p-adic Abelian Integrals: from Theory to Practice

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Abstract

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Leonardo COLÒ

Let K be a complete subfield of $\mathbb{C}_p$. Consider a rigid space X over K with good reduction and a differential of the second kind ω over X. Coleman theory of p-adic integration tells us how to give a meaning to an expression of the form

$$\int_P^Q \omega \qquad\qquad P, Q \in X(K)\backslash(\omega)_\infty$$

The work of Coleman relies on using the Dwork principle of continuation along Frobenius to overcome the topological problems coming from the ultrametric nature of K.

Between 2006 and 2011, K.S. Kedlaya and J. Balakrishnan have constructed algorithms to compute explicitly Coleman's integrals on hyperelliptic curves and, together with R. Bradshaw, they have implemented these methods in `SAGE`.

In this thesis, I study the theory of Coleman both from the theoretical and the algorithmic point of view and I provide the results of some explicit computations.

After a review of some fundamental ideas in rigid geometry, I present the theory of Coleman as it appears in his original articles. The second part of this work is devoted to the computational approach: I describe the ideas of Kedlaya and Balakrishnan and I produce some concrete examples. Finally, the last Chapter deals with one application of Coleman's integrals: I study the method of Chabauty and Coleman and I show how it can be used to effectively detect rational points on curves.

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Introduction

Over the last century, p-adic numbers and p-adic analysis have started playing an important role in number theory. Suppose we have an algebraic variety X over $\mathbb{R}$; it is well known that $X(\mathbb{R})$ or $X(\mathbb{C})$, the set of points on X valued on $\mathbb{R}$ or $\mathbb{C}$ respectively, form a real or complex variety on which we are perfectly able to integrate.

The situation changed with the introduction of p-adic numbers by K. Hensel in 1897. The German mathematician developed his theory, based on the work of E. Kummer, in analogy with the relation between the ring of polynomials $\mathbb{C}[x]$ and its field of fractions $\mathbb{C}(x)$. The result was a set of formal expansions

$$\sum_{n \geqslant n_0}^{+\infty} a_n p^n \qquad a_i \in \mathbb{Z},\ 0 \leqslant a_i < p$$

which turns out to have the structure of a field and, nowadays, we denote $\mathbb{Q}_p$, the field of p-adic numbers.

The definition of Hensel was improved by A. Ostrowski (1918) who proved that any absolute value on $\mathbb{Q}$ is equivalent either to the standard one or to a p-adic absolute value. Completing $\mathbb{Q}$ with respect to the latter, we recover the field of p-adic numbers discovered by Hensel.

The work of Ostrowski allowed mathematicians to ask whether the well-known properties of real numbers (the completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$) were inherited also by $\mathbb{Q}_p$. In particular, one of the most important area of study was the possibility of constructing a theory of analytic functions over p-adic fields. If at the beginning the question might have been motivated only by curiosity, with the progress of Algebraic Geometry it became clear that the question was of central importance.

However, in 1944, M. Krasner introduced the concept of ultrametric fields, to which p-adic numbers belong, and proved (1954) that the topology of these objects is totally disconnected making clear the fact that to obtain a global theory of functions over p-adic fields there was the need of different ideas from the ones used in the real and complex case.

The turning point was the work of the American mathematician J. Tate. He understood that algebraic methods were somehow unsuitable to such a description and focused the attention on an analytic approach. Let K be a complete non-archimedean field; Tate introduced the algebra

$$\mathcal{O}\left(K^{\times}\right) = \left\{ \sum_{\nu \in \mathbb{Z}} c_\nu \zeta^\nu \,\middle|\, \lim_{|\nu| \to +\infty} |c_\nu| r^\nu = 0 \quad \forall r > 0 \right\}$$

of Laurent expansions converging on $K^{\times}$. He constructed its field of fractions $\mathcal{M}(K^{\times}) = \mathrm{Frac}\left(\mathcal{O}(K^{\times})\right)$ and defined, for every $q \in K^{\times}$ with $|q| < 1$, the set

$$\mathcal{M}^q(K^{\times}) = \left\{ f \in \mathcal{M}(K^{\times}) \,\middle|\, f(q\zeta) = f(\zeta) \right\}$$

He observed that $\mathcal{M}^q(K^{\times})$ is an elliptic function field and the set of K-points of the associated elliptic curve $\mathcal{E}_K$ coincides with the quotient $K^{\times}/q^{\mathbb{Z}}$ (Tate elliptic curve). To describe the nature of this elliptic curve, Tate realized that algebraic geometry was ineffective. In 1961 he published an important article [Tat] developing a theory that was able to give a meaning to the quotient $K^{\times}/q^{\mathbb{Z}}$. Nowadays we use to indicate his work as the birth of Rigid Geometry.

The theory of Tate was eventually enriched and expanded by D. Mumford, who generalized the construction of Tate to higher dimensional abelian varieties (1972), M. Raynaud, who introduce Formal Geometry (1974) and G. Faltings (1990).

Although a theory of analytic functions over an ultrametric field K had been developed, the nature of the topology on K still made difficult to construct a theory of integration: the fact that the p-adic topology is totally disconnected makes impossible to pass from local to global as we do in the complex case.

The situation changed in the 80's with the work of the American mathematician R. Coleman. He was the first to propose a solution to the problem of constructing a global theory of integration on rigid spaces using the Dwork principle of "continuation along Frobenius".

The idea appeared first in [Col1] (1982) under the form of integration on subsets of $\mathbb{P}^1$; at the end of the article Coleman announced his intention to develop this work to a theory of p-adic abelian integrals on arbitrary varieties.[1]

In fact, in [Col2] (1985), he established the basis for an integration theory for differential of the II kind on abelian varieties of arbitrary dimension having good reduction at p.

Finally, in a joint work with E. de Shalit (1988, [CdS]), the approach was enlarged to a wider range of differentials on curves.

[1]"In a subsequent paper we intend to show how the ideas in this paper lead to a theory of p-adic abelian integrals" - R.F. Coleman [Col1].

Coleman theory was then extended by Y. Zarhin (*"Local heights and abelian varieties"*, 1988/1989) and P. Colmez (*"Périodes p-adiques des variétés abéliennes"*, 1992 and [Colm], 1998); they were able to eliminate the hypothesis of good reduction and, equally remarkable, they did not pass through rigid geometry.

During the last 20 years, the work of Coleman has been generalized by A. Besser (*"A generalization of Coleman's p-adic integration theory"*, 1999) using methods of p-adic cohomology, V. Vologodsky. (*"Hodge structure on the fundamental group and its application to p-adic integration"*, 2003) and V.G. Berkovich (*"Integration of one-forms on p-adic analytic spaces"*, 2007).

Apart from their purely theoretical interest, Coleman integrals have a great importance because of several applications introduced during the years:

- *Torsion Points on Curves.* This was Coleman's original application of p-adic integration. He proved (after Raynaud) the Manin-Mumford conjecture asserting that any curve, of genus at least two in an abelian variety, contains only finitely many torsion points. (R.F. Coleman, *"Torsion points on curves and p-adic abelian integrals"*, 1985)

- *Rational Points on Curves.* Coleman used the theory of p-adic abelian integrals to show that it was possible to give effective bounds to the number of rational points on an algebraic curve over a number field K, provided that the Mordell–Weil rank of the Jacobian of the curve is not too large. (R.F. Coleman, *"Effective Chabauty"*, 1985)

- *p-adic Regulators, Polylogarithms and Multiples of Zeta Values.* If $\mathcal{C}$ is a smooth complete curve over $\mathbb{Q}_p^{\mathrm{alg}}$ whose Jacobian J has good reduction, Coleman and de Shalit constructed a p-adic analogue of regulator pairing in the form of a homomorphism

$$r_{p,\mathcal{C}} : K_2\left(\mathbb{Q}_p^{\mathrm{alg}}(\mathcal{C})\right) \longrightarrow \mathrm{Hom}\left(\mathrm{H}^0(\mathcal{C}, \Omega^1_{\mathcal{C}}), \mathbb{Q}_p^{\mathrm{alg}}\right)$$

whose value at the Steinberg symbol $\{f, g\}$ is the linear functional

$$r_{p,\mathcal{C}}\left(\{f, g\}\right)(\omega) = \sum_{i=1}^{t} \int_{P_i}^{Q_i} \mathrm{Log}(g) \cdot \omega$$

where $\mathrm{div}(f) = \sum_{i=1}^{t}(Q_i) - (P_i)$ and Log denotes a fixed branch of the p-adic logarithm.

This can be used to compute special values of the p-adic $\mathcal{L}$-function associated to an elliptic curve over $\mathbb{Q}$ having good reduction at p. (R.F. Coleman and E. de Shalit, *"p-adic regulators on curves and special values of p-adic $\mathcal{L}$-functions"*, 1988)

- *p-adic heights on Curves.* Coleman and Gross proposed a new definition of a p-adic height pairing on curves over number fields with good reduction at primes above p (based on the work of Mazur, Tate and Schneider). The pairing was defined as a sum of local terms; the ones corresponding to primes above p depend on Coleman's theory of p-adic integration. (R.F. Coleman and B.H. Gross, *p-adic heights on curves"*, 1989)

- *p-adic Periods* It was Coleman (inspired by the work of Fontaine) who first proposed to use p-adic integrals to define p-adic period on varieties having good reduction at p. His ideas were eventually formalized by Colmez:

  Theorem. *The map* $(\omega, \gamma) \to \int_\gamma \omega$ *of* $H^1_{dR}(X) \times T_p(X)$ *to* B^+_{dR} *is bilinear, commutes with the action of Galois, respects filtrations and it is non degenerate when extending scalars to* B_{dR}.

  (R.F. Coleman, *"Hodge-Tate periods and p-adic abelian integrals"*, 1984 - P. Colmez, *"Périodes p-adiques des variétés abéliennes"*, 1992)

The great variety of potential applications of Coleman integrals has resulted, in the 2000's, in the spread of a more concrete line of investigation.

A first explicit method for the computation of Coleman integrals on hyperelliptic curves was described in the M.Sc. thesis of Igor Gutnik *"Coleman Integration on hyperelliptic curves using Kedlaya algorithm"* (Ben-Gurion University of the Negev, 2005). Gutnik produced an implementation in `MAGMA` based on previous works of K.S. Kedlaya on Frobenius computations. Unfortunately, his work was not tested, optimized, distributed or used for any application.

Few years later Kedlaya proposed the numerical calculation of Coleman integrals on hyperelliptic curves first at Banff (2/2007) and then at the Arizona Winter School (3/2007). An implementation for the case $g = 1$ was developed in `SAGE` mostly by R. Bradshaw, using the implementation of the Frobenius calculations developed at MSRI (6/2006) by himself, J. Balakrishnan, D. Harvey and L. Xiao.

This work was eventually extended to arbitrary g by K. Kedlaya, J. Balakrishnan and R. Bradshaw.

During the last ten years there have been several attempts to construct algorithms for the applications we have illustrated before: p-adic heights have been studied by Balakrishnan (*"Local heights on hyperelliptic curves"*), Besser (*"On the computation of p-adic height pairings on Jacobians of hyperelliptic curves"*) and Harvey (*"Efficient computation of p-adic heights"*). Besser and R. de Jeu (*"An algorithm for computing p-adic polylogarithms"*) have done some computations for p-adic regulators; H. Furusho have introduced some methods to study p-adic multiple zeta values (*"p-adic multiple zeta values. II. Tannakian interpretations"*) and, finally, rational points on curves have been studied among the others by W. McCallum - B. Poonen (*"The method of Chabauty and Coleman"*) and M. Kim.

In conclusion, we would like to mention some more recent contributions: since Kedlaya's formulation of the algorithm, his work has been extended among the others by J. Denef and F. Vercauteren (introducing the computations in characteristic 2), P. Gaudry and N. Gürel (superelliptic curves), W. Castryck, T.G. Abbot, D. Roe and D. Harvey.

Structure of the Thesis

The aim of this Master thesis is to give an overview on the theory developed by Coleman of p-adic abelian integrals, to discuss the computational methods introduced in recent times by Kedlaya and Balakrishnan and to produce some concrete computations.

This presentation is articulated into 6 chapters.

Chapter 1 - We present some preliminary results about valued fields and normed space. The purpose of this chapter is to highlight the setting in which we will be working for the rest of the thesis.

Chapter 2 - This chapter deals with the construction of Tate that are the basis of rigid geometry. In particular, we give the definition of affinoid algebras and affinoid spaces carrying on a comparison with the objects of study in classical algebraic geometry.

Chapter 3 - We use the dictionary developed in the previous chapter to construct general rigid spaces. Firstly, we define a suitable topology on affinoid spaces and we use it to glue them together; then, we show how rigid spaces and algebraic varieties are related illustrating the techniques of analytification and reduction. Finally, we glance at the construction of formal geometry.

Chapter 4 - The goal of this part is to describe the theory of p-adic abelian integrals developed by Coleman in [Col3]. After a brief motivation, we describe the objects coming into play and we prove the main Theorem of Coleman's theory.

Chapter 5 - We discuss here the explicit algorithms for computing Coleman integrals on hyperelliptic curves; we also present some concrete examples.

Chapter 6 - In this last chapter we give an example of application of Coleman integrals; in particular, we describe the Chabauty-Coleman method for counting rational points on curves.

Chapter 1

Valued Fields and Normed Spaces

In this first chapter we recall some basic definitions and results about valued fields which will be useful for the study of rigid geometry. In particular, we introduce the notion of *non-archimedean* valuation and we study the associated topology. In the second section we deduce the behavior of this kind of valuations in fields extensions and we give an overview on the problem of field completion. In the last part of the chapter we introduce the idea of Banach spaces and Banach algebras which will be used to study Tate's algebras and Affinoid Algebras in the second chapter.

The main References are [Gou, Chapter 2], [BGR, §1.5], [Ser2, Chapter II] and [EP, Chapter 3].

1.1 Non Archimedean Fields

Let K be a field.

Definition. An absolute value on K is a map $|\cdot| : K \to \mathbb{R}$ satisfying the following conditions:

1. $|x| \geqslant 0, \ \forall x \in K$ and $|x| = 0$ if and only if $x = 0$.

2. $|x \cdot y| = |x| \cdot |y|$ for all $x, y \in K$.

3. $|x + y| \leqslant |x| + |y|$ for all $x, y \in K$.

We say that an absolute value on K is non-archimedean if it satisfies the additional condition:

4. $|x + y| \leqslant \max\{|x|, |y|\}$ for all $x, y \in K$.

It turns out that one can associate a valuation on K to any non-archimedean absolute value.

A valuation is a map $\nu : K \to \mathbb{R} \cup \{\infty\}$ satisfying the following conditions:

1. $\nu(x) = \infty$ if and only if $x = 0$.

2. $\nu(x \cdot y) = \nu(x) + \nu(y)$ for all $x, y \in K$.

3. $\nu(x + y) \geqslant \min\{\nu(x), \nu(y)\}$ for all $x, y \in K$.

Indeed, we can set $\nu(x) = -\log|x|$ and, in the other direction, $|x| = e^{-\nu(x)}$. This gives a one-to-one correspondence between non-archimedean absolute values and valuations and it allows us to talk indiscriminately about absolute values and valuations. A field K with a valuation is called a valued field.

Example. Fix a prime number $p \in \mathbb{Z}$. The p-adic valuation on $\mathbb{Z}$ is the map

$$\nu_p : \mathbb{Z}\backslash\{0\} \to \mathbb{R}$$

defined as follows: if n is an integer, $\nu_p(n)$ is the unique positive integer such that n can be written as

$$n = p^{\nu_p(n)} n' \quad \text{with} \quad p \nmid n'$$

We can extend ν_p to $\mathbb{Q}$ as follows: if $x = \frac{n}{m} \in \mathbb{Q}^{\times}$, then

$$\nu_p(x) = \nu_p(n) - \nu_p(m)$$

with the convention that $\nu_p(0) = \infty$.

For any $x \in \mathbb{Q}$ we define the p-adic absolute value as

$$|x|_p = p^{-\nu_p(x)}$$

with the convention that $|0| = 0$.

It is easy to prove that the p-adic absolute value is non-archimedean.

Definition. Given an absolute value $|\cdot|$ on a field K, we define the distance $\mathrm{d}(x, y)$ between two elements $x, y \in K$ by

$$\mathrm{d}(x, y) = |x - y|$$

The function $\mathrm{d}(x, y)$ is called a metric and it induces a topology on K.

Metrics arising from non-archimedean absolute values are called ultrametrics.

Lot of the properties of usual metric spaces do not remain true when we study non-archimedean fields. In particular, the notion of open balls which is of great importance in metric spaces turns out to be pretty strange in the ultrametric setting.

Definition. Let K be a field with an absolute value $|\cdot|$. Take an element $a \in K$ and a positive real number $r \in \mathbb{R}$. The open ball centered at a with radius r is the set

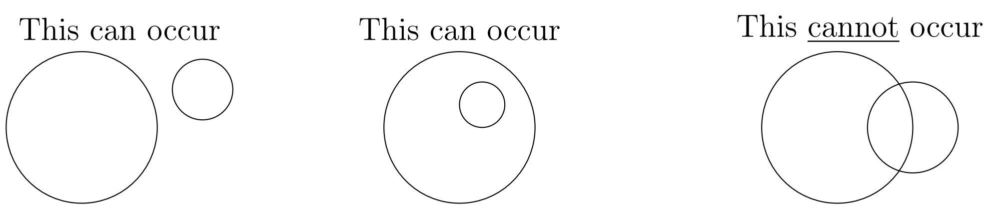
$$\mathrm{B}(a, r) = \{x \in K \mid \mathrm{d}(x, a) = |x - a| < r\}$$

Proposition 1.1.1. *Let K be a field endowed with a non-archimedean absolute value.*

(a) *If $b \in B(a, r)$, then $B(a, r) = B(b, r)$; in other words, every point that is contained in an open ball is a center of that ball.*

(b) *If $a, b \in K$ and $r, s \in \mathbb{R}_+^\times$ we have $B(a, r) \cap B(b, s) \neq \varnothing$ if and only if $B(a, r) \subseteq B(b, s)$ or $B(a, r) \supseteq B(b, s)$; in other words, any two open balls are either disjoint or contained in one another.*

This situation is completely different from the one we are used to when we work with metric spaces. The proposition is saying the following:



Proposition 1.1.2. *The topology of K induced by a non-archimedean absolute value is totally disconnected, i.e., any subset in K consisting of more than just one point is not connected.*

We want now to take a more algebraically flavored point of view.

Definition. Let K be a field with a non-archimedean absolute value $|\cdot|$. The value group of K is the set of values assumed by $|\cdot|$:

$$|K^\times| = \{|a| \mid a \in K\}$$

The valuation ring of K is the set

$$\mathcal{O}_K = \{a \in K \mid |a| \leqslant 1\}$$

The maximal ideal of $\mathcal{O}_K$ is

$$\mathfrak{p} = \{a \in K \mid |a| < 1\}$$

Finally, the residue field is

$$k = \mathcal{O}_K / \mathfrak{p}$$

1.2 Completions

The problem giving rise to the theory of completions is that, in some cases, a field K presents some "missing points", i.e., it is possible to construct some convergent sequences whose limit is not in K.

In particular, there are special sequences, called Cauchy sequences, which somehow "should have" a limit because their terms get crowded into balls with smaller and smaller radius. The idea of completing a field consists in "filling the gaps" in such a way that the sequences that should have a limit do have a limit.

Definition. Let K be a field with an absolute value $|\cdot|$. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements $x_n \in K$ is called a Cauchy sequence if the x_n's become arbitrarily close to each other as n grows. More precisely $(x_n)_n$ is a Cauchy sequence if for every $\epsilon > 0$ one can find N such that $|x_n - x_m| < \epsilon$ whenever $n, m \geqslant N$.

Definition. A field K is said to be complete with respect to the absolute value $|\cdot|$ if every Cauchy sequence of elements of K has a limit in K.

One can refer to [Lan2, §IV.4] for a precise description on how to complete a given metric space. The idea is to consider the set of all the Cauchy sequences in K with an equivalence relation identifying two sequences which are "converging to the same missing point" and to show that this set has the desired properties.

The standard example is the completion of $\mathbb{Q}$. It is well known that the field of rational numbers is not complete with respect to the standard absolute value $|\cdot|_\infty$. The completion of $(\mathbb{Q}, |\cdot|_\infty)$ yields $\mathbb{R}$, the field of real numbers, which turns out to be complete with respect to the metric given by the extension of $|\cdot|_\infty$ and to contain a copy of $\mathbb{Q}$ which is dense.

On the other hand, we have seen that there is another absolute value on $\mathbb{Q}$ coming from the p-adic valuation.

Lemma 1.2.1. *A sequence $(x_n)_{n\in\mathbb{N}}$ in a non-archimedean field K is a Cauchy sequence if and only if $|x_{n+1} - x_n|$ tends to 0 as $n \to +\infty$.*

Lemma 1.2.2. *The field $\mathbb{Q}$ of rational numbers is not complete with respect to the p-adic absolute value.*

The proofs of these two lemmas can be found in [Gou, Lemmas 3.2.2 and 3.2.3]. The completion of $\mathbb{Q}$ with respect to the p-adic absolute value is called the field of p-adic numbers and it is denoted by $\mathbb{Q}_p$.

As we said, we can think of points of $\mathbb{Q}_p$ as represented by Cauchy sequences of rational numbers; we can introduce an absolute value on $\mathbb{Q}_p$, extending (the meaning of this will be explained in the next section) the one on $\mathbb{Q}$, in the following way: if $\lambda \in \mathbb{Q}_p$ and $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence representing λ, then

$$|\lambda|_p = \lim_{n\to+\infty} |x_n|_p$$

The valuation ring of $\mathbb{Q}_p$ is $\mathcal{O}_{\mathbb{Q}_p}$, usually denoted by $\mathbb{Z}_p$; the maximal ideal of $\mathbb{Z}_p$ is $p\mathbb{Z}_p$ and the residue field is $\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$, the field with p elements.

1.3 Extension of Valuation

From now on we will suppose that K is a non-archimedean field. Consider a field extension $K \subseteq L$ and suppose that $|\cdot|_K$ is a valuation on K while $|\cdot|_L$ is a valuation on L. We say that $|\cdot|_L$ is an extension of $|\cdot|_K$ if $|\alpha|_L = |\alpha|_K$ for every $\alpha \in K$. The problem of existence for extension of valuations is solved by the following result:

Theorem 1.3.1 (Chevalley)**.** *Consider a field K, a subring $R \subseteq K$ and a prime ideal $\mathfrak{p} \subseteq R$. There exists a valuation ring $\mathcal{O}$ of K such that $R \subseteq \mathcal{O}$ and $\mathfrak{m} \cap R = \mathfrak{p}$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$.*

Corollary 1.3.2. *Let L/K be a field extension and let $\mathcal{O}_1 \subseteq K$ be a valuation ring. There exists an extension $\mathcal{O}_2 \subseteq L$ of $\mathcal{O}_1$.*

In general, we are very far from having uniqueness of extension of valuation. The situation becomes more pleasant when we consider algebraic or finite extensions.

Theorem 1.3.3. *Let K be a valued field with valuation ν and L a finite extension of K. Then ν has only finitely many inequivalent extensions $\omega_1, \ldots, \omega_t$ to L.*

Finally, if K is complete and L/K is finite, then there is only one extension of the valuation on K:

Theorem 1.3.4. *Suppose that K is a field that is complete with respect to $|\cdot|$ and that L is a finite extension of K of degree $n = [L : K]$. Then there is precisely one extension of $|\cdot|$ to L, namely*

$$|a| = |Norm_{L/K}(a)|^{1/n}$$

and L is complete with respect to this absolute value.

Notice that, if K is complete, then there exists a unique extension of the valuation on K to K^{alg}, the algebraic closure of K. In general, K^{alg} is an infinite extension of K and in this case K^{alg} is not complete. We will denote $\widehat{K^{\text{alg}}}$ the completion of K^{alg}.

Remark. This is another important difference between the two different kind of absolute values that we defined over $\mathbb{Q}$. The completion with respect to $|\cdot|_\infty$ is $\mathbb{R}$ whose algebraic closure $\mathbb{C}$ is finite dimensional over $\mathbb{R}$ and, in fact, the absolute value on $\mathbb{R}$ extends uniquely to $\mathbb{C}$.

On the other hand, if we complete $\mathbb{Q}$ with respect to the p-adic absolute value, we obtain $\mathbb{Q}_p$ whose algebraic closure is an infinite extension. Since $\mathbb{Q}_p$ is complete, the p-adic valuation extends uniquely to $\mathbb{Q}_p^{\text{alg}}$ but this turns out to be not complete ([Gou, Theorem 5.7.4]). We can construct the completion of $\mathbb{Q}_p^{\text{alg}}$ playing again with Cauchy sequences.

Proposition 1.3.5. *There exists a field $\mathbb{C}_p$ and an absolute value $|\cdot|$ on $\mathbb{C}_p$ such that $\mathbb{C}_p$ contains $\mathbb{Q}_p^{alg}$, and the restriction of $|\cdot|$ to $\mathbb{Q}_p^{alg}$ coincides with the p-adic absolute value. Further, $\mathbb{C}_p$ is complete with respect to $|\cdot|$ and $\mathbb{Q}_p^{alg}$ is dense in $\mathbb{C}_p$.*

$\mathbb{C}_p$ has the desirable property of being algebraically closed (as well as complete) but the price we have to pay is the loss of locally compactness and maximally completeness: in $\mathbb{C}_p$ there is a decreasing set of closed disks $D_n = \{a \in K \mid |a - c_n| \leqslant r_n\}$ having the properties $c_n \in K$, $r_n \in |K^\times|$, $r_{n+1} < r_n$ and $D_n \supset D_{n+1}$ $\forall n \geqslant 1$, has an empty intersection, $\cap D_n = \varnothing$.

1.4 Banach Algebras

We denote by K a complete non-archimedean field.

Definition. A normed space over K is a vector space V over K with a map $\| \; \| : V \to \mathbb{R}$ such that

1. $\|v\| \geqslant 0$.

2. $\|v\| = 0$ if and only if $v = 0$.

3. $\|a \cdot v\| = |a| \cdot \|v\|$ for all $a \in K$ and for all $v \in V$.

4. $\|v + w\| \leqslant \max\{\|v\|, \|w\|\}$ for all $v, w \in V$.

The map $\| \; \|$ is called a norm. A seminorm is a map satisfying properties **1**, **3** and **4** (possibly not **2**).

As for field extensions, one verifies that if K is complete, then every finite dimensional vector space over K possesses only one norm (up to equivalence) and it is a Banach space with respect to it.

Remark. It is possible to prove that classical theorems on Banach spaces in functional analysis over archimedean fields still hold in the non-archimedean case ([Tia, §1.2]).

Definition. A Banach algebra A over K is a commutative K-algebra having an identity element and a norm $\| \ \|$ such that:

1. A is a Banach space with respect to $\| \ \|$.

2. $\|1\| = 1$

3. $\|a \cdot b\| \leqslant \|a\| \cdot \|b\|$

Definition. A Banach Module M over a Banach algebra A is an A-module provided with a norm $\| \ \|$ such that M is a Banach space with respect to $\| \ \|$ and $\|a \cdot m\| \leqslant \|a\| \cdot \|m\|$ for every $a \in A$ and $m \in M$.

Let A be a Banach K-algebra and E, F be two Banach A-module. Consider the usual tensor product $E \otimes_A F$. For any $x \in E \otimes_A F$, we can define

$$\|x\| = \inf_{x=\sum_{i=1}^{r} e_i \otimes f_i} \max_i \{\|e_i\|, \|f_i\|\}$$

where $x = \sum_{i=1}^{r} e_i \otimes f_i$ runs through all possible representations of x. This defines a seminorm on the tensor product $E \otimes_A F$. We define the completed tensor product $E \widehat{\otimes}_A F$ as the completion of $E \otimes_A F$. There are two natural maps $\iota_1 : E \to E \widehat{\otimes}_A F$ and $\iota_2 : F \to E \widehat{\otimes}_A F$. The completed tensor product has following universal property: if M is a Banach A-module and $\phi : E \to M$ and $\psi : F \to M$ are two continuous A-linear maps, then there exists a unique A-linear map $\phi \widehat{\otimes} \psi : E \widehat{\otimes}_A F \to M$ such that $\phi = \phi \widehat{\otimes} \psi \circ \iota_1$ and $\psi = \phi \widehat{\otimes} \psi \circ \iota_2$.

Chapter 2

Affinoid Algebras and Affinoid Spaces

In classical algebraic geometry we construct affine varieties starting from polynomial algebras over fields with archimedean absolute values. The idea is to consider the prime spectrum of these algebras and to define a Zariski topology over it. An algebraic variety is obtained by gluing together in a suitable way some affine varieties.

When considering a field K, complete with respect to a *non-archimedean* valuation, we can still try to work with polynomial algebras but this approach has some inconveniences. In particular, it turns out that $K[\zeta_1, \dots, \zeta_n]$ is not complete with respect to the Gauss norm. The completion of the space of polynomials is a Tate Algebra.

An affinoid algebra is the quotient of some Tate algebra and plays the role of a finitely generated algebra in algebraic geometry. An affinoid space is the set of maximal ideals of an affinoid algebra and it will be the analogue of an affine variety.

In this chapter we present the definition of Tate Algebras and we prove some basic results about them. We will then explain why the canonical constructions coming from algebraic geometry are not the suitable tools to approach the study of varieties over non-archimedean fields and we will give the definitions of affinoid spaces and affinoid subdomains. In the last section we will start constructing a topology on affinoid spaces which will allow us to glue them together.

The references are [Bos, Chapters 2 and 3], [Tia, Chapter 1], [BGR, Chapter 5], [FvdP, Chapter 3] and the original article by Tate [Tat].

2.1 Tate Algebras

Before starting, let us fix the notation. K will denote a complete field with respect to a non-archimedean valuation. The valuation ring of K will be written as $\mathcal{O}_K = \{a \in K \mid |a| \leqslant 1\}$. The maximal ideal of the valuation ring is $\mathfrak{p} = \{a \in K \mid |a| < 1\}$ and the residue field $\mathcal{O}_K/\mathfrak{p}$ of K is denoted by k.

According to the philosophy of Algebraic geometry, the study of an affine variety is equivalent to the study of its ring of algebraic functions ([Har]). One might try to figure out what could be a good notion of analytic functions over a non-archimedean field.

Tate's idea is to mimic the Weierstrass' definition of holomorphic functions.

Lemma 2.1.1. *A formal power series*

$$f = \sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu = \sum_{(\nu_1, \ldots, \nu_n) \in \mathbb{N}^n} c_{\nu_1, \ldots, \nu_n} \zeta_1^{\nu_1} \cdot \ldots \cdot \zeta_n^{\nu_n} \in K[\![\zeta_1, \ldots, \zeta_n]\!]$$

converges in $\{(a_1, \ldots, a_n) \in \left(K^{alg}\right)^n \mid |a_i| \leqslant 1\}$ *if and only if* $\lim_{|\nu| \to \infty} |c_\nu| = 0$ *(where* $|\nu| = \sum \nu_i$*).*

In the following, we will use the notation

$$\mathbb{B}^n(K) = \{(a_1, \ldots, a_n) \in K^n \mid |a_i| \leqslant 1\}$$

Definition. The Tate algebra (or standard affinoid algebra) over K is the set of all formal power series $\sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu \in K[\![\zeta_1, \ldots, \zeta_n]\!]$ such that $\lim_{|\nu| \to \infty} |c_\nu| = 0$. It is denoted by

$$T_n = K\langle \zeta_1, \ldots, \zeta_n \rangle = \left\{ \sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu \in K[\![\zeta_1, \ldots, \zeta_n]\!] \,\middle|\, \lim_{|\nu| \to \infty} |c_\nu| = 0 \right\}$$

Hence, the Tate algebra is the subring of $K[\![\zeta_1, \ldots, \zeta_n]\!]$ consisting of formal power series with coefficients in K that converge in $\mathbb{B}^n(K^{\mathrm{alg}})$.

This algebra can be endowed with a norm, called the *Gauss Norm*, defined by

$$\left\| \sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu \right\| = \max_\nu |c_\nu|$$

The Gauss norm has the following properties:

- $\|f\| = 0 \iff f = 0$.

- $\|cf\| = |c| \, \|f\|$ for all $c \in K$ and $f \in T_n$.
- $\|fg\| = \|f\| \, \|g\|$ for all $f, g \in T_n$.
- $\|f + g\| \leqslant \max\{\|f\|, \|g\|\}$ for all $f, g \in T_n$.

It follows from the third property that T_n is an integral domain.

The Gauss Norm gives to T_n the structure of K-Banach algebra:

Proposition 2.1.2. *T_n is complete with respect to the Gauss norm.*

Sketch of Proof. Suppose we have a Cauchy sequence $(f_i)_{i\in\mathbb{N}}$ in T_n. We take

$$f_i = \sum_{\nu} c_{\nu}^{(i)} \zeta^{\nu}$$

Then, for each fixed $\nu \in \mathbb{N}^n$, the sequence $(c_{\nu}^{(i)})_{i\in\mathbb{N}}$ is a Cauchy sequence. We set c_ν to be its limit; it can be proved that $f = \sum_{\nu} c_\nu \zeta^\nu$ is the limit of $(f_i)_{i\in\mathbb{N}}$ and it lives in T_n. For more details we can refer to [Bos, Proposition 2.2.3] □

Proposition 2.1.3 (Maximum Modulus Principle)**.** *Let $f \in T_n$. Then $|f(x)| \leqslant \|f\|$ for all points $x \in \mathbb{B}^n(K^{alg})$, and there exists a point $x \in \mathbb{B}^n(K^{alg})$ such that the equality $|f(x)| = \|f\|$ holds.*

The Tate algebra has lot of properties in common with the usual polynomial ring in n variables over K. The key result for proving it is the Weierstrass Preparation and Division Theorem. Before stating it, we need to introduce some notation. Let $f \in T_n$ be of the form $\sum_{\nu=0}^{\infty} g_\nu \zeta_n^\nu$ with $g_\nu \in T_{n-1}$. f is called ζ_n-distinguished of order $s \in \mathbb{N}$ if g_s is a unit in T_{n-1} and $\|g_s\| = \|g\|$ and $\|g_s\| > \|g_\nu\|$ for $\nu > s$. If, in addition, f has norm 1, then we say that f is regular of order s. Notice that in case f is regular, the two conditions above are equivalent to require that the reduction of f in $\overline{T}_n = k[\zeta_1, \ldots, \zeta_n]$ ([BGR, Proposition 5.1.2.2]) is of the form $\overline{f} = \overline{g}_s \zeta_n^s + \overline{g}_{s-1}\zeta_n^{s-1} + \ldots + \overline{g}_0$ with $\overline{g}_s \in \overline{k}^{\times}$.

Theorem 2.1.4 (Weierstrass Division)**.** *Let $g \in T_n$ be ζ_n-distinguished of degree s. Then, for each $f \in T_n$ there exist uniquely determined elements $q \in T_n$ and $r \in T_{n-1}[\zeta_n]$ with $\deg r < s$ such that f can be written as $f = qg + r$. Further, $\|f\| = \max\{\|q\| \, \|g\|, \|r\|\}$*

Proof. Without loss of generality we may assume $\|g\| = 1$.

(Uniqueness) Suppose we have two different decompositions of the same f:

$$gq + r = f = gq' + r' \Longrightarrow (q - q')g = r' - r \xRightarrow{\|g\|=1} \|q - q'\| = \|r' - r\|$$

Take some $c \in K$ with $|c| = \|q - q'\|^{-1}$. Then $c(q - q')g = c(r' - r)$ and so the same is true with overbars everywhere; but this contradicts the uniqueness in the ordinary division algorithm for polynomials.

(Estimate) If $f = qg + r$, then clearly $\|f\| \leqslant \max\{\|qg\|, \|r\|\}$. Suppose that $\|f\| \lneq \max\{\|qg\|, \|r\|\}$, then we may assume $\max\{\|qg\|, \|r\|\} = 1$. Thus $\|f\| \lneq 1$ which means that $0 = \overline{qg} + \overline{r}$ and $\overline{q} \neq 0$, $\overline{r} \neq 0$ but this contradicts the Euclid's division in $\overline{K}[\zeta_1, \ldots, \zeta_{n-1}][\zeta_n]$.

(Existence) Define

$$B = \{qg + r \mid r \in T_{n-1}[\zeta_n], \deg r < s, q \in T_n\}$$

It can be deduced that B is a closed subgroup of T_n. Let's write g in the form $g = \sum_{\nu=0}^{\infty} g_\nu(\zeta_1, \ldots, \zeta_{n-1})\zeta_n^\nu$; we define $\epsilon = \max_{\nu > s}\{|g_\nu|\}$, where $\epsilon < 1$. Further, we set $K_\epsilon = \{x \in K \mid |x| \leqslant \epsilon\}$ and $k_\epsilon = \mathcal{O}_K/K_\epsilon$. Then, there is a natural ring epimorphism $\sigma_\epsilon : T_n \to k_\epsilon[\zeta_1, \ldots, \zeta_n]$ with $\ker \sigma_\epsilon = \{f \in T_n \mid \|f\| \leqslant \epsilon\}$, and $\sigma_\epsilon(g)$ is a unitary polynomial in ζ_n of degree s. Therefore, Euclid's division with respect to $\sigma_\epsilon(g)$ is possible in the ring $k_\epsilon[\zeta_1, \ldots, \zeta_{n-1}][\zeta_n]$ and so, for all $f \in T_n$, we can find $q \in T_n$ and $r \in T_{n-1}[\zeta_n]$ with $\deg r < s$ such that $\sigma_\epsilon(f) = \sigma_\epsilon(g)\sigma_\epsilon(q) + \sigma_\epsilon(r)$. Hence, for all $f \in T_n$, there is an element $b \in B$ such that $|f - b| < \epsilon|f|$. Therefore, B is dense in T_n and, since B is closed in T_n, we get $B = T_n$. Hence, every $f \in T_n$ admits the desired decomposition.

□

Theorem 2.1.5 (Weierstrass Preparation)**.** *Let $g \in T_n$ be ζ_n-distinguished of degree s. Then there are a unique monic polynomial $w \in T_{n-1}[\zeta_n]$ of degree s and a unique unit $e \in T_n$ such that $g = e \cdot w$. Further, $\|w\| = 1$ so that w is distinguished of degree s.*

Proof. **(Existence)** We start by applying the Weierstrass Division Theorem 2.1.4 to the monomial ζ_n^s; we get

$$\zeta_n^s = qg + r$$

with $q \in T_n$ and $r \in T_{n-1}[\zeta_n]$ of degree < s. Now $\omega = qg = \zeta_n^s - r$ is ζ_n-distinguished of degree s. Assuming $\|g\| = \|q\| = 1$, we can look at the reduction $\overline{qg} = \overline{\omega}$. Since both $\overline{\omega}$ and $\overline{g}$ are polynomials of degree s in ζ_n, it follows that $\overline{q}$ is a unit in $k^\times$ (ω is monic).

(Uniqueness) If $g = e\omega$ and $r = \zeta_n^s - \omega$ as before, then $\zeta_n^s = e^{-1}g + r$ which, by the uniqueness of Weierstrass Division, shows the uniqueness of e^{-1} and r and, therefore, of e and ω.

□

Theorem 2.1.6 (Weierstrass Distinction). *If $f_1, \ldots, f_m \in T_n$ all have norm 1, then there exists an automorphism σ of T_n (preserving the Gauss norms) such that $f_1^\sigma, \ldots, f_m^\sigma$ are ζ_n-distinguished.*

The proof of this can be found in [Bos, Lemma 2.2.7]. The Weierstrass Preparation and Division Theorems immediately yield a lot of properties of Tate Algebras.

Proposition 2.1.7. *The Tate Algebra T_n is Nöetherian.*

Proof. We will work by induction on n. We consider a non trivial ideal $\mathfrak{I}$ of T_n. Choose a non zero element $f \in \mathfrak{I}$. We use Weierstrass Distinction Theorem 2.1.6 and we select an automorphism σ of T_n such that f^σ is ζ_n-distinguished of degree s. Now we apply Weierstrass Division Theorem 2.1.4 and we obtain that $\mathfrak{I}^\sigma$ is generated by f^σ and $I^\sigma \cap T_{n-1}[\zeta_n]$. By the induction hypothesis, T_{n-1} is Nöetherian, and so is $T_{n-1}[\zeta_n]$ by the usual Hilbert basis theorem. Thus, $\mathfrak{I}^\sigma$ is finitely generated, and then so is $\mathfrak{I}$. □

Proposition 2.1.8. *The Tate Algebra T_n is a U.F.D.*

Proof. Also in this situation we proceed by induction; we may assume that T_{n-1} is a unique factorization domain. It follows that $T_{n-1}[\zeta_n]$ is a U.F.D. by a result of Gauss. We consider a non-zero element $f \in T_n$ that is not a unit. Applying Weierstrass Distinction Theorem 2.1.6 (where necessary), we may assume that f is ζ_n-distinguished. Modulo the use of Weierstrass Division Theorem 2.1.4, we can take f to be in $T_{n-1}[\zeta_n]$; thus f has a factorization. Now consider a factorization $f = \omega_1 \cdot \ldots \cdot \omega_t$ into prime elements $\omega_i \in T_{n-1}[\zeta_n]$. Since f is a monic polynomial in ζ_n, we can assume the same for $\omega_1, \ldots, \omega_t$. Then, as $\|\omega_i\| \geqslant 1$, we must have $\|\omega_i\| = 1$ for all i, since $\|f\| = 1$. It remains to show that the ω_i are prime in T_n (they are prime in $T_{n-1}[\zeta_n]$). Now it suffices to observe that there is an isomorphism

$$T_{n-1}[\zeta_n]/(\omega) \simeq T_n/(\omega)$$

and both sides are free T_{n-1} modules. It follows that T_n is a U.F.D. □

Corollary 2.1.9. *The Tate Algebra T_n is normal.*

For details in the proof one can refer to [Bos, Proposition 2.2.15].

Proposition 2.1.10. *The Krull dimension of the Tate Algebra T_n is n.*

Proof. Clearly the Krull dimension is at least n since we have the sequence of ideals

$$(\zeta_1) \subset (\zeta_1, \zeta_2) \subset \ldots \subset (\zeta_1, \zeta_2, \ldots, \zeta_n)$$

To prove the inverse inequality we can observe that, by Weierstrass Distinction and Preparation Theorems (2.1.6 and 2.1.5), for any irreducible $f \in T_n$, $T_n/(f)$ is finite over T_{n-1} and so it has Krull dimension $n-1$. But then, since finite ring extensions of Nöetherian rings do not increase Krull dimensions, T_n has dimension at most n. $\square$

Proposition 2.1.11. *Let $\| \ \|$ be a norm on the Tate algebra T_n making T_n into a Banach algebra. Then every ideal $\mathfrak{I}$ of T_n is closed with respect to $\| \ \|$.*

This is a particular application of a more general Lemma: [FvdP, Lemma 1.2.3].

Proof. Choose generators $g_1, \ldots, g_r$ of $\mathfrak{I}$ such that $\|g_i\| = 1$ for all i and every $f \in \mathfrak{I}$, $f = \sum_{i=1}^r f_i g_i$ for some $f_i \in T_n$ with $\|f_i\| \leqslant \|f\|$ (We can always do this choice - [Bos, Corollary 2.2.7]). If$f = \sum_{\nu=0}^\infty f_\nu$, $f_\nu \in \mathfrak{I}$ is convergent in T_n, there are equations $f_\nu = \sum_{i=1}^r f_{\nu,i} g_i$, $f_{\nu,i} \in T_n$. But then $f = \sum_{i=1}^r \left(\sum_{\nu=0}^\infty f_{\nu_i}\right) g_i$ belongs to $\mathfrak{I}$ and we are done. $\square$

2.2 Affinoid Algebras

In this section we introduce a generalization of Tate algebras: affinoid algebras. Affinoids algebras play a parallel role to the one of finitely generated algebras in algebraic geometry. Hence, as we do over archimedean fields, we start by proving a non-archimedean version of the Nöether Normalization Theorem.

Definition. A K-algebra A is called an affinoid K-algebra if there is an epimorphism of K-algebras $T_n \to A$ for some $n \in \mathbb{N}$.

Theorem 2.2.1 (Nöether Normalization)**.** *Let $\mathfrak{I}$ be an ideal of T_n, and let $A = T_n/\mathfrak{I}$ be the corresponding affinoid algebra. Then there exists a finite injective map $T_d \hookrightarrow A$ for some d; moreover, A is Nöetherian and its Krull dimension is d.*

Proof. We will proceed by induction on n. After applying Weierstrass Preparation and Division Theorems, we may assume that $\mathfrak{I}$ contains a monic polynomial $f \in T_{n-1}[\zeta_n]$. Then $T_n/(f)$ is free over T_{n-1} with basis $\zeta_n, \ldots, \zeta_n^{d-1}$; now we can set $\mathfrak{J} = \mathfrak{I} \cap T_{n-1}$. Now we have an injective and finite map $T_{n-1}/\mathfrak{J} \to T_n/\mathfrak{I}$.

By induction, we have a finite injective K-algebra homomorphism $T_d \to T_{n-1}/\mathfrak{J}$ which gives a finite and injective map

$$T_d \longrightarrow T_{n-1}/\mathfrak{J} \longrightarrow T_n/\mathfrak{J}$$

To prove the statement about the Krull dimension, we only need to recall that for a finite injective homomorphism $A \to B$ of Nöetherian rings, the rings A and B have the same Krull dimension. The statement now follows from Proposition 2.1.10.

Finally, the proof that A is Nöetherian is the same given for the Tate Algebra T_n in Proposition 2.1.7. $\square$

Corollary 2.2.2. *For any maximal ideal $\mathfrak{m}$ of A, the field $A/\mathfrak{m}$ is a finite extension of K.*

Proof. It follows directly from Nöether Normalization Theorem: If $\mathfrak{m}$ is a maximal ideal, then $A/\mathfrak{m}$ is a field. We know the existence of a finite injective map

$$T_d \hookrightarrow A/\mathfrak{m}$$

but, since the Krull dimension of $A/\mathfrak{m}$ is zero, then we must have $d = 0$. Hence, we have a finite injective homomorphism $K \to A/\mathfrak{m}$. $\square$

Proposition 2.2.3. *An affinoid K-algebra A is Jacobson (i.e., every prime ideal is the intersection of maximal ideals).*

Let A be a K-affinoid algebra. We want to introduce an intrinsic semi-norm on A. Let $\mathfrak{J}$ be a maximal ideal in A. Then $A/\mathfrak{J}$ is a finite extension of K by what we have just proved. This means, thanks to the discussion in Section 1.3, that it carries a unique extension of the valuation on K: we will denote it by $|\cdot|$. Further, we denote by $f(\mathfrak{J})$ the image of $f \in A$ in $A/\mathfrak{J}$.

Definition. The spectral semi-norm on A is defined by:

$$\|f\|_{\mathrm{sp}} = \sup_{\mathfrak{m} \in \mathrm{Max}(A)} |f(\mathfrak{m})|$$

This is, in fact, a semi-norm (we will see that in some cases it will be a norm).

We present now a list of properties of the spectral semi-norm. For more details one can see [FvdP, §3.4], [BGR, §6.2] or [Bos, §3.1].

Lemma 2.2.4. *The spectral norm satisfies $\|f^n\|_{sp} = \|f\|_{sp}^n$ for any element f of an affinoid K-algebra.*

Lemma 2.2.5. *Let $\phi : A \to B$ be a morphism of affinoid K-algebras. Then $\|\phi(f)\|_{sp} \leqslant \|f\|_{sp}$ for all $f \in A$.*

Proposition 2.2.6. *On a Tate algebra T_n, the spectral (or supremum) norm coincides with the Gauss norm.*

Proof. Using the Maximum Modulus Principle (Proposition 2.1.3) we see that

$$\|f\|_{\mathrm{sp}} = \sup_{x \in \mathbb{B}^n(K^{\mathrm{alg}})} \{|f(x)|\} \qquad \forall f \in T_n$$

It can be proved [Bos, Corollary 2.2.13] that we can construct a surjective map

$$\begin{aligned} \mathbb{B}^n(K^{\mathrm{alg}}) &\longrightarrow \mathrm{Max}(T_n) \\ x &\longrightarrow \mathfrak{m}_x = \{f \in T_n \mid f(x) = 0\} \end{aligned}$$

Thus, we get an embedding $T_n/x \hookrightarrow K^{\mathrm{alg}}$ and we see that

$$f(\mathfrak{m}_x) = f(x) \implies |f(\mathfrak{m}_x)| = |f(x)|$$

Now the result follows from the surjectivity of the map defined above. $\square$

Proposition 2.2.7. *Let $T_d \hookrightarrow A$ be a finite monomorphism into some K-algebra A. Let $f \in A$ and assume that A, as a T_d-module, has no zero divisors.*

(i) *There is a unique monic polynomial $P_f = \zeta_r + a_{r-1}\zeta^{r-1} + \ldots + a_1\zeta + a_0 \in T_d[\zeta]$ of minimal degree such that $P_f(f) = 0$ (Minimal Polynomial).*

(iii) *The supremum norm of f is given by*

$$\|f\|_{sp} = \max_{i=1,\ldots,r} \|a_i\|^{1/i}$$

Sketch of Proof. Take $f \in A$. f satisfies a minimal polynomial P with coefficients in the field of fractions of T_n. Since T_d is a U.F.D. (Proposition 2.1.8), then it is integrally closed and therefore P has coefficients in T_d. Now, for any $\mathfrak{m} \in \mathrm{Max}(T_n)$ and any root λ of $\zeta^r + a_{r-1}(\mathfrak{m})\zeta^{n-1} + \ldots + a_0(\mathfrak{m})$ there is $x \in \mathrm{Max}(A)$ such that $x \cap T_n = \mathfrak{m}$ and $f(x) = \lambda$. Hence

$$\|f\|_{\mathrm{sp}} = \max\{\max_i |a_i(\mathfrak{m})| \,\Big|\, \mathfrak{m} \in \mathrm{Max}(T_n)\} = \max_i \|a_i\|^{1/i} \qquad \square$$

Theorem 2.2.8 (Maximum Principle)**.** *For any affinoid K-algebra A and any $f \in A$, there exists a point $\mathfrak{m} \in Max(A)$ such that $\|f\|_{sp} = |f(\mathfrak{m})|$.*

A proof of this can be found in [Bos, Theorem 3.1.15] or [FvdP, Proposition 3.4.3].

Corollary 2.2.9. *Let A be an affinoid algebra. The intersection of all maximal ideals of A is the ideal of nilpotent elements of A. In particular, if A is a reduced affinoid algebra, the spectral semi-norm on A is a norm.*

Theorem 2.2.10. *Let A be a reduced affinoid algebra. The spectral norm is equivalent to any other norm which makes A into a Banach algebra.*

For a proof of this see [FvdP, Theorem 3.4.9].

2.3 Affinoid Spaces

Let us consider now an affinoid K-algebra A. We have seen that if we have an element $f \in A$ and a maximal ideal x of A, then we have an embedding $A/x \hookrightarrow K^{\mathrm{alg}}$ yielding a good definition of valuation of $f(x)$ (the image of f in A/x). Thus, we can think of A as the set of functions on its maximal spectrum Max(A).

We will denote by $\mathrm{Sp}(A)$ the maximal spectrum $\mathrm{Max}(A) = \{\mathfrak{m} \subseteq A \mid \mathfrak{m} \text{ maximal ideal}\}$ together with its K-algebra of functions A. It is clear that we are somehow mimicking the constructions of algebraic geometry where we have affine varieties with their rings of regular functions.

Considering this parallelism with algebraic geometry, one might ask why we are restricting ourselves to consider only maximal ideals instead of extending the study to the whole spectrum $\mathrm{Spec}(A)$. There are several reasons to do that. Let us give a look at some of them:

- First of all, as noticed in [Bos], for a prime ideal $\mathfrak{q}$ of an affinoid K-algebra A, the the field of fractions of $A/\mathfrak{q}$, will, in general, be of infinite degree over K. Hence, $K_{\mathfrak{q}}$ cannot be viewed as an affinoid K-algebra (since, otherwise, $K_{\mathfrak{q}}$ would be finite over K).

- Secondly, we have seen that there is a natural norm on affinoid K-algebras. This is defined valuating $f \in A$ over maximal ideals and considering the absolute value of these objects. This valuation is well defined once that we know how to embed $A/ideal \hookrightarrow K^{\mathrm{alg}}$ and this is possible for maximal ideals and not, in general, for prime ideals.

- Further, since an affinoid K-algebra is Jacobson, this implies that, if we have an element $f \in A$ that vanishes on all the maximal ideals, this must be nilpotent. We can observe that we do not have any need of considering prime ideals. (To be honest, this is also the case of finitely generated algebras over fields or algebraic varieties and in fact it is possible to study algebraic varieties considering their maximal spectrum).

- Finally, it is possible to prove that any homomorphism of K-affinoid algebras induces a map between their maximal spectra (in the reverse order - in the sense that this functor is contravariant) by sending a maximal ideal to its contraction. This is a similar situation to algebraic geometry where $\phi : A \to B$ induces $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ sending $B \supseteq \mathfrak{q} \to \phi^{-1}(\mathfrak{q})$.

Definition. Let A be an affinoid K-algebra. We associate to A the set

$$X = \mathrm{Sp}\,(A) = \mathrm{Max}(A)$$

of its maximal ideals. We call X an affinoid space.

In principle, we could define a Zariski topology on X and study the properties of the resulting topological space.

Definition. We define the Zariski topology on $\mathrm{Sp}\,(A)$ such that the closed subsets of $\mathrm{Sp}\,(A)$ are of the form

$$V(\mathfrak{a}) = \{x \in \mathrm{Sp}\,(A) \mid f(x) = 0 \ \forall f \in \mathfrak{a}\} \qquad \text{for any ideal } \mathfrak{a} \text{ of } A$$

Alternatively, we can see $V(\mathfrak{a})$ as:

$$V(\mathfrak{a}) = \{x \in \mathrm{Sp}\,(A) \mid \mathfrak{a} \subset \mathfrak{m}_x\}$$

where $\mathfrak{m}_x$ is the maximal ideal of A corresponding to $x \in \mathrm{Sp}\,(A)$.

Example. Let K be an algebraically closed field and consider an affinoid K-algebra A. If $\phi : A \to K$ is a homomorphism of K-algebras, then $\ker(\phi)$ is a maximal ideal and $\phi(f) = f(x)$. Thus, $\mathrm{Sp}\,(A)$ can be viewed as the set of K algebra homomorphisms $A \to K$.

In particular, for the free Tate Algebra T_n, we can identify

$$\mathrm{Sp}\,(T_n) = \mathbb{B}^n(K) = \{(x_1, \ldots, x_n) \in K^n \mid |x_i| \leqslant 1\}$$

Thus, T_n can be regarded as the space of "functions" over the set of its maximal ideals.

Example. The affinoid space associated to an affinoid algebra $A = T_n/\mathfrak{J}$ can be seen as

$$\mathrm{Sp}\,(A) = V(\mathfrak{J}) = \{x \in \mathbb{B}^n(K) \mid f(x) = 0 \text{ for all } f \in \mathfrak{J}\}$$

One can think of A as the set of "functions" on $\mathrm{Sp}\,(A)$.

It is easy to prove that the topology just defined satisfies similar properties to the Zariski topology defined for affine schemes:

Lemma 2.3.1. *Let A be an affinoid K-algebra and $\mathfrak{a}, \mathfrak{b}, (\mathfrak{a}_i)_{i \in I}$ be ideals of A. Then,*

(i) *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.*

(ii) $\bigcap_{i \in I} V(\mathfrak{a}_i) = V\left(\sum_{i \in I} \mathfrak{a}_i\right)$.

(iii) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(iv) *The sets*

$$D(f) = \{x \in Sp\,(A) \mid f(x) \neq 0\}$$

form a basis for the Zariski topology on $Sp\,(A)$.

The proof is straightforward and very similar to the one given for classical affine spaces. As usual, for $U \subset \mathrm{Sp}\,(A)$, one can define the sets

$$I(U) = \{f \in A \mid f(u) = 0 \text{ for all } y \in Y\}$$

This yields the Hilbert Nullstellensatz Theorem:

Theorem 2.3.2 (Hilbert Nullstellensatz)**.** *Let A be an affinoid K-algebra and $\mathfrak{a} \subseteq A$ an ideal. Then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$*

Corollary 2.3.3. *The maps $V(\cdot)$ and $I(\cdot)$ induce a bijection between the set of reduced ideals in A and the set of Zariski-closed subsets of $Sp\,(A)$.*

A complete discussion about Zariski topology on affinoid spaces can be found in [Tia, §1.5].

Remark. As pointed out in the introduction to this section, a morphism

$$\phi : \mathrm{Sp}\,(A) \to \mathrm{Sp}\,(B)$$

of affinoid K-spaces can be induced by a morphism $\phi^* : B \to A$ of affinoid K-algebras by:

$$\phi(\mathfrak{m}) = \left(\phi^*\right)^{-1}(\mathfrak{m})$$

Note that $\phi(\mathfrak{m})$ is maximal since we have a chain of injections

$$K \hookrightarrow B/\left(\phi^*\right)^{-1}(\mathfrak{m}) \hookrightarrow A/\mathfrak{m}$$

and $A/\mathfrak{m}$ is a field which is finite over K.

We want to conclude this section with a brief remark about fiber products of affinoid K-spaces ([BGR, Proposition 7.1.4/4]).

Proposition 2.3.4. *In the category of K-affinoid spaces the fiber product of two spaces $Sp\,(B_1)$ and $Sp\,(B_2)$ over a space $Sp\,(A)$ can be constructed in this way:*

$$\begin{tikzcd} Sp\left(B_1 \hat{\otimes} B_2\right) = Sp\,(B_1) \times_{Sp(A)} Sp\,(B_2) \arrow[r] \arrow[d] & Sp\,(B_1) \arrow[d] \\ Sp\,(B_2) \arrow[r] & Sp\,(A) \end{tikzcd}$$

2.4 Affinoid Subdomains

We have observed that A can be regarded as a space of functions over $\mathrm{Sp}\,(A)$. In general, it would be nice to extend this notion of "analytic functions" to (open) subsets of $\mathrm{Sp}\,(A)$. For this purpose, the Zariski topology turns out to be too rough since it does not take into account the non-archimedean nature of A. Therefore, we have to provide $\mathrm{Sp}\,(A)$ with an extra topological structure. This was first done by Tate in [Tat] and then simplified by Gerritzen and Grauert with the introduction of Rational subsets.

Let A be an affinoid K-algebra and $X = \mathrm{Sp}\,(A)$. For $f \in A$ and $\epsilon \in \mathbb{R}_{>0}$ we define the set

$$X(f, \epsilon) = \{x \in X \mid |f(x)| \leqslant \epsilon\}$$

Definition. The canonical (Tate) topology on $\mathrm{Sp}\,(A)$ is the topology generated by sets of the form $X(f, \epsilon)$. In particular, a subset $\mathcal{U} \subseteq X$ is open if and only if it is the union of sets of the form $X(f_1, \epsilon_1) \cap \ldots \cap X(f_r, \epsilon_r)$.

For simplicity we will write

$$X(f) = X(f, 1) \qquad \text{and} \qquad X(f_1, \ldots, f_r) = X(f_r) \cap \ldots \cap X(f_r)$$

Lemma 2.4.1. *For any affinoid K-space $X = Sp(A)$, the canonical topology is generated by all subsets $X(f)$ with f varying in A, i.e., a subset $\mathcal{U} \subseteq Sp(A)$ is open if and only if it is a union of sets of type $X(f_1, \ldots, f_r) = X(f)$.*

Proof. Since $|f(x)| \in |\left(K^{\mathrm{alg}}\right)^{\times}|$, we can write

$$X(f, \epsilon) = \bigcup_{\substack{\epsilon' \leqslant \epsilon \\ \epsilon' \in |\left(K^{\mathrm{alg}}\right)^{\times}|}} X(f, \epsilon')$$

Because of Theorem 1.3.4, for all $\epsilon' \in |\left(K^{\mathrm{alg}}\right)^{\times}|$ we can find $c \in |K^{\times}|$ such that $|c| = \epsilon'^{s}$. Thus,

$$X(f, \epsilon') = X(f^s, \epsilon'^s) = X(c^{-1} f^s)$$

□

We state now a technical Lemma which will help us in determining the openness of subsets. The proof can be found in [Bos, Lemma 3.3.3].

Lemma 2.4.2. *For an affinoid K-space $X = Sp(A)$, consider an element $f \in A$ and a point $x \in Sp(A)$ such that $\epsilon = |f(x)|$. Then, there is an element $g \in A$ satisfying $g(x) = 0$ such that $|f(y)| = \epsilon$ for all $y \in X(g)$. In other words, $X(g)$ is an open neighborhood of x contained in $\{y \in X \mid |f(y)| = \epsilon\}$.*

Corollary 2.4.3. *The following sets are open with respect to the canonical topology:*

$$\{x \in Sp(A) \mid f(x) \neq 0\}$$
$$\{x \in Sp(A) \mid f(x) \leqslant \epsilon\}$$
$$\{x \in Sp(A) \mid f(x) = \epsilon\}$$
$$\{x \in Sp(A) \mid f(x) \geqslant \epsilon\}$$

Lemma 2.4.4. *Let $\phi^* : A \to B$ be a morphism of affinoid K-algebras. We consider the associated morphism $\phi : Y = Sp(B) \to Sp(A) = X$. Then, for every choice $f_1, \ldots, f_r \in A$, we have*

$$\phi^{-1}(X(f_1, \ldots, f_r)) = Y(\phi^*(f_1), \ldots, \phi^*(f_r))$$

In particular, ϕ is continuous with respect to the canonical topology.

Proof. For each $y \in Y$ the map $\phi^* : B \to A$ gives rise to a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[d] & A \arrow[d] \\ B/\mathfrak{m}_{\phi(y)} \arrow[r, hook] & A/\mathfrak{m}_y \end{tikzcd}$$

which implies the result. □

Definition. Let X be an affinoid K-space. A subset in X of the form

$$X(f_1, \ldots, f_r) = \{x \in X \mid |f_i(x)| \leqslant 1\}$$

is called a Weierstrass domain in X.

Definition. Let X be an affinoid K-space. A subset in X of the form

$$X(f_1, \ldots, f_r, g_1^{-1}, \ldots, g_s^{-1}) = \{x \in X \mid |f_i(x)| \leqslant 1, |g_j(x)| \geqslant 1\}$$

is called a Laurent domain in X.

Definition. Let X be an affinoid K-space. A subset in X of the form

$$X\left(\frac{f_1}{f_0}, \ldots, \frac{f_r}{f_0}\right) = \{x \in X \mid |f_i(x)| \leqslant |f_0(x)|\}$$

for functions $f_0, \ldots, f_r \in A$ without common zeros, is called a Rational domain in X.

Lemma 2.4.5. *Weierstrass, Laurent and Rational domains are open in $X = Sp(A)$ with respect to the canonical topology. Further, the Weierstrass domains form a basis of this topology.*

This is a straightforward application of Lemma 2.4.2.

Definition. Let $X = \mathrm{Sp}(A)$ be an affinoid K-space. A subset $\mathcal{U} \subseteq X$ is called an affinoid subdomain of X if there exists a morphism of affinoid K-spaces

$$\iota : X' = \mathrm{Sp}(A') \to \mathrm{Sp}(A) = X$$

such that $\iota(X') \subseteq \mathcal{U}$ and the following universal property holds: for any morphism of affinoid K-spaces $\gamma : Y \to X$ with $\gamma(Y) \subseteq \mathcal{U}$, there exists a unique morphism $\gamma' : Y \to X'$ such that $\gamma = \iota \circ \gamma'$.

$$\begin{tikzcd} & Y = \mathrm{Sp}\,(B) \arrow[dl, dashed, "\exists! \gamma'"'] \arrow[dr, "\gamma"] & \\ X' = \mathrm{Sp}\,(A') \arrow[rr, "\iota"'] & & \mathcal{U} \subseteq X = \mathrm{Sp}\,(A) \end{tikzcd}$$

Example (Weierstrass domains are affinoid subdomains). Let us consider a Weierstrass domain

$$X(f_1, \dots, f_r) = \{x \in X \mid |f_i(x)| \leqslant 1,\ i = 1, \dots, r\}$$

There is a natural morphism of affinoid K-algebras:

$$\iota^* : A \longrightarrow A\langle f_1, \dots, f_r\rangle = \frac{A\langle \zeta_1, \dots, \zeta_r\rangle}{(\zeta_i - f_i)_{i=1,\dots,r}}$$

and associated to it a morphism of affinoid K-spaces:

$$\iota : \mathrm{Sp}\,(A\langle f_1, \dots, f_r\rangle) \longrightarrow \mathrm{Sp}\,(A) = X$$

We want to show that it satisfies the universal property; let $\phi : \mathrm{Sp}\,(B) = Y \to X = \mathrm{Sp}\,(A)$ be a morphism of affinoid K-spaces such that $\phi(Y) \subseteq X(f_1, \dots, f_r)$. There is a morphism $\phi^* : A \to B$ corresponding to ϕ. Now

$$\phi(Y) \subseteq X(f_1, \dots, f_r) \iff \left\|\phi^*(f_i)\right\| \leqslant 1 \quad \forall i$$

Indeed, from the inclusion $A/\mathfrak{m}_{\phi(y)} \hookrightarrow B/\mathfrak{m}_y$ of finite extensions of K we obtain the equality $|\phi^*(f_i)(y)| = |f_i(\phi(y))|$ for each i.

Thus, there exists a morphism $\psi^* : A\langle \zeta_1, \dots, \zeta_n\rangle \to B$ such that $\psi^*|_A = \phi^*$ and $\psi^*(\zeta_i) = \phi^*(f_i)$, i.e, the morphism ψ^* factors through $A\langle \zeta_1, \dots, \zeta_r\rangle/(\zeta_i - f_i)_{i=1,\dots,r}$.

Example (Laurent domains are affinoid subdomains). Let us consider a Laurent domain

$$X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}) = \{x \in X \mid |f_i(x)| \leqslant 1,\ , |g_i(x)| \geqslant 1\ i = 1, \dots, r\}$$

There is a natural morphism of affinoid K-algebras:

$$\iota^* : A \longrightarrow A\langle f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}\rangle = \frac{A\langle \zeta_1, \dots, \zeta_r, \xi_1, \dots, \xi_s\rangle}{(\zeta_i - f_i, 1 - g_j\xi_j)_{\substack{i=1,\dots,r\\ j=1,\dots,s}}}$$

and associated to it a morphism of affinoid K-spaces:

$$\iota : \mathrm{Sp}\left(A\langle f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}\rangle\right) \longrightarrow \mathrm{Sp}\,(A) = X$$

Now, proving that this satisfies the universal property is similar to the previous example.

Example (Rational domains are affinoid subdomains). A rational domain is a set of the form

$$X\left(\frac{f_1}{f_0},\dots,\frac{f_r}{f_0}\right) = \{x \in X \;\mid\; |f_i(x)| \leqslant |f_0(x)|\}$$

First of all we observe that f_0 cannot have zeros in $X\left(\frac{f_1}{f_0},\dots,\frac{f_r}{f_0}\right)$: if x_0 was such a zero we would have $|f_i(x_0)| \leqslant |f_0(x_0)| = 0$ and this contradicts the hypothesis that $f_0,\dots,f_r$ have no common zeros. We observe that there is a morphism of K-affinoid algebras

$$\iota^* : A \longrightarrow A\langle\frac{f_1}{f_0},\dots,\frac{f_r}{f_0}\rangle = \frac{A\langle\zeta_1,\dots,\zeta_r\rangle}{(f_i - f_0\zeta_i)_{i=1,\dots,r}}$$

which gives a morphism of K-affinoid spaces $\iota : \mathrm{Sp}\left(A\langle\frac{f_1}{f_0},\dots,\frac{f_r}{f_0}\rangle\right) \to \mathrm{Sp}\,(A) = X$ and again this satisfies the universal property.

We present now a series of results about affinoid subdomains without proof. A complete description can be found in [BGR, §7.2] and [Bos, §3.3].

Lemma 2.4.6. *Let $\mathcal{U}$ be a subset of $Sp\,(A)$ and let $\iota : X' = Sp\,(A') \to Sp\,(A) = X$ be a K-affinoid map. Then*

(1) *ι is injective and it satisfies $\iota(Sp\,(A')) = \mathcal{U}$. Thus, it induces a bijection of sets $X' \xrightarrow{\sim} \mathcal{U}$.*

(2) *For $x \in Sp\,(A')$ and $n \in \mathbb{N}$, the map $\iota^* : A \to A'$ induces an isomorphism of affinoid K-algebras $A/\mathfrak{m}^n_{\iota(x)} \simeq A'/\mathfrak{m}^n_x$.*

(3) *For $x \in Sp\,(A')$, we have $\mathfrak{m}_x = \iota^*(\mathfrak{m}_{\iota(x)})A'$.*

Proposition 2.4.7. *[Transitivity of Affinoid Subdomains] For an affinoid K-space X, consider an affinoid subdomain $\mathcal{U} \subseteq X$, and an affinoid subdomain $\mathcal{V} \subseteq \mathcal{U}$. Then $\mathcal{V}$ is an affinoid subdomain in X as well.*

Proposition 2.4.8. *Let $\phi : Y \to X$ be a morphism of affinoid K-spaces and let $X' \hookrightarrow X$ be an affinoid subdomain. Then $Y' = \phi^{-1}(X')$ is an affinoid subdomain of Y, and there is a unique morphism of affinoid K-spaces $\phi' : Y' \to X'$ such that the diagram*

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, hook] & X' \arrow[d, hook] \\ Y \arrow[r] & X \end{tikzcd}$$

is commutative.

In fact, the diagram is Cartesian in the sense that it characterizes Y' as the fiber product $X' \times_X Y$. Further If X' is Weierstrass (respectively Laurent, Rational) domain in X, the corresponding fact is true for Y' as an affinoid subdomain of Y.

Proposition 2.4.9. *Let X be an affinoid K-space and let $\mathcal{U}, \mathcal{V} \subset X$ be affinoid subdomains. Then $\mathcal{U} \cap \mathcal{V}$ is an affinoid subdomain of X. If $\mathcal{U}$ and $\mathcal{V}$ are Weierstrass, respectively Laurent or rational domains, the same is true for $\mathcal{U} \cap \mathcal{V}$.*

Lemma 2.4.10. *Let $\mathcal{U} \hookrightarrow X$ be a morphism of affinoid K-spaces defining $\mathcal{U}$ as an affinoid subdomain of X. Then $\mathcal{U}$ is open in X, and the canonical topology of $\mathcal{U}$ equals the canonical topology of X restricted to $\mathcal{U}$.*

Theorem 2.4.11 (Gerritzen-Grauert)**.** *Let X be an affinoid K-space and $\mathcal{U} \hookrightarrow X$ an affinoid subdomain. Then $\mathcal{U}$ is a finite union of rational subdomains of X.*

Note that, in general, the converse is not true: a counterexample can be found in [FvdP, Remarks 4.1.5(7)].

Remark. In general, points $\{x\} \subseteq X$ are not affinoid subdomains. Indeed, one can consider the map $\{x\} \hookrightarrow X$ which gives rise to the K-algebras homomorphism

$$A \longrightarrow A/\mathfrak{m}_x \qquad \mathfrak{m}_x \text{ is the ideal corresponding to } x$$

The problem is that this homomorphism fails to satisfy the universal property. For instance we can construct $\gamma : A \to A/\mathfrak{m}_x^2$ and this does not factor through $A \to A/\mathfrak{m}_x$

2.5 Affinoid Functions

Now that we have defined a topology on an affinoid K-space $X = \mathrm{Sp}\,(A)$, we would like to construct a sheaf on it. The idea is again to mimic algebraic geometry and to define a sheaf of functions.

For any affinoid subdomain $\mathcal{U} \subseteq X$ we denote $\mathcal{O}_X(\mathcal{U})$ the corresponding K-affinoid algebra. Given $\mathcal{V}$ an affinoid subdomain of $\mathcal{U}$ we know from Proposition 2.4.7 that $\mathcal{V}$ is an affinoid subdomain of X and we have a canonical morphism of affinoid K-algebras given by the universal property:

$$\mathcal{O}_X(\mathcal{U}) \longrightarrow \mathcal{O}_X(\mathcal{V})$$

Remark. This map can be regarded as a sort of restriction map of affinoid functions on $\mathcal{U}$ to affinoid functions on $\mathcal{V}$.

What we obtain is a presheaf of affinoid K-algebras on the category of affinoid subdomains of X.

Definition. The presheaf $\mathcal{O}_X$ is called the presheaf of affinoid functions on X.

Definition. For every point $x \in X$, the ring

$$\mathcal{O}_{X,x} = \varinjlim_{\mathcal{U} \ni x} \mathcal{O}_X(\mathcal{U})$$

where the limit runs over all the affinoid subdomains containing x, is called the stalk of $\mathcal{O}_X$ at x. Every element $f_x \in \mathcal{O}_{X,x}$ is called germ of $f \in \mathcal{O}_X(\mathcal{U})$ at x.

Proposition 2.5.1. *Let $x \in X = Sp(A)$ be a point corresponding to the maximal ideal $\mathfrak{m}_x$ of $A = \mathcal{O}_X(X)$. Then $\mathcal{O}_{X,x}$ is a local ring with maximal ideal*

$$\mathfrak{m}_x \mathcal{O}_{X,x} = \{f_x \in \mathcal{O}_{X,x} \mid f_x(x) = 0\}$$

Proof. Let $\mathcal{U} \subseteq X$ be an affinoid subdomain. From Lemma 2.4.6 we get an isomorphism $\mathcal{O}_X(X)/\mathfrak{m}_x\mathcal{O}_X(X) \simeq \mathcal{O}_X(\mathcal{U})/\mathfrak{m}_x\mathcal{O}_X(\mathcal{U})$. Since the direct limit preserves exact sequences, we obtain another isomorphism

$$\frac{\mathcal{O}_X(X)}{\mathfrak{m}_x O_X(X)} \simeq \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x\mathcal{O}_{X,x}}$$

which shows that $\mathfrak{m}_x\mathcal{O}_{X,x}$ is a maximal ideal since $\mathcal{O}_X(X)/\mathfrak{m}_x\mathcal{O}_X(X)$ is a field.

Now to see the uniqueness of the maximal ideal, we consider $f_x \in \mathcal{O}_{X,x} \backslash \mathfrak{m}_x\mathcal{O}_{X,x}$. This germ is represented by $f \in \mathcal{O}_X(\mathcal{U})$ for some affinoid subdomain $x \in \mathcal{U} \subseteq X$. Hence, $f(x) \neq 0$ and, up to multiplication by a scalar, we may assume $|f(x)| \geqslant 1$. But then $\mathcal{U}(f^{-1})$ is a Laurent affinoid subdomain of X containing x and the restriction of f to $\mathcal{U}(f^{-1})$ is a unit in $\mathcal{O}_X(\mathcal{U}(f^{-1}))$. Thus, f_x is a unit in $\mathcal{O}_{X,x}$ and $\mathfrak{m}_x\mathcal{O}_{X,x}$ is the unique maximal ideal of $\mathcal{O}_{X,x}$. $\square$

Proposition 2.5.2. *For any point x of an affinoid variety X, the local ring $\mathcal{O}_{X,x}$ is Nöetherian*

We conclude this section stating a result about local properties of affinoid varieties [BGR, §7.3.2]. We say that an affinoid variety X is "reduced", "normal" or "smooth" at a point $x \in X$, if the local ring $\mathcal{O}_{X,x}$ is reduced, normal or regular, respectively.

Lemma 2.5.3. *An affinoid space $X = Sp(A)$ is reduced or normal if and only if A is reduced or normal, respectively.*

Let $Sp(A')$ be an affinoid subdomain of $X = Sp(A)$. Then if A is reduced or normal, A' is reduced or normal, respectively.

2.6 Tate's Acyclicity Theorem

We let X be an affinoid K-space and $\mathbf{Aff}(X)$ be the category of affinoid subdomains of X; here we take the morphisms to be the inclusions. We have seen that, in general, O_X (the functor associating to an affinoid subdomain its affinoid algebra) is a presheaf. It is natural to ask whether this is the maximum we can obtain or, under some conditions, we may expect to obtain a sheaf.

Recall. A presheaf $\mathcal{F}$ on an affinoid K-space X is a sheaf if the sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{F}(\mathcal{U}) & \longrightarrow & \prod_{i\in I}\mathcal{F}(\mathcal{U}_i) & \longrightarrow & \prod_{i,j}\mathcal{F}(\mathcal{U}_i\cap\mathcal{U}_j) \\ & f & \longrightarrow & (f|_{\mathcal{U}_i}) & & \\ & & & (f_i)_{i\in I} & \longrightarrow & (f_i|_{\mathcal{U}_i\cap\mathcal{U}_j} - f_j|_{\mathcal{U}_i\cap\mathcal{U}_j})_{i,j} \end{array}$$

is exact for every $\mathcal{U}\in\mathbf{Aff}(X)$ and every covering $\mathfrak{U}=\{\mathcal{U}_i\}$ of $\mathcal{U}$. A family of morphisms $(\mathcal{U}_i\to\mathcal{U})_{i\in I}$ in $\mathbf{Aff}(X)$ is a covering of $\mathcal{U}$ if $\bigcup_{i\in I}\mathcal{U}_i=\mathcal{U}$.

It is possible to prove that $\mathcal{O}_X$ satisfies the uniqueness condition, i.e., the first morphism is injective ([Bos, Corollaries 4.1.4 and 4.1.5]):

Lemma 2.6.1. *An affinoid function f on some affinoid K-space X is zero if and only if all its germs $f_x\in\mathcal{O}_{X,x}$ at points $x\in X$ are zero.*

Lemma 2.6.2. *Let X be an affinoid K-space and $X=\bigcap_{i\in I}X_i$ a covering by affinoid subdomains. Then the restriction maps $\mathcal{O}_X(X)\to\mathcal{O}_X(X_i)$ define an injection*

$$\mathcal{O}_X(X)\hookrightarrow\prod_{i\in I}\mathcal{O}_X(X_i)$$

The problem is that, in general, $\mathcal{O}_X$ is far from satisfying the gluing condition (exactness at $\prod_{i\in I}\mathcal{F}(\mathcal{U}_i)$).

Definition. For a presheaf $\mathcal{F}$ on X and a covering $\mathfrak{U}=\{\mathcal{U}_i\}_{i\in I}$ of X by affinoid subdomains $\mathcal{U}_i\subseteq X$, we say that $\mathcal{F}$ is a $\mathfrak{U}$-sheaf, if for all affinoid subdomains $\mathcal{U}\subseteq X$ the sequence above applied to the covering $\mathfrak{U}|_{\mathcal{U}}=\{\mathcal{U}\cap\mathcal{U}_i\}_{i\in I}$ is exact.

The best result we can obtain is the following:

Theorem 2.6.3 (Tate)**.** *Let X be an affinoid K-space. The presheaf $\mathcal{O}_X$ of affinoid functions is a $\mathfrak{U}$-sheaf on X for all finite coverings $\mathfrak{U}=\{\mathcal{U}_i\}_{i\in I}$ of X by affinoid subdomains $\mathcal{U}_i\subseteq X$.*

The proof of this theorem relies on restricting computations on simpler and simpler coverings. Here we mention just few intermediate lemmas (A complete discussion can be found in [Bos, §4.3]).

Definition. If we have two coverings $\mathfrak{U} = \{\mathcal{U}_i\}_{i\in I}$ and $\mathfrak{V} = \{\mathcal{V}_j\}_{j\in J}$ of X by affinoid subdomains, we say that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ if there exists a map $\tau : J \to I$ such that $\mathcal{V}_j \subseteq \mathcal{U}_{\tau(j)}$ for any $j \in J$.

Lemma 2.6.4. *Let $\mathcal{F}$ be a presheaf on an affinoid K-space X and $\mathfrak{U}, \mathfrak{V}$ two coverings of X where $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. If $\mathcal{F}$ is a $\mathfrak{V}$-sheaf then it is a $\mathfrak{U}$-sheaf.*

Lemma 2.6.5. *Every affinoid covering $\mathfrak{U}$ of X admits a rational covering as a refinement.*

The next steps consists in reducing the discussion to the case of Laurent domains

Lemma 2.6.6. *Let $\mathcal{F}$ be a presheaf on an affinoid K-space X. If $\mathcal{F}$ is a $\mathfrak{U}$-sheaf for all Laurent coverings $\mathfrak{U}$ of X, then it is a $\mathfrak{V}$-sheaf for all affinoid coverings $\mathfrak{V}$ of X.*

Finally, to prove Tate's Theorem it suffices to do computations for Laurent coverings. Instead of Theorem 2.6.3, we want to focus on a slightly stronger result which is known as Tate's acyclicity Theorem; this will give us the opportunity to introduce the idea of Čech cohomology. A detailed approach to Čech cohomology can be found in [Liu, §5.2] or [BGR, §8.1 and 8.2].

Let X be an affinoid K-space, $\mathcal{F}$ a presheaf (of abelian groups for instance) on X and $\mathfrak{U} = \{\mathcal{U}_i\}_{i\in I}$ a finite covering of X made of affinoid subdomains. We denote

$$\mathcal{U}_{i_0,\dots,i_q} = \mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_q} \qquad (i_0, \dots, i_q) \in I^{q+1}$$

and we set

$$\mathcal{C}^q(\mathfrak{U}, \mathcal{F}) = \prod_{(i_0,\dots,i_q)\in I^{q+1}} \mathcal{F}(\mathcal{U}_{i_0,\dots,i_q})$$

An element of $\mathcal{C}^q(\mathfrak{U}, \mathcal{F})$ is called a q-cochain (of $\mathfrak{U}$ in $\mathcal{F}$). We say that a q-cochain is alternating if, for every permutation σ of the indices, $f_{\sigma(i_0),\dots,\sigma(i_q)} = \mathrm{sgn}(\sigma) f_{i_0,\dots,i_q}$ and $f_{i_0,\dots,i_q} = 0$ as soon as two indices are equal. $\mathcal{C}^q_a(\mathfrak{U}, \mathcal{F})$ denotes the module of alternating q-cochains.

We naturally have two graded modules

$$\mathcal{C}(\mathfrak{U}, \mathcal{F}) = \bigoplus_{q\geqslant 0} \mathcal{C}^q(\mathfrak{U}, \mathcal{F}) \qquad \mathcal{C}_a(\mathfrak{U}, \mathcal{F}) = \bigoplus_{q\geqslant 0} \mathcal{C}^q_a(\mathfrak{U}, \mathcal{F})$$

Let us define a coboundary map $d : \mathcal{C}^q(\mathfrak{U}, \mathcal{F}) \longrightarrow \mathcal{C}^{q+1}(\mathfrak{U}, \mathcal{F})$. If $f \in \mathcal{C}^q(\mathfrak{U}, \mathcal{F})$, then

$$(df)_{i_0,\dots,i_{q+1}} = \sum_{i=0}^{q+1} (-1)^k \, f_{i_0,\dots,\widehat{i_k},\dots,i_{q+1}}\Big|_{\mathcal{U}_{i_0,\dots,i_{q+1}}}$$

It can be proved that $d^2 = 0$. Thus, we obtain a complex, called complex of Čech cochains on $\mathfrak{U}$ with values in $\mathcal{F}$.

$$\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}) : \quad 0 \longrightarrow \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^1} \mathcal{C}^2(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^2} \dots$$

We define the Čech cohomology groups as the cohomology groups of $(\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}), d^\bullet)$:

$$H^q(\mathfrak{U}, \mathcal{F}) = \frac{\ker(d^q)}{\operatorname{Im}(d^{q-1})} \qquad\qquad H^q_a(\mathfrak{U}, \mathcal{F}) = \frac{\ker(d^q_a)}{\operatorname{Im}(d^{q-1}_a)}$$

Lemma 2.6.7. *The inclusion $\mathcal{C}^\bullet_a(\mathfrak{U}, \mathcal{F}) \hookrightarrow \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ induces an isomorphism*

$$H^q_a(\mathfrak{U}, \mathcal{F}) \simeq H^q(\mathfrak{U}, \mathcal{F})$$

Corollary 2.6.8. *If the covering $\mathfrak{U}$ is made of n elements, we have*

$$H^q(\mathfrak{U}, \mathcal{F}) = 0 \qquad\qquad \forall q \geqslant n$$

Definition. A covering $\mathfrak{U}$ is called $\mathcal{F}$-acyclic if the sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\epsilon} \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^1} \mathcal{C}^2(\mathfrak{U}, \mathcal{F}) \xrightarrow{d^2} \dots$$

is exact. Here ϵ is the argumentation map sending $f \to (f|_{\mathcal{U}_i})_{i \in I}$. In other words $\mathfrak{U}$ is $\mathcal{F}$-acyclic if $\mathcal{F}$ is a $\mathfrak{U}$-sheaf and $H^q(\mathfrak{U}, \mathcal{F}) = 0$ for all $q > 0$.

Theorem 2.6.9 (Tate Acyclicity Theorem)**.** *Let X be an affinoid K-space and $\mathfrak{U}$ a finite covering of X by affinoid subdomains. Then $\mathfrak{U}$ is acyclic with respect to the presheaf $\mathcal{O}_X$ of affinoid functions on X.*

Even if nowadays we refer to Theorem 2.6.9 as Tate's Acyclicity Theorem, it is curious to notice that while Tate did introduce the concept of an affinoid subspace, he did not even formulate the question of whether an arbitrary finite covering by affinoid subspaces is acyclic ([Ked]). This was done by Gerritzen and Grauert.

Theorem 2.6.10. *Let $X = Sp(A)$ be an affinoid K-space, M an A-module, and $\mathfrak{U}$ a finite covering of X by affinoid subdomains. Then $\mathfrak{U}$ is acyclic with respect to the presheaf $M \otimes_A \mathcal{O}_X$.*

Chapter 3

Rigid Spaces

In this chapter we introduce the notion of rigid spaces: these objects can be obtained gluing some affinoid spaces together.

As we have seen, Tate's acyclicity Theorem is the best result we can obtain. Thus, $\mathcal{O}_X$ cannot be a sheaf unless we reduce the choices for affinoid coverings; for this purpose, we start by defining Grothendieck topologies. In the second section we will see how to glue affinoid K-spaces together using an approach similar to the one used in Algebraic geometry to glue schemes.

In section 3 we will explain how to associate to a K-scheme a K rigid space.

We will then introduce some constructions we can do on rigid spaces such as cohomology and reductions. Finally, we will give a brief look at the ideas of formal schemes and formal geometry.

The main references for the chapter are [FvdP, Chapter 4], [BGR, Chapter 9], [Bos, Chapter 5], [Con] and [Ked].

3.1 Grothendieck Topology

We have seen that $\mathcal{O}_X$, the presheaf of affinoid functions on an affinoid K-space X endowed with the canonical topology, will usually not be a sheaf. To solve this problem, we will try to introduce a different definition of sheaf or, more precisely, of open cover. We present here some generalities about Grothendieck topologies. A complete reference for this section is [Art].

Definition. A Grothendieck topology $\mathfrak{T}$ on X (or $\mathcal{G}$-topology) consists of

(a) A category $\mathrm{Cat}(\mathfrak{T})$; we can think of the objects in this category as the open subsets in X. We call each object an admissible open (or $\mathfrak{T}$-open) subsets of X.

(b) A set $\mathrm{Cov}(\mathfrak{T})$ of families $(\mathcal{U}_i \to \mathcal{U})_{i\in I}$ of morphism in $\mathrm{Cat}(\mathfrak{T})$, called admissible coverings (or $\mathfrak{T}$-coverings).

Further, $\mathrm{Cat}(\mathfrak{T})$ and $\mathrm{Cov}(\mathfrak{T})$ satisfy the following conditions

(i) If $\mathcal{U}$ and $\mathcal{V}$ are in $\mathrm{Cat}(\mathfrak{T})$, then $\mathcal{U} \cap \mathcal{V}$ is also an object in $\mathrm{Cat}(\mathfrak{T})$.

(ii) If $\Phi : \mathcal{V} \xrightarrow{\sim} \mathcal{U}$ is an isomorphism in $\mathrm{Cat}(\mathfrak{T})$, then $\Phi \in \mathrm{Cov}(\mathfrak{T})$.

(iii) If $(\mathcal{U}_i \to \mathcal{U})_{i\in I}$ and $(\mathcal{V}_{ij} \to \mathcal{U}_i)_{j\in J_i}$ are in $\mathrm{Cov}(\mathfrak{T})$, then the same is true for the composition $(\mathcal{V}_{ij} \to \mathcal{U})_{i\in I,\ j\in J_i}$.

(iv) If $(\mathcal{U}_i \to \mathcal{U})_{i\in I}$ is an admissible covering and $\mathcal{V} \to \mathcal{U}$ is a morphism in $\mathrm{Cat}(\mathfrak{T})$, then the fiber products $\mathcal{U}_i \times_{\mathcal{U}} \mathcal{V}$ exist in $\mathrm{Cat}(\mathfrak{T})$, and $(\mathcal{U}_i \times_{\mathcal{U}} \mathcal{V} \to \mathcal{V})$ belongs to $\mathrm{Cov}(\mathfrak{T})$.

A set X endowed with a Grothendieck topology $\mathfrak{T}$ will be called a $\mathcal{G}$-topological space.

Definition. Let $\mathfrak{T} = (\mathrm{Cat}(\mathfrak{T}), \mathrm{Cov}(\mathfrak{T}))$ be a Grothendieck topology. A presheaf on $\mathfrak{T}$ with values in $\mathrm{Cat}(\mathfrak{T})$ is a contravariant functor $\mathcal{F} : \mathrm{Cat}(\mathfrak{T}) \to \mathbf{Sets}$. A presheaf $\mathcal{F}$ on $\mathfrak{T}$ is a sheaf if the sequence

$$0 \longrightarrow \mathcal{F}(\mathcal{U}) \longrightarrow \prod_{i\in I} \mathcal{F}(\mathcal{U}_i) \longrightarrow \prod_{i,j} \mathcal{F}(\mathcal{U}_i \times_{\mathcal{U}} \mathcal{U}_j)$$

is exact for any admissible covering $(\mathcal{U}_i \to \mathcal{U})_{i\in I}$.

Now we specialize the definition to the case of affinoid K-spaces.

Definition. Let $X = \mathrm{Sp}\,(A)$ be an affinoid K-space. Let $\mathrm{Cat}(\mathfrak{T})$ be the category of affinoid subdomains of X with the inclusions as morphisms. The set $\mathrm{Cov}(\mathfrak{T})$ consists of all finite families $(\mathcal{U}_i \to \mathcal{U})_{i\in I}$ of inclusions of affinoid subdomains of X such that $\mathcal{U} = \bigcup_{i\in I} \mathcal{U}_i$. We call $\mathfrak{T}$ the weak Grothendieck topology on X.

By Tate's Acyclicity Theorem we know that $\mathcal{O}_X$ is a sheaf for the weak Grothendieck Topology.

There is a canonical way of enlarging and refining this topology:

Definition. Let X be an affinoid K-space. The strong Grothendieck topology on X is defined by:

(i) A subset $\mathcal{U} \subseteq X$ is an admissible open if there is a covering $\mathfrak{U} = \{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{U}$ by affinoid subdomains $\mathcal{U}_i \subseteq X$ such that for all morphisms of affinoid K-spaces $\phi : \mathcal{W} \to X$ satisfying $\phi(\mathcal{W}) \subseteq \mathcal{U}$, the covering $\{\phi^{-1}(\mathcal{U}_i)\}_{i\in I}$ of $\mathcal{W}$ admits a refinement that is a finite covering of $\mathcal{W}$ by affinoid subdomains.

(ii) A covering $(\mathcal{V}_i \to \mathcal{V})_{i\in I}$ of some admissible open subset $\mathcal{V} \subseteq X$ by admissible open sets $\mathcal{V}_i$ is called admissible if for each morphism of affinoid K-spaces $\phi : \mathcal{W} \to X$ satisfying $\phi(\mathcal{W}) \subseteq \mathcal{U}$, the covering $\{\phi^{-1}(\mathcal{U}_i)\}_{i\in I}$ of $\mathcal{W}$ admits a refinement by finitely many affinoid subdomains.

We need to prove that the strong $\mathcal{G}$-topology is indeed a Grothendieck topology:

Proposition 3.1.1. *Let X be an affinoid K-space. The strong Grothendieck topology is a Grothendieck topology on X satisfying the following completeness conditions:*

(G0) *$\varnothing$ and X are admissible opens.*

(G1) *If $(\mathcal{U}_i \to \mathcal{U})_{i\in I}$ is an admissible covering of an admissible open subset $\mathcal{U}$ and $\mathcal{V} \subseteq \mathcal{U}$ is a subset such that $\mathcal{V} \cap \mathcal{U}_i$ is an admissible open for all $i \in I$, then $\mathcal{V}$ is an admissible open in X.*

(G2) *If $(\mathcal{U}_i \to \mathcal{U})_{i\in I}$ is a covering of an admissible open set $\mathcal{U} \subseteq X$ by admissible open subsets $\mathcal{U}_i \subseteq X$ such that $(\mathcal{U}_i)_{i\in I}$ admits an admissible covering of $\mathcal{U}$ as refinement, then $(\mathcal{U}_i)_{i\in I}$ is admissible too.*

We already know, from Proposition 2.4.8, that every morphism of affinoid K-spaces $\phi : X \to Y$ is continuous with respect to the weak Grothendieck topology. The next result shows that this is also true for the strong Grothendieck topology:

Proposition 3.1.2. *Let $\phi : X \to Y$ be a morphism of affinoid K-spaces. Then ϕ is continuous with respect to the strong Grothendieck topologies on X and Y.*

We also want to mention that the strong $\mathcal{G}$-topology is related to the Zariski topology in this sense:

Lemma 3.1.3. *Let X be an affinoid K-space. Then, the strong Grothendieck topology on X is finer than the Zariski topology.*

We have seen that $\mathcal{O}_X$, the presheaf of affinoid functions, is a sheaf for the weak $\mathcal{G}$-topology. Further, we have observed that many of the properties of the weak $\mathcal{G}$-topology are inherited by the strong one. It is therefore natural to ask whether this construction preserves the structure of sheaves or not. For a complete approach one can refer to [BGR, §9.2].

Proposition 3.1.4. *Let X be an affinoid K-space. Then, any sheaf $\mathcal{F}$ on X with respect to the weak Grothendieck topology admits a unique extension with respect to the strong Grothendieck topology.*

The proof consists in studying the sheafification of some extension of the sheaf with respect to the strong $\mathcal{G}$-topology and then to prove that any other extension is isomorphic to it. The idea relies on the fact that the strong Grothendieck topology $\mathfrak{T}$ is *slightly finer* than the weak one $\mathfrak{T}_w$ which means that the admissible opens of $\mathfrak{T}_w$ form a basis for $\mathfrak{T}$ and each $\mathfrak{T}_w$-covering admits a $\mathfrak{T}$-covering as a refinement [Bos, §5.2].

This last result shows that there is a unique way to extend the sheaf O_X in the weak Grothendieck topology to the strong Grothendieck topology. The resulting sheaf is called the sheaf of rigid analytic functions on X and it is denoted by $\mathcal{O}_X$ as well.

3.2 Rigid Analytic Spaces

Finally, we are ready to give the definition of general rigid spaces.

Definition. A $\mathcal{G}$-ringed K-space is a pair $(X, \mathcal{O}_X)$ consisting of a $\mathcal{G}$-topological space X and a sheaf of K-algebras $\mathcal{O}_X$ on X.

Definition. A locally $\mathcal{G}$-ringed space is a $\mathcal{G}$-ringed space $(X, \mathcal{O}_X)$ over K such that for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

The trivial example is the case of affinoid K-spaces. If $X = \mathrm{Sp}\,(A)$ is an affinoid space, then we can see X as a $\mathcal{G}$-ringed K-space if we endow it with the strong Grothendieck topology and we consider the sheaf of rigid analytic functions $\mathcal{O}_X$. Further, thanks to Proposition 2.5.1, we conclude that $(\mathrm{Sp}\,(A)\,, \mathcal{O}_{\mathrm{Sp}(A)})$ is a locally $\mathcal{G}$-ringed space over K.

Definition. A morphism of $\mathcal{G}$-ringed spaces over K is a pair

$$\left(\phi, \phi^*\right) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

where ϕ is a continuous map and ϕ^* is a collection of K-algebra homomorphisms $\phi^*_{\mathcal{V}} : \mathcal{O}_Y(\mathcal{V}) \to \mathcal{O}_X(\phi^{-1}(\mathcal{V}))$, for each admissible open $\mathcal{V} \subseteq Y$, such that this family is compatible with restriction homomorphisms induced by inclusions $\mathcal{W} \subseteq \mathcal{V}$.

In other words, we want to have a commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_Y(\mathcal{V}) \arrow[r, "\phi^*_{\mathcal{V}}"] \arrow[d] & \mathcal{O}_X(\phi^{-1}(\mathcal{V})) \arrow[d] \\
\mathcal{O}_Y(\mathcal{W}) \arrow[r, "\phi^*_{\mathcal{W}}"'] & \mathcal{O}_Y(\phi^{-1}(\mathcal{W}))
\end{tikzcd}$$

Taking the direct limit, one gets a ring homomorphism

$$\phi^*_x : \mathcal{O}_{Y,\phi(x)} \longrightarrow \mathcal{O}_{X,x} \qquad \forall x \in X$$

Definition. The pair (ϕ, ϕ^*) is a morphism of locally $\mathcal{G}$-ringed spaces if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally $\mathcal{G}$-ringed spaces and ϕ^*_x is a map of local rings for each $x \in X$.

It is more or less clear that we are somehow reconstructing the theory of schemes that is used in algebraic geometry in the non-archimedean setting.

Coming back to our example, we have seen that affinoid spaces have a natural structure of locally $\mathcal{G}$-ringed spaces. The problem now is how to extend the notion of morphisms of K-affinoid spaces to morphisms of locally $\mathcal{G}$-ringed spaces over K. Let $\phi : X \to Y$ be a morphism of affinoid K-spaces. We know from Proposition 3.1.2 that ϕ is continuous and then we can take it as first component. We only have to describe how ϕ^* acts. If $\mathcal{V}$ is an affinoid subdomain of Y, we know from Proposition 2.4.8 that $\phi^{-1}(\mathcal{V})$ is an affinoid subdomain of X and ϕ induces a unique affinoid map $\phi_{\mathcal{V}} : \phi^{-1}(\mathcal{V}) \to \mathcal{V}$. Now we denote

$$\phi^*_{\mathcal{V}} : \mathcal{O}_Y(\mathcal{V}) \to \mathcal{O}_X(\phi^{-1}(\mathcal{V}))$$

the associated map of affinoid algebras.

Since affinoid subdomains form a basis for the strong Grothendieck topology, we can now extend the construction to all the admissible open subsets of Y using admissible coverings; namely, if $\mathfrak{V} = \{\mathcal{V}_i\}_i$ is an affinoid covering of an admissible open $\mathcal{V} \subseteq Y$, then

$$\forall f \in \mathcal{O}_Y(\mathcal{V}) \qquad \phi^*_{\mathcal{V}}(f)\big|_{\phi^{-1}(\mathcal{V}_i)} = \phi^*_{\mathcal{V}_i}(f|_{\mathcal{V}_i}) \qquad i \in I$$

Proposition 3.2.1. *If X and Y are affinoid K-spaces, there is a one-to-one correspondence between K-affinoid maps $X \to Y$ and maps of locally $\mathcal{G}$-ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. In other words, the functor from the category of K-affinoid spaces to the category of locally $\mathcal{G}$-ringed spaces is fully faithful.*

This result allows us to see the category of affinoid K-spaces as a subcategory of the category of locally $\mathcal{G}$-ringed spaces. In particular, we will use this fact to think of rigid spaces as locally $\mathcal{G}$-ringed spaces that, locally, are affinoid spaces in a similar fashion as schemes that locally are affine schemes.

Definition. A rigid analytic K-space is a locally $\mathcal{G}$-ringed space $(X, \mathcal{O}_X)$ over K such that

1. The $\mathcal{G}$-topology of X satisfies conditions **(G0)**, **(G1)** and **(G2)** of Proposition 3.1.1.

2. X admits an admissible covering $(X_i)_{i\in I}$ where $\left(X_i, O_X|_{X_i}\right)$ is an affinoid K-space for all $i \in I$.

A morphism of Rigid analytic spaces is a morphism of locally $\mathcal{G}$-ringed spaces.

Example. Any admissible open subset of an affinoid K-space X is a rigid space. This follows immediately from the axioms of Grothendieck topologies ([Con, Example 2.4.2])

Proposition 3.2.2. *Let X be a rigid K-space and Y an affinoid K-space. Then the canonical map*

$$\begin{aligned} Hom(X, Y) &\longrightarrow Hom(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \\ \phi &\longrightarrow \phi_Y^* \end{aligned}$$

is bijective.

We present here the general technique to glue rigid spaces together:

Proposition 3.2.3 (Pasting Analytic Spaces)**.** *Suppose that the following data are given:*

(i) *Rigid K-spaces $(X_i)_{i\in I}$.*

(ii) *Open subspaces $X_{i,j} \subseteq X_i$ and isomorphisms $\phi_{i,j} : X_{i,j} \to X_{j,i}$, for all $i, j \in I$*

Assume that the following conditions hold:

(a) *$\phi_{i,j} \circ \phi_{j,i} = Id$, $X_{i,i} = X_i$ and $\phi_{i,i} = Id$.*

(b) *The map $\phi_{i,j}$ induces isomorphisms $\phi_{i,j,l} : X_{i,j} \cap X_{i,l} \to X_{j,i} \cap X_{j,l}$ such that $\phi_{i,j,l} = \phi_{l,j,i} \circ \phi_{i,l,j}$ for all $i, j, l \in I$.*

Then the X_i's can be glued by identifying $X_{i,j}$ with $X_{j,i}$ via $\phi_{i,j}$ to yield a rigid K-space X admitting $(X_i)_{i\in I}$ as an admissible covering. More precisely, there exists a rigid K-space X with an admissible covering $(X_i')_{i\in I}$ and isomorphisms $\psi_i : X_i \xrightarrow{\sim} X_i'$ giving rise to isomorphisms $\psi_{i,j} : X_{i,j} \to X_i' \cap X_j'$ such that the diagram

$$\begin{tikzcd}
X_{i,j} \arrow[d, "\phi_{i,j}"', "\wr"] \arrow[dr, "\psi_{i,j}"] & \\
& X'_i \cap X'_j \\
X_{j,i} \arrow[ur, "\psi_{j,i}"'] &
\end{tikzcd}$$

commutes for all $i, j, \in I$; the analytic space X is unique up to isomorphism.

Corollary 3.2.4. *For two rigid K-spaces X and Y over Z, the fiber product $X \times_Z Y$ can be constructed.*

Proof. Since we know how to construct the fiber product between affinoid K-spaces (Proposition 2.3.4), we can work locally and then glue together the affinoid spaces. $\square$

3.3 Analytification

In this section we will describe how to associate (in a functorial way) to any K-scheme of locally finite type X a rigid analytic space X^{rig} called the rigid analytification of X. Some good references are [Bos, §5.4], [BGR, §9.3.4], [Tia, §2.3] and [Ber1, §0.3]. Some examples can also be found in [Con, §2.4].

We start by presenting how to construct a rigid analytic analogue for the n-dimensional affine space over K.

Choose a constant $c \in K$ with $|c| > 1$. For $i \in \mathbb{N} \cup \{0\}$ we denote by A_i the K-algebra of power series

$$\sum_\nu a_\nu \zeta^\nu \in K[\![\zeta]\!] \qquad \text{where } \zeta = (\zeta_1, \ldots, \zeta_n)$$

converging on the ball of center 0 and radius $|c|^i$ in $\left(K^{\mathrm{alg}}\right)^n$:

$$A_i = T_{n,\rho} = \{\sum_\nu a_\nu \zeta^\nu \in K[\![\zeta]\!] \ \mid \ \lim_\nu \rho^\nu |a_\nu| = 0\} = K\langle c^{-i}\zeta_1, \ldots, c^{-i}\zeta_n\rangle$$

where $\rho = (|c|^i, \ldots, |c|^i)$.

The A_i's occur in a decreasing sequence

$$A_0 \supset A_1 \supset A_2 \supset \ldots \supset K[\zeta]$$

and they are all affinoid algebras. There is a canonical isomorphism $A_i = A_{i+1}\langle c^{-1}\zeta\rangle$ and hence, we have an inclusion $A_{i+1} \hookrightarrow A_i$ inducing a map of K-affinoid spaces $\mathrm{Sp}\,(A_i) \to \mathrm{Sp}\,(A_{i+1})$ which identifies $\mathrm{Sp}\,(A_i)$ with an affinoid subdomain of $\mathrm{Sp}\,(A_{i+1})$.

This yields an increasing sequence of affinoid subdomains

$$\begin{array}{ccccccc} \mathbb{B}^n = \mathrm{Sp}\,(A_0) & \hookrightarrow & \mathrm{Sp}\,(A_1) & \hookrightarrow & \mathrm{Sp}\,(A_2) & \hookrightarrow & \dots \\ \shortparallel & & \shortparallel & & \shortparallel & & \\ X_0 & & X_1 & & X_2 & & \end{array}$$

Set $X_{i,j} = X_{\min\{i,j\}}$ and $\phi_{i,j} : X_{i,j} \to X_{j,i}$ to be the identity map. Now we apply Proposition 3.2.3 and we glue together the X_i's. The resulting space is a rigid analytic space which is not an affinoid. We denote it by

$$\mathbb{A}_K^{n,\mathrm{rig}} = \bigcup_r \mathbb{B}^n\left(0, |c|^r\right)$$

Remark. Observe that the construction is independent of the choice of c.

Now we would like to generalize this process to arbitrary algebraic varieties over K.

Let B be a finitely generated K-algebra: $B = K[\zeta_1, \dots, \zeta_n]/\mathfrak{a}$ where $\mathfrak{a} \subseteq K[\zeta_1, \dots, \zeta_n]$ is an ideal. Using the notation introduced before, we get a sequence of K-algebras homomorphisms

$$A_0/\mathfrak{a}A_0 \longleftarrow A_1/\mathfrak{a}A_1 \longleftarrow A_2/\mathfrak{a}A_2 \longleftarrow A_3/\mathfrak{a}A_3 \longleftarrow \dots \longleftarrow B$$

which yields an increasing sequence of affinoid subdomains

$$\mathrm{Sp}\,(A_0/\mathfrak{a}A_0) \hookrightarrow \mathrm{Sp}\,(A_1/\mathfrak{a}A_1) \hookrightarrow \mathrm{Sp}\,(A_2/\mathfrak{a}A_2) \hookrightarrow \mathrm{Sp}\,(A_3/\mathfrak{a}A_3) \hookrightarrow \dots$$

One can construct a rigid analytic space X^{rig} by gluing together these affinoids:

$$X^{\mathrm{rig}} = \bigcup_{r \geqslant 0} \mathrm{Sp}\,(A_r/\mathfrak{a}A_r)$$

Remark. X^{rig} admits $\{\mathrm{Sp}\,(A_r/\mathfrak{a}A_r)\}_{r\in\mathbb{N}}$ as covering.

Definition. Let $(X, \mathcal{O}_X)$ be a K-scheme of locally finite type. A rigid analytification of $(X, \mathcal{O}_X)$ is a rigid K-space $\left(X^{\mathrm{rig}}, \mathcal{O}_{X^{\mathrm{rig}}}\right)$ together with a morphism of locally $\mathcal{G}$-ringed K-spaces $(\iota, \iota^*) : \left(X^{\mathrm{rig}}, \mathcal{O}_{X^{\mathrm{rig}}}\right) \longrightarrow (X, \mathcal{O}_X)$ satisfying the following universal property: given a rigid K-space $(Y, \mathcal{O}_Y)$ and a morphism of locally $\mathcal{G}$-ringed K-spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$, the latter factors through (ι, ι^*) via a unique morphism of rigid K-spaces $(Y, \mathcal{O}_Y) \to \left(X^{\mathrm{rig}}, \mathcal{O}_{X^{\mathrm{rig}}}\right)$.

Proposition 3.3.1. *Let X and Y be two K-schemes of locally finite type.*

1. *X admits a unique structure of rigid analytic space X^{rig} on X satisfying the following properties:*

(a) *The underlying map of sets identifies the points of X^{rig} with the closed points of X.*

(b) *For every open subset $\mathcal{U} \subseteq X$ (respectively open covering of $\mathcal{U}$), $\mathcal{U} \cap X^{rig}$ is an admissible open of X^{rig} (resp. an admissible covering of $\mathcal{U} \subseteq X^{rig}$).*

(c) *For every affine open subset $\mathcal{U} \subseteq X$, the structure of rigid space induced on $\mathcal{U} \cap X^{rig}$ coincide with the one of $\mathcal{U}^{rig}$.*

2. *The rigid analytification defines a functor from the category of K-schemes of locally finite type to the category of rigid K-spaces.*

Example (Projective space). $\mathbb{P}_K^{n,\mathrm{rig}}$ as a rigid analytic variety can be obtained by pasting n copies of the rigid affine $\mathbb{A}_K^{n,\mathrm{rig}}$. Then $\mathbb{P}_K^{n,\mathrm{rig}}$ is covered by $n+1$ copies of $\mathbb{B}_K^n$ -balls of radius one-; these are isomorphic to

$$X_i = \mathrm{Sp}\left(K\langle\frac{\zeta_0}{\zeta_i},\ldots,\frac{\zeta_n}{\zeta_i}\rangle\right) \qquad\qquad i = 0,\ldots,n$$

where we identify ζ_i/ζ_i with the constant 1.

Example (Elliptic curves). Given an elliptic curve $\mathcal{E}$ over K with split multiplicative reduction ($j(\mathcal{E}) > 1$), Tate showed that there exists a unique $q \in K^\times$ with $|q| < 1$ such that

$$\mathcal{E}^{\mathrm{rig}} \simeq \mathbb{G}_m/q^{\mathbb{Z}}$$

and every such q occurs. Here $\mathbb{G}_m$ is the multiplicative group scheme over K which can be cut out of $\mathbb{A}^2$ ([BGR, Example 9.3.4(4)] and [FvdP, §5.1]).

3.4 Coherent Sheaves

In this section we provide a brief description of coherent sheaves on rigid spaces. We'll mainly follow [FvdP, §4.4-4.5].

Let $X = (X, \mathfrak{T}_X, \mathcal{O}_X)$ be a rigid space provided with the Grothendieck topology. It has an admissible affinoid covering $\{X_i\}_{i\in I}$ and we can use it to construct sheaves on X by gluing sheaves on X_i. Suppose the following data are given:

a) On each X_i a sheaf $\mathcal{F}_i$.

b) For every i, j, an isomorphism of sheaves $\psi_{i,j} : \mathcal{F}_i|_{X_{i,j}} \to \mathcal{F}_i|_{X_{i,j}}$ where $X_{i,j} = X_i \cap X_j$.

c) For every $i, j, k \in I$, $\psi_{i,j} \circ \psi_{j,k} = \psi_{i,k}$ on $X_{i,j,k}$.

Then there exist a (unique up to isomorphism) sheaf $\mathcal{F}$ on X and isomorphisms

$$\psi_i : \mathcal{F}|_{X_i} \to \mathcal{F}_i \text{ such that } \psi_{i,j} \circ \psi_j = \psi_i$$

Definition. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on a rigid space X is called coherent if there exist an admissible affinoid covering $\{X_i\}_{i\in I}$ of X and, for every $i \in I$, a finitely generated $\mathcal{O}_X(X_i)$-module M_i such that the restrictions of $\mathcal{F}$ to X_i is isomorphic (as a sheaf of $\mathcal{O}_X$-modules) to $\tilde{M}$ where $\tilde{M}(\mathcal{U}) = M_i \otimes_{\mathcal{O}_X(X_i)} \mathcal{O}_X(\mathcal{U})$

Definition. The sheaf $\mathcal{F}$ is called locally free of rank r (or rigid vector bundle of rank r) if the admissible affinoid covering $\{X_i\}$ can be chosen such that each M_i is a free $\mathcal{O}_X(X_i)$ module of rank r.

Theorem 3.4.1 (Kiehl). *For every coherent sheaf $\mathcal{F}$ on an affinoid space $X = Sp\,(A)$ there is a finitely generated A-module M such that $\mathcal{F}$ is isomorphic to the sheaf of $\mathcal{O}_X$-modules $\tilde{M}$.*

Definition. We present some definitions relative to morphism of rigid spaces:

- A morphism of rigid K-spaces $\phi : X \to Y$ is called a closed immersion if there exists an admissible affinoid covering $\{Y_i\}_{i\in I}$ such that, for all $i \in I$, the induced morphism $\phi_i : \phi^{-1}(Y_i) \to Y_i$ is a closed immersion of affinoid K-spaces, i.e., ϕ_i is a morphism of affinoid spaces (for instance $Y_i = \mathrm{Sp}\,(B_i)$ and $\phi^{-1}(Y_i) = \mathrm{Sp}\,(A_i)$), and the corresponding morphism of affinoid K-algebras $B_i \to A_i$ is an epimorphism.

- A rigid K-space X is called quasi-compact if it admits a finite admissible affinoid covering. A morphism of rigid K-spaces is called quasi-compact if for each quasi-compact open subspace $Y_0 \subseteq Y$, its inverse image $\phi^{-1}(Y_0)$ is quasi-compact.

- A morphism of rigid K-spaces $\phi : X \to Y$ is called separated (resp. quasi- separated) if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed immersion (resp. a quasi-compact morphism).

- A rigid K-space X is called separated (resp. quasi-separated) if the structural morphism $X \to \mathrm{Sp}\,(K)$ is separated (resp. quasi-separated).

- A morphism $f : X \to Y$ of rigid spaces is said to be finite if Y has an admissible affinoid covering $\{Y_i\}_{i\in I}$ such that each $f^{-1}(Y_i)$ is an affinoid and $\mathcal{O}_Y(Y_i) \to \mathcal{O}_X(f^{-1}(Y_i))$ is a finite morphism of affinoid algebras.

Definition. A morphism of rigid K-spaces $\phi : Y \to X$ is called smooth if there exist admissible affinoid coverings $\{Y_i\}_{i\in I}$ and $\{X_i\}_{i\in I}$ of Y and X respectively, such that

(i) $f(Y_i) \subseteq X_i$.

(ii) If $A_i = \mathcal{O}_X(X_i)$ and $B_i = \mathcal{O}_Y(Y_i)$, then there exist an isomorphism

$$B_i = \frac{A_i\langle T_1 \dots, T_n\rangle}{(f_i, \dots, f_m)}$$

such that

$$\det\left(\frac{\partial f_j}{\partial T_k}\right)_{1\leqslant j,k\leqslant m}$$

is invertible in B_i. If, further, $m = n$, then ϕ is said to be étale.

Definition. A rigid K-space X is said to be connected if one of the following equivalent conditions hold:

(a) $\mathcal{O}_X(X)$ has no nilpotent elements outside 0 and 1.

(b) There is no admissible covering of X consisting of two disjoint opens.

We conclude this section by mentioning that there is a technique to associate to a rigid space X a reduced rigid space X^{red} ([FvdP, Exercise 4.6.2]). In general, this is similar to the one used to associate to a scheme its reduced scheme ([Liu, §2.4.1]). If $X = (X, \mathfrak{T}_X, \mathcal{O}_X)$ is a rigid space, then

$$X^{\mathrm{red}} = (X, \mathfrak{T}_X, \mathcal{O}_X/\mathcal{N})$$

where $\mathcal{N} \subset \mathcal{O}_X$ is a coherent sheaf of ideals on X such that for every admissible affinoid $\mathcal{U} \subseteq X$, $\mathcal{N}(\mathcal{U})$ is the ideal of nilpotent elements of $\mathcal{O}_X(\mathcal{U})$.

Proposition 3.4.2. *X^{red} has a structure of rigid space and it satisfies the following universal property: for every morphism $g : Y \to X$ with Y reduced, there exists a unique morphism $h : Y \to X^{red}$ such that $g = \iota \circ h$ where $\iota : X^{red} \to X$ is the canonical morphism.*

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, dashed, "\exists! h"'] & & X \\ & X^{red} \arrow[ur, bend right, "\iota"'] & \end{tikzcd}$$

3.5 Analytic Reductions

In this section we want to attach to a rigid K-space X its analytic reduction, i.e., a reduced algebraic variety over the residue field k. In order to be able to do that we recall some definitions. If $X = \mathrm{Sp}\,(A)$ is an affinoid K-space, then we denote by

$$A^o = \{a \in A \mid \|a\| \leqslant 1\}$$

the $\mathcal{O}_K$-algebra of functions over X with norm less than or equal to 1. It contains the ideal

$$A^{oo} = \{a \in A \mid \|a\| < 1\}$$

Finally, we denote by $\overline{A}$ the quotient A^o/A^{oo}. The latter is a finitely generated k-algebra which turns out to be reduced since the spectral norm is power multiplicative.

Definition. The reduced affine k-scheme $\overline{X}^c = \mathrm{Max}(\overline{A})$ is called the canonical reduction of the affinoid K-space X.

Remark. We are using the set of maximal ideals instead of the prime spectrum because this allows us to define a canonical reduction map

$$\mathrm{Red} = \mathrm{Red}_X^c : X \longrightarrow \overline{X}$$

defined by

$$x \longrightarrow \ker(\overline{A} \longrightarrow \overline{A/\mathfrak{m}_x})$$

where we recall that $A/\mathfrak{m}_x$ is a finite extension of K and, therefore, it carries a unique valuation extending the one of K. The map whose kernel appears in the definition of Red_X^c is the reduction of the quotient map $A \to A/\mathfrak{m}_x$.

Proposition 3.5.1. *The map Red_X^c is surjective.*

Proposition 3.5.2. *The preimage $(Red_X^c)^{-1}(\mathcal{U})$ of a Zariski open subset $\mathcal{U}$ of $\overline{X}^c$ is an admissible open in X.*

Sketch of Proof. If $f \in A^o$ with $\|f\| = 1$, then $\overline{f} \notin \mathrm{Red}_X^c(x)$ if and only if $|f(x)| \geqslant 1$. This means that

$$(\mathrm{Red}_X^c)^{-1}(\mathrm{Max}(\overline{A}_{\overline{f}})) = X\left(\frac{1}{f}\right)$$

□

3.6 Towards Formal Geometry

So far, we have worked over the field K. The idea of formal rigid geometry is to replace K by its ring of integers $\mathcal{O}_K$. In this way we obtain $\mathcal{O}_K$-algebras that can be regarded as $\mathcal{O}_K$-models for affinoid K-algebras.

The idea is that, taking generic fibers (tensoring over $\mathcal{O}_K$ with K), one should obtain affinoid K-algebras.

In this section we will try to follow a more general approach working with general topological rings. Some good references are [Ber1, §0], [Bos, Chapter 7], [Tia, Chapter 3], [FvdP, §4.8], [Con, §3.3] and [Lüt, Chapter 3].

Let's consider a topological ring $\mathcal{A}$.

Definition. We say that $\mathcal{A}$ is *adic* if the topology on $\mathcal{A}$ is $\mathfrak{a}$-adic for some ideal $\mathfrak{a} \subseteq \mathcal{A}$ and $\mathcal{A}$ is separated and complete for this topology.

The ideal $\mathfrak{a}$ is usually called the ideal of definition.

Remark. The first property is equivalent to: for any $a \in \mathcal{A}$, $(a + \mathfrak{a}^n)_{n \geqslant 0}$ form a basis of open neighborhoods of $a \in \mathcal{A}$. The second request means that $\mathcal{A} = \varprojlim_n \mathcal{A}/\mathfrak{I}^n$.

One can associate to $\mathcal{A}$ an affine formal scheme

$$X = \operatorname{Spf}(\mathcal{A}) = \varinjlim_n \operatorname{Spec}(\mathcal{A}/\mathfrak{a}^n)$$

One can see that X, as a topological space, consists of all open prime ideals $\mathfrak{p} \subseteq \mathcal{A}$. An ideal $\mathfrak{p} \subseteq \mathcal{A}$ is open if and only if it contains a power of $\mathfrak{a}$ which means that $\operatorname{Spf}(\mathcal{A})$ is canonically identified with $\operatorname{Spec}(\mathcal{A}/\mathfrak{a})$.

In this way the Zariski topology on $\operatorname{Spec}(\mathcal{A}/\mathfrak{a})$ induces a topology on $\operatorname{Spf}(\mathcal{A})$: for any $f \in \mathcal{A}$, we denote $D(f) = \{x \in \operatorname{Spf}(\mathcal{A}) \mid f(x) \neq 0\} = \{\mathfrak{p} \subseteq \mathcal{A} \mid \mathfrak{p} \text{ prime}, \ f \notin \mathfrak{p}\}$.

Then

$$D(f) \longrightarrow \mathcal{A}\langle f^{-1}\rangle := \varprojlim_n (\mathcal{A}/\mathfrak{a}^n)\left[f^{-1}\right]$$

defines a presheaf $\mathcal{O}$ of topological rings on the category of subsets $D(f) \subseteq \operatorname{Spf}(\mathcal{A})$, $f \in \mathcal{A}$.

Lemma 3.6.1. *$\mathcal{O}$ is, in fact, a sheaf.*

Proof. Let $D(f_i)$ be an open covering of $D(f)$. The sequence

$$0 \longrightarrow \mathcal{A}/\mathfrak{a}^n\left[f^{-1}\right] \longrightarrow \prod_i \mathcal{A}/\mathfrak{a}^n\left[f_i^{-1}\right] \longrightarrow \prod_{i,j} \mathcal{A}/\mathfrak{a}^n\left[(f_i f_j)^{-1}\right]$$

is exact. Now we use the fact that the inverse limit is left exact. If we apply $\varprojlim$ we obtain another left exact sequence

$$0 \longrightarrow \mathcal{A}\langle f^{-1}\rangle \longrightarrow \prod_i \mathcal{A}\langle f_i^{-1}\rangle \longrightarrow \prod_{i,j} \mathcal{A}\langle (f_i f_j)^{-1}\rangle$$

which implies that $\mathcal{O}$ is a sheaf. □

For any $x \in X = \mathrm{Spf}(\mathcal{A})$, let $\mathfrak{p}_x \subseteq \mathcal{A}$ be the corresponding open prime ideal. We define

$$\mathcal{O}_{X,x} = \varinjlim_{x \in D(f)} A\langle f^{-1}\rangle$$

The following Lemma ([Tia, p. 3.1.2]) tells us that our construction is the correct one:

Lemma 3.6.2. *The ring $\mathcal{O}_{X,x}$ is local with maximal ideal $\mathfrak{p}_x\mathcal{O}_{X,x}$.*

Definition. A formal scheme is a locally topologically ringed space $(X, \mathcal{O}_X)$ such that every point $x \in X$ admits an open neighborhood $\mathcal{U}$ such that $(\mathcal{U}, \mathcal{O}_X|_{\mathcal{U}})$ is isomorphic to an affine formal scheme $\mathrm{Spf}(\mathcal{A})$ for some adic ring $\mathcal{A}$.

Now we fix a non-archimedean field K and we consider its ring of integers $\mathcal{O}_K$. We also choose an element π in $\mathcal{O}_K$ such that $0 < |\pi| < 1$. Define

$$\mathcal{O}_K\langle \zeta_1, \dots, \zeta_n\rangle = \varprojlim_n \mathcal{O}_K/(\pi^n)\,[\zeta_1, \dots, \zeta_n]$$

This is a separated complete π-adic $\mathcal{O}_K$-algebra and it can be regarded as the integral model of the Tate algebra $K\langle \zeta_1, \dots, \zeta_n\rangle$.

Remark. If $\mathcal{O}_K$ is not discrete valued, then $\mathcal{O}_K\langle \zeta_1, \dots, \zeta_n\rangle$ is not Nöetherian.

Definition. We say that a topological $\mathcal{O}_K$-algebra $\mathcal{A}$ is

i. of topologically finite type if $\mathcal{A}$ is isomorphic to $\mathcal{O}_K\langle \zeta_1, \dots, \zeta_n\rangle/\mathfrak{b}$, equipped with the π-adic topology where $\mathfrak{b} \subseteq \mathcal{O}_K\langle \zeta_1, \dots, \zeta_n\rangle$ is an ideal;

ii. of topologically finite presentation if, in addition to **i.**, the ideal $\mathfrak{b}$ is finitely generated;

iii. admissible if $\mathcal{A}$ is topologically of finite presentation and $\mathcal{A}$ is flat over $\mathcal{O}_K$.

We present here a series of results that might be useful in the proceeding The proofs can be found in [Tia, §3.2].

Theorem 3.6.3. *Let $\mathcal{A}$ be an $\mathcal{O}_K$-algebra of topologically finite type, and $\mathcal{M}$ be a finitely generated $\mathcal{A}$-module which is flat over $\mathcal{O}_K$. Then, $\mathcal{M}$ is an $\mathcal{A}$-module of finite presentation.*

Corollary 3.6.4. *Let $\mathcal{A}$ be an $\mathcal{O}_K$-algebra of topologically finite type. If $\mathcal{A}$ is flat over $\mathcal{O}_K$, then $\mathcal{A}$ is of topologically finite presentation over $\mathcal{O}_K$.*

Proposition 3.6.5. *Let $\mathcal{A}$ be an $\mathcal{O}_K$-algebra of topologically finite type and $\mathcal{M}$ be a finite $\mathcal{A}$-module. Then $\mathcal{M}$ is π-adic separated and complete.*

Corollary 3.6.6. *Every $\mathcal{O}_K$-algebra of topologically finite type is separated and complete for the π-adic topology.*

3.7 Analytic Spaces and Formal Schemes

To every formal scheme $\mathfrak{X}$ of locally finite type over $\mathcal{O}_K$ one can associate a rigid analytic K-space $\mathfrak{X}_K$ thanks to the work of Raynaud ([Ber1, §0.2]).

Let $\mathfrak{X} = \mathrm{Spf}(\mathcal{A})$ a formal affine $\mathcal{O}_K$-scheme of finite type. Since $\mathcal{A}$ is of topologically finite type, $\mathcal{A} \otimes K$ is a Tate algebra and, then, the affinoid K-space $\mathfrak{X}_K$ can be defined by:

$$\mathfrak{X}_K = \mathrm{Sp}\left(\mathcal{A} \otimes_{\mathcal{O}_K} K\right)$$

Definition. If $\mathfrak{X}$ is an admissible formal $\mathcal{O}_K$-scheme, a rigid point of $\mathfrak{X}$ is a morphism $j : \mathcal{Y} \to \mathfrak{X}$ of admissible formal schemes such that j is a closed immersion and

$$\mathcal{Y} = \mathrm{Spf}(\mathcal{B})$$

with $\mathcal{B}$ a local integral domain of dimension 1.

Two rigid points $j_1 : \mathcal{Y}_1 \to \mathfrak{X}$ and $j_2 : \mathcal{Y}_2 \to \mathfrak{X}$ are said to be equivalent if there exists an isomorphism $\iota : \mathcal{Y}_1 \to \mathcal{Y}_2$ such that $j_1 = j_2 \circ \iota$.

We denote $\mathrm{Pts}_{\mathrm{rig}}(\mathfrak{X})$ the set of isomorphism classes of rigid points on $\mathfrak{X}$.

Lemma 3.7.1. *Let $\mathfrak{X} = Spf(\mathcal{A})$ be an affine admissible formal $\mathcal{O}_K$-scheme. Then there exist canonical bijections between the following sets of points:*

(a) *Isomorphism classes of rigid points on $\mathfrak{X}$.*

(b) *Non open prime ideals $\mathfrak{p} \subseteq \mathcal{A}$ such that $\dim(\mathcal{A}/\mathfrak{p}) = 1$.*

(c) *Maximal ideals in $\mathcal{A} \otimes_{\mathcal{O}_K} K$.*

Moreover, the bijections between the three sets of points are given as follows:

1. *Given a rigid point $j : \mathcal{Y} = Spf(\mathcal{B}) \to Spf(\mathcal{A})$ defined by a surjection $j^* : \mathcal{A} \to \mathcal{B}$, we associate to j the prime ideal $\mathfrak{p} = \ker(j^*)$ which is a point of type (b).*

2. *Given a non open prime ideal $\mathfrak{p} \subseteq \mathcal{A}$, we associate to it the prime ideal $\mathfrak{p}_K = \mathfrak{p} \otimes_{\mathcal{O}_K} K \subseteq \mathcal{A} \otimes_{\mathcal{O}_K} K$, which is a maximal ideal of $\mathcal{A} \otimes_{\mathcal{O}_K} K$.*

3. *Given a maximal ideal $\mathfrak{m} \subseteq \mathcal{A} \otimes_{\mathcal{O}_K} K$, let $\mathfrak{p} = \mathfrak{m} \cap \mathcal{A}$; we associate to it the canonical morphism $Spf(\mathcal{A}/\mathfrak{p}) \to Spf(A)$.*

Proof. Firstly, we focus on the map defined in **1**:

$$\begin{aligned} \mathrm{Pts}_{\mathrm{rig}}(\mathfrak{X}) &\longrightarrow \{\mathfrak{p} \subseteq \mathcal{A} \,|\, \mathfrak{p} \text{ non open}, \ \dim(\mathcal{A}/\mathfrak{p}) = 1\} \\ (j : \mathrm{Sp}(\mathcal{B}) \to \mathfrak{X}) &\longrightarrow \ker(j^*) \end{aligned}$$

Clearly $\ker(j^*)$ is a prime ideal and it is not open since $\mathcal{B}$ is flat over $\mathcal{O}_K$.

Now let's have a look at the map defined in **2.**

$$\begin{aligned} \{\mathfrak{p} \subseteq \mathcal{A} \,|\, \mathfrak{p} \text{ non open}, \ \dim(\mathcal{A}/\mathfrak{p}) = 1\} &\longrightarrow \mathrm{Sp}(\mathcal{A} \otimes_{\mathcal{O}_K} K) \\ \mathfrak{p} &\longrightarrow \mathfrak{p}_K = \mathfrak{p} \otimes_{\mathcal{O}_K} K \end{aligned}$$

we need to prove that $\mathfrak{p}_K$ is a maximal ideal. Suppose that there is a prime ideal $\mathfrak{q} \subseteq \mathcal{A}$ with $\mathfrak{p} \subset \mathfrak{q}$. Then $\mathfrak{q}$ is a maximal ideal (since $\dim(\mathcal{A}/\mathfrak{p}) = 1$). Furthermore, $\mathfrak{q}$ must be open in $\mathcal{A}$ (otherwise we would have $\pi\mathcal{A} + \mathfrak{q} = \mathcal{A}$ for $\pi \in \mathcal{O}_K$, $0 < |\pi| < 1$, which yields an equation of type $1 - a\pi = q$ for some $a \in \mathcal{A}$, $q \in \mathfrak{q}$ and this would imply the invertibility of q which is a contradiction). It follows that $\mathfrak{p}_K$ is a maximal ideal in $\mathcal{A} \otimes_{\mathcal{O}_K} K$.

Finally,

$$\begin{aligned} \mathrm{Sp}(\mathcal{A} \otimes_{\mathcal{O}_K} K) &\longrightarrow \mathrm{Pts}_{\mathrm{rig}}(\mathfrak{X}) \\ \mathfrak{m} &\longrightarrow (j : \mathrm{Spf}(\mathcal{A}/\mathfrak{p}) \to \mathrm{Spf}(\mathcal{A})) \qquad \mathfrak{p} = \mathfrak{m} \cap \mathcal{A} \end{aligned}$$

We have a natural inclusion $\mathcal{B} = \mathcal{A}/\mathfrak{p} \hookrightarrow K' = (\mathcal{A} \otimes_{\mathcal{O}_K} K)/\mathfrak{m}$ where K' is a finite extension of K. We denote by $\mathcal{B}$ the image of $\mathcal{A}$ in K'. It is possible to prove that $\mathcal{O}_{K'}$ is the integral closure of $\mathcal{O}_K$ in K' and $\mathcal{B} \subseteq \mathcal{O}_{K'}$ ([Bos, Lemma 8.3(6)]). It follows that $\mathcal{A} \to \mathcal{B}$ gives rise to a rigid point $\mathrm{Spf}(\mathcal{B}) \to \mathrm{Spf}(\mathcal{A})$ and the quotient $\mathcal{A}/\mathfrak{p}$ is isomorphic to $\mathcal{B}$.

All these maps are injective by construction and, if we compose them, we get the identity on $\mathrm{Pts}_{\mathrm{rig}}(\mathfrak{X})$. This is enough to prove that they are all bijections. $\square$

Corollary 3.7.2. *If $\mathfrak{X}$ is a formal $\mathcal{O}_K$-scheme, then there is a bijection of sets*

$$Pts_{rig}(\mathfrak{X}) \longleftrightarrow \mathfrak{X}_K$$

Remark. In general, if $\mathfrak{X}$ is an arbitrary admissible formal $\mathcal{O}_K$-scheme, we can construct $\mathfrak{X}_K$ by gluing together the rigid spaces associated to $\mathfrak{X}_i = \mathrm{Spf}(\mathcal{A}_i)$ where the $\mathfrak{X}_i$'s form a formal affine covering of $\mathfrak{X}$.

Let $j : \mathcal{Y} = \mathrm{Spf}(\mathcal{B}) \to \mathfrak{X}$ be a rigid point. It induces a closed immersion

$$j_k : \mathrm{Spec}(\mathcal{B} \otimes_{\mathcal{O}_K} k) \to \mathfrak{X}_k = \mathfrak{X} \otimes_{\mathcal{O}_K} k$$

Since $\mathcal{B}$ is a local integral domain, $\mathrm{Spec}(\mathcal{B} \otimes_{\mathcal{O}_K} k)$ consists only of a single point. Thus, the image j_k is a well-defined closed point on the special fiber $\mathfrak{X}_k$, usually called the specialization of j. Using the previous corollary, we conclude that there exists a canonical specialization map

$$\mathrm{sp} : \mathfrak{X}_K \longrightarrow \mathfrak{X}_k$$

Proposition 3.7.3. *Let $\mathfrak{X}$ and $\mathfrak{X}'$ be two formal $\mathcal{O}_K$-schemes of locally finite presentation.*

(a) *There exists a unique structure of rigid analytic space on $\mathfrak{X}_K$ satisfying the following properties:*

  (i) *The inverse image of every open (resp. open covering) of $\mathfrak{X}_k$ is an admissible open (resp. admissible covering) of $\mathfrak{X}_K$.*

  (ii) *For every open affine subscheme $\mathcal{U} \subseteq \mathfrak{X}$ with reduction $\mathcal{U}_k \subseteq \mathfrak{X}_k$, the structure induced on $\mathcal{U}_K = sp^{-1}(\mathcal{U}_k)$ by $\mathfrak{X}_K$ is the same as the one induced by $\mathcal{U}$, i.e., $sp^{-1}(\mathcal{U}_0) = \mathcal{U}^{rig}$.*

(b) *The construction of $\mathfrak{X}_K$ is functorial and, for every morphism of formal $\mathcal{O}_K$-schemes $\sigma : \mathfrak{X} \to \mathfrak{X}'$, the following diagram commutes:*

$$\begin{array}{ccc}
\mathfrak{X}_K & \xrightarrow{sp} & \mathfrak{X}_k \\
{\scriptstyle \sigma_K}\downarrow & & \downarrow{\scriptstyle \sigma_k} \\
\mathfrak{X}'_K & \xrightarrow[sp]{} & \mathfrak{X}'_k
\end{array}$$

(c) *This functorial construction preserves open and closed immersions.*

The space $\mathfrak{X}_K$ is called generic fiber of $\mathfrak{X}$.

We conclude this section with few facts about the generic fiber ([Ber1, Remarks 0.2.4]).

Remark. The rigid analytic space $\mathfrak{X}_K$ is quasi-separated.

Remark. If $\mathfrak{X}$ is of finite type, then $\mathfrak{X}_K$ is quasi-compact.

Remark. $\mathfrak{X}_K$ depends only on the biggest formal flat subscheme of $\mathfrak{X}$.

Chapter 4

p-adic Abelian Integrals

In this chapter we introduce the theory of p-adic integration developed by Coleman in [Col1] and [Col3].

In the first section we will try to give an intuition of the methods used starting from a classical example; we will then review (Sections 4.2 - 4.4) some basic notions required to fully understand the ideas of Coleman. Finally, in Section 4.5 we will prove the main Theorem of Coleman's integration.

The main references are the original articles by Coleman: in [Col1] we can find an explicit theory of p-adic integrals on $\mathbb{P}^1$, in [Col3] (which is the article we will mainly follow) we have a general theory of p-adic integrals for differentials of the II kind on varieties of any dimension having good reduction at p and, finally, in [CdS] the theory is extended to arbitrary forms on curves with good reduction.

In addition, one can refer to [Bre, §1.2], for a more expository treatise of the theory, or [Bes].

4.1 Battle Plan

We give here a naïve description of the Coleman theory referring to the following sections for all the details.

Suppose that K is a complete subfield of $\mathbb{C}_p$ and suppose we are give a variety X over K with good reduction and a closed holomorphic one form ω on X.

Question. *How can we define*

$$\int_P^Q \omega$$

for $P, Q \in X(K)$ or $X(K^{alg})$?

More in general, we would like to construct a coherent Theory of integration on X, i.e., we would like to have

- Additivity at points: $\displaystyle\int_P^Q \omega + \int_Q^R \omega = \int_P^R \omega.$
- Linearity on forms: $\displaystyle\lambda_1 \int_P^Q \omega_1 + \lambda_2 \int_P^Q \omega_2 = \int_P^Q (\lambda_1\omega_1 + \lambda_2\omega_2).$
- Change of variables: if $\Phi : X \to X'$ is a morphism with some "good properties" and ω' is a one form on X', then $\displaystyle\int_P^Q \Phi^*(\omega') = \int_{\Phi(P)}^{\Phi(Q)} \omega'.$
- Fundamental Theorem of Calculus: $\displaystyle\int_P^Q df = f(Q) - f(P).$

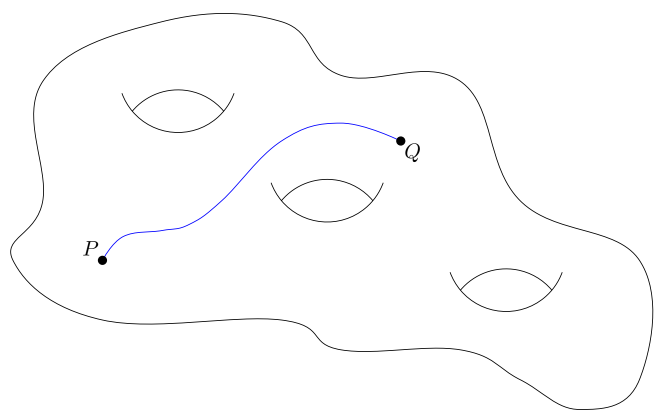


Figure 2: Integration path from P to Q

A first attempt can be mimicking the standard integration theory developed for varieties over $\mathbb{C}$:

1) Cover X with (in the case of rigid spaces) affinoid subdomains.

2) Integrate on each affinoid.

3) Adjust the constant of integration and do analytic continuation.

We have seen in the previous chapter that point 1 cause non problem. Also point 3 is not an issue since affinoids have plenty of intersections. The problem is that there is no natural way of integrating on affinoids.

We present here a classical example following [Bes].

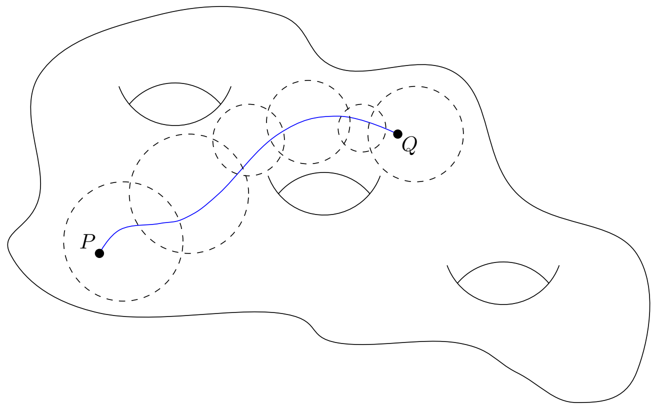


Figure 3: Affinoid Covering

Example. Suppose we are given the space $X = \{z \in K \mid |z| = 1\}$ where K is a complete subfield of $\mathbb{C}_p$. Now consider the form $\omega = dz/z$. Ideally, we would like to obtain the logarithmic function $\log(z)$ as primitive of ω.

- Choose $\alpha \in X$.
- Expand ω in power series centered at α:

$$\omega = \frac{d(\alpha + x)}{\alpha + x} = \frac{dx}{\alpha + x} = \frac{1}{\alpha}\frac{dx}{1 + \frac{x}{\alpha}} = \frac{1}{\alpha}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{x}{\alpha}\right)^n dx$$

- Integrate term by term:

$$F_\omega(\alpha + x) = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n+1}\left(\frac{x}{\alpha}\right)^{n+1} + \mathrm{C}$$

  which converges for $|x| < 1$ [Gou, §4.5].

Now the strategy in $\mathbb{C}$ would be to play with the constant C doing analytic continuation: cover X with open disks and adjust C so that the two expansions agree on the intersection. In the p-adic word this is not possible since two open disks are either disjoint or one contained in the other (Proposition 1.1.1).

Our first idea yields nothing. Trying to solve this problem, we can cover our affinoid by residue disks. We have seen that there is a way of integrating on residue disks:

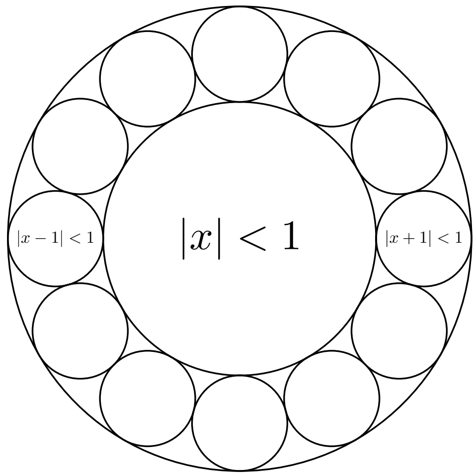


Figure 4: Covering of $\mathcal{O}_K$ by residue disks

Back to the example. We recall that $\alpha \in X = \{z \in K \mid |z| = 1\}$ and $\omega = dz/z$. $z = \alpha + x$, $|z| < 1$

$$F_\omega(\alpha + x) = \log(\alpha + x) + \mathrm{C}_\alpha = \log(1 + \frac{x}{\alpha}) + \mathrm{C}_\alpha$$

Unfortunately, the price we pay is the loss of intersections since residue disks are completely disjoint. Here intervenes Coleman: his solution consists in using Frobenius endomorphims to connect integrals in disjoint residue disks.

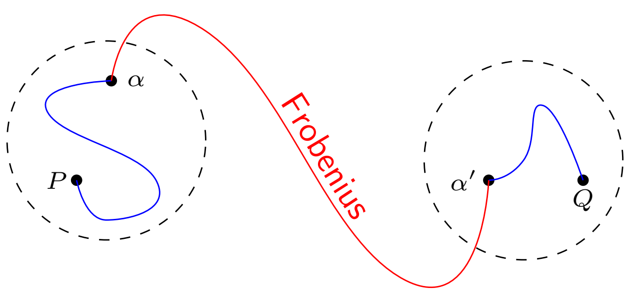


Figure 5: Continuation along Frobenius between two residue disks

Back to the example. If $X = \{z \in K \mid |z| = 1\}$, we can take

$$\begin{aligned} \Phi : X &\longrightarrow X \\ z &\longrightarrow z^p \end{aligned}$$

Coleman's idea is to use the fact that $\Phi^*\omega = p\omega$ to find a relation between integrals:

$$\Phi^* F_\omega = pF_\omega + \mathrm{C}$$

Supposing C $= 0$ (we can just adjust F_ω by constants), we get

$$F_\omega(z^p) = pF_\omega(z)$$

Now, if α satisfies $\alpha^{p^k} = \alpha$ (we will see that this is reasonable), then

$$F_\omega(\alpha) = F_\omega(\alpha^{p^k}) = p^k F_\omega(\alpha) \Longrightarrow F_\omega(\alpha) = 0$$

This determines F_ω on the disk $|z - \alpha| < 1$ and therefore on the whole X.

In this first section we have assumed without proving many facts (for instance the existence of these Frobenius endomorphisms or of some points fixed by powers of Frobenius); in the following sections we will fix all the details.

4.2 Lifting Morphisms

Let X be an affinoid over K (our complete subfield of $\mathbb{C}_p$) and $A(X)$ be the algebra of rigid analytic functions on X.

Notation. Here we follow the notation in [Col3]: $A(X) := \mathcal{O}_X(X)$.

Consider, as in Section 3.5, $A_0(X) = \{f \in A(X) \mid \|f\| \leqslant 1\}$ with respect to the spectral seminorm $\|\ \|$. We recall that, if X is reduced, then $\|\ \|$ is, in fact, a norm (Corollary 2.2.9 and Lemma 2.5.3) and $A(X)$ is complete with respect to this norm (Theorem 2.2.10).

We define

$$\tilde{A}(X) = \frac{A_0(X)}{\mathfrak{p}A_0(X)}$$

$$\tilde{X} = \mathrm{Spec}(\tilde{A}(X))$$

If $A_0(X)$ is of topologically finite type, then $\tilde{X}$ is a scheme of finite type over $\mathbb{F} = \mathcal{O}_K/\mathfrak{p}$.

Definition. We say that X has good reduction if $A_0(X)$ is of topologically finite type over $\mathcal{O}_K$ and $\tilde{X}$ is smooth over $\mathbb{F}$ ($\tilde{X}_{\mathbb{F}^{\mathrm{alg}}}$ is regular).

We have a natural reduction map

$$\begin{aligned} \mathrm{red} : X &\longrightarrow \tilde{X}(\mathbb{F}) \\ x &\longrightarrow \tilde{x} = x \cap A_0(X) \mod \mathfrak{p}A_0(X) \end{aligned}$$

and if we extend the scalars, we get $\mathrm{red}_{\mathbb{C}_p} : X_{\mathbb{C}_p} \longrightarrow \tilde{X}(\mathbb{F}^{\mathrm{lg}})$.

Lemma 4.2.1 ([Ber1, Lemma 1.1.1]). *If $u \in \tilde{X}(\mathbb{F}^{alg})$, then $red^{-1}(u)$ (residue class in X) has a natural structure of rigid space over $\mathbb{C}_p$.*

Let us consider a morphism of affinoid K-spaces $\phi : X \to Y$; we denote $\tilde{\phi} : \tilde{X} \to \tilde{Y}$ its reduction. Viceversa, if $\psi : \tilde{X} \to \tilde{Y}$ is a morphism over $\mathbb{F}$, we say that $\bar{\psi} : X \to Y$ is a lifting of ψ if $\tilde{\bar{\psi}} = \psi$.

Theorem 4.2.2.A. *Let K be discretely valued or $K = \mathbb{C}_p$. Suppose that*

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} \tag{$*$}$$

is a commutative diagram of reduced K-affinoids such that $W \to Y$ is a closed immersion (with the meaning of Section 4.4) and $\tilde{X}$ is smooth over $\tilde{Z}$. If $h : \tilde{X} \to \tilde{Y}$ is a morphism commuting with the reduction of $(*)$, then there exists a lift $\bar{h} : Y \to X$ commuting with $(*)$.

$$\begin{tikzcd} \tilde{W} \arrow[r] \arrow[d, hook] & \tilde{X} \arrow[d] \\ \tilde{Y} \arrow[r] \arrow[ur, "h"] & \tilde{Z} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} W \arrow[r] \arrow[d, hook] & X \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "\bar{h}"] & Z \end{tikzcd}$$

We will prove an analogue of Theorem 4.2.2.A for affinoid algebras (recall that in section 2.3 we have seen how to go back a forth from maps of affinoid algebras and morphisms of affinoid spaces).

Definition. A Tate $\mathcal{O}_K$-algebra is an $\mathcal{O}_K$-algebra of the form

$$\mathcal{O}_K\langle\zeta_1, \ldots, \zeta_n\rangle/\mathfrak{I}$$

for some finitely generated ideal $\mathfrak{I}$ of $\mathcal{O}_K\langle\zeta_1, \ldots, \zeta_n\rangle$, the completion of $\mathcal{O}_K[\zeta_1, \ldots, \zeta_n]$.

Let A be a Tate $\mathcal{O}_K$-algebra and set $\tilde{A} = A/\mathfrak{p}A$.

Definition. The annihilator in A of $r \in \mathcal{O}_K$ is

$$\mathrm{Ann}_A(r) = \{a \in A \mid ra = 0\}$$

Definition. Given a homomorphism of Tate $\mathcal{O}_K$-algebras $A \to B$, we say that B is $\mathcal{O}_K$-torsion free over A if

$$\mathrm{Ann}_B(r) = \mathrm{Ann}_A(r) \cdot B \qquad \forall r \in \mathcal{O}_K$$

Definition. We say that B is formally smooth over A if $\tilde{B}$ is smooth over $\tilde{A}$ and B is $\mathcal{O}_K$-torsion free over A.

Theorem 4.2.2.B. *Let K be discretely valued or $K = \mathbb{C}_p$. Suppose that*

$$\begin{tikzcd} D & B \arrow[l] \\ C \arrow[u] & A \arrow[l] \arrow[u] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd} \tag{$*'$}$$

is a commutative diagram of Tate $\mathcal{O}_K$-algebras such that $\tilde{C} \to \tilde{D}$ is surjective and B is formally smooth over A. If $s : \tilde{B} \to \tilde{C}$ is a homomorphism commuting with the reduction of $(*')$, then there exists a lift $\bar{s} : B \to C$ which makes the following into a commutative diagram:

$$\begin{tikzcd} D & B \arrow[l] \arrow[dl, dashed, "\bar{s}"'] \\ C \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

Proof. The proof consists of several lemmas patched together.

Lemma 4.2.3. *If $\psi : A \twoheadrightarrow B$ is a surjective homomorphism of Tate $\mathcal{O}_K$-algebras, then its kernel is finitely generated.*

Proof. Without loss of generality we can assume that $A = \mathcal{O}_K\langle\zeta_1, \ldots, \zeta_m\rangle$ (A is a quotient of $\mathcal{O}_K\langle\zeta_1, \ldots, \zeta_m\rangle$). By hypothesis, we know that B is a quotient of $\mathcal{O}_K\langle\zeta_1, \ldots, \zeta_n\rangle$ by some finitely generated ideal $\mathfrak{J}$:

$$0 \longrightarrow \mathfrak{J} \longrightarrow \mathcal{O}_K\langle\zeta_1, \ldots, \zeta_n\rangle \longrightarrow B \longrightarrow 0$$

Consider now the following diagram

$$\begin{tikzcd} & \mathcal{O}_K\langle\zeta_1, \ldots, \zeta_n\rangle \arrow[dl, "h"'] \arrow[d, "\pi"] \\ \mathcal{O}_K\langle\zeta_1, \ldots, \zeta_m\rangle \arrow[r, two heads, "\psi"'] & B \end{tikzcd}$$

where $h : \mathcal{O}_K\langle\zeta_1, \ldots, \zeta_n\rangle \to \mathcal{O}_K\langle\zeta_1, \ldots, \zeta_m\rangle$ makes it into a commutative diagram. Now, consider $\zeta'_1, \ldots, \zeta'_m$ to be elements in $\mathcal{O}_K\langle\zeta_1, \ldots, \zeta_n\rangle$ such that $\pi(\zeta'_i) = \psi(\zeta_i)$ for any $i = 1, \ldots, m$.

Then, the kernel of ψ is generated by $h(\mathfrak{J})$ together with the set $\{\zeta_i - h(\zeta'_i)\}_{i=1}^m$. In conclusion, this kernel is finitely generated. $\square$

Remark. Since B is of topologically finite type over $\mathcal{O}_K$ then it is so over A. In particular, this implies that there exists an integer t and a surjective map

$$A\langle \zeta_1, \ldots, \zeta_t \rangle \twoheadrightarrow B$$

which, together with the previous lemma, gives us an isomorphism of A-algebras

$$A\langle \zeta_1, \ldots, \zeta_t \rangle / (g_1, \ldots, g_s) \simeq B \qquad \text{for } g_1, \ldots, g_s \in A\langle \zeta_1, \ldots, \zeta_t \rangle$$

We denote by G the column vector $(g_1; \ldots; g_s) \in (A\langle \zeta_1, \ldots, \zeta_t \rangle)^s$. Let g be the composition:

$$g : A\langle \zeta_1, \ldots, \zeta_t \rangle \longrightarrow B \longrightarrow D$$

and $\tilde{V}$ be the map $\tilde{V} : \tilde{A}_t \to \tilde{B} \to \tilde{D}$ where $A_t = A\langle \zeta_1, \ldots, \zeta_t \rangle$.

We notice that, in the hypothesis of Theorem 4.2.2.B, the map $\tilde{C} \twoheadrightarrow \tilde{D}$ is surjective; this implies that $C \twoheadrightarrow D$ is surjective and, therefore, $D = C/\mathfrak{I}$ for some ideal $\mathfrak{I}$.

Lemma 4.2.4. *There exists a map $V : A_t \to C$ lifting $\tilde{V}$ such that $V \equiv g \mod \mathfrak{I}$.*

$$\begin{tikzcd}
A_t \arrow[r] \arrow[d] \arrow[rr, bend left, "g"] & B \arrow[r] \arrow[d] & D = C/\mathfrak{I} \arrow[d] \\
\tilde{A}_t \arrow[r] \arrow[rr, bend right, "\tilde{V}"'] & \tilde{B} \arrow[r] & \tilde{D}
\end{tikzcd}$$

Proof. Observe that we are in the following situation:

$$\begin{tikzcd}
D & B \arrow[l] & & D & B \arrow[l] \\
C \arrow[u, two heads] & A_t \arrow[u, two heads] \arrow[ul, "g"'] & {} \arrow[r, squiggly, "\substack{\text{Find } g' \text{ such that} \\ g' \equiv g \mod \mathfrak{I}}"] & C \arrow[u, two heads] & A_t \arrow[u, two heads] \arrow[ul, "g"'] \arrow[l, dashed, "g'"]
\end{tikzcd}$$

Where g' is defined in the following way: consider the vector $\zeta = (\zeta_1, \ldots, \zeta_t)$ and set $g'(\zeta)$ to be any lift of $g(\zeta)$ (i.e., $g'(\zeta) \equiv g(\zeta) \mod \mathfrak{I}$); then extend to all A_t using the map $A \to C$ defined in $(*')$.

In the same way one define a morphism $V' : A_t \to C$ lifting $\tilde{V}$. Now observe that

$$V'(\zeta) - g'(\zeta) \in (\mathfrak{p} + \mathfrak{I})^t \subseteq C^t$$

because of the commutativity of the diagram:

$$
\begin{tikzcd}
& & & C \arrow[dl, two heads] \arrow[ddd] \\
A_t \arrow[r] \arrow[d] \arrow[rr, bend left, "g"] \arrow[urrr, bend left, "g'"] & B \arrow[r] \arrow[d] & D \arrow[d] & \\
\tilde{A}_t \arrow[r] \arrow[rr, bend right, "\tilde{V}"'] \arrow[drrr, bend right, "\tilde{V}'"'] & \tilde{B} \arrow[r] & \tilde{D} & \\
& & & \tilde{C} \arrow[ul, two heads]
\end{tikzcd}
$$

Now it suffices to consider $a \in \mathfrak{p}^t \subseteq C^t$ and $b \in \mathfrak{I}^t \subseteq C^t$ such that

$$V'(\zeta) - g'(\zeta) = a - b$$

Set $d = V'(\zeta) - a = g'(\zeta) - b$ and clearly $\tilde{d} = \tilde{V}$ and $d \equiv g \mod \mathfrak{I}$. The desired map V now is the unique homomorphism $A_t \to C$ such that $V(\zeta) = d$. □

The homomorphism V we have just found is the first approximation of the lifting we were looking for. The idea is to consider a sequence of approximations tending to the desired lift.

Lemma 4.2.5 ([Col3, Lemma A-3])**.** *There exist $t \times s$ matrix N and $s \times s$ matrices M and Q over A_t such that*

$$G(\zeta + NG) = G^T MG + QG$$

where the entries of Q are in $\mathfrak{p}A_t$.

We set $V_0 = V$ and we define recursively V_k by

$$V_{k+1}(\zeta) = V_k(\zeta) + N(V_k(\zeta))G(V_k(\zeta))$$

Since $V_{k+1}(\zeta) \in C^s$, it determines a unique homomorphism $V_{k+1} : A_t \to C$. Applying Lemma 4.2.5, we immediately see that $V_{k+1}(\zeta) - V_k(\zeta) \to 0$ and therefore (by the completeness of Tate algebras), the sequence $\{V_k\}$ converges to the desired lifting. □

So far we have proved Theorem 4.2.2.B; Theorem 4.2.2.A follows taking $A = A_0(W)$, $B = A_0(X)$, $C = A_0(Y)$ and $D = A_0(Z)$.

Remark. This is enough since, if X is a reduced affinoid, then $A_0(X)$ is a Tate $\mathcal{O}_K$-algebra ([Col3, Lemma A-1.5]).

4.3 Frobenius Endomorphisms

Suppose now that S is a scheme over a field F and let σ be an automorphism of F. In the canonical way, we can consider the scheme S^σ obtained from S with the technique of base change:

$$\begin{tikzcd} S^\sigma = S \times_{\mathrm{Spec}(F)} \mathrm{Spec}(F^\sigma) \arrow[r, "\sigma"] \arrow[d] & S \arrow[d] \\ \mathrm{Spec}(F^\sigma) \arrow[r, "\mathrm{Spec}(\sigma)"'] & \mathrm{Spec}(F) \end{tikzcd}$$

If f is a form on S, we denote by f^σ its pullback via σ.

If X is an affinoid over $F = K$ and σ is an automorphism of K, then we consider

$$S = \mathrm{Spec}(A(X))$$

(which is a scheme over K). We define X^σ to be the affinoid characterized by

$$S^\sigma = \mathrm{Spec}(A(X^\sigma))$$

Next, we consider the case $F = \mathbb{F}_p$ and σ =Frobenius automorphism of $\mathbb{F}$:

$$\sigma : \mathbb{F}_p \longrightarrow \mathbb{F}_p$$
$$x \longrightarrow x^p$$

If S is a scheme over $\mathbb{F}_p$, then the absolute Frobenius morphism on S is $\phi : S \to S^\sigma$ which is the identity on points and the map $f \to f^\sigma$ on sections:

$$\phi^* f^\sigma = f^p \qquad \forall f \in \mathcal{O}_S(\mathcal{U})$$

In general, for any integer $n \in \mathbb{Z}_{>0}$, $\phi : S \to S^{\sigma^n}$ is characterized by $\phi^* f^{\sigma^n} = f^{p^n}$.

If S is of finite type over $\mathbb{F}_p$, then it has a finite affine covering $\{S_i\}$ such that $\{\mathcal{O}_S(S_i)\}$ are finitely generated $\mathbb{F}_p$-algebras; this implies that there exists $n \in \mathbb{Z}_{>0}$ such that $S \simeq S^{\sigma^n}$. If

$$\rho : S^{\sigma^n} \xrightarrow{\sim} S$$

is such an isomorphism, we call

$$\rho \circ \phi : S \longrightarrow S$$

the Frobenius Endomorphism of S.

Finally, suppose that X is a K-affinoid space and $\overline{\sigma}$ is a continuous automorphism of K which restricts to the Frobenius endomorphism of $\mathbb{F} = \mathcal{O}_K/\mathfrak{p}$. Since $\tilde{X}$ is of finite type over $\mathbb{F}$, $\tilde{X}$ possesses a Frobenius endomorphism and an endomorphism of X lifting one of those is called Frobenius endomorphism of X.

Corollary 4.3.1. *Suppose that X is a reduced affinoid over K with good reduction. Then:*

1. *X possesses a Frobenius endomorphism.*

2. *There is a morphism from X to $X^{\overline{\sigma}}$ lifting the Frobenius morphism $\tilde{X} \to \tilde{X}^{\sigma}$.*

3. *$X \simeq X^{\overline{\sigma}^n}$ for some positive integer n.*

Let now X be a reduced affinoid over K with good reduction.

$$\begin{tikzcd}
\tilde{X}_{\mathbb{F}^{\mathrm{alg}}} = \tilde{X} \times_{\mathrm{Spec}(F)} \mathrm{Spec}(F^{\mathrm{alg}}) \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\
\mathrm{Spec}(F^{\mathrm{alg}}) \arrow[r] & \mathrm{Spec}(F)
\end{tikzcd}$$

$\tilde{X}_{\mathbb{F}^{\mathrm{alg}}}$ is a scheme over $\mathbb{F}^{\mathrm{alg}}$. If $\mathcal{U}$ is a residue class in X, then $\tilde{\mathcal{U}}$ is in $\tilde{X}(\mathbb{F}^{\mathrm{alg}})$ ($\mathbb{F}^{\mathrm{alg}}$ does not have finite extensions) and therefore, $\tilde{\mathcal{U}}$ is defined over some finite extension of $\mathbb{F}$. In particular, this implies that there is an integer $m \in \mathbb{Z}_{>0}$ such that

$$\tilde{\phi}(\tilde{\mathcal{U}}) = \tilde{\mathcal{U}}$$

Lemma 4.3.2. *For each residue class $\mathcal{U}$ of X, there exist some $m \in \mathbb{Z}_{>0}$ and some $\xi \in \mathcal{U}$ such that:*

$$\phi^m(\xi) = \xi$$

The point ξ is called a Teichmüller point.

Sketch of proof. Since $\mathcal{U}$ reduces to one point in $\tilde{X}$, $\mathcal{U}$ is isomorphic to the open ball

$$\mathbb{B}^d(0, 1_-) = \{(z_1, \ldots, z_d) \in K^d \mid |z_i| < 1 \ \forall i\}$$

where d is the dimension of $\tilde{X}$ ([Ber1, Proposition 1.1.1]).

Now we apply Lemma 3.0 in [Dwo] which tells us that ϕ^m has a unique fixed point in $\mathbb{B}^d(0, 1_-)$. The proof consists in showing that, given any $x \in \mathcal{U}$, the sequence $\{\phi^{mn}(x)\}_n$ is convergent and the limit does not depend on the choice of the point x: increasing n we can make $\phi^{mn}(x)$ arbitrarily close to $\phi^{mn}(y)$ for any other $y \in \mathcal{U}$.

Finally the Teichmüller point is defined to be

$$\xi = \lim_{n\to+\infty} \phi^{mn}(x) \qquad \text{for any } x \in \mathcal{U}$$

□

4.4 Differentials

In this section we give some details about the theory of differentials on rigid spaces. The main references are [MW, §4], [VdP] and [FvdP, §3.6].

Let A be an affinoid K-algebra and M a finitely generated A-module. A derivation for A is a K-linear map $D : A \to M$ such that $D(ab) = aD(b) + bD(a)$. Theorem 4.1 in [MW] guarantees the existence of a module of differentials $\Omega_{A/K}$ with a derivation

$$d : A \to \Omega_{A/K}$$

satisfying the following universal property: "*if E is another A-module with a derivation $\lambda : A \to E$, then there exists a unique A-linear differential homomorphisms $\Omega_{A/K} \to E$*".

Similarly, if X is an affinoid K-space and $A = A(X)$, we have a module of rigid differentials $\Omega_{X/K}$ with a standard derivation map

$$d : A(X) \longrightarrow \Omega_{X/K}$$

We define $\Omega^i_{X/K}$ as the i-th exterior power of $\Omega_{X/K}$:

$$\Omega^i_{X/K} = \Omega^{\otimes i}_{X/K}/I$$

where I is the ideal generated by the objects $\omega_{j_1} \otimes \ldots \otimes \omega_{j_i}$ where $\omega_{j_l} = \omega_{j_k}$ for some $l \neq k$. The derivation map d extends naturally to a derivation of the complex $\Omega^\bullet_{X/K}$. If Y is any rigid space over K, we can construct a natural complex of rigid differentials $(\Omega^\bullet_{Y/K}, d)$ on Y gluing together the differential sheaves of the affinoids covering Y ([FvdP, §4.4]).

Definition. A closed differential on Y is an element $\omega \in \mathrm{H}^0(Y, \Omega^\bullet_{Y/K})$.

Let X be a connected affinoid K-space with good reduction. Consider the diagonal $\tilde{\Delta}$ in $\tilde{X} \times \tilde{X}$.

Remark. $D = \mathrm{red}^{-1}\tilde{\Delta}$ has the structure of rigid space: since $\tilde{X}$ is an affine scheme over $\mathbb{F}$, then it is separated and, therefore, $\tilde{\Delta}$ is affine. ([Liu, §3.3.1]). Now we can apply [Ber1, 0.2.2.1].

Denote by $A(D)$ the ring of rigid analytic functions on D and consider two projections

$$\rho_1, \rho_2 : D \to X$$

Proposition 4.4.1 ([Col3, Proposition 1.2])**.** *If ω is a closed differential on X, then*

$$\rho_1^*\omega - \rho_2^*\omega \in dA(D)$$

Corollary 4.4.2. *Suppose that $f_1, f_2 : X' \to X$ are two morphisms of reduced connected affinoid spaces over K with good reduction such that $\tilde{f}_1 = \tilde{f}_2$ and consider a closed one form ω on X. Then, the following hold:*

1) $f_1^*\omega - f_2^*\omega \in dA(X')$.

2) *If λ is a function on $X(\mathbb{C}_p)$ which is analytic on each residue class of X and such that $d\lambda = \omega$, then $f_1^*\lambda - f_2^*\lambda \in A(X')$.*

Proof. Point **1** follows directly from the proposition: if $f = (f_1, f_2) : X' \to X \times X$, we immediately notice that $f(X') \subseteq D$ (the two reductions are equal, i.e., $\tilde{f}_1 = \tilde{f}_2$)

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[dr, bend right, "f_i"'] & D \arrow[d, shift right, "\rho_1"'] \arrow[d, shift left, "\rho_2"] \\ & X \end{tikzcd}$$

This implies that

$$f_1^*\omega - f_2^*\omega = f^*\left(\rho_1^*\omega - \rho_2^*\omega\right)$$

Let's now prove point **2**. Consider a function $F \in A(D)$ such that $dF = \rho_1^*\omega - \rho_2^*\omega$ whose existence is given by Proposition 4.4.1. Since F is now constant on the diagonal D, we may assume that $F = 0$ on D.

If $\mathcal{U}$ is a residue disk in X, then $\rho_1^*\lambda - \rho_2^*\lambda$ is analytic on $\mathcal{U} \times \mathcal{U}$ and vanishes on $\Delta \cap \mathcal{U} \times \mathcal{U}$. Further,

$$d(\rho_1^*\lambda - \rho_2^*\lambda) = \rho_1^*\omega - \rho_2^*\omega$$

This means that $F = \rho_1^*\lambda - \rho_2^*\lambda$ on $\mathcal{U} \times \mathcal{U}$ and, since D is a union of $\mathcal{U} \times \mathcal{U}$, we conclude that

$$\rho_1^*\lambda - \rho_2^*\lambda = F \in A(D)$$

Now the Corollary follows applying the pullback via f. $\square$

We have now most of the ingredients that we need to construct our theory of integration:

- Lifting of Frobenius morphisms.
- Teichmüller points.
- Differentials on affinoids.

The last ingredient we have to introduce is a suitable covering of our space X.

Let $\mathfrak{X}$ be a proper scheme of finite type over $\mathcal{O}_K$ and $\tilde{\mathfrak{X}}$ be its special fiber over $\mathbb{F}$. If $Y \subseteq \tilde{\mathfrak{X}}$ is an affine open set, then $W = \mathrm{red}^{-1}Y \subseteq \mathfrak{X}_K$ has the structure of rigid space. If $\mathfrak{X}$ is smooth, then W has good reduction and $\tilde{W} = Y$. In this case we call W a Zariski affinoid open subset of $\mathfrak{X}_K$.

Definition. If $\mathfrak{X}_K$ is smooth, then a differential of the second kind on $\mathfrak{X}_K$ is an element $\omega \in \Omega_{\mathfrak{X}_K/K}(\mathcal{U})$, for some dense open subset $\mathcal{U}$ of $\mathfrak{X}_K$, such that:

(i) $d\omega = 0$.

(ii) There exists a Zariski open covering $\mathcal{C}$ of $\mathfrak{X}_K$ such that for every $\mathcal{W} \in \mathcal{C}$,

$$\rho^{\mathcal{U}}_{\mathcal{U}\cap\mathcal{W}}(\omega) \in \rho^{\mathcal{W}}_{\mathcal{U}\cap\mathcal{W}}(\Omega_{\mathfrak{X}_K/K}(\mathcal{W})) + d\mathcal{O}_{\mathfrak{X}_K}(\mathcal{U}\cap\mathcal{W})$$

where ρ denotes the restriction map.

In other words, the covering we want is such that

$$\omega|_{\mathcal{W}} = \omega_{\mathcal{W}} + d\left(f|_{\mathcal{W}}\right) \qquad \omega_{\mathcal{W}} \in \Omega_{\mathcal{W}/\mathbb{C}_p}\ , \quad f \in \mathcal{O}(\mathfrak{X}_K)$$

Question. *How do Zariski open subsets behave under the action of Frobenius endomorphisms?*

Definition. If $\mathfrak{X}$ is proper and smooth over $\mathcal{O}_K$ we say that Frobenius acts properly on $\mathfrak{X}$ if, for each Frobenius endomorphism ϕ of $\tilde{\mathfrak{X}}$ there exists a polynomial $Z(T) \in \mathbb{C}_p[T]$ such that

(i) No root of $Z(T)$ in $\mathbb{C}_p$ is a root of unity.

(ii) For each Zariski affinoid open $\mathcal{W}$ of $\mathfrak{X}$ such that $\phi\tilde{\mathcal{W}} = \tilde{\mathcal{W}}$, there exists a lifting $\overline{\phi} : \mathcal{W} \to \mathcal{W}$ of $\phi|_{\mathcal{W}}$ such that $Z(\overline{\phi}^*)\omega \in dA(\mathcal{W})$ for each differential of the second kind ω on $\mathfrak{X}_K$ regular on $\mathcal{W}$.

Theorem 4.4.3 ([Col3, Theorem 1.4])**.** *If K is discretely valued and $\mathfrak{X}$ is a smooth projective scheme over $\mathcal{O}_K$, then any Frobenius endomorphism acts properly on $\mathfrak{X}$.*

4.5 p-adic Integrals

In this section we describe how to effectively integrate differentials of the second kind on affinoids. The main reference is [Col3, §2].

Let $\mathfrak{X}$ be a smooth proper connected scheme of finite type over $\mathcal{O}_K$ and ω a differential of the second kind on $\mathfrak{X}_K$; we know that Frobenius acts properly on $\mathfrak{X}$ (Theorem 4.4.3).

Now we want to consider $\mathcal{D}$, the collection of Zariski open subsets X in $\mathfrak{X}_K$ such that

$$\omega|_X - dg_X = \omega_X \in \Omega_K(X)$$

for some $g_X \in K(X)$, the function field of X.

Lemma 4.5.1. *$\mathcal{D}$ is, in fact, a covering.*

Proof. This is straightforward using the fact that ω is of II kind: by the definition, there exists a Zariski open covering $\mathcal{C}$ of $\mathfrak{X}_K$ such that for every $X \in \mathcal{C}$

$$\rho_X^{\mathfrak{X}_K}(\omega) \in \rho_X^X(\Omega_{\mathfrak{X}_K/K}(X)) + d\mathcal{O}_{\mathfrak{X}_K}(X)$$

and therefore $\mathcal{C} \subseteq \mathcal{D}$ and the latter is a covering. □

Now we consider $(\omega)_\infty$ the set of poles of ω and we write

$$\mathfrak{X}'_K = \mathfrak{X}_K - (\omega)_\infty$$

Let ϕ be a power of the Frobenius endomorphism of $\tilde{\mathfrak{X}}$ and consider $\mathcal{D}' \subseteq \mathcal{D}$ consisting of those X such that

$$\phi\tilde{X} = \tilde{X}$$

Lemma 4.5.2. *$\mathcal{D}'$ is a covering.*

Proof. $\mathfrak{X}$ is smooth, then every Zariski open affinoid X of $\mathfrak{X}_K$ have good reduction and $\tilde{X}$ is affine. This, in particular, means that we can cover $\tilde{\mathfrak{X}}$ by affines $\tilde{X}$. But now we know that $\mathfrak{X}_K$ is quasi compact since $\mathfrak{X}$ is of finite type (Remarks at the end of Section 3.7 based on [Ber1, 0.2.4]). Thus, we can extract from $\mathcal{D}$ a finite subcovering $\mathcal{D}'$ which implies that $\tilde{\mathcal{D}}'$ is also finite. Now $\tilde{\mathfrak{X}}$ is covered by a finite number of affine subspaces. Hence, we can choose a sufficiently large power of ϕ fixing all the $\tilde{X}$'s. In conclusion, modulo replacing ϕ with one of its power, $\mathcal{D}'$ is a covering. □

Choose now a polynomial $Z(T) \in \mathbb{C}_p[T]$ associated to $\mathfrak{X}$ and to ϕ as in the definition of proper action of Frobenius.

If $X \in \mathcal{D}'$, let $\bar{\phi} = \bar{\phi}_X$ be a lifting of the restriction $\phi|_{\tilde{X}}$ to $\tilde{X}$. It follows that

$$Z(\bar{\phi}^*)\omega_X \in dA(X)$$

Theorem 4.5.3 (Coleman)**.** *With the notation as above, there exists a locally analytic function f_ω on $\mathfrak{X}'_K(\mathbb{C}_p)$ unique up to additive constant such that*

I) $df_\omega = \omega$.

II) $\forall X \in \mathcal{D}'$ *there exists* $g_X \in K(X)$ *such that* $(f_\omega - g_X)|_X$ *extends to a locally analytic function on* X *and*

$$Z(\bar{\phi}_X^*)(f_\omega - g_X) \in A(X)$$

Further, f_ω is independent of all the choices (the covering $\mathcal{D}'$, the polynomial Z and the power of the Frobenius endomorphism of $\tilde{\mathfrak{X}}$ fixing the elements of $\mathcal{D}'$).

For ω and f_ω as above and for two points $P, Q \in \mathfrak{X}_K(\mathbb{C}_p)$, we define

$$\int_P^Q \omega = f_\omega(Q) - f_\omega(P)$$

the integral of ω from P to Q.

Proof of Theorem 4.5.3

The proof of the Theorem can be articulated into 4 steps: we'll first show that the conditions $df_X = \omega_X$ and $Z(\overline{\phi}_X^*)(f_X) \in A(X)$ determine a unique function f_X which is locally analytic on X. Then, we will show that $f_X + g_X$ and $f_{X'} + g_{X'}$ agree on the intersection $X \cap X'$ and, finally, we will see how to glue together all these $f_X + g_X$.

Step I: Determining $(f_\omega - g_X)|_X$

In this first step we focus our attention on one Zariski open affinoid.

Theorem 4.5.4. *Let X be a smooth connected affinoid over K with good reduction $\tilde{X}$. Let ω be a closed one form on X and ϕ be a Frobenius endomorphism of X. Suppose $Z(T)$ is a polynomial over $\mathbb{C}_p$ such that*

$$Z(\phi^*)\omega \in dA(X)$$

and such that no root of Z is a root of unity.

Then, there exists a unique (up to additive constant) locally analytic function f_ω on $X(\mathbb{C}_p)$ such that

(a) $df_\omega = \omega$.

(b) $Z(\phi^*)f_\omega \in A(X)$.

Proof. Modulo multiplying Z by a constant, we may assume

$$Z(T) = T^n + a_{n-1}T^{n-1} + \ldots + a_0 \in \mathbb{C}_p$$

hence,

$$Z(\phi^*)(h) = \sum_k a_k(h \circ \phi^k)$$

Let $\Omega = \left(\omega, \phi^*\omega, \ldots, (\phi^{n-1})^*\right)$ regarded as a column vector and consider the matrix M over $\mathbb{C}_p$ defined as the companion matrix of Z:

$$M = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix}$$

Then $\phi^*\Omega \equiv M\Omega \mod (dA(X))^n$ - by hypothesis $Z(\phi^*)\omega \in dA(X)$.

(Case $\deg P = 1$*).* In the special case $Z(T) = T - a$, we have $\Omega = \omega$ and $M = a$ which shows that $(\phi^* - a)\omega = \phi^*\omega - a\omega \in dA(X)$.

(Case $\deg P = 2$*).* In case $Z(T) = T^2 + aT + b$, we have

$$\Omega = \begin{pmatrix} \omega \\ \phi^*\omega \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}$$

Thus,

$$M\Omega = \begin{pmatrix} \phi^*\omega \\ -a\omega - b\omega \end{pmatrix} \qquad \text{and} \qquad \phi^*\Omega = \begin{pmatrix} \phi^*\omega \\ (\phi^*)^2\,\omega \end{pmatrix}$$

Therefore, we have equality on the first component while on the second component we use the definition of Z.

Hence, Theorem 4.5.4 is equivalent to:

Claim. *There exists a locally analytic function $f : X(\mathbb{C}_p) \to \mathbb{C}_p^n$ unique up to additive constant such that*

(a') $dF = \Omega$.

(b') $\phi^* F = MF \mod (A(X))^n$

Indeed, $F = (f_\omega, f_\omega \circ \phi, \dots, f_\omega \circ \phi^{n-1})$.

(Uniqueness) Suppose we have two solutions; their difference would be a function

$$G = (g_1, \dots, g_n)$$

locally analytic satisfying $dG = 0$ and $\phi^* G \equiv MG \mod (A(X))^n$. Observe that the former implies that G is locally constant and therefore, $\phi^* G - MG$ is locally constant too. Since X is connected, we conclude that

$$\phi^* G - MG = \mathrm{C} \qquad \text{some } \mathrm{C} \in \mathbb{C}_p$$

We would like to conclude that $G = (1 - M)^{-1}\mathrm{C}$. By hypothesis, we know that 1 is not a root of Z; then, $1 - M \in \mathrm{GL}_n(\mathbb{C}_p)$ and so

$$\left(\phi^*\right)^k G - M^k G = \left(1 - M^k\right)(1 - M)^{-1}\mathrm{C} \tag{1}$$

Now let $\mathcal{U}$ be a residue disk in X and $\xi_{\mathcal{U}}$ be its Teichmüller point with respect to ϕ (Lemma 4.3.2):

$$\phi^m(\xi_{\mathcal{U}}) = \xi_{\mathcal{U}}$$

Now fix $k = m$ and evaluate equation (1) at $\xi_{\mathcal{U}}$:

$$\underbrace{\phi^* G(\xi_{\mathcal{U}}) - M^k G(\xi_{\mathcal{U}})}_{\overset{=}{G(\xi_{\mathcal{U}}) - M^k G(\xi_{\mathcal{U}})}} = \left(1 - M^k\right)(1 - M)^{-1}\mathrm{C}$$

which yields

$$\left(1 - M^k\right) G(\xi_{\mathcal{U}}) = \left(1 - M^k\right)(1 - M)^{-1}\mathrm{C} \implies G(\xi_{\mathcal{U}}) = (1 - M)^{-1}\mathrm{C} \tag{2}$$

Observe that, in fact, also $1 - M^k \in \mathrm{GL}_n(\mathbb{C}_p)$.

Since G is locally constant and, for every $x \in \mathcal{U}$, $\phi^{mn}(x) \xrightarrow[n\to+\infty]{} \xi_{\mathcal{U}}$, we can find an integer k_x such that

$$G(\phi^{mk_x}(x)) = G(\xi_{\mathcal{U}})$$

Using again equation (1), we get

$$\begin{aligned}\left(1 - M^{mk_x}\right)(1-M)^{-1}\mathrm{C} = \left(\phi^*\right)^{mk_x} G(x) - M^{mk_x} G(x) = \\ = G(\xi_{\mathcal{U}}) - M^{mk_x} G(x) \overset{(2)}{=} (1-M)^{-1}\mathrm{C} - M^{mk_x} G(x)\end{aligned}$$

Thus,

$$M^{mk_x} G(x) = M^{mk_x}(1-M)^{-1}\mathrm{C} \implies M^{mk_x}\left(G(x) - (1-M)^{-1}\mathrm{C}\right) = 0$$

Regarding M^{mk_x} as a linear map on $\mathbb{C}_p^n$, we see that

$$G(x) - (1-M)^{-1}\mathrm{C} \in \ker M^{mk_x}$$

Remark. $\mathbb{C}_p$ is algebraically closed and, therefore, $\phi^m : \mathcal{U} \to \mathcal{U}$ is surjective and, for any positive integer r, there exists y_r such that $\phi^{mr}(y_r) = x$.

Using this remark we re-read equation (1):

$$G(x) - (1-M)^{-1}\mathrm{C} = M^{mr}\left(G(y_r)(1-M)^{-1}\mathrm{C}\right)$$

which means that $G(x) - (1-M)^{-1}\mathrm{C} \in \mathrm{Im}(M^{mr})$. We conclude that

$$G(x) - (1-M)^{-1}\mathrm{C} \in \ker(M^{mk_x}) \cap \left(\bigcap_{r\geqslant 0}\right) \mathrm{Im}(M^{mr}) = \{0\}$$

and therefore $G(x) = (1-M)^{-1}\mathrm{C}$ as we wanted. In particular, G is constant and, therefore, F is unique.

(Existence) Suppose

$$\phi\Omega = M\Omega + dh \qquad \text{some } h \in A(X)^n$$

Again $\mathcal{U}$ will be a residue class of X and $\xi_{\mathcal{U}}$ its Teichmüller point with respect to ϕ. Let m be the minimum integer such that $\phi^m(\xi_{\mathcal{U}}) = \xi_{\mathcal{U}}$.

Since Ω is closed and $\mathcal{U}$ is an open ball , there exists a unique function $F_{\mathcal{U}}$ analytic on $\mathcal{U}$ such that $dF_{\mathcal{U}} = \Omega|_{\mathcal{U}}$ and (we fix this value at $\xi_{\mathcal{U}}$)

$$F_{\mathcal{U}}(\xi_{\mathcal{U}}) = (1-M^m)^{-1}\sum_{i=0}^{m-1} M^i h\left(\phi^{m-(i+1)}(\xi_{\mathcal{U}})\right)$$

Let F be the unique locally analytic function defined by $F|_{\mathcal{U}} = F_{\mathcal{U}}$ (recall that the residue disks are disjoint). Clearly F satisfies $dF = \Omega$; further,

$$\begin{aligned}\left(\phi^* F\right)(\xi_{\mathcal{U}}) = (F\circ\phi)(\xi_{\mathcal{U}}) = (1-M^m)^{-1}\sum_{i=0}^{m-1} M^i h\left(\phi^{m-i}(\xi_{\mathcal{U}})\right) &= \\ = M(1-M^m)^{-1}\sum_{j=-1}^{m-2} M^j h\left(\phi^{m-(j+1)}(\xi_{\mathcal{U}})\right) &=\end{aligned}$$

$$= M(1-M^m)^{-1}\sum_{j=0}^{m-1} M^j h\left(\phi^{m-(j+1)}(\xi_{\mathcal{U}})\right) - M(1-M^m)^{-1}M^{m-1}h\left(\phi^{m-m}(\xi_{\mathcal{U}})\right) +$$

$$+ M(1-M^m)^{-1}M^{-1}h\left(\phi^m(\xi_{\mathcal{U}})\right) =$$

$$= MF(\xi_{\mathcal{U}}) + (1-M^m)^{-1}h\left(\phi^m(\xi_{\mathcal{U}})\right) - M^m(1-M^m)^{-1}h\left(\xi_{\mathcal{U}}\right) =$$

$$= MF(\xi_{\mathcal{U}}) + (1-M^m)^{-1}\left(h\left(\phi^m(\xi_{\mathcal{U}})\right) - M^m h(\xi_{\mathcal{U}})\right) =$$

$$= MF(\xi_{\mathcal{U}}) + (1-M^m)^{-1}\left(h(\xi_{\mathcal{U}}) - M^m h(\xi_{\mathcal{U}})\right) =$$

$$= MF(\xi_{\mathcal{U}}) - (1-M^m)^{-1}(1-M^m)h(\xi_{\mathcal{U}}) = MF(\xi_{\mathcal{U}}) - h(\xi_{\mathcal{U}})$$

This means that

$$\left(\phi^* F - MF\right)(\xi_{\mathcal{U}}) = h(\xi_{\mathcal{U}})$$

and

$$d\left(\phi^* F - MF\right) = \Omega^* - M\Omega = dh \qquad \text{on } \mathcal{U}$$

In conclusion, $\phi^* F - MF = h$ on each residue disk $\mathcal{U} \subseteq X$ and, hence, on all X.

Corollary 4.5.5. *The function f_ω is analytic on each residue class of X.*

This concludes the first step of the proof of Theorem 4.5.3. □

Step II: Independence of f_ω of all choices

Lemma 4.5.6. *The function f_ω depends modulo constants only on ω and not on the choice of P.*

Proof. Consider the vector space

$$V_\phi = \langle (\phi^*)^n \omega \mod dA(X) \rangle_{n \geqslant 0}$$

Because of the hypothesis $Z(\phi^*\omega) \in dA(X)$ of Theorem 4.5.4, we know that V_ϕ is finite dimensional. If P_ϕ is the minimal polynomial of ϕ acting on V_ϕ, then $P_\phi(T) \mid Z(T)$ which implies that if $f'_\omega = \omega$ and $P_\phi(\phi^*)f'_\omega \in A(X)$, then $Z(\phi^*)f'_\omega \in A(X)$ and so $f_\omega - f'_\omega$ is constant by the uniqueness of f_ω. □

Lemma 4.5.7. *With the notation introduced before, if ω' is another closed one form on X such that $Z(\phi^*)\omega' \in dA(X)$, then*

i. $f_{\omega+\omega'} = f_\omega + f_{\omega'} + \mathrm{C}$ *for* $\mathrm{C} \in \mathbb{C}_p$.

ii. *If ω is exact, $f_\omega \in A(X)$.*

Lemma 4.5.8 ([Col3, Corollary 2.1.d])**.** *The function f_ω is independent (modulo constants) of the choice of ϕ.*

Sketch of Proof. It suffices to show that, replacing ϕ by ϕ^t, we do not change f_ω. It can be verified that

$$P_{\phi^t}(T^t) = \prod_{\zeta^t=1} P_\phi(\zeta T)$$

Thus, the fact that $P_\phi(\phi^*)(f_\omega) \in A(X)$ implies that $P_{\phi^t}((\phi^t)^*)f(\omega)$ and now the result follows from the uniqueness of f_ω. □

Lemma 4.5.9 ([Col3, Corollary 2.1.e])**.** *Let σ be a continuous automorphism of $\mathbb{C}_p$ and ω^σ the pullback of ω to X^σ. Let f^σ_ω be the function on $X^\sigma(\mathbb{C}_p)$ defined by*

$$f^\sigma_\omega(x) = \sigma f_\omega(\sigma^{-1}(x))$$

then

(a) *ω^σ satisfies the hypothesis of Theorem 4.5.4 on X^σ.*

(b) *If f_{ω^σ} is a solution of Theorem 4.5.4, then $f^\sigma_\omega - f_{\omega^\sigma}$ is constant. In particular, if σ fixes K, then $f^\sigma_\omega - f_\omega$ is constant.*

Step III: Comparing functions on $X \cap X'$

So far we have focused on one affinoid. Let's enlarge the picture recovering the notation of 4.5.3. In the previous steps we have seen that on each $X \in \mathcal{D}'$ there exists a locally analytic function f_X unique up to additive constant such that

(I) $df_X = \omega_X = \omega|_X - dg_X$

(II) $Z(\overline{\phi}^*)f_X \in A(X)$

Now we set $h_X = f_X + g_X$; this is a function on $X \backslash (X \cap (\omega)_\infty)$.

Lemma 4.5.10. *h_X is independent of all the choices of g and f up to additive constants.*

Proof. Take $g'_X \in K(X)$ such that $\omega|_X - dg'_X = \omega'_X \in \Omega_{X/K}$, then

$$\omega'_X - \omega_X = dg_X - dg'_X = d(g_x - g'_X) \Longrightarrow g_X - g'_X \in A(X)$$

Now choose f'_X to be a solution of $df'_X = \omega'_X$ and $Z(\overline{\phi}^*)f'_X \in A(X)$ (a solution for Theorem 4.5.4), then

$$f'_X - f_X = g_X - g'_X$$

modulo constants (uniqueness applied to $\omega_X - \omega'_X$). Thus,

$$f_X + g_X = f'_X + g'_X + \mathrm{C}$$

and this concludes the proof. □

Step IV: Gluing the integrals

We claim that $h_X - h_{X'}$ is constant on $X \cap X'$ for $X, X' \in \mathcal{D}'$. First note that $X \cap X' \in \mathcal{D}$; hence, it suffices to prove the claim in case $X' \subseteq X$. In this case we can take $g_X = g_{X'}$. Thus, $\omega_{X'}$ is the restriction of ω_X to X' and if we restrict f_{ω_X} to X' we get a solution to Theorem 4.5.4 for X'. In conclusion $h_{X'} = h_X|_{X'}$ (modulo constants).

This means that, in order to glue the integrals, we only have to adjust the constants so that h_X and $h_{X'}$ agree on the intersection.

Finally, as mentioned before, we define

$$\int_P^Q \omega \overset{\text{def}}{:=} f_\omega(Q) - f_\omega(P)$$

This concludes the proof of the main Theorem. □

Properties of the Integral

Proposition 4.5.11. *Suppose that ω and ω' are two differentials of the second kind on $\mathfrak{X}_K$. Then,*

(a) *Additivity on forms:* $\lambda_1 \int_P^Q \omega_1 + \lambda_2 \int_P^Q \omega_2 = \int_P^Q (\lambda_1\omega_1 + \lambda_2\omega_2)$.

(b) *Fundamental Theorem of Calculus: if $dg = \omega$ for a meromorphic function $g \in \mathbb{C}_p(\mathfrak{X})$, then,* $\int_P^Q \omega = g(Q) - g(P)$.

(c) *Change of variables: if $\tau : \mathfrak{X}' \to \mathfrak{X}$ is a morphism of smooth proper schemes over $\mathcal{O}_K$ on which Frobenius acts properly, then,*

$$\int_P^Q \tau^*(\omega) = \int_{\tau(P)}^{\tau(Q)} \omega$$

if $\tau(P), \tau(Q) \notin (\omega)_\infty$.

In his article [Col3], Coleman proved that the change of variable formula holds even if we soften and relax some hypothesis. In order to be able to state this result, we need a small digression on Albanese variety.

Given a variety X which is smooth over K, one can associate to it, in a functorial way, its Albanese Variety. As a reference one can look at [Ser1], [Lan1, §II.3] or [Colm, §I.5].

In general, the Albanese variety associated to X is a pair $(\mathrm{Alb}(X), f)$ consisting of an abelian variety $\mathrm{Alb}(X)$ and a natural map

$$f : X \longrightarrow \mathrm{Alb}(X)$$

such that:

(1) There exists a non-negative integer $n \in \mathbb{Z}_{\geqslant 0}$ such that the map

$$F : \underbrace{X \times X \times \ldots \times X}_{n\text{-times}} \longrightarrow \mathrm{Alb}(X)$$

equal to the sum of f with itself n-times, is generically surjective.

(2) For every rational map $g : X \to Y$ with Y an abelian variety, there exists a homomorphism $g_* : A \to B$ and a constant $\mathrm{C} \in Y$ such that $g = g_* f + \mathrm{C}$.

Example. If X is proper and smooth over $\mathbb{C}$, then

$$\mathrm{Alb}(X) \simeq \frac{\mathrm{H}^1_{\mathrm{dR}}(X)^*}{\mathrm{H}_1\left(X(\mathbb{C}), \mathbb{Z}\right)}$$

Observe that any point $a \in X(K)$ gives rise to a morphism

$$f_a : X \longrightarrow \mathrm{Alb}(X) \quad \text{such that} \quad f_a(a) = 0$$

A morphism of this type is called an Albanese morphism.

Proposition 4.5.12 ([Colm, Proposition I.5.3])**.** *If X is proper, then an Albanese morphism f_a induces an isomorphism*

$$H^1_{dR}(Alb(X)) \simeq H^1_{dR}(X)$$

Now we can re-state the Change of variables property:

Theorem 4.5.13 (Changes of variables)**.** *Suppose that $\mathfrak{X}$ and $\mathfrak{X}'$ are two smooth and proper schemes over $\mathcal{O}_K$ of finite type on which Frobenius acts properly. Consider a rational map*

$$f : \mathfrak{X}'_K \longrightarrow \mathfrak{X}_K$$

and let ω be a differential of the second kind on $\mathfrak{X}_K$. Then

$$\int_P^Q f^*\omega = \int_{f(P)}^{f(Q)} \omega$$

for $P, Q \in \mathfrak{X}'_K(\mathbb{C}_p)$ in the domain of regularity of f such that $f(P), f(Q) \notin (\omega)_\infty$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathfrak{X}'_K \arrow[r, "f"] \arrow[d] & \mathfrak{X}_K \arrow[d] \\
\mathfrak{X}' \arrow[d] & \mathfrak{X} \arrow[d] \\
\mathrm{Alb}(\mathfrak{X}') \arrow[r] & \mathrm{Alb}(\mathfrak{X})
\end{tikzcd}$$

where the two maps $\mathfrak{X}' \to \mathrm{Alb}(\mathfrak{X}')$ and $\mathfrak{X} \to \mathrm{Alb}(\mathfrak{X})$ are Albanese morphisms and the bottom arrow is the morphism induced functorially by f.

Recall. Observe that $\mathrm{Alb}(\mathfrak{X})$ and $\mathrm{Alb}(\mathfrak{X}')$ are the models of $\mathrm{Alb}(\mathfrak{X}_K)$ and $\mathrm{Alb}(\mathfrak{X}'_K)$.

By Proposition 4.5.12, we know that

$$\mathrm{H}^1_{\mathrm{dR}}(\mathrm{Alb}(\mathfrak{X}_K)) \simeq \mathrm{H}^1_{\mathrm{dR}}(\mathfrak{X}_K)$$

and this implies that any one form of II kind on $\mathfrak{X}_K$ comes (modulo an exact differential) from a one form of II kind on $\mathrm{Alb}(\mathfrak{X}_K)$. Now the Theorem follows from Proposition 4.5.11 $\square$

There is an important Corollary to this Theorem:

Corollary 4.5.14. *The integral $\int_P^Q \omega$ does not depend on the model $\mathfrak{X}$ of $\mathfrak{X}_K$.*

For a detailed discussion about models, one can refer to [BLR].

Chapter 5

p-adic Integrals on Curves

The general theory of integration has been developed in the previous chapter following [Col3]. Now we want to specialize our approach to curves; in particular to hyperelliptic curves.

Hystorically, Coleman did the inverse road developing first a theory of integration on $\mathbb{P}^1$ in [Col1] and then constructing the theory in higher dimensions.

The reason why we postponed the chapter on curves is twofold:

- We would like to use it as an example of the abstract theory developed before.
- Secondly, we want to approach the problem from a different point of view introducing in the picture some concrete computations.

In recent times (around 2007 – 2011), Jennifer S. Balakrishnan, Robert W. Bradshaw and Kiran S. Kedlaya have constructed explicit algorithms thanks to which it is possible to do some very concrete computations using Coleman's integrals. The motivation of their work was the possibility of exploring the various application of this theory.

In this chapter we want to present these algorithmic methods; we will mainly follow [Col1] for the first three sections while [Bal], [BaT] and [BBK] for the second part. We also want to mention a video of a seminar given by K.S. Kedlaya and R.W. Bradshaw at Clay Institute in 2007: [KB].

5.1 Preliminary Definitions

In this section we give some introductory definitions following [Col1].

Let's consider $\mathbb{C}_p$, the completed algebraic closure of $\mathbb{Q}_p$. We will denote by $\mathcal{O}$ its ring of integers and by $\mathfrak{P}$ the maximal ideal of $\mathcal{O}$.

Definition. An open affinoid subspace of $\mathbb{P}^1_{\mathbb{C}_p}$ is a set of the form

$$X = \{z \in \mathbb{P}^1_{\mathbb{C}_p} \mid |f(z)| \leqslant 1,\ f \in S\}$$

where S is a finite subset of $\mathrm{Rat}(\mathbb{C}_p)$, the set of rational functions over $\mathbb{C}_p$, containing at least one non-constant function.

One can refer to [FvdP, Chapter 2] for a discussion about $\mathbb{P}^1$ over non-archimedean fields.

We define $A(X)$ to be the set of analytic functions over X; this is the completion of the set of rational functions which are regular over X with respect to the supremum norm.

For simplicity, we will denote

$$\mathbb{B}[a,r] = \{z \in \mathbb{A}^1 \mid |z-a| \leqslant r\} \qquad \mathbb{B}^1 = \mathbb{B}[0,1]$$

$$\mathbb{B}(a,r) = \{z \in \mathbb{A}^1 \mid |z-a| \lneq r\}$$

$$A[a,r,R] = \{z \in \mathbb{A}^1 \mid r \leqslant |z-a| \leqslant R\}$$

Definition. A wide open set is a subset of $\mathbb{P}^1$ of the form

$$\mathcal{U} = \{z \in \mathbb{P}^1 \mid |f(z)| < e_f,\ f \in S\}$$

where, again, S is a finite set of rational functions over $\mathbb{C}_p$ containing at least one non-constant function and $e_f \in \{1, \infty\}$.

Example. The open balls $\mathbb{B}(a,r)$, with $a \in \mathbb{A}^1(\mathbb{C}_p)$, are wide open subsets.

If $X \subseteq \mathbb{P}^1$ is an affinoid and $\mathcal{U}$ is a wide open containing X, we say that $\mathcal{U}$ is a wide open neighborhood of X.

Notation. If $V \subseteq \mathbb{P}^1$ is an open subset, we write

$$\Omega(V) = A(V)dz \qquad \text{and} \qquad \Omega_{\mathcal{L}}(V) = \underbrace{\mathcal{L}(V)}_{\substack{\text{locally analytic} \\ \text{functions on } V}} dz$$

$$\mathrm{H}^1(V) = \Omega(V)/dA(V)$$

where $d : A(V) \to \Omega(V)$ is the canonical derivation.

It is interesting to notice that there exist canonical derivations making the following into a commutative diagram:

$$\begin{tikzcd} \mathcal{L}(V) \arrow[r, "d"] & \Omega_{\mathcal{L}}(V) \\ A(V) \arrow[r, "d"] \arrow[u, hook] & \Omega(V) \arrow[u, hook] \end{tikzcd}$$

Now, if V is an annulus about $a \in \mathbb{A}^1$, it can be proved ([FvdP, §2.2]) that every $f \in A(V)$ admits a unique expression in power series around a.

For $\omega \in \Omega(V)$ we write

$$\omega = \sum_{n \in \mathbb{Z}} \alpha_n (z-a)^n dz$$

and we define ([FvdP, §2.3]) the residue of ω at a as

$$\mathrm{Res}_a \omega = \alpha_{-1}$$

Lemma 5.1.1. $\omega \in dA(V) \iff Res_a = 0$

Proof. **(⇒)** Suppose

$$f(z) = \sum_{n \geqslant n_0} \alpha_n (z-a)^n$$

Thus,

$$df \overset{\text{has expansion}}{\rightsquigarrow} \sum_{n \geqslant n_0} n\alpha_n (z-a)^{n-1} dz$$

and, therefore, $\mathrm{Res}_a df = 0 \cdot \alpha_{-1} = 0$.

(⇐) If ω has residue 0, then we can integrate term by term obtaining a function f such that $df = \omega$. It can be proved that the convergence of the expansion of ω is inherited by the expansion of f.

□

5.2 The Logarithm

Definition. A branch of the Logarithm is a locally analytic function $l : \mathbb{C}_p^\times \longrightarrow \mathbb{C}_p^+$ such that

$$\frac{d}{dz} l(1) = 1$$

Lemma 5.2.1 ([Col1, Theorem 2.1])**.** *$l(z)$ is analytic on $\mathbb{B}(x, |x|)$ for any $x \in \mathbb{C}_p$. Further,*

$$l(x) = -\sum_{n=0}^{+\infty} \frac{(1-x)^n}{n} \qquad \textit{in } \{x \in \mathbb{C}_p \mid |x-1| < 1\}$$

Lemma 5.2.2 ([Col1, Theorem 2.2]). *Let V be an annulus about a, and suppose that $g \in A(V)^\times$. If $n \in \mathbb{C}_p$ is defined by*

$$n = Res_a \frac{dg}{g}$$

then $n \in \mathbb{Z}$ and g can be written as $g = \mathrm{c}(z-a)^n(1+h)$, where $\mathrm{c} \in \mathbb{C}_p$, $h \in A(V)$, and $|h(z)| < 1$ for all $z \in V$.

For an open subset $V \subseteq \mathbb{P}^1$, we can fix a branch of the logarithm $\mathrm{Log}(z)$. We define

$$A_{\mathrm{Log}}(V) = A(V)\left[\mathrm{Log}(f) \mid f \in A(V)^\times\right]$$

Lemma 5.2.2 has an important corollary:

Corollary 5.2.3. *If V is an annulus about a, then*

$$A_{Log}(V) = A(V)\left[Log(z-a)\right]$$

The importance of working on wide open annulus is given by the following "uniqueness principle".

Proposition 5.2.4. *Let V be a wide open annulus, and $f \in A_{Log}(V)$. If f vanishes on a non-empty open subset of V, then f vanishes identically on V .*

In particular, the previous proposition enables us to compute the cohomology of a wide open annulus: we denote by $\mathrm{H}^i_{\mathrm{Log}}(V)$ the i-th cohomology group of the complex

$$0 \longrightarrow A_{\mathrm{Log}}(V) \xrightarrow{d} \Omega^1_{\mathrm{Log}}(V) \longrightarrow 0$$

Lemma 5.2.5. *If V is a wide open annulus, then*

(i) *$H^0_{Log}(V) = \mathbb{C}_p$, i.e., if $f, f' \in A_{Log}(V)$ such that $df = df'$, then $f = f' + \mathrm{c}$, $\mathrm{c} \in \mathbb{C}_p$.*

(ii) *$H^1_{Log}(V) = 0$, i.e., $\forall \omega \in \Omega^1_{Log}(V)$, there exists $f \in A_{Log}$ such that $df = \omega$.*

The conclusion of this brief discussion is that there exists a theory of integration on wide open annulus, modulo admitting the use of Logarithms:

Definition. If $A(a,r,R) = \{z \in \mathbb{A}^1 \mid r < |x| < R\}$ is an open annulus (or disk if $r = 0$), then for $P, Q \in A(a,r,R)$, we define

$$\int_P^Q \sum_{n\in\mathbb{Z}} c_n t^n dt = c_{-1} l\left(\frac{Q}{P}\right) + \sum_{n\in\mathbb{Z}\setminus\{-1\}} \frac{c_n}{n+1}\left(Q^{n+1} - P^{n+1}\right)$$

Observe that this integral is taken over a wide *open* annulus as the denominator $n+1$ would affect the convergence on the boundary if the domain of integration was closed.

5.3 Curves with Good Reduction

Definition. A curve over $\mathcal{O}$ is a smooth proper connected scheme over $\mathcal{O}$ of relative dimension 1.

We keep the same notation adopted in Chapter 4. If X is a curve over $\mathcal{O}$, $\tilde{X}$ will be the reduction of X over $\mathcal{O}/\mathfrak{P}$ while $X_{\mathbb{C}_p}$ will denote the generic fiber of X.

Again we consider the canonical reduction map $\mathrm{red} : X_{\mathbb{C}_p} \longrightarrow \tilde{X}$

Recall. The inverse image of a point of $\tilde{X}$ is a unitary open disk in $X_{\mathbb{C}_p}$ ([Bre, 1.2.1.2]): this follows from [Ber1, Proposition 1.1.1] taking $y = (y_1, y_2) \in \tilde{X}(\mathbb{F})$; then

$$\{y\} = V(\zeta_1 - y_1, \zeta_2 - y_2) = V(f_1, f_2)$$

and $\mathrm{red}^{-1} V(\zeta_1 - y_1, \zeta_2 - y_2) = \{x \in X_{\mathbb{C}_p} \mid |f_i(x)| < 1\}$.

Definition. We call such an open disk a residue disk of X.

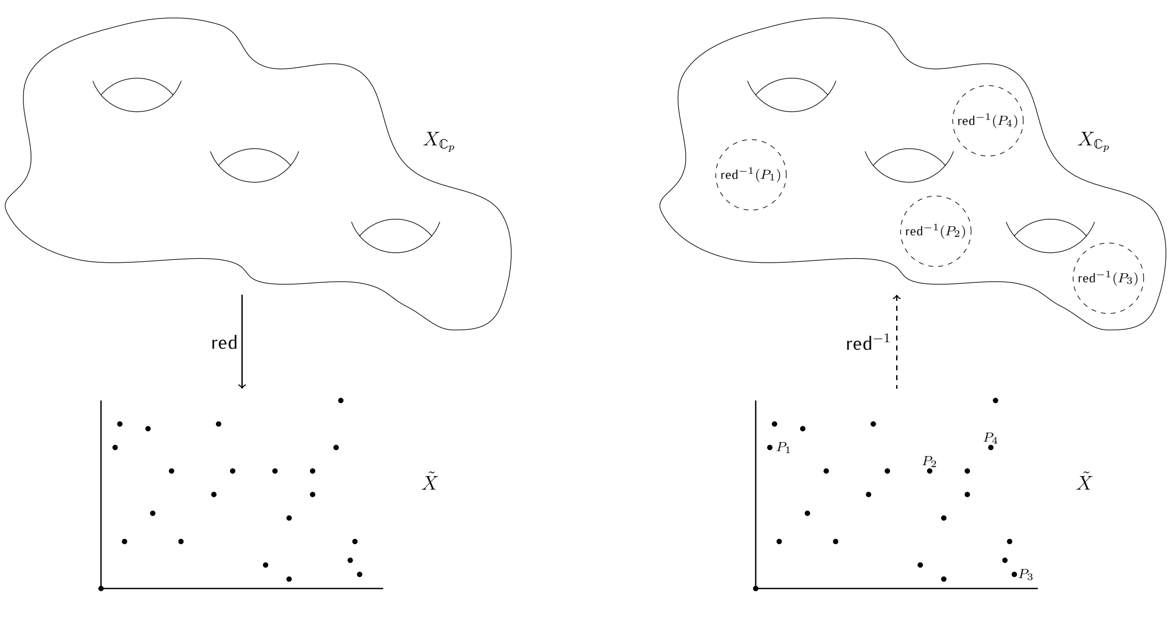


Figure 6: Reduction of a curve and residue disks

Definition. Let X be a curve over $\mathcal{O}$. A wide open subspace of X is a rigid analytic subspace of $X_{\mathbb{C}_p}$ that is the complement of the union of a finite collection of disjoint closed disks of radius $r < 1$.

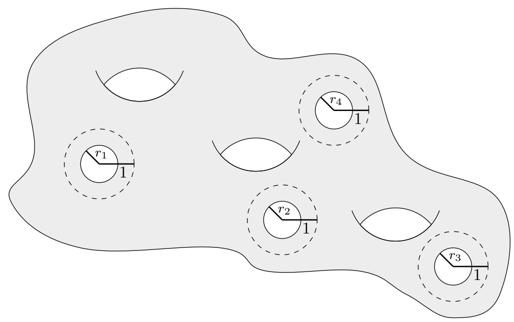


Figure 7: Wide open subspace

Remark. Wide open subsets are obtained cutting out certain closed disks and keeping everything else.

This will allow us to consider differential forms which are not holomorphic on all X but just on a wide open subset $\mathcal{U}$ (in a certain sense we are eliminating the problematic points $(\omega)_\infty$).

In the previous chapter, we have seen the properties of the Coleman integral. Let's us re-state the result in the case of curves using the terminology of [BBK]:

Theorem 5.3.1. *For every curve X over $\mathcal{O}$ and every wide open subspace V of $X_{\mathbb{C}_p}$, there exists a unique map*

$$\mu_V : Div^0(V) \times \Omega^1_{V/\mathbb{C}_p} \longrightarrow \mathbb{C}_p$$

such that:

(Linearity) *The map μ_0 is linear on $Div^0(V)$ (linearity on points) and $\mathbb{C}_p$-linear on $\Omega^1_{V/\mathbb{C}_p}$ (linearity on forms).*

(Compatibility) *For any residue disk D of X and any isomorphism $\phi : V \cap D \to A(r, R)$, the restriction of μ_V to $Div^0(V \cap D) \times \Omega^1_{V/\mathbb{C}_p}$ is compatible with the definition in Section 5.2 of integral on an open wide annulus.*

(Change of variables) *Let X' be another curve over $\mathcal{O}$, V' be a wide open subspace of X', and let $\phi : V \to V'$ be any morphism of rigid spaces relative to a continuous* *automorphism of* $\mathbb{C}_p$. *Then*

$$\mu_{V'}\left(\phi(\cdot),\ \cdot\ \right) = \mu_V\left(\ \cdot\ , \phi^*(\cdot)\right)$$

(Fundamental Theorem of Calculus) *If* $f \in A(V)$ *and* $\sum_i \gamma_i P_i \in Div^0(V)$, *then*

$$\mu_V\left(\sum_i \gamma_i P_i\ ,\ df\right) = \sum_i \gamma_i f(P_i)$$

In the following sections we will try to explain the explicit method of Kedlaya and Balakrishnan for computing the Coleman's integrals.

5.4 Hyperelliptic Curves

We start by introducing the principal ingredient of the algorithm: hyperelliptic curves.

Let K be a field of characteristic $\neq 2$.

Definition. An hyperelliptic curve is a smooth projective curve given by an equation of the form

$$\mathcal{C} : y^2 = F(x)$$

where $F \in K[x]$ is a monic polynomial of degree $2g+1$ such that $\bar{F}(x)$ has no repeated roots.

Remark. This gives us a curve of genus g with good reduction.

We denote by

$$\iota : (x, y) \longrightarrow (x, -y)$$

the hyperelliptic involution.

Definition. A Weierstrass point P is a K^{alg}-rational point fixed by ι. We denote

$$Z = \{P \in \mathcal{C}\left(K^{\text{alg}}\right) \mid \iota(P) = P\}$$

the set of Weierstrass points.

Remark. Here is where we are explicitly making holomorphic functions on wide opens.

Now let K be an unramified extension of $\mathbb{Q}_p$ and $\mathcal{C}/K$ be an hyperelliptic curve with good reduction. Let $\mathcal{C}'$ be the affine curve obtained from $\mathcal{C}$ by eliminating the Weierstrass points.

The coordinate ring of $\mathcal{C}'$ is given by

$$A = \frac{K[x,y,z]}{(y^2 - F(x), yz - 1)} = \frac{K[x,y,y^{-1}]}{(y^2 - F(x))}$$

Let's try to compute the de Rham cohomology of $\mathcal{C}'$; we have to consider the complex

$$0 \longrightarrow A \xrightarrow{d} \Omega_{A/K} \xrightarrow{d} \bigwedge^2 \Omega_{A/K} \xrightarrow{d} \dots \xrightarrow{d} \bigwedge^{\dim \mathcal{C}'} \Omega_{A/K} \longrightarrow 0$$

where $\Omega_{A/K}$ is the module of Kähler differentials, i.e., the A-module generated by the symbols dr, for $r \in A$, modulo the relations $dc = 0$ for $\mathrm{c} \in K$ and $d(ab) = a \cdot db + b \cdot da$.

Clearly $\mathrm{H}^0_{\mathrm{dR}}(\mathcal{C}') = K$ and $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{C}') = 0$ for $i > 1$. To determine $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{C}')$ it is necessary to use the relations defining $\mathcal{C}'$ ([Hrt, Example 3.1.2]):

$$\mathrm{H}^1_{\mathrm{dR}}(\mathcal{C}') = \left\langle x^i \frac{dx}{y}, x^i \frac{dx}{y^2} \right\rangle_{i=0,\dots,2g-1}$$

The problem is that the underlying coordinate ring does not admit a proper Frobenius lifting:

Example. Suppose that $\bar{X}$ is the affine space over $\mathbb{F}_p$ defined by the equation $xy = 1$. Its coordinate ring is $\bar{A} = \mathbb{F}_p[x, x^{-1}]$. If one wants to construct the de Rham cohomology immediately bump into the problem of lifting forms $x^{p-1}dx$ (the cohomology group being independent on the choice of the lifting). For instance, we can lift $\bar{A}$ to $\mathbb{Z}_p$ in two ways:

$$A_1 = \mathbb{Z}_p[x, x^{-1}] \qquad \text{and} \qquad A_2 = \mathbb{Z}_p[x, (x + px^2)^{-1}]$$

and the cohomology groups are not isomorphic:

$$\mathrm{H}^1_{\mathrm{dR}}(A_1) = \left\langle \frac{dx}{x} \right\rangle \qquad \text{and} \qquad \mathrm{H}^1_{\mathrm{dR}}(A_2) = \left\langle \frac{dx}{x}, \frac{dx}{1 + px} \right\rangle$$

Therefore, the de Rham cohomology is not the suitable tool to study integrals on hyperelliptic curves.

5.5 Monsky-Washnitzer Cohomology

As we have seen, computing the de Rham cohomology of a rigid space might be somehow problematic.

In general, if X is a smooth variety over K and A is its coordinate ring, one can consider A^∞, the $\mathfrak{p}$-adic completion of A (where $\mathfrak{p} \subseteq \mathcal{O}$ is the maximal ideal). Unfortunately, the price we pay is that now the de Rham cohomology of A^∞ is bigger than the one of A. To solve this problem, in 1968, the two American mathematicians P. Monsky and G. Washnitzer introduced a subring of A^∞ consisting of power series converging fast enough that their integrals also converge. This subject has been first developed in the 60's by Monsky and Washnitzer [MW] motivated by the work of Dwork, and then refined in the 80's by van der Put [VdP].

Definition. If A is an $\mathcal{O}_K$-algebra, its $\mathfrak{p}$-adic completion is

$$A^\infty = \lim_{\substack{\longleftarrow \\ i}} \frac{A}{\mathfrak{p}^i A}$$

The weak completion of A is the subset $A^\dagger$ of A^∞ consisting of elements having representation

$$\sum_{i=0}^{+\infty} P_i(\zeta_1, \ldots, \zeta_n)$$

where $\zeta_1, \ldots, \zeta_n \in A$, $P_i \in \mathfrak{p}^i [\zeta_1, \ldots, \zeta_n]$ and there exists a constant C such that

$$\deg P_i \leqslant \mathrm{C}(i+1) \quad \forall i$$

If one consider

$$T_n^\dagger = \left\{ \sum_\nu a_\nu \zeta^\nu \,\middle|\, a_\nu \in \mathcal{O}_K,\ \exists r > 1 \text{ such that } \lim_{|\nu| \to +\infty} |a_\nu| r^\nu = 0 \right\}$$

the algebra of overconvergent power series, then a weakly complete finitely generated $\mathcal{O}_K$-algebra, is the homomorphic image of $T_n^\dagger$ for some n.

Remark. $T_n^\dagger$ is a set of power series converging on a space which is slightly bigger than a unitary ball.

Theorem 5.5.1 ([VdP, Proposition 2.2])**.** *$T_n^\dagger$ satisfies Weierstrass Preparation and Division Theorem.*

Theorem 5.5.2 ([MW, Theorem 2.1])**.** *Any weakly complete finitely generated algebra is Nöetherian.*

Now let $A^\dagger = T_n^\dagger/\mathfrak{I}$ be a weakly complete finitely generated algebra. We define the module of differentials as

$$\Omega^1(A^\dagger) = \frac{A^\dagger d\zeta_1 + \ldots + A^\dagger d\zeta_n}{\substack{\text{Submodule generated by} \\ \left\{\frac{\partial f_i}{\partial \zeta_1} d\zeta_1 + \ldots + \frac{\partial f_i}{\partial \zeta_n} d\zeta_n \,\middle|\, i=1,\ldots,m\right\}}}$$

where $(f_1, \ldots, f_m) = \mathfrak{I}$.

This is the universal finite module of differentials of $A^\dagger$ over $\mathcal{O}_K$ ([VdP, §2]). As usual, we define

$$\Omega^i(A^\dagger) = \bigwedge^i \Omega^1(A^\dagger)$$

the i-the exterior power of $\Omega^1(A^\dagger)$ and we denote with d^i the exterior differentiation:

$$d(xdy_1 \wedge \ldots \wedge dy_i) = dx \wedge dy_1 \wedge \ldots \wedge dy_i$$

We obtain the de Rham complex

$$0 \longrightarrow \Omega^0(A^\dagger) \xrightarrow{d^0} \Omega^1(A^\dagger) \xrightarrow{d^1} \Omega^2(A^\dagger) \xrightarrow{d^2} \ldots$$

Notation. Let $A^\dagger$ be a weakly complete finitely generated algebra, then we set

$$\bar{A} = A^\dagger/\pi A^\dagger$$

where π is a uniformizer of $\mathfrak{p}$.

Remark. Observe that $T_n^\dagger/\pi T_n^\dagger$ is isomorphic to the polynomial algebra $\mathbb{F}[\zeta_1, \ldots, \zeta_n]$.

We deduce from this that, if $A^\dagger$ is a weakly complete finitely generated algebra (i.e., there exists a surjective morphism $T_n^\dagger \twoheadrightarrow A^\dagger$), then $\bar{A}$ is a finitely generated $\mathbb{F}$-algebra.

On the other hand, we also know [Bes, §1.3.2] that any finitely generated smooth $\mathbb{F}$-algebra can be obtained as the reduction $\bar{A}$ of a suitable $A^\dagger$.

In particular, the weak completion depends, up to isomorphisms, only on $\bar{A}$.

Definition. The Monsky-Washnitzer cohomology of $\bar{A}$ is the cohomology of the de Rham complex $\Omega^\bullet(A^\dagger) \otimes K$:

$$\mathrm{H}^i_{\mathrm{MW}}(\bar{A}, K) = \mathrm{H}^i_{\mathrm{dR}}(\Omega^\bullet(A^\dagger) \otimes K)$$

Remark. $\mathrm{H}^i_{\mathrm{MW}}(\bar{A}, K)$ is a finite dimensional K-vector space [Ber2, §3].

Question. *The reason why we introduced the Monsky-Washnitzer cohomology was to overcome the lifting problems arising when computing the de Rham cohomology in characteristic p. Do we have, in fact, solved the issue?*

Lemma 5.5.3 ([VdP, Theorem 2.4.4]). *With the notation as above, the following hold:*

(a) *Any two lifts are isomorphic.*

(b) *Any morphism $\bar{f} : \bar{A} \to \bar{B}$ can be lifted to a morphism $f^\dagger : A^\dagger \to B^\dagger$.*

(c) *Any two maps $f^\dagger, g^\dagger : A^\dagger \to B^\dagger$ with the same reduction induce homotopic maps $f^\dagger_*, g^\dagger_* : \Omega^\bullet(A^\dagger) \to \Omega^\bullet(B^\dagger)$.*

Let's go back to hyperelliptic curves. Consider again $\mathcal{C} : y^2 = F(x)$ and $\mathcal{C}' = \mathcal{C} \backslash Z$. If

$$A = \frac{K[x, y, y^{-1}]}{(y^2 - F(x))}$$

is the coordinate ring of $\mathcal{C}'$, then its Monsky-Washnitzer weak completion is

$$A^\dagger = \left\{ \sum_{n=-\infty}^{+\infty} \frac{B_n(x)}{y^n} \,\middle|\, B_n \in K[x],\ \deg B_n \leqslant 2g \right\}$$

with the further condition that $\nu_p(B_n(x))$ grows faster than some linear function of $|n|$ as $|n| \to \pm\infty$.

Remark. We are "allowing singularities" near the Weierstrass points but, with the additional condition, we force the elements of $A^\dagger$ to be holomorphic not only out of the Weierstrass residue disks but also in some annulus around Weierstrass points.

Now we consider the derivation

$$d : A^\dagger \longrightarrow \Omega$$

and we observe that

$$dy = F'(x)\frac{dx}{2y} \implies \Omega = A^\dagger \frac{dx}{2y}$$

which means

$$\begin{aligned}
d : A^\dagger &\longrightarrow A^\dagger \frac{dx}{2y} \\
\sum_{i,j} a_{i,j} \frac{x^i}{y^j} &\longrightarrow \sum_{i,j} a_{i,j} d\left(\frac{x^i}{y^j}\right) = \sum_{i,j} a_{i,j} \frac{ix^{i-1}y^j dx - jx^i y^{j-1} dy}{y^{2j}} = \\
&= \sum_{i,j} a_{i,j} \frac{2ix^{i-1}y^j - 2jx^i y^{j-1}\frac{F'}{2y}}{y^{2j-1}} \cdot \frac{dx}{2y} = \\
&= \sum_{i,j} a_{i,j} \left(\frac{2ix^{i-1}}{y^{j-1}} - \frac{jx^i F'}{y^{j+1}}\right) \cdot \frac{dx}{2y}
\end{aligned}$$

We denote

$$\mathrm{H}^0_{\mathrm{MW}}(\mathcal{C}') = \ker d = \{h \in A^\dagger \mid dh = 0\}$$

$$\mathrm{H}^1_{\mathrm{MW}}(\mathcal{C}') = \operatorname{coker} d = A^\dagger \frac{dx}{y} \Big/ dA^\dagger$$

Remark. The hyperelliptic involution induces a map ι^* on cohomology decomposing $\mathrm{H}^1_{\mathrm{MW}}(\mathcal{C}')$ into two eigenspaces on which it acts respectively as I and $-I$:

$$\mathrm{H}^1_{\mathrm{MW}}(\mathcal{C}')^+ = \{\text{Even 1-forms}\} \quad , \quad \mathrm{H}^1_{\mathrm{MW}}(\mathcal{C}')^- = \{\text{Odd 1-forms}\}$$

Lemma 5.5.4 ([Bal, Lemma 2.2.2])**.** *The two eigenspaces have the following description:*

$$H^1_{MW}(\mathcal{C}')^+ \text{ has basis } \left\{x^i \frac{dx}{y^2}\right\}_{i=0}^{2g} \quad \text{and} \quad H^1_{MW}(\mathcal{C}')^- \text{ has basis } \left\{x^i \frac{dx}{2y}\right\}_{i=0}^{2g-1}$$

Remark. We notice that even 1-forms can be written in terms of x alone; thus, they can be integrated directly as in the definition in Section 5.2. Consequently, we will focus our attention on odd 1-forms.

Now, any differential $\omega \in \Omega$ can be written uniquely as

$$\omega = df + \gamma_0\omega_0 + \gamma_1\omega_1 + \ldots + \gamma_{2g-1}\omega_{2g-1}$$

where $f \in A^\dagger$, $\gamma_i \in K$ and the ω_i's are the elements of the basis:

$$\omega_i = x^i \frac{dx}{2y}$$

Remark. The process of writing ω in terms of elements of the basis can be made algorithmic thanks to Kedlaya [Bal, §2.2.2]: in Section 5.8 we will give an intuition of how this method works (Algorithm 6).

5.6 Lifting of Frobenius

We recall that K is an unramified extension of $\mathbb{Q}_p$. Thus, we have a unique automorphism ϕ_K lifting the Frobenius automorphism $\phi : x \to x^p$ on its residue field $\mathbb{F}$. Now we extend this automorphism to $A^\dagger$. Clearly $\phi(x) = x^p$; then,

$$\begin{aligned}\phi(y) &= (\phi_K(F)(x^p))^{1/2} = (\phi_K(F)(x^p) - F(x)^p + F(x)^p)^{1/2} = \\ &= F(x)^{p/2}\left(1 + \frac{\phi_K(F)(x^p) - F(x)^p}{F(x)^p}\right)^{1/2} = y^p\left(1 + \frac{\phi_K(F)(x^p) - F(x)^p}{F(x)^p}\right)^{1/2} = \\ &= y^p \sum_{i=0}^{\infty} \binom{1/2}{i} \frac{(\phi_K(F)(x^p) - F(x)^p)^i}{y^{2ip}}\end{aligned}$$

$$
\begin{aligned}
\phi(z) = \phi(y^{-1}) = y^{-p} \sum_{i=0}^{\infty} \binom{-1/2}{i} \frac{(\phi_K(F)(x^p) - F(x)^p)^i}{y^{2ip}} &= \\
&= z^p \sum_{i=0}^{\infty} \binom{-1/2}{i} (\phi_K(F)(x^p) - F(x)^p)^i \, z^{2ip}
\end{aligned}
$$

Let's us compute the action of Frobenius on the basis of $\mathrm{H}^1_{\mathrm{MW}}(\mathcal{C}')^-$:

$$
\begin{aligned}
\phi^* \left(x^i \frac{dx}{2y} \right) = x^{ip} d(x^p) \phi \left(\frac{1}{2y} \right) = x^{ip} d(x^p) \frac{1}{2\phi(y)} &= \\
&= p x^{ip+p-1} \left(2y^p \sum_{k=0}^{+\infty} \binom{-1/2}{i} \frac{\phi_K(F)(x^p) - F(x)^p}{y^{2pk}} \right)^{-1} dx
\end{aligned}
$$

Note that we need y^{-1} to be an element of $A^\dagger$: this is why we work on $\mathcal{C}'$ instead of $\mathcal{C}$.

At this point we must make an important consideration about the effectiveness of the Kedlaya algorithm: as all the elements in $A^\dagger$ are infinite series, a practical computation will be made with a suitable approximation. Hence, one should keep track of the precision used.

5.7 Local Parameters

Before being able to integrate, we need to compute a parametrization of the path between the two endpoints of integration. In this section we describe the algorithms needed to compute local parameters in residue disks. A good discussion about this can be found in [Bal, §2.1] or [BaT, §3.1].

Algorithm 1. Local coordinate at a point in a non-Weierstrass residue disk

Input: A point $P = (x_P, y_P) \in \mathcal{C}(K)$ in a non-Weierstrass disk and an integer n.

Output: A parametrization $(x(t), y(t))$ at P in terms of a local coordinate.

1. We set $x(t) = t + x_P$ for a local coordinate t.

2. We approximate a solution of $y(t) = \sqrt{F(x(t))}$ using the Newton-Raphson method of tangents with $y_0 = y_P$. Thus,

$$
y_{i+1} = y_i - \frac{y_i^2 - F(x(t))}{2y_i} = \frac{1}{2}\left(y_i + \frac{F(x(t))}{y_i} \right) \qquad \forall i \geqslant 0
$$

3. The integer n gives us the precision. Hence, the number of iterations depends on n: for precision $O(t^n)$, one can take i to be $\lceil \log_2(n) \rceil$.

Example. Let us consider the hyperelliptic curve

$$y^2 = x(x-1)(x-2)(x-5)(x-6) = x^5 - 14x^4 + 65x^3 - 112x^2 + 60x$$

which has good reduction at 7. Let us consider the point $P = (3, 6)$. The local coordinates at P are given by

$$x(t) = 3 + t$$

For $y(t)$ we obtain:

i	$y_i(t)$
0	6
1	$6 + 3t - \frac{13}{12}t^2 - \frac{13}{12}t^3 + \frac{1}{12}t^4 + \frac{1}{12}t^5$
2	$6 + 3t - \frac{11}{6}t^2 - \frac{1}{6}t^3 - \frac{115}{1728}t^4 - \frac{169}{3456}t^5 + O(t^6)$
3	$6 + 3t - \frac{11}{6}t^2 - \frac{1}{6}t^3 - \frac{49}{462}t^4 - \frac{77}{864}t^5 + O(t^6)$

hence,

$$y(t) = 6 + 3t - \frac{11}{6}t^2 - \frac{1}{6}t^3 - \frac{49}{462}t^4 - \frac{77}{864}t^5 + O(t^6)$$

ALGORITHM 2. Local coordinate at a finite Weierstrass point

INPUT: A finite Weierstrass point $P = (x_P, 0) \in \mathcal{C}(K)$ and an integer n.

OUTPUT: A parametrization $(x(t), y(t))$ at P in terms of a local coordinate.

1. We set $y(t) = t$ for a local coordinate t.

2. We approximate $x(t)$ using the Newton-Raphson method of tangents. Take

$$G(x) = \frac{F(x)}{(x - x_P)}$$

which is a polynomial in x since $F(x_P) = 0$. Set

$$x_0 = x_P + \frac{t^2}{G(x_P)} \qquad \text{and} \qquad h(x,t) = F(x) - t^2$$

Then the Newton-Raphson method yields

$$x_{i+1}(t) = x_i(t) - \frac{h(x_i(t), t)}{h'(x_i(t), t)} \qquad \text{where} \qquad h'(x,t) = \frac{\partial h(x,t)}{\partial x}$$

3. The integer n gives us the precision. Hence, the number of iterations depends on n: for precision $O(t^n)$, one can take i to be $\lceil \log_2(n) \rceil$.

Example. Consider again the hyperelliptic curve

$$y^2 = x(x-1)(x-2)(x-5)(x-6) = x^5 - 14x^4 + 65x^3 - 112x^2 + 60x$$

Now take the point $P = (2,0)$. The local coordinates at P are given by

i	$x_i(t)$
0	$2 + \frac{1}{24}t^2$
1	$2 + \frac{1}{24}t^2 - \frac{11}{6912}t^4 + O(t^6)$
2	$2 + \frac{1}{24}t^2 - \frac{11}{6912}t^4 + O(t^6)$
3	$2 + \frac{1}{24}t^2 - \frac{11}{6912}t^4 + O(t^6)$

Thus,

$$x(t) = 2 + \frac{1}{24}t^2 - \frac{11}{6912}t^4 + O(t^6) \qquad \text{and} \qquad y(t) = t$$

At last, we have to consider the case of infinity. We know that $\deg F(x) = 2g+1$ and $y^2 = F(x)$; this implies that x has a pole of order 2 at ∞ while y has a pole of order $2g+1$.

ALGORITHM 3. Local coordinate at infinity

INPUT: The point P_∞ above $x = \infty$ and an integer n

OUTPUT: A parametrization $(x(t), y(t))$ at P_∞ such that t has a zero at ∞.

1. Take $x_0 = t^{-2}$ and let

$$h(x,t) = \left(\frac{x^g}{t}\right)^2 - F(x) \rightsquigarrow h'(x,t) = \frac{\partial h(x,t)}{\partial x}$$

Now approximate a solution for $x(t)$ using the Newton-Raphson method

$$x_{i+1}(t) = x_i(t) - \frac{h(x_i(t),t)}{h'(x_i(t),t)}$$

2. The integer n gives us the precision. Hence, the number of iterations depends on n: for precision $O(t^n)$, one can take i to be $\lceil \log_2(n) \rceil$.

3. Set

$$y(t) = \frac{(x(t))^g}{t}$$

Example. Let

$$y^2 = x(x-1)(x-2)(x-5)(x-6) = x^5 - 14x^4 + 65x^3 - 112x^2 + 60x$$

be our favorite hyperelliptic curve. At ∞ we have

i	$x_i(t)$
0	$\frac{1}{t^2}$
1	$\frac{1}{t^2} + \frac{14}{61} - \frac{3181}{3721}t^2 + \frac{72086}{226981}t^4 + \frac{39925007}{13845841}t^6 + O(t^8)$
2	$\frac{1}{t^2} + 14 + \frac{2580535}{3721}t^2 + \frac{8071768502}{226981}t^4 + \frac{25224060548463}{13845841}t^6 + O(t^8)$
3	$\frac{1}{t^2} + 14 - 65t^2 + 1022t^4 + \frac{31584621039599}{13845841}t^6 + O(t^8)$

And, therefore,

$$x(t) = \frac{1}{t^2} + 14 - 65t^2 + 1022t^4 + O(t^6)$$

$$y(t) = \frac{1}{t^5} + \frac{28}{t^3} + \frac{66}{t} + 224t + 32841t^3 - 132860t^5 + O(t^6)$$

5.8 Explicit Integrals

Finally, we present the Algorithm to compute Coleman integrals on hyperelliptic curves.

The idea is to first integrate on residue disks and then use the Frobenius lifting to connect integrals on different residue disks.

Step I - Integrating on residue disks

This first step involves the so called "tiny integrals". Let P and Q be in the same residue disk; then we can use the fact (guaranteed by Coleman Theorem) that the function f_ω is locally analytic.

ALGORITHM 4. Tiny Coleman integrals

INPUT: Two points P, Q in the same residue disk and a basis of differentials $\{\omega_i\}_{i=0}^{2g-1}$.
OUTPUT: The integrals $\int_P^Q \omega_i$.

1. Using one of the algorithms (1), (2) or (3), compute a parametrization $(x(t), y(t))$ at P in terms of a local coordinate t.

2. Formally integrate

$$\int_P^Q \omega_i = \int_P^Q x^i \frac{dx}{2y} = \int_0^{t(Q)} \frac{x(t)^i}{2y(t)} \frac{dx(t)}{dt} dt$$

as a power series in t.

Example. Take the hyperelliptic curve

$$y^2 = x(x-1)(x-2)(x-5)(x-6) = x^5 - 14x^4 + 65x^3 - 112x^2 + 60x$$

over $\mathbb{Q}_7$ and consider the point $P = (3, 6)$. We can find the Teichmüller point T living in the same residue disk of P (Section 4.3): this is a point fixed by Frobenius ϕ:

$$\phi(T) = T \qquad \text{and} \qquad \begin{cases} x(T) \equiv x(P) \mod 7 \\ y(T) \equiv y(P) \mod 7 \end{cases}$$

To find T, one can just take the Teichmüller lift of $x(P)$ and then solve

$$y^2 = x(x-1)(x-2)(x-5)(x-6)$$

to find the y-coordinate.

Remark. We should be careful in choosing the correct sign of the y-coordinate.

```
sage: R.<x>=QQ['x']
sage: E= HyperellipticCurve(x^5-14*x^4+65*x^3-112*x^2+60*x)
sage: K=Qp(7,8)
sage: EK=E.change_ring(K)
sage: P=(K(3),K(6));
sage: EK.frobenius(P) == P
False
sage: TP = EK.teichmuller(P); TP
(3+4*7+6*7^2+3*7^3+2*7^5+6*7^6+2*7^7 +O(7^8) :
6+5*7+6*7^2+6*7^3+3*7^4+7^5+2*7^6+5*7^7+O(7^8) :
1+O(7^8))
sage: E.frobenius(TP) == TP
True
```

This gives us the Teichmüller point

$$T = \big(3 + 4\cdot 7 + 6\cdot 7^2 + 3\cdot 7^3 + 2\cdot 7^5 + 6\cdot 7^6 + 2\cdot 7^7 + O(7^8); \\ 6 + 5\cdot 7 + 6\cdot 7^2 + 6\cdot 7^3 + 3\cdot 7^4 + 7^5 + 2\cdot 7^6 + 5\cdot 7^7 + O(7^8)\big)$$

Now we compute the integral

$$\int_P^T \omega_0 = \int_P^T \frac{dx}{2y} = 5\cdot 7 + 3\cdot 7^2 + 3\cdot 7^3 + 3\cdot 7^4 + 6\cdot 7^6 + 7^7 + O(7^8)$$

```
sage: K = pAdicField(7, 8)
sage: x = polygen(K)
sage: C = HyperellipticCurve(x^5-14*x^4+65*x^3-112*x^2+60*x)
sage: P = C(3,6);
sage: TP = C.teichmuller(P);
sage: x, y = C.monsky_washnitzer_gens()
sage: C.tiny_integrals([1],P,TP)
5*7 + 3*7^2 + 3*7^3 + 3*7^4 + 6*7^6 + 7^7 + O(7^8)
```

Remark. One can even integrate any holomorphic differential ω.

Remark. Since P and Q are in the same residue disk, all the power series involved are, in fact, power series in *pt*

Remark. This works either on Weierstrass or non-Weierstrass residue disks.

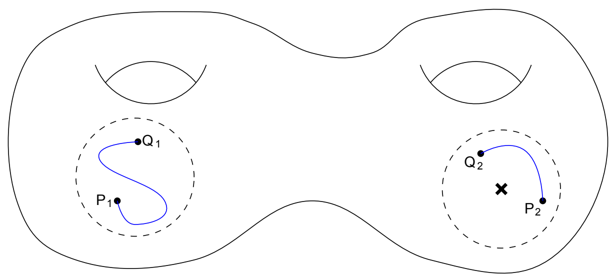


Figure 8: Tiny Integrals in non-Weierstrass and Weierstrass residue disks

Step II - Connecting two integrals

Suppose now that P and Q lie in two different residue disks. We cannot use the method of tiny integrals anymore: the problem is that now the series expansion does not converge everywhere.

In this case, we have to do a distinction between the case of Weierstrass and non-Weierstrass disks.

We essentially follow the construction of Coleman.

We indicate with $\mathcal{U}_P$ the residue disk of the point P and with $\mathcal{U}_Q$ the residue disk of Q. As we have seen, in each residue disk there exists a unique Teichmüller point fixed by Frobenius. The idea is to perform the two tiny integrals between P and ξ_P (the Teichmüller point of $\mathcal{U}_P$) and from ξ_Q to Q and then to connect them using Frobenius:

$$\int_P^Q \omega = \int_P^{\xi_P} \omega + \int_{\xi_P}^{\xi_Q} \omega + \int_{\xi_Q}^{Q} \omega$$

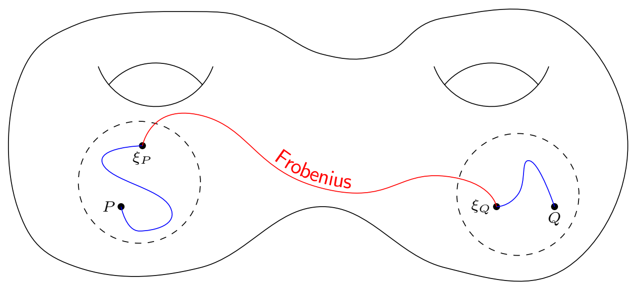


Figure 9: Coleman Integral via Teichmüller points

Hence, we have reduced the problem to compute

$$\int_{\xi_P}^{\xi_Q} \omega$$

Connecting two integrals: non-Weierstrass disks

The idea relies in computing the action of Frobenius on the elements of the basis of $\mathrm{H}^1_{\mathrm{MW}}(\mathcal{C}')$.

Algorithm 5. Coleman integral with endpoints in two non-Weierstrass disks

Input: Two points P, Q in different residue disks, a basis of differentials $\{\omega\}_{i=0}^{2g-1}$ and an integer m such that the residue fields of P and Q are contained in $\mathbb{F}_{p^m}$.
Output: The integrals $\int_P^Q \omega_i$.

1. Compute Teichmüller points ξ_P and ξ_Q.

2. Calculate the action of the m-th power of Frobenius on each basis element:

$$(\phi^m)^* \omega_i = df_i + \sum_{i=0}^{2g-1} M_{ij}\omega_j$$

3. By change of variables, we get the fundamental linear system

$$\sum_{i=0}^{2g-1} (M-I)_{i,j} \int_{\xi_P}^{\xi_Q} \omega_j = f_i(\xi_P) - f_i(\xi_Q)$$

4. One can prove that the matrix $M - I$ is invertible. Thus, we can solve the system above and find the desired integral.

Remark (Action of Frobenius). To compute the action of the m-th power of Frobenius, first we have to compute the action of Frobenius on the basis $\{\omega_i\}_{i=0}^{2g-1}$. As already mentioned, we can use the Kedlaya's algorithm [Bal, §2.2.2]:

ALGORITHM 6. Kedlaya's Algorithm

INPUT: The basis of differentials $\{\omega\}_{i=0}^{2g-1}$.

OUTPUT: Functions $h_i \in A^\dagger$ and a $2g \times 2g$ matrix B such that

$$\phi^*\omega_i = dh_i + \sum_{i=0}^{2g-1} B_{i,j}\omega_j$$

for all $i = 0, \ldots, 2g-1$

1. Compute $\phi(x)$ and $\phi(y)$ as infinite series in $A^\dagger$.

2. Use Newton iteration method to approximate

$$\frac{y}{\phi(y)}$$

3. Write

$$\phi^*\omega_i = \phi^*\left(x^i\frac{dx}{y}\right) = px^{pi+p-1}\frac{y}{\phi(y)}\frac{dx}{2y} = dh_i + \sum_{i=0}^{2g-1} B_{i,j}\omega_j$$

Denote by f (respectively h) the column vector whose i-th component is f_i (h_i). Once that we have the action of Frobenius, we can define:

$$f = \phi^{m-1}(h) + B\phi^{m-2}(h) + B\phi_K(B)\phi^{m-3}(h) + \ldots + B\phi_K(B)\cdot\ldots\cdot\phi_K^{m-2}(B)h$$

$$M = \phi_K(B)\cdot\ldots\cdot\phi_K^{m-1}(B)$$

Remark (Change of variables). To obtain the fundamental linear system we observe that

$$\int_{\phi^m(\xi_P)}^{\phi^m(\xi_Q)} \omega_i \overset{\text{Theorem (5.3.1)}}{=} \int_{\xi_P}^{\xi_Q} (\phi^m)^*\omega_i = \int_{\xi_P}^{\xi_Q}\left(df_i + \sum_{i=0}^{2g-1} M_{i,j}\omega_j\right) =$$

$$= f_i(\xi_Q) - f_i(\xi_P) + \sum_{i=0}^{2g-1} M_{ij}\int_{\xi_P}^{\xi_Q}\omega_j$$

but now
$$\int_{\phi^m(\xi_P)}^{\phi^m(\xi_Q)} \omega_i = \int_{\xi_P}^{\xi_Q} \omega_i$$
from which
$$\int_{\xi_P}^{\xi_Q} \omega_i = f_i(\xi_Q) - f_i(\xi_P) + \sum_{i=0}^{2g-1} M_{ij} \int_{\xi_P}^{\xi_Q} \omega_j$$

Connecting two integrals: Weierstrass disks

Finally, it only remains to study the situation in which P and Q live in two different residue disks of which at least one is Weierstrass.

We will assume that ω is everywhere meromorphic with no poles at P and Q. This is because otherwise we cannot even define $\int_P^Q \omega$.

Lemma 5.8.1. *Let $P, Q \in \mathcal{C}(\mathbb{C}_p)$ with P a Weierstrass point. Let ω be an odd differential which is everywhere meromorphic on $\mathcal{C}$ and has no poles in P or Q. Then for ι, the hyperelliptic involution,*
$$\int_P^Q \omega = \frac{1}{2} \int_{\iota(Q)}^Q \omega$$
In particular, if Q is also a Weierstrass point, then
$$\int_P^Q \omega = 0$$

Proof. Observe that, since P is Weierstrass, then
$$\begin{aligned} \int_P^Q \omega &= \int_P^{\iota(P)} \omega + \int_{\iota(P)}^{\iota(Q)} \omega + \int_{\iota(Q)}^Q \omega = \\ &= \underbrace{\int_P^P \omega}_{=0} + \int_P^Q \iota^*(\omega) + \int_{\iota(Q)}^Q \omega = \int_P^Q (-\omega) + \int_{\iota(Q)}^Q \omega \end{aligned}$$

Thus,
$$2\int_P^Q \omega = \int_{\iota(Q)}^Q \omega \qquad \square$$

Therefore, in order to compute $\int_P^Q \omega$, we find the Weierstrass point P' in the residue disk of P and then
$$\int_P^Q \omega = \int_P^{P'} \omega + \int_{P'}^Q \omega = \int_P^{P'} \omega + \frac{1}{2} \int_{\iota(Q)}^Q \omega$$

where the first is a tiny integral, while the second integral involves points lying in non-Weierstrass disks.

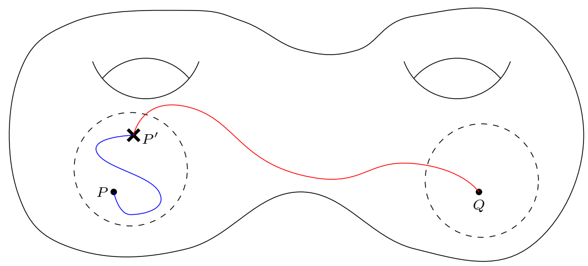


Figure 10: Coleman integral involving Weierstrass disks

Example. Let us consider again the hyperelliptic curve

$$y^2 = x(x-1)(x-2)(x-5)(x-6) = x^5 - 14x^4 + 65x^3 - 112x^2 + 60x$$

and the points $P = (3,6)$, $Q = (10,120)$:

$$\int_P^Q \omega_0 = \int_P^Q \frac{dx}{2y} = 6\cdot 7 + 6\cdot 7^2 + 3\cdot 7^3 + 3\cdot 7^4 + 2\cdot 7^5 + 6\cdot 7^7 + O(7^8)$$
$$\int_P^Q \omega_1 = \int_P^Q x\frac{dx}{2y} = 4\cdot 7 + 2\cdot 7^2 + 6\cdot 7^3 + 4\cdot 7^5 + 5\cdot 7^7 + O(7^8)$$
$$\int_P^Q \omega_2 = \int_P^Q x^2\frac{dx}{2y} = 2 + 5\cdot 7 + 2\cdot 7^2 + 4\cdot 7^3 + 7^4 + 5\cdot 7^5 + 2\cdot 7^6 + 4\cdot 7^7 + O(7^8)$$
$$\int_P^Q \omega_3 = \int_P^Q x^3\frac{dx}{2y} = 6 + 3\cdot 7 + 3\cdot 7^3 + 6\cdot 7^5 + 6\cdot 7^6 + O(7^8)$$

```
sage: K=Qp(7,8)
sage: x=polygen(K)
sage: E=HyperellipticCurve(x^5 - 14*x^4 + 65*x^3 - 112*x^2 + 60*x)
sage: P=E(3,6)
sage: Q=E(10,120)
sage: w=E.invariant_differential()
sage: x, y=E.monsky_washnitzer_gens()
sage: (w).coleman_integral(P,Q)
6*7+6*7^2+3*7^3+3*7^4+2*7^5+6*7^7+O(7^8)
sage: (x*w).coleman_integral(P,Q)
4*7+2*7^2+6*7^3+4*7^5+5*7^7+O(7^8)
sage: (x^2*w).coleman_integral(P,Q)
2+5*7+2*7^2+4*7^3+7^4+5*7^5+2*7^6+4*7^7+O(7^8)
sage: (x^3*w).coleman_integral(P,Q)
6+3*7+3*7^3+6*7^5+6*7^6+O(7^8)
```

Example. Let us consider the elliptic curve '11.a' (http://www.lmfdb.org/EllipticCurve/Q/11/a/2):

$$\mathcal{E}' : y^2 + y = x^3 - x^2 - 10^x - 20 \xrightarrow{\text{Weierstrass form}} \mathcal{E} : y^2 = x^3 - 13392x - 1080432$$

$\mathcal{E}$ has good reduction at 19. Consider the point $P = (168, 1188)$ which is a 5-torsion point on $\mathcal{E}$.

```
sage: K=Qp(19,15)
sage: EE=EllipticCurve(K,'11a')
sage: E=EE.short_weierstrass_model()
sage: P=E(K(168),K(1188))
sage: 5*P
(0 : 1 + O(19^8) : 0)
sage: w=E.invariant_differential();
sage: x, y=E.monsky_washnitzer_gens()
sage: w.coleman_integral(P,2*P)
O(19^8)
```

$$\int_P^{2P} \omega_0 = \int_P^{2P} \frac{dx}{2y} = O(19^8)$$

which is consistent with the fact that ω_0 is holomorphic and P is a torsion point ([Col3, Proposition 3.1]).

```
sage: (x*w).coleman_integral(P,2*P)
9+2*19+15*19^2+3*19^3+15*19^4+3*19^5+15*19^6+3*19^7+O(19^8)
```

$$\int_P^{2P} \omega_1 = \int_P^{2P} x\frac{dx}{2y} = 9 + 2\cdot 19 + 15\cdot 19^2 + 3\cdot 19^3 + 15\cdot 19^4 + 3\cdot 19^5 + 15\cdot 19^6 + 3\cdot 19^7 + O(19^8)$$

This reflects the fact that ω_1 is not holomorphic on $\mathcal{E}$.

```
sage: (x*w).coleman_integral(2*P,3*P)
18+5*19+15*19^2+3*19^3+15*19^4+3*19^5+15*19^6+3*19^7+O(19^8)
sage: (x*w).coleman_integral(3*P,4*P)
9+2*19+15*19^2+3*19^3+15*19^4+3*19^5+15*19^6+3*19^7+O(19^8)
```

$$\int_{3P}^{4P} \omega_1 = \int_{-2P}^{-P} x\frac{dx}{2y} = 9 + 2\cdot 19 + 15\cdot 19^2 + 3\cdot 19^3 + 15\cdot 19^4 + 3\cdot 19^5 + 15\cdot 19^6 + 3\cdot 19^7 + O(19^8)$$

and this is consistent with the linearity on $\mathrm{Div}^0(\mathcal{E})$ of the map $\mu_{\mathcal{E}}$ (Theorem 5.3.1).

The detection of torsion points is one of the original applications of the theory of Coleman integrals [Col3]. Another very important example of the power of this tool is illustrated in [Col2], a very influential paper in which Coleman resumed an idea of Chabauty and proved that p-adic abelian integrals could be used to produce effective bounds for the number of rational points on curves.

Proposition 5.8.2. *Suppose that K is a complete discretely valued subfield of $\mathbb{C}_p$. Suppose $j : (\mathcal{C}, c_0) \to (A, O)$ is a morphism over K of a pointed curve into an abelian variety, both with good reduction. Suppose G is a subgroup of $A(K)$, then $j(\mathcal{C}(K)) \cap G$ is contained in the set of all $x \in \mathcal{C}(K)$ such that*

$$\int_{c_0}^{x} \omega = 0$$

for all $\omega \in j^* V_G$ where $V_G = \left\{ \omega \in H^0(A, \Omega^1_{A/K}) \,\middle|\, f_\omega(x) = 0, \;\forall x \in G \right\}$.

Example. Let us consider the hyperelliptic curve

$$\mathcal{E} : y^2 = x^7 + \frac{9}{4}x^6 - 2x^5 - 9x^4 - 4x^3 + 8x^2 + 8x + 2$$

having good reduction at 7 and whose Jacobian has rank 1. $\mathcal{E}$ has the following five known rational points:

$$\mathcal{E}(\mathbb{Q}) \supseteq \mathcal{E}(\mathbb{Q})_{\text{known}} = \left\{ \infty, \left(-1, \frac{1}{2}\right), \left(-1, -\frac{1}{2}\right), \left(1, \frac{5}{2}\right), \left(1, -\frac{5}{2}\right) \right\}$$

```
sage: p=7; K=Qp(p,10)
sage: x=polygen(K)
sage: E=HyperellipticCurve(x^7+9/4*x^6-2*x^5-9*x^4-4*x^3+8*x^2+8*x+2)
sage: w=E.invariant_differential()
sage: x, y=E.monsky_washnitzer_gens()
sage: INFTY=E(K(0),K(1),K(0))
sage: P=E(K(-1),K(-1/2))
sage: P1=E(K(-1),K(+1/2))
sage: P2=E(K(1),K(-5/2))
sage: P3=E(K(1),K(5/2))
```

We compute the coleman integrals from ∞ to $\left(-1, -\frac{1}{2}\right)$ on the basis of $\mathrm{H}^1_{\mathrm{MW}}(\mathcal{E})$

```
sage: A=E.coleman_integrals_on_basis(INFTY,P)
sage: a=A[0]
sage: b=A[1]
sage: c=A[2]
```

$$A = \begin{pmatrix} 5 \cdot 7^3 + 5 \cdot 7^4 + 4 \cdot 7^6 + 4 \cdot 7^7 + 2 \cdot 7^8 + O(7^9) \\ 3 \cdot 7 + 5 \cdot 7^2 + 7^3 + 2 \cdot 7^4 + 2 \cdot 7^6 + 5 \cdot 7^7 + 4 \cdot 7^8 + O(7^9) \\ 6 \cdot 7 + 7^2 + 5 \cdot 7^4 + 4 \cdot 7^5 + 6 \cdot 7^6 + 5 \cdot 7^7 + 4 \cdot 7^8 + O(7^9) \end{pmatrix}$$

Define the following differentials

```
sage: alpha=b*w-a*x*w
sage: beta=c*w-a*(x^2)*w
```

We observe that the integrals of both α and β vanish at every rational point:

```
sage: alpha.coleman_integral(P,P1)
O(7^10)
sage: alpha.coleman_integral(P,P2)
O(7^10)
sage: alpha.coleman_integral(P,P3)
O(7^10)
sage: beta.coleman_integral(P,P1)
O(7^10)
sage: beta.coleman_integral(P,P2)
O(7^10)
sage: beta.coleman_integral(P,P3)
O(7^10)
```

α and β play the role of ω in Proposition 5.8.2.

Remark. Actually, it is also possible to prove that $\mathcal{E}(\mathbb{Q})_{\text{known}}$ is all $\mathcal{E}(\mathbb{Q})$ [Bal+, Algorithm 3.3 and Example 4.1].

Remark. Potentially, the problem of computing rational points on curve can be made effective (provided that the curve respects the hypothesis of Chabauty-Coleman). In Chapter 6 we'll describe an algorithm based on [Bal, Algorithm 6.2.1].

References. For a complete overview of the potential of the algorithms described above, one may have a look at the `SAGE` Reference Manual "Hyperelliptic curves over a p-adic field": http://doc.sagemath.org/html/en/reference/curves/sage/schemes/hyperelliptic_curves/hyperelliptic_padic_field.html.

5.9 Implementation Analysis

In the previous section we have given some examples of the implementation of the algorithm in `SAGE`. In the following, we present some results about the precision of the computations. The proofs can be found in [BBK, §4].

Proposition 5.9.1. *Let $\int_P^Q \omega$ be a tiny integral in a non-Weierstrass residue disc (the discussion about Weierstrass disks is similar), with $P, Q \in \mathcal{C}(K)$ with an accuracy of n digits. Let $(x(t), y(t))$ be the local interpolation between P and Q defined by*

$$x(t) = x_P \cdot (1-t) + x_Q \cdot t \qquad\qquad y(t) = \sqrt{F(X(t))}$$

Let $\omega = g(x, y)$ be a differential of the second kind such that $h(t) = g(x(t), y(t))dx$ belongs to $\mathcal{O}[\![t]\!]$. If we truncate $h(t)$ modulo t^m, then the computed value of the integral $\int_P^Q \omega$ will be correct up to

$$\min\{n, m+1-\lfloor \log_p(m+1) \rfloor\}$$

digits of absolute precision.

Proposition 5.9.2. *Let $\int_P^Q \omega$ be a Coleman integral, with ω a differential of the second kind and with P and Q living in two different non-Weierstrass residue disks and a precision of n digits. Let M be the matrix of the action of Frobenius on the basis differentials (Algorithm 6). Set $B = M^{-1}$, and let $m = \nu_p(\det(B))$. Then the computed value of the integral $\int_P^Q \omega$ will be accurate up to*

$$n - \max\{m, \lfloor \log_p(n) \rfloor\}$$

digits of precision.

We conclude this chapter with a note about the running time of the algorithm. For instance, let us observe how `SAGE` uses the time in computing the following function:

```
sage: w.coleman_integral(P,Q)
```

$< 0.0001\%$	–	Setup.
10.88647%	–	Tiny Integrals.
67.75531%	–	Monsky-Washnitzer Computations.
$< 0.0001\%$	–	Evaluating f (Remark of Algorithm 6).
0.103681%	–	Evaluating f over the Rationals.
20.47693%	–	Changing Rings.
0.570244%	–	Evaluating f on p-adic Field with Capped Precision.
0.207361%	–	Solve the Fundamental Linear System.

We immediately notice that the great majority of the time is spent doing Monsky-Washnitzer computations and changing rings.

This is because there is no "good" (in the sense of fast) linear algebra over $\mathbb{Q}_p$ and, therefore, we work over the rationals where linear algebra is much faster.

Essentially, changing rings consists in pretending that the polynomials $B_n(x)$ in the definition of $A^\dagger$ are defined over $\mathbb{Q}$ instead of $\mathbb{Q}_p$.

Example. Consider the Hyperelliptic curve with even model

$$\mathcal{E} : y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1$$

This has good reduction at 3 and its Jacobian has Mordell-Weil rank 1.

We compute the Matrix of Frobenius appearing in the fundamental linear system

```
sage: p = 3
sage: prec = 10
sage: R.<x> = QQ['x']
sage: A,forms=monsky_washnitzer.matrix_of_frobenius_hyperelliptic(
x^6+8*x^5+22*x^4+22*x^3+5*x^2+6*x+1,p,prec)
```

Because of the dimension of the matrix we report here only the first column

$$\begin{pmatrix} 3^2 + 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + 2 \cdot 3^7 + 2 \cdot 3^9 + O(3^{10}) \\ 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 + 2 \cdot 3^8 + 2 \cdot 3^9 + O(3^{10}) \\ 3 + 2 \cdot 3^2 + 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + O(3^{10}) \\ 2 \cdot 3 + 3^2 + 3^3 + 3^5 + 3^8 + 2 \cdot 3^9 + O(3^{10}) \\ 3 + 3^3 + 2 \cdot 3^4 + 3^5 + O(3^{10}) \end{pmatrix}$$

In the following, we change the ring of definition of the matrix A:

```
sage: EQ=HyperellipticCurve(x^6+8*x^5+22*x^4+22*x^3+5*x^2+6*x+1)
sage: K=Qp(p,prec)
sage: E=EQ.change_ring(K)
sage: M=A.change_ring(ZZ)
```

we obtain

$$M = \begin{pmatrix} 45774 & 23097 & 37179 & 49839 & 26815 \\ 53580 & 13467 & 16317 & 15091 & 41178 \\ 2046 & 44625 & 3159 & 17202 & 4756 \\ 46212 & 45531 & 52146 & 46726 & 30348 \\ 435 & 33288 & 52140 & 31222 & 8975 \end{pmatrix}$$

```
sage: V = VectorSpace(K,5)
sage: R = forms[0].base_ring()
sage: PP=E(K(0),K(1))
sage: QQ=E(K(0),K(-1))
```

These two are non-Weierstrass rational points on $\mathcal{E}$ and they turn out to be the Teichmüller points in their residue disks:

```
sage: E.is_same_disc(PP,QQ)
False
sage: E.is_weierstrass(PP)
False
sage: E.is_weierstrass(QQ)
False
sage: E.frobenius(PP) == PP
True
sage: E.frobenius(QQ) == QQ
True
```

We evaluate the functions f_i at the two points PP and QQ

```
sage: L=[f(R(PP[0]),R(PP[1]))-f(R(QQ[0]),R(QQ[1])) for f in forms]
sage: b=V(L)
```

Finally, we solve the fundamental linear system

```
sage: M_sys = matrix(K, A).transpose() - 1
sage: M_sys**(-1) * b
```

obtaining

$$\int_{(0,1)}^{(0,-1)} \omega_0 = 3^{-1} + 2 + 2 \cdot 3^2 + 3^3 + 3^4 + 3^6 + O(3^7)$$
$$\int_{(0,1)}^{(0,-1)} \omega_1 = 1 + 3^3 + 3^5 + 2 \cdot 3^6 + 3^7 + O(3^8)$$
$$\int_{(0,1)}^{(0,-1)} \omega_2 = 1 + 3^2 + 3^3 + 2 \cdot 3^4 + 3^5 + O(3^8)$$
$$\int_{(0,1)}^{(0,-1)} \omega_3 = 2 \cdot 3^{-2} + 1 + 3 + 3^3 + 3^5 + O(3^6)$$
$$\int_{(0,1)}^{(0,-1)} \omega_4 = 2 \cdot 3^{-1} + 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + 3^4 + 2 \cdot 3^5 + O(3^7)$$

References. One can have a look at the complete source code developed by Bradshaw at `https://github.com/sagemath/sage/blob/master/src/sage/schemes/hyperelliptic_curves/hyperelliptic_padic_field.py` - The previous example roughly follows the code around line 650.

Chapter 6

Rational Points on Curves

In this chapter we give a brief description of the method of Chabauty and Coleman; this is a p-adic tool for determining the set of rational points on a curve $\mathcal{C}$ defined over $\mathbb{Q}$ of genus $g \geqslant 2$. We provide a theoretical introduction as well as some numerical examples of the implementation of this method. The purpose is to study one of the applications of Coleman integration theory.

The main reference is the article of McCallum and Poonen [MP] but of course we also keep an eye on the original works of Chabauty [Cha] and Coleman [Col2]. For the description of the algorithm at the end of Section 6.3, we refer to the PhD thesis of Balakrishnan [Bal, Chapter 6].

6.1 Formulation of the Problem

Let us consider a curve $\mathcal{C}$ defined over $\mathbb{Q}$, the field of rational numbers. The problem of finding the set of rational points on $\mathcal{C}$ is one of the fundamental questions arising in algebraic geometry.

Despite its appearance, the question is very difficult and the numerous attempts of solving it have led to the development of many new techniques in geometry and number theory. For centuries, mathematicians have tried to find a general method to compute $\mathcal{C}(\mathbb{Q})$ but even nowadays we do not know if there is an algorithm suitable to approach this problem; in fact, we do not even know if there is an algorithm deciding whether $\mathcal{C}(\mathbb{Q})$ is finite or not.[1]

Fortunately, in some cases, we have quantitative results giving at least the finiteness of the number of rational points on a curve: in 1983, G. Faltings proved the Mordell conjecture

[1] Some of these questions have been solved for other base rings such as $\mathbb{C}$, $\mathbb{R}$, $\mathbb{F}_p$, $\mathbb{Q}_p$ and $\mathbb{Z}$ (see, for instance, the Hilbert 10^{th} problem).

(which is now known as Faltings' Theorem) stating that a curve of genus greater than 1 over $\mathbb{Q}$ has only finitely many rational points (*Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, 1983).

However, Faltings' proof is not effective in the sense that it does not yield an explicit method of finding $\mathcal{C}(\mathbb{Q})$. Nevertheless, this can be done, thanks to the work of C. Chabauty before and eventually R. Coleman, in the case when the Jacobian of the curve has Mordell-Weil rank strictly less than the genus of $\mathcal{C}$.

In particular, in the article of Coleman [Col2], the integration theory developed so far plays a central role.

Definition. The Jacobian of a curve $\mathcal{C}$ is

$$J(\mathcal{C}) = \frac{\mathrm{H}^0(\Omega^1_{\mathcal{C}})^*}{\mathrm{H}_1(\mathcal{C})} \text{ where } \mathrm{H}_1(\mathcal{C}) \hookrightarrow \mathrm{H}^0(\Omega^0_{\mathcal{C}})^* \text{ via the map } [\gamma] \to \int_\gamma -$$

6.2 The Theorem of Chabauty

Let J be the Jacobian of our curve $\mathcal{C}$. By the Abel-Jacoby Theorem, we know that J is an abelian variety of dimension g over $\mathbb{Q}$.

Suppose that we know a point $O \in \mathcal{C}(\mathbb{Q})$; then, we can identify our curve with a subvariety of its Jacobian using the Abel-Jacobi embedding

$$\begin{aligned} \mathcal{C} &\hookrightarrow J \\ P &\longrightarrow [P - O] \end{aligned}$$

sending P to the class of the divisor of $P - O$.

The idea is to perform the following steps:

1. Compute $J(\mathbb{Q})$.

2. Determine which points in $J(\mathbb{Q})$ lie on $\mathcal{C}$.

Observe that computing $J(\mathbb{Q})$ is, in general, a difficult problem. However, by the Mordell-Weil Theorem, we know that $J(\mathbb{Q})$ is a finitely generated abelian group. Hence, describing it can be read as "finding generators and relations". From now on we'll suppose that $J(\mathbb{Q})$ is known.

Theorem 6.2.1 (Mordell-Weil)**.** *If $\mathcal{C}$ is defined over a number field K, then $J(K)$ is a finitely generated abelian group.*

Example. Consider the curve

$$y^2 = x(x-1)(x-2)(x-5)(x-6)$$

all the computation are implemented in `MAGMA`.

```
> P<x> := PolynomialRing(Rationals());
> C := HyperellipticCurve(x*(x-1)*(x-2)*(x-5)*(x-6));
> BadPrimes(C);
> ptsC := Points(C : Bound := 1000); ptsC;
> J := Jacobian(C);
> RankBound(J);
```

We get

$$J(\mathbb{Q}) \simeq \mathbb{Z} \times J(\mathbb{Q})_{\mathrm{tors}}$$

```
> PJ:= J! [ ptsC[5], ptsC[1] ];
> Order(PJ);
> heightconst := HeightConstant(J : Effort:=2, Factor);
> LogarithmicBound := Height(PJ) + heightconst;
> AbsoluteBound := Ceiling(Exp(LogarithmicBound));
> PtsUpToAbsBound := Points(J : Bound:=AbsoluteBound );
> ReducedBasis([ pt : pt in PtsUpToAbsBound ]);
> Height(PJ);
```

The generator for the the infinite part is given by $[(3,6)-\infty]$.

```
> TT,mm:=TwoTorsionSubgroup(J); TT;
> T,m:=TwoTorsionSubgroup(J); T;
> m(T.1); m(T.2); m(T.3); m(T.4);
> PJ1:= J! [ ptsC[3], ptsC[1] ]; Order(PJ1);
> PJ2:= J! [ ptsC[4], ptsC[1] ]; Order(PJ2);
> PJ3:= J! [ ptsC[7], ptsC[1] ]; Order(PJ3);
> PJ4:= J! [ ptsC[8], ptsC[1] ]; Order(PJ4);
```

Hence,

$$J(\mathbb{Q})_{\mathrm{tors}} \simeq \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$

and it is generated by

$$[(1,0)-\infty] \qquad [(2,0)-\infty] \qquad [(5,0)-\infty] \qquad [(6,0)-\infty]$$

Remark. In some cases, it suffices something less than the full knowledge of $J(\mathbb{Q})$ [MP, Remark 2.2].

Chabauty idea was to study the lie Group $J(\mathbb{Q}_p)$. Consider $J_{\mathbb{Q}_p}$, the base change of J to $\mathbb{Q}_p$ (naively, we are considering the variety defined by the same equations of J but we think at them as defined over $\mathbb{Q}_p$). We take

$$\mathrm{H}^0\left(J_{\mathbb{Q}_p}, \Omega^1\right)$$

the g dimensional $\mathbb{Q}_p$-vector space of 1-forms on $J_{\mathbb{Q}_p}$, and we consider $\omega_J \in \mathrm{H}^0\left(J_{\mathbb{Q}_p}, \Omega^1\right)$; we can construct a map

$$\begin{aligned} \eta_J : J(\mathbb{Q}_p) &\longrightarrow \mathbb{Q}_p \\ P &\longrightarrow \int_0^P \omega_J \end{aligned}$$

uniquely characterized by:

- η_J is a homomorphism.
- There exists an open subset $\mathcal{U} \subseteq J(\mathbb{Q}_p)$ such that if $Q \in \mathcal{U}$, then $\int_0^Q \omega_J$ can be computed by expanding ω_J in power series in local coordinates, finding a formal antiderivative and evaluating the resulting formal expansion at the local coordinates of Q.

We obtain a bilinear pairing

$$\begin{aligned} J(\mathbb{Q}_p) \times \mathrm{H}^0\left(J_{\mathbb{Q}_p}, \Omega^1\right) &\longrightarrow \mathbb{Q}_p \\ (P, \omega_J) &\longrightarrow \int_O^P \omega_J \end{aligned}$$

that we can re-write as

$$\log : J(\mathbb{Q}_p) \longrightarrow \left(\mathrm{H}^0\left(J_{\mathbb{Q}_p}, \Omega^1\right)\right)^*$$

Definition. We denote by $\widetilde{J(\mathbb{Q})}$ the p-adic closure of $J(\mathbb{Q})$ in $J(\mathbb{Q}_p)$.

Lemma 6.2.2 ([MP, Lemma 4.2]). *If $r' = \dim \widetilde{J(\mathbb{Q})}$ and $r = \dim J(\mathbb{Q})$, then $r' \leqslant r$.*

Now we recall that $\mathcal{C}(\mathbb{Q}_p)$ lies in $J(\mathbb{Q}_p)$ and, in particular, it is a one dimensional submanifold.

Theorem 6.2.3 ([Cha]). *If $\mathcal{C}$ is a curve of genus $g \geqslant 2$ defined over $\mathbb{Q}$ and $r' < g$, then $\mathcal{C}(\mathbb{Q}_p) \cap \widetilde{J(\mathbb{Q})}$ is finite and, therefore, so is $\mathcal{C}(\mathbb{Q})$.*

6.3 The Method of Coleman

Now we turn the previous Theorem into a practical method of computing rational points following [Col2]. The idea is to find functions on $J(\mathbb{Q}_p)$ (here intervenes Coleman integration theory) vanishing on $\widetilde{J(\mathbb{Q})}$ and restrict them to parametrizations of $\mathcal{C}(\mathbb{Q})$.

Suppose now that our curve $\mathcal{C}$ has good reduction at p; thus, J has good reduction at p as well and the embedding

$$\mathcal{C} \hookrightarrow J$$

induces an embedding of the special fiber of $\mathcal{C}$ into the reduction of J.

One can show that the embedding above induces an isomorphism of $\mathbb{Q}_p$-vector spaces

$$\mathrm{H}^0\left(J_{\mathbb{Q}_p}, \Omega^1\right) \simeq \mathrm{H}^0\left(\mathcal{C}_{\mathbb{Q}_p}, \Omega^1\right)$$

Suppose that ω is the image of ω_J via this isomorphism; then, we have

$$\int_P^Q \omega = \int_0^{[Q-P]} \omega_J$$

and we can recover some properties of the integral on the right from the theory of integration on J.

(i) If $P_i, Q_i \in \mathcal{C}(\mathbb{Q}_p)$ are such that $[\sum(Q_i - P_i)]$ is a torsion element of $J(\mathbb{Q}_p)$, then

$$\sum \int_{P_i}^{Q_i} \omega = 0$$

(ii) If P and Q have the same reduction in $\mathbb{F}_p$, then $\int_P^Q \omega$ can be computed by expanding in power series in a local parameter t on the curve $\mathcal{C}$.

By the definition of η_J, its restriction to $\mathcal{C}(\mathbb{Q}_p)$, is the function

$$\begin{aligned} \eta = \eta_J\big|_{\mathcal{C}(\mathbb{Q}_p)} : \mathcal{C}(\mathbb{Q}_p) &\longrightarrow \mathbb{Q}_p \\ P &\longrightarrow \int_O^P \omega \end{aligned}$$

One can see that

$$\log\left(\widetilde{J(\mathbb{Q})}\right) \subseteq \mathrm{H}^0\left(J_{\mathbb{Q}_p}, \Omega^1\right)^* \simeq \mathbb{Q}_p^{\oplus g}$$

is a $\mathbb{Z}_p$-submodule of rank r'.

If $r' < g$, there exists a non-zero $\mathbb{Q}_p$-linear functional

$$\lambda : \mathrm{H}^0\left(J_{\mathbb{Q}_p}, \Omega^1\right)^* \twoheadrightarrow \mathbb{Q}_p$$

that vanishes on $\log\left(\widetilde{J(\mathbb{Q})}\right)$. By duality, λ corresponds to a particular $\omega_J \in \mathrm{H}^0\left(J_{\mathbb{Q}_p}, \Omega^1\right)$. Hence, λ gives rise to η_J, ω, η as above:

$$\eta_J : J(\mathbb{Q}_p) \xrightarrow{\log} \mathrm{H}^0\left(J_{\mathbb{Q}_p}, \Omega^1\right)^* \xrightarrow{\lambda} \mathbb{Q}_p$$

and so η_J vanishes on $\widetilde{J(\mathbb{Q})}$. It follows that ω satisfies also

(iii) If $P_i, Q_i \in \mathcal{C}(\mathbb{Q}_p)$ are such that $[\sum(Q_i - P_i)] \in \widetilde{J(\mathbb{Q})}$, then $\sum \int_{P_i}^{Q_i} \omega = 0$.

and η vanishes on $\mathcal{C}(\mathbb{Q}_p) \cap \widetilde{J(\mathbb{Q})}$. Now we only need to bound the number of zeros of η.

Theorem 6.3.1 (Coleman). *Let $\mathcal{C}, J, p, r'$ be as above. Suppose also that p is a prime of good reduction for $\mathcal{C}$.*

a. *Let ω be a nonzero 1-form in $H^0\left(\mathcal{C}_{\mathbb{Q}_p}, \Omega^1\right)$ satisfying conditions (i)-(iii). Scale ω by an element of $\mathbb{Q}_p^\times$ so that it reduces to a nonzero 1-form $\overline{\omega} \in H^0\left(\mathcal{C}_{\mathbb{F}_p}, \Omega^1\right)$. Suppose $\overline{Q} \in \mathcal{C}(\mathbb{F}_p)$. Let $m = \mathrm{ord}_{\overline{Q}}\overline{\omega}$. If $m < p-2$, then the number of points in $\mathcal{C}(\mathbb{Q})$ reducing to $\overline{Q}$ is at most $m+1$.*

b. *If $p > 2g$, then $\#\mathcal{C}(\mathbb{Q}) \leqslant \#\mathcal{C}(\mathbb{F}_p) + (2g-2)$.*

The proof requires some technical lemmas in p-adic analysis involving the theory of Newton Polygons [Gou, §6.4].

Coleman's Theorem can be improved by choosing the "best" ω for each residue disk.

Theorem 6.3.2 (M. Stoll). *If $r < g$ and $p > 2r + 2$ is a prime of good reduction then*

$$\#\mathcal{C}(\mathbb{Q}) \leqslant \#\mathcal{C}(\mathbb{F}_p) + 2r$$

Explicit computations on the map η can potentially give lot of informations about the set $\mathcal{C}(\mathbb{Q})$.

In the following, we suppose that $\mathcal{C}$ is a hyperelliptic curve of equation $y^2 = F(x)$ where F is a monic and nonsingular polynomial defined over $\mathbb{Q}$.

Algorithm 7. Single Coleman integrals on basis elements from a Weierstrass point to a parameter z.

Input: A Weierstrass basepoint P, a non-Weierstrass point Q (whose residue disk will be the object of investigation) and a holomorphic basis of differentials.
Output: A power series $f_Q(z) = \int_Q^{Q_z} \omega_i$, where $Q_z = \left(z + x(Q), \sqrt{f(z + x(Q))}\right)$ is taken so that Q_z is in the residue disk of Q.

1. Compute $Q_z = \left(x(Q) + z, \sqrt{f(z + x(Q))}\right)$, choosing the correct square root.

2. Compute $\phi(Q_z)$, choosing the right square root.

3. Compute the local coordinate at Q_z: $x(t) = t + z + x(Q)$, $y(t) = \sqrt{f(x(t))}$.

4. This gives us

$$\int_{\phi(Q_z)}^{Q_z} \omega_i = \int_{x(\phi(Q))-x(Q)}^{0} x(t)^i \frac{dx(t)}{2y(t)} dt$$

5. Using the fundamental linear system, compute

$$f_Q(z) = \int_P^{Q_z} \omega_i = (M - I)^{-1} \left(-f_i(Q_z) - \int_{\phi(Q_z)}^{Q_z} \omega_i \right)$$

Remark (Effectiveness). It is necessary to point out that the algorithm presents some limitations (for a more detailed analysis of the algorithm one can refer to [MP, §7].):

- It may be difficult to bound r' and r.
- In the case when $r' = g$, there is no chance of finding a bound for $\#\mathcal{C}(\mathbb{Q})$.
- Even if $\#\left(\mathcal{C}(\mathbb{Q}_p) \cap \widetilde{J(\mathbb{Q})}\right)$ is known, the true value of $\#\mathcal{C}(\mathbb{Q})$ could be smaller.

6.4 Examples

Example. Let's consider our Hyperelliptic curve

$$\mathcal{E} : y^2 = x(x-1)(x-2)(x-5)(x-6) = x^5 - 14x^4 + 65x^3 - 112x^2 + 60x$$

whose Jacobian has Mordell-Weil rank 1. We have already observed that the curve has good reduction at 7.

Suppose we want to find all the rational points on $\mathcal{E}$. The purpose is to see the method of Coleman in action.

Since $\mathcal{E}$ has good reduction at 7, we reduce the equation defining $\mathcal{E}$ over $\mathbb{F}_7$:

$$\bar{\mathcal{E}} : y^2 = x^5 + 2x^3 + 4x$$

We consider the following table:

$x \in \mathbb{F}_7$	0	1	2	3	4	5	6
$x^5 \in \mathbb{F}_7$	0	1	4	5	2	3	6
$2x^3 \in \mathbb{F}_7$	0	2	2	5	2	5	5
$x^5 + 2x^3 + 4x \in \mathbb{F}_7$	0	0	0	1	6	0	0

Remark. $\alpha \in \mathbb{F}_7$ is a square if and only if $\alpha = 0, 1, 2, 4$ which means that $y = 0, 1, 6$.

$$\mathcal{E}(\mathbb{F}_7) = \{\overline{(0,0)}; \overline{(1,0)}; \overline{(2,0)}; \overline{(5,0)}; \overline{(6,0)}; \overline{(3,6)}; \overline{(3,1)}; \overline{\infty}\}$$

Notice that $\mathcal{E}(\mathbb{F}_7)$ has 8 elements and we can represent them in the following way:

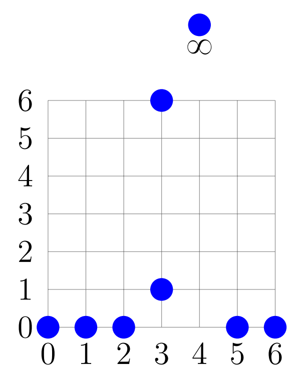


Figure 11: Representation of $\mathcal{E}(\mathbb{F}_7)$

We can try to lift these points and we find:

$$\mathcal{E}(\mathbb{Q}) \supseteq \{(0,0); (1,0); (2,0); (5,0); (6,0); (3,6); (3,-6); \infty\}$$

We now want to study if there are other rational points in the residue disks of the two non-Weierstrass points $P_\pm = (3, \pm 6)$. First of all we consider $\mathcal{E}$ defined over $\mathbb{Q}_7$ and we compute the Coleman integrals

$$a = \int_{\infty}^{(3,6)} \omega_0 \qquad\qquad b = \int_{\infty}^{(3,6)} \omega_1$$

where $\omega_0 = dx/2y$ and $\omega_1 = xdx/2y$ are the first elements of the basis of differentials.

```
sage: Campo=QQ; x=polygen(Campo)
sage: EQ=HyperellipticCurve(x^5-14*x^4+65*x^3-112*x^2+60*x)
sage: p=7; prec=10; K=Qp(p,prec); E=EQ.change_ring(K)
sage: w=E.invariant_differential()
sage: x, y=E.monsky_washnitzer_gens()
sage: INFTY=E(K(0),K(1),K(0)); P=E(K(3),K(6))
sage: A=E.coleman_integrals_on_basis_hyperelliptic(INFTY,P)
sage: a=A[0]; b=A[1]
```

We obtain:

$$a = 6 \cdot 7 + 6 \cdot 7^2 + 3 \cdot 7^3 + 3 \cdot 7^4 + 2 \cdot 7^5 + 6 \cdot 7^7 + 4 \cdot 7^8 + 4 \cdot 7^9 + O(7^{10})$$

$$b = 4 \cdot 7 + 2 \cdot 7^2 + 6 \cdot 7^3 + 4 \cdot 7^5 + 5 \cdot 7^7 + 3 \cdot 7^8 + O(7^{10})$$

Let us define a new differential α (living in the annihilator of the Jacobian $J(\mathcal{E})$ as in Proposition 5.8.2):

$$\alpha = b\omega_0 - a\omega_1$$

Remark. It is immediate to verify that $\int_{P_+}^{Q} \alpha$ vanishes on each rational point Q (the situation is similar to the one in the example following Proposition 5.8.2).

Now we want to integrate α from the base point P_+ to a generic point $Q_t \equiv (t,s)$ in the same residue disk. In particular, since this will be a tiny integral, we can compute integrals expanding the equation defining $\mathcal{E}$ in power series in the uniformizing parameter x:

$$\int_{(3,6)}^{(t,s)} \alpha = \int_{(3,6)}^{(t,s)} (b-ax)\frac{dx}{2y} = \int_3^t \frac{(b-ax)dx}{2(x^5-14x^4+65x^3-112x^2+60x)^{1/2}} =$$

$$= \int_3^t \frac{b-ax}{2}\left(\frac{1}{6} - \frac{x-3}{12} + \frac{5(x-3)^2}{54} - \frac{29(x-3)^3}{432} + \right.$$
$$\left. + \frac{352(x-3)^4}{5184} + O\left((x-3)^5\right)\right) dx =$$

$$= \frac{1}{2}\int_3^t \left[\left(-\frac{a}{2}+\frac{b}{6}\right) + \left(\frac{a}{12}-\frac{b}{12}\right)(x-3) + \left(-\frac{7a}{36}+\frac{5b}{54}\right)(x-3)^2 + \right.$$
$$\left. + \left(\frac{47a}{432}-\frac{29b}{432}\right)(x-3)^3 + O((x-3)^4)\right] dx =$$

$$= \frac{1}{2}\left[\left(-\frac{a}{2}+\frac{b}{6}\right)(t-3) + \left(\frac{a}{24}-\frac{b}{24}\right)(t-3)^2 + \left(-\frac{7a}{108}+\frac{5b}{162}\right)(t-3)^3 + \right.$$
$$\left. + \left(\frac{47a}{1728}-\frac{29b}{1728}\right)(t-3)^4 + \ldots\right]$$

Notice that we are working in the residue disk of $(3,6)$: this means that $t \equiv 3 \mod 7$. Hence, we can write $t = 3 + 7z$ for $z \in \mathbb{Z}_7$. We get

$$\left(-\frac{a}{4}+\frac{b}{12}\right)(7z)+\left(\frac{a}{48}-\frac{b}{48}\right)(7z)^2+\left(-\frac{7a}{216}+\frac{5b}{324}\right)(7z)^3+\left(\frac{47a}{3456}-\frac{29b}{3456}\right)(7z)^4+\ldots$$

Now we substitute the exact values of a and b:

```
sage: alpha=b*w-a*x*w
sage: z=polygen(K)
sage: f=1/2*((-1/2*a+1/6*b)*(7*z)+(1/24*a-1/24*b)*(7*z)^2+
(-7/108*a+5/162*b)*(7*z)^3+(47/1728*a-29/1728*b)*(7*z)^4)
```

We find

$$f(z) = (2\cdot 7^3 + 7^4 + O(7^5))z + (5\cdot 7^3 + 2\cdot 7^4 + O(7^5))z^2 + \sum_{j\geqslant 3} O(7^4)z^j$$

Theorem 6.4.1 (Strassmann). *Let* $f = \sum_{i\geqslant 0} a_i z^i$ *be a power series with* $a_i \in \mathbb{Z}_p$ *such that* $\lim a_i = 0$. *Let* $k = \min \nu_p(a_i)$ *and let*

$$N = max\,\{j \mid \nu_p(a_j) = k\}$$

Then, the number of zeros of f *in* $\mathbb{Z}_p$ *is at most* N.

In our case, the Theorem says that we have at most two rational points in the residue disk of $P_+ = (3,6)$. It is not difficult to observe that 1 is a zero of f:

```
sage: f(x=K(1))
O(7^6)
```

this zero yields a second rational point reducing to $(3,6)$ in $\bar{\mathcal{E}}$: if $z = 1$, then $t = 10$ and therefore $Q = (10, -120)$.

Repeating the argument above for the other Weierstrass residue disk, we obtain the following table

P_0	Bound on the Number of Rational Points $P \equiv P_0 \pmod 7$	Rational Points $P \equiv P_0 \pmod 7$
$(3,6)$	2	$(3,6), (10,-120)$
$(3,-6)$	2	$(3,-6), (10,120)$

and this gives:

$$\mathcal{E}(\mathbb{Q}) \supseteq \{\infty, (0,0), (1,0), (2,0), (5,0), (6,0), (3,6), (3,-6), (10,120), (10,-120)\}$$

As we have seen at the beginning of the example, $\#\mathcal{E}(\mathbb{F}_7) = 8$.

Hence, applying Theorem 6.3.1.b,

$$10 \leqslant \#\mathcal{E}(\mathbb{Q}) \leqslant \#\mathcal{E}(\mathbb{F}_7) + 2g - 2 = 8 + 2 \cdot 2 - 2 = 10 \Longrightarrow \#\mathcal{E}(\mathbb{Q}) = 10$$

In conclusion, we have:

$$\mathcal{E}(\mathbb{Q}) = \{\infty, (0,0), (1,0), (2,0), (5,0), (6,0), (3,6), (3,-6), (10,120), (10,-120)\}$$

Example. Let us consider the hyperelliptic curve

$$\mathcal{E} : y^2 = x(x-3)(x-4)(x-6)(x-7)$$

As in the previous example, the curve has genus $g = 2$. Its Jacobian has now rank 0:

$$J(\mathbb{Q}) \simeq \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^4$$

Finally, the curve has good reduction at 5; we observe that $5 > 2r + 2 = 2$. Hence, we can apply Stoll's Theorem 6.3.2:

$$\#\mathcal{E}(\mathbb{Q}) \leqslant \#\mathcal{E}(\mathbb{F}_5) + 2r = \#\mathcal{E}(\mathbb{F}_5)$$

An easy computation shows that

$$\mathcal{E}(\mathbb{F}_5) = \{\overline{\infty}, \overline{(0,0)}, \overline{(1,0)}, \overline{(2,0)}, \overline{(3,0)}, \overline{(4,0)}\} \qquad \#\mathcal{E}(\mathbb{F}_5) = 6$$

Hence, $\mathcal{E}(\mathbb{Q})$ consists only of the 6 Weierstrass points:

$$\mathcal{E}(\mathbb{Q}) = \{\infty, (0,0), (3,0), (4,0), (6,0), (7,0)\}$$

We want to conclude this section with an example where Coleman method is not sufficient to write a complete list of rational points.

Example. Consider the hyperelliptic curve

$$\mathcal{C} : y^2 = x^5 + 4x^4 - x^2 + 1$$

This is a genus 2 curve, with good reduction at 5, whose Jacobian has rank 1.

We reduce the curve over $\mathbb{F}_5$: $y^2 = x^5 + 4x^4 + 4x^2 + 1$:

$x \in \mathbb{F}_5$	0	1	2	3	4
$x^5 \in \mathbb{F}_5$	0	1	2	3	4
$4x^4 \in \mathbb{F}_5$	0	4	4	4	4
$4x^2 \in \mathbb{F}_5$	0	4	1	1	4
$x^5 + 4x^4 + 4x^2 + 1 \in \mathbb{F}_5$	1	0	3	4	3

$y \in \mathbb{F}_5$	0	1	2	3	4
$y^2 \in \mathbb{F}_5$	0	1	4	4	1

and we find

$$\mathcal{C}(\mathbb{F}_5) = \left\{\overline{\infty}, \overline{(0,1)}, \overline{(0,4)}, \overline{(1,0)}, \overline{(3,2)}, \overline{(3,3)}\right\} \implies \#\mathcal{C}(\mathbb{F}_5) = 6$$

Using Theorem 6.3.1, we have the following bound on the number of rational points:

$$\#\mathcal{C}(\mathbb{Q}) \leqslant 8$$

$\overline{(0,1)}$ and $\overline{(0,4)}$ lift respectively to $(0,1)$ and $(0,-1)$ in $\mathcal{C}(\mathbb{Q})$ but the other 3 points have no obvious lift.

x is a local coordinate on the residue disk of $(0,1)$:

$$\frac{1}{y} = \frac{1}{\sqrt{F(x)}} = 1 + \frac{1}{2}x^2 - \frac{13}{8}x^4 - \frac{1}{2}x^5 - \frac{43}{16}x^6 - \frac{3}{4}x^7 + \frac{323}{128}x^8 + \frac{33}{16}x^9 + O(x^{10})$$

Again we compute the two Coleman integrals

$$a = \int_{\infty}^{(0,1)} \frac{dx}{2y} \qquad\qquad b = \int_{\infty}^{(0,1)} x\frac{dx}{2y}$$

and we define the differential $\alpha = b\omega_0 - a\omega_1$. Following the method of Coleman, we integrate α between $(0,1)$ and a generic point (t,s) living in the same residue disk of $(0,1)$. Noticing that $t \equiv 0 \mod 5$, t can be written as $5z$ (for $z \in \mathbb{Z}_5$) and so we get the function

$$f(z) = b \cdot (5z) - \frac{a}{2} \cdot (5z)^2 + \frac{b}{6} \cdot (5z)^3 \frac{a}{8} \cdot (5z)^4 - \frac{13b}{40} \cdot (5z)^5 + \dots$$

Substituting the exact values of a and b we find

$$f(z) = (4 \cdot 5^2 + 3 \cdot 5^3 + O(5^4))z + (2 \cdot 5^2 + 3 \cdot 5^3 + O(5^4))z^2 + \sum_{j \geqslant 3} O(5^4)z^j$$

Now we apply Strassmann Theorem 6.4.1 and we see that in the residue disk of $(0,1)$ there are at most two rational points. The second one is given by $\left(-\frac{15}{4}, -\frac{193}{32}\right)$. In the same way we find $\left(-\frac{15}{4}, \frac{193}{32}\right)$ in the residue disk of $(0,-1)$.

Question. How can we study the residue disks of $\overline{(1,0)}, \overline{(3,2)}$ and $\overline{(3,3)}$?

We have still not used Theorem 6.3.1.a: we consider again our differential α:

$$\alpha = \left[-\left(5+5^2+5^3+O(5^5)\right)+\left(1+3\cdot 5+4\cdot 5^2+4\cdot 5^3+2\cdot 5^4+O(5^5)\right)x\right]\frac{dx}{2y}$$

We have to study the order of vanishing of the reduction $\overline{\alpha}$ of α modulo 5. Notice that

$$\overline{\alpha} = \frac{x}{2y}dx$$

We list the order of vanishing of $\overline{\alpha}$ at the elements in $\mathcal{C}(\mathbb{F}_5)$ in the following table:

$\overline{Q}$	$\overline{(0,1)}$	$\overline{(0,4)}$	$\overline{(1,0)}$	$\overline{(3,2)}$	$\overline{(3,3)}$	$\overline{\infty}$
$\mathrm{ord}_{\overline{Q}}\overline{\alpha}$	1	1	0	0	0	0

By Theorem 6.3.1.a, we can say that in the residue disks given by the preimages of $\overline{(1,0)}, \overline{(3,2)}$ and $\overline{(3,3)}$ there is at most one rational point.

First of all we observe that, by Hensel's Lemma, the residue class of $\overline{(1,0)}$ contains one Weierstrass point W. This is defined over $\mathbb{Q}_5$ but it is not rational. With an easy computation in `SAGE` we can find that

$$W \equiv (1+5+2\cdot 5^2+3\cdot 5^3+5^4+3\cdot 5^5+3\cdot 5^6+O(5^7)\ ,\ 0)$$

```
sage: R.<x> = QQ['x']
sage: E = HyperellipticCurve(x^5+4*x^4-x^2+1)
sage: K = Qp(5,10); EK = E.change_ring(K)
sage: EK.weierstrass_points()
```

It only remains to study the residue disks of $\overline{(3,2)}$ and $\overline{(3,3)}$.

As noticed before, Coleman method is not completely effective in this situation: we do not have any evidence of the existence of a rational point in the remaining residue classes. To study these disks one have to describe more in details the Jacobian of the curve making more explicit the idea of Chabauty. Here, we only give an idea of how this works.

Notation. We denote by $\mathcal{J}$ the Neron model of J (note that $\mathcal{J} = \mathrm{Pic}^0(\mathcal{C}/\mathbb{Z}_5)$) and by $\overline{J(\mathbb{Q})}$ the reduction of $J(\mathbb{Q})$ over $\mathbb{F}_5$.

The idea is to study $\overline{J(\mathbb{Q})} \subseteq \overline{\mathcal{J}}(\mathbb{F}_5)$.

Idea ([Wet, §1.8]). Suppose that we know generators for a finite index subgroup $G \leqslant J(\mathbb{Q})$. Since the $\mathbb{Q}_5$-linear spaces spanned by $\log(G)$ and $\log(J(\mathbb{Q}))$ are equal, we will often be able to determine the generators of $V = \mathbf{Ann}(J(\mathbb{Q}))$.

We denote $\overline{G}$ and $\overline{J(\mathbb{Q})}$ the images of G and $J(\mathbb{Q})$ under the reduction map. If the index of $\overline{G}$ in $\overline{J(\mathbb{Q})}$ is coprime to the order of $\mathcal{J}(\mathbb{F}_5)$, then $\overline{G} = \overline{J(\mathbb{Q})}$.

Let $P \in \mathcal{C}(\mathbb{Q}_5)$ and let $\overline{P} \in \mathcal{C}(\mathbb{F}_5)$ be its reduction. If $P \in \mathcal{C}(\mathbb{Q})$, then we see that $\left[r\overline{P} - \overline{D}\right] \in \overline{J(\mathbb{Q})}$ where D is a rational divisor of positive degree r; conversely, if $\left[r\overline{P} - \overline{D}\right] \notin \overline{J(\mathbb{Q})}$ then there is no rational point in the residue class of P.

A good example of this method can be found in [Wet, Chapter 1].

In our situation, it turns out that there is no rational point in the residue disk of $\overline{(3,2)}$ and the same is true for the residue disk of $\overline{(3,3)}$. Thus,

$$\mathcal{C}(\mathbb{Q}) = \left\{\infty, (0,1), (0,-1), \left(-\frac{15}{4}, \frac{193}{32}\right), \left(-\frac{15}{4}, -\frac{193}{32}\right)\right\}$$

We notice that, in this case, the Coleman's bound is not sharp anymore.

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