Understanding Inquiry, an Inquiry into Understanding: a conception of Inquiry Based Learning in Mathematics

Iulian Frasinescu

A Thesis

in

the Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements

For the degree of Master in the Teaching of Mathematics at

Concordia University

Montreal, Quebec, Canada

August 2018

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: Iulian Frasinescu

Entitled: Understanding Inquiry, an Inquiry into Understanding: a conception of Inquiry Based Learning in Mathematics

and submitted in partial fulfillment of the requirements for the degree of

Master in the Teaching of Mathematics

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final Examining Committee:

______________________________	Chair
Dr. Alina Stancu	Examiner
Dr. Viktor Freiman	Examiner
Dr. Nadia Hardy	Supervisor
Dr. Anna Sierpinska	Co-supervisor

Approved by ____

Chair of Department or Graduate Program Director

2018 ____

Date Dean of Faculty

ABSTRACT

Understanding Inquiry, an Inquiry into Understanding:
a conception of Inquiry Based Learning in Mathematics

Iulian Frasinescu

IBL (Inquiry Based Learning) is a group of educational approaches centered on the student and aiming at developing higher-level thinking, as well as an adequate set of Knowledge, Skills, and Attitudes (KSA). IBL is at the center of recent educational research and practice, and is expanding quickly outside of schools: in this research we propose such forms of instruction as Guided Self-Study, Guided Problem Solving, Inquiry Based Homeschooling, IB e-learning, and particularly a mixed (Inquiry-Expository) form of lecturing, named IBLecturing. The research comprises a thorough review of previous research in IBL; it clarifies what is and what is not Inquiry Based Learning, and the distinctions between its various forms: Inquiry Learning, Discovery Learning, Case Study, Problem Based Learning, Project Based Learning, Experiential Learning, etc. There is a continuum between Pure Inquiry and Pure Expository approaches, and the extreme forms are very infrequently encountered. A new cognitive taxonomy adapted to the needs of higher-level thinking and its promotion in the study of mathematics is also presented. This research comprises an illustration of the modeling by an expert (teacher, trainer, etc.) of the heuristics and of the cognitive and metacognitive strategies employed by mathematicians for solving problems and building proofs. A challenging problem has been administered to a group of gifted students from secondary school, in order to get more information about the possibility of implementing Guided Problem Solving. Various opportunities for further research are indicated, for example applying the recent advances of cognitive psychology on the role of Working Memory (WM) in higher-level thinking.

Table of Contents

List of Figures

List of Tables

Acknowledgements

I would like to thank my supervisors Dr. Sierpinska and Dr. Hardy for helping me to finish my work and the MTM program. Dr. Sierpinska had an important contribution to the structuring of the thesis and greatly encouraged me in this research. I wish to thank my parents for giving me the taste for scientific inquiry and providing me with the gene of divergent thinking. I also owe a debt of gratitude to my sister for her support during my studies. A very special thanks to Dr. Adelina Manea from Brasov for her kind help and especially for the opportunity to propose my quiz to a wonderful mathematical circle. Thanks be to God for the inspiration, faith and assistance provided all along my research!

1 INTRODUCTION

"The voyage of discovery is not in seeking new landscapes but in having new eyes." - Proust

This research is the result of a relentless quest for meaning of Inquiry-based education, and aims at giving some sense and structure to the findings of a large number of researches on this approach to teaching.

During my review of the literature I noticed the insistence of international institutions and governments on the implementation of Inquiry-based learning (IBL) in public education and the fostering of higher-order skills and attitudes. There is a strong trend toward approaches by competencies in postsecondary education, which also drives the methodologies employed at secondary level. In 1997, OECD member countries launched the Programme for International Student Assessment (PISA) for monitoring the extent to which students near the end of compulsory schooling have attained the targeted competencies, and started the DeSeCo Project directed by Switzerland and linked to PISA for providing a conceptual framework that would help identify key competencies, define the goals for education systems, and strengthen international assessments (OECD, 2015):

"A competency is more than just knowledge and skills... For example, the ability to communicate effectively is a competency that may draw on an individual's knowledge of language, practical IT skills and attitudes towards those with whom he or she is communicating... In most OECD countries, value is placed on flexibility, entrepreneurship and personal responsibility. Not only are individuals expected to be adaptive, but also innovative, creative, self-directed and self-motivated... Coping with today's challenges calls for better development of individuals' abilities to tackle complex mental tasks, going well beyond the basic reproduction of accumulated knowledge. An underlying part of this framework is reflective thought and action. Thinking reflectively demands relatively complex mental processes and requires the

subject of a thought process to become its object. For example, having applied themselves to mastering a particular mental technique, reflectiveness allows individuals to then think about this technique, assimilate it, relate it to other aspects of their experiences, and to change or adapt it... Reflectiveness implies the use of metacognitive skills (thinking about thinking), creative abilities and taking a critical stance."

The reasons for this urge can be expressed in two words: Informational Age. Industrial revolution, an offshoot of Reformation, rationalism and positivism, brought about a society based on profit and productivity, but aroused a new anxiety among skilled workers: the risk of being replaced by machines. Eventually, production was assigned to machines or robots, and most of the workforce moved to services. Today, as artificial intelligence replaces human reasoning, there is only one way for the society: up on the cognitive spectrum. Authorities compel educational systems to change their curricula and teaching methods in order to promote higher-order thinking and to provide graduates with the set of Knowledge-Attitudes-Skills (KSA) required by the corporations: autonomy, ability to solve/to complete complex, poorly structured, open problems/tasks, lifelong training and self-improvement, reflectiveness, etc.

The "literacy" concept of OECD actually refers to "information literacy" and involves students' capacity to pose, solve and interpret problems in a variety of subject matter areas. PISA assessments began with comparing students' knowledge and skills in the areas of reading, mathematics, science and problem solving. Students have to acquire expertise in dealing with information: identifying the missing knowledge and searching for it, keeping the useful part and discarding the useless one, structuring, processing and communicating the information in an appropriate way. In order to develop such skills, learner-centered approaches such as IBL have to be used, since expository teaching generally aims at a quick delivery of a vast content to a larger audience and thus it cannot take care of each student's mental constructions and cognitive development.

My interest in inquiry comes from secondary school and was elicited by the TV series "Connections"[1] (by J. Burke). It demonstrated how various discoveries, technological advancements and historical events derive from seemingly unrelated events, actions or innovations, and may be triggered by casual happenings. The main idea is that we cannot understand the development of any concept, event, or of the society as a whole if we consider it in isolation. History and innovation are often driven by actions and motivations that would normally lead to different outcomes, by a "law" of hazard and unintended consequences.

I first encountered IBL in middle school, where the physics teacher employed an experiential/exploratory approach. His classes were very engaging, and for helping students remember the laws of physics he used funny acronyms. My first teacher in geometry had an easy-going but conceptual approach that helped students to get into a "flow", as Csikszentmihalyi (1997) would say. The second teacher had an opposite approach and I got a falling grade on the first test since I did not justify much of my work. The teacher was surprised and asked me if in my native city geometry is taught without proofs. She required students to write two-column proofs following a prescribed format. So I had to learn such formal proving, which did not take much but inhibited the "flow" in a way. When reviewing students' attempts at solving the problem presented in the section 3.5 of this thesis, I noticed that some of them, trying to use the Backward-Forward technique of proving, got confused and produced circular reasoning. Thus, training students in writing the two-column proofs may perhaps be justified at a very basic level, but not later since formalism hinders intuition:

> *"Intuition cannot give us exactness, nor even certainty... As certainty was required, it has been necessary to give less and less place to intuition... But we must not imagine that the science of mathematics has attained absolute exactness without making any sacrifice. What it has gained in exactness it has lost in objectivity. We can now move freely over its whole domain, which*

---

[111] https://en.wikipedia.org/wiki/Connections_(TV_series)

> *formerly bristled with obstacles. But these obstacles have not disappeared; they have only been removed to the frontier, and will have to be conquered again if we wish to cross the frontier and access the realms of practice... Pure logic cannot give us this view of the whole (structure); it is to intuition we must look for it... The chief aim of mathematical education is to develop certain faculties of mind and among these, intuition is by no means the least precious... It is by logic that we prove, but by intuition that we discover."*
> (Poincaré, 1908, pp. 123-129)

I met the "DTPC" (definition-theorem-proof-corollary) approach at the university, but it was not purely expository since there were also practice hours (problem-solving sessions or tutorials). From a learner's perspective, this approach provides more fluidity and flexibility by using gaps in proofs, but makes achieving understanding, operational skills and longtime retention more difficult. The main issue regarding purely expository teaching is that the teachers do not test students' understanding or their "mental constructions" (in the sense of the constructivist framework) during lectures, so they cannot evaluate on the spot students' learning and adapt their lecture to the development of students' cognition. Thus, lessons may be simply recorded and used by the students or by other teachers. Moreover, it is less costly to teach online with such an approach, since there would be no overhead costs: classrooms, maintenance, etc. As a result, expository teaching is moving from classroom to the web and soon only some IBL courses will be given in the class. The change is striking in professional training, where e-learning thrives.

This thesis aims at reflecting on the possibility of incorporating some valuable elements of inquiry-based learning in the teaching of mathematics at postsecondary level, in order to induce and to stimulate students' higher-order thinking and skills. Such elements are:

- Modeling by the teacher of the heuristics involved in building a proof or solving a problem;

- Description of the approaches, strategies and thinking processes that lead to a successful resolution of the task;

- Introduction of students' inquiry phases with flexible guiding according to their needs;

- Feedback from the teacher and classroom discussion;

- Development of learning materials and evaluation tools that promote understanding and help students practice inquiry through gradual cognitive and metacognitive guiding.

I will also propose a conception of Inquiry-based Lecturing (IBLecturing) that is a compromise between two radically opposite approaches in the teaching of mathematics:

- The DTPC approach, composed of definition-theorem-proof-corollary style expository lectures, standard exercises for homework and limited time (1 – 3 hours) tests and examination using questions also similar to those that students have seen before in class or online.

- The pure IBL approach or inquiry-based learning, where lecturing is reduced to a minimum and students learn or invent new mathematics as they explore non-standard problems; assessment is based on reports from longer term projects or activities.

The DTPC approach, which belongs to the "tell and drill" teaching methodologies (Adler, 1993), has long been criticized for being ineffectual in terms of students' learning: students can learn to solve the standard exam problems, but are often helpless when faced with new types of problems and there is little transfer from what they learn in one such course to another. On the other hand, pure inquiry approach is not realistic in mathematics because students are being prepared for knowing and applying advanced theories and techniques and, at the graduate level, even inventing new mathematics, and for that they have to be quickly acquainted with the preliminaries. Expository approaches offer a low-cost solution but the trade-off is students' poor training in high-level thinking, as well as a lack of integration, structure and reliability of their mental constructions.

In practice, purely expository or inquiry-based approaches are very seldom employed. Belsky (1971) replaced the binary classification of teaching approaches with a continuous range from 0% to 100% of inquiry activities for each of them:

> *"…classroom presentations are not classified as exclusively representative of either 'expository' or 'inquiry' methods. Rather, a continuum is envisioned in which the method is evaluated as tending toward one or the other of these categories, based on the proportion of time the teacher resorts to one method or the other. It goes against common experience to believe that actual classroom activity can accurately be represented as exclusively the product of any system of methodology. Rather, all teachers alter their presentation styles."* (Belsky, 1971, pp. 10-11)

Belsky (p. 92) recorded the proportion of lesson time teachers spent on each of these approaches over a period of two months. Time spent in extraneous activities during the lessons was excluded from calculations and only the periods of effective learning were counted. Below are the results for the teachers who were not asked to change their behavior. Those who identified themselves as "Inquiry-oriented" had an average of 53.2% of the teaching time spent in inquiry approach according to their self-perception, but their observed average of inquiry learning was only 36.4%. Those who identified themselves as "Expository-oriented" had an average proportion of 74.5% of the time in expository teaching according to their self-perception, but their observed average of expository teaching was 90.2% . Those who were asked to change their approach from expository to inquiry estimated the proportion of time spent in inquiry approach, after the change, at 67.5% on average; however, the observed proportion was only 46.8% . In conclusion, teachers that believe they follow IBL spend just a bit more than a third of the teaching time in inquiry approaches, and those who perceive themselves as expository teachers increase the proportion of IBL in their lessons from 10% to less than 50% after being asked to change their method.

So there is a middle way. I propose to keep the lecturing by allowing it half of the lesson and enrich the exposition of the results of mathematicians' research with elements of the inquiry that led to them, connections with other subjects and openings toward novel results. There would be standard homework exercises and exams, but also "challenges" where students would be given more time and could engage in inquiries of their own.

Teachers and institutions need some systematic way to plan their teaching and assess students' learning. For the traditional expository teaching, a popular aid for these purposes has been a classification of cognitive learning objectives called Bloom's taxonomy. Students have to become acquainted with some "Knowledge"; e.g., in mathematics, they have to know some definition. For example – the definition of a limit of a sequence. They must then develop some "Comprehension" of this knowledge. Further, they must be able to engage in an "Application" of this knowledge.

In the proposed IBLecturing approach, students observe and become familiar with the new content presented by the teacher. Such content is carefully selected by the teacher in order to allow students' exploration and completion of omitted parts. For example, proofs can be presented in an abridged form, and the teacher asks students to complete the details as homework or in the classroom, in a team or under individual guidance. Alternatively, learners can be properly guided in the resolution of problems and the completion of proofs (which are also problems) by allowing them to try at first without any hint or help from the teacher. After each step of the exploratory phase, students ask and receive some individual or group scaffolding and feedback from the teacher, then continue their exploration and so on until completing the task. Such activities may take place in the math lab (physical or virtual) or in the classroom. In this way, students get a taste of the subject by "Operating" with it under the guidance of an expert. Guiding is partially provided in the form of teacher's expository explanations of his own inquiry or a heuristic approach (that is, modeling by an expert), partially via hints, suggestions and individual metacognitive feedback. The key element is to provide the right amount of guiding to each learner, in order to continually keep students "in the flow". They are encouraged to pose questions and to seek an answer for themselves with gradual teacher's help. As they progress, more complex or difficult tasks are given to them and the context or the form of the particular knowledge is modified in order to help them learn when it works and when it does not, in which situations it applies or does not apply, when and why one procedure is better than another, etc.

In Bloom's taxonomy, the cognitive actions that students are expected to engage in are measurable: memorizing factual knowledge, understanding what it is/does, where and how it can be employed, applying it to given problems, etc. In IBLecturing, outcomes are measurable by using numerical scoring for standard assessment based on short answer questions and a set of performance criteria aligned with targeted outcomes (competencies) for inquiry tasks: projects, research reports, etc.

IBL aspires to engage students in cognitive actions that resemble more those of a research mathematician at work than a student preparing for an exam. IBLecturing's ambitions are adapted to students' knowledge, skills, and attitudes (KSA) but still, it is assumed that the lecturer would be able to convey the main elements of the mathematical inquiry. Naming and describing these elements and giving examples of how they can be interwoven in the lectures will be my way of presenting this approach.

The thesis is structured as follows. In Chapter II of my thesis, I present the IBL approach and the difficulties with its characterization on the one hand, and with its implementation on the other. Chapter III contains a description of the IBLecturing, with several illustrative examples. In Chapter IV, I offer some conclusions and reflect on avenues of further research.

2 CURRENT CONCEPTIONS OF IBL IN THE LITERATURE

2.1 INTRODUCTION

"There is a great satisfaction in discovering a difficult thing for one's self... and the teacher does the scholar a lasting injury who takes this pleasure from him. The teacher should be simply suggestive." - David Page (1847, p.85)

"Children should be able to do their own experimenting and their own research. Teachers, of course, can guide them by providing appropriate materials, but the essential thing is that in order for a child to understand something, he must construct it himself, he must re-invent it. Every time we teach a child something, we keep him from inventing it himself. On the other hand, that which we allow him to discover by himself will remain with him visibly." Piaget (1972 a, p.27)

Curiosity is an essential human trait, and inquiry (investigation, exploration, quest, or research) is the natural expression of this mental drive. Although British English makes a distinction between "enquiry" (i.e. questioning, request for information) and "inquiry" (i.e. investigation), in American English these words have the same meaning. In science education, instructional tools such as investigation and questioning cannot be separated, because any research starts with one or several questions about the subject of investigation, and questioning is generally induced by the desire to understand something or by curiosity, as the first step of an investigation. Moreover, question generating is a main feature of inquiry learning. Hence, we may see scientific inquiry as a blend of receptive and active search for an answer, a fusion between questioning (seeking answers among familiar knowledge or available resources, including tutors) and exploration (seeking outside of readily available resources, venture into the unknown).

From the earliest age of humanity, questioning has been considered a main element of both formal and informal education. A classic example of questioning as a teaching tool is Plato's "Meno", where a detailed account of the Socratic dialogue is presented.

Xun Kuang, a Confucian philosopher (312-230 B.C.), was an early promoter of the so-called "hands-on" or experiential approach in education:

> *"Not having heard something is not as good as having heard it; having heard it is not as good as having seen it; having seen it is not as good as knowing it; knowing it is not as good as putting it into practice."* (Knoblock, 1990, p.81)

The saying has been restated by Dr. Herb True as

> *Tell me, and I'll forget. Show me, and I may remember. Involve me, and I'll understand.* (True, 1978)

In this form, it became a slogan among education researchers, some of them wrongly citing Confucius (Hmelo-Silver et al. 2007, p. 105) or Benjamin Franklin as its originators.

Inquiry approaches in education have also been promoted by educationalists such as Comenius (1592–1670), Rousseau (1712–1778), Pestalozzi (1746–1827) or Dewey (1859–1952).

Dewey was an early promoter of problem-based learning (PBL), as part of his student-centered, interactive, "hands-on" approach, which he called "learning by doing". In particular, his emphasis on "experience" is in vogue again (EduTech Wiki, n.d.). Far West Lab's report on Experience-Based Career Education was one the first attempts to extend Dewey's method to a "hands-on, minds-on" approach, based on exploration and investigation (Johnson's 1976, p.140). Finally, the formula became "hands-on, minds-on, hearts-on" in a review of technical education in Singapore, indicating a holistic approach that provides motivation, assisted learning, and integral training for the students. The learners were expected to acquire strong technical skills, flexible and independent thinking and passion for what they do, as well as confidence and care for the community and society (Lee 2008, p. 126).

Another type of inquiry instruction has been designed by Maria Montessori (1870-1952). It is a student-centered approach which generates individual learning opportunities and encourages child's involvement in the learning process, fostering his autonomy and motivation. Play, which allows children to conduct a thorough exploration of the world by self-directed and stimulating activities, has been strongly advocated by Froebel, the father of kindergarten, and by Piaget (Gallagher & Reid, 2002). Great education theorists and psychologists such as Pólya, Piaget, Ausubel, and Bruner, proposed a challenging and exploratory type of instruction, based on student's engagement in self-directed activities, problem solving and discovery (Maaß & Artigue, 2013). Lakatos (1976) considered mathematical inquiry a cornerstone of mathematical practice. A convincing plea for inquiry has also been made by Papert (1990): "You can't teach people everything they need to know. The best you can do is position them where they can find what they need to know when they need to know it."

Colburn (2006) remarked that science education community has embraced no idea more than that called "inquiry", or "inquiry-based instruction". At the same time, discovery (a former label for inquiry learning) has been considered "one of the most advocated if not most popular teaching strategies of the past three or four decades" (Brooks & Shell, 2006). Yet, discovery has also been one of the most contested topics in science education (e.g. Ausubel, 1964; Novak, 1973; McDaniel & Schlager, 1990; Taconis et al., 2001; Mayer, 2004), and the controversy continues now regarding inquiry learning (Palmer, 1969; Rogers, 1990; Kirschner et al., 2006; Hmelo-Silver et al., 2007; Smith et al., 2007).

In official frameworks and national standards, there are many calls for a greater emphasis on student inquiry in science education. Developing an inquiry-based science program is the central tenet of the National Science Education Standards for K-12 education (NRC, 1996), but the stage has already been set by the Project 2061, a long-term initiative to reform U.S. school education (Bybee, 2000). The National Academy of Sciences, which shapes U.S. government policies in science education has stated that the main focus of science education should be inquiry (NRC, 1996). A large number of government-sponsored programs and projects have been undertaken in North America, Europe and Asia in order to promote inquiry at all education levels: FIRST and POGIL projects in the U.S.A. (Ebert-May & Hodder, 1995; POGIL, 2015), CREST program in the U.K. and in Australia, Fibonacci, PRIMAS, and MASCIL in Europe (Baptist & Raab, 2012; Maaß & Artigue, 2013), High Scope Program in Taiwan (APSE, 2011), PBI@School in Singapore (Wong et al., 2012) and SEAMEO QITEP in Indonesia (SEAMEO, 2015).

The National Science Foundation supported a review on the role of inquiry in science teaching (Project Synthesis, concluded in 1981), which revealed that the term "inquiry" has been used by the education community in a variety of ways, either as content or as instructional technique, and thus a confusion developed about the term's meaning (Bybee, 2000). Further, Hammer (2000) remarks that, as a general nicety, student inquiry seems a simple, desirable goal. Yet, implementing inquiry is not a simple matter at all. No one understands clearly how to discern and assess it, or how to coordinate such progressive agenda with the traditional one of covering the content. Moreover, this is not for lack of trying, since various attempts by philosophers of science to define what the specific method is - e.g. Popper (1963), or by educators to specify "process" skills as appropriate educational objectives - starting with Gagné (1965) - largely proved to be unsuccessful. According to Hammer (2000), if it is possible to capture the essence of scientific reasoning - and some authors contend that it is not, e.g., Feyerabend (1988) - such an attempt remains to be done.

This chapter tries to clarify in the first place the meaning of IBL by identifying the key features that distinguish it from traditional approaches and by providing a comparative survey of inquiry-based ways of teaching. It reviews the various implementations of IBL and exposes the benefits as well as the obstacles related to its use in school education. The final part is focused on the relevance of inquiry approaches in pure geometry, by pointing out the crucial role of proof and inquiry in this field and the benefits of using IBL in conjunction with geometry for developing students' higher cognitive skills. It reviews the research and the implementation of inquiry approaches in teaching geometry, and presents several software and IBL textbooks that could be employed with school students.

2.2 Inquiry-based learning – a characterization

2.2.1 What is IBL?

> *"Every truth has four corners: as a teacher I give you one corner, and it is for you to find the other three."* – Confucius

Providing a clear, thorough definition of IBL is a difficult task. Spronken-Smith (2007, p. 4) contends that the nature of IBL is contested and any search for studies on this topic must include such terms as "inquiry" (or "enquiry"), "discovery learning", "research-based teaching", "inductive teaching and learning". Although the inquiry approach is becoming pervasive throughout all levels of education, there is a paucity of research that provides a clear overview and synthesis of IBL. While each author seems to choose his own working definition of IBL, there is a commonality of opinion about what it constitutes.

Discovery learning, the catchphrase of mathematics instruction in the years 1960s (Fey, 1969), has been the term of choice for inquiry education for almost thirty years, although nowadays it usually refers to investigations where a specific mathematical content has to be "discovered". By contrast, in open inquiry the teacher does not have such an agenda, and leaves to his students the freedom of exploration. Gagné (1966, p. 135) characterized discovery learning as "something the learner does, beyond merely sitting in his seat and paying attention" - a very broad description, comprising all kinds of active learning and including inquiry. The recent prevalence of the term "inquiry" could be explained by the fact that minimally guided discovery has generally been discarded as ineffective in mathematics education (Kirschner et al, 2006, pp. 12-13; Sweller, 1999; Mayer, 2004), while strongly guided discovery may be too restrictive (Clark, 1988, p. 339).

Finley & Pocovi (2000) argue that scientific inquiry does not necessarily involve experimental discovery, since most of the greatest scientific accomplishments consisted of describing, finding rules or laws, explaining and modeling various phenomenon. Indeed, in many cases (such as history, geology, astronomy, social sciences, etc.) experimentation may be impossible, and a scientific finding can only be validated by the accuracy of its predictions.

Hence, investigation skills are fostered not only in discovery, but in inquiry learning, too. As the label "discovery" gradually lost its appeal, IBL has come to designate open and guided inquiry, previously covered by discovery approaches.

Yet, the confusion between discovery and inquiry learning persisted, and not long ago there were still studies claiming that "one of the most advocated it not most popular teaching strategies of the past three or four decades has been discovery" (Brooks & Shell, 2006) or asserting that "the nature of IBL is contested and even the term itself is not in widespread use throughout the educational literature" (Spronken-Smith, 2007).

Dorier & Maaß (2012) define Inquiry-Based Education (IBE) as a student-centered paradigm of teaching science, in which students are invited to work in ways similar to scientists' work. Students are guided to observe phenomena, ask questions, seek scientific ways of answering related questions (e.g. carrying out experiments, systematically controlling variables, drawing diagrams, looking for patterns and relationships, making conjectures and generalizations), interpret and evaluate their solutions, communicate and discuss these effectively. According to Hussain et al. (2012, p. 286), the term "inquiry mathematics" is often associated with Western reform movements from the 1980s onward. For instance, NCTM (1991) provided this description of IBL:

> *"Students should engage in making conjectures, proposing approaches and solutions to problems, and arguing about the validity of particular claims.... They should be the audience for one another's comments... . Discourse should be focused on making sense of mathematical ideas... and solving problems".*

Wells (1999, p. 122) views inquiry as an active quest for knowledge that arises in activity, often collaborative investigative activity, in mathematics classroom: "[inquiry is a] willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them". A similar viewpoint has been adopted by Hussain et al. (2012): IBL involves a re-evaluation of the nature of mathematics, and could be seen as an "ethical consequence" of valuing students' mathematical investigations, so strongly promoted by the current reforms in education.

To distinguish among conceptions of IBL, some authors (Staver & Bay, 1987; Colburn, 2000; Ako, 2008) distinguished three levels of inquiry:

- **Structured** inquiry (problem & method given)

- **Guided** inquiry (problem only given)

- **Open** inquiry (students formulate & solve the problems)

Characteristics of pure (open) inquiry are:

1. students are involved in their learning, symptom of situational interest (Mitchell, 1993)
2. students pose/formulate questions (Bruce & Davidson, 1994; NRC, 1996; Alberta Education, 2004; Colburn, 2006; Beairsto, 2011; EduTech Wiki)
3. students investigate widely (Alberta Education, 2004)
4. the knowledge students build is new to them (Chan et al., 1997; Alberta Education, 2004)
5. students communicate their solutions to others (Alberta Education, 2004; EduTech Wiki)
6. tasks given to students are open-ended (Colburn, 2006; EduTech Wiki)
7. teaching is student-centered, implying student's interest as person-object relation (Schiefele et al., 1979; Krapp et al., 1992; Bruce & Davidson, 1994; Hidi et al., 2004; Colburn, 2006)
8. activities for students are hands-on (Colburn, 2006; Bruce, 2008a; EduTech Wiki)
9. students solve problems (Colburn, 2006; Beairsto, 2011)
10. students develop their own ways towards solutions (Beairsto, 2011)
11. students' questions should be scientifically oriented (NRC, 2000)

12. students give priority to evidence in responding to questions (NRC, 2000)
13. students formulate explanations based on evidence (NRC, 2000)
14. students connect explanations to scientific knowledge (NRC, 2000)
15. students communicate and justify explanations (NRC, 2000)
16. students create tentative generalizations (Colburn, 2006)
17. students exercise reflective practice (Spronken-Smith, 2007)

The above characteristics have been organized according to the subject of their postulates: the relation of students with their own knowledge; the students' attitude towards mathematics (4, 10, 11, 17); students' mathematical activities (5, 12, 13, 14, 15); students' rapport with teachers (2, 3) and teachers' rapport with students (8); teachers' rapport with mathematics (7, 9) (Figure 1).

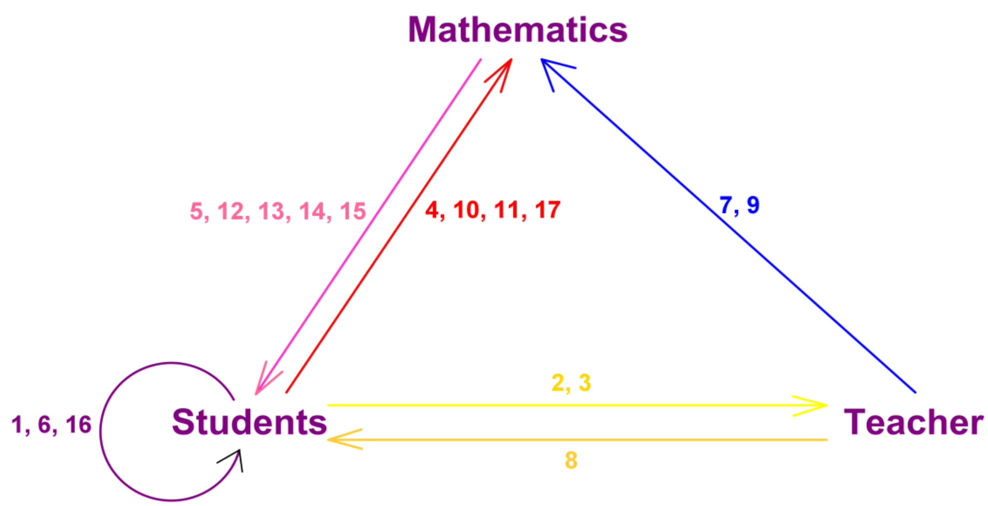

Figure 1. IBL characteristics on the didactic triangle

Lave and Wenger (1991) proposed a "situated learning" model, where learning takes place in a "community of practice." Wenger (1998) listed three modes of belonging to such a community:

- engagement
- imagination
- alignment

Wenger's ideas have been integrated by Jaworski into her vision of an "inquiry community", with a specific emphasis on the *critical alignment* of the participants, such that it is possible for them "to align with aspects of practice while critically questioning roles and purposes as a part of their participation for ongoing regeneration of the practice." (Jaworski, 2006, p. 190) Critical alignment has also been identified as a main feature of inquiry learning in the study of Goodchild et al. (2013).

Vulliamy & Webb (1992) discuss the process-product distinction in collaborative inquiry, and show that in their teacher development programme the process has been rated as, or more, important than the product (the degree, etc.). Although teachers may have registered with certification in mind, the professional learning that resulted from their inquiry became more significant for many of them. In mathematics education, many such programs have reported the importance of process of engagement in research or inquiry for professional learning and development (e.g., Krainer, 1993; Britt et al., 1993). Thus, engagement in individual inquiry for each teacher, results in knowledge growth that enhances that individual's teaching." (Jaworski, 2003, p. 258)

The role of student's engagement and participation in challenging and meaningful activities was emphasized by many researchers. According to Bishop (1991), reform-oriented approaches claim that doing mathematics should involve sense-making activities by using tasks that provide students with a variety of challenging experiences through which they can actively construct their mathematical meanings. Within this "active learning" approach, associated with experiential, collaborative and inquiry-based instruction (Anthony, 1996), students gain autonomy and take control over the direction of their learning. Research has shown that inquiry learning is positively correlated with students' goal-direction and satisfaction (Fresko et al., 1986). Bruce et al. (1994) claim that students' engagement in meaningful activities is the key to deep, effective learning:

> *"Children are able to learn enormously complex things through immersion in the world, through their participation in meaningful activity. When they see a reason to participate, which depends in part upon an understanding of the*

activity as a whole, their learning proceeds at an amazing pace. When they do not, major contortions in schooling practices are required to produce even minimal behavioral changes. Moreover, there is little evidence that the piecemeal learning that results from those contortions can be reintegrated into the whole activity later on." (Bruce & Davidson, 1994, p. 9)

The importance of students' engagement has also been emphasized by Adler (1997), who used the term "participatory-inquiry approach" for IBL and stressed that it is often driven by the twin goals of: **1)** moving away from authoritarian, teacher-centered approaches to learning and teaching and to mathematical knowledge itself, and **2)** improving socially unequal distribution of access and success rates. In this approach, pupils are expected to take responsibility for their learning. Typically, they engage with challenging mathematical tasks, either alone, but more likely in pairs or small groups. The knowledge pupils bring to class is recognized and valued. Diverse and creative responses are encouraged, and justifications for mathematical ideas sought, often through having pupils explain their ideas to the rest of the class. The task-based, interactive mathematical activity that is provided in such a class offers learners a qualitatively different mathematical experience, and hence possibilities for knowledge development that extend beyond traditional "telling and drilling" of procedures (Adler, 1993).

In IBL, mathematics is seen as a *practice* (Adler 1997, p. 237). According to Adler, there is a bridge to cross between every day and school mathematical discourse, since "good mathematics teaching entails chains of signification in the classroom" (Walkerdine, 1988).

Jaworski (2006) describes three types of inquiry practices:

- inquiry in <u>mathematics</u>: students' learning of mathematics through exploration in classroom
- inquiry in <u>mathematics teaching</u>: design of tasks for students by teachers and other educators

- inquiry in research: research of the inquiry process carried out in the first two levels

In the field of teachers' education, Jaworski (2003, p. 256) identifies inquiry with research and emphasizes the distinction between learning *as a process* and learning *as a product*, an issue already explored by Vulliamy et al. (1992). A similar distinction has been done by Lakatos (1976, p. 42) regarding mathematics itself, since it develops as a process of "conscious guessing" about relationships among quantities and shapes, where proof follows a "zig-zag" path starting from conjectures and moving to the examination of premises through the use of counterexamples. This activity of doing mathematics is different from what is recorded once it is done: naive conjectures and their testing (validation or refutation) do not appear in the fully fledged deductive structure: "The zig-zag of discovery cannot be discerned in the end product."

Regarding the deductive way in which mathematics is taught at all school levels, Pólya (1957, p. 7) said: "Mathematics 'in statu nascendi' - in the process of being invented - has never before been presented in quite this manner to the student, or to the teacher himself, or to the general public".

Lampert contends that

> *"...the product of mathematical activity might be justified with a deductive proof, but the product does not represent the process of coming to know. Nor is knowing final or certain, even with a proof, for the assumptions on which the proof is based continue to be open to re-examination in the mathematical community of discourse. It is this vulnerability to re-examination that allows mathematics to grow and develop".* (Lampert, 1990, p. 30)

It could be added that proofs themselves are continuously re-evaluated, improved or enriched, and the final product may include different proofs for the same theorem - each one having specific mathematical or pedagogical advantages. Likewise, Klein (1932, p. 208) remarked that "the investigator himself ... does not work in a rigorous deductive fashion. On the contrary, he makes use of fantasy and proceeds inductively, aided by heuristic expedients". As a matter of fact, inquiry and inductive learning are so closely related, that the distinction between them is still debated. Lehrer et al. (2013) define induction as a new form of practice to students, one in which questioning was installed as a norm in the classroom.

An inductive, inquiry-based approach in mathematics education that would be more meaningful and could spark student's interest in proving has been proposed by

Moore (1903) in a speech before the Mathematical Association of America on December 29, 1902:

> *"The teacher should lead up to an important theorem gradually in such a way that the precise meaning of the statement in question ... is fully appreciated ... and furthermore, the importance of the theorem and indeed the desire for formal proof is awakened, before the formal proof itself is developed. Indeed, much of the proof (of the theorem) should be secured by the research of the students themselves". (Moore, 1903, p. 419)*

Teachers should put more emphasis on students' understanding of the process by which mathematical knowledge is generated, in order to promote their ability to recognize patterns and to find general rules:

> *"In our mathematics classes we ought to concentrate less on covering a certain body of knowledge and more on thinking about what we have done, how that can be generalized and applied to other problems, and how to go about finding general principles."* (Willoughby, 1963)

A gradual transition from particular to general, which would elicit students' desire to go on to the abstract level, has been suggested long time ago by Durell (1894) when he advocated the "New Education" in school mathematics - a label similar to the "New Math" from the 1960s:

"In each new advance, the student should begin with the concrete object, something which he can handle and perhaps make, and go on to abstractions only for the sake of realized advantages." (Durell, 1894, p. 15)

Pólya (1957) has underlined the importance of problem-posing, an activity closely related to questioning and specific to discovery/inquiry approaches:

"The mathematical experience of a student is incomplete if he has never had the opportunity to solve a problem invented by himself." (Pólya, 1957, p. 68)

The need to move towards an active, student-centered, and more stimulating way of teaching mathematics has been recognized for a long time: "... it is first necessary to arouse his (the student's) interest and then let him think about the subject in his own way (Young, 1911, p. 5). As stated by Ivey (1960, p. 152), "The premise here is that education has a great teaching facility which as yet is unused - the student."

The meaning of "active learning" in inquiry-based instruction is finding, generating and structuring the information. Therefore, investigative tasks have been classified by Calleja (2013) according to the degree of structure/guidance provided to students, the mathematics embedded within the task, and the time devoted for students' activity:

a. At the basic level, the investigations are <u>structured</u> tasks that lead students to mathematical discoveries. The given instructions guide students, who worked individually or in pairs, to use particular predetermined mathematical concepts and apply them to arrive at a solution.

b. At the next level, the investigations are <u>semi-structured</u>. This means that they are either less structured or students are initially given some guidance in their work but later they are free to explore and engage with the task using their own conceptual mathematical understanding and reasoning. In order to benefit from discussing ideas and solutions when working on these more challenging tasks, the students are instructed to work in small groups of two or three.

c. At the third and higher level, the students encounter unstructured investigations that are more process-oriented activities. These required students to investigate the problem posed or the situation presented in as many different ways as they wished and through different methods. These investigations place greater demands on students to think through a solution, to make inferences and to test their own conjectures. As this type of investigation requires students to challenge, argue about and justify their reasoning, the unstructured investigations are set as a group activity involving three to four students.

Other than the level of structure, the investigations may be classified along the three "reality levels" identified by Skovsmose (2001). Skovsmose sees mathematical investigations as a landscape that ranges across the following levels of real-life contexts:

i. pure mathematics which simply involves working with numbers or geometric figures;
ii. semi-reality which refers to an everyday-life problem that is rendered artificial as it is tackled in a classroom situation where variables can be controlled;
iii. real-life situations where students are directly involved in carrying out the exercise in the actual setting.

By combining these two classifications we get nine types of investigations (Calleja, 2013, p. 166).

Observations of informal acquisition of knowledge and skills that occur outside of school settings, such as children *learning to ride their skateboards* with a group of friends, offer compelling models of learning that are not task-dependent, rather they are participant or learner-determined. Children can be seen to flourish within these forms of *self-selected* and *self-directed* experiential learning. The learning that occurs in student-centered approaches is a form of playing around. It is socially valued and seen as worthwhile.

The learners feel supported by a self-selected social group. They learn at their own pace, in their own time, and in a place of their choosing. Children are free to make mistakes which they accept as a natural and even humorous part of learning. They challenge each other to take risks, and they provide each other with informal feedback, helpful hints, and encouragement. They are free to discover and invent, they can start and stop whenever they like, and they gain intrinsic satisfaction from their growing accomplishments. Above all, the learning engages the whole child - the cognitive, affective, motor-sensory and social "self" (Walls, 2005).

Communication in the classroom under the form of questioning, answering, and presenting in a clear, rigorous way his own reasoning or ideas to the other participants is strongly promoted in inquiry learning. Also, team work is a key element in collaborative inquiry, even if it is not limited to inquiry approaches. As Ben-Chaim et al. (1990, p. 415) remarked, even if instruction tends to be more individual than collective, it does not occur in isolation but rather in interaction with the teacher and the peers. Lastly, creativity and critical thinking are essential skills for scientific inquiry, because they deal with the processes of generating and testing of hypothesis, respectively. "Good research is not about good methods as much as it is about good thinking" (Stake 1995, p. 19).

Pollard (1997, p. 182) describes how teachers might provide for *negotiated curriculum*, arguing that rather than reflect the judgments of the teacher alone, it builds on the interests and enthusiasms of the class and noting that "Children rarely fail to rise to the occasion if they are treated seriously. The motivational benefits of such an exercise are considerable". According to Ernest (1991, p. 288), the role of the teacher should be to support this student-centered pedagogy, as manager of the learning environment and learning resources, and as a "facilitator of learning". Yet, the implied dichotomy between teaching and "facilitating" has been strongly contested by authors like Stewart (1993). Neyland (2004, p. 69) argues that a postmodern ethical orientation to mathematics education will shift the focus away from procedural compliance and onto direct ethical relationship between teachers and their students. In a participant-determined pedagogy, the learner would be seen as a growing and valued member of a local community, instead of an educational product. Within such a discourse, mathematics education might embrace some of the following principles:

1. mathematics curriculum is locally negotiated between schools, parents, and children
2. flexible learning situations are collaboratively shaped between teachers and children
3. learning situations are not constrained by specific learning outcomes – only broad goals are stated
4. children engage in learning situations at their own pace and in a manner of their choosing
5. children choose with whom to engage in the learning situations
6. children seek information and assistance from a variety of sources, not just the teacher/textbook
7. children assess their own learning according to collaboratively constructed assessment criteria
8. all learning and assessment operates to enhance the physical and social well-being of children

2.2.2 What IBL is not?

"There ain't no rules around here! We're trying to accomplish something!" - *Edison*

In traditional education, *soft skills* such as communication, collaboration, critical thinking and creativity (21st Century Skills), necessary for applying academic learning in real world contexts, are addressed only tangentially, if at all (Beairsto, 2011). The same can be said of higher-order skills in the revised cognitive taxonomy (Anderson et al., 2001).

Although from the early 1900s the challenge of integrating thinking and content area knowledge concerned only elite education, currently it is expected that thinking and reasoning be included in all students' education (Kinder et al., 1991).

One aspect of higher-order skills is the organization of knowledge, notably the richness of connections: "The ability to access knowledge varies dramatically as a function of how well linked the knowledge is." (Prawat, 1989, p. 4) However, as indicated by Kinder et al. (1991, p. 207), most instruction does not provide indications, let alone specific instruction, to facilitate the linkage of knowledge. Inquiry-based approaches are powerful means for improving this situation, especially in mathematics education.

IBL stands in sharp contrast to the traditional approach, where teacher-initiated recall-type questions and I-R-F interactions (initiation-response-feedback) predominate and where pupils "go for an answer" (Campbell, 1986). Inquiry lessons are better described by pupils' "going for a question" (Adler, 1997, p. 243). In traditional classrooms, "pupils all do the same thing in the same way" (Adler, 1997, p. 236), while in IBL teaching and learning are differentiated, and the tasks are customized.

In Table 1 I summarize the main differences between traditional and inquiry learning:

Table 1. Comparison of traditional and inquiry approaches

Traditional instruction	Inquiry instruction
1. teacher-centered 2. information/procedures have priority 3. promotes knowledge volume 4. memorization/procedural skills are fostered 5. students learn by seeing 6. many students are not observed/supported in the classroom, non-participants are "left behind" 7. learning of recipes, rules and procedures 8. local/tactical thinking and perspective 9. communication is initiated/led by the teacher 10. students are receivers of the information dispensed by the teacher 11. teacher is the only source of relevant knowledge, only the information provided by him is required 12. the validity of knowledge is based on teacher's authority 13. promotes the basic 3 levels of Bloom's taxonomy 14. accurate recall of information and fidelity to the prescribed rules/procedures are the most desired outcomes 15. develops mostly hard skills 16. fosters cognition	1. student-centered 2. understanding has priority 3. promotes knowledge depth 4. reasoning/critical thinking skills are fostered 5. students learn by doing 6. every student is involved and each small group is observed/supported in cooperative inquiry 7. learning of adaptive/flexible methods 8. global/strategic thinking and perspective 9. communication is initiated/led by the students 10. students are posing questions, they are helped/guided by the pairs/by the teacher 11. students construct/create their own knowledge, the teacher is a mentor/coach, providing expertise, modeling, guidance 12. the validity of knowledge is based on evidence 13. promotes the top 3 levels of Bloom's taxonomy 14. creation of new and useful knowledge, invention, adapting/improving the available procedures are the most desired outcomes 15. develops hard skills and soft skills 16. fosters cognition and metacognition

17. collaborative work is not encouraged 18. instruction follows a very precise schedule 19. linear teaching approach and curriculum 20. summative evaluation is done by traditional tests/exams, quizzes 21. promotes extrinsic motivation - teaching/learning to the test, grading is the main motivator 22. uniform tasks; low ability students get frustrated by lack of progress, gifted students become bored 23. rigid, detailed and uniform curriculum	17. team work and discussion are promoted 18. instruction follows a flexible/adaptive schedule 19. holistic teaching approach and curriculum 20. assessment by creative tasks: projects, open tasks (requiring modeling, proof, inventing problems) 21. promotes intrinsic motivation - challenge and curiosity are the main motivators 22. open tasks, customized; the width and depth of student's work is according to his skills/drive 23. advisory/flexible curriculum (it may be missing)

Table 2. Comparison of common inductive approaches in education

Feature	IBL	PBL	POL	CBL	DL
Questions or problems provide context for learning	1	2	2	2	2
Complex, ill-structured, open-ended real-world problems provide context for learning	4	1	3	2	4
Major projects provide context for learning	4	4	1	3	4
Case studies provide context for learning	4	4	4	1	4
Student discover course material for themselves	2	2	2	3	1
Primarily self-directed learning	4	3	3	3	2
Active learning	2	2	2	2	2
Collaborative/cooperative learning	4	3	3	4	4

Note: 1 – by definition, 2 – always, 3 – usually, 4 – possibly

Various features of the most common inductive approaches were presented by Spronken-Smith (2007) in Table 2 , adapted from Prince & Felder (2006). The acronyms in the first row have the following meanings: IBL = Inquiry Based Learning, PBL = Problem Based Learning, POL = Project Oriented Learning, CBL = Case Based Learning, DL = Discovery Learning.

The diagram in Figure 2 describes the mutual inclusions between IBL, PBL (Problem-based learning), CBL (Case-based learning), and Active learning.

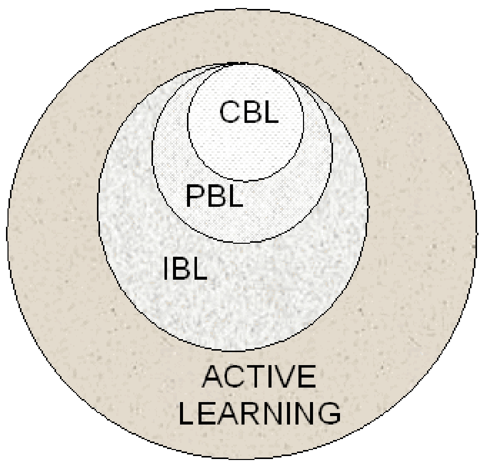

Figure 2. Relations between IBL, PBL, CBL, and Active Learning (Ako, 2008)

IBL falls under the realm of "inductive" approaches in education, which have the following features: they begin with a set of observations/data to interpret or a complex real-world problem, and as the learners study the data or the problem, they generate a need for facts, procedures and guiding principles.

Prince & Felder (2006, p. 123) state that inductive learning encompasses various teaching approaches, such as IBL, PBL, POL, CBL, and DL.

The timescale for IBL (over weeks or months) is typically much longer than for either PBL (hours to weeks) or CBL (minutes to hours). In PBL and CBL, the content and skills to be learned are usually far more thoroughly prescribed than in open inquiry; hence, they may be considered structured and guided forms of IBL (Ako, 2008).

Maaß & Artigue (2013) identify IBL with discovery and distinguish it from PBL, when discussing about "more student-centered ways of teaching, such as inquiry-based learning or discovery learning, problem based learning, and mathematical modeling". Regarding IBL, DL, constructivist learning, problem solving or PBL, they say "all these are sometimes even said to be synonymous".

Most versions of inquiry learning see a continuing cycle or spiral of inquiry (Bruner, 1965). There is usually a strong caution against interpreting steps in the cycle as all being necessary or in a rigid order. In fact, inquiry learning is less well characterized by a series of learning steps than by the concept of *situated learning* (Lave & Wenger, 1991). This notion has been introduced by Lave & Wenger in order to emphasize that learning happens as a function of the activity, context and culture in which it occurs, rather than through abstract and decontextualized presentations. People learn through their participation in a *community of practice*, and learning is a social process of moving from the periphery of the community (apprenticeship) to its center (mastery). Most of this process is incidental rather than deliberate.

A five-steps cyclic model of inquiry learning has been proposed by Bruce & Davidson (1994) as shown in Figure 3:

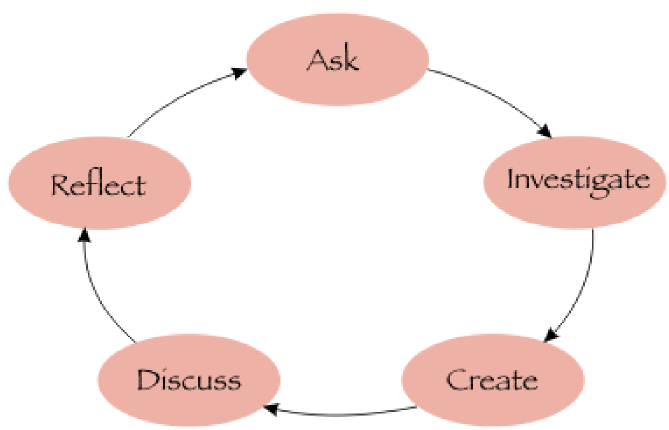

Figure 3. A cyclic model of inquiry (Bruce & Davidson, 1994)

Actually, inquiry is not a sequential process. Two phases may interact even if they are not adjacent, and the influence may work in both senses when they are adjacent: for instance, preliminary findings (Create) may result in a decision to revise the original question (Ask) or to alter data collection procedures (Investigate). Therefore, non-cyclic models of inquiry have been proposed, e.g. by Krajcik et al. (2000, p. 284) as in Figure 4:

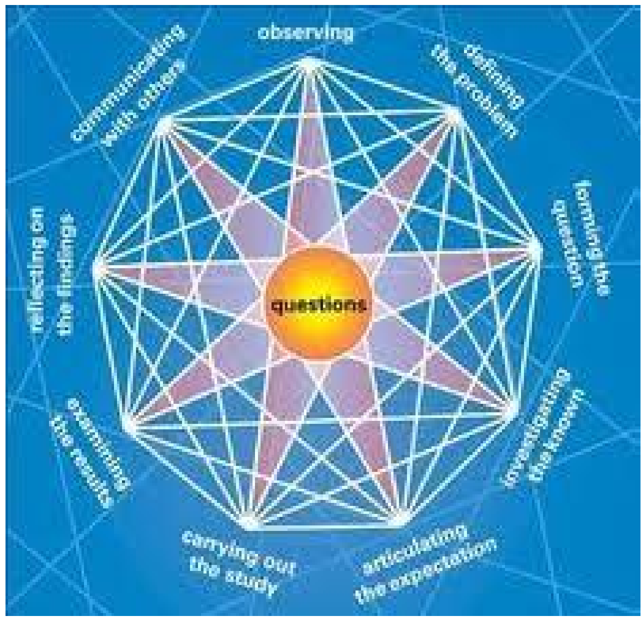

Figure 4. A non-cyclic model of inquiry (Krajcik & al, 2000)

IBL is manifested in a variety of curricular and instructional approaches, which can be roughly grouped according to the aspects of the inquiry cycle they emphasize:

Problem-based learning sets the formulation of questions as a task for the learner. An emphasis on rich, authentic materials for investigation can be seen in materials-based and research-based curricular approaches.

Project-based learning emphasizes the creative aspects of learning through extended projects and performances. Discussion and collaboration are important in cooperative learning and in much of the writing process work. Response-centered classrooms highlight the reflective and constructive aspects of meaning-making. (Bruce, 2008a).

Various inquiry approaches and the stages emphasized by them are listed below:

- Ask: open school; problem-based learning
- Investigate: materials-based, open-world; resource-based learning; investigation-based, research-based learning
- Create: project-based learning
- Discuss: cooperative learning; writing process
- Reflect: constructivist learning; reader response; service learning

It seems appropriate to add an exploratory stage at the beginning of this cycle; such a stage would become the focus of the discovery learning, i.e. an inquiry approach "concerned with the initial development of understanding" (Beairsto, 2011).

Froyd et al. (2012) mention POL and PBL among the branches of IBL: "Inquiry-based learning methods including problem-based and project-based learning (...) are products of research in cognitive psychology". However, Chan (2007, p. F3C-2) asserts that IBL is different from general PBL: "the former emphasizes the inquiry processes throughout the entire project while the later focuses on the development of the ultimate deliverables." Thus, IBL is a process-oriented approach, while POL is rather product-oriented.

Additionally, Crick (2012, p. 689) claims that "some forms of project-based learning do not allow for this sort of enquiry at all if they begin with predetermined problems or questions which already have predetermined answers. The danger then is that the learner is more concerned with finding the right answer than formulating a solution."

The confusion involving IBL, POL, and PBL has not been cleared up by Allan (2007, p. 80), as well as Bransford & Stein (1993), when they described PBL as a

> *"...comprehensive instructional approach to engage students in sustained, cooperative investigation. It is most commonly found in secondary education in the USA. It has been referred to also as problem-based learning and*

inquiry-based learning. In the PBL approach, students are required to answer a question or develop a product for example. In doing this it is felt that they are able to take control of the learning environment and process, working in groups to complete a series of tasks to each the project outcome (Brogan 2006). Because the project involves complex tasks, a range of inter-disciplinary skills is developed as distinct from focusing on one aspect of knowledge or skill development – mathematics, for example (Blumenfeld et al., 1991)". (Bransford & Stein, 1993)

Gordon (2008, p. 24) agrees that "the distinction between inquiry based learning and other approaches to learning - such as problem based learning or project based learning - is in some respects quite fine."

The differences between several levels of inquiry and hands-on approaches have been presented by Bonnstetter (1998) in the form of the comparative Table 3.

Table 3. Inquiry levels (Bonnstetter, 1998)

	Traditional Hands-on	**Structured Inquiry**	**Guided Inquiry**	**Student Directed Inquiry**	**Student Research**
Topic	Teacher	Teacher	Teacher	Teacher	Teacher/Student
Question	Teacher	Teacher	Teacher	Teacher/Student	Student
Materials	Teacher	Teacher	Teacher	Student	Student
Procedure/ Design	Teacher	Teacher	Teacher/Student	Student	Student
Result/Analysis	Teacher	Teacher/Student	Student	Student	Student
Conclusions	Teacher	Student	Student	Student	Student

Teacher Controlled -- Student Controlled

Exogenous --------------------------Cognitive Development------------------------- Endogenous

Focus on Teaching --- Focus on Learning

2.2.3 Implementation of IBL

2.2.1.1 *Strategies of implementation*

Maaß & Artigue (2013) classify the strategies employed in the implementation and dissemination of IBL into two main groups: Top-down approaches and Bottom-up approaches.

The top-down strategies are generally considered ineffective (Tirosh & Graeber, 2003; Ponte et al., 1994). In Europe for example, an integration of IBL in science curricula has been achieved, but implementation of inquiry into school practice has not.

Bottom-up strategies, on the other hand, refer to groups of teachers working together, identifying their needs, developing their own questions, and dealing with them in a collaborative way (Joubert & Sutherland, 2009). These approaches risk neglecting organizational aspects of the change process and the planned expansion on a large scale. If they are conducted in isolation, school-based development is in danger of becoming introspective (OECD, 1998).

There is also a variety of combinations between these two main categories.

The implementation of IBL employs resources for the teaching, for professional development, and for assessment: videos of lessons, digital resources, curricular materials, etc. These resources are important for the dissemination of IBL, but they do not guarantee the success of such undertaking.

Another component of the transition to an inquiry approach is the training of teachers, including pre-service and post-service education (Ponte, 2008).

Typical strategies for implementing IBL are:

- Project-oriented pedagogical approach

- Cyclic inquiry model

- Practical inquiry model

Implementing investigative tasks essentially involves four-phases: tasks as planned and designed by the teacher; tasks as presented to the students; tasks as negotiated by students; and tasks as concluded by the students and the teacher (Ponte et al., 2003).

Some guiding principles for implementing inquiry (Ministry of Education Ontario, 2013):

- "start with questions and problems that students want to find out more about
- place ideas at the center
- work toward a common goal of understanding in the classroom
- guide the inquiry toward valuable ideas, do not let go of the class
- remain faithful to the student's line of inquiry when introducing him to new ideas
- use expository teaching when needed, and inquiry approach when appropriate

Getting it started...

- Make ideas the "central currency" of the classroom – the work of everyday teaching and learning.
- Model classroom norms of respectful discussion.
- Intervene to build momentum and to make sure all students understand and are invested in the ideas being discussed.
- Build on spontaneous questions that cause students to wonder and to ask further questions.
- Connect student questions and ideas to the big ideas of the curriculum.
- Keep student thinking at the center by involving students in initial planning of the inquiry

Keeping it going…

- Engage students in knowledge-building by bringing them together frequently to share thinking and
- Discuss the big ideas of an inquiry.
- Teach "on-the-spot" direct instruction mini-lessons when you see that students need to know certain pieces of information and have certain skills to move forward.
- Balance content-specific language with everyday student talk.
- Continually assess what's happening in the inquiry to make judgments about when and when not to intervene.
- Revisit initial theories and ideas about a question and reflect on the ways that the initial understanding differs from current understanding.

Reflecting on learning…

- Explicitly teach students what metacognition or reflective thinking is – talk about how learning deepens when we plan for it, analyze it and monitor our progress.
- Make sure students have time every day to practice metacognitive habits, such as reflecting on how they are progressing, how they are dealing with problems and how they are coming to new understandings.
- Have students put the reflection questions into their everyday language to make them their own."

Several strategies have been proposed and tested in order to overcome the difficulties related to the scaling-up in the implementation and dissemination of IBL: the

Cascade Model, the setting up of local learning communities (e.g. networks of teachers with similar objectives), the setting-up of e-learning communities involving asynchronous communication through e-forums, etc.

Examples of successful implementation and dissemination projects from Europe:

- **Lamap** ("La main à la pâte"), launched in 1996 by the French Academy of Science (http://www.fondationlamap.org ; http://www.lamap.fr)
- The national Austrian project **IMST**, launched in 1998 (http://www.imst.ac.at)
- **PRIMAS**, an international project launched in 2009 by the European Union (http://www.primas-project.eu)
- **Fibonacci**, a project launched in 2010 by the European Union (http://www.fibonacci-project.eu)

Lamap aimed at promoting inquiry-based pedagogy in science education at primary school level. Its activities cover the whole country, thanks to a network of 20 pilot centers having 3000 associated classes. In 2006, Lamap extended its activities toward junior high school.

IMST (Innovations in Mathematics, Science and Technology Teaching) aims at implementing at nation-wide scale innovative teaching, including IBL.

PRIMAS (Promoting Inquiry in Mathematics and Science Education) aims at a large-scale implementation of IBL within a funding period of 4 years. PRIMAS has a focus on pre-service and in-service teachers' development, with the participation of 14 universities from 12 countries.

Fibonacci's duration was 38 months and it aimed at disseminating IBL through the development of twinning between Reference Centers (RC) and Twin Centers (TC), and the involvement of local community by creating a Community Board which would ensure the sustainability of developed actions. The project involves 25 members from 21 countries: Academies of Sciences, Universities, Teacher Education Institutions, etc. In 2013, there were 60 centers participating (RC and TC).

Several inquiry teaching approaches have been proposed and tested in the past in order to improve this situation, among them the renowned Moore method, implemented since 1911 by the professor Robert Lee Moore at the University of Pennsylvania in a strong form, and then at the University of Texas (Zitarelli, 2004) as guided discovery. Arguably, it was the most successful large-scale training for mathematics PhD ever used; however, the fact that only elite students participated, and even among them, only the most fitted were retained in the program, precludes any inference about the effectiveness of such approach in other contexts, especially with average students. It is a very student-centered approach in mathematics education. All the work is individual, guidance is minimal, and there is a fierce competition between students. Moore, himself, was highly competitive and felt that the competition among the students was a healthy motivator. In his classes, there was a limited to no use of books, only instructor notes were handed out throughout the semester. Such an approach is very research-intensive. Since research is the top level of inquiry (Bonnstetter, 1998), it requires a high degree of autonomy and proficiency in mathematical investigation from the student. The skills needed in order to operate at such level are acquired only after many years of training and are typically neglected in school education, up to graduate university level. Hence, Moore method (also called "Texas method") should be modified in order to be implemented in normal classrooms, and several versions of it have been proposed (e.g. Asghari, 2012; Mahavier, 1997; Chalice, 1995; McLoughlin, 2008). In some of them, even small-group collaborative investigation is allowed (Davidson, 1971; Salazar, 2012).

Mahavier successfully used a modified Moore method for 15 years of teaching at all school levels, from fifth grade through college sophomore. He claims that "genuine Texas-style teaching gets the student to achieve his maximum potential not only in mathematics course but beyond it as well, to appreciate the power of his own mind, and to recognize the beauty of learning". According to Mahavier (1999), three elements are crucial for the successful implementation of Moore method: caring about the students, respect for learning, and enthusiasm in the classroom. In the period of writing his article, Mahavier employed Texas-rooted elements at a large, suburban, public high school. Students were generally of low economic status, and the results were encouraging, especially from the point of view of engagement and motivation. McLoughlin (2008) used this method for a long time at various undergraduate courses taught at Kutztown University of Pennsylvania, for example Introduction to Mathematics (given as a general education liberal arts course in mathematics, and required as a minimum level one). The results were rather mixed, and the author opines that even if it is more organic and natural to leave the student free to work on creating a proof or a counter-example and not be concerned with pace or how long a student takes to grasp a concept and produce a refutation, in practice such a thing is not possible and some scaffolding must be used.

2.2.1.2 *Requirements*

First of all, the implementation of IBL requires:

- school's support for IBL (Adler 1997, p. 242)
- teacher's skills in listening to, valuing, and pushing pupils in their interactions (Adler 1997, p. 243)

Regarding the students, researchers generally agree that:

"(1) Inquiry skills often require some form of hypothetical-deductive reasoning as in Piagetian formal operations, and (2) students capable of using only concrete operational thought cannot develop an understanding of formal concepts. Thus, students lacking formal operational thinking abilities for a topic being studied in class will have a great deal of difficulty understanding inquiry-based activities related to the topic. The more familiar the activity, materials, and context of the investigation, the less likely students will have this difficulty. Students more easily learn observable ideas via inquiry-based instruction than theoretical ideas. For example, IBL is likely to be effective for showing many students that chemical reaction rates depend on the

concentrations of reactants. On the other hand, inquiry-based methods are poor as a means toward helping most students understand how scientists explain the phenomena, via the kinetic-molecular theory. Inquiry-based instruction is probably most effective in developing content achievement when the content is more concrete than theoretical." (Colburn, 2006)

Pólya and Lakatos identified two specific attitudes required from the students when performing a mathematical inquiry: courage and modesty. Pólya (1954) thought intellectual courage and modesty to be essential to the activity of acquiring mathematical knowledge. He asserted that the doer of mathematics must assume an "inductive attitude" and be willing to question both observations and generalizations:

"In our personal life we often cling to illusions. That is we do not dare to examine certain beliefs which could be easily contradicted by experience, because we are afraid of upsetting the emotional balance. [In doing mathematics] we need to adopt the inductive attitude [which] requires a ready descent from the highest generalizations to the most concrete observations. It requires saying "maybe" and "perhaps" in a thousand different shades. It requires many other things, especially the following three:

Intellectual courage: we should be ready to revise any one of our beliefs

Intellectual honesty: we should change a belief when there is a good reason to change it

Wise restraint: we should not change a belief wantonly, without some good reason, without serious examination" (Pólya, 1954, p. 7)

Pólya (1954) called these the "moral qualities" required in order to do mathematics and claimed that although examining one's assumptions is an emotionally risky matter, such an attitude is essential for the practice of good mathematics. Also, Lakatos (1976) argued that making a conjecture (i.e. "conscious guessing") involves taking a risk: it requires the admission that one's assumptions are open to revision, that one's insights may have been limited, or that one's conclusions may have been inappropriate. Exposing one’s own conjectures to others' review increases personal vulnerability. Hence, in the midst of an argument among his students about a theorem in geometry, the teacher in Lakatos' book (1976, p. 30) announced: "I respect conscious guessing, because it comes from the best human qualities: courage and modesty."

Another requirement pertains to students’ cognitive skills. Kuhn et al. (2000, p. 496) claim that the arguments supporting IBL merits rest on a critical assumption, namely that students possess the cognitive skills that enable them to engage in these activities in a way that is profitable with respect to the objectives identified previously. If students lack these skills, inquiry learning could in fact be counterproductive, leading learners to frustration and to the conclusion that the world, in fact, is not analyzable and worth trying to understand.

Mayer (2004) has shown the danger of equating a constructivist vision of active learning (i.e., the idea that deep learning occurs when students engage in active cognitive processing during learning) with a seemingly corresponding vision of active methods of instruction (i.e., instructional methods emphasizing learning by doing such as discovery learning). Mayer (2004, p. 15) refers to this confusion as the constructivist teaching fallacy, namely the idea that active learning requires active behavior. Instead, the goal of constructivist methods is to elicit appropriate cognitive activity during learning - a goal that does not necessarily require behavioral activity:

> *“The formula constructivism = hands-on activity is a formula for educational disaster.” Mayer (2004, p. 17)*

In a review of constructivism (D'Angelo et al., 2009), Mayer argues that according to various studies pure discovery methods lead to poorer learning than guided discovery or direct teaching (Shulman et. al, 1966; Sweller, 1999; Brainerd, 2003; Kirschner et al., 2006). Based on Sweller's (1999) cognitive load, Mayer (2001) thinks that discovery methods of instruction can encourage learners to engage in extraneous cognitive processing—that does not support the instructional goal. Because cognitive resources are limited, when a learner wastes precious cognitive capacity on extraneous processing, they have less capacity to support essential cognitive processing (to mentally represent the target material) and generative cognitive processing (to mentally organize and integrate the material). Guidance (scaffolding, coaching, and modeling) and direct instruction are effective when they help guide the learner's essential and generative processing while minimizing extraneous processing. Discovery learning is particularly ineffective when students do not naturally engage in appropriate cognitive processing during learning, a situation that characterizes most novice learners.

Yet, other studies indicate that discovery may be useful as a first stage in knowledge-building (e.g., in primary education) and that it greatly enhances motivation. In general, similar arguments may be stated regarding open inquiry vs. guided inquiry or vs. direct teaching, which explains why pure inquiry is seldom encountered in education outside of science fairs.

Regarding teacher's abilities required in order to successfully implement IBL, Makar (2014, p.76) contends that a key element was "teachers' skill in provoking students' reasoning and developing a class culture which valued substantive conversation". A cultural/institutional element, which also involves teacher's ability and willingness to promote an inquiry environment for the instruction is the development of mathematical inquiry attitudes and norms in the classroom. Makar (2014, p. 66) emphasizes that the classroom culture was one in which the students repeatedly shared and discussed emerging ideas and were encouraged to debate and articulate their reasonings as they evolved. Cobb (1999) stresses that norms of collaboration and public debate are central to an inquiry-based environment. In Makar's experiment, students had been developing these norms through the year and this could be observed in the way they critiqued and probed each other's ideas, built their ideas on other’s and created new ways of talking about their emerging understandings that were “sensible”, “in the range” and “typical around the world” (Makar 2014, p. 75). In order to use effectively the inquiry approach, the teacher needs to be able to change their role "from an instructor to a facilitator", which is a challenging requirement and may explain why IBL does not seem to be widespread in Europe (Maaß & Artigue, 2013). According to Schaumburg et al. (2009), successfully carrying out such a shift on a large scale requires supporting measures such as professional development courses, but these measures were not taken into account when IBL was being implemented in Europe.

2.2.1.3 *Limitations/Obstacles in the implementation of IBL*

A serious obstacle encountered when implementing IBL is the <u>inherent complexity</u> of genuine research and inquiry, which are quite similar to scientific investigation:

> *"Inquiry activities targeted to young children may have simple goals that do not extend beyond description, classification, or measurement of familiar phenomena. However, inquiry activities designed for older children typically have, as their goal, the identification of causes and effects. The context is multivariate, and the goal becomes one of identifying which variables are responsible for an outcome or how a change in the level of one variable causes a change in the system. Equally important is the identification of non-causal variables. Are students of the elementary and middle school grades (in which inquiry activities are most commonly used) capable of inferring such relations based on investigations of a multivariable system? The literature on scientific reasoning indicates significant strategic weaknesses that have implications for inquiry activity (Klahr, 2000; Klahr et al., 1993; Kuhn et al., 1988, 1992, 1995; Schauble, 1990, 1996). But the most critical aspect is that students at the middle school level, and sometimes well beyond, may have an incorrect mental model that underlies strategic weaknesses, and that impedes the multivariate analysis required in the most common forms of inquiry learning."* (Kuhn et al., 2000, p. 497)

The "strategic weaknesses" mentioned above are essentially higher-order cognitive skills, which are not stimulated in secondary school and even beyond, at undergraduate level. Part of the difficulty students have with proof as a mathematical method may stem from its apparent redundancy: proof is often first emphasized in geometry, a field where many proofs seem unnecessary because the visual representation itself makes the result so obvious. For instance, Healy (1993) recounted the story of his class in which groups of students presented their own experimental results and reasoning to the class and voted on whether or not to include these in a book representing the class's work over the year. de Villiers (1998) remarks that students' difficulty of perceiving a need for proof is well known to all high-school teachers and has been identified without exception in all educational research as a major problem in the teaching of proof. He argues: who has not yet experienced frustration when confronted by students asking: "Why do we have to prove this?" de Villiers, like many other researchers, explained this fact by the visually character of geometry, which makes many results seem obvious: "the students do not recognize the necessity of the logical proof of geometric theorems, especially when these proofs are of a visually obvious character or can easily be established empirically" (Gonobolin 1975, p. 61).

Further, Goldenberg et al. (1998) argue that in the most common curricula, both in and out of the U.S.A., geometry represents the only visually oriented mathematics that students are offered. Curricula tend to present an otherwise visually impoverished, nearly totally linguistically mediated mathematics, a mathematics that does not use, train, or even appeal to the "metaphorical right-brain", and this choice has significant side effects because visual thinking can play a key role in developing students' understanding (Tall, 1991). Goldenberg et al. (1998, p. 5) mention that only 50% of U.S. students ever take high-school geometry, and even within geometry more emphasis is put on verbal rather than on visually based reasoning.

Goodchild et al. (2013, p. 402) identified an obstacle for inquiry in the official mathematics syllabus prescribed for the course. The syllabus may lack challenge and could be boring, inadequate for investigation or discovery. For example, an introductory course in linear functions where the content has already been met before does not bring excitement and fails to provoke students' engagement. The task proposed to the students may be boring too, as was the case with the research conducted by Goodchild et al. (2013, p. 403). The teachers participating at the experiment chose a task that was artificial, not challenging and unrelated to real world, so it did not make much sense for the students Moreover, the content to be learned may be unsuitable for open-ended tasks. Goodchild et al. identified a specific obstacle for IBL in students' inability to make links between related topics and to take knowledge from one task within a topic to another. Less specific obstacles are students' behavior (restlessness) and the lack of space within the classroom. Since investigations typically require a lot of persistence and reflection, such behavioral issues rule out prolonged inquiry tasks and hinder the implementation of IBL.

The lack of space within the classroom is an obstacle for all collaborative work and prevents experimenting with alternative forms of organization and grouping. A serious obstacle for inquiry learning is the pressure of standard evaluations or assessments, since students are mostly motivated by grades. As Goodchild et al. (2013) remarked, the curriculum hangs as a cloud over all thoughts of development because students must be prepared for their exams, and the teachers spend a substantial amount of time in preparations for these tests. Such focus on grades and exams has already generated a trend toward an extreme form of test-oriented instruction, also known as "teaching to the test". Actually, inquiry learning is a completely different paradigm from traditional instruction, and not just an alternative way of presenting the mathematics content - that is only the lowest form of inquiry! Inquiry learning requires a different curriculum and specific forms of assessment, so it is not compatible with traditional evaluations. Actually, the main obstacles for the implementation of inquiry are the presence of standard evaluations and the pressure of uniform tests at secondary level - which shape the entire mathematics education system. Higher order skills are desirable outcomes, but they are seldom tested in standard evaluation, and a basic principle in education says: that which is not assessed, is not learned (Bain, 2004).

Adler (1997, p. 242) describes two obstacles in the implementation of IBL:

- broader schooling system where traditional approaches to mathematics teaching are dominant
- canonical school mathematics curriculum

Bruce et al. (1994, p. 9) hold a similar view:

> *"The curriculum must be student-centered in an Inquiry Model, because the meaningfulness of experiences depends on the student's own knowledge, values, and goals. Nevertheless, teachers have vital roles to play as supporters of inquiry and as experienced people engaging in inquiry themselves. The key question is: do the teacher's actions support the inquiry and open up possibilities, or do they establish constraints and limits?"*

Nicol (1998) describes the challenges of questioning, listening, and responding that are met by teachers when they try to introduce IBL in their classrooms. She mentions the tensions often experienced by prospective teachers with the kinds of questions posed and the reasons for posing them, with what they are listening for, and with how they respond to students' thinking and ideas.

During the experiment performed by Nicol, some teachers displayed an inadequate scaffolding (forceful guiding), a lack of visualization tools (no diagrams to help understanding), or they missed the opportunities for discovery due to a misconception regarding IBL (a belief that the teaching is successful if the student manages to arrive at the desired result, even if this is done only with a strong support from the teacher, and that the teacher must keep a full control of student's learning process). A too strong guidance has also been displayed in several instances during the experimental research of Elbers (2003). Since excessive scaffolding spoils the pleasure of discovery and decreases intrinsic motivation, too much guiding or teacher's inability to provide the right amount of guiding to the students represents an obstacle for the implementation of IBL.

Another issue revealed by the research of Nicol (1998) is teacher's lack of intellectual courage and honesty: even if the attempt to perform an inquiry-based lesson has been a total failure, the teacher evaluates their teaching in superlative terms, and congratulates herself for making a good-looking pedagogy. The lack of impartiality and the refusal to face her own errors and shortcomings are serious obstacles for an adequate use of inquiry instruction. When introducing a new pedagogical approach, the teacher is also a learner; and in order to successfully learn anything we need to have an attitude of learners, not one of "knowers." Also, in the classroom, a teacher should display the appropriate attitude toward pupils - and not one of competitor, judge, etc. They should be ready to confront negative feelings about their competence, self-doubt, and a sentiment of inadequacy. The research of Nicol (1998) indicated that a main obstacle to the implementation of IBL is the teacher's aversion to the loss of control, which leads to a teacher-centered approach, incompatible with inquiry.

Zack & Graves (2002) indicate classroom size as an obstacle for IBL. They recommend a reduction of the classroom to only 12-13 students in order to make inquiry effective. Another obstacle that occurred in collaborative inquiry during their research was the problematic participation of a member of a team, namely when his participation dropped down dramatically. In general, one of the main problems in collaborative learning is the lack of involvement of some students. Hence, teacher's lack of skills in collaborative learning and an inadequate grouping of students may become significant obstacles for the successful implementation of collaborative inquiry.

Walls (2005, p. 754) emphasizes teachers' difficulty in passing from traditional to inquiry approach:

> *"Although recent shifts in mathematics education have strongly encouraged teachers to select or design tasks for interest or relevance, and increasingly expect or even compel children to participate by sharing their thinking as they undertake these tasks, it is seldom considered essential that children are consulted about the context, content or efficacy of such tasks. Irrespective of how open or closed the tasks may be, task-oriented pedagogies construct mathematical learning as a form of compulsory labor divided into discrete units of work which must be at least attempted and preferably completed by the learners, and by which learners' performances might be judged. International moves toward more expansive and connected mathematics*

have been tempered by increasing specificity of learning outcomes. It is believed that armed with the correct training and diagnostic tools, teachers will be better able to make the most significant decisions about what mathematics their pupils will learn, when they will learn it, and how that learning will take place. Such approaches diminish opportunities for learners to select learning contexts and to direct their own learning, and overlook significant learning factors such as children's social networks, first languages, current understandings of the world, sensitivities, interests, passions, and aversions."

Standard teaching practices are a huge obstacle when implementing IBL:

"Data from different studies and reports show that highly structured teaching practices are dominant, to the detriment of student-oriented practices (OECD, 2009). These act as difficult obstacles to the introduction of interdisciplinary oriented tasks that follow an inquiry-based learning approach." (Maaß et al., p. 373)

Other challenges are common to many student-centered or collaborative pedagogical approaches:

- issues of *power and control* (Adler 1997, p. 240)
- teacher's *dilemmas of mediation*: listening to and validating diverse perspectives vs developing mathematical communicative competence; moving effectively between learners' informal expression of their thinking and a more formalized mathematical discourse (Adler 1997, p. 241)

One initial obstacle to the adequate implementation of IBL in the classroom is the existence of a so-called "illusory zone of promoted action", defined by Blanton et al. (2005, p. 14) as "a zone of permissibility that the teacher appears to establish through behaviors and routines used in instruction, but in actuality, does not allow". A study by

Hussain et al. (2013, p.209) has revealed a teacher behavior that hinders the effectiveness of inquiry lessons: "In the early lesson what appears to be promoted (inquiry learning) is only realized at a surface level, such as students working in small groups... What the teacher appears to allow is not allowed in actuality."

Another issue when implementing inquiry instruction is the difficulty of extending students' Zone of Free Movements (ZFM) and Zone of Promoted Actions (ZPA), when the students have internalized an (old) ZFM/ZPA system which shapes their values, actions and expectations about mathematics and mathematics learning (Hussain et al., 2013). Such inertial forces manifest not only in students', but also in teachers' behavior. Therefore, teacher's deep-seated routines or habits, as well as their prior beliefs and misconceptions about what constitutes "good teaching" and what is inquiry learning, are serious obstacles for the implementation of IBL. Further, Hussain et al. (2013) remark that the crux of the problem is twofold: students need time/routines in order for their values, actions and expectations about mathematics and mathematics learning to be transformed, and the teacher has to extend the habitual ZPA and ZFM in the classroom. Hence, time constraints represent a major obstacle and a source of limitations for IBL: inquiry can only be implemented on a long term basis, since it attempts to build habits of mind (e.g. higher order thinking) and attitudes (e.g. autonomy, persistence, openness to criticism, investigative attitude). In contrast to theoretical and procedural knowledge, attitudes are the most difficult to shape or to change, therefore authentic inquiry learning requires a steady use of investigation in the classroom and, preferably, a school policy that encourages the use of inquiry instruction at all science classes. In this respect, Dorier & Garcia (2013) believe that factors present at the systemic level represent serious obstacles for the implementation of inquiry learning, and could explain the poor dissemination of IBL in Europe.

Since research has shown that inquiry is negatively correlated with speed and difficulty (Fresko & Ben-Chaim, 1986), teachers should not attempt to cover a large content area or to deal with difficult topics by using inquiry approaches.

According to Goodchild et al. (2013), the use of inquiry starts off as a mediating tool in the practice (e.g., an inquiry-based task is used as a tool to engage students in mathematical thinking) and shifts over time to become an inquiry stance or an inquiry way of being in practice - when teachers and/or students become “inquirers” as one of the norms of practice. Goodchild et al. discuss about the necessary move toward an inquiry "way of being", and conclude: "inquiry is slow to develop, this we certainly learned". Thus, wholesale implementation of inquiry is a long term undertaking. Moreover, the teachers involved in the experiment realized that the exploratory approach was hugely demanding on their time, both in taking time for planning and in valuable classroom time. This is an additional obstacle related to IBL, namely the incertitude and the higher risk assumed by the teachers, compared to traditional instruction. Implementation of IBL has to be judged according to the ratio between the time/effort invested and the perceived pedagogical benefits, i.e. the "return on investment". In the experiment of Goodchild & al. (2013) the implementation was unsuccessful and the results were disappointing, mainly because the tasks presented to the students did not manage to elicit any excitement or interest from them. This situation emphasizes the importance of instructional design in IBL.

An example of successful implementation of inquiry, with an excellent choice of the tasks assigned to the students, is the experiment presented in Makar (2014). The researcher proposed a challenging, real-life, ill-defined, open-ended and authentic problem, which aroused children's interest and led to their engagement. Also, Elbers (2003) suggests that teachers should organize their inquiry lessons around problems which are topical and meaningful for the students. This requirement implies that teacher's lack of ability in choosing stimulating tasks would be an obstacle to the implementation of IBL, a fact which has been proven by the studies of Goodchild et al. (2013) and Makar (2014).

Hussain et al. (2013, p. 300) acknowledge that although Vygotsky's Zone of Proximal Development (ZPD) is widely recognized as an important construct, the practical use of this concept is somehow problematic because it is not possible to determine the limits of a learner's ZPD and thus, the limits of his ZPA or ZFM. Moreover, ZPD can only be interpreted as an attribute of an individual; and classrooms have many individuals, with large differences between them from the point of view of skills, interests, self-drive, habits of mind and attitudes. As a possible solution, Hussain et al. suggest the use of Mercer's intermental development zone, a collective zone related to ZPD which may be useful for understanding how interpersonal communication can aid learning. There is still the danger, when dealing with ZPD, that the "object of knowledge" is viewed by the teacher as a static, acultural object, a sort of Platonic ideal form to be "transmitted" to students. Indeed, the ZPD is frequently thought of and applied in a one-sided manner that juxtaposes a more knowledgeable teacher or peer and a less capable learner (Hussain et al., 2013, p. 301). ZPD can be defined only for a maximum cognitive load that can be sustained by the learner over a given time frame, which in turn depends heavily on learner's motivation, personal interest in performing that specific task, and also on his compatibility (mental, emotional, etc.) with the more knowledgeable tutor. This obstacle concerns not only IBL, but all approaches where ZPD is employed.

Inquiry is inherently open-ended, therefore it is not possible to plan in detail anticipated trajectories of learning, as is commonly advocated for instructional design. Traditional lesson planning does not operate in an inquiry-based approach. In IBL, the teacher has to adopt an "opportunistic", adaptive, situated (i.e. context-dependent) strategy, and this is clearly a serious challenge for many teachers.

A significant difficulty related to the implementation of IBL is the scaling-up from school level to large-scale, especially when extending the action to international level (Maaß & Artigue, 2013).

2.2.4 IBL in Geometry

> *"A youth who had begun to read geometry with Euclid, when he had learned the first proposition, inquired: 'What do I get by learning these things?' So Euclid called his slave and said, 'Give him threepence, since he must make gain out of what he learns.'"* - Stobæus (Gow, 1884, p. 195)

2.2.4.1 Geometry and proof, a historical perspective

Since early antiquity, the great civilizations of Babylon, Egypt, China, India and Greece have used geometry for practical purposes such as the measuring of areas and volumes, civil engineering and religious activities. For thousands of years, geometry has been the main "applied science", but only the Greeks managed to develop a system of thinking based on abstract geometry. The axiomatic construction of classical geometry has been accomplished by Euclid in his work "Elements of geometry", which remained for two millennia the main textbook used by school students.

> *Ptolemy Soter founded the great Library of Alexandria and probably personally sponsored Euclid is his mathematical activity. (Gow, 1884, p.195) He found Euclid's Elements too difficult, and asked if there were an easier way to master it. Euclid famously replied: "Sire, there is no Royal Road to geometry."*
> (Proclus, 1970, p. 57)

Any philosopher of science had to acquire deductive reasoning proficiency by the thorough study of Euclid's "Elements", and some manuscripts even include the annotations of famous scientists such as Galileo Galilei (Euclid, 1558). According to Grabiner (2015), Euclid's work is the earliest example we have of a systematic approach to geometry. The method consisted in proving statements (theorems) by deriving them from a set of obvious truths or axioms, through the use of logic. A modern expression of this approach is the rationalist belief of Descartes that if we start with self-evident truths and then proceed logically deducing more and more complex truths from these, then there's nothing we couldn't come to know. A contemporary of Descartes, Spinoza, wrote an *Ethics Demonstrated in Geometrical Order* where a discussion about God and the divine nature is led in the form of definitions, axioms, propositions and corollaries, with the Q.E.D. duly appended after the proofs. Newton demonstrated Euclid's influence too when he called his principles of motion "axioms" and deduced the law of gravity in the form of two mathematical theorems. He also stated that "it's the glory of geometry that from so few principles it can accomplish so much."

"I cannot say that I ever saw him laugh but once... It was upon occasion of asking a friend to whom he had lent Euclid to read: what progress he had made in that author, and how he liked him? He answered: (...) of what use and benefit in life that study would be to him? Upon which Sir Isaac was very merry." (Whiteside, 1974, p. XIII)

Even the politician Thomas Jefferson, who was not ignorant in mathematics, stated the American Declaration of Independence as the conclusion of a logical argument, where he employed formal expressions such as: "We hold these truths to be self-evident", "therefore...", and the word "prove" (Grabiner, 2015).

In Ancient Greece, geometry has been revered as the queen of sciences, a realm of perfection where beauty, order and truth come together. A distinctive trait of Greek geometry was that its goals were much broader than learning a specific content, or acquiring a needed tool for practical applications (such as engineering), as is the case nowadays. Geometry was regarded as the key for understanding harmony, beauty, and higher philosophy. In Pythagoras's School, apprenticeship was based on a thorough study of geometry, in order to achieve proficiency in logical reasoning and argumentation. For the Greeks, proof was a form of argument, and not a ritual as it is perceived now by many students (Lehrer & Chazan, 1998, p. x). In modern times, the outlook has become one determined by a kind of philosophical rationalism, with the formalist assumption that mathematics in general (and proof in particular) is absolutely precise, rigorous, and certain. Such a view is still dominant among mathematics teachers and mathematicians.

Hence, validation (verification, conviction) is often seen as the only role or purpose of the proof, which is narrowly regarded merely as a means to remove personal doubt or that of skeptics (de Villiers, 1998). With very few exceptions, teachers of mathematics seem to hold the naive view described by Davis & Hersh (1986) that behind each theorem there stands a sequence of logical transformations moving from hypothesis to conclusion. As pointed out by Bell (1976), this view avoids consideration of the real nature of proof, because conviction in mathematics is often obtained by quite other means than that of following a logical proof.

Indeed, proof is not necessarily a prerequisite for conviction - to the contrary, conviction is far more frequently a prerequisite for the finding of a proof. A mathematician simply does not think: "Hmm... this result looks very doubtful and suspicious; therefore, let's try to prove it." For what other reasons would we spend sometimes months or years to prove certain conjectures, if we weren't already reasonably convinced of their truth? (de Villiers, 1998, p. 375). Mathematicians usually make discoveries inductively, but prove them deductively:

> *"Having verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial suspicion and gave us a strong confidence in the theorem. Without such confidence we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is true, you start proving it."* (Pólya 1954, pp. 83-84)

Absolute certainty is virtually missing in mathematical research, and a high level of conviction may be reached even in the absence of a proof. For instance, in support of still unproven twin prime pair theorem and Riemann hypothesis, the "heuristic evidence" is so strong that it carries conviction even without rigorous proof (Davis & Hersh, 1983, p. 369). Proofs themselves are not the absolute truth, as they may contain errors (sometimes even fatal). For instance, Arenstorf published in 2004 a purported proof of the twin primes theorem. Unfortunately, a serious error was found in the proof, so the paper was retracted and the twin prime conjecture remains fully open. A former editor of the Mathematical Reviews disclosed that <u>half of the proofs</u> published in it were incomplete and/or contained errors, although the theorems they were purported to prove were essentially true (Hanna, 1983, p. 71).

The fundamental issue with mathematical proving is not the risk of making errors, since wrong statements can be disproved relatively easy with counterexamples, but the fact that gaps in demonstration cannot be detected by logical means. Indeed, if A and B are both true, then A implies B is logically true and we can formally write $A \rightarrow B$, but this is not a mathematical proof by any means since it does not say anything about the path from A to B, how we get to B when we know that A is true. For example, if Fermat would have written the following "proof" of his last theorem: 1 + 1 = 2 implies the theorem, then from a logical point of view he would have been absolutely correct, since both statements (1+1 = 2, and Fermat's last theorem) are true, but from a mathematical point of view he would not have proven anything, as the path from 1 + 1 = 2 to the theorem is still missing, the gap remains the same.

Moreover, the "acceptable" steps in a rigorous proof for a novice would be much smaller than for an expert, so the amount of details necessary to convince an audience of beginners must be far larger than at a mathematical congress. As a result, even correct proofs are seldom detailed and "complete" - complete simply means providing enough details to convince the intended audience (Davis & Hersh 1986, p. 73).

Attempts to construct rigorously complete proofs lead to such long, complicated demonstrations that their evaluation becomes impossible. Even the proof for the relatively simple theorem of Pythagoras would take up at least 80 pages, according to Renz (1981, p. 85).

To conclude, when the result is intuitively self-evident or supported by empirical evidence, proof is not concerned with "making sure", but rather with "explaining why". For most mathematicians, the clarification/explanation aspect of a proof is probably of greater importance than the aspect of verification (de Villiers 1998, p. 378). For instance, Halmos noted that although the computer-assisted proof of the four-color theorem convinced him that the theorem was true, he would still personally have preferred a demonstration which also gave an understanding (Albers, 1982). A "good" proof has been defined by Manin (1981, p. 107) as one which "makes us wiser" and by Bell (1976, p. 24) as one which conveys "an insight into why the proposition is true", therefore explanatory power is a good criterion for judging proofs.

2.2.4.2 *Theoretical researches and models*

> *"It has often been said that geometry is the art of applying good reasoning to bad drawings. This is not a jest, but a truth which deserves reflection."* - *Poincaré (1920, pp. 59-60)*

Although since the age of Euclid it had been widely acknowledged that geometry involves specific ways of thinking, there are few theoretical models for the geometric reasoning. It may be argued that part of geometry's specificity is that it deals with concepts and generic objects rather than with instances or particular cases of abstract notions. The saying of Poincaré highlights the fact that in geometry, when trying to build a proof, one must be careful not to be entrapped by some particularity of a "bad" drawing, which is not among the premises. Also, direct measurements on a drawing cannot be used in demonstrations.

The research about how children develop understanding in geometry has been rather limited, as compared with research involving other concepts, such as numbers. Even if Piaget and his coworkers published two significant studies relating to this area, *The child's conception of space* (Piaget & Inhelder, 1956) and *The child's conception of geometry* (Piaget, Inhelder, & Szeminska, 1960), little impact on classroom practice has resulted. Part of the problem lies with Piaget's questionable "topological primacy theory", around which it has proven difficult to build a school syllabus (Pegg & Davey, 1998, p. 109).

The result of all these problems and of their own classroom experience has been the development by van Hiele of a theory describing students' growth in understanding geometry by means of five levels or stages, and proposing five corresponding teaching phases, such that instruction takes into account the development of student's thinking inside geometry, and not just his general cognitive stages (as prescribed by Piaget). Progression from one level to the next is not the result of maturation or natural development. According to van Hiele (1986, p. 41), "the levels are situated not in the subject matter but in the thinking of man", and it is not by exposure to a higher level content that students' progress in their geometric thinking; rather, it is the nature and quality of the experience in the teaching/learning process that influences a genuine advancement from a lower to a higher stage. Research has proven that van Hiele's stages are not always sequential and clear-cut, that growth in the period between two levels is somehow continuous, that students do not always have a fully developed set of objects at a given level before they move to the next, and that objects of an earlier level are not subsumed completely by a higher level (Pegg & Davey, 1998, p. 111).

2.2.4.3 Implementation

Despite its glorious past and perennial importance, dating back from the origins of civilization, and a recent resurgence as cutting-edge mathematics, geometry and space visualization in school are often compressed into a caricature of Greek geometry, generally reserved to the second year of high school (Lehrer & Chazan 1998, p. ix).

The result is not only an impoverished understanding of space, but also a general lack of mathematical reasoning, argumentation, and investigation skills among students and adults alike. Indeed, geometry has traditionally been considered the ideal field for acquiring and developing such skills, but formalist views of mathematics as a "game" in which abstract symbolism, algebraic shorthand and formulas are manipulated, prevailed in the second half of the 19th and the early part of 20th century. More recently, various education reforms have put exploration, sense-making, and empirical understanding of concepts in the center of mathematical instruction. As stated by Sanni (2007), learning geometry is an investigative rather than instructive process.

According to Lehrer et al. (2013, p. 366), geometry is a very promising site for inquiry learning because spatial reasoning supports the development of skills and attitudes that are essential for generating and revising mathematical knowledge. Moise (1975, p. 477) even claimed that traditional Euclidean geometry course is “the only mathematical subject that young students can understand and work with in approximately the same way as a mathematician”. Moise pointed out that mathematicians work deductively (actually, they prove and present mathematical statements in this way) , and studying Euclidean geometry gives students a unique opportunity to experience deductive development of an axiomatic system.

Goldenberg et al. (2012, p. 3) hold that geometry can help students connect with mathematics, and geometry can be an ideal vehicle for building a "habits-of-mind perspective" (Goldenberg et al., 1998). Posing questions is an essential habit-of-mind in IBL, and its promotion through investigative geometry has been explored in a research by Lehrer et al. (2013). Their study shows that questioning is an often neglected aspect of mathematics, and that posing productive questions is a difficult task. As one student reflected, "asking good questions is hard".

A modern trend in mathematical storytelling focuses on applications - how a body of facts and ideas is used. Although most prevalent curricular story is a tale of logic, in actual implementation this tale is often so abridged that the original logic is apparent only to the teacher, not to the student. In such a story, mathematical facts and ideas are presented in a linear sequence, with each rung of the ladder building directly from the previous one. However, mathematical discoveries rarely occur in such an orderly fashion, nor does this logical sequence accord well with what is known about how students learn mathematics. How common it is to read a problem solution and feel, "What a clever trick to use here! Why, it makes things so simple, even I can understand it, but how on earth did anyone ever think of it in the first place?" (Goldenberg et al., 1998, p. 4).

Goldenberg et al. argue that **a mathematics course should never neglect *telling a story* about thinking - powerful, mathematical thinking**. Such an approach would help students acquire valuable "habits-of-mind", because mathematical ways of thinking have important application outside of mathematics as well as within it. For students to understand mathematics, they must learn how to think from a mathematical point of view. For those of them who pursue advanced mathematical study, they must spend some of their time learning to "think like the professionals". They claim that, within mathematics, geometry is particularly suitable for helping people develop these ways of thinking, being an ideal intellectual territory within which to perform experiments, develop visually based reasoning styles, learn to search for patterns, and use these to spawn constructive arguments. It is also ideally placed for expanding a student's conception of mathematics, by virtue of its rich hooks or connections with the rest of mathematics, with other sciences, and with real-world. According to Goldenberg et al. (2012, p. 3), the study of geometry could help students connect with mathematics. It can be an ideal vehicle for building a "habits-of-mind perspective" (Goldenberg et al., 1998).

Such habits-of-mind that support inquiry and which could be developed through the study of geometry are generalizing, reasoning with relationships, and identifying invariants (Driscoll et al., 2007). For example, the distinction between a drawing, which exemplifies an instance, and a figure, which exemplifies a class with associated properties, suggests a pathway for seeking broader classes or patterns (Goldenberg & Cuoco, 2012). Generalization can be promoted by asking students to justify the grounds of a claim beyond a single instance (Ellis, 2011). As pointed out by van Hiele (1986), geometry is replete with visual concepts such as planar concepts and relations among them, such as congruency, that provide opportunities to construct relationships. This feature is better embodied through the dragging operation of dynamic geometry tools, but can also be visualized with traditional tools (Lehrer et al., 2013).

Lehrer & Chazan (1998) remark that spatial reasoning has been reintegrated into the mathematical mainstream and placed at the core of K-12 environments that promote learning with understanding. Traditional topics like measure, dimension, and form, are reinvigorated and receive increased attention by virtue of their connection with modeling, structure, and design. In order to successfully implement an investigative approach in school geometry, specific learning environments have to be designed. Some basic tips for implementing investigative geometry have been proposed by Hitchman (2015). The definite reference in this field has been the book of Lehrer & Chazan (1998, reprinted in 2012).

In order to support students' visualization of shapes and forms in a Cartesian framework, a prominent role has been assigned to technology - for example, the use of computer algebra systems like Geogebra or Cabri. This trend has gained even more strength with the recent move toward investigative approaches in school mathematics, and the greater emphasis put on the study of geometry, starting from primary school, in the official curriculum. Chazan & Yerushalmy (1998) are among the authors who advocate the use of computers in the learning of geometry by inquiry. They justify such choice with the excitement of teachers and researchers alike for the opportunity to use geometry software as an effective instructional tool by proposing construction activities to the students. Moreover, such explorative tasks could be completed collaboratively. Students would have to invent a solution, test it, and justify it mathematically – in this way, both inductive and deductive reasoning are promoted and combined. This approach would help alleviate the current situation of high school geometry, where "students view themselves as passive consumers of others' mathematics" (Schoenfeld, 1988). According to Schoenfeld, there has been "little sense of exploration, of the possibility that the students could make sense of the mathematics for themselves" (p. 18).

Goldenberg & Cuoco (1998) have presented a review of the development of dynamic geometry software. The authors use the term dynamic geometry in order to emphasize its distinctive feature with respect to other geometry software: the continuous real-time transformation often called "dragging". Dynamic geometry establish an experimentation environment where the student performs some construction by drawing points, lines, curves, and polygons, then make observations about the result and conjecture about how their observations might be affected if the same construction was performed on another triangle, for example. For example, these operations can be done with the Geometric Supposer software, developed since 1983 by two pioneers of geometry experimentation environments, Schwartz and Yerushalmy. The greatest benefit arises when the students come to test their conjectures, because dynamic geometry allows them to see what seems like a continuum of intermediary states. It also helps generating the locus of some object as another is transformed in a continuous way, and conjecturing about it, while the final proof may invoke classical Euclidean methods. This is essentially an inductive way of learning mathematics, and its motivating quality is revealed by the shock and delight that students often express at some unexpected behavior of the figure they are dragging.

Geometer's Sketchpad is a dynamic geometry software that could be very useful in fostering identification of patterns and geometric properties in shapes, a skill which is generally underdeveloped at young students. Many of them do not realize that a square is also a rectangle and a rhombus, or that all three are also parallelograms. Most have never heard of a "line of symmetry", and are unable to identify perpendicular lines (Olive, 1998). Geometer's Sketchpad has also been used successfully for exploring trigonometry (Shaffer, 1995).

Design software such as KidCAD is another promising tool for the implementation of investigative geometry in a computer-rich environment. It greatly helps develop spatial visualization, scale and proportional reasoning, measurement and unit conversion skills, and understanding isometries, via hands-on activities and real-world scenarios such as designing or rearranging the classroom (Watt, 1998).

A thorough discussion about spatial skills and how digital tools such as the Geographic Information Systems could be used in developing them through mathematical inquiry has been done by Hagevik (2003).

Unfortunately, in school education there is still a lack of connection between geometry and other subjects, such as physics. Raghavan et al. (1998) contend that despite recurrent calls in the official reform programs (for example, NCTM) to reinforce and exploit interdisciplinary connections, the mutual supportive nature of mathematics and science is often under-emphasized or even ignored in school curricula. They cite the issue of area and volume, which measure basic properties of matter and are central concepts in science and are commonly presented only in fifth and sixth grade mathematics classes. Moreover, instruction is mostly quantitative in nature, emphasizing rote application of formulas rather than fostering qualitative understanding that supports meaningful application of concepts within a variety of contexts. No explicit link is made to science concepts for which area and volume are components, such as surface force and mass. In order to improve this situation, Raghavan et al. (1998) advocate the implementation of a computer-supported, integrated approach in learning science. In such a curriculum, labeled MARS (Model-based Analysis and Reasoning in Science), topics from physics, chemistry, biology, and earth science are introduced and revisited in successive years. This framework would allow students, for instance, to investigate the notion of volume by immersing objects and measuring the displaced liquid, or to relate concepts such as volume, surface area, weight, density, water pressure, and buoyancy by using firstly material cubes and prisms, and finally the computer model to draw boundaries within a container of liquid and to examine the forces exerted on bounded areas. Such an integrated inquiry approach in sciences would fit particularly well with the Realistic Mathematics Education (REM) system adopted in the Netherlands.

Geometry lessons are among the earliest opportunities for students to engage in modeling. As stated by the NCTM (1989), "geometric models can be applied to real-world problems to simplify complex situations, and many algebraic and numeric ideas can be fostered by looking at them through a geometric perspective. The complex spatial patterns of the real world can be simplified into component relationships such as points, lines, angles, transformations, similarity, and dimensionality, and these physically simpler (but cognitively more abstract) ideas can be operated on mentally - changed, recombined, transformed - whereas the physical objects themselves may not be" (Middleton & Corbett, 1998). Modeling implies generalization and recognition of patterns in a process of solving real-world problems, not of developing a new theory, where exercises are a more useful tool. Freudenthal (1986) discussed this process as the *mathematization* of physical reality, holding that the development of students' knowledge of geometry reflects a tension between experience in an irregular world and the mathematical system that represents and explains physical reality in terms of abstract regularities.

A very effective implementation of inquiry into geometry via everyday situations has been used in the Netherlands. The idea behind such an approach is that students have a great deal of informal geometrical knowledge at their disposal, and even young children can model real life situations by using elementary geometry. The theoretical framework of this approach is represented by the Realistic Mathematics Education (RME), a framework and a reform movement developed by a group of Dutch researchers according to the ideas of Freudenthal and van Hiele. The reform was started in part in response to critiques of Dutch mathematics curriculum, and particularly with the way Euclidean geometry was taught in the Netherlands. It brought about a radical break with the traditional teaching of Euclidean geometry, an approach whose shortcomings have been exposed by van Hiele and other Dutch researchers. This move toward a reinvention approach in geometry has also been advised by Freudenthal. Finally, implementation has reached an advanced stage, and present-day geometry education in the Netherlands is based on everyday scenarios involving reasoning about spatial relations. A slogan of Dutch reform in teaching geometry is "looking at the world from a geometrical perspective" (Gravemeijer 1998, p. 46). For example, as a preparation for the calculus course, students have to describe the variation of the shadow of a man walking away from a light pole at constant speed, or moving away from the pole in equal jumps. A physical device is used in order to show the change that occurs.

According to Freudenthal (1971), reflecting on geometrical aspects of everyday life situations is an important feature of a mathematical attitude. Such attitudes define the so-called "mathematical literacy", which is usually described for number sense and statistics. Similarly, "geometrical literacy" would be indicated by a person's ability to recognize geometric objects and relations behind common objects and contexts such as vision lines, shadows, side or top views, or maps. Freudenthal (1973) proposed a philosophy of "mathematics as a human activity", centered on finding problems and solving them, but it is also an activity of organizing a subject matter. The matter can be theoretical, for example arising from mathematics itself, but it can also be a matter from reality. Such organizing activity could be labeled "mathematizing", and according to Freudenthal mathematics education for young children should start with the mathematization of everyday reality. Freudenthal also endorsed a combination of activities and content learning in school mathematics, under the principle of guided reinvention. Such ideas could define a productive and effective way of teaching geometry through inquiry, as proven by the Dutch education system which gradually introduced this approach after 1973 and has successfully employed it since then.

Activities are central to the process of learning geometry. Indeed, expository teaching is hardly compatible with this field, which involves specific skills that can be developed only by practice, i.e. "learning by doing". Therefore, constructivist approaches are to be employed in geometry. It could be also argued that no other subject in school mathematics has generated so much literature and practice regarding instruction through activities. Beyond the traditional tools (ruler, compass, straightedge, protractor), various physical and virtual educational resources, manipulative materials (games, puzzles, tiles, Logo), and computer software (Cabri, Geometer's Sketchpad, Geogebra*)* have been specifically designed for teaching and learning geometry. This situation shows that exploration and inquiry can play a key role in school geometry and confirms the assumption that investigative approaches are the natural way of instruction in this field. For example, building three-dimensional structures with cubes and describing such arrays is an effective way to develop spatial reasoning (Battista & Clements, 1996). Bar models of various materials (bamboo, rubber, foam, etc.) have been used in teaching stability/rigidity (Middleton & Corbett, 1998). Various curve-drawing devices are really helpful in learning analytic geometry through exciting hands-on activities and investigation (Dennis & Confrey, 1998). Computer tools such as *Geo-Logo* have been successfully used for helping primary school students explore paths, polygons, turns, angles, and lengths (Clements & Battista, 1998). Such exploratory activities are an excellent opportunity for children to have a taste of scientific investigation, because they involve modeling, prediction, testing of hypothesis, cognitive conflict, reflection, trial and error (heuristics), and they also stimulate team working. According to Battista & Clements (1996), the account of such an activity "nicely illustrates the constructivist claim that, like scientists, students are theory builders. Cognitive restructuring is engendered when students' current knowledge fails to account for certain happenings, or results in obstacles, contradictions, or surprises. The difference between the scientist and the student is that the student interacts with a teacher, who can guide his or her construction of knowledge as the student attempts to complete instructional activities" (Cobb, 1988, p. 92).

An investigative curriculum in geometry could be developed around authentic problems and meaningful problem-solving situations, with realistic stories or scenarios acting as hooks - hence the term "anchored instruction" (Zech et al., 1998). Communication with the teacher and collaboration between the students could be improved by using online environments such as Math Forums with separate "rooms" for each class (Renninger et al., 1998). Online solutions also include Web-based instruction, which has been successfully tested in teaching geometry at 3rd to 6th grade (Chan, 2006).

Geometry is also the ideal site for conjecturing, and it connects the most elementary, visual objects (available to the youngest children) with abstract thinking. Visualization and visual thinking are at the heart of what makes geometry a special case within mathematics.

Proving is one of the highest-order mental skills (since it involves creativity) and one of the most neglected in school mathematics. By its very nature, synthetic geometry is based on proofs and heuristics; hence the only way it can be taught effectively is through inquiry. Euclid's "Elements" implicitly proposed such an approach, since a large part of its proofs are not provided, and have to be completed by the reader. Actually, this is the earliest known form of PBL, one of the most popular inquiry approaches. Proving and geometry are intimately connected; if other subjects in mathematics can be taught with only few, easy proofs sketched by the teacher or with the statements justified intuitively, in geometry rigorous proving is a central part of any undertaking.

Geometry is essentially non-algorithmic, and almost any claim must be supported by some proof. This has a huge effect on the development of child's cognitive skills, but it is also very demanding - especially for the beginners. The student has to struggle with the assumptions and the conclusions of a theorem, until connecting them via mathematical deduction. It is a difficult task, especially if he has no prior experience with deductive reasoning, and it occurs frequently in inductive learning environments. That is one of the reasons for the declining quantity and quality of geometry taught at all school levels in North America and U.K. McLoughlin (2008) claims that a mathematics student need to learn to conjecture and prove or disprove his conjecture. He argues that

> *"Understanding mathematics - really understanding it - is not something that is learned through reading other people's work or watching a master teacher demonstrate his or her great skill at doing a proof (no errors, elegant, and complete). The student learns by getting his 'hands dirty' just as an apprentice plumber gets his hands dirty taking a sink apart, fixing it, and then putting it back together. He must explore the parts of the sink, learn how the parts interact, take things apart, put the part or the whole system together again, make mistakes, learn from the triumphs and mistakes (but most especially we learn from our mistakes). The nuts-and-holts of a skill can be taught; but initiative, discipline, imagination, patience, creativity, perseverance, and hard work cannot be taught - such must be nurtured, encouraged, suggested, and cultivated".* (McLoughlin, 2012, p. 2)

On the other hand, Goldenberg et al. (1998) point out that we cannot reasonably expect beginning students to become aware of the subtleties of mathematical proof before having experienced several refutations of their conjectures in order to feel a need for further analysis (beyond visual evidence). Middle school students are unlikely to behave like Lakatos's (1976) "advanced class", and for them proof might need to be motivated only by the uncertainties that remain without a mathematical justification, or by a need for an explanation for why a phenomenon occurs. Proof of something too obvious would likely feel ritualistic and empty for average students. Developing a questioning habit-of-mind, the desire to validate empirical results with mathematical proofs, and the ability to develop such creative proofs, is costly in terms of time and must start as early as possible, especially if mathematical proficiency is to be achieved by the student. Experimental research has shown that conjecturing tasks can be implemented in teaching and learning geometry from primary school level (Lin, 2013). Also, Kim & Ju (2012) conducted an experimental study involving investigative geometry and proof learning in middle school. As a result of the experimental instruction, the attitudes of the students toward mathematics ameliorated, their thinking and communication skills improved. They learned how to listen and how to verbalize their ideas smoothly. They displayed essential inquiry qualities such as courage and modesty. Kim & Ju consider their experiment a useful starting point for designing proof instruction by a persuasive, investigative approach. Combining inductive and deductive approaches in activities where students discover geometric properties and then prove them in a rigorous way is a promising instructional approach that could improve proving and conjecturing skills together, as shown by Fairbairn (2008). Diagrams are an effective tool that could be used in order to develop heuristic skills by inventing proofs and refutations (Komatsu, 2013).

A research of Zacharie (2009) has revealed that students emerge from proof-oriented courses such as high-school geometry unable to construct anything beyond very trivial proofs; moreover, most university students do not know what constitutes a proof and cannot determine whether a purported proof is valid. As the author points out, when making a comparison between mathematics and other subjects, we can say with certainty that in mathematics things are proved, while in other subjects they are not. Indeed, in physics, in biology, and in other fields a theory is validated or rejected by experimentation and data evidence, while in mathematics claims are validated by rigorous proofs, which are the hard and ultimate evidence. Zacharie contends that students' difficulties related to mathematical proof manifest in three points: appreciation why proofs are important, the relation between verification and understanding, and proof construction. He remarks that it is common for undergraduate students to say that they like mathematics but they hate proofs. However, the ability to understand and construct proofs is transformative, both in perceiving old ideas and making new and exciting mathematical discoveries. In many cases, it appears that negative attitudes toward proofs result from certain teaching practices, the selection of statements to be proven, and from teacher's inability to explain conceptually difficult concepts in simple terms. These aspects have to be carefully taken into account when implementing investigative geometry instruction. In the research of Zacharie (2009), several prospective high school teachers from the Department of Pure

Mathematics were asked to reflect on their own experiences of learning proofs, and their answers reveal the critical role of the teacher in shaping students' attitudes toward proofs.

The comments indicate some common flaws, such as:

- the teacher went too fast and did not know how to explain difficult concepts in simple terms;

- I had a bad teacher, who admitted he disliked proofs;

- the teacher did not give a reason why each step of a proof was correct;

- most mathematics lessons were boring and made me asleep;

- the teacher did not state all the theorems involved in the proof;

- the teacher did not convince me about the necessity of the proofs.

An important group of abilities developed through investigative geometry is represented by the visual skills. Senechal (1991) emphasized the distinction between visualization, which brings inherently visible things to mind (e.g. spatial visualization) and visual thinking, which refers to a visual rendering of ideas that are not inherently spatial (e.g. the visual representation of geometric sequences, or diagrams in topology). While the first category is extremely useful in real life, and is traditionally promoted in geometry courses (but also in design or other non-mathematical courses), the last one is central to mathematical reasoning and can be cultivated through inquiry. Goldenberg et al. (2012) proposed the use of stimulating tasks such as drawing "impossible figures" in order to help students engage and develop spatial vision. This may involve a rich and fun combination of drawing, manipulating, imagining, paper folding, building, and using computerized enhancements or simulations. The activities may be physical, but the essential skills are mental. A very interesting and useful type of activities in geometry is the so-called "proofs without words", regularly presented in the Mathematics Magazine. One example is a geometric proof of formula for the partial sum of a geometric sequence, by using right triangles. Other tasks presented by Goldenberg et al. (2012) involve fractals, billiards, infinite sums, and dynamic geometry tools.

Borasi (1992) conducted an instructional experiment in humanistic inquiry with two sixteen-year-old students from "School Without Walls", a second-rate alternative school in Rochester. The official presentation mentions that it offers a "small school setting and project-based learning approach" where "all curriculum, courses, and student experiences are planned, developed, and implemented with the clear intention of helping students become self-sufficient lifelong learners". Such statements clearly indicate an inquiry-learning approach, supported by a small class setting and a non-traditional curriculum with more emphasis put on attitudes (autonomy, perseverance, broad outlook) and reasoning skills (reflection, critical thinking) than on procedural skills and content knowledge. Actually, the instruction in School Without Walls is very close to a PBL (Project-based learning) approach. Evaluation is conducted in an alternative way, with grades replaced by detailed individual reports on how each student is progressing. These reports are prepared jointly with the student after periodic teacher-student conferences.

The experiment conducted by Borasi consisted of a ten-lesson "mini-course" on mathematical definitions - mostly in geometry, but also in algebra. It involved a series of thought-provoking activities which helped the students appreciate some of the special characteristics of these definitions such as ambiguity or the difficulty to provide a "perfect definition". The instruction took place outside of regular mathematics courses, and resembled to a tutoring experiment. Taxicab geometry scenarios were used in order to challenge the participants and to encourage them to work with nonstandard mathematical situations, to manifest creativity, and to engage in genuine mathematical debate. Lakatos (1976) provided a theoretical background for the experiment of Borasi. Arguably, students' investigations were so strictly guided by the researcher, that her study is of little use from the point of view of inquiry learning implementation.

2.2.4.4 *Textbooks for investigative geometry*

- Clark’s IBL textbook for geometry

Clark (2012) has designed a guide for teaching investigative Euclidean geometry, under the form of a short textbook. It does not have much graphics, except geometric figures and diagrams, and it uses a rather abstract language and way of presentation. Thus, it seems to target rather advanced students, who are more familiar with abstract thinking, formal logic, and proofs. The book has some similarities with the Elements of Euclid, by starting each section with a sequence of definitions and axioms, followed by propositions and theorems which have to be proved by the student. The difference is that Clark's book does not present any proof, but only hints (here and there) and problems (not exercises), as it follows an exploratory approach. It also has some guidelines for the instructor in the final section.

Overall, the book is useful for implementing inquiry-based axiomatic geometry, but rather as an auxiliary resource and not as a textbook. One of its greatest weaknesses is the absence of exercises and the abstract approach, with very few connections to real life applications or to other subjects. Another one is the lack of details regarding the steps of the investigations, or how to guide the students into the resolution of each problem. For these reasons, it has a limited pedagogical value and a reduced audience compared to widely acknowledged textbooks such as *Discovering Geometry*.

- Serra - "Discovering Geometry: An Investigative Approach"

The widely recognized textbook for learning geometry by investigation has been *Discovering Geometry* (Serra, 2008). According to Serra, the origins of his book date back to more than 35 years ago, during his first ten years of teaching. He believed that students learn with a greater depth of understanding when they are actively engaged in the process of discovering concepts and we should delay the introduction of proof in geometry until students are ready. Serra was also involved in a Research In Industry grant where he repeatedly heard that the skills valued in all working environments were the ability to express ideas verbally and in writing, and the ability to work as part of a team. Thus, he wanted his students to be engaged daily in doing mathematics and exchanging ideas in small cooperative groups. Until *Discovering Geometry*, no textbook followed this philosophy, therefore Serra created his own daily lesson plans and a classroom management system. This led to the first edition of the book, titled *Discovering Geometry: An Inductive Approach* (1989). The title of third (2003) and fourth edition (2008), *Discovering Geometry: An Investigative Approach,* emphasized the inquiry approach by using the fashionable catchword "Investigation". It also avoids the controversial identity between inquiry and inductive approaches.

The book is richly illustrated with images from real-world which exemplify and provide concrete meaning to the theoretical notions presented alongside. It manages to situate the concepts in their evolution, by including short historical remarks, and tries to do a work of popularization by explaining higher mathematical concepts in elementary and sometimes funny, everyday terms. It is a very friendly, humorous, and gentle approach to geometry, with lots of explanations and exercises. The geometry content is presented in a rather informal way, being more or less hidden in the tasks. It reveals itself little by little, as the student performs these activities and completes the exercises. Actually, every good teaching propose a specific sequence of examples, definitions, exercises, in a conceptual order. In Serra's book, not every step is inquiry but the sequence is. The exercises proposed are either computational (direct or indirect application of a formula, such as Ex. 1 and 4 at p. 37, respectively) or conceptual (connecting different concepts, such as Ex. 5 at p. 37). Computational exercises are algorithmic, while conceptual ones are not algorithmic but use definitions and properties of concepts.

Lessons usually start with a brief introduction, and continue either with some worked examples, or with a strongly guided investigation, where each step is clearly stated and intermediary questions are formulated. Each activity is followed by its discussion, followed by examples. There may be up to five investigation activities in a lesson, which concludes with a sequence of exercises. Some exercises may be activities or mini-investigations. Theoretical elements (concepts, rules, theorems, conjectures) are formulated or disclosed only at the relevant point during or after an activity, when they naturally emerge - as the student has acquired an intuitive or empirical grasp of that notion or idea. There are also connection examples relating the content to other fields (science, technology, arts, history, career, and recreation), review exercises, and activities specially designed for improving algebra, reasoning, or visual thinking skills.

In order to deepen and to enrich students' understanding of the new content with conditional knowledge and connections with other content from geometry, the book proposes a series of open exercises titled "Take another look" and several projects, where the student has more freedom to experiment and can investigate further. Each chapter starts with the list of learning objectives and concludes with an account of the available tools for assessing what has been learned: update your portfolio, organize your notebook, write in your journal, performance assessment, write test items, and give a presentation.

Overall, *Discovering Geometry* is rightfully acknowledged as an outstanding textbook and the best available resource for introducing geometry by investigation at secondary or even elementary level. It could be also used as a model for developing investigative mathematics textbooks, in higher geometry or in other subjects.

From a researcher's point of view, there is still a lack of evidence regarding the better quality of learning acquired by using *Discovering Geometry* (and implicitly, the inquiry approach) with respect to traditional textbooks and approaches. One such comparative study was done by Koedinger, and it did not find any substantial difference:

> *"Although it remains possible that a more complete and detailed quantitative analysis might yield some differences, we did not see the kind of qualitative differences that might be expected from the non-traditional approach of Discovering Geometry. This result should not be interpreted as critical of this particular text, as at least three mediating factors reduced the likelihood of an effect:*
>
> *a) greater teacher experience using the traditional text*

b) great variability in the way different teachers implement the Discovering Geometry curriculum

c) high variability and generally poor preparation of this urban student population.

This result ***should*** *be considered as evidence for substantial difficulties in implementing curriculum reform in a way that yields substantial student achievement gains. It takes much more than a textbook."* (Koedinger, 1998)

3 A CONCEPTION OF IBL

3.1 INTRODUCTION

> *"These developmental years are not just a time to educate but they are your obligation to form a brain and if you miss them you have missed them forever." - Michael Phelps, co-inventor of the Positron Emission Tomography (CMA, 2015)*

When trying to understand inquiry, we encounter a major obstacle: the lack of an adequate model describing the cognitive processes that occur during this activity and the structure of cognition itself. Developing a theoretical framework that describes properly and completely the structure of knowledge remains a very challenging task, and it will remain a topic of much debate in cognitive psychology as well as in the theory of education.

Various models have been proposed for a specific use in education, e.g. assessment (Bloom, 1956), or even for broad “learning, teaching, and assessing” purposes (Anderson et al., 2001), but they are unsuitable for explaining the whole spectrum of understanding, from simple comprehension to illumination. This is especially true in mathematics, a field with a mainly vertical organization of knowledge. Bloom's cognitive taxonomy, famous but also contested (de Landsheere, 1975, pp. 73-94; Sugrue, 2002), was designed by a team of educational psychologists before the recent advances in cognitive psychology. Anderson et al. (2001) proposed an updated version of Bloom’s classification, by replacing nouns with verbs and Synthesis with “Creating”. The original Bloom's classification did not involve any visual representation, and the addition of a triangular diagram was just a convenient means to emphasize a hierarchical structure of thinking, namely the idea that each new, more complex level builds on top of previous, simpler ones:

"The triangle does not appear anywhere in either Taxonomy.... [Q]uite likely designed by someone as part of a presentation made to educational practitioners.... I believe that the triangular representation was developed in order to indicate that, in the original Taxonomy, the six categories formed a cumulative hierarchy. That is, it was believed by the authors of the original Taxonomy that mastery of each lower category was necessary before moving to the next higher category." (Anderson, 2017)

Even models specifically developed for describing the processes involved in mathematical inquiry such as the CPiMI (Model for Cognitive Processes in Mathematical Investigation) proposed by Yeo (2013; 2017) don't deal with the vertical structure of mathematical knowledge and the multiple layers of understanding, except the basic one: understanding the task. This is why I had to develop a different model, with better explanatory power regarding essential processes in cognition such as inquiry, understanding and discovery.

3.2 A TAXONOMY OF COGNITIVE PROCESSES

"I have hardly ever known a mathematician who was capable of (dialectic) reasoning." - Plato

Two aspects that distinguish the cognitive model proposed here from the original taxonomy of educational learning objectives designed by Bloom and his associates (Bloom et al., 1956) are: my model lists seven cognitive processes rather than six objectives and the processes resume in a spiral way after having reached the most advanced one. With respect to the revised Bloom's taxonomy (Anderson et al., 2001) which also uses processes, there are also significant differences: my model lists seven cognitive processes rather than six and they are different in nature.

For simplicity, a triangular representation of these processes is provided in Figure 5 .

Figure 5. Cognitive processes in learning through inquiry

To name the cognitive processes involved in inquiry, we have used verbs: Inventing, Integrating, Operating, etc. We will use nouns such as Invention, Integration, Operation, etc. to refer to a phase in inquiry where the corresponding process is dominant, or to the product of this cognitive activity.

We will now describe each cognitive process in turn.

3.2.1 Definitions of cognitive processes involved in inquiry

3.2.1.1 *Recording*

At this basic level, knowledge is acquired by simple observation through the use of senses (listening to a teacher's lecture, watching a demonstration for modeling purposes, etc.) or it is retrieved from the learner's memory. With each recall the knowledge is refreshed, until it expands into the unconscious, from which it may emerge via associations.

Keywords: observation, memorization, data acquisition, recall.

3.2.1.2 *Operating*

This process consists in active exploration, practical work and hands-on experience, where one learns the functional aspects of a subject by testing them in various contexts, for example performing a method in all kinds of situations in order to acquire empirical know-how and an awareness of its strengths and limitations. Procedural skills are typically promoted here; however, this process need not be triggered by drill. Drill does not allow for active exploration; it develops fast, automatic performance of routine tasks via repetition and memorization.

Keywords: exploration, experiment, practice, procedure, application, classification, data mining, automatic translation.

3.2.1.3 *Reasoning*

This is the process of rationalization, whereby logical comprehension and basic discernment are attained. One is able to anticipate the direct outcome of an operation and of one's acts in general. The learner can distinguish when a procedure can and when it cannot be used. At this level, learners can handle causality, logical implication (syllogism) and contradiction. They are able to follow, to check and to validate the logical correctness of a proof. They can use Mathematical induction as a method, a technique for proving a certain type of mathematical statements.

Keywords: comprehension, logic, causality, implication, discernment, formal proof, contextual translation.

3.2.1.4 *Targeting*

In this process, chains of procedures targeting a specific outcome are generated, with each step getting closer to the final goal. The sequence of operations is straightforward, but the success is not ensured, and the attempt may lead to a dead end. If the target has not been reached, after a trial one may look further and build an additional chain that will hopefully achieve the desired outcome, and so on until a deadlock is met and the general strategy has to be revised. Designing a path towards a problem's solution by choosing familiar procedures and combining them in an adequate sequence involves this process. Each procedure is selected from the available toolbox according to its effectiveness, familiarity and easiness. Thinking is focused on the final goal and the resolution path is targeting it, but there is a risk of falling into a "tunnel vision" and lack of perspective as a result of a focused, uni-dimensional view. At this stage, one is able not only to comprehend demonstrations but also to build simple proofs. Yet, more difficult proofs may involve higher levels.

Keywords: focused vision, foresight, analytical/convergent thinking, gazing, aiming.

3.2.1.5 *Integrating*

This is the processes where associations, links and connections inside and outside of a subject are developed. In contrast to the linear, sequential thinking of the Targeting process, here the mind may jump from one piece of knowledge to another without the presence of a deductive chain as required in formal logic, but by similarity and analogy. In this way, relations are built in all directions until they form a *multidimensional* structure like a neural network. This applies not only to static knowledge but also to processes, principles or approaches - which may be “recycled” and used in a new context. At this level, thinking has a holistic quality but without adequate control it entails the risk of distraction from the goal, lack of focus, dissipation in unproductive directions, confusion. Ramanujan, “the man who knew infinity”[2], appeared to engage in the processes of Integrating and Inventing in their highest form.

Keywords: global vision, holistic/divergent thinking, joining, comparing, connection, association, classification, network, similarity, analogy, allegory, metaphor, synthesis.

3.2.1.6 *Structuring*

After having integrated the available knowledge, a complete evaluation is performed and all the inessential information is discarded. If a structure becomes visible, a simplified

---

[2] Reference to a film under this title, about Ramanujan: https://en.wikipedia.org/wiki/The_Man_Who_Knew_Infinity_(film)

model may be designed. The model is tested, evaluated, validated or possibly rejected and replaced according to new information, as in Bayesian statistics. Approaches, strategies and knowledge are subjected to a comprehensive judgment here, after proper treatment at lower cognitive stages. Gaps in knowledge or in a model's structure are also identified, which may lead to the discovery of new knowledge or to invention. If the lowest level of understanding can be described as a puzzle or totally unstructured information, complete understanding or insight is reached when all the knowledge has been integrated and the whole structure with its internal and external connections is revealed, like a relief map.

Keywords: evaluation/judgment, weight of evidence, hypothesis, to understand/to realize, wisdom, modeling, structure, organization, system, scheme, abstraction, symbol.

3.2.1.7 *Inventing*

New, distinct knowledge is born by means of this process, through creative thinking, by inventing tools, techniques/approaches, objects, structures, concepts, etc. The knowledge may not be new for others, but since it has been unknown for the inventor it motivates him or her to start a new seven-step cycle of cognitive processes: Recording, Operating, etc. Creativity may be expressed by inventing strategies for solving a problem or for completing a task, or more powerfully by designing new tools and novel approaches, models or structures. For example, in his periodic table, Mendeleev combined a horizontal linear structure defined by the atomic mass of elements with a vertical grouping based on their similar physical and chemical properties. Regarding the various types of creativity, Arnold contrasted the approaches of the Soviet Union's most famous mathematicians Gelfand and Kolmogorov:

> *"Suppose they both arrived in a country with a lot of mountains...*
> *Kolmogorov would immediately try to climb the highest mountain.*
> *Gelfand would immediately start to build roads." (Chang, 2009)*

Keywords: creation, discovery, originality, illumination, enlightenment, revelation, inspiration, insight, idea.

3.2.2 A hierarchy among the cognitive processes

The Québec school system has adopted a classification of knowledge in three categories (MELS, 2007, p. 6):

- Declarative
- Procedural
- Conditional

Declarative knowledge refers to the information that a person records in her mind, and that can be spoken or written (theoretical knowledge). Learning declarative knowledge corresponds to Recording. Procedural knowledge, which corresponds to Operating, comprises information on how to do something or how to perform the procedural steps that make up a task, and the ability to actually perform that task (know-how, practical knowledge). Conditional knowledge, which corresponds to Reasoning, refers to the awareness about when to use a procedure or when not to, information as to why and under what conditions a procedure works, in addition to why one procedure is better than another (Tardif, 1992). So, pre-university education targets only the three basic layers of cognition: Declarative, procedural and conditional knowledge naturally correspond to the three basic cognitive processes: Recording, Operating and Reasoning. This justifies the grouping of the lower cognitive stages into a single category, Knowledge. Higher education aims at developing deeper understanding (less observable), with invention as its visible, ultimate outcome.

My teaching practice with science classes at top level colleges in Bucharest and Montreal confirmed the structure of basic knowledge in three layers and their hierarchy: students acquire procedural skills and speed of execution before achieving discernment, which includes the ability to distinguish the contexts where a procedure can be applied from those where it can not. They often manifest a tendency to automatically perform familiar procedures without a basic understanding of their meaning and opportunity.

For example, at a final test administered to a class of students, most of them proficient in Differential Calculus, one of the questions asked to evaluate the limit at 2 of the function $f\colon \mathbb{N} \to \mathbb{N}$, given by $f(x) = x^2 + 2$. All the class computed it by substituting 2 for x in the expression above, neglecting the fact that a limit is defined only in an accumulation point of a set.

Other students, at the end of their one-term course in Differential Calculus, were unable to give a definition of the derivative, although they had acquired some skill in the calculation of limits and derivatives. In order to introduce L'Hôpital's rule, I showed them a simple indeterminate form, $\lim_{x\to 0} \frac{\sin x}{x}$, which also happens to define the derivative at zero of a familiar function. Even after being told that the limit can be written $\lim_{x\to 0} \frac{\sin x - \sin 0}{x-0}$, which is the derivative of sine function at zero, they were still unable to recognize in the next example, $\lim_{x\to 0} \frac{e^x - 1}{x}$, the derivative at zero of the exponential function. Maybe, after having learned L'Hôpital's rule, students will use it to calculate $\lim_{x\to 0} \frac{\sin x}{x}$ as well, since they have not realized the sequence of knowledge, where the derivative of sine function in zero is found by means of areas from the start, and thus it would be circular reasoning to use the value of a derivative in order to compute the derivative itself.

This supports the idea that Reasoning is a cognitive process of higher level than Operation or application of rules and procedures, which can be executed more or less blindly - with little awareness or discernment regarding what is behind them. On the contrary, genuine comprehension of a method requires a lot of practice and experience in order to know exactly how it works in various contexts and to employ it efficiently; also, hands-on experience is an essential component of reliable knowledge according to Dewey (1938), Kolb (1984), and other educationalists. Active exploration, experimentation and practice are related and comprised in Operating, which follows and expands Recording:

> *"The teacher and the book are no longer the only instructors; the hands, the eyes, the ears, in fact the whole body, become sources of information, while*

teacher and textbook become respectively the starter and the tester. No book or map is a substitute for personal experience; they cannot take the place of the actual journey." (Dewey, 1915, p. 74)

The above structure is also justified by the traditional sequence of teaching a procedure:

- Presentation of some relevant examples, in order to show the utility of the procedure

- Execution of the procedure by the teacher, followed by its detailed description

- Execution of the procedure by the students: first step by step, then completely, and drill

- Displaying various examples, with instances when the procedure can or cannot be used.

Drill, which is the automation of a procedure, relates to Sfard's interiorization and Dubinsky's process stage (Tall, 1999), the first step toward reification (Sfard, 1991) and encapsulation (Dubinsky, 1986 & 1991), respectively. It allows a learner to treat a sequence of operations as a single large operation and to foresee its outcome, thus being able to choose instantly between several procedures that could complete a given task.

In conclusion, we can group the cognitive processes in two categories:

- **Knowledge** : Recording, Operating, Reasoning
- **Understanding** : Targeting, Integrating, Structuring, Inventing

Knowledge forms the basis of school education and involves a procedural way of reasoning. By contrast, higher layers require searching for and trying different strategies and there is no guarantee that they will work.

At the undergraduate level, Targeting is often involved. Master's level typically requires Integrating or Structuring activity (project, thesis, etc.) and creative work at Invention stage is usually required only at a doctoral level. Invention is an expression of deepest understanding, acquired via enlightenment or revelation, as Piaget (1972 b, p. 20) said:

"To understand is to discover, or reconstruct by rediscovery, and such conditions must be complied with if in the future individuals are to be formed who are capable of production and creativity and not simply repetition."

It should be remarked that **Inquiry**, which essentially means **Research**, is much more than simple exploration of available information, classification and data mining, often labeled as Investigation. Such processes would be more properly assigned to Operation and Reasoning stages of cognition. Inquiry starts when there is no obvious path, made up of familiar procedural steps, to the target. This initially occurs in Targeting, when a procedural chain leads to a dead end, and the person is wondering what other method to employ in order to finally reach the target. Therefore, IBL addresses Understanding layers starting with Targeting and proceeding up to Inventing.

Depending on the difficulty of the task, the amount of time available, and the knowledge, skills or cognitive development of the learner, inquiry may require more or less guiding or scaffolding. However, Structured inquiry (problem & method given) naturally involves Reasoning, and Open inquiry (students formulate & solve the problems) expects students to engage in Structuring and Inventing. Guided inquiry (problem only given) may involve any cognitive process beyond Reasoning, from Targeting to Inventing. Rusu (1972) proposed a self-study by problem solving that allows learners to decide when to access guiding.

3.2.3 Higher-order cognitive processes require probabilistic rather than deterministic thinking

"For every complex problem, there is a solution that is clear, simple and wrong." – H.L. Mencken

At the three basic stages of cognition, Recording, Operating and Reasoning, learners operate inside their safety zone since they are dealing only with tasks that can be completed by following a procedure or an obvious sequence of standard operations. In such cases, when an algorithm or a recipe is available, successful completion of the task is ensured and there is no risk of failure: one does not break away from a deterministic cognitive framework, also known as "**deterministic thinking**", in contrast to the "**probabilistic thinking**" (Bolisani et al., 2018, pp. 86-88). Newton learned the hard way this difference when he lost a fortune after venturing in the stock market (Holodny, 2017).

Beyond those Knowledge stages, one deals with **approaches** that may or may not be successful rather than with **methods** where the success is certain when they are properly used.

Starting with Targeting, we can only talk about **perceived probability** of success: several strategies or approaches may be available, and one will choose the easiest one, the most familiar, or that which is most likely to succeed. Hence, at these stages we deal with probabilistic thinking or "**plausible reasoning**" described by Pólya (1954), which is far more sophisticated, flexible and difficult to master than the deterministic thinking usually met in school and in the society. The command-and-control, deterministic mindset of the industrial management means order and discipline, and it provides some major benefits: efficiency, predictability, reliability. However, deterministic thinking dramatically reduces organizational entropy, diminishing the potential of creativity and innovation (Bolisani et al., 2018).

The difficulty of breaking away from a deterministic mindset and dealing with the unknown instead of rejecting or just ignoring it via explicit or implicit assumptions respectively can be illustrated by the famous controversy involving indeterminacy principle in quantum physics. A probabilist may be amused by Einstein's apparent misunderstanding of probabilities when he argued that "God does not play dice with the universe", because randomness involves **our** lack of information, not God's! Probability is the "proportion" or a measure of the unknown and it specifies the degree of certainty of an event: from 0 (improbable) to 1 (certain). It is a completely subjective and relative concept, since what is perceived as probability prior to knowing the result of an event becomes certainty afterwards. A person who plays dice may use a probabilistic model based on symmetry, agreeing that while it is impossible to predict the result of an individual dice rolling, when the number of rollings is very large each face of the dice will be up in roughly the same proportion. The lack of information is almost total, but when the dice is regular there is still some uniformity and a "statistical" degree of predictability, valid only for huge numbers of similar trials. In contrast, if a person would know all the factors influencing a particular rolling, she would be able to predict its result with absolute certainty (Laplace, 1902, p. 4). When quantum mechanics has shown that we cannot determine both the position and the momentum of an atomic particle, Heisenberg (1927, p. 197) declared the final failure of classical causality and determinism:

> *"But what is wrong in the sharp formulation of the law of causality, 'when we know the present precisely, we can predict the future', it is not the conclusion but the assumption that is false. Even in principle we cannot know the present in all detail."*

The difference between deterministic and probabilistic thinking is just the admission of the unknown in one's representations. A higher stage of probabilistic mindset is reached when algorithms and models themselves are subjected to correction or reform once new data or information contradicts the current representation, as in Bayesian probabilities. A model or a representation (which may involve an approach) will be seen as a hypothesis with some degree of likelihood, hence Pólya's "plausible reasoning". It also relates to the heuristic approach or "educated guessing", which hopefully leads to a solution or to discovery and thus it reaches the Structuring level. Actually, the word "heuristic" derives from the Greek εὑρίσκω ("I discover"), recalling Archimedes' legendary exclamation "I found (it)!" (εὕρηκα) when he discovered his principle in hydrostatics.

Probabilistic thinking is an important aspect that distinguishes inductive from deductive reasoning. **Inductive thinking** is a way of constructing general propositions by deriving them from specific examples. This reasoning is probabilistic: it only states that, given the premises, the conclusion is probable. It may be correct, incorrect, correct to within a certain degree of accuracy, or correct in certain situations. The degree of confidence may be increased through testing, or by additional observations. By contrast, in **deductive reasoning** specific examples are derived from general propositions, and the conclusion is always true when the premise is true. Two important categories of inductive reasoning are employed in Integration: **associative thinking** - based on associations and connections, and **analogical thinking** - based on analogies and similarities.

As already shown, prominent figures in the area of mathematics education, such as Klein (1932, p. 208) and Pólya (1954, pp. 7 & 83-84), asserted that mathematicians think inductively, but prove their results deductively. Induction in mathematics labels both a method and a way of thinking, but these involve different cognitive levels. The method of mathematical induction is a form of deduction and only involves Knowledge processes, since a sample plus a rule about the unexamined cases actually give us information about every member of the set (Chowdhary, 2015, p. 26). By contrast, inductive thinking begins when the rule is not yet proven or certain, but just a plausible hypothesis. So, for example, a problem such as, "Prove that the number of diagonals in a convex n-gone is $\frac{(n-3)n}{2}$." will trigger the "Knowledge" processes; there is a chance that the problem "Conjecture a formula for the number of diagonals in a convex n-gone and prove it." will engage some students in the "Understanding" processes.

A different way of thinking, **abductive thinking**, is developed and employed in the process of Structuring. It starts from a set of observations then seeks to find the simplest and most likely explanation. It is involved in the formation of hypothesis and model building in applied mathematics, where models are judged according to their explanatory power and simplicity. By using criteria of simplicity and elegance, Abduction relates to Ockham's razor principle in heuristics. The concept of abduction was introduced by Peirce and included in his methodology of inquiry that inspired Peirce's student, Dewey and the book Logic: The theory of inquiry (Dewey, 1938). According to Peirce (1976, pp. 62-63), a hypothesis is judged and selected for testing when it offers to quicken and to reduce the "cost" of the inquiry process towards new truths:

> *"Methodeutic (Speculative rhetoric) has a special interest in Abduction, or the inference which starts a scientific hypothesis. Any hypothesis which*

explains the facts is justified critically. But among justifiable hypotheses we have to select that one which is suitable for being tested by experiment."

Abduction is a step further from inductive thinking to approaches that define Structuring, and which are often called **structural** and **conceptual thinking**. Structural thinking would try to simplify, to organize and to reveal the underlying structure of the information already interconnected in Integration. It also adjusts the existing mental model when necessary as new knowledge is acquired and tested. Critical reasoning is a prerequisite, but structural thinking goes far beyond it by using holistic thinking and abduction for detecting the hidden order. Conceptual thinking uses abstraction and conceptualization for building general models, theories and frameworks that explain and give meaning to information, these constructions becoming themselves objects of study. Such ways of thinking were powerfully displayed by Riemann in his notion of variety and Grothendieck in his concept of schemas.

Many students don't really understand the principle of induction but they apply it as a method, successfully when the context allows doing it easily and unsuccessfully when it does not. Informal evidence from the 1990s revealed that among pre-service teachers of mathematics at a Canadian university (personal communication), only half managed to understand mathematical induction, and the others eventually learned it as a more or less magic rule. However, the idea of induction toward infinity is deeply ingrained in human mind, for example the concept of natural numbers is an infinite construction: one, two, three... infinity. This is a dynamic, transcendental step that cannot be performed by computers, which by their nature operate only with finite sets and processes. Ancient Greeks were confronted themselves with the issue of distinguishing between "potential infinity" i.e. the mental abstraction of a dynamic process, and "actual infinity", defined by its intermediary, finite stages, as in the well-known Dichotomy paradox of Zeno. The notion of infinity illustrates the difference between algorithmic and conceptual thinking, between artificial and human intelligence. Another obstacle encountered by computers is to imitate the “reasoning by contradiction”, which means trying to reach a contradiction by starting from a false assumption. This is a non-algorithmic task, very difficult to implement on a computer. No wonder that from the sample of pre-service teachers mentioned above, only half managed to understand the reasoning by contradiction.

Pólya (1957; 1981), Engel (1999), Posamentier (2015) and other authors tried to classify the various strategies used in problem solving, many of these involving inductive thinking. Engel, who developed a very effective way of teaching heuristics and problem solving by training students in the main principles of mathematical thinking, argued that:

> *"A successful research mathematician has mastered <u>a dozen general heuristic principles of large scope and simplicity</u>, <u>which he/she applies over and over again</u>. These principles are not tied to any subject but are applicable in all branches of mathematics. He usually does not reflect about them but knows them subconsciously." (Engel, 1999, p. 39)*

3.2.4 A summary

> *"For the mind does not require filling like a bottle, but rather, like wood, it only requires kindling to create in it an impulse to think independently and an ardent desire for the truth. Imagine, then, that a man should need to get fire from a neighbour, and, upon finding a big bright fire there, should stay there continually warming himself; just so it is if a man comes to another to share the benefit of a discourse, and does not think it necessary to kindle from it some illumination for himself and some thinking of his own, but, delighting in the discourse, sits enchanted; he gets, as it were, a bright and ruddy glow in the form of opinion imparted to him by what is said, but the mouldiness and darkness of his inner mind he has not dissipated nor banished by the warm glow of philosophy."*- Plutarch (1927, p. 259)

In Table 4*: Cognitive stages, mindset and thinking* we review the classification of cognitive processes and their specific ways of thinking, with some examples of methods or approaches typically used.

Table 4: Cognitive stages, mindset and thinking

Phase	Layer	Mindset	Reasoning Thinking	Examples
Knowledge	Recording	Deterministic	-	
	Operating		-	
	Reasoning		Deductive	- Syllogisms - Method of mathematical induction: forward, backward - Reasoning by contradiction - Checking and following proofs
Understanding	Targeting	Deterministic steps/tactics Probabilistic path/strategy	Deductive	- Building proofs of the statements - Choosing the most plausible path towards a solution - Selecting the most adequate method
	Integrating	Probabilistic	Inductive	Associative thinking: combining various approaches such as algebraic, geometric or arithmetic Analogic thinking: "recycling" proofs, finding invariants
	Structuring		Abductive Reflexive/ Contemplative	Heuristics as "educated guess" Structural thinking: developing models, conjecturing on the structure, making hypothesis Conceptual thinking: thinking in terms of concepts; abstraction, conceptualization
	Inventing	Faith	Fiery/Sparkling	Combinatorial thinking Enlightenment/revelation

3.3 EXAMPLES OF IBLECTURES

"A person who can, within a year, solve $x^2 - 92y^2 = 1$ is a mathematician."
Brahmagupta (598-668 CE)

The teacher shows the students how mathematicians conduct their inquiries. Every proof is the result of an inquiry. We remark the "**creative load**" of the problems presented.

3.3.1 Lecture 1: Backward induction – Principle of infinite descent (Fermat)

While mathematical induction is a common method used for proving positive statements, backward induction is rarely encountered, and employed almost exclusively for proving negative propositions. Backward induction, first used by the Pythagoreans and called Infinite descent by Fermat (1659) who rediscovered it, reduces a statement depending on the natural number *n* to a similar proposition where *n* is replaced by a smaller value. By repeating this procedure, either we end up with a basic set of values of *n* for which the property has been tested (and thus we can state with certainty if it is true or false), or we get an infinite chain of decreasing natural numbers, which is impossible and thus the initial statement must be false. Backward and forward induction are equivalent and they can be reduced to proofs by contradiction by choosing the whole number *n* minimal such that the statement to be proved does not hold. Fermat used infinite descent to prove his last theorem for *n* = 4, Euler for *n* =3, Legendre and Dirichlet for *n* = 5, and Lamé for *n* = 7. The same method was used to easily prove Sylvester's line problem in Euclidean spaces. Backward induction has been revived by von Neumann (1944), who used it as an essential tool in game theory. In finance, the pricing of American options is based on this method.

Next, I will present some examples of the way mathematicians think when solving problems and how the cognitive processes are involved.

- Problem 6 of IMO **1988** (Stephan Beck, Germany)

Let a and b be positive integers such that $1 + ab$ divides $a^2 + b^2$. Prove that $\frac{a^2+b^2}{1+ab}$ is the square of an integer.

This was the most difficult problem at the IMO 1988 (Knežević et al., 2013): only 11 of the 268 competitors completely solved it. It became a famous challenging problem, widely used for training students in backward induction and for exercising creativity. The standard proof presented online or in the books on "the art of problem solving" is not very insightful and does not allow any generalization. It uses some tricky operations involving a second degree equation with integer coefficients and roots.

Solution 1 (standard)

If $\frac{a^2+b^2}{1+ab} = k \in \mathbb{Z}$, with $a, b \in \mathbb{N}\backslash\{0\}$, then $k \geq 1$. Fix $k \in \mathbb{N}\backslash\{0\}$ and define

$$M := \{(a,b) \in \mathbb{N}^2 | \frac{a^2+b^2}{1+ab} = k;\ 1 \leq b \leq a\}.$$

Let $(\alpha, \beta) \in M$ such that β is minimal. Then we have $\alpha^2 - k\alpha\beta + \beta^2 - k = 0$, i.e. α is a root of the quadratic equation $X^2 - k\beta X + \beta^2 - k = 0$. Let α_1 be the other root; we get by Viète relations $\alpha_1 = k\beta - \alpha \in \mathbb{Z}$ and $\alpha\alpha_1 = \beta^2 - k$. If $\alpha_1 > 0$, since $\beta \leq \alpha$ we have $\beta\alpha_1 \leq \alpha\alpha_1 < \beta^2$, which implies $\beta\alpha_1 < \beta^2$, i.e. $\alpha_1 < \beta$, contradicting the minimality of β. Therefore, $\alpha_1 \leq 0$. We have:

$$\alpha(\alpha_1 + 1) = \alpha\alpha_1 + \alpha = \beta^2 - k + k\beta - \alpha_1 \geq \beta^2 - k + k\beta = \beta^2 + k(\beta - 1) \geq 1,$$

which implies $\alpha_1 + 1 > 0$ i.e. $\alpha_1 > -1$. We conclude that $\alpha_1 = 0$, hence $\beta^2 - k = 0$ and $k = \beta^2$.

We can model the process of building this solution by the sequence of mental activities detailed in the left column of Table 5. For each activity, the outcome and the cognitive process involved are indicated in the middle and the right column, respectively. Such tables will not be used in actual IBLectures, but are presented here only for purposes of research.

Table 5. Cognitive analysis of the standard solution to IMO 1988 Problem 6.

Description of the mental activity	**Outcome**	**Type of cognitive process involved**
Reading the problem statement	Memorization of the content	Recording
Trying various arithmetic and/or algebraic methods	No successful approach, dead end	Targeting [1]
Comparing the form of the expression $\frac{a^2+b^2}{1+ab}$ with familiar functions	The expression is a rational function, symmetric in a and b. The numerator and the denominator are second and first degree polynomials, respectively	Operating [3]
Deriving implications of the symmetry in a and b of the expression $\frac{a^2+b^2}{1+ab}$	Finding that a and b are interchangeable and that we may suppose without loss of generality that $a \geq b$. Alternatively, we may suppose without loss of generality that $b \geq a$.	Reasoning [5]
Idea coming up: transforming the expression in order to use the properties of second degree functions	Reducing the problem to a second degree equation with root : $X^2 - kbX + b^2 - k = 0$. Consider the other root of the equation, a_1.	Inventing [6]
Trying the "root flipping" approach	Viète's relations $a + a_1 = kb$ and $aa_1 = b^2 - k$ are found, also the fact that a and a_1 are interchangeable and that $a_1 = kb - a \in \mathbb{Z}$.	Targeting [1]
Deriving implications of the Viète relations	The relation $aa_1 < b^2$ is obtained.	Reasoning [5]
Evaluating the relation $a_1 < \frac{b^2}{a}$	Understanding that the relation $aa_1 < b^2$, rewritten as $a_1 < \frac{b^2}{a}$, implies a very simple relation: $a_1 < b$, if $a \geq b$.	Structuring[7]
Recalling the symmetry in a and b of the initial expression	We may suppose without loss of generality that $a \geq b$	Recording
Supposing that $a \geq b$, which implies $a_1 < a$, and selecting a useful method	Finding that backward induction can be used in this case, since a and a_1 are interchangeable	Targeting [2]
Supposing that $a \geq b$, and using backward induction	Defining $M := \{(a,b) \in \mathbb{N}^2 \| \frac{a^2+b^2}{1+ab} = k;\ 1 \leq b \leq a\}$, $(\alpha, \beta) \in M$ such that β is minimal	Operating [4]

The reasoning used for a, b and a_1 is followed for α, β and α_1	Finding that α is root of $X^2 - k\beta X + \beta^2 - k = 0$. Considering the other root of the polynomial, α_1. The relation $\alpha_1 = k\beta - \alpha \in \mathbb{Z}$ is found. Deduction of the fact that $\alpha_1 \leq 0$.	Reasoning [5]
Resolving the case $\alpha_1 = 0$, by using the fact that α_1 is root of the polynomial $X^2 - k\beta X + \beta^2 - k$	We get $k = \beta^2$, and the problem is solved in this case.	Reasoning [5]
Trying to solve the case $\alpha_1 < 0$ by reaching a contradiction. Using for this purpose the knowledge already found that $\alpha_1 \in \mathbb{Z}$.	We get $\alpha_1 + 1 \leq 0$ in this case. Also, we find $0 \geq \alpha(\alpha_1 + 1) = \alpha\alpha_1 + \alpha = \beta^2 - k + k\beta - \alpha_1 > \beta^2 - k + k\beta = \beta^2 + k(\beta - 1) \geq 1$, which is a contradiction.	Targeting [2]

(1) This has been classified as Targeting because the problem solver tries various approaches, strategies, methods in order to get closer to the solution.

(2) This has also been classified as Targeting because the solver tries one method knowing that it might not apply (tentative approach).

(3) Here the solver classifies the expression using familiar categories.

(4) The solver uses backward induction as a standard method.

(5) This action has been classified as Reasoning because it is just deriving conclusions from given premises (syllogism).

(6) This action has been classified as Inventing because the problem solver thinks outside of the box and introduces a new variable k, which connects the numerator and the denominator into a second degree polynomial.

(7) Here the solver evaluates the outcome of a previous action and decides on which direction to continue further.

Solution 2 (alternative):

Since $\frac{a^2+b^2}{1+ab}$ is symmetric in a and b, we may suppose without loss of generality that $a \geq b$. Then we can write $a = bc + r$ for some integers c and r, with $0 \leq r < b$. We get:

$$\frac{a^2+b^2}{1+ab} = \frac{(\frac{a}{b})^2+1}{\frac{a}{b}+\frac{1}{b^2}} = \frac{c^2+2\frac{cr}{b}+\frac{r^2}{b^2}+1}{c+\frac{r}{b}+\frac{1}{b^2}} = c + \frac{\frac{cr}{b}-\frac{c}{b^2}+\frac{r^2}{b^2}+1}{c+\frac{r}{b}+\frac{1}{b^2}} = c + \frac{\frac{c}{b}\left(r-\frac{1}{b}\right)+\frac{r^2}{b^2}+1}{c+\frac{r}{b}+\frac{1}{b^2}}$$

We show that $c - 1 < c + \frac{\frac{c}{b}\left(r-\frac{1}{b}\right)+\frac{r^2}{b^2}+1}{c+\frac{r}{b}+\frac{1}{b^2}} < c + 2$. This is equivalent to

$$-(c + \frac{r}{b} + \frac{1}{b^2}) < \frac{c}{b}\left(r - \frac{1}{b}\right) + \frac{r^2}{b^2} + 1 < 2c + \frac{2r}{b} + \frac{2}{b^2}, \text{ i.e.}$$

$$-cb - r - \frac{1}{b} < c\left(r - \frac{1}{b}\right) + \frac{r^2}{b} + b < 2cb + 2r + \frac{2}{b}.$$

The left inequality is equivalent to $0 < c\left(r + b - \frac{1}{b}\right) + \frac{r^2}{b} + b + r + \frac{1}{b}$, which is true since $b \geq r + 1 \geq 1$.

The right inequality is equivalent to $c\left(2b - r + \frac{1}{b}\right) - (b - r) + r - \frac{r^2}{b} + \frac{2}{b} > 0$,

which can be rewritten $(c-1)(b-r) + bc + r(1 - \frac{r}{b}) + \frac{c+2}{b} > 0$, true since $0 \leq r < b$.

Since $\frac{a^2+b^2}{1+ab} \in \mathbb{Z}$, we must have then

$$c + \frac{\frac{c}{b}\left(r-\frac{1}{b}\right)+\frac{r^2}{b^2}+1}{c+\frac{r}{b}+\frac{1}{b^2}} \in \{c, c+1\}.$$

Case a): $c + \frac{\frac{c}{b}\left(r-\frac{1}{b}\right)+\frac{r^2}{b^2}+1}{c+\frac{r}{b}+\frac{1}{b^2}} = c$, i.e. $\frac{c}{b}\left(r - \frac{1}{b}\right) + \frac{r^2}{b^2} + 1 = 0$, that is $c(1 - br) = r^2 + b^2$.

This is only possible if $br < 1$, but $b \geq 1$, and thus $r = 0$. We get $c = b^2$, perfect square.

Case b): $c+\frac{\frac{c}{b}\left(r-\frac{1}{b}\right)+\frac{r^2}{b^2}+1}{c+\frac{r}{b}+\frac{1}{b^2}}=c+1$, i.e. $\frac{\left(\frac{a}{b}\right)^2+1}{\frac{a}{b}+\frac{1}{b^2}}=\frac{\left(c+\frac{r}{b}\right)^2+1}{c+\frac{r}{b}+\frac{1}{b^2}}=c+1$, equivalent to

$\frac{c+1}{b^2}+\frac{r}{b}+c=\frac{cr}{b}+\frac{r^2}{b^2}+1$, i.e. $\frac{c+1}{b^2}+c+1=\frac{(c+1)r}{b}+\frac{r^2}{b^2}-2\frac{r}{b}+2$. This is rewritten:

$(c+1)\left(1-\frac{r}{b}+\frac{1}{b^2}\right)=\frac{r^2}{b^2}-2\frac{r}{b}+2$, i.e. $c+1=\frac{r^2-2rb+2b^2}{b^2-rb+1}=\frac{(b-r)^2+b^2}{1+b(b-r)}$, which is of the same form as $\frac{a^2+b^2}{1+ab}$ but with smaller sum of variables:

$(b-r)+b<a+b$, unless $a=b$; in that case, $\frac{a^2+b^2}{1+ab}=\frac{2a^2}{a^2+1}=2-\frac{2}{a^2+1}$, so if it is integer we must have $a^2+1\leq 2$ i.e. $a=1$, hence $b=1$ and $\frac{a^2+b^2}{1+ab}=1$, perfect square.
If $a>b$, by mathematical induction on $a+b$ we reach the desired result.

Remark: By backward induction on $b=\min\{a,b\}$ we get the same result since $b-r<b$, unless $r=0$. But in such case, $c+1=\frac{2b^2}{b^2+1}$, which cannot be an integer unless $b=1$, as we have shown above, and if $b=1$ we get $c+1=1$, i.e. $\frac{a^2+b^2}{1+ab}=c=0$, contradiction.

The cognitive processes involved in the above alternative solution are detailed in Table 6:

.

Table 6. Cognitive analysis of the alternative solution of IMO 1988 Problem 6.

Description of the mental activity	**Outcome**	**Type of cognitive process involved**
Reading the problem statement	Memorization of the content	Recording
Comparing the form of the expression $\frac{a^2+b^2}{1+ab}$ with familiar functions	The expression is a rational function, symmetric in a and b	Operating
Trying to solve the problem by arithmetic means	No successful approach, dead end	Targeting
Evaluating the structure of the expression $\frac{a^2+b^2}{1+ab}$	Realization of the fact that the numerator has one degree more than the denominator in each variable and for large values of a or b it could be approximated by a first degree function	Structuring
Supposing that $a \geq b$,estimate $\frac{a^2+b^2}{1+ab}$ by using the division of a by b with remainder	Without loss of generality, suppose $a \geq b$ and then put $a = bc + r$ for some whole numbers c, r with $1 \leq c$ and $r < b$. Express $\frac{a^2+b^2}{1+ab}$ as a rational function in c	Inventing
Reflecting on the form of the rational function $\frac{c^2+2\frac{cr}{b}+\frac{r^2}{b^2}+1}{c+\frac{r}{b}+\frac{1}{b^2}}$ in c	The expression can be related to a rational function asymptotically close to c from differential calculus	Integrating
Estimating the rational function in c	Conjecturing that the rational function in c is very close to c	Structuring
Testing how far from c can the rational function go by forcing c out of it and by estimating the residual	$\frac{\frac{c}{b}\left(r-\frac{1}{b}\right)+\frac{r^2}{b^2}+1}{c+\frac{r}{b}+\frac{1}{b^2}}$ found to be in the interval $(-1,2)$	Targeting
Evaluating the first degree equation in c obtained after simplifications	The linear equation $r^2 = c + 1 + b(c-1)(b-r)$ suggests expressing it in $c+1$, c or $c-1$	Structuring
For the two possible values of c, solving the first degree equation in c	**Case a)** solved. **Case b)** leading to a dead end when trying to factor $c-1$. Calculation error identified and corrected at **Case a)**	Targeting

Reflecting at the resulting form of c, solution of the linear equation	A similarity with the initial expression $\frac{a^2+b^2}{1+ab}$ is identified, only the "+" sign in the numerator expression changed to "−".This suggests that backward induction is somehow involved	Integrating
Evaluating the structure of the expression $\frac{a^2+b^2}{1-ab}$	Conjecturing that backward induction can be used for solving the more general problem: $\frac{a^2+b^2}{1+ab} \in \mathbb{Z}$, with $a, b \in \mathbb{Z}$. By replacing b with $-b$, the problem for $\frac{a^2+b^2}{1-ab} \in \mathbb{Z}$ and that for $\frac{a^2+b^2}{1+ab} \in \mathbb{Z}$ in the general case $a, b \in \mathbb{Z}$ are shown to be equivalent	Structuring
Returning to the initial rational fraction in c in order to factor $c+1$	The relation $c + 1 = \frac{(b-r)^2+b^2}{1+b(b-r)}$ is obtained	Targeting
Reflecting at the form of the above expression in b and r	The analogy with the initial problem is identified. Backward induction can be employed at **Case b)**	Integrating
Completing, checking and writing the proof	A formal proof is obtained and written down	Reasoning

After completing the proof, I continued my inquiry by formulating questions or conjectures about the possible generalizations and by trying to prove or to reject them. The result of this additional research, which states the initial problem as an open task (unstructured inquiry) and leads to other research questions, is presented in Table 7:

.

Table 7. Cognitive analysis of the generalization of the IMO 1988 Problem 6.

Description of the mental activity	**Outcome**	**Type of cognitive process involved**
Evaluating the structure of the initial problem's solution	The solution is based on the reduction of the problem to a linear equation in $c := [\frac{a}{b}] \in \mathbb{N}\backslash\{0\}$, with coefficients that are second degree expressions in $x := \frac{a-bc}{b} \in [0, \frac{b-1}{b}]$	Structuring
Based on the evaluation of the existing solution, constructing related expressions of arbitrary degree, for which a similar solution could be built	A general expression, related to the initial one, is obtained: $\frac{a^n+b^n}{1+a^{n-1}b+a^{n-2}b^2+\cdots+b^{n-1}a}$, $n \geq 2$.	Integrating
"Recycling" and adapting the proof of the initial problem ($n = 2$) for the general case	A first step toward a solution when $n \geq 3$ is obtained. The problem is reduced to a single equation of degree $n - 1$ in c, with coefficients polynomials of degree n in $\frac{r}{b}$, where $r = a - bc$.	Targeting
"Recycling" and adapting the proof of the initial problem ($n = 2$) for $n \geq 3$	Complete solution for the case $n = 3$. Dead end encountered when trying to solve the general case.	Targeting
Recalling the results of previous evaluation of the initial problem's solution	The conditions imposed to the rational expression in a and b such that the initial solution may be adapted and reused are recalled.	Recording
By using the results of this evaluation, building related expressions of arbitrary degree, for which a similar solution could be developed	A general expression, related to the initial one, but with non-homogenous numerator, is obtained: $\frac{a^n+b^n+a^{n-2}b^{n-2}-1}{1+a^{n-1}b^{n-1}}$, $n \geq 2$.	Integrating
Trying to solve the new problem in the general case	Problem solved successfully by employing, for the first steps, a similar approach to that used for the initial problem, and a combined arithmetic-algebraic approach for the last steps.	Integrating
Completing, checking and writing the proof	A formal proof is obtained and written down.	Reasoning
Based on the evaluation of the existing solution, constructing related expressions of arbitrary degree, for which a similar solution could be built	A general expression, related to the initial one, is obtained: $\frac{a^n+b^n}{1+ab(a^{n-2}+b^{n-2}-1)}$, $n \geq 2$.	Integrating

"Recycling" and adapting the proof of the initial problem ($n = 2$) for the general case	A first step toward a solution when $n \geq 3$ is obtained. The problem is reduced to a single equation of degree $n - 1$ in c, with coefficients polynomials of degree n in $\frac{r}{b}$, where $r = a - bc$.	Targeting
Trying to complete the last steps of the general case by using an arithmetic-algebraic approach.	Algebraic approach is unsuccessful for $n \geq 3$. By using a combined arithmetic-algebraic approach, a complete solution for $n = 3$ is obtained. For $n > 3$, it leads to a relation $c = \frac{b^n + r^n}{1 + be}$ for some $e \in \mathbb{N}\backslash\{0\}$, wrongly seen as similar to initial one and allowing backward induction. Mistake due to fatigue.	Integrating
Checking, completing and writing the proof	Error found in the case of $n > 3$. This step cannot be completed. Complete solution only for $n = 3$.	Reasoning

3.3.2 Lecture 2: Proving the irrationality of π by zooming into the number line

The following example illustrates how a creative approach used in a particular proof can be "recycled" via inductive thinking and used as a more general approach in research.

Below is a sketch of the Niven's (1947) proof of the irrationality of π.

- **Irrationality of π** (Niven)

Suppose $\pi = \frac{a}{b}$ with $a, b \in \mathbb{N}$. Define, for every positive integer n:

$$f(x) = \frac{x^n(a - bx)^n}{n!}$$

and

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) + \cdots + (-1)^n f^{(2n)}(x)$$

Note that $f(x)$ and its derivatives $f^{(j)}(x)$ have integral values for $x = 0$. This is also true for $x = \pi = \frac{a}{b}$, since $f(x) = f(\frac{a}{b} - x)$.

We have $\frac{d}{dx}(F'(x)\sin x - F(x)\cos x) = F''(x)\sin x + F(x)\sin x = f(x)\sin x$

whence $\int_0^\pi f(x)\sin x \,\mathrm{d}x = (F'(x)\sin x - F(x)\cos x)\big|_0^\pi = F(\pi) + F(0) \in \mathbb{Z}$.

But for $0 < x < \pi$ we have $0 < f(x)\sin x < \frac{\pi^n a^n}{n!}$, which tends to zero as n approaches infinity, so $\int_0^\pi f(x)\sin x\, dx$ is a whole number that tends to zero as $n \to \infty$, contradiction.

Remark: In the above proof, there is no hint about the way Niven arrived at these specific forms of the functions f and F, and how everything fits perfectly in order to give the desired result. After having read it, my feeling was that something is missing and I tried to really understand what's behind this trick, to uncover its heuristics and to make sense of Niven's approach. Now I will recall my own inquiry that ensued at that point.

After reflecting on the meaning of π, how it occured in history and how it is introduced in elementary mathematics (and thus engaging in the process of Structuring), I concluded that π is defined by means of integration, namely by calculating the length of a circle or the area of a disc. By recalling from high school (Recording) the construction of circular trigonometric functions via radian measure by using the length of a circle in order to derive the length of an arc, it became clear to me (Structuring) that saying "π is the first positive zero of sine function" is equivalent to saying that the length of a semicircle is π, or that the integral of sine between 0 and π is 2. It is natural to combine in a single expression (Integrating) the two functions that cancel in 0 and π, namely $f = f_n$ and sine, in order to get the expression $f_n(x)\sin x$, whose integral from 0 to π obviously tends to zero as $n \to \infty$.

Also, choosing $f_n(x) = \frac{x^n(a-bx)^n}{n!}$ is natural, because it uniformly tends to zero on any interval and all its derivatives in 0 and $\frac{a}{b}$ are integers (Reasoning using mathematical induction). One step remained obscure (Structuring): how did Niven "guess" that the primitive of f_n involves such a nice combination of f_n's derivatives, namely $F(x)$? The primitive can be computed via repeated integration by parts (Reasoning: mathematical induction), but the calculations are longer than the proof itself (Structuring: evaluation), perhaps that's why Niven preferred to skip the heuristics and only provided the clean, end product (Structuring: conjecturing). By such reflections, I started to get some understanding (Structuring) into the deeper meaning of the result.

At that point, I decided to go straight to the core of the property and to clean the proof of all the artificial tricks in order to make it more insightful and to include it in my research for the thesis, if possible. I thought initially that the proof was just a brief and imperfect translation of the ideas developed in the article of Niven (which I had not read) and that it might be reformulated in order to be more accessible to high school students. I started from the integral and by trial and error looked for a nice function, similar to f, that would cancel along with its first n derivatives at the ends of the integration interval (Targeting). For reasons of symmetry (Integrating stage: similarity), I replaced the limits of the integral with $\pm\frac{\pi}{2}$, sine function with the even function cosine, and changed the integrand with the even expression $\left(\frac{\pi^2}{4}-x^2\right)^n$. Two integrations by parts provided me a recurrence of integers, which after a simple transformation led to the desired result (Targeting). A sketch of the final proof is presented below:

- **Irrationality of π (alternative proof)**

Suppose $\frac{\pi}{2} = \frac{a}{b}$ with $a, b \in \mathbb{N}$, and let $I_n := \int_{-\pi/2}^{\pi/2} \left(\frac{\pi^2}{4}-x^2\right)^n \cos x \; dx$, where $n \in \mathbb{N}$.

Repeated integration by parts gives us for $n \geq 2$:

$$I_n = 2n(2n-1)I_{n-1} - 2n(2n-2)\frac{\pi^2}{4}I_{n-2} = 2n(2n-1)I_{n-1} - 4n(n-1)\frac{a^2}{b^2}I_{n-2},$$

and by taking $J_n := I_n \frac{b^n}{n!}$ $\forall\, n \in \mathbb{N}$ we get $J_n = 2b(2n-1)J_{n-1} - 4a^2 J_{n-2}$, $\forall\, n \geq 2$.

Since $J_0 = 2$ and $J_1 = 4b$, we get $J_n \in \mathbb{Z}$, $\forall\, n \in \mathbb{N}$.

Moreover, $0 < I_n \leq \int_{-a/b}^{a/b} \left(\frac{a^2}{b^2}\right)^n = 2\left(\frac{a}{b}\right)^{2n+1}$, $\forall\, n \in \mathbb{N}$, hence $0 < J_n \to 0$ as $n \to \infty$,

which is impossible because $J_n \in \mathbb{Z}$, $\forall\, n \in \mathbb{N}$.

We remark that it is a proof by contradiction (Reasoning), and the details of the so-called "repeated integration by parts" are not provided, but the procedure to be used is clearly stated and "completing the dots" only requires basic skills in integration and accurate calculations (Operating). A mathematical induction step is implicit when saying that we get $J_n \in \mathbb{Z}, \forall n \in \mathbb{N}$ (Reasoning: mathematical induction). Also, the final step, $J_n \to 0 \; as \; n \to \infty$, is not justified in this sketch of proof, but can be completed by using familiar procedures, namely by breaking the process into several elementary operations indicated by general rules and principles of calculus (Operating).

Thus, completing the proof only involves Knowledge phase and is usually “left to the reader”. So, this task would not be considered a challenging question in IBLecturing approach. In order to connect the defining property of $\frac{\pi}{2}$ as the first positive zero of cosine function and the supposition that it is rational, which means it is a zero of a proper linear function with integer coefficients, I built a combination of suitable functions that are canceled by it (Integrating). Employing a recurrence of integrals that are positive integers and converge to zero, in order to get a contradiction, is an approach seldom used; but given the mathematical culture of Niven suggested by his expertise in number theory (Structuring: conjecturing), probably he had seen it elsewhere; otherwise, he must have invented it by some amazing inspiration. In any case, Integrating and Structuring skills are necessary but not sufficient in order to construct this proof: creativity (Inventing) is strongly required. The fact that such a simple demonstration has not been devised for about two hundred years and that all the proofs available until 1947 were very complicated and technical, requiring sophisticated tools from higher mathematics, certifies the difficulty of performing at the Invention level and the unpredictable working of creativity.

On the other hand, the task of proving the irrationality of e by using a strategy similar to that used for π is a challenging question because we meet several issues and a deep understanding (Structuring) of the Niven's method is necessary in order to make the necessary adjustments. Firstly, e can be defined as a zero of various functions, but all are unsuitable for a good recurrence of integrals (Targeting: reaching a dead end). Then, an exponential could be employed in order to get such a recurrence, but the exponent and the integrals must be multiplied by some factors if we want to keep them as integers (Structuring). By heuristic means (Targeting: trial and error/educated guess), I found a suitable form of the integrand, $f_n(x) = e^{ax}\left(\frac{1}{e^2} - x^2\right)^n$. Below is the result, in the form of a formal proof (Reasoning).

- **Irrationality of e**

Suppose $e = \frac{a}{b}$ with $a, b \in \mathbb{N}\setminus\{0\}$, and let $I_n := \int_{-1/e}^{1/e} e^{ax}\left(\frac{1}{e^2} - x^2\right)^n \mathrm{d}x$, where $n \in \mathbb{N}$.

Repeated integration by parts gives us for $n \geq 1$, since $e^{ax} = \frac{d}{dx}\left(\frac{e^{ax}}{a}\right)$,

$$I_n = \frac{2n}{a} \int_{-\frac{1}{e}}^{\frac{1}{e}} e^{ax} x \left(\frac{1}{e^2} - x^2\right)^{n-1} \mathrm{d}x$$

$$= -\frac{2n}{a^2} \int_{-1/e}^{1/e} e^{ax}\left[\left(\frac{1}{e^2} - x^2\right)^{n-1} - 2(n-1)x^2\left(\frac{1}{e^2} - x^2\right)^{n-2}\right]\mathrm{d}x \quad (\text{for } n \geq 2)$$

and we get

$$I_n = -\frac{2n}{a^2}\left[2(n-1)I_{n-1} + I_{n-1} - \frac{2(n-1)}{e^2} I_{n-2}\right] = -\frac{2n(2n-1)}{a^2} I_{n-1} + \frac{4n(n-1)}{e^2 a^2} I_{n-2}, \quad \forall\, n \geq 2 \ (*)$$

By defining

$$J_n := \frac{I_n}{n!} \cdot \frac{a^{2n}}{2^n} > 0 \quad \forall n \in \mathbb{N},$$

we get the recurrence

$$J_n = -(2n-1)J_{n-1} + b^2 J_{n-2}$$

for any $n \geq 2$ (since $e^2a^2 = b^2$), with

$$J_0 = I_0 = \frac{e^b - e^{-b}}{a}, \qquad J_1 = \frac{2}{a^3}\left[(b-1)\left(\frac{a}{b}\right)^b + (b+1)\left(\frac{b}{a}\right)^b\right].$$

By taking $C := a^3(ab)^b$ we get $CJ_0, CJ_1 \in \mathbb{Z}$ and thus $CJ_n \in \mathbb{Z}, \forall n \in \mathbb{N}$ because the sequence $(CJ_n)_{n\geq 0}$ satisfies the same recurrence (*) as $(J_n)_{n\geq 0}$.

Since $J_n > 0 \ \forall n \in \mathbb{N}$, we have then

$$0 < CJ_n = C\frac{I_n}{n!}\left(\frac{a^2}{2}\right)^n \to 0 \text{ as } n \to \infty,$$

because

$$0 < I_n \leq \int_{-1/e}^{1/e} e^{ax}\left(\frac{1}{e^2}\right)^n \mathrm{d}x \leq \int_{-1/e}^{1/e} e^{ax} \mathrm{d}x = \text{constant (bounded)}$$

and we have a contradiction.

Such proof, which uses analogy (Integrating), holistic thinking (Integrating) and evaluation (Structuring) in order to build upon the ideas of Niven, involves in a much lesser degree the Inventing stage, but still requires a good understanding of the approach employed for π in order to make it effective in a different setting. Showing the irrationality of e in this way may be considered useless from a mathematical point of view, since an almost trivial demonstration could be done through a serial expansion of the exponential function, yet it is very useful from the point of view of the cognitive approach in pedagogy because it shows not only some valuable techniques, but also the functioning of inductive thinking, and how a rewarding idea could be adapted and "extrapolated" or “recycled” in various mathematical contexts.

3.3.3 Lecture 3: A classical geometric puzzle

The following problem will illustrate the thinking processes and the cognitive stages involved in an attempt to find a solution. Rusu (1970, p. 18) proposed it at a summer training program to a group of sixty in-service teachers of mathematics, and none of them was able to solve it immediately. After its publication in 1962, purely geometric proofs were given by Rusu, Coxeter (1967) and it was also generalized (Tudor, 2009, pp. 225-226).

- **Problem**: Let *ABC* be an isosceles triangle with *AB* = *AC* and the angle $\widehat{BAC}$ of 20°. Let *M* and *N* be on the sides *AC* and *BC* respectively, such that $\widehat{ABM}$ has 20° and $\widehat{ACN}$ has 30°. Find the measure of $\widehat{BMN}$.

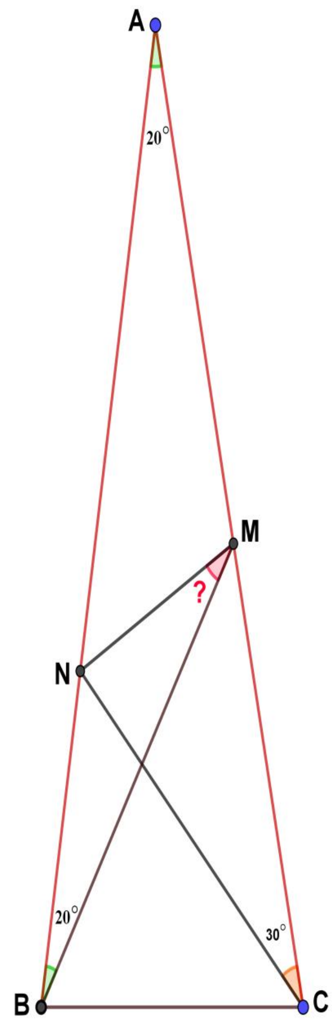

Figure 6. Problem statement

In Figure 6, all the data provided in the problem's statement have been entered.

Looking at the drawing, we realize there is no familiar method that could give us the result (Reasoning), and we cannot see any sequence of procedures that would lead us closer to the target (Targeting). Therefore, we have to identify the connections between the various knowledge available (Integrating). The first step involves breaking the content into small pieces, comprehending the information at hand and that which can be acquired via routine operations or procedural chains (Reasoning) and selecting the useful one. We find that $\widehat{BCN}$ and $\widehat{BNC}$ have both 50°, thus ΔBCN is isosceles with $BN = BC$. Also, $\widehat{MBC}$ has 60° and $\widehat{BMC}$ has 40°. All this information is entered in Figure 7.

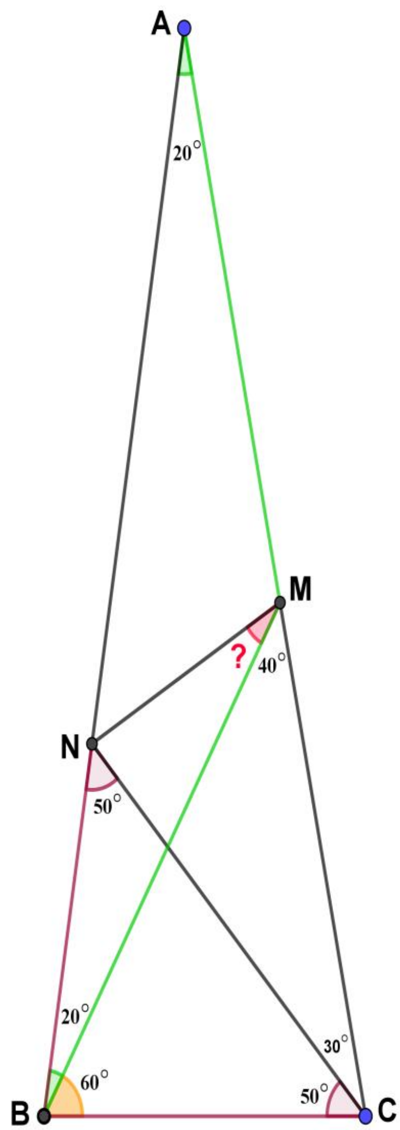

Figure 7. Simple deductions from the givens

Here we get to the Integrating stage, where associations and connections are done. We have the isosceles triangles *NBC* and *BMA*. Thus, *BC* is connected with *BN*, and similarly *BM* is connected with *MA*. But we cannot solve the problem with the knowledge available at this stage. So, we go up to the Structuring level where we systematize the knowledge that has been gathered, in order to make an evaluation or a judgment regarding it. Here, we infer that some construction has to be done, since nothing valuable can be stated further by using only the current configuration. There are several angles whose measures are known, but we don't have any specific property for an angle of 20°, 40° or 50°.

An angle of 30° has such property, but it requires a right triangle, and since it is missing here it should be constructed. Yet, such a construction would complicate the figure without revealing anything useful, at least at first sight. In conclusion (Structuring), the most suitable strategy is to employ an angle of 60°, which can be used not only in combination with a right angle but in isosceles triangles as well, since an isosceles triangle with an angle of 60° must be equilateral. $\widehat{MBC}$ has 60° and it is placed inside $\widehat{NBC}$, with *BC* = *BN*. Then, if we take, on the side *BM,* a segment *BP* = *BC*, it connects with both *BC* and *BN,* so we get three isosceles triangles with a common vertex in *B*, since *BP* = *BC* = *BN* (Figure 8).

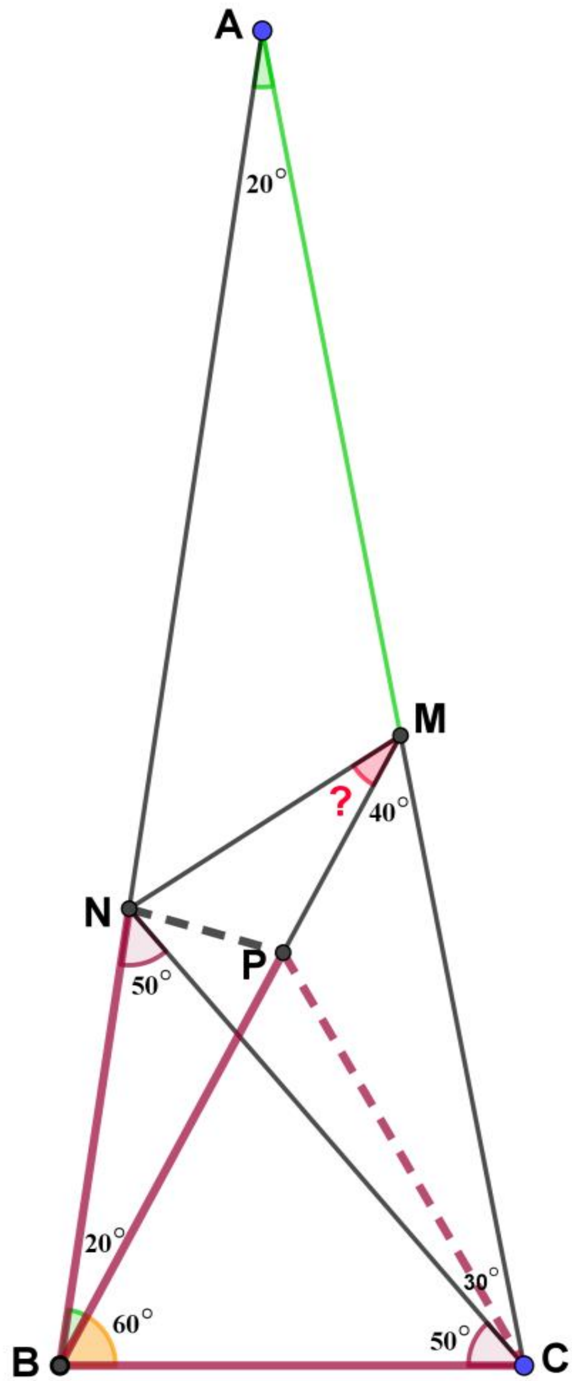

Figure 8. Insight

We employ familiar procedures (Operating) and chain reasoning (Targeting) in order to determine the angles that arise. Careful evaluation of the drawing (Structuring) suggests us to extend the segment *CP* until it cuts *AB* in a point *Q* in order to get additional information. The main knowledge available now is entered in Figure 9

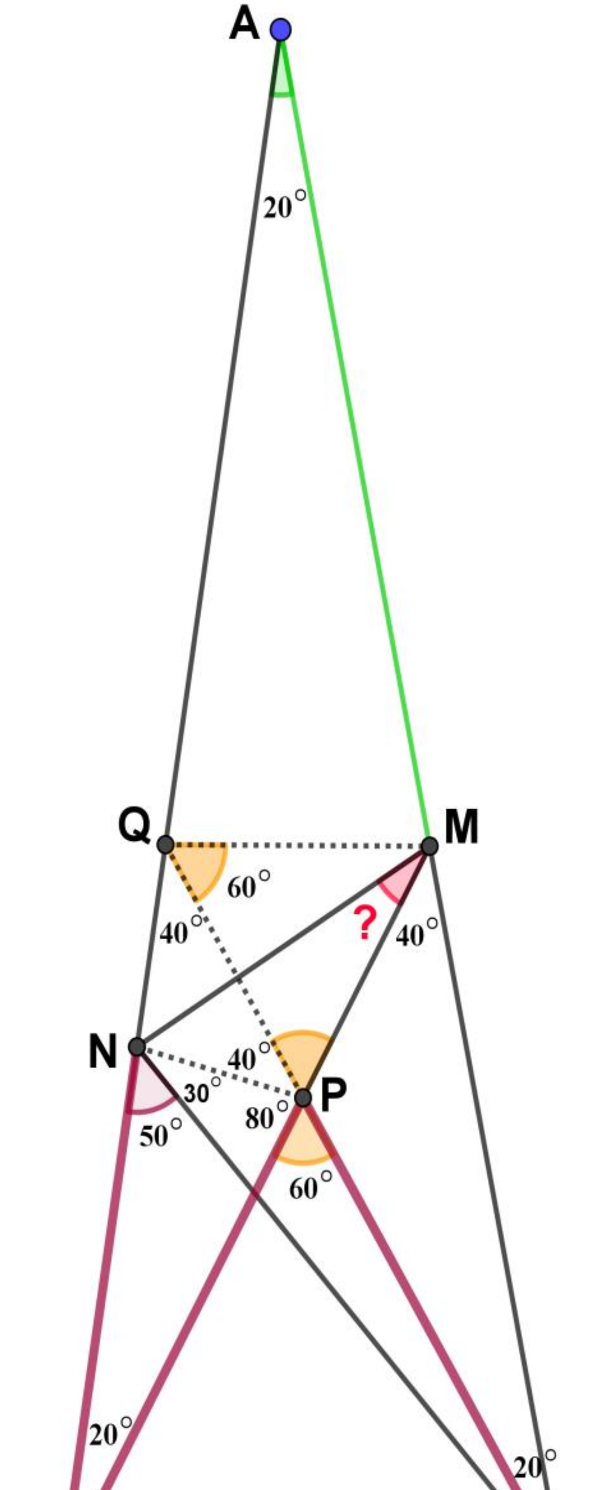

Figure 9. Solution

By the general symmetry of the configuration made up by the points A, B, C, P, M, Q (Integrating) we get that PQM is an isosceles triangle with an angle of 60°, hence equilateral. PNQ is an isosceles triangle, too, since $\widehat{NPQ}$ and $\widehat{NQP}$ are both 40°. Thus, the triangles MNP and MQN are congruent and P, Q are symmetric with respect to MN, which proves that $\widehat{PMN}$ is 30°, since $\widehat{PMQ}$ is 60°. The details of the proof can be completed by using only familiar procedures, for example in order to show rigorously that ΔPQM is isosceles we develop a chain reasoning (Targeting) as follows: the triangles AMB and AQC are congruent since their angles are the same and $AB = AC$, hence $BM = CQ$; but $BP = CP$ in the equilateral triangle BCP, so we get $PM = PQ$.

Another way to solve the problem by using the property that isosceles triangles with an angle of 60° are equilateral is to construct the angle $\widehat{NBP}$ of 60° inside $\widehat{ABC}$ as in Figure 10Figure 9:

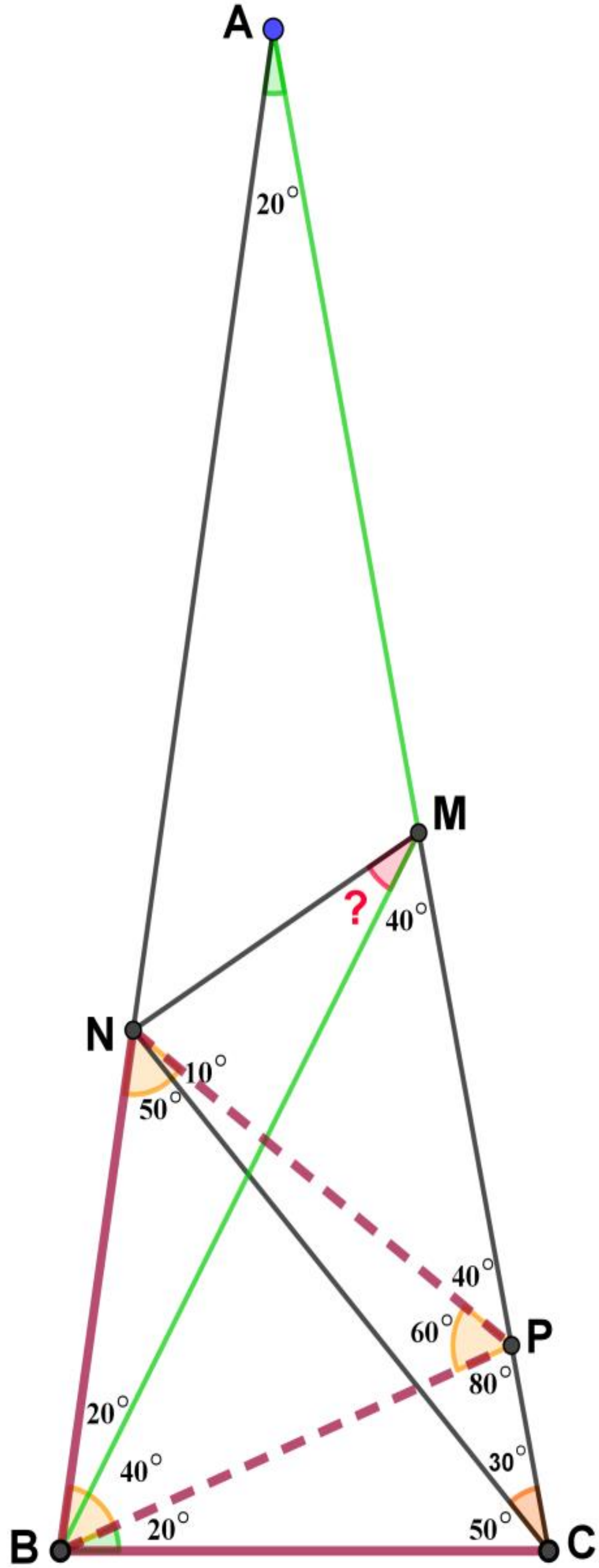

Figure 10. Alternative solution

After entering the main information into the figure, we find that BNP is an equilateral triangle with $BC = BP = BN = NP$; moreover, $\widehat{MBP}$ and $\widehat{BMP}$ are both 40°, hence BMP is an isosceles triangle with $MP = BP$. This implies $MP = NP$, and the triangle NMP will be isosceles with $\widehat{NPM}$ of 40°, therefore $\widehat{NMP}$ has 70° and the angle $\widehat{NMB}$ will be of 30° (Targeting – because a choice of the angle to look for was made), which completes the proof.

We remark that, once we have built the appropriate angle of 60° inside $\widehat{ABC}$, everything flows easily and only requires using logical deduction (Reasoning) or generating chains of procedures that aim at finding the measure of $\widehat{BMN}$ (Targeting). But for discovering this construction (Inventing), we need first to combine (Integrate) and then to evaluate (by Structuring) all the information that has been found by using familiar operations and sequences of procedures. Then, after carefully selecting the relevant information among the data already acquired, we have to organize it (Structure) with the help of valuable tools such as visual diagrams. At this point, much reflection is necessary in order to achieve deep understanding (through Structuring) and thus to be able to reach the highest cognitive layer, creation (Invent), i.e. to generate new, distinctive knowledge through invention.

For the problem presented above, the segment BP, constructed with the same length as both BC and BN at an angle of 60° from one of them, is an ad hoc tool created by us (Inventing stage) in order to connect some parts of the geometric configuration and thus to open a novel path towards a solution.

3.4 EXAMPLE OF A CHALLENGING QUESTION FOR STUDENTS

In order to see if the cognitive stages that I have assumed in my model and that I have identified through introspection mainly appear also in the solutions of more challenging questions by the learners, I gave such a problem to twenty secondary school students participating at a mathematical circle in Brașov, Romania. Most of them were from grade

7 and grade 8, they mastered the basic procedures of plane geometry and thus were able to solve the problem by synthetic means. There were only two grade 9 students in the group, they had asked me the permission to participate and I accepted them in order to get a sample of more mature thinking (Structuring), even if they had the advantage of more advanced tools at their disposal such as trigonometry, Cartesian coordinates and complex numbers.

The task consisted in proving a plane geometry property, a maximum time given for completing it was around one hour and a half and there was no minimum time, so they could leave immediately if they found the problem too difficult. However, there was a reward for the best productions and the students were acquainted with challenging tasks, so I did not expect them to quit immediately. After explaining the purposes of this test, I asked them to write down not only the operations performed but also their thinking in the attempt to solve the problem, since my research focuses on it. I also specified that the description of student's own thinking will be taken into account at the evaluation. The analysis of students' metacognition is an important source of valuable information, which could lead to novel findings in cognitive psychology and education.

Each student received a copy of the problem statement with a Geogebra diagram that included the grid, similar to a drawing on mathematics grid paper (Figure 11). The goal was to provide some guiding to the students, in order to facilitate the discovery of a certain structure in the drawing and to help them find a synthetic solution by inventing a construction made by using grid nodes.

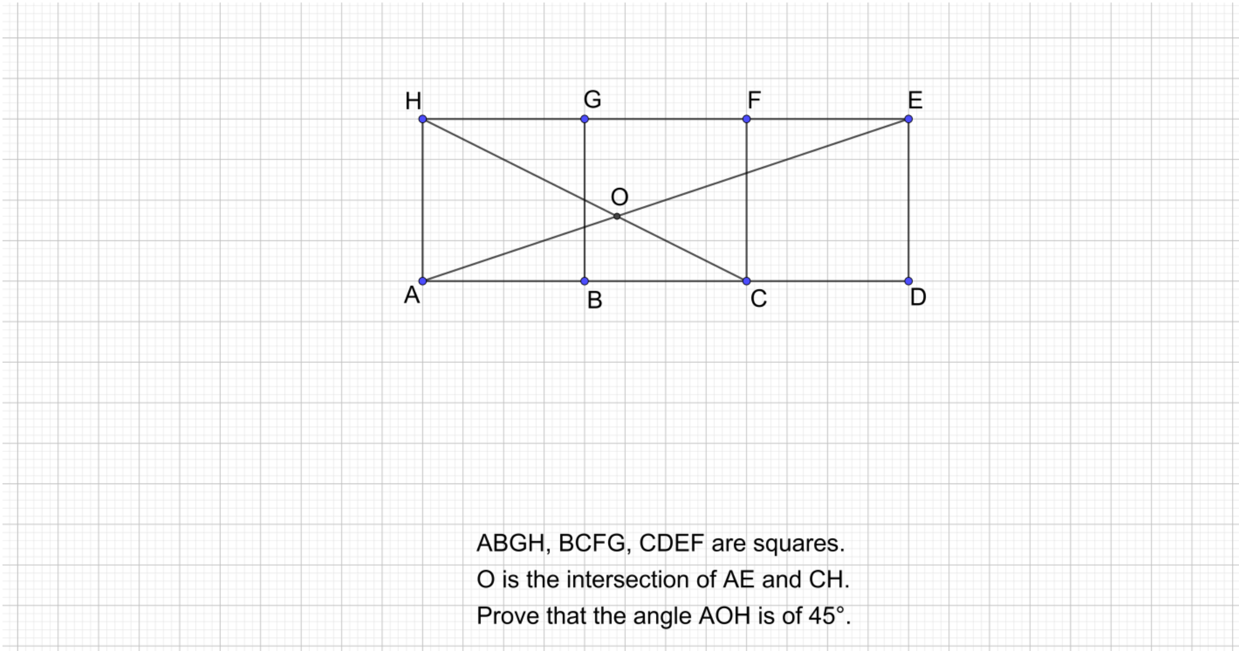

Figure 11. The challenging problem statement

In Table 8, I present students' comments on their solutions and my assessment of the cognitive level they represent in terms of the highest level of processes the student seems to have reached.

Table 8. Cognitive analysis of students' solutions

Label of the student	Approach	Cognitive stage	Student's comments and my remarks on the student's solution
A	Using the cosine theorem (Solution 1) Using sine formula for the area of a triangle (Solution 2)	Targeting	"1. Since I had previously done similar problems, the solution came to me instantly. I did not have other ideas for trial, since the problem can always be solved in this way and the method is more general than a geometric trick which would not work for an angle of, say 17.5°. [3] [**Remark**: correct and complete proof.] 2. A second way of immediately solving the problem, similar to the first, is to compute the scalar product of the vectors $\overrightarrow{OH}$ and $\overrightarrow{OA}$. These two approaches are the first that I thought about, because they always lead to the result. If the problem was more difficult, I would have also used a rotation of the triangle, but since the problem is simple, it was not necessary. Another immediate proof is using the property that for similar triangles of ratio k, the ratio of their heights is also k. (If X is the intersection of AE and CH), by the similarity of ΔAOH and ΔXOC we get the ratio of their heights l_1 and l_2 (from O). But $l_1 + l_2 = 2$, so we can find the area of ΔAOH and thus the angle by the sine (formula for area)." [**Remark**: on his draft, the student almost found the similarity ratio to be $3/2$.]
B	Trying to use similarity	Targeting	"I thought that I could use similarity: extending HC until it cuts ED in M, hence $\Delta AOH \sim \Delta EOM$. My first idea was to employ the fundamental theorem of similarity. We use it for ΔAOC and ΔHOE." [**Remark**: by using $\Delta AOC \sim \Delta HOE$, she found that $OA = 2/3\ OE$ and $HC = 5/2\ OC$. After finding the lengths of OC, HM and OH, complete deadlock. The student has not noticed that by knowing $\Delta AOH \sim \Delta EOM$and HM, we can find HO and AO (after calculating AE), so the triangle AOH is completely determined.]
C	Making a suitable construction	Targeting	"Teachers train you to think of just one method when dealing with a problem. Right triangle is the method of which you think when hearing about 45°, and from that derive sine, cosine, tan, cotan". [**Remark**: the student constructed ΔOCM with $M \in OE$ such that $\widehat{OCM}$ has 90° and tried to prove that it is isosceles by taking OP and MN altitudes in ΔAOC and ΔAMC, respectively. He proved that $\widehat{OCP} \equiv \widehat{CMN}$, then invoked "ruler" (measurement) for the critical step $BP = MQ$ in order to show that $\Delta COP \equiv \Delta MCN$, which solves the problem. Proving that $BP = MQ$ (which is true) is not trivial, it could be done by calculating CP and OP by using $\Delta AOC \sim \Delta HOE$ and $\Delta COP \sim \Delta CHA$, then using $\Delta CMN \sim \Delta HCA$ and $\Delta AOP \sim \Delta AMN$ in order to get an equation in $x := MQ$. Inadequate proof, the path is very unlikely to be completed by an 8th grade student.]

[3] My translation from Romanian.

D	Making a suitable construction	Targeting	"We take M (on the line HC) such that $\widehat{EMO}$ = 45° " [**Remark**: the student constructed the altitude EM in ΔHOE and tried to prove that ΔEOM is isosceles. It is $\widehat{OEM}$ not $\widehat{EMO}$ that is 45°. Being a 7^{th} grade student, she did not have enough knowledge for making any progress and finally used direct measurements of EM and OM (and also of some angles), in order to justify the result: "We proceed by observations... We measure with the protractor". Insufficient Knowledge does not allow the student to reach a solution, in spite of engaging in higher level cognitive process.
E	Making a suitable construction	Targeting	"We consider altitude EM in ΔHOE, $\{N\} = CD \cap ME$ and $\{P\} = BG \cap OH$". [**Remark**: the student proved that N is the middle of CD by using $\Delta END \equiv \Delta HPG$, then tried to derive the measure of $\widehat{AOH}$ by flipping other angles and concluded: "The idea was not good". She built parallels from C to AE and from A to HC, finding that: "Again we do not have enough elements to reach the desired result." Finally, she used the protractor for measuring the angle. It looks like 7^{th} grade students learned somewhere to use the "observation method" i.e. making measurements if all other methods fail.]
F	Calculating lengths and using inverse trigonometric functions	Structuring (a little)	"ΔAOH seems to be right isosceles, but it is not. All the triangles formed here are ordinary. I am not able to find any theorem that I could use. I do not know well enough trigonometry, which could be used almost everywhere in the configuration given here. I cannot find any helpful construction. What I could use: I remark that all the sides of the squares are equal, so we can compute the diagonals (of all rectangles) and there are many similar triangles." [**Remark**: the 7^{th} grade student found that tan $\widehat{ACH}$ =1/2, tan $\widehat{AEH}$ =1/3, and mentioned "from these, we can find the angles by using a calculator". He even wrote the inverse function (maybe with some help): "$\tan^{-1} \frac{1}{2} = 27°$, $\tan^{-1} \frac{1}{3} = 18°$ ", mentioning that he found these values "by using the protractor on the drawing". In Romania, inverse trigonometric functions are studied only in grade 10.]
G	Using the law of cosines	Targeting	"I tried to use generalized Pythagorean theorem... We try to find all the angles." [**Remark**: since $\Delta AOC \sim \Delta EOH$ and by calculating AE and HC with the help of the Pythagorean law, she got a first degree equation in HO and another one in AO that allowed her to find HO and AO. We notice even in good students' a lack of training in using the properties of proportions. Finally, cos $\widehat{ACH}$ was determined by employing the cosine law in ΔAOH, but the lack of skills in calculations and in checking the accuracy of operations (the "numerical common sense") led to a wrong result.]
H	Using inscribed quadrilaterals	Integrating (a little)	[**Remark**: the student considered $AO \cap BH = \{P\}$, $AO \cap BG = \{R\}$, $HO \cap BG = \{I\}$ and $HC \cap ED = \{L\}$ in order to prove that $BOIP$ is an inscribed quadrilateral: "Since $\widehat{HBR}$ has 45°, we have to prove that $\Delta IOR \sim \Delta PBR$." The 7^{th} grade student realized that solving the problem has been reduced to proving that $\widehat{APH} \equiv \widehat{BIC}$. After rightly considering the altitude HH' in ΔAOH and finding that $\Delta AHH' \sim \Delta RAB$, she met a deadlock. She did not notice that all the sides of $\Delta HH'P$ can be (painfully) determined now, in order to show that $\Delta HH'P \sim \Delta HGI$, which solves the problem.]

I	Making a construction	Targeting	[**Remark**: the 7th grade student considered the altitude AS in ΔAOH, and simplified the problem: “We have to prove that ΔAOS is isosceles”, then she met a dead end due to lack of skills. She concluded: “We look after angles of 30° and 60° in a right triangle, and we cannot find.”]
J	Measuring the lengths	Operating	[**Remark**: the 7th grade student thought that using squared paper for the drawing was an invitation to employ the measuring i.e. the “observation method”.]
K	No systematic approach	Targeting	“I cannot compute $\widehat{HAO}$ and $\widehat{AHO}$ (I would need a trigonometric table).” [**Remark**: the 10th grade student knew how to calculate the height in a right triangle by using area. Lack of structure and strategy.]
L	Three approaches: trigonometric, Cartesian, and via inscribed polygons	Structuring	1. Trigonometric solution: “We have 3 squares of equal sides in which two lines are taken, so with the Pythagorean theorem we can find any length and by using trigonometric functions we can find the measures of angles. The target angle is 45°, so we will use the trigonometric functions since the final values will be exact. If the target angle was e.g. of 65°, we would have met an obstacle.” [**Remark**: the student employed the formula for the sine of a sum to get sin $\widehat{HAO}$ by expressing it in function of sine and cosine of $\widehat{CHA}$ and $\widehat{EAH}$, found by using Pythagorean theorem.] 2. Analytic approach: “Since we have some data about AE and HC, we can calculate their slopes, and thus we can find the angle between them, $\widehat{AOH}$.” [**Remark**: the 9th grade student did not know the tangent formula for the angle between two lines with known slopes. It seems he did not know yet the formula for the tangent of a sum of angles. He found the coordinates of O (the intersection of lines), which he used to get OA and OH, then by employing the law of cosines he found $\widehat{AOH}$.] 3. Classical geometry: “We have a square in which two adjacent vertices must form a 45° angle with a point from the same half-plane.” [**Remark**: in order to prove that O is on the same circle as A, B, G, H he considered $\{R\} = AG \cap BH$, the center of the circle and reduced the problem to calculating RO. Via Cartesian coordinates, completing this final step would be an easy task.]
M	Trying to use similarity	Targeting	“Initially, I tried to get some measure of an angle via trigonometry, but I did not find anything useful. Also, I thought about solving the problem by using $\Delta GAE \sim \Delta OHE$, since $\widehat{AGE}$ has 135°, and in order to have $\widehat{AOH}$ of 45°, $\widehat{HOE}$ must have 135°, too. Thus, I tried to get equal ratios in order to prove the similarity.” [**Remark**: the 8th grade student found the ratio between AE and HE, and reduced the problem to finding OE. She did not notice that OE can be determined by using $\Delta AOC \sim \Delta EOH$, and compared $\frac{OE}{GE}$ instead of $\frac{GE}{OE}$ with $\frac{AE}{HE}$, reaching the wrong conclusion that if $\widehat{AOH}$ was 45°, the height from O in ΔAOC would be $\frac{AH}{3}$, which is not true (checking via measurement). “Tunnel vision” by focusing too much on a

			useless construction; also, poor control / validation, so lack of Structuring.]
N	Trying to use similarity	Reasoning	[**Remark**: lack of mental vision (Targeting)]
O	Express triangle's area in two ways in order to find the sine	Integrating	"I thought of using $\Delta AOC \sim \Delta EOH$. I had to find AO, OE in function of AE, and HO, OC in function of HC." [**Remark**: the 8th grade student correctly determined AO, OE, HO, OC, and the ratio between OE and OF. She expressed the area of ΔAOH in two ways: by using sine, and with Heron formula. It only remained to plug in the values of the sides, but she did not finish the calculations. Lack of skills in transforming proportions, but solved them via linear equations.]
P	No systematic approach	Reasoning	"We notice that no triangle is isosceles. We cannot use the Pythagorean theorem since we don't have the lengths of the sides". [**Remark**: poor mental vision (Targeting). The 8th grade student seemed to be "overtrained" on procedures where the lengths are given (in a right triangle) and even if she noticed the largest number of useful similarities of triangles, she was unable to get anything out of them.]
Q	Trying to use angles	Reasoning	"We notice that $\widehat{GOA}$ = 90°." [**Remark**: this has to be proved, the problem is solved if we do this. The 8th grade student seems to have "noticed" it and other details by measurements on the drawing. Tunnel vision (focus on angles) or lack of training in using other methods.]
R	Express triangle's area in two ways in order to find the sine	Targeting	"The obstacle that I encounter is the asymmetry of the configuration". [**Remark**: the 8th grade student built the altitude AT in ΔAOH and wrote the area of ΔAOH in two ways, by using sine and by using AT, which is useless since it is equivalent to expressing sine in a right triangle in function of its sides. He tried to make a construction, but used symmetry instead of parallels when extending the drawing. Deadlock encountered.]
S	Expressing all the angles and lengths in function of two angles	Targeting	"By denoting $m(\widehat{AHO}) = x$ and $m(\widehat{HAO}) = y$, some angles become x or y. In several right triangles we apply sine and cosine of x and y and we get ratios of sides that can be plugged into those obtained via similarities." [**Remark**: many angles are x or y or their complementary. But no similarity ratio with a simple value was noticed by the student. The approach is not systematic or clear and cannot lead to the result.]
T	Trying to prove that a right triangle is isosceles, by calculating its sides	Targeting	[**Remark**: the 8th grade student correctly calculated AE, HC and the height from A in ΔAOH, but missed the direction and did not notice that she only needed to find AO, which can be determined easily by using $\Delta AOC \sim \Delta HOE$. A typical source of circular reasoning: writing a targeted result as a side note, not clearly separated from the proof itself, and then mixing it up.]

The participants worked between 50 and 80 minutes on the task, so the average was about one hour. The two students that had complete and correct solutions, were 9th grade students who participated in the final stage of the national mathematical contest, and the difference of skills, maturity of thinking and depth of understanding between them and the other students was really striking. No student was able to provide a purely geometric proof via a construction or by using inscribed quadrilaterals, but the best performer specified the first steps of such a solution.

4 Discussion and Conclusions

> *"I hate books, for they only teach people to talk about what they don't understand" - Jean Jacques Rousseau*

This research has tried to propose IBLecturing, the most "expository" teaching approach from a larger set of IBL approaches, labeled IES (Inquiry Enriched Schooling/Study) and which comprise among others:

- GSS – Guided Self-Study
- GPS – Guided Problem Solving
- IBH – Inquiry Based Homeschooling
- IB e-learning
- IBLecturing

IES is the core of the IBL (Inquiry-Based Learning) framework and aims at developing students' higher-level thinking, which involves the higher cognitive processes in the taxonomy proposed in the section 3.2 of this thesis. A triangular representation of this taxonomy has been provided in order to highlight in an intuitive way the fact that each stage is built on the basis of lower stages and if the foundation is not large enough, it is not possible to construct a high edifice when no external support (guiding/scaffolding) is provided. But in this case, when the support is removed, the construction becomes precarious and risks to collapse when challenged. Moreover, as we mount the triangle, each level is smaller, and the final one is a small fraction of the base. Edison has already shown that: "None of my inventions came by accident. I see a worthwhile need to be met and I make trial after trial until it comes. What it boils down to is one per cent inspiration and ninety-nine percent perspiration." (Newton, 1987, p. 24), and an outcome of such an attitude is the reliability of Edison's products (Paoletti, 2018).

While other approaches and even some inquiry-based methodologies such as PBL (Problem-Based Learning) try to focus on problem solving, comprehensive IBL equally targets problem posing skills and attitudes, which overlap with the much promoted (at least, formally) “Critical thinking”. The “art of problem posing”, at least under the form of generalizations or by “recycling” ideas from proofs, can be supported by teacher’s modeling as shown in Section 3.3 would add “The art of building insightful proofs”, which is an essential skill for IBLearning.

This thesis has shown that inquiry is inherent in any non-routine task, and students’ attempts to solve the problem from Section 3.4 confirmed the importance of “divergent thinking” when dealing with challenging problems. The excellent research of Belsky (1971, p. 49) recalls the studies of Gallagher et al. (1967), based on Guilford’s (1956) analysis of intellectual operations, which consist of:

1) Cognitive memory

2) Convergent thinking

3) Divergent thinking

4) Evaluative thinking

5) Routine categories (a catch all of miscellaneous verbal activity)

When I found Belsky’s work, the Chapter 3 and Table 4 of this thesis were already completed and I was pleasantly surprised to find a confirmation of my ideas about the different categories of thinking, especially “Structural reasoning” and Abduction, which are strongly related to the Evaluative thinking mentioned by him. Moreover, Belsky as well as many other researchers insist on the importance of Divergent thinking as a main element of creativity. I would like to call it “Combinatorial thinking”, but since the pair Divergent - Convergent thinking is very well-known I used it. However, it would be interesting to see if there are specific features of Combinatorial thinking with respect to the more general Divergent thinking, in order to distinguish further the various subclasses of Divergent thinking. Another confirmation of the importance of “divergence” which may manifest through unconventional attitudes or ways of life at highly creative persons such as Galois, considered an anarchist (Taton, 1947), Grothendieck (whose parents were anarchists), or von Neumann who could not create unless there was much noise, loud German march music, agitation, parties. Einstein was highly exasperated by this behavior, as he lived in the same house at Princeton and needed a lot of tranquility in order to reflect, since he was a contemplative scientist and listened to classical music. So there must be opposite or at least very different types of thinking at higher levels, especially Invention. My conjecture is that persons with strong Combinatorial thinking need first to demolish a structure in small pieces in order to recombine them in a new, original and often more useful structure, which immediately results in Inventing.

By analyzing students' attempts at solving the problem from Section 3.4 I found that my cognitive taxonomy is largely confirmed, especially regarding the Targeting level, which could be also called "Aiming". It has been quite difficult to find groups of students able to solve challenging problems in geometry - or at least to have a chance at solving them - even in the best schools. This is why I had to use a bit of scaffolding by giving them a drawing on squared paper; nevertheless, no student was able to use it for finding a purely geometric proof by construction, and some were even entrapped by such "hint" since they tried to use the "measurement method". Otherwise, I was very fortunate to meet such a group of gifted students; they only need more metacognitive and inquiry modeling in order to fulfill their potential. Anyway, it is clear that their 1-2 hours attendance each Saturday at the University in the Mathematics circle sessions was very helpful for their development of higher-level thinking and skills.

The importance of the Structuring phase (which includes Bloom's Evaluation) as a stage of deep understanding and thus a prerequisite for Inventing, was confirmed by my own metacognitive recall of finding novel solutions through inquiry in Section 3.3 and also by other examples from mathematics: for instance, Lagrange reached this stage in the problem of finding the roots of polynomials via formulas involving radicals when he remarked that the theory of permutations is the "true philosophy of the whole question", then Galois upgraded to Inventing level by finding a solution to the problem following Lagrange's "guidance" (Lagrange died when Galois was about one year old).

As also suggested by A. J. Green, "America's top SAT tutor" according to Business Insider, Guided Self Study (GSS) could be one of the most effective approaches (Green, 2016). It is much more complex than giving a hint, a strategy often used in problem books for reducing the "creative load" of a difficult problem. In my view, GSS should be based on materials that allow students to gradually access the guidance, according to their skills and needs. Ideally, there should be as many guiding stages as possible, but the highest number that I found in the literature was only three, in the excellent book of Rusu (1972):

- "How we think" (Heuristics)
- "Idea"
- (Complete) "Solution"

Another outstanding book from the same category of GSS "textbooks" is the famous "Problem solving strategies" of Engel (1999), where the author remarks that:

> *"Problem solving* ***can be learned only by solving problems****. But it must be supported by strategies supplied by the trainer." (Engel, 1999, p. 1)*

To this point of view I would add that metacognitive support, including modeling by the teacher, is extremely important since not only strategies but also attitudes and "setting dispositions"/mindsets can be modeled or demonstrated by the trainer. The advantage of using small groups of students for IBL has been illustrated by the experimental research of Borasi (1992), which I mentioned in Chapter 2.

A deficiency of most experimental researches in education is the fact that the teachers are not replaced after a while, in order to compare the efficiency of each teaching approach. At least, if teachers would be permutated from a group to another for a period of, say, one semester, the results would be more reliable since they would eliminate the distortions brought by teachers' abilities and student groups' average skills. This issue has been emphasized by Belsky (1971, pp. 7-8):

"There are two basic reasons why the innumerable studies comparing one teaching method with another fail to provide the reliable data which is needed. First, Metcalf's argument (1963 , p. 937) that the issue will remain unresolved until research records both the degree to which the method is applied and the quality of its employment appears justified. The second and related reform that is necessary lies in the area of research design. Most research has failed to specify the behavior of teachers that falls within the method being investigated (Wallen et al., 1963, p. 485), to observe directly whether such behavior does in fact take place and, only then, to relate the differences in the characteristics measured to the change in student achievement (Medley et al., 1963, pp. 249-250)."

Literature review revealed that a common misconception among teachers is the confusion regarding the meaning of the constructivist slogan: "an instructor does not teach the student a syllabus, but facilitates students' learning". When implementing IBL, facilitating student's learning <u>should not</u> be understood as facilitating their task! It is essential to let student struggle for a while, otherwise there is no IBL.

Some students/teachers think that what suits them suits all. Expository teaching is easier for teachers, but students don't enjoy it and become less motivated. Implementing and conducting IBL sessions is quite demanding for a teacher, but enjoyable and highly motivating for the students. Since inquiry promotes motivation through challenge and autonomy, this thesis provides practical solutions for implementing the ideas of Deci and Ryan regarding self-determination and intrinsic motivation (Ryan & Deci, 2000; Deci, 2015, 2016, 2017). I propose employing the concept of optimal challenge from the flow approach (Csikszentmihalyi, 1997) for the best results in IBL. Another issue involves the inductive strategy of Generalization. There is a danger that such strategy employed for complex problems such in real life e.g. in Social Sciences or in applied mathematics may lead to faulty models since they are built by destroying a more complex structure and by overlooking the subclasses, the differences and the specificities of various groups or individuals.

I made a distinction between Knowledge and Understanding groups of processes. However, in common language the distinction between the words labeling these two complementary phases is minor - for example the fact that only the understanding can be “deep”, and not the comprehension. In any case, such labels must not be regarded as completely describing these categories.

In any case, a distinction must be made because Anderson’s taxonomy and the CPiMI (Model for Cognitive Processes in Mathematical Investigation) proposed by Yeo (2013; 2017) equate Understanding with Bloom’s “Comprehension” of the task or of the method. I would argue that there is a world of difference between comprehension in Bloom’s sense of Fermat’s last Theorem statement and its deep Understanding in the sense of the conception proposed here. Perhaps due to such confusion between the simplicity of a statement and its deepness or difficulty there were so many attempts at solving it by more or less elementary means. When Wolfskehl offered in 1908 a large prize for its solution, there were 621 supposed solutions sent to the Göttingen Academy in the first year alone, and the total amount of wrong proofs received until Wiles won the prize has been estimated at over 5000. (Barner, 1997)

For the implementation of IBL, and especially of IES, I suggest the use of Collins et al.’s (1980, 2009 a, 2009 b) ideas and principles. Also, Chazan et al. (1998) is a good guide for designing geometry instruction. Serra (2008) has been the best available textbook in IBL learning of geometry, and should be definitely used as a reference.

I suggest the use of such inquiry-intensive subjects as Ancient history, archeology, economy/finances, forensic science/literature, games such as Go, chess, puzzles involving combinatorial thinking, Scrabble, Sokoban, etc. “The Art of Game Design” by Shell (2008) is an excellent reference for employing games in IBL. Countries like South Korea have achieved strong results in the promotion of higher-level thinking through the use of games (Go) and the intensive study of mathematics.

As a research opening, there is the also the involvement of other cognitive layers in each single phase, as I have noticed in Introduction. It has been acknowledged that each level requires the assistance of lower levels, but what is less known is the involvement of higher layers in the processes occurring at a particular layer.

One of the questions for future research is the connection between the highest and the lowest cognitive layers, namely, Invention and Recording. Another aspect that needs to be emphasized is that, at any stage, the other stages are more or less present, even if insignificantly or in a latent form. For example, Operating involves some reasoning, since usually it is not 100% robotic. Also, the Targeting phase involves some Integrating, Structuring, and Inventing when designing the deductive chain, otherwise there would be only chaotic trials, as in the work of many participants in the quiz presented in Section 3.4. For a long-term recording of information, it has to be integrated and structured, which requires reflection and time.

Quick learning is the enemy of reliability, since the learner does not have the time to make the necessary connections in his or her mind. If the content is too structured, the student will not learn how to structure the information, and will also be untrained in making connections (Integrating level). Regarding education, it has been argued that "One only has to be fast when catching flees." (Gelfand, n.d.). This is why expository teaching leads to unreliable, short-term learning, which, coupled with 'learning to the test' and infrequent evaluation leads to deplorable educational results. There is the exception of photographic memory, but even persons with such an ability often have extremely powerful high-level thinking, for example Euler who at age 70 could recite the whole text of Virgil's Aeneid by specifying the first and last sentence on each page of the edition he owned, or the amazing von Neumann (Macrae, 1992), who memorized in "image format" and forever whatever he saw, including entire telephone books or encyclopedias of 21 volumes; he also did not see any value in programming languages, since he could do everything in machine code (Lee, 1995).

Yet, there must be several types of long-time memory in the same brain, since von Neumann could barely remember a visitor's name - therefore he was not using names in introducing people (Life, 1957). In public education speed must be adapted to a lowest common denominator, but gifted students understand very fast and become easily bored. There are also different ways of high-level thinking - for example, although Einstein and von Neumann worked near each other in the same building, they were not intimate and never formally collaborated:

> *"Einstein's mind was slow and contemplative. He would think about something for years. Johnny's mind was just the opposite. It was lightning quick – stunningly fast. If you gave him a problem he either solved it right away or not at all. If he had to think about it a long time and it bored him, his interest would begin to wander. And Johnny's mind would not shine unless whatever he was working on had his undivided attention"*(Life, 1957)

However, the problems von Neumann did care about, such as his "theory of games", absorbed him for much longer periods. Partly because of this quicksilver quality von Neumann was not an outstanding teacher to many of his students. But for the advanced students, who could ascend to his level he was inspirational. His lectures were brilliant, although at times difficult to follow because of his rush. (Life, 1957) This suggests that motivation is the main element of learning, and it has been proved that in this aspect IBL gives the best results.

Regarding the speed of thinking at Invention level, just after the death of von Neumann, a second edition of "How to solve it" was published, where Pólya said:

> *"(von Neumann was) the only student of mine I was ever intimidated by. He was so quick. There was a seminar for advanced students in Zürich that I was teaching and von Neumann was in the class. I came to a certain theorem, and I said it is not proved and it may be difficult. Von Neumann didn't say anything but after five minutes he raised his hand. When I called on him he went to the blackboard and proceeded to write down the proof. After that I was afraid of von Neumann."*- Pólya (1957, p. xv)

Another topic for future research involves the use of IBLecturing for teaching proving and also for learning through insightful proofs, according to the ideas of Moore:

"The teacher should lead up to an important theorem gradually in such a way that ... the desire for formal proof is awakened, before the formal proof itself is developed. Indeed, much of the proof (of the theorem) should be secured by the research of the students themselves". (Moore, 1903, p. 419)

A very productive direction of research would be to integrate the most advanced results of cognitive psychology into the theory and the implementation of IBL, for example the recent findings regarding the role of Working Memory (WM) in the mental processes involved in high-level thinking. The main hints for this field are provided by the works of Mammarella et al. (2013, 2017) and Geary et al. (2017).

IBLecturing tries to combine the best aspects of Expository and Inquiry methodologies into a single approach, where the proportion and the timing of lecturing and research phases are adjustable according to students' preliminaries and their feedback:

Expository advantages

- Content acquisition
- Modeling by an expert

IBL advantages

- Motivation
- Individual feedback and guidance
- Higher order thinking

In order to optimize learning, each IBLecturing session should end with an open task proposed to the students as self-study or homework to be completed until the next meeting. The task should involve inquiry into the topic which will be addressed by the trainer in his next lecture and will be done individually outside of the classroom because it is time-consuming; nevertheless, students may use online communication with the trainer in order to request and to get suitable guiding, according to their needs and wishes. This is the most student-centered phase of IBLecturing, and it not only motivates students for the topic which will be presented by the teacher but also helps them acquire some basic, intuitive perception of its meaning through exploration and personal involvement (by getting their hands "dirty"); such activity is similar to a good preparation of the soil for seeding by a farmer. For example, a teacher may propose to his students an open task involving global and instant variation of a continuous function in order to build an intuitive basis before introducing the notion of derivative. The task may follow the historical development of the concept to be introduced or it may involve some relevant applications or connections with real life.

Open tasks have the advantage of allowing each student to go as far as he can without any pressure to achieve a definite task until the deadline and even without formal assessment. Alternatively, the teacher may reward via formal grades the best researches in order to stimulate students' involvement, but the goal should always be a preparation for deeper understanding of the subject and not the attainment of some performance standards.

At the next gathering, the class starts with a group discussion of the various approaches, ways of thinking, mistakes and findings occurring in students' attempts to solve the task. Only when he clarified all these issues and after having derived the suitable conclusions, the trainer starts his lecture and presents the new content.

IBL mainly deals with causes (WHY?) and conjectures (WHAT IF?) rather than effects/correlations (SINCE/AS X HAPPENED, Y OCCURRED) or data (WHAT? WHEN? WHERE?), on which expository approaches are mainly focused.

The main limitation is a lack of validation via standard qualitative and quantitative methods due to the significant obstacles regarding the completion of such tests or interviews, especially in grade school.

The main contributions of this thesis are: a systematization of IBL methodologies, a practical classification of the cognitive processes involved in learning and doing mathematics, and the proposal of a novel approach, which would combine inquiry and lectures: IBLecturing.

This approach is quite realistic when correctly implemented by competent teachers. Modeling by an expert is an essential part of IBLecturing, so the instructor must have cognitive and metacognitive proficiency. Appropriate timing and duration of each phase are essential for successful implementation of IBLecturing: keeping students "in the flow"!

IBLecturing is de facto already implemented in many schools by the splitting of mathematical instruction in two phases: laboratory and lectures. Unfortunately, laboratory period is seen by many teachers as just an exercise and drill session and thus inquiry is more or less ignored.

The triangular model of cognitive processes could be validated by group testing of secondary, undergraduate and graduate students as in Section 3.3 (the challenging problem administered to twenty students from secondary school) - supplemented with individual testing as well, but following the standard rules of quantitative and qualitative scientific research.

This research is a bottom-up initiative, supported by several strategies, at implementing IBL. It is centered on the easiest and the most acceptable step available for public education, IBLecturing. Three institutional forces are shaping the reform of public education worldwide: PISA quantitative research - by far the most important due to its over exposure in media (Sjøberg, 2017; Baroutsis et al., 2018), TIMSS assessment in science education, and DESECO - the only study that specifically targets higher order thinking and inquiry.

PISA was initiated by OECD and only evaluates 15 years old students' skills at applying basic knowledge from primary and middle school. TIMSS tests science knowledge and skills of grades 4 and 8 children in a larger set of countries, some of them not members of OECD; TIMSS Advanced evaluates students in the final year of secondary school. DESECO is an OECD general statement of educational goals for the future, with no details regarding the practical way of achieving them in public schools.

Any improvement in a certain direction requires, if the instruction time remains unchanged, reducing the amount of time available for the development of other knowledge or skills. For instance promoting students' skills at applying knowledge in real-life contexts (aimed by PISA) is generally done by showing them various practical examples and by a thorough training in the translation of elementary real-life problems into mathematical language; the trade-off is that the time available for developing theoretical skills and even higher-order thinking will be reduced correspondingly. It is not possible to achieve both PISA and DESECO requirements without expanding the instruction time or the amount of homework, or both - as it happens in many Eastern Asian countries.

In contrast to conventional education, IBL is not so much about building a CV and acquiring diplomas, but about acquiring vital skills in our age of information, manipulation and social marketing. Moreover, IBL is not about quantity, but about quality of learning and understanding. Many researches into the effectiveness of IBL have missed important issues:

- IBL and expository instruction cannot replace one another but are complementary

- Inquiry is time-consuming, so most of it should be done outside of the classroom as self-study, team work or homework

- Modeling and guidance are very necessary and have to be provided by the instructor

- Guidance has to be well-balanced, customized and adaptive in order to keep each student "in the flow"

- Inquiry should involve the main concepts targeted by the curriculum, in order to motivate learners in mastering them.

In the past, such requirements were very difficult to fulfill, but now modern technologies allow distant learners to show their work and to receive individual guidance from an instructor upon request, via online communication. There is still an issue with the choice of suitable tasks for inquiry, due to a lack of inquiry-based textbooks in all the branches of mathematics except geometry. Experienced teachers have already built a personal "database" of good problems for introducing the main concepts from the curriculum, and it can be expanded further by using the various collections of challenging problems and strategies for solving them, published recently.

This thesis has started as a research into the possible implementation of IBL in the study of geometry via Computer Algebra Software (CAS), but new findings such as the cognitive structure involved in mathematical thinking as well as the role of modeling and guidance provided by an expert trainer in IBL eventually led to a larger perspective, where IBLecturing would integrate these results into a practical and effective teaching approach for any branch of mathematics education. Still, geometry has been and remains the natural field for implementing IBL and for promoting higher-order thinking; not only that Euclid's “Elements” were designed with such vision in mind 23 centuries ago, but recent textbooks such as Serra's “Discovering geometry” went even further with the promotion of inquiry and successfully helped introducing IBL in school teaching of mathematics. Moreover, geometry is the most suitable field of school mathematics for calling up students' higher-order skills such as Structuring and Inventing, so I chose an elementary geometry task in order to identify the upper levels of the cognitive model proposed in Chapter 2.

One of the simplest and most common ways of introducing inquiry into expository teaching is via properly designed "gaps" in the proofs or in the problems' solutions. Students are required to complete these gaps at home, until the next meeting. Some advantages of this approach are: the presentations are more fluid and the main ideas are more visible; precious classroom time is saved by the teacher and more content can be presented in a similar amount of time; students practice inquiry. A trade-off is that part of the training is transferred to homework and the teacher must be available for some guidance beyond classroom hours, which may increase institutional costs. In order to save teacher's time, such homework could be used as part of students' summative evaluation - even without percentage grades, such as in the evaluation of graduate students' research (projects, thesis, etc.) If there is no guidance at all, many students will work in teams or will ask for guidance from the best achievers among their colleagues - if available, which may reduce the quality but also the institutional cost of their training. This approach can be used with any group of students, starting in high school. With gifted students, it can be employed even earlier. The only initial requirement from the learners is to be able to understand a proof intuitively, for example a property justified on a drawing in basic geometry.

A second way of introducing inquiry in traditional settings is IBLecturing, which is an improvement with respect to the previous approach since it includes modeling by an expert, in addition to the guidance. The number of participants at the expository lessons is not limited, in principle; however, in order to obtain efficient, customized and adaptive guiding from the trainer, group size should be minimal. From my experience, it would be very difficult to implement properly any form of guided inquiry with groups larger than 20 students. The degree of challenge has to be balanced by a suitable amount of individual guidance - according to each student's skills, in order to keep him "in the flow"; nevertheless, the tasks should not require too much scaffolding, otherwise students may feel they had no significant contribution to the solution, which will reduce their motivation and learning.

In conclusion, IBLecturing does not require gifted students, but the tasks should be accessible with a reasonable amount of guidance. A common approach used when presenting sophisticated, more involved proofs or solutions, is to break them into several steps or stages. Additional guidance can be provided at each step, at student's request. For example, solving the general second degree equation is an excellent task for all students and it only requires an initial hint, except the final discussion regarding the existence of real solutions; on the contrary, solving the general third degree equation requires several stages and more guidance. Open problems are very suitable tasks, and can be given at the end of each meeting; ideally, they should prepare the field for introducing the content of the next meeting. The best achievements may be considered in the overall summative grading of students; alternatively, the teacher may use such tasks only for training, as formative assessment. A final project involving higher-order skills could be proposed, to be assessed as an essay: for example, presenting Cavalieri's principle in high school or Michelson's measurement of the speed of light. Masterful lessons consisting of inquiry-enriched presentations have been given by Tom Leighton at MIT and are available online, for example Leighton (2010 a; 2010 b). While such lectures don't comprise a modeling of mathematicians' thinking when trying to solve challenging problems, they demonstrate the effectiveness of introducing inquiry in the teaching of mathematics and may be used as a valuable reference when designing IBLectures.

Several undergraduate programs focused on research in mathematics have already been implemented successfully with small groups of students by the professors Gallian from the University of Minnesota - Duluth (Gallian, 2015) and Hildebrand from the University of Illinois (Hildebrand, 2018).

When I administered the quiz presented in Section 3.4 to the group of gifted students, I did not know much about their skills, except that they had been more exposed to challenging problems than other students of their age. The problem had already been chosen, and I tried at first to propose it to a regular class from my former high school but teachers were reluctant to collaborate. They maintained that since classical plane geometry is studied in grade 7, followed by space geometry in grade 8 and vectors in grade 9, students quickly forget plane geometry after passing to grade 8 and thus I should target middle school classes for such quiz. Moreover, I was advised to choose gifted students for solving it, since school standards dropped very much in the last years and proposing such a problem in a secondary school would lead to failure. In order to make the task more accessible, the drawing was done on squared paper; however, this scaffolding was not useful for the participants, who were not able to find a suitable construction. I also remarked that contrary to teachers' remarks, 8th grade students performed better than 7th grade students, and the two 9th grade students were far above their younger fellows with respect to higher-order thinking, and especially Structuring.

This suggests that maturity of thinking is much more important than the freshness of knowledge and that around the age of 15 higher-order thinking can flourish suddenly if properly stimulated. Students were really involved in the challenging task proposed to them, and I felt that IBLecturing would help them greatly in developing strategic thinking, holistic view, structuring skills and creativity at this critical age. Unfortunately, they lack modeling and individual guidance; even in mathematical circles, lessons are given in an expository way due to the time available and the volume of content. Inquiry is only present as self-study and when solving challenging problems proposed as homework, but teacher's monitoring and feedback would be extremely useful. In order to assess the effectiveness of IBL for all students (not only for the gifted ones), specific testing, aiming at higher-order thinking, should be designed. Much of IBL value is the promotion of specific attitudes and metacognitive skills, which need a lot of time to develop (e.g., the probabilistic mindset). If IBL is used only in mathematics, and the other school disciplines are taught and evaluated in conventional, expository ways, the desired attitudes and mindsets are not allowed to grow and the benefits of such narrow implementation of IBL are really small.

5 REFERENCES

Adler, J. (1993). Beyond 'tell and drill' maths. *The Weekly Mail* and *Guardian*: Review/Education Supplement, 2.

Adler, J. (1997). A participatory-inquiry approach and the mediation of mathematical knowledge in a multilingual classroom. *Educational Studies in Mathematics* 33(3), 235-258.

Ako Aotearoa (2008). Inquiry-Based Learning Report. Retrieved from: https://akoaotearoa.ac.nz/project/inquiry-based-learning/resources/books/inquiry-based-learning-report

Albers, D. J. (1982). Paul Halmos: Maverick mathologist. *Two-Year College Mathematics Journal* 13(4), 234-241.

Alberta Education (2004). Focus on Inquiry. Alberta Learning. Retrieved from: https://education.alberta.ca/media/313361/focusoninquiry.pdf

Allan, J. (2007). Snapshot of a generation: Bridging the theory-practice divide with project based learning. *Australian Journal of Adult Learning* 47(1), 78.

Anderson, L. W., Krathwohl, D. R., Airasian, P. W., Cruikshank, K. A., Mayer, R. E., Pintrich, P. R., Raths, J., & Wittrock, M. C. (2001). *A Taxonomy for Learning, Teaching, and Assessing: A Revision of Bloom's Taxonomy of Educational Objectives*, abridged edition. White Plains, NY: Longman.

Anderson, L. W. (2017). Guest post on Larry Ferlazzo's educational blog. Retrieved from: http://larryferlazzo.edublogs.org/2017/04/15/guest-post-from-lorin-w-anderson-co-author-of-the-revised-blooms-taxonomy/

Anthony, G. (1996) Active learning in a constructivist framework. *Educational Studies in Mathematics* 31(4), 349-369.

APSE (2011). High Scope Program Conference. Asia-Pacific Science Education. Retrieved from: http://apse2011.lssh.tp.edu.tw/about-the-conference

Asghari, A. (2012). Moore and Less! *PRIMUS* 22(7), 509-524.

Ausubel, D. P. (1964). Some psychological and educational limitations of learning by discovery. *The Arithmetic Teacher* 11(5), 290-302.

Bain, K. (2004). *What the Best College Teachers Do*. Cambridge, MA: Harvard University Press.

Baptist, P., & Raab, D. (2012). Implementing Inquiry in Mathematics Education. Retrieved from http://fibonacci.uni-bayreuth.de/resources/resources-for-implementing-inquiry.html

Barner, K. (1997). Wolfskehl and the Wolfskehl Prize, *Notices AMS* 44 (10), November 1997, pp. 1294-1303. Retrieved from: http://www.ams.org/notices/199710/barner.pdf

Baroutsis, A. & Lingard, B. (2018). PISA-shock: how we are sold the idea our PISA rankings are shocking and the damage it is doing to schooling in Australia. Retrieved from: http://www.aare.edu.au/blog/?p=2714

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education* 27(3), 258-292.

Beairsto, B. (2011). Necessary Disruption. Part 5: Inquiry-Based Learning. Retrieved from: http://www.cea-ace.ca/blog/bruce-beairsto/2011/04/5/necessary-disruption-part-5-inquiry-based-learning

Bell, A. W. (1976). A study of students' proof-explanations in mathematical situations. *Educational Studies in Mathematics* 7, 23-40.

Belsky, T. B. (1971). *The Development of an "Expository-Inquiry Continuum" to Measure Relationships between "Expository" and "Inquiry" Teaching Methods and Student Cognitive Characteristics in Secondary Social Studies Classes*. Doctoral dissertation. University of Massachusetts Amherst.

Ben-Chaim, D., Fresko, B., & Carmeli, M. (1990). Comparison of teacher and pupil perceptions of the learning environment in mathematics classes. *Educational Studies in Mathematics* 21(5), 415-429.

Berndt, B., C. A. Reddi. (2004). Two Exams Taken by Ramanujan in India. *The American Mathematical Monthly* 111(4), 330-339.

Bishop, A. J. (1991). *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*. Dordrecht: Kluwer Academics Publishers.

Blair, C. Jr. (1957). Passing of a great mind. *Life*, Feb 25, 1957, pp. 89-104. Retrieved from: https://books.google.ca/books?id=rEEEAAAAMBAJ&printsec=frontcover&source=gbs_ge_summary_r&cad=0#v=onepage&q&f=false

Bloom, B.S. (Ed.). Engelhart, M.D., Furst, E.J., Hill, W.H., Krathwohl, D.R. (1956). *Taxonomy of Educational Objectives, Handbook 1: The Cognitive Domain*. New York: David McKay Co Inc.

Blumenfeld, P. C., Soloway, E., Marx, R. W., Krajcik, J. S., Guzdial, M. & Palincsar, A. (1991). Motivating project-based learning: sustaining the doing, supporting the learning. *Educational Psychologist* 26 (3 & 4); 369–398.

Bolisani, E., Bratianu, C. (2018). *Emergent Knowledge Strategies: Strategic Thinking in Knowledge Management*. New York: Springer International Publishing.

Bonnstetter, R. J. (1998). Inquiry: Learning from the past with an eye on the future. *Electronic Journal of Science Education* 3(1).

Borasi, R. (1992). *Learning mathematics through inquiry*. Portsmouth, NH: Heinemann Educational Publishers.

Brainerd, C. J. (2003). Jean Piaget, learning research, and American education. In B. J. Zimmerman & D. H. Schunk (Eds.), *Educational Psychology: A Century of Contributions*. Mahwah, NJ: Erlbaum.

Bransford, J. D. & Stein, B. S. (1993). *The IDEAL Problem Solver* (second edition), New York: Freeman.

Britt, M.S., Irwin, K.C., Ellis, J., & Ritchie, G. (1993). *Teachers Raising Achievement in Mathematics: Final Report to the Ministry of Education*. Centre for Mathematics Education, Auckland College of Education, Auckland NZ.

Brogan, M. (2006). What you do first is get them into groups: project-based learning and teaching of employability skills. *Fine Print* 29(2): 11–16.

Brooks, D. W., & Shell, D. F. (2006). Working memory, motivation, and teacher-initiated learning. *Journal of Science Education and Technology* 15(1), 17-30.

Bruce, B. C., & Davidson, J. (1994). *An Inquiry Model for Literacy across the Curriculum*. Technical Report No. 598. University of Illinois at Urbana-Champaign.

Bruce, B. C. (2008a). What is inquiry-based learning? Retrieved from: http://chipbruce.net/resources/inquiry-based-learning/defining-inquiry-based-learning

Bruce, B. C. (2008b). The inquiry cycle. Retrieved from: http://chipbruce.net/resources/inquiry-based-learning/the-inquiry-cycle

Bruce, B. C. (2008c). Other definitions of inquiry-based learning. Retrieved from: http://chipbruce.net/resources/inquiry-based-learning/definitions-of-inquiry-based-learning

Bybee, R. W. (2000). Teaching Science as Inquiry. In J. *Minstrell* and E. van Zee (Eds.), *Inquiring into Inquiry Learning and Teaching in Science* (pp. 20-46). Washington, DC: American Association for the Advancement of Science.

Calleja, J. (2013). Mathematical Investigations: The Impact of Students' Enacted Activity on Design, Development, Evaluation and Implementation. *Proceedings of ICMI Study 22*. Oxford.

Chalice, D. R. (1995). How to teach a class by the modified Moore method. *American Mathematical Monthly* 102(4), 317-321.

Chan, C., Burtis, J., & Bereiter, C. (1997). Knowledge-building as a mediator of conflict in conceptual change. *Cognition and Instruction* 15, 1-40.

Chan, H., Tsai, P., & Huang, T. Y. (2006). Web-based learning in a geometry course. *Journal of Educational Technology & Society* 9(2), 133-140.

Chan, Y. Y. (2007, October). Teaching queueing theory with an inquiry-based learning approach: A case for applying webquest in a course in simulation and statistical analysis. In *Frontiers In Education Conference-Global Engineering: Knowledge Without Borders, Opportunities Without Passports*. FIE '07. 37th Annual (pp. F3C-1). IEEE.

Chang, K. (2009). Israel Gelfand, math giant, dies at 96. New York Times, Oct. 8, 2009. Retrieved from: https://www.nytimes.com/2009/10/08/science/08gelfand.html?_r=2&scp=1&sq=Gelfand&st=cse

Chazan, D., & Yerushalmy, M. (1998). Charting a course for secondary geometry. In Lehrer, R., & Chazan, D. (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 67-90). Hillsdale, NJ: Lawrence Erlbaum Associates.

Clark, C. (1988). The necessity of curricular objectives. *Journal of Curriculum Studies* 20(4), 339-349.

Clark, D. M. (2012). *Euclidean Geometry: A Guided Inquiry Approach* (Vol. 9). Providence, RI: American Mathematical Society.

Clements, D. H., Battista, M. T., & Sarama, J. (1998). Development of geometric and measurement ideas. In Lehrer, R., & Chazan, D. (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 201-226). Hillsdale, NJ: Lawrence Erlbaum Associates.

CMA (2015). Theories of learning. Children's Museum Alliance. Retrieved from: http://www.thespotmuseum.org/theories

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist* 23(2), 87-103.

Cobb, P. (1999). Individual and collective mathematical development: The case of statistical data analysis. *Mathematical Thinking and Learning* 1, 5–43.

Colburn, A. (2000). An inquiry primer. *Science Scope* 23(6), 42-44.

Colburn, A. (2006). What teacher educators need to know about inquiry-based instruction. Annual meeting of the Association for the Education of Teachers in Science, Akron, OH. Retrieved from: http://web.csulb.edu/~acolburn/AETS.htm

Collins, A., & Halverson, R. (2009 b). *Rethinking Education in the Age of Technology: the Digital Revolution and Schooling in America*. New York: Teachers College Press.

Collins, A., & Stevens A. L. (1980). *Goals and Strategies of Interactive Teachers*. Cambridge, Mass.: Bolt, Beranek and Newman, Inc.

Collins, A., White B., & Frederiksen J. (2009 a). The interplay of scientific inquiry and metacognition. In D. J. Hacker, J. Dunlosky, A. C. Graesser (Eds.) *Handbook of Metacognition in Education* (p. 175-205). New York and London: Routledge.

Coxeter, H. S. M., Greitzer, S. L. (1967). *Geometry Revisited*. Washington, DC: The Mathematical Association of America (MAA).

Crick, R. D. (2012). Deep engagement as a complex system: Identity, learning power and authentic enquiry. In S. L. Christenson, A. L. Reschly, C. Wylie (Eds.) *Handbook of Research on Student Engagement* (pp. 675-694). New York: Springer.

Csikszentmihalyi, M., & Csikszentmihalyi, I., S. (1997). *Optimal Experience*. Cambridge UK: Cambridge University Press.

D'Angelo, C. M., Touchman, S., Clark, D. B., O'Donnell, A., Mayer, R., Dean, D., & Hmelo-Silver, C. (2009). Constructivism. Retrieved from: http://www.education.com/reference/article/constructivism/

Davidson, N. (1971). The small group-discovery method as applied in calculus instruction. *American Mathematical Monthly*. August-September, 789-791.

Davis, P. J., Hersh, R. (1986). *Descartes' Dream*. New York: Harcourt, Brace, & Jovanovich.

Deci, E. L. (2015). Intrinsic motivation. Retrieved from: https://www.youtube.com/watch?v=-Ba7bpEUONM

Deci, E. L. (2016). Intrinsic Motivation: The Inherent Tendency to Be Active. In R. Sternberg, S. Fiske, & D. Foss (Eds.), *Scientists Making a Difference: One Hundred Eminent Behavioral and Brain Scientists Talk about their Most Important Contributions* (pp. 288-292). Cambridge University Press. https://doi.org/10.1017/CBO9781316422250.063

Deci, E. L. (2017). Self-Determination Theory. Retrieved from: https://www.youtube.com/watch?v=m6fm1gt5YAM

Dennis, D., & Confrey, J. (1998). Geometric Curve Drawing Devices as an Alternative Approach to Analytic Geometry: An Analysis of the Methods, Voice, and Epistemology of a High School Senior. *Designing Learning Environments for Developing Understanding of Geometry and Space.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Dewey, J., & Dewey, E. (1915). *Schools of Tomorrow*. New York: Dutton.

Dewey, J. (1938). *Experience and Education*. New York: Macmillan.

Dewey, J. (1938). Logic: *The Theory of Inquiry*. New York: Henry Holt & Co.

Dorier, J.-L., & Maaß, K. (2012). Inquiry-based mathematics education. In *Encyclopedia of Mathematics Education*. Springer.

Dorier, J. & Garcia, F.J. (2013). Challenges and opportunities for the implementation of inquiry-based learning in day-to-day teaching. *ZDM - The International Journal on Mathematics Education* 45(6), 837–849.

Driscoll, M., DiMatteo, R. W., Nikula, J. E., & Egan, M. (2007). *Fostering geometric thinking: A guide for teachers grades 5–10*. Portsmouth, NH: Heinemann.

Dubinsky, Ed (1986), Reflective Abstraction and Computer Experiences: A new approach to teaching theoretical mathematics, In Lappan, Glenda, Even, Ruhama, (Eds), *Proceedings of the Eighth Annual PME-NA Meeting*, E. Lansing, Michigan: Michigan State University.

Dubinsky, Ed (1991). Reflective Abstraction in Advanced Mathematical Thinking. In David O. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 95–123). Kluwer: Dordrecht.

Durell, F. (1894). Applications of the New Education to the differential and Integral Calculus. *American Mathematical Monthly,* 1, January.

Ebert-May D., & Hodder, J. (1995). *Faculty institutes for reforming science teaching through field stations*. Final report for a project support by the National Science Foundation, Division of Undergraduate Education.

EduTech Wiki (n.d.). Inquiry-based learning. Retrieved from: http://edutechwiki.unige.ch/en/Inquiry-based_learning

Elbers, E. (2003). Classroom interaction as reflection: Learning and teaching mathematics in a community of inquiry. *Educational Studies in Mathematics* 54(1), 77-99.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education* 42(4), 308–345.

Engel, A. (1999). *Problem-Solving Strategies*. Second edition. Springer.

Ernest, P. (1991). *The Philosophy of Mathematics Education*. Basingstoke: Falmer Press.

Euclid (1558). *Euclidis elementorum libri XV. Graece et Latine, quibus, cum ad omnem Mathematicae scientiae partem, tum ad quam libet Geometriae tractationem, facilis comparatur aditus.* Lutetiae, apud Gulielmum Canellat. Biblioteca Nazionale Centrale di Firenze. Postillati 111. Manuscript annotations by Galileo Galilei. Image retrieved from: https://www.liberquarterly.eu/articles/10.18352/lq.8003

Fairbairn, D. M., Dover, R., & Doyle, D. (2008). What Did One Angle Say to the Other Angles?. *The Mathematics Teacher*, 102(1), 62-68.

de Fermat, P. (1659). Letter to Carcavi. August 1659. Retrieved from: https://science.larouchepac.com/fermat/

Fey, J. (1969). Classroom teaching of mathematics. *Review of Educational Research* 39(4), 535-551.

Finley, F. N., & Pocovi, M. C. (2000). Considering the scientific method of inquiry. In J. Minstrell and E. van Zee (Eds.), *Inquiring into Inquiry Learning and Teaching in Science* (pp. 47-62). Washington, DC: American Association for the Advancement of Science.

Fresko, B., & Ben-Chaim, D. (1986). Learning environment differences in the mathematics classroom. In *10th International Conference for the Psychology of Mathematics Education, London, England*.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Springer Science & Business Media.

Freudenthal, H. (2006). *Revisiting Mathematics Education: China Lectures* (Vol. 9). Springer Science & Business Media.

Froyd, J. E., Wankat, P. C., & Smith, K. (2012). Five major shifts in 100 years of engineering education. *Proceedings of the IEEE* 100 (Special Centennial Issue), 1344-1360.

Gagné, R. M. (1965). *The Psychological Bases of Science: A Process Approach.* Washington, DC: American Association for the Advancement of Science.

Gagné, R. M. (1966). Varieties of Learning and the Concept of Discovery. In L. S. Shulman and E. R. Keislar (Eds.), *Learning by Discovery: A Critical Appraisal*. Chicago: Rand McNally.

Gallagher, J. J., Ashner, M. J., & Jenné W. (1967). *Productive Thinking of Gifted Children in Classroom Interaction*. Washington, D.C.: NEA, Council for Exceptional Children.

Gallagher, M. J., & Reid, K. (2002). *The Learning Theory of Piaget and Inhelder*. Lincoln, NE: iUniverse.

Gallian, J. A. (2012). Undergraduate Research in Mathematics Has Come of Age. *Notices of the American Mathematical Society* 59(8).

Gallian, J. A. (2015). Enhanced opportunities for highly motivated undergraduates. A Panel 5 presentation from the regional meeting at the University of Chicago, September 20. Retrieved from: https://www.tpsemath.org/chicago2015 ; pdf slides link: https://d3n8a8pro7vhmx.cloudfront.net/math/pages/156/attachments/original/1443020333/Chicago-Gallian.pdf?1443020333

Geary, D. C., Berch, D. B., Ochsendorf, R., & Koepke, K. M. (Eds.). (2017). *Acquisition of Complex Arithmetic Skills and Higher-Order Mathematics Concepts* (Vol. 3). Academic Press.

Gelfand, I. (n.d.). Work in Education - Israel M. Gelfand - Official Website. Retrieved from: http://israelmgelfand.com/edu_work.html

Goldenberg, E. P., & Cuoco, A. A. (1998). What is dynamic geometry? In R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 351–367). Mahwah, NJ: Erlbaum.

Goldenberg, E. P., Cuoco, A. A., & Mark, J. (1998). A role for geometry in general education. In R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 3–44). Mahwah, NJ: Erlbaum.

Gonobolin, F. N. (1975). Students' comprehension of geometric proofs. In J. W. Wilson (Ed.), *Soviet Studies in the Psychology of Learning and Teaching Mathematics* (Vol. 12), *Problems of Instruction*. Chicago: University of Chicago Press. (Original work published in 1954).

Goodchild, S., Fuglestad, A. B., & Jaworski, B. (2013). Critical alignment in inquiry-based practice in developing mathematics teaching. *Educational Studies in Mathematics* 84(3), 393-412.

Gordon, N., & Brayshaw, M. (2008). Inquiry based Learning in Computer Science teaching in Higher Education. *Innovation in Teaching and Learning in Information and Computer Sciences* 7(1), 22-33.

Gow, J. (1884). *A Short History of Greek Mathematics*. Cambridge University Press.

Grabiner, J. (2015). How Euclid once ruled the world. Based on a talk given in December 2015 at Ada Lovelace Symposium, University of Oxford. Retrieved from: https://plus.maths.org/content/how-euclid-once-rules-world

Gravemeijer, K. P. (1998). From a different perspective: Building on students' informal knowledge. In R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space 1*, 45-66.

Green, A. J. (2016). I made $ 1,000 an hour as a SAT tutor. My students did better without me. Retrieved from: https://www.vox.com/2016/1/8/10728958/sat-tutor-expensive

Guilford, J. P. (1956). The structure of the intellect. *Psychological Bulletin* 53, 267-293.

Hagevik, R. A. (2003). *The effects of online science instruction using geographic information systems to foster inquiry learning of teachers and middle school science students*. Unpublished PhD dissertation, North Carolina State University, Raleigh.

Hammer, D. (2000). *Teacher inquiry*. In J. *Minstrell* and E. van Zee (Eds.), *Inquiring into Inquiry Learning and Teaching in Science* (pp. 184-215). Washington, DC: American Association for the Advancement of Science.

Hanna, G. (1983). *Rigorous proof in mathematics education*. Toronto: Ontario Institute for Studies in Education.

Healy, C. C. (1993). *Build-a-Book Geometry: A Story of Student Discovery*. Key Curriculum Press.

Heisenberg, W. (1927). Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik, 43 (3),* 172–198. English translation: The actual content of quantum theoretical kinematics and mechanics. Washington, DC: NASA, Dec. 1983.

Hidi, S., Renninger, K. A., & Krapp, A. (2004). Interest, a motivational variable that combines affective and cognitive functioning. In D. Y. Dai & R. J. Sternberg (Eds.), *The educational psychology series. Motivation, emotion, and cognition: Integrative perspectives on intellectual functioning and development* (pp. 89-115). Mahwah, NJ, US: Lawrence Erlbaum Associates Publishers.

van Hiele, P. M. (1986). *Structure and Insight: A Theory of Mathematics Education*. New York: Academic Press.

Hildebrand, A. J. (2018). Undergraduate Research in Mathematics. Official website: https://faculty.math.illinois.edu/~hildebr/ugresearch/

Hitchman, T. J. (2015). Euclidean geometry: an introduction to mathematical work. Retrieved from: http://theronhitchman.github.io/euclidean-geometry/EG_main01.pdf

Hmelo-Silver, C. E., Duncan, R. G., & Chinn, C. A. (2007). Scaffolding and achievement in problem-based and inquiry learning: A response to Kirschner, Sweller, and Clark (2006). *Educational Psychologist* 42(2), 99-107.

Holodny, E. (2017). Isaac Newton was a genius, but even he lost millions in the stock market. *Business Insider*. Nov. 10, 2017. Retrieved from: https://www.businessinsider.com/isaac-newton-lost-a-fortune-on-englands-hottest-stock-2016-1

Hussain, M. A., Monaghan, J., & Threlfall, J. (2013). Teacher-student development in mathematics classrooms: Interrelated zones of free movement and promoted actions. *Educational Studies in Mathematics* 82(2), 285-302.

Ilyas, Mohammed. (2016). Language quotient (LQ): new models of language learning. *International Journal of Advanced and Applied Sciences* 3(9), 44-50.

Ivey, J. E. Jr. (1960). Matching Quantity with Reality in Education. *The Ohio State University Educational Research Bulletin XXXIX* (6). September 14, I960.

Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: Towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics* 54(2-3), 249-282.

Jaworski, B. (2006). Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design. In P. Liljedahl (Ed.), *Proceedings of the 2006 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3–16). Calgary, Canada: University of Calgary.

Johnson, J. N. (1976). The Far West Model. Experience-Based Career Education. Retrieved from: http://files.eric.ed.gov/fulltext/ED387629.pdf

Joubert, M., & Sutherland, R. (2009). *A Perspective on the Literature: CPC for Teachers of Mathematics*. London: National Centre for Excellence in the Teaching of Mathematics.

Kamiński, M. M. (2017). Backward induction: merits and flaws. *Studies in Logic, Grammar and Rhetoric* 50(63), 9-24.

Kim, D., & Ju, M. K. (2012). A changing trajectory of proof learning in the geometry inquiry classroom. *ZDM* 44(2), 149-160.

Kinder, D., & Carnine, D. (1991). Direct instruction: What it is and what it is becoming. *Journal of Behavioral Education* 1(2), 193-213.

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist* 41(2), 75-86.

Klahr, D. (2000). *Exploring Science: The Cognition and Development of Discovery Processes*. Cambridge, MA: MIT Press.

Klahr, D., Fay, A. L., & Dunbar, K. (1993). Heuristics for scientific experimentation: A developmental study. *Cognitive Psychology* 25, 111-146.

Klahr, D., & Nigam, M. (2004). The equivalence of learning paths in early science instruction: Effects of direct instruction and discovery learning. *Psychological Science* 15, 661–667.

Klein, F. (1932). *Elementary Mathematics from an Advanced Standpoint*. New York: Macmillan.

Knežević, M., & Krtinić, Đ. (2013). A note on infinite descent principle. *The Teaching of Mathematics* 16(2), 67–78.

Knoblock, J. (1990). *Xunzi: A Translation and Study of the Complete Works, Books 7-16*. Book 8. Chapter 11. Stanford University Press.

Koedinger , K. R. (1998). Conjecturing and argumentation in high school geometry students. In Lehrer, R. and Chazan, D. (Eds.) *New Directions in the Teaching and Learning of Geometry.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Kolb, D. A. (1984). *Experiential Learning: Experience as the Source of Learning and Development* (Vol. 1). Englewood Cliffs, NJ: Prentice-Hall.

Komatsu, K., & Tsujiyama, Y. (2013). Principles of task design to foster proofs and refutations in mathematical learning: Proof problem with diagram. In Margolinas, C. (Ed.), Task Design in Mathematics Education. *Proceedings of ICMI Study 22*. Oxford.

Krainer, K. (1993). Understanding students' understanding: On the importance of cooperation between teachers and researchers. In P. Boero (ed.), *Proceedings of the 3rd Bratislava International Symposium on Mathematical Teacher Education*. Comenius University Bratislava.

Krajcik, J., Blumenfeld, B., & Marx, R. Soloway. E. (2000). Instructional, curricular, and technological supports for inquiry in science classrooms. In J. *Minstrell* and E. van Zee (Eds.), *Inquiring into Inquiry Learning and Teaching in Science* (pp. 283-315). Washington, DC: American Association for the Advancement of Science.

Krapp, A., Hidi, S., Renninger, K. A., Valsiner, J., Deci, E. L., & Prenzel, M. (1992). General questions in the study of interest. In K. A. Renninger, S. Hidi, A. Krapp, A. Renninger (Eds.). *The Role of Interest in Learning and Development*. Hillsdale, NJ: Erlbaum.

Kuhn, D., Amsel, E., & O'Loughlin, M. (1988). *The Development of Scientific Thinking Skills*. New York: Academic Press.

Kuhn, D., Schauble, L., & Garcia-Mila, M. (1992). Cross-domain development of scientific reasoning. *Cognition and Instruction* 9(4), 285-327.

Kuhn, D., Garcia-Mila, M., Zohar, A., & Andersen, C. (1995). Strategies of knowledge acquisition. *Monographs of the Society for Research in Child Development* 60 *(4*, Serial No. 245).

Kuhn, D., Black, J., Keselman, A., & Kaplan, D. (2000). The development of cognitive skills to support inquiry learning. *Cognition and Instruction* 18(4), 495-523.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal* 27(1), 29-63.

de Landsheere V., de Landsheere G. (1975). *Définir les objectifs de l'éducation*. Paris: Presses Universitaires de France.

Laplace, P. S. (1902). *A Philosophical Essay on Probabilities*. New York: J. Wiley & Sons.

Lave, J., Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge University Press.

Lee, J. A. N. (1995). Computer pioneers. IEEE Computer Society. Retrieved from: http://history.computer.org/pioneers/von-neumann.html

Lee, S. K. (Ed.). (2008). *Toward a better future: Education and training for economic development in Singapore since 1965*. World Bank Publications.

Lehrer, R., & Chazan, D. (Eds.). (2012). *Designing Learning Environments for Developing Understanding of Geometry and Space*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Lehrer, R., Kobiela, M., & Weinberg, P. J. (2013). Cultivating inquiry about space in a middle school mathematics classroom. *ZDM* 45(3), 365-376.

Leighton, T. (2010 a). Mathematics for Computer Science. Lecture 2: Induction. MIT OpenCourseWare. Massachusetts Institute of Technology. Retrieved from: https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2010/video-lectures/lecture-2-induction/

Leighton, T. (2010 b). Mathematics for Computer Science. Lecture 7: Matching Problems. MIT OpenCourseWare. Massachusetts Institute of Technology. Retrieved from: https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-042j-mathematics-for-computer-science-fall-2010/video-lectures/lecture-7-matching-problems/

Lin, P. J., & Tsai, W. H. (2013). A Task Design for Conjecturing in Primary Classroom Contexts. In Margolinas, C. (Ed.), Task Design in Mathematics Education. *Proceedings of ICMI Study 22*. Oxford.

Maaß, K., Garcia, J., Mousoulides, N., & Wake, G. (2013). Designing Interdisciplinary Tasks in an International Design Community. In Margolinas, C. (Ed.). Task Design in Mathematics Education. *Proceedings of ICMI Study 22*. Oxford.

Maaß, K., & Artigue, M. (2013). Implementation of inquiry-based learning in day-to-day teaching: a synthesis. *ZDM* 45(6), 779-795.

Macrae, N. (1992). *John von Neumann: The Scientific Genius Who Pioneered the Modern Computer, Game Theory, Nuclear Deterrence, and Much More* (p.32). Pantheon Press.

Mahavier, W. T. (1997). A gentle discovery method: (the modified Texas method). *College Teaching* 45(4), 132-135.

Mahavier, L. (1999). On Three Crucial Elements of Texas-Style Teaching as Shown to be Successful in the Secondary Mathematics Classroom. *The Legacy of RL Moore Project*. Retrieved from: http://legacyrlmoore.org/

Makar, K. (2014). Young children's explorations of average through informal inferential reasoning. *Educational Studies in Mathematics* 86(1), 61-78.

Mammarella, I. C., Giofrè, D., & Cornoldi, C. (2014). The relationship between geometry, working memory and intelligence in children. *Journal of Experimental Child Psychology* 123, 112–128.

Mammarella, I. C., Giofrè, D., & Caviola, S. (2017). Learning Geometry: the Development of Geometrical Concepts and the Role of Cognitive Processes. In D. Geary, D. Berch, R. Ochsendorf, K. Mann Koepke (Eds.) (pp. 221-246). *Acquisition of Complex Arithmetic Skills and Higher-Order Mathematics Concepts* . Cambridge, Massachusetts: Academic Press.

Manin, Y. I. (1981). A digression on proof. *Two-Year College Mathematics Journal* 12(2), 104-107.

Mayer, R. E. (2001). *Multimedia Learning*. New York: Cambridge University Press.

Mayer, R. E. (2004). Should there be a three-strikes rule against pure discovery learning? *American Psychologist* 59, 14–19.

McDaniel, M. A., & Schlager, M. S. (1990). Discovery learning and transfer of problem-solving skills. *Cognition and Instruction* 7(2), 129-159.

McLoughlin, M. P. M. M. (2008). Inquiry Based Learning: A Modified Moore Method Approach To Encourage Student Research. Retrieved from: https://files.eric.ed.gov/fulltext/ED502664.pdf

McLoughlin, M. P. M. M. (2012). *On doing mathematics: Why we should not encourage 'feeling', 'believing', or 'interpreting' mathematics*. Paper presented at the annual meeting of the American Mathematical Society. January 7, 2012. Boston, MA.

MELS (2007). Gouvernement du Québec. Ministère de l'Éducation, du Loisir et du Sport. Situations d'apprentissage et d'évaluation. Intégration linguistique, scolaire et sociale. Retrieved from: http://www.education.gouv.qc.ca/fileadmin/site_web/documents/education/jeunes/pfeq/RPD_PFEQ_ILSS-sec_SAE.pdf

Metcalf, L. E. (1963). Research on teaching the Social studies. Gage, N. L. (Ed.) *Handbook on Research on Teaching*. Chicago: Rand McNally and Co.

Middleton, J. A., & Corbett, R. (1998). Sixth-grade students' conceptions of stability in engineering contexts. In Lehrer, R., & Chazan, D. (Eds.). (2012). *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 249-265). Hillsdale, NJ: Lawrence Erlbaum Associates.

Ministry of Education Ontario (2013). Inquiry-based Learning. Capacity Building Series. Retrieved from: https://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_InquiryBased.pdf

Mitchell, M. (1993). Situational interest: Its multifaceted structure in the secondary school mathematics classroom. *Journal of Educational Psychology* 85(3), 424.

Moise, E. E. (1975). The meaning of Euclidean geometry in school mathematics. *The Mathematics Teacher* 68(6), 472-477.

Moore, E. H. (1903). On the Foundations of Mathematics. *Bulletin of the American Mathematical Society*. IX, May 1903.

von Neumann, J., & Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton, NJ: Princeton University Press.

Newton, J. D. (1987). *Uncommon Friends: Life with Thomas Edison, Henry Ford, Harvey Firestone, Alexis Carrel & Charles Lindbergh*. Toronto: Thomson Learning.

Neyland, J. (2004). Toward a postmodern ethics of mathematics education. In M. Walshaw (Ed.), *Mathematics education within the postmodern* (pp. 55-73). Charlotte, North Carolina: Information Age Publishing.

Nicol, C. (1998). Learning to teach mathematics: Questioning, listening, and responding. *Educational Studies in Mathematics* 37(1), 45-66.

Niven, I. (1947). A simple proof that π is irrational. *Bulletin of the AMS* 53 (1947), 509.

Novak, J. D. (1973). Inquiry, Discovery Learning, and Problem Solving. In: A Summary of Research in Science Education - 1972. Retrieved from: https://trove.nla.gov.au/work/152017846?q&versionId=165706191

NRC (1996). *Inquiry. Principles and Definitions. National Science Education Standards.* Retrieved from: http://www.nap.edu/openbook.php?record_id=4962&page=23

NRC (2000). *Inquiry and the National Science Education Standards. A Guide for Teaching and Learning.* Washington, DC: National Academy Press.

OECD (1998). *Staying ahead: In-service training and teacher professional development*. Paris: OECD.

OECD (2009). *Creating Effective Teaching and Learning Environments. First Results from TALIS*. Retrieved from: www.oecd.org/education/preschoolandschool/43023606.pdf

OISE (2011). *Natural Curiosity: Building Children's Understanding of the World through Environmental Inquiry. A resource for teachers*. University of Toronto. Retrieved from: www.naturalcuriosity.ca/pdf/NaturalCuriosityManual.pdf

Olive, J. (1998). Opportunities to explore and integrate mathematics with the Geometer's Sketchpad. In Lehrer, R., & Chazan, D. (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 395-418). Hillsdale, NJ: Lawrence Erlbaum Associates.

Page, D. P. (1847). *Theory and Practice of Teaching, or, The Motives and Methods of Good School-keeping*. Syracuse: Hall & Dickson.

Palmer, J. R. (1969). Text Book Materials and Social Inquiry: Problems and Prospects. *The High School Journal* 53(2), 112-121.

Paoletti, G. (2018). *This Lightbulb Was Switched On In 1901 - And It's Still Going*. Retrieved from: https://allthatsinteresting.com/centennial-bulb

Papert, S. A., Harel, I. (1990). *Introduction: Constructionist Learning*. Cambridge, MA: MIT Media Laboratory.

Pegg, J., Davey, G. (1998). Interpreting student understanding in geometry: a synthesis of two models. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 109-135). Mahwah, NJ: Erlbaum.

Peirce, C. S. (1976). *The New Elements of Mathematics. Vol. 4. Mathematical Philosophy*. Originally published in 1902. Reprint by: Mouton Publishers & Humanities Press.

Piaget, J. (1972 a). Some Aspects of Operations. In M. W. Piers, L. B. Murphy, E. H. Erikson, R. A. Spitz, K. Lorenz (Eds.) *Play and Development: A Symposium with Contributions by Jean Piaget, Peter H. Wolff and Others* (pp. 15-27). New York: W. W. Norton.

Piaget, J. (1972 b). *To Understand Is To Invent*. New York: The Viking Press, Inc.

Plutarch, L. M. (1927). *Moralia, volume 1. On Listening to Lectures*. Loeb Classical Library. Harvard University Press. Retrieved from: http://penelope.uchicago.edu/Thayer/E/Roman/Texts/Plutarch/Moralia/De_auditu*.html

POGIL (2015). *Process Oriented Guided Inquiry Learning*. Retrieved from: https://pogil.org/about

Poincaré, H. (1914). *Science and Method.* Nashville, Tennessee: Thomas Nelson and Sons.

Poincaré, H. (1920). Dernières pensées. Paris : Flammarion.

Pollard, A. (1997). *Reflective Teaching in the Primary School: A Handbook for the Classroom* (third edition). London: Cassell.

Pólya, G. (1954). *Mathematics and Plausible Reasoning*. Volume 1: Induction and Analogy in Mathematics. Volume 2: Patterns of Plausible Inference. Princeton, NJ: Princeton University Press.

Pólya. G. (1957). *How to Solve It*. New York: Doubleday Anchor Books.

Pólya, G. (1981). *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving.* Combined edition. Originally published: 1962 (vol. 1) and 1965 (vol. 2). New York: Wiley.

Ponte, J. P., Matos, J. F., Guimaraes, H. M., Leal, L. C., & Canavarro, A. P. (1994). Teachers' and students' views and attitudes towards a new mathematics curriculum: a case study. *Educational Studies in Mathematics* 26, 347–365.

Ponte, J. P., Segurado, M. I., & Oliveira, H. (2003). A collaborative project using narratives: what happens when pupils work on mathematical investigations? In Collaboration in Teacher Education: Examples from the context of mathematics education. Dordrecht: Kluwer. Retrieved from: http://repositorio.ul.pt/bitstream/10451/4164/1/03-Ponte-Segurado-Oliveira.pdf

Ponte, J. P. (2008). Research and practice: bridging the gap or changing the focus? In M. Menghini, F. Furinghetti, L. Giarcardi, & F. Arzarella (Eds.), *The First Century of the International Commission on Mathematics Instruction. Reflecting and Shaping the World of Mathematics Education* (pp. 1908–2008). Roma: Instituto della Enciclopedia Italiana foundata da Giovanni Treccani.

Popper, K. (1963). *Conjectures and Refutations* (Vol. 7). London: Routledge and Kegan Paul.

Posamentier, A. S., Krulik, S. (2015). *Problem-solving Strategies in Mathematics: From Common Approaches to Exemplary Strategies*. Book 1. Singapore: World Scientific Publishing Co.

Prawat, R. S. (1989). Promoting access to knowledge, strategy, and disposition in students: A research synthesis. *Review of Educational Research* 59(1), 1-41.

Proclus (1970). *A Commentary on the First Book of Euclid's Elements*. Translated with Introduction and Notes by G. R. Morrow. Princeton University Press.

Raghavan, K., Sartoris, M. L., & Glaser, R. (1998). Interconnecting science and mathematics concepts. In Lehrer, R., & Chazan, D. (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 267-295). Hillsdale, NJ: Lawrence Erlbaum Associates.

Renninger, K. A., Weimar, S. A., & Klotz, E. A. (1998). and Communicating About Geometry: The Math Forum. In Lehrer, R., & Chazan, D. (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space* (p. 465). Hillsdale, NJ: Lawrence Erlbaum Associates.

Renz, P. (1981). Mathematical proof: What it is and what it ought to be. *Two-Year College Mathematics Journal* 12(2), 83-103.

Rider, Vanessa (n.d.). Mount Vernon City School District. Bloom's revised taxonomy. Retrieved from: http://p2cdn4static.sharpschool.com/UserFiles/Servers/Server_87286/Image/Vridder/Staff/BloomRevisedTaxonomy.jpg

Rogers, P. (1990). 'Discovery', learning, critical thinking, and the nature of knowledge. *British Journal of Educational Studies* 38(1), 3-14.

Rusu, E. (1970). *Matematica în liceu - probleme de metodică*. Bucureşti: Editura Didactică şi Pedagogică.

Rusu, E. (1972). *Cum gîndim şi rezolvăm 200 de probleme*. Bucureşti: Editura Albatros.

Ryan, R. M., & Deci, E. L. (2000). Self-determination theory and the facilitation of intrinsic motivation, social development, and well-being. American psychologist, 55(1), 68.

Salazar, D. A. (2012). Enhanced-Group Moore Method: Effects on van Hiele Levels of Geometric Understanding, Proof-Construction Performance and Beliefs. *Online Submission*. Retrieved from: https://files.eric.ed.gov/fulltext/ED535502.pdf

Salim, K. & Tiawa D.H. (2015). Implementation of Structured Inquiry Based Model Learning Toward Students' Understanding of Geometry. *International Journal of Research in Education and Science* 1(1), 75-83.

Sanni, R. (2007). Teaching geometry in schools: an investigative rather than instructive process. *Pythagoras* 65, 39-44.

Schauble, L. (1990). Belief revision in children: The role of prior knowledge and strategies for generating evidence. *Journal of Experimental Child Psychology* 49, 31-57.

Schauble, L. (1996). The development of scientific reasoning in knowledge-rich contexts. *Developmental Psychology* 32, 102-119.

Schell, J. (2008). *The Art of Game Design: A Book of Lenses*. San Francisco: Morgan Kaufmann Publishers.

Schiefele, H., Hausser, K., & Schneider, G. (1979). „Interesse“ als Ziel und Weg der Erziehung. Überlegungen zu einem vernachlässigten pädagogischen Konzept. *Zeitschrift für Pädagogik* 25, 1-20.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of'well-taught'mathematics courses. *Educational Psychologist* 23(2), 145-166.

SEAMEO (2015). High-Level Policy Forum on IBSE. Quality Improvement of Teachers and Education Personnel (QITEP). Southeast Asian Ministers of Education Organization (SEAMEO). Retrieved from: http://www.qitepscience.org/?p=4452

Serra, M. (2008). *Discovering Geometry: An Investigative Approach* (4th edition). Emeryville, CA: Key Curriculum Press.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics* 22(1), 1-36.

Shaffer, D. (1995). *Exploring trigonometry with the Geometer's Sketchpad*. Emeryville, CA: Key Curriculum Press.

Shulman, L. S., & Keislar, E. R. (Eds.). (1966). *Learning by Discovery: A Critical Appraisal*. Chicago: Rand McNally.

Sjøberg, S. (2017). PISA testing – a global educational race? *Europhysics News* 48(4):17-20.

Skovsmose, O. (2001). Landscapes of Investigation. *ZDM* 33(4), 123-132.

Smith, T. M., Desimone, L. M., Zeidner, T. L., Dunn, A. C., Bhatt, M., & Rumyantseva, N. L. (2007). Inquiry-oriented instruction in science: Who teaches that way?. *Educational Evaluation and Policy Analysis* 29(3), 169-199.

Spronken-Smith, R. (2007). Experiencing the process of knowledge creation: The nature and use of inquiry-based learning in higher education. *Journal of Geography in Higher Education* 2, 183.

Stake, R. E. (1995). *The Art of Case Study Research*. Thousand Oaks, CA: Sage Publications.

Staver, J. R., & Bay, M. (1987). Analysis of the project synthesis goal cluster orientation and inquiry emphasis of elementary science textbooks. *Journal of Research in Science Teaching* 24(7), 629-643.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning* 10, 313–340.

Stewart, D. (1993). Teaching or facilitating: A false dichotomy. *Canadian Journal of Education/Revue canadienne de l'education* 18(1), 1-13.

Sugrue, B. (2002). Problems with Bloom's taxonomy. Retrieved from: https://eppicinc.files.wordpress.com/2011/08/sugrue_bloom_critique_perfxprs.pdf

Sweller, J (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science* 12(2), 257–285.

Sweller, J. (1999). *Instructional Design in Technical Areas*. Camberwell, Australia: Australian Council for Educational Research.

Taconis, R., Ferguson-Hessler, M. G., & Broekkamp, H. (2001). Teaching science problem solving: An overview of experimental work. *Journal of Research in Science Teaching* 38(4), 442-468.

Tall, D. (1991). Intuition and rigour: the role of visualization in the calculus. *Visualization in teaching and learning mathematics*. Retrieved from: https://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot1991a-int-rigour-maa.pdf

Tall, D., Thomas, M., Davis, G., Gray, E., Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior* 18(2), 223-241.

Tardif, J. (1992). *Pour un Enseignement Stratégique: L'apport de la Psychologie Cognitive*. Les Éditions Logiques.

Taton, R. (1947). Les relations d'Évariste Galois avec les mathématiciens. *Revue d'histoire des sciences* 1-2, 114-130.

Tirosh, D., & Graeber, A. O. (2003). Challenging and changing mathematics teaching practices. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second International Handbook of Mathematics Education* (pp. 643–688). Dordrecht: Kluwer.

True, G. H. (1978). Not Funny, Reader Says. In E. Wilson (Ed.) *Morning News*. (p.14B, col.2). Dallas: 29 March 1978.

Tudor, I. (2009). Generalizarea unei probleme a profesorului Eugen Rusu. *Gazeta Matematica* Seria B, Vol.CXIV (5), mai 2009.

de Villiers, M. (1998). An alternative approach to proof in dynamic geometry. In Lehrer, R., & Chazan, D. (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 369-394). Hillsdale, NJ: Lawrence Erlbaum Associates.

Walkerdine, V. (1988). *The Mastery of Reason: Cognitive Development and the Production of Rationality*. London: Routledge.

Walls, F. (2005). Challenging task-driven pedagogies of mathematics. *Proceedings of the 28th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 751-758).

Wenger, E. (1998). *Communities of Practice: Learning, Meaning and Identity*. Cambridge: Cambridge University Press.

Whiteside, D. T. (Editor) (1974). *The Mathematical Papers of Isaac Newton.* Volume VI, 1684-1691. Cambridge: Cambridge University Press.

Willoughby, S. (1963). Discovery. *The Mathematics Teacher* LVI, January 1963, 22-25.

Wong, D. J. S., Lau, C. Y., & Lee, P. C. K. (2012). *PbI@ School: A Large-scale Study on the Effect of "Physics by Inquiry" Pedagogy on Secondary 1 Students' Attitude and Aptitude in Science*. Retrieved from: https://www.nie.edu.sg/files/NIE_research_brief_12-006.pdf

Woolnough, B. (2000). Appropriate practical work for school science: Making it practical and making it science. In J. *Minstrell* and E. van Zee (Eds.), *Inquiry into Inquiry Learning and Teaching in Science* (pp. 434-446). Washington, DC: American Association for the Advancement of Science.

Yeo, J. B. (2013). *The nature and development of processes in mathematical invest*igation. Singapore: National Institute of Education, Nanyang Technological University.

Yeo, J. B. (2017). Specialising and conjecturing in mathematical investigation. *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (pp. 337-344). Singapore: PME.

Young, A. E. (1909). On the Teaching of Analytic Geometry. *The American Mathematical Monthly* 16(12), 205-212.

Zacharie, M. (2009). Why College or University Students Hate Proofs in Mathematics?. *Journal of Mathematics and Statistics* 5(1), 32.

Zack, V., & Graves, B. (2002). Making mathematical meaning through dialogue:"Once you think of it, the Z minus three seems pretty weird". In C. Kieran, E. A. Forman, A. Sfard (Eds.) *Learning Discourse: Discursive approaches to research in mathematics education* (pp. 229-271). New York: Springer.

Zech, L., Vye, N. J., Bransford, J. D., Goldman, S. R., Barron, B. J., Schwartz, D. L., Kisst-Hackett, R., & Mayfield-Stewart, C. (1998). An introduction to geometry through anchored instruction. In Lehrer, R., & Chazan, D. (Eds.) *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 439-463). Hillsdale, NJ: Lawrence Erlbaum Associates.

Zitarelli, D. E. (2004). The origin and early impact of the Moore Method. *American Mathematical Monthly* 111(6), 465-486.