On Certain Techniques in Convex Geometry

Mariam AlHilani

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By: Mariam AlHilani

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Signed by the final examining committee:

____________________ Cody Hyndman (Chair)

____________________ Ronald Stern (Examiner)

____________________ Harald Walter Proppe (Examiner)

____________________ Alina Stancu (Supervisor)

Approved by ____

Arusharka Sen (Graduate Program Director)

____________ 2018 ____

André Roy (Dean of Faculty of Arts and Science)

ABSTRACT

On Certain Techniques in Convex Geometry

Mariam AlHilani

We recall a proof of Mahler's conjecture in $\mathbb{R}^2$ and the technique employed to prove it. This technique shows that, by adding new vertices to a convex polygon K, one increases the value of Mahler's functional $K \mapsto V(K) \cdot V(K^*)$, thus the minimum of the functional is reached for the convex polygon with least number of vertices. We then study similar techniques in connection to Petty's conjecture in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Petty's conjecture states that the functional $K \mapsto \dfrac{V(\Pi K)}{V^{n-1}(K)}$, where K is a compact convex set in $\mathbb{R}^n$, reaches the maximum for the polytope of least vertices in $\mathbb{R}^n$. In $\mathbb{R}^2$, we prove that the inequality holds for any convex body K by a similar technique with that of Mahler's problem, which is different from the original proof of Petty's inequality in $\mathbb{R}^2$. In $\mathbb{R}^3$, we validate the conjecture for a few specific cases. More precisely, we compare the value of Petty's functional of a convex body K in $\mathbb{R}^n$, for $n = 2$ and $n = 3$, with that of another convex body K' that is obtained by cutting off a vertex of K with a plane, thus introducing more vertices. However, we provide an example that shows that this technique cannot be applied to arbitrary polytopes in $\mathbb{R}^3$ to prove Petty's conjecture in this class and then, by approximation, in general.

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Contents

Chapter 1

Introduction and Prerequisites

1.1 Introduction

We start with a brief introduction on convex geometry and summarize the prerequisites needed for the rest of the thesis such as convex hull, duality, projection bodies and other selected topics used in our study. We focus our attention on two famous extremal problems: Mahler's conjecture and Petty's conjecture that we state at the end of the first chapter. Both of these problems conjecture certain inequalities to hold for all convex bodies in $\mathbb{R}^n$ and claim that the equality cases is reached for simplices. Simplices are the simplest polytopes in any Euclidean space. These conjectures have been proved only in $\mathbb{R}^2$ and they are open in the general case.

In Chapter 2, we examine the proof of the planar Mahler's conjecture following the techniques used by Mahler himself for the symmetric case in dimension 2, [5], as presented by Henze [4]. Our main objective is recalling the proof (different other proofs were given later) that shows that by adding new vertices to a polytope, the value of Mahler's functional $V(K) \cdot V(K^*)$ increases, therefore the simplest polytope, the simplex, has the minimal Mahler's functional.

In the next chapters, we examine some problems related to Petty's conjecture using similar techniques inspired by Mahler's proof. In Chapter 3, we consider Petty's conjecture in $\mathbb{R}^2$ and we give a proof by induction to the inequality $\dfrac{V(\Pi K)}{V(K)} \leq \dfrac{V(\Pi T)}{V(T)}$, where $T \subset \mathbb{R}^2$ is a triangle. In Chapter 4, we study using the same techniques for the upper bound of $\dfrac{V(\Pi K)}{V^2(K)}$ for any convex body $K \in \mathbb{R}^3$ and we prove that the inequality holds for few specific cases of convex bodies.

Finally, in Chapter 5, we compare the value of Petty's functional, $\dfrac{V(\Pi K)}{V^{n-1}(K)}$, for $n = 2$ and $n = 3$, of a convex body K in $\mathbb{R}^n$ with that of another convex body K' that is obtained by cutting a vertex of K. In the last part of this chapter, we provide a counterexample that this technique can be applied to arbitrary polytopes in $\mathbb{R}^3$ to prove Petty's conjecture in this class and then, by approximation, in general.

1.2 Convex Bodies, Minkowski Sum of Convex Bodies

Throughout the thesis, the ambient space is the real vector space $\mathbb{R}^n$.

Definition 1.2.1. *[9] A set $K \subset \mathbb{R}^n$ is convex if for any two points x and $y \in K$, the line segment*

$$(1-\lambda)\, x + \lambda\, y \in K, \quad \forall \lambda \in [0,1], \tag{1.1}$$

belongs to K.

Half spaces, ellipses with their interior, and triangles, with their interior are examples of convex sets. A convex set and, respectively, a non-convex set are illustrated in Figure 1.1.

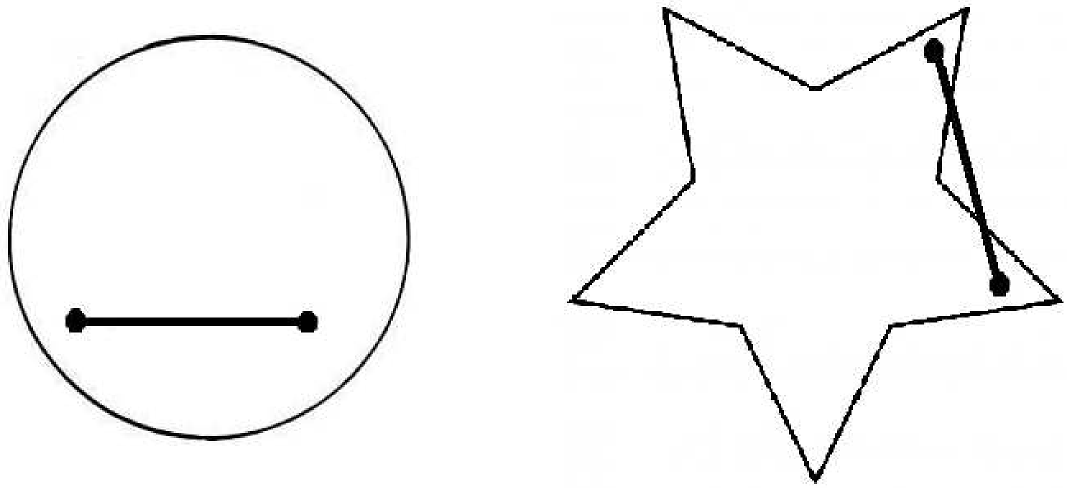

Figure 1.1: convex and non-convex body

Example 1.2.1. *A half-space M is defined by $w \cdot x \leq \alpha$, where w is a fixed vector, and α is a fixed real number. Let $x, y \in M$ and let $\lambda, \beta \geq 0$ such that $\lambda + \beta = 1$, arbitrary otherwise. Thus, $w \cdot x \leq \alpha$ and $w \cdot y \leq \alpha$ and so*

$$w \cdot (\lambda x + \beta y) = \lambda w \cdot x + \beta w \cdot y \leq \lambda \alpha + \beta \alpha = \alpha \tag{1.2}$$

Then, $\lambda x + \beta y \in M$ and M is convex.

A convex set can be constructed from a set of arbitrary points by taking their convex hull.

Definition 1.2.2. *[9] A point x is said to be a convex combination of $x_1, \ldots, x_p$ if there exists $\lambda_1, \ldots, \lambda_p$ with $\lambda_1 + \ldots + \lambda_p = 1$ and $\lambda_i \geq 0$, $i = 1, \ldots, p$, such that*

$$x = \lambda\, x_1 + \ldots + \lambda_p\, x_p. \tag{1.3}$$

Definition 1.2.3. *[9] For an arbitrary set $K \subset \mathbb{R}^n$, the set of all convex combinations* *of any finitely many elements of K is called the convex hull of K and is denoted by $conv\,K$.*

Theorem 1.2.1. *[9] The convex hull of the points $x_1, x_2, \ldots, x_p$ is the set of points of the form*

$$x = \lambda\, x_1 + \ldots + \lambda_p\, x_p \quad \textit{where} \quad \lambda_1 + \ldots + \lambda_p = 1,\ \lambda_i \geq 0,\ i = 1, \ldots, p. \tag{1.4}$$

Corollary 1.2.1. *Let $x_1, \ldots, x_p \in \mathbb{R}^n$. Then,*

$$conv\,(x_1, \ldots, x_p) = \{\lambda_1 x_1 + \ldots + \lambda_p x_p \mid \lambda_1 + \ldots + \lambda_p = 1, \lambda_i \geq 0,\ i = 1, \ldots, p\}. \tag{1.5}$$

A set T and conv$\,T$ are illustrated in Figure 1.2

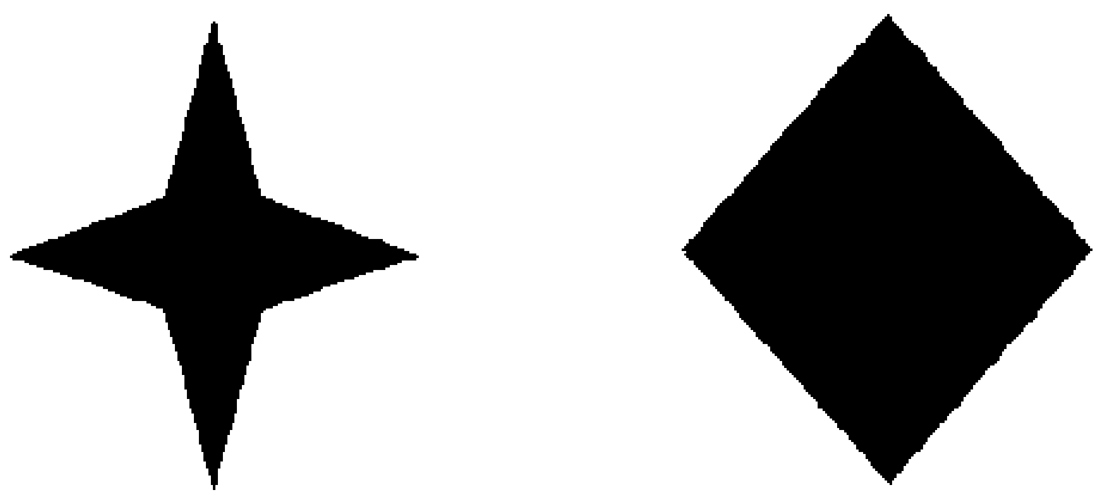

Figure 1.2: T and conv T

Theorem 1.2.2. *[9] Let $x_1, x_2, \ldots, x_p$ be points of K, a convex body in $\mathbb{R}^n$. Let $\lambda_i \geq 0$, $i = 1, \ldots, p$, such that $\lambda_1 + \ldots + \lambda_p = 1$ be arbitrary otherwise. Then*

$\lambda_1 x_1 + \ldots + \lambda_p x_p \in K$.

Proof. We prove the theorem by induction. For $p = 1$, the claim is trivial. Suppose it is true for some positive integer k, we will prove that it is also true for $k+1$. let $y = \lambda_1 x_1 + \ldots + \lambda_{k_1} x_{k+1}$ where $x_1, x_2, \ldots, x_{k+1} \in K$ and $\lambda_1 + \ldots + \lambda_{k+1} = 1$. At least one λ_i should be < 1, let's assume that $\lambda_{k+1} < 1$. Let

$$z = \frac{\Lambda_1}{\lambda} \cdot x_1 + \ldots + \frac{\Lambda_k}{\lambda} \cdot x_k \tag{1.6}$$

where $\lambda = \lambda_1 + \ldots + \lambda_k = 1 - \lambda_{k+1} > 0$.

By the hypothesis of induction, $z \in K$ and, since K is convex and contains z and x_{k+1}, we get that the equality $y = \lambda z + \lambda_{k+1} x_{k+1}$ implies $y \in K$. □

Proposition 1.2.1. *If $K \subset \mathbb{R}^n$ is convex, then $\mathrm{conv}\, K = K$.*

Definition 1.2.4. *[8] For any convex sets $K, L \subset \mathbb{R}^n$, the Minkowski sum of K and L is the convex set obtained by vector addition:*

$$K + L := \{k + l \mid k \in K, l \in L\}. \tag{1.7}$$

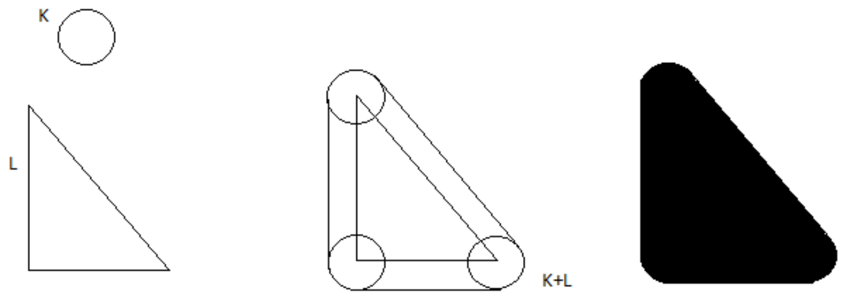

Figure 1.3: Minkowsi sum of K and L

Definition 1.2.5. *[8] Let K be a convex compact set in $\mathbb{R}^n$. The support function of K, $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, is defined by*

$$h_K(v) = \sup\{v \cdot x \mid x \in K\}. \tag{1.8}$$

Note that if K contains the origin in its interior, then h_K is positive for all directions v, $h_K(v)$ being the distance from the origin to the hyperplane of normal v supporting K. Moreover, a convex body K is completely determined by its support function. Lastly, note that $h_{K+L}(v) = h_K(v) + h_L(v)$ for all $v \in \mathbb{S}^{n-1}$.

Proposition 1.2.2. *[3] If $\phi \in GL_n$ where GL_n is a non-singular linear transformations, then $h_{\phi K}(v) = ||\phi^t v||\, h_K(\phi^t v)$.*

Proof.

$$\begin{aligned} h_{\phi K}(v) &= \sup\{v \cdot x \mid x \in \phi K\} \\ &= \sup\{v \cdot \phi y \mid y \in K\} \\ &= \sup\{\phi^t v \cdot y \mid y \in K\} \\ &= ||\phi^t v||\, h_K(\phi^t v/||\phi^t v||). \end{aligned} \tag{1.9}$$

□

The support function can be in fact extended to $\mathbb{R}^n \setminus \{0\}$ by homogeneity via the formula $h_K(x) = h_K(x/||x||)$. We will see this formula again later.

Definition 1.2.6. *[8] A polytope $P \subset \mathbb{R}^n$ is the convex hull of a finite subset of $\mathbb{R}^n$. Let P be a nonempty polytope in $\mathbb{R}^n$, then $P = conv\{x_1, \ldots, x_m\}$ whenre $x_1, \ldots, x_m$ are points in $\mathbb{R}^n$.*

Theorem 1.2.3. *[8] Every polytope $P \subset \mathbb{R}^n$ is the intersection of finitely many closed half-spaces.*

Proof. Let $\mathbb{P}^n$ be the set of polytopes with nonempty interior and let $P \in \mathbb{P}^n$. We can assume that dim $P = n$ because flats and half-flats can be described as intersections of finitely many closed half spaces. We denote by $F_1, \ldots, F_m$ the faces of P and thus $F_i = H_i \cap P$ where H_i is a support plane of P. We also denote by H_i^- the closed half-space bounded by H_i for $i = 1, \ldots, m$ and containing P. Therefore, in order to prove the theorem, we will have to prove

$$P = H_1^- \cap \ldots \cap H_m^-. \tag{1.10}$$

The first inclusion $P \subset H_1^- \cap \ldots \cap H_m^-$ is trivial.

For the other inclusion, let $x \in \mathbb{R}^n \setminus P$ and let M be the union of the affine hulls of the $n-1$ vertices of P and x. We choose y such that $y \subset int\,(P) \backslash M$. Then, there exists a point z such that $z \in bd\,P \cap\ [x, y]$. In other words, z is in a support plane of P which is equivalent to saying that z lies in some face F of P. Suppose that dim $F =: k \leq n-2$. Caratheodory's convex hull shows that z is in the convex hull of some $k+1 \leq n-1$ vertices of P and thus to M. But then we have $y \in M$ which is a contradiction. This shows that F is a facet and $F = F_i$ for $i = 1, \ldots, m$. x doesn't belong to H_i^- since $y \in H_i^-$. Equation (1.10) is thus proven. □

Let $\mathbb{K}^n$ be the set of nonempty compact, convex subsets of $\mathbb{R}^n$. In fact, we consider $\mathbb{K}^n$ the set of compact, convex subsets of $\mathbb{R}^n$ with nonempty interior, as if the interior of K is the empty set, then as $K \neq \emptyset$, then $K \in \mathbb{K}^l$ with $1 \leq l \leq n-1$.

Definition 1.2.7. *[8] The Hausdorff distance between two convex sets L and K in $\mathbb{K}^n$ is defined by*

$$\delta(L, K) := \max\{\sup_{x\in L}\inf_{y\in K}\ |x-y|,\quad \sup_{x\in K}\inf_{y\in L}\ |x-y|\}. \tag{1.11}$$

Alternatively, it can be defined by

$$\delta(L, K) = \min\{\lambda \geq 0 \mid L \subseteq K + \lambda B^n\}. \tag{1.12}$$

One can easily check that δ is a metric on $\mathbb{K}^n$ which is called the Hausdorff metric.

Theorem 1.2.4. *[8] Let $\epsilon > 0$. Then, for any $K \in \mathbb{K}^n$ there exists a polytope $P \in \mathbb{K}^n$ such that $P \subset K \subset P + \epsilon B$ thus, consequently, $\delta(P, K) \leq \epsilon$.*

Proof. Let B_i be the the balls with radius ϵ, and with their centers in K, that cover K. By the definition of polytopes, we can find a polytope P that is the convex hull of the centers of B_i. It is easy to see that this polytope P has the property claimed by the theorem. □

Corollary 1.2.2. *[8] For any $K \in \mathbb{K}^n$ there exists a sequence of polytopes, P_i, converging to K in the Hausdorff metric.*

Proof. According to Theorem 1.2.4 we can find a sequence of polytopes, P_n, such that $P_n \subset K \subset P_n + \epsilon B$ by taking, successively, $\epsilon = \frac{1}{n}$. $\square$

Lemma 1.2.1. *[8] Let $K_1, K_2 \in \mathbb{K}^n$ and let $K_2 \subset int\, K_1$. Then, there exists a number η such that for any $K \in \mathbb{K}^n$ with $\delta(K_1, K) < \eta$ satisfies the fact that $K_2 \subset K$.*

Proof. We have $K_2 \subset int\, K_1$, thus the function $h_{K_1}(\,.\,) - h_{K_2}(\,.\,)$ is positive on $\mathbb{R}^n \setminus \{0\}$ and, consequently, since the function is continuous on $\mathbb{S}^{n-1}$ (compact), it attains a minimum, η, that is positive on $\mathbb{S}^{n-1}$. Now, let $K \in \mathbb{K}^n$ be such that $\delta(K_1, K) < \eta$. Thus, $|h_{K_1}(u) - h_{K_2}(u)| \leq \eta$, $\forall u \in \mathbb{S}^{n-1}$. Then $h_{K_2}(u) \leq h_{K_1}(u) - \eta < h_K(u)$ where $u \in \mathbb{S}^{n-1}$ and, finally, $K_2 \subset K$. $\square$

Theorem 1.2.5. *[8] The volume functional, V_n, is continuous on $\mathbb{K}^n$ with respect to Hausdorff metric.*

Proof. Let $K \in \mathbb{K}^n$ and let $\bar{K} \in \mathbb{K}^n$. Without loss of generality, if $V_n(K) = 0$ satisfies $\delta(K, \bar{K}) = \beta \leq 1$, then K is contained in a hyperplane and $\bar{K} \subset K + \beta B$. Thus, $V_n(\bar{K}) \leq V_n(K + \beta B_n) \leq C(K) \cdot \beta$, and using Fubini's theorem we can find $C(K)$ such that $C(K)$ is independent of β. Now, we suppose that $0 \in int\, K$. Let $\epsilon > 0$, we choose $\lambda > 1$ such that $(\lambda^n - 1) \cdot \lambda^n \cdot V_n(K) < \epsilon$ and $\sigma > 0$ such that $\sigma B_n \subset int\, K$. According to Lemma 1.2.1, we can find a number $\beta > 0$ such that $\beta \leq (\lambda - 1)\sigma$ and such that $\sigma B_n \subset \bar{K}$ for any $\bar{K} \in \mathbb{K}^n$ while satisfying the fact that $\delta(k, \bar{K}) < \beta$. Assuming that the latter is true, we have

$$K \in \bar{K} + \beta B_n \subset \bar{K} + (\lambda - 1)\,\sigma B_n \subset \bar{K} + (\lambda - 1)\bar{K} = \lambda \bar{K}. \tag{1.13}$$

Also, $\bar{K} \in \lambda K$. Then,

$$V_n(K) \leq V_n(\bar{K}) = \lambda^n V_n(\bar{K}). \tag{1.14}$$

Thus,

$$\begin{aligned} V_n(K) - V_n(\bar{K}) &\le (\lambda^n - 1)V_n(\bar{K}) \le (\lambda^n - 1)\lambda^n V_n(K), \\ V_n(\bar{K}) - V_n(K) &\le (\lambda^n - 1)V_n(K) \le (\lambda^n - 1)\lambda^n V_n(K). \end{aligned} \tag{1.15}$$

Therefore,

$$|V_n(K) - V_n(\bar{K})| \le (\lambda^n - 1)\lambda^n V_n(K) \le \epsilon, \tag{1.16}$$

concluding the proof. □

For simplicity, in our thesis, we omit the index n in V_n unless there is a risk of confusion.

Corollary 1.2.3. *Let K be a nonempty compact convex set in $\mathbb{R}^n$. Then there exist a sequence of nonempty polytopes P_i in $\mathbb{R}^n$, and another sequence of nonempty polytopes Q_i in $\mathbb{R}^n$, $i \in \mathbb{N}$, such that $P_i \subseteq K \subseteq Q_i$ and $P_i \to K$ and $Q_i \to K$ in the Hausdorff metric.*

Corollary 1.2.4. *Let K be a nonempty compact convex set in $\mathbb{R}^n$, then there exist in $\mathbb{R}^n$ sequences of nonempty polytopes P_i and Q_i, $i \in \mathbb{N}$, such that $P_i \subseteq K \subseteq Q_i$, $\Pi P_i \to \Pi K$ and $\Pi Q_i \to \Pi K$ in the Hausdorff metric.*

1.3 Special Convex Bodies: the Polar and the Projection Body of K

Definition 1.3.1. *[9] Let K be a set in $\mathbb{R}^n$ containing the origin. The polar or dual, K^*, of the set K is defined by*

$$K^* = \{x \in \mathbb{R}^n \mid v \cdot x \le 1 \ \ \textit{for all} \ \ v \in K\}. \tag{1.17}$$

Note that K^* is always a convex set even if K is not convex.

Theorem 1.3.1. *[7] If $K \subset \mathbb{R}^n$ is a convex body containing the origin, then $K = K^{**}$.*

Proof. Let an arbitrary $y \in K \Rightarrow$ for any $x \in \mathbb{K}^*$, we have $x \cdot y \leq 1 \Rightarrow y \in \mathbb{K}^{**}$. Now, it is enough to prove

$$K^{**} \subset K. \tag{1.18}$$

Let $x \in \mathbb{R}^n \setminus K$. Then there exists a hyperplane H that separates x and K, each of them being in a different half-space. As $0 \in K$ and

$$H = \{w \in \mathbb{R}^n \mid w \cdot v = 1\}$$

for $v \neq 0$, then

$$K \subset \{w \in \mathbb{R}^n \mid w \cdot v < 1\} \text{ and } x \cdot v > 1. \tag{1.19}$$

From the two previous equations respectively we conclude that $v \in K^*$ and that $x \notin K^{**}$. Therefore, inclusion (1.18) has been proved via complements. $\square$

Example 1.3.1. *[9] Let $C \subset \mathbb{R}^n$ be the unit n-cube centered at the origin. We want to find its polar C^*. Thus*

$$C = \{(c_1, \ldots, c_n) \in \mathbb{R}^n \mid |c_1| \leq 1, \ldots, |c_n| \leq 1\}. \tag{1.20}$$

For any $(x_1, \ldots, x_m) \in C^*$, we have $(c_1, \ldots, c_n)$ of C such that $c_i = 1$ if $x_i \geq 0$ or $c_i = -1$ if $x_i < 0$. Thus,

$$(x_1, \ldots, x_m).(c_1, \ldots, c_n) = (c_1 \cdot x_1, \ldots, c_n \cdot x_n) = |x_1| + \ldots + |x_n| \leq 1. \tag{1.21}$$

On the other hand, suppose that $(x_1, \ldots, x_m)$ satisfies $|x_1| + \ldots + |x_n| \leq 1$, then for *any* $(c_1, \ldots, c_n)$ *in* C*, we have*

$$\begin{aligned}(x_1, \ldots, x_m) \cdot (c_1, \ldots, c_n) &= (c_1.x_1, \ldots, c_n.x_n) \\ &\leq |c_1| \cdot |x_1| + \ldots + |c_n| \cdot |x_n| \\ &\leq |x_1| + \ldots + |x_n| \leq 1. \end{aligned} \tag{1.22}$$

Thus $(x_1, \ldots, x_m) \in C^*$*. The convex body* C^* *is the regular* n*-cross-polytope defined by*

$$C^* = \{(x_1, \ldots, x_m) \mid |x_1| + \ldots + |x_n| \leq 1\}. \tag{1.23}$$

Now, we will find C^{**}*, the polar of* C^**.*

Define $(x_1, \ldots, 0) \in C^*$ *for every* $(c_1, \ldots, c_n) \in C^{**}$ *such that* $x_1 = 1$ *if* $c_1 \geq 0$ *and* $x_1 = -1$ *if* $c_1 < 0$*. So,*

$$(x_1, \ldots, 0) \cdot (c_1, \ldots, c_n) = |c_1| \leq 1. \tag{1.24}$$

In the same way, we prove that $|c_2| \leq 1, \ldots, |c_n| \leq 1$*. Hence* $(c_1, \ldots, c_n) \in C$ *and* $C^{**} \subset C$*, while* $C \subset C^{**}$ *holds from the definition of the polar. Thus* $C = C^{**}$*.*

Definition 1.3.2. *[4] A non-zero vector* $p = (p_1, p_2)$ *in* $\mathbb{R}^2$ *is said to be polar to the line* $l_p = \{x \in \mathbb{R}^2 \quad | \quad p_1 \cdot x_1 + p_2 \cdot x_2 = 1\}$ *and vice versa.*

The above definition leads us to the following (simplified) definition of the polar set of a convex polygon in $\mathbb{R}^2$:

Definition 1.3.3. *[4] The polar set* K^* *of a convex polygon* $K = conv\,\{x_1, \ldots, x_m\}$ *is the convex hull of* $\{v_{ij} \mid [x_i, x_j]$ *is an edge of* $K\}$ *where* v_{ij} *is the polar point to the line through* x_i *and* x_j *of the edge* $[x_i, x_j]$*.*

The goal of the rest of the section is to explain the projection body of a polytope K, denoted by ΠK. We will see that, for any polytope $K \subset \mathbb{R}^n$, ΠK is the Minkowski sum of line segments orthogonal to the faces of K having length equal to the $(n-1)$-dimensional volume of the correspondent face.

Definition 1.3.4. *[3] Let K be a convex body in $\mathbb{R}^n, n \geq 2$. Then ΠK, the projection body of K, is a centered convex body defined via its support function by*

$$h_{\Pi K}(u) = V_{n-1}(K|u^{\perp}) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| \, dS_K(v), \tag{1.25}$$

for all $u \in S^{n-1}$. Here $dS_K(.)$ is the surface area measure of K as the $(n-1)$-Hausdorff measure of the boundary of K.

Definition 1.3.5. *[3] Cauchy's projection formula is*

$$c_{n-1,i}V_i(K|u^{\perp}) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| \, dS_i(K, v), \tag{1.26}$$

for all $u \in S^{n-1}$, $i = 1, \ldots, n-1$. Here $dS_i(K,v)$ is the i-th surface area measure defined in [8], with $dS_{n-1}(K, v) = dS_K(v)$ and $c_{n-1,i} = \binom{n-1}{i}$.

Proposition 1.3.1. *[3] $\Pi K = \Pi(-K)$.*

The proof is immediate due to the fact that, in each direction, the projection of K coincides with the projection of $-K$, the reflection of K.

Theorem 1.3.2. *[3] Let K be a convex body in $\mathbb{R}^2$. Then ΠK is the rotation by $\frac{\pi}{2}$ about the origin of $2\triangle K := 2(K + (-K))$, the symmetric difference of K, that is the Minkowski sum of K with its reflection through the origin. Thus, every centered convex body in $\mathbb{R}^2$ is a projection body.*

Proof. If ΠK is a convex body in $\mathbb{R}^2$, and $u, v \in S^1$ are unit vectors such that v is orthogonal to u, then

$$h_{\Pi K}(u) = V_1(K|u^{\perp}) = w_K(u) = w_{\triangle K}(v) = h_{2\triangle K}(v). \tag{1.27}$$

where w_k is the width of K in the direction u, in other words the distance between the two supporting lines of K with normals u and $-u$.
Therefore, the projection body ΠK is the rotation by $\frac{\pi}{2}$ about the origin of the convex body $2\triangle K$.
If K is centered at the origin, then $\triangle K = K \Rightarrow \Pi K_1 = K : K_1$ is $\frac{1}{2}K$ rotated by $\frac{\pi}{2}$ about the origin. $\square$

Example 1.3.2. *Projection bodies; Some simple examples:*

- *If C is the unit disk in $\mathbb{R}^2$. Then, ΠC is the centered disk of radius 2.*
- *If S is the centered unit square. Then, ΠS is the centered square such that $\Pi K = 2\,K$*

Let C be the centered unit cube in $\mathbb{R}^n$. Then, $\Pi K = 2\,K$. We conclude this by using the projection formula that defines the support function of the projection body of C

$$h_{\Pi C}(u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v|\, dS(C, v). \tag{1.28}$$

Given the piecewise linear structure of the boundary of C, here $S(C, .)$ is the sum of point masses of weight 1 at the intersection of the coordinate axes with S^{n-1}. We reduce the integral to a sum of the n terms $|u.e_i|$, $1 \leq i \leq n$ with e_i being the unit vector in the i-th coordinate direction. Each term is the support function of the [-e_i,e_i]. Thus, ΠC is the vector sum of all the [-e_i,e_i]. In other words, ΠC is the centered unit cube expanded by a factor of 2.

The previous reasoning can be applied to any arbitrary polytope in $\mathbb{R}^n$ concluding that the projection of a polytope is the Minkowski sum of line segments, or vectors, orthogonal to the faces of K having length equal to the $(n-1)$-dimensional volume of the correspondent face.

Lemma 1.3.1. *[2] Let $F_1, \dots, F_n$ be the faces of a polytope K, let the outward facing unit normal of F_i be v_j and let $A(F_i)$ be the area of each F_i. As ΠK is the Minkowski sum of the area segments of K, then*

$$V(\Pi K) = \sum_{1 \le i < j < k \le n} |w_i, w_j, w_k| \tag{1.29}$$

where $w_i = A(F_i)\, v_i$ for $i = 1, \dots, n$ and $|w_i, w_j, w_k|$ is the determinant of the matrix that has w_i, w_j, w_k as columns.

In other words, $V(\Pi K)$ is equal to sum of all volumes of parallelepipeds that can be formed by the 3-combination of vectors that are normal to the faces of K and have length equal to the corresponding area of the face they are orthogonal to.

In what follows we will use the definition of the support function of a convex body both as a function on $\mathbb{S}^{n-1}$ and its extension by homogeneity, $h_K(x) = h_K(x/||x||)$, to $\mathbb{R}^n \setminus \{0\}$.

Theorem 1.3.3. *[3] The projection bodies of affinely equivalent convex bodies are also affinely equivalent. If $\phi \in GL_n$ where GL_n is any non-singular linear transformation from $\mathbb{R}^n$ to itself, then*

$$\Pi(\phi K) = |\det \phi|\, \phi^{-t}(\Pi K). \tag{1.30}$$

Proof. As the name implies, two convex bodies are affinely equivalent if and only if there is an affine transformation of $\mathbb{R}^n$ that sends one convex body into another, where recall that an affine transformation is a linear transformations composed with a translation, possibly by the zero vector. Note that the linear transformation involve must be invertible as each convex body is sent into another convex body, thus a compact convex set with non-empty interior is sent into another set, compact and convex, with non-empty interior.

We will start by proving formula (1.30). Let K be a convex body in $\mathbb{R}^n$, $\phi \in GL_n$ an invertible linear transformation, $u \in \mathbb{S}^{n-1}$ and let w be such that $\phi \cdot w = u$. Then

$$\begin{aligned} h_{\Pi(\phi K)}(u) &= \frac{1}{2}\int_{\mathbb{S}^{n-1}} |u \cdot v|\, dS(\phi K, v) \\ &= \frac{n}{2}\, V(\phi K, n-1; [-u; u]) \\ &= \frac{n}{2}\, V(\phi K, n-1; \phi[-w; w]) \\ &= \frac{n}{2}\, |\det \phi|\, V(K, n-1; [-w; w]) \\ &= |\det \phi|\, h_{\Pi K}(w) \\ &= |\det \phi|\, h_{\Pi K}(\phi^{-1} u). \end{aligned} \tag{1.31}$$

Above, we have used Cauchy's projection formula from Definition 1.3.5 with $i = n-1$, the invariance of mixed volumes under volume-preserving linear transformations, and the following equation

$$nV(K, n-1; [0, u]) = V_{n-1}(K \mid u^{\perp}), \tag{1.32}$$

where the mixed volumes $V(K, [n-i], L, [i])$ are defined, up to some constant, as the t^i coefficients in $V(K + tL)$, the volume of the Minkowski sum of K with the dilation tL, as a polynomial in t, [8].

For the change in support functions under linear transformations, we use Proposition 1.2.2 and get

$$h_{\phi K}(u) = h_K(\phi^t u) = ||\phi^t(u)|| h_K(\frac{\phi^t u}{||\phi^t u||}). \tag{1.33}$$

Thus,

$$h_{\Pi K}(\phi^{-1} u) = h_{\phi^{-t}(\Pi K)}(u). \tag{1.34}$$

Equation (1.31), together with equation (1.34), completes the proof of the second part of Theorem 1.3.3.

Since it is obvious that any translation of a convex body leaves its projection body unchanged, as only the areas of projections matter, the proof of the rest of the theorem follows immediately. □

Finally, we note another corollary of Theorem 1.2.4:

Corollary 1.3.1. *If a sequence of polytopes, P_i, converges to $K \in \mathbb{K}^n$ in the Hausdorff metric, then, ΠP_i, converges to ΠK in the Hausdorff metric where ΠP_i and ΠK are the projection bodies of P_i and K, respectively.*

Proof. Given that P_i converges to $K \in \mathbb{K}^n$, it follows that $S(P_i)$ converges to $S(K)$, where $S(K)$ denotes the surface area of K, as $i \to \infty$. Given that the support function of the projection body of a convex body K in a given direction $u \in \mathbb{S}^{n-1}$ is the area of the projection of K on a hyperplane orthogonal to u, we thus obtain the corollary. □

1.4 Statement of Mahler's Conjecture and, respectively, Petty's Conjecture

Let $\mathbb{K}_0^n$ be the set of all compact, convex sets in $\mathbb{R}^n$ containing the origin in their interior. The volume product functional, also known as the Mahler product, is the map that assigns to each $K \in \mathbb{K}_0^n$, the value $M(K) = V(K) \cdot V(K^*)$, where recall that K^* is the polar of K and that the polar depends on the choice of the origin.

It is worth noting that $M(TK) = M(K)$, for any general linear transformation T of $\mathbb{R}^n$. We thus say that $M(\ .\ )$ is linearly invariant. For an extensive discussion on Mahler's functional, including Mahler's conjecture, we refer the reader to [4].

Note also that if the origin is taken closer and closer to the boundary of K, then $M(K)$ becomes larger and larger and is, thus, unbounded. Therefore, generally, one cannot have an upper bound for Mahler's functional. However, it was proved by Santaló that for centrally symmetric convex bodies whose center of symmetry coincides with the origin, the maximum of Mahler's product is reached for ellipsoids and is equal to ω_n^2, where ω_n denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$. It can be shown that if K is not centrally symmetric, there exists a choice of the origin in the interior of K, choice called Santaló point, such that the same bound holds.

The lower bound of Mahler's functional remains unknown except for dimension $n = 2$. It has been conjectured, and proved by Mahler in the planar case that, for any centrally convex body $K \in \mathbb{K}_0^n$, we have

$$M(K) \geq \frac{4^n}{(n!)^2}, \tag{1.35}$$

with equality if and only if K is a parallelotope, [5].

The lower bound remained an open problem despite many attempts and it is called Mahler's Conjecture. Only some very special cases of Mahler's conjecture have been proved.

The conjecture has a non-symmetric analogue in which the lower bound is claimed to be reached for simplices. Mahler has shown that the method used in the plane for the centrally symmetric case, which we will present in Chapter 2, works also to prove the non-symmetric planar case.

We will now focus on Petty's conjecture. For this, recall that $\mathbb{K}^n$ is the set of all compact, convex sets with non-empty interior. Petty's functional is the map that assigns to each $K \in \mathbb{K}^n$ the value $P(K) := \dfrac{V(\Pi K)}{V^{n-1}(K)}$, where recall that ΠK is the projection body of K.

It is clear from the definition of the projection body, as well as the properties of volume as the n-dimensional Lebesgue measure in $\mathbb{R}^n$, that Petty's functional is translation invariant. Moreover, by Theorem 1.3 and, again the properties of volume, Petty's functional is linearly invariant. Combining the previous two facts, we conclude that the value of the functional is unchanged under any affine transformation applied to K. Thus, $P(\, . \,)$ is an affine invariant of K.

Both bounds of Petty's functional, the lower one and the upper one, in $\mathbb{K}^n$ for $n \geq 3$ are not yet known. In 1971, Petty conjectured [2] that

$$\omega_{n-1}^n \omega_n^{2-n} \leq \frac{V(\Pi K)}{V^{n-1}(K)} \tag{1.36}$$

with equality if and only if K is an ellipsoid.

Regarding the upper bound, for centrally symmetric convex bodies $K \in \mathbb{K}^n$, Schneider [1] conjectured that 2^n is the upper bound and that it is achieved, in particular, for parallelotopes, like a reverse of the symmetric case of Mahler's conjecture. In other words, Schneider hypothesized that

$$\frac{V(\Pi K)}{V^{n-1}(K)} \leq 2^n \tag{1.37}$$

for any convex body $K \in \mathbb{K}^n$ symmetric with respect to the origin and we have equality for direct sums of planar centrally symmetric convex bodies.

However, Brannen gave a counterexample where Petty's functional, $\dfrac{V(\Pi K)}{V^{n-1}}$, exceeds 2^n where $n \geq 3$. In fact, he found centrally symmetric convex bodies $K \subset \mathbb{R}^n$ such that $P(K) = \dfrac{9}{8} \cdot 2^n$ for every $n \geq 3$, [1].

Finally, Brannen [1] conjectured that, for all convex bodies $K \in \mathbb{K}^n$, we have

$$\frac{V(\Pi K)}{V^{n-1}} \leq \frac{(n+1) \cdot n^n}{n!} \tag{1.38}$$

and equality is satisfied if and only if K is a simplex.

It is this latter bound that we refer to as Petty's conjecture and the one for which we address Mahler's technique in this thesis.

Chapter 2

Mahler's Conjecture in $\mathbb{R}^2$

The aim of this chapter is to present a particular proof of the symmetric case of Mahler's conjecture in dimension two. This proof stands out for a certain technique in which it is shown that decreasing the number of vertices of a polygon, Mahler's functional decreases as well. Consequently, one can use this fact to deduce that, in the centrally symmetric planar case, the minimum of Mahler's functional is reached for the parallelogram,

In the next chapters, we will investigate uses of similar techniques, although not identical, for other problems such as Petty's conjectured inequality in dimension 2 and 3. Therefore, we regard this proof as the starting prototype.

Finally, let us mention that the proof that we will present below dates from 1939 and is due to Mahler himself [5], but our presentation follows a more modern update of Henze, [4].

Proposition 2.0.1. *Let $T \subset \mathbb{R}^2$ be a triangle containing the origin in its interior and let T^* be its polar. Denote $T=$ conv $\{x, y, z\}$, $T^*=$ conv $\{x^*, y^*, z^*\}$ and assume that no two of x, y and z are linearly dependent. Then,*

$$V(T^*) = \frac{2V(T)^2}{d_{xy}d_{yz}d_{zx}}, \tag{2.1}$$

where $d_{xy} = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ *with* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$.

Proof. We have that d_{xy}, d_{yz} and d_{zx} are all different than zero since no two of x, y and z are linearly dependent.

As we saw in the introduction, the definition of the polar body implies that x^* is the intersection of the lines that are polar to the points x and y that we will call l_x and l_y, respectively. By solving the system of the two linear equations representing the lines $l_x = \{x \in \mathbb{R}^2 \mid a_1 \cdot x_1 + a_2 \cdot x_2 = 1\}$ and $l_y = \{y \in \mathbb{R}^2 \mid a_1 \cdot y_1 + a_2 \cdot y_2 = 1\}$, we get

$$x^* = \frac{1}{d_{xy}}(y_2 - x_2, x_1 - y_1). \tag{2.2}$$

Similar calculations for y^* and z^* lead to

$$y^* = \frac{1}{d_{yz}}(z_2 - y_2, y_1 - z_1) \text{ and } z^* = \frac{1}{d_{zx}}(x_2 - z_2, z_1 - x_1). \tag{2.3}$$

Given that the area of a parallelogram can be represented by means of a determinant, we deduce the value of $V(T^*)$ as follows:

$$
\begin{aligned}
V(T^*) &= \frac{1}{2}\,\det(y^*-x^*, z^*-x^*) \\
&= \frac{1}{2}\,(d_{x^*y^*}+d_{y^*z^*}d_{z^*x^*}) \\
&= \frac{1}{2}\left(\frac{1}{d_{xy}d_{yz}}\cdot\begin{vmatrix} y_2-x_2 & z_2-y_2 \\ x_1-y_1 & y_1-z_1\end{vmatrix} + \frac{1}{d_{yz}d_{zx}}\cdot\begin{vmatrix} z_2-y_2 & x_2-z_2 \\ y_1-z_1 & z_1-x_1\end{vmatrix}\right. \\
&\qquad\left. + \frac{1}{d_{zx}d_{xy}}\cdot\begin{vmatrix} x_2-z_2 & y_2-x_2 \\ z_1-x_1 & x_1-y_1\end{vmatrix}\right) \qquad (2.4) \\
&= \frac{1}{2d_{xy}d_{yz}d_{zx}}\cdot(d_{xy}^2+d_{yz}^2+d_{zx}^2+2d_{xy}d_{yz}+2d_{yz}d_{zx}+2d_{zx}d_{xy}) \\
&= \frac{(d_{xy}+d_{yz}+d_{zx})^2}{2d_{xy}d_{yz}d_{zx}} \\
&= \frac{2\cdot V(T)^2}{d_{xy}d_{yz}d_{zx}}.
\end{aligned}
$$

□

We will now investigate where the maximality of $V(T^*)$ is attained.

Let's assume that d_{xy}, d_{yz} are strictly positive and that $d_{zx} < 0$. The assumption is reasonable without any loss of generality because no two of x, y and z are linearly independent. Consequently, supposing that the origin does not belong to $T = \operatorname{conv}\{x, y, z\}$, d_{xy}, d_{yz} and d_{zx} are not all of the same sign, thus the assumption.

We define a line d parallel to $[x, z]$ and we choose y arbitrarily on d by the equation,

$$d_{xy}+d_{yz}-|d_{zx}| = 2\cdot V(T) \qquad (2.5)$$

Thus, $0 < d_{xy} < 2\cdot V(T)+|d_{zx}|$ since $d_{xy} > 0$. Let us define α and β such that $0 < \alpha \leq d_{xy} \leq \beta < 2\cdot V(T)+|d_{zx}|$ which implies a stretch S of d.

Substituting our results in equation (2.1), we get

$$V(T^*) = \frac{2 \cdot V(T)}{d_{xy}(2 \cdot V(T) + |d_{zx}| - d_{xy})|d_{zx}|}. \tag{2.6}$$

Therefore, $V(T^*)$ depends on $y \in S$ and reaches a maximum on S. Additionally,

$$\begin{aligned}
\frac{2 \cdot V(T)}{V(T^*) \cdot |d_{zx}|} &= d_{xy}(2 \cdot V(T) + |d_{zx}| - d_{xy}) \\
&= (V(T) + \frac{|d_{zx}|}{2})^2 - ((V(T) + \frac{|d_{zx}|}{2})^2 - 2 \cdot d_{xy} \cdot (V(T) + \frac{|d_{zx}|}{2}) + d_{xy}^2) \\
&= (V(T) + \frac{|d_{zx}|}{2})^2 - (V(T) + \frac{|d_{zx}|}{2} - d_{xy})^2. \qquad (2.7)
\end{aligned}$$

Therefore $\frac{2 \cdot V(T)}{V(T^*) \cdot |d_{zx}|}$ attains a maximum when $d_{xy} = V(T) + \frac{|d_{zx}|}{2}$ and, furthermore, $V(T^*)$ reaches its maximum when y is a boundary point of S.

Proposition 2.0.2. *Given $P \subset \mathbb{R}^2$, a centrally symmetric polygon with $2m \geq 6$ vertices containing the origin in its interior, we can find a centrally symmetric polygon $H \subset \mathbb{R}^2$ with $2(m-1)$ vertices and containing the origin in its interior, such that*

$$V(H) \cdot V(H^*) < V(P) \cdot V(P^*), \tag{2.8}$$

where P^* and H^* are the polars of P and H, respectively.

Proof. Let $v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m}$ be the vertices of P such that $v_i = -v_{m+i}$, where $1 \leq i \leq m$, so that P is centrally symmetric. Let $T = \operatorname{conv}\{v_1, v_2, v_3\}$, $T' = \operatorname{conv}\{v_{m+1}, v_{m+2}, v_{m+3}\}$ and $M = \operatorname{conv}\{v_1, v_3, \ldots, v_m, v_{m+1}, v_{m+3}, \ldots, v_{2m}\}$. Thus, $P = M \cup T \cup T'$. Notice that T and T' are symmetric to each other with respect to the origin and do not contain the origin. We could always find such T and T' because the number of vertices of P is $2m \geq 6$.

To illustrate the procedure, see Figure 2.1 where we choose an example with

8 vetices, thus $m = 4$, $T = \operatorname{conv}\{v_1, v_2, v_3\}$, $T' = \operatorname{conv}\{v_5, v_6, v_7\}$ and $M = \operatorname{conv}\{v_1, v_3, v_4, v_5, v_7, v_8\}$.

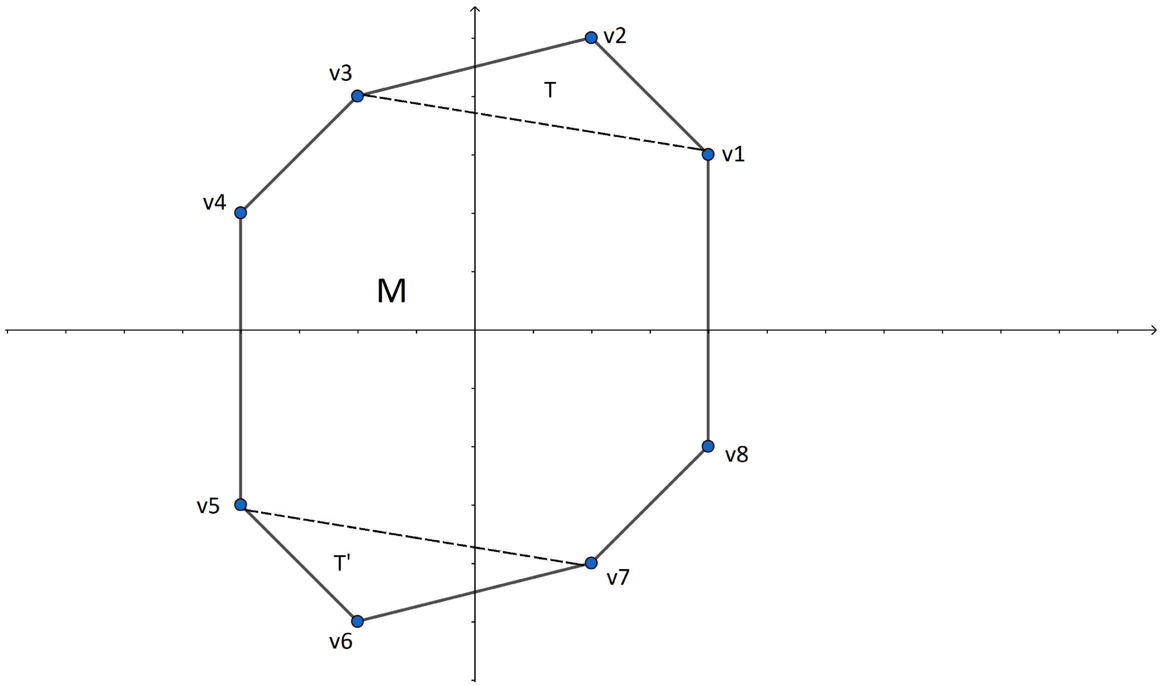

Figure 2.1: P

Furthermore, consider P^*, M^*, T^* and T'^* the polars of P, M, T and T', respectively:

$P^* = \operatorname{conv}\{v_1^*, \ldots, v_m^*, v_{m+1}^*, \ldots, v_{2m-1}^*\}$ where v_j^* corresponds to the segment, or edge, $[v_j, v_{j+1}]$ of P with $1 \leq j \leq 2(m-1)$;

$T^* = \operatorname{conv}\{v_1^*, z^*, v_3^*\}$ where z^* corresponds to the edge $[v_1, v_3]$;

$T'^* = \operatorname{conv}\{v_{m+1}^*, -z^*, v_{m+3}^*\} = \operatorname{conv}\{v_5^*, -z^*, v_7^*\} = \operatorname{conv}\{-v_1^*, -z^*, -v_3^*\}$;

$M^*= \operatorname{conv}\{\ z^*, v_3, \ldots, v_m, -z^*, v_{m+3}, \ldots, v_{2m}\}= \operatorname{conv}\{\ z^*, v_3, v_4, -z^*, v_7, v_8\}$.

We notice that T^* and T'^* are in M^* and that $P^* = M^* \setminus (T^* \cup T'^*)$.

The above definitions are illustrated in Figure 2.2.

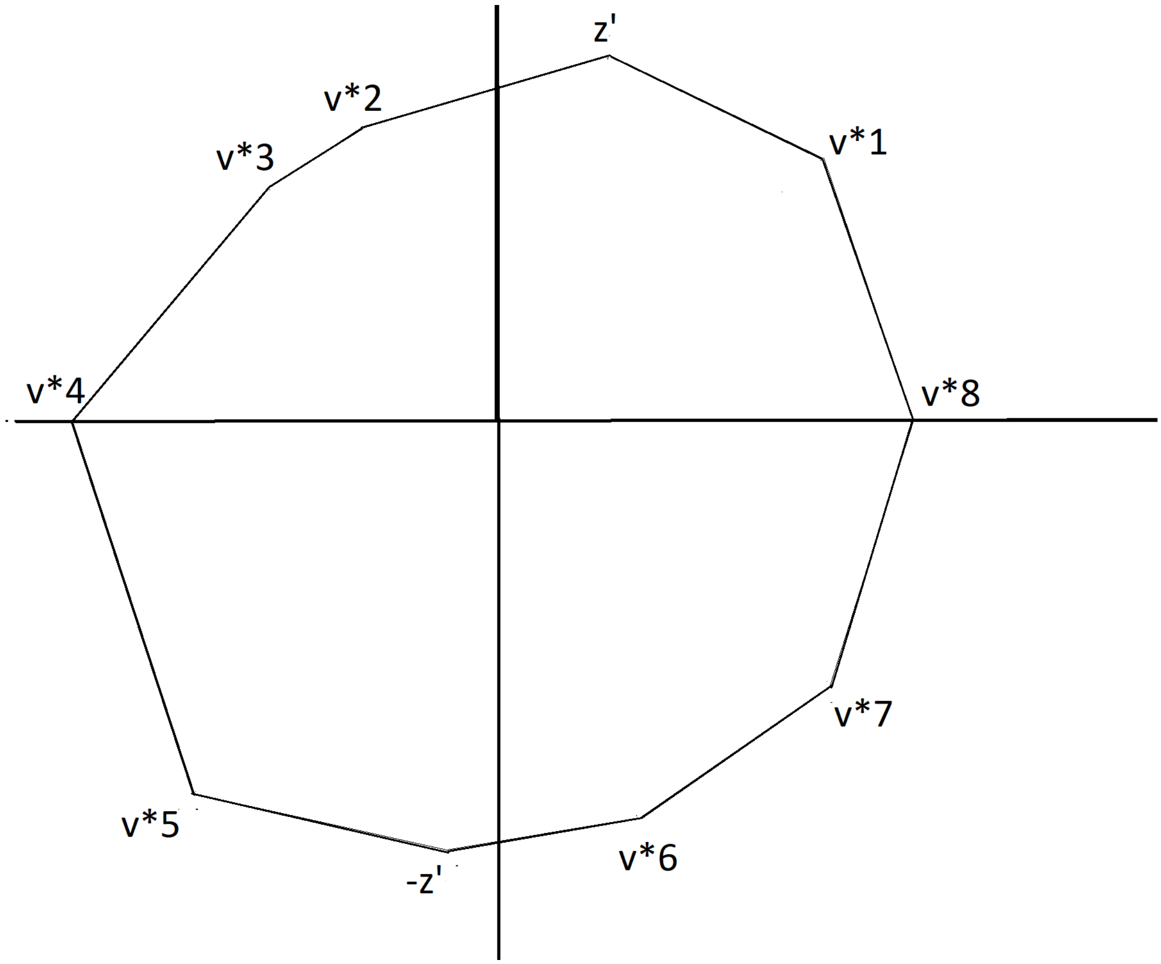

Figure 2.2: $P*$

Let l be a line parallel to the edge $[v_1, v_3]$ and passing through v_2. Now, we extend $[v_3, v_4]$ and $[v_{2m}, v_1]$ toward l. If we move v_2 on the part of l that is cut by the extension of $[v_3, v_4]$ and $[v_{2m}, v_1]$, P will conserve its convexity and its area. Let us call Tv the convex hull of v_1, v_2' and v_3, where v_2' is a any position of v_2 on on the part of l that is cut by the extension of $[v_3, v_4]$ and $[v_{2m}, v_1]$. The above procedure is illustrated below in Figure 2.3.

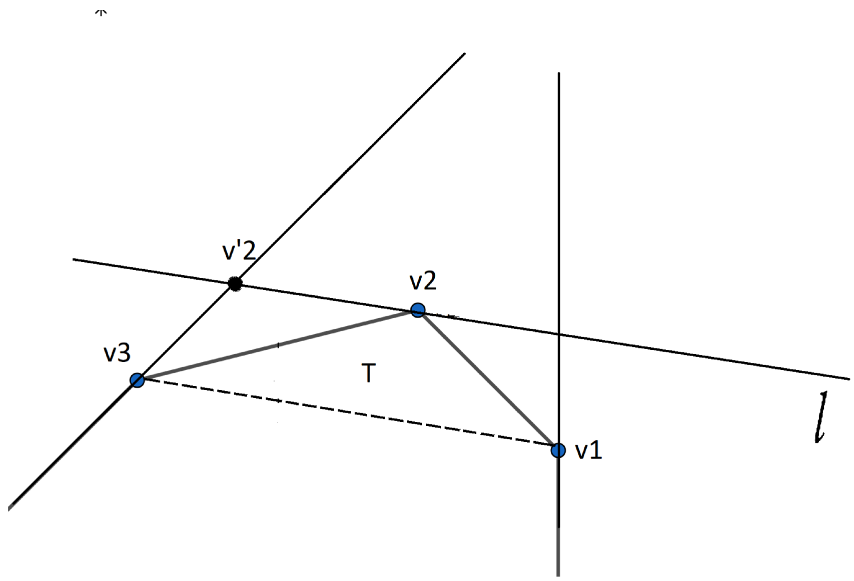

Figure 2.3:

Now, if we place v_2 on the intersection of l with $[v_3, v_4]$ or $[v_{2m}, v_1]$, the area of the polar of Tv, denoted by Tv^*, attains it maximal value as seen from Proposition 2.0.1. We do the same with $v_{m+2} = -v_2$ (in our case $-v_2 = v_6$).

This way P has been reduced to $2(m-1)$ vertices, let us call it H. Thus, H is a centrally symmetric polygon with $2(m-1)$ vertices such that

$H = \text{conv}\,\{v_2', v_3, \ldots, v_{m-1}, -v_2', v_{m+2}, \ldots, v_{2m}\}$ and $V(P) = V(H)$. The latter comes from the geometry of the triangles, the bases of the new triangles, as well as their heights, are the same as those of the old triangles, thus leaving their area unchanged.

Finally, knowing that $V(Tv^*) \geq V(T^*) = V(T'^*)$, we have:

$$V(H^*) = V(P^*) + V(T^*) + V(T'^*) - 2 \cdot V(Tv^*) < V(P^*). \tag{2.9}$$

Consequently, we have just proved inequality (2.8) claimed by the proposition.

□

Theorem 2.0.1. *Let P be centrally symmetric convex polygon in $\mathbb{R}^2$, then $M(P) \geq 8$*

and the equality is satisfied if and only if P is a parallelogram.

Proof. Consider first $P = \text{conv}\{v_1, v_2, -v_1, -v_2\}$, thus P is a parallelogram. Therefe-ore, we can divide P into 4 triangles, T_1, T_2, T_3 and T_4, that have the same area such that $T_1 = \text{conv}\{v_1, v_2, 0\}$, $T_2 = \text{conv}\{-v_1, v_2, 0\}$, $T_3 = \text{conv}\{-v_1, -v_2, 0\}$ and $T_4 = \text{conv}\{v_1, -v_2, 0\}$. The information above is illustrated in Figure 2.4.

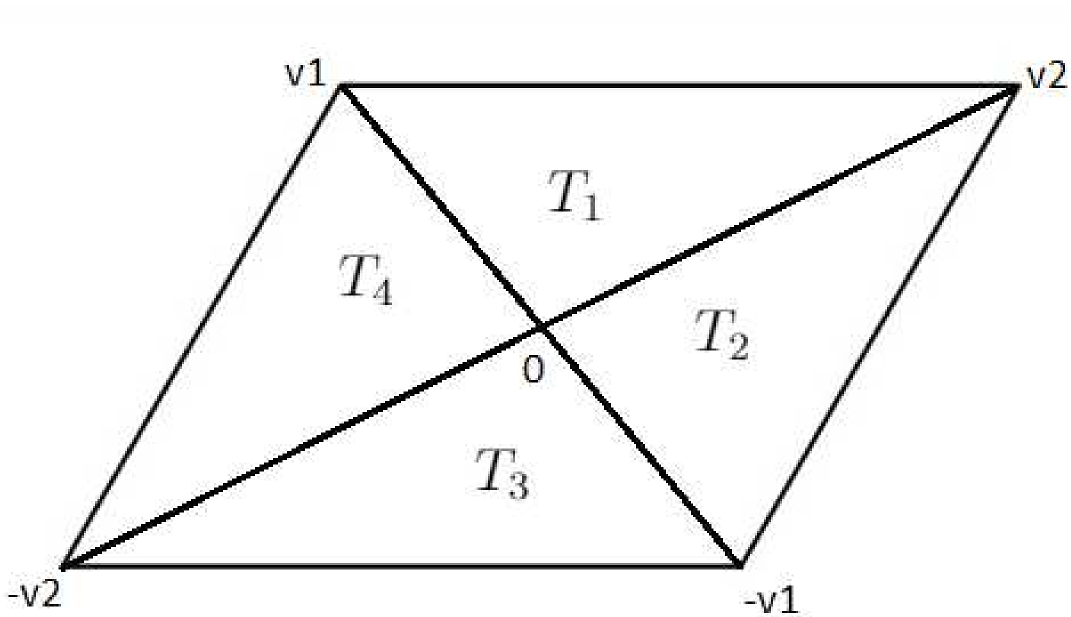

Figure 2.4: P

Therefore,

$$V(P) = \frac{1}{2}d_{v_1,v_2} + \frac{1}{2}d_{v_1,-v_2} + \frac{1}{2}d_{v_1,v_2} + \frac{1}{2}d_{-v_1,v_2} = 2d_{v_1,v_2}. \tag{2.10}$$

Using the same notation as in the previous lemma, we have $P^* = \text{conv}\{v_1^*, v_2^*, -v_1^*, -v_2^*\} = \text{conv}\{\pm v_1^*, \pm v_2^*\}$, where $V_1^* = \dfrac{1}{d_{v_1v_2}}\begin{vmatrix} v_{22} & -v_{12} \\ v_{11} & -v_{21} \end{vmatrix}$ and $V_2^* = \dfrac{1}{d_{v_1v_2}}\begin{vmatrix} -v_{12} & -v_{22} \\ v_{11} & v_{21} \end{vmatrix}$ given that v_1= (v_{11}, v_{12}) and v_2= (v_{11}, v_{12}). Thus, $V(P^*) = \dfrac{4}{d_{v_1v_2}}$ which implies that $M(P) = 8$.

We will prove the remaining part by induction. Let $P = \text{conv}\{\pm v_1, \ldots \pm v_m\}$ such that $m \geq 3$. By the induction hypothesis, and the previous lemma, there exists a q-gon Q with $2(m-1)$ vertices such that

$$M(P) > M(Q) \geq 8. \tag{2.11}$$

This also settles the fact that strict inequality occurs if Q has more than 4 sides and, thus, we conclude the proof. $\square$

Finally, Mahler noticed that for any centrally symmetric convex body K in $\mathbb{R}^2$, one can find a sequence of centrally symmetric polygons in $\mathbb{R}^2$ that converges to K. Since $M(K)$ is a continuous functional, see Theorem 1.2.5, we have $M(K) \geq 8$, but the equality is lost in this case.

As observed by Mahler himself, the same argument may be used in the non-symmetric planar case, but it would be more subtle, because we do not control the choice of the origin and the polar of a set depends on the choice of the origin. In the symmetric case, we have used the fact that, by eliminating opposite vertices, the origin remains the center of symmetry, and/or mass, of the resulting convex polygon.

For simplicity, to illustrate the method, we presented here Mahler's proof only in the symmetric case as, for further analysis, in Petty's conjecture the position of the origin is irrelevant.

Chapter 3

Petty's Conjecture in $\mathbb{R}^2$

3.1 Calculation of the Upper Bound (Triangle)

Proposition 3.1.1. *Let T be a triangle in $\mathbb{R}^2$, then*

$$\frac{V(\Pi T)}{V(T)} = 6. \tag{3.1}$$

Proof. Let T be a triangle as presented in Figure 3.1.

Denoting by $h_1,\ h_2$ and h_3 the heights issued from each vertex onto $a,\ b$ and c, respectively, we have $V(T) = \dfrac{a \cdot h_1}{2} = \dfrac{b \cdot h_2}{2} = \dfrac{c \cdot h_3}{2} =: m$.

Next we will use the fact that the projection body is the sum of Minkowski sum of segments to construct ΠT and calculate its volume.

Step 1: We take the side of length a by its middle point and place it at the origin as in Figure 3.2.

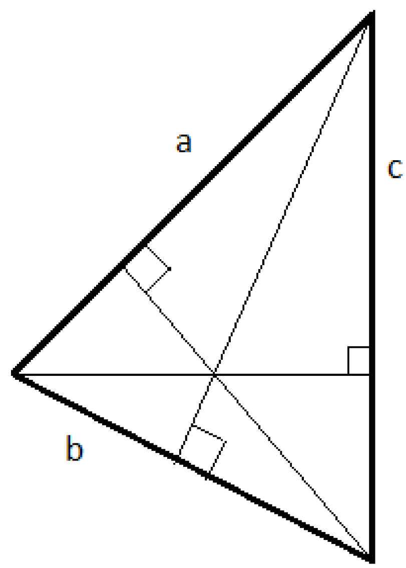

Figure 3.1: T

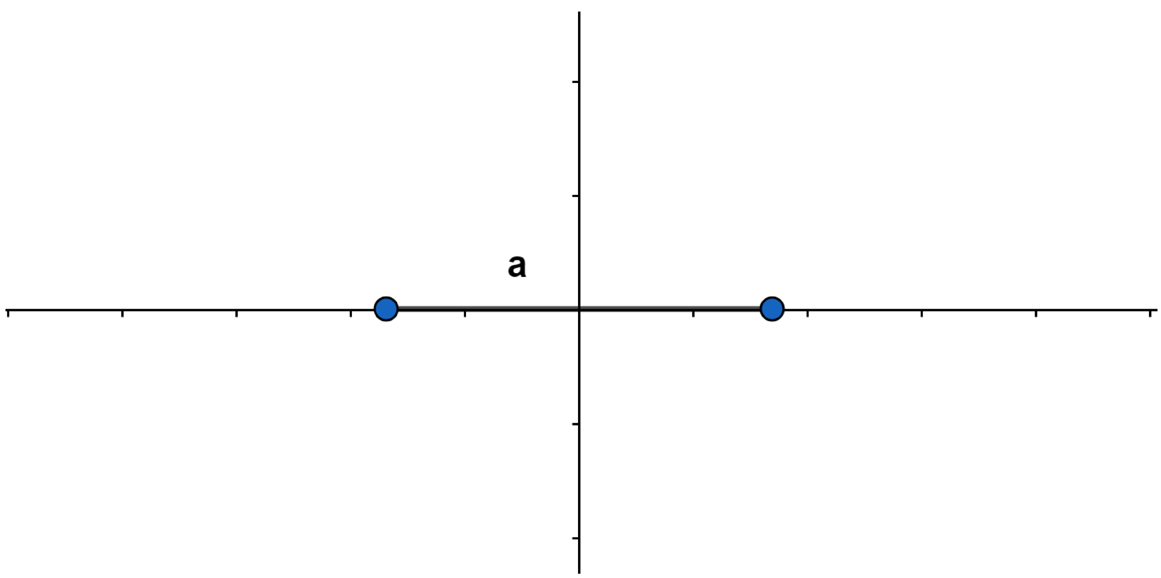

Figure 3.2: Step 1

Step 2: We take the side of length b by its middle point and moved it alongside a, obtaining the parallelogram, P_{ab}, as in Figure 3.3, whose area is $V(P_{ab}) = a \cdot h_1 = b \cdot h_2 = 2 \cdot m$.

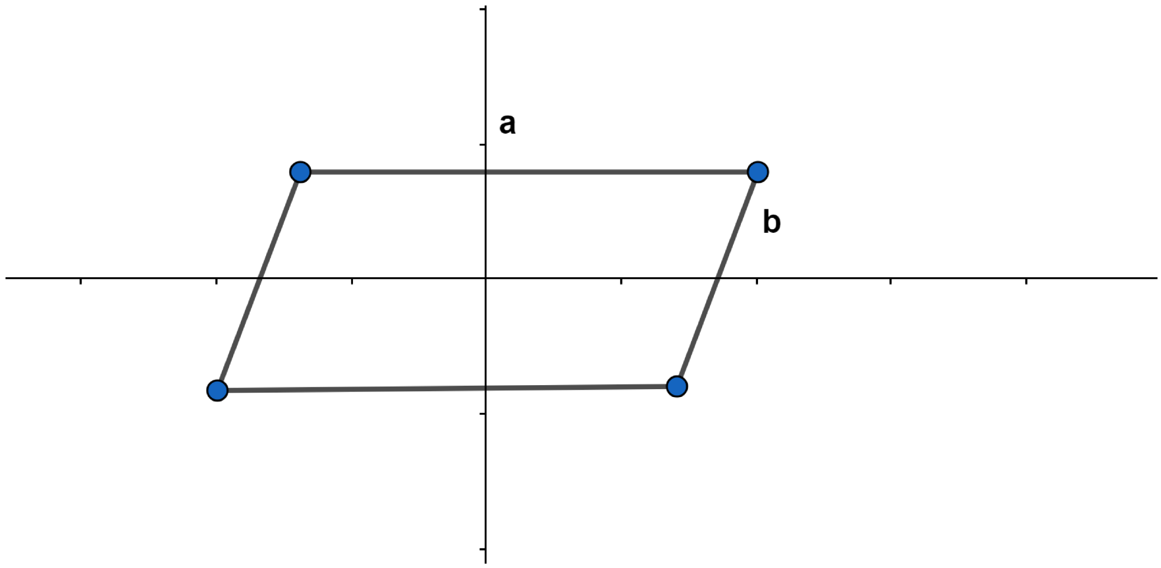

Figure 3.3: Step 2

Step 3: We take the side of length c by its middle and move it along the parallelogram obtained in Step 2. We thus obtain ΠT, as in Figure 3.4, it is formed of P_{ab}, $P1$, $P'1$, $P2$ and $P'2$.

We notice that $P1$ and $P'1$ form together the parallelogram of sides b and c, preserving the angle between the two sides. Thus, $V(P1) + V(P'1) = b \cdot h_2 = c \cdot h_3 = 2 \cdot m$. Similarly for $P2$ and $P'2$, $P1$ and $P'1$ form the parallelogram of sides a and c, preserving the angle between the two sides and $V(P2) + V(P'2) = a \cdot h_1 = c \cdot h_3 = 2 \cdot m$.

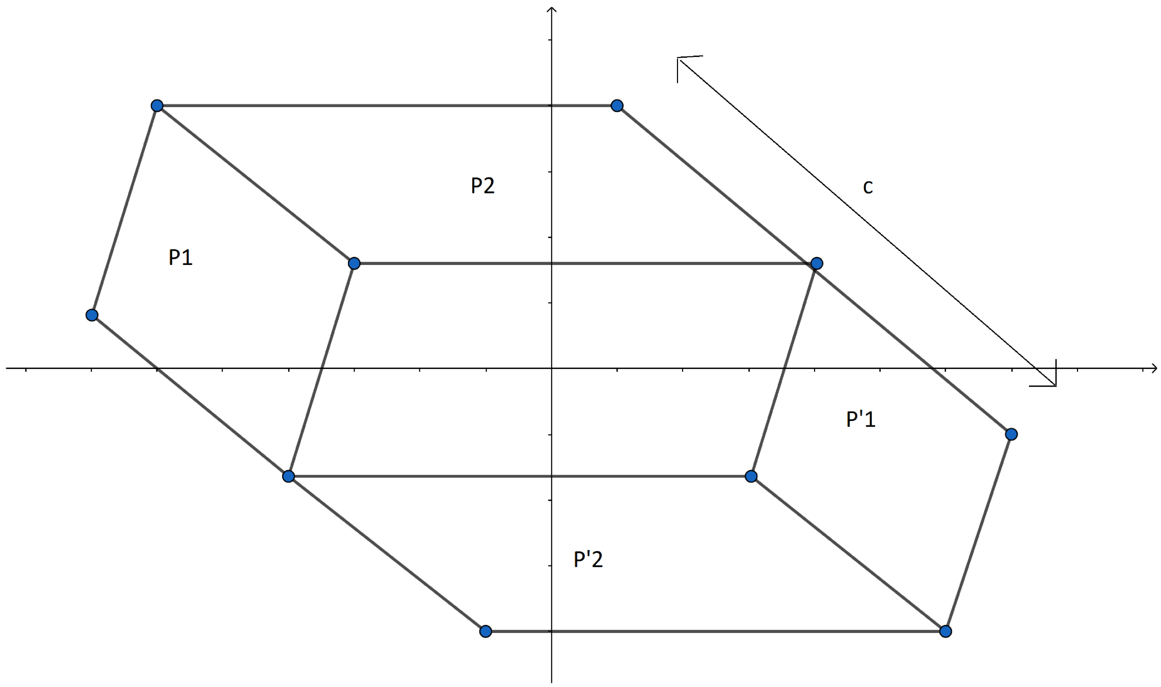

Figure 3.4: The hexagon ΠT, the projection body of T.

Therefore, $V(\Pi T) = V(P_{ab}) + V(P1) + V(P'1) + V(P2) + V(P'2) = 6 \cdot m$.

Substituting these results in equation (3.1), we obtain

$$\frac{V(\Pi T)}{V(T)} = \frac{6 \cdot m}{m} = 6, \tag{3.2}$$

concluding the proof. $\square$

3.2 Proof for a Convex Quadrilateral

Proposition 3.2.1. *Let $K \subset \mathbb{R}^2$ be a convex polygon with 4 sides. Then, K satisfies Petty's conjecture in $\mathbb{R}^2$, namely*

$$\frac{V(\Pi K)}{V(K)} \leq \frac{V(\Pi T)}{V(T)}, \tag{3.3}$$

where $T \subset \mathbb{R}^2$ is a triangle.

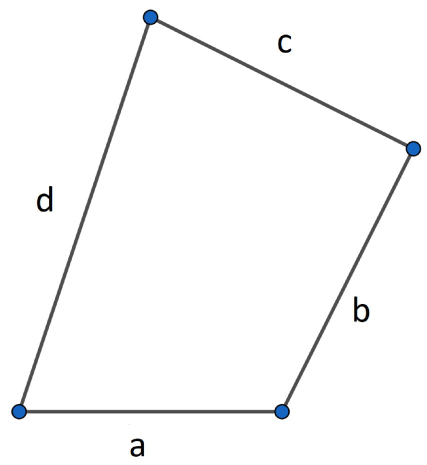

Figure 3.5: Convex quadrilateral body K

Proof. We have showed in Section 3.1 that

$$\frac{V(\Pi T)}{V(T)} = 6.$$

Now, we will prove that $\dfrac{V(\Pi K)}{V(K)} \leq 6$ for any K, a convex quadrilateral in $\mathbb{R}^2$ as shown in Figure 3.5.

Let us follow the steps for constructing the projection body ΠK similarly with what has been performed in the previous section for a triangle:

Step 1: Take the side of length a and place it by its middle point at the origin.

Step 2: Take the side of length b by its middle point and move it along a. We will get a parallelogram of area P_1 as in Figure 3.6.

Step 3: Take the side of length c by its middle point and move it along the parallelogram that we got in Step 2. We will get the shape shown in Figure 3.7.

We notice that the last shape is actually formed by the parallelogram of Step 2, and 4 half-parallelograms. Two halves of the parallelogram formed by b and c (1' and 1")

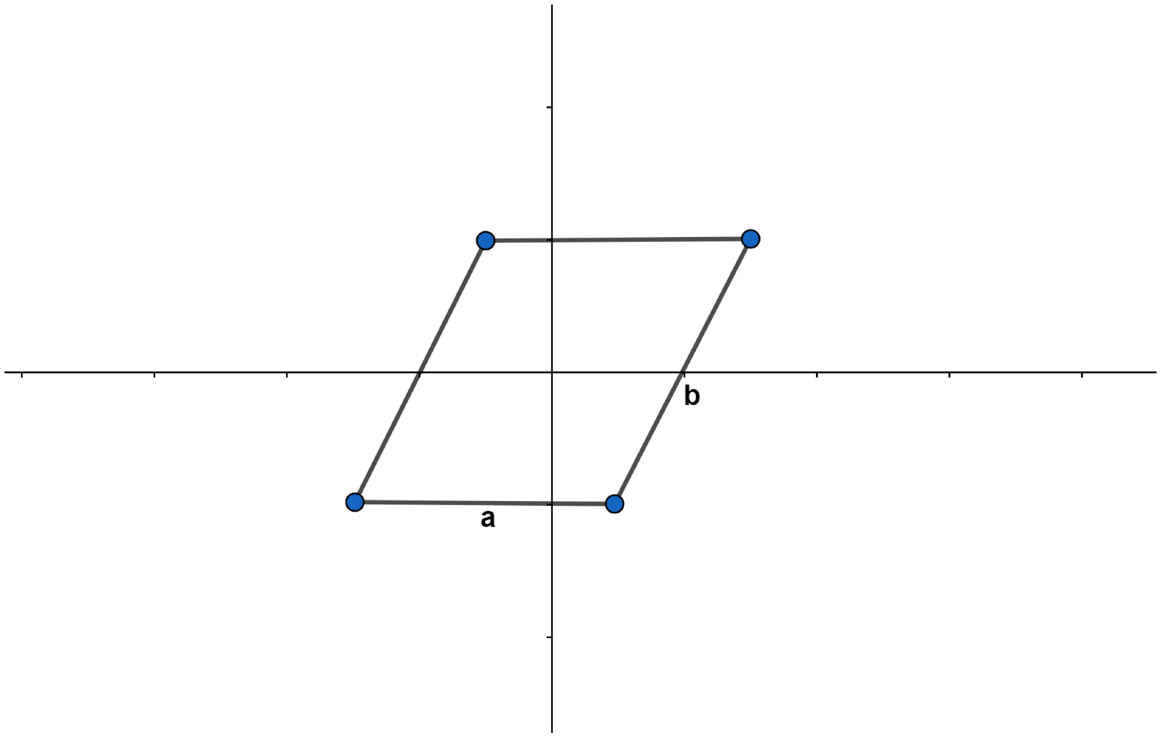

Figure 3.6:

and two halves of the parallelogram formed by a and c (2' and 2"). Therefore, the area of the body we got in this step is equal to P_1+P_2+P_3

P_2 is the area of the parallelogram formed by a and c,

P_3 is the area of the parallelogram formed by b and c.

Step 4: Move the side of length d by its middle point along the shape we got in Step 3. The result is shown in Figure 3.8.

By the same reasoning, the area of the last shape is equal to P_1+P_2+P_3+P_4+P_5+P_6, where

P_4 is the area of the parallelogram formed by a and d (3' and 3");

P_5 is the area of the parallelogram formed by b and d (5' and 5");

P_6 is the area of the parallelogram formed by c and d (4' and 4").

Thus, $V(\Pi K) = P_1 + P_2 + P_3 + P_4 + P_5 + P_6$.

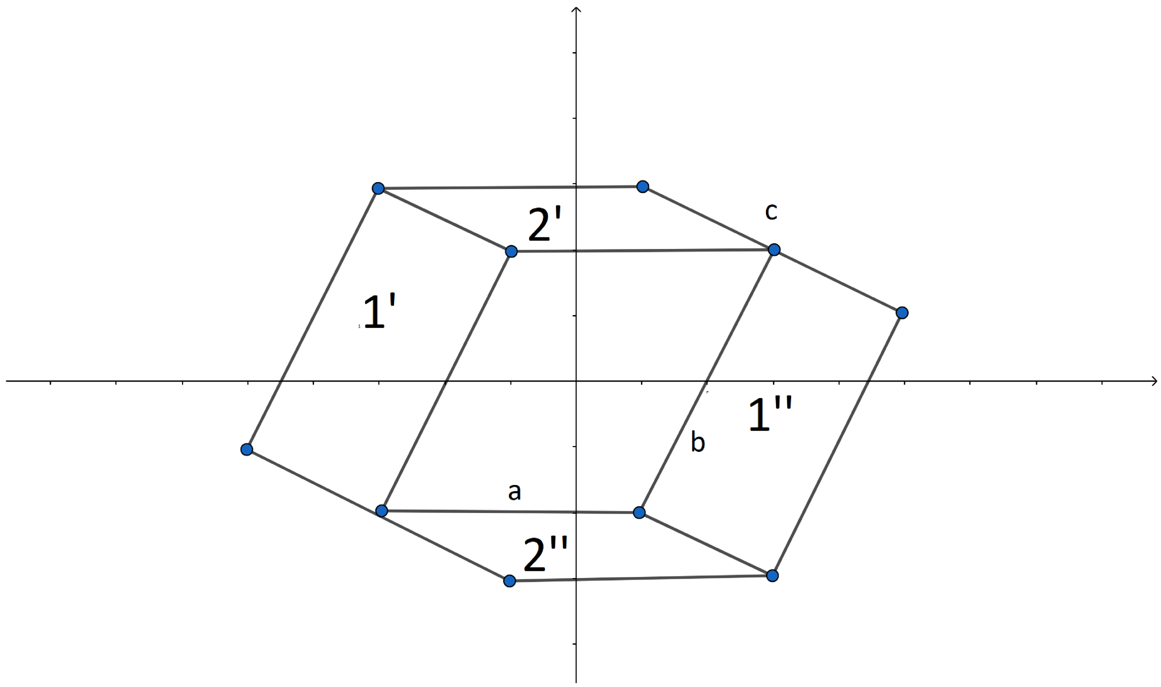

Figure 3.7:

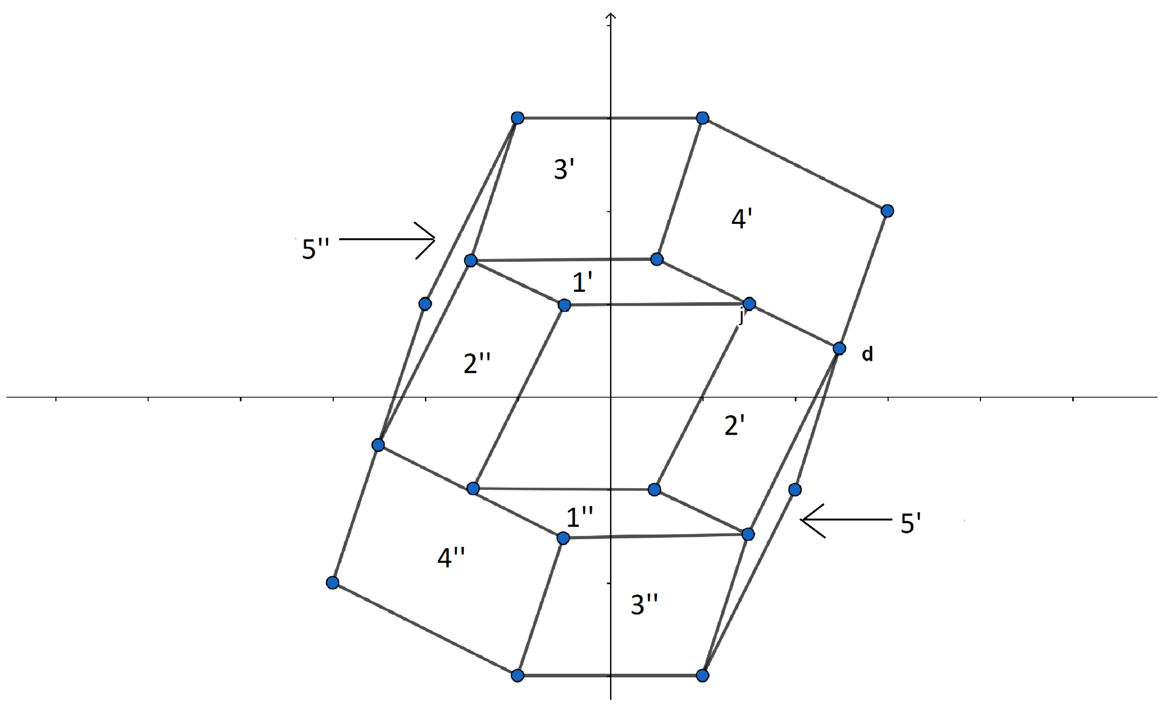

Figure 3.8: ΠK

Now, we notice that $V(K) = (P_6 + P_1)/2$.

By plugging in our results in $\frac{V(\Pi K)}{V(K)} \leq 6$, which we want to show, we have that the inequality is equivalent to

$$\begin{aligned} &\frac{P_1 + P_2 + P_3 + P_4 + P_5 + P_6}{(1/2) \cdot (P_6 + P_1)} \leq 6 \\ &P_1 + P_2 + P_3 + P_4 + P_5 + P_6 \leq 3 \cdot (P_6 + P_1) \qquad (3.4) \\ &P_2 + P_3 + P_4 + P_5 \leq 2 \cdot P_6 + 2 \cdot P_1. \end{aligned}$$

Note further that $P_3 + P_4 = 2 \cdot V(K) = P_6 + P_1$, so we can reduce the inequality to the following

$$P_2 + P_5 \leq P_6 + P_1 = 2 \cdot V(K).$$

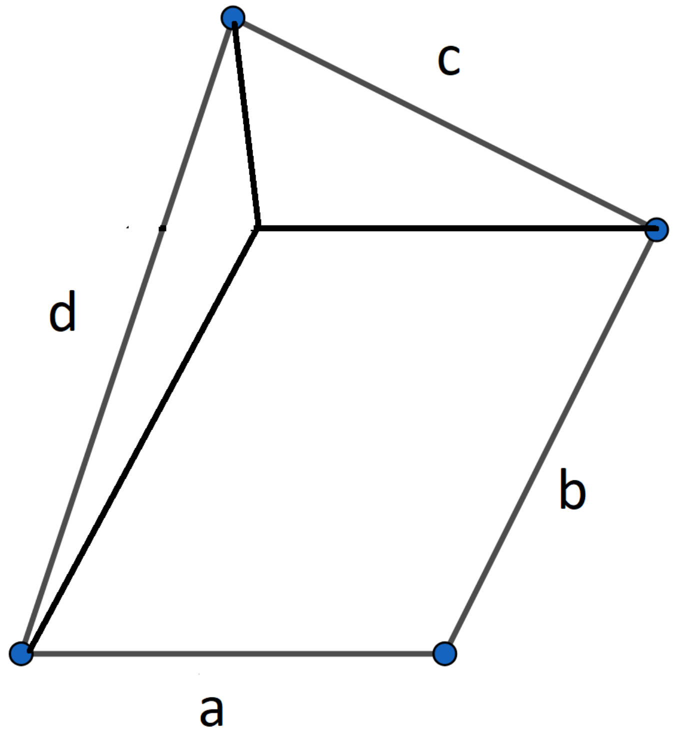

Figure 3.9: dissertation of K

As we can see in Figure 3.9, we have $V(K) = P_1 + (1/2) \cdot P_2 + (1/2) \cdot P_5$.

Thus, $2 \cdot V(K) = 2 \cdot P_1 + P_2 + P_5$.

Therefore, our main inequality becomes equivalent to $0 \leq 2 \cdot P_1 \Rightarrow 0 \leq P_1$, which is always true, concluding the proof.

$\square$

3.3 General Proof by Induction

Proposition 3.3.1. *Let $Q \subset \mathbb{R}^2$ be a convex planar polygon. Then, Q satisfies Petty's conjecture in $\mathbb{R}^2$, namely*

$$\frac{V(\Pi Q)}{V(Q)} \leq \frac{V(\Pi T)}{V(T)}, \tag{3.5}$$

for any triangle $T \subset \mathbb{R}^2$.

Proof. In Section 3.2, we proved that $\frac{V(\Pi K)}{V(K)} \leq \frac{V(\Pi T)}{V(T)} = 6$ for any convex polygonal body K in $\mathbb{R}^2$ with 4 sides.

Now, supposing that $\frac{V(\Pi K)}{V(K)} \leq 6$ for any convex polygonal body K in $\mathbb{R}^2$ with n sides, we will prove that $\frac{V(\Pi Q)}{V(Q)} \leq 6$ (or, equivalently, that $V(\Pi Q) \leq 6 \cdot V(Q)$) for any convex body Q with $n+1$ sides in $\mathbb{R}^2$.

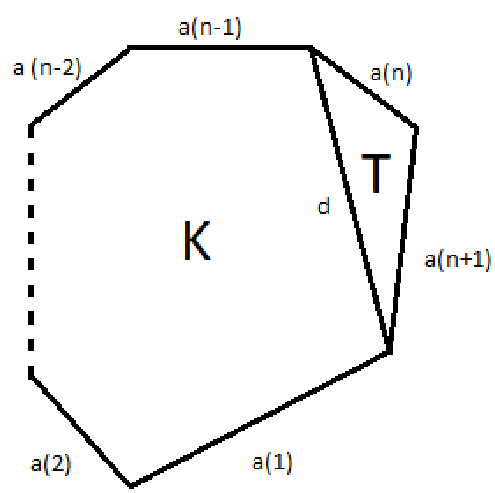

Figure 3.10: Q

As it is shown in Figure 3.10, the polygonal body Q can be divided into 2 convex bodies, K and T, so $V(Q) = V(K) + V(T)$.

We have started with Q having $n+1$ sides; $a_1, \ldots, a_{n+1}$.

Now, from the way we have cut Q, the convex polygonal body K has n sides; $a_1, \ldots, a_{n-1}, d$. Thus, due to our assumption, $\dfrac{V(\Pi K)}{V(K)} \leq 6$ and, consequently, $V(\Pi K) \leq 6 \cdot V(K)$

Furthermore, T has 3 sides: d, a_n, a_{n+1} and, as shown in Section 3.1, $\dfrac{V(\Pi T)}{V(T)} = 6$, or $V(\Pi T) = 6 \cdot V(T)$.

Recall that $V(\Pi Q)$ is equal to the sum of the area of the parallelograms formed by the 2-combinations of the lengths of the sides of Q. Denote by $V(ij)$ the area of the parallelogram formed by the 2 sides i and j, where i and j are any of the sides shown in Figure 3.10. From the additivity property of the area, we thus obtain the following equality:

$V(\Pi Q) = V(\Pi K) - V(da_1) - \ldots - V(da_{n-1}) + V(\Pi T) - V(da_n) - V(da_{n+1}) + V(a_n a_1) + \ldots + V(a_n a_{n-1}) + V(a_{n+1} a_1) + \ldots + V(a_{n+1} a_n)$.

Replacing our above result in $V(\Pi Q) \leq 6 \cdot V(Q)$, we obtain the equivalent claim:

$$V(\Pi K) - V(da_1) - \ldots - V(da_{n-1}) + V(\Pi T) - V(da_n) - V(da_{n+1}) + V(a_n a_1) + \ldots + V(a_n a_{n-1}) + V(a_{n+1} a_1) + \ldots + V(a_{n+1} a_n) \leq 6 \cdot V(K) + 6 \cdot V(T).$$

Since $V(\Pi K) \leq 6 \cdot V(K)$ and $V(\Pi T) = 6 \cdot V(T)$, this latter inequality becomes:

$$V(a_n a_1) + \ldots + V(a_n a_{n-1}) + V(a_{n+1} a_1) + \ldots + V(a_{n+1} a_n) \leq V(da_1) + \ldots + V(da_{n+1}).$$

Note that $V(da_1) = V(a_n a_1) + V(a_{n+1} a_1) \ldots V(da_{n-1}) = V(a_n a_{n-1}) + V(a_{n+1} a_{n-1})$ and $V(da_n) = V(a_{n+1} a_n)$ as these parallelograms share the same base and same height.

Thus, our inequality is equivalent now to

$$0 \leq V(da_{n+1})$$

which is always true. $\square$

Due to Corollary 1.3.1, Proposition 3.3.1 can be extended to arbitrary convex bodies in $\mathbb{R}^2$:

Corollary 3.3.1. *Let $Q \subset \mathbb{R}^2$ be a compact convex set with nonempty interior. Then, Q satisfies Petty's conjecture in $\mathbb{R}^2$, namely*

$$\frac{V(\Pi Q)}{V(Q)} \leq \frac{V(\Pi T)}{V(T)}, \quad \forall T \textit{ triangle } \subset \mathbb{R}^2. \tag{3.6}$$

Chapter 4

Petty's Conjecture in $\mathbb{R}^3$

4.1 Calculation of the Upper Bound (Tetrahedron)

Proposition 4.1.1. *Let $T \subset \mathbb{R}^3$ be an arbitrary, non-degenerate tetrahedron. Then,*

$$\frac{V(\Pi T)}{V^2(T)} = 18. \tag{4.1}$$

Proof. We will prove Proposition 4.1.1 using a right tetrahedron T that has 3 faces as right isosceles triangles. Consequently, the value of Petty's functional would be the same for any tetrahedron since Petty's functional is affine invariant as we have showed earlier.

Thus let T be the tetrahedron with the following vertices as in Figure 4.1:

$A = (0, 0, 0)$

$B = (1, 0, 0)$

$C = (0, 1, 0)$

$D = (0, 0, 1)$.

In order to calculate V(ΠT), we first need the unit normals to each face. The directions of the normals to the faces of this tetrahedron are the same for any right tetrahedron T that has 3 faces as right isosceles triangles.

Now, we calculate the normals:

- Normal to the triangle ABC:

  $(A - B) = (-1, 0, 0)$ and $(A - C) = (0, -1, 0)$

  Thus, the normal to the face ABC is $(A - B) \times (A - C) = (0, 0, 1)$.

- Normal to the triangle ADC:

  $(A - D) = (0, 0, -1)$ and $(A - C) = (0, -1, 0)$

  Thus, the normal to the face ACD is $(A - D) \times (A - C) = (1, 0, 0)$.

- Normal to the triangle ADB:

  $(A - B) = (-1, 0, 0)$ and $(A - D) = (, 0, -1)$

  Thus, the normal to the face ABD is $(A - B) \times (A - D) = (0, 1, 0)$.

- Normal to the triangle DBC:

  $(B - D) = (1, 0, -1)$ and $(B - C) = (1, -1, 0)$

  Thus, the normal to the face BCD is $(B - D) \times (B - C) = (1, 1, 1)$.

  This last vector is the only one that is not normalized to have unit length one.

  Thus, we do so and after normalizing it, we obtain the unit normal to the face BCD as $\dfrac{(1,1,1)}{(1^2+1^2+1^2)^{1/2}} = \dfrac{(1,1,1)}{3^{1/2}}$.

Thus, for any such tetrahedron where

$$\text{length of } \ AB = \text{length of } \ AC = \text{length of } \ AD = a, \tag{4.2}$$

we have $A = \dfrac{a^2}{2} = \text{Area}(ABC) = \text{Area}(ABD) = \text{Area}(ACD)$.

Additionally, $\text{Area}(BCD) = \dfrac{3^{1/2}}{4} \cdot (a \cdot 2^{1/2})^2 = a^2 \cdot \dfrac{3^{1/2}}{2} = 3^{1/2} \cdot A$, since BCD is an equilateral triangle with side a.

Thus, for calculating $V(\Pi T)$, we will use the following vectors, called area vectors:

- $v_1 = (A, 0, 0)$;

- $v_2 = (0, A, 0)$;

- $v_3 = (0, 0, A)$;

- $v_4 = \dfrac{(1,1,1)}{3^{1/2}} \cdot 3^{1/2} \cdot A = (A, A, A)$.

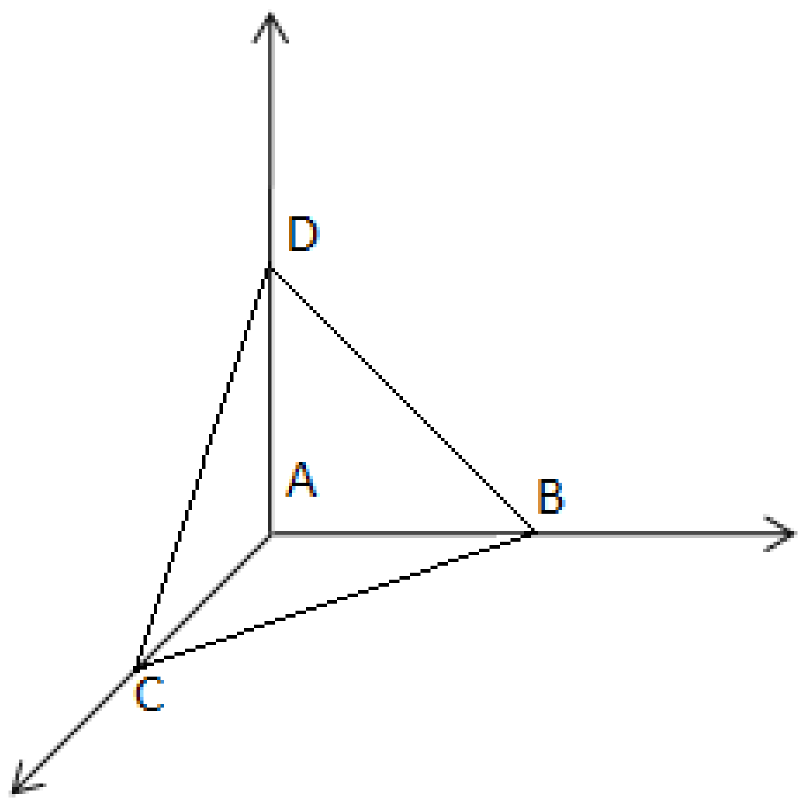

Figure 4.1: Right Tetrahedron T

Recall that $V(\Pi T)$ is equal to the sum of volumes of the parallelepipeds formed by the 3-combination of the area vectors v_1, v_2, v_3 and v_4, volumes which we list below:

- Volume of the parallelepiped formed by v_1, v_2 and v_3 is $V_1 = \|v_1,\, v_2,\, v_3\| = A^3$.

- Volume of the parallelepiped formed by v_1, v_2 and v_4 is $V_2 = \|v_1, v_2, v_4\| = A^3$.

- Volume of the parallelepiped formed by v_1, v_3 and v_4 is $V_3 = \|v_1, v_3, v_4\| = A^3$.

- Volume of the parallelepiped formed by v_2, v_3 and v_4 is $V_4 = \|v_2, v_3, v_4\| = A^3$.

  We have denoted by $\|u, v, w\|$ the absolute value of the determinant whose rows are the coordinates of the vectors u, v, w in $\mathbb{R}^3$.

Therefore, $V(\Pi T) = V_1 + V_2 + V_3 + V_4 = 4 \cdot A^3$

On the other hand, $V(T) = \dfrac{1}{6} \cdot a^3 = \dfrac{2 \cdot 2^{1/2}}{6} \cdot A^{3/2}$ and, consequently, $V(T)^2 = \dfrac{8}{36} \cdot A^3$.

Finally, substituting the above values in equation (4.1), we get

$$\frac{V(\Pi T)}{V^2(T)} = \frac{4 \cdot A^3}{\frac{8}{36} \cdot A^3} = 18, \tag{4.3}$$

which concludes the proof. □

4.2 Validation of Petty's Conjecture for K a Parallelepiped

Proposition 4.2.1. *Let $K \subset \mathbb{R}^3$ be an arbitrary parallelepiped. Then K satisfies Petty's conjecture in $\mathbb{R}^3$, i.e.*

$$\frac{V(\Pi K)}{V^2(K)} \leq \frac{V(\Pi T)}{V^2(T)}, \tag{4.4}$$

where $T \subset \mathbb{R}^3$ is a tetrahedron.

Proof. We have calculated in Section 4.1 that the value of Petty's functional is

$$\frac{V(\Pi T)}{V^2(T)} = 18, \tag{4.5}$$

for any tetrahedron $T \subset \mathbb{R}^3$.

Thus, given any parallelepiped K, it suffices to prove that

$$\frac{V(\Pi K)}{V^2(K)} \leq 18. \tag{4.6}$$

Recall that $V(\Pi K)$ is equal to the sum of areas of the parallelepipeds formed by the 3-combination of the 6 vectors that are perpendicular to each face of K and that have a length equal to the area of the correspondent face. Since a parallelepiped has pairs of parallel faces with same area, we have 3 different area vectors, each of which is repeated twice. Thus, we define $V(\Pi K')$, the area of the parallelepiped formed by the combination of the 3 vectors as shown in Figure 4.2. Consequently, since we are in $\mathbb{R}^3$,

$$V(\Pi K) = V(\Pi K') \cdot 2^3. \tag{4.7}$$

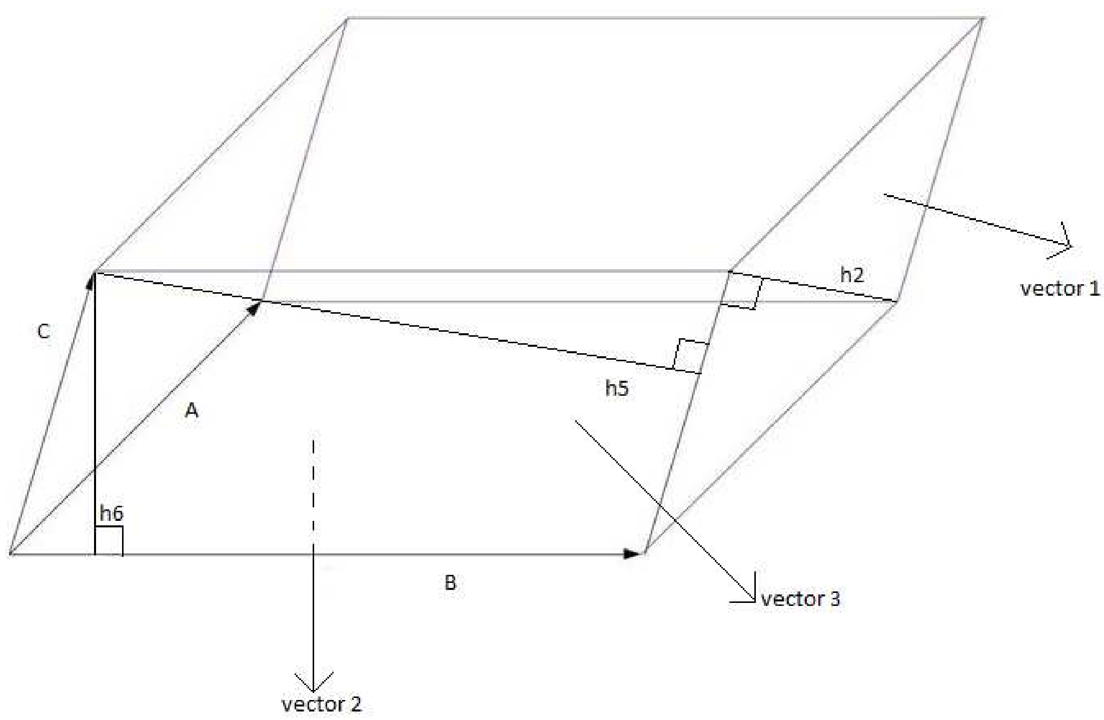

Figure 4.2: Parallelepiped K

In order to prove equation (4.6), we will write $V(\Pi K)$ in terms of $V(K)$.

First, we note the difference in volume between a parallelepiped M and another parallelepiped $\tilde{M}$ that is formed by changing only the length of the three vectors that form M, thus multiply each vector by a positive constant α, β and, respectively, γ as shown in Figure 4.3.

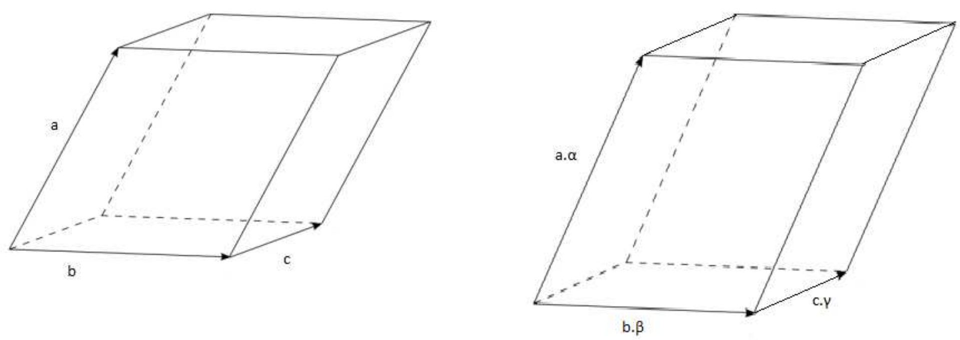

Figure 4.3: The parallelepipeds M and $\tilde{M}$

We get, directly from the volume formula as determinant, that

$$V(\tilde{M}) = 2^3 \cdot \alpha \cdot \beta \cdot \gamma \cdot V(M). \tag{4.8}$$

The body $\Pi K'$ is the parallelepiped presented in Figure 4.4. From the definition of projection bodies in Section 1.3, and the fact that $C \cdot h_2$, $A \cdot h_5$ and $B \cdot h_6$, are the areas of the faces of K (h_2, h_5, h_6 are chosen accordingly and presented in Figure 4.2), we have the following results:

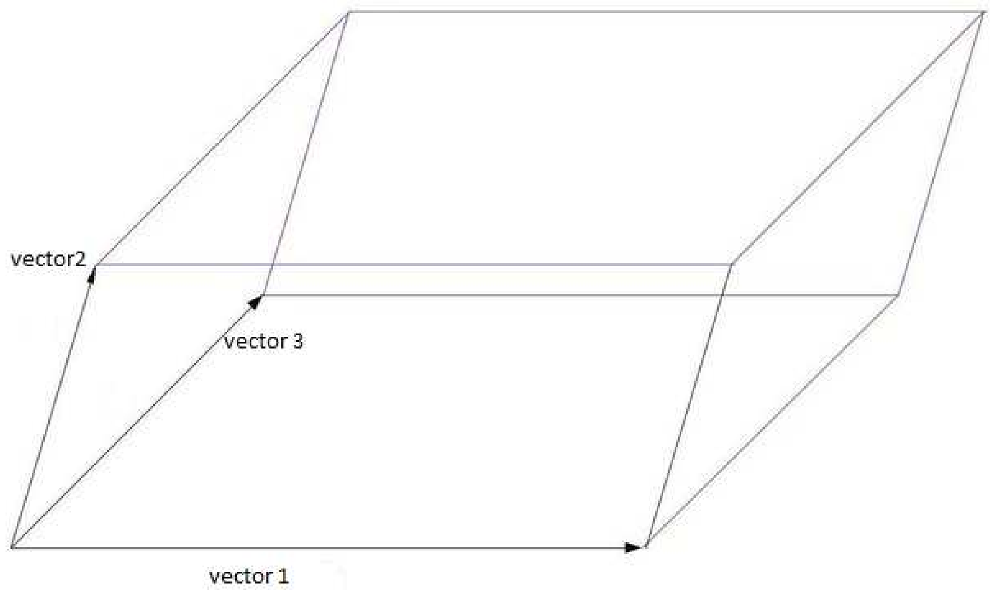

Figure 4.4:

$$\text{length of vector } 1 = C \cdot h_2 \tag{4.9}$$

$$\text{length of vector } 2 = A \cdot h_5 \tag{4.10}$$

$$\text{length of vector } 3 = B \cdot h_6. \tag{4.11}$$

Consequently, we find α, β and γ in order to write $V(\Pi K')$ in terms of $V(K)$ and, further, $V(\Pi K)$ in terms of $V(K)$.

We notice that $\Pi K'$ and K only differ by the lengths of the 3 vectors that form the two parallelepipeds (the projection has conserved the angles between the normals to the faces).

Now let α, β and γ be such that,

$$\text{length of vector } 1 = \alpha \cdot B \tag{4.12}$$

$$\text{length of vector } 2 = \beta \cdot C \tag{4.13}$$

$$\text{length of vector } 3 = \gamma \cdot A. \tag{4.14}$$

Using equations (4.9) to (4.14), we get,

$$\alpha = \frac{C \cdot h_2}{B} \tag{4.15}$$

$$\beta = \frac{A \cdot h_5}{C} \tag{4.16}$$

$$\gamma = \frac{B \cdot h_6}{A}. \tag{4.17}$$

Therefore, since $V(\Pi K') = \alpha \cdot \beta \cdot \gamma \cdot V(K)$ and $V(\Pi K) = 2^3 \cdot V(\Pi K')$, we have: $V(\Pi K) = 2^3 \cdot \alpha \cdot \beta \cdot \gamma \cdot V(K) = 2^3 \cdot \frac{C \cdot h_2}{B} \cdot \frac{A \cdot h_5}{C} \cdot \frac{B \cdot h_6}{A} \cdot V(K) = 2^3 \cdot h_2 \cdot h_5 \cdot h_6 \cdot V(K)$.

Finally, we will prove that $h_2 \;.\; h_5 \;.\; h_6 =$ V(K).

Proof. To prove the above claim, it is enough to show that $V(K)$ is equal to the volume of the rectangular box (a box with right dihedral angles) formed by the sides h_2, h_5 and h_6.

Let K_1 be same parallelepiped as K, but with the side C replaced by h_6. Then, V(K)=V(K_1). Let K_2 be same parallelepiped as K_1 with side B replaced by h_2. Then, $V(K) = V(K_1) = V(K_2)$. Thirdly, let K_3 be same parallelepiped as K_2 with side A replaced by h_5. Then, $V(K) = V(K_1) = V(K_2) = V(K_3) = h_2 \cdot h_5 \cdot h_6$.

□

We now obtain that

$$\frac{V(\Pi K)}{V^2(K)} = \frac{2^3 . V^2(K)}{V^2(K)} = 2^3, \tag{4.18}$$

which confirms that Petty's Conjecture is satisfied for K:

$$2^3 = \frac{V(\Pi K)}{V^2(K)} \le \frac{V(\Pi T)}{V^2(T)} = 18. \tag{4.19}$$

$\square$

Chapter 5

The Cut-off Vertex Method and Petty's Functional

5.1 The problem in $\mathbb{R}^2$

Proposition 5.1.1. *Let T be a triangle in $\mathbb{R}^2$ and let Q be a quadrilateral formed by cutting off one of the vertices of T', a triangle in $\mathbb{R}^2$. Then*

$$\frac{V(\Pi Q)}{V(Q)} \leq \frac{V(\Pi T)}{V(T)}. \tag{5.1}$$

Proof. We have calculated in Section 3.1 that, for any triangle $T \subset \mathbb{R}^2$, the value of Petty's functional is

$$\frac{V(\Pi T)}{V(T)} = 6. \tag{5.2}$$

Thus, it suffices to prove that

$$\frac{V(\Pi Q)}{V(Q)} \leq 6. \tag{5.3}$$

We thus calculate $V(Q)$ and $V(\Pi Q)$. Note that T and Q are illustrated in Figure

5.1.

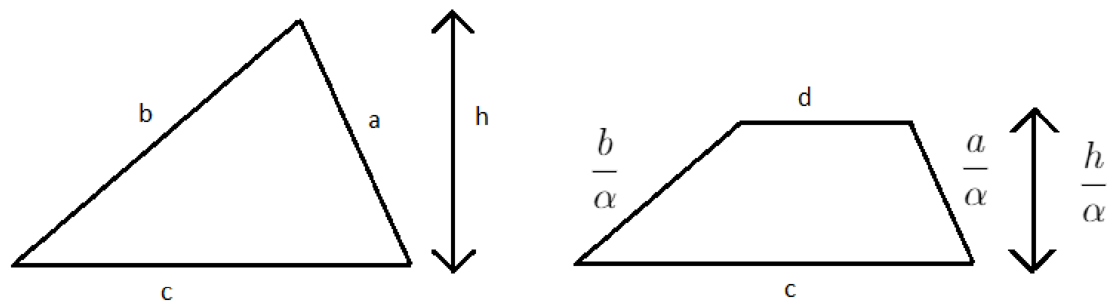

Figure 5.1: The polygons T and Q

Given the affine invariance of Petty's functional, we may choose to cut off $\dfrac{\alpha-1}{\alpha}$, $\alpha \geq 1$, of the two sides of T attached to one of the vertices. This means that our new side d is parallel to the base c and that the measures of Q are $d,\ c,\ \dfrac{a}{\alpha}$ and $\dfrac{b}{\alpha}$.

Consequently, $V(T) = \dfrac{c \cdot h}{2} = m$.

Furthermore, from the trapezoid's area formula, we have:
$V(Q) = \dfrac{c+d}{2} \cdot \dfrac{h}{\alpha} = \dfrac{h \cdot c}{2 \cdot \alpha} + \dfrac{d \cdot h}{2 \cdot \alpha} = \dfrac{m}{\alpha} + \dfrac{d \cdot h}{2 \cdot \alpha}$.

Note that ΠQ is formed by 6 parallelograms, as $\binom{4}{2} = 6$:

Parallelogram 1 is formed by c and d and, since the two sides are parallel, we have $V(P_1) = 0$.

Parallelogram 2 is formed by $\dfrac{a}{\alpha}$ and c and so $V(P_2) = \dfrac{2 \cdot m}{\alpha}$.

Parallelogram 3 is formed by $\dfrac{b}{\alpha}$ and c and so $V(P_3) = \dfrac{2 \cdot m}{\alpha}$.

Parallelogram 4 is formed by $\dfrac{a}{\alpha}$ and $\dfrac{b}{\alpha}$ and so $V(P_4) = \dfrac{2 \cdot m}{\alpha^2}$.

Parallelogram 5 is formed by $\dfrac{a}{\alpha}$ and d and so $V(P_5) = \dfrac{h}{\alpha} \cdot d$.

Parallelogram 6 is formed by $\frac{b}{\alpha}$ and d and so $V(P_6) = \frac{h}{\alpha} \cdot d$.

Thus,

$$\begin{aligned} V(\Pi Q) &= V(P1) + V(P2) + V(P3) + V(P4) + V(P5) + V(P6) \\ &= 4 \cdot \frac{m}{\alpha} + \frac{2 \cdot m}{\alpha^2} + \frac{2 \cdot h \cdot d}{\alpha}. \end{aligned}$$

We substitute our results in equation (5.3) and obtain

$$\frac{4 \cdot m}{\alpha} + \frac{2 \cdot m}{\alpha^2} + \frac{2 \cdot h \cdot d}{\alpha} \leq \frac{6 \cdot m}{\alpha} + \frac{6 \cdot d \cdot h}{2 \cdot \alpha} \tag{5.4}$$

$$\frac{2 \cdot m}{\alpha^2} + \frac{2 \cdot h \cdot d}{\alpha} \leq \frac{2 \cdot m}{\alpha} + \frac{3 \cdot d \cdot h}{\alpha} \tag{5.5}$$

$$\frac{2 \cdot m}{\alpha} + 2 \cdot h \cdot d \leq 2 \cdot m + 3 \cdot d \cdot h \tag{5.6}$$

$$\frac{2 \cdot m}{\alpha} \leq 2 \cdot m + d \cdot h. \tag{5.7}$$

Knowing that $\frac{2 \cdot m}{\alpha} \leq 2 \cdot m$ since $\alpha \geq 1$ and that $d \cdot h \geq 0$, equation (5.7) is satisfied and so is then equation (5.1). $\square$

5.2 An example of the problem in $\mathbb{R}^3$

In this section, we will give an example that by adding new vertices to a convex polytope in $\mathbb{R}^3$, Petty's functional does not necessarily decrease. Specifically, we show that

$$\frac{V(\Pi Q)}{V^2(Q)} > \frac{V(\Pi P)}{V^2(P)} \tag{5.8}$$

where P is a convex polytope in $\mathbb{R}^3$ and Q is P cut by a hyperplane eliminating one vertex, but introducing this way more vertices.

Example 5.2.1. *Let P be a convex body in $\mathbb{R}^3$ that is the union of C_1 and C_2 where C_1 is a cube and C_2 is a regular pyramid with a base square and whose lateral faces*

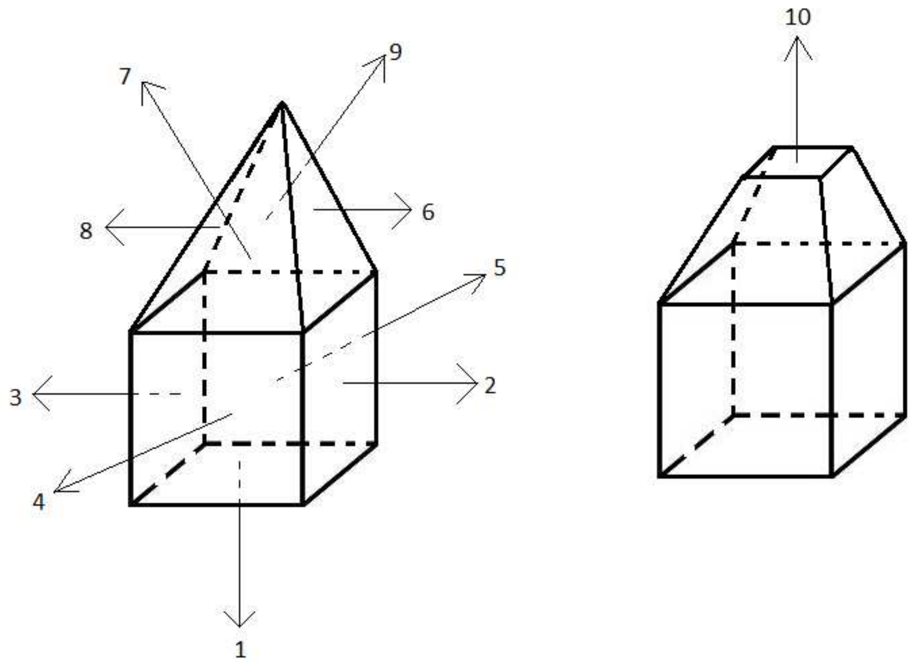

Figure 5.2: P and Q

are isosceles triangles. Now, we cut P by the vertex of C_2 by a plane parallel to its base and we call it Q. We illustrate the convex bodies considered in Figure 5.2.

Let h' be the height of C_2. In order to form Q, we cut off with a parallel plane to the base of C_2 an amount $(1-\alpha)$ of h', $0 < \alpha < 1$, so that $\alpha h'$ is left and the same proportion goes to every side that is attached to the vertex that is cut. Thus we obtain the following results:

$$V(P) = V(C_1) + V(C_2) \text{ and } V(Q) = V(C_1) + (1-(1-\alpha)^3)V(C_2).$$

Let a be the length of each side of the cube C_1 and b be the length of the side of the triangle that is formed by the projection of the tetrahedron C_2 on the xz-plane. Let θ be the angle between each face of C_2 and the base and let β be such that $V(C_2) = \beta V(C_1)$, then:

$$\beta = \frac{\tan\theta}{6}.$$

Proof. In Figure 5.3, we see the projection of the tetrahedron C_2 on the xz-plane assuming that the origin is placed at the center of symmetry of the base of C_2.

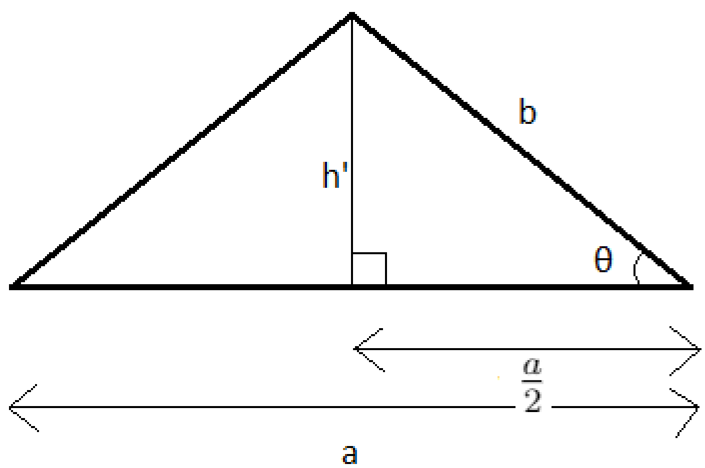

Figure 5.3: The projection of C_2 on the xz-plane

We have that $\sin\,\theta = \dfrac{h'}{b}$ and $\cos\,\theta = \dfrac{a}{2 \cdot b}$. We substitute the previous two equations in $V(C_2) = \dfrac{a^2 \cdot h'}{3}$ and we obtain

$$V(C_2) = \frac{a^2 \cdot b \cdot \sin\theta}{3} = \frac{a^3 \cdot \sin\theta}{6 \cdot \cos\theta} = \frac{a^3 \cdot \tan\theta}{6} = V(C_1) \cdot \frac{\tan\theta}{6}. \tag{5.9}$$

Note that $V(C_1) = a^3$. □

Thus, now we have,

$$V(P) = \left(1 + \frac{\tan\theta}{6}\right) V(C_1) \quad \textit{and} \quad V(Q) = \left(1 + \frac{\tan\theta}{6} \cdot (1 - (1-\alpha)^3)\right) V(C_1).$$

Therefore, the previously conjectured inequality

$$\frac{V(\Pi Q)}{V^2(Q)} \leq \frac{V(\Pi P)}{V^2(P)} \quad \textit{is equivalent to} \quad V(\Pi Q) \cdot V^2(P) \leq V(\Pi P) \cdot V^2(Q) \tag{5.10}$$

and becomes

$$V(\Pi Q)\cdot\left(1+\frac{\tan\theta}{6}\right)^2 \le V(\Pi P)\cdot\left(1+\frac{\tan\theta}{6}\cdot(1-(1-\alpha)^3)\right)^2. \tag{5.11}$$

We now calculate $V(\Pi P)$ and $V(\Pi Q)$. As shown in Figure 5.2 the vectors $1,\dots,9$ are the vectors that are perpendicular to each face of P and that have length equal to the area of the corresponding face. Let $V(ijk)$ represent the volume of the parallelepiped formed by the vectors i, j and k where the vectors i,j,k can be any of the indices $1,\dots,9$.

Recall that $V(\Pi P)$ is equal to sum of all volumes of parallelepipeds that can be formed by the 3-combination of vectors$1,\dots,9$. Consequently, $V(\Pi P)$ is the sum of the following volumes that we divided into sets:

Set X: $V(124)$, $V(125)$, $V(134)$, $V(135)$. Let V_X be such that all the volumes in this set are equal to V_X, thus they all are equal to each other.

Set Y_1: $V(127)$, $V(129)$, $V(137)$, $V(139)$, $V(146)$, $V(148)$, $V(156)$, $V(158)$. Let V_{Y_1} be such that all the volumes in this set are equal to V_{Y_1}.

Set Z_1: $V(167)$, $V(169)$, $V(178)$, $V(189)$. Let V_{Z_1} be such that all the volumes in this set are equal to V_{Z_1}.

Set Y: $V(246)$, $V(247)$, $V(248)$, $V(249)$, $V(256)$, $V(257)$, $V(258)$, $V(259)$, $V(346)$, $V(347)$, $V(348)$, $V(349)$, $V(356)$, $V(357)$, $V(358)$, $V(359)$. Let V_Y be such that all the volumes in this set are equal to V_Y.

Set Z: $V(267)$, $V(269)$, $V(278)$, $V(279)$, $V(289)$, $V(367)$, $V(369)$, $V(378)$, $V(379)$, $V(389)$, $V(467)$, $V(468)$, $V(469)$,$V(478)$,$V(489)$, $V(567)$, $V(568)$, $V(569)$, $V(578)$, $V(589)$. Let V_Z be such that all the volumes in this set are equal to V_Z.

Set W: $V(678)$, $V(679)$, $V(689)$, $V(789)$. Let V_W be such that all the volumes in this set are equal to V_W.

Notice that $V(ijk)$=0 in the following cases:

Case 1: ijk is a $3-$combination of vectors $1, 4, 5, 7, 9$ because these vectors are in the same plane.

Case 2: ijk is a $3-$combination of vectors $1, 2, 3, 6, 8$ because these vectors are in the same plane also.

Case 4: ijk consists of the vectors 4 and 5 and any other third vector (because the vectors 4 and 5 are in the same direction so the height of the parallelepiped is 0).

Case 5: ijk consists of the vectors 1 and 10 and any other third vector (because the vectors 1 and 10 are in the same direction so the height of the parallelepiped is 0) where vector 10 is the corresponding vector to the face that is formed by cutting P in order to form Q.

Case 6: ijk consists of the vectors 2 and 3 and any other third vector (because the vectors 2 and 3 are in the same direction so the height of the parallelepiped is 0).

Now, knowing that the volume is the absolute value of the determinant, we will calculate V_X, V_{Y_1}, V_{Z_1}, V_Y, V_Z and V_W:

$$V_X = a^2 \cdot a^2 \cdot a^2 = a^6;$$

$$V_{Y_1} = \begin{vmatrix} 0 & 0 & -a^2 \\ 0 & a^2 & 0 \\ h_1 & 0 & h_2 \end{vmatrix} = a^4 \cdot h_1;$$

$$V_{Z_1} = \begin{vmatrix} 0 & 0 & -a^2 \\ 0 & h_1 & h_2 \\ -h_1 & 0 & h_2 \end{vmatrix} = a^2 \cdot h_1^2;$$

$$V_Y = \begin{vmatrix} 0 & -a^2 & 0 \\ a^2 & 0 & 0 \\ 0 & h_1 & h_2 \end{vmatrix} = a^4 \cdot h_2;$$

$$V_Z = \begin{vmatrix} 0 & a^2 & 0 \\ 0 & h_1 & h_2 \\ h_1 & 0 & h_2 \end{vmatrix} = a^2 \cdot h_1 \cdot h_2;$$

$$V_W = \begin{vmatrix} 0 & h_1 & h_2 \\ h_1 & 0 & h_2 \\ 0 & h_1 & h_2 \end{vmatrix} = 2 \cdot h_1^2 \cdot h_2,$$

where h_1 is the height of vectors $6, 7, 8$ and 9 projected on the xy-plane and h_2 is the height of vectors $6, 7, 8$ and 9 projected on the z-axis as shown in Figure 5.4.

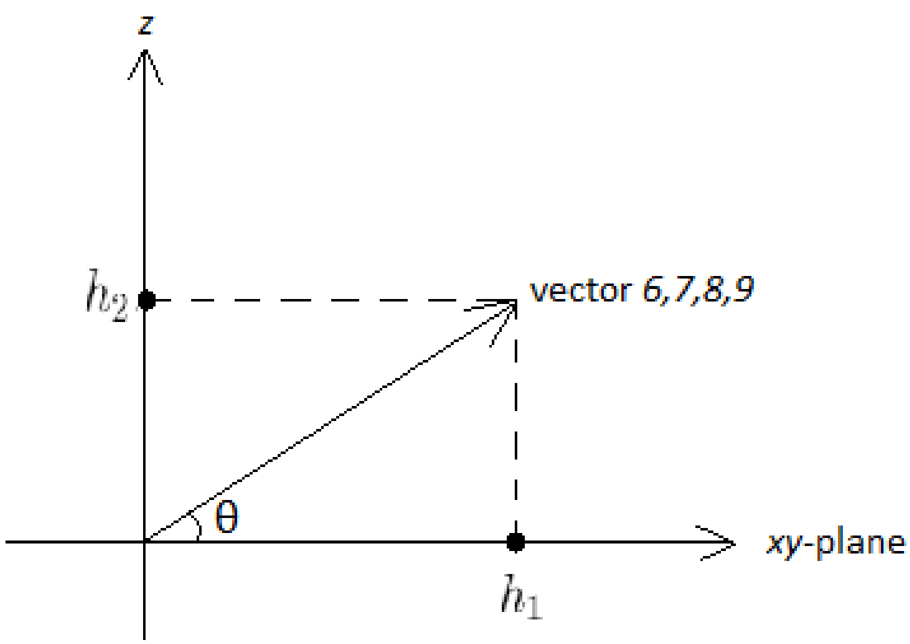

Figure 5.4: vectors $6, 7, 8, 9$

Now, we will express h_1 and h_2 in terms of a and θ:

length of vector $6 =$ *length of vector* $7 =$ *length of vector* $8 =$ *length of vector* $9 = \dfrac{a \cdot b}{2}$

where b is the height of the triangular faces of P.

Thus, $\cos\theta = \dfrac{2 \cdot h_1}{a \cdot b}$ *and* $\sin\theta = \dfrac{2 \cdot h_2}{a \cdot b}$.

Consequently, $h_1 = \dfrac{a \cdot b \cdot \cos\theta}{2}$ *and* $h_2 = \dfrac{a \cdot b \cdot \sin\theta}{2}$.
Since $\cos\theta = \dfrac{a}{2 \cdot b}$, *we have*

$$h_1 = \frac{a^2}{4} \text{ and } h_2 = \frac{a^2}{4} \cdot \tan\theta.$$

Therefore,

$$\begin{aligned} V(\Pi P) &= V(SetX) + V(SetY_1) + V(SetZ_1) + V(SetY) + V(SetZ) + V(SetW) \\ &= 4 \cdot V_X + 8 \cdot V_{Y_1} + 4 \cdot V_{Z_1} + 16 \cdot V_Y + 20 \cdot V_Z + 4 \cdot V_W \\ &= 4 \cdot a^6 + 8 \cdot \frac{a^6}{4} + 4 \cdot \frac{a^6}{16} + 16 \cdot \frac{a^6 \cdot \tan\theta}{4} + 20 \cdot \frac{a^6 \cdot \tan\theta}{16} + 4 \cdot \frac{a^6 \cdot \tan\theta}{32} \\ &= a^6 \cdot (\frac{25}{4} + \frac{43}{8} \cdot \tan\theta). \end{aligned} \tag{5.12}$$

The projection body ΠQ *is formed by the vectors* $1, 2, 3, 4, 5, 6', 7', 8', 9', 10$ *where vectors* $1, 2, 3, 4, 5$ *are same for* ΠP *and* ΠQ *and vectors* $6', 7', 8', 9', 10$ *have same direction as vectors* $6, 7, 8, 9, 1$ *respectively, but with different magnitudes.* $V(\Pi Q)$ *is the sum of the following volumes that we divided into sets:*
Set X'*:* $V(124)$, $V(125)$, $V(134)$, $V(135)$.
Set Y_1'*:* $V(127')$, $V(129')$, $V(137')$, $V(139')$, $V(146')$, $V(148')$, $V(156')$, $V(158')$.
Set Z_1'*:* $V(16'7')$, $V(16'9')$, $V(17'8')$, $V(18'9')$.
Set Y'*:* $V(246')$, $V(247')$, $V(248')$, $V(249')$, $V(256')$, $V(257')$, $V(258')$, $V(259')$, $V(346')$, $V(347')$, $V(348')$, $V(349')$, $V(356')$, $V(357')$, $V(358')$, $V(359')$.
Set Z'*:* $V(26'7')$, $V(26'9')$, $V(27'8')$, $V(27'9')$, $V(28'9')$, $V(36'7')$, $V(36'9')$, $V(37'8')$, $V(37'9')$, $V(38'9')$, $V(46'7')$, $V(46'8')$, $V(46'9')$,$V(47'8')$,$V(48'9')$, $V(56'7')$, $V(56'8')$, $V(56'9')$, $V(57'8')$, $V(58'9')$.
Set W'*:* $V(6'7'8')$, $V(6'7'9')$, $V(6'8'9')$, $V(7'8'9')$.
Set X''*:* $V(10\ \ 2\ \ 4)$, $V(10\ \ 2\ \ 5)$, $V(10\ \ 3\ \ 4)$, $V(10\ \ 3\ \ 5)$, *or equivalently,* $V(1'24)$, $V(1'25)$, $V(1'34)$, $V(1'35)$.
Set Y_1''*:* $V(10\ \ 2\ \ 7')$, $V(10\ \ 2\ \ 9')$, $V(10\ \ 3\ \ 7')$, $V(10\ \ 3\ \ 9')$, $V(10\ \ 4\ \ 6')$,

$V(10\ \ 4\ \ 8')$, $V(10\ \ 5\ \ 6')$, $V(10\ \ 5\ \ 8')$, or equivalently, $V(1'27')$, $V(1'29')$, $V(1'37')$, $V(1'39')$, $V(1'46')$, $V(1'48')$, $V(1'56')$, $V(1'58')$.

Set Z_1'': $V(10\ \ 6'\ \ 7')$, $V(10\ \ 6'\ \ 9')$, $V(1\ \ 7'\ \ 8')$, $V(10\ \ 8'\ \ 9')$, or equivalently, $V(1'6'7')$, $V(1'6'9')$, $V(1'7'8')$, $V(1'8'9')$.

As vectors 6 and $6'$, 7 and $7'$, 8 and $8'$, 9 and $9'$, 1 and 10 or 1 and $1'$ have the same direction as each other, but different length because of the cut, we have:

$$\textit{length of } 6' = (1-(1-\alpha)^2) \textit{ length of } 6,$$

$$\textit{length of } 7' = (1-(1-\alpha)^2) \textit{ length of } 7,$$

$$\textit{length of } 8' = (1-(1-\alpha)^2) \textit{ length of } 8,$$

$$\textit{length of } 9' = (1-(1-\alpha)^2) \textit{ length of } 9,$$

$$\textit{length of } 1' = (1-\alpha)^2 \textit{ length of } 1.$$

We get the later result from Thalès theorem after projection C_2 on the xz-plane and then by calculating the area of the square.

Therefore, we can write the value associated to each of the sets above as the follows:

Set X'= Set $X = 4 \cdot V_X$

Set Y_1'= $(1-(1-\alpha)^2)$ Set $Y_1 = 8 \cdot (1-(1-\alpha)^2) \cdot V_{Y_1}$

Set Z_1'= $(1-(1-\alpha)^2)^2$ Set $Z_1 = 4 \cdot (1-(1-\alpha)^2)^2 \cdot V_{Z_1}$

Set Y'= $(1-(1-\alpha)^2)$ Set Y =$16 \cdot (1-(1-\alpha)^2) \cdot V_Y$

Set Z'= $(1-(1-\alpha)^2)^2$ Set $Z = 20 \cdot (1-(1-\alpha)^2)^2 \cdot V_Z$

Set W'= $(1-(1-\alpha)^2)^3$ Set $W = 4 \cdot (1-(1-\alpha)^2)^3 \cdot V_W$

Set X''= $(1-\alpha)^2$ Set $X = 4(1-\alpha)^2 V_X$

Set Y_1''= $(1-\alpha)^2(1-(1-\alpha)^2)$ Set $Y_1 = 8(1-\alpha)^2(1-(1-\alpha)^2)V_{Y_1}$

Set Z_1''= $(1-\alpha)^2(1-(1-\alpha)^2)^2$ Set $Z_1 = 4(1-\alpha)^2(1-(1-\alpha)^2)^2 V_{Z_1}$.

Therefore,

$$
\begin{aligned}
V(\Pi Q) &= V(Set\ X') + V(Set\ Y_1') + V(Set\ Z_1') + V(Set\ Y') + V(Set\ Z') + V(Set\ W') \\
&+ V(Set\ X'') + V(Set\ Y_1'') + V(Set\ Z_1'') \\
&= 4 \cdot V_X + 8 \cdot (1-(1-\alpha)^2) \cdot V_{Y_1} + 4 \cdot (1-(1-\alpha)^2)^2 \cdot V_{Z_1} \\
&+ 16 \cdot (1-(1-\alpha)^2) \cdot V_Y + 20 \cdot (1-(1-\alpha)^2)^2 \cdot V_Z + 4 \cdot (1-(1-\alpha)^2)^3 \cdot V_W \\
&+ 4(1-\alpha)^2 V_X + 8(1-\alpha)^2(1-(1-\alpha)^2) V_{Y_1} + 4(1-\alpha)^2(1-(1-\alpha)^2)^2 V_{Z_1} \\
&= 4 \cdot a^6 + 8 \cdot (1-(1-\alpha)^2) \cdot \frac{a^6}{4} + 4 \cdot (1-(1-\alpha)^2)^2 \cdot \frac{a^6}{16} \\
&+ 16 \cdot (1-(1-\alpha)^2) \cdot \frac{a^6 \cdot \tan\theta}{4} + 20 \cdot (1-(1-\alpha)^2)^2 \cdot \frac{a^6 \cdot \tan\theta}{16} \\
&+ 4 \cdot (1-(1-\alpha)^2)^3 \cdot \frac{a^6 \cdot \tan\theta}{32} + 4(1-\alpha)^2 a^6 \\
&+ 8(1-\alpha)^2(1-(1-\alpha)^2)\frac{a^6}{4} \\
&+ 4(1-\alpha)^2(1-(1-\alpha)^2)^2\frac{a^6}{16} \\
&= 4 \cdot a^6 + 2 \cdot (1-(1-\alpha)^2) \cdot a^6 + (1-(1-\alpha)^2)^2 \cdot \frac{a^6}{4} \\
&+ 4 \cdot (1-(1-\alpha)^2) \cdot a^6 \cdot \tan\theta + 5 \cdot (1-(1-\alpha)^2)^2 \cdot \frac{a^6 \cdot \tan\theta}{4} \\
&+ (1-(1-\alpha)^2)^3 \cdot \frac{a^6 \cdot \tan\theta}{8} + 4(1-\alpha)^2 a^6 \\
&+ 2(1-\alpha)^2(1-(1-\alpha)^2) \cdot a^6 \\
&+ (1-\alpha)^2(1-(1-\alpha)^2)^2\, \frac{a^6}{4}. \qquad (5.13)
\end{aligned}
$$

Plugging in our previous results in equation (5.11) and, simplifying both sides by a^6*, we get*

$$
\begin{aligned}
[\quad & 4 + 2 \cdot (1-(1-\alpha)^2) + (1-(1-\alpha)^2)^2 \cdot \frac{1}{4} \\
+ \quad & 4 \cdot (1-(1-\alpha)^2) \cdot \tan\theta + 5 \cdot (1-(1-\alpha)^2)^2 \cdot \frac{\tan\theta}{4} \\
+ \quad & (1-(1-\alpha)^2)^3 \cdot \frac{\tan\theta}{8} + 4(1-\alpha)^2 \\
+ \quad & 2(1-\alpha)^2(1-(1-\alpha)^2) \\
+ \quad & (1-\alpha)^2(1-(1-\alpha)^2)^2\frac{1}{4} \quad] \cdot \left(1 + \frac{\tan\theta}{6}\right)^2 \\
\leq \quad & (\frac{25}{4} + \frac{43}{8} \cdot \tan\theta) \cdot \left(1 + \frac{\tan\theta}{6} \cdot (1-(1-\alpha)^3)\right)^2. \qquad (5.14)
\end{aligned}
$$

Denote by $f(\alpha, \theta)$ the left-hand side of the previous inequality and by $g(\alpha, \theta)$ its right-hand side. Note that these are continuous functions on $(0,1) \times (0, \pi/2)$ for any $0 < \alpha < 1$ and $0 < \theta < \pi/2$.

We expand $f(\alpha, \theta) - g(\alpha, \theta)$ using Wolfram Alpha in order to make the calculation easier. We get

$$
\begin{aligned}
& f(\alpha, \theta) - g(\alpha, \theta) = \\
& [\alpha^6 \cdot \frac{2 - \tan\theta}{8} + \alpha^5 \cdot \frac{3\tan\theta - 6}{4} + \alpha^4 \cdot \frac{6 - \tan\theta}{4} + \alpha^3 \cdot (4 - 4\tan\theta) \\
+ \quad & \alpha^2 \cdot (\tan\theta - 6) + 8\alpha\tan\theta + 8] \cdot (1 + \frac{\tan\theta}{6})^2 \\
- \quad & (\frac{25}{4} + \frac{43}{8} \cdot \tan\theta) \cdot (1 + \frac{\tan\theta}{6} \cdot (\alpha^3 - 3 \cdot \alpha^2 + 3 \cdot \alpha))^2. \qquad (5.15)
\end{aligned}
$$

We solve for α the equation $f(\alpha, \theta) - g(\alpha, \theta) = 0$ using Wolfram Alpha and we get a unique solution that is $\alpha = 1$ which is, a priori, known because in this case we did not cut any subset from P and consequently $P = Q$ and $P(P) = P(Q)$. Thus, for $\alpha = 1$, inequality (5.14) is satisfied for any θ.

Using the same software, we solve $f(\alpha, \theta) - g(\alpha, \theta) \leq 0$ when $\alpha \in [0,1]$ and $\theta \in [0, \frac{\pi}{2}]$ and we get the following solution:

- $\alpha_1 = 0$ *and* $0.684611 < \tan\theta < 11.5029$ *which is equivalent to* $34.395974245° < \theta < 85.03152046°$ *(in degrees).*

- $\alpha_2 = 0.321719$ *and* $0.660805 < \tan\theta$ *which is equivalent to* $33.456953714° < \theta$ *(in degrees).*

For $0 \leq \alpha \leq 1$*, only for* α_1 *and* α_2*, the inequality is satisfied for some* $\tan\theta$ *that does not depend on* α*. For* $0 < \alpha < 0.321719$ *and* $0.321719 < \alpha < 1$*,* $\tan\theta$ *depends on* α *for the inequality to be satisfied. We provide below some cases:*

- *Example 1: Let* $\alpha = 0.1$*,* $f(\alpha, \theta) - g(\alpha, \theta) \leq 0$ *is satisfied when* $0.681372 < \tan\theta < 14.8048$ *or consequently* $34.269407868° < \theta < 86.13561013°$.

- *Example 2: Let* $\alpha = 0.3$*,* $f(\alpha, \theta) - g(\alpha, \theta) \leq 0$ *is satisfied when* $0.663332 < \tan\theta < 151.812$ *or consequently* $33.557565103° < \theta < 89.62263127°$.

- *Example 3: Let* $\alpha = 0.4$*,* $f(\alpha, \theta) - g(\alpha, \theta) \leq 0$ *is satisfied when* $0.651013 < \tan\theta$ *or consequently* $33.064649512° < \theta$.

- *Example 4: Let* $\alpha = 0.5$*,* $f(\alpha, \theta) - g(\alpha, \theta) \leq 0$ *is satisfied when* $0.637098 < \tan\theta$ *or consequently* $32.50114552° < \theta$.

- *Example 5: Let* $\alpha = 0.7$*,* $f(\alpha, \theta) - g(\alpha, \theta) \leq 0$ *is satisfied when* $0.604926 < \tan\theta$ *or consequently* $31.170852112° < \theta$.

- *Example 6: Let* $\alpha = 0.9$*,* $f(\alpha, \theta) - g(\alpha, \theta) \leq 0$ *is satisfied when* $0.56687 < \tan\theta$ *or consequently* $29.547605382° < \theta$.

We notice that as $\alpha \nearrow 1$ *the range of* θ *for which* $f(\alpha, \theta) - g(\alpha, \theta) \leq 0$ *is satisfied increases.*

For instance, if we plug in $\theta = \dfrac{\pi}{4}$ *in the inequality, the conjecture will be satisfied for any* $0 \leq \alpha \leq 1$ *and* $0 \leq \theta \leq \dfrac{\pi}{2}$.

Furthermore, if we plug in $\theta = \dfrac{\pi}{4}$ *in the inequality, we will get*

$$
\begin{aligned}
[\quad & 4+2\cdot(1-(1-\alpha)^2)+(1-(1-\alpha)^2)^2\cdot\frac{1}{4} \\
+ \quad & 4\cdot(1-(1-\alpha)^2)+5\cdot(1-(1-\alpha)^2)^2\cdot\frac{1}{4} \\
+ \quad & (1-(1-\alpha)^2)^3\cdot\frac{1}{8}+4(1-\alpha)^2 \\
+ \quad & 2(1-\alpha)^2(1-(1-\alpha)^2) \\
+ \quad & (1-\alpha)^2(1-(1-\alpha)^2)^2\frac{1}{4} \quad]\cdot\left(1+\frac{1}{6}\right)^2 \\
\leq \quad & (\frac{25}{4}+\frac{43}{8})\cdot\left(1+\frac{1}{6}\cdot(1-(1-\alpha)^3)\right)^2. \qquad (5.16)
\end{aligned}
$$

which can be reduced to:

$$
\begin{aligned}
[\quad & \frac{\alpha^6}{8}+\frac{-3\alpha^5}{4}+\frac{5\alpha^4}{4}-5\cdot\alpha^2+8\alpha+8]\cdot(\frac{7}{6})^2 \qquad (5.17) \\
\leq \quad & (\frac{93}{8})\cdot(1+\frac{1}{6}\cdot(1-(1-\alpha)^3))^2.
\end{aligned}
$$

Using Wolfram Alpha, the above inequality is satisfied for any $\alpha < 1$ and, in fact, $\alpha > 1$. Also, since we know that for $\alpha = 1$ we have equality between $f(\alpha, \theta)$ and $g(\alpha, \theta)$, the monotonicity of Petty's functional is satisfied, as in the planar case, when $\theta = \frac{\pi}{4}$.

However, we will present now a counter example for the specific case when $\alpha = 0.5$, $a = 1$ and $\theta = \frac{\pi}{6}$. In this case, after substituting the corresponding values, as above, in

$$\frac{V(\Pi Q)}{V^2(Q)} \leq \frac{V(\Pi P)}{V^2(P)}, \qquad (5.18)$$

we get,

$$\frac{1805}{256} \leq \frac{25}{4} \qquad (5.19)$$

which is false.

Thus, we have proved:

Theorem 5.2.1. *There exists a polytope P in $\mathbb{R}^3$, and there exists an affine hyperplane $H \subset \mathbb{R}^3$, such that $P \cap H^+$ contains exactly one vertex of P and the convex body $Q = P \cap H^-$ has a larger Petty ratio than that of P:*

$$\frac{V(\Pi Q)}{V^2(Q)} \geq \frac{V(\Pi P)}{V^2(P)}. \tag{5.20}$$

Thus, the cut-off method does not always decrease the value of Petty's functional and hence, unlike the planar case, cannot be used to prove Petty's conjecture in $\mathbb{R}^3$.

$\square$

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