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PARRONDO'S PARADOX FOR GAMES WITH THREE PLAYERS

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ABSTRACT. Parrondo's paradox appears in game theory which asserts that playing two losing games, A and B (say) randomly or periodically may result in a winning expectation. In the original paradox the strategy of game B was capital-dependent. Some extended versions of the original Parrondo's game as history dependent game, cooperative Parrondo's game and others have been introduced. In all of these methods, games are played by two players. In this paper, we introduce a generalized version of this paradox by considering three players. In our extension, two games are played among three players by throwing a three-sided dice. Each player will be in one of three places in the game. We set up the conditions for parameters under which player one is in the third place in two games A and B. Then paradoxical property is obtained by combining these two games periodically and chaotically and (s)he will be in the first place when (s)he plays the games in one of the mentioned fashions. Mathematical analysis of the generalized strategy is presented and the results are also justified by computer simulations.

Key words and phrases: Parrondo's paradox, Combined game, Game theory.

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1 Introduction

Ajdari and Prost [1] introduced a *Brownian ratchet,* which was named as the *flashing ratchet* by Astumian and Bier [5]. They showed that by switching the potential on and off alternately, a Brownian particle in a periodic potential has a net drift in a given direction [5]. In fact in each case of the switch on or switch off, the particle does not move systematically but by combining these actions, directed motion is resulted. The mechanism of flashing ratchet system has been shown in Fig. 1 [24]. There are two states of potential, U_{off} and U_{on} which are flat and sawtooth, respectively. Particles in the potentials tend to move to the direction with minimum energy. When the potential U_{on} is applied, particles will concentrate around one of the potential valleys. On the other hand when the potential is flashed *off*, particles will move freely and a Gaussian distribution for the movement of particles will explain the movement around the minima as shown in Fig. 1 [24]. When the potential is flashed *on*, some particles will move to the right of αL with probability P_{fwd} and some particles will move to the left of $-(1-\alpha)L$ with probability P_{bck}. In fact, asymmetry of the sawtooth potential leads to directed movement of particles. This asymmetry depends on the value of α where $0 \leq \alpha \leq 1$. When $\alpha > 1/2$ then $P_{fwd} > P_{bck}$ and this leads to the movement of particles to the right. Flashing ratchet system will still work if there is gradient [5],[24].
In Parrondo's paradox, playing two losing games in a random or periodic strategy leads to a winning game. The original game was introduced by Parrondo [37], as a discrete-time version of flashing Brownian ratchet. It was resulted from discretization of the Fokker-Planck equation [3]. It has been applied to several fields such as economics [43], physical quantum systems [20], [40], [30],[22], [39], population genetics [42], [46], [31], reliability theory [16], controlling chaos [2],[6], [14], [15], [13], spin systems [34], [17], chaotic dynamical systems [4], noise induced synchronization [28]. Meyer and Blumer model the Parrondo's games by probabilistic lattice gas automata [32]. Buceta, Lindenberg and Parrondo used Parrondo' paradox to introduce a new system for pattern formation [7]. Lee and Johnson [29] showed that random switching has a useful role for implementing quantum algorithms. Challet and Johnson [8] showed that perfect or near-perfect devices can be obtained by combination of many imperfect objects. New Parrondo games are introduced using discrete-time quantum walk on a line by Chandrashekar and Banerjee [9].

Pinsky and Schuetzow [41] showed that combining two transient diffusion processes randomly, will result in a positive recurrent process and this can be considered as a continuous-time version of original Parrondo's paradox. Key [27] produced an increasing branching process by combining two decreasing branching processes. There are some speculation about the application of Parrondo's paradox to biology [24]. However, Fotoohinasab et al. [21] applied the paradox to model denoising of genetic switches. They showed that the robustness of genetic switches to noise can be increased by combining the noisy switches.
As mentioned earlier, in Parrondo's paradox two losing games can be combined in a manner which results in a winning game. The combined rule is obtained by switching between two games periodically or randomly.

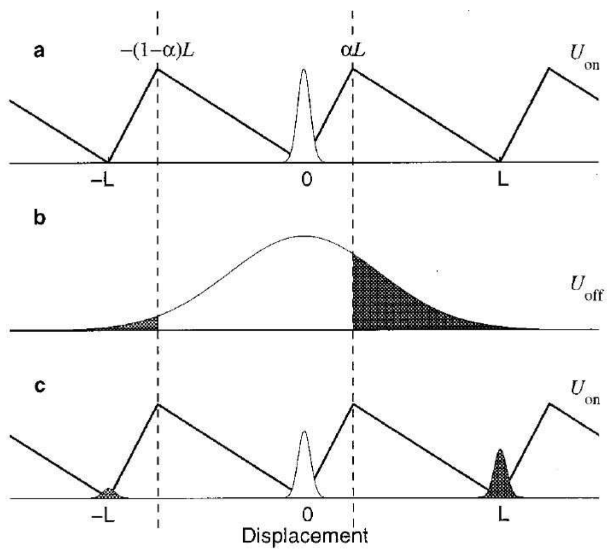

Figure 1: Brownian ratchet system.(Reprinted from[24] with permission)

Every time a coin is tossed, time increases by one unit and the capital increases or decreases by one. For game A there is a probability of winning p. The probability of winning in game B depends on the present capital $X(t)$, if it is a multiple of M, the probability of winning will be p_1, if it is not the probability of winning will be p_2. It is obvious that the game A is a losing game, when $p < 0.5$. According to [24], game B is a losing game when the inequality $(1-p_1)(1-p_2)^{M-1} > p_1 p_2^{M-1}$ holds, and when we combine the two games randomly, a winning game will be obtained if the inequality $(1-p'_1)(1-p'_2)^{M-1} < p'_1(p'_2)^{M-1}$ holds, where $p'_1 = \gamma p + (1-\gamma)p_1$ and $p'_2 = \gamma p + (1-\gamma)p_2$. In fact each time game A is played with probability γ and game B is played with probability $1-\gamma$, where $0 < \gamma < 1$. Game B is a fair game, when $(1-p_1)(1-p_2)^{M-1} = p_1 p_2^{M-1}$ holds [24]. One solution of this equation is $p_1 = 1/10$ and $p_2 = 3/4$, when $M = 3$. So according to the original Parrondo's game for the choice, $M = 3$, $p = 1/2 - \varepsilon$, $p_1 = 1/10 - \varepsilon$ and $p_2 = 3/4 - \varepsilon$ and $0 < \varepsilon < 0.1$, both the games are loosing games (the capital is a decreasing function of the number of runs). However, the combined game with $\gamma = 0.5$ is a winning game for ε small enough[24]. It is shown that combining two games periodically is also a winning game.

There are some similarities between Brownian ratchet and Prrondo's paradox [23],[24],[25]. In the Brownian ratchet, particles will move to a direction if the potential switches on and off. In Parrondo's paradox one will win the game if (s)he plays two loosing games alternately. Two states of Brownian ratchet are analogous to two games in Parrondo's game. Capital has the role of particles in Brownian ratchet and probabilities are like the energy profile. When the potential is off, particles move to both directions with equal probability and the condition is similar to the simple game A. Game B is analogous to the case when the potential is on. Probabilities p_1 and p_2 in game B are similar to the parameter α in Brownian ratchet. Noise parameter, ε, is analogous to the gradient in a Brownian ratchet [23],[24],[25]. Table 1 shows these similarities [24]. One of the properties of game B in the original Parrondo paradox is that it depends on a modulo rule based on the capital of the player. This strategy is appropriate to explain the paradox in terms of energy level, but it is not common for biological and biophysical processes [38]. Therefore several strategies have been presented to extend the range of applications of the original paradox. In [38], capital dependent rule was replaced by history dependent rule. In fact the outcome of each play depends on the winning or losing in the last two runs.

Brownian ratchet	**Parrondo's paradox**
Electrostatic, Gravity	Rules of games
Potential field gradient	Parameter ε
U_{on} and U_{off} applied	Games A and B played
Displacement x	Capital or gain
Switching U_{on} and U_{off}	Alternating games
Depends on α	Branching of B to p_1 and p_2
Work done$<$energy in	Total gain$<$ gain with p_2 alone
FokkerPlanck equation	Discrete-time Markov chains

Table 1: Similarities between the Brownian ratchet and the Parrondo's paradox. (Reprinted from [24] with permission)

Another version of paradox is called cooperative Parrondo's game [45]. In this version the game A is the same as in the original paradox game, but game B is played by N number of players [45]. Mihailovic and Rajkovic [33] in 2006 extended the cooperative Parrondo's games on a two-dimensional lattice.
Cheong and Soo [10] analyzed the original paradox with different approach. They started with one process and derived its complementary process. Some other extended Parrondo's games are proposed in [12], [48], [47], [49], [18].

In this paper, we introduce an extended version of Parrondo's paradox by considering three players and show that paradox could occur for three players. Here, we assume that there are three states for each player, for example, player one can be in the first or the second or the third place. Paradox occurs when player one is in the third place in two games A and B after long run, but (s)he will be in the first place when (s)he plays the games alternatively.
In section two, our new strategies for game A and game B are introduced. The conditions of taking different places by the players are obtained theoretically. We present ranges for parameters to obtain the desired condi-

tions also, in this section. Section three presents the results of several computer simulations and conclusions are given in section four.

2 A Generalized Model

In our model, two games are played among three players. So a three-sided dice is tossed in each run. Every time a dice is thrown, time increases by one unit and the capital of each player increases by two units or decreases by one. All three players have their own states at the end of each round, depending on the outcome of the dice rolling. These states are namely; first, second and third. However, our strategy for game A is modified as follows: we have three players in our model and call them as player 1, player 2 and player 3. Consider a three-sided dice where the probabilities of rolling side 1, 2 and 3 are p_1^A, p_2^A and p_3^A, respectively. Then the probabilities of taking the first place for each of the players at time t will be as follows:

- The probability of increasing the capital of player 1 by two will be p_1^A. If the outcome of dice rolling is one, the capital of player 1 will be increased by two.
- The probability of increasing the capital of player 2 by two will be p_2^A. If the outcome of dice rolling is two, the capital of player 2 will be increased by two.
- The probability of increasing the capital of player 3 by two will be p_3^A. If the outcome of dice rolling is three, the capital of player 3 will be increased by two.

First, following [24] we obtain the conditions for which one can be in different places for the game A. Then we derive the conditions under which the paradox happens.

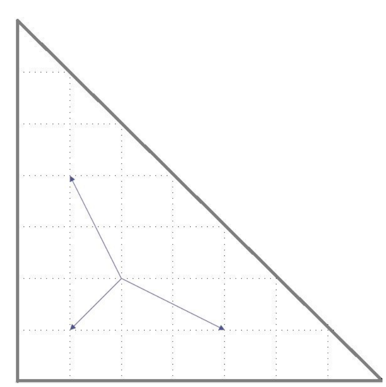

Figure 2: Random walk on a triangle.

As mentioned above, game A is a dice rolling or a random walk on a triangle in the plane (see Fig. 2). Now assume for Game A,

$$p(\text{dice rolling ends up with one}) = \mathrm{p}_1^{\mathrm{A}},$$
$$p(\text{dice rolling ends up with two}) = \mathrm{p}_2^{\mathrm{A}},$$
$$p(\text{dice rolling ends up with three}) = \mathrm{p}_3^{\mathrm{A}}.$$

For game A, we assume that α_j^1 is the probability of the player 1's capital reaches zero in a finite number of plays, given that (s)he starts with a capital of j units. As in the case of player 1, we denote these values for player 2 and player 3 by α_j^2 and α_j^3 respectively. Using Markov chain properties [26], we have the following cases:

1) If $\alpha_j^1 = 1$, $\alpha_j^2 = 1$ and $\alpha_j^3 = 1$ for all $j \geq 0$, the game will be fair or the player 1 will be in the first or the third places.
2) If $\alpha_j^1 < 1$, $\alpha_j^2 = 1$ and $\alpha_j^3 = 1$ for all $j > 0$, the player 1 will be in the first place.
3) If $\alpha_j^1 < 1$, $\alpha_j^2 < 1$ and $\alpha_j^3 = 1$ for all $j > 0$, the player 1 will not be in the third place.
4) If $\alpha_j^1 < 1$, $\alpha_j^2 = 1$ and $\alpha_j^3 < 1$ for all $j > 0$, the player 1 will not be in the third place.
5) If $\alpha_j^1 = 1$, $\alpha_j^2 < 1$ and $\alpha_j^3 < 1$ for all $j > 0$, the player 1 will be in the third place.
6) If $\alpha_j^1 = 1$, $\alpha_j^2 = 1$ and $\alpha_j^3 < 1$ for all $j > 0$, the player 1 will not be in the first place.
7) If $\alpha_j^1 = 1$, $\alpha_j^2 < 1$ and $\alpha_j^3 = 1$ for all $j > 0$, the player 1 will not be in the first place.
8) If $\alpha_j^1 < 1$, $\alpha_j^2 < 1$ and $\alpha_j^3 < 1$ for all $j > 0$, the player 1 will be in the first place or will not be in the first place.

The conditions for other two players are similar. For α_j^1, we can write the following recursive formula:

$$\alpha_j^1 = p_1^A \alpha_{j+2}^1 + p_2^A \alpha_{j-1}^1 + p_3^A \alpha_{j-1}^1, \tag{2.1}$$

for $j \geq 0$ with the initial condition $\alpha_0^1 = 1$.
The recursive equations for player 2 and player 3, similar to that for player 1 are:

$$\alpha_j^2 = p_1^A \alpha_{j-1}^2 + p_2^A \alpha_{j+2}^2 + p_3^A \alpha_{j-1}^2,$$
$$\alpha_j^3 = p_1^A \alpha_{j-1}^3 + p_2^A \alpha_{j-1}^3 + p_3^A \alpha_{j+2}^3,$$

for $j \geq 0$ with the initial conditions $\alpha_0^2 = 1$ and $\alpha_0^3 = 1$.
The general solution of equation (2.1) with initial condition $\alpha_0^1 = 1$ is

$$\alpha_j^1 = C_1\left(\left(\frac{-p_1^A + \sqrt{(p_1^A)^2 + 4p_1^A(p_2^A + p_3^A)}}{2p_1^A}\right)^j - 1\right) + C_2\left(\left(\frac{-p_1^A - \sqrt{(p_1^A)^2 + 4p_1^A(p_2^A + p_3^A)}}{2p_1^A}\right)^j - 1\right) + 1,$$

in which C_1 and C_2 are constants. The solutions for the case of two other players, are analogous to this case. Therefore we can conclude that game A is fair if

$$\left.\begin{array}{l}\dfrac{-p_1^A+\sqrt{(p_1^A)^2+4p_1^A(p_2^A+p_3^A)}}{2p_1^A}=1,\\ \dfrac{-p_2^A+\sqrt{(p_2^A)^2+4p_2^A(p_1^A+p_3^A)}}{2p_2^A}=1,\\ \dfrac{-p_3^A+\sqrt{(p_3^A)^2+4p_3^A(p_1^A+p_2^A)}}{2p_3^A}=1,\end{array}\right. \implies \quad p_1^A=p_2^A=p_3^A=1/3.$$

Now, we consider game B. The strategy of game B can be considered mathematically as follows:

$$p(\text{dice rolling ends up with one}|\text{the capital of player1 is a multiple of } 3)=\mathrm{p}_{1,1}^{\mathrm{B}}.$$
$$p(\text{dice rolling ends up with one}|\text{the capital of player1 is not a multiple of } 3)=\mathrm{p}_{2,1}^{\mathrm{B}}.$$
$$p(\text{dice rolling ends up with two}|\text{the capital of player1 is a multiple of } 3)=\mathrm{p}_{1,2}^{\mathrm{B}}.$$
$$p(\text{dice rolling ends up with two}|\text{the capital of player1 is not a multiple of } 3)=\mathrm{p}_{2,2}^{\mathrm{B}}.$$
$$p(\text{dice rolling ends up with three}|\text{the capital of player1 is a multiple of } 3)=\mathrm{p}_{1,3}^{\mathrm{B}}.$$
$$p(\text{dice rolling ends up with three}|\text{the capital of player1 is not a multiple of } 3)=\mathrm{p}_{2,3}^{\mathrm{B}}.$$

As in the case of the game A, for game B we assume that β_j^1 is the probability of the capital of the player 1 reaches zero in a finite number of plays, given that (s)he starts with a capital of j units. We show these values for player 2 and player 3 by β_j^2 and β_j^3 respectively. As for game A, using Markov chain theory we have:

1) If $\beta_j^1=1$, $\beta_j^2=1$ and $\beta_j^3=1$ for all $j\geq 0$, the game will be fair or the player 1 will be in the first or the third places.
2) If $\beta_j^1<1$, $\beta_j^2=1$ and $\beta_j^3=1$ for all $j>0$, the player 1 will be in the first place.
3) If $\beta_j^1<1$, $\beta_j^2<1$ and $\beta_j^3=1$ for all $j>0$, the player 1 will not be in the third place.
4) If $\beta_j^1<1$, $\beta_j^2=1$ and $\beta_j^3<1$ for all $j>0$, the player 1 will not be in the third place.
5) If $\beta_j^1=1$, $\beta_j^2<1$ and $\beta_j^3<1$ for all $j>0$, the player 1 will be in the third place.
6) If $\beta_j^1=1$, $\beta_j^2=1$ and $\beta_j^3<1$ for all $j>0$, the player 1 will not be in the first place.
7) If $\beta_j^1=1$, $\beta_j^2<1$ and $\beta_j^3=1$ for all $j>0$, the player 1 will not be in the first place.
8) If $\beta_j^1<1$, $\beta_j^2<1$ and $\beta_j^3<1$ for all $j>0$, the player 1 will be in the first place or will not be in the first place.

According to the definition of game B, β_j^1 could be obtained from the following recursive formulas.
For $i\geq 0$ and $j\varepsilon\{1,2\}$ we have:

$$\beta_{3i}^1=p_{1,1}^B\beta_{3i+2}^1+p_{1,2}^B\beta_{3i-1}^1+p_{1,3}^B\beta_{3i-1}^1,$$

and

$$\beta^1_{3i+j} = p^B_{2,1}\beta^1_{3i+j+2} + p^B_{2,2}\beta^1_{3i+j-1} + p^B_{2,3}\beta^1_{3i+j-1},$$

for which we take the initial condition as $\beta^1_0 = 1$. We have similar recursive equations for β^2_j and β^3_j with initial conditions $\beta^2_0 = 1$ and $\beta^3_0 = 1$. So with these initial conditions after some calculations we arrive at the following solution:

$$\begin{aligned}
\beta^1_{3i+j} = {} & D_{1i}((-\frac{(1-p^B_{2,1})}{p^B_{2,1}} - \frac{1}{2p^B_{1,1}} - \\
& \sqrt{(\frac{(1-p^B_{2,1})}{p^B_{2,1}} + \frac{1}{2p^B_{1,1}})^2 - \frac{(1-p^B_{1,1})(1-p^B_{2,1})^2}{p^B_{1,1}(p^B_{2,1})^2}})^j - 1) \\
& + D_{2i}((-\frac{(1-p^B_{2,1})}{p^B_{2,1}} - \frac{1}{2p^B_{1,1}} + \\
& \sqrt{(\frac{(1-p^B_{2,1})}{p^B_{2,1}} + \frac{1}{2p^B_{1,1}})^2 - \frac{(1-p^B_{1,1})(1-p^B_{2,1})^2}{p^B_{1,1}(p^B_{2,1})^2}})^j - 1) + 1,
\end{aligned}$$

in which D_{1i} and D_{2i} are constants. Using the above approach for player 2 and player 3, we arrive at the following equations to get conditions for fairness in game B.

$$\begin{aligned}
& 1 + \frac{(1-p^B_{2,1})}{p^B_{2,1}} + \frac{1}{2p^B_{1,1}} = \\
& \sqrt{(\frac{(1-p^B_{2,1})}{p^B_{2,1}} + \frac{1}{2p^B_{1,1}})^2 - \frac{(1-p^B_{1,1})(1-p^B_{2,1})^2}{p^B_{1,1}(p^B_{2,1})^2}}, \\
& 1 + \frac{(1-p^B_{2,2})}{p^B_{2,2}} + \frac{1}{2p^B_{1,2}} = \\
& \sqrt{(\frac{(1-p^B_{2,2})}{p^B_{2,2}} + \frac{1}{2p^B_{1,2}})^2 - \frac{(1-p^B_{1,2})(1-p^B_{2,2})^2}{p^B_{1,2}(p^B_{2,2})^2}}, \\
& 1 + \frac{(1-p^B_{2,3})}{p^B_{2,3}} + \frac{1}{2p^B_{1,3}} = \\
& \sqrt{(\frac{(1-p^B_{2,3})}{p^B_{2,3}} + \frac{1}{2p^B_{1,3}})^2 - \frac{(1-p^B_{1,3})(1-p^B_{2,3})^2}{p^B_{1,3}(p^B_{2,3})^2}}.
\end{aligned}$$

Thus game B will be fair in the following regions:

$$\begin{aligned}
& 0 < p^B_{1,1} < 1, \quad p^B_{2,1} = \frac{1-p^B_{1,1}}{2} \\
& 0 < p^B_{1,2} < 1 - p^B_{1,1}, \quad p^B_{2,2} = 1 - p^B_{1,1}/2 - p^B_{2,1} - p^B_{1,2}/2, \\
& p_{1,3} = 1 - p^B_{1,1} - p^B_{1,2}, \quad p^B_{2,3} = 1 - p^B_{2,1} - p^B_{2,2}. \qquad (2.2)
\end{aligned}$$

As mentioned earlier, game A is fair when $p_1^A = 1/3$, $p_2^A = 1/3$ and $p_1^A = 1/3$, by considering positive noise parameters ε_1 and ε_2 the probabilities could be written as $p_1^A = 1/3 - \varepsilon_1$, $p_2^A = 1/3 - \varepsilon_2$ and $p_3^A = 1/3 + \varepsilon_1 + \varepsilon_2$. So the player 1 in game A will be in the third place after long run.
For game B, we consider the subset of the parameter space which leads to a fair game and denote it by:

$$F_B = \{p_{m,n}^B, 1 \leq m,n \leq 3 : 0 < p_{1,1}^B < 1, p_{2,1}^B = \frac{1 - p_{1,1}^B}{2},$$
$$0 < p_{1,2}^B < 1 - p_{1,1}^B, p_{2,2}^B = 1 - p_{1,1}^B/2 - p_{2,1}^B - p_{1,2}^B/2,$$
$$p_{1,3}^B = 1 - p_{1,1}^B - p_{1,2}^B, p_{2,3}^B = 1 - p_{2,1}^B - p_{2,2}^B\}.$$

As for game A we consider two noise parameters ε_1 and ε_2, so that the corresponding probabilities for game B are as follows:

$$p_{1,1}^B - \varepsilon_1, p_{2,1}^B - \varepsilon_1, p_{1,2}^B - \varepsilon_2,$$
$$p_{2,2}^B - \varepsilon_2, p_{1,3}^B + \varepsilon_1 + \varepsilon_2, p_{2,3}^B + \varepsilon_1 + \varepsilon_2.$$

Hence, we have a subset of parameters which leads to a fair game and by considering positive noises ε_1 and ε_2, we have a subset of games in which player 1 will be in the third place.
In the next section, we show that the paradox holds for certain parameters through computer simulations, that is, for these parameters player 1 is in the third place in two games, but the player will ultimately be in the first place when the games are combined periodically. `Check the accuracy of the previous paragraph`

3 Computer Simulation

We performed the following computer simulation for our generalized model, to investigate the results of games A, B and the combined game. For game A the simulation was performed with $\varepsilon_1 = 0.005$ and $\varepsilon_2 = 0.001$. Therefore the probabilities are $p_1^A = 1/3 - 0.005$, $p_2^A = 1/3 - 0.001$ and $p_3^A = 1/3 + 0.006$. It is obvious that player 1 will ultimately be in the third place. This result is shown in Fig. 3. According to this figure the capital of player 1 is decreasing so (s)he will ultimately go to the third place.

We simulated the fair game B, by choosing the parameters from the subset F_B. Fig. 4 shows the conditions for fairness for game B with probabilities $p_{1,1}^B = 0.6$, $p_{2,1}^B = 0.2$, $p_{1,2}^B = 0.3$, $p_{2,2}^B = 0.35$, $p_{1,3}^B = 0.1$ and $p_{2,3}^B = 0.45$.

Noise parameters $\varepsilon_1 = 0.005$ and $\varepsilon_2 = 0.001$ are considered for game B, in which player 1 would be in the third place. Fig. 5 shows that the capital of player 1 is decreasing in this case.
As shown by Fig. 3 and Fig. 5, player 1 would be in the third place in two games A and B, but by combining these two games periodically this player will be in the first place (see Fig. 6). So, one can play two games A and B in such an order as player 1 takes the first place after long run. If one plays twice B and once A

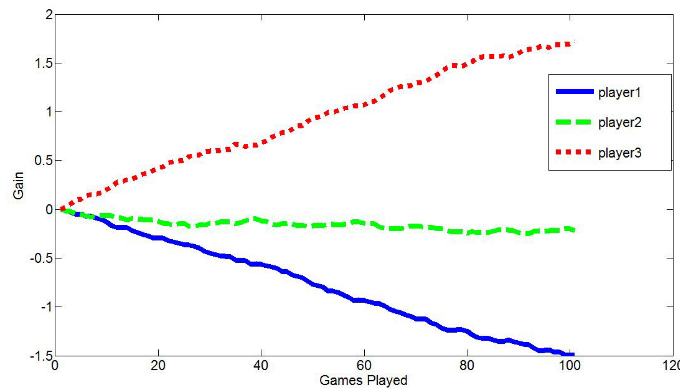

Figure 3: Game A. The simulation of playing game A, 100 times.This game is played with probabilities $p_1^A = 1/3 - \varepsilon_1$, $p_2^A = 1/3 - \varepsilon_2$ and $p_3^A = 1/3 + \varepsilon_1 + \varepsilon_2$ and noise parameters are assumed to be $\varepsilon_1 = 0.005$ and $\varepsilon_2 = 0.001$. The capital of player 1 is decreasing so (s)he will ultimately go to the third place.

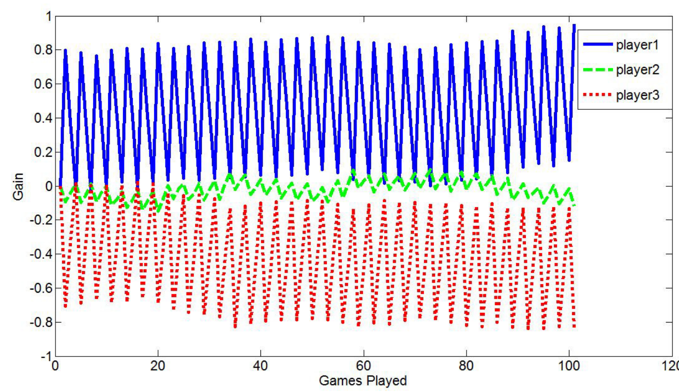

Figure 4: Game B. Game B is played 100 times with probabilities $p_{1,1}^B = 0.6$, $p_{2,1}^B = 0.2$, $p_{1,2}^B = 0.3$, $p_{2,2}^B = 0.35$, $p_{1,3}^B = 0.1$ and $p_{2,3}^B = 0.45$.

subsequently the paradoxical result will be obtained. The results of simulation are shown in Fig. 6. Now we combine two games using a new strategy. Tang et al. [44] proposed a new strategy in switching from one game to the another game based on various chaotic sequences. These numeric sequences are generated by different chaotic systems that strongly depend on the initial conditions [44]. We implemented these series to combine two new games A and B with three players. Some maps to generate chaotic time series sequences

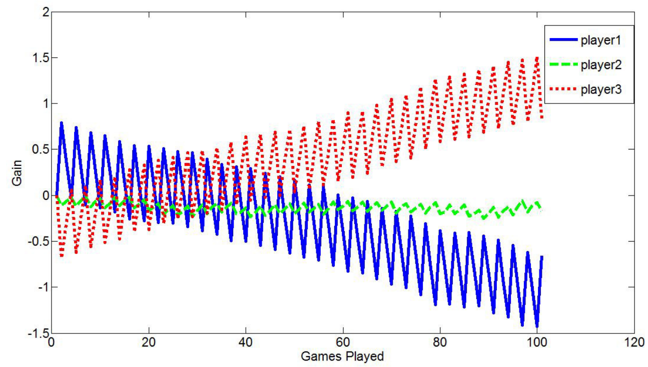

Figure 5: Simulation results for the unfair conditions in game B. Game B is played 100 times with probabilities $p^B_{1,1} = 0.6 - \varepsilon_1$, $p^B_{2,1} = 0.2 - \varepsilon_1$, $p^B_{1,2} = 0.3 - \varepsilon_2$, $p^B_{2,2} = 0.35 - \varepsilon_2$, $p^B_{1,3} = 0.1 + \varepsilon_1 + \varepsilon_2$ and $p^B_{2,3} = 0.45 + \varepsilon_1 + \varepsilon_2$. With noise parameters $\varepsilon_1 = 0.005$ and $\varepsilon_2 = 0.001$. The capital of player 1 is decreasing so (s)he is in the third place.

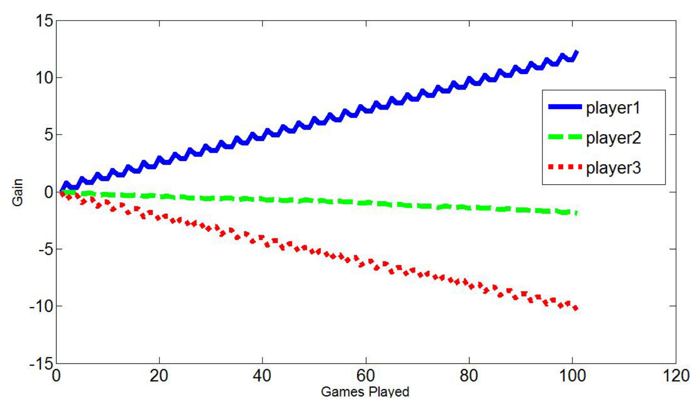

Figure 6: Simulation results for the combined game. The simulation was performed by switching between game A and game B. The simulation starts by playing game B, this game is played twice and game A is played once, upto 100 runs. The capital of player 1 is increasing so (s)he is in the first place.

are the Logistic map, the sinusoidal map, and the Tent map [44]. We apply them to combine two games in this paper. The series are defined as:

1) Logistic Map

$$x_{n+1} = ax_n(1-x_n)$$

2) Sinusoidal Map

$$x_{n+1} = ax_n^2 \sin(\pi x_n)$$

3)Tent Map

$$\begin{cases} ax_n & if \quad x_n \geq 0.5. \\ a(1-x_n) & otherwise \end{cases}$$

The coefficients of these chaotic generators determine their stability. It is shown that under the stable regions, the systems behave periodically. According to Tang et al. [44] there are many ways to switch from the game A to game B based on this strategy, but the easiest and the most used strategy is to compare each value of a chaotic sequence with an appropriate constant γ.
As mentioned above, the chaotic sequence generators depend on the initial conditions so it is obvious that we can find different solutions. Some simulations and results using these strategies are shown as follows. Player 1 in all of the following example is in the third place in two games but by combining two games (s)he will be in the first place.
The paradox results for probabilities $p^B_{1,1} = 0.9061$, $p^B_{2,1} = 0.0470$, $p^B_{1,2} = 0.0316$, $p^B_{2,2} = 0.4842$, $p^B_{1,3} = 0.0623$ and $p^B_{2,3} = 0.4689$ and $\gamma = 0.2$, $a = 4$, $x_0 = 0.1$ are shown in Fig. 7, in which we compare the results of periodic runs by logistic map switching. Fig. 8 shows the result using periodic strategy [BBABBA...] and Tent map strategy, for probabilities $p^B_{1,1} = 0.7866$, $p^B_{2,1} = 0.1067$, $p^B_{1,2} = 0.0226$, $p^B_{2,2} = 0.4887$, $p^B_{1,3} = 0.1907$ and $p^B_{2,3} = 0.4046$ and $\gamma = 0.4$, $a = 1.9$, $x_0 = 0.1$. Using periodic strategy, better gain is produced for player 1 compared to the chaotic time series sequences case.

Fig. 9 shows the results using periodic strategy [BBABBA...] and Sinusoidal map strategy, for probabilities $p^B_{1,1} = 0.7207$, $p^B_{2,1} = 0.1397$, $p^B_{1,2} = 0.1231$, $p^B_{2,2} = 0.4384$, $p^B_{1,3} = 0.1562$ and $p^B_{2,3} = 0.4219$ and $\gamma = 0.6$, $a = 2.3$, $x_0 = 0.5$.

4 Conclusions

In this paper, we have extended the original Parrondo's paradox game by considering the games which are played among the three players using three-sided dice. We have obtained analytical expressions for our generalized model using discrete-time Markov chain theory and all conditions are investigated by solving difference equations. We have shown that the paradox also occurs in this new extended model by switching between two games periodically and chaotically. We have done comprehensive calculations and simulations

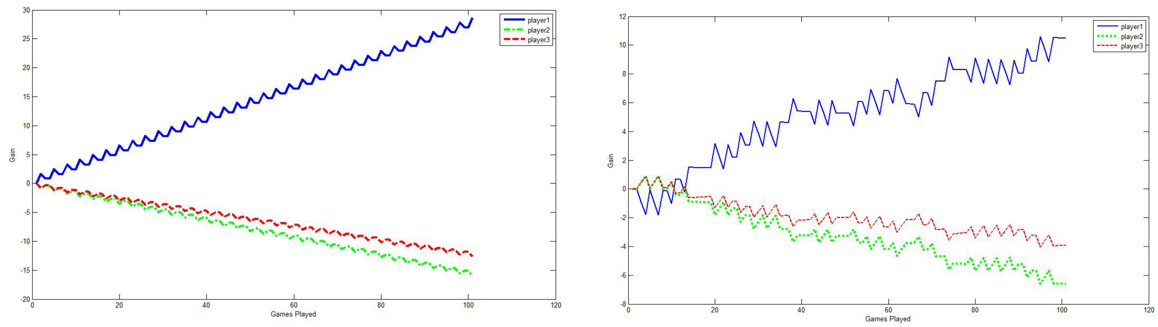

Figure 7: Simulation result for the combined game using periodic strategy[BBABBA...] (left) and logistic map switching (right). In the both cases the player 1 will be in the first place.

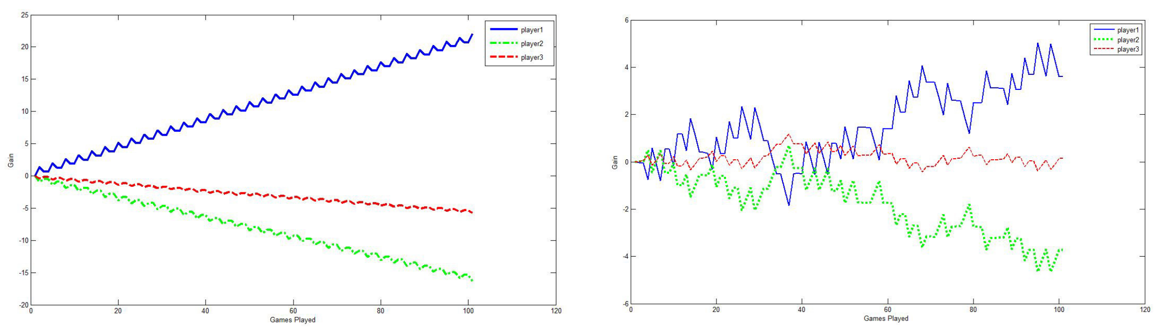

Figure 8: Simulation result for the combined game using periodic strategy[BBABBA...] (left) and Tent map switching (right). The player 1 is in the first place.

and considered several values for parameters and various combinations of game A and game B, but we could only find the combination [BBABBA...] to obtain paradoxical results. Also, we have considered different values for parameters and combined two games randomly, but paradox did not occur in any of these situations. So it might be concluded that the conditions to arrive at paradox in this extended model is much more limited than those of original Parrondo's games.

An speculative example of Parrondo's paradox in molecular biology is sexual reproduction [36],[19],[35]. As in Muller ratchet, in a sexual population the fitness is declined since the harmful mutation accumulation only proceeds in one direction and sexual reproduction breaks Muller ratchet by recombination, which allows selection of beneficial mutation. This model has been used to explain why there are two sexes. In this model recombination as second game in Parrondo's game may not be precise and the combination of two losing

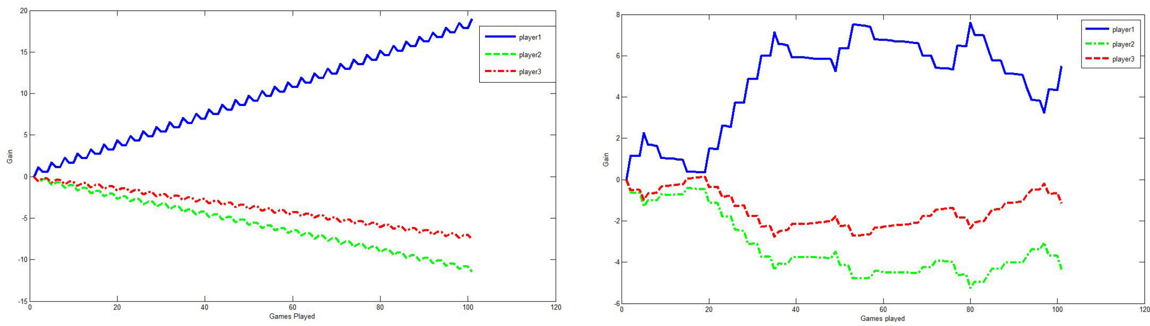

Figure 9: Simulation result for the combined game using periodic strategy[BBABBA...] (left) and Sinusoidal map switching (right). The player 1 is in the first place.

games (mutation and recombination) can reduce errors and more information is correctly transformed. Although most cells of organisms with sexual reproduction are diploid and have paired set of chromosomes but polyploidy is found in some organisms, especially in plants [11]. Extension of Parrondo's paradox may be considered as triploid cell with three players. But as our results show, in the game with more than two players, the tolerance to noise and order of combination of two games are reduced and this might explain why polyploidy is not common in most of sexual organism and there are two sexes and not more.
The paradox property may appear in games in which the number of players is more than three, therefore an extension of the Parrondo's paradox to these games is proposed to be investigated in a future work.

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