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ON SOME CIRCULAR DISTRIBUTIONS INDUCED BY INVERSE STEREOGRAPHIC PROJECTION

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ON SOME CIRCULAR DISTRIBUTIONS INDUCED BY INVERSE STEREOGRAPHIC PROJECTION[1]

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ABSTRACT. In earlier studies of circular data, the corresponding probability distributions considered were mostly assumed to be symmetric. However, the assumption of symmetry may not be meaningful for some data. Thus there has been increased interest, more recently, in developing skewed circular distributions. In this article we introduce three skewed circular models based on *inverse stereographic projection*, originally introduced by Minh and Farnum (2003), by considering three different versions of skewed-*t* considered in the literature, namely skewed-*t* by Azzalini (1985), two-piece skewed-*t* (Fernández and Steel, 1998) and skewed-*t* by Jones and Faddy (2003). Shape properties of the resulting circular distributions along with estimation of parameters using maximum likelihood are also discussed in this article. Further, real data sets are used to illustrate the application of the new models. It is found that Azzalini and Jones-Faddy skewed-*t* versions are good competitors, however, the Jones-Faddy version is computationally more tractable.

Key words and phrases: Circular data; Skewed-t distribution; Inverse stereographic projection.

1 Introduction

In many scientific fields, the observations are 'directions'. For example, a biologist may be interested in studying the direction of flight of a bird or the orientation of an animal while a geologist may be interested in measuring the direction of earth's magnetic pole. Directional data are often met in Biology, Geography, Geology, Geophysics, Medicine, Meteorology and Oceanography, such as in analysing the origins of comets, solving bird navigational problems, assessing variation in the onset of leukaemia, investigating wind directions etc. (see [13], [22] and [23]).

[1] Part of this research is based on Shamal Karmaker's M.Sc. thesis presented in the Department of Mathematics and Statistics at Concordia University.

The directions are considered as points on the circumference of a circle in two dimensions or on the surface of a sphere in three dimensions. A typical observation as a circular observation may be assumed to belong to $[0,2\pi)$ or to $[-\pi,\pi)$. Earlier attempts in modeling circular data involved mostly symmetric circular distributions (see [10], [22]). More recently, alternative circular models exhibiting asymmetry and multimodality have been investigated (see [1], [17], [19], [20], [24] and [28]).

This article focuses on asymmetric unimodal circular distributions generated by inverse stereographic projection (ISP) of $\mathbb{R} \to \mathbb{C}$, a transformation that maps points on the real line to those on a unit circle and was originally considered by Minh and Farnum [24], as described below (however, they termed it as 'bilinear transformation').

Stereographic projection of a circle to the real line is defined by a one-to-one mapping from $[-\pi,\pi)$ to $(-\infty,\infty)$ given by

$$x = T(\theta) = u + v\frac{\sin\theta}{1+\cos\theta} = u + v\tan\left(\frac{\theta}{2}\right), \tag{1.1}$$

where $x \in (-\infty,\infty)$, $\theta \in [-\pi,\pi)$, $u \in \mathbb{R}$, and $v > 0$. Then as considered in [24], the *inverse stereographic projection* (ISP) is given by the functional relation (involving two parameters u and v)

$$\theta = T^{-1}(x) = 2\tan^{-1}\left(\frac{x-u}{v}\right) \tag{1.2}$$

is a random point on the unit circle. Circular distributions may be generated by probability distributions on the real line using the standard theory of transformations.

Let $f(x)$ and $F(x)$ be the density function and distribution function of the random variable X respectively. Also let $g(\theta)$ and $G(\theta)$ denote the density function and distribution function of the random point θ on the unit circle respectively. Then $G(\theta)$ and $g(\theta)$ can be written in terms of $F(x)$ and $f(x)$ using the following equations as stated below.

For $v > 0$,

$$G(\theta) = F\left(u + v\tan\left(\frac{\theta}{2}\right)\right) = F(T(\theta)) \tag{1.3}$$

$$\begin{aligned} g(\theta) &= \frac{v}{2}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right) f\left(u + v\tan\left(\frac{\theta}{2}\right)\right) \\ &= \frac{v}{2}\left(1+\left(\frac{T(\theta)-u}{v}\right)^2\right) f(T(\theta)) \end{aligned} \tag{1.4}$$

In the next section we describe three well known skewed distributions on the real line, namely Azzalini [3] type, Jones and Faddy [16] and two-piece Student's t [26] providing alternative choices for $f(x)$, that may be useful in practice. Note that other alternative skewed densities considered in the literature, for example those due to Wang and Shimizu [29] and Azzalini and Capitano [4] may also be considered, however, we have restricted ourselves to the three densities mentioned above as they seem simple yet tractable.

The resulting circular distributions are described in §3, where as §4 studies their skewness properties. In §5, we use these models on some standard data sets in terms of investigating their usefulness in practice and the final section §6 gives conclusions.

2 Skewed-t distribution on the real line

Sometimes, it is useful to consider an alternative to the normal or t-distribution which is both heavy tailed and skewed. There are different skewed-t distributions which were proposed by different authors following different techniques. Amongst these the following distributions figure promptly: the skewed-t distribution of [16], obtained by transforming a beta random variable; the skewed-t distribution based on [3] general form of creating skewed distributions that is obtained by a specific weighting function [4, 5, 11, 21, 26, 27], the two-piece skewed-t distribution [8, 9, 26].

Jones and Faddy [16] proposed a family of distributions which includes the symmetric $t-$distribution as an special case, and also includes extension of the $t-$distribution, on the real line, with non-zero skewness. The corresponding probability density function is given by

$$f_{JF}(x;a,b) = \frac{1}{\beta(a,b)\, 2^{a+b-1}\, \sqrt{a+b}} \left(1+\frac{x}{\sqrt{a+b+x^2}}\right)^{a+\frac{1}{2}} \left(1-\frac{x}{\sqrt{a+b+x^2}}\right)^{b+\frac{1}{2}} \tag{2.1}$$

where $a>0$ and $b>0$ be parameters and $\beta(.,.)$ is the beta function. When $a=b$ then f reduces to the t-distribution on $2a$ degrees of freedom. When $a>b$ or $a<b$, f is positively or negatively skewed respectively.

The skewed-t family proposed by Azzalini [3] is obtained from a general formula for constructing skewed distributions from a symmetric distribution as given by

$$f(x;\alpha) = 2g(x)G(\alpha x), \qquad x,\alpha \in \mathbb{R}, \tag{2.2}$$

where g and G are the density and distribution functions, respectively, of a symmetric distribution. When g is the standard normal density ϕ then f provides the well-known skew-normal distribution. The parameter α is the skewness parameter, with positive and negative α leading to positive and negative skewness respectively. Also $\alpha = 0$ corresponds to symmetric density g.

The density of the Azzalini type skewed-t distribution, considered in [26], is obtained by replacing $g(x)$ and $G(\alpha x)$ in Eq. (2.2) by $f_\nu(x)$, and $F_{\nu+1}\left(\alpha x\sqrt{\frac{\nu+1}{\nu+x^2}}\right)$ respectively that gives

$$f_A(x;\nu,\alpha) = 2f_\nu(x)\,F_{\nu+1}\left(\alpha x\sqrt{\frac{\nu+1}{\nu+x^2}}\right), \qquad x,\alpha \in \mathbb{R}, \tag{2.3}$$

where $f_\nu(.)$ and $F_\nu(.)$ denote the density and distribution functions, respectively of the Student's $t-$ distribution with ν degrees of freedom.

Another type of the skewed-t family comprises of 'two-piece' distributions made up of differently scaled halves of a symmetric distribution. In general, the two-piece t distribution is given by

$$f_{TP}(x;\nu,\gamma) = f_\nu\left(\frac{x}{1+\gamma}\right) I(x<0) + f_\nu\left(\frac{x}{1-\gamma}\right) I(x\geq 0), \tag{2.4}$$

where I denotes the indicator function and $-1<\gamma<1$. The parameter γ is a skewness parameter, with positive and negative γ leading to negative and positive skewness and $\gamma = 0$ corresponds to symmetric density.

3 Inverse Stereographic Projection (ISP) skewed-t distributions on the unit circle

By applying ISP method defined by a one-to-one mapping in equation (1.1) in the three different skewed-t distribution on the real line we will obtain three different circular ISP skewed-t distribution.

3.1 Jones and Faddy ISP skewed-t distribution

Considering the Inverse Stereographic Projection (1.1) of the Jones and Faddy skewed-t distribution (2.1), the corresponding circular p.d.f. by using equation (1.4) is given by

$$g_{JF}(\theta;a,b,v) = \frac{v\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}{2^m\beta(a,b)\sqrt{m}} \left(1+\Psi(\theta)\right)^{a+\frac{1}{2}} \left(1-\Psi(\theta)\right)^{b+\frac{1}{2}} \tag{3.1}$$

where $u=0$, $v>0$, $-\pi\leq\theta<\pi$ and $m=a+b$ and

$$\Psi(\theta) = \frac{v\tan\left(\frac{\theta}{2}\right)}{\sqrt{m+v^2\tan^2\left(\frac{\theta}{2}\right)}}$$

For $a=b=\frac{m}{2}$, using the identity

$$\frac{1}{1+x^2} = 1-\frac{x^2}{1+x^2} = \left(1+\frac{x}{\sqrt{1+x^2}}\right)\left(1-\frac{x}{\sqrt{1+x^2}}\right)$$

and the Legendre duplication formula for the Gamma function,

$$\Gamma(2a)\sqrt{2\pi} = 2^{2a-\frac{1}{2}}\Gamma(a)\Gamma(a+\frac{1}{2})$$

the equation (3.1) reduces to the modified Minh-Farnum symmetric circular distribution [2] given as

$$f_{MMF}(\theta) = \frac{v}{2\sqrt{m}\beta\left(\frac{m}{2},\frac{1}{2}\right)} \frac{\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}{\left(1+\frac{v^2\tan^2\left(\frac{\theta}{2}\right)}{m}\right)^{\frac{(m+1)}{2}}} \tag{3.2}$$

The skewed version of the Cartwright's [7] power-of-cosine pdf is obtained by substituting $v=\sqrt{m}$ in equation (3.1), namely

$$f(\theta;a,b) = \frac{1}{2^m\beta(a,b)}\left(1+\sin\left(\frac{\theta}{2}\right)\right)^{a-\frac{1}{2}}\left(1+\sin\left(\frac{\theta}{2}\right)\right)^{b-\frac{1}{2}} \tag{3.3}$$

Both of these versions can adapt the additional location parameter that we omit for simplicity. An alternative form of given density in equation (3.1) can be written as

$$g_{JF}(\theta;a,b,v) = \frac{v_0}{2^m\beta(a,b)}\frac{2}{1+\cos\theta}(1+\Psi(\theta))^{a+\frac{1}{2}}(1-\Psi(\theta))^{b+\frac{1}{2}} \tag{3.4}$$

where, $a+b=m$, $v_0=\frac{v}{\sqrt{m}}$ and $\Psi(\theta)=\frac{v_0\sin\theta}{\sqrt{(1+\cos\theta)^2+v_0^2\sin^2\theta}}$ *and* $-\pi\leq\theta<\pi$.

3.1.1 Reparametrization of Jones and Faddy ISP skewed-*t* distribution

Reparametrization does not produce a different distribution family but simply re-expresses the original parameters into new ones, so that these can be better interpreted [15]. The skewness on the circle is naturally defined to have the opposite sign from the skewness of its usual linear representation [18]. For Jones and Faddy ISP skewed-*t* family, skewness depends on both the parameters a and b, through the ratio a/b. Hence we define the new parameters in the case of Jones and Faddy ISP skewed-*t* family as:

$$\gamma = \frac{a}{b} \text{ and } m = a+b$$

Hence the reparametrized form of the Jones and Faddy ISP skewed-*t* family is obtained from Eq (3.1) by choosing

$$a = \frac{m\gamma}{1+\gamma} \text{ and } b = \frac{m}{1+\gamma}$$

The p.d.f. of the reparametrized Jones and Faddy skewed-t is

$$g_{JF}(\theta;m,v,\gamma)=\frac{v\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}{2^m\beta\left(\frac{m\gamma}{1+\gamma},\frac{m}{1+\gamma}\right)\sqrt{m}}(1+\Psi(\theta))^{\frac{m\gamma}{1+\gamma}+\frac{1}{2}}(1-\Psi(\theta))^{\frac{m}{1+\gamma}+\frac{1}{2}},\qquad v,m,\gamma>0 \quad (3.5)$$

where,

$$\Psi(\theta)=\frac{v\tan\left(\frac{\theta}{2}\right)}{\sqrt{\left(m+v^2\tan^2\left(\frac{\theta}{2}\right)\right)}}\quad and -\pi\leq\theta<\pi$$

Here γ is interpreted as the skewness parameter; $\gamma>1$ leads to positive skewness and $\gamma<1$ leads to negative skewness, where as $\gamma=1$ corresponds to a symmetric density.

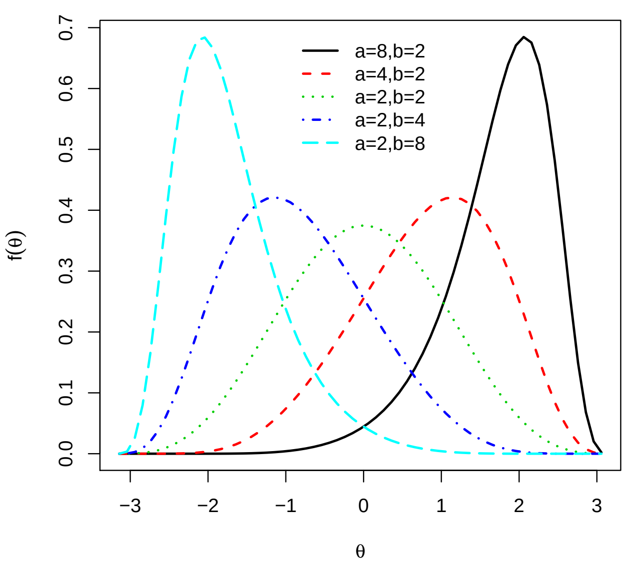

Figure 1: Probability density functions of Jones and Faddy ISP skewed-t distribution for different values of a and b

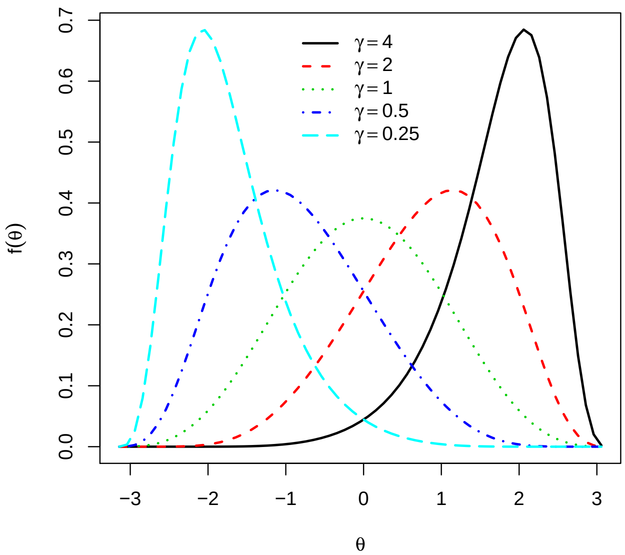

Figure 2: Probability density functions of reparametrized JF ISP skewed-t distribution for different values of γ

3.2 Azzalini type ISP skewed-t distribution

By applying the Inverse Stereographic Projection (1.1) on the Azzalini type skewed-t distribution (2.3), the pdf of the circular Azzalini type ISP skewed-t distribution by using equation (1.4) is given by

$$g_A(\theta;\nu,v,\alpha)=\frac{v\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}{\sqrt{\nu}\,\beta\left(\frac{\nu}{2},\frac{1}{2}\right)}\frac{1}{\left(1+\dfrac{v^2\tan^2\left(\frac{\theta}{2}\right)}{\nu}\right)^{\frac{\nu+1}{2}}}\left[1-\frac{1}{2}I_{(\nu+1)/2,1/2}\left(z(\theta),\frac{\nu+1}{2},\frac{1}{2}\right)\right] \tag{3.6}$$

where

$$z(\theta)=\frac{(\nu+1)(\nu+v^2\tan^2\left(\frac{\theta}{2}\right))}{\nu^2+v^2\tan^2\left(\frac{\theta}{2}\right)(\nu+\alpha^2(\nu+1))}$$

and $-\pi\leq\theta<\pi$, $\alpha\in\mathbb{R}$, $I_{\alpha,\beta}(.)$ is the regularized incomplete beta function (see [14]), $v>0$ and ν is the degrees of freedom. The parameter α controls skewness and positive and negative α leading to positive and negative skewness respectively. Also $\alpha=0$ corresponds to symmetric density.

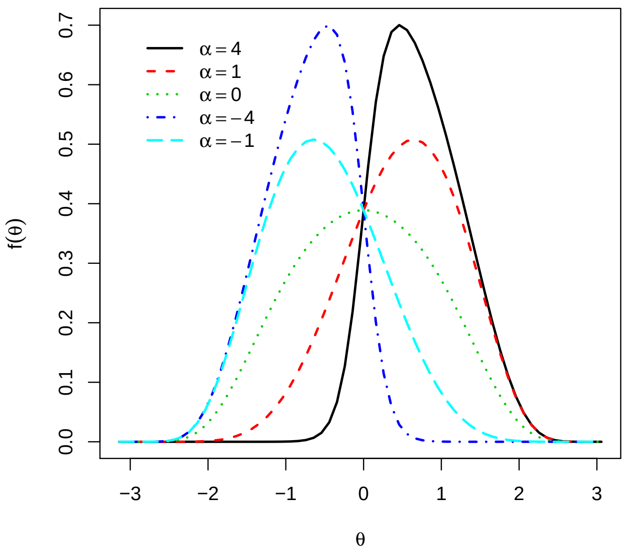

Figure 3: Probability density functions of Azzalini type ISP skewed-t distribution for different values of α

3.3 Two-piece ISP skewed-t distribution

By using the Inverse Stereographic Projection (1.1) on the two-piece skewed-t distribution (2.4), the pdf of the circular two-piece ISP skewed-t distribution by using equation (1.4) is given by

$$g_{TP}(\theta;\nu,v,\gamma) = \frac{v\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}{2\sqrt{\nu}\beta\left(\frac{\nu}{2},\frac{1}{2}\right)}\left[\frac{1}{(1+\kappa_1(\theta,\gamma))^{\frac{\nu+1}{2}}}I(\theta<0)+\frac{1}{(1+\kappa_2(\theta,\gamma))^{\frac{\nu+1}{2}}}I(\theta\geq 0)\right] \qquad (3.7)$$

where, $\kappa_1(\theta,\gamma) = \dfrac{v^2\tan^2\left(\frac{\theta}{2}\right)}{\nu(1+\gamma)^2}$, $\kappa_2(\theta,\gamma) = \dfrac{v^2\tan^2\left(\frac{\theta}{2}\right)}{\nu(1-\gamma)^2}$, $-\pi \leq \theta < \pi$, $v > 0$, $-1 < \gamma < 1$, ν is the degrees of freedom and I is the indicator function. The parameter γ controls skewness and negative and positive γ leading to the positive and negative skewness respectively. Also $\gamma = 0$ corresponds to symmetric density.

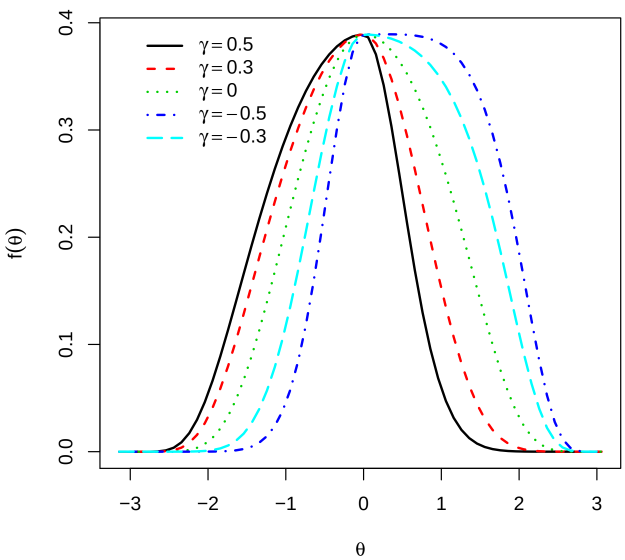

Figure 4: Probability density functions of two-piece ISP skewed-t distribution for different values of γ

4 Numerical Examples

In order to illustrate the applications of the new distributions discussed in the previous sections and, we consider three different data sets that have been in the literature, the first is known as 'Bird Migration Heading Data' the second one known as the 'Drosophila Larval Locomotion Data' and the third one is 'Ants data'(see Bruderer and Jenni [12] and Fisher [10] respectively).

We will fit our three different proposed ISP circular models to the data by estimating the parameters by the maximum likelihood method. Note that the maximum likelihood estimators are not available explicitly; we can find the estimates for a particular sample by using numerical methods. We will also consider mixtures of these models with the circular uniform distribution in order to avoid zero value of the density at the tails.

Our comparison will be based on the values of Chi-square goodness-of-fit statistic and AIC that is defined as

$$AIC = -2\log_e \text{Likelihood} + 2p \tag{4.1}$$

where p is the number of parameters in the model. Even though, log-likelihood values are good enough for respective comparison of different circular distributions with the same number of parameters, AIC values are useful for cross comparison among the models with different number of parameters, such as mixture and non-mixture models. Hence, we have displayed both of these in the tables. Additionally, we have also displayed the $P-$ values corresponding to the χ^2-values that may enable us to judge the adequacy of the fitted distributions.

4.1 Bird migration headings data

The bird migration headings data set introduced to the ornithological literature by [6]. The data consists of the 'headings' of 1827 migrating birds recorded at an observational post near Stuttgart during the autumnal migration period of 1987. Here, the term 'heading' refers to the direction, measured in a clockwise direction from North, of a bird's body during flight.

Table 1: Parameters estimated for Bird migration headings data

Distribution	ML Estimates
Jones and Faddy ISP skewed-*t*	$\widehat{m} = 5.08$, $\widehat{\gamma} = 2.11$, $\widehat{\nu} = 2.62$
Azzalini type ISP skewed-*t*	$\widehat{\nu} = 1.88$, $\widehat{\alpha} = 1.58$, $\widehat{\nu} = 2.98$
Two-piece ISP skewed-*t*	$\widehat{\nu} = 2.08$, $\widehat{\gamma} = -0.52$, $\widehat{\nu} = 3.36$

Table 2: Comparison of fit for Bird migration headings data

Distribution	Max Log Likelihood	AIC	χ^2	$P-$value
Jones and Faddy ISP skewed-*t*	-2186.9	4379.9	208.30	0.000
Azzalini type ISP skewed-*t*	-2163.9	4333.9	154.71	0.000
Two-piece ISP skewed-*t*	-2354.1	4714.1	428.77	0.000

Tables 1 and 2 shows the parameter estimates, maximized log-likelihood, AIC and chi-square values for three different distributions. The histogram of the data with estimated densities is given in Figure 5 for visual inspection.

According to Table 2, Azzalini type ISP skewed-*t* model gives better fit compared to other distributions, based on the likelihood, AIC as well as chi-square values. However, visually, ISP Jones and Faddy skewed-*t* and Azzalini type skewed-*t* is very similar and it is clear that neither provides good fit to the data. Specifically, neither is capable of simultaneously modelling the peakedness and long 'tails' of the histogram. Moreover, zero density at the tails that is necessitated by all the new distributions creates a problem for modeling.

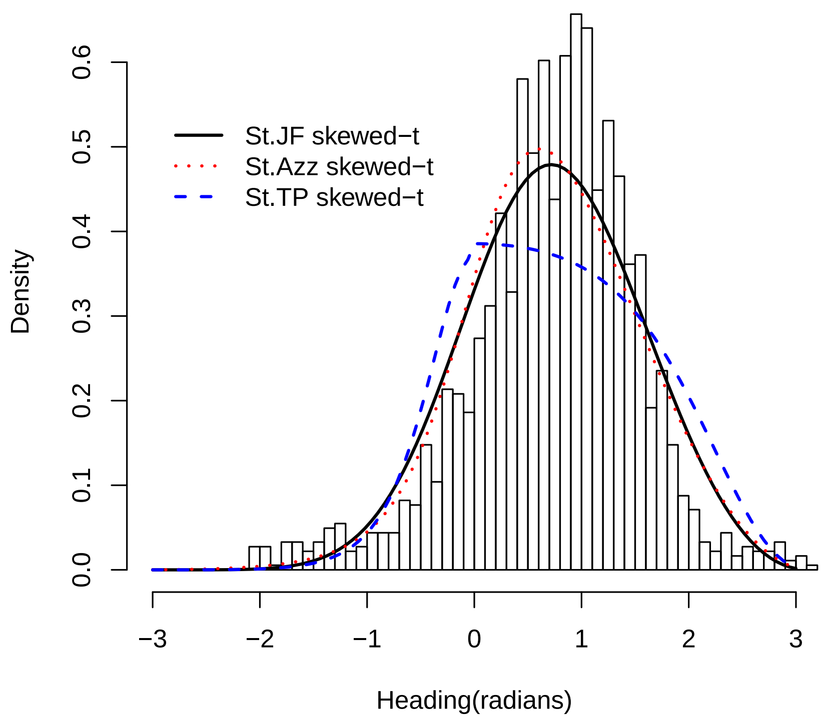

Figure 5: Histogram of the 1827 bird-flight headings together with fitted densities

4.1.1 Mixture Distributions

To overcome the problem of modelling the peakedness and long 'tails' simultaneously, we consider mixing the circular uniform distribution with the proposed ISP skewed-t distribution as in [25]. The density of the empirical model which corresponds to mixture distribution with circular uniform and ISP skewed-t components.

$$g_{MJF}(\theta;m,v,\gamma,w) = \frac{w}{2\pi} + (1-w)g_{JF}(\theta;m,v,\gamma) \tag{4.2}$$

$$g_{MA}(\theta;\nu,v,\alpha,w) = \frac{w}{2\pi} + (1-w)g_{A}(\theta;\nu,v,\alpha) \tag{4.3}$$

$$g_{MTP}(\theta;\nu,v,\gamma,w) = \frac{w}{2\pi} + (1-w)g_{TP}(\theta;\nu,v,\gamma) \tag{4.4}$$

where $g_{MJF}(\theta;m,v,\gamma,w)$, $g_{MA}(\theta;\nu,v,\alpha,w)$ and $g_{MTP}(\theta;\nu,v,\gamma,w)$ are the mixture ISP Jones and Faddy, Azzalini type and two-piece skewed-*t* distribution, respectively and w being the mixing probability associated with the two components. The ML solution obtained by using log-likelihood derived from equation (4.2) to (4.4) and their comparison of fit is presented in Tables 3 and 4.

Table 3: Parameters estimated for mixture distribution for Bird migration headings data

Distribution	ML Estimates
Jones and Faddy ISP skewed-*t*	$\widehat{m} = 17.38$, $\widehat{\gamma} = 1.75$, $\widehat{v} = 2.86$, $\widehat{w} = 0.096$
Azzalini type ISP skewed-*t*	$\widehat{v} = 1.65$, $\widehat{\alpha} = 2.29$, $\widehat{\nu} = 801.4$, $\widehat{w} = 0.103$
Two-piece ISP skewed-*t*	$\widehat{v} = 2.34$, $\widehat{\gamma} = -0.67$, $\widehat{\nu} = 1599.9$, $\widehat{w} = 0.117$

Table 4: Comparison of fit for mixture distribution for Bird migration headings data

Distribution	Max Log Likelihood	AIC	χ^2-value	$P-$value
Jones and Faddy ISP skewed-*t*	-2098.1	4204.3	28.57	0.195
Azzalini type ISP skewed-*t*	-2111.6	4231.2	37.62	0.028
Two-piece ISP skewed-*t*	-2225.0	4458.1	193.35	0.000

Table 4 shows that the mixture of the circular uniform distribution with Jones and Faddy ISP skewed-*t* gives a better fit compared to others distributions, according to the likelihood, AIC as well as chi-square values where as the two-piece ISP skewed-*t* provides a very poor fit. The mixture modeling improves both Jones and Faddy ISP skewed-*t* and Azzalini type ISP skewed-*t* distributions as plausible models, however, only Jones and Faddy type ISP skewed$-t$ seems meaningful according to the $P-$value. Note that two-piece ISP skewed-*t* does not provide a good fit even after mixing with circular uniform.

4.2 Drosophila larval locomotion data

In our second example, we present an analysis of the $n = 180$ changes in direction of a single Drosophila fly larva, measured once per second over a period of three minutes [12]. The maximum likelihood estimates of the parameters including maximized log-likelihood, AIC and chi-square values for three proposed ISP circular distribution are presented in Tables 5 and 6. A histogram of the data with fitted densities is portrayed in Figure 7.

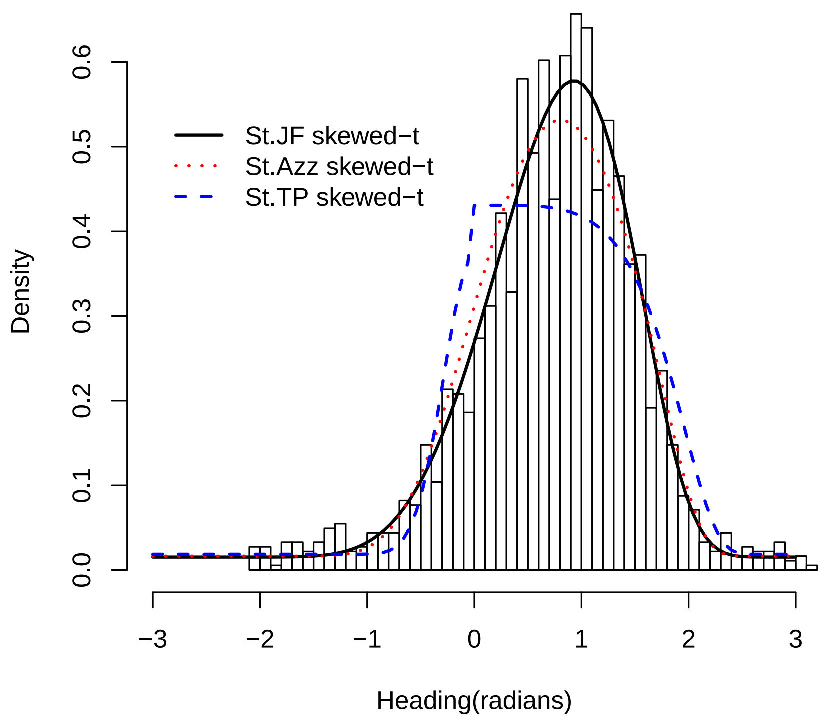

Figure 6: Histogram of the 1827 bird-flight headings together with fitted mixture densities

Table 5: Parameters estimated for Drosophila larval locomotion data

Distribution	ML Estimates
Jones and Faddy ISP skewed-*t*	$\widehat{m} = 1.11$, $\widehat{\gamma} = 0.79$, $\widehat{\nu} = 10.64$
Azzalini type ISP skewed-*t*	$\widehat{\nu} = 10.36$, $\widehat{\alpha} = -0.227$, $\widehat{\nu} = 1.08$
Two-piece ISP skewed-*t*	$\widehat{\nu} = 2.61$, $\widehat{\gamma} = 0.201$, $\widehat{\nu} = 2.57$

Table 6: Comparison of fit for Drosophila larval locomotion data

Distribution	Max Log Likelihood	AIC	χ^2-value	$P-$value
Jones and Faddy ISP skewed-*t*	-113.98	233.96	1.29	0.731
Azzalini type ISP skewed-*t*	-114.86	235.71	1.31	0.726
Two-piece ISP skewed-*t*	-228.64	463.27	19.10	0.000

Table 6 illustrates that Jones and Faddy ISP skewed-*t* model provides better fit compared to other distributions, according to the likelihood, AIC as well as chi-square values. The visual inspection of Figure 7 shows that both Jones and Faddy type ISP skewed-*t* and Azzalini type ISP skewed-*t* provide similar fits (actually they overlap each other), where as the two-piece ISP skewed-*t* provides a very poor fit. This is also confirmed using the AIC and χ^2- values. Both Jones and Faddy skewed-*t* and Azzalini type ISP skewed-*t* distributions are accepted as good models where as the two-piece ISP skewed-*t* is rejected.

Table 7: Parameters estimated for mixture distribution for Drosophila larval locomotion data

Distribution	ML Estimates
Jones and Faddy ISP skewed-*t*	$\widehat{m} = 1.27$, $\widehat{\gamma} = 0.77$, $\widehat{v} = 10.51$, $\widehat{w} = 0.0266$
Azzalini type ISP skewed-*t*	$\widehat{v} = 10.31$, $\widehat{\alpha} = -0.212$, $\widehat{\nu} = 0.911$, $\widehat{w} = 0.06$
Two-piece ISP skewed-*t*	$\widehat{v} = 2.79$, $\widehat{\gamma} = 0.2307$, $\widehat{\nu} = 1215$, $\widehat{w} = 0.0727$

Table 8: Comparison of fit for mixture distribution for Drosophila larval locomotion data

Distribution	Max Log Likelihood	AIC	χ^2-value	$P-$value
Jones and Faddy ISP skewed-*t*	-113.85	235.70	1.02	0.797
Azzalini type ISP skewed-*t*	-114.64	237.27	4.17	0.360
Two-piece ISP skewed-*t*	-207.21	422.41	18.34	0.000

Tables 7 and 8 present analogous results to those presented in Tables 5 and 6. Moreover, Figure 8 looks similar to Figure 7. because of the ISP Jones and Faddy skewed-*t* and Azzalini type skewed-*t* provides good fit except two-piece, mixing ISP skewed-*t* with circular uniform do not have significant effect for this data.

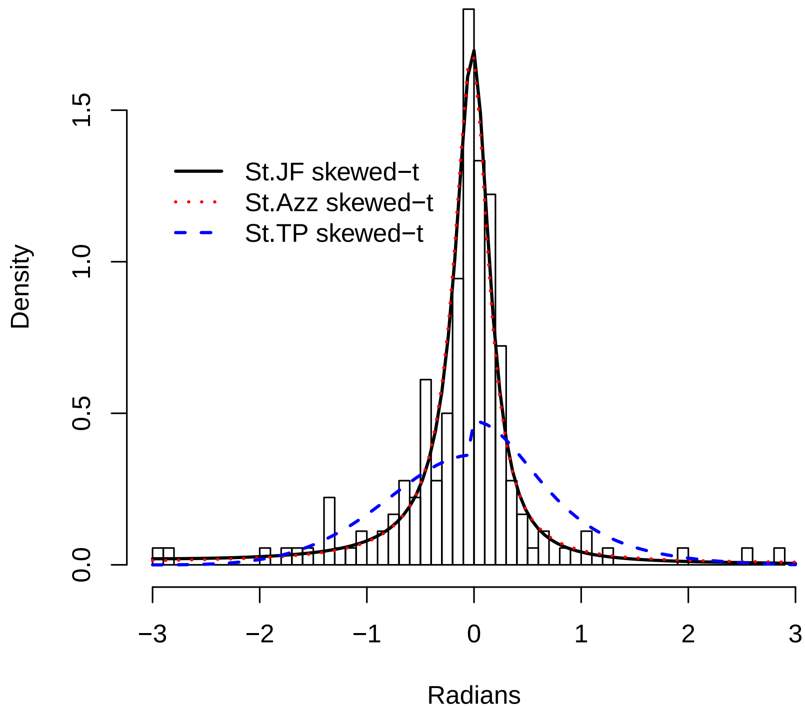

Figure 7: Histogram of the Drosophila larval locomotion data together with fitted densities

5 Conclusions

The basic objective of this study is to provide flexible and numerically tractable probability models for circular data. In this regard we find that *inverse stereographic projection* (ISP) method of transforming a distribution on the real line to that on a unit circle is quite useful. There have been many families of asymmetric distributions proposed earlier in the literature which offer a comprehensive varieties of forms such as that due to Kato and Jones [19]. However, such families involve intensive computations where as for the new families of distributions based on various skewed-t families, explicit form of the density function is available.

In order to judge the flexibility of shapes, skewness of each of the resulting distribution is investigated. On this account we have some success, however, it is not able to incorporate heavy tails as the density at the tails is necessarily zero. In order to alleviate this problem a mixture model that mixes the resulting ISP circular distribution with the circular uniform distribution, is proposed. The skewed-t families chosen for

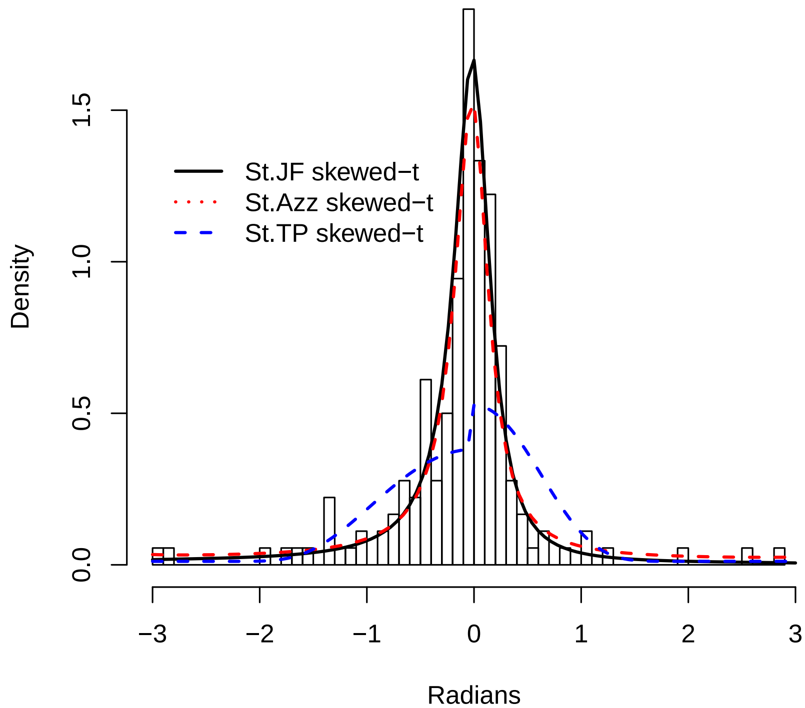

Figure 8: Histogram of the Drosophila larval locomotion data together with fitted mixture densities

investigation are those due to Jones and Faddy [16], Azzalini [3] and two-piece skewed-t considered by Rosco et al. [26].

We have also studied three practical examples with respect to the suitability of the new models. We compared the rankings for the fitted models based on different criteria (maximized log-likelihood, AIC and chi-square values), and showed that they can provide somewhat different rankings. Example 1 (Bird migration headings data) illustrates that the Jones and Faddy ISP skewed-t distribution provides a flexible model for asymmetrically distributed circular data, particularly when used as a component in a finite mixture modelling. On the other hand, two-piece ISP skewed-t distribution does not fit well. Example 2 (Drosophila Larval Locomotion data) shows that ISP Jones and Faddy skewed-t and Azzalini type skewed-t fit very well considering peakedness and long ‘tails’ of the histogram without finite mixture modelling. Azzalini type ISP skewed-t and Jones and Faddy ISP skewed-t are overlapping each other. However, ISP circular distribution obtained from two-piece skewed-t does not fit good for any of the examples considered. The graph of this distribution is not of very appealing nature as the two pieces seem to be fitted forcibly.

The incorporation of a mixture with the uniform circular distribution basically lifts the tails and provides a very flexible model as seen from the examples. The ISP circular distribution obtained from the Azzalini type skewed-t involves the distribution function of the Student's t distribution, hence is not computationally attractive over that obtained by Jones and Faddy skewed-t. Hence we may conclude that the ISP circular distribution obtained from Jones and Faddy skewed-t is an attractive alternative to other asymmetric unimodal circular distributions, especially when combined with a mixture of uniform circular distribution.

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