangle\right] + \left[\left\langle 0, 0, \frac{-1}{2}\right\rangle, \left\langle 0, 0, \frac{1}{2}\right\rangle\right]$
$+ \left[\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle, \left\langle \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\rangle\right].$

Example 2: $K = T + I_{[0,1,0]}\epsilon$, a deformation of the right tetrahedron in $\mathbb{R}^3$.

In this case, $I_{[0,1,0]}$ is parallel to u_1. Thus I is perpendicular to the face F_1.

Here, we add to T a line segment of length ϵ in the positive y-direction. As seen in Chapter 2, K has 5 faces.

$\Rightarrow 2\Pi K = \sum_{i=1}^{5} [-a_iu_i, a_iu_i]$ = convex hull of, at most, $2^5 = 32$ vectors.

$\Rightarrow 2\Pi K = \left[\left\langle 0, \frac{-1}{2}, 0\right\rangle, \left\langle 0, \frac{1}{2}, 0\right\rangle\right] + \left[\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle, \left\langle \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\rangle\right] + \left[\langle \epsilon, 0, \epsilon\rangle, \langle -\epsilon, 0, -\epsilon\rangle\right]$
$+ \left[\left\langle 0, 0, \frac{-1}{2} - \epsilon\right\rangle, \left\langle 0, 0, \frac{1}{2} + \epsilon\right\rangle\right] + \left[\left\langle \frac{1}{2} + \epsilon, 0, 0\right\rangle, \left\langle \frac{-1}{2} - \epsilon, 0, 0\right\rangle\right] = \left[\left\langle 0, \frac{-1}{2}, 0\right\rangle, \left\langle 0, \frac{1}{2}, 0\right\rangle\right] +$
$\left[\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle, \left\langle \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\rangle\right] + \left[\langle \epsilon, 0, \epsilon\rangle, \langle -\epsilon, 0, -\epsilon\rangle\right] + (1 + 2\epsilon)\left[\left\langle 0, 0, \frac{-1}{2}\right\rangle, \left\langle 0, 0, \frac{1}{2}\right\rangle\right] +$
$(1 + 2\epsilon)\left[\left\langle \frac{1}{2}, 0, 0\right\rangle, \left\langle \frac{-1}{2}, 0, 0\right\rangle\right].$

$\Rightarrow 2\Pi K = 2\Pi T - \left[\left\langle 0, 0, \frac{-1}{2}\right\rangle, \left\langle 0, 0, \frac{1}{2}\right\rangle\right] - \left[\left\langle \frac{1}{2}, 0, 0\right\rangle, \left\langle \frac{-1}{2}, 0, 0\right\rangle\right] + \left[\langle \epsilon, 0, \epsilon\rangle, \langle -\epsilon, 0, -\epsilon\rangle\right] +$
$(1 + 2\epsilon)\left[\left\langle 0, 0, \frac{-1}{2}\right\rangle, \left\langle 0, 0, \frac{1}{2}\right\rangle\right] + (1 + 2\epsilon)\left[\left\langle \frac{1}{2}, 0, 0\right\rangle, \left\langle \frac{-1}{2}, 0, 0\right\rangle\right].$

$\Rightarrow 2\Pi K = 2\Pi T + \left[\langle \epsilon, 0, \epsilon\rangle, \langle -\epsilon, 0, -\epsilon\rangle\right] + 2\epsilon\left[\left\langle 0, 0, \frac{-1}{2}\right\rangle, \left\langle 0, 0, \frac{1}{2}\right\rangle\right] + 2\epsilon\left[\left\langle \frac{1}{2}, 0, 0\right\rangle, \left\langle \frac{-1}{2}, 0, 0\right\rangle\right].$

$\Rightarrow 2\Pi K = 2\Pi T + \left[\langle \epsilon, 0, \epsilon\rangle, \langle -\epsilon, 0, -\epsilon\rangle\right] + \left[\langle 0, 0, -\epsilon\rangle, \langle 0, 0, \epsilon\rangle\right] + \left[\langle \epsilon, 0, 0\rangle, \langle -\epsilon, 0, 0\rangle\right].$

Let U be the (degenerate) parallelepiped formed by the Minkowski sum of the vectors $\{\langle 1,0,1\rangle, \langle 0,0,1\rangle, \langle 1,0,0\rangle\}$, or equivalently,

$U = \text{conv}(\{2,0,0\},\{2,0,2\},\{0,0,0\},\{0,0,2\},\{0,0,-2\},\{-2,0,-2\},\{-2,0,0\})$,

Then, $2\Pi K = 2\Pi T + \epsilon \cdot U$.

The plot of U viewed with *Mathematica* using the command ListPointPlot3D.

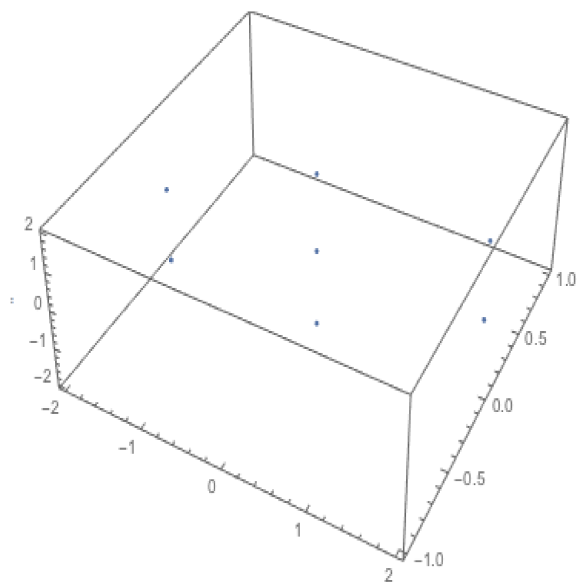


Figure 3.1: Plot of U

To evaluate the volume of the projection body of K, we use mixed volumes and obtain: $\text{Vol}_3(2\Pi K) = 2^3\text{Vol}_3(\Pi K) = 2^3\text{Vol}_3(\Pi T + \epsilon \cdot U) = 2^3[V^{(3)}(\Pi T, \Pi T, \Pi T) + \epsilon \text{V}^{(3)}(\Pi T, U, \Pi T) + \epsilon \text{V}^{(3)}(\Pi T, \Pi T, U) + \epsilon^2 V^{(3)}(\Pi T, U, U) + \epsilon \text{V}^{(3)}(U, \Pi T, \Pi T) + \epsilon^2 V^{(3)}(U, U, \Pi T) + \epsilon^2 V^{(3)}(U, \Pi T, U) + \epsilon^3 V^{(3)}(U, U, U)] = 2^3\,[\text{Vol}_3(\Pi T) + 3\epsilon \text{V}^{(3)}(\Pi T, \Pi T, U) + 3\epsilon^2 V^{(3)}(\Pi T, U, U) + \epsilon^3 \text{Vol}_3(U)] = 2^3[\frac{1}{2} + 3\epsilon \text{V}^{(3)}(\Pi T, \Pi T, U) + 3\epsilon^2 V^{(3)}(\Pi T, U, U) + 8\sqrt{2}\epsilon^3]$.

The coefficients of ϵ^2 and ϵ^3 are negligible when ϵ is close to zero. We thus conclude that $\text{Vol}_3(\Pi K) \approx \frac{1}{2} + 3\epsilon V^{(3)}(\Pi T, \Pi T, U)$.

Example 3: $K = T + I_{[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}]}\epsilon$**, a deformation of the right tetrahedron in** $\mathbb{R}^3$**.**

Here, $I_{[\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}]}$ is the segment added to T, parallel to u_4. Thus I is perpendicular to the face F_4.

In this case, we add to T a factor of $\frac{\epsilon}{\sqrt{3}}$ in the $(1,1,1)$-direction. As seen in Chapter 2, this produces a polyhedron K with 7 faces. Consequently,

$\Rightarrow 2\Pi K = \sum_{i=1}^{7} [-a_i u_i, a_i u_i]$ = convex hull of, at most, $2^7 = 128$ vectors

$\Rightarrow 2\Pi K = \left[\left\langle 0, \frac{-1}{2}, 0\right\rangle, \left\langle 0, \frac{1}{2}, 0\right\rangle\right] + \left[\left\langle 0, 0, \frac{-1}{2}\right\rangle, \left\langle 0, 0, \frac{1}{2}\right\rangle\right] + \left[\left\langle \frac{1}{2}, 0, 0\right\rangle, \left\langle \frac{-1}{2}, 0, 0\right\rangle\right]$
$+\left[\left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle, \left\langle \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right\rangle\right] + \left[\left\langle \frac{-\epsilon}{\sqrt{3}}, \frac{2\epsilon}{\sqrt{3}}, \frac{-\epsilon}{\sqrt{3}}\right\rangle, \left\langle \frac{\epsilon}{\sqrt{3}}, \frac{-2\epsilon}{\sqrt{3}}, \frac{\epsilon}{\sqrt{3}}\right\rangle\right] + \left[\left\langle \frac{\epsilon}{\sqrt{3}}, \frac{\epsilon}{\sqrt{3}}, \frac{-2\epsilon}{\sqrt{3}}\right\rangle, \left\langle \frac{-\epsilon}{\sqrt{3}}, \frac{-\epsilon}{\sqrt{3}}, \frac{2\epsilon}{\sqrt{3}}\right\rangle\right] +$
$\left[\left\langle \frac{2\epsilon}{\sqrt{3}}, \frac{-\epsilon}{\sqrt{3}}, \frac{-\epsilon}{\sqrt{3}}\right\rangle, \left\langle \frac{-2\epsilon}{\sqrt{3}}, \frac{\epsilon}{\sqrt{3}}, \frac{\epsilon}{\sqrt{3}}\right\rangle\right].$
$\Rightarrow 2\Pi K = 2\Pi T + \left[\left\langle \frac{-\epsilon}{\sqrt{3}}, \frac{2\epsilon}{\sqrt{3}}, \frac{-\epsilon}{\sqrt{3}}\right\rangle, \left\langle \frac{\epsilon}{\sqrt{3}}, \frac{-2\epsilon}{\sqrt{3}}, \frac{\epsilon}{\sqrt{3}}\right\rangle\right] + \left[\left\langle \frac{\epsilon}{\sqrt{3}}, \frac{\epsilon}{\sqrt{3}}, \frac{-2\epsilon}{\sqrt{3}}\right\rangle, \left\langle \frac{-\epsilon}{\sqrt{3}}, \frac{-\epsilon}{\sqrt{3}}, \frac{2\epsilon}{\sqrt{3}}\right\rangle\right] +$
$\left[\left\langle \frac{2\epsilon}{\sqrt{3}}, \frac{-\epsilon}{\sqrt{3}}, \frac{-\epsilon}{\sqrt{3}}\right\rangle, \left\langle \frac{-2\epsilon}{\sqrt{3}}, \frac{\epsilon}{\sqrt{3}}, \frac{\epsilon}{\sqrt{3}}\right\rangle\right].$

Let U be the parallelepiped formed by the Minkowski sum of the vectors
$\{\langle \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle, \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\rangle, \langle \frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle\}$.

We can immediately see that U is degenerate because its volume (given via the determinant of the three vectors) is zero. To conclude,

$U = \text{conv}(\{\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{-4}{\sqrt{3}}\}, \{\frac{-2}{\sqrt{3}}, \frac{4}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\}, \{0,0,0\}, \{\frac{-4}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\}, \{\frac{4}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\}, \{\frac{2}{\sqrt{3}}, \frac{-4}{\sqrt{3}}, \frac{2}{\sqrt{3}}\},$
$\{\frac{-2}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{4}{\sqrt{3}}\})$,

and $2\Pi K = 2\Pi T + \epsilon \cdot U$.

The plot of U, the parallelepiped, with *Mathematica* using the command ListPointPlot3D

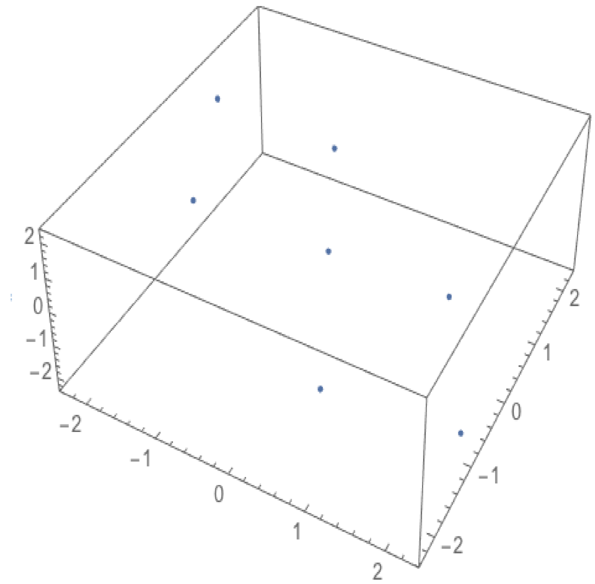


Figure 3.2: Plot of U

Volume of U:

We calculate Det $\begin{pmatrix} \frac{-2}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{4}{\sqrt{3}} \\ \frac{-4}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{-2}{\sqrt{3}} & \frac{4}{\sqrt{3}} & \frac{-2}{\sqrt{3}} \end{pmatrix} = 0$ and conclude that all three vectors lie in the same plane (coplanar vectors).

As above, we get: $\mathrm{Vol}_3(2\Pi K) = 2^3\mathrm{Vol}_3(\Pi K) = 2^3\mathrm{Vol}_3(\Pi T + \epsilon\cdot U) = 2^3[V^{(3)}(\Pi T, \Pi T, \Pi T) + \epsilon \mathrm{V}^{(3)}(\Pi T, U, \Pi T) + \epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U) + \epsilon^2 V^{(3)}(\Pi T, U, U) + \epsilon \mathrm{V}^{(3)}(U, \Pi T, \Pi T) + \epsilon^2 V^{(3)}(U, U, \Pi T) + \epsilon^2 V^{(3)}(U, \Pi T, U) + \epsilon^3 V^{(3)}(U, U, U)] = 2^3\,[\mathrm{Vol}_3(\Pi T) + 3\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U) + 3\epsilon^2 V^{(3)}(\Pi T, U, U) + \epsilon^3 \mathrm{Vol}_3(U)] = 2^3[\frac{1}{2} + 3\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U) + 3\epsilon^2 V^{(3)}(\Pi T, U, U) + \epsilon^3 \mathrm{Vol}_3(U)]$,

and the term $V^{(3)}(\Pi T, U, U)$ becomes negligible as ϵ is close to 0.

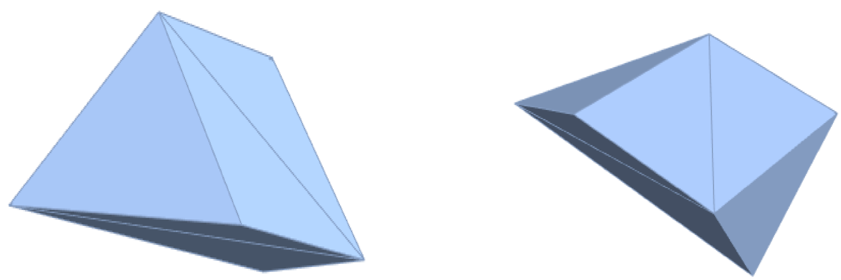
Similarly, we obtain $\mathrm{Vol}_3(\Pi K) \approx \frac{1}{2} + 3\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U)$.

3.2.3 How are new faces formed when adding a direction to T?

Let us consider $K = T + I_{[-1,0,0]}\epsilon$.

We will present two different views for the plot of K.

Using *Mathematica*, we present two views of K for $\epsilon = 1$, using the command
K = ConvexHullMesh[{0,0,0},{1,0,0},{0,1,0},{0,0,1},{-1,0,0},{-1,0,1},{-1,1,0}]



(a) View 1 (b) View 2

Figure 3.3: Two different views for $K = T + [-1, 0, 0]$ in $\mathbb{R}^3$

When we add to T a segment of ϵ length I in the form of either $[\pm 1, 0, 0]$, $[0, \pm 1, 0]$ or $[0, 0, \pm 1]$, then $K = T + \epsilon \cdot I$ will have at most 7 faces such that two faces are degenerate, by which we mean that they lie in the same plane. Thus, in these cases, we count a total of 5 non-degenerate faces. This situation only happens when some of the normals of K are the same as those to T whose faces meeting at the origin are two-by-two perpendicular to each other along the axes of coordinates.

We consider a segment of unit length $I = [a, b, c]$ such that $\sqrt{a^2 + b^2 + c^2} = 1$, and at least two of a, b or $c \neq 0$. Then $K = T + \epsilon \cdot I$ will always be composed of 3 of the original faces of T, 3 rectangles and an equilateral triangle whose vertices are dependent on a, b and c. The new faces are derived from, at most, the edges of T, and together with the translation of the faces of T, we obtain at most 7 faces for

$K = T + \epsilon \cdot I$. By definition, $\Pi(T + \epsilon \cdot I)$ will be the Minkowski sum of at most 7 area vectors.

To see this, recall that, for any polyhedron K, its projection polyhedron, ΠK, is the Minkowski sum of segments whose direction is normal to a corresponding face of K and length equal to the area of that face. As seen in Chapter 2, for $K = T + \epsilon \cdot I$, some faces of K have the same normals as T, while others are new faces with new normals. However, we note that the faces of K having the same normals as T may not have the same area as those of T. For example, in $K = T + [0, 1, 0] \cdot \epsilon$, two faces whose normals are the same as those of T have areas larger than the corresponding faces of T.

Normals/Areas of the faces of $K = T$:

Face 1: $u_1 = \langle 1, 0, 0 \rangle$, $a_1 = \frac{1}{2}$

Face 2: $u_2 = \langle 0, 1, 0 \rangle$, $a_2 = \frac{1}{2}$

Face 3: $u_3 = \langle 0, 0, 1 \rangle$, $a_3 = \frac{1}{2}$

Face 4: $u_4 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$, $a_4 = \frac{\sqrt{3}}{2}$.

Normals/Areas of the faces of $K = T + I_{[0,1,0]}\epsilon$:

Face 1: $u_1 = \langle 0, 1, 0 \rangle$, $a_1 = \frac{1}{2}$

Face 2: $u_2 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$, $a_2 = \frac{\sqrt{3}}{2}$

Face 3: $u_3 = \left\langle \frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right\rangle$, $a_3 = \sqrt{2}\epsilon$

Face 4: $u_4 = \langle 0, 0, 1 \rangle$, $a_4 = \frac{1}{2} + \epsilon$

Face 5: $u_5 = \langle 1, 0, 0 \rangle$, $a_5 = \frac{1}{2} + \epsilon$.

Normals/Areas of the faces of $K = T + I_{[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]}\epsilon$:

Face 1: $u_1 = \langle 0, 1, 0 \rangle$, $a_1 = \frac{1}{2}$

Face 2: $u_2 = \langle 0, 0, 1 \rangle$, $a_2 = \frac{1}{2}$

Face 3: $u_3 = \langle 1, 0, 0 \rangle$, $a_3 = \frac{1}{2}$

Face 4: $u_4 = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$, $a_4 = \frac{\sqrt{3}}{2}$

Face 5: $u_5 = \left\langle \frac{\sqrt{6}}{6}, \frac{-\sqrt{6}}{3}, \frac{\sqrt{6}}{6} \right\rangle$, $a_5 = \sqrt{2}\epsilon$

Face 6: $u_6 = \left\langle \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{-\sqrt{6}}{3} \right\rangle$, $a_6 = \sqrt{2}\epsilon$

Face 7: $u_7 = \left\langle \frac{-\sqrt{6}}{3}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6} \right\rangle$, $a_7 = \sqrt{2}\epsilon$.

3.2.4 How do we define U in $\Pi K = \Pi T + \epsilon \cdot U$?

Essentially, U is the Minkowski sum of the leftover segments, those which are added to the segments forming ΠT. Be definition, U is a zonotope, which is precisely a set of points resulting from the Minkowski sum of segments.

a. For $K = T + I_{[0,1,0]}\epsilon$, we have:

$$2\Pi K = 2\Pi T + \Big([\langle 1, 0, 1 \rangle, \langle -1, 0, -1 \rangle] + [\langle 0, 0, -1 \rangle, \langle 0, 0, 1 \rangle] + [\langle 1, 0, 0 \rangle, \langle -1, 0, 0 \rangle] \Big) \cdot \epsilon$$

$$\Rightarrow U = \frac{1}{2}\Big([\langle 1, 0, 1 \rangle, \langle -1, 0, -1 \rangle] + [\langle 0, 0, -1 \rangle, \langle 0, 0, 1 \rangle] + [\langle 1, 0, 0 \rangle, \langle -1, 0, 0 \rangle] \Big)$$

$\Rightarrow U$ is the Minkowski sum of three segments having length equal to $\sqrt{2}$, two of which are normals to the associated faces of T. U is composed of three segments, each of which is perpendicular to I:

$$\langle 0, 1, 0 \rangle \cdot \langle 1, 0, 1 \rangle = 0$$

$$\langle 0, 1, 0 \rangle \cdot \langle 0, 0, 1 \rangle = 0$$

$$\langle 0, 1, 0 \rangle \cdot \langle 1, 0, 0 \rangle = 0.$$

b. For $K = T + I_{[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]}\epsilon$, we have:

$$2\Pi K = 2\Pi T + \Bigg(\left[\left\langle \frac{-1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle, \left\langle \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \right] + \left[\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}} \right\rangle, \left\langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right\rangle \right] + \left[\left\langle \frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle, \left\langle \frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \right] \Bigg) \cdot \epsilon$$

$\Rightarrow U = \frac{1}{2}\Bigg(\Big[\Big\langle \frac{-1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \Big\rangle, \Big\langle \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}} \Big\rangle\Big] + \Big[\Big\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}} \Big\rangle, \Big\langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{2}{\sqrt{3}} \Big\rangle\Big] +$
$\Big[\Big\langle \frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \Big\rangle, \Big\langle \frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \Big\rangle\Big] \Bigg)$

$\Rightarrow U$ is the Minkowski sum of three segments having length equal to $\sqrt{2}$, none of which are normals to the associated faces of T. U is composed of three segments, each of which is perpendicular to I:

$\Big\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \Big\rangle \cdot \Big\langle \frac{-\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}, \frac{-\sqrt{3}}{3} \Big\rangle = 0$
$\Big\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \Big\rangle \cdot \Big\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{-2\sqrt{3}}{3} \Big\rangle = 0$
$\Big\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \Big\rangle \cdot \Big\langle \frac{2\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}, \frac{-\sqrt{3}}{3} \Big\rangle = 0.$

Back to our objective

We want to show: $\mathrm{Vol}_3(\Pi(T + \epsilon \cdot I)) \cdot \mathrm{Vol}_3(T + \epsilon \cdot I)^{-2} \leq 18$

We have:

$\mathrm{Vol}_3(\Pi(T + \epsilon \cdot I)) \approx \mathrm{Vol}_3(\Pi T) + 3\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U) = \frac{1}{2} + 3\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U)$

$\mathrm{Vol}_3(T + \epsilon \cdot I)^{-2} \approx 216\mathrm{Vol}_3(T) - 1296\epsilon \mathrm{V}^{(3)}(T, T, I) = 36 - 1296\epsilon \mathrm{V}^{(3)}(T, T, I)$

Thus, we can show that $(\frac{1}{2} + 3\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U)\Big) \cdot \Big(36 - 1296\epsilon \mathrm{V}^{(3)}(T, T, I)\Big) \leq 18$ or, equivalently,

$18 - 648\epsilon \mathrm{V}^{(3)}(T, T, I) + 108\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U) - 3888\epsilon^2 V^{(3)}(\Pi T, \Pi T, U) V^{(3)}(T, T, I) \leq 18.$

As ϵ is close to 0, the term $-3888\epsilon^2 V^{(3)}(\Pi T, \Pi T, U) V^{(3)}(T, T, I)$ is negligible, thus it suffices to show that: $18 - 648\epsilon \mathrm{V}^{(3)}(T, T, I) + 108\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U) \leq 18$

$\Leftrightarrow -648\epsilon \mathrm{V}^{(3)}(T, T, I) + 108\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U) \leq 0$

$\Leftrightarrow 108\epsilon \mathrm{V}^{(3)}(\Pi T, \Pi T, U) - 648\epsilon \mathrm{V}^{(3)}(T, T, I) \leq 0$

$\Leftrightarrow 108\epsilon\left(V^{(3)}(\Pi T,\Pi T,U)-6V^{(3)}(T,T,I)\right)\leq 0$

$\Leftrightarrow V^{(3)}(\Pi T,\Pi T,U)-6V^{(3)}(T,T,I)\leq 0.$

By definition, we have:

$V^{(3)}(\Pi T,\Pi T,U)=$

$=\frac{1}{3}\sum_{u\in\text{Norm}(\Pi T,\Pi T)}h_U(u)V^{(2)}(\Pi T(u),\Pi T(u))=\frac{1}{3}\sum_{u\in\text{Norm}(\Pi T)}h_U(u)\cdot A(\Pi T(u)),$

and

$V^{(3)}(T,T,I)=\frac{1}{3}\sum_{u\in\text{Norm}(T,T)}h_I(u)V^{(2)}(T(u),T(u))=\frac{1}{3}\sum_{u\in\text{Norm}(T)}h_I(u)\cdot A(T(u)).$

The unit normals to the faces of T are: $\langle 1,0,0\rangle,\langle 0,1,0\rangle,\langle 0,0,1\rangle,\left\langle\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle.$

We want to show that the inequality holds to imply that T is a local maximizer of Petty's functional under the given transformations.

We assume that $A\in\text{SL}(n)$ is a special linear transformation with $\det(A)=1$ such that A transforms T into a regular tetrahedron with the same volume as the right tetrahedron, thus $\text{Vol}(T)=\frac{1}{6}$. It is known that, in that case, ΠT is the rhombic dodecahedron (12 faces) with $\text{Vol}(\Pi T)=\frac{1}{2}$ [6].

Given that all projection bodies are centrally symmetric, regardless of the original convex body, we apply to ΠT the special case (for $n=3$) of the reverse isoperimetric inequality for centrally symmetric convex bodies [2]:

$$\frac{(\text{Area}(\Pi T))^3}{(\text{Vol}(\Pi T))^2}\leq\frac{(\text{Area}(C))^3}{(\text{Vol}(C))^2},$$

where C is a unit cube in $\mathbb{R}^3$.

Since $(\text{Area}(\Pi T))^3=(12\cdot\text{Area}(F(\Pi T)))^3$, where $F(\Pi T)$ denotes a face of ΠT,

$(\text{Area}(C))^3=(2\cdot a_b+p_b\cdot h)^3=(2\cdot 1+4\cdot 1)^3=6^3,$

$(\mathrm{Vol}(\Pi T))^2 = (\frac{1}{2})^2 = \frac{1}{4}$,

$(\mathrm{Vol}(C))^2 = 1^2 = 1$,

we get: $4(12 \cdot \mathrm{Area}(F(\Pi T)))^3 \leq 6^3 \Leftrightarrow$

$(12 \cdot \mathrm{Area}(F(\Pi T)))^3 \leq \frac{6^3}{4} \Leftrightarrow 12 \cdot \mathrm{Area}(F(\Pi T)) \leq \frac{6}{2^{\frac{2}{3}}}$.

Therefore, we have: $\mathrm{Area}(F(\Pi T)) \leq \frac{6}{2^{\frac{2}{3}}} \cdot \frac{1}{12} = 2^{\frac{-5}{3}}$.

Note that we may obtain the exact value of the area of a face of ΠT using the surface area formula of the rhombic dodecahedron:

$$\mathrm{Area}(\Pi T) = 8\sqrt{2} \cdot e^2, \tag{3.4}$$

where e = side length of ΠT (e may be obtained from the volume formula of the rhombic dodecahedron, $\mathrm{Vol}(\Pi T) = \frac{16\sqrt{3}}{9} \cdot e^3$, knowing that $\mathrm{Vol}(\Pi T)$ is $\frac{1}{2}$).

i.e., $\frac{1}{2} = \left(\frac{16\sqrt{3}}{9}\right) e^3 \Rightarrow e = \left(\frac{\frac{9}{2}}{16\sqrt{3}}\right)^{\frac{1}{3}} \approx 0.545562$.

Thus, $\mathrm{Area}(\Pi T) = 8\sqrt{2}\left(\frac{\frac{9}{2}}{16\sqrt{3}}\right)^{\frac{2}{3}} \approx 3.367386$.

Finally we get : $\mathrm{Area}(F(\Pi T)) \approx \frac{3.367386}{12} \approx 0.280616$.

Now, the volume of the regular tetrahedron may be expressed as a function of its side length as follows: $\mathrm{Vol}(T) = \frac{l^3}{6\sqrt{2}}$, where l is the side length of T. Since $\mathrm{Vol}(T) = \frac{1}{6}$, we have: $l^3 = \sqrt{2} \Rightarrow l = 2^{\frac{1}{6}}$. This means that the length of each of the three vectors generating U is $2^{\frac{1}{6}}$. Then, the Minkowski sum of each of the vectors generating U forms a regular hexagon whose side length is $2^{\frac{1}{6}}$. The hexagon can be circumscribed to a disk of radius $2^{\frac{1}{6}}$. Thus, the support function of U is at most the support function of the disk, equal to its radius $r = 2^{\frac{1}{6}}$ and this in, at most, 6 directions. Hence we get: $h_U(u) \leq 2^{\frac{1}{6}}$ in, at most, 6 directions. These cannot be all normal directions to the faces of ΠT due the dihedral angle of the rhombic dodecahedron which is $\frac{2\pi}{3}$. Since I is orthogonal to U, two faces of ΠT are parallel to U and therefore will not contribute to the mixed volume $V^{(3)}(\Pi T, \Pi T, U)$ because h_U in the I-direction is zero. Thus, $h_U(u) \leq 2^{\frac{1}{6}}$ in at most 1 direction of a face of ΠT, and in the other nine directions of its faces we have that $h_U(u) \leq 2^{\frac{1}{6}} \mid \cos(\frac{2\pi}{3}) \mid = \frac{2^{\frac{1}{6}}}{2} = 2^{\frac{-5}{6}}$.

The four vectors that are normal to the faces of T are uniformly distributed. The direction of vertices and faces can be interchanged. Consider α to be the angle between the normal planes to the faces of T. Then it is known that the angle α (dihedral angle) is: $\alpha = \cos^{-1}\left(-\frac{1}{3}\right)$.

If I is any unit segment, it will be close to at least two of the vectors that are normal to the faces of T because there is no room to put it further than that (this is true for all vectors). Therefore, I makes an angle β smaller than $\frac{1}{2}\alpha = \frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)$ for at least two of the faces of T. However, $\cos(\beta) \geq \cos\left(\frac{1}{2}\alpha\right) = \left(\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)\right)$ since the cosine function is decreasing in the first quadrant $\Rightarrow \frac{1}{2}\cos(\alpha) \geq \frac{1}{2}\cos\left(\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)\right.$. Thus, using trigonometry and by definition of the support function, there are at least two faces of T with direction u such that $h_I(u) \geq \frac{1}{2}\left(\cos\left(\frac{1}{2}\cos^{-1}\left(-\frac{1}{3}\right)\right)\right) = \frac{1}{2} \mid \cos\left(\pi - \frac{1}{2}\left(\cos^{-1}\left(-\frac{1}{3}\right)\right)\right) \mid = \frac{1}{2} \cdot \frac{1}{\sqrt{3}}$ (by symmetry of the cosine function).

Thus, $h_I(u) \geq \frac{1}{2} \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$.

Putting everything together, we get:
$3V^{(3)}(\Pi T, \Pi T, U) = \sum_{u \in \text{Norm}(\Pi T)} h_U(u) \cdot A(\Pi T(u)) \leq (2^{\frac{1}{6}} + 2^{\frac{-5}{6}} \cdot 9) \cdot 0.280616 \approx 1.732394$.

Now $3V^{(3)}(T, T, I) = \sum_{u \in \text{Norm}(T)} h_I(u) \cdot A(T(u)) \geq 2\frac{\sqrt{3}}{6} \cdot A(T(u))$, where it is known that $\text{Area}(F(T)) = \frac{\sqrt{3}}{4} \cdot l^2 = \frac{\sqrt{3}}{4} \cdot \left(2^{\frac{1}{6}}\right)^2 = \frac{\sqrt{3}}{4} \cdot 2^{\frac{1}{3}}$
$\Rightarrow 3V^{(3)}(T, T, I) \geq 2\frac{\sqrt{3}}{6}\frac{\sqrt{3}}{4} \cdot 2^{\frac{1}{3}} = 2^{\frac{4}{3}} \cdot \frac{3}{24} = \frac{2^{\frac{4}{3}}}{8} = \frac{2^{\frac{4}{3}}}{2^3} = 2^{\frac{-5}{3}}$

$\Leftrightarrow -3V^{(3)}(T,T,I) \leq -\left(2^{\frac{-5}{3}}\right)$

$\Rightarrow V^{(3)}(\Pi T, \Pi T, U) - 6V^{(3)}(T,T,I) \approx \frac{1}{3} \cdot 1.732394 - 2\left(2^{\frac{-5}{3}}\right) \approx -0.052496.$

In conclusion, we have proved the following theorem:

Theorem 3.2.1. *Let T be any tetrahedron in $\mathbb{R}^3$. Then, for any unit segment $I \subset \mathbb{R}^3$ centred at the origin and any small $\epsilon > 0$, we have that*

$$P_n(T + \epsilon I) < P_n(T). \tag{3.5}$$

Chapter 4

Insight on Petty's Projection Inequality for Polar Bodies

4.1 Petty's Projection Inequality

Broadly speaking, inequalities are synonymous to relationships. The scope of geometric inequalities encompasses several analogies in relation to the inequalities themselves, many of which are unsurprisingly interconnected. Among other parallels, Petty's projection inequality is one such example, in that it is equivalent to Busemann-Petty centroid inequality and is a strengthened form of the classical isoperimetric inequality [9], [14]. Petty's projection inequality is of fundamental importance in the framework of affine isoperimetric inequalities. Petty's projection inequality relating the volume of a convex body K and that of its polar projection body states the following: $\mathrm{Vol}_n(K)^{n-1}\mathrm{Vol}_n(\Pi\mathrm{K})^{\star} \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n$ with equality $\Leftrightarrow$ K is an ellipsoid. Both Lutwak and Zhang provide generalizations and consequences of Petty's result, yet with different approaches - Lutwak makes use of mixed volumes and projection measures (brightness, girth, width functions) [9], whereas Zhang incorporates compact domains to strengthen Petty's original inequality [15].

Advances in the literature show that Petty's projection inequality has been applied to, and is a consequence of, various affine isoperimetric inequalities in the hope of obtaining more powerful results within the area of Minkowski geometry. It is shown by Petty that affinely equivalent convex bodies give rise to affinely equivalent projection bodies. From this point of view, the two functionals, $\mathrm{Vol}_n(\Pi K)\mathrm{Vol}_n(K)^{1-n}$ and $\mathrm{Vol}_n(K)^{n-1}\mathrm{Vol}_n(\Pi K)^\star$, are affine invariants. Zhang generalizes from these results an affine invariant Sobolev inequality that is stronger than the classical Sobolev inequality [15]. Furthermore, an application to stochastic geometry is founded by Petty's projection inequality via Schneider. The incentive to develop Petty's projection inequality for polar bodies was due to the difficulty in proving the upper bound of Petty's conjecture. By introducing the concept of polarity, it became possible to simplify and derive existing inequalities into some where conclusions are more easily drawn and the implications are similar. In this regard, variations of interesting results, new proofs and conjectures emerged naturally.

Although duality generally entails a direct equivalence, the one existing between $\mathrm{Vol}_n(\Pi K)^\star$ and $\mathrm{Vol}_n(\Pi K)$ is not a direct relation; rather, it is merely a similarity. We have seen that $(\Pi K)^\star$ may be represented as the intersection of the halfspaces whose normals are precisely the vertices of ΠK. In this way, we will validate both Petty's projection inequality and the lower bound introduced by Zhang [14]. In brief, among bodies of given volume, the polar projection bodies have maximal volume for ellipsoids and minimal volume for simplices, whereas ordinary projection bodies are conjectured to have maximal volume for simplices and minimal volume for ellipsoids.

4.2 Validation of Petty's Projection Inequality and its Reverse [14]

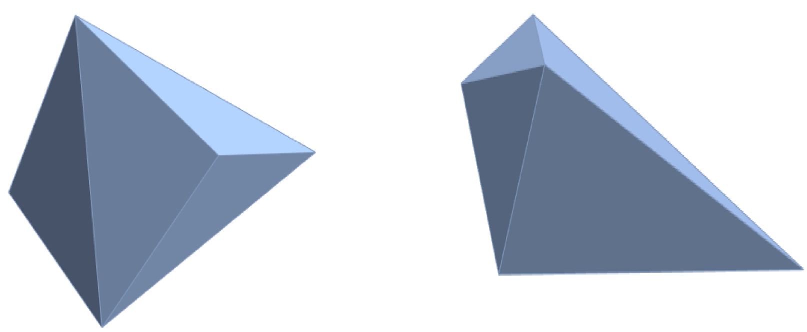
4.2.1 Example: we consider $P = \textbf{conv}(T \cup [2,2,2]) \subset \mathbb{R}^3$, P is the convex hull of the union of a simplex and a line segment of length $2\sqrt{3}$]

$P = \text{conv}(T \cup [2,2,2])$

Two different views for the plot of P

Using *Mathematica*, we present two views of P with the command

P = ConvexHullMesh[{0,0,0},{1,0,0},{0,1,0},{0,0,1},{2,2,2}]



(a) View 1 (b) View 2

Figure 4.1: Two different views for the plot of P in $\mathbb{R}^3$

Thus P is a polytope with six two-dimensional faces (all triangles).

Using *Mathematica*, we calculate the volume of P with the command

$\text{Vol}_3(P)$ = Volume[ConvexHullMesh[{0,0,0},{1,0,0},{0,1,0},{0,0,1},{2,2,2}]] = 1

Breakdown for the faces of P:

a. Right triangle formed by the points: (0,0,1), (1,0,0), (0,0,0);

b. Right triangle formed by the points: (1,0,0), (0,1,0), (0,0,0);

c. Right triangle formed by the points: (0,0,1), (0,1,0), (0,0,0);

d. Isosceles triangle formed by the points: (0,0,1), (0,1,0), (2,2,2);

e. Isosceles triangle formed by the points: (0,0,1), (1,0,0), (2,2,2);

f. Isosceles triangle formed by the points: (1,0,0), (0,1,0), (2,2,2).

Calculations for ΠP:

We denote by: u_i the outward-unit normal vector with respect to face i, a_i the area of face i, and $a_i u_i$ the area segment corresponding to face i. Using this notation, we will analyze each of the faces of P.

- Right triangle with vertices $(0,0,1), (1,0,0), (0,0,0)$

$(0,0,0) - (0,0,1) = \langle 0,0,-1 \rangle$

$(0,0,0) - (1,0,0) = \langle -1,0,0 \rangle$

$\langle 0,0,-1 \rangle \times \langle -1,0,0 \rangle = \langle 0,1,0 \rangle$

$||\langle 0,1,0 \rangle|| = 1$

$a_1 = \frac{1}{2}||\langle 0,1,0 \rangle|| = \frac{1}{2}$

$u_1 = \langle 0,1,0 \rangle$

$-u_1 = \langle 0,-1,0 \rangle$

$a_1(u_1) = \left\langle 0, \frac{1}{2}, 0 \right\rangle; a_1(-u_1) = \left\langle 0, -\frac{1}{2}, 0 \right\rangle.$

- Right triangle with vertices $(1,0,0),(0,1,0),(0,0,0)$

$(0,0,0)-(1,0,0)=\langle -1,0,0\rangle$

$(0,0,0)-(0,1,0)=\langle 0,-1,0\rangle$

$\langle -1,0,0\rangle \times \langle 0,-1,0\rangle = \langle 0,0,1\rangle$

$||\langle 0,0,1\rangle|| = 1$

$a_2 = \frac{1}{2}||\langle 0,0,1\rangle|| = \frac{1}{2}$

$u_2 = \langle 0,0,1\rangle$

$-u_2 = \langle 0,0,-1\rangle$

$a_2\left(u_2\right) = \left\langle 0,0,\frac{1}{2}\right\rangle; a_2\left(-u_2\right) = \left\langle 0,0,\frac{-1}{2}\right\rangle.$

- Right triangle with vertices $(0,0,1),(0,1,0),(0,0,0)$

$(0,0,0)-(0,0,1)=\langle 0,0,-1\rangle$

$(0,0,0)-(0,1,0)=\langle 0,-1,0\rangle$

$\langle 0,0,-1\rangle \times \langle 0,-1,0\rangle = \langle -1,0,0\rangle$

$||\langle -1,0,0\rangle|| = 1$

$a_3 = \frac{1}{2}||\langle -1,0,0\rangle|| = \frac{1}{2}$

$u_3 = \langle -1,0,0\rangle$

$-u_3 = \langle 1,0,0\rangle$

$a_3\left(u_3\right) = \left\langle \frac{-1}{2},0,0\right\rangle; a_3\left(-u_3\right) = \left\langle \frac{1}{2},0,0\right\rangle.$

- Isosceles triangle with vertices $(0,0,1),(0,1,0),(2,2,2)$

$(2,2,2)-(0,0,1)=\langle 2,2,1\rangle$

$(2,2,2)-(0,1,0)=\langle 2,1,2\rangle$

$\langle 2,2,1\rangle \times \langle 2,1,2\rangle = \langle 3,-2,-2\rangle$

$||\langle 3,-2,-2\rangle|| = \sqrt{3^2+(-2)^2+(-2)^2} = \sqrt{9+4+4} = \sqrt{17}$

$a_4 = \frac{1}{2}||\langle 3,-2,-2\rangle|| = \frac{1}{2}\sqrt{17}$

$u_4 = \frac{1}{\sqrt{17}}\langle 3,-2,-2\rangle$

$-u_4 = \frac{1}{\sqrt{17}}\langle -3, 2, 2\rangle$

$a_4 (u_4) = \langle \frac{3}{2}, -1, -1\rangle; a_4 (-u_4) = \langle \frac{-3}{2}, 1, 1\rangle.$

- Isosceles triangle with vertices $(0, 0, 1), (1, 0, 0), (2, 2, 2)$

$(2, 2, 2) - (0, 0, 1) = \langle 2, 2, 1\rangle$

$(2, 2, 2) - (1, 0, 0) = \langle 1, 2, 2\rangle$

$\langle 2, 2, 1\rangle \times \langle 1, 2, 2\rangle = \langle 2, -3, 2\rangle$

$||\langle 2, -3, 2\rangle|| = \sqrt{2^2 + (-3)^2 + 2^2} = \sqrt{4 + 9 + 4} = \sqrt{17}$

$a_5 = \frac{1}{2}||\langle 2, -3, 2\rangle|| = \frac{1}{2}\sqrt{17}$

$u_5 = \frac{1}{\sqrt{17}}\langle 2, -3, 2\rangle$

$-u_5 = \frac{1}{\sqrt{17}}\langle -2, 3, -2\rangle$

$a_5 (u_5) = \left\langle 1, \frac{-3}{2}, 1\right\rangle; a_5 (-u_5) = \left\langle -1, \frac{3}{2}, -1\right\rangle.$

- Isosceles triangle with vertices $(1, 0, 0), (0, 1, 0), (2, 2, 2)$

$(2, 2, 2) - (1, 0, 0) = \langle 1, 2, 2\rangle$

$(2, 2, 2) - (0, 1, 0) = \langle 2, 1, 2\rangle$

$\langle 1, 2, 2\rangle \times \langle 2, 1, 2\rangle = \langle 2, 2, -3\rangle$

$||\langle 2, 2, -3\rangle|| = \sqrt{2^2 + 2^2 + (-3)^2} = \sqrt{4 + 4 + 9} = \sqrt{17}$

$a_6 = \frac{1}{2}||\langle 2, 2, -3\rangle|| = \frac{1}{2}\sqrt{17}$

$u_6 = \frac{1}{\sqrt{17}}\langle 2, 2, -3\rangle$

$-u_6 = \frac{1}{\sqrt{17}}\langle -2, -2, 3\rangle$

$a_6 (u_6) = \left\langle 1, 1, \frac{-3}{2}\right\rangle; a_6 (-u_6) = \left\langle -1, -1, \frac{3}{2}\right\rangle.$

Thus, by the definition of the projection body of a polytope, we have $2\Pi P = [-a_1u_1, a_1u_1] + [-a_2u_2, a_2u_2] + [-a_3u_3, a_3u_3] + [-a_4u_4, a_4u_4] + [-a_5u_5, a_5u_5] + [-a_6u_6, a_6u_6].$

$$\Rightarrow \Pi P = \left[\left\langle 0, \tfrac{-1}{4}, 0\right\rangle, \left\langle 0, \tfrac{1}{4}, 0\right\rangle\right] + \left[\left\langle 0, 0, \tfrac{-1}{4}\right\rangle, \left\langle 0, 0, \tfrac{1}{4}\right\rangle\right] + \left[\left\langle \tfrac{1}{4}, 0, 0\right\rangle, \left\langle \tfrac{-1}{4}, 0, 0\right\rangle\right]$$

$$+ \left[\left\langle \tfrac{-3}{4}, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle, \left\langle \tfrac{3}{4}, \tfrac{-1}{2}, \tfrac{-1}{2}\right\rangle\right] + \left[\left\langle \tfrac{-1}{2}, \tfrac{3}{4}, \tfrac{-1}{2}\right\rangle, \left\langle \tfrac{1}{2}, \tfrac{-3}{4}, \tfrac{1}{2}\right\rangle\right] + \left[\left\langle \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{3}{4}\right\rangle, \left\langle \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{-3}{4}\right\rangle\right].$$

Alternatively, we describe ΠP as the convex hull of, at most, $2^6 = 64$ possible vectors.

$$\Pi P = \mathrm{conv}(\{1, 1, -2\}, \left\{1, 1, \tfrac{-3}{2}\right\}, \left\{\tfrac{1}{2}, 1, -2\right\}, \left\{\tfrac{1}{2}, 1, \tfrac{-3}{2}\right\}, \left\{1, \tfrac{1}{2}, -2\right\}, \left\{1, \tfrac{1}{2}, \tfrac{-3}{2}\right\},$$
$$\left\{\tfrac{1}{2}, \tfrac{1}{2}, -2\right\}, \left\{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{-3}{2}\right\}, \left\{0, 0, \tfrac{-1}{2}\right\}, \{0, 0, 0\}, \left\{\tfrac{-1}{2}, 0, \tfrac{-1}{2}\right\}, \left\{\tfrac{-1}{2}, 0, 0\right\}, \left\{\tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}\right\},$$
$$\left\{\tfrac{-1}{2}, \tfrac{-1}{2}, 0\right\}, \left\{0, \tfrac{-1}{2}, \tfrac{-1}{2}\right\}, \left\{0, \tfrac{-1}{2}, 0\right\}, \left\{\tfrac{-1}{2}, 2, -1\right\}, \left\{\tfrac{-1}{2}, 2, \tfrac{-1}{2}\right\}, \{-1, 2, -1\}, \left\{-1, 2, \tfrac{-1}{2}\right\},$$
$$\left\{\tfrac{-1}{2}, \tfrac{3}{2}, -1\right\}, \left\{\tfrac{-1}{2}, \tfrac{3}{2}, \tfrac{-1}{2}\right\}, \left\{-1, \tfrac{3}{2}, -1\right\}, \left\{-1, \tfrac{3}{2}, \tfrac{-1}{2}\right\}, \left\{\tfrac{-3}{2}, 1, \tfrac{1}{2}\right\}, \left\{\tfrac{-3}{2}, 1, 1\right\}, \left\{-2, 1, \tfrac{1}{2}\right\},$$
$$\{-2, 1, 1\}, \left\{\tfrac{-3}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\}, \left\{\tfrac{-3}{2}, \tfrac{1}{2}, 1\right\}, \left\{-2, \tfrac{1}{2}, \tfrac{1}{2}\right\}, \left\{-2, \tfrac{1}{2}, 1\right\}, \left\{2, \tfrac{-1}{2}, -1\right\}, \left\{2, \tfrac{-1}{2}, \tfrac{-1}{2}\right\},$$
$$\left\{\tfrac{3}{2}, \tfrac{-1}{2}, -1\right\}, \left\{\tfrac{3}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}\right\}, \{2, -1, -1\}, \left\{2, -1, \tfrac{-1}{2}\right\}, \left\{\tfrac{3}{2}, -1, -1\right\}, \left\{\tfrac{3}{2}, -1, \tfrac{-1}{2}\right\}, \left\{1, \tfrac{-3}{2}, \tfrac{1}{2}\right\},$$
$$\left\{1, \tfrac{-3}{2}, 1\right\}, \left\{\tfrac{1}{2}, \tfrac{-3}{2}, \tfrac{1}{2}\right\}, \left\{\tfrac{1}{2}, \tfrac{-3}{2}, 1\right\}, \left\{\tfrac{1}{2}, -2, \tfrac{1}{2}\right\}, \left\{\tfrac{1}{2}, -2, 1\right\}, \left\{1, -2, \tfrac{1}{2}\right\}, \{1, -2, 1\}, \left\{\tfrac{1}{2}, \tfrac{1}{2}, 0\right\},$$
$$\left\{\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right\}, \left\{0, \tfrac{1}{2}, 0\right\}, \left\{0, \tfrac{1}{2}, \tfrac{1}{2}\right\}, \left\{\tfrac{1}{2}, 0, 0\right\}, \left\{\tfrac{1}{2}, 0, \tfrac{1}{2}\right\}, \left\{0, 0, \tfrac{1}{2}\right\}, \left\{\tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{3}{2}\right\}, \left\{\tfrac{-1}{2}, \tfrac{-1}{2}, 2\right\},$$
$$\left\{-2, \tfrac{-1}{2}, \tfrac{3}{2}\right\}, \left\{-1, \tfrac{-1}{2}, 2\right\}, \left\{\tfrac{-1}{2}, -1, \tfrac{3}{2}\right\}, \left\{\tfrac{-1}{2}, -1, 2\right\}, \left\{-1, -1, \tfrac{3}{2}\right\}, \{-1, -1, 2\}).$$

This allows us to visualize the projection body ΠP using *Mathematica*.

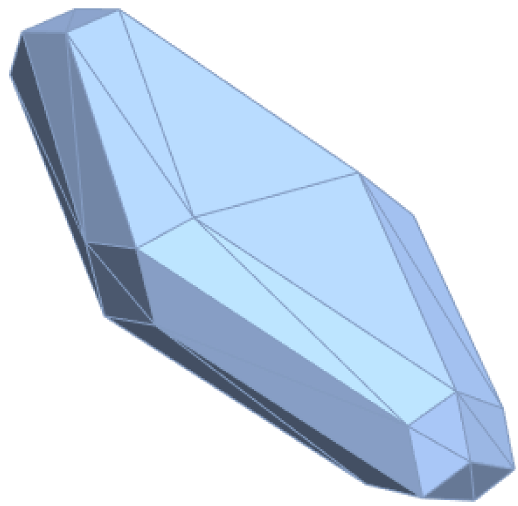


Figure 4.2: Plot of ΠP for $P = T \cup [2,2,2] \subset \mathbb{R}^3$

Calculations for $(\Pi P)^\star$

Knowing that $(\Pi P)^\star$ may be represented as the intersection of the halfspaces whose normals are precisely the vectors whose convex hull forms ΠP, we use *Mathematica* to obtain the plot and volume of $(\Pi P)^\star$. In what follows, for each of the vectors, vertices of ΠP, we find the halfspace corresponding to $(\Pi P)^\star$.

1. $v_1 = \langle 1, 1, -2\rangle$

   $\|v_1\|^2 = 1^2 + 1^2 + (-2)^2 = 6$

   $\left(x - \frac{1}{6}\right) + \left(y - \frac{1}{6}\right) - 2\left(z + \frac{2}{6}\right) \leq 0 \Leftrightarrow x + y \leq 1 + 2z.$

2. $v_2 = \langle 1, 1, \frac{-3}{2}\rangle$

   $\|v_2\|^2 = 1^2 + 1^2 + (\frac{-3}{2})^2 = 2 + \frac{9}{4} = \frac{17}{4}$

   $\left(x - \frac{1}{\frac{17}{4}}\right) + \left(y - \frac{1}{\frac{17}{4}}\right) - \frac{3}{2}\left(z + \frac{\frac{3}{2}}{\frac{17}{4}}\right) \leq 0 \Leftrightarrow x + y \leq 1 + \frac{3z}{2}.$

3. $v_3 = \langle \frac{1}{2}, 1, -2\rangle$

   $\|v_3\|^2 = (\frac{1}{2})^2 + 1^2 + (-2)^2 = \frac{1}{4} + 1 + 4 = \frac{21}{4}$

   $\frac{1}{2}(x - \frac{\frac{1}{2}}{\frac{21}{4}}\Big) + \left(y - \frac{1}{\frac{21}{4}}\right) - 2\left(z + \frac{2}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow x + 2y \leq 2 + 4z.$

4. $v_4 = \langle \frac{1}{2}, 1, \frac{-3}{2} \rangle$

   $\|v_4\|^2 = (\frac{1}{2})^2 + 1^2 + (\frac{-3}{2})^2 = \frac{1}{4} + 1 + \frac{9}{4} = \frac{14}{4}$

   $\frac{1}{2}(x - \frac{\frac{1}{2}}{\frac{14}{4}}\Big) + \left(y - \frac{1}{\frac{14}{4}}\right) - \frac{3}{2}\left(z + \frac{\frac{3}{2}}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow x + 2y \leq 2 + 3z.$

5. $v_5 = \langle 1, \frac{1}{2}, -2 \rangle$

   $\|v_5\|^2 = 1^2 + (\frac{1}{2})^2 + (-2)^2 = 1 + \frac{1}{4} + 4 = \frac{21}{4}$

   $\left(x - \frac{1}{\frac{21}{4}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{21}{4}}\right) - 2\left(z + \frac{2}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow 2x + y \leq 2 + 4z.$

6. $v_6 = \langle 1, \frac{1}{2}, \frac{-3}{2} \rangle$

   $\|v_6\|^2 = 1^2 + (\frac{1}{2})^2 + (\frac{-3}{2})^2 = 1 + \frac{1}{4} + \frac{9}{4} = \frac{14}{4}$

   $\left(x - \frac{1}{\frac{14}{4}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{14}{4}}\right) - \frac{3}{2}\left(z + \frac{\frac{3}{2}}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow 2x + y \leq 2 + 3z.$

7. $v_7 = \langle \frac{1}{2}, \frac{1}{2}, -2 \rangle$

   $\|v_7\|^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 + (-2)^2 = \frac{1}{4} + \frac{1}{4} + 4 = \frac{18}{4}$

   $\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{18}{4}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{18}{4}}\right) - 2\left(z + \frac{2}{\frac{18}{4}}\right) \leq 0 \Leftrightarrow x + y \leq 2 + 4z.$

8. $v_8 = \langle \frac{1}{2}, \frac{1}{2}, \frac{-3}{2} \rangle$

   $\|v_7\|^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{-3}{2})^2 = \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \frac{11}{4}$

   $\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{11}{4}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{11}{4}}\right) - \frac{3}{2}\left(z + \frac{\frac{3}{2}}{\frac{11}{4}}\right) \leq 0 \Leftrightarrow x + y \leq 2 + 3z.$

9. $v_9 = \langle 0, 0, \frac{-1}{2} \rangle$

   $\|v_9\|^2 = \frac{1}{4}$

   $-\frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{1}{4}}\right) \leq 0 \Leftrightarrow 2 + z \geq 0.$

10. $v_{10} = \langle 0, 0, 0 \rangle.$

11. $v_{11} = \langle \frac{-1}{2}, 0, \frac{-1}{2} \rangle$

    $\|v_{11}\|^2 = (\frac{-1}{2})^2 + (\frac{-1}{2})^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

    $-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{1}{2}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{1}{2}}\right) \leq 0 \Leftrightarrow 2 + x + z \geq 0.$

12. $v_{12} = \langle \frac{-1}{2}, 0, 0 \rangle$

    $\|v_{12}\|^2 = \frac{1}{4}$

    $-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{1}{4}}\right) \leq 0 \Leftrightarrow 2 + x \geq 0.$

13. $v_{13} = \langle \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2} \rangle$

$\|v_{13}\|^2 = (\frac{-1}{2})^2 + (\frac{-1}{2})^2 + (\frac{-1}{2})^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{3}{4}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{3}{4}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{3}{4}}\right) \leq 0 \Leftrightarrow 2 + x + y + z \geq 0.$

14. $v_{14} = \langle \frac{-1}{2}, \frac{-1}{2}, 0 \rangle$

$\|v_{14}\|^2 = (\frac{-1}{2})^2 + (\frac{-1}{2})^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{1}{2}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{1}{2}}\right) \leq 0 \Leftrightarrow 2 + x + y \geq 0.$

15. $v_{15} = \langle 0, \frac{-1}{2}, \frac{-1}{2} \rangle$

$\|v_{15}\|^2 = (\frac{-1}{2})^2 + (\frac{-1}{2})^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$-\frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{1}{2}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{1}{2}}\right) \leq 0 \Leftrightarrow 2 + y + z \geq 0.$

16. $v_{16} = \langle 0, \frac{-1}{2}, 0 \rangle$

$\|v_{16}\|^2 = \frac{1}{4}$

$-\frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{1}{4}}\right) \leq 0 \Leftrightarrow 2 + y \geq 0.$

17. $v_{17} = \langle \frac{-1}{2}, 2, -1 \rangle$

$\|v_{17}\|^2 = (\frac{-1}{2})^2 + 2^2 + (-1)^2 = \frac{1}{4} + 4 + 1 = \frac{21}{4}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{21}{4}}\right) + 2\left(y - \frac{2}{\frac{21}{4}}\right) - \left(z + \frac{1}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow 4y \leq 2 + x + 2z.$

18. $v_{18} = \langle \frac{-1}{2}, 2, \frac{-1}{2} \rangle$

$\|v_{18}\|^2 = (\frac{-1}{2})^2 + 2^2 + (\frac{-1}{2})^2 = \frac{1}{4} + 4 + \frac{1}{4} = \frac{18}{4}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{18}{4}}\right) + 2\left(y - \frac{2}{\frac{18}{4}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{18}{4}}\right) \leq 0 \Leftrightarrow 4y \leq 2 + x + z.$

19. $v_{19} = \langle -1, 2, -1 \rangle$

$\|v_{19}\|^2 = (-1)^2 + 2^2 + (-1)^2 = 6$

$-\left(x + \frac{1}{6}\right) + 2\left(y - \frac{2}{6}\right) - \left(z + \frac{1}{6}\right) \leq 0 \Leftrightarrow 2y \leq 1 + x + z.$

20. $v_{20} = \langle -1, 2, \frac{-1}{2} \rangle$

$\|v_{20}\|^2 = (-1)^2 + 2^2 + (\frac{-1}{2})^2 = 1 + 4 + \frac{1}{4} = \frac{21}{4}$

$-\left(x + \frac{1}{\frac{21}{4}}\right) + 2\left(y - \frac{2}{\frac{21}{4}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow 4y \leq 2 + 2x + z.$

21. $v_{21} = \langle \frac{-1}{2}, \frac{3}{2}, -1\rangle$

$\|v_{21}\|^2 = (\frac{-1}{2})^2 + (\frac{3}{2})^2 + (-1)^2 = \frac{1}{4} + \frac{9}{4} + 1 = \frac{14}{4}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{14}{4}}\right) + \frac{3}{2}\left(y - \frac{\frac{3}{2}}{\frac{14}{4}}\right) - \left(z + \frac{1}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow 3y \leq 2 + x + 2z.$

22. $v_{22} = \langle \frac{-1}{2}, \frac{3}{2}, \frac{-1}{2}\rangle$

$\|v_{22}\|^2 = (\frac{-1}{2})^2 + (\frac{3}{2})^2 + (\frac{-1}{2})^2 = \frac{1}{4} + \frac{9}{4} + \frac{1}{4} = \frac{11}{4}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{11}{4}}\right) + \frac{3}{2}\left(y - \frac{\frac{3}{2}}{\frac{11}{4}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{11}{4}}\right) \leq 0 \Leftrightarrow 3y \leq 2 + x + z.$

23. $v_{23} = \langle -1, \frac{3}{2}, -1\rangle$

$\|v_{23}\|^2 = (-1)^2 + (\frac{3}{2})^2 + (-1)^2 = 1 + \frac{9}{4} + 1 = \frac{17}{4}$

$-\left(x + \frac{1}{\frac{17}{4}}\right) + \frac{3}{2}\left(y - \frac{\frac{3}{2}}{\frac{17}{4}}\right) - \left(z + \frac{1}{\frac{17}{4}}\right) \leq 0 \Leftrightarrow \frac{3y}{2} \leq 1 + x + z.$

24. $v_{24} = \langle -1, \frac{3}{2}, \frac{-1}{2}\rangle$

$\|v_{24}\|^2 = (-1)^2 + (\frac{3}{2})^2 + (\frac{-1}{2})^2 = 1 + \frac{9}{4} + \frac{1}{4} = \frac{14}{4}$

$-\left(x + \frac{1}{\frac{14}{4}}\right) + \frac{3}{2}\left(y - \frac{\frac{3}{2}}{\frac{14}{4}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow 3y \leq 2 + 2x + z.$

25. $v_{25} = \langle \frac{-3}{2}, 1, \frac{1}{2}\rangle$

$\|v_{25}\|^2 = (\frac{-3}{2})^2 + 1^2 + (\frac{1}{2})^2 = \frac{9}{4} + 1 + \frac{1}{4} = \frac{14}{4}$

$-\frac{3}{2}\left(x + \frac{\frac{3}{2}}{\frac{14}{4}}\right) + \left(y - \frac{1}{\frac{14}{4}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow 2y + z \leq 2 + 3x.$

26. $v_{26} = \langle \frac{-3}{2}, 1, 1\rangle$

$\|v_{26}\|^2 = (\frac{-3}{2})^2 + 1^2 + 1^2 = \frac{9}{4} + 1 + 1 = \frac{17}{4}$

$-\frac{3}{2}\left(x + \frac{\frac{3}{2}}{\frac{17}{4}}\right) + \left(y - \frac{1}{\frac{17}{4}}\right) + \left(z - \frac{1}{\frac{17}{4}}\right) \leq 0 \Leftrightarrow 2(-1 + y + z) \leq 3x.$

27. $v_{27} = \langle -2, 1, \frac{1}{2}\rangle$

$\|v_{27}\|^2 = (-2)^2 + 1^2 + (\frac{1}{2})^2 = 4 + 1 + \frac{1}{4} = \frac{21}{4}$

$-2\left(x + \frac{2}{\frac{21}{4}}\right) + \left(y - \frac{1}{\frac{21}{4}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow 2y + z \leq 2 + 4x.$

28. $v_{28} = \langle -2, 1, 1\rangle$

$\|v_{28}\|^2 = (-2)^2 + 1^2 + 1^2 = 6$

$-2\left(x + \frac{2}{6}\right) + \left(y - \frac{1}{6}\right) + \left(z - \frac{1}{6}\right) \leq 0 \Leftrightarrow y + z \leq 1 + 2x.$

29. $v_{29} = \langle \frac{-3}{2}, \frac{1}{2}, \frac{1}{2} \rangle$

$\|v_{29}\|^2 = (\frac{-3}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}$

$-\frac{3}{2}\left(x + \frac{\frac{3}{2}}{\frac{11}{4}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{11}{4}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{11}{4}}\right) \leq 0 \Leftrightarrow y + z \leq 2 + 3x.$

30. $v_{30} = \langle \frac{-3}{2}, \frac{1}{2}, 1 \rangle$

$\|v_{30}\|^2 = (\frac{-3}{2})^2 + (\frac{1}{2})^2 + 1^2 = \frac{9}{4} + \frac{1}{4} + 1 = \frac{14}{4}$

$-\frac{3}{2}\left(x + \frac{\frac{3}{2}}{\frac{14}{4}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{14}{4}}\right) + \left(z - \frac{1}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow y + 2z \leq 2 + 3x.$

31. $v_{31} = \langle -2, \frac{1}{2}, \frac{1}{2} \rangle$

$\|v_{31}\|^2 = (-2)^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 = 4 + \frac{1}{4} + \frac{1}{4} = \frac{18}{4}$

$-2\left(x + \frac{2}{\frac{18}{4}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{18}{4}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{18}{4}}\right) \leq 0 \Leftrightarrow y + z \leq 2 + 4x.$

32. $v_{32} = \langle -2, \frac{1}{2}, 1 \rangle$

$\|v_{32}\|^2 = (-2)^2 + (\frac{1}{2})^2 + 1^2 = 4 + \frac{1}{4} + 1 = \frac{21}{4}$

$-2\left(x + \frac{2}{\frac{21}{4}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{21}{4}}\right) + \left(z - \frac{1}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow y + 2z \leq 2 + 4x.$

33. $v_{33} = \langle 2, \frac{-1}{2}, -1 \rangle$

$\|v_{33}\|^2 = 2^2 + (\frac{-1}{2})^2 + (-1)^2 = 4 + \frac{1}{4} + 1 = \frac{21}{4}$

$2\left(x - \frac{2}{\frac{21}{4}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{21}{4}}\right) - \left(z + \frac{1}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow 4x \leq 2 + y + 2z.$

34. $v_{34} = \langle 2, \frac{-1}{2}, \frac{-1}{2} \rangle$

$\|v_{32}\|^2 = 2^2 + (\frac{-1}{2})^2 + (\frac{-1}{2})^2 = 4 + \frac{1}{4} + \frac{1}{4} = \frac{18}{4}$

$2\left(x - \frac{2}{\frac{18}{4}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{18}{4}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{18}{4}}\right) \leq 0 \Leftrightarrow 4x \leq 2 + y + z.$

35. $v_{35} = \langle \frac{3}{2}, \frac{-1}{2}, -1 \rangle$

$\|v_{35}\|^2 = (\frac{3}{2})^2 + (\frac{-1}{2})^2 + (-1)^2 = \frac{9}{4} + \frac{1}{4} + 1 = \frac{14}{4}$

$\frac{3}{2}\left(x - \frac{\frac{3}{2}}{\frac{14}{4}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{14}{4}}\right) - \left(z + \frac{1}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow 3x \leq 2 + y + 2z.$

36. $v_{36} = \langle \frac{3}{2}, \frac{-1}{2}, \frac{-1}{2} \rangle$

$\|v_{36}\|^2 = (\frac{3}{2})^2 + (\frac{-1}{2})^2 + (\frac{-1}{2})^2 = \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}$

$\frac{3}{2}\left(x - \frac{\frac{3}{2}}{\frac{11}{4}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{11}{4}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{11}{4}}\right) \leq 0 \Leftrightarrow 3x \leq 2 + y + z.$

37. $v_{37} = \langle 2, -1, -1 \rangle$

$\|v_{37}\|^2 = 2^2 + (-1)^2 + (-1)^2 = 4 + 1 + 1 = 6$

$2\left(x - \frac{2}{6}\right) - \left(y + \frac{1}{6}\right) - \left(z + \frac{1}{6}\right) \leq 0 \Leftrightarrow 2x \leq 1 + y + z.$

38. $v_{38} = \langle 2, -1, \frac{-1}{2} \rangle$

$\|v_{38}\|^2 = 2^2 + (-1)^2 + (\frac{-1}{2})^2 = 4 + 1 + \frac{1}{4} = \frac{21}{4}$

$2\left(x - \frac{2}{\frac{21}{4}}\right) - \left(y + \frac{1}{\frac{21}{4}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow 4x \leq 2 + 2y + z.$

39. $v_{39} = \langle \frac{3}{2}, -1, -1 \rangle$

$\|v_{39}\|^2 = (\frac{3}{2})^2 + (-1)^2 + (-1)^2 = \frac{9}{4} + 1 + 1 = \frac{17}{4}$

$\frac{3}{2}\left(x - \frac{\frac{3}{2}}{\frac{17}{4}}\right) - \left(y + \frac{1}{\frac{17}{4}}\right) - \left(z + \frac{1}{\frac{17}{4}}\right) \leq 0 \Leftrightarrow 3x \leq 2(1 + y + z).$

40. $v_{40} = \langle \frac{3}{2}, -1, \frac{-1}{2} \rangle$

$\|v_{40}\|^2 = (\frac{3}{2})^2 + (-1)^2 + (\frac{-1}{2})^2 = \frac{9}{4} + 1 + \frac{1}{4} = \frac{14}{4}$

$\frac{3}{2}\left(x - \frac{\frac{3}{2}}{\frac{14}{4}}\right) - \left(y + \frac{1}{\frac{14}{4}}\right) - \frac{1}{2}\left(z + \frac{\frac{1}{2}}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow 3x \leq 2 + 2y + z.$

41. $v_{41} = \langle 1, \frac{-3}{2}, \frac{1}{2} \rangle$

$\|v_{41}\|^2 = 1^2 + (\frac{-3}{2})^2 + (\frac{1}{2})^2 = 1 + \frac{9}{4} + \frac{1}{4} = \frac{14}{4}$

$\left(x - \frac{1}{\frac{14}{4}}\right) - \frac{3}{2}\left(y + \frac{\frac{3}{2}}{\frac{14}{4}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow 2x + z \leq 2 + 3y.$

42. $v_{42} = \langle 1, \frac{-3}{2}, 1 \rangle$

$\|v_{42}\|^2 = 1^2 + (\frac{-3}{2})^2 + 1^2 = 1 + \frac{9}{4} + 1 = \frac{17}{4}$

$\left(x - \frac{1}{\frac{17}{4}}\right) - \frac{3}{2}\left(y + \frac{\frac{3}{2}}{\frac{17}{4}}\right) + \left(z - \frac{1}{\frac{17}{4}}\right) \leq 0 \Leftrightarrow x + z \leq 1 + \frac{3y}{2}.$

43. $v_{43} = \langle \frac{1}{2}, \frac{-3}{2}, \frac{1}{2} \rangle$

$\|v_{43}\|^2 = (\frac{1}{2})^2 + (\frac{-3}{2})^2 + (\frac{1}{2})^2 = \frac{1}{4} + \frac{9}{4} + \frac{1}{4} = \frac{11}{4}$

$\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{11}{4}}\right) - \frac{3}{2}\left(y + \frac{\frac{3}{2}}{\frac{11}{4}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{11}{4}}\right) \leq 0 \Leftrightarrow x + z \leq 2 + 3y.$

44. $v_{44} = \langle \frac{1}{2}, \frac{-3}{2}, 1 \rangle$

$\|v_{44}\|^2 = (\frac{1}{2})^2 + (\frac{-3}{2})^2 + 1^2 = \frac{1}{4} + \frac{9}{4} + 1 = \frac{14}{4}$

$\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{14}{4}}\right) - \frac{3}{2}\left(y + \frac{\frac{3}{2}}{\frac{14}{4}}\right) + \left(z - \frac{1}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow x + 2z \leq 2 + 3y.$

45. $v_{45} = \langle \frac{1}{2}, -2, \frac{1}{2} \rangle$

$\|v_{45}\|^2 = (\frac{1}{2})^2 + (-2)^2 + (\frac{1}{2})^2 = \frac{1}{4} + 4 + \frac{1}{4} = \frac{18}{4}$

$\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{18}{4}}\right) - 2\left(y + \frac{2}{\frac{18}{4}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{18}{4}}\right) \leq 0 \Leftrightarrow x + z \leq 2 + 4y.$

46. $v_{46} = \langle \frac{1}{2}, -2, 1 \rangle$

$\|v_{46}\|^2 = (\frac{1}{2})^2 + (-2)^2 + 1^2 = \frac{1}{4} + 4 + 1 = \frac{21}{4}$

$\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{21}{4}}\right) - 2\left(y + \frac{2}{\frac{21}{4}}\right) + \left(z - \frac{1}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow x + 2z \leq 2 + 4y.$

47. $v_{47} = \langle 1, -2, \frac{1}{2} \rangle$

$\|v_{47}\|^2 = 1^2 + (-2)^2 + (\frac{1}{2})^2 = 1 + 4 + \frac{1}{4} = \frac{21}{4}$

$\left(x - \frac{1}{\frac{21}{4}}\right) - 2\left(y + \frac{2}{\frac{21}{4}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow 2x + z \leq 2 + 4y.$

48. $v_{48} = \langle 1, -2, 1 \rangle$

$\|v_{48}\|^2 = 1^2 + (-2)^2 + 1^2 = 1 + 4 + 1 = 6$

$\left(x - \frac{1}{6}\right) - 2\left(y + \frac{2}{6}\right) + \left(z - \frac{1}{6}\right) \leq 0 \Leftrightarrow x + z \leq 1 + 2y.$

49. $v_{49} = \langle \frac{1}{2}, \frac{1}{2}, 0 \rangle$

$\|v_{49}\|^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{1}{2}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{1}{2}}\right) \leq 0 \Leftrightarrow x + y \leq 2.$

50. $v_{50} = \langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle$

$\|v_{50}\|^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

$\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{3}{4}}\right) + \frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{3}{4}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{3}{4}}\right) \leq 0 \Leftrightarrow x + y + z \leq 2.$

51. $v_{51} = \langle 0, \frac{1}{2}, 0 \rangle$

$\|v_{51}\|^2 = (\frac{1}{2})^2 = \frac{1}{4}$

$\frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{1}{4}}\right) \leq 0 \Leftrightarrow y \leq 2.$

52. $v_{52} = \langle 0, \frac{1}{2}, \frac{1}{2} \rangle$

$\|v_{52}\|^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\frac{1}{2}\left(y - \frac{\frac{1}{2}}{\frac{1}{2}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{1}{2}}\right) \leq 0 \Leftrightarrow y + z \leq 2.$

53. $v_{53} = \langle \frac{1}{2}, 0, 0 \rangle$

$\|v_{53}\|^2 = (\frac{1}{2})^2 = \frac{1}{4}$

$\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{1}{4}}\right) \leq 0 \Leftrightarrow x \leq 2.$

54. $v_{54} = \langle \frac{1}{2}, 0, \frac{1}{2} \rangle$

$\|v_{54}\|^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\frac{1}{2}\left(x - \frac{\frac{1}{2}}{\frac{1}{2}}\right) + \frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{1}{2}}\right) \leq 0 \Leftrightarrow x + z \leq 2.$

55. $v_{55} = \langle 0, 0, \frac{1}{2} \rangle$

$\|v_{55}\|^2 = (\frac{1}{2})^2 = \frac{1}{4}$

$\frac{1}{2}\left(z - \frac{\frac{1}{2}}{\frac{1}{4}}\right) \leq 0 \Leftrightarrow z \leq 2.$

56. $v_{56} = \langle \frac{-1}{2}, \frac{-1}{2}, \frac{3}{2} \rangle$

$\|v_{56}\|^2 = (\frac{-1}{2})^2 + (\frac{-1}{2})^2 + (\frac{3}{2})^2 = \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \frac{11}{4}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{11}{4}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{11}{4}}\right) + \frac{3}{2}\left(z - \frac{\frac{3}{2}}{\frac{11}{4}}\right) \leq 0 \Leftrightarrow 3z \leq 2 + x + y.$

57. $v_{57} = \langle \frac{-1}{2}, \frac{-1}{2}, 2 \rangle$

$\|v_{57}\|^2 = (\frac{-1}{2})^2 + (\frac{-1}{2})^2 + 2^2 = \frac{1}{4} + \frac{1}{4} + 4 = \frac{18}{4}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{18}{4}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{18}{4}}\right) + 2\left(z - \frac{2}{\frac{18}{4}}\right) \leq 0 \Leftrightarrow 4z \leq 2 + x + y.$

58. $v_{58} = \langle -1, \frac{-1}{2}, \frac{3}{2} \rangle$

$\|v_{58}\|^2 = (-1)^2 + (\frac{-1}{2})^2 + (\frac{3}{2})^2 = 1 + \frac{1}{4} + \frac{9}{4} = \frac{14}{4}$

$-\left(x + \frac{1}{\frac{14}{4}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{14}{4}}\right) + \frac{3}{2}\left(z - \frac{\frac{3}{2}}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow 3z \leq 2 + 2x + y.$

59. $v_{59} = \langle -1, \frac{-1}{2}, 2 \rangle$

$\|v_{59}\|^2 = (-1)^2 + (\frac{-1}{2})^2 + 2^2 = 1 + \frac{1}{4} + 4 = \frac{21}{4}$

$-\left(x + \frac{1}{\frac{21}{4}}\right) - \frac{1}{2}\left(y + \frac{\frac{1}{2}}{\frac{21}{4}}\right) + 2\left(z - \frac{2}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow 4z \leq 2 + 2x + y.$

60. $v_{60} = \langle \frac{-1}{2}, -1, \frac{3}{2} \rangle$

$\|v_{60}\|^2 = (\frac{-1}{2})^2 + (-1)^2 + (\frac{3}{2})^2 = \frac{1}{4} + 1 + \frac{9}{4} = \frac{14}{4}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{14}{4}}\right) - \left(y + \frac{1}{\frac{14}{4}}\right) + \frac{3}{2}\left(z - \frac{\frac{3}{2}}{\frac{14}{4}}\right) \leq 0 \Leftrightarrow 3z \leq 2 + x + 2y.$

61. $v_{61} = \langle \frac{-1}{2}, -1, 2 \rangle$

$\|v_{61}\|^2 = (\frac{-1}{2})^2 + (-1)^2 + 2^2 = \frac{1}{4} + 1 + 4 = \frac{21}{4}$

$-\frac{1}{2}\left(x + \frac{\frac{1}{2}}{\frac{21}{4}}\right) - \left(y + \frac{1}{\frac{21}{4}}\right) + 2\left(z - \frac{2}{\frac{21}{4}}\right) \leq 0 \Leftrightarrow 4z \leq 2 + x + 2y.$

62. $v_{62} = \langle -1, -1, \frac{3}{2} \rangle$

$\|v_{62}\|^2 = (-1)^2 + (-1)^2 + (\frac{3}{2})^2 = 1 + 1 + \frac{9}{4} = \frac{17}{4}$

$-\left(x + \frac{1}{\frac{17}{4}}\right) - \left(y + \frac{1}{\frac{17}{4}}\right) + \frac{3}{2}\left(z - \frac{\frac{3}{2}}{\frac{17}{4}}\right) \leq 0 \Leftrightarrow \frac{3z}{2} \leq 1 + x + y.$

63. $v_{63} = \langle -1, -1, 2 \rangle$

$\|v_{63}\|^2 = (-1)^2 + (-1)^2 + 2^2 = 1 + 1 + 4 = 6$

$-\left(x + \frac{1}{6}\right) - \left(y + \frac{1}{6}\right) + 2\left(z - \frac{2}{6}\right) \leq 0 \Leftrightarrow 2z \leq 1 + x + y.$

We use the following input in *Mathematica* to generate the plot of $(\Pi P)^\star$ for $-1 \leq x, y, z \leq 1$:

RegionPlot3D[$x+y \leq 1+2z$ && $x+y \leq 1+\frac{3}{2}z$ && $x+2y \leq 2+4z$ && $x+2y \leq 2+3z$ && $2x+y \leq 2+4z$ && $2x+y \leq 2+3z$ && $x+y \leq 2+4z$ && $x+y \leq 2+3z$ && $2+z \geq 0$ && $2+x+z \geq 0$ && $2+x \geq 0$ && $2+x+y+z \geq 0$ && $2+x+y \geq 0$ && $2+y+z \geq 0$ && $2+y \geq 0$ && $4y \leq 2+x+2z$ && $4y \leq 2+x+z$ && $2y \leq 1+x+z$ && $4y \leq 2+2x+z$ && $3y \leq 2+x+2z$ && $3y \leq 2+x+z$ && $\frac{3}{2}y \leq 1+x+z$ && $3y \leq 2+2x+z$ && $2y+z \leq 2+3x$ && $2(-1+y+z) \leq 3x$ && $2y+z \leq 2+4x$ && $y+z \leq 1+2x$ && $y+z \leq 2+3x$ && $y+2z \leq 2+3x$ && $y+z \leq 2+4x$ && $y+2z \leq 2+4x$ && $4x \leq 2+y+2z$ && $4x \leq 2+y+z$ && $3x \leq 2+y+2z$ && $3x \leq 2+y+z$ && $2x \leq 1+y+z$ && $4x \leq 2+2y+z$ && $3x \leq 2(1+y+z)$ && $3x \leq 2+2y+z$ && $2x+z \leq 2+3y$ && $x+z \leq 1+\frac{3}{2}y$ && $x+z \leq 2+3y$ && $x+2z \leq 2+3y$ && $x+z \leq 2+4y$ && $x+2z \leq 2+4y$ && $2x+z \leq 2+4y$ && $x+z \leq 1+2y$ && $x+y \leq 2$ && $x+y+z \leq 2$ && $y \leq 2$ && $y+z \leq 2$ && $x \leq 2$ && $x+z \leq 2$ && $z \leq 2$ && $3z \leq 2+x+y$ && $4z \leq 2+x+y$ && $3z \leq 2+2x+y$ && $4z \leq 2+2x+y$ && $3z \leq 2+x+2y$ && $4z \leq 2+x+2y$ && $\frac{3}{2}z \leq 1+x+y$ &&

$2z \leq 1 + x + y, \{x, -1, 1\}, \{y, -1, 1\}, \{z, -1, 1\}]$

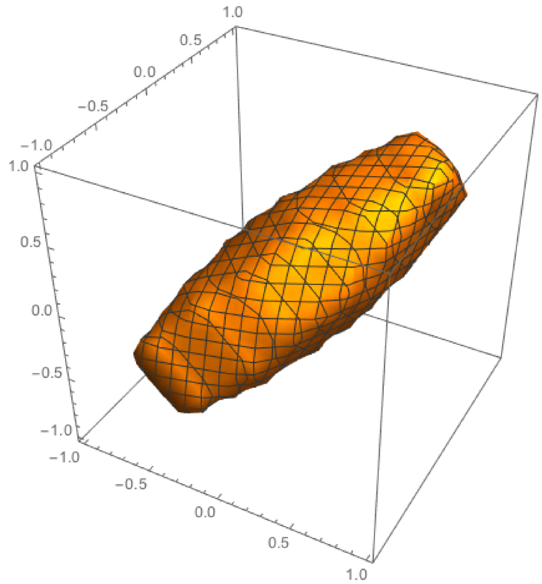


Figure 4.3: Plot of $(\Pi P)^\star$ for $P = T \cup [2, 2, 2] \subset \mathbb{R}^3$

We use the following input in *Mathematica* to calculate the volume of $(\Pi P)^\star$ for $-1 \leq x, y, z \leq 1$:

$\mathrm{Vol}_3(\Pi P)^\star = \mathrm{NIntegrate}[\mathrm{Boole}[x + y \leq 1 + 2z \;\&\&\; x + y \leq 1 + \frac{3}{2}z \;\&\&\; x + 2y \leq 2 + 4z \;\&\&\; x + 2y \leq 2 + 3z \;\&\&\; 2x + y \leq 2 + 4z \;\&\&\; 2x + y \leq 2 + 3z \;\&\&\; x + y \leq 2 + 4z \;\&\&\; x + y \leq 2 + 3z \;\&\&\; 2 + z \geq 0 \;\&\&\; 2 + x + z \geq 0 \;\&\&\; 2 + x \geq 0 \;\&\&\; 2 + x + y + z \geq 0 \;\&\&\; 2 + x + y \geq 0 \;\&\&\; 2 + y + z \geq 0 \;\&\&\; 2 + y \geq 0 \;\&\&\; 4y \leq 2 + x + 2z \;\&\&\; 4y \leq 2 + x + z \;\&\&\; 2y \leq 1 + x + z \;\&\&\; 4y \leq 2 + 2x + z \;\&\&\; 3y \leq 2 + x + 2z \;\&\&\; 3y \leq 2 + x + z \;\&\&\; \frac{3}{2}y \leq 1 + x + z \;\&\&\; 3y \leq 2 + 2x + z \;\&\&\; 2y + z \leq 2 + 3x \;\&\&\; 2(-1 + y + z) \leq 3x \;\&\&\; 2y + z \leq 2 + 4x \;\&\&\; y + z \leq 1 + 2x \;\&\&\; y + z \leq 2 + 3x \;\&\&\; y + 2z \leq 2 + 3x \;\&\&\; y + z \leq 2 + 4x \;\&\&\; y + 2z \leq 2 + 4x \;\&\&\; 4x \leq 2 + y + 2z \;\&\&\; 4x \leq 2 + y + z \;\&\&\; 3x \leq 2 + y + 2z \;\&\&\; 3x \leq 2 + y + z \;\&\&\; 2x \leq 1 + y + z \;\&\&\; 4x \leq 2 + 2y + z \;\&\&\; 3x \leq 2(1 + y + z) \;\&\&\; 3x \leq 2 + 2y + z \;\&\&\; 2x + z \leq 2 + 3y \;\&\&\; x + z \leq 1 + \frac{3}{2}y \;\&\&\; x + z \leq 2 + 3y \;\&\&\; x + 2z \leq 2 + 3y \;\&\&\; x + z \leq 2 + 4y \;\&\&\; x + 2z \leq 2 + 4y \;\&\&\; 2x + z \leq 2 + 4y \;\&\&\; x + z \leq 1 + 2y \;\&\&\; x + y \leq 2 \;\&\&\; x + y + z \leq 2 \;\&\&\; y \leq 2 \;\&\&$

$y + z \leq 2$ && $x \leq 2$ && $x + z \leq 2$ && $z \leq 2$ && $3z \leq 2 + x + y$ && $4z \leq 2 + x + y$ && $3z \leq 2 + 2x + y$ && $4z \leq 2 + 2x + y$ && $3z \leq 2 + x + 2y$ && $4z \leq 2 + x + 2y$ && $\frac{3}{2}z \leq 1 + x + y$ && $2z \leq 1 + x + y], \{x, -1, 1\}, \{y, -1, 1\}, \{z, -1, 1\}]$
$= 0.863472.$

Validation of Petty's projection inequality for $P = T \cup [2, 2, 2] \subset \mathbb{R}^3$:

It is known that for any convex body P in $\mathbb{R}^3$, the following inequality holds:

$$V(P)^{n-1}V(\Pi P)^* \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n, \tag{4.1}$$

with equality if and only if P is an ellipsoid [9].

In our example, we have a convex polytope P whose volume $V(P) = 1 \Rightarrow V(P)^2 = 1$ and $V(\Pi P)^* = 0.863472$.

Therefore, since

$\omega_n = \omega_3 =$ volume of unit ball in $\mathbb{R}^3 = \frac{4}{3}\pi$;

$\omega_{n-1} = \omega_2 =$ volume (area) of unit ball in $\mathbb{R}^3 = \pi$;

$\therefore 1 \cdot 0.863472 \leq \left(\frac{\frac{4}{3}\pi}{\pi}\right)^3 = \frac{64}{27} \approx 2.3704.$

Validation of the reverse of Petty's projection inequality for $P = T \cup [2, 2, 2] \subset \mathbb{R}^3$:

It is known that for any convex body P in $\mathbb{R}^3$, the following inequality holds

$$V(P)^{n-1}V(\Pi P)^* \geq \frac{(2n)!}{n^n(n!)^2}, \tag{4.2}$$

with equality if and only if P is a simplex [14].

In $\mathbb{R}^3$, we must thus have the upper bound: $V(P)^2V(\Pi \mathrm{P})^* \geq \frac{6!}{3^3(3!)^2} = \frac{720}{27(6)^2} = \frac{720}{972} = \frac{20}{27} \approx 0.740741.$

Inspecting the inequality for our specific example, we have $V(P) = 1 \Rightarrow V(P)^2 = 1$;

and $V(\Pi P)^* = 0.863472$.

$\Rightarrow V(P)^2 V(\Pi \mathrm{P})^* = 0.863472 \geq \frac{20}{27}$.

While the latest validations do not produce a new result, they illustrate the possibilities available with *Mathematica* to work with polar and projection bodies. We think that the calculations in this direction may also lead to new ideas on attacking the conjectured inequality.

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