ACIMs for Non-Autonomous Discrete Time Dynamical Systems;
A Generalization of Straube's Theorem

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A Thesis
In the Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
For the Degree of
Master of Science (Mathematics)at
Concorida University
Montreal, Quebec, Canada

March 2018

Concordia University

School of Graduate Studies

This is to certify that the thesis prepared

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Abstract

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ACIMs for Non-Autonomous Discrete Time Dynamical Systems;
A Generalization of Straube's Theorem

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This Master's thesis provides sufficient conditions under which a Non-Autonomous Dynamical System has an absolutely continuous invariant measure. The main results of this work are an extension of the Krylov-Bogoliubov theorem and Straube's theorem, both of which provide existence conditions for invariant measures of single transformation dynamical systems, to a uniformly convergent sequence of transformations of a compact metric space, which we define to be a non-autonomous dynamical system.

Acknowledgements

This process was both challenging and rewarding, and I wouldn't have made it through without the support of a great many kind people in my life.

First and foremost, I would like to thank my supervisors, Dr. Pawel Gora and Dr. Abraham Boyarsky, for their continuing support and guidance on this project. Thank you both for allowing me the privilege to be part of your research group. I learned a lot and had fun doing it.

I owe many thanks to Marie-France Leclere for her guidance and advice during my time at Concordia. I wouldn't have survived a month in Montreal without you, Marie-France.

I've been fortunate enough to make plenty of new friends during my graduate studies, and I'd like to thank all my fellow graduate students for helping to keep things from getting too serious. Specifically, I'd like to call out Aleksander Danielski, Patrick Meisner and Lucas Berry for their friendship and for always playing in suit, as well as Jake Chinis for his strong friendship and support, especially in the late days of this process.

Much of the encouragement I needed to complete this project I found in my family back home in Sydney, Nova Scotia. I'd like to thank my mother, Shelley, my sister, Sarah, and of course my father Dale, who has always pushed me to succeed.

Finally, I'd like to thank Krystel Arpin-Gemme for her endless support and kindness. You'll never know just how much you helped me complete this, Krystel. Thank you and I love you.

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Dedicated to my father, Dr. Dale Keefe, the greatest teacher I've ever known…

Chapter 1

Background

1.1 Review of Measure Theory

In this section, we provide a brief review of some key concepts in measure theory. For additional details on this subject, we refer the reader to Rudin's introductory text [9]. These ideas will form the foundation for the work to be presented in this text.

Definition 1.1.1. *Let Σ be the collection of subsets of a compact metric space X with the following properties:*

- $\emptyset \in \Sigma$.
- $E \in \Sigma \implies (X \setminus E) \in \Sigma$.
- $\{E_n\}_{n=1}^{\infty} \subset \Sigma \implies \cup_{n=1}^{\infty} E_n \in \Sigma$.

*Then we say Σ is a **σ-algebra**.*

Definition 1.1.2. *Let X be a compact metric space. Then the **Borel σ-algebra**, $\mathscr{B}$, of X is the collection of all sets containing:*

- *all open subsets of X.*
- *$X \setminus E$, if E is an open set.*
- *$E = \cup_{n=1}^{\infty} E_n$, where $\{E_n\} \subset X$ is a sequence of open sets.*
- *$E = \cap_{n=1}^{\infty} E_n$, where $\{E_n\} \subset X$ is a sequence of open sets.*

We now define a measure, a fundamental tool used in the study of dynamical systems. A measure can be thought of as a function assigning a notion of "size" to the sets in a σ-algebra.

Definition 1.1.3. *Let μ be a set-function defined on $\mathscr{B}$ such that,*

- $|\mu(E)| < \infty$, *for all* $E \in \mathscr{B}$
- $\mu(\emptyset) = 0$

and suppose that for every sequence of pairwise disjoint sets, $\{E_n\}_{n=1}^{\infty} \subset \mathscr{B}$*,*

$$\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n),$$

then we say μ *is a measure on* $\mathscr{B}$*.*

We call the last criterion of the definition of a measure *additivity*, in that we can simply add the sizes of disjoint sets together to get the size of their union. Specifically in this definition, we say the measure is countably additive, as its additive property holds for unions of countably many sets. However, as we further discuss below, we can further sub-divide the notion of a measure based on its additivity.

Definition 1.1.4. *Let* μ *be a set-function defined on* $\mathscr{B}$ *such that,*

- $|\mu(E)| < \infty$, *for all* $E \in \mathscr{B}$.
- $\mu(\emptyset) = 0$.
- $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$ *for all disjoint* $E_1, E_2 \in \mathscr{B}$.

then we say μ *is a* ***finitely additive measure****.*

If μ follows the original definition of a measure (it has the additive property for countably many disjoint sets), we say it is a **countably additive measure**. Thus we may further expand the notion of a measure to require it be at least finitely additive.

Finally, we provide a definition highlighting a specific desired property of measures, namely purely finitely additive.

Definition 1.1.5. *Let* μ *be a measure such that* $\mu \geq 0$*. Then we say* μ *is* ***purely finitely additive*** *if and only if every countably additive measure* ν *for which* $0 \leq \nu \leq \mu$ *is identically zero.*

For a given measure μ, we call the collection $(X, \mathscr{B}, \mu)$ a measure space. We call the collection $(X, \mathscr{B}, \mu)$ a **compact measure space** when X is compact. This definition is important as the spaces we consider in this text are exclusively compact. When $\mu(X) = 1$, we say that the measure μ has been *normalized* or that μ is a **probability measure**. The connection being that all probabilities on a space of events must sum to 1, and so $\mathscr{B}$ can be thought of a set of events, with μ providing the probability of each event occurring.

We now wish to define a space of measures, and establish a few results for spaces of measures. First, however, we must start with a foundation of normed linear spaces.

Definition 1.1.6. *Let $\mathscr{L}$ be a linear space. Then we define a* ***norm*** *on $\mathscr{L}$ to be a function, $\|\cdot\|$ satisfiying,*

- $\|f\| = 0 \iff f \equiv 0$.
- $\|\alpha f\| = |\alpha|\|f\|$.
- $\|f + g\| \leq \|f\| + \|g\|$.

for all $f, g \in \mathscr{L}$ and any scalar $\alpha \in \mathbb{R}$. We say the combination, $(\mathscr{L}, \|\cdot\|)$ is a ***normed linear space****.*

Definition 1.1.7. *Let X be a normed linear space. Then, we call the space of all linear functionals on X, the* ***adjoint or dual space*** *of X and denote it as X^*.*

A normed linear space and its adjoint are connected in the sense of convergence. The following definition establishes how this connection works.

Definition 1.1.8. *We say a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges weakly to $x \in X$ if and only if for any $F \in X^*$, we have that,*

$$\lim_{n\to\infty} F(x_n) = F(x).$$

In the other direction, we say that a sequence of functionals, $\{F_n\}_{n=1}^{\infty}$ converges in the weak topology to a functional $F \in X^*$ if and only if for all $x \in X$,*

$$\lim_{n\to\infty} F_n(x) = F(x).$$

We now formally define a space of measures, and equip it with a norm, hence creating a normed linear space of great interest for this text.

Definition 1.1.9. *Let $\mathscr{M}$ denote the space of all measures on $(X, \mathscr{B})$. Then we define the norm of $\mu \in \mathscr{M}$, as*

$$\|\mu\| = \sup_{\mathscr{P}} \sup_{P=\cup_{n=1}^{N} P_n} \{|\mu(P_1)| + ... + |\mu(P_N)|\},$$

where $\mathscr{P}$ is the set of all finite partitions of X and $P = \cup_{n=1}^{N} P_n$ is a partition of X. This norm is known as the "total variation norm".

Interestingly, as demonstrated in [4] the space of measures on X can be considered as the space of linear functionals on $C(X)$, which is the space of all continuous functions on X.

Theorem 1.1.1. *Let X be a compact metric space. Let $\mathscr{M}$ denote the space of all measures on $(X, \mathscr{B})$. Then the adjoint space of $C(X)$, $C^*(X)$, is equivalent to $\mathscr{M}$.*

Next, we present the Banach-Alaoglu theorem, a staple of functional analysis, which provides a key property of compact spaces, and which when applied to the space of measures, gives us a powerful tool that we make prolific use of throughout this text.

Theorem 1.1.2. *(**Banach-Alaoglu**) Let X be a compact normed space. Then the closed unit ball of X^* is compact with respect to the weak* topology.*

Corollary 1.1.2.1. *The closed unit ball of $\mathscr{M}$ is compact in the weak* topology.*

Finally, we conclude the section by providing the definition of an absolutely continuous measure, a central concept of this text.

Definition 1.1.10. *Let $(X, \mathscr{B}, \mu)$ be a measure space. Let ν be a measure defined on all sets in $\mathscr{B}$. Then, we say ν is **absolutely continuous** with respect to μ if and only if for every set $B \in \mathscr{B}$,*

$$\mu(B) = 0 \implies \nu(B) = 0.$$

We write $\nu << \mu$.

Indeed, this is not the form of an absolutely continuous measure that we will typically be working with. This definition forms a rudimentary basis for a topic which we will expand upon in greater detail in Chapter 2. In particular, we note that the Radon-Nikodym theorem, Theorem 2.1.1, which builds further upon the idea of an absolutely continuous measure, will play a key role in the development of the Frobenius-Perron operator, an essential tool in dynamical systems.

1.2 Measure-Preserving Transformations

In the study of dynamical systems, we are generally working with functions which, under iteration, "transform" the space X. We aptly call these functions transformations. To expand upon this idea, we now define what it means for a transformation to be measurable and for a measure to be invariant under iterations of a transformation.

Definition 1.2.1. *Let $(X, \mathscr{B}, \mu)$ be a measure space and $\tau : X \to X$ be a transformation of X. Then τ is said to be a **measurable transformation** if and only if,*

$$B \in \mathscr{B} \implies \tau^{-1}(B) \in \mathscr{B}.$$

Definition 1.2.2. *Let $(X, \mathscr{B}, \mu)$ be a measure space and $\tau : X \to X$ be a measurable transformation. Then μ is said to be τ**-invariant** if and only if for every $B \in \mathscr{B}$,*

$$\mu(\tau^{-1}(B)) = \mu(B). \tag{1.1}$$

Although, in reality, it is very difficult or sometimes impossible to verify equation (1.1) for every set in the collection $\mathscr{B}$. This is why in the following definition we establish the notion of a π-system. Having established the π-system, we will be able to prove invariance of τ for a collection of simpler sets and then through a theorem which appears in [1], conclude that τ is invariant for all sets in $\mathscr{B}$.

Definition 1.2.3. *We say that $\mathscr{P}$ is a π**-system** generating the σ-algebra $\mathscr{B}$ if and only if, for every E_1 and E_2 in $\mathscr{P}$, $E_1 \cap E_2$ is in $\mathscr{P}$ and $\mathscr{B}$ is the smallest σ algebra containing all sets of $\mathscr{P}$.*

Theorem 1.2.1. *Let $(X, \mathscr{B}, \mu)$ be a compact measure space and τ be a measurable transformation. Let $\mathscr{P}$ be a π-system generating $\mathscr{B}$. If*

$$\mu(\tau^{-1}(P)) = \mu(P),$$

for every $P \in \mathscr{P}$, then τ preserves μ on $\mathscr{B}$.

We now provide an alternate condition for τ to admit an invariant measure. It is one we will make extensive use of in the main two results of this chapter as well as the primary results of this text.

Theorem 1.2.2. *Let $(X, \mathscr{B}, \mu)$ be a compact measure space. Let τ be a measurable transformation of X. Let $g \in C(X)$ be arbitrary. If,*

$$\int_X g(x) d\mu = \int_X g(\tau(x)) d\mu,$$

then μ is a τ-invariant measure.

We now define a *dynamical system*, one of the central topics in this text.

Definition 1.2.4. *Let $(X, \mathscr{B}, \mu)$ be a measure space. Let τ be a transformation of X, with μ being a τ-invariant measure. Then we call the collection, $(X, \mathscr{B}, \mu, \tau)$ a **dynamical system**.*

Next, we note that a transformation $\tau : X \to X$ induces a transformation of the space of measures on X, $\mathscr{M}$. This transformation, which we denote as $\tau_*(\mu)$, is defined as $(\tau_*\mu)(A) = \mu(\tau^{-1}(A))$. In this vein, we note that in the context of this text, it is critical that transformations have a property known as non-singuarity.

Definition 1.2.5. *We say a transformation $\tau : X \to X$ is non-singular if and only if,*

$$\tau_*\mu << \mu.$$

In other words, if and only if for all $A \in \mathscr{B}$,

$$\mu(A) = 0 \implies \mu(\tau^{-1}(A)) = 0.$$

Finally, we conclude this section by defining an *absolutely continuous invariant measure* or ACIM. Since many invariant measures are uninteresting (point measures, zero measure, etc.) we often look for ACIMs of dynamical system.

Definition 1.2.6. *Let $(X, \mathscr{B}, \mu, \tau)$ be a dynamical system. Let ν be a measure defined on all sets in $\mathscr{B}$ which is invariant of τ and absolutely continuous with respect to μ. Then we say ν is an **absolutely continuous invariant measure (ACIM)** for τ.*

1.3 The Krylov-Bogoliubov Theorem

We now present the Krylov-Bogoliubov theorem. This result establishes that every continuous transformation on a compact metric space is guaranteed to have an invariant measure.

Theorem 1.3.1. *:(**Krylov-Bogoliubov**) Let X be a compact metric space and $\tau : X \to X$ a continuous map. Then there exists a τ-invariant probability measure, μ on X.*

Proof. We follow the original proof found in [2]. Let ν be a probability measure on X. Consider the sequence,

$$\mu_n = \frac{1}{n}(\nu + \tau_*\nu + \tau_*^2\nu + ... + \tau_*^{n-1}\nu).$$

Since ν is a probability measure, μ_n is also a probability measure. By Theorem 1.1.2, the unit ball of $\mathscr{M}$ (probability measures) is compact. Hence, the sequence $\{\mu_n\}$ contains a convergent subsequence, $\mu_{n_k} \to \mu$.
We will prove that μ is our τ-invariant measure. By Theorem 1.2.1, It is enough to show that,

$$\int g d\mu = \int g(\tau) d\mu,$$

for any continuous g. Let $g \in C(X)$. Then,

$$\begin{aligned}\left| \int g d\mu - \int g(\tau) d\mu \right| &= \lim_{k\to\infty} \left| \int g d\mu_{n_k} - \int g(\tau) d\mu_{n_k} \right| \\ &= \lim_{k\to\infty} \left| \frac{1}{n_k}(\nu + \tau_*\nu + ... + \tau_*^{n_k-1}\nu)(g) - \frac{1}{n_k}(\tau_*\nu + ... + \tau_*^{n_k-1}\nu + \tau_*^{n_k}\nu)(g) \right| \\ &= \lim_{k\to\infty} \frac{1}{n_k} |\nu(g) - \tau_*^{n_k}\nu(g)|,\end{aligned}$$

where, $\nu(g) = \int g d\nu$. Hence,

$$\begin{aligned}\left| \int g d\mu - \int g(\tau) d\mu \right| &= \lim_{k\to\infty} \frac{1}{n_k} |\nu(g) - \tau_*^{n_k}\nu(g)| \\ &\leq \lim_{k\to\infty} \frac{1}{n_k} \cdot 2 \cdot \sup|g| = 0.\end{aligned}$$

Therefore μ is τ-invariant. ■

While the Krylov-Bogoliubov Theorem guarantees the existence of an invariant measure, it does not tell us how to find it or guarantee that it is "interesting", it could simply be a point measure (a measure with all density placed upon one point in X). This is a concept we expand upon in the following section.

1.4 Straube's Theorem

In this section, we will present Straube's theorem, as proven by Emil Straube in [10]. Straube's theorem is a result which provides equivalent conditions for a dynamical system to have an ACIM. It is a powerful result which guarantees that certain dynamical systems have interesting measures. First however, we establish a few smaller results from the theory of additive measures which we will make use of in the proof of Straube's theorem.

Lemma 1.4.1. *Let* $(X, \mathscr{B}, \mu)$ *be a compact measure space such that* μ *is purely finitely additive and* $\mu \geq 0$*. Let* ν *be a countably additive measure defined on* $(X, \mathscr{B})$ *such that* $\nu \geq 0$*. Then, there exists a decreasing sequence* $\{E_n\} \subset \mathscr{B}$ *such that* $\lim_{n\to\infty} \nu(E_n) = 0$ *and* $\mu(E_n) = \mu(X)$ *for all* n*. Conversely, if* μ *is a measure and the above conditions hold for all countably additive* ν*, then* μ *is purely finitely additive.*

Lemma 1.4.2. *Let* μ *be a measure such that* $\mu \geq 0$*. Then there exist measures* μ_p *and* μ_c *such that* $\mu_p \geq 0$*,* $\mu_c \geq 0$*,* μ_p *is purely finitely additive,* μ_c *is countably additive and* $\mu = \mu_p + \mu_c$*.*

Lemma 1.4.3. *Let* μ *be a measure. Then the decomposition of* μ *in lemma 1.4.2 is unique.*

The proofs of these Lemmas are rather simple when framed in the context provided in [11]. However, in the interest of remaining concentrated on the subject of this text, we omit them here.

Lemma 1.4.4. *Let* μ *be a measure decomposed as* $\mu = \mu_c + \mu_p$*. Then* μ_c *is the greatest of the measures* ν*, such that* $0 \leq \nu \leq \mu$*.*

Proof. Let ν be a countably additive measure such that $\nu \leq \mu$. Then,

$$\mu - \nu \geq 0.$$

Since μ and ν are both measures, the set function $(\mu - \nu)$ will also be a measure. Hence by lemma 1.4.2, we can decompose this new measure as,

$$(\mu - \nu) = (\mu - \nu)_c + (\mu - \nu)_p \geq 0.$$

Furthermore, Lemma 1.4.3 guarantees that this decomposition is unique. Now we write,

$$\begin{aligned}\mu &= \nu + (\mu - \nu) \\ &= \nu + (\mu - \nu)_c + (\mu - \nu)_p.\end{aligned}$$

Since ν is countably additive and our decomposition is unique, it follows that,

$$\mu_c = \nu + (\mu - \nu)_c,$$

and,

$$\mu_p = (\mu - \nu)_p.$$

However, as we saw above,

$$(\mu - \nu)_c \geq 0.$$

Thus,

$$\nu \leq \mu_c.$$

■

Lemma 1.4.5. *Let ν be a non-negative finitely additive measure and let,*

$$\int_X g d\nu = 0,$$

for any continuous function, g, on X. Then ν is purely finitely additive.

Proof. As in Definition 1.1.5, we have to show that any countably additive measure μ which satisfies

$$0 \leq \mu \leq \nu, \tag{1.2}$$

is identically the zero measure. Hence, let μ satisfy (1.2). Then for any continuous function g, we have,

$$0 \leq \mu(g) \leq \nu(g) = 0.$$

Therefore, $\mu(g) = 0$ for any continuous function g. Since μ is countably additive, we have $\mu = 0$. ■

Finally, we include one final result from [11], establishing $\mathscr{L}_\infty^*$ as a space of finitely additive measures.

Theorem 1.4.6. *Let $\mathscr{N} \subset \mathscr{B}$ such that $\mathscr{N}$ is closed under countable unions and having the property that*

$$N \in \mathscr{N}, A \in \mathscr{B} \text{ and } A \subset N \implies A \in \mathscr{N}$$

Let F be a real-valued functional defined on $\mathscr{L}_\infty$ such that,

- *$F(x+y) = F(x) + F(y)$, for all $x, y \in \mathscr{L}_\infty$.*
- *$F(\alpha x) = \alpha F(x)$, for all $\alpha \in \mathbb{R}$ and $x \in \mathscr{L}_\infty$.*
- *$|F(x)| \leq A\|x\|_\infty$, for some $A \geq 0$ and all $x \in \mathscr{L}_\infty$.*

Then there exists a finitely additive measure, σ on $\mathscr{M}$, such that,

$$F(f) = \int_X f(x) d\sigma(x), \tag{1.3}$$

for all $f \in \mathscr{L}_\infty$, and $\sigma(N) = 0$ for all $N \in \mathscr{N}$. Conversely, if σ is a finitely additive measure on $\mathscr{M}$ such that $\sigma(N) = 0$ for all $N \in \mathscr{N}$, then 1.3 defines a bounded linear functional on $\mathscr{L}_\infty$.

We now present Straube's theorem.

Theorem 1.4.7. ***(Straube)*** *Let $(X, \mathscr{B}, \mu)$ be a normalized measure space with τ a non-singular transformation. Then, there exists a τ invariant normalized measure which is absolutely continuous with respect to μ if and only if there exists $\delta > 0$ and $0 < \alpha < 1$ such that,*

$$\mu(E) < \delta \implies \sup_{k \in \mathbb{N}} \mu(\tau^{-k}(E)) < \alpha,$$

for all $E \in \mathscr{B}$ *and* $k \geq 0$.

Proof. We demonstrate the proof given in [10].

$\Longrightarrow$

Let ν be τ-invariant, normalized and absolutely continuous with respect to μ. Choose $\delta > 0$ such that for all $E \in \mathscr{B}$, $\mu(E) < \delta$ implies $\nu(E) < \frac{1}{4}$. Then, we claim that δ and $\alpha = 1 - \frac{\delta}{2}$ are as desired. We proceed by contradiction. Assume there exists $E \in \mathscr{B}$ and an index k such that $\mu(E) < \delta$ while $\mu(\tau^{-k}(E)) > 1 - \delta$. Therefore,

$$\mu(X \setminus \tau^{-k}(E)) < \delta.$$

Now by our choice of δ,

$$\mu(E) < \frac{1}{4},$$

while simultaneously,

$$\mu(X \setminus \tau^{-k}(E)) < \frac{1}{4}.$$

However, $X \setminus \tau^{-k}(E) = \tau^{-k}(X \setminus E)$, so the invariance of ν with respect to τ yields,

$$\nu(X \setminus E) = \nu(\tau^{-k}(X \setminus E)) = \nu(X \setminus \tau^{-k}(E)) < \frac{1}{4}.$$

$\Longleftarrow$

Define the measures

$$\mu_n(E) = \frac{1}{n} \sum_{k=0}^{n-1} \mu(\tau^{-k}(E)),$$

for any $E \in \mathscr{B}$. Then for all n,

- $\mu_n(X) = 1$.

- $\mu_n << \mu$ (τ is non-singular).

- $\mu_n \geq 0$.

Hence, $\{\mu_n\}$ is a sequence of probability measures, so it is in the unit ball of the L_∞^* weak* topology by Theorem 1.4.6, which is compact as a result of Theorem 1.1.2. So we let $\tilde{z}$ be the limit point of $\{\mu_n\}$. Define the set function,

$$z(E) = \tilde{z}(\chi_E).$$

Then, z is finitely additive, bounded ($0 \leq z(E) \leq z(X) = 1$), and it vanishes on sets of μ-measure zero. We now use z to construct a countably additive function. By Lemmas 1.4.2 and 1.4.3, we can uniquely decompose z into,

$$z = z_c + z_p,$$

where z_c is countably additive and z_p is purely finitely additive.

First we prove that z_c is τ-invariant. Consider the finitely additive measure

$$\nu = z - z \circ \tau = z_c - z_c \circ \tau + z_p - z_p \circ \tau.$$

In Theorem 1.3.1, we showed that for any continuous function g on X we have,

$$\mu_{n_k}(g) - \mu_{n_k}(\tau^{-1}(g)) \to 0,$$

as $k \to 0$. Meaning for any continuous g on X (which is bounded as X is compact), we have the following,

$$\nu(g) = z(g) - z \circ \tau(g) = 0.$$

Therefore by Lemma 1.4.5, ν is purely finitely additive. Or in other words,

$$z_c - z_c(\tau) = 0.$$

Thus, z_c is τ-invariant.

Now we claim that $z_c \neq 0$. Since if otherwise, by Lemma 1.4.1, there would exist a decreasing sequence $\{E_n\} \subset \mathscr{B}$ such that,

$$\lim_{n\to\infty} \mu(E_n) = 0,$$

and,

$$z(E_n) = z(X) = 1.$$

Since $\mu(E_n) \to 0$, there exists an n_0 such that $n > n_0 \implies \mu(E_n) < \delta$, for any $\delta > 0$. Now by our assumptions, there is an $\alpha < 1$ so that,

$$\sup_k \mu(\tau^{-k}(E_n)) < \alpha.$$

Thus, $\mu(\tau^{-k}(E_n)) < \alpha$ for all k. So,

$$z(E_n) < \alpha < 1.$$

This is a contradiction. So we have demonstrated that $z_c \neq 0$.

Thus far, we have,

$$z(E) = z(\tau^{-1}(E)),$$

and that z can be decomposed into two positively additive set functions so that,

$$z(E) = z_c(\tau^{-1}(E)) + z_p(\tau^{-1}(E)).$$

Furthermore, since z_c is a countably additive set function, Lemma 1.4.4 assures us that,

$$z_c(\tau^{-1}(E)) \leq z_c(E),$$

for all $E \in \mathscr{B}$. Where here we treat $z_c(\tau^{-1}(E))$ as a measure induced by the pre-image of E. Hence,

$$z_c(E) - z_c(\tau^{-1}(E)) \geq 0.$$

In other words, it is a positive measure. But since $\tau^{-1}(X) = X$, the total weight is precisely 0. Thus it is the zero measure. This shows that z_c is invariant.

Since z vaishes on sets of μ measure zero, and $z_c \leq z$, z_c will also vanish on sets of μ measure zero. So $z_c << \mu$. Therefore, normalizing we have,

$$\nu(E) = \frac{z_c(E)}{z_c(X)},$$

is a normalized measure invariant of τ and absolutely continuous with respect to μ. ∎

Chapter 2

The Frobenius-Perron Operator

The Frobenius-Perron operator is a linear operator which describes the probabilistic behaviour of successive iterations of a dynamical system. The operator was first studied by Kuzmin in [6] and [7]. Developing the Frobenius-Perron operator will provide us with an essential tool for uncovering the absolutely continuous invariant measures of dynamical systems.

2.1 The Radon-Nikodym Theorem

The existence and uniqueness of the Frobenius-Perron operator follows as a result of the Radon-Nikodym theorem, a theorem in measure theory which establishes the existence of a function called the "Radon-Nikodym derivative". The Radon-Nikodym derivative can be used as a density function, or one which generates the measures of sets in a σ-algebra.

Theorem 2.1.1. ***The Radon-Nikodym Theorem*** *Let $(X, \mathscr{B}, \mu)$ be a measure space with μ being a σ-finite measure and ν be a σ finite measure absolutely continuous with respect to μ. Then there exists a unique measurable function $f : X \to [0, \infty)$ such that for every $A \in \mathscr{B}$,*

$$\nu(A) = \int_A f d\mu.$$

We call the function f the "Radon-Nikodym derivative".

2.2 Motivation

We use the same motivation given in [3]. Suppose we are working with some random variable X, on an interval $[a, b]$. Suppose that the random variable has probability density function f. Then for all measurable sets $A \in I$,

$$\text{Prob}\{X \in A\} = \int_A f d\lambda,$$

where λ is the normalized Lebesque measure on I. Now suppose that we have a transformation $\tau : I \to I$, then $\tau(X)$ is also a random variable, and thus we may inquire about its probability density function. To obtain the density function for $\tau(X)$, we must be able to write

$$\text{Prob}\{\tau(X) \in A\} = \int_A \phi d\lambda,$$

for some function ϕ. If such a ϕ exists, it would depend on τ and f.

We begin our derivation by assuming X is a random variable having probability density function $f \in \mathscr{L}^1$, τ is non-singular and defining for any measurable set A,

$$\begin{aligned} \mu(A) &= \text{Prob}\{\tau(X) \in A\} \\ &= \text{Prob}\{X \in \tau^{-1}(A)\} \\ &= \int_{\tau^{-1}(A)} f d\lambda. \end{aligned}$$

Since τ is non-singular, $\lambda(A) = 0 \implies \lambda(\tau^{-1}(A) = 0$.Thus,

$$\begin{aligned} \mu(A) &= \int_{\tau^{-1}(A)} f d\lambda \\ &= 0. \end{aligned}$$

Therefore, μ is absolutely continuous with respect to λ. Now it follows by Theorem 2.1.1 that there exists a $\phi \in \mathscr{L}^1$ such that,

$$\mu(A) = \int_A \phi d\lambda$$

for any measurable set A. Furthermore, as a result of Theorem 2.1.1, ϕ is unique up to sets of zero measure. Thus we set,

$$P_\tau f = \phi$$

and call this function the Frobenius-Perron operator associated with the transformation τ. As our motivating derivation implies, the Frobenius-Perron operator transforms the probability density function of a random variable, X, which has been transformed by τ into the density function for $\tau(X)$. As such, it can be used as a tool in our analysis of dynamical systems to find absolutely continuous invariant measures.

2.3 The Frobenius-Perron Operator

We now formally define the Frobenius-Perron operator.

Definition 2.3.1. *Let $(X, \mathscr{B}, \mu)$ be a measure space and let $\tau : X \to X$ be a non-singular transformation. Then the Frobenius-Perron operator, $P_\tau : \mathscr{L}^1 \to \mathscr{L}^1 \to$, is defined to be the almost everywhere unique $\mathscr{L}^1$ function satisfying:*

$$\int_A P_\tau f d\mu = \int_{\tau^{-1}(A)} f d\mu,$$

for any $A \in \mathscr{B}$.

We continue by establishing several properties of the Frobenius-Perron operator.

Proposition 2.3.1. ***Linearity**: $P_\tau : \mathscr{L}^1 \to \mathscr{L}^1$ is a linear operator.*

Proof. Let $f, g \in \mathscr{L}(I)$. Suppose α, a scalar, and $A \subset I$, a measurable set, are arbitrary. Then,

$$\begin{aligned}
\int_I \big(P_\tau(\alpha f + g)\big) d\lambda &= \int_{\tau^{-1}(I)} (\alpha f + g) d\lambda \\
&= \alpha \int_{\tau^{-1}(I)} f d\lambda + \int_{\tau^{-1}(I)} g d\lambda \\
&= \int_I \big(\alpha P_\tau f + P_\tau g\big) d\lambda.
\end{aligned}$$

Thus it follows that

$$P_\tau(\alpha f + g) = \alpha P_\tau f + P_\tau g,$$

λ almost everywhere. ■

Proposition 2.3.2. ***Positivity***: *Let $f \in \mathscr{L}^1(I)$ with $f \geq 0$. Then, $P_\tau f \geq 0$.*

Proof. Let $A \in \mathscr{B}$ be arbitrary. Then,

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}(A)} f d\lambda \geq 0,$$

since $f \geq 0$. Therefore since A was arbitrary, $P_\tau f \geq 0$. ■

Proposition 2.3.3. ***Preservation of Integrals*** *Let $f \in \mathscr{L}^1(I)$. Then,*

$$\int_I P_\tau f d\lambda = \int_I f d\lambda.$$

Proof. Let $f \in \mathscr{L}^1(I)$. Then,

$$\begin{aligned} \int_I P_\tau f d\lambda &= \int_{\tau^{-1}(I)} f d\lambda \\ &= \int_I f d\lambda. \end{aligned}$$

Since f was arbitrary, the proposition holds. ■

Proposition 2.3.4. ***Contraction***: *The inequality,*

$$\|P_\tau f\|_1 \leq \|f\|_1$$

holds for all $f \in \mathscr{L}^1(I)$. i.e. $P_\tau f$ is a contraction.

Proof. Let $f \in \mathscr{L}^1(I)$ be arbitrary. Then it is possible to decompose f into two separate functions,

$$\begin{aligned} f_- &= \min(0, f) \\ f_+ &= \max(0, f), \end{aligned}$$

both of which must also be in $\mathscr{L}^1(I)$. Furthermore, we can write f as,

$$f = f_+ - f_-,$$

and,

$$|f| = f_+ + f_-.$$

So applying Proposition 2.3.1, the property of linearity, we have that,

$$\begin{aligned} P_\tau f &= P_\tau\big(f_+ - f_-\big) \\ &= P_\tau f_+ - P_\tau f_-. \end{aligned}$$

Therefore,

$$\big|P_\tau f\big| \leq \big|P_\tau f_+\big| + \big|P_\tau f_-\big|,$$

which as a result of Proposition 2.3.2, positivity, is

$$\begin{aligned} &= P_\tau f_+ + P_\tau f_- \\ &= P_\tau(f_+ + f_-) \\ &= P_\tau|f|. \end{aligned}$$

Thus, taking the norm of the operator, we get that,

$$\begin{aligned} \|P_\tau f\|_1 &= \int_I P_\tau f d\lambda \\ &\leq \int_I P_\tau|f| d\lambda. \end{aligned}$$

Finally, applying Proposition 2.3.3, the preservation of integrals, we get,

$$\begin{aligned} \|P_\tau f\|_1 &\leq \int_I |f| d\lambda \\ &= \|f\|_1, \end{aligned}$$

which proves the result. ■

A direct result of Proposition 2.3.4 is that the Frobenius-Perron operator is continuous.

Corollary 2.3.4.1. ***Continuity in the norm topology:*** *$P_\tau f : \mathscr{L}^1 \to \mathscr{L}^1$ is a continuous operator with respect to the norm topology.*

Proof. Let $f, g \in \mathscr{L}^1$. Then,

$$\|P_\tau f - P_\tau g\|_1 \leq \|f - g\|_1,$$

which implies $\|P_\tau f - P_\tau g\|_1 \to 0$ as $\|f - g\|_1 \to 0$. ■

We now present the property of composition for the Frobenius-Perron operator, which will be useful when we apply it to non-autonomous dynamical systems.

Proposition 2.3.5. ***Composition:*** *Let $(X, \mathscr{B}, \mu)$ be a measure space and $\tau : X \to X$ and $\sigma : X \to X$ be non-singular transformations. Then, $P_{\tau\circ\sigma} f = P_\tau \circ P_\sigma f$ (μ almost everywhere), and in particular, $P_{\tau^n} f = P_\tau^n f$.*

Proof. Let τ and σ be non-singular transformations. Then, $\tau \circ \sigma$ is non-singular as, for every $A \in \mathscr{B}$ such that $\mu(A) = 0$, $\mu(\tau^{-1}(A)) = 0$ since τ is non-singular. As such,

$$\begin{aligned} \mu((\tau \circ \sigma)^{-1}(A)) &= \mu(\sigma^{-1}(\tau^{-1}(A))) \\ &= 0, \end{aligned}$$

as σ is also non-singular. Now let $f \in \mathscr{L}^1 and A \in \mathscr{B}$ and,

$$\begin{aligned} \int_A P_{\tau\circ\sigma} f d\mu &= \int_{(\tau\circ\sigma)^{-1}A} f d\mu \\ &= \int_{\sigma^{-1}(\tau^{-1}A)} f d\mu \\ &= \int_{\tau^{-1}A} P_\sigma f d\mu \\ &= \int_A P_\tau(P_\sigma f) d\mu. \end{aligned}$$

Therefore, $P_{\tau\circ\sigma} f = P_\tau P_\sigma f$, μ almost everywhere. It then follows by induction that $P_{\tau^n} f = P_\tau^n f$, μ almost everywhere. ■

Proposition 2.3.6. ***Adjoint***: *If $f \in \mathscr{L}^1, g \in \mathscr{L}^\infty$, then $< P_\tau f, g >=< f, U_\tau g >$. i.e.*

$$\int_I (P_\tau f) g d\lambda = \int_I f U_\tau(g) d\lambda, \tag{2.1}$$

where $U_\tau(g)$ is the Koopman operator, defined as $U_\tau(g) = g \circ \tau$.

Proof. Let $A \in \mathscr{B}$ and set $g = \chi_A$. Then,

$$\begin{aligned} \int_I (P_\tau f) g d\lambda &= \int_I P_\tau f d\lambda \\ &= \int_A P_\tau f d\lambda \\ &= \int_{\tau^{-1}(A)} f d\lambda \\ &= \int_I f \chi_{\tau^{-1}(A)} d\lambda \\ &= \int_I f \cdot (\chi_A \circ \tau) d\lambda. \end{aligned}$$

This verifies equation (2.1) for characteristic functions. Since linear combinations are dense in $\mathscr{L}^\infty$, we can conclude that the result holds for any $f \in \mathscr{L}^\infty$. ■

The following proposition is particularly useful in our research on absolutely continuous invariant measures. It states that a measure μ, absolutely continuous with respect to λ is τ-invariant if and only if it is a fixed point of the Frobenius-Perron operator. i.e. $P_\tau f^* = f^*$. In other words, it provides an equivalent definition for an a.c.i.m. in terms of the Frobenius-Perron operator.

Proposition 2.3.7. *Let $\tau : I \to I$ be non-singular. Let the measure μ be defined by,*

$$\mu(A) = \int_A f^* d\lambda,$$

where $f^* \in \mathscr{L}^1$, $f^* \geq 0$, and $\|f\|_1$. Then μ is τ-invariant (Definition 1.2.2) if and only if,

$$P_\tau f^* = f^*$$

Note: the definition of μ makes it absolutely continuous with respect to λ.

Proof. $\Longrightarrow$

Assume μ is τ-invariant. Then,

$$\mu(A) = \mu(\tau^{-1}(A)),$$

for every measurable set A. Therefore, on an arbitrary measurable set A, we have,

$$\mu(\tau^{-1}(A)) = \int_{\tau^{-1}(A)} f^* d\lambda$$
$$= \int_A P_\tau f^* d\lambda,$$

and,

$$\mu(A) = \int_A f^* d\lambda.$$

Thus, by assumpition,

$$\int_A P_\tau f^* d\lambda = \int_A f^* d\lambda.$$

Since A was arbitrary, $P_\tau f^* = f^*$, λ almost everywhere.

$\Longleftarrow$

Assume $P_\tau f^* = f^*$, λ almost everywhere.Then,

$$\int_A P_\tau f^* d\lambda = \int_{\tau^{-1}(A)} f^* d\lambda$$
$$= \mu(\tau^{-1}(A)),$$

which by assumption,

$$= \int_A f^* d\lambda$$
$$= \mu(A).$$

Thus,

$$\mu(A) = \mu(\tau^{-1}(A)).$$

And since A was arbitrary, we conclude μ is τ-invariant. ∎

Proposition 2.3.8. ***Continuity in $\mathscr{L}^1$:*** *Let $(X, \mathscr{B}, \mu)$ be a normalized measure space and let $\tau : X \to X$ be nonsingular. Then, $P_\tau : \mathscr{L}^1 \to \mathscr{L}^1$ is continuous in the weak topology on $\mathscr{L}^1$.*

Proof. In order for P_τ to be continuous in the weak topology of $\mathscr{L}^1$, we must have the following condition:

$$f_n \to f \text{ weakly} \implies P_\tau f_n \to P_\tau f \text{ weakly}.$$

Where we say $f_n \to f$ weakly in $\mathscr{L}^1$ if and only if,

$$\int_X f_n g d\mu \to \int_X f g d\mu,$$

for all $g \in \mathscr{L}^1$. Thus, we assume that $f_n \to f$ weakly and use Proposition 2.3.6, giving,

$$\int_X (P_\tau f_n) g d\mu = \int_X f_n (g \circ \tau) d\mu.$$

Now the composition $g \circ \tau \in \mathscr{L}^\infty$ and by assumption, $f_n \to f$ weakly. Thus,

$$\begin{aligned} \int_X f_n (g \circ \tau) d\mu &\to \int_X f (g \circ \tau) d\mu \\ &= \int_X (P_\tau f) g d\mu. \end{aligned}$$

Therefore,

$$\int_X (P_\tau f_n) g d\mu \to \int_X (P_\tau f) g d\mu,$$

as $n \to \infty$. Hence, $P_\tau f_n \to P_\tau f$ weakly in $\mathscr{L}^1$. ∎

Chapter 3

The Lasota-Yorke Inequality

3.1 Piecewise Monotonic Transformations

We will now expand our study of the Frobenius-Perron operator to a specific class of dynamical systems known as piecewise monotonic transformations.

Definition 3.1.1. *Let $X = [a, b]$. Then $\tau : X \to X$ is called **piecewise monotonic** if and only if there exists a partition, $P(X) = \cup_{i=1}^{N}[a_{i-1}, a_i]$ and $r \in \mathbb{N}$ such that:*

- *$\tau|_{(a_{i-1},a_i)}$ is a C^r function, for all $i \in \{1, ..., n\}$, which can be extended to a C^r function on $[a_{i-1}, a_i]$ for $i \in \{1, ..., n\}$.*
- *For each τ_i, there exists a constant M_i such that $|\tau_i'(x) - \tau_i'(y)| \leq M_i|x - y|$, for all $x, y \in X_i$. This is known as the Lipschitz condition.*
  and,
- *$|\tau'| > s_i$>0 on (a_{i-1}, a_i), for all $i \in \{1, ..., n\}$.*

Additionally, we define the following constants:

$$s := \min_{1 \leq i \leq N} s_i \text{ and, } M := \max_{1 \leq i \leq N} M_i.$$

*If $s > 1$ wherever the derivative exists, then τ which is known as an **expanding** transformation. Finally, we say $\tau \in \mathrm{T}(X)$, the class of piecewise expanding maps on X.*

We now task ourselves with finding the Frobenius-Perron operator for an expanding piecewise monotonic transformation. By Definition 2.3.1,

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}A} f d\lambda, \tag{3.1}$$

for all $A \in \mathscr{B}$.

Since τ is piecewise monotonic, we can define an inverse function for each restriction $\tau|_{(a_{i-1},a_i)}$. Thus, we define a family of functions,

$$\phi_i = \tau^{-1} B_i,$$

where $B_i = \tau([a_{i-1}, a_i])$. Then $\phi_i : B_i \to [a_{i-1}, a_i]$ and,

$$\tau^{-1}(A) = \cup_{i=1}^{N} \phi_i(B_i \cap A). \tag{3.2}$$

Note that the sets $\{\phi_i(B_i \cap A\}_{i=1}^{N}$ are mutually disjoint, and depending on A, may even be empty. Substituting (3.2) into (3.1),

$$\int_A P_\tau f d\lambda = \sum_{i=1}^{N} \int_{\phi(B_i \cap A)} f d\lambda.$$

Now we apply the change of variable, $u = \phi_i(x)$,

$$= \sum_{i=1}^{N} \int_{B_i \cap A} f(\phi_i(x)) |\phi_i'(x)| d\lambda.$$

Thus,

$$\begin{aligned}\int_A P_\tau f d\lambda &= \int_A \sum_{i=1}^{N} f(\phi_i(x)) |\phi_i'(x)| \chi_{B_i} d\lambda \\ &= \int_A \sum_{i=1}^{N} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} |\phi_i'(x)| \chi_{\tau(a_{i-1},a_i)}(x) d\lambda,\end{aligned}$$

for any $f \in \mathscr{L}^1$ and any $A \in \mathscr{B}$. Thus,

$$P_\tau f = \sum_{i=1}^{N} f(\phi_i(x)) |\phi_i'(x)| \chi_{B_i} \tag{3.3}$$

We now introduce the concept of variation for a function in two forms, total variation and essential variation.

Definition 3.1.2. *We define the **total variation** of a function $f : X \to \mathbb{R}$ as,*

$$\overline{\bigvee}_X f = \sup_{P(X)} \sup_{x_i \in P(x)} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|,$$

where $P(x)$ is a partition of X.

Definition 3.1.3. *We define the **essential variation** of a function $f : X \to \mathbb{R}$ as,*

$$\bigvee_X f = \inf_{g \approx f} \overline{\bigvee_X} g,$$

where $g \approx f$ denotes equality almost everywhere on X with respect to λ.

We now extend our study of the Frobenius-Perron operator to the space of functions of bounded essential variation.

Definition 3.1.4. *We define the space of functions of bounded essential variation as,*

$$BV(X) = \{f \in \mathscr{L}^1(X) : \bigvee_X f < \infty\},$$

modulo equality almost everywhere.

Furthermore, we equip $BV(X)$ with the following norm.

Proposition 3.1.1. *The function,*

$$\|f\|_{BV} = \|f\|_{\mathscr{L}^1} + \bigvee_X f,$$

is a norm on $BV(X)$.

Proof. As per Definition 1.1.6, we must satisfy the following criteria.

i)$\|f\|_{BV} = 0 \iff f = 0$

Assume $\|f\|_{BV} = 0$ then,

$$\begin{aligned} 0 &= \|f\|_{BV} \\ &= \|f\|_{\mathscr{L}^1} + \bigvee_X f, \end{aligned}$$

if and only if $\|f\|_{\mathscr{L}^1} = 0$ and $\bigvee_X f = 0$. The former satisfies the criterion trivially as $\|f\|_{\mathscr{L}^1}$ is a norm. Thus,

$$
\begin{aligned}
0 &= \inf_{g \approx f} \overline{\bigvee_X} g \\
&= \|f\|_{\mathscr{L}^1} + \inf_{g \approx f} \sup_{P(X)} \sup_{x_i \in P(X)} \sum_{i=1}^{N} |g(x_i) - g(x_{i-1})|, \\
&\iff g = 0,
\end{aligned}
$$

on all partitions of X. Thus, $g = 0$, λ almost everywhere, and since $g \approx f$, $f = 0$. Thus we have shown the first condition.

ii) $\|\alpha f\|_{BV} = |\alpha| \|f\|_{BV}$ for any $\alpha \in \mathbb{R}$.

Let $\alpha \in \mathbb{R}$. Then,

$$
\begin{aligned}
\|\alpha f\|_{BV} &= \|\alpha f\|_{\mathscr{L}^1} + \bigvee_X \alpha f \\
&= \alpha \|f\|_{\mathscr{L}^1} + \inf_{g \approx f} \sup_{P(X)} \sup_{x_i \in P(X)} \sum_{i=1}^{N} |\alpha g(x_i) - \alpha g(x_{i-1})| \\
&= \alpha \|f\|_{\mathscr{L}^1} + \alpha \inf_{g \approx f} \sup_{P(X)} \sup_{x_i \in P(X)} \sum_{i=1}^{N} |g(x_i) - g(x_{i-1})| \\
&= \alpha \|f\|_{BV}
\end{aligned}
$$

which satisfies the second condition.

iii)$\|f + g\|_{BV} \leq \|f\|_{BV} + \|g\|_{BV}$.

$$
\begin{aligned}
\|f+g\|_{BV} &= \|f+g\|_{\mathscr{L}^1} + \bigvee_X (f+g) \\
&\leq \|f\|_{\mathscr{L}^1} + \|g\|_{\mathscr{L}^1} + \bigvee_X (f+g) \\
&\leq \|f\|_{\mathscr{L}^1} + \|g\|_{\mathscr{L}^1} + \bigvee_X (f+g) \\
&= \|f\|_{\mathscr{L}^1} + \|g\|_{\mathscr{L}^1} + \inf_{h_1+h_2 \approx f+g} \sup_{P(X)} \sup_{x_i \in P(X)} \sum_{i=1}^{N} |h_1(x_i) + h_2(x_i) - h_1(x_{i-1}) - h_2(x_{i-1})| \\
&\leq \|f\|_{\mathscr{L}^1} + \|g\|_{\mathscr{L}^1} + \inf_{h_1+h_2 \approx f+g} \sup_{P(X)} \sup_{x_i \in P(X)} \sum_{i=1}^{N} \big(|h_1(x_i) - h_1(x_{i-1})| + |h_2(x_i) - h_2(x_{i-1})|\big) \\
&= \|f\|_{\mathscr{L}^1} + \|g\|_{\mathscr{L}^1} + \inf_{h_1 \approx f} \sup_{P(X)} \sup_{x_i \in P(X)} \sum_{i=1}^{N} |h_1(x_i) - h_1(x_{i-1})| \\
&+ \inf_{h_2 \approx g} \sup_{P(X)} \sup_{x_i \in P(X)} \sum_{i=1}^{N} |h_2(x_i) - h_2(x_{i-1})| \\
&= \|f\|_{\mathscr{L}^1} + \|g\|_{\mathscr{L}^1} + \bigvee_X f + \bigvee_X g \\
&= \|f\|_{BV} + \|g\|_{BV},
\end{aligned}
$$

which shows the third condition and concludes the proof. ∎

3.2 The Lasota-Yorke Inequality

We will now, using [5] as a reference, offer a stronger Lasota-Yorke inequality, after first defining a δ-function which indicates whether or not a specific branch of τ touches the extrema of the considered interval (1, or 0 if $X = [0,1]$) at the end points of a specific sub-interval. This function will be an essential short-hand in the following proof.

Definition 3.2.1. *Let,*

$$
\delta_i^{\pm} := \delta_{\{\tau(a_i^{\pm}) \notin \{0,1\}\}} = \begin{cases} 0, & \tau(a_i^{\pm}) \in \{0,1\}, \\ 1, & \tau(a_i^{\pm}) \notin \{0,1\}, \end{cases}
$$

where $\tau(a_i^{\pm}) := \lim_{x \to a_i^{\pm}} \tau(x)$. If $\delta_i^{-} = 1$, then we say the left end point of the ith branch is hanging.

Now define the following constant based on the preceding δ-function. Let,

$$\eta_i := \begin{cases} \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\}, & i = 1 \\ \max\left\{\frac{\delta_{N-1}^-}{s_{N-1}}, \frac{\delta_N^-}{s_q}\right\}, & i = N \\ \max\left\{\frac{\delta_{i-1}^-}{s_{i-1}}, \frac{\delta_i^+}{s_{i+1}}\right\}, & i \in \{2, ..., N-1\}. \end{cases}$$

Then the new Lasota-Yorke inequality is stated as follows.

Proposition 3.2.1. ***A stronger Lasota-Yorke inequality:*** *Let $\tau \in \mathrm{T}(X)$. Then, for every $f \in BV(X)$,*

$$\bigvee_X P_\tau f \leq \max_{1 \leq i \leq N} \left\{\frac{1}{s_i} + \eta_i\right\} \bigvee_X f + \left[\frac{M}{s^2} + \frac{2\max_{1\leq i\leq N} \eta_i}{\min_{1\leq i\leq N} m(X_i)}\right] \int_X |f| dm. \tag{3.4}$$

Proof. We estimate $P_\tau f$. Begin by defining a partition of X, $\{x_0, x_1, ..., x_{N^*}\}$, where without loss of generality, we assume that:

- $\tau_i(a_i^-)$ and $\tau(a_i^+)$ are both in the partition for all $i \in \{1, ..., N\}$, and,
- $\max\{|x_j - x_{j-1}|, j \in \{1, ..., N^*\}\} \to 0$ as $N^* \to \infty$.

Now define, $X_i = (a_{i-1}, a_i)$ and the following functions which will both be supported on $\tau(X_i)$,

- $g_i(x) = g(\tau_i^{-1}(x))\chi_{\tau(X_i)(x)}$,
- $f_i(x) = f(\tau_i^{-1}(x))\chi_{\tau(X_i)(x)}$.

Now, let J_i denote the set of indices j such that, x_{j-1} and x_j are both members of the set $\tau(X_j)$. Then we have,

$$\begin{aligned} \sum_{j=1}^{N^*} |P_\tau f(x_j) - P_\tau f(x_{j-1})| &\leq \sum_{j=1}^{N^*} \sum_{i=1}^{N} |g_i(x_j) f_i(x_j) - g_i(x_{j-1}) f_i(x_{j-1})| \\ &\leq \sum_{i=1}^{N} \sum_{j \in J_i} |g_i(x_j) f_i(x_j) - g_i(x_{j-1}) f_i(x_{j-1})| \\ &\quad + \sum_{i=1}^{N} (|g(a_{i-1}^+ f(a_{i-1}^+ \delta_{i-1}^+| + |g(a_i^- f(a_i^-)\delta_i^-|) \\ &\leq \sum_{i=1}^{N} \sum_{j \in J_i} |f_i(x_j)(g_i(x_j) - g_i(x_{j-1}))| + \sum_{i=1}^{N} \sum_{j \in J_i} |g_i(x_{j-1})(f_i(x_j) - f_i(x_{j-1}))| \\ &\quad + \sum_{i=1}^{N} \left(\frac{\delta_{i-1}^+}{s_i}|f(a_{i-1})| + \frac{\delta_i^-}{s_i}|f(a_i)|\right). \end{aligned}$$

Now using the Lipschitz condition on τ', we can estimate the first sum in the previous inequality as,

$$\begin{aligned}\sum_{j\in J_i}|f_i(x_j)(g_i(x_j)-g_i(x_{j-1}))| &\leq \sum_j \left| f_i(x_j)\frac{\tau'(\tau_i^{-1}(x_{j-1}))-\tau'(\tau_i^{-1}(x_j))}{\tau'(\tau_i^{-1}(x_j))\tau'(\tau_i^{-1}(x_{j-1}))}\right| \\ &\leq \frac{M}{s^2}\sum_j |f(\tau_i^{-1}(x_j))||\tau_i^{-1}(x_j)-\tau_i^{-1}(x_{j-1})| \\ &\leq \frac{M}{s^2}\int_{X_i}|f|dm+\varepsilon_i(N^*).\end{aligned}$$

This follows as a result of interpreting the last sum above as a Riemann sum, which leads to the follow approximation by an integral and an error term, namely $\varepsilon(N^*)$. Note that by our assumptions on the partition, $\varepsilon(N^*)\to 0$ as $N^*\to\infty$. Thus, using this estimate,

$$\begin{aligned}\sum_{j=1}^{N^*}|P_\tau f(x_j)-P_\tau f(x_{j-1})| \leq \frac{M}{s^2}\sum_{i=1}^{N}\int_{X_i}|f|dm+\sum_{i=1}^{N}\varepsilon_i(N^*)+\sum_{i=1}^{N}\frac{1}{s_i}\bigvee\nolimits_{X_i} f \\ +\sum_{i=1}^{N}\left(\frac{\delta_{i-1}^{+}}{s_i}|f(a_{i-1})|+\frac{\delta_i^{-}}{s_i}|f(a_i)|\right).\end{aligned}$$

The last sum above can be estimated by grouping it into three parts as follows,

$$\begin{aligned}\frac{\delta_0^+}{s_1}|f(a_0)|+\frac{\delta_1^+}{s_2}|f(a_1)| &\leq \max\left\{\frac{\delta_0^+}{s_1},\frac{\delta_1^+}{s_2}\right\}\left(\bigvee\nolimits_{X_i} f+2\inf_{X_i}|f|\right) \\ &\leq \max\left\{\frac{\delta_0^+}{s_1},\frac{\delta_1^+}{s_2}\right\}\left(\bigvee\nolimits_{X_1} f+\frac{2}{m(X_1)}\int_{X_1}|f|\right).\end{aligned}$$

Similarly, for the last index,

$$\frac{\delta_{N-1}^-}{s_{N-1}}|f(a_{N-1})|+\frac{\delta_N^-}{s_N}|f(a_N)| \leq \max\left\{\frac{\delta_{N-1}^-}{s_{N-1}},\frac{\delta_N^-}{s_N}\right\}\left(\bigvee\nolimits_{X_N} f+\frac{2}{m(X_N)}\int_{X_N}|f|\right).$$

And, for the middle indices ($i=2,...,N-1$), we have,

$$\frac{\delta_{i-1}^-}{s_{i-1}}|f(a_{i-1})|+\frac{\delta_i^+}{s_{i+1}}|f(a_i)| \leq \max\left\{\frac{\delta_{i-1}^-}{s_{i-1}},\frac{\delta_i^+}{s_{i+1}}\right\}\left(\bigvee\nolimits_{X_i} f+\frac{2}{m(X_i)}\int_{X_N}|f|\right).$$

Therefore, our estimate becomes,

$$\sum_{j=1}^{N^*}|P_\tau f(x_j)-P_\tau f(x_{j-1})| \le \frac{M}{s^2}\sum_{i=1}^{N}\int_{X_i}|f|dm + \sum_{i=1}^{N}\varepsilon_i(N^*) + \sum_{i=1}^{N}\frac{1}{s_i}\overline{\bigvee}_{X_i} f$$
$$+\max\left\{\frac{\delta_0^+}{s_1},\frac{\delta_1^+}{s_2}\right\}\left(\overline{\bigvee}_{X_1} f + \frac{2}{m(X_1)}\int_{X_1}|f|\right)$$
$$+\max\left\{\frac{\delta_{N-1}^-}{s_{N-1}},\frac{\delta_N^-}{s_N}\right\}\left(\overline{\bigvee}_{X_N} f + \frac{2}{m(X_N)}\int_{X_N}|f|\right)$$
$$+\sum_{i=2}^{N-1}\max\left\{\frac{\delta_{i-1}^-}{s_{i-1}},\frac{\delta_i^+}{s_{i+1}}\right\}\left(\overline{\bigvee}_{X_i} f + \frac{2}{m(X_i)}\int_{X_N}|f|\right).$$

Now if we estimateX_i by the $\min_i m(X_i)$, we get,

$$\sum_{j=1}^{N^*}|P_\tau f(x_j)-P_\tau f(x_{j-1})| \le \max_{1\le i\le N}\left\{\frac{1}{s_i}+\eta_i\right\}\bigvee_X f$$
$$+\left[\frac{M}{s^2}+\frac{2\max_{1\le i\le N}\eta_i}{\min_i m(X_i)}\right]\int_X |f|dm + \sum_{i=1}^{N}\varepsilon_i(N^*),$$

which if we take the limit when $N^*\to\infty$, we arrive at inequality (3.4). ■

Note that the original Lasota-Yorke inequality follows immediately from Theorem 3.2.1.

Corollary 3.2.0.1. ***Original Lasota-Yorke inequality:*** *Let $s=\min_{1\le i\le N} s_i > 2$, then inequality (3.4) becomes,*

$$\bigvee_X P_\tau f \le 2s^{-1}\bigvee_X f + (K+2\beta^{-1})\|f\|_{\mathscr{L}^1},$$

*where $K=\frac{M}{s^2}$ and $\beta=\min_{1\le i\le N} m(X_i)$. Thus we have obtained the **original Lasota-Yorke inequality** as presented in [8].*

Finally, we conclude this chapter with a result which, similarly to Straube's theorem, establishes the existence of an acim for a dynamical system.

Theorem 3.2.1. ***Existence of an acim:*** *Suppose a dynamical system $\tau\in \mathrm{T}(X)$ which satisfies inequality (3.2.1) with the coefficient,*

$$\max_{1\le i\le N}\left\{\frac{1}{s_i}+\eta_i\right\}\le\alpha<1,$$

for some $\alpha > 0$*, then for any* $f \in BV(X)$ *and* $n \in \mathbb{N}$*,*

$$\|P_\tau^n f\|_{BV} \leq \alpha^n \|f\|_{BV} + \left(1 + \frac{K + 2\beta^{-1}}{1 - \alpha}\right) \|f\|_{\mathscr{L}^1},$$

where $K = \frac{M}{s^2}$ *and* $\beta = \min_{1 \leq i \leq N} m(X_i)$*. Moreover, there exists an acim for the transformation* τ *whose density is of bounded variation.*

Proof. Using the norm of $BV(X)$, $\|f\|_{BV}$ as defined in Definition 3.1.1, and Proposition 3.2.1 we get,

$$\|P_\tau^n f\|_{BV} \leq \alpha^n \|f\|_{BV} + \left(1 + \frac{K + 2\beta^{-1}}{1 - \alpha}\right) \|f\|_{\mathscr{L}^1}.$$

Since the set $\{f \in BV : \|f\|_{BV} \leq 1\}$ is relatively compact in the $\|\cdot\|_{\mathscr{L}^1}$ norm, it follows by similar arguments to those used in the proof of Theorem 1.3.1 that P_τ has a fixed point within $BV(X)$. Thus we have proven the result. ∎

Chapter 4

Generalization of Straube's Theorem

4.1 Non-Autonomous Dynamical Systems

We begin this chapter by defining a non-autonomous dynamical system.

Definition 4.1.1. *Let $(X, \mathscr{B}, \mu)$ be a compact measure space. Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of transformations $\tau_n : X \to X$, $\tau_0 = id$ converging uniformly to a transformation $\tau : X \to X$ and $\tau_0 = id$. Then we call the collection $(X, \mathscr{B}, \mu, \{\tau_n\})$ a **non-autonomous dynamical system** and denote its orbits as following:*

$$x_n = \tau_n \circ ... \circ \tau_1 \circ \tau_0(x_0) = \tau_{(0,n)}(x_0),$$

4.2 Extension of the Krylov-Bogoliubov Theorem

Next we will extend the first main result of Chapter 1 to cover non-autonomous dynamical systems, namely the Krylov-Bogoliubov theorem.

Theorem 4.2.1. *Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of transformations defining a non-autonomous dynamical system on $(X, \mathscr{B})$. Define the measures $z_i = (\tau_{(0,i)})_* \nu$ where ν is a fixed probability measure defined for all $B \in \mathscr{B}$. Let z be a *-weak limit point of the sequence $\frac{1}{n}\sum_{i=1}^{n} z_i$. Then $z = \tau_* z$. i.e. z is a τ-invariant measure.*

Proof. We use the proof of the Krylov-Bogoliubov Theorem for guidance. Let ν be a probability measure on defined for all $B \in \mathscr{B}$. Then, the sequence,

$$\left[\frac{1}{n}\sum_{i=1}^{n} z_i\right]_n = \left[\frac{1}{n}\sum_{i=1}^{n}(\tau_{(0,i)})_*\nu\right] = \left[\frac{1}{n}\Big((\tau_{(0,1)})_*\nu + (\tau_{(0,2)})_*\nu + ... + (\tau_{(0,n)})_*\nu\Big)\right],$$

is a sequence of probability measures and is hence inside the *-weak unit ball. So it has a convergent subsequence $\{z_{n_k}\}$. Let z be the limit point of this subsequence. We shall prove z is τ-invariant. Let $g \in C_0(X)$ and let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that, if $|x - y| < \delta$ we have,

$$\omega_g(\delta) = \sup_{|x-y|<\delta} |g(x) - g(y)| < \varepsilon,$$

Then, since $\tau_n \to \tau$ uniformly, there exists some M so that,

$$\|\tau_s - \tau\|_\infty < \delta, \text{ for all } s > M.$$

Furthermore,

$$\begin{aligned} |\tau_{(0,s+1)}(x) - \tau \circ \tau_{(0,s)}(x)| &= |\tau_s \circ \tau_{(0,s)}(x) - \tau \circ \tau_{(0,s)}(x)| \\ &= |(\tau_s - \tau) \circ \tau_0^s(x)| \\ &\leq \|\tau_s - \tau\|_\infty \\ &< \delta. \end{aligned}$$

Now, since $\|\tau_s - \tau\|_\infty < \delta$,

$$|g(\tau \circ \tau_{(0,s)}) - g(\tau_{(0,s+1)})| \leq \omega_g(\delta) < \varepsilon, \text{ for all } s > M.$$

Thus,

$$\begin{aligned} \left| \int g dz - \int g(\tau) dz \right| &= \lim_{k\to\infty} \left| \int g dz_{n_k} - \int g(\tau) dz_{n_k} \right| \\ &= \lim_{k\to\infty} \left| \frac{1}{n_k} \Big((\tau_{(0,1)})_* \nu + (\tau_{(0,2)})_* \nu + \ldots + (\tau_{(0,n_k)})_* \nu \Big)(g) \right. \\ &\quad \left. - \frac{1}{n_k} \Big((\tau_{(0,1)})_* \nu + (\tau_{(0,2)})_* \nu + \ldots + (\tau_{(0,n_k)})_* \nu \Big)(g(\tau)) \right|. \end{aligned}$$

Now, using the fact that,

$$(\tau_{(0,i)})_* \nu(g \circ \tau) = (\tau \circ \tau_{(0,i)})_* \nu(g),$$

we get,

$$\begin{aligned}\left|\int gdz - \int g(\tau)dz\right| &= \lim_{k\to\infty}\left|\frac{1}{n_k}\Big((\tau_{(0,1)})_*\nu + (\tau_{(0,2)})_*\nu + \dots + (\tau_{(0,n_k)})_*\nu\Big)(g)\right.\\
&\left. - \frac{1}{n_k}\Big((\tau\circ\tau_{(0,1)})_*\nu + (\tau\circ\tau_{(0,2)})_*\nu + \dots + (\tau\circ\tau_{(0,n_k)})_*\nu\Big)(g)\right|\\
&\le \lim_{k\to\infty}\frac{1}{n_k}\Big(|(\tau_{(0,1)})_*\nu(g)| + |(\tau_{(02)})_*\nu(g) - (\tau\circ\tau_{(0,1)})_*\nu(g)|\\
&+ \dots + |(\tau_{(0,n_k)})_*\nu(g) - \tau\circ\tau_{(0,n_k-1)})_*\nu(g)| + |(\tau\circ\tau_{(0,n_k)})_*\nu(g)|\Big)\\
&= \lim_{k\to\infty}\frac{1}{n_k}\Big(|(\tau_{(0,1)})_*\nu| + |(\tau\circ\tau_{(0,n_k)})_*\nu(g)| + |(\tau_{(0,2)})_*\nu(g) - (\tau\circ\tau_{(0,1)})_*\nu(g)|\\
&+ \dots + |(\tau_{(0,n_k)})_*\nu(g) - \tau\circ\tau_{(0,n_k-1)})_*\nu(g)|\Big).\end{aligned}$$

And since $\|\tau_s - \tau\|_\infty < \delta$, for all $s > M$,

$$\begin{aligned}\left|\int gdz - \int g(\tau)dz\right| &\le \lim_{k\to\infty}\frac{1}{n_k}\Big(|(\tau_{(0,1)})_*\nu(g)| + |(\tau\circ\tau_{(0,n_k)})_*\nu(g)|\\
&+ |(\tau_{(0,2)})_*\nu(g) - (\tau\circ\tau_{(0,1)})_*\nu(g)| + \dots + |(\tau_{(0,M+1)})_*\nu(g) - (\tau\circ\tau_{(0,M)})_*\nu(g)|\\
&+ |(\tau_{(0,M+2})_*\nu(g) - (\tau\circ\tau_{(0,M+1)})_*\nu(g)|\\
&+ \dots + |(\tau_{(0,n_k)})_*\nu(g) - \tau\circ\tau_{(0,n_k-1)})_*\nu(g)|\Big)\\
&< \lim_{k\to\infty}\frac{1}{n_k}\Big(|(\tau_{(0,1)})_*\nu(g)| + |(\tau\circ\tau_{(0,n_k)})_*\nu(g)|\\
&+ |(\tau_{(0,2)})_*\nu(g) - (\tau\circ\tau_{(0,1)})_*\nu(g)| + \dots + |(\tau_{(0,M+1)})_*\nu(g) - (\tau\circ\tau_{(0,M)})_*\nu(g)|\\
&+ (n_k - M - 2)\varepsilon\Big)\\
&< \lim_{k\to\infty}\left(\frac{1}{n_k}(M+2)\text{sup}|g| + (n_k - M - 2)\varepsilon\right)\\
&= \varepsilon.\end{aligned}$$

Since ε was arbitrary, $z = \tau_* z$.

■

4.3 Extension of Straube's Theorem

As mentioned in Chapter 1, the Krylov-Bogoliubov Theorem and its extension above only guarantee the existence of an invariant measure. However, beyond the measure being invariant, not much else can be known about it. It may simply be a point measure. A more interesting result guarantees the existence of an absolutely continuous invariant measure. The following result extends the results of Straube's theorem to non-autonomous dynamical systems.

Theorem 4.3.1. *Let* $(X, \mathscr{B}, \mu)$ *be a measure space and* $\{\tau_n\}_{n=0}^{\infty}$ *be a sequence of non-singular transformations generating a non-autonomous dynamical system. If there exists* $\delta > 0$ *and* $0 < \alpha < 1$ *such that,*

$$\mu(E) < \delta \implies \sup_{k \in \mathbb{N}} \mu(\tau_{(0,k)}^{-1}(E)) < \alpha,$$

for all $E \in \mathscr{B}$. *Then, there exists a* τ *invariant normalized measure which is absolutely continuous with respect to* μ.

Proof. Define the measures,

$$\mu_n(E) = \frac{1}{n} \sum_{k=0}^{n-1} \mu(\tau_{(0,k)}^{-1}(E)),$$

for any $E \in \mathscr{B}$. Then for all n,

- $\mu_n(X) = 1$.
- $\mu_n << \mu$ (τ_n is non-singular for every n).
- $\mu_n \geq 0$.

Hence, $\{\mu_n\}$ is a sequence of probability measures, so it is in the unit ball of the L_∞^* weak* topology, which is compact as a result of Theorem 1.1.2. So we let $\tilde{z}$ be the limit point of $\{\mu_n\}$. Define the set function,

$$z(E) = \tilde{z}(\chi_E).$$

Then, z is finitely additive, bounded ($0 \leq z(E) \leq z(X) = 1$), and it vanishes on sets of μ-measure zero.

We now use z to construct a countably additive function. By lemmas 1.4.2 and 1.4.3, we can uniquely decompose z into,

$$z = z_c + z_p,$$

where z_c is countably additive and z_p is purely finitely additive.

We claim that $z_c \neq 0$. Otherwise, by Lemma 1.4.1, there exists a decreasing sequence $\{E_n\} \subset \mathscr{B}$ such that,

$$\lim_{n\to\infty} \mu(E_n) = 0,$$

and,

$$z(E_n) = z(X) = 1.$$

Since $\mu(E_n) \to 0$, there exists an n_0 such that $n > n_0 \implies \mu(E_n) < \delta$, for any $\delta > 0$. Now by our assumptions, there is an $\alpha < 1$ so that,

$$\sup_k \mu(\tau_{(0,k)}^{-1}(E_n)) < \alpha.$$

Thus, $\mu(\tau_{(0,k)}^{-1}(E_n)) < \alpha$ for all k. So,

$$z(E_n) < \alpha < 1.$$

This is a contradiction. So we have demonstrated that $z_c \neq 0$.

Finally, we demonstrate that z_c is τ invariant. Let $\nu = z - z(\tau)$, then ν cn be decomposed as $\nu = z_c + z_p - z_c(\tau) - z_p(\tau)$. As a result of Theorem 4.3.1, we have that for any continuous g,

$$\mu_{n_k}(g) - \mu_{n_k}(g(\tau)) \to 0.$$

Thus for any continuous g on X,

$$\nu(g) = z(g) - z \circ \tau(g) = 0.$$

Hence as a result of Lemma 1.4.5, ν is purely finitely additive and so,

$$z_c - z_c(\tau) = 0.$$

So z_c is τ-invariant.

Thus far, we have,

$$z(E) = z(\tau^{-1}(E)),$$

and that z can be decomposed into two positively additive set functions so that,

$$z(E) = z_c(\tau^{-1}(E)) + z_p(\tau^{-1}(E)).$$

Furthermore, since z_c is a countably additive set function,

$$z_c(\tau^{-1}(E)) \leq z_c(E),$$

for all $E \in \mathscr{B}$. Where here we treat $z_c(\tau^{-1}(E))$ as a measure induced by the pre-image of E. Hence,

$$z_c(E) - z_c(\tau^{-1}(E)) \geq 0.$$

In other words, it is a positive measure. But since $\tau^{-1}(X) = X$, the total weight is precisely 0. Thus it is the zero measure. This shows that z_c is invariant.

Since z vanishes on sets of μ measure zero, and $z_c \leq z$, z_c will also vanish on sets of μ measure zero. So $z_c << \mu$. Therefore, normalizing we have,

$$\nu(E) = \frac{z_c(E)}{z_c(X)}$$

is a normalized measure invariant of τ and absolutely continuous with respect to μ. ■

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