Global Hedging using Options

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Abstract

The classical global hedging approach presented in the literature (see Schweizer [1995]) involves using only the underlying asset to hedge a given contingent claim. The current thesis extends this approach by allowing for the use of a portfolio comprised of the underlying as well as other options written on that same underlying to be used as hedging instruments. Classical quadratic global hedging results such as the dynamic programming solution approach are adapted to this framework and are used to solve the global hedging problem presented here. The performance of this methodology is then investigated and benchmarked against the classical global hedging, as well as the traditional delta and delta-gamma hedging approaches. Various numerical analyses of the hedging errors, turnover and the shapes of quantities involved in dynamic programming solution approach are performed. It is found that option-based global hedging, where options are used as hedging instruments, outperforms other methodologies by yielding the lowest quadratic hedging error as expected. Situations where option-based global hedging has the most significant advantage over the other hedging methodologies are identified and discussed.

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Chapter 1

Introduction

Hedging plays a central role within the financial industry. It is used to offset potential losses of an investment. The most well-known hedging strategy for financial derivatives is delta hedging based on the modelling approach of Black and Scholes [1973]. In that paper, it is shown that under certain assumptions, one can take away the investment risk by continuously trading the underlying and the risk-free asset. Due to the unrealistic assumptions of the Black and Scholes model for the underlying price process and the impossibility of continuous trading, which leads to market incompleteness this method cannot fully eliminate risk in practice. Therefore, researchers have proposed hedging frameworks that are applicable under market incompleteness and more realistic features of the underlying asset model.

Among the recently proposed hedging approaches from the literature, one framework that allows minimizing the final hedging loss rather than locally neutralizing the delta or gamma of the option is global hedging. The idea of global hedging was first introduced by Duffie et al. [1991] and Schweizer [1995]. The objective of global hedging is to build a self-financing hedging portfolio which minimizes the hedging error at the end of a fixed time horizon. The hedging errors are defined as hedging losses penalized by a loss function. This loss function determines the risk aversion.

Schweizer [1995] suggests a one-dimensional global hedging framework using a quadratic loss function. Many papers including Černỳ et al. [2007], Rémillard and Rubenthaler [2013], Augustyniak et al. [2017] and Godin [2018] have worked within this framework. Bertsimas et al. [2001] solve the optimal replication problem of a European derivative security while considering an approximation error, denoted by ε. The approximation error is the square root of the mean-squared replication error of the optimal strategy, which is given recursively when working on quadratic loss functions. The authors interpret it as the degree of market incompleteness. Černỳ et al. [2007] generalize global quadratic hedging into a multi-dimensional semi-martingale setting and Rémillard and Rubenthaler [2013] use this result to find an iterative method to calculate the weights of the assets held in the self-financing hedging portfolio for each hedging period. Augustyniak et al. [2017] compare discrete-time global quadratic hedging and discrete-time local risk-minimization under the general class of GARCH models. They present simulation results showing that global hedging outperforms local risk-minimization when hedging options with a very long maturity. In this thesis, it is shown that this framework can be expressed in the similar way as the framework used in

Augustyniak et al. [2017]. Godin [2018] obtains a closed-form solution for the discrete-time global quadratic hedging presented in Schweizer [1995] applied to vanilla European options when the underlying asset follows a geometric Gaussian random walk.

Another approach, called local risk-minimization, aims at minimizing the incremental costs of hedging at the next time step. This approach is proposed by Schweizer [1988] and then continued in Schweizer [1991]. Later on, Coleman et al. [2007] show that under a Merton jump diffusion model, hedging with standard options is superior to hedging solely with the underlying in terms of risk reduction. They also provide numerical results suggesting that risk minimization using standard options is potentially more effective in terms of both volatility and jump risk reduction when the volatility model suitably reflects the dynamics of the data. Coleman et al. [2006] investigate the effectiveness of jointly modelling the interest rate and the underlying asset in the local risk-minimization setting when hedging options with long maturity. They use both the underlying and liquid standard options as hedging instruments to compare their performance. Their numerical experiments show that hedging with standard options leads to a considerably better performance than annual or monthly hedging with the underlying in terms of hedging error minimization and risk reduction.

In contrast to dynamic hedging, some researchers have proposed using a large span of derivatives with flexible payoffs as hedging instruments which allows keeping the initial weights of the hedging instruments fixed until the expiration of the option being hedged. In particular, Carr and Wu [2013] suggest using a continuum of vanilla options to statically hedge another vanilla option. They first show that when there is only one source of risk and a Markovian underlying, the target vanilla option payoff with a given time to maturity can be spanned using shorter maturity options with strikes varying in $\mathbb{R}^+$. They also talk about semi-static hedging which involves the idea of using static hedging with infrequent adjustments to the hedging portfolio. In this method, they proceed by applying sequential static hedging using shorter maturity vanilla options to hedge a long term path dependent option where the path is discretely monitored.

In practice, financial institutions use options in the framework of delta-gamma hedging as mentioned in Hull [2003] to locally neutralize the convexity of an option using other options. However, most of the literature on global hedging focuses on using only the underlying as the hedging instrument to hedge a given option. In the current thesis, the impact of using standard options combined with the underlying as hedging instruments on risk reduction using discrete-time global hedging is investigated. The global hedging method including options and the underlying as hedging instrument is subsequently referred to as option-based global hedging, whereas the global hedge using only the underlying is referred to as underlying-based global hedging. The prime contribution of the current thesis is to study the extent of the outperformance of global hedging using options as hedging instruments versus delta-gamma hedging, which has never been investigated in the literature to the best of the author's knowledge.

In a simulation study, the current work shows that the option-based global hedging always performs best compared to the benchmark methods in terms of mean squared error. Moreover, it is found to yield considerably better results for higher levels of volatility and drift, as well as when the frequency of hedging portfolio rebalancing is lower. The investigation of the turnover of the option-based global hedging shows that having more accuracy does not necessarily lead to higher transaction costs. It is worth noting that option-based global hedging is more accurate for common scenarios, but is more prone to far left tail risk (extreme losses) compared to delta-gamma hedging. In the end of the thesis, the positions obtained from option-based global hedging and delta-gamma hedging are compared and are shown to have a very similar behaviour up to some correction term adjusting for the departure of hedging portfolio value from the option value.

The thesis is structured as follows. Chapter 2 introduces terminology and presents market dynamics used within the current work. Chapter 3 formulates the quadratic global hedging problem and presents both the theoretical solution to the problem and the computational algorithm allowing for its implementation. Chapter 4 presents simulation experiments that are analyzed to assess the performance of the proposed global hedging framework. Chapter 5 summarizes results obtained.

Chapter 2

Market Model

In this section the main tools and assumptions needed for the global hedging framework are introduced. An arbitrage-free financial market with no transaction costs is considered. This market is comprised of one risky asset S, which acts as the underlying asset, and one risk-free asset B. The underlying is assumed to pay no dividends to keep this exposition simple. Moreover, it is assumed that at each time step, there is a set of m options written on the underlying that can be traded in the market. We assume that the trading occurs in equally spaced discrete time with a fixed financial horizon $T \in \mathbb{N}$, i.e. $t = 0, \ldots, T$.

In the underlying-based global hedging framework, the hedging instruments making up the hedging portfolio are the underlying asset and the risk-free asset. These are traded, in the hedging portfolio, from the beginning until the maturity of the option. However, in this thesis, shorter maturity options are also used to hedge an option with a longer maturity. Therefore, if the options held in the hedging portfolio are maturing at the end of the hedging period, then they will either be exercised or left to expire. On the other hand, if they are not maturing at the end of the period, they are liquidated and replaced with shares of newly available options.

In order to be able to precisely define the framework used, the letters b and e standing for *beginning* and *end* will be used to differentiate between the value of the options held in the hedging portfolio at the beginning and at the end of each hedging period. $\bar{S}_t^b := [S_t^b, D_t^{1,b}, \ldots, D_t^{m,b}]$ consists of the value of the underlying asset, whose time-t value is denoted by S_t, and the options, whose time t value is denoted by $D_t^1, \ldots, D_t^m$ at the beginning of the hedging period $[t, t+1)$, and $\bar{S}_t^e = [S_t^e, D_t^{1,e}, \ldots, D_t^{m,e}]$ consists of the price of the underlying and the options at the end of the hedging period $[t, t+1)$ right before the next hedging period starts. As for the underlying $S_{t+1}^b = S_t^e$ for all $t = 0, \ldots, (T-1)$, but this does not necessarily hold for the options as the options currently being held in the portfolio might be replaced with other options.

The risky asset and the options written on the risky asset follow a square integrable process, meaning that each $\{\bar{S}_t^{i,b}\}_{t=0}^T$ [1] and $\{\bar{S}_t^{i,e}\}_{t=0}^T$ are square integrable processes for $i = 1, \ldots, (m+1)$. These processes are defined on the probability space $(\Omega, \mathscr{F}_T, \mathbb{P})$ where $\mathscr{F}_t := \sigma(S_0^b, S_1^b, \ldots, S_t^b)$ is the sigma-algebra generated by the underlying up to time t. The ordered set of all the sigma-algebras, $\{\mathscr{F}_0, \ldots, \mathscr{F}_T\}$, defines a filtration $\mathbb{F}$ on the probability space. The measure $\mathbb{P}$ is the real-world measure.

The risk-free asset follows a deterministic process $\{B_t\}_{t=0}^T$ that is assumed

[1] $\bar{S}_t^{i,b}$ is the ith member of $\bar{S}_t^b$. A Similar convention holds for $\bar{S}_t^e$.

to be the following: $B_t := e^{rt\delta}$, where r is a constant number representing the annualized risk-free rate and δ is the time span between time steps. The discount factor process $\{\beta_t\}_{t=0}^T$ is defined as: $\beta_t = B_t^{-1} = e^{-rt\delta}$. In the following section, the terminology and the assumptions needed for the global hedging framework are explained.

2.1 Terminology

A few standard definitions in the theory of hedging, as appearing in Schweizer [1988] and Augustyniak et al. [2017] will now be reviewed. These formalize the types of strategies which are considered both mathematically tractable and financially reasonable.

Definition 1. (Trading Strategy) A trading strategy θ is a pair of stochastic processes $\theta = (\theta^B, \theta^{\bar{S}})$ such that $\theta_t = (\theta_t^B, \theta_t^{\bar{S}})$ for $t = 0, \ldots, T$ where θ_0 is predefined as being equal to θ_1. Here $\theta^{\bar{S}}$ and θ^B are both predictable i.e. $\theta_t^{\bar{S}} \in \mathscr{F}_{t-1}$ and $\theta_t^B \in \mathscr{F}_{t-1}$ for $t = 1, \ldots, T$. $\theta_t^{\bar{S}}$ and θ_t^B respectively represent a vector and a scalar containing the number of shares of the risky assets and the risk-free asset held in the hedging portfolio over the time period $(t-1, t]$ for $t = 1, \ldots, T$.

Definition 2. (Admissible Trading Strategy) Let $\bar{S}_t^b$, $\bar{S}_t^e$ and β_t be defined for $t = 0, \ldots, T$ as discussed before. Then define $\Delta_t = \beta_t \bar{S}_{t-1}^e - \beta_{t-1} \bar{S}_{t-1}^b$ for $t = 1, \ldots, T$. An admissible hedging strategy is a strategy such that $\left(\theta_t^{\bar{S}}\right)^T \Delta_t$ is square-integrable for all $t = 1, \ldots, T$.

Definition 3. (Value of Hedging Portfolio) Let V_t^θ be the value of the hedging strategy θ at time t (after re-balancing). We can define V_t^θ as below:

$$V_t^\theta = \left(\theta_t^{\bar{S}}\right)^\top \bar{S}_{t-1}^e + \theta_t^B B_t.$$

Definition 4. (Self-Financing Hedging Strategy) A hedging strategy is self-financing if:

$$V_t^\theta = \left(\theta_{t+1}^{\bar{S}}\right)^\top \bar{S}_t^b + \theta_{t+1}^B B_t = \left(\theta_t^{\bar{S}}\right)^\top \bar{S}_{t-1}^e + \theta_t^B B_t \qquad \text{for} \quad t = 1, \ldots, T,$$

meaning that the purchase of a new asset should be financed by selling an old one. Hedging using a self-financing strategy, V_t^θ can be written as follows:

$$V_t^\theta = V_0^\theta + \sum_{n=1}^{t} \left(\theta_n^{\bar{S}}\right)^\top \left(\bar{S}_{n-1}^e - \bar{S}_{n-1}^b\right) + \sum_{n=1}^{t} \left(\theta_n^B\right)^\top \left(B_n - B_{n-1}\right).$$

Discounting the values of the portfolio at each time, the risk-free asset would always have a value of 1 and:

$$\beta_t V_t^\theta = \left(\theta_{t+1}^{\bar{S}}\right)^\top \beta_t \bar{S}_t^b + \theta_t^B = \left(\theta_t^{\bar{S}}\right)^\top \beta_t \bar{S}_{t-1}^e + \theta_{t-1}^B \qquad \text{for} \quad t = 1, \ldots, T.$$

Therefore, V_t^θ can be written as the following sum:

$$\beta_t V_t^\theta = V_0^\theta + \sum_{n=1}^{t+1} \left(\theta_n^{\bar{S}}\right)^\top \left(\beta_n \bar{S}_{n-1}^e - \beta_{n-1} \bar{S}_{n-1}^b\right) = V_0^\theta + \sum_{n=1}^{t+1} \left(\theta_n^{\bar{S}}\right)^\top \Delta_n.$$

See Lamberton and Lapeyre [2011], Chapter 1 for more details.

It is well known that any $\{\theta_t^{\bar{S}}\}_{t=1}^T$ defines a unique self-financing strategy . This result allows us to reduce the complexity of the numerical study since we only need to keep track of the shares of the risky assets.

Next, the model specified for the underlying and the pricing scheme for the options used as hedging instrument are discussed.

2.2 Black-Scholes Model

The model considered for the dynamics of the underlying asset is the Black-Scholes model where the underlying follows a discrete Geometric Brownian Motion, meaning:

$$S_t = S_0 e^{\sum_{i=1}^t \varepsilon_i} \tag{2.1}$$

with $\{\varepsilon_t\}_{t=1}^T$ denoting the log-returns given by:

$$\varepsilon_t = \left(\mu - \frac{\sigma^2}{2}\right)\delta + \sigma\sqrt{\delta}z_t, \quad t \in \{1, \ldots, T\}.$$

Here, μ is the annualized drift of the underlying, σ is the standard deviation of the underlying returns and $\{z_t\}_{t=1}^T$ is a strong Gaussian white noise under the measure $\mathbb{P}$.

In order to price an option in a discrete-time setting using risk-neutral measures, one has to choose which risk-neutral measure to work with as the market is incomplete hence an infinite number of martingale measures exist. In this thesis, the classic risk-neutral measure obtained by the discrete-time Girsanov theorem is used. This measure will assist us in recovering the traditional Black-Scholes pricing scheme. This risk-neutral measure is defined through the following process:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{t=1}^{T} \xi_t,$$
$$\xi_t = exp\left(z_t^{\mathbb{P}}\lambda - \frac{1}{2}\lambda^2\right),$$
$$\lambda = -\frac{\mu - r}{\sigma}. \tag{2.2}$$

Defining $z_t^{\mathbb{Q}} := z_t^{\mathbb{P}} - \lambda$, $\{\varepsilon_t\}_{t=1}^{T}$ has the following dynamics:

$$\varepsilon_t = \left(r - \frac{\sigma^2}{2}\right)\delta + \sigma\sqrt{\delta}z_t^{\mathbb{Q}}, \quad t \in \{1, \ldots, T\},$$

where $\{z_t^{\mathbb{Q}}\}_{t=1}^{T}$ is a standard Gaussian white noise under the measure $\mathbb{Q}$. The proofs for this section can be found in Godin et al. [2018].

Under the risk-neutral measure $\mathbb{Q}$, a European call option with strike price K and maturity T years is priced according to the following risk-neutral pricing formula:

$$C(S_t, t, T, K) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)\delta}(S_T - K)^+ \middle| \mathscr{F}_t\right].$$

An explicit formula for pricing the European call option can be found in Black and Scholes [1973]. The formula is as follows:

$$\begin{aligned} C(S_t,t,T,K) &= N(d_1)S_t - N(d_2)Ke^{-r(T-t)\delta} \qquad (2.3) \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)\delta\right] \\ d_2 &= d_1 - \sigma\sqrt{(T-t)\delta} \end{aligned}$$

where $N(\cdot)$ is the standard normal cumulative distribution function. A European put option can be priced using the put-call parity as follows:

$$\begin{aligned} P(S_t,t,T,K) &= Ke^{-r(T-t)\delta} - S_t + C(S_t,t,T,K) \\ &= N(-d_2)Ke^{-r(T-t)\delta} - N(-d_1)S_t. \end{aligned}$$

Definition 5. (Delta of an option) The delta of an option, denoted by Δ, measures the rate of change of the options value with respect to changes in the price of the underlying. In the Black-Scholes setting

$$\Delta^{Call}(S_t,t,T,K) = \frac{\partial C}{\partial S_t} = N(d_1),$$

and

$$\Delta^{Put}(S_t,t,T,K) = \frac{\partial P}{\partial S_t} = 1 - N(d_1).$$

(See *Hull* [2003] for more details.)

Definition 6. (Gamma of an option) The Gamma of an option, denoted by Γ, measures the rate of change of the Δ with respect to changes in the price of the underlying. In the Black-Scholes setting

$$\Gamma(S_t, t, T, K) = \frac{\partial \Delta}{\partial S_t} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}.$$

(See *Hull* [2003] for more details.)

Chapter 3

Quadratic Global Hedging

In this chapter, the quadratic global hedging problem is reviewed.

3.1 Hedging Problem

In this thesis, a market participant who takes a short position on a European option is considered. We assume that the market participant attempts to reduce risk associated with the hedging portfolio shortfall at the maturity of the option. A quadratic penalty function is considered in this work, as this is standard in the literature.

Quadratic global hedging aims at finding an admissible self-financing strategy θ and an initial capital V_0 that solves:

$$\underset{V_0,\theta}{\arg\min}\ \mathbb{E}\left[(\Phi(S_T) - V_T^{\theta})^2\right]. \tag{3.1}$$

The function Φ represents the payoff of the option at the terminal time step T. In this thesis, European vanilla call options are considered and therefore,

$$\Phi(S_T) = \max(S_T - K, 0)$$

where K is the strike of the European call option being hedged. Conditions for the existence of the solution and the solution itself will be provided in the next sections.

In this thesis, a quadratic loss function is chosen. It is important to note that quadratic loss functions do not differentiate between losses and gains since they penalize the gains as much as the losses. This is not optimal in the view of a market participant. However, using quadratic loss functions provides us with semi closed-form solutions for the problem which facilitates the computation. Therefore, quadratic loss functions will be used in this thesis, as is in the standard literature.

3.2 Solution to the Hedging Problem

The following results are based on those of Rémillard and Rubenthaler [2013]. The details of the modifications are provided in the Appendix A.

Theorem 1. The solution to the global quadratic hedging problem defined in equation (3.1) is fully determined by $V_0 = C_0$ and the following backward recursive scheme scheme, beginning at time $t = T$, where $C_T = \Phi(\bar{S}_T)$ and $\nu_{T+1} = 1$:

$$\theta_t^{\bar{S}} = \alpha_t - \beta_{t-1} V_{t-1}^{\theta} b_t, \tag{3.2}$$

where $A_t, \mu_t, b_t, \nu_t, \alpha_t, \beta_t$, and C_t are defined by

$$\begin{aligned}
A_t :=& \mathbb{E}\left[\Delta_t \Delta_t^\top \nu_{t+1} \mid \mathscr{F}_{t-1}\right], \\
\mu_t :=& \mathbb{E}\left[\Delta_t \nu_{t+1} | \mathscr{F}_{t-1}\right] \\
b_t :=& A_t^{-1} \mu_t, \\
\nu_t :=& \mathbb{E}\left[(1 - b_t^\top \Delta_t) \nu_{t+1} | \mathscr{F}_{t-1}\right] \\
\alpha_t :=& A_t^{-1} \mathbb{E}[\beta_t C_t \Delta_t \nu_{t+1} | \mathscr{F}_{t-1}] \\
\beta_{t-1} C_{t-1} :=& \frac{1}{\nu_t} \mathbb{E}[\beta_t C_t (1 - b_t^\top \Delta_t) \nu_{t+1} | \mathscr{F}_{t-1}]
\end{aligned} \tag{3.3}$$

and Δ_t is defined in Definition 2.

Proof. See Appendix A. □

The next lemma details the conditions for the existence of the solution proposed in Theorem 1.

Lemma 2. Suppose that $\mathbb{E}\left[\nu_{t+1} \middle| \mathscr{F}_{t-1}\right] A_t^\top - \mu_t \mu_t^\top$ is invertible $\mathbb{P}$-a.s., for every $t = 1, \ldots, T$. Then $\nu_t \in (0, 1]$ and A_t is invertible for all $t = 0, \ldots, T$. In addition, $\{\nu_{t+1}\}_{t=0}^{\top}$ is a positive submartingale.

Proof. See Rémillard and Rubenthaler [2013]. □

Rémillard and Rubenthaler [2013] mention that C_t can be interpreted as the value of the option at time t and that there is no obvious method to validate if the assumptions of Lemma 2 hold. Therefore, in most applications, one has to verify these conditions, often using brute force calculation.

3.3 Optimal Replication Error

As mentioned in the introduction, Bertsimas et al. [2001] also solved the optimal quadratic replication problem. Their results will now be presented. Let $\varepsilon(V_0)$ be defined as follows:

$$\varepsilon(V_0) = \min_{\{\theta_t^*\}} \mathbb{E}\left[\left(\Phi(S_T) - V_T^{\theta}\right)^2\right]$$

where $\{\theta_t^*\}$ is the optimal hedging strategy and $\mathbb{E}\left[\left(\Phi(S_T) - V_T^{\theta}\right)^2\right]$ is the term mentioned in equation (3.1). $\varepsilon(V_0)$ is the smallest mean square error of hedging if the hedger starts with a portfolio value V_0. The latter error measure can be minimized with respect to the initial wealth V_0 to yield the least-cost-optimal-replication strategy and a corresponding measure of the minimum replication error ε^*:

$$\varepsilon^* := \min_{\{V_0\}} \varepsilon(V_0).$$

Theorem 3. Using the assumptions and the variables defined in Theorem 1, ε^* can be calculated using a backward recursive scheme initiated at $t = T$ defining $c_T = 0$ and

$$c_{t-1} = \mathbb{E}[c_t|\mathscr{F}_{t-1}] + \mathbb{E}\left[\nu_{t+1}(\beta_t C_t - \alpha_t \Delta_t)^2 |\mathscr{F}_{t-1}\right] - \nu_t(\beta_{t-1}C_{t-1})^2.$$

Moreover, $\varepsilon^* = e^{2r\delta T} c_0$.

Proof. See the Appendix A. □

The performance of the global hedging schemes and relevant benchmarks are examined and compared in the next section.

Chapter 4

Numerical Experiment

In this section, the effectiveness of the proposed quadratic global hedging with options, which we call option-based quadratic global hedging, is assessed through a simulation study. The results are analyzed and then compared to other hedging frameworks. These hedging frameworks include traditional quadratic global hedging referred to as underlying-based quadratic global hedging, delta-gamma hedging (see Raju [2012] for details), and delta hedging (see Hull [2003] for details).

4.1 Simulation Analysis

In this simulation study, the underlying price process is modelled using a Geometric Brownian motion, following Section 2.2. A single European call option will be hedged using the proposed option-based global hedging framework. The hedging simulations will explore the impact of the following parameters: moneyness (m), drift (μ), volatility (σ), years to maturity (τ) and the number of hedging periods. Throughout this study the initial underlying price is $S_0 = 100$ and the annual risk-free rate is set to $r = 0.05$. Firstly, 100,000 Monte-Carlo simulation paths are generated using equation (2.1). Then, they have been used for a single baseline case to study the distribution of the hedging errors in each hedging framework. For the rest of the simulation study, for each Monte-Carlo simulation performed, $10,000$ paths have been generated to compare the performance of the hedging frameworks mentioned before.

The solution to the global hedging algorithms is calculated through a dynamic programming scheme as explained in details in Appendix B. In summary, there are three phases to this calculation. Firstly, the quantities mentioned in equation (3.3) are calculated on a grid for the values of the underlying. Secondly, cubic splines are used to interpolate between the values of the grid, as the values of the underlying on simulation paths may fall between the discrete values of the underlying on the grid. Lastly, equation (3.2) is used to calculate $\theta_t^{\bar{S}}$, for $t = 1, \ldots, T$. The details of the analysis will be examined in the next section.

4.2 Performance Assessment

In this section, descriptive statics of the hedging errors are considered to compare the performance of all hedging frameworks With a baseline set of parameters. The baseline case uses all the hedging frameworks to hedge a European call option, rebalancing monthly. In the baseline case, the initial underlying price is $S_0 = 100$, the strike is $K = 100$, the drift is equal to $\mu = 0.1$, volatility is equal to $\sigma = 0.2$, and the time to maturity of the option being hedged is $T = 1$ year.

Firstly, the effect of the grid resolution of the underlying on the results obtained by global hedging algorithms is investigated and the results are shown in Table 4.1. The results in Table 4.1 are calculated using the baseline parameters and 100,000 Monte-Carlo paths. In Table 4.1, Model, Avg, MSE, $\hat{c}^*$ and C_0 respectively represent the framework used for hedging, the sample average, the sample mean squared hedging errors, an approximation of the optimal error MSE provided by the dynamic programming scheme in Theorem 3, and the initial capital needed for each hedging portfolio. VaR(α) is defined as below:

$$VaR(\alpha) = \inf\left\{x \in \mathbb{R} : F_X(x) > \alpha\right\}$$

where $F_X(X)$ is the empirical cumulative distribution of hedging errors. When $\alpha > 0.5$ ($\alpha < 0.5$), CVaR(α) is the average of hedging errors larger (smaller) than $VaR(\alpha)$. When referring to the models in tables, Glbl 2A, Glbl 1A, $\Delta - \Gamma$ and Δ respectively stand for option-based global hedging, underlying-based global hedging, delta-gamma hedging and delta hedging. Results in Table 4.1 show that refining the grid for the price of the underlying past a certain point provides diminishing returns. It can be seen that for the baseline case, there is no significant change in hedging accuracy obtained from the Monte-Carlo simulations or the quantities obtained within the global hedging frameworks when considering mesh size[1] below $\mathscr{M} = 1$. Therefore, for the rest of the study a grid with mesh size of 1 will be used as it is more time efficient and essentially just as accurate as refinements of the grid beyond that point.

Params	Model	Avg	MSE	$\hat{c}^*$	C_0
$\mathscr{M} = 1$	Glbl 2A	0.048	0.476	0.498	10.462
	Glbl 1A	0.281	2.993	3.135	10.413
$\mathscr{M} = 0.5$	Glbl 2A	0.051	0.477	0.490	10.465
	Glbl 1A	0.284	2.995	3.121	10.417
$\mathscr{M} = 0.1$	Glbl 2A	0.051	0.477	0.489	10.465
	Glbl 1A	0.284	2.996	3.120	10.471

Table 4.1: Statistics of hedging errors for the baseline case for varying mesh sizes.

Next, a close look is taken at the distribution of hedging errors in the simulation with baseline parameters under the mesh size $\mathscr{M} = 1$. Figure 4.1 gives the box plot for the hedging errors corresponding to each of the hedging frameworks in the baseline case. The statistics and tail measures of the hedging errors obtained from the same simulation for all four hedging algorithms are summarized in Table 4.2.

Results in Table 4.2 confirm that the MSE obtained by the option-based global hedging is the smallest as expected. It is readily observed that the mean squared hedging errors obtained through global hedging algorithms within Monte-Carlo simulations is indeed close to the approximation to optimal replication error(MSE) $\hat{c}^*$, obtained within the dynamic programming scheme. This confirms that suf-

[1]The mesh size is the distance between two consecutive points on the grid for the underlying.

ficient accuracy is achieved by the numerical solution to the hedging problem. $^{RMSE}/_{C_0}$ is considered as a measure of risk per unit of exposure in the hedging portfolio. This is particularly useful for situations where different frameworks yield notably different starting value for the hedging portfolio. In Table 4.2, the $^{RMSE}/_{C_0}$ is the smallest for option-based global hedging. This shows the latter hedging framework is less risky for common scenarios compared to the other hedging frameworks. Based on the values of *CVaR* in Table 4.2 and the boxplot in Figure 4.1, it can be concluded that the distribution of hedging errors obtained through delta-hedging has fatter tails among all the frameworks. Comparing the distribution of hedging errors obtained from delta-gamma hedging and option-based global hedging, delta-gamma hedging has a fatter right tail associated with the profits, while option-based global hedging has fatter left tail associated with loss. Therefore, in this case, option-based global hedging is more accurate for common scenarios but delta-gamma hedging is exposed to less risk in the far left tail.

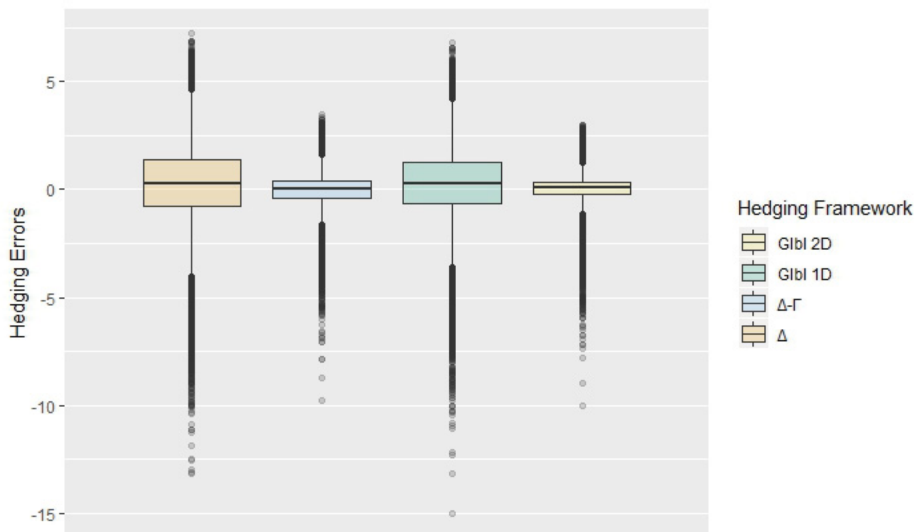

Figure 4.1: Boxplot of Hedging Errors for the baseline case.

Stats	Glbl 2A	Glbl 1A	$\Delta - \Gamma$	Δ
Avg	0.049	0.282	0.009	0.291
MSE	0.446	3.002	0.568	3.430
c^*	0.550	3.465	-	-
C_0	10.462	10.414	10.451	10.451
$^{RMSE}/_{C_0}$	0.064	0.166	0.072	0.177
VaR(0.99)	1.775	4.366	2.092	4.628
CVaR(0.99)	2.065	4.879	2.894	5.163
VaR(0.01)	-1.945	-4.446	-1.896	-4.639
CVaR(0.01)	-2.867	-5.73	-2.226	-5.987
VaR(0.999)	2.42	5.525	4.113	5.822
CVaR(0.099)	2.598	5.8	5.076	6.122
VaR(0.001)	-4.212	-7.361	-2.639	-7.871
CVaR(0.001)	-5.195	-8.656	-2.855	-9.123

Table 4.2: Statistics of hedging errors for the baseline case.

We now turn to additional Monte-Carlo simulations where some of the parameter values are altered in comparison to the baseline case. First, the effect of different levels of drift(μ) on each hedging framework is investigated. For this investigation all the parameters are kept as constant and the same as the baseline case, and the drift varies between 0.15, 0.27 and 0.47. The results are presented in Table 4.3. In Table 4.3, it is seen that as the drift increases, the mean squared of hedging errors for all the hedging methods decreases. However, the improvement obtained through the global hedging algorithms compared to the traditional methods is more significant. This happens since as mentioned in equation (3.2),

$$\theta_t^{\bar{S}} = \alpha_t - \beta_{t-1} V_{t-1} b_t = \alpha_t - \beta_{t-1}(V_{t-1} - C_{t-1}) b_t - \beta_{t-1} C_{t-1} b_t. \tag{4.1}$$

Equation (4.1) shows that the term $\{\beta_{t-1}(V_{t-1} - C_{t-1}) b_t\}$ acts as a correction term adjusting for the deviation of the value hedging portfolio from the value of the option, C_t, obtained in global hedging frameworks. The correction term which is proportional to b_t, grows as the drift(μ) increases and therefore allows for more extensive corrections of the hedging errors. In this way, global hedging algorithms have more pronounced corrections that allow for more reduction of previously incurred hedging errors. The improvement obtained when using options in the global hedging framework, as compared to the underlying-based global hedging algorithm, is less significant when the drift increases. This is also due to the relation between the correction term and the drift, meaning that as the correction term adjusts the hedging portfolio weight, the use of the second asset provides less incremental performance.

Params	Model	Avg	MSE	$\hat{c}^*$	C_0
$\mu = 0.15$	Glbl 2A	0.046	0.379	0.485	10.461
	Glbl 1A	0.253	2.566	2.991	10.340
	$\Delta - \Gamma$	0.010	0.519	-	10.451
	Δ	0.233	3.185	-	10.451
$\mu = 0.27$	Glbl 2A	0.006	0.206	0.250	10.377
	Glbl 1A	0.145	1.407	1.600	10.000
	$\Delta - \Gamma$	0.002	0.393	-	10.451
	Δ	0.021	2.608	-	10.451
$\mu = 0.47$	Glbl 2A	0.001	0.032	0.037	10.249
	Glbl 1A	0.001	0.261	0.239	8.895
	$\Delta - \Gamma$	-0.013	0.252	-	10.451
	Δ	-0.536	1.987	-	10.451

Table 4.3: Statistics of hedging errors for various μ.

The performance of the different hedging strategies is now investigated for various values of volatility when the rest of the parameters are fixed to the value of the baseline case. The findings are summarized by Table 4.4 where the statistics for the hedging errors are calculated using the baseline parameters when the volatility(σ) varies between 0.05, 0.1, and 0.3.

Params	Model	Avg	MSE	$\hat{c}^*$	C_0	$^{\text{RMSE}}/_{C_0}$
$\sigma = 0.05$	Glbl 2A	-0.001	0.005	0.005	5.271	0.013
	Glbl 1A	0.014	0.034	0.033	5.231	0.035
	$\Delta - \Gamma$	0.000	0.008	-	5.283	0.017
	Δ	0.002	0.055	-	5.283	0.044
$\sigma = 0.1$	Glbl 2A	0.003	0.100	0.112	7.135	0.044
	Glbl 1A	0.125	0.673	0.713	7.105	0.115
	$\Delta - \Gamma$	0.006	0.134	-	7.148	0.051
	Δ	0.121	0.816	-	7.148	0.126
$\sigma = 0.3$	Glbl 2A	0.003	1.060	1.152	14.196	0.073
	Glbl 1A	0.441	7.097	7.511	14.192	0.188
	$\Delta - \Gamma$	0.019	1.344	-	14.231	0.081
	Δ	0.459	8.039	-	14.231	0.199

Table 4.4: Statistics of hedging errors for various values of σ.

Table 4.4 shows that when the underlying is more volatile, the option-based global hedging outperforms delta-gamma hedging more significantly in terms of mean squared error. Moreover, an increase in the volatility leads to an increase in $^{RMSE}/_{C_0}$ for all the hedging frameworks. In Table 4.4, it is seen $^{RMSE}/_{C_0}$ for delta-gamma hedging progressively departs from its corresponding optimal framework as volatility increases.

Next, a simulation study is performed to investigate the effect of varying time to maturity of the option being hedged between 1,2 and 5 years. All other quantities are fixed to be the values in the baseline case, including the use of monthly rebalancing. The results are summarized by the Table 4.5.

Params	Model	Avg	MSE	c^*	C_0	$^{\text{RMSE}}\!/\!_{C_0}$
$\tau = 1$	Glbl 2A	0.040	0.430	0.550	10.462	0.063
	Glbl 1A	0.282	2.933	3.465	10.414	0.164
	$\Delta - \Gamma$	0.012	0.555	-	10.451	0.071
	Δ	0.290	3.361	-	10.451	0.175
$\tau = 2$	Glbl 2A	0.033	0.390	0.455	16.140	0.060
	Glbl 1A	0.173	2.666	2.990	16.082	0.101
	$\Delta - \Gamma$	0.008	0.554	-	16.127	0.046
	Δ	0.185	3.461	-	16.127	0.115
$\tau = 5$	Glbl 2A	0.037	0.211	0.241	29.17	0.016
	Glbl 1A	0.087	1.494	1.665	29.065	0.042
	$\Delta - \Gamma$	0.005	0.443	-	29.139	0.023
	Δ	0.085	2.956	-	29.139	0.059

Table 4.5: Statistics of hedging errors for various maturities of the option being hedged.

Firstly, one can observe that the option price and the sample mean squared error increase substantially as T increases while the rebalancing frequency remains fixed. Since the price of options with different maturities varies substantially, the $^{RMSE}\!/_{C_0}$ obtained by the algorithms will be compared. As seen in Table 4.5, $^{RMSE}\!/_{C_0}$ is the smallest for all the algorithms performed on options with longest maturity. In all the cases, $^{RMSE}\!/_{C_0}$ is the smallest for option-based global hedging. However, the biggest improvement obtained in terms of $^{RMSE}\!/_{C_0}$ is obtained by underlying-based global hedging. This happens since in the case of global hedging algorithms, more hedging periods implies having more steps to correct the departure of the hedging portfolio from the value of the option calculated in global hedging frameworks. However, the improvement obtained by option-based global hedging is not as significant compared to underlying based and delta hedging. This happens since the use of options has already reduced the mean squared error by a notable amount even for shorter maturities.

As a next step, Table 4.6, examines the results of the hedging algorithms, while keeping the number of rebalancing periods constant and allowing the years to maturity of the option being hedged to vary. The rest of parameters are kept fixed and equal to the quantities in the baseline case.

Params	Model	Avg	MSE	$\hat{c}^*$	C_0	$^{\text{RMSE}}/_{C_0}$
$\tau = 1$	Glbl 2A	0.040	0.430	0.498	10.462	0.063
	Glbl 1A	0.282	2.933	3.136	10.414	0.164
	Δ-Γ	0.012	0.555	-	10.451	0.071
	Δ	0.290	3.361	-	10.451	0.175
$\tau = 2$	Glbl 2A	0.095	0.733	0.856	16.176	0.053
	Glbl 1A	0.361	4.924	5.224	16.033	0.138
	Δ-Γ	0.013	1.055	-	16.127	0.064
	Δ	0.381	6.427	-	16.127	0.157
$\tau = 5$	Glbl 2A	0.004	1.064	1.185	29.069	0.035
	Glbl 1A	0.394	7.126	7.569	28.832	0.093
	Δ-Γ	0.026	2.200	-	29.139	0.051
	Δ	0.447	13.854	-	29.139	0.128

Table 4.6: Statistics of hedging errors for fixed number of hedging periods and varying time to maturity.

Comparing the results in Table 4.5 to those of Table 4.6, it is seen that the estimated ${RMSE}/{C_0}$ significantly increases as the rebalancing frequency decreases for delta hedging as well as underlying-based hedging. For example, ${RMSE}/{C_0}$ increases by 0.69 i.e from 0.059 to 0.128, for the 5 year option in the case of delta hedging. This measure also comparatively increases for delta-gamma hedging, and option-based global hedging. However, between the two, the increase is more notable for delta-gamma hedging. For example, the increase in ${RMSE}/{C_0}$ for the 5 year option for the case of delta-gamma hedging and option-based global hedging is respectively 0.28 and 0.019 i.e from 0.023 to 0.051 and from 0.016 to 0.035. This shows that by using a smaller number of hedging periods when performing the option-based global hedging algorithm, one can achieve the same level of mean squared of errors obtained by delta-gamma hedging using more hedging periods. Therefore, two-dimensional global hedging leads to lower transaction costs than delta-gamma hedging, while still having a comparable accuracy.

Next, the effect of the moneyness of the option being hedged on the performance of the hedging algorithms is investigated. Moneyness is i.e. $m = {S_0}/{K}$. Therefore, higher (lower) values of the moneyness for a European call option represent an in-the-money (out-of-the-money) option. All the parameters are kept fixed and the performance of the hedging frameworks is summarized for different levels of moneyness in Table 4.7. For this study, the moneyness varies between 1.1, 1 and 0.9. The rest of the parameters are kept fixed and equal to those in the baseline case.

Params	Model	Avg	MSE	$\hat{c}^*$	C_0	$^{\text{RMSE}}/_{C_0}$
$m = 1.1$	Glbl 2A	0.025	0.257	0.328	16.066	0.032
	Glbl 1A	0.201	1.691	2.040	16.030	0.081
	$\Delta - \Gamma$	0.015	0.333	-	16.057	0.036
	Δ	0.207	1.948	-	16.057	0.087
$m = 1$	Glbl 2A	0.040	0.430	0.550	10.462	0.063
	Glbl 1A	0.282	2.933	3.465	10.414	0.164
	Δ-Γ	0.012	0.555	-	10.451	0.071
	Δ	0.290	3.361	-	10.451	0.175
$m = 0.9$	Glbl 2A	0.055	0.571	0.723	5.670	0.133
	Glbl 1A	0.309	3.717	4.387	5.619	0.343
	$\Delta - \Gamma$	0.002	0.717	-	5.657	0.150
	Δ	0.320	4.347	-	5.657	0.362

Table 4.7: Statistics of hedging errors for various levels of moneyness.

It is seen in Table 4.7, that all the models comparatively exhibit the worst $^{RMSE}/_{C_0}$ when $m = 0.9$. Delta-gamma hedging departs the most from option-based global hedging in terms of $^{RMSE}/_{C_0}$ when the moneyness decreases. This explains increasing values of $^{RMSE}/_{C_0}$ as moneyness decreases, which confirms the common known fact that hedging out-of-the-money options is harder than hedging in-the-money options.

In summary, the option-based global hedging algorithms always yield the best results in terms of mean squared error and $^{RMSE}/_{C_0}$. However, this is more significant at higher levels of σ and μ, as well as when the number of rebalancing periods is lower. It is also important to note that the option-based global hedging is more accurate for common scenarios but more prone to risk in the tails.

Next, the turnover for the underlying (U_{tn}) and the options used as hedging instruments (opt_{tn}) is examined. The turnovers are defined as:

$$U_{tn} = \sum_{i=2}^{n} |\theta_i^U - \theta_{i-1}^U|,$$
$$opt_{tn} = \sum_{i=2}^{n} |\theta_i^{opt} - 0| = \sum_{i=2}^{n} |\theta_i^{opt}|.$$

where θ_i^U and θ_i^{opt} respectively represent the number of shares of the underlying and options used as hedging instruments in the hedging portfolio between time steps $i-1$ and i. In Table 4.8, the turnover of the hedging positions using the baseline parameters while varying the volatility(σ) between 0.1, 0.2 and 0.3 have been calculated. The turnover defined for the options is specific to the case where the options used as hedging instrument are liquidated at then end of every period, and then replaced with a new option. As an example, Table 4.8 shows the average turnover of each hedging algorithm for the underlying and the option if traded by the hedging framework.

Table 4.8 shows that the turnover for the underlying is smaller for option-based global hedging and underlying-based global hedging respectively compared to delta-gamma hedging, and delta hedging. Regarding the option turnover, one can see that the option-based global hedging leads to slightly smaller transaction sizes than delta-gamma hedging. Therefore, it would be slightly less exposed to transaction costs.

In order to guarantee that the conclusion made from comparing the average

Params	Model	Underlying Turnover	Option Turnover
$\sigma = 0.1$	Glbl 2A	1.026	2.45
	Glbl 1A	0.759	-
	Δ-Γ	1.048	2.754
	Δ	0.790	-
$\sigma = 0.2$	Glbl 2A	1.105	2.904
	Glbl 1A	0.941	-
	Δ-Γ	1.125	3.309
	Δ	0.954	-
$\sigma = 0.3$	Glbl 2A	1.111	3.055
	Glbl 1A	0.979	-
	Δ-Γ	1.122	3.425
	Δ	0.987	-

Table 4.8: The turnover of the hedging frameworks.

turnover of the underlying is accurate, the percentage of the paths for which each algorithm has higher turnover in underlying is examined in Table 4.9. The comparison of percentage of the paths that have higher turnover in Table 4.9 is computed using the same parameters as in Table 4.8. For example, when $\sigma = 0.1$, the turnover of the underlying in option-based global hedging is higher than underlying-based global hedging for 89.16% of the paths. One can observe that for all values of σ, the underlying turnover for option-based global hedging and underlying-based global hedging is respectively lower than the underlying turnover in delta-gamma hedging and delta hedging in all cases. This confirms that using the global hedging strategies to obtain more accuracy does not necessarily come at the cost of higher transaction costs.

Params	Model	Glbl 1A	Δ-Γ	Δ
$\sigma = 0.1$	Glbl 2A	0.8916	0.1816	0.8832
	Δ-Γ	0.8937		0.8867
$\sigma = 0.2$	Glbl 2A	0.7276	0.2374	0.7249
	Δ-Γ	0.7315		0.7296
$\sigma = 0.3$	Glbl 2A	.6553	0.3266	0.6539
	Δ-Γ	0.6560		0.6546

Table 4.9: The comparison of percentage of the paths that have higher underlying asset turnover for all the framework.

4.3 Analysis of the Strategy

In this section, an analysis of the hedging strategies for delta-gamma hedging and option-based global hedging is presented for the baseline parameters. The goal is to provide a deeper insight into why option-based global hedging outperforms delta-gamma hedging. Also, some of the quantities mentioned in equation (1) are graphed for the option-based global hedging strategy in order to compare them with their analogous shape in the case of underlying-based global hedging.

Godin [2018] argues that the quantity C_t found in equation (1) looks like the Black-Scholes price of the option at time t in the case of underlying-based global hedging. He also notes that the quantity $\alpha_{t+1} - \beta_{t+1} b_{t+1} C_t$ looks like the delta of the option being hedged at time t.

Remark 4. The number of shares of the underlying at time t in delta hedging is equal to the delta of the option being hedged at time t.

Rémillard and Rubenthaler [2013] also obtains the closed-form formulas for underlying-based global quadratic hedging for quantities b_t and v_t under the assumption that the underlying model follows a geometric Gaussian random walk which is compatible with the assumption made in this thesis. Denoting the length of time of each hedging period by d, the quantities b_t and v_t are found to be

$$b_t = \frac{1}{\beta_{t-1}S_{t-1}}\frac{\xi_1}{\xi_2}, \tag{4.2}$$

$$v_t = \bar{\gamma}^{T-t+1}, \tag{4.3}$$

where $\eta_1, \eta_2, \xi_1, \xi_2$, and $\bar{\gamma}$ are defined by

$$\begin{aligned}
\eta_1 &= \mathbb{E}[\beta e^{d\delta} e^{\varepsilon_t}] = \beta e^{d\delta} e^{\left(\mu+0.5\sigma^2\right)},\\
\eta_2 &= \mathbb{E}[(\beta e^{d\delta} e^{\varepsilon_t})^2] = \beta^2 e^{2d\delta} e^{\left(2\mu+2\sigma^2\right)},\\
\xi_1 &= (\eta_1 - 1), \quad \xi_2 = (\eta_2 - 2\eta_1 + 1), \quad \bar{\gamma} = 1 - \frac{\xi_1^2}{\xi_2}.
\end{aligned}$$

However, explicit formulas for the case of option-based quadratic global hedging are currently unknown. In order to gain a better conceptual understanding of the behaviour of b_t and v_t, their quantities are plotted for various values of the underlying and time steps t, in Figure 4.2.

In the case of underlying-based global hedging, as illustrated by equation (4.3), v_t is seen to only be a function of the time step. Although, the closed form formula

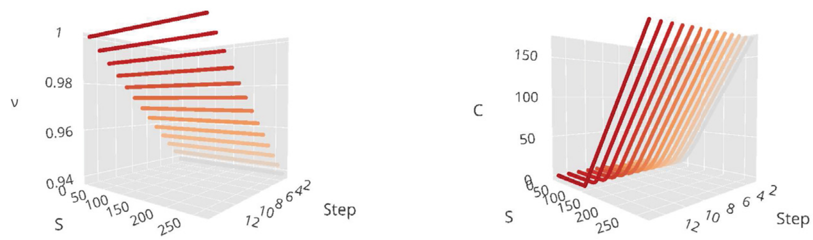

Figure 4.2: 3D surface of the quantities v and C of the option-based global hedging.

for option-based global hedging has not been derived, the graph of v_t in Figure 4.2 indicates that v follows the same pattern as it does in the underlying-based case. This happens since at-the-money options are used as hedging instruments and therefore gains obtained from these options are homogeneous with respect to the value of the underlying at each time step. Therefore, v_t does not depend on the value of the underlying in this special case of option-based global hedging. The quantity C_t, which is interpreted as the price of the option at time t, exhibits the same shape as that of the call option's payoff, with a time-dependent vertex[2] as in the case of the underlying-based global hedging.

In the case of underlying-based global hedging, b_t is seen to be inversely proportional to S_{t-1} as in equation (4.2). Therefore as S_{t-1} increases, b_t decreases. For option-based global hedging, b_t is a two-dimensional process, where its first component is associated with the number of underlying shares in the hedging portfolio, whereas the second component is associated to the number of option positions within the portfolio. In Figure 4.3, We can observe that b_t for the under-

[2]The point at which the slope suddenly changes.

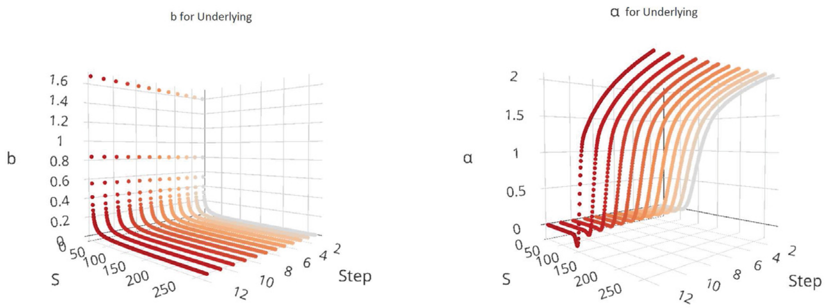

Figure 4.3: 3D surface of the quantities α and b of the option-based global hedging.

lying component in the case of option-based global hedging behaves the same as b_t for underlying-based global hedging. In Figure 4.4, it can be seen that the overall shape of b_t for options is totally in the opposite direction of the underlying. The quantity b_t for the options used as hedging instrument is always negative since the variation possible in the at-the-money call option value is limited from below. As for the quantity α_t, it can be seen in Figure 4.3 and Figure 4.4 that its shape for the underlying component and the option component also goes in totally different direction as if option-based hedging hedges itself against its shares of underlying in its hedging portfolio.

Next, the behaviour of the difference of the shares of hedging instruments obtained by option-based global hedging and delta-gamma hedging is investigated. Figure 4.5 presents a path-by-path simulation of the evolution of such a difference. Figure 4.5 shows that as the time steps get closer to maturity, delta-gamma hedging departs more and more from the optimal solution and that the variance of the difference in the number of shares held through both strategies increases. This shows that with more hedging periods, option-based global hedging has more op-

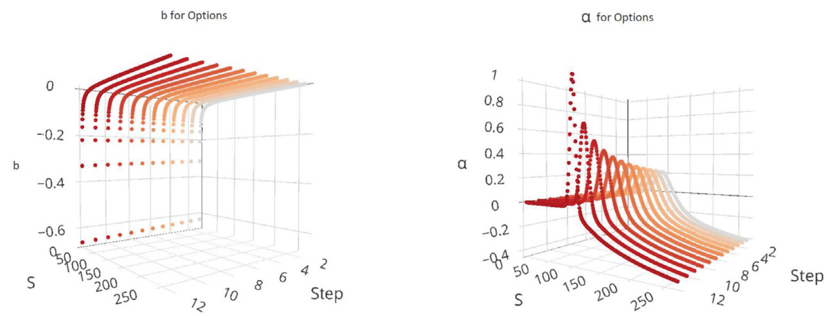

Figure 4.4: 3D surface of the quantities α and b of the option-based global hedging for options.

portunities to correct the departure of the hedging portfolio value from the option value C_t, provided in equation (1).

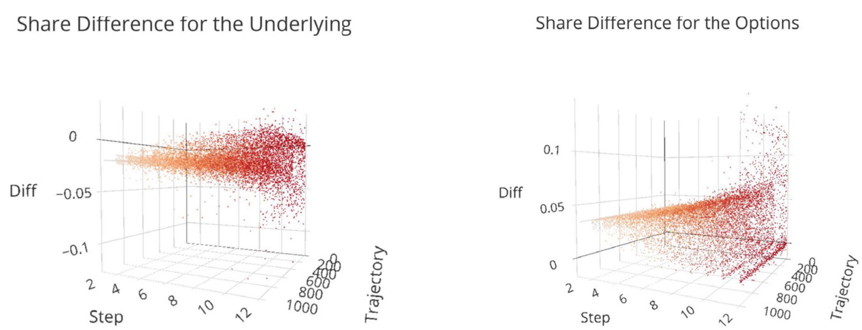

Figure 4.5: 3D surface of the difference in hedging shares for the underlying.

Next, the number of underlying shares and option positions obtained by the two frameworks are compared for selected individual paths. The path selected is the one where the largest difference of hedging was observed; this provides insight on where the variation between these two frameworks comes from. The number of shares obtained, value of hedging portfolios and the underlying process are plotted in Figure 4.6. It can be seen that the pattern for the number of shares and the evolution of the portfolio is very similar. Indeed at time step 11, there is a sudden drop in number of options obtained by both algorithms. Moreover, the distance between the number of share for the option used as hedging instrument has significantly increased. This happens since there is a huge jump in the value of the portfolios between time step 10 and 11 due to the departure of the value of the option-based global hedging portfolio from the value of the option estimated by the dynamic programming at that time step. This is the correction term which impacts the global hedging portfolio due to the previously incurred hedging error, which explains the departure between both methods.

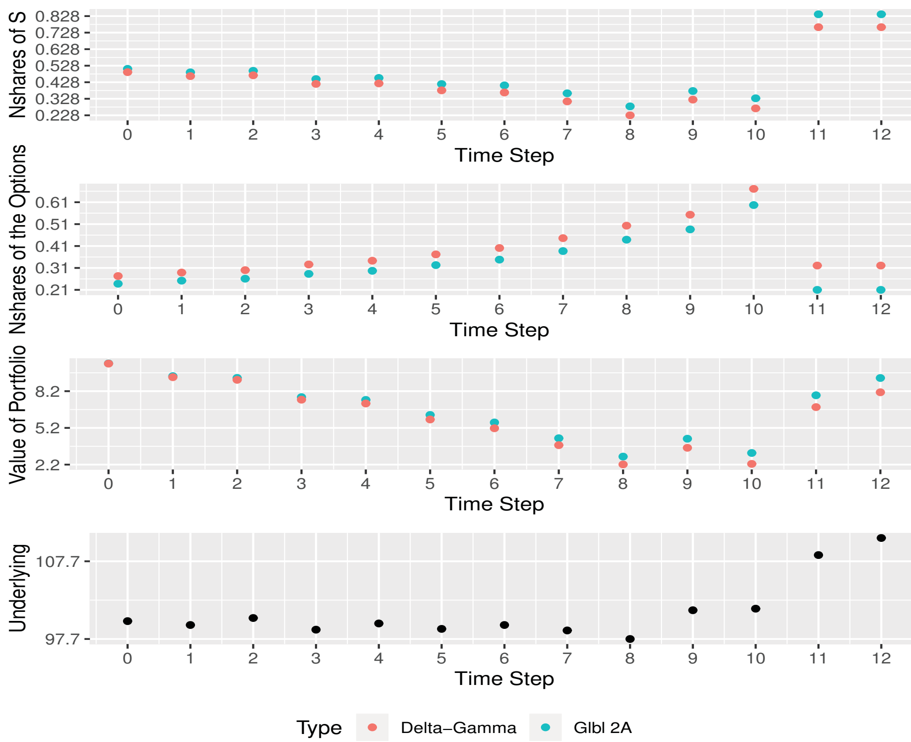

Figure 4.6: A comparison between the shares of underlying and options used as hedging instruments obtained by delta-gamma hedging and option-based global hedging while looking at the value of the portfolio they make.

As mentioned earlier, in the case of underlying-based global hedging, the number of shares assuming no previous error i.e. $\theta_t^S = \alpha_t^S - \beta_{t-1}C_{t-1}b_t^S$, looks like the delta of the option being hedged at time step t. It is good to note that the number of shares of the underlying at time t obtained by delta hedging is equal to the delta of the option being hedged at time t. The number of shares for the underlying and the options used as hedging instrument obtained from option-based global hedging are respectively plotted against and compared to the analogous quantities for delta-gamma hedging in Figure 4.7 and in Figure 4.8 assuming that $V_t = C_t$

for all t. It can be seen in Figure 4.7 that the shape of both the number of shares for the underlying and the options used as hedging instruments behave the same for both algorithms. We can also see that the number of options positions for option-based global hedging still exhibits a material departure from the numbers obtained for delta-gamma hedging. Hence, it is reasonable to assume that option-based quadratic global hedging is roughly similar to delta-gamma hedging plus a correction term when the underlying follows the Black-Scholes model.

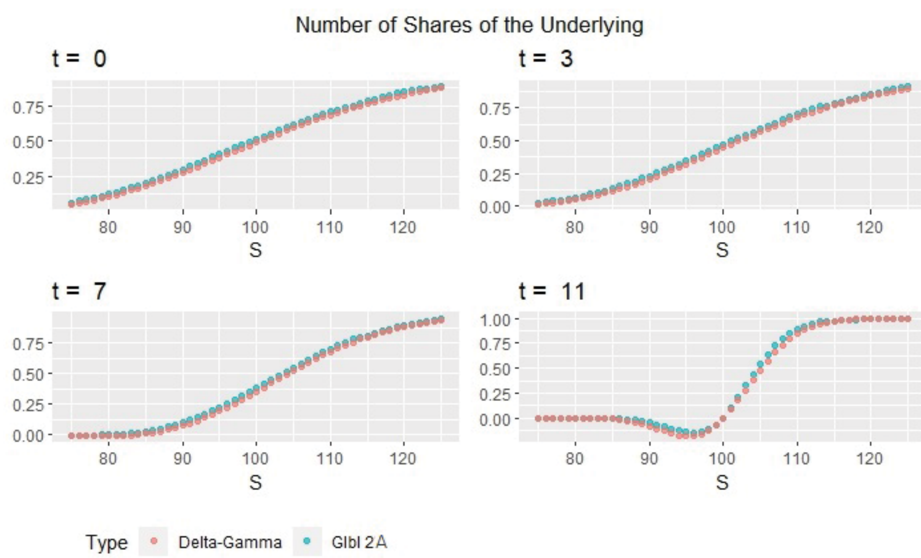

Figure 4.7: Number of shares of underlying obtained by option-based global hedging and delta-gamma hedging for various values of underlying, S, and time steps, t.

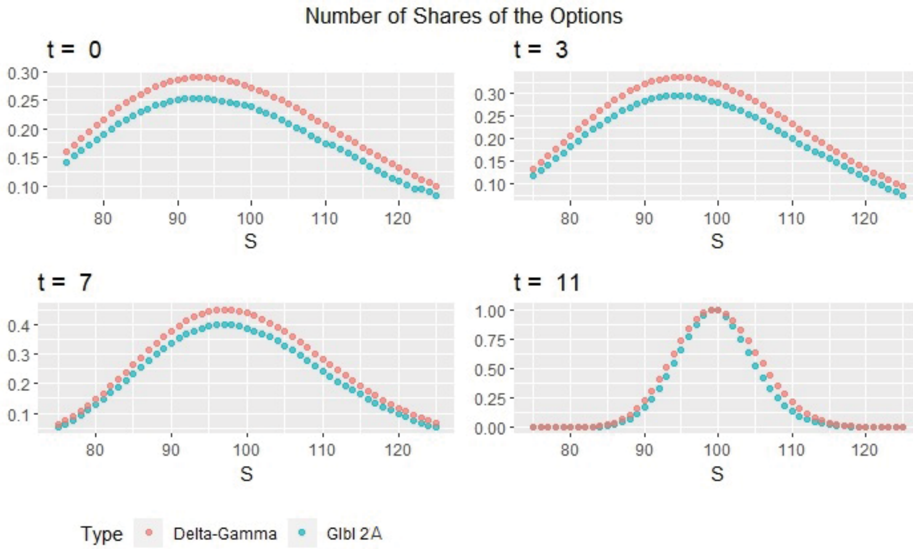

Figure 4.8: Number of shares of options used as hedging instruments obtained by option-based global hedging and delta-gamma hedging for various values of underlying, S, and time steps, t.

Chapter 5

Conclusion and Future Work

This thesis began by reviewing the classical approach to global quadratic hedging which involves hedging a European option only with the underlying. The current work then proposed to use options in combination with the underlying as hedging instruments, as a mean of decreasing the quadratic hedging errors. Specifically, at-the-money options with one-period of maturity were considered. This first required adaptation of the theoretical results in Schweizer [1995] and Rémillard and Rubenthaler [2013]. The optimal replication error (MSE) calculation is then adapted using the results in Bertsimas et al. [2001]. Such a quantity was used to ensure sufficient accuracy was reached within the simulation studies that were conducted.

An extensive numerical study was performed in Chapter 4. Throughout the whole study, it was assumed that the underlying follows the Black-Scholes model. This study showed that the average of quadratic hedging errors obtained by the option-based global hedging framework is always lower than for the underlying-based global hedging, delta hedging, and delta-gamma hedging approaches. However, option-based global hedging is comparatively riskier in tails compared to delta-gamma hedging. It was also demonstrated that for higher drift (μ) and volatility (σ) levels, option-based global hedging outperforms the other classical hedging approaches by even large margins. Moreover, it was seen that the performance advantage of option-based global hedging over alternative methods becomes even greater as the number of hedging periods decreases. Option-based global hedging is then analyzed in order to investigate its turnover. The latter is then compared to turnover of the other frameworks.It is found that obtaining more accuracy in terms of the quadratic hedging errors does not come at the expense of a higher turnover.

The shapes of the quantities used to calculate the global hedging portfolio composition were analyzed. This analysis allowed comparing the quantities obtained from option-based global hedging to the closed form of the quantities obtained for underlying-based global hedging obtained from Godin [2018]. The figures reflected that the shape of the quantities examined both in the underlying-based global hedging and the option-based approaches is similar, and that these quantities behave in an analogous manner.

Lastly, the evolution of the number of shares of the hedging instruments through time was investigated. It was seen that option-based global hedging seems to behave analogously to delta-gamma hedging with an added correction term. It was argued that at each time step, this correction term adjusts the portfolio weights such that the hedging portfolio value will tend to become closer the option value implied by the option-based global hedging framework in subsequent time steps.

Future research directions of interest which build on the results of this thesis include considering other models for the underlying, such as jump-diffusion or GARCH models. Another possible research direction is to use other types of options with different moneyness levels and times to maturity. This could lead to further improvements of the performance of option-based global hedging. On the theoretical side, one can aim at providing closed-form formulas for the quantities used in the dynamic programming framework for option-based global hedging. Moreover, the incorporation of transaction costs and stochastic interest models into the framework could be investigated, leading to even more effective extensions. These are all interesting future research topics to be explored in future work.

Appendix A

Proof of Theorem 1

Proof. Rémillard and Rubenthaler [2013] have suggested a recursive framework for solving the problem stated in equation (3.1). The aforementioned framework is transformed into the statement in Theorem 1. Here is the solution stated in Rémillard and Rubenthaler [2013]: Set $P_{T+1} = 1$ and $\nu_{T+1} = 1$ and for $t = T, ..., 1$ define:

$$A_t = \mathbb{E}[\Delta_t \Delta_t^\top P_{t+1} | \mathscr{F}_{t-1}],$$

$$\mu_t = \mathbb{E}[\Delta_t P_{t+1} | \mathscr{F}_{t-1}],$$

$$b_t = A_t^{-1} \mathbb{E}[\Delta_t P_{t+1} | \mathscr{F}_{t-1}],$$

$$P_t = \prod_{j=t}^{\top} (1 - b_j^\top \Delta_j),$$

$$\nu_t = \mathbb{E}[P_t | \mathscr{F}_{t-1}]$$

provided that these expressions exist. If the expressions exist, the solution set (V_0, θ) of the minimization problem (3.1) is as follows for $t = T, \ldots, 1$:

$$V_0 = \frac{\mathbb{E}[\beta_T C P_1]}{\nu_1}$$

$$\theta_t^{\bar{S}} = \alpha_t - \beta_{t-1} V_{t-1} b_t$$

where,

$$\alpha_t := A_t^{-1} \mathbb{E}[\beta_T C \Delta_t P_{t+1} | \mathscr{F}_{t-1}].$$

The transformation of the formulas is as follows:

$$\begin{aligned} A_t &= \mathbb{E}[\Delta_t \Delta_t^\top P_{t+1} | \mathscr{F}_{t-1}] = \mathbb{E}[\mathbb{E}[\Delta_t \Delta_t^\top P_{t+1} | \mathscr{F}_t] | \mathscr{F}_{t-1}] \\ &= \mathbb{E}[\Delta_t \Delta_t^\top \mathbb{E}[P_{t+1} | \mathscr{F}_t] | \mathscr{F}_{t-1}] = \mathbb{E}[\Delta_t \Delta_t^\top \nu_{t+1} | \mathscr{F}_{t-1}], \end{aligned}$$

$$\mu_t = \mathbb{E}[\Delta_t \nu_{t+1} | \mathscr{F}_{t-1}],$$

$$b_t = A_t^{-1} \mathbb{E}[\Delta_t \nu_{t+1} | \mathscr{F}_{t-1}],$$

$$\begin{aligned} \nu_t &= \mathbb{E}[P_t | \mathscr{F}_{t-1}] = \mathbb{E}[(1 - b_t^\top \Delta_t) P_{t+1} | \mathscr{F}_{t-1}] \\ &= \mathbb{E}[\mathbb{E}[(1 - b_t^\top \Delta_t) P_{t+1} | \mathscr{F}_t] | \mathscr{F}_{t-1}] \\ &= \mathbb{E}[(1 - b_t^\top \Delta_t) \mathbb{E}[P_{t+1} | \mathscr{F}_t] | \mathscr{F}_{t-1}] = E[(1 - b_t^\top \Delta_t) \nu_{t+1} | \mathscr{F}_{t-1}]. \end{aligned}$$

Rémillard and Rubenthaler [2013] show that:

$$\beta_{t-1} C_{t-1} = \frac{1}{\nu_t} \mathbb{E}[\beta_t C_t (1 - b_t^\top \Delta_t) \nu_{t+1} | \mathscr{F}_{t-1}].$$

and after they prove that $\alpha_t = A_t^{-1}\mathbb{E}[\beta_t C_t \Delta_t \nu_{t+1} | \mathscr{F}_{t-1}]$ and $V_0 = C_0$.

□

Appendix B

Sketch of the dynamic programming algorithm

This section explains the procedure to estimate the variables in equation (3.3) needed for solving the quadratic global hedging problem under the following assumptions:

(1) The underlying follows the Black-Scholes model meaning $S_t = f_t(S_{t-1}, z_t)$ where $f(x,y) = xe^{(r-\sigma^2/2)\delta+\sigma\sqrt{\delta}y}$ as explained in equation (2.3).

(2) The options used as hedging instrument are written on the underlying.

The procedure for approximating variable A is detailed and the approximation for the rest of the variables can be obtained through a similar procedure. $\hat{v}_{t+1}(\cdot)$ stands for the approximated value for $v_{t+1}(\cdot)$ obtained from the same procedure.

For all $t = 1, \ldots, T$:

$$\begin{aligned} A_t &= \mathbb{E}\Big[\Delta_t \Delta_t^\top \nu_{t+1} | \mathscr{F}_{t-1}\Big] \\ &\approx \mathbb{E}\Big[\Delta_t \Delta_t^\top \hat{\nu}_{t+1}(\bar{S}_t) | \mathscr{F}_{t-1}\Big] \\ &\overset{(2)}{=} \mathbb{E}\Big[\Delta_t \Delta_t^\top \hat{\nu}_{t+1}(S_t) | S_{t-1}\Big] \\ &\overset{(1)}{=} \mathbb{E}\Big[\Delta_t \Delta_t^\top \hat{\nu}_{t+1}(S_{t-1}, z_t) | S_{t-1}\Big] \\ &= \mathbb{E}\Big[\Psi_t(S_{t-1}, z_t) | S_{t-1}\Big] \end{aligned}$$

where,

$$\Psi_t(S_{t-1}, z_t) = \left(\begin{bmatrix} \beta_t S_t^b - \beta_{t-1} S_{t-1}^b \\ \beta_t D_{t-1}^{1,e} - \beta_{t-1} D_{t-1}^{1,b} \\ \vdots \\ \beta_t D_{t-1}^{m,e} - \beta_{t-1} D_{t-1}^{m,b} \end{bmatrix} \Big[\beta_t S_t^b - \beta_{t-1} S_{t-1}^b, \quad \ldots\ldots \quad , \beta_t D_{t-1}^{m,e} - \beta_{t-1} D_{t-1}^{m,b}\Big] \right) \times$$
$$\hat{\nu}_{t+1}(S_{t-1}, z_t).$$

As it is not feasible to approximate A_t for all the possible values of the underlying S, $\left(S^1, \ldots, S^n\right)$ are chosen as the grid values of the underlying values for which the variables in equation (3.3) are calculated. At each time step t and for each $S^i \in \left(S^1, \ldots, S^n\right)$, $A_t(S^i,t)$ can be estimated as follows:

$$\begin{aligned}
\hat{A}_t(S^i,t) &\approx \mathbb{E}\left[\Delta_t \Delta_t^\top \hat{v}_{t+1} | \mathscr{F}_{t-1}\right] \\
&= \mathbb{E}\left[\Psi_t(S^i, z_t) | S_{t-1}\right] \\
&= \int \Psi_t(S^i, z) f_{z_t}(z) dz
\end{aligned}$$

Then using the Gaussian-Hermite Quadrature approximation as shown in Steen et al. [1969], the integral can be approximated as:

$$\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{m} w_k^{gh} \Psi(exp(\sqrt{2}\sigma_t^i x_k^{gh} + \mu_t^i))$$

where x_k^{gh} and w_k^{gh} are respectively the Gaussian Hermite nodes and weights associated to them, $\sigma_t^i = \sigma\sqrt{\delta}$ and $\mu_t^i = ln(S_{t-1}) + (\mu - \frac{\sigma^2}{2})\delta$. μ and σ are the same variables as in Section 2.2.

Appendix C

Proof of Theorem 3

Bertsimas et al. [2001] show that the solution to the optimal quadratic replication problem is obtained through the following backward recursive scheme:

$$\theta_t^{\bar{S}} = p_t - q_t V_t$$

paired with

$$V_0 = b_0,$$

intiated at $t = T$ where $b_T = \Phi(S_T)$, $a_{T+1} = 1$, $c_{T+1} = 0$ and

$$
\begin{aligned}
p_t =& \frac{\mathbb{E}\left[a_{t+1} b_{t+1} \Delta_t | \mathscr{F}_{t-1}\right]}{\mathbb{E}\left[a_{t+1} \Delta_t \Delta_t^\top | \mathscr{F}_{t-1}\right]}, \\
q_t =& \frac{\mathbb{E}\left[a_{t+1} \Delta_t | \mathscr{F}_{t-1}\right]}{\mathbb{E}\left[a_{t+1} \Delta_t \Delta_t^\top | \mathscr{F}_{t-1}\right]}, \\
a_t =& \mathbb{E}\left[a_{t+1}\left(1 - q_t \Delta_t\right)^2 | \mathscr{F}_{t-1}\right], \\
b_{t-1} =& \frac{1}{a_t} \mathbb{E}\left[a_{t+1}\left(b_t - p_t \Delta_t\right)\left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right] \\
c_t =& \mathbb{E}\left[c_{t+1} | \mathscr{F}_{t-1}\right] + \mathbb{E}\left[a_{t+1}\left(b_t - p_t \Delta_t^\top \Delta_t\right) | \mathscr{F}_{t-1}\right] - a_t b_{t-1}^2
\end{aligned}
$$

and

$$\varepsilon^* = c_0.$$

Looking at the variable a_t we can see that:

$$a_t = \mathbb{E}\left[\prod_{j=t}^{\top}\left(1 - q_j \Delta_j\right)^2 | \mathscr{F}_{t-1}\right].$$

Using the following result from Schweizer et al. [1996]:

$$\mathbb{E}\left[\prod_{j=t}^{\top}\left(1 - q_j \Delta_j\right)^2 | \mathscr{F}_{t-1}\right] = \mathbb{E}\left[\prod_{j=t}^{\top}\left(1 - q_j \Delta_j\right) | \mathscr{F}_{t-1}\right],$$

a_t can be written as follows:

$$a_t = \mathbb{E}\left[\prod_{j=t}^{\top}\left(1 - q_j \Delta_j\right) | \mathscr{F}_{t-1}\right] = \mathbb{E}\left[a_{t+1}\left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right].$$

and using the following again from Schweizer et al. [1996]:

$$\left[\Delta_t \prod_{j=1}^{\top} \left(1 - q_j \Delta_j\right) | \mathscr{F}_{t-1}\right] = 0$$

for $t = 1, \ldots, T$,

$$\begin{aligned}
b_{t-1} =& \frac{1}{a_t} \mathbb{E}\left[a_{t+1}\left(b_t - p_t \Delta_t\right)\left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right] \\
=& \frac{1}{a_t} \mathbb{E}\left[a_{t+1} b_t \left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right] - \frac{1}{a_t} \mathbb{E}\left[a_{t+1} p_t \Delta_t \left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right] \\
=& \frac{1}{a_t} \mathbb{E}\left[a_{t+1} b_t \left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right] - \frac{1}{a_t} p_t \mathbb{E}\left[a_{t+1} \Delta_t \left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right] \\
=& \frac{1}{a_t} \mathbb{E}\left[a_{t+1} b_t \left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right] - \frac{p_t}{a_t} \mathbb{E}\left[\Delta_t \prod_{j=t}^{\top} \left(1 - q_j \Delta_j\right) | \mathscr{F}_{t-1}\right] \\
=& \frac{1}{a_t} \mathbb{E}\left[a_{t+1} b_t \left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right] - \frac{p_t}{a_t} \frac{\mathbb{E}\left[\Delta_t \prod_{j=1}^{\top} \left(1 - q_j \Delta_j\right) | \mathscr{F}_{t-1}\right]}{\prod_{j=1}^{t-1} (1 - q_j \Delta_j)} \\
=& \frac{1}{a_t} \mathbb{E}\left[a_{t+1} b_t \left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right] - \frac{p_t}{a_t} \frac{0}{\prod_{j=1}^{t-1} (1 - q_j \Delta_j)} \\
=& \frac{1}{a_t} \mathbb{E}\left[a_{t+1} b_t \left(1 - q_t \Delta_t\right) | \mathscr{F}_{t-1}\right]
\end{aligned}$$

In our framework, we approximate the discounted value of the option at the hedging period $t = T$ and therefore we have the following setting, initialized by:

$$b_T = \beta_T C_T = \beta_T \Phi\left(S_T\right)$$

$$a_{T+1} = \nu_{T+1} = 1$$

$$c_{T+1} = 0$$

and for $t = T, \dots, 1$:

$$p_t = \alpha_t$$

$$q_t = b_t$$

$$a_t = \nu_t$$

$$b_{t-1} = \beta_{t-1} C_{t-1}$$

and hence:

$$\begin{aligned} c_t &= \mathbb{E}\left[c_{t+1} | \mathscr{F}_{t-1}\right] + \mathbb{E}\left[a_{t+1} \left(b_t - p_t \Delta_t\right)^2 | \mathscr{F}_{t-1}\right] - a_t b_{t-1}^2 \\ &= \mathbb{E}\left[c_{t+1} | \mathscr{F}_{t-1}\right] + \mathbb{E}\left[\nu_{t+1} \left(\beta_t C_t - \alpha_t \Delta_t\right)^2 | \mathscr{F}_{t-1}\right] - \nu_t (\beta_{t-1} C_{t-1})^2. \end{aligned}$$

Appendix D

Tables

In Table D.1, the turnover of the hedging positions calculated using 10,000 paths when hedging a European call option using monthly rebalancing where $S_0 = 100$, $K = 100$, $\sigma = 0.2$, $T = 1$ and the μ varies between 0.15, 0.27 and 0.47.

Also, in Table D.2, the turnover of the hedging positions calculated using 10,000 paths when hedging a European call option using monthly rebalancing where $S_0 = 100$, $K = 100$, $\mu = 0.1$, $\sigma = 0.2$, and the years to maturity varies between 1,2 and 5.

Params	Model	Underlying Turnover	Option Turnover
$\mu = 0.15$	Glbl 2A	1.094	2.760
	Glbl 1A	0.883	-
	Δ-Γ	1.112	3.151
	Δ	0.911	-
$\mu = 0.27$	Glbl 2A	1.016	2.382
	Glbl 1A	0.716	-
	Δ-Γ	1.024	2.662
	Δ	0.788	-
$\mu = 0.47$	Glbl 2A	0.815	1.679
	Glbl 1A	0.438	-
	Δ-Γ	0.818	1.841
	Δ	0.592	-

Table D.1: The turnover of the hedging positions for varying μ.

Params	Model	Underlying Turnover	Option Turnover
$T = 1$	Glbl 2A	1.105	2.904
	Glbl 1A	0.941	-
	Δ-Γ	1.125	3.309
	Δ	0.954	-
$T = 2$	Glbl 2A	1.506	3.812
	Glbl 1A	1.303	-
	Δ-Γ	1.534	4.351
	Δ	1.319	-
$T = 5$	Glbl 2A	1.896	4.684
	Glbl 1A	1.648	-
	Δ-Γ	1.934	5.366
	Δ	1.669	-

Table D.2: The turnover of the hedging positions for varying time to maturity.

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