Enhancing Seventh-Graders' Relational Thinking Through Mental Mathematics

Alexandra N. Kindrat

A Thesis
In the Department
of
Education

Presented in Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy (Education) at
Concordia University
Montreal, Quebec, Canada

January 2018

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: Alexandra N. Kindrat

Entitled: Enhancing Seventh-Graders' Relational Thinking Through Mental Mathematics

and submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY (Education)

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final examining committee:

______________________________ Chair
Dr. Laura Collins

______________________________ External Examiner
Dr. Ana C. Stephens

______________________________ External to Program
Dr. Diane Poulin-Dubois

______________________________ Examiner
Dr. Robert Bernard

______________________________ Examiner
Dr. Saul Carliner

______________________________ Thesis Supervisor
Dr. Helena P. Osana

Approved by ______________________________
Dr. Saul Carliner, Department Chair

March 2, 2018 (Date of Defence)

____
Dr. André Roy, Dean of Faculty of Arts and Science

Abstract

Enhancing Seventh-Graders' Relational Thinking Through Mental Mathematics

Alexandra N. Kindrat, Ph.D

Concordia University, 2018

This dissertation sought to investigate the effects of a mental mathematics intervention on seventh graders' relational thinking in mathematics, an important topic as relational thinking is a precursor to algebraic reasoning. In this dissertation, improvements in students' understanding of the equal sign, reasoning about true-false number sentences, mental mathematics, and working memory were addressed over the course of the three studies. The topic is important because there is lack of empirically-validated instructional methods to support students' relational thinking beyond an understanding of the equal sign. Moreover, the majority of studies investigating ways to improve relational thinking have been conducted at the primary level.

All three studies examined the relational thinking of seventh-graders before and after a mental mathematics intervention. In Study 1, students were assessed at three time points on their ability to solve equivalence problems and their reasoning about true-false number sentences. Study 2 replicated Study 1 but also extended it by examining long-term effects and including a measure of students' mental computation. Study 3 examined whether the effects on relational thinking of a mental mathematics intervention could be augmented beyond what was observed in Studies 1 and 2 if students were permitted to write down parts of their mental computation strategies, thereby reducing their cognitive load. Overall, mean mental mathematics, equivalence, and reasoning about true-false number sentences scores improved following the intervention, suggesting an important link between mental computation and relational thinking. Reducing the cognitive load during mental computational did not have an effect on students' relational thinking in Study 3.

More research is needed on how mental mathematics instruction can serve as an effective tool at the primary and senior high school levels to improve relational thinking. Further studies are also needed to determine the role of working memory in mental mathematics related to relational thinking. Educators can learn from the present studies that mental mathematics contributes positively to students' relational thinking in the seventh-grade, and that even short mental computation interventions may be effective. Limitations of this dissertation include the focus on only one grade level, as well as possible experimental bias.

Acknowledgements

I would like to thank my supervisor, Dr. Helena P. Osana, for her support, guidance, encouragement, and friendship that she has so generously given to me over the course of my studies. I am very grateful for the amount of time she has spent explaining to me various concepts, helping me organize my studies, and discussing my research. Her involvement in this project has been indispensable to me.

I am grateful and appreciative to my committee members Dr. Robert Bernard and Dr. Saul Carliner for their direction, support, and insightful comments over the course of my doctoral studies and research.

Thank you to Dr. Diane Poulin-Dubois and Dr. Ana C. Stephens for agreeing to sit on my Examination Committee.

Dedication

Thank you to my family for their continual love and support throughout all of my academic endeavours.

Contribution of Authors

All three studies of this dissertation were designed and carried out in collaboration with my supervisor, Dr. Helena P. Osana. Specifically, both Dr. Osana and I designed the studies, created the instruction, constructed new measures and modified existing ones, and analyzed the data. As the lead author, I was the primary researcher on all three studies, and was responsible for delivering the interventions, collecting participant data, scoring the data, creating the scoring rubrics, and providing training for inter-rater reliability. The first study is in press in the *Fields Mathematics Education Journal* (Kindrat, A. N., & Osana, H. P., in press).

Table of Contents

List of Tables

List of Figures

Chapter 1: General Introduction

Introduction

The manuscripts presented in this dissertation aim to improve our understanding of how relational thinking in mathematics can be improved through mental mathematics instruction. Beyond the domain of mathematics, relational thinking is predicated on the ability to detect a common structure between two situations (Gentner & Colhoun, 2010). In fact, mental agility in humans relies on analogical reasoning, which can be seen as a form of relational thinking (i.e., Gentner, 2003; Gentner & Christie, 2008; Kurtz, Gentner, & Gunn, 1999). For example, relational thinking in biology would allow for an individual to understand the relationship between a fire burning oxygen to create energy, which is analogous to a mitochondria combining glucose and oxygen to create energy. Both situations, which are superficially dissimilar, require oxygen to produce energy (Gentner & Calhoun, 2010). Moreover, an astronaut in a dire situation aboard the International Space Station without the appropriate tools would undoubtedly look for analogous equipment (to the typical equipment used that would be required) at his or her disposal to remedy the catastrophic situation. Both of these examples require relational thinking: the ability to see the commonalities between two situations. As such, relational thinking in the real world is valuable, and it is possible that through mathematics instruction, students' relational thinking abilities are improved even beyond the domain of mathematics.

In mathematics, relational thinking is a type of thinking that requires that mathematics expressions, such as numbers in algebraic form in an equation, are considered in their entirety, prior to beginning any computation or other procedure (Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Stephens, 2006). It entails that an individual visualize patterns and understand generalizations that involve number properties (Britt & Irwin, 2011; Cooper & Warren, 2011; Mason, Stephens & Watson, 2009). Moreover, relational thinking requires that students have an understanding of the equal sign as indicating that an equivalence relationship exists between numbers (Jacobs et al., 2007) as well as seeing this relationship in decomposing, transforming, and substituting a variety of numerical expressions in a given equation (Britt & Irwin, 2011; Jones, Inglis, Gilmore, & Evans, 2012). Encouraging students to think about the numerical relationships between quantitatively equivalent expressions allows for a seamless transition from relational thinking to algebra (Britt & Irwin, 2011; Knuth, Stephens, Blanton & Gardiner, 2016).

Relational thinking is seen as the precursor to algebra (Carpenter, Franke, & Levi, 2003), a topic that often causes students considerable difficulty (Cai & Knuth, 2005; Kieran, 1981). For this reason, many educators and policymakers believe that teaching for relational thinking should begin in the early primary and middle school grades (Lester, 2007; NCTM, 2000).

Despite the apparent relationship between algebra and relational thinking, the literature on teaching for relational thinking instruction is sparse. Among the literarure that does exist, the majority focus on students' abilities to notice and articulate common patterns across problems, notice generalizations within operations, and generate explanations for why certain generalizations are true or not (Carpenter et al., 2003; Russell, Schifter, & Bastable. 2003). Although such thinking skills are critical to mathematics, these specific instructional initiatives are not specific to relational thinking While attempting to find further literature that could be helpful in teaching for relational thinking, I noticed in much of the mathematics education literature that both relational thinking and mental mathematics share several theoretical features (Jacobs et al., 2007; Stephens & Ribeiro, 2012; Thompson, 2010; Threlfall, 2002). As such, the aim of the following studies was to investigate whether and how relational thinking could be improved by using a mental mathematics intervention as a vehicle for improving relational thinking and understanding of mathematical equivalence in seventh-grade students.

Relational Thinking

Many students struggle to understand the relationship between numbers in mathematical equations (Molina & Ambrose, 2006; Stephens & Ribeiro, 2012). In many ways, this is contingent on their understanding of mathematical equivalence (Byrd, McNeil, Chesney, & Matthews, 2015; Carpenter, Levi, Franke & Zeringue, 2005; Powell, Kearns, & Driver, 2016; Stephens et al., 2013). The majority, if not all, of the research on relational thinking indicates that an understanding of equivalence is critical. Many researchers have demonstrated that instruction centred equivalence encourages relational thinking, as it ensures that students can see the relationships between numbers on both sides of the equal sign (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; McNeil & Alibali, 2005a; Molina & Ambrose, 2006).

The literature also describes the importance of understanding number properties (Carpenter et al., 2003, 2005; Irwin & Britt, 2005; MacGregor & Stacey, 1999; Stephens & Ribeiro, 2012), including the commutative property, the associative property, and the distributive property in relational thinking. Student success in relational thinking depends on their understanding of the fundamental properties of numbers, which allows them to restructure arithmetic operations to transform number sentences into more manageable expressions (Carpenter et al., 2005; Empson, Levi, & Carpenter, 2011; Jacobs et al., 2007).

Flexibility is also an important component of relational thinking (Carpenter et al., 2005; Irwin & Britt, 2005; Mason & Stephens, 2006; Stephens & Ribeiro, 2012). An ability to be flexible with numbers is defined as choosing or generating an appropriate strategy to solve a given problem, and applying knowledge about numbers and properties in different ways depending on the context (Proulx, 2013; Star & Newton, 2009; Threlfall, 2002). Flexibility is important in relational thinking as it allows students to decide which approach (i.e., partitioning, transforming, substituting, re-ordering, or decomposing numbers, as necessary) might be best to change a given equation into one that is more manageable to solve (Britt & Irwin, 2011, Jones, et al., 2012).

Mental Mathematics

Mental mathematics is defined as the process of carrying out arithmetical operations in one's head (Maclellan, 2001; Reys, 1984), although this does not preclude recording parts of the mathematical reasoning that produced the results (Harries & Spooner, 2000). Although mental computation can include rote memorization of numerical facts, it requires that individuals are able to recall and retrieve these facts when necessary. It also requires that students possess an understanding of number structure and number properties (Reys, 1984; Varol & Farran, 2007), and, much like relational thinking, it promotes flexibility as students solve a variety of mathematical problems across domains (Varol & Farran, 2007). Knowledge of number properties and flexibility in transforming, compensating, and re-ordering numbers has been shown to be important in mental mathematics (Heirdsfeld & Cooper, 2002; Proulx, 2013; Thompson, 2010, Threlfall, 2002; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). These processes involve an understanding of *sameness* (i.e., that two numbers or expressions are equivalent when they represent the same quantity), as well as the concept of *substitution* (i.e., that a number, or an expression, can be replaced for another because the two are equivalent; Heirdsfield, 2011; Jones et al., 2012). If successful in their ability to be flexible, recognize number properties, and understand equivalence through the concepts of sameness and substitution, students can create manageable and meaningful mental substitutions, which I hypothesize could support their mental computation.

Improving Relational Thinking

When researchers describe relational thinking at the primary and middle school levels, its focus is often on improving students' understanding of equivalence, in isolation of the other relational thinking components discussed earlier. Students often inaccurately believe that the equal sign is an indication to write down the answer to an operation performed on the numbers that precede it (i.e., an operational view of the equal sign), rather than understanding the correct equivalence relationship on either side of it in a mathematical equation (Alibali et al., 2007; Kieran, 1981). This operational view is prevalent even at the grade eight level (Alibali et al., 2007).

Classroom activities focusing on the broader conceptualization of relational thinking – that is, beyond an understanding of equivalence, including knowledge of number properties and flexibility – has been linked to students' relational thinking (Carpenter et al., 2003; Stephens & Ribeiro, 2012). Such research includes teacher-guided discussions with groups of students (Blanton & Kaput, 2011; Carpenter et al., 2003, 2005; Molina & Ambrose, 2006; Russell et al., 2011), one-on-one interviews with students to gain insight into their thinking about relational thinking (Molina, Castro, & Mason, 2008; Subramaniam & Banerjee, 2011), longitudinal assessments to determine students' relational thinking improvements over time (Britt & Irwin, 2011), questionnaires (Stephens & Ribeiro, 2012; Stephens & Wang, 2008), and the use of manipulatives to describe relational thinking (Kiziltoprak & Köse, 2017), to name a few. However, with much of this research, there are often no baseline measures of students' relational thinking before any implemented activities which makes it difficult to examine any changes in their thinking over time. Moreover, the activities undertaken to foster relational thinking in the above mentioned studies often involve intact classes, and often with no comparison group with which to compare results.

As relational thinking and mental mathematics are thought to share many common theoretical components, the studies presented in this dissertation aimed to investigate the effect of a mental mathematics intervention on students' relational thinking. All three studies made use of comparison groups, as well as baseline assessment measures prior to any intervention being administered. In the third study, I attempted to establish causation between improvements in mental mathematics and relational thinking because of the random assignment of the students.

Studies 1 and 2: The Relationship between Mental Mathematics and Relational Thinking

Study 1 investigated the impact of a mental mathematics intervention on seventh-graders' relational thinking and their understanding of the equal sign. Using two seventh-grade classes, students were assessed on their understanding of the equal sign and their reasoning abilities about true-false number sentences immediately following the conclusion of the mental mathematics intervention and at a delayed time point.

Study 2 had similar goals to Study 1: I investigated the impact of a mental mathematics intervention on seventh-graders' relational thinking. In this sense, Study 2 was a replication of Study 1, but also an extension because it also included a mental mathematics assessment to provide a manipulation check for the mental mathematics intervention. Study 2 also aimed to determine whether the improvement in relational thinking could be maintained over a longer time period (12 weeks) compared to Study 1, which only verified maintenance for four weeks after the intervention. In addition, two classes participated in Study 1, whereas three classes participated in Study 2. In Study 2, students were assessed at five points on their ability to solve mental mathematics problems, solve equivalence problems, and reason about true-false number sentences.

Both Study 1 and 2 suggested a link between improved relational thinking and mental mathematics instruction. The literature on mental mathematics has shown, however, that computation demands extensive working memory resources (DiStefano & LeFevre, 2004; Hitch, 1978). Thus, Study 3 aimed at investigating whether improvements in relational thinking could be enhanced even more than what was observed in Study 1 and 2 by reducing students' working memory loads during the mental mathematics intervention.

Working Memory and Relational Thinking

Working memory has been described as a mental workspace that plays a role in the monitoring, regulation, and maintenance of pertinent information to carry out complex cognitive tasks (Miyake & Shah, 1999). As students solve complex problems, they are required simultaneously to retain existing information while they process new information (Raghubar, Barnes, & Hecht, 2010). This requires working memory resources. An understanding of how working memory is related to how students learn mathematics may be important for instruction

(Raghubar et al., 2010), and may have implications for designing relational thinking instruction as well.

Research suggests than working memory may be related to mental mathematics performance (Adams & Hitch, 1997; Hitch, 1978; Logie, Gilhooly & Wynn, 1994), which may in turn also have an impact on relational thinking. For example, Adams and Hitch (1997) conducted a study aimed at investigating the link between children's mental arithmetic and their working memory to determine whether it was working memory or arithmetical competence that caused difficulty in the children's mental mathematics performance. When young children (aged 7 to 11 years old) were told to add pairs of multi-digit numbers, children's accuracy increased when the addends were delivered visually and remained visible for the duration of the calculation. The authors concluded that when the working memory load was reduced by allowing the addends in the mental addition task were permanently visible, performance improved (Adams & Hitch, 1997).

In another study, Hitch (1978) conducted a series of experiments investigating the impact of working memory on performing mental arithmetic. Participants were assessed on their ability to solve verbally presented multi-digit addition problems, where a three digit number was added to a two digit number. Participants were required to solve 48 mental addition problems using two different strategies. In the first strategy, they were asked to add from right to left (i.e., units, tens, and then hundreds), recording their partial addition results immediately as they progressed through their calculation. In the second strategy, participants were asked to add from left to right (i.e., hundreds, tens, and then ones), but were only permitted to write down their final answer. The results indicated that for the second strategy, when participants were not permitted to write down their partial results as they progressed through their mental calculation, their computational accuracy was impeded (Hitch, 1978), suggesting that working memory issues may have affected their mental computation performance. Therefore, because working memory is critical in mental mathematics, it is possible that reducing students' cognitive load as they perform mental mathematics may serve to improve their relational thinking beyond what was observed in Study 1 and 2.

Study 3. Study 3 investigated the effects of a mental mathematics intervention on seventh-graders' relational thinking, but unlike Study 1 and 2, students' cognitive load was alleviated while they perform mental computations. This was achieved by requiring them to write down certain elements of the problem during the mental mathematics instruction. The study aimed to test the added effects of reducing the cognitive load that is inherent in mental computation on students' equivalence problem solving and relational thinking, as well as establish causality between the mental mathematics intervention and the improvement in relational thinking.

Using two seventh-grade classes, students were assessed at two time points on their understanding of equivalence, their ability to solve mental mathematics problems, and their reasoning abilities about true-false number sentences. A delayed assessment also identified the nature of students' strategies when solving mental mathematics problems. Students in each class were randomly assigned to two conditions: a Regular Mental Mathematics (RMM) condition and a Reduced Cognitive Load (RCL) condition. Both conditions received the identical instruction with identical content, but the RMM condition was instructed to solve the expression entirely mentally (with no tools for recording thinking), while the RCL condition was instructed to solve the expressions in the same manner as the RMM condition, but with the added requirement of writing out specific parts of the mental calculations. Study 3 therefore investigated whether the suggested improvement in relational thinking seen in Study 1 and 2 could be enhanced even further with a reduction in cognitive load.

Conclusion

It is my hope that the research conducted and presented in this dissertation can provide the foundation for improving relational thinking at the middle school level. In pursuing my research, it was my goal that not only my own students could benefit from this research, but that fellow educators could also incorporate mental mathematics into their classrooms so that their students, too, could benefit from this novel approach to encouraging success in algebra.

Chapter 2: Study One

The Relationship Between Mental Computation and Relational Thinking in the Seventh-Grade

Published in *Fields Mathematics Education Journal*

Introduction

Among the many concepts included in the middle and secondary school curriculum is algebra, a topic that often causes much difficulty for many students (Carpenter et al., 2003; Jacobs et al., 2007; National Council of Teachers of Mathematics, NCTM, 2000). As algebra is known to be a gatekeeper to students' future academic and professional opportunities (Knuth, Stephens, McNeil, & Alibali, 2006; Spielhagen, 2006), alleviating their struggles in algebra is of critical importance. While algebra allows mathematical generalizations and solving for unknown quantities, critical to its success is its precursor, relational thinking (Carpenter et al., 2003). Mathematics educators have argued that exposure to relational thinking should begin in kindergarten and thread through the mathematics curriculum to the end of high school (Falkner, Levi, & Carpenter, 1999; Kaput, 1998; NCTM, 2000).

Relational thinking includes the ability to look at a mathematical expression or equation in its entirety instead of in a manner where a prescribed sequence of procedures is to be followed to arrive at an answer (Carpenter et al., 2005; Jacobs et al., 2007; Stephens, 2006). It entails making explicit generalizations that are based on the fundamental properties of number operations (Jacobs et al., 2007; Stephens & Ribeiro, 2012). It involves attending to patterns and rules in creating mathematical generalizations (Britt & Irwin, 2011; Cooper & Warren, 2011; Mason et al., 2009). It requires making strategic decisions, and requires that students think before they act. Sometimes, when thinking prior to acting, students can use relational thinking to simplify a calculation before proceeding. Students who are able to view the equal sign as an indication of a relationship between two expressions (Jacobs et al., 2007), are engaging in relational thinking (Carpenter et al., 2003; Jacobs et al., 2007; Molina & Ambrose, 2006). Individuals who are successful at relational thinking are able to discover patterns in a variety of mathematical expressions and further generalize these patterns from familiar to unfamiliar situations (Cooper & Warren, 2011; Mason et al., 2009).

An understanding of the relationships between numbers allows students to generalize their strategies to a variety of numerical expressions (Britt & Irwin, 2011). To demonstrate relational thinking, compensation and equivalence can be used in transforming number sentences such as 99 + 78 to 100 + 77, which allows students to think about the relationships that exist between numbers (Britt & Irwin, 2011), and reveal their understanding that both expressions are equivalent. Moreover, the mathematical relationships in arithmetic expressions such as 8 + __ = 12 could be used to reflect on 1986 + 8 + __ = 1986 + 12, and eventually to $x + 8 + y = x + 12$ (Carpenter et al., 2005). Encouraging students to reflect on the numerical relationships between structurally similar expressions promises a seamless transition from relational thinking to algebra. Success in algebra, therefore, is predicated on instruction of arithmetic in ways that supports students' relational thinking skills and creates a solid foundation for learning algebra with meaning (Britt & Irwin, 2011; Knuth et al., 2016).

Equivalence is important in relational thinking, as it allows students to be flexible in their approach to solving a mathematical equation or evaluating a mathematical expression (Knuth et al., 2006; Stephens & Ribeiro 2012). Referring to the previous example of 1986 + 8 + __ = 1986 + 12, for instance, students with a relational view of the equal sign are able to determine that the answer is 4 by reflecting on the relationship between the numbers. Because of the meaning of the equal sign, students can disregard the 1986 on both sides of the equal sign as they balance both sides of the expression to equal the same total. Students can then focus on only the resultant equation 8 + __ = 12, which would either reduce the level of computation or allow it to be circumvented altogether (Carpenter et al., 2003).

Despite the apparent relationship between algebra and relational thinking, the literature on teaching for relational thinking instruction is incomplete. In an attempt to understand how relational thinking instruction might be improved, I noticed in much of the mathematics education literature that both relational thinking and mental mathematics share several features (Jacobs et al., 2007; Stephens & Ribeiro, 2012; Thompson, 2010; Threlfall, 2002). As such, the aim of this study was to determine whether relational thinking could be improved by using a mental mathematics intervention as a vehicle for improving relational thinking in seventh-grade students.

Relational Thinking

Many students struggle in their understanding of the relationships between numbers in mathematical expressions (Molina & Ambrose, 2006; Stephens & Ribeiro, 2012). Several researchers have reported that a lack of relational thinking in mathematics is a result of students' deficiencies in (a) their understanding of the equal sign (Behr, Erlwanger, & Nichols, 1980, Kieran, 1981; Matthews et al., 2012; McNeil & Alibali, 2005a; Molina, Castro, & Castro, 2009, (b) their ability to apply fundamental number properties (Pillay, Wilss & Boulton-Lewis, 1998; Steffe, 2001); and (c) their ability to be flexible in their strategy choice and in their repertoire of strategies for solving problems (Empson, 1999; Star & Newton, 2009).

The first and arguably the most important of the theoretical underpinnings of relational thinking is an understanding of equivalence (Byrd et al., 2015; Carpenter et al., 2005; Powell et al., 2016; Stephens et al., 2013). The majority, if not all, of the literature describing relational thinking describes an understanding of equivalence as critical to relational thinking, and many researchers indicate that instruction centred on an understanding of equivalence encourages relational thinking (Alibali et al., 2007; McNeil & Alibali, 2005a; Molina & Ambrose, 2006).

In essence, the equal sign symbol indicates that a relationship exists between the expressions present on either side of the equal sign and that this relationship is to be maintained (Matthews et al., 2012; Stephens et al., 2013). Beyond an understanding of equivalence, to be successful in relational thinking, students are also required to first consider the number sentence as a whole, and then uncover relevant components and numerical relationships that may exist (Molina et al., 2008), while keeping in mind that the equal sign must be respected despite any transformation or substitution that may occur (Jones, Inglis, Gilmore, & Evans, 2013). This entails an understanding of accurate conceptions of the equal sign in both canonical (e.g., 3 + 4 = 7) and non-canonical (e.g., 11 = 2 + 3 + 6) contexts. Because misconceptions about the equal sign are notoriously common (Kieran, 1981; Powell et al., 2016; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011), relational thinking is often compromised.

To illustrate, when asking a student to justify whether an equation such as 19 x 4 = 19 x 2 + 19 + 19 is true or not, he or she must first understand that what is on the left side of the equal sign must be equal to what is on the right of the equal sign. A student who possesses an incorrect "operator view" of the equal sign (Alibali, 1999; Kieran, 1981; McNeil & Alibali, 2004, 2005b; Seo and Ginsburg, 2003) – that is, viewing the symbol as an indication that the answer belongs immediately to the right of it – might instead claim that the expression is incorrect because 19 x 4 does not equal 19, disregarding the rest of the expression to its right (i.e., x 2 + 19 + 19). Misconceptions about the equal sign are prevalent: Most children in primary school do not have a complete understanding of equivalence, as indicated by their inability to solve basic non-canonical equivalence problems such as 5 + 3 + 4 = __ + 5 (Baroody & Ginsburg, 1983; Perry, 1985).

Indeed, an understanding of equivalence is important in relational thinking. In fact, many researchers define relational thinking as the ability to view the equal sign as an indicator for a balanced relationship between the two expressions on either side of the symbol (Alibali et al., 2007; McNeil & Alibali, 2005; Molina & Ambrose, 2006). However, others argue that relational thinking involves not only an understanding of equivalence, but also an understanding of fundamental number properties, as well as an ability to be flexible in one's choice of strategy, among other components (Carpenter et al., 2005; Irwin & Britt, 2005; MacGregor & Stacey, 1999; Stephens & Ribeiro, 2012).

As such, beyond an understanding of equivalence, much of the literature describes a second theoretical component of relational thinking: understanding of number properties (Carpenter et al., 2003, 2005; Irwin & Britt, 2005; MacGregor & Stacey, 1999; Stephens & Ribeiro, 2012). These include the commutative property ($a + b = b + a$ or $a \text{ x } b = b \text{ x } a$); the associative property ($(a + b) + c = a + (b + c)$ and $(a \text{ x } b) \text{ x } c = a \text{ x } (b \text{ x } c)$); the distributive property ($a \text{ x } (b + c) = a \text{ x } b + a \text{ x } c$); the identity property ($a + 0 = a$; $a \text{ x } 1 = a$); the inverse property ($a + (-a) = 0$ and $a \text{ x } 1/a = 1$); and the zero property ($a \text{ x } 0 = 0$). Student success in relational thinking is indeed dependent on one's understanding, whether implicit or explicit, of the above mentioned fundamental properties (Carpenter et al., 2005; Empson et al., 2011; Jacobs et al., 2007). To illustrate using the same equation above, a student could transform the expression as follows, using the distributive property in the second step:

$$19 \text{ x } 4 = 19 \text{ x } 2 + 19 + 19$$

$$19 \text{ x } (2 + 2) = 19 \text{ x } 2 + 19 \text{ x } 2,$$

thereby concluding that the expression is true.

The third theoretical component of relational thinking is flexibility (Carpenter et al., 2005; Irwin & Britt, 2005; Stephens & Ribeiro, 2012). An ability to be flexible with numbers is defined as choosing or generating an appropriate strategy to solve a given problem, and applying knowledge about numbers and properties in different ways depending on the context (Proulx, 2013; Star & Newton, 2009; Threlfall, 2002). Such flexibility requires an understanding of how to transform, partition, and substitute numbers (i.e., splitting numbers into more manageable or smaller units to make reasoning easier), as well as re-ordering (e.g., 3 + 5+ 460 = 460 + 3 + 5) and decomposing numbers (e.g., 349 = 300 + 40 + 9) (Britt & Irwin, 2011). Again in the context of the previously mentioned example (i.e., 19 x 4), a student who is flexible in his or her approach is able to transform the expression in multiple ways.

For example, a student may transform as previously indicated above to determine whether the expression is correct:

$$19 \times 4 = 19 \times 2 + 19 + 19$$

$$19 \times (2 + 2) = 19 \times 2 + 19 \times 2.$$

Alternatively, a student who is flexible in his or her knowledge might instead perform the following:

$$19 \times 4 = 19 \times 2 + 19 + 19$$

$$(20 - 1) \times 4 = 20 \times 4 - 1 \times 4$$

and can therefore conclude, without computation, that the expression is true (Jones et al., 2012, 2013).

Mental Mathematics

Mental mathematics is defined as the process of carrying out arithmetical operations in one's head (Maclellan, 2001; Reys, 1984), although this does not exclude the process of recording aspects of mathematical reasoning (Harries & Spooner, 2000). According to Maclellan (2001), however, the key components of mental mathematics is number sense rather than simply applying a learned algorithm to a given mathematics problem (Maclellan, 2001). Mental computation encourages a deeper understanding of number structure and number properties (Reys, 1984; Varol & Farran, 2007), and promotes flexibility as students solve mathematical problems (Varol & Farran, 2007).

The literature has identified a number of theoretical components involved in mental mathematics (Heirdsfield, 2011; Heirdsfield & Cooper, 2002, Lemonidis, 2015; Maclellan, 2001; Reys, 1984; Thompson, 2010). I suggest in this paper that two of these components overlap with the conceptualization of relational thinking as described above. They are knowledge of number properties and flexibility in transforming, compensating, and re-ordering numbers (Heirdsfield,

2011) depending on the context, and in one's approach to solving a mathematical problem (Heirdsfeld & Cooper, 2002; Proulx, 2013; Thompson, 2010, Threlfall, 2002; Verschaffel et al., 2009).

Students who do well at mental computation use, either at an implicit or explicit level, number properties to perform transformations on numbers, creating more manageable and meaningful mental computations (Reys & Barger, 1994; Reys, 1984;Varol & Farran, 2007). Consider the student who is faced with the task of mentally computing 230 x 4. He or she could transform the expression as follows:

$$230 \times 4 = 230 \times (2 \times 2) = (230 \times 2) \times 2,$$

using the property of associativity to double 230 twice, thus making the computation more manageable. For example, flexibility in transforming and reordering numbers also plays a large role in mental computation. For example, a student who is asked to compute 47 + 28 could transform the expression into 40 + 20 + 7 + 3 + 5 or into 50 + 30 – 3 – 2 (Britt & Irwin, 2011; Heirdsfield, 2011; Thompson, 2010). In this way, transforming the expression in such ways creates new ones that are more manageable and often more efficient (Proulx, 2013).

Improving Relational Thinking Through Instruction

There is a growing literature describing the nature of children's relational thinking with a focus on improving students' understanding of equivalence, but few studies exist at the K-12 level that focus on instructional strategies to improve the many other important components of relational thinking such as flexibility, as well as applying knowledge of number properties to reason about number sentences.

One of the few studies that did target relational thinking instruction beyond students' ability to solve equivalence problems was conducted by Jacobs et al. (2007), who delivered a yearlong professional development course on relational thinking to teachers in the first through fifth-grade. The course focused on understanding the equal sign, number relations to simplify calculations, as well as creating and justifying conjectures about fundamental properties of number operations (Jacobs et al., 2007). After having received the professional development, the participating teachers were able to generate more relational thinking strategies to solve a given problem, and their students had a better understanding of relational thinking, as assessed through written assessments and interviews.

In another study, Irwin and Britt (2005) describe the Numeracy Project as an effective curriculum for improving relational thinking in New Zealand. The Numeracy Project is a mathematics curriculum designed for primary school students (ages 5-14) to improve their relational thinking by focusing on flexibility with mental computation on rational numbers. The objective of the curriculum was to allow students to gain a deeper understanding of number operations, and to assist them to be flexible in their mental problem solving execution through mental mathematics instruction (Irwin & Britt, 2005). According to Irwin and Britt (2005), relational thinking is an ability to understand an underlying relationship that exists between numerical expressions, which is to be maintained despite any transformations that are undertaken. They also contend that students who are able to apply the properties in their computation and be flexible in their approach to solving across various contexts are undertaking relational thinking (Irwin & Britt, 2005).

Students participating in the Numeracy Project were encouraged to formulate and experiment with a variety of mental strategies for the duration of the curriculum, and were instructed on how to solve mathematics problems such as 27 + 15 or 34 + 19, using other means than computation or paper-and-pencil algorithms. For example, the curriculum reinforced how to transform expressions such as 27 + 15 into 30 + 12 to make the mental computation less arduous (Britt & Irwin, 2011). Strategies not limited to "compensating, factorising, and maintaining equivalence" (Irwin & Britt, 2005, p. 170) were reinforced throughout the curriculum, and students were exposed to applying the above and similar strategies before becoming acquainted with any algebraic symbols.

When comparing assessments of 12 year old students who participated in the Numeracy Project for at least one year to those who did not, students who participated in the Numeracy Project used strategies that were congruent with Irwin and Britt's (2005) view of relational thinking more often than students who had not participated in the project. The assessment evaluated compensation in addition (e.g., 47 + 25 = 50 + 22), compensation in subtraction (e.g., 87 − 48 = 89 – 50), compensation in the distributive law of multiplication over addition (e.g., 3 x 88 = 3 x 90 – 6), equivalence with sums and differences (e.g., __ + 26 = 431), compensation in multiplication (e.g., 5 x 18 = 10 x __), and equivalence with fractional values (e.g., 3/4 = 15/__) (Irwin & Britt, 2005). These results suggest, in line with my hypothesis, that improvements in mental mathematics may be related to students' relational thinking. Because Irwin and Britt's methodology did not allow them to explain the students' improvement other than that they participated in the Numeracy Project, further research is required to investigate the mechanisms that may have been responsible for the increases in relational thinking.

In possibly the most relevant study for my purposes, Osana, Proulx, Adrien, & Nadeau (2013) carried out a mental mathematics unit in a university-level elementary mathematics methods course for preservice teachers that consisted of mental mathematics activities, discussion, and practice problems. The authors administered a written test of relational thinking before and after the three-week mental mathematics unit. The authors' hypothesis was that the students' relational thinking scores would improve after having engaged in the mental mathematics activities. Indeed, their predictions were borne out: The students' relational thinking improved, but again, methodological weaknesses in their study, such as the absence of a comparison group, prevented them from explaining their results. In addition, the authors' study only involved university level students, and as such, the generalizability of their findings to seventh-graders is tentative at best.

Present Study

The present study investigated the impact of a mental mathematics intervention on seventh-graders' relational thinking and their understanding of the equal sign. This study was conducted because the literature assessing the effects of a mathematical intervention as a means to improve relational thinking is lacking and because studies describing the relationship between mental mathematics and relational thinking are virtually non-existent, despite the theoretical consistencies across both domains. Using two seventh-grade classes, students were assessed at three time points (Time 1, Time 2, and Time 3) on their (a) ability to solve equivalence problems, and (b) reasoning abilities about true-false number sentences. One class received a mental mathematics intervention between Time 1 and Time 2, and the second class received the same intervention between Time 2 and Time 3. Because relational thinking and mental mathematics share common attributes, such as an understanding of equivalence, number properties and an ability to be flexible in one's selection of computational strategies, I predicted that following the mental mathematics intervention in each class, students' relational thinking and understanding of the equal sign would improve. I also predicted that the performance of the students in the first class would be maintained from Time 2 to Time 3.

Method

Participants

Seventh-grade students from two intact classes (24 students from Class 1, 28 students from Class 2) from a suburban public high school in Quebec, Canada were asked to participate in the study. Eight students were excluded: one was transferred to another class, two did not return the consent form, one did not have parental consent, and the remaining four students did not complete all of the assessments. The final sample consisted of 20 students from Class 1 and 24 students from Class 2.

The high school was composed of middle- to high-income families, and was rated on the higher end of the school board's socio-economic index, measured by family income levels and mother's education (Ministère de l'éducation, enseignement supérieur, et recherche, MEESR, 2014). Both classes followed the identical seventh-grade mathematics curriculum, as mandated by the Quebec Education Plan of the Ministère de l'éducation, enseignement supérieur, et recherche (MEESR, 2016). *Canadian Mathematics 7* (Paholek, 1993) was used as the textbook for the delivery of the mathematics curriculum.

Design

The study design is presented in Figure 1. A paper-and-pencil test of prior mathematics knowledge (PK) was first administered to ensure that no differences existed between the two classes. The next day (Time 1) two assessments, the Equivalence Test (ET) and Relational Thinking Test (RTT), were administered to both classes prior to any intervention. The following day, the mental mathematics (MM) intervention began in Class 1. The MM instructional sessions took place in the first 20 minutes of each of 15 mathematics classes over a four-week period. During these four weeks, the students in Class 2 did not receive any MM instructional sessions, but rather received the regular seventh-grade curriculum. The day after the MM intervention was completed in Class 1, isomorphic versions on the ET and RTT assessments were administered at Time 2 to both classes. The following day, the MM intervention began in Class 2 using the same procedures, while Class 1 received the regular curriculum. The day after the second delivery of the intervention, isomorphic versions of the ET and RTT assessments were then administered a third time, at Time 3, to all students in both Classes 1 and 2. Students who were absent completed their assessments immediately upon their return, up to three days after the official assessment day, provided their class intervention had not yet begun.

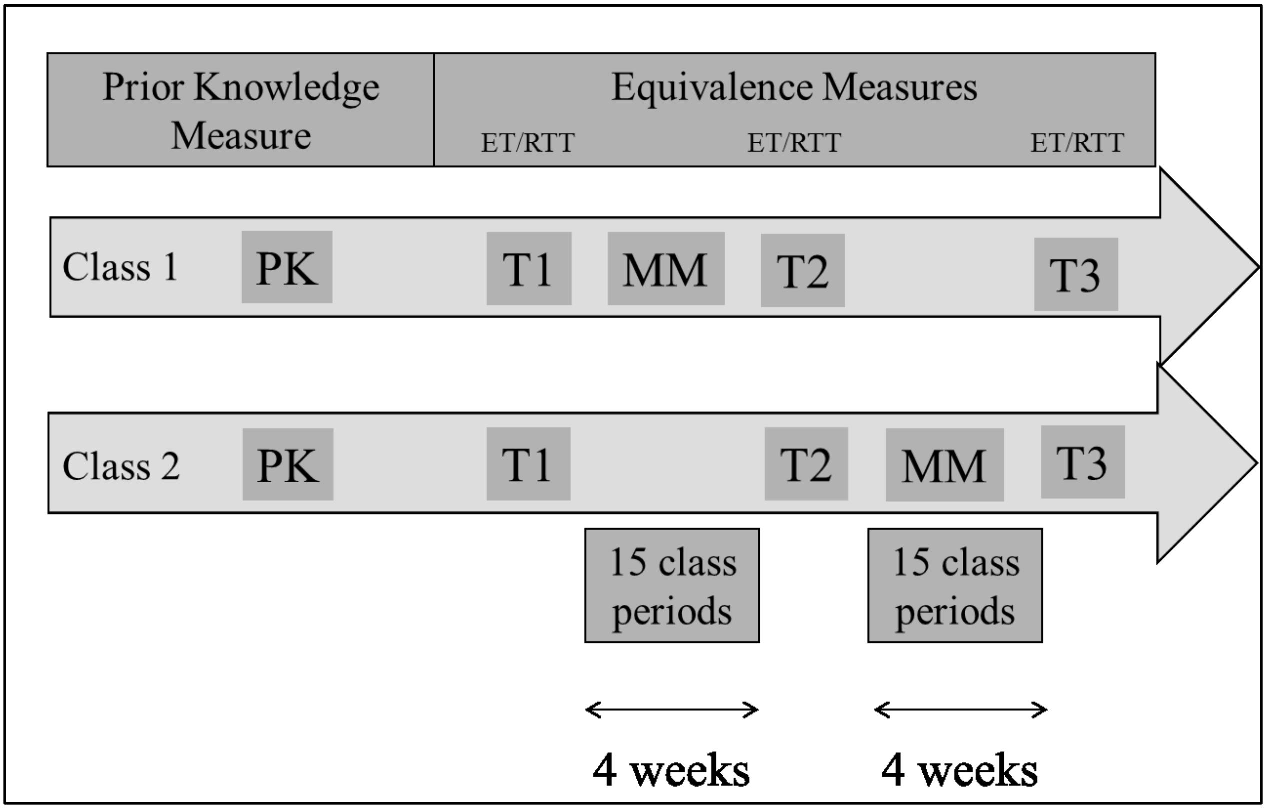


Figure 1. Design of Study 1. T1 = Time 1, T2 = Time 2, T3 = Time 3. PK = Prior Knowledge Test, ET = Equivalence Test; RTT = Relational Thinking Test, MM = Mental Mathematics Intervention.

Mental mathematics intervention. I was the students' regular mathematics teacher, and I delivered the intervention to both Class 1 and 2. At the beginning of each MM instructional session, the students were seated at their own desks with their own individual white board and dry erase marker. No other materials were provided. I began the session by writing a mathematical expression on the white board (e.g., 37 + 58) and then gave the students 30 seconds to compute the answer mentally. The students were instructed to remain silent during the 30 seconds and not to write anything down but their final answer on their white boards. Once the 30-second time period was up, I asked the students to hold up their white boards so that I could view the answers.

After raising their white boards, the students then shared their strategies with their peers as I guided the whole-class discussions. First, I asked a student with an incorrect answer to describe how he or she had reached the answer. I then asked a student who had computed a correct answer to describe his or her strategy. The discussion centered on both the incorrect and correct responses, with students discussing the merits of one strategy relative to another. The discussion for each expression lasted no more than four minutes, during which time the equal sign was never displayed. I then erased the mathematical expression from the board, and the session continued with the next mathematical expression.

During the discussions, I pointed out how the students rearranged, transformed, and substituted numbers to make the computations easier to compute mentally. In addition, I pointed out that certain strategies were particularly suitable for specific operations. For example, I illustrated how dividing large numbers by factors of the divisor made dividing mentally easier, and I also encouraged students to estimate what their answer should be before they used any strategy. Additionally, I highlighted applications of the fundamental properties of arithmetic without directly naming them. For example, while discussing 22 x 6, I underscored how the distributive property was used in one of the student's strategies. Specifically, I explained how the 6 was substituted with 4 + 2, yielding 22 x (4 + 2), which then resulted in the sum of two products, 22 x 4 and 22 x 2, yielding 88 + 44. The latter computation was also discussed; the students were able to transform it into the equivalent expression of 80 + 40 + 8 + 4, ultimately leading to 132 as a final answer.

Before the study began, I created 26 sets of expressions for the mental mathematics intervention, which are presented in Appendix A. Each set contained four expressions, one for each operation. The first set, for example, contained the expressions 62 + 38; 73 – 31; 21 x 9; and 225 ÷ 25. The second set of expressions contained the same four operations, but presented in a different order, namely 77 – 26; 17 x 5; 600 ÷ 4; and 42 + 58. Each subsequent set contained all four operations in a different order from the preceding set. This ensured that the discussion time for each operation over the 15-day intervention was as similar as possible within and across classes.

I began the intervention in each class by starting with Set 1 and continued across the 15 sessions through as many of the 26 sets as possible. In each mental mathematics session, I continued through the sets until 20 minutes were completed. During the next session, I picked up where I had left off in the previous session and again continued through the sets in the specified order. In any given session, between six and seven expressions were computed mentally and discussed. Across the 15-day intervention, 92 mental mathematics expressions were used for Class 1, which included 23 addition problems, 23 subtraction problems, 23 multiplication problems, and 23 division problems. For Class 2, 95 mental mathematics expressions were used, which included 24 addition problems, 24 subtraction problems, 24 division problems, and 23 multiplication problems.

Assessment measures. The study consisted of three assessments: the Prior Knowledge test (PK), the Equivalence Test (ET), and the Relational Thinking Test (RTT). The PK test was designed to assess the students' procedural knowledge in arithmetic, calculating with exponents, converting decimals to fractions, and ordering rational numbers. The ET was designed to measure the students' understanding of the equal sign (based on Watchorn & Bisanz, 2005), and the RTT was designed to assess students' ability to determine the truth value of equations using relational thinking.

At the start of each day of testing (Day 1: PK; Day 2: ET and RTT), I indicated to the students that they would be completing tests. I told them that the assessment(s) would not count in terms of their mathematics course grade, but that I would be very happy if they could complete the tests to the best of their ability and to take the exercise seriously. No feedback or clarification was provided to any student at any time during any of the assessments, and the students completed the tests independently.

Prior Knowledge test (PK). The PK was a paper-and-pencil multiple-choice measure consisting of 16 items, presented in Appendix B, and was used to assess the students' knowledge of the sixth-grade mathematics curriculum. The assessment consisted of procedural knowledge items because they were similar to the types of mental computation skills required of the students during the intervention. Students were required to circle their response from a list of four choices.

At the start of the PK test, I delivered a paper copy to each student, placing it face down on his or her desk. Once every student had a test, the students were asked to turn them over, and I stated that the class had 30 minutes to complete the PK. The students were permitted to use the margins for computations and other written work, but I told them that only their multiple-choice selections would be graded. I also instructed the students to look over their answers once they were finished, and to remain silent for the duration of the assessment. I indicated that calculators were not permitted, and that any and all calculations were to be done on the test paper. After the 30 minutes were completed, students were asked to turn over their papers, and I collected them.

Correct answers received 1 point and incorrect answers 0 points. The points were summed to obtain a total PK score, which was out of a possible 16 points. The scores were then converted to percent.

Equivalence Test (ET). A paper-and-pencil equivalence test (created by Watchorn & Bisanz, 2005) was administered to both classes at three time points. The test consisted of 29 problems, including 9 canonical and 20 non-canonical problems, and each contained only single-digit numbers. Examples of problem types were: 7 + 8 = 6 + __, 7 + 3 = 7 + __, 4 + 7 = 7 + __. Three isomorphic versions of the ET, presented in Appendix C, were used for counterbalancing purposes. Each version contained the same number of each type of problem (i.e., nine addition, $a + b =$ __; four identity, $a + b = a +$ __; four commutativity, $a + b = b +$ __; four part–whole, $a + b = c +$ __; and eight combination, $a + b + c = a +$ __; Watchorn & Bisanz, 2005). The numbers in the problems and the order of the problem types varied in each version.

At the start of testing, I delivered the ET to each student, placing it face down on his or her desk. Once every student had a test, I went through the instructions orally in front on the class. Specifically, I stated that the class had 15 minutes to complete the ET, and that they were to complete the assessment by writing down their answer on a blank line provided in each equation. I also instructed the students to look over their answers once they were finished, and to remain silent for the duration of the assessment. After the 15 minutes were completed, students were asked to turn over their papers, and I collected them.

Only the responses for the 20 non-canonical problems were used in the analysis. Correct answers received 1 point and incorrect answers 0 points. The points were summed to obtain a total ET score, which was then divided by 20 for a mean score between 0 and 1.

Relational Thinking Test (RTT). A relational thinking test, based on Osana et al. (2013) and Carpenter et al. (2003), was administered to both classes at three time points. There were four items on the test, one for each operation (addition, subtraction, multiplication, and division). Each item consisted of a number sentence, such as 228 ÷ 6 = 456 ÷ 12, and the students were asked to indicate whether the sentence was true or false by circling the word "true" or "false" on the test paper. The students were then asked to provide a written justification for their responses in a blank space provided on the test. Examples of RTT items were: 65 + 36 = 67 + 38, 105 - 45 = 106 – 46, 228 ÷ 6 = 456 ÷ 12, and 29 x 52 = 28 x 53.

Three isomorphic versions of this assessment, presented in Appendix D, were administered for the repeated administrations across the study. In each version, all four operations were used, but the order of the items varied across versions. The numbers used in each item were also different, but the structure of the numerical relationships across versions remained the same for each operation (e.g., Version 1: 45 + 26 = 43 + 24; Version 2: 55 + 36 = 53 + 34; Version 3: 67 + 48 = 65 + 46).

Immediately following the collection of the ET, I delivered the RTT to each student. The students were given 20 minutes to complete the assessment. Again, I reviewed the instructions orally with the class. I told the students to indicate in each question if the number sentence was true or false and to justify their answers in the space provided. The students were also told that they were not permitted to communicate during the test. After 20 minutes, students were asked to turn over their tests, and I collected them from each student.

Students' written justifications were coded using the following rubric (created by Osana et al., 2013): (a) Category 1: Relational thinking without computation or with computation only as a means to justify a written relational response; (b) Category 2: Relational thinking with Computation; (c) Category 3: Other. Examples of student responses belonging in each Category of the rubric can be found in Figure 2. Responses that were placed in Category 1 demonstrated that the student engaged in relational thinking by considering the relationship between the numbers without computing the quantities on both sides of the equal sign to determine the truth value of the equation. Computation in this category was permitted only if the student had first justified the response relationally and only if the computation was used to support or illustrate the relational response. Student responses that were placed in Category 2 demonstrated that the student had an understanding of the equal sign and that they were able to determine if the response was true or false with the use of computation only. Category 3 responses were those where the student either had an operator view of the equal sign, did not supply any justification, or provided responses that were impossible to interpret.

Category 1 responses received 2 points, Category 2 responses received 1 point, and Category 3 responses received 0 points. Category 1 responses were awarded more points than Category 2 and 3 responses because they indicated that the students responded relationally and did not need to compute to arrive at their answer. Category 2 responses were also considered to be relational, as students appeared to understand the equal sign, but chose to compute rather than to consider the relationships between the numbers to arrive at their answer. Category 3 responses received 0 points because they contained no evidence of relational thinking or an understanding of the equal sign.

Student scores were the mean number of points assigned to each question, with a minimum score of 0 and a maximum score of 2. A random sample of 20% of the responses was coded by a second rater, and inter-rater reliability of 90% agreement was achieved.

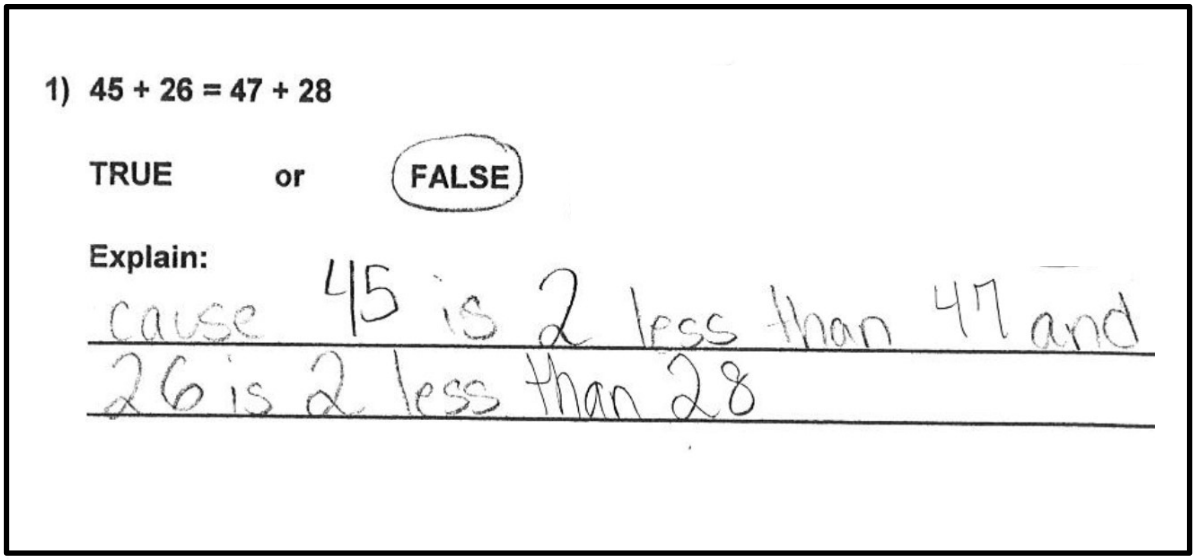


Figure 2a: Category 1: Relational thinking without computation or with computation only as a means to justify a written relational response (2 points)

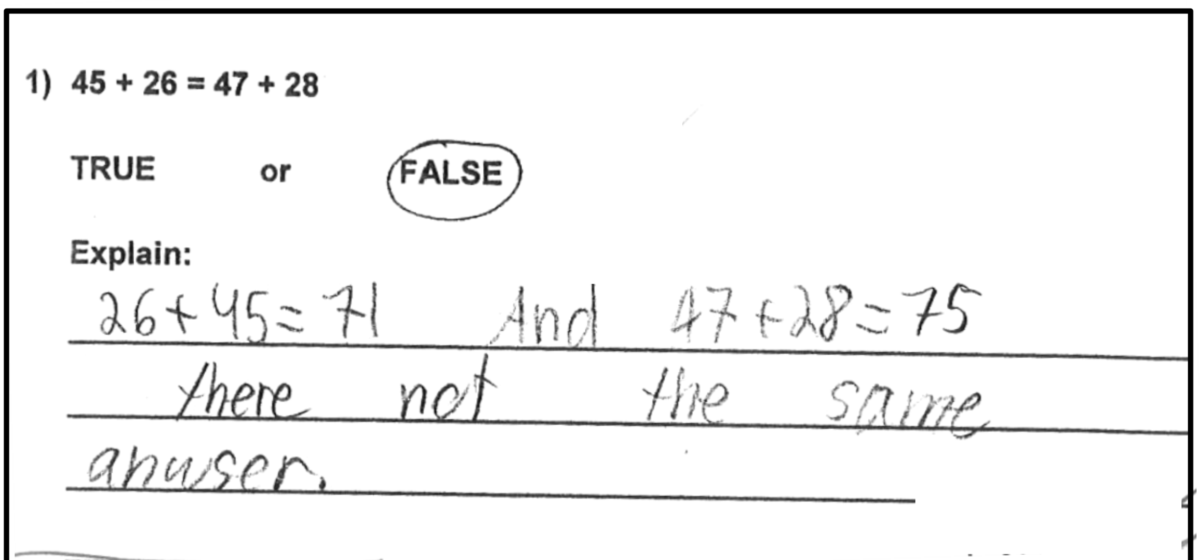


Figure 2b: Relational Thinking with Computation (1 point)

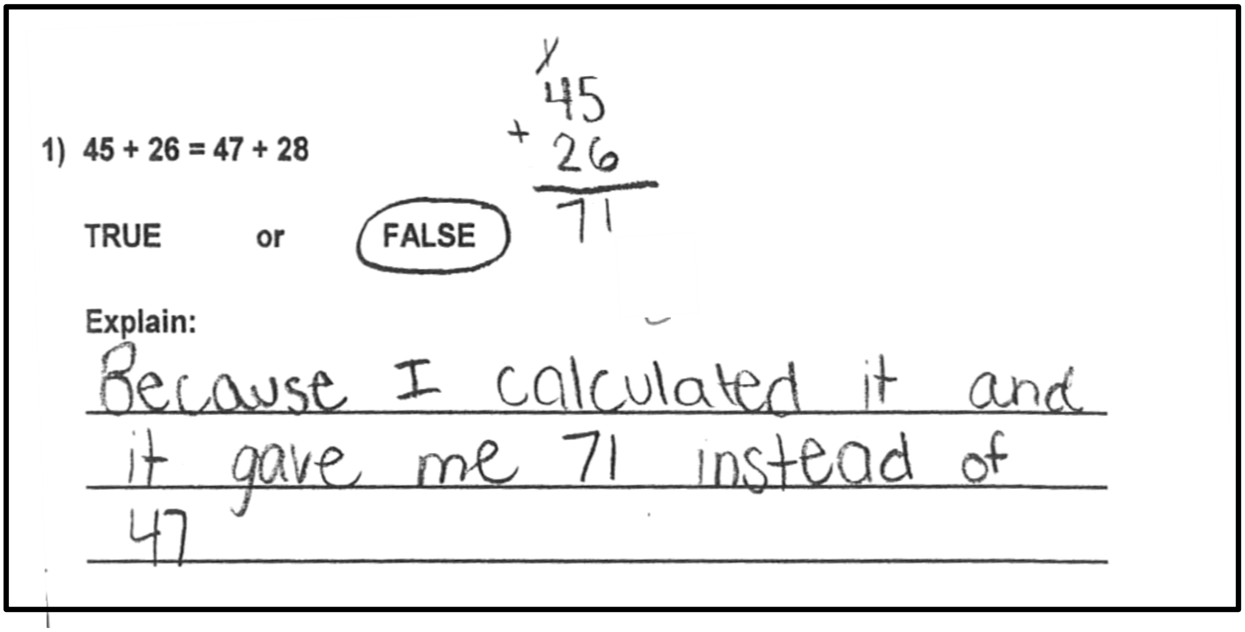


Figure 2c: Other (operational, no sense, uninterpretable) (0 points)

Results

Descriptive Statistics

The means and standard deviations of the Prior Knowledge test (PK), the Equivalence Test (ET), and the Relational Thinking Test (RTT) scores as a function of class and time are presented in Table 1.

Table 1

Means and (Standard Deviations) of Prior Knowledge Test (PK), Equivalence Test (ET), and Relational Thinking Test (RTT) as a Function of Class and Time

	T1			T2		T3	
	PK	ET	RTT	ET	RTT	ET	RTT
Class 1[a]	42.35	58.26	1.03	95.43	1.59	93.04	1.39
	(22.56)	(39.53)	(.40)	(9.09)	(.44)	(19.05)	(.45)
Class 2[b]	38.33	61.41	.91	75.91	.99	94.57	1.17
	(16.45)	(37.85)	(.47)	(34.90)	(.40)	(17.59)	(.42)

Note. All PK and ET measures scores are reported in percent. T1 = Time 1, T2 = Time 2, T3 = Time 3. RTT measures scores are minimum = 0 and maximum = 2. [a]N = 20. [b]N = 24.

The patterns of the means for each class at each time point were in the expected direction. Specifically, at Time 1, both classes performed similarly on the tests of Prior Knowledge, Equivalence, and Relational Thinking. Following each class's respective MM intervention, the ET and RTT mean scores increased, with Class 1 outperforming Class 2 at Time 2 on both the ET and RTT. At Time 3, after Class 2 had received the intervention, the ET and RTT mean scores for both classes were comparable.

Correlations between assessment measures (PK, ET, RTT). The correlation coefficients between the PK, ET, and RTT at the three time points for each class are reported in Table 2. The data show that for Class 1, the PK scores were correlated with RTT at Time 1, Time

2, and Time 3. In addition, the ET and RTT scores at Time 2 were significantly correlated. For Class 2, the PK correlated with no other assessment, but there was a correlation between the ET at Time 2 and the RTT at Time 1. Moreover, the ET at Time 3 was correlated with the RTT at all three time points. Together, these correlations provide evidence of the construct validity of the measures.

The correlations allowed for me to determine which baseline measures could serve as suitable covariates in subsequent analyses. Specifically, although the PK scores were correlated with RTT at all time points for Class 1, it was not correlated with the ET or RTT measures at any time point in Class 2. With respect to RTT, the scores in Class 1 were correlated between Time 1 and Time 2, and between Time 2 and Time 3 for Class 2. Finally, the ET mean scores at Time 1 for Class 1 were not correlated with their scores at Time 2. Similarly for Class 2, the ET scores at neither Time 1 nor Time 2 were correlated with the Time 3 ET scores. As such, RTT was the only suitable baseline measure that could serve as a suitable covariate in subsequent analyses.

Table 2

Class 1 and Class 2 Intercorrelations Among the Prior Knowledge Test (PK), the Equivalence Test (ET), and the Relational Thinking Test (RTT) at the three time points

	PK	ET T1	ET T2	ET T3	RTT T1	RTT T2	RTT T3
PK	--	.32	.18	.11	.45*	.48*	.58**
ET T1	-.12	--	.25	.30	.00	.00	.25
ET T2	-.20	.78**	--	.87**	.13	.48*	.36
ET T3	.20	.27	.39	--	.00	.29	.14
RTT T1	.05	.59	.49*	.41*	--	.52*	.41
RTT T2	-.21	.29	.45	.49*	.13	--	.66**
RTT T3	.17	.13	.35	.54**	.25	.76**	--

Note. Class 1 ($N = 20$) correlations are located above the diagonal, and Class 2 ($N = 24$) correlations are located below the diagonal. T1 = Time 1, T2 = Time 2, T3 = Time 3. ∗ indicates significance at the $p < .05$ level; ∗∗indicates significance at the $p < .01$ level.

Prior Knowledge test. A *t*-test revealed no difference in prior knowledge between Class 1 and Class 2 at Time 1, $t(42) = 0.68$, $p = .50$.

Performance as a Function of the MM Intervention

Equivalence Test. To test the hypothesis that students' understanding of the equal sign would improve after the MM intervention, a 3 time (Time 1, Time 2, Time 3) x 2 class (Class 1, Class 2) mixed ANOVA was performed, using class as the between groups factor and time as the within factor. Alpha was set at .05. A graphical representation of the ET means is presented in Figure 3.

The ANOVA revealed a main effect of time, $F(2, 84) = 26.9$, $p < .001$, $\eta^2 = .39$. Three post hoc tests were conducted to test for differences across both classes between Time 1 and Time 2, Time 2 and Time 3, and Time 1 and Time 3. For each of these tests, I used an alpha of .05/3 (= .0167). The only difference found was between Time 1 ($M = 59.8$, $SD = 38.5$) and Time 2 ($M = 85.7$, $SD = 26.5$), $p < .0167$, $d = 0.80$.

A significant time x class interaction was also found, $F(2, 84) = 3.43$, $p = .037$, $\eta^2 = .08$. Tests of simple effects with Bonferroni corrections revealed that there was no difference in mean ET scores between Class 1 and Class 2 at Time 1 ($p > .05$). For Class 1, the ET scores were higher at Time 2 compared to Time 1 ($p < .001$, $d = 1.3$), and this improvement was maintained from Time 2 to Time 3 ($p > .05$). The tests also revealed that at Time 2, Class 1 outperformed Class 2 ($p < .05$, $d = .74$). Further, Class 2 improved from Time 2 to Time 3 ($p < .01$, $d = .68$), but no improvements were observed between Time 1 and Time 2 ($p > .05$), providing some evidence that the improvement in Class 1 between Time 1 and Time 2 was likely not due to maturation. At Time 3, there was no difference in scores between Class 1 and Class 2 ($p > .05$), suggesting that the students in Class 2 caught up to their peers in Class 1 after having received the MM intervention. In sum, these results indicate that the scores on the ET for each class improved immediately following their respective MM intervention and were maintained four weeks later in Class 1.

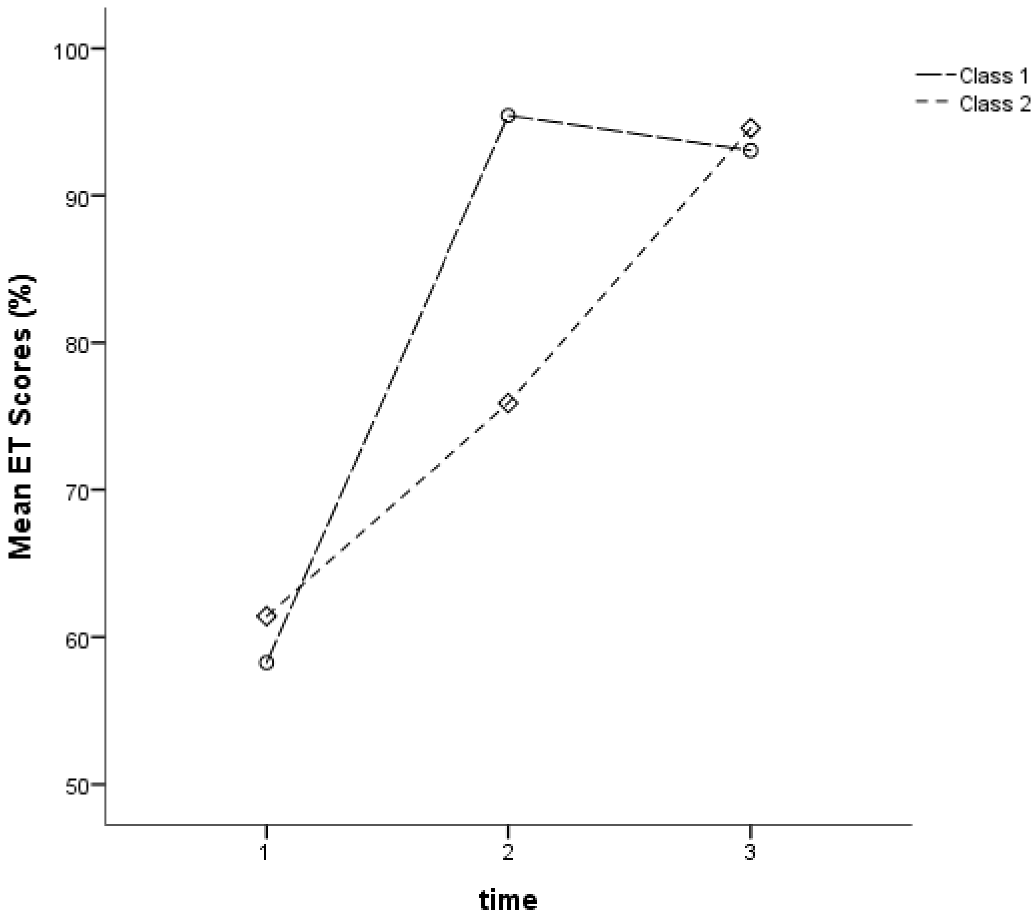


Figure 3. Mean Equivalence Test scores as a function of Time and Class. Means are reported in percent.

Relational Thinking Test. A t-test revealed no difference on the RTT between Class 1 and Class 2 at Time 1, $t(42) = 0.89$, $p = .38$. Although the RTT was the only suitable baseline covariate (see correlations section above), we did not use it as such because there was no difference on this measure between the two classes at Time 1.

To verify the hypothesis that students' relational thinking would improve after the MM intervention, a 3 time (Time 1, Time 2, Time 3) x 2 class (Class 1, Class 2) mixed ANOVA was again performed using the RTT as the dependent measure, with class as the between groups factor and time as the within groups factor. A graphical representation of the RTT means is presented in Figure 4.

The ANOVA revealed a main effect of time, $F(2, 84) = 14.22$, $p = .001$, $\eta^2 = .25$. To test for mean differences between each pair of time points across both classes, an alpha of 0.167 was again used for each pairwise comparison. There was a difference in mean RTT scores between Time 1 ($M = .96$, $SD = .44$) and Time 2 ($M = 1.26$, $SD = .51$), $p < .0167$, $d = 0.63$. No other pairwise differences were found.

A significant time x class interaction was also found, $F(2, 84) = 6.75$, $p = .002$, $\eta^2 = .14$. Follow-up tests of simple effects with Bonferroni corrections indicated that for Class 1, the RTT scores were higher at Time 2 compared to Time 1 ($p < .001$, $d = 1.34$), and at Time 2, Class 1 outperformed Class 2 ($p < .05$, $d = 1.46$). No improvements were observed for Class 2 between Time 1 and Time 2 ($p > .05$), but students in Class 2 did improve from Time 2 to Time 3 ($p < .05$, $d = .45$). At Time 3, the RTT scores for Class 1 decreased from those at Time 2 ($p < .05$, $d = .45$) to the point where there was no difference between the two classes ($p > .05$). These results suggest that the scores on the RTT improved immediately following the mental mathematics intervention for each class, but no maintenance was observed on the RTT for Class 1.

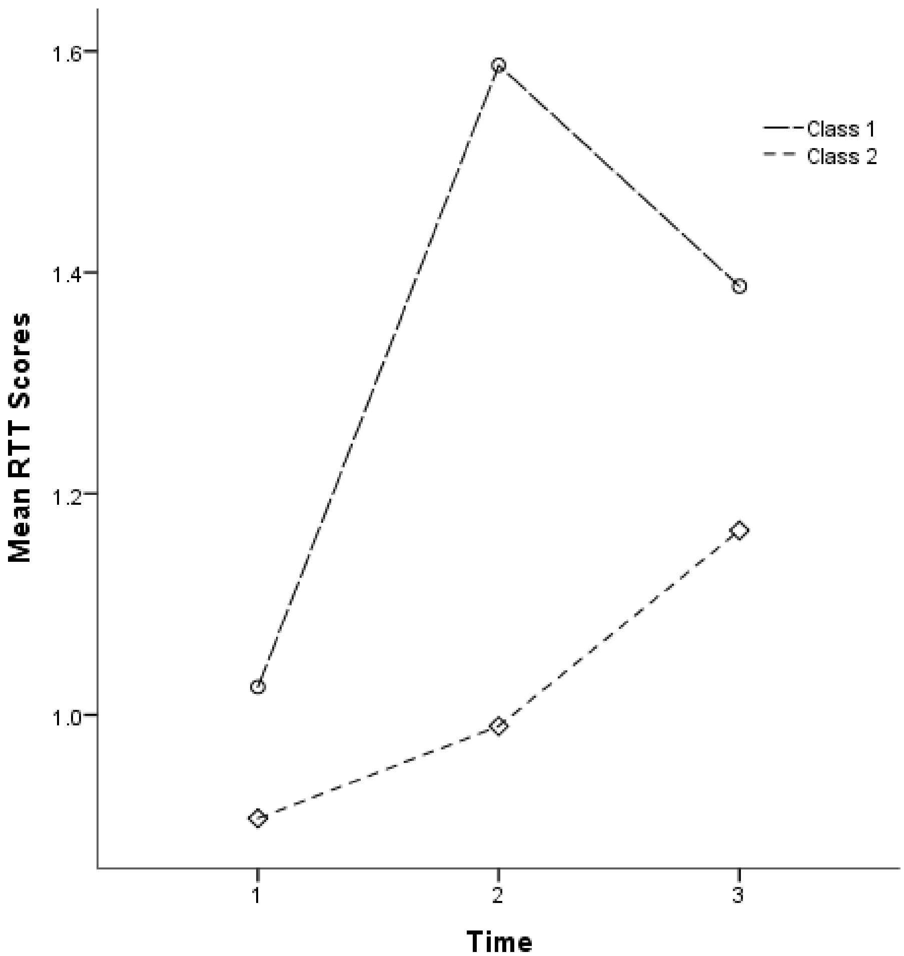


Figure 4. Mean Relational Thinking Test scores as a function of Time and Class. Min = 0, Max = 2.

Discussion

The objective of the present study was to investigate whether seventh-graders' relational thinking improved following a mental mathematics intervention. Using two intact seventh-grade classes, students were assessed at three time points on their ability to solve equivalence problems and on their reasoning about true-false number sentences. A first class (Class 1) received a mental mathematics intervention, and following its conclusion, the same mental mathematics intervention was delivered to a second class (Class 2). In line with my predictions, the results revealed that, immediately following the intervention in each class, the students' understanding of the equal sign and their relational thinking improved. Moreover, for Class 1, the level of performance on equivalence problems persisted for four weeks after the intervention ended, but, contrary to my expectations, the students' relational thinking was not maintained.

The immediate improvement in students' understanding of the equal sign and relational thinking is especially important as researchers have previously argued that students who struggle to understand the equal sign appear to engage in meaningless computations instead of reflecting on the relationships that exist between numbers (Jacobs et al., 2007; Kieran, 1992). As such, without viewing the equal sign symbol as a relational one, further relational thinking can be impeded. I speculate that the students in my study improved their understanding of the equal sign because they were encouraged to create equivalent and more manageable expressions when attempting to solve mental mathematics problems. For example, without the explicit use of the equal sign, students could transform 49 + 51 into the equivalent expression 50 + 50, thereby making the expression more manageable to solve mentally. Despite the fact that the equal sign was not demonstrated while such mathematical transformations were shared and discussed during the intervention, the equal sign is implicit in this transformation, as the two expressions are equivalent and each can be substituted for the other (Jones et al., 2013).

Because students' scores on the relational thinking assessment improved immediately following the mental mathematics intervention, I speculate that this improvement can be explained by the theoretical similarities between mental mathematics and relational thinking. As indicated previously, facility in both domains are contingent on a solid understanding of number properties and being flexible in one's knowledge and selection of problem solving strategies (Heirdsfield, 2011; Heirdsfield & Cooper, 2002; Thompson, 2010, Threlfall, 2002; Verschaffel et al., 2009). During the course of the mental mathematics intervention, students engaged in whole-class discussions, guided by me, in which they shared the strategies they used to mentally solve a given problem. By explaining their strategies and being exposed to different approaches strategies adopted by their peers, applications of the fundamental properties of arithmetic were made visible. Moreover, the wide variety of strategies that were presented and discussed during the mental mathematics intervention encouraged students to become more flexible in selecting their own strategies for a given mathematical expression. As indicated previously, students improved immediately on both the equivalence and relational thinking assessments, and the results for the Equivalence Test were maintained for Class 1. In light of this improvement, I speculate a possible causal link between mental mathematics and relational thinking. Future studies are indeed required to investigate the connection is indeed causal.

Unlike students' understanding of the equal sign, which persisted four weeks after the intervention concluded in Class 1, students' relational thinking in Class 1 was not maintained over time. I speculate that long-term relational thinking may have been hindered because the students no longer received any mental mathematics instruction between their second relational thinking assessment (immediately following the mental mathematics intervention) and their third relational thinking assessment (four weeks after the conclusion of the mental mathematics intervention). The lack of maintenance in their scores is likely due to the complex nature of relational thinking relative to what is involved in solving mathematical equivalence problems; it requires that students examine a given numerical expression as a unit, and then evaluate the mathematical structure and corresponding components to arrive at a solution (Molina et al., 2008). Relational thinking relies on such strategies as transforming, partitioning, and re-arranging numbers, and an ability to be flexible in a variety of contexts (Carpenter et al., 2005; Empson et al., 2011; Stephens & Ribeiro, 2012). Therefore, in my view, it is not entirely surprising that relational thinking was not maintained four weeks after the intervention, but future research is needed that would investigate promising ways to achieve this goal.

Limitations

Although this study provides support for the relationship between mental mathematics and relational thinking, there are limitations to the current study. For example, the students that participated in the study were from two intact classes, as random assignment was not feasible. Therefore, it is possible that the results obtained in this study are a result of factors other than the mental mathematics intervention. It is possible, for example, that the two classes may have had pre-existing differences that were not accounted for, such as private mathematics tutoring or previous exposure to algebra, which may have confounded the results. Moreover, there may have existed differences in prior knowledge such as students' understanding of the equal sign, and previous exposure to number properties, which may have not been assessed by the Prior Knowledge assessment.

To mitigate this weakness, I designed the study so that each of the two classes had their interventions delivered using a staggered treatment design (i.e., the first class received the intervention while the second class did not, followed by the delivery of the intervention to the second class). This staggered design was intended to moderate the effects of the lack of random assignment by providing some evidence that any observed improvements were likely not a result of maturation.

Moreover, I was the students' regular mathematics teacher, and I was responsible for the delivery of all of the mental mathematics sessions in both classes. Despite my best efforts to deliver equally effective instruction in both classes regardless of content (i.e., the mental mathematics intervention or the regular mathematics curriculum), it is possible that experimenter bias may have occurred. For future studies, an observer who is blind to the study's hypotheses could verify, based on predetermined criteria, that my instructional practices were equivalent in all groups.

Further to this, the study took place in a public suburban high school with a student population from middle- to high-income families. As such, it is possible that these results are not generalizable to other high schools or other populations, as it is conceivable that students from different or less affluent populations may respond differently to the intervention.

In the absence of a true experimental design, it would be useful in future studies to assess students' mental mathematics abilities before and after the intervention, not only as an index of treatment integrity, but also to verify that the improvements in mental mathematics ability corresponded in time to the improvements on the Equivalence and Relational Thinking assessments. In accordance with my hypothesis, I would predict that patterns of performance on a mental mathematics assessment would mirror performance on outcome measures of equivalence and relational thinking.

Instructional Implications

Educators can take from this study that mental mathematics may serve as an effective way to enhance relational thinking in middle school students. As my results suggest, students' abilities in relational thinking improved following a mental mathematics intervention of just 20 minutes per day over a 15 day period. Therefore, to the extent that the results allow for pedagogical prescriptions, teachers who aim to improve students' understanding of the equal sign and relational thinking should aim to incorporate mental mathematics instruction into their daily mathematics routine.

To facilitate student improvement in relational thinking, however, the results of my study also imply that teachers should have a solid foundation of number properties themselves, and be able to teach mathematical equivalence in a meaningful way and reflect on numerical relationships (Jacobs et al., 2007). From my perspective, educators also need to understand the similarities between mental mathematics and relational thinking, as described in the present chapter, so that the way in which they encourage mental mathematics in their classrooms can encourage relational thinking. I argue that once this foundation is established, educators should use similar strategies as those used in the present study to engage their students' mental mathematics activities and to foster their relational thinking. During the intervention delivered in this study, the types of classroom discussions, and the ways in which I guided these discussions, appeared critical in establishing the links between mental mathematics strategies and the processes required for relational thinking. I was instrumental in underscoring the commonalities in the variety of strategies that were presented and in the implicit use and demonstration of fundamental number properties. I encouraged the creation of equivalent expressions as I directed the whole-class discussion towards transforming, rearranging, and substituting various numbers within a given problem, thereby supporting students' understanding of equivalence and increasing their problem solving flexibility.

Although there are many studies that investigate the nature of students' relational thinking, the results of the present study add to the existing literature as there is paucity of research on how to improve relational thinking in real classroom settings with the use of a clearly prescribed instructional intervention. As far as I am aware, the present study is the first to investigate the link between mental computation and relational thinking, with one notable exception demonstrating a similar correlation with preservice teachers (Osana et al., 2013). In their research, Osana et al. (2013) implemented a mental mathematics unit in a university-level mathematics methods course. Their findings revealed an increase in relational thinking following the mental mathematics activities. Despite their results, however, the study involved only one class, and the intervention was only a third as long as the present study. Moreover, their study focused on university students, while the present study is the first to describe a potentially effective way to improve the relational thinking of seventh-graders by implementing a mental mathematics intervention. Indeed, further research on how to improve relational thinking is needed. Regular mental computation activities in the mathematics classroom could have important implications for students' mathematics achievement thinking beyond the seventh-grade.

Connecting Study 1 to Study 2

Study 1 examined the relational thinking of seventh-graders before and after a 15-day mental mathematics intervention in the context of whole number arithmetic. Students from two seventh-grade classes were assessed at three time points on their ability to solve equivalence problems, and their reasoning abilities about true-false number sentences (i.e., relational thinking assessment). Results indicated that the students' performance improved on both measures after the mental mathematics intervention. Maintenance of performance on the equivalence problems was observed four weeks after the conclusion of the intervention. The results suggested a link between mental mathematics and relational thinking.

Despite the improvement in students' understanding of equivalence and relational thinking, the study design did not include a mental mathematics assessment before and after the mental mathematics intervention. Therefore, no information was available as to whether the students actually improved in their mental mathematics abilities following the intervention. It is possible that the improvement in relational thinking was due to some other reason, such as an increase in conceptual knowledge and not mental computation in and of itself. Study 2 was therefore conducted to address whether students' mental mathematics scores increased following the intervention. If the students' mental mathematics scores increased, this would provide further support for the finding that improvements in mental mathematics are linked to improvements in relational thinking. Moreover, Study 2 was designed with assessments occurring over five time points to determine if relational thinking could be maintained over the long-term.

Chapter 3: Study Two

Exploring the Long-Term Effects of a Mental Mathematics Intervention on Seventh Graders' Understanding of the Equal Sign and Relational Thinking

Paper accepted at the American Educational Research Association annual meeting, New York, NY, 2018

Introduction

It is well documented that algebra often causes a great amount of difficulty for many students among the topics covered in the high school mathematics curriculum (Booth, 1988; Cai & Moyer, 2008; Jacobs et al., 2007; NCTM, 2000). As such, many researchers and educators indicate that instruction for algebra should begin as early as possible (Falkner et al., 1999; NCTM, 2000; Kaput, 1998), beginning with relational thinking, which is thought to provide the foundation for success in algebra at the middle and high school levels (Carpenter et al., 2003; Falker et al., 1999; Jacobs et al., 2007).

Relational Thinking

Relational thinking includes the ability to view a mathematical expression or equation in its entirety rather than in a manner where a prescribed sequence of procedures is to be followed to arrive at an answer (Carpenter et al., 2005; Jacobs et al., 2007; Stephens, 2006). It entails making explicit generalizations that are based on number properties (Jacobs et al., 2007; Stephens & Ribeiro, 2012). It also involves paying attention to patterns and rules in creating mathematical generalizations (Britt & Irwin, 2011; Cooper & Warren, 2011; Mason et al., 2009) and it requires making strategic decisions, including the ability to think before acting. Sometimes, when thinking prior to acting, students can use relational thinking to simplify a calculation before proceeding. Students who are able to view the equal sign as an indication of a relationship between two expressions (Jacobs et al., 2007) are engaging in relational thinking (Carpenter et al., 2003; Jacobs et al., 2007; Molina & Ambrose, 2006).

Relational thinking depends on (a) a solid understanding of mathematical equivalence – that both expressions on either side of the equal sign need to be equivalent (Carpenter et al., 2003; 2005; Stephens, 2006; Knuth et al., 2006); (b) flexible reasoning, which is required to select among a variety of approaches to solve a given problem and to execute a selected approach (Britt & Irwin, 2011; Empson et al., 1999; Star & Newton, 2009); and (c) an awareness of the properties of numbers and the ability to generalize these properties to transform mathematical expressions into new expressions that are easier to work with (Britt & Irwin, 2011; Carpenter et al., 2005).

Despite the importance of relational thinking at the primary level, the literature on teaching for relational thinking is incomplete. In my attempt to find existing research on this topic, I noticed in much of the mathematics education literature that both relational thinking and mental mathematics share several features (Jacobs et al., 2007; Stephens & Ribeiro, 2012; Thompson, 2010; Threlfall, 2002). I describe these commonalities below.

Mental Mathematics

Mental mathematics is defined as the process of carrying out arithmetical operations mentally (Maclellan, 2001; Reys, 1984), without excluding the process of recording aspects of mathematical reasoning on paper or elsewhere (Harries & Spooner, 2000). Mental mathematics has been described as a means to encourage knowledge of number properties (Heirdsfield, 2011; Thompson, 2010). It also supports students in becoming flexible in their reasoning skills as they are required to select from a variety of appropriate strategies to solve a given problem, and as they work within their selected strategy. For example, a student, if asked to solve an expression such as 16 x 4, may choose to initially select a certain strategy to transform the expression to 4 x 4 x 2 x 2. But, if asked to solve 16 x 5, the same student, who is flexible in their strategy selection, could pick a different strategy (e.g., 15 x 5 + 5), rather than reverting to the same strategy that was used to solve 16 x 4. (Heirdsfield, 2011; Heirdsfeld & Cooper, 2002; Proulx, 2013; Thompson, 2010, Threlfall, 2002; Verschaffel et al., 2009). As such, it is apparent that these components required in mental mathematics – namely an understanding of equivalence, number properties, and the ability to have flexible reasoning -- are similar to those in relational thinking.

Improving Relational Thinking Through Instruction

Several studies have investigated primary school students' relational thinking by focusing on improving students' understanding of equivalence (Carpenter et al., 2005; Stephens, 2006). Despite this, studies aimed at investigating instructional strategies beyond an understanding of equivalence – that is, to improve other central components of relational thinking – is incomplete.

These components include teaching students to (a) become aware of the mathematical expression or equation in its entirety instead of executing a prescribed sequence of procedures (Carpenter et al., 2005), (b) apply knowledge of number properties to reason about number sentences, and (c) improve their flexibility in mathematics when choosing an appropriate strategy and when working within a given strategy.

A small number of studies have undertaken the objective to improve relational thinking with a focus beyond the meaning of equivalence. For example, Jacobs et al. (2007) delivered a year-long professional development course on relational thinking for teachers. The course focused on understanding the equal sign, number relations to simplify calculations, as well as creating and justifying conjectures about fundamental properties of number operations (Jacobs et al., 2007). The results indicated that teachers and their students were better able to use relational thinking strategies to solve a given problem after having participated in the professional development. Further to this, Irwin and Britt (2005) conducted research involving a curriculum focused on improving relational thinking through mental mathematics instruction, which focused on teaching students to become flexible with mental computation on rational numbers, an understanding of number operations, and execution of mental problem solving. The results indicated that participating in the curriculum improved relational thinking in students (Irwin & Britt, 2005). In addition, Osana et al. (2013) also carried out a mental mathematics unit in a university-level mathematics methods course for preservice teachers and their results suggested an improvement in relational thinking following the unit. Despite this, locating further literature to evaluate or describe instructional methods to improve relational thinking continues to be difficult.

However, while examining the existing mathematics education research, I remarked that many concepts that were indicated as predictors of success in relational thinking were also predictors of success in mental mathematics. For example, both mental mathematics and relational thinking involve an understanding of number properties and equivalence, and depend on an ability to be flexible in one's reasoning skills in performing transformations, compensations, as well as re-ordering numbers depending on the situation. In addition, students must also become flexible in their method to solving a mathematical problem (Carpenter et al., 2005; Heirdsfield, 2011; Heirdsfield & Cooper, 2002; Molina & Ambrose, 2006; Proulx, 2013; Stephens, 2006; Thompson, 2010; Threlfall, 2002; Verschaffel et al., 2009).

In my previous study (Study 1), students were assessed at three time points on their (a) equivalence problem solving, and (b) ability to reason relationally about true-false number sentences (relational thinking). Results indicated that each class improved on both assessments immediately following the mental mathematics intervention. Students in one of the participating classes were able to maintain their scores on the test of equivalence problems four weeks after the conclusion of the intervention. However, the results of this study only suggest a link between mental mathematics and relational thinking, as random assignment was not possible. In addition, despite the improvement in their understanding of equivalence and relational thinking, the absence of a mental mathematics assessment prior to and following the mental mathematics intervention did not provide information as to whether students improved in their mental mathematics abilities following the intervention. It is possible that the improvement in relational thinking scores was due to some other reason, such as an increase in conceptual knowledge and not mental computation in and of itself. For this reason, Study 2 was conducted to address whether students' mental mathematics scores increased following their mental mathematics intervention, and would allow further support for the finding that improvements in mental mathematics are linked to improvements in relational thinking.

Present Study

The present study investigated the impact of a mental mathematics intervention on seventh-graders' relational thinking. This study was conducted because (a) the literature assessing the effects of a mathematical intervention as a means to improve relational thinking is incomplete, (b) studies describing the relationship between mental mathematics and relational thinking are virtually non-existent, despite the theoretical consistencies across both domains, and (c) the first study I conducted (Study 1) did not include a mental mathematics assessment to provide a treatment integrity assessment for the mental mathematics intervention. This study differs from Study 1 in the following ways. First, it includes a mental mathematics assessment following the mental mathematics intervention to assess whether students' mental mathematics abilities increased after the intervention was delivered. Study 2 also aims to determine whether the improvement in relational thinking can be maintained over a longer time period (12 weeks), compared to Study 1 which was designed to verify maintenance for only four weeks after the intervention. In addition, two classes participated in Study 1, whereas three classes participated in Study 2.

Using three intact seventh-grade classes, students were assessed at five points (Time 1, Time 2, Time 3, Time 4, and Time 5) on their ability to solve mental mathematics problems, solve equivalence problems, and reason about true-false number sentences. The first class received a mental mathematics intervention between Time 1 and Time 2, the second class received the same intervention between Time 2 and Time 3, and the third class received the same intervention between Time 3 and Time 4. Based on the findings of Study 1, and because relational thinking and mental mathematics share common attributes, such as an understanding of number properties and an ability to be flexible in one's selection of computational strategies, it was predicted that following the mental mathematics intervention in each class, students' relational thinking and understanding of the equal sign would improve immediately. Moreover, I predicted that the performance of the students following their respective mental mathematics intervention would increase immediately, and would be maintained at least four weeks following the conclusion of their respective intervention. Moreover, in accordance with the results in Study 1, I predicted maintenance on the students' understanding of the equal sign for at least four weeks.

Method

Participants

Students from three intact seventh-grade classes (27 students from Class 1, 24 students from Class 2, 30 from Class 3) from a suburban public high school in Quebec, Canada were asked to participate in the study. The students had not participated in any previous study conducted by the authors. The final sample consisted of 23 students from Class 1, 15 students from Class 2, and 28 students from Class 3. Of the 15 students who were excluded, two were transferred to another non-participating class, three did not return the consent form, two students did not receive parental consent, and the remaining eight students did not complete all of the assessments and were thus removed from the analyses. All three classes followed the identical grade seven curriculum, and Class 3 was comprised of students admitted in the International Baccalaureate Program following an admissions exam. The International Baccalaureate Program aims to develop inquisitive, well-informed young people who aid in creating a multi-cultural and respectful world (International Baccalaureate Organization, 2017).

The participants were from a high school that was comprised of middle- to high-income families, and the high school was rated on the higher end of the school board's socio-economic index, measured by family income levels and mother's education (Ministère de l'éducation, enseignement supérieur, et recherche, MEESR, 2014). All three classes followed the same seventh-grade mathematics curriculum, as mandated by the Quebec Education Plan of the Ministère de l'éducation, enseignement supérieur, et recherche (MEESR, 2016). *Canadian Mathematics 7* (Paholek, 1993) was the textbook used for delivery of the mathematics curriculum.

Design

Figure 5 depicts the study design. Prior mathematics knowledge was first assessed via a paper-and-pencil test (the Prior Knowledge test), which was administered to confirm that there were no differences in prior knowledge between the three classes. The following day, at Time 1, two assessments, the Equivalence Test (ET) and the Relational Thinking Test (RTT), were administered to all three classes. The next day, a Mental Mathematics Test (MMT) was administered to all three classes, prior to any intervention. Because of an administrative error with the numbers used in two of the RTT problems which was noticed immediately at Time 1, two additional questions were delivered to the students immediately following the Mental Mathematics (MM) assessment. For each of the subsequent time point assessments, five RTT questions were delivered to all students on the first day, and two additional questions were delivered on the second day (i.e., the correction to the administrative error was carried out at all five time points). Students who were absent were permitted to complete their assessments immediately upon their return, up to a maximum of three days after the official assessment day, provided that their class intervention had not yet started. If they returned later, or if their class's intervention had already begun, their assessment scores were excluded from the analysis.

On the subsequent day, the MM intervention began in Class 1. Mental mathematics instruction occurred in the first 20 minutes of each of 15 mathematics classes over a four-week period. During these four weeks, the students in Class 2 and Class 3 did not receive any MM instructional sessions, but instead proceeded with the regular seventh-grade curriculum, which focused on teaching students about the order of operations.

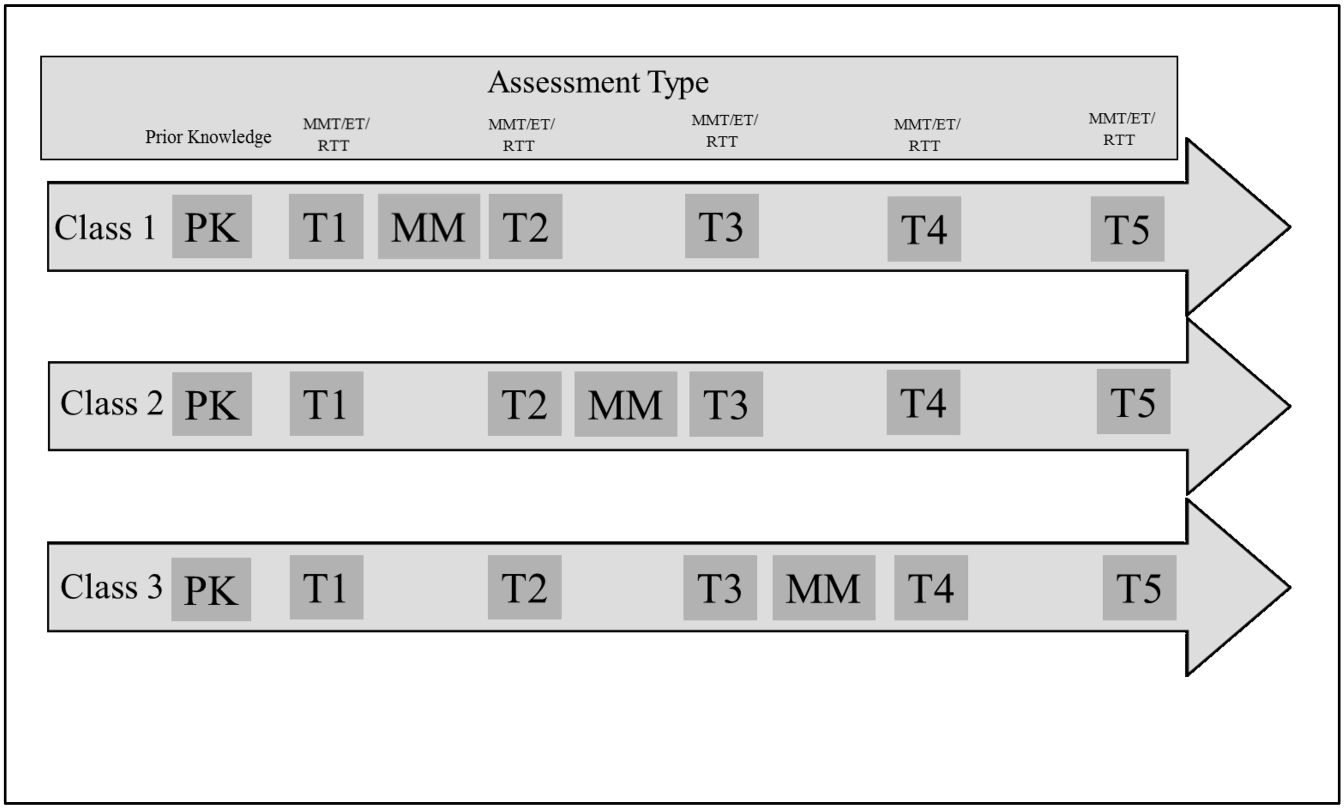


Figure 5: Design of Study 2. T1 = Time 1, T2 = Time 2, T3 = Time 3, T4 = Time 4, T5 = Time 5. PK = Prior Knowledge Test, ET = Equivalence Test; RTT = Relational Thinking Test, MMT = Mental Mathematics Test, MM = Mental Mathematics Intervention.

The day after the MM intervention was completed in Class 1, isomorphic versions of the ET and RTT were administered at Time 2 to all classes, as was the MMT the following day. On the subsequent day, the MM intervention began in Class 2 using the same procedures as in Class 1, while Classes 1 and 3 received the regular curriculum consisting of instruction on the order of operations. The day after the delivery of the intervention to Class 2 concluded, isomorphic versions of the ET and RTT assessments were then administered a third time, at Time 3, to all students in all three classes. The following day, an isomorphic version of the MMT was delivered to all classes. The next day, the MM intervention began in Class 3 using the same procedures, while Class 1 and 2 received the regular curriculum which consisted of instruction on the order of operations. The day after the delivery of the MM intervention to Class 3, isomorphic versions of the ET and RTT assessments were then administered a fourth time, at Time 4, to all students. The following day, an isomorphic version of the MMT was delivered to all classes. Four weeks later, all three assessments were delivered again over a two-day period at Time 5 in all three classes.

Mental mathematics intervention. I was the students' regular mathematics teacher, and I delivered the intervention in all three classes. At the beginning of each MM instructional session, the students were seated at their own desks with their own individual white board and dry erase marker. No other materials were provided. I began the session by writing a mathematical expression on the white board (e.g., 47 + 38) and then allowed the students 30 seconds to mentally compute the answer. The students were asked to stay quiet during the 30 seconds and not to write anything but their final answer on their white boards. Once the 30-second time period concluded, I indicated to the students to hold up their white boards so that I could see their answers.

After raising their white boards, the students then shared their strategies with their peers as I guided the whole-class discussions. The general format of the whole-class discussions was as follows. First, I requested that two or three students with an incorrect answer explain how they had obtained their answers, followed by two or three students with correct answers explaining how they obtained their answer. The discussion was focused on both the incorrect and correct responses, with students debating why one strategy may have worked better than another. For each problem, the discussion lasted at most four minutes, during which time I never displayed the equal sign. I then removed the mathematical expression from the board, and the MM session continued with the next mathematical expression. The discussion time for each arithmetic operation (i.e., addition, subtraction, multiplication, division) over the 15-day intervention was as similar as possible within each class and across all three classes.

During the discussions, I highlighted ways in which the students explained how they manipulated the mathematical expressions. I focused on how they rearranged, transformed, and substituted numerical expressions to create expressions that could be more easily computed mentally. I also indicated that certain strategies might be more appropriate for specific operations. For example, I described how dividing large numbers by factors of the divisor could make the division easier. I focused on how the fundamental properties of arithmetic could be used to compute mentally, without explicitly naming the properties. For example, while discussing 18 + (22 + 37), I described that the associative property could be used. I explained how the expression could be rearranged to (18 + 22) + 37, thus allowing the 18 and 22 to be transformed easily to 40, which then could be more easily added to 37.

Prior to the start of the study, 26 sets of expressions were created for the mental mathematics intervention, presented in Appendix A. Each set consisted of four expressions, one for each operation. The first set, for example, consisted of the expressions 62 + 38; 73 - 31; 21 x 9; and 225 ÷ 5. The second set of expressions contained the same four operations, but was presented in a different order, namely 77 – 26; 17 x 5; 600 ÷ 4; and 42 + 58. Each subsequent set of expressions included all four operations in a different order from the previous set.

On Day 1 of each class' respective MM intervention, I began by starting with Set 1 and continued through as many of the 26 sets as possible over the 15 MM sessions. In each MM session, I continued through the sets until 20 minutes were concluded. For the subsequent session, I continued where I had stopped on the previous session, and continued again through the sets in the given order. In any given session, between six and seven expressions were computed mentally and discussed. Across the 15-day intervention, 93 mental mathematics expressions were used for Class 1, which included 23 addition problems, 22 subtraction problems, 24 multiplication problems, and 24 division problems. For Class 2, 95 mental mathematics expressions were used, which included 23 addition problems, 23 subtraction problems, 24 multiplication problems, and 25 division problems. For Class 3, 94 mental mathematics expressions were used, which included 23 addition problems, 22 subtraction problems, 24 multiplication problems, and 25 division problems.

Assessment measures. The study consisted of four assessments: the Prior Knowledge test (PK), the Equivalence Test (ET), the Relational Thinking Test (RTT), and the Mental Mathematics Test (MMT). The PK test was created to assess the students' procedural knowledge in arithmetic: their skills working with exponents, converting decimals to fractions, and ordering rational numbers. The ET was designed to measure the students' understanding of the equal sign (based on Watchorn & Bisanz, 2005), and the RTT was designed to assess students' ability to determine the truth value of equations using relational thinking. The MMT was designed to assess students' abilities to compute mentally with whole numbers using each of the four arithmetic operations.

At the start of each day of testing (Day 1: PK; Day 2: ET and RTT, Day 3: MMT and RTT), I told each class that they would be completing tests. I indicated that the assessments would not be used for the calculation of their mathematics course grade, but that I would be very happy if the students would take the tests seriously and did the best that they could. No feedback or clarification was given to any student throughout the assessments, and each student was required to complete the tests alone.

Prior Knowledge test (PK). The PK was a paper-and-pencil multiple-choice measure consisting of 16 items, presented in Appendix B, which assessed the students' knowledge of the sixth-grade mathematics curriculum. The assessment consisted of procedural knowledge items, as I believed that these items would indicate whether students entered the study with basic number sense. Students were asked to circle the correct answer from a list of four choices.

At the beginning of the PK test, I placed a paper copy of the assessment face down on each student's desk. Once every student had a test, the students were instructed to turn it over, and I indicated that the class had 30 minutes to complete the PK. The students were allowed to use the margins of the paper for computations and other written work, but I told them that only their multiple-choice selections would be graded. I also encouraged the students to review their responses once they were finished, and to remain silent for the full 30 minute period. I indicated that calculators were not permitted but that any and all calculations were to be done on the test paper. After the 30 minutes were completed, students were asked to turn over their papers, and I collected them.

Correct answers received 1 point and incorrect answers 0 points. The points were summed to obtain a total PK score, which was out of a possible 16 points. The scores were then converted to percent.

Equivalence Test (ET). A paper-and-pencil equivalence test (created by Watchorn & Bisanz, 2005) was administered to both classes at three time points. The test consisted of 29 problems, including 9 canonical and 20 non-canonical problems, and each contained only single-digit numbers. Examples of problem types were: 7 + 8 = 6 + __, 7 + 3 = 7 + __, 4 + 7 = 7 + __. Three isomorphic versions of the ET, presented in Appendix C, were used for counterbalancing purposes. Each version contained the same number of each type of problem (i.e., nine addition, $a + b =$ __; four identity, $a + b = a +$ __; four commutativity, $a + b = b +$ __; four part–whole, $a + b = c +$ __; and eight combination, $a + b + c = a +$ __; Watchorn & Bisanz, 2005). The numbers in the problems and the order of the problem types varied in each version

At the start of testing, I delivered the ET to each student, placing it face down on his or her desk. I stated that the class had 15 minutes to complete the ET, and that they were to complete the assessment by writing down their answer on a blank line provided in each equation. I encouraged the students to review their answers once they were finished, and to stay quiet until the 15 minute period was terminated. After the 15 minutes were completed, students were asked to turn over their papers, and I collected them.

Only the responses for the 20 non-canonical problems were used in the analyses. Correct answers received 1 point and incorrect answers 0 points. The points were summed to obtain a total ET score, which was then divided by 20 for a score between 0 and 1.

Relational Thinking Test (RTT). A relational thinking test, based on Osana et al. (2013) and Carpenter et al. (2003), consisted of five items. As indicated previously, because of an administrative error with two of the RTT problems which was noticed immediately at T1, two additional questions were delivered to the students on the following day immediately following the MMT. For each of the subsequent time point assessments, five RTT questions were delivered on the first day (which included two erroneous questions), and two additional RTT questions were delivered on the second day. The two erroneous questions were omitted, so that only five of the questions were used in the analyses. Each item consisted of a number sentence, such as 228 ÷ 6 = 456 ÷ 12, and the students were required to indicate whether the sentence was true or false by circling the word “true” or “false” on the test paper. The students were then asked to provide a written justification for their responses in a blank space provided on the test. Examples of RTT items were: 45 + 26 = 47 + 28, 105 - 45 = 106 – 46, 228 ÷ 6 = 456 ÷ 12, 29 x 52 = 28 x 53, and (28 x 11) - 28 = 27 x 11.

Five isomorphic versions of this assessment were administered, presented in Appendix D. In each version, the numbers used in each item were different, but the structure of the numerical relationships across versions remained the same for each operation (e.g., Version 1: 67 + 48 = 65 + 46; Version 2: 55 + 36 = 53 + 34; Version 3: 73 + 57 = 71 + 55; Version 4: 84 + 37 = 82 + 35; Version 5: 77 + 49 = 75 + 47).

Once I had collected all of the ET assessments on the first testing day, I then delivered the RTT to each student. The students were given 20 minutes to complete the assessment. On the second testing day, once I had collected all of the MMTs, I then delivered the remaining two RTT questions to each student. The students were given 8 minutes to complete the two RTT questions. I instructed the students to indicate in each question if the number sentence was true or false and to justify their answers in the space provided. The students were also told that they were not permitted to communicate during the test. After the allotted time concluded, the students were asked to turn over their tests, and I collected them from each student.

Students' written justifications were coded using the following rubric (designed by Osana et al., 2013): (a) Category 1: Relational thinking without computation or with computation only as a means to justify a written relational response; (b) Category 2: Relational thinking with Computation; (c) Category 3: Other. Student sample responses are shown in Figure 2. Responses that were placed in Category 1 demonstrated that the student engaged in relational thinking by considering the relationship between the numbers without computing the quantities on both sides of the equal sign to determine the truth value of the equation. Computation in this category was permitted only if the student had first justified the response relationally and if the computation was used to support or illustrate the relational response. Student responses that were placed in Category 2 demonstrated that the student had an understanding of the equal sign and that they were able to determine if the response was true or false but only with the use of computation. Category 3 responses were those where the student either had an operator view of the equal sign, did not supply any justification, or provided responses that were not interpretable.

Category 1 responses received 2 points, Category 2 responses received 1 point, and Category 3 responses received 0 points. Category 1 responses were awarded more points than

Category 2 and 3 responses because they indicated that the students responded relationally and did not need to compute to justify their answer. Category 2 responses were also considered to be relational, as students appeared to understand the meaning of the equal sign, but computed the quantities on each side of the equal sign rather than considering the relationships between the numbers. Category 3 responses received 0 points because their justifications contained no evidence of relational thinking or an understanding of the equal sign.

Student scores were the mean number of points assigned to each question, with a minimum score of 0 and a maximum score of 2. A random sample of 20% of the responses was coded by a second rater, and inter-rater reliability of 93% agreement was achieved.

Mental Mathematics Test (MMT). A mental mathematics test was administered to all three classes at all five time points. Five isomorphic versions of this assessment were administered to students, and each test consisted of 12 questions (see Appendix E). Each isomorphic version of the MMT included 12 standard questions of similar difficulty covered during the MM intervention (3 addition, 3 subtraction, 3 multiplication, 3 division). Examples of problem types were: 57 + 59; 107 – 38; 14 x 4; 248 ÷ 8. The numbers in the problems and the order of the problem types varied in each version.

At the start of testing, a MM answer sheet consisting of 12 numbered blank lines, presented in Appendix F, was delivered face down to each student at his or her desk. I explained to the students that I would present a mathematical expression on the board and that they were to compute the expression mentally. I indicated that the students were not allowed to write anything down except their final answer on the corresponding numbered line on the answer sheet. Students were instructed that they had 30 seconds to arrive at their answer, after which the expression would be removed from the board. I encouraged the students to review their answers after each question if time allowed, and that everyone would remain silent throughout the assessment. At the termination of the assessment, students were asked to turn over their papers, and I collected them.

Students received 1 point for each correct answer and 0 points for each incorrect answer. The score for each item was added and divided by the number of questions (i.e., 12) to obtain a total score between 0 and 1.

Results

Descriptive Statistics

The means and standard deviations of the Prior Knowledge test (PK), the Mental Mathematics Test (MMT), the Equivalence Test (ET), and the Relational Thinking Test (RTT) scores as a function of class and time are presented in Table 1.

Table 3

Means and (Standard Deviations) of Prior Knowledge Test (PK), Equivalence Test (ET), Relational Thinking Test (RTT), and Mental Mathematics Test (MM) as a Function of Class and Time

	T1				T2			T3			T4			T5		
	PK	MM	ET	RTT	MM	ET	RTT	MM	ET	RTT	MM	ET	RTT	MM	ET	RTT
C1[a]	57.1	63.4	78.3	1.02	78.6	97.8	1.40	79.0	98.7	1.56	80.8	98.0	1.57	78.6	99.3	1.40
	(17.4)	*(18.3)*	*(39.4)*	*(.48)*	*(15.9)*	*(3.94)*	*(.30)*	*(17.9)*	*(2.24)*	*(.29)*	*(16.4)*	*(3.61)*	*(.41)*	*(12.3)*	*(1.72)*	*(.59)*
C2[b]	47.3	65.6	74.3	1.16	64.9	82.0	1.08	86.7	94.0	1.49	85.6	98.3	1.53	82.2	97.3	1.37
	(20.2)	*(13.0)*	*(30.7)*	*(.32)*	*(17.2)*	*(33.7)*	*(.33)*	*(11.3)*	*(16.5)*	*(.32)*	*(10.7)*	*(2.44)*	*(4.05)*	*(14.8)*	*(3.72)*	*(.38)*
C3[c]	70.5	83.3	97.7	1.28	82.7	97.9	1.26	83.3	98.2	1.19	91.7	99.3	1.54	90.0	99.3	1.64
	(15.8)	*(15.0)*	*(3.19)*	*(.39)*	*(14.5)*	*(3.45)*	*(.35)*	*(16.4)*	*(3.11)*	*(.29)*	*(10.7)*	*(2.24)*	*(.36)*	*(10.5)*	*(2.24)*	*(.37)*

Note. All PK, ET, and MMT scores are reported in percent. C1 = Class 1, C2 = Class 2, C3 = Class 3. T1 = Time 1, T2 = Time 2, T3 = Time 3, T4 = Time 4, T5 = Time 5. For RTT, minimum score = 0 and maximum score = 2. [a]$N = 23$, [b]$N = 15$, [c]$N = 28$.

At Time 1, both Class 1 and Class 2 performed similarly on the tests of prior knowledge, mental mathematics, equivalence, and relational thinking. The patterns of the means for both Class 1 and Class 2 at each time point were in the expected direction (i.e., following the mental mathematics intervention, their respective MM, ET, RTT scores improved). However, at Time 1, Class 3 outperformed both Classes 1 and 2 on the prior knowledge, mental mathematics, and equivalence tests. Despite this, Class 3's relational thinking scores were similar to both Class 1 and Class 2 at Time 1.

For Class 1 and 2, following each class's respective MM intervention, the MM, ET and RTT mean scores all increased, with Class 1 outperforming Class 2 at Time 2 on all three measures. Following Class 3's MM intervention, RTT scores increased. At T5, all increases in RTT scores following each class's respective MM intervention were maintained.

Correlations between assessment measures (PK, MMT, ET, RTT).

To determine which, if any, of the initial data (i.e., baseline assessment measures) could serve as suitable covariates for all three classes, the correlations between the assessment measures at Time 1 with each class's respective assessments immediately following their intervention were computed. The results indicated that for Class 1, which had its intervention between Time 1 and Time 2, the ET at Time 2 was correlated with the RTT at Time 1 ($r = .55$, $p < .01$). Further, the MMT scores at T1 were correlated with the RTT ($r = .63$, $p < .01$) and MMT ($r = .45$, $p < .05$) scores at Time 2. For Class 2 which had its intervention between Time 2 and Time 3, these same correlations observed for Class 1 were not observed at Time 3. No significant correlations were seen at Time 4 in Class 3, which had its intervention between Time 3 and Time 4. In order to use any of the assessments as a covariate in the data analysis, the correlation between the baseline measure and each subsequent outcome measure (immediately after each class's respective MM intervention) must be consistent within all three classes. As this was not the case, it was determined that none of the assessment measures was a suitable covariate for the subsequent analyses.

Prior Knowledge Test. In order to determine whether there was a difference in mean prior knowledge test scores between classes at Time 1, a one-way ANOVA was performed. Alpha was set at 0.05. The ANOVA revealed a statistically significant difference between the three classes $F(2,63) = 9.35$, $p < .001$. Post hoc tests revealed that there was no difference between Class 1 and Class 2 ($p > 0.05$), but there was a difference between Class 1 and Class 3 ($p < .05$, $d = .81$), as well as between Class 2 and Class 3 ($p < 0.001$, $d = 1.29$).

Mental Mathematics Test. To verify the hypothesis that students' mental mathematics abilities would improve after the MM intervention, a 5 time (Time 1, Time 2, Time 3, Time 4, Time 5) x 3 class (Class 1, Class 2, Class 3) mixed ANOVA was performed using the MMT as the dependent measure, with class as the between groups factor and time as the within groups factor. A graphical representation of the MMT means is presented in Figure 6.

The ANOVA revealed a main effect of time, $F(4, 252) = 20.84$, $p = .001$, $\eta^2 = .25$, as well as a main effect of class, $F(2, 63) = 6.63$, $p = .002$, $\eta^2 = .17$. A significant time x class interaction was also found, $F(8, 252) = 5.45$, $p < .001$, $\eta^2 = .15$. Tests of simple effects with Bonferroni corrections revealed that at T1, differences existed between Class 1 and Class 3 ($p < .05$), and Class 2 and Class 3 ($p < .05$). In addition, for Class 1, after their MM intervention, the MMT scores were higher at Time 2 compared to Time 1 ($p < .05$, $d = .89$), and this improvement was maintained through to Time 5 (all *p*s > .05). Class 2 did not improve from Time 1 to Time 2 ($p > .05$), providing some evidence that the improvement in Class 1 between Time 1 and Time 2 was likely not due to maturation. Following the MM intervention in Class 2, their scores increased compared to Time 2 ($p < .001$, $d = 1.39$), and this improvement was maintained through to Time 5 (all *p*s > .05). For Class 3, their scores did increase from Time 3 ($M = 83.3$, $SD = 16.4$) to Time 4 ($M = 91.7$, $SD = 10.7$), although not significantly. Because Class 3 began the study with mean MMT score that was near ceiling, there was little room for improvement in their mean MMT scores despite the MM intervention. In sum, these results indicate that the scores on the MMT for both Class 1 and Class 2 improved immediately following their respective MM intervention, which were maintained for twelve weeks in Class 2 and eight weeks in Class 3.

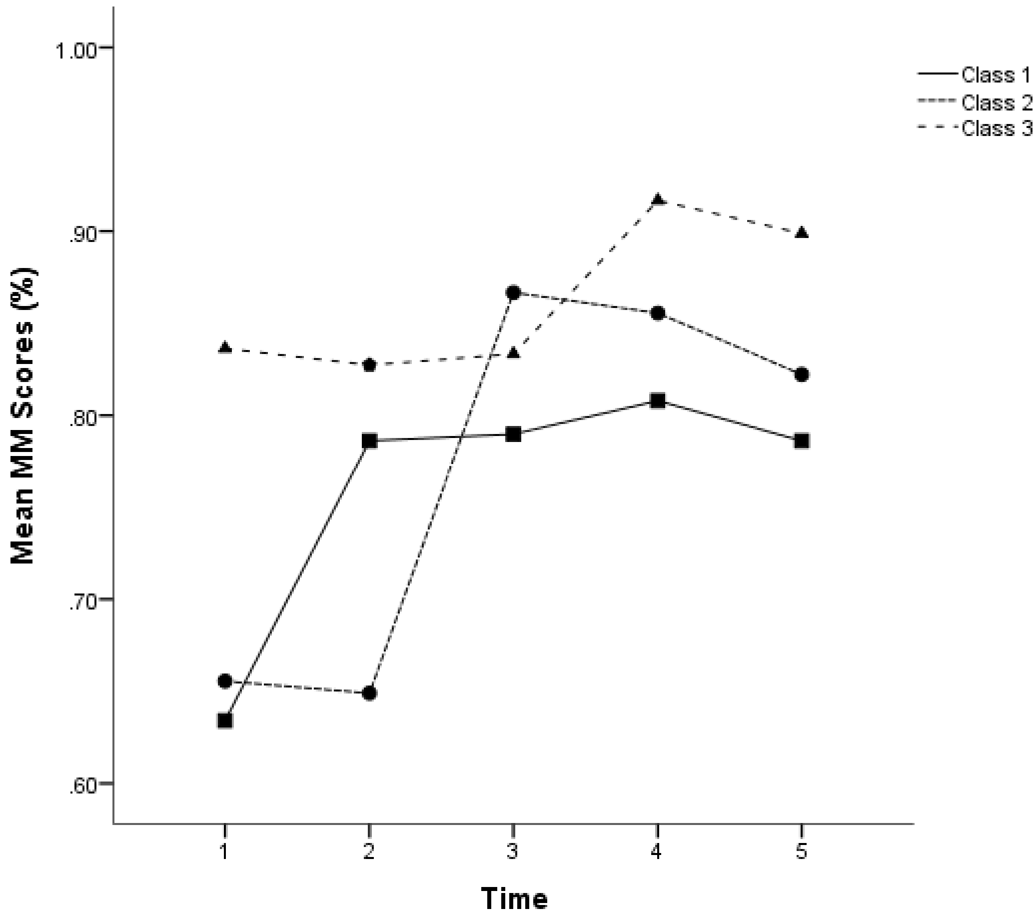


Figure 6. Mean Mental Mathematics Test scores as a function of Time and Class. Means are reported in percent.

Equivalence Test. To test the hypothesis that students' understanding of the equal sign would improve after the MM intervention, a 5 time (Time 1, Time 2, Time 3, Time 4, Time 5) x 3 class (Class 1, Class 2, Class 3) mixed ANOVA was performed, using class as the between groups factor and time as the within factor. Alpha was set at .05. A graphical representation of the ET means is presented in Figure 7.

The ANOVA revealed a main effect of time, $F(4, 252) = 13.21$, $p = .001$, $\eta^2 = .17$, as well as a main effect of class, $F(2, 63) = 6.46$, $p = .003$, $\eta^2 = .17$. A significant time x class interaction was also found, $F(8, 252) = 4.01$, $p = .0001$, $\eta^2 = .11$. Tests of simple effects with Bonferroni corrections revealed that at T1, differences existed between Class 1 and Class 3 ($p < .05$), and Class 2 and Class 3 ($p < .05$). In addition, for Class 1, after their MM intervention, the ET scores were higher at Time 2 compared to Time 1 ($p < .05$, $d = .90$), and this improvement was maintained through to Time 5 (all *p*s > .05). Class 2 did not improve from Time 1 to Time 2 ($p > .05$), providing some evidence that the improvement in Class 1 between Time 1 and Time 2 was likely not due to maturation. At Time 3, following their MM intervention, Class 2 improved ($p < 0.01$, $d = .48$), and this improvement was maintained through to Time 5 (all *p*s > .05). Despite their intervention that occurred between Time 3 and Time 4, Class 3 did not improve following their MM intervention. Because Class 3 began the study with a high mean score near ceiling ($M = 97.7$, $SD = 3.19$), there was little room for improvement in their mean ET scores despite the MM intervention. In sum, these results indicate that the scores on the ET for both Class 1 and Class 2 improved immediately following their respective MM intervention and were maintained for twelve weeks in Class 1 and eight weeks in Class 2.

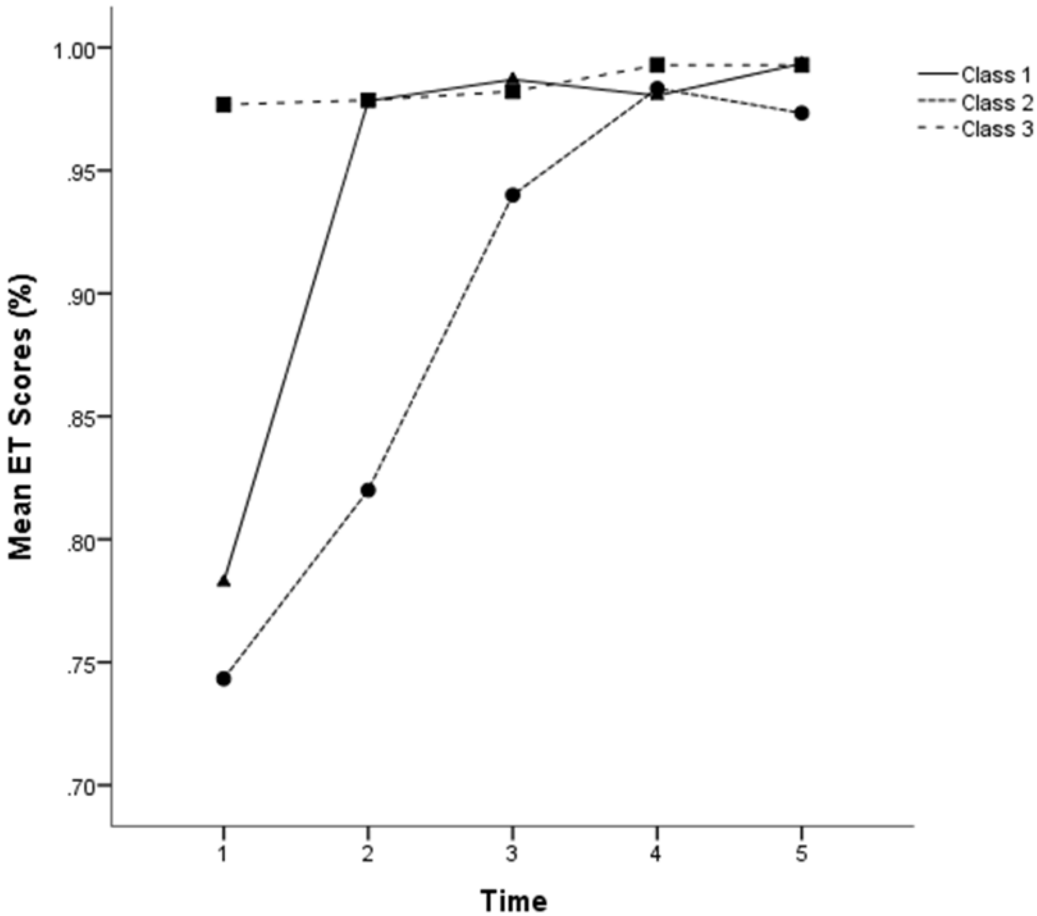


Figure 7. Mean Equivalence Test scores as a function of Time and Class. Means are reported in percent.

Relational Thinking Test. To verify the hypothesis that students' relational thinking would improve after the MM intervention, a 5 time (Time 1, Time 2, Time 3, Time 4, Time 5) x 3 class (Class 1, Class 2, Class 3) mixed ANOVA was again performed using the RTT as the dependent measure, with class as the between groups factor and time as the within groups factor. A graphical representation of the RTT means is presented in Figure 8.

The ANOVA revealed a main effect of time, $F(4, 252) = 16.45$, $p = .001$, $\eta^2 = .21$. There was no effect of class. A significant time x class interaction was also found, $F(8, 252) = 5.79$, $p = .002$, $\eta^2 = .16$. Tests of simple effects with Bonferroni corrections revealed that there was no difference in mean RTT scores between the three classes at T1 (all *p*s > .05). Following the MM intervention, Class 1's RTT mean scores improved from Time 1 to Time 2 ($p < 0.01$, $d = 0.97$), and this improvement was maintained through to Time 5 (all *p*s > .05). No improvements were observed for Class 2 or Class 3 between Time 1 and Time 2 (*p*s > .05). However, following their MM intervention, students in Class 2 did improve from Time 2 to Time 3 ($p < .001$, $d = 1.26$), and this improvement was maintained through to Time 5 (all *ps* > .05). Following their MM intervention, Class 3 improved from Time 3 to Time 4, ($p < .001$, $d = 1.08$), and maintained their scores through to Time 5 (all *p*s < .05). These results suggest that the scores on the RTT improved on the first assessment immediately following the mental mathematics intervention for each class, and maintenance of the RTT was observed after the MM intervention for twelve weeks in Class 1, eight weeks in Class 2, and four weeks in Class 3.

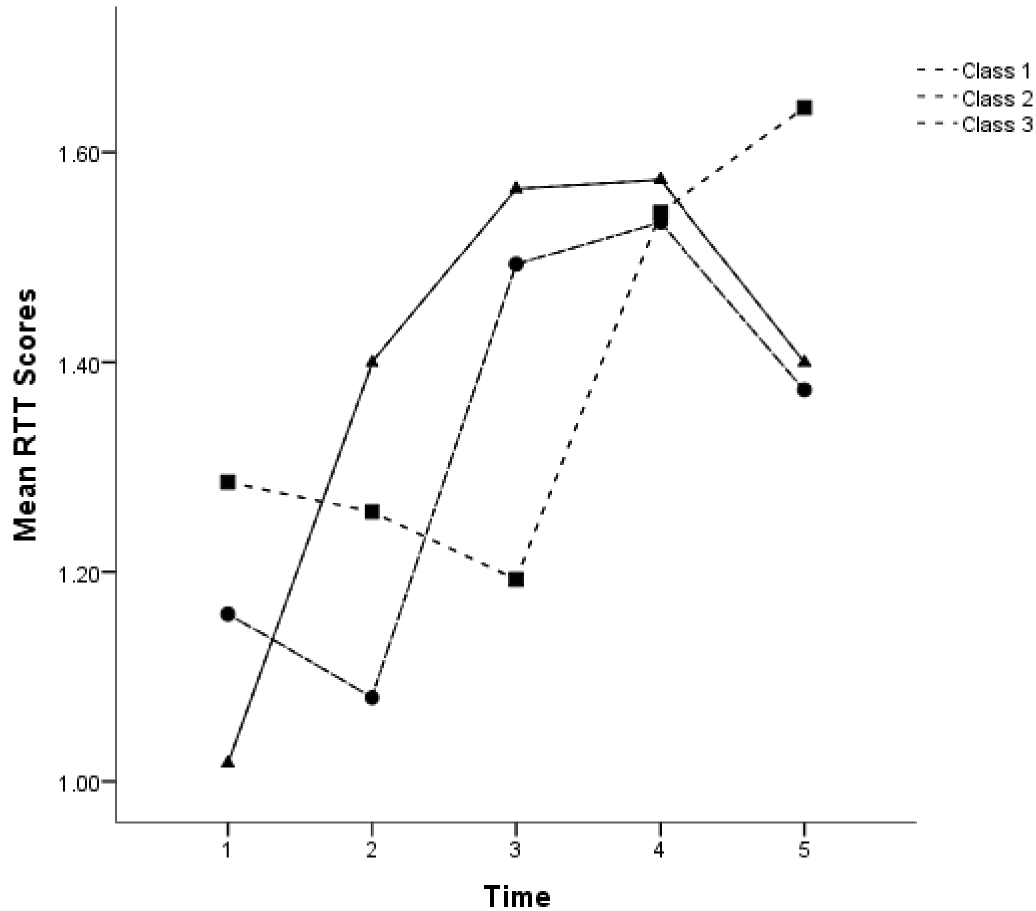


Figure 8. Mean Relational Thinking Test scores as a function of Time and Class. Min = 0, Max = 2.

Patterns in Mental Mathematics and Other Assessments

When observing the patterns between the assessments for Class 1 and Class 2, the improvements in ET and RTT correspond in time to the MMT improvements. Specifically, for Class 1, improvements on all three measures -- MMT, ET, and RTT -- occurred at Time 2 immediately following their MM intervention. Similarly, the scores obtained at Time 2 were maintained for all three assessments through to Time 5. For the students in Class 2, improvements in MMT, ET, and RTT scores occurred from Time 2 to Time 3, immediately following their MM intervention. Similar to Class 1, the scores on all three measures at Time 3 were maintained to Time 5. For Class 3, after the MM intervention, there was no improvement on the MMT nor the ET assessments, as Class 3's scores began and remained near ceiling throughout the study on these two assessments. However, Class 3's RTT scores did improve following the MM intervention, and this improvement was maintained through to Time 5.

Discussion

The objective of the present study was to investigate whether seventh-graders' relational thinking, assessed through a measure of equivalence problem solving and a test of reasoning about non-canonical equations, improved following a mental mathematics intervention. The study builds on Study 1, which also verified whether a mental mathematics intervention was linked to improvements in relational thinking. However, unlike Study 1, Study 2 assessed students' performance in mental mathematics both before and after the mental mathematics intervention to provide evidence that the mental mathematics intervention was effective in improving mental mathematics abilities. Further to this, Study 2 assessed students' mental mathematics, understanding of equivalence, and relational thinking at five time points to verify if students could maintain their performance for a longer period of time than Study 1. Moreover, Study 2 involved a larger sample – that is, students from three classes participated, while for Study 1, only two classes participated.

As I projected, Class 1 and 2 improved on their performance on the test of equivalence problems immediately following the mental mathematics intervention. For these two classes, these findings replicated the results found in Study 1, but the design of Study 2 permitted me to test for maintenance, which was found for Class 1 and 2, who maintained their level of performance for twelve weeks in Class 1, and eight weeks in Class 2. In contrast, Class 3 showed no improvement after their intervention, as their mean scores were near ceiling for the duration of the study. However, because two of the classes did not improve until after their mental mathematics intervention had concluded, I argue that that the mental mathematics intervention may have been responsible for the students' improvement in solving equivalence problems, despite the fact that the lack of random assignment prevented me from establishing causality.

Students' scores on the relational thinking assessment improved immediately following the mental mathematics intervention in all three classes, and these scores were maintained for twelve weeks in Class 1, eight weeks in Class 2, and four weeks in Class 3. This long-term maintenance was not observed in Study 1: Students improved immediately, but the design did not allow me to test the level of performance over time. Similar to performance on equivalence problems, because each class did not improve until after their mental mathematics intervention had concluded, this may support the notion that the mental mathematics intervention is responsible for the observed improvement in relational thinking.

Performance on the mental mathematics assessment improved following the mental mathematics intervention in both Class 1 and 2, and was maintained in both classes through to Time 5. No improvements on the mental mathematics assessment were observed at any time point in Class 3, as this class began the study with scores already close to ceiling. Class 1 and 2 maintained their improved mental mathematics scores after their respective intervention. In Study 1, I speculated that the increases in relational thinking were due to the students' improved mental computation skills, despite the fact that they were not assessed on their mental mathematics. A stronger case can be made in the present study because I incorporated a treatment check – that is, I provided evidence from Class 1 and 2 that the intervention achieved what was intended – to increase students' mental computation skills.

Moreover, the improvement in mental mathematics scores parallels in time the improvements in both equivalence problem solving and relational thinking in both Class 1 and 2. The corresponding increases in mental computation further supports the hypothesis that mental computation itself is responsible for the observed improvements in relational thinking, as performance on all three assessments in Class 1 and 2 increased and were maintained simultaneously following the mental mathematics assessment.

The data allow me to conclude that students' improved relational thinking may be attributed to a number of distinct, yet related, mental processes they had developed during the mental mathematics intervention. First, the students were encouraged to construct equivalent expressions without the explicit use or mention of the equal sign. They may have begun to change their view of the equal sign from an operator view to the relational meaning “can be substituted for” (e.g., substituting 67 for 60 + 7) (Jones et al., 2013), which likely encouraged students to create transformed equivalent expressions, thereby reinforcing a relational view (Carpenter et al., 2003; 2005; Stephens & Ribeiro, 2012). For example, students asked to solve 23 + 47 mentally could substitute the 23 with 20 + 3, transforming the expression to 20 + 3 + 47, then further substitute the 3 + 47 with 50 to create 20 + 50, to finally arrive at the answer of 70 (Jacobs et al., 2007). Throughout the mental mathematics intervention, I guided students in whole-class discussions, during which they shared the strategies they used to mentally solve a given problem. By describing their strategies, as well as being shown different strategies used by their peers, they learned to construct equivalent expressions which likely played a role in the improvement of their relational thinking scores (Carpenter et al., 2003).

During the mental mathematics intervention, students were also encouraged, without the mention of them by name, to use the properties of numbers implicitly in their transformations. For example, 8 x 12 could be transformed to (8 x 10) + (8 x 2), to arrive at 80 + 16, which equals 96. Over the course of the mental mathematics intervention, I guided discussions with the students during which I explained and encouraged the application of fundamental number properties. Students learned to apply these properties, which may have played a role in the increase in their relational thinking scores.

Furthermore, students were encouraged to undertake a variety of strategies during the whole-class discussions and thus presumably became more flexible in their approach to selecting an appropriate strategy for a given problem. By describing their own individual strategies during the whole-class discussions, as well as being exposed to different strategies used by their classmates, students may have become more flexible, resulting in an improvement in their relational thinking.

The students in Class 3 were placed together in the same class at the beginning of the academic year because they had performed above a school-mandated cut-off on an International Baccalaureate (IB) Program admissions exam. Therefore, as a result of admission into this program, I was aware that differences may have been observed in mathematical ability between students in Class 3 and those in Classes 1 and 2. Class 3 began the study with a significantly higher Prior Knowledge mean score than both Class 1 and 2, indicating that indeed there were differences between Class 3 and the other two classes at the onset of the study. Possibly because of their placement in the IB program, Class 3 also outperformed the students in the other two classes on both the equivalence test and the mental mathematics assessment at Time 1. As such, students in Class 3 did not improve on these two measures following the intervention, possibly because there was little room for improvement. Their performance on the relational thinking assessment was comparable to that of the other two classes, however, and their improvement in relational thinking mean scores immediately following the mental mathematics intervention may add support to the hypothesis that the intervention is responsible for improvements in relational thinking.

Limitations

Despite this study providing a link between mental mathematics and relational thinking, there are some limitations. For example, the student participants were from three intact classes, as random assignment was not possible, and therefore the results obtained may be a result of factors other than the mental mathematics intervention. It is possible, for example, that the three classes may have had pre-existing differences that were not accounted for, such as private mathematics tutoring or previous exposure to algebra, in addition to possible differences in prior knowledge that were not captured by the Prior Knowledge assessment, such as their interpretations of the equal sign and knowledge of number properties. These differences may have confounded the results. However, in Class 1 and 2, the patterns of improvement on the mental mathematics assessment that mirrored the students' improvement on equivalence problem solving and relational thinking provide further support that it is the mental mathematics intervention, rather than possible confounding factors, that is responsible for the observed increase in scores on each outcome measure. In addition, to moderate possible confounds, I designed the study so that each of the classes had their interventions delivered using a staggered treatment design, which was intended to mitigate the effects of the lack of random assignment by providing some evidence that any observed improvements were likely not a result of maturation.

Moreover, I was the students' regular mathematics teacher, and I was responsible for the delivery of all the mental mathematics sessions in both classes. Despite my efforts to deliver equally effective instruction in both classes, it is possible that experimenter bias may have occurred, causing me to unknowingly and unintentionally instruct or create a learning environment that would result in confirming my predictions. For future studies, an observer who is blind to the study's hypotheses could verify, based on predetermined criteria, that the instructional practices of a teacher-researcher were equivalent in all groups.

Finally, the study took place in a public suburban high school with a student population from middle- to high-income families. This reduces the external validity of the study. It is possible that these results are not applicable to other high school populations, as students from different populations may respond differently to the intervention.

Instructional Implications

Educators can appreciate from this study that mental mathematics may serve as a valuable way to improve relational thinking in middle school students. As the results suggest, students' abilities in relational thinking improved following a daily 20 minute mental mathematics intervention over 15 days. Therefore, teachers seeking to improve students' understanding of the equal sign, mathematical equivalence, and relational thinking could rather easily incorporate mental mathematics instruction into their daily mathematics routine. From my perspective, fellow educators need to understand the similarities between mental mathematics and relational thinking so that that their mental mathematics instruction can maximize the likelihood of students' development of relational thinking. With this in mind, educators should use similar instructional strategies as those implemented in the present study to improve mental mathematics abilities and to foster relational thinking in their students. The results of the present study add to the existing literature as there is a lack of research on how to improve relational thinking in real classroom settings with the use of a clearly prescribed instructional intervention. As far as I am aware, the present study is one of the first (with the notable exceptions of Kindrat & Osana, in press, and Osana et al., 2013) to investigate the link between mental computation and relational thinking. Further research on how to improve relational thinking is indeed required. As relational thinking has been promoted as critical to academic success in mathematics (Carpenter et al., 2003), regular mental computation in the mathematics classroom could have significant implications for students' mathematics achievement thinking in seventh-grade and beyond.

Connecting Study 2 to Study 3

Study 2 examined the relational thinking of seventh-graders before and after a 15-day mental mathematics intervention in the context of whole number arithmetic. Using the same staggered treatment design as Study 1, students from three classrooms were assessed at five time points on their (a) ability to solve equivalence problems, (b) reasoning abilities about true-false number sentences, and (c) mental mathematics abilities. Results indicated that the students in the first and second classes improved and maintained their performance on the equivalence assessment after the intervention through to the final time point, but there was no improvement after the intervention in Class 3, likely because of ceiling effects on the measure at baseline. Students in all three classes improved their reasoning abilities about true-false number sentences after the mental mathematics intervention. Performance was maintained on this measure at the same level through all subsequent time points, including on a delayed posttest four weeks later.

Mental mathematics performance for the first and second classes increased significantly after the mental mathematics interventions, providing some indication that the intervention had the desired effect. Improved mental computation performance remained at the same levels at all subsequent time points for the students in these two classes. For the third class, however, no improvements on the mental mathematics measure were found at any time point, likely because of ceiling effects for this group at Time 1.

Despite the improvement in students' understanding of equivalence and relational thinking, the findings in Study 1 and Study 2 do not provide a causal link between improvements in mental mathematics and relational thinking, as random assignment was not possible. Study 3 was designed to include random assignment to test a possible causal link between mental mathematics and relational thinking.

Moreover, Study 3 also addressed whether the effects of a mental mathematics intervention on relational thinking could be augmented beyond what was observed in Studies 1 and 2 if the students were permitted to write down some of the mental strategies to keep track of their thinking. Writing down specific elements of the mental mathematics computation was hypothesized to reduce the load on students' working memory, thereby further enhancing relational thinking. There is a large body of literature that suggests that working memory plays an important role in mathematical performance (Ashcraft & Krause, 2007; Ayres, 2001; LeFevre, DeStefano, Coleman, & Shanahan, 2005) and in mental mathematics specifically

(Adams & Hitch, 1997; Ashcraft & Kirk, 2001; DeStefano & LeFevre, 2004; Fürst & Hitch, 2000). For this reason, Study 3 was designed to address whether reducing the load on students' working memory could enhance their relational thinking.

Chapter 4: Study Three

The Role of Mental Mathematics Instruction and Working Memory in Seventh Graders' Relational Thinking

Introduction

Students are instructed on a variety of mathematical concepts over the course of their academic careers. Included in these concepts is algebra, a topic that can create difficulty for many students (Booth, 1988; Jacobs et al., 2007; Thwaites, 1982). Many studies report that, in an aim to decrease the difficulty associated with the learning of algebra, its instruction should begin in primary school (Carpenter & Levi, 2000; Carraher, Schliemann, & Brizuela, 2006; Kaput, 1998), with an emphasis on generalizations and number properties, which provide the foundation for relational thinking (Carpenter et al., 2003; Nathan & Koedinger, 2000; Slavitt, 1999).

Relational thinking includes the ability to look at a mathematical expression or equation in its entirety instead of in a manner where a prescribed sequence of procedures is to be followed (Carpenter et al., 2005; Jacobs et al., 2007; Stephens, 2006). It entails making explicit generalizations that are based on the fundamental properties of number operations (Jacobs et al., 2007; Stephens & Ribeiro, 2012). It involves attending to patterns and rules in creating mathematical generalizations (Britt & Irwin, 2011; Cooper & Warren, 2011; Mason et al., 2009). It also requires making strategic decisions, and that students think before they act. Sometimes, when thinking prior to acting, students can use relational thinking to simplify the calculation before proceeding. The hallmarks of relational thinking are (a) a solid understanding of equivalence (Carpenter et al., 2003, 2005; Stephens, 2006), (b) flexible reasoning when selecting among a variety of approaches to solve a given problem and while undertaking a selected approach (Britt & Irwin, 2005; Empson et al., 1999; Star & Newton, 2009), and (c) an awareness of the properties of numbers and the ability to generalize these properties to transform mathematical expressions into new expressions that are easier to manage (Britt & Irwin, 2011; Carpenter et al., 2005).

Improving Relational Thinking Through Instruction

Several studies have investigated primary school students' relational thinking by focusing on improving students' understanding of mathematical equivalence (Carpenter et al., 2005; Stephens, 2006). However, not much research has examined ways to enhance the various components of relational thinking beyond an understanding of equivalence. Such efforts might include encouraging students to (a) become aware of the mathematical expression or equation in its entirety instead of the simple execution of a prescribed sequence of procedures (Carpenter et al., 2005); (b) apply knowledge of number properties to reason about number sentences; and (c) improve their flexibility in mathematics, including when they choose an appropriate strategy and when they work within a given strategy.

There exist only a limited number of studies that focus on instruction aimed at improving relational thinking. Jacobs et al. (2007) conducted a yearlong professional development course on relational thinking for teachers, which allowed the teachers and their students to come up with a greater variety of strategies to solve a given equation by looking at the relationships between the numbers. In another study, Irwin and Britt (2005) implemented a curriculum as a means to improve relational thinking by encouraging students to become flexible with mental computations on rational numbers (Irwin & Britt, 2005). Osana et al. (2013) also carried out a mental mathematics unit in a university-level mathematics methods course for elementary preservice teachers and their results suggested an improvement in relational thinking following the unit.

In my attempt to find literature reporting the effects of instructional methods on relational thinking, I noticed that many concepts that were indicated as predictors of success in relational thinking were also predictors of success in mental mathematics. For example, both mental mathematics and relational thinking appear to depend on an understanding of number properties, equivalence, and on flexible reasoning skills, including transforming, compensating, and re-ordering numbers during problem solving (Carpenter et al., 2005; Heirdsfield, 2011; Heirdsfeld & Cooper, 2002; Molina & Ambrose, 2006; Proulx, 2013; Stephens, 2006; Thompson, 2010, Threlfall, 2002; Verschaffel et al., 2009). As such, the objective of Study 1 was to examine the relationship between mental computation and relational thinking in seventh-graders before and after a mental mathematics intervention. It was predicted that if students improved in mental mathematics, they would likewise improve in relational thinking.

Using two intact seventh-grade classes in Study 1, students were assessed at three time points on their (a) ability to solve equivalence problems, and (b) reasoning about true-false number sentences. Results indicated that each class improved on both assessments immediately following the mental mathematics intervention. Students in one class were able to maintain their scores on the test of equivalence problems four weeks after the conclusion of the intervention. However, the results of this study only suggest a link between mental mathematics and relational thinking, as random assignment was not possible. In addition, despite the improvement in relational thinking, the absence of a mental mathematics assessment prior to and following the mental mathematics intervention did not provide information as to whether students actually improved in their mental mathematics abilities following the mental mathematics intervention. A verification of the integrity of the mental mathematics intervention, together with improvements in relational thinking, would further support the hypothesis that mental mathematics is responsible for the growth. For example, it is possible that the conceptual knowledge that students learned during the intervention in Study 1 was what aided them in their relational thinking, rather than their presumed improvement in mental computation. For this reason, Study 2 was conducted to address this issue in the design of Study 1.

Study 2 also examined the relational thinking of seventh-graders before and after a mental mathematics intervention. Using three intact seventh-grade classes, students were assessed at five time points on their (a) ability to solve equivalence problems, (b) reasoning abilities about true-false number sentences, and (c) mental mathematics abilities. Results indicated that following their respective mental mathematics intervention, two classes improved their performance on the mental mathematics assessment. Moreover, the same two classes also improved and maintained their performance on the equivalence problems and reasoning about true-false number sentences. Although the results of this study still can only suggest a link between mental mathematics and relational thinking, the presence of a mental mathematics assessment both prior to and immediately following the intervention allowed me to establish whether the mental mathematics intervention was indeed an effective tool for improving students' mental mathematics abilities. The parallel pattern of improvement observed in the mental mathematics assessments with respect to both the Equivalence assessment and the Relational Thinking assessment provides further evidence that the improvement in mental mathematics is likely responsible for the improvement in relational thinking.

The objective of Study 3 was to determine whether the effects on relational thinking of a mental mathematics intervention could be augmented beyond what was observed in Studies 1 and 2. That is, if the students were permitted to write down some of the mental strategies to keep track of their thinking, this might reduce the load on their working memory, thereby further enhancing relational thinking. There is an abundance of literature that suggests that working memory plays an important role in mathematical performance (Ashcraft & Krause, 2007; Ayres, 2001; LeFevre et al., 2005; Seyler, Kirk, & Ashcraft, 2003; Siegler & Booth, 2005; Zbrodoff & Logan, 2005) and specifically in mental mathematics (Adams & Hitch, 1997; Ashcraft & Kirk, 2001; DeStefano & LeFevre, 2004; Fürst & Hitch, 2000). For this reason, Study 3 was designed to address whether reducing the load on students' working memory could enhance their relational thinking. I predicted that if students' working memory is relieved while performing mental mathematics, their relational thinking would improve even beyond what was observed in Studies 1 and 2.

Improving Relational Thinking by Reducing Working Memory

Working memory has been described as a mental workspace that plays a role in the monitoring, regulation, and maintenance of pertinent information in order to carry out complex cognitive tasks (Miyake & Shah, 1999). The ability to solve complex problems usually requires that students retain partial information while they process new information simultaneously (Raghubar et al., 2010). This requires working memory resources. An understanding of how working memory is related to how students learn mathematics may be important for instruction in mathematics (Raghubar et al., 2010), and this may have implications for designing relational thinking instruction through mental mathematics.

Working memory is comprised of three main subsystems: the visuospatial sketch pad, for retaining and working with visual-spatial information; the phonological loop, for maintaining and practicing verbal information (Baddeley, 1992); and the central executive (executive functioning system), a complex controlling system which plays a role in the synchronization of execution on distinct tasks, selective attention, task-switching, and inhibition (Baddeley, 2007). Working memory is predictive of school achievement, including reading comprehension and mathematics, in both children and adults (Hitch, 1978; Swanson & Beebe-Frankeberger, 2004). In research conducted by Heathecote (1994), when adults were required to solve mental addition problems, presented either visually or auditorily, spatial interference (a deficiency in remembering the location of numerical items with respect to one another) and articulatory suppression (speaking while being presented with numbers to remember) were able to disrupt the working storage of mathematical problem information (Heathcote, 1994). In children, students who have difficulty in mathematics often also demonstrate poor achievement on measures that assess the visuospatial sketch pad and the central executive components of the working memory (Gathercole, Pickering, Ambridge, & Wearing, 2004; Geary, Hoard, & Hamson, 1999; McLean & Hitch, 1999; Siegal & Ryan, 1989).

In research conducted by Bull and Scerif (2001), the authors showed that in 7-year-old students, those who performed well on measures that assessed executive functioning were also more likely to have success in mathematics, particularly in the Counting Span Task (where students must count dots on two screens and remember the two totals separately (i.e., 8, 3). Moreover, 7-year-old students with low achievement in mathematics also had difficulty on the Wisconsin Card Sorting Task (WCST), a test of executive functioning (Bull, Johnson, & Roy, 1999). Bull et al. (1999) concluded that children of low mathematical skill have significant difficulty maintaining information in working memory.

Working memory has also been shown to be critical for success on mathematical word problems (Passolunghi, Cornoldi, & De Liberto, 1999; Passolunghi & Siegel, 2001; Swanson & Beebe-Frankenberger, 2004). Research conducted on fourth-grade students who had difficulty with problem solving showed that the same students had difficulty with central executive tests of working memory (Passolunghi & Siegel, 2001). Likewise, Passolunghi et al. (1999) found that fourth-grade students whose abilities in solving arithmetic problems were poor also had difficulty on working memory tasks that required them to dismiss irrelevant information.

There exists some evidence that working memory is indeed related to mental mathematics performance (Adams & Hitch, 1997; Hitch, 1978; Logie et al., 1994). For example, Adams and Hitch (1997) conducted experiments to investigate the link between children's mental arithmetic and their working memory, to determine whether it was working memory or arithmetical competence that caused difficulty in their mental mathematics success. Children (aged 7 to 11 years old) were asked to add pairs of multi-digit numbers. The results indicated that children had more success when the numbers to be added were delivered visually and were available for the duration of the calculation than when they were presented verbally, suggesting a working memory constraint (Adams & Hitch, 1997). Therefore, when working memory load was reduced by permitting the addends in the mental addition task to be permanently visible, performance improved (Adams & Hitch, 1997).

Hitch (1978) also conducted a series of experiments designed to investigate the role of working memory in performing mental arithmetic. When assessing participants on their ability to solve verbally presented multi-digit addition problems, all subjects reported breaking down the problems into a series of elementary stages, which required the temporary storage of information (i.e., working memory) (Hitch, 1978). Further studies examined the effects of delaying participants from indicating their partial results as they progressed through their calculations: This impeded their calculation accuracy and demonstrated that the answers to interim calculations are often forgotten if they are not used immediately (Hitch, 1978). To conclude, Hitch (1978) also demonstrated that not remembering the problem's initial information could also be a source of error, the frequency of which can increase as the number of calculations increases, as the individual progresses between the initial presentation of the problem and the final answer (Hitch, 1978).

Another study (Logie et al., 1994) involved adults (aged 18 to 65 years) who were asked to mentally add two-digit numbers delivered to them verbally. The participants' ability to perform the mental computations was significantly disrupted by articulatory suppression. When the experiment was repeated but the problem was presented visually instead of verbally, disruption of the mental addition was still observed with articulatory suppression. To sum, mental addition, despite the distractions, was better overall when the initial probe was presented visually. These results are consistent with a role for the central executive component of working memory in performing the calculations required for mental addition (Logie et al., 1994).

Because the majority of research focusing on working memory related to mental mathematics has focused on mental addition, and not on other arithmetical operations (Adams & Hitch, 1998; Hitch, 1978; Logie et al., 1994), the present study aimed to address whether a reduction in working memory during mental subtraction could enhance relational thinking beyond what was observed in Study 1 and 2. This reduction in working memory would be somewhat alleviated by allowing students to record their steps, as previous research has indicated that fewer mental calculation errors are made when the information appearing in a problem is made available through a written record (Hitch, 1978).

Present Study

The present study investigated the effects of a mental mathematics intervention on seventh-graders' relational thinking, but in comparison to Study 1 and 2, I attempted to alleviate the students' cognitive load while they performed mental computations by requiring them to write down certain elements of the problem during the mental mathematics instruction. The study tested the added effects of reducing the cognitive load that is inherent in mental computation on students' equivalence problem solving and relational thinking. The intervention in Study 3 differed from that delivered in Studies 1 and 2 in that whole-class discussions did not take place during the mental mathematics instruction. Instead, the interventional instruction was directed solely by the teacher to maintain equivalent instructional sessions between both conditions except for the manipulation of the experimental variable (i.e., manipulation of cognitive load during mental computation). Without these constraints, it would have been difficult to ensure that the students in the RCL condition would have recorded elements of mental computation as opposed to more procedural strategies for solving each problem (e.g., the standard algorithm). In other words, allowing the students in the RCL condition the freedom to choose which elements of the mental computation to record could have prevented any mental mathematics from occurring at all.

Using two seventh-grade classes, students were assessed at two time points (Time 1, Time 2) on their (a) ability to solve mental mathematics problems (Mental Mathematics Accuracy Test; MMAT); (b) ability to solve equivalence problems (Equivalence test; ET), and; (c) reasoning abilities about true-false number sentences (Relational Thinking Test; RTT). A week after the conclusion of the Time 2 assessments, a Mental Mathematics Strategy Test (MMST) was administered, in order to determine students' accuracy and strategy selection in solving mental mathematics problems. At the beginning of the study, students in each class were randomly assigned to two conditions: a Regular Mental Mathematics (RMM) condition and a Reduced Cognitive Load (RCL) condition. Both the RMM and RCL conditions received the identical instruction with identical expressions, but the RMM condition was instructed to solve the expression entirely mentally (with no tools for recording thinking), while the RCL condition was instructed to solve the expression in the same manner as the RMM condition, but was required to write out specific parts of the mental calculations. In this way, I was able to investigate if the suggested improvement in relational thinking seen in Study 1 and 2 could be enhanced even further with a reduction in cognitive load. I predicted that following the mental mathematics intervention, all students would improve in their relational thinking, but those in the RCL condition would improve to a greater degree than those in the RMM condition.

Method

Participants

Seventh-grade students from two classes in a suburban public high school in Quebec, Canada were asked to participate in the study. At the time of data collection, the high school was composed of middle- to high-income families, and was rated on the higher end of the school board's socio-economic index, measured by family income levels and mother's education (Ministère de l'éducation, enseignement supérieur, et recherche, MEESR, 2014). Both classes followed the identical seventh-grade mathematics curriculum, as mandated by the Quebec Education Plan of the Ministère de l'éducation, enseignement supérieur, et recherche (MEESR, 2016). *Canadian Mathematics 7* (Paholek, 1993) was used as the textbook for the delivery of the mathematics curriculum.

The sample consisted of 35 students. The students had not participated in previous studies conducted by the authors. To be included in the study, students were required to: provide parental consent; be present for at least half of the eight mental mathematics (MM) intervention sessions; complete all Time 1 assessments before the intervention began; complete Time 2 assessments (MMAT, ET, RTT) within four days of the conclusion of the MM intervention; and to complete the MMST (one week after Time 2) either on the scheduled testing day or within two days following that day. All students provided parental consent to participate in the study. All students were present on all assessment days at Time 1, Time 2, and one week after Time 2. Over the course of the two week intervention period, 28 students completed all eight intervention sessions, four students completed seven sessions, two students completed six sessions, and one student completed five sessions. Therefore, none of the 35 students was excluded from the sample.

Design

The study design is presented in Figure 9.

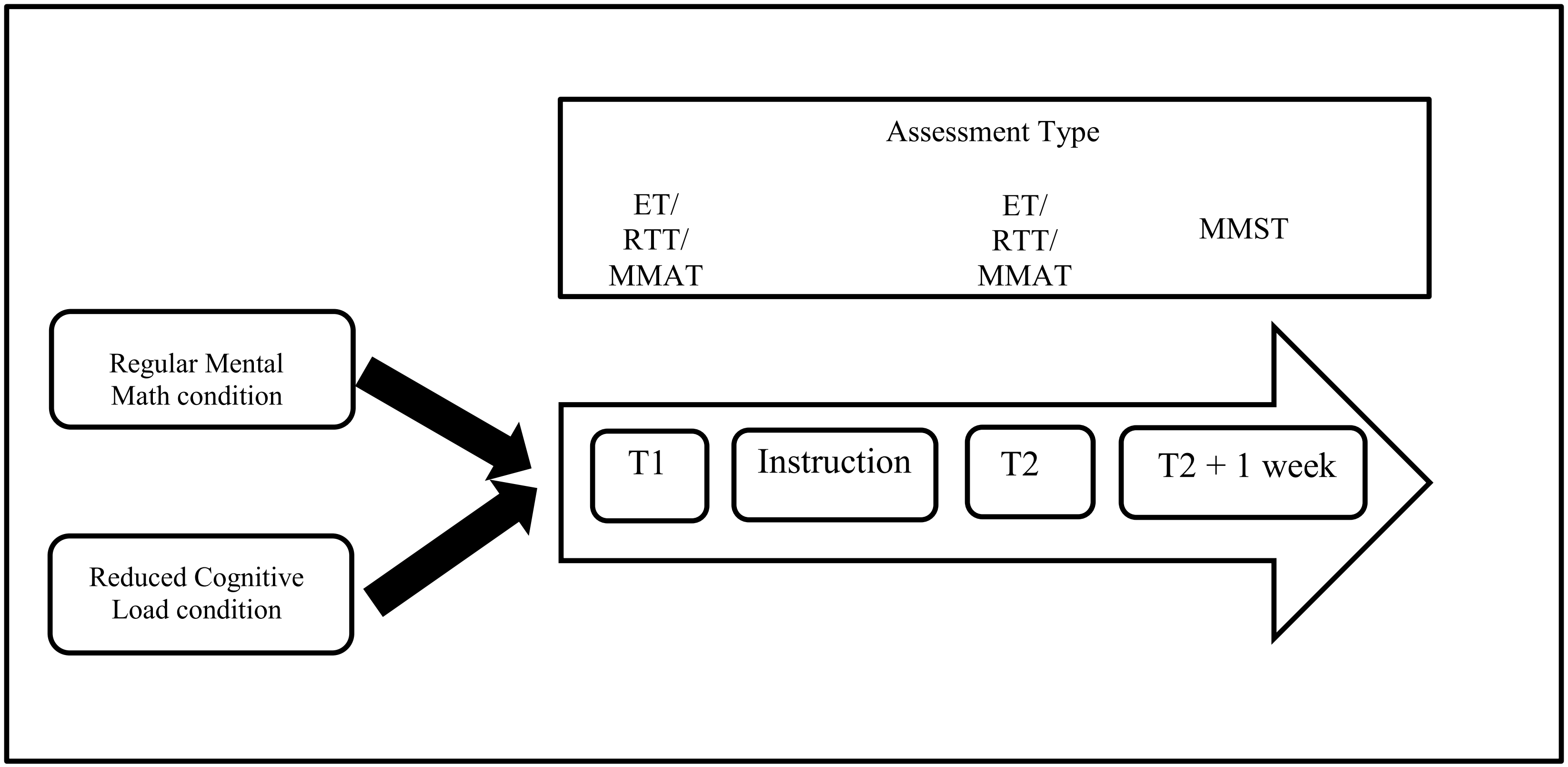


Figure 9. Design of Study 3. T1 = Time 1, T2 = Time 2. MMAT = Mental Mathematics Accuracy Test; ET = Equivalence Test; RTT = Relational Thinking Test; MMST = Mental Mathematics Strategy Test.

The study consisted of a two group, pretest posttest design. Students were randomly assigned to two conditions in each class: a Regular Mental Mathematics (RMM) condition (18 students: 10 students from Class 1 and 8 students from Class 2) and a Reduced Cognitive Load (RCL) condition (17 students: 9 students from Class 1 and 8 students from Class 2). Over the course of the study, I administered three assessments to both conditions at both Time 1 and Time 2: the Mental Mathematics Accuracy Test (MMAT), the Equivalence Test (ET), and the Relational Thinking Test (RTT). The Mental Mathematics Strategy Test (MMST) was administered one week after the three other assessments were administered at Time 2.

The MMAT was designed to assess students' accuracy in performing mental mathematics. The MMST was designed to determine the types of strategies used in computing mental mathematics problems. The ET was designed to measure the students' understanding of the equal sign (based on Watchorn & Bisanz, 2005) and the RTT (based on Osana et al., 2013) was designed to assess students' ability to determine the truth value of equations using relational thinking.

The instructional intervention took place in eight mathematics classes over a two-week period. The first four intervention sessions occurred on the first four consecutive days during the first week, and the second four intervention sessions occurred during the first four consecutive days of the second week. Both conditions in each class followed the identical schedule on each intervention day. As such, each intervention session was delivered four times per day (e.g., on Day 1, in Class 1: RMM followed by RCL; and in Class 2: RMM followed by RCL).

The delivery of the instructional intervention occurred during the first 40 minutes of each class, during which one of the two conditions had their intervention first during the first 20 minutes, and the other condition began their intervention in the second 20-minute segment of the class. The order in which the conditions received their specific instruction alternated by day (e.g., on Day 1, the RMM condition received their intervention first, and on Day 2, the RCL condition received their intervention first.

Mental Mathematics Intervention

Schedule. As indicated in Figure 10, during each instructional session, anywhere from 8 to 12 mental computations involving subtraction were performed. During the first few sessions in each condition, I began by first demonstrating each strategy, after which students practiced each strategy on their own, with my feedback, as I circulated around the class. As the days progressed, my demonstrations decreased in frequency, and the frequency of independent practice increased. Feedback was decreased over the course of the intervention.

On Days 1, 2, 3, and 5, I began the session with a demonstration of the Expanding the Subtrahend strategy for a given expression, followed by a demonstration of the Combining Like Units strategy for a second expression. Students then solved a new expression on their own, with feedback, and then again solved a new expression using the other strategy, also with feedback. On Day 6, I began with a demonstration of the Combining Like Units strategy for a given expression, followed by a demonstration of the Expanding the Subtrahend strategy. Again, students then solved a new expression on their own with feedback and then again solved a new expression using the other strategy, also with feedback. On Days 4, 7, and 8, students practiced using their choice of the two instructed strategies without any feedback. Feedback at all times involved demonstrating the strategy on the board, identical to what was done at the beginning as a demonstration.

Problems involved four types of subtraction: simple (using numbers under 100 with no regrouping), simple/regrouping (using numbers under 100 with regrouping), complex (using numbers over 100), and complex/regrouping (using numbers over 100 with regrouping).

As seen in Figure 10, as the days progressed, the types of problems changed from simple to simple/regrouping, and then further to complex and finally complex/regrouping problems. Over the course of the eight sessions in each condition, a total of 84 mental computations were performed, either as demonstrations or computed individually. Students performed 29 simple computations, 23 simple/regrouping computations, 6 complex computations, and 26 complex/regrouping computations over the course of the eight day intervention.

Day 1	DS1	DS2	PS1	PS2	DS1	DS2	PS1	PS2	DS1	DS2
Problem	78 - 54	99 - 64	74 - 53	74 – 41	65 - 41	58 - 27	46 - 21	67 - 42	68 - 43	93 - 41
Type	S	S	S	S	S	S	S	S	S	S

Day 2	DS1	DS2	PS1	PS2	DS1	DS2	PS1	PS2	DS1	DS2
Problem	79 - 42	84 – 58	87 – 49	88 – 41	63 - 26	54 - 26	49 - 23	98 - 42	79 - 42	84 - 58
Type	S	SR	SR	S	SR	SR	S	S	S	SR

Day 3	DS1	DS2	PS1	PS2	DS1	DS2	PS1	PS2
Problem	67 - 29	84 – 27	58 - 42	57 – 43	87 – 36	74 – 47	65 – 42	63 – 28
Type	SR	SR	S	S	S	SR	S	SR

Day 4	PS	PS	PS	PS	PS	PS	PS	PS	PS	PS	PS	PS
Problem	87 - 24	63 - 37	67 -41	63 – 36	47 - 24	58 – 37	94 – 23	95 – 67	78 – 47	67 - 26	61 - 37	91 - 49
Type	S	SR	S	SR	S	S	S	SR	S	S	SR	SR

Day 5	DS1	DS2	PS1	PS2	DS1	DS2	PS1	PS2	DS1	DS2
Problem	126 - 84	187 - 54	158 - 61	195 – 57	187 - 62	148 - 79	163 - 92	174 - 37	142 - 63	167 - 49
Type	CR	C	CR	CR	C	CR	CR	CR	CR	CR

Day 6	DS2	DS1	PS2	PS1	DS2	DS1	PS2	PS1	DS2	DS1
Problem	264 - 86	273 - 94	186 - 71	243 – 64	245 - 67	152 - 86	249 - 68	157 - 63	386 - 242	487 - 394
Type	CR	C	C	CR	C	CR	CR	CR	C	CR

Day 7	PS	PS	PS	PS	PS	PS	PS	PS	PS	PS	PS	PS
Problem	79 - 43	126 - 74	84 - 47	148 - 92	77 - 28	97 - 39	158 - 89	104 - 33	74 - 38	97 - 54	111 - 36	137 - 52
Type	S	CR	SR	CR	SR	SR	CR	CR	SR	S	CR	CR

Day 8	PS	PS	PS	PS	PS	PS	PS	PS	PS	PS	PS	PS
Problem	69 - 26	136 - 84	94 - 57	158 - 67	87 - 38	87 - 49	168 - 97	106 - 36	77 - 39	87 - 68	113 - 37	147 - 62
Type	S	CR	SR	CR	SR	SR	CR	CR	SR	SR	CR	CR

Figure 10. Mental Mathematics Intervention daily schedule (for both RMM and RCL) with daily problems and problem type. DS1 = Demonstration of Strategy 1, DS2 = Demonstration of Strategy 2, PS1 = Practice of Strategy 1 with feedback, PS2 = Practice of Strategy 2 with feedback, PS = Practice of either Strategy without feedback. Types of problem are indicated (S = simple, SR = simple/regrouping, C = complex, CR = complex/regrouping).

Experimental conditions. The intervention for both conditions consisted of teaching the students two mental computational strategies for subtraction: the Expanding the Subtrahend strategy, and the Combining Like Units strategy. Both strategies were incorporated in the intervention to ensure that students could flexibly choose a method for solving mental computation problems. This would increase the likelihood that the students would engage in mental computation rather than simply applying one assigned procedure. As flexibility has been described as a component of relational thinking (Knuth et al., 2006; Stephens & Ribeiro 2012), by describing how they arrived at their answer using one of the two instructed strategies, students may have become more flexible and improved their relational thinking. Both strategies required students to decompose expressions, permitting practice with substitution, as well as with the creation of equivalent expressions, both components critical in relational thinking (Carpenter et al., 2003, 2005; Jones et al., 2013; Stephens & Ribeiro, 2012).

At the beginning of each instructional session, the students sat at their own desks, facing the front of the classroom. If the students were not part of the condition-specific instruction, the students turned their desks towards the back wall away from the front of the class, and wore headphones and listened to music while doing an assigned task related to the regular mathematics curriculum. The students receiving the intervention each received a handout. Students in the RMM condition received a one-page handout with 12 numbered blank lines (see Appendix G) for them to record their final answers to each mental computation. The RCL condition received a workbook consisting of 20 pages, with alternating pages dedicated to each strategy, so that on days where students were to choose between the two strategies, they already had the requisite workbook pages. Each workbook page was used for one mental computation, with specific locations on the page to record specific numbers used during my demonstrations of the mental mathematics strategies (see Appendix H). No other materials were provided.

In both conditions, on every day of the intervention, I indicated that I would show the students two ways to perform mental computations. On Day 1, I first demonstrated how to compute a subtraction mentally using the Expanding the Subtrahend strategy. To illustrate this strategy, I wrote an expression (e.g., 58 – 39) on the white board. The Expanding the Subtrahend strategy (see Figure 11a) involved decomposing the subtrahend of the expression. I indicated to the students that the subtrahend will be split into tens and ones. I then showed them how to mentally compute another problem by using a Combining Like Units strategy (see Figure 11b), which involved working with the tens in both the minuend and the subtrahend and then adding and subtracting the ones back into the expression, in that order, as necessary.

a)

58 – 39
Step 1: 58 – 30 – 9
Step 2: 28 – 9
Answer = 19.

b)

68 - 37
Step 1: 60 – 30 + 8 – 7
Step 2: 30 + 1
Answer = 31.

Figure 11. Strategies instructed during the Mental Mathematics Intervention. (a) The Expanding the Subtrahend strategy. (b) The Combining Like Units strategy.

In the RMM condition, once the first expression was written on the board, I began by demonstrating the first step of the Expanding the Subtrahend strategy, which was to decompose the subtrahend into its tens and ones (i.e., 39 decomposed into 30 and 9 as in Figure 11a). This yielded Step 1: 58 – 30 – 9, which I wrote on the board underneath the original expression. Once this step was completed, I then showed how to perform 58 – 30 (equalling 28), leading to Step 2: 28 – 9, which I wrote below the first step. I then told the students to compute the final answer mentally which was 19. I wrote this answer below Step 2. Students did not write anything down except the final answer on line 1 on their handout (see Appendix G).

Next, I showed the students in the RMM condition the Combining Like Units strategy. In this strategy, using the example of 68 – 37 as in Figure 11b, I told the students that the first step was to deal only with the tens of the minuend and subtrahend (i.e., 60 – 30), and then to compensate by adding or subtracting the ones back in from both the minuend and subtrahend. In this example, the compensation involved adding 8 and subtracting 7. This yielded Step 1 (60 – 30 + 8 – 7), which I recorded on the board below the initial expression. Following Step 1, I then instructed the students to subtract 30 from 60 (60 – 30), and 7 from 8 (8 – 7), to yield Step 2: 30 + 1, which I wrote below Step 1. Finally, students mentally solved 30 + 1, and wrote their final answer of 31 on the line below Step 2. Students then recorded the final answer on line 2 on their handout (see Appendix G).

In the RCL condition, the same expressions and strategies were demonstrated in the same order as in the RMM condition, but the students used their workbooks to record specific numbers at each step of the strategy. Regardless of the strategy being demonstrated, the students were told to write down the expression that I wrote on the board on the top line of the workbook page. For example, when teaching the Expanding the Subtrahend strategy for the expression 58 – 39, I first asked the students to write the 59 and 39 on the top line of page 1 of their workbook. The arithmetic operation signs were already present on the handout (see Appendix H). Then, I showed the first step (i.e., 58 – 30 – 9), explaining how to decompose the second number into tens and ones. Students were told to write the three numbers (i.e., 58, 30, 9), in that order, on the second line. I then explained that the subtraction 58 – 30 could be computed, to give 28, to arrive at the third step (i.e., 28 – 9). They recorded these numbers (i.e., 28 and 9) on the third line. Finally, I instructed the students to mentally compute the final answer, which they were told to write on the final line in the answer space for the problem.

I gave the same explanation of the Combining Like Units strategy to the RCL condition as I gave to the students in the RMM condition, except that the students in the RCL condition recorded specific parts of the strategy. However, similar to the way they that the RCL condition recorded specific numbers for the Expanding the Subtrahend strategy, students were told to use the same procedure for the Combining Like Units strategy on the second page of their workbook.

Following the demonstration of both strategies, I told the students in both conditions to try to solve the next expression that I wrote on the board on their own, exactly in the way that I had instructed them, using the Expanding the Subtrahend strategy. I allowed them two minutes to mentally compute the answer, allowing the students in the RCL condition to record each step in their workbooks. After this, I demonstrated the strategy for that problem on the board and gave students feedback. For the next problem, I required the students to solve the problem on their own using the Combining Like Units strategy. I allowed them two minutes to compute the answer after which I demonstrated the strategy on the board and gave feedback.

In each condition, after the students had solved each expression, regardless of whether I performed a demonstration or if the students practiced on their own, I circulated throughout the class to verify that the students in the RMM condition only wrote down the answer on the blank line, and that the RCL condition wrote down specific numbers and final answer in the workbooks. At the conclusion of each instructional session, I collected all of the students' handouts (RMM) and workbooks (RCL). Before the next scheduled session in each condition, I verified the students' work to ensure that they appropriately carried out the activities specific to their respective conditions. Upon verification, all students had correctly followed my instructions.

Addressing Experimenter Bias

As the students' regular mathematics teacher, I delivered the intervention to all students. In order to mitigate any possible experimenter bias as a result of delivering all of the instructional sessions, another teacher in the school, who was blind to the study's hypotheses used pre-determined criteria to verify that my instructional practices were equivalent for both conditions. The pre-determined criteria included ensuring that the verbal instructions, verbal and written delivery of demonstrations and feedback, and sequence of intervention items were identical between conditions. The observer was required to verify all of the components on the Equivalent Instruction Criteria checklist (see Appendix J).

Assessment measures

Testing took place before the intervention (over two days at Time 1: ET, RTT, MMAT), which occurred in the final two days of the week preceding the start of the intervention. At the conclusion of the intervention, testing took place at Time 2 on the first two days of the week following the final week of the intervention (ET, RTT, MMAT), as well as one week later for the MMST assessment. At the start of each day of testing, I indicated to the students that they would be writing tests. I told them that the tests would not count in terms of their mathematics course grade, but that I would be very happy if they would complete the tests to the best of their ability and to take the exercise seriously. No feedback or clarification was provided to the students at any time during the assessments, and the students completed the tests independently.

Mental Mathematics Accuracy Test (MMAT). The Mental Mathematics Test (MMAT) consisted of 20 subtraction questions of four different types, as shown in Appendix E. At Time 1, there were seven problems with both minuend and subtrahend less than 100 not involving regrouping (simple problems); three problems with the minuend greater than 100 with no regrouping (complex); seven problems with both the minuend and subtrahend less than 100 with regrouping (simple/regrouping); and three problems with the minuend greater than 100 with regrouping (complex/regrouping) (one of which involved double regrouping). At Time 2, an isomorphic version of the MMAT included seven simple problems; two complex problems; seven simple/regrouping problems; and four complex/regrouping problems.

At the start of testing, I delivered a MMAT answer sheet (see Appendix F) face down to each student at his or her desk. The answer sheet consisted of 20 numbered blank lines where students would write their answers to each item. Once each student had an answer sheet, I told the students that they would be presented with a mathematical expression on the board and that they were to compute the expression mentally. I indicated that the students were not allowed to write anything down except their final answer on the numbered line corresponding to the question asked. Students were instructed that they had 30 seconds to arrive at their answer, after which the expression would be removed from the board. I encouraged the students to review their answers after each question if time allowed, and that they were to remain silent throughout the assessment. At the end of the assessment, students were asked to turn over their papers, and I collected them. Students received 1 point for each correct answer and 0 points for each incorrect answer. The score for each item was summed and divided by 20 to obtain a total MMAT score between 0 and 1. The score was then converted to percent.

Mental Mathematics Strategy Test (MMST). The Mental Mathematics Strategy Test (MMST) was designed to determine the types of strategies students used when computing mental mathematics problems. It consisted of five subtraction questions: two simple problems (e.g., 84 – 32, 96 – 49), two simple/regrouping problems (e.g., 75 – 31, 92 – 37), and one complex problem (e.g., 153 – 32). Students were required to write down all the steps they used to mentally compute a problem.

At the start of testing, I delivered an MMST answer booklet (see Appendix I) face down to each student at his or her desk. The booklet consisted of five pages, with 10 lines on each page. At the top of each page, the student wrote the question as seen on the board. Below, they wrote down the steps they used to mentally compute the answer. On the last step, students were asked to write down their final answer.

Students were not constrained to any specific mental computation strategy. My specific instructions to the students were: “Today I am asking you to solve five subtraction problems in your head. I will write the first one down on the board and you will have one minute to write down all of the steps you are taking in your head, showing me one step per line. Once you have shown all of your steps, put your final answer on the next line. You are free to solve the problem any way you want, but please be sure to show me how you came to your answer. Don’t only write down the final answer to the problem. We will do a total of five of these problems today. Once the minute is over, I will erase the problem and go on to the next one.”

I encouraged the students to review their steps and answers after each question if time allowed, and told them that they were to remain silent throughout the assessment. At the end of the assessment, students were asked to turn over their papers, and I collected them.

The students’ written work on the MMST was coded for strategy type. The strategy on each problem was coded as either Expanding the Subtrahend, Combining Like Units, or Other. A random sample of 20% of the responses was coded by a trained second rater, and inter-rater reliability of 94% agreement was achieved.

Equivalence Test (ET). A paper-and-pencil equivalence test (created by Watchorn & Bisanz, 2005) was administered to both classes at two time points. The test consisted of 29 problems, including 9 canonical and 20 non-canonical problems, and each contained only single-digit numbers. Examples of problem types were: 7 + 8 = 6 + __, 7 + 3 = 7 + __, 4 + 7 = 7 + __. Two isomorphic versions of the ET, presented in Appendix C, were used for counterbalancing purposes. Each version contained the same number of each type of problem (i.e., nine addition, $a + b =$ __; four identity, $a + b = a +$ __; four commutativity, $a + b = b +$ __; four part–whole, $a + b = c +$ __; and eight combination, $a + b + c = a +$ __; Watchorn & Bisanz, 2005). The numbers in the problems and the order of the problem types varied in each version.

At the start of testing, I delivered the ET to each student, placing it face down on his or her desk. Once every student had a test, I went through the instructions orally in front on the class. Specifically, I stated that the class had 15 minutes to complete the ET, and that they were to complete the assessment by writing down their answer on a blank line provided in each equation. I also instructed the students to look over their answers once they were finished, and to remain silent for the duration of the assessment. After the 15 minutes were completed, students were asked to turn over their papers, and I collected them.

Only the responses for the 20 non-canonical problems were used in the analyses. Correct answers received 1 point and incorrect answers 0 points. The points were summed to obtain a total score, which was then divided by 20 for an ET score between 0 and 1. The score was then converted to percent.

Relational Thinking Test (RTT). A relational thinking test, based on Osana et al. (2013) and Carpenter et al. (2003), was administered to assess students' relational thinking. There were four items on the test, two involving addition and two involving subtraction (see Appendix D). Each item consisted of a number sentence, such as 45 + 26 = 47 + 28. The students were asked to indicate whether the sentence was true or false by circling the word "true" or "false" on the test paper. The students were then asked to provide a written justification for their responses in a blank space provided on the test.

Two isomorphic versions of the RTT were administered for counterbalancing purposes. In each version, the order of the items varied. The numbers used in each item were also different, but the structure of the numerical relationships across versions remained the same for each operation (e.g., Version 1: 67 + 48 = 65 + 46; Version 2: 55 + 36 = 53 + 34).

Students were given 20 minutes to complete the test. Again, I reviewed the instructions orally with the class before they began. I instructed the students to indicate in each question if the number sentence was true or false and to justify their answers in the space provided. The students were also told that they were not permitted to communicate during the test. After 20 minutes, students were asked to turn over their tests, and I collected them.

Students' written justifications were coded using the following rubric: (a) Category 1: Relational thinking without computation or with computation only as a means to justify a written relational response; (b) Category 2: Relational thinking with computation; (c) Category 3: Other. Student sample responses are shown in Figure 2. Responses that were placed in Category 1 demonstrated that the student engaged in relational thinking by considering the relationship between the numbers without computing the quantities on both sides of the equal sign to determine the truth value of the equation. Computation in this category was permitted only if the student had first justified the response relationally and only if the computation was used to support or illustrate the relational response. Student responses that were placed in Category 2 demonstrated that the student had an understanding of the equal sign and that they were able to determine if the response were true or false with the use of computation only. Category 3 responses were those where the student either had an operator view of the equal sign, did not supply any justification, or provided responses that too difficult to interpret with any confidence.

Category 1 responses received 2 points, Category 2 responses received 1 point, and Category 3 responses received 0 points. Category 1 responses were awarded more points than Category 2 and 3 responses because they indicated that the students responded relationally and did not need to compute to make a decision about the truth value of the equation. Category 2 responses were also considered to be relational, as students appeared to understand the meaning of the equal sign, but chose to compute rather than to consider the relationships between the numbers to arrive at their answer. Category 3 responses received 0 points because they contained no evidence of relational thinking or understanding of the equal sign.

Student scores were the total number of points assigned, with a minimum score of 0 and a maximum score of 8. The sums were divided by 8 to obtain an RTT score between 0 and 1. These scores were then converted to percent. A random sample of 20% of the responses was coded by a trained second rater, and inter-rater reliability of 91% agreement was achieved.

Results

The present study was conducted to determine if writing down certain components of a mental mathematics calculation during a mental mathematics intervention would offer students an advantage over students who were not permitted to write anything down. Students in both RMM and RCL conditions were assessed at the beginning of the study and immediately after on the Equivalence Test (ET) and the Relational Thinking Test (RTT). These two assessments were administered to determine whether there were condition effects on students' understanding of the equal sign and their relational thinking. In addition, students in both conditions were assessed on mental mathematics computation problems using the Mental Mathematics Accuracy Test (MMAT) before and after the intervention to determine if their accuracy had improved following the intervention. Moreover, a week after the conclusion of the Mental Mathematics (MM) intervention, students completed the Mental Mathematics Strategy Test (MMST), a written test used to assess the strategies students used to solve mental mathematics problems. These strategy data were collected to gain insight on the reasons behind any potential condition effects. The effects of the RCL condition in relation to the RMM condition on the four assessments measures (ET, RTT, MMAT, and MMST) are presented in the following sections.

Performance on ET as a Function of the MM Intervention

On the ET, I predicted that both conditions would improve. However, as the RCL condition's cognitive load was reduced during the intervention, I also predicted that they would outperform the RMM condition at posttest, as the RCL condition would be more likely to attend to the manipulation of the numbers during the instruction. The means and standard deviations of the Equivalence Test (ET) scores as a function of condition, question type, and time are presented in Table 4.

Table 4

Means and (Standard Deviations) of the Equivalence Test (ET) as a Function of Condition, Question Type, and Time

	T1		T2	
Problem Type	RMM[a]	RCL[b]	RMM[a]	RCL[b]
Combination	56.7 (49.6)	72.9 (41.8)	95.1 (10.6)	94.9 (6.34)
Commutative	56.9 (49.9)	85.3 (29.3)	100.0 (0.0)	100.0 (0.0)
Identity	58.3 (49.3)	83.8 (33.0)	100.0 (0.0)	98.5 (6.06)
Part-Whole	55.6 (48.9)	76.5 (36.9)	95.8 (9.59)	92.7 (14.7)
Total	55.8 (48.5)	79.1 (33.0)	97.2 (5.21)	96.2 (4.52)

Note. All scores are reported in percent. RMM = Regular Mental Mathematics condition, RCL = Reduced Cognitive Load condition. T1 = Time 1, T2 = Time 2. [a]$N = 18$, [b]$N = 17$.

A 2 time (Time 1, Time 2) x 2 condition (RMM, RCL) x 4 problem type (combination, commutative, identity, part-whole) mixed ANOVA was performed using condition as the between groups factor and time and problem type as the within factors. The ANOVA did not reveal any effects of problem type, and therefore the levels of this factor were collapsed for further analysis. A 2 time (Time 1, Time 2) x 2 condition (RMM, RCL) mixed ANOVA was then performed, using condition as the between groups factor and time as the within factor. A graphical representation of the ET means is presented in Figure 12. The ANOVA revealed a main effect of time, $F(1, 33) = 18.04$, $p < .001$, $\eta^2 = .65$, with Time 2 mean scores ($M = 96.7$, $SD = .05$) higher than Time 1 mean scores ($M = 67.1$, $SD = .43$). However, no main effect of condition or time by condition interaction was found. Taken together, these results do not support my prediction that the RCL would improve more over time than the RMM condition, but the results show that both conditions improved over time regardless of condition.

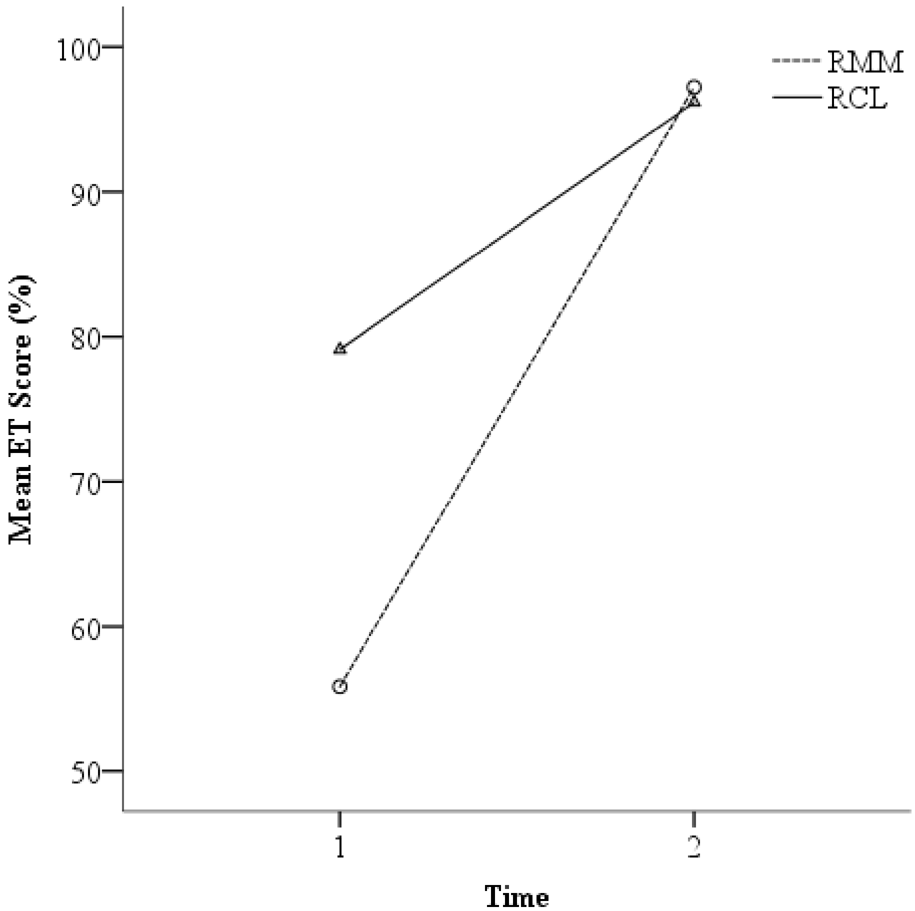


Figure 12. Mean Equivalence Test scores as a function of Time and Condition. Means are reported in percent. RMM = Regular Mental Mathematics condition, RCL = Reduced Cognitive Load condition.

Performance on RTT as a Function of the MM Intervention

On the RTT, I predicted that both conditions would improve, but because their cognitive load was alleviated, I predicted additionally that the RCL condition would improve to a greater extent than the RMM condition over time. The means and standard deviations of the Relational Thinking Test (RTT) scores as a function of condition, question type, and time are presented in Table 5.

Table 5

Means and (Standard Deviations) of the Relational Thinking Test (RTT) as a Function of Condition, Question Type, and Time

	T1		T2	
Problem Type	RMM[a]	RCL[b]	RMM[a]	RCL[b]
Addition	43.1 (16.7)	50.0 (26.5)	81.9 (24.0)	91.2 (20.0)
Subtraction	41.7 (19.2)	48.5 (22.5)	73.6 (26.4)	85.3 (19.9)
Total	42.4 (17.7)	49.3 (25.0)	77.8 (23.3)	88.2 (19.0)

Note. All scores are reported in percent. RMM = Regular Mental Mathematics condition, RCL = Reduced Cognitive Load condition. T1 = Time 1, T2 = Time 2. [a]*N* = 18, [b]*N* = 17.

A 2 time (Time 1, Time 2) x 2 condition (RMM, RCL) x 2 problem type (addition, subtraction) mixed ANOVA was performed using condition as the between groups factor and time and problem type and as the within factors. A graphical representation of the mean RTT scores over time is presented in Figure 13. The ANOVA revealed a main effect of time, $F(1, 33) = 8.51$, $p < .001$, $\eta^2 = .21$, with mean RTT scores across both conditions and both problem types higher at Time 2 ($M = 82.9$, $SD = 21.7$) compared to Time 1 ($M = 45.71$, $SD = 21.9$). There was no condition effect. The ANOVA also revealed a main effect of problem type, $F(1, 33) = 86.40$, $p = .006$, $\eta^2 = .72$, with students scoring higher on addition problems ($M = 66.5$, $SD = 18.1$) than subtraction problems ($M = 62.3$, $SD = 18.6$) across both conditions and time points. The main effect of problem type is uninterpretable in this context, however, given that the means are averaged across conditions and time points. No interactions were found. Taken together, these results do not support my prediction that the RCL would improve more over time then the RMM condition. Again, however, the results revealed that both conditions improved over time regardless of condition.

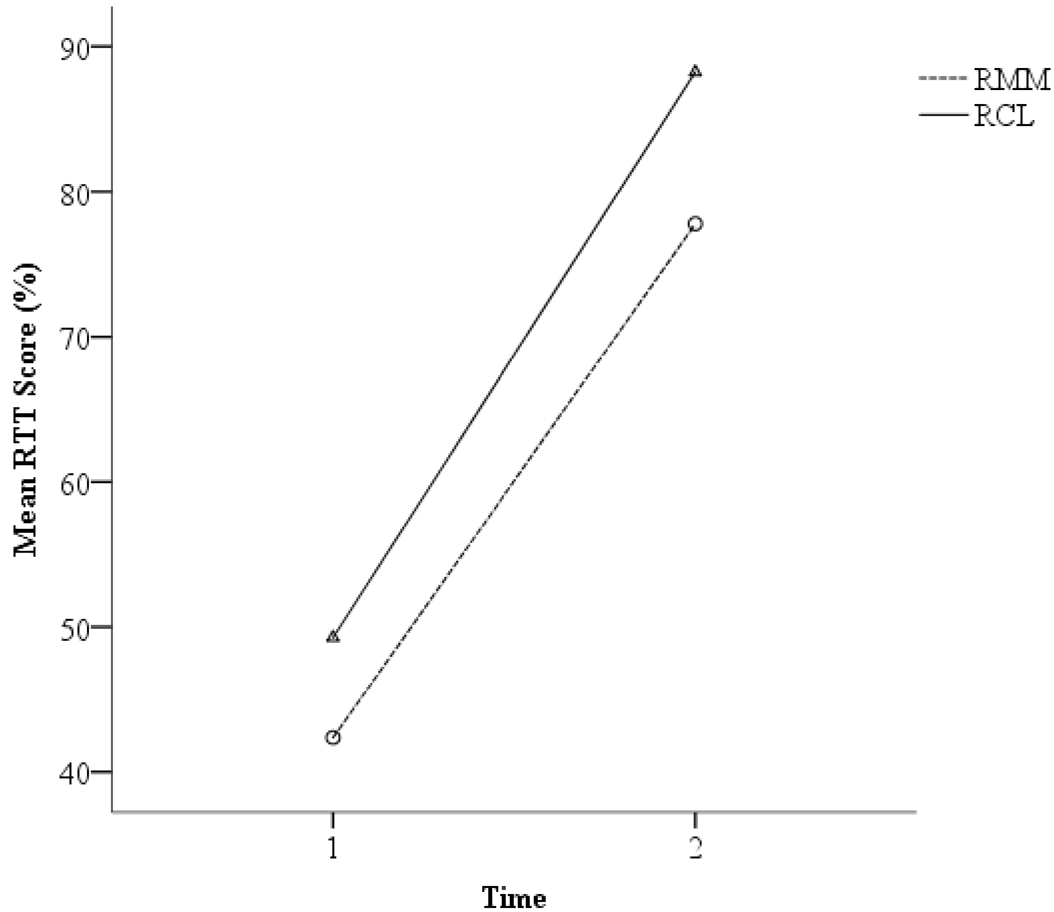


Figure 13. Mean Relational Thinking Test scores as a function of time and condition. Means are reported in percent. RMM = Regular Mental Mathematics condition, RCL = Reduced Cognitive Load condition.

Performance on MMAT as a Function of the MM Intervention

On the MMAT, I predicted that both conditions would improve, but that the RMM condition would outperform the RCL condition at posttest because of the additional practice in mental computation (without tools) they received during the intervention. The means and standard deviations of the MMAT scores as a function of condition, question type, and time are presented in Table 6.

Table 6

Means and (Standard Deviations) of the Mental Mathematics Test (MMAT) as a Function of Condition, Question Type, and Time

	T1		T2	
Problem Type	RMM[a]	RCL[b]	RMM[a]	RCL[b]
Simple	91.3 (14.8)	90.8 (12.3)	91.3 (14.8)	90.8 (12.3)
Simple/regrouping	65.9 (40.0)	62.2 (27.6)	84.9 (24.2)	77.0 (25.9)
Complex	90.7 (19.2)	92.2 (14.6)	88.9 (21.4)	91.2 (19.7)
Complex/regrouping	57.4 (33.9)	51.0 (37.5)	69.4 (32.7)	77.9 (31.7)
Total	77.2 (23.3)	75.0 (16.4)	86.9 (15.0)	84.9 (18.0)

Note. All scores are reported in percent. RMM = Regular Mental Mathematics condition, RCL = Reduced Cognitive Load condition. T1 = Time 1, T2 = Time 2. [a]$N = 18$, [b]$N = 17$.

A 2 time (Time 1, Time 2) x 2 condition (RMM, RCL) x 4 problem type (simple, simple/regrouping, complex, complex/regrouping) mixed ANOVA was performed using condition as the between groups factor and time and problem type as the within group factors. A graphical representation of the MMAT mean scores by time for each condition is presented in Figure 14, with the top panel referring to the RMM condition and the bottom panel referring to the RCL condition. The ANOVA revealed no main effect of condition, but there was a main effect of time, $F(1, 33) = 9.09$, $p = .005$, $\eta^2 = .21$, with mean MMAT scores higher at Time 2 (*M*

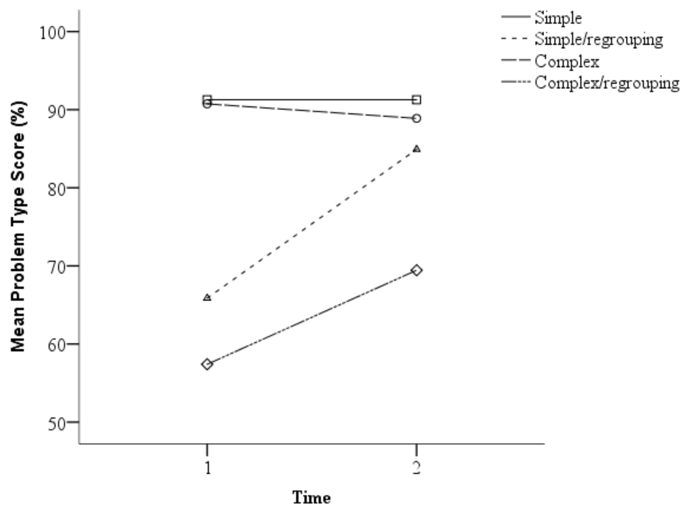
= 85.9, *SD* = 16.3) compared to Time 1 (*M* = 76.1, *SD* = 20.0) across both conditions and problem types.

The ANOVA also revealed a main effect of problem type, $F(3, 99) = 25.37$, $p < .001$, $\eta^2 = .43$. Post-hoc analyses with Bonferroni corrections indicated that performance on simple problems (*M* = 91.0, *SD* = .14) was significantly higher than performance on simple/regrouping problems (*M* = 72.6, *SD* = .26), $p < .001$, and also higher than complex/regrouping problems (*M* = 64.0, *SD* = .26), $p < .001$. Performance on complex problems (*M* = 90.7, *SD* = .14) was also significantly higher than simple/regrouping problems (*M* = 72.6, *SD* = .26), $p < .001$, and complex/regrouping problems (*M* = 64.0, *SD* = .26), $p < .001$. Finally, performance on simple/regrouping problems (*M* = 72.6, *SD* = .26) was significantly higher than complex/regrouping problems (*M* = 64.0, *SD* = .26), $p < .01$). In sum, regardless of condition and time, performance on problems that did not involve regrouping was higher than those involving regrouping.

In addition, there was a time by problem type interaction, $F(3, 99) = 6.204$, $p < .001$, $\eta^2 = .15$, indicating that students' performance over time across conditions depended on problem type. Tests of simple effects indicated that for simple/regrouping problems, students showed improvement between Time 1 and Time 2, $p < .001$, and the same effect was found for complex/regrouping problems, $p < .001$. There was no change on simple or complex problems, possibly because of ceiling effects at both time points.

Taken together, these results do not support my prediction that the RMM would improve more over time than the RCL condition. However, the results reveal that both conditions improved over time regardless of condition, with performance on problems involving regrouping improving more than those without.

a)



b)

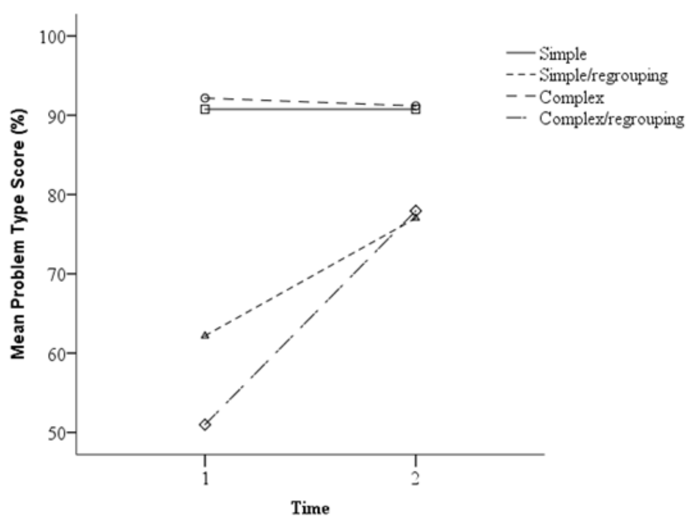


Figure 14. Mental Mathematics Accuracy Test problem type mean scores as a function of time for each condition. a) Regular mental math (RMM) condition. b) Reduced cognitive load (RCL). Means are reported in percent.

Mental Mathematics Strategy Test

Strategy use on the MMST was analyzed using 2 x 2 chi-square analysis, with item as the unit of analysis (*N* = 170). This test was conducted to test for a relationship between condition and strategy type (i.e., Expanding the Subtrahend, Combining Like Units, or Other). Table 7 reports the frequencies and proportions (within condition) of strategy use on the MMST. The proportion of strategy use within each condition are graphed in Figure 15.

Table 7

Frequency of Strategy Type on MMST by Condition and (Percent within Condition) (N =170)

Condition	Expanding the Subtrahend	Combining Like Units	Other
RMM	4 (4.7)	23 (27.1)	58 (68.2)
RCL	5 (5.9)	47 (55.3)	33 (38.8)
Total	9 (5.3)	70 (41.2)	91 (53.5)

Note. RMM = Regular Mental Mathematics condition, RCL = Reduced Cognitive Load condition.

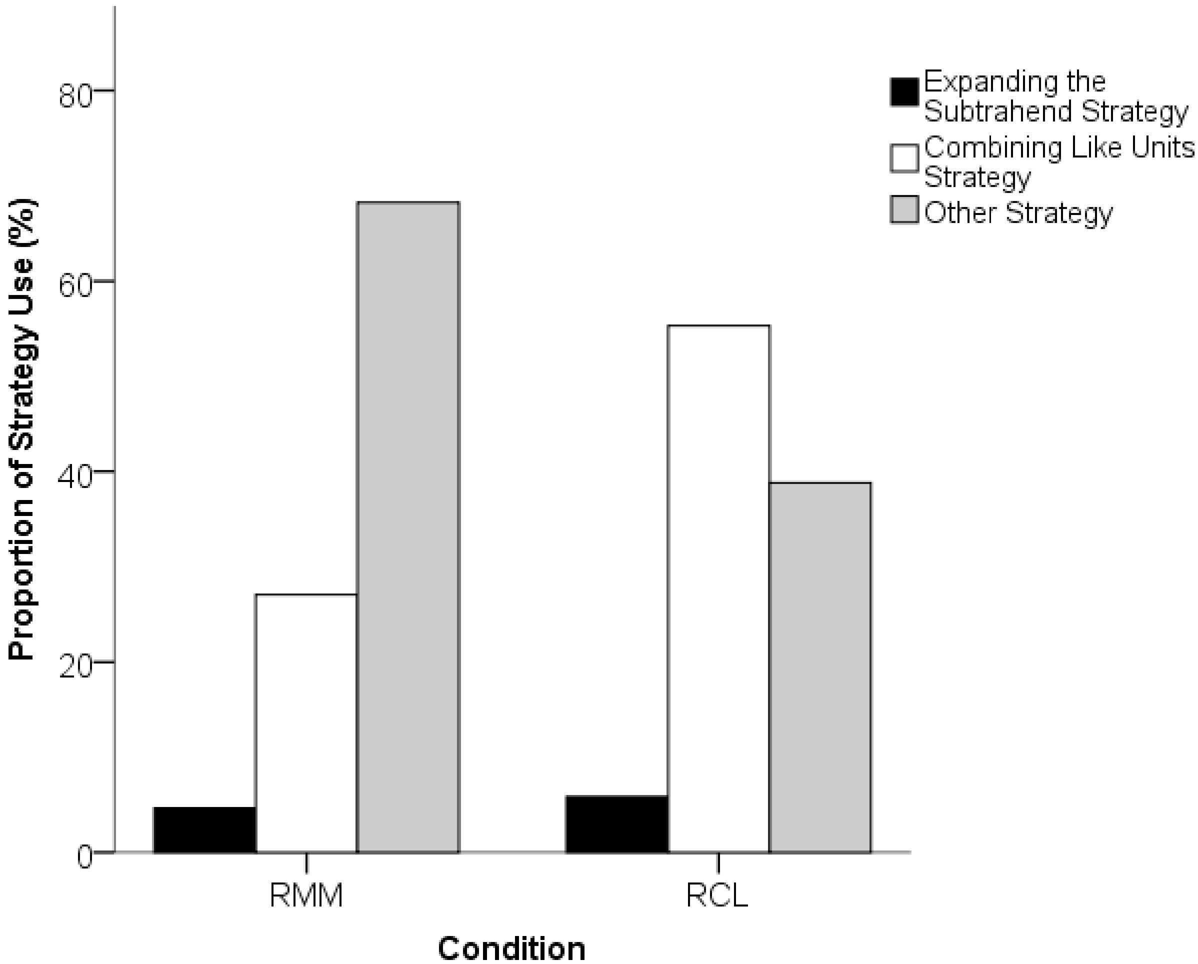


Figure 15. Mental Mathematics Strategy Assessment: Proportion of strategy use by condition. RCL = Reduced Cognitive Load condition, RMM = Regular Mental Mathematics condition.

Results indicated that strategy type was significantly related to condition, $\chi^2(2) = 15.21$, $p < .001$. As indicated in table 7, in the RMM condition, over two-thirds of the items were solved using strategies other than the two taught during the intervention. In contrast, over half of the items in the RCL condition were solved using the Combining Like Units strategy, and about 40% of items were solved using other strategies. These results may explain the lack of condition effects on the ET and RTT. Despite my assumption that students in both conditions would use the strategies taught during the intervention, it appears from the MMST results that, relative to students in the RCL condition, those in the RMM condition may have relied on their own personal strategies rather than those I taught during the MM intervention.

On the ET and RTT, I predicted that the RCL condition would outperform the RMM condition at posttest. However, it appears from the MMST results that RMM students may have struggled with the MM intervention strategies because their cognitive load may have been taxed more than their RCL counterparts as they performed the mental computations. As such, the burden on cognitive load may have resulted in the RMM condition reverting to their own more meaningful strategies, which in turn may have yielded better performance on the posttest than expected. Such a boost in performance may have removed the interaction effect I predicted.

For the MMAT, it was predicted that the RMM condition would outperform the RCL condition at posttest because they had more opportunities to practice mental computation (without any tools) during the intervention. This interaction was not obtained: Both conditions performed equally well at posttest. Thus, despite the potential advantageous practice effect for RMM, the reduction in cognitive load for the RCL condition may have allowed the RCL condition to perform better than expected at posttest, thereby removing the predicted time x condition interaction.

Discussion

The objective of the present study was to investigate if reducing seventh-graders' cognitive load during a mental mathematics intervention (by allowing them to write down certain elements of a mental computation problem) would allow them to improve more, on equivalence problem solving and a test of reasoning about non-canonical equations than students who were not able to write down anything during the intervention. The study builds on Study 1 and Study 2, both of which verified that a mental mathematics intervention was linked to improvements in equivalence problem solving and relational thinking. Study 3 extends both Study 1 and 2 in that it assessed whether students' performance could be augmented if their cognitive load was reduced by being permitted to write down certain elements during the MM intervention. In addition, Study 3 also allowed for the determination of a possible causal effect between mental mathematics and relational thinking, as random assignment was part of the study design.

Summary of Findings

As I predicted, both conditions improved on their performance on the test of equivalence problems following the mental mathematics intervention. These findings were consistent with the results found in Study 1 and Study 2, but, despite the finding that all students improved on the ET, the results do not support my prediction that the RCL condition would outperform the RMM condition at posttest. Also in line with my predictions, both conditions also improved on their performance on the relational thinking measure following the mental mathematics intervention. Again, for the two conditions, these findings were consistent with Study 1 and Study 2. However, similar to the findings on the ET, the results on the RTT do not support my prediction of a time by condition interaction, namely that the RCL condition would outperform the RMM condition at posttest.

Finally, as I had predicted, both conditions improved in their mental computations following the intervention, which provided evidence that the objectives of the intervention were reached. For the two conditions, the overall improvement in mean scores on the mental computation assessment was consistent with the results found in Study 2. A time by problem type interaction was found for problems that involved more difficult regrouping items, while no interaction was found for items without regrouping.

The results on the Mental Mathematics Accuracy Assessment (MMAT) do not support my prediction, however, that the RMM condition would outperform the RCL condition at posttest. Originally, I had hypothesized that because the RMM condition would have practiced performing mental computations during the intervention without writing anything down, they would have outperformed the RCL condition. However, this was not the case, as both conditions performed similarly on the MMAT. In addition, across both conditions, the intervention impacted the performance on the most challenging problems.

Explaining the Results

Equivalence (ET) and relational thinking (RTT) tests. The data from the MMST (which revealed students' strategy use for mental computation) allowed me to speculate on the lack of condition effects on the ET and RTT. Specifically, students in the RMM condition performed better at posttest than I had originally predicted. According to the MMST results, students in the RMM condition solved the majority of the problems on the MMST using strategies other than the ones I instructed during the intervention, while in the RCL condition, the majority of the items were solved using the instructed interventional strategies. I speculate that because the RMM students may have struggled with the instructed strategies, perhaps because they could not document parts of their work, they consequently reverted back to their own more meaningful strategies.

Previous research suggests that the type of instruction being provided to the student can have a strong impact on the individual student's willingness to accept or abandon a newly instructed strategy (Alibali, 1999). The literature supports the possibility that students may not always adopt new strategies taught to them. Students' learning of any subject matter arises from the interaction between what they are taught and what they previously know. It is well documented that students' previous knowledge and beliefs about a given topic can strongly influence the ways in which they make sense of new ideas, particularly in mathematics (Ball, 1988; Siegler, 2003). As a result, the RMM condition's performance may have been enhanced relative to what I had originally predicted, creating equivalent results on the ET and RTT at posttest between both RCL and RMM conditions. As a result of their cognitive load being reduced, I expected that the RCL condition would outperform the RMM condition at posttest, but the use of personal and meaningful strategies by the RMM condition may have had the effect of removing the predicted time by condition interaction.

Therefore, with regard to the strategies undertaken by the RMM condition, it is possible that they struggled with the newly instructed strategies as a result of their previous knowledge (or the strategies they were previously instructed), or the instruction of the strategies which possibly did not resonate with the RMM condition as much as had been originally predicted.

Mental Mathematics Accuracy Test. I had originally predicted that the RMM condition would outperform the RCL condition at posttest because they had more opportunities to practice mental computation (without any tools) during the intervention. The predicted time by condition interaction was not obtained: Both conditions performed equally well at posttest. Despite the potential advantageous practice effect for RMM, the reduction in cognitive load for the students in the RCL condition may have permitted them to perform better than expected at posttest, thereby removing the predicted interaction.

Overall Improvements in Relational Thinking

The general improvement by both conditions in their understanding of the equal sign and relational thinking abilities may be attributed to a variety of mental processes that both conditions had developed during the mental mathematics intervention. First, the students were required to construct equivalent expressions within the confines of each instructed strategy during the mental mathematics intervention, without the explicit use or mention of the equal sign. They may have begun to change their view of the equal sign from an operator view to the relational meaning "can be substituted for" (e.g., substituting 63 + 42 by 60 + 40 + 3 – 2 in the Combining Like Units strategy) (Jones et al., 2013), which likely encouraged students to create transformed equivalent expressions, thereby establishing conditions conducive to viewing the symbol as relational (Carpenter et al., 2003, 2005; Stephens & Ribeiro, 2012). By gaining experience using the instructed or other strategies, students in both conditions may have learned to construct equivalent expressions, which may have been key to the improvement of their relational thinking scores (Carpenter et al., 2003).

Limitations

As in Study 1 and 2, I was the students' regular mathematics teacher, and I was responsible for the delivery of all the mental mathematics sessions in both conditions. Despite my efforts to deliver equally effective instruction in both conditions, it is possible that experimenter bias may have occurred, causing me to unknowingly and unintentionally instruct or create a learning environment that would result in confirming my predictions. To mitigate this possible experimenter bias, I had an observer who was blind to the study's hypotheses verify, based on predetermined criteria, that my instructional practices were equivalent for both conditions. Although the observer indicated that indeed the instructional practices were comparable, she was only able to attend the interventions on two of the eight intervention days, and therefore it is possible that there was experimental bias on the days she was not present.

In addition, the study took place in a public suburban high school with a student population from middle- to high-income families which may have reduced the external validity of the study. It is possible that these results are not applicable to other high school populations, as students from different populations, with different backgrounds, may respond differently to the intervention.

Moreover, the design of the study was such that the RMM condition may have had an advantage of a practice effect as the requirements of the mental mathematics outcome measure were identical to the activities they practiced during the intervention. The format of the MMAT assessment was the same as what they had practiced during the intervention. Despite this limitation of the study design, the practice effect did not appear to advantage the RMM condition as their mean MMAT scores were comparable to the RCL condition at posttest.

Another possible limitation of the study design was that the MMST was delayed by one week following the conclusion of the MM intervention. Despite my best attempt to administer the MMST as soon as possible after the intervention, constraints in the students' schedule prevented earlier administration. It is possible that had the MMST been administered with the other assessments immediately following the conclusion of the intervention, the results may have more accurately represented the mental computation strategies used on the MMAT items. Furthermore, the MMST did not include any complex/regrouping items, and therefore no data are available to determine which strategies students used for these types of problems. In the future, interviewing students to investigate the nature of mental computation strategies and the reasons they were used may also be advantageous when attempting to understand the development of students' thinking in mental arithmetic.

Finally, in terms of the MMAT, it would have been beneficial to include more items on the assessment, and more of each type of problem. The number of each type of problem on the pretest and posttest were not identical, and it would have been beneficial to assess students on an equal number of problem types at both pre and posttest.

Instructional Implications

Educators can appreciate from Study 3 that mental mathematics may serve as a valuable way to improve relational thinking in middle school students. As the results suggest, students' abilities in relational thinking improved following a daily 20 minute mental mathematics intervention over eight days. Therefore, teachers seeking to improve students' understanding of the equal sign symbol and relational thinking could rather easily incorporate mental mathematics instruction into their daily mathematics routine. From my perspective, fellow educators need to understand the similarities between mental mathematics and relational thinking so that that their mental mathematics instruction can maximize the likelihood of students' learning. The results of the present study add to the existing literature as there is a lack of research on how to improve relational thinking in real classroom settings with the use of a clearly prescribed instructional intervention. Moreover, literature to address how to improve relational thinking by decreasing the role of working memory in mental mathematics is also sparse. Further research on how to improve relational thinking is indeed required, including research on reducing cognitive load, during mental mathematics instruction.

Chapter 5: Conclusion

The studies in this dissertation sought to address the need to improve students' relational thinking and its instruction in the context of whole number arithmetic in seventh-grade students as a means to prepare them for algebra. Relational thinking in mathematics is important for primary- and middle-school students, as it sets the foundation for future success in middle- and high-school algebra (Lacampagne, Blair, & Kaput, 1999; NCTM, 2000; Stacey, Chick, & Kendal, 2004). With this in mind, educators, researchers, and policy makers in North America and beyond have stressed the importance of developing students' relational thinking in earlier grades (Carraher & Schliemann, 2007; Cai & Knuth, 2005; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; NCTM, 2000). As suggested in this dissertation, relational thinking can be developed through mental mathematics instruction, as all three studies demonstrate an overall improvement in relational thinking following a classroom-based mental mathematics intervention. Although no causal link was established, the results are encouraging.

Improvements in Relational Thinking through Mental Mathematics

The present dissertation was based on the common theoretical underpinnings that exist between mental mathematics and relational thinking. These include an understanding of equivalence, number properties, and an ability to be flexible with numbers and their operations (Carpenter et al., 2003; Stephens & Ribeiro, 2012; Thompson, 2010; Threlfall, 2002). However, the manner by which improvements in mental mathematics is linked to improving relational thinking needs to be further addressed.

In the present dissertation, the results can be explained by the transfer of knowledge from mental mathematics to relational thinking – that is, the notions and skills that students acquired likely transferred to relational thinking tasks. In general, transfer is defined as the process where learning that occurs in a given context is able to enhance or weaken one's abilities in another context (Perkins & Salomon, 1992). Analogical reasoning is a model of transfer that can be useful for explaining the findings in the three studies. The theory of analogical reasoning involves several components including recognizing common elements between two situations and mapping or aligning the two situations and proposing inferences from one to the other (Gentner, Loewenstein, & Thompson, 2003; Gentner & Smith, 2012). According to Gentner and Calhoun (2010), analogical mapping is critical for success in relational thinking.

Analogical reasoning is not limited to the mathematical domain (Gentner & Calhoun, 2010). For example, students who learn about equivalence through a mental mathematics intervention could also possibly extend their new knowledge to an understanding of balancing atoms and molecules in chemical formulas in their chemistry class, or to an understanding that when interpreting a musical score, every bar of music must contain an equivalent amount of beats per bar. As such, relational thinking within and between course subjects could theoretically be improved through mental mathematics, as students may be able to transfer their understanding of equivalence to not only other mathematical contexts, but to other academic domains as well.

During the mental mathematics interventions that took place in the present studies, it is likely that students noticed and created analogs as they mentally computed. Students transformed, substituted, compared, and created equivalent expressions without any explicit mention of the equal sign from the instructor. It is likely that they used analogical reasoning skills to map a given expression onto an equivalent (i.e., analogous) expression that may have been more manageable for them to compute. For example, given the expression 24 x 4, students could create or notice their peers generating the analogous expression (24 x 2) + (24 x 2). With time, I speculate that the structural mapping that occurred between the two expressions caused the induction of a schema, which is an idea or concept that can be transferred to a different context (Gick & Holyoak, 1983). In the three studies conducted in this dissertation, I speculate that the induced schema was the notion of equivalence. The notion of equivalence, induced from the mental mathematics sessions were transferred to novel relational thinking tasks, resulting in improved performance after the intervention.

Because the literature describing mental mathematics as a vehicle for improvements in relational thinking are virtually non-existent, it is not surprising that the literature to support a bidirectional effect – that is, relational thinking as a means to improve mental mathematics – is also sparse. In several domains, analogical reasoning in both directions (i.e., bidirectional effects) is common. It would seem logical that if mental mathematics instruction can support improvements relational thinking (as suggested in this dissertation), instruction in relational thinking could likewise foster improvements in mental mathematics.

Many domains highlight bidirectional transfer in their descriptions of learning. For example, bidirectional transfer of learning has been documented in language studies performed by Pavlenko and Jarvis (2002), who conducted research requiring native Russian speakers learning English as a second language to produce oral narratives in both languages. The narratives indicated that cross linguistic influence worked in a bidirectional fashion: The Russian language influenced the English oral narratives, while the English language influenced their Russian narratives. Therefore, transfer of learning occurred not only as participants transferred their linguistic knowledge of Russian into their English narratives, but likewise their native Russian language began to change as they transferred their knowledge of English onto their Russian language narratives.

In mathematics, a bidirectional transfer of learning between procedural and conceptual knowledge has also been discussed at length (Resnick & Ford 1981; Rittle-Johnson, Schneider & Star, 2015). Students need to acquire both conceptual and procedural knowledge in mathematics, but much debate has occurred as to whether there is a bidirectional transfer of knowledge between the two types of knowledge, and which occurs first. Most mathematics educators confer that conceptual knowledge can often foster and provide the foundation for procedural knowledge, as conceptual knowledge can help students create and invent their own procedures (Hiebert & Lefevre 1986; Kamii & Dominick, 1997). However, it is also argued that a solid foundation in procedural knowledge can support students' conceptual knowledge (Karmiloff-Smith, 1992; Siegler & Stern 1998). In studies conducted by Schneider, Rittle-Johnson, and Star (2011) investigating the relationship between procedural and conceptual knowledge of linear equations in seventh- and eighth-graders, it was shown that following an intervention where students were instructed only on conceptual knowledge of linear relations, students performed equally as well on an assessment that isolated conceptual knowledge as on an assessment designed to assess procedural knowledge in linear relations. The opposite was also found to be true, suggesting that instruction centered on either conceptual or procedural knowledge can be transferred bidirectionally between the two domains. Therefore, learning one type of knowledge (i.e., conceptual or procedural) led to improvements in the other type of knowledge (i.e., procedural or conceptual).

Although no research exists to support the transfer of knowledge from relational thinking to mental mathematics, it is likely that, just as my research suggested improvements in mental mathematics because of improvements in relational thinking, the converse may also be true.

Throughout the studies in this dissertation, I found that instruction centered on mental mathematics was linked to improvements in relational thinking. This was predicted and explained by the theoretical consistencies across both domains. I speculate, therefore, that instruction in relational thinking could also be linked to improvements in mental mathematics. More specifically, it is possible that relational thinking instruction centered on the notion of mathematical equivalence, substitution, as well as on compensation and transforming equivalent expressions, could foster improvements in mental mathematics. This is an important area for future research.

Instructional Implications

The present three studies contribute the literature as they are the first to investigate the relationship between mental mathematics instruction and improvements in relational thinking at the seventh-grade level. Moreover, they are the first to provide a correlational link between mental mathematics instruction and relational thinking, regardless of level. The studies provide novel and explicit instructional strategies on how to foster improvements in relational thinking overall, beyond an improvement in students' understanding of the equal sign, which is the most commonly studied component of relational thinking that exists in the literature at present.

Educators can learn from the present studies that mental mathematics serves as an effective way to enhance relational thinking at the seventh-grade level. As my results suggest, students' abilities in relational thinking can improve following a mental mathematics intervention of just 20 minutes per day over a short period. Therefore, educators hoping to improve their students' relational thinking should aim to incorporate mental mathematics instruction into their daily mathematics routine.

To have the same effects as was observed in the research I conducted, however, the results of my studies also imply that teachers are required to have a solid foundation of number properties and to be able to use the equal sign meaningfully to reflect on numerical relationships (Jacobs et al., 2007). It is important that middle school educators understand the similarities between mental mathematics and relational thinking, as described in the present dissertation, so that they are able to connect mental mathematics to relational thinking through their instruction.

Limitations and Directions for Future Research

The studies I conducted revealed a relationship between mental mathematics instruction and improvements in relational thinking. However, the investigations in this dissertation were conducted only with seventh-graders. In the future, it would be beneficial to investigate the effects of mental mathematics instruction on primary school students, as well as middle school students in higher grades. Moreover, as I was the students' regular math teacher, it would be necessary to replicate the studies with another teacher conducting the mental mathematics interventions. As much as I tried to remove any experimental bias, it is possible that my relationship with my students and my desired hypotheses may have confounded the results. In the future, the individual conducting the intervention should be blind to the hypotheses and ideally not the students' mathematics teacher. In addition, the results would have given more insight into the nature of students' relational thinking and mental computation strategies if one-on-one interviews had been conducted. Future research should include interviews for such fine-grained outcome measures.

In both Study 1 and 2, an assessment of prior knowledge assessment was administered to determine if there were any differences between the classes at the onset of the study. In Study 3, however, baseline equivalence in working memory was not verified prior to the study. In future research investigating working memory with regard to relational thinking and mental mathematics, a working memory assessment should be included in the preliminary assessment to control for alternative explanations of the data.

Regardless of these limitations, the results obtained from all three studies are promising and future research building on these results is necessary. As relational thinking has been promoted as critical to academic success in mathematics (Carpenter et al., 2003), regular mental computation instruction in the mathematics classroom should be incorporated into all seventh-grade mathematics curricula as it could have significant implications for students' mathematics achievement thinking across the grade levels and beyond mathematics itself.

References

Adams, J. W., & Hitch, G. J. (1997). Working memory and children's mental addition. *Journal of Experimental Child Psychology,* 67(1), 21-38.

Alibali, M. W. (1999). How children change their minds: Strategy change can be gradual or abrupt. *Developmental Psychology,* 35, 127–145.

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning*, *9*(3), 221-247.

Ashcraft, M. H., & Kirk, E. P. (2001). The relationships among working memory, math anxiety, and performance. *Journal of Experimental Psychology: General*, *130*(2), 224.

Ashcraft, M. H., & Krause, J. A. (2007). Working memory, math performance, and math anxiety. *Psychonomic Bulletin & Review, 14(*2), 243-248.

Ayres, P. L. (2001). Systematic mathematical errors and cognitive load. *Contemporary Educational Psychology*, *26*(2), 227-248.

Baddeley, A. (1992). Working Memory and Conscious Awareness. *Theories of memory*, 11-20.

Baddeley A.D. (2007). *Working Memory, Thought and Action*. Oxford, UK: Oxford Univ. Press

Ball, D. L. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics*, *8*(1), 40–48.

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the “equals” sign. *The Elementary School Journal*, *84*(2), 199-212.

Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching, 92,* 13-15.

Blanton, M., & Kaput, J. (2005). Characterizing a Classroom Practice That Promotes Algebraic Reasoning. *Journal for Research in Mathematics Education, 36*(5), 412-446. Retrieved from http://www.jstor.org/stable/30034944

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 20-32). Reston, VA: National Council of Teachers of Mathematics

Britt, M., & Irwin, K. (2011). Algebraic thinking with and without algebraic representation: A pathway for learning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 137-160). Dordrecht, The Netherlands: Springer.

Bull, R., & Scerif, G. (2001). Executive functioning as a predictor of children's mathematics ability: Inhibition, switching, and working memory. *Developmental Neuropsychology, 19(*3), 273-293.

Bull, R., Johnston, R. S., & Roy, J. A. (1999). Exploring the roles of the visual-spatial sketch pad and central executive in children's arithmetical skills: Views from cognition and developmental neuropsychology. *Developmental Neuropsychology, 15(*3), 421-442.

Byrd, C. E., McNeil, N. M., Chesney, D. L., & Matthews, P. G. (2015). A specific misconception of the equal sign acts as a barrier to children's learning of early algebra. *Learning and Individual Differences*, *38*, 61-67.

Cai, J., & Knuth, E. J. (2005). Introduction: The development of students' algebraic thinking in earlier grades from curricular, instructional and learning perspectives. *ZDM*, *37*(1), 1- 4.

Cai, J., & Knuth, E. (2011). *Early algebraization*. New York: Springer

Cai, J., & Moyer, J. (2008). Developing algebraic thinking in earlier grades: Some insights from international comparative studies. *Algebra and Algebraic Thinking in School Mathematics*, *70*, 169-182.

Carpenter, T. P., & Levi, L. (2000). *Developing Conceptions of Algebraic Reasoning in the Primary Grades*. Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Carpenter, T. P., Levi, L., Franke, M. L., & Zeringue, J. K. (2005). Algebra in elementary school: Developing relational thinking. *Zentralblatt für Didaktik der Mathematik*, *37*(1), 53-59.

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, 87-115.

Cooper, T. & Warren, E. (2011). Year 2 to Year 6 students' ability to generalize: Models, representations and theory for teaching and learning. In J. Cai & E. Knuth (Eds.), *Early Algebraization: A global dialogue from multiple perspectives* (pp. 187-214). Heidelberg, Germany: Springer.

DeStefano, D., & LeFevre, J. A. (2004). The role of working memory in mental arithmetic. *European Journal of Cognitive Psychology*, *16*(3), 353-386.

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction, 17(*3), 283-342.

Empson, S. B., Levi, L., & Carpenter, T. P. (2011). The algebraic nature of fractions: Developing relational thinking in elementary school. In J. Cai, & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp.409-426). New York, NY: Springer.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, *6*(4), 232.

Fürst, A. J., & Hitch, G. J. (2000). Separate roles for executive and phonological components of working memory in mental arithmetic. *Memory & Cognition*, 28, 774-782

Gathercole, S. E., Pickering, S. J., Ambridge, B., & Wearing, H. (2004). The structure of working memory from 4 to 15 years of age. *Developmental Psychology*, *40*(2), 177.

Geary, D. C., Hoard, M. K., & Hamson, C. O. (1999). Numerical and arithmetical cognition: Patterns of functions and deficits in children at risk for a mathematical disability. *Journal of Experimental Child Psychology, 74(*3), 213-239.

Gentner, D. (2003). Why we're so smart. In D. Gentner & S. Goldin-Meadow (Eds.), *Language in mind: Advances in the study of language and cognition* (pp. 195-236). Cambridge, MA: MIT Press.

Gentner, D., & Christie, S. (2008). Relational language supports relational cognition in humans and apes. *Behavioral and Brain Sciences, 31*, 137-183.

Gentner, D. & Colhoun, J. (2010). Analogical processes in human thinking and learning. In A. von Müller, E. Pöppel, B. Glatzeder, V. Goel, & A. von Müller (Eds.), *Towards a Theory of Thinking*. Berlin, Germany: SpringerVerlag.

Gentner, D., Loewenstein, J., & Thompson, L. (2003). Learning and transfer: A general role for analogical encoding. *Journal of Educational Psychology*, *95*(2), 393.

Gentner, D., & Smith, L. (2012). Analogical reasoning. *Encyclopedia of human behavior*, *1*, 130-136.

Gick, M. L., & Holyoak, K. J. (1983). Schema induction and analogical transfer. *Cognitive psychology*, *15*(1), 1-38.

Harries, T., & Spooner, M. (2013). *Mental mathematics for the numeracy hour*. London, UK: Routledge.

Heathcote, D. (1994). The role of visuo-spatial working memory in the mental addition of multi-digit addends. *Cahiers de Psychologie Cognitive/Current Psychology of Cognition, 13*, 207–245.

Heirdsfield, A. M. (2011). Teaching mental computation strategies in early mathematics. *Young Children, 66*(2), 96-102.

Heirdsfield, A. M., & Cooper, T. J. (2002). Flexibility and inflexibility in accurate mental addition and subtraction: Two case studies. *The Journal of Mathematical Behavior*, *21*(1), 57-74.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. *Conceptual and procedural knowledge: The case of mathematics*, *2*, 1-27.

Hitch, G. J. (1978). The role of short-term working memory in mental arithmetic. *Cognitive Psychology*, *10*(3), 302-323.

International Baccalaureate Organization. 2014. *Program information.* Retrieved from http://www.ibo.org/

Irwin, K. C., & Britt, M. S. (2005). The algebraic nature of students' numerical manipulation in the New Zealand Numeracy Project. *Educational Studies in Mathematics*, *58*(2), 169-188.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258-288.

Jones, I., Inglis, M., Gilmore, C., & Dowens, M. (2012). Substitution and sameness: Two components of a relational conception of the equals sign. *Journal of Experimental Child Psychology*, *113*(1), 166-176.

Jones, I., Inglis, M., Gilmore, C., & Evans, R. (2013). Teaching the substitutive conception of the equals sign. *Research in Mathematics Education, 15*(1), 34-49.

Kamii, C., & Dominick, A. (1997). To teach or not to teach algorithms. *The Journal of Mathematical Behavior*, *16*(1), 51-61.

Kaput, J. (1998, May). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. Paper presented at the Algebra Symposium, Washington, DC.

Karmiloff-Smith, A. (1992). *Beyond modularity: A developmental perspective on cognitive science*. Cambridge, MA: MIT press.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317-326.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York, NY: Macmillan.

Kieran, C. (2007). Interpreting and assessing the answers given by the CAS expert: A reaction paper. *International Journal for Technology in Mathematics Education*, *14*(2), 103.

Kindrat, A. N., & Osana, H.P. (in press). The relationship between mental computation and relational thinking in the seventh-grade. *Fields Mathematics Education Journal*.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Kızıltoprak, A., & Köse, N. Y. (2017). Relational thinking: The bridge between arithmetic and algebra. *International Electronic Journal of Elementary Education*, *10*(1), 131-146.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence & variable. *ZDM*, *37*(1), 68-76.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37(*4), 297-312.

Knuth, E., Stephens, A., Blanton, M., & Gardiner, A. (2016). Build an early foundation for algebra success. *Phi Delta Kappan, 97*(6), 65-68.

Kurtz, K. J., Gentner, D., & Gunn, V. (1999). Reasoning. In D. E. Rumelhart & B. M. Bly (Eds.), *Cognitive science: Handbook of perception and cognition* (2nd ed., pp. 145-200). San Diego, CA: Academic Press.

Lacampagne, C., Blair, W., & Kaput, J. (1995). *The algebra initiative colloquium*. Washington, DC: US Department of Education & Office of Educational Research and Improvement

Lester, F. K., Jr. (Ed.). (2007). *Second handbook of research on mathematics teaching and learning.* Charlotte, NC: Information Age.

LeFevre, J.-A., DeStefano, D., Coleman, B., & Shanahan, T. (2005). Mathematical cognition and working memory. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 361-378). New York: Psychology Press.

Lemonidis, C. (2015). *Mental computation and estimation: Implications for mathematics education research, teaching and learning*. New York, NY: Routledge.

Logie, R. H., Gilhooly, K. J., & Wynn, V. (1994). Counting on working memory in arithmetic problem solving. *Memory & Cognition*, *22*(4), 395-410.

MacGregor, M. and Stacey, K. (1999). A flying start to algebra. *Teaching Children Mathematics 6(*2), 78–85.

Maclellan, E. (2001). Mental calculation: Its place in the development of numeracy. *Westminster Studies in Education, 24(*2), 145-154.

Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal, 21(*2), 10-32.

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Taylor, R. (2012). Measure for measure: What combining diverse measures reveals about children's understanding of the equal sign as an indicator of mathematical equality. *Journal for Research in Mathematics Education*, *43*(3), 316-350.

McLean, J. F., & Hitch, G. J. (1999). Working memory impairments in children with specific arithmetic learning difficulties. *Journal of Experimental Child Psychology, 74(*3), 240-260.

McNeil, N. M., & Alibali, M. W. (2004). You'll see what you mean: Students encode equations based on their knowledge of arithmetic. *Cognitive Science, 28*, 451–466.

McNeil, N. M., & Alibali, M. W. (2005a). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development, 6*, 385-206.

McNeil, N. M., & Alibali, M. W. (2005b). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development, 76*, 883–899.

Ministère de l'éducation, enseignement supérieur, et recherche. (2014). *Compilation spéciale des données du recensement canadien*. Retrieved from http://www.education.gouv.qc.ca/fileadmin/site_web/documents/PSG/statistiques_info_decisionnelle/Indices_defavorisation_ecoles_2014_2015.pdf

Ministère de l'éducation, enseignement supérieur, et recherche. (2016). *Quebec Education Program*. Retrieved from http://www1.education.gouv.qc.ca/sections/programmeFormation/secondaire1/index_en.asp

Miyake, A., & Shah, P. (1999). *Models of working memory: Mechanisms of active maintenance and executive control*. New York, NY: Cambridge University Press.

Molina, M., & Ambrose, R. (2006). What is that equal sign doing in the middle? Fostering relational thinking while negotiating the meaning of the equal sign. *Teaching Children Mathematics, 13*(2), 111-117.

Molina, M., Castro, E., & Castro, E. (2009). Elementary students understanding of the equal sign in number sentences. *Electronic Journal of Research in Educational Psychology*, *17*, 341-368.

Molina, M., Castro, E., & Mason, J. (2008). Elementary students' approaches to solving true/false number sentences. *Revista de Investigacion en Didactica de la Matematica PNA, 2(2)*, 75-86.

Nathan, M. J., & Koedinger, K. R. (2000). An investigation of teachers' beliefs of students' algebra development. *Cognition and Instruction, 18(*2), 209-237.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Osana, H. P., Proulx, J, Adrien, E., & Nadeau, D. (2013). Developing Relational Thinking in Pre-Service Teachers. In M. Martinez & A. Castro Superfine (Eds.), (2013). *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 845-847). Chicago, IL: University of Illinois at Chicago.

Paholek, E. (1993). *Canadian mathematics 7*. Edmonton, Canada: Rational Publications.

Passolunghi M.C., Cornoldi C., & Deliberto S. (1999). Working memory and intrusion of irrelevant information in a group of specific poor problem solvers. *Memory and Cognition, 27(*5), 779-90.

Passolunghi, M. C., & Siegel, L. S. (2001). Short-term memory, working memory, and inhibitory control in children with difficulties in arithmetic problem solving. *Journal of Experimental Child Psychology, 80(*1), 44-57.

Pavlenko, A., & Jarvis, S. (2002). Bidirectional transfer. *Applied linguistics*, *23*(2), 190-214.

Perkins, D. N., & Salomon, G. (1992). Transfer of learning. *International encyclopedia of education*, *2*, 6452-6457.

Perry, M. (1985, April). Children's strategies in interpreting symbolic equivalence. Paper presented at the biennial meeting of the *Society for Research in Child Development*, Toronto, Canada.

Pillay, H., Wilss, L. & Boulton-Lewis, G. (1998). Sequential development of algebra knowledge: A cognitive analysis. *Mathematics Education Research Journal 10*, 87– 102.

Powell, S. R., Kearns, D. M., & Driver, M. K. (2016). Exploring the connection between arithmetic and prealgebraic reasoning at first and second grade. *Journal of Educational Psychology, 108(7*), 943-959.

Proulx, J. (2013). Mental mathematics, emergence of strategies, and the enactivist theory of cognition. *Educational Studies in Mathematics*, *84*(3), 309-328.

Raghubar, K. P., Barnes, M. A., & Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and Individual Differences*, *20*(2), 110-122.

Resnick, L.B. & Ford, W.W. (1981). *The psychology or mathematics for instruction*. Hillsdale (NJ): Lawrence Erlbaum Ass..

Reys, B. J., & Barger, R. H. (1994). Mental computation: Issues from the United States perspective. In R. E. Reys & N. Nohda (Eds.), *Computational alternatives for the twenty-first century* (pp. 31-47). Reston, VA: The National Council of Teachers of Mathematics.

Reys, R. E. (1984). Mental computation and estimation: Past, present, and future. *The Elementary School Journal, 84(*5), 547-557.

Rittle-Johnson, B., Matthews, P. G., Taylor, R. S., & McEldoon, K. L. (2011). Assessing knowledge of mathematical equivalence: A construct-modeling approach. *Journal of Educational Psychology, 103*, 85-104.

Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review*, *27*(4), 587-597.

Russell, S. J., Schifter, D., & Bastable, V. (2011). Developing algebraic thinking in the context of arithmetic. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp., 43-69. Heidelberg, Germany: Springer.

Seo, K., & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence . . . .": Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 161-187). Mahwah, NJ: Erlbaum.

Seyler, D. J., Kirk, E. P., & Ashcraft, M. H. (2003). Elementary subtraction. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *29*(6), 1339.

Sherman, J., & Bisanz, J. (2009). Equivalence in symbolic and nonsymbolic contexts: Benefits of solving problems with manipulatives. *Journal of Educational Psychology*, *101*(1), 88.

Siegel, L. S., & Ryan, E. B. (1989). The development of working memory in normally achieving and subtypes of learning disabled children. *Child Development*, 973-980.

Siegler, R. S. (2003). Implications of cognitive science research for mathematics education. In Kilpatrick, J., Martin, W. B., & Schifter, D. E. (Eds.), *A research companion to principles and standards for school mathematics* (pp. 219-233). Reston, VA: National Council of Teachers of Mathematics.

Siegler, R. S., & Booth, J. L. (2005). Development of numerical estimation. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition (*pp. 197-212). New York: Psychology Press.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General*, *127*(4), 377.

Slavitt, D. (1999). The role of operation sense in transitions from arithmetic to algebraic thought. *Educational Studies in Mathematics,* 37, 251-274.

Spielhagen, F. R. (2006). Closing the achievement gap in math: The long-term effects of eighth-grade algebra. *Journal of Advanced Academics*, *18*(1), 34-59.

Stacey, K., Chick, H., & Kendal, M. (Eds.). (2006). *The future of the teaching and learning of algebra: The 12th ICMI study* (Vol. 8). New York: Springer Science & Business Media.

Star, J., & Newton, K. J. (2009). The nature and development of experts' strategy flexibility for solving equations. *ZDM, 41*(5), 557-567.

Steffe, L.P. (2001). What is algebraic about children's numerical operating? In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.). *Proceedings of the 12th Conference of the International Commission on Mathematical Instruction,Vol. 2* (pp 556 – 563). Melbourne, Australia.

Stephens, A. C. (2006). Equivalence and relational thinking: Preservice elementary teachers' awareness of opportunities and misconceptions. *Journal of Mathematics Teacher Education*, *9*(3), 249-278.

Stephens, A. C., Knuth, E. J., Blanton, M. L., Isler, I., Gardiner, A. M., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *The Journal of Mathematical Behavior*, *32*(2), 173-182.

Stephens, M. (2007). Students' emerging algebraic thinking in the middle school years. *Mathematics: Essential Research, Essential Practice*, 2, 678-687.

Stephens, M., & Ribeiro, A. (2012). Working towards algebra: The importance of relational thinking. *Revista Latinoamericano de Investigacion en Matematica Educativa, 15*(3), 373–402.

Stephens, M., & Wang, X. (2008). Investigating some junctures in relational thinking: a study of year 6 and year 7 students from Australia and China. *Journal of Mathematics Education*, *1*(1), 28-39.

Subramaniam, K., & Banerjee, R. (2011). The arithmetic-algebra connection: A historical-pedagogical perspective. In J. Cai & E. Knuth (Eds.), *Early Algebraization: A global dialogue from multiple perspectives* (pp. 87-107). Heidelberg, Germany: Springer.

Swanson, H. L., & Beebe-Frankenberger, M. (2004). The relationship between working memory and mathematical problem solving in children at risk and not at risk for serious math difficulties. *Journal of Educational Psychology*, *96*(3), 471.

Thompson, I. (2010). Getting your head around mental calculation. In I. Thompson (Ed.), *Issues in teaching numeracy in primary schools* (pp. 145-156). New York, NY: Open University Press.

Threlfall, J. (2002). Flexible mental calculation. *Educational Studies in Mathematics, 50,* 29-47.

Thwaites, G. N. (1982). Why do children find algebra difficult. *Mathematics in School, 11(*4), 16-17.

Varol, F., & Farran, D. (2007). Elementary school students' mental computation proficiencies. *Early Childhood Education Journal*, *35*(1), 89-94.

Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education, 24*(3), 335-359.

Watchorn, R., & Bisanz, J. (2005, October). Enduring instructional effects on equivalence problems. Poster presented at the meeting of the Cognitive Development Society, San Diego, CA..

Zbrodoff, N. J., & Logan, G. D. (2005). What everyone finds: The problem-size effect. In J. I. D. Campbell (Ed.), *Handbook of mathematical cognition* (pp. 331-345). New York, NY: Psychology Press.

Appendix A

Study 1 and 2: Expressions used during Mental Mathematics intervention

1) 62 + 38	27) 34 + 15	53) 46 - 11	79) 61 x 9
2) 73 - 31	28) 100 ÷ 5	54) 27 x 11	80) 81 ÷ 3
3) 21 x 9	29) 15 + 36	55) 324 ÷ 12	81) 1296 ÷ 4
4) 225 ÷ 5	30) 78 - 56	56) 42 + 35	82) 45 + 79
5) 77 - 26	31) 15 x 6	57) 21 x 8	83) 21 x 4
6) 17 x 5	32) 45 ÷ 5	58) 98 - 43	84) 90 - 15
7) 600 ÷ 4	33) 100 ÷ 20	59) 800 ÷ 80	85) 88 ÷ 4
8) 42 + 58	34) 74 - 12	60) 22 + 12	86) 13 + 46
9) 13 x 4	35) 45 + 73	61) 36 + 37	87) 79 x 2
10) 625 ÷ 25	36) 20 x 11	62) 36 - 16	88) 96 - 87
11) 81 - 27	37) 13 + 11	63) 78 ÷ 3	89) 78 + 34
12) 462 + 119	38) 130 ÷ 5	64) 17 x 9	90) 79 x 4
13) 300 ÷ 50	39) 21 x 5	65) 672 ÷ 3	91) 78 - 64
14) 56 + 48	40) 85 - 45	66) 26 + 27	92) 99 ÷ 3
15) 15 x 5	41) 79 - 35	67) 97 - 41	93) 101 x 5
16) 62 - 13	42) 80 ÷ 5	68) 68 x 2	94) 125 ÷ 5
17) 14 + 11	43) 75 x 4	69) 18 x 12	95) 57 + 19
18) 17 x 11	44) 45 + 72	70) 21 + 14	96) 45 - 16
19) 200 ÷ 5	45) 12 x 13	71) 95 ÷ 5	97) 99 + 61
20) 46 - 23	46) 45 - 31	72) 36 - 19	98) 65 - 44
21) 21 - 12	47) 124 ÷ 4	73) 140 ÷ 10	99) 31 x 3

22) 15 + 75	48) 58 + 19	74) 85 x 3	100) 98 ÷ 2
23) 15 x 8	49) 1225 ÷ 25	75)26 + 17	101) 32 + 48
24) 84 ÷ 4	50) 87 - 43	76) 72 - 13	102) 67 - 34
25) 19 X 11	51) 28 + 85	77) 41 + 59	103) 200 ÷ 4
26) 34 - 17	52) 79 x 3	78) 48 - 31	104) 39 x 4

Appendix B

Prior Knowledge Test (Studies 1 and 2)

1) Use the Order of Operations to evaluate the following:
   14 + 3 x 8

a) 18
b) 38
c) 72
d) None

2) Use the Order of Operations to evaluate the following:
   9 – 3 x 2 + 6

a) 18
b) 48
c) 9
d) None

3) Choose the greatest number.

a) 0.315
b) 0.0325
c) 0.3025
d) None

4) Choose the greatest number.

a) 0.799
b) 0.08
c) 0.8
d) None

5) Add
0.46 + 0.72

a) 1.08
b) 1.18
c) 118
d) None

6) Subtract
0.2154 – 0.1526

a) 0.0628
b) 0.628
c) 6.28
d) None

7) Multiply
8.54 x 3.27

a) 2.79258
b) 27.9258
c) 279.258
d) None

8) Multiply
0.2 x 0.079

a) 0.0158
b) 1.58
c) 15.8
d) None

9) Evaluate the following exponent
8^7

a) 56
b) 2 097 152
c) 87
d) None

10) Evaluate the following exponent
4^5

a) 1024
b) 20
c) 45
d) None

11) Find the quotient and round to the nearest hundredths place
$28.96 \div 8$

a) 0.28
b) 0.36
c) 3.62
d) None

12) Find the quotient and round to the nearest hundredths place
$145.7 \div 47$

a) 3.1
b) 0.32
c) 0.31
d) None

13) Convert to a reduced fraction

0.625

a) 6/25
b) 5/8
c) 625/1000
d) None

14) Convert to a reduced fraction

0.2

a) 2/10
b) 1/5
c) ½
d) None

14) Add and simplify

$$5\frac{5}{8} + 7\frac{5}{12}$$

a) $12\frac{10}{20}$
b) 12 ½
c) $13\frac{1}{24}$
d) None

15) Subtract and simplify

$$12\frac{2}{3} - 5\frac{7}{9}$$

a) $6\frac{8}{9}$
b) $7\frac{5}{6}$
c) 7 ¾
d) None

Appendix C

Equivalence Test –Version 1 (Studies 1, 2, and 3 at Time 1)

3 + 4 = ___

13 – 5 = ___

11 + 4 – 5 = ___

8 – 5 = ___

9 + 4 – 2 = ___

3 + 4 = 4 + ___

6 + 9 = ___ + 9

6 + 4 = 5 + ___

4 + 5 + 3 = ___ + 6

6 + 4 + 3 = 6 + ___

5 + 6 = ___

4 + 5 = ___ + 5

4 + 3 + 6 = 5 + ___

4 + 7 = ___ + 4

3 + 4 + 5 = ___ + 3

7 + 8 = 6 + ___

7 + 5 – 2 = ___

5 +3 + 7 = 5 + ___

6 + 4 + 5 = ___ + 3

6 + 2 = 3 + ___

7 + 3 = 7 + ___

6 + 8 = ___ + 6

15 – 9 = ___

5 + 6 = ___ + 3

4 + 5 + 6 = ___ + 4

5 + 3 = ___ + 5

3 + 7 + 5 = 2 + ___

9 + 5 = 9 + ___

6 + 2 = ___

Equivalence Test –Version 2 (Studies 1, 2, and 3 at Time 2)

6 + 9 = ___

15 - 4 = ___

8 + 7 - 2 = ___

14 - 9 = ___

10 + 3 - 5 = ___

6 + 4 + 5 = ___ + 6

3 + 6 + 4 = 2 + ___

3 + 4 = 2 + ___

6 + 4 = 6 + ___

6 + 9 = ___ + 6

9 + 5 = ___

4 + 5 = ___ + 4

7 + 8 = 7 + ___

4 + 7 = ___ + 3

5 + 3 + 4 = ___ + 7

3 + 7 + 5 = 3 + ___

8 + 4 – 7 = ___

6 + 8 = ___ + 7

4 + 5 + 3 = 4 + ___

7 + 3 = 3 + ___

7 + 5 + 3 = ___ + 6

6 + 2 = 6 + ___

10 - 5 = ___

5 + 6 = ___ + 6

4 + 6 + 5 = 2 + ___

9 + 5 = 5 + ___

4 + 3 + 6 = ___ + 4

5 + 3 = ___ + 2

6 + 8 = ___

Equivalence Test –Version 3 (Studies 1 and 2 at Time 3)

6 + 4 = ___

12 – 2 = ___

12 + 3 - 6 = ___

11 - 6 = ___

7 + 6 - 4 = ___

3 + 4 = 3 + ___

3 + 5 + 4 = 2 + ___

6 + 4 = 4 + ___

7 + 5 + 3 = 7 + ___

6 + 9 = ___ + 7

4 + 7 = ___

4 + 5 = ___ + 3

3 + 6 + 4 = ___ + 3

7 + 8 = 8 + ___

4 + 5 + 6 = ___ + 2

4 + 7 = ___ + 7

9 + 3 – 4 = ___

6 + 2 = 2 + ___

6 + 8 = ___ + 8

7 + 3 = 4 + ___

6 + 4 + 3 = 5 + ___

5 + 6 + 4 = ___ + 5

9 – 3 = ___

5 + 3 + 4 = 5 + ___

5 + 7 + 3 = ___ + 4

9 + 5 = 6 + ___

5 + 3 = ___ + 3

5 + 6 = ___ + 5

7 + 3 = ___

Equivalence Test –Version 4 (Study 2 at Time 4)

7 + 4 = ___

10 – 2 = ___

13 + 4 - 7 = ___

12 - 7 = ___

8 + 6 - 3 = ___

5 + 4 = 5 + ___

4 + 6 + 4 = 3 + ___

9 + 4 = 4 + ___

6 + 4 + 2 = 6 + ___

3 + 6 = ___ + 8

5 + 3 = ___

5 + 6 = ___ + 4

4 + 7 + 5 = ___ + 4

3 + 9 = 9 + ___

8 + 4 + 5 = ___ + 2

6 + 7 = ___ + 7

8 + 4 – 2= ___

7 + 5 = 5 + ___

8 + 6 = ___ + 6

5 + 3 = 7 + ___

5 + 3 + 2 = 4 + ___

7 + 8 + 6 = ___ + 7

7 – 2 = ___

7 + 8 + 4 = 7 + ___

7 + 8 + 4 = ___ + 3

8 + 4 = 7 + ___

1 + 5 = ___ + 5

8 + 6 = ___ + 8

9 + 3 = ___

Equivalence Test –Version 5 (Study 2 at Time 5)

4 + 1 = ___

13 – 5 = ___

11 + 4 - 2 = ___

13 - 9 = ___

8 + 5 - 3 = ___

9 + 8 = 9 + ___

7 + 5 + 3 = 4 + ___

8 + 5 = 5 + ___

9 + 3 + 4 = 9 + ___

8 + 9 = ___ + 6

8 + 7 = ___

2 + 6 = ___ + 4

2 + 5 + 3 = ___ + 2

9 + 7 = 4 + ___

3 + 7 + 8 = ___ + 1

3 + 7 = ___ + 7

8 + 3 – 2 = ___

8+ 2 = 2 + ___

2 + 7 = ___ + 7

9 + 3 = 5 + ___

8 + 4 + 5 = 6 + ___

2 + 5+ 8 = ___ + 7

6 – 3 = ___

7 + 4 + 5 = 6 + ___

8 + 6 + 4 = ___ + 5

8 + 4 = 5 + ___

4 + 2 = ___ + 2

7 + 6 = ___ + 7

2 + 7 = ___

Appendix D: Relational Thinking Tests (RTT)

Study 1

Time 1.

1) 45 + 26 = 47 + 28 True or False. Justify your answer.
2) 104 – 44 = 105 – 45 True or False. Justify your answer.
3) 114 ÷ 3 = 228 ÷ 6 True or False. Justify your answer.
4) 23 x 17 = 16 x 23 True or False. Justify your answer.

Time 2.

1) 324 ÷ 12 = 648 ÷ 24 True or False. Justify your answer.
2) 25 x 19 = 18 x 24 True or False. Justify your answer.
3) 55 + 36 = 57 + 38 True or False. Justify your answer.
4) 113 – 41 = 116 - 44 True or False. Justify your answer.

Time 3.

1) 67 + 48 = 65 + 46 True or False. Justify your answer.
2) 108 – 55 = 109 - 56 True or False. Justify your answer.
3) 972 ÷ 6 = 486 ÷ 12 True or False. Justify your answer.
4) 26 x 22 = 25 x 21 True or False. Justify your answer.

Study 2

Time 1.

1) 45 + 26 = 47 + 28 True or False. Justify your answer.
2) 104 – 44 = 105 – 45 True or False. Justify your answer.
3) 114 ÷ 3 = 228 ÷ 6 True or False. Justify your answer.
4) 22 x 18 = 19 x 21 True or False. Justify your answer.
5) (28 x 11) – 28 = 27 x 11 True or False. Justify your answer.

Time 2.

1) 65 + 36 = 67 + 38 True or False. Justify your answer.
2) 33 x 18 = 19 x 32 True or False. Justify your answer.
3) (38 x 11) – 38 = 37 x 11 True or False. Justify your answer.
4) 105 – 45 = 106 – 46 True or False. Justify your answer.
5) 228 ÷ 6 = 456 ÷ 12 True or False. Justify your answer.

Time 3.

1) 106 – 46 = 107 – 47 True or False. Justify your answer.
2) (37 x 11) – 37 = 36 x 11 True or False. Justify your answer.
3) 42 + 24 = 44 +26 True or False. Justify your answer.
4) 36 x 18 = 35 x 19 True or False. Justify your answer.
5) 225 ÷ 5 = 550 ÷ 10 True or False. Justify your answer.

Time 4.

1) 144 ÷ 6 = 288 ÷ 12 True or False. Justify your answer.
2) (42 x 11) – 42 = 41 x 11 True or False. Justify your answer.
3) 25 x 16 = 17 x 24 True or False. Justify your answer.
4) 63 + 42 = 64 + 44 True or False. Justify your answer.
5) 107 – 47 = 108 – 48 True or False. Justify your answer.

Time 5.

1) 125 ÷ 5 = 250 ÷ 10 True or False. Justify your answer.
2) (52 x 11) – 52 = 51 x 11 True or False. Justify your answer.
3) 36 x 16 = 17 x 35 True or False. Justify your answer.
4) 73 + 52 = 75 + 54 True or False. Justify your answer.
5) 207 – 47 = 208 – 48 True or False. Justify your answer.

Study 3

Time 1.

1) 45 + 26 = 47 + 28 True or False. Justify your answer

2) 97 – 44 = 94 – 41 True or False. Justify your answer.

3) 37 + 58 = 41 +56 True or False. Justify your answer.

4) 68 - 36 = 65 – 39 True or False. Justify your answer.

Time 2.

1) 68 – 33 = 63 – 28 True or False. Justify your answer

2) 43 + 67 = 41 + 65 True or False. Justify your answer.

3) 67 - 34 = 69 – 32 True or False. Justify your answer

4) 52 + 38 = 54 + 36 True or False. Justify your answer.

Appendix E

Study 2: Mental Mathematics Assessment Expressions

Time 1

1) 57 + 59
2) 107 - 38
3) 14 x 4
4) 248 ÷ 8
5) 115 - 32
6) 45 x 4
7) 3600 ÷ 60
8) 112 + 48
9) 59 x 4
10) 325 ÷ 25
11) 114 - 92
12) 68 + 52

Time 2

1) 145 - 47
2) 19 x 5
3) 444 ÷ 11
4) 113 + 117
5) 49 x 4
6) 89 - 68
7) 54 + 146
8) 64 ÷ 4
9) 61 x 4
10) 115 - 87
11) 119 + 121
12) 729 ÷ 9

Time 3

1) 162 - 65
2) 21 x 6
3) 1025 ÷ 5
4) 257 + 153
5) 51 x 4
6) 91 - 72
7) 128 ÷ 4
8) 64 + 87
9) 62 x 2
10) 117 - 19
11) 253 + 147
12) 256 ÷ 8

Time 4

1) 62 + 38
2) 29 x 3
3) 112 ÷ 4
4) 257 - 158
5) 49 x 8
6) 101 - 42
7) 125 ÷ 5
8) 64 + 87
9) 61 x 6
10) 112 - 15
11) 456 + 404
12) 216 ÷ 8

Time 5

1) 128 ÷ 4
2) 49 x 11
3) 154 + 136
4) 854 - 655
5) 548 - 447
6) 256 ÷ 8
7) 356 + 144
8) 65 x 3
9) 101 x 6
10) 258 - 157
11) 541 + 659
12) 216 ÷ 4

Study 3: Mental Mathematics Assessment Expressions

Time 1

1) 67- 39
2) 79 - 54
3) 186 - 48
4) 89 - 23
5) 92 - 46
6) 51 – 27
7) 153 – 41
8) 196 - 43
9) 87 - 51
10) 68 - 42
11) 63 - 31
12) 95 - 48
13) 248 - 34
14) 58 - 42
15) 81- 49
16) 67 - 38
17) 172 - 48
18) 253 - 78
19) 65 - 41
20) 72 - 37

Time 2

1) 77 – 48
2) 89 - 64
3) 196 - 58
4) 79 - 33
5) 93 - 47
6) 61 - 47
7) 253 - 47
8) 186 - 53
9) 97 - 51
10) 78 - 42
11) 73 - 51
12) 95 - 39
13) 237 - 36
14) 68 - 43
15) 91 - 39
16) 77 - 29
17) 172 - 49
18) 257 - 68
19) 75 - 31
20) 77 - 39

Appendix F

Study 2: Mental Mathematics Assessment Answer Sheet

1)
2)
3)
4)
5)
6)
7)
8)
9)
10)
11)
12)

Study 3: Mental Mathematics Assessment Answer Sheet

1)
2)
3)
4)
5)
6)
7)
8)
9)
10)
11)
12)
13)
14)
15)
16)
17)
18)
19)
20)

Appendix G

Study 3: Student daily handout for Regular Mental Mathematics condition

1)
2)
3)
4)
5)
6)
7)
8)
9)
10)
11)
12)

Appendix H

Study 3: Page 1 and 2 of Reduced Cognitive Load condition workbook

(Page 1) Expression # 1

Strategy # 1

____ - ____

Step 1: ____ - ____ -____

Step 2: ____ - ____

Answer: ____

(Page 2) Expression # 1

Strategy # 2

____- ____

Step 1: ____ - ____ +____ - ____

Step 2: ____ - ____

Answer: ____

Appendix I

Study 3: Page 1 of Mental Mathematics Strategy Assessment

Step __: ____

Step __ : ____

Step __: ____

Step __ : ____

Step __: ____

Step __ : ____

Step __: ____

Step __ : ____

Step __: ____

Step __ : ____

Appendix J

Equivalent Instruction Criteria checklist

Please indicate Yes or No as to whether the instructional sessions were equivalent for all of today's instruction sessions, according to the following criteria:

1) Did the instructor provide the instructions in the identical way in all sessions?

2) Did the instructor deliver any documentation in the identical way in all sessions?

3) Did the instructor provide the same amount of time to each session to complete each expression?

4) During the intervention, were the strategies delivered in the same order for all sessions?

5) During the intervention, was the same number of strategies delivered in each session?

6) During the intervention, did the instructor provide the same general level of encouragement and energy to her instruction?

7) At the conclusion of the intervention, did the instructor collect all documents the same way in all sessions?

8) Did each session last the same amount of time?

9) Did the instructor circulate with the same frequency in all sessions?