Location and Location-Routing Problems with Disruption Risks

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This is to certify that the thesis prepared

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Abstract

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Doctor of Philosophy in Industrial Engineering

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The academic literature on logistics network disruptions has increased sharply recently. Disruptions are random events that cause an element of a logistics network to stop functioning, either completely or partially, for a (typically random) given amount of time. Because of today's globalized threats such as, labor disruptions or failures resulting from harsh weather conditions, there has been a renewed interest in resilient facility location. Design of reliable logistics networks to avoid disruption can be accomplished by fortification of existing facilities and defining backup facilities.

In this thesis, we will look at two components of a logistics system that can be affected by a disruption: the locations of the facilities, and the routes between a customer and a facility. We study the following three designs of logistics networks under disruption: (i) Reliable Capacitated Facility Location under Disruption, (ii) Shared Capacitated Reliable Facility Location in Presence of Disruption , and (iii) Reliable Facility Location and Routing in Logistics Network in presence of disruption considering backup sharing.

A column generation approach is proposed to model and solve all three logistics problems. Results show the effectiveness of the decomposition schemes for solving exactly much larger facility location instances than in the literature. In addition, shared backup is shown to be a very effective scheme for the design of reliable facility locations/roads.

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I am grateful to all the people who have been part of my PhD journey. First and foremost, my supervisor Dr. Brigitte Jaumard for her continuous advices, support, teaching and training throughout these years. I have very much enjoyed working with you and learned a great deal of knowledge and experience during this time. Thank you for being patient, wise and endlessly giving during this journey. I would not have imagined completing my PhD without your support. I am grateful to my parents and my family who support me through all stages of my life. Thanks to my beloved wife whom has given me an endless, unconditional love through these years.

M. Badakhshian

Contribution of Authors

This dissertation is presented in a manuscript-style. It contains three papers that are currently under review or on the point to be submitted in different journals. They are presented here as follows. The first article entitled "Reliable Capacitated Facility Location under Disruptions: A Column Generation Approach" in the Journal of Computers and Industrial Engineering and co-authored with Dr. Brigitte Jaumard. The second article entitled "Shared Capacitated Reliable Facility Location in Presence of Disruption: A Column Generation Approach" in the journal of Computer Operation Research and is co-authored with Dr. Brigitte Jaumard. The third article entitled "Designing Reliable Facility Location and Routing in Logistics Network in presence of disruption considering backup sharing" in the Journal of European Journal of Operational Research and is co-authored with Dr. Brigitte Jaumard.

The author of this thesis acted as the principal researcher with the corresponding duties of validating mathematical formulations, proofs, designing and implementing the algorithms and making the analysis of computational experiments along with the writing of the first drafts of all articles.

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Chapter 1

Introduction

1.1 Overview

In recent years, the design of resilient logistics networks has gained a lot of global attention in the context of the severe effects of disruption. Facility location and routing are the main elements of a logistics network, and the decisions made about the location of suppliers and the routes between customers and supplier facility are part of strategic decision making in a logistics network.

The basic Facility Location (FLP) or p-median problem involves the location of the p facilities on a logistics network, which the total transportation cost of serving all demands is minimized. In a p-median problem, the Euclidean distance between the facility and the user (customer) is considered as the transportation cost. In the real world the transportation cost is based on the route between the users and the facility. In the Location and Routing Problem (LRP), the routes between the facility and the users are considered and the decisions on facilities and routes are made simultaneously.

Capacitated Facility Location Problem (CPMP) is a branch of FLP. In this problem, there are constraints on the capacity of locations, so that the location can supply limited users demands.

Real-world environments are dynamic in nature and they are subject to various disruptions, which can affect the logistics networks performance. Disruption is the result of an event that causes an unplanned and negative deviation on the objectives of organizations. For instance, natural disaster and industrial plant fires are two events that result in disruption. These types of disruptions affect different elements of the logistics network. For instance, disruption may affect the availability of facilities or it may affect the route between a user and a supplier facility.

Different events cause different levels of disruptions. Two levels of disruptions have been studied: partial and complete disruption. Partial disruption is the result of an event that may cause delays in logistics networks. For instance, the disruption causes delay on route between the user and the facility. Complete disruption causes the logistics network elements to become unavailable [1][2]. For instance, this level of disruption makes supplier facility completely unavailable. A reliable logistics network is designed to consider disruption effects. Different approaches have been proposed to design reliable logistics networks, such as the fortification of logistics network elements. One of the main elements in logistics network is the supplier facility. The facility fortification approach is used to protect the facility in the event of disruption, so its offering protection to the facility under disruption [3]. In this case, the facility is reliable and protected in the event of a disruption, and it can supply users demands in the presence of a disruption. Usually a facility fortification costs a lot and all facilities cannot be affordably fortified. In this case, we fortify selected facilities based on the available budget. Another approach, creating a reliable facility location is considering a backup facility for those users connected to a disrupted facility. In this case, if the primary or main supplier is disrupted, the backup facilities supply the users demands. Having a backup for the disrupted facility makes the facility location reliable in the presence of disruption [4][5]. In FLP, when we consider backup facility, we have additional transportation cost (backup transportation cost) based on Euclidean distance between the user and the backup facility. In LRP, the transportation cost is based on the route between the user and the facility. In this case, the backup transportation cost is based on the route between the user and the backup facility (backup route).

We must consider that a backup facility requires greater capacity. In this case, the constraints on facility capacity play a critical role in reliable facility location problems (FLP) or location and routing problems (LRP). The required backup capacity is equal to total backup demands. In this problem, we considered one disruption at a time and we have one disrupted facility. So, the required backup capacity is equal to the maximum of backup demands. Then we can share the backup capacity for facilities under disruption.

In this thesis, we proposed models for reliable FLP and LRP under disruption. We considered complete and independent disruption. There is only one disruption at a time, i.e. we have time to recover from the current disruption before another one occurs. In the first two parts of this research (Chapters two and three), we assume that there is a disruption occurring only to a facility only, while in the third part (Chapter four), there is a disruption happening to either a facility or a route. We solve the problem without sharing backup and, then based on primary user-supplier assignment, we look into sharing backup capacity. Then we look into backup sharing and primary assignment at the same time, when there is only disruption occurring to the facility lactation (Chapters Two and Three). In LRP, we consider routes between the facilities and the users, and there is a disruption occurring either on route or at the facility. In continuation, we describe the FLP and LRP under disruption. Afterward, we express the contribution of this thesis. Then we remark on the scope and the main objectives of this thesis.

1.2 Reliable Facility Location and Routing Problem

The FLP is usually modeled as a set of facility, which supplies a set of users. We select a specific number of facilities to open and we assign the users with optimal transportation cost (time, distance). In the literature, the transportation cost is calculated based on the Euclidean distance between the user and the facility. Authors considered a specific route between the user and the facility to calculate the transportation cost. In this research, we consider Euclidean distance in the first two parts (Chapters two and three). We consider the route between the users and the facilities for our last part (Chapter four). The FLP is modeled as capacitated or uncapacitated facilities. In capacitated FLP, there is limitation on the available product quantity to supply users demand. To decrease the complexity of the problem, authors considered uncapacitated facilities within their problem. In this research, we consider capacitated facilities when there are constraints on the facilities capacity. The capacitated location and routing problem (CLRP) is a branch of Capacitated FLP. In CLRP, the routes are a set of arcs that connect the user to the facility and there are capacity constraints for the facilities.

In this thesis, we focused on the FLP and the LRP problem in logistics network in the presence of disruption. We study the capacitated FLP under disruption. We consider the facility fortification and the backup facility to make the logistics network reliable. Since there are constraints on capacity, we study backup capacity sharing. In CLRP, we considered the routes between the user and the facilities and there is a disruption on the facility or the route. In continuation, we discuss the problem in detail.

1.2.1 The Studied Facility Location Problems

In this thesis, we denote by the set of customers, the set of potential facility locations for supply users, and the limitation on the number of facilities to open. Each customer has demand, and there is capacity limitation for each location. The transportation cost between facility location and the customer is known. In this problem, each customer is assigned to a primary supplier. If the primary supplier is not fortified, the customer needs a different backup supplier.

1.2.1.1 Capacitated FLP under Disruption

In this problem, there is one disruption at the time, which makes the facility unavailable. There are two ways to make facilities reliable under disruption: (i) fortify the facilities and (ii) assign a backup facility to users, so that if the facility is under disruption, there is another facility to supply the users connected to the disrupted facility.

In this problem, we fortify the facilities as much as the budget allows. Then we consider the backup facilities, so that if the facility is not fortified, there is a backup available for users.

1.2.1.2 Shared Backup Capacitated FLP under Disruption

Since the capacity of facilities is limited, it is not possible to assign many users to the opened facility as backup. Also, we know that we have one disruption at the time. Then we can share the capacity for backup assignment. As it is shown in the figure 1.1, we are looking for a model to find the optimal cost in case that we share the backup capacity.

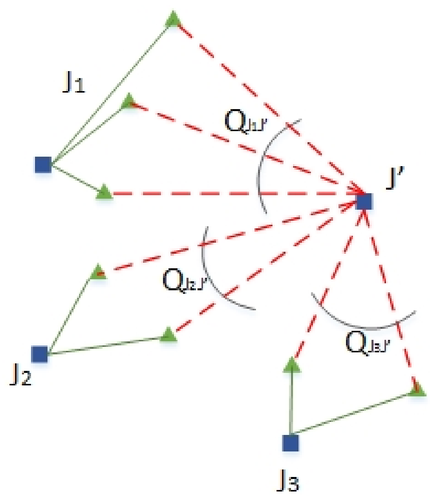


FIGURE 1.1: Shared Backup Resource Requirements

In this problem, we consider fortification and the shared backup facility. We look for the primary assignments for the users and also find a backup assignment for non-fortified ones.

1.2.1.3 Capacitated LRP under Disruption

Disruption can affect the route between the facility and its users. Then it makes that route unavailable. In this problem, we consider routes between the facilities and users (LRP) instead of the Euclidean distance. There is a disruption occurring on a route or to a supplier. There is one disruption happening at the time. Since there is one disruption at this time, we consider the primary and backup facilities and the route for each user. Details are shown in Figure 1.2.

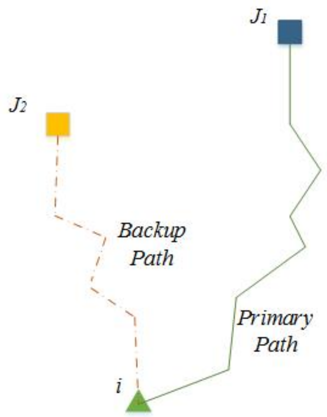


FIGURE 1.2: Primary and Backup Route in Case of Route Disruption

1.3 Contribution of the PhD Thesis

In this thesis, we proposed new models and their solutions for the facility location problem under disruption. For the first problem, which is explained in the Section 1.2.1.1, the column generation (CG) decomposition is designed to find the primary and backup assignments.

Then by using of the solution of this model, the primary assignment is recorded and used in the shared backup model. In the case of the shared backup model, the solution process has two main steps. First, we find the primary assignment for users without considering backup sharing. In the second step, we use the primary assignment in previous step, and we find the shared backup assignment for users in cases where their primary facility is not fortified.

The problem explained in Section 1.2.1.2 is shared backup capacitated FLP such that the Primary and backup assignments can be found in one model. In this case, the model finds the primary and shared backup assignment. The shared backup capacitated FLP is proposed to find the Primary and backup assignments, while considering fortification based on the budget and shared backup for facilities that are not fortified.

The third problem of this thesis, which we explain in Section 1.2.1.3, is modeled as a CLRP under disruption. The CG decomposition model is proposed to find primary assignments and their routes, as well as shared backup assignment and their routes.

Also, the heuristics methods are proposed in the different problems to improve the column generation. The heuristic methods help to improve the column generations quality of solutions and its processing time. In continuation, the scope and the objectives of thesis are explained.

1.4 Scope and Objectives

The overall objective of the thesis is to investigate the optimal transportation cost for reliable facility locations under disruptions in logistics networks. The main objectives of this thesis are explained below.
The objective of this thesis can be summarized as the following:

- To introduce a new decomposition formulation of reliable facility location under disruption.
- Capacity sharing of facilities for backup assignment.
- Shared backup facility assignment considering the route between the facility and the user.
- To introduce a decomposition model to solve facility and routing problems under disruption.
- An improvement of column generation (in terms of processing time and quality of solution.)

1.5 Organization of the Thesis

In this thesis, capacitated FLP and LRP under disruption are studied. Three main models are proposed to solve the problems in Section 1.2.1. In chapter two, the objective is to find the optimal solution for a reliable capacitated facility location, while considering fortification and the backup facility, either with backup sharing or without backup sharing. Chapter three studies a model to find an optimal solution for reliable capacitated facility location in the event of decision about the primary and backup sharing assignment at the same time. The objective in chapter four is to find an optimal solution when, instead of Euclidean distance, there is a route between the user and the facility and disruption occurs on the route and in the facility. The column generation (CG), a well-suited decomposition method for integer linear problem (ILP), or mixed integer linear problem (MILP) are used to solve this problem. Heuristics methods are used to improve the CG solution. In the Chapter five, we state the conclusion of this thesis and future works are suggested.

Chapter 2

Reliable Capacitated Facility Location under Disruptions: A Column Generation Approach

Abstract

Because of today's globalized threats that come in addition to, e.g., electricity disruptions or harsh weather conditions, there has been a renewed interest in resilient facility location. In this paper, we revisit the capacitated p-median facility location problem subject to a single facility disruption (i.e., never more than one facility disruption at a time) and explore the concept of shared protection. This last concept has been widely studied in the context of communication networks, but is fairly new in the context of facility locations. In order to address the scalability limitations of the previous formulations and solution schemes, we also propose two decomposition formulations and algorithms, using column generation techniques. Extensive numerical experiments complete the study, and quantify the capacity savings when shared backup capacity is considered: up to 82% for data instances with 40 selected facility locations (out of 150 potential locations) and 150 users. In addition, thanks to the decomposition schemes, we can solve exactly capacitated p-median facility location problem subject to a single facility disruption with up to 150 potential facility location, $p = 40$, and 150 users within reasonable computational times.

keyword

Reliable Facility Location, Disruptions, Column Generation, Backup Facility, Shared Resources, Single Failure.

2.1 Introduction

Logistics networks refer to the entire chain of distribution centers and transportation of goods or services from a supplier to the final customers or users. On time production and delivery, reliable suppliers and limitation on inventory are the main concerns of companies in today logistics systems. Resilient logistics systems were first studied within a defense or military context (Yoho *et al.* [6]). But then, with the continuously increasing size and complexity of the distribution networks, resiliency is now also a major concern in today logistic systems. Examples of different types of disruptions, ranging from a fire event, to a terrorist attack, or the SARS outbreak can be found in, e.g., Li *et al.* [5].

In this paper, we examine a column generation approach for an extension of the capacitated p-median facility location problem in the presence of disruptions. The p-median problem (pMP) is a classical location problem in which the goal is to locate p facilities while minimizing the sum of the distances from each demand node (user) to its nearest facility on a network (or a graph). In the presence of disruptions, the objective of the classical pMP is extended to include the sum of the distances for the backup facilities. The capacitated p-median problem (CpMP) considers capacities for the product or services to be offered by each facility. The total user demand cannot exceed the total facility capacities. We also assume that a fortification budget is given, which may allow some facilities to be fortified, but not all of them. Therefore, protection must be provided to the unfortified ones, in the form of backup facilities for their users.

We propose a decomposition scheme to solve the resulting capacitated p-median problem under disruption. The Master Problem (MP) optimizes a reliable covering of the users by at most p facilities satisfying capacity constraints within the limits of a fortification budget. The so-called pricing problem is a facility configuration generator, i.e., generates a potential reliable user coverage for a given possible facility location.

Two original optimization models are proposed for the capacitated p-median problem under disruption. In the first model, called (RpMP), we assign a primary facility to each user and a backup supplier to each user primarily connected to a non-fortified facility. Primary and backup capacities for suppliers are considered independently. In the second model, called (SB2_RpMP), we share the backup capacity of facilities among their primary users, under the assumption of a single facility disruption at a time (meaning we have time to restore the failing facility before another failure occurs). We then compare capacity savings achieved with the backup sharing capacities.

The paper is organized as follows. Literature review is discussed in Section 2.2. A detailed problem statement and the notations are defined in Section 2.3. We next propose two models of resilient facility location: the first model, RpMP, is presented in Section 2.4. Therein, we consider capacity constraints, facility fortification and backup facility for users assigned to non fortified facilities. Solution process is presented in Section 2.4.4. The second model, SB2_RpMP, is presented in Section 2.5. It adds shared backup capacity to model RpMP. Numerical results are presented for three sets of data instances of facility location. In Section 2.6, results show that up to 84% of capacity can be saved for the backup capacity when sharing is sought, for data instances with 150 users and 40 facility locations, selected among 150 potential ones.

2.2 Related Work

Facility location problems have been widely studied subject to various objectives and different sets of constraints, e.g., p-median problem, p-center problem, max-covering problem for some of the most classical ones.

More recently, researchers have started to investigate the same facility location problems under disruption(s). Different reactive and proactive resilient schemes have been investigated, e.g., Albareda-Sambola [7], Qin *et al.* [8], Scaparra *et al.* [9].

Albareda-Sambola *et al.* [7] proposed a model in which facilities can fail with independent failures and use an extra dummy non-failing facility with large assignment costs. Qin *et al.* [8] considered a limited protection budget in order to select and fortify some facilities. Scaparra *et al.* [9] proposed a fortification/interdiction model so that the disruptive effects of possible intentional attacks to the system are minimized.

We next review the reliable facility location problem classification and discuss the models of the literature for resilient facility location, with a special attention to the p-median problem or variants of it, under disruption.

2.2.1 Generalities

Decisions about reliable facility location are either costly or difficult to reverse. The impact of decisions will remain for a long time horizon (Snyder [10]). In addition, parameter estimation, e.g., costs, demands, transportation times, may be inaccurate due to poor measurements. To have an accurate measurement, we have to consider the failure event effects on the reliability of facilities as disruption affects the availability of facilities in logistics networks. In the literature, different classes of reliable facility location under disruption problems are proposed. In the sequel, we review the classifications of works on facility location under disruption, as well as the solution methods.

2.2.2 Classification of Facility Location Problems

There are various studies on reliable facility location problems in logistics networks. Most of them considered the un-capacitated p-median problem (UFLP) (Lim *et al.*[4]). The capacitated p-median problem (CpMP) has been studied by e.g., Lorena and Senne [11]. However the p-median problem in the context of facility location under disruption has not yet been studied.

Disruption leads to uncertainty on reliability of facility locations. Facility disruptions are of two types: *(i)* disruption on customer demand, and travel time (cost) between facility locations and customers, *(ii)* unavailability of facility locations.

Key papers related to the first category (category *(i)*) are reviewed by Snyder [10] and can be categorized as follows. Category *(i-a)* refers to the classical facility location problem (e.g., Ricciardi *et al.* [12]). Category *(i-b)* refers to stochastic facility location problems (for instance, Louveaux and Peeters [13], Chen *et al.* [14]), and *(i-c)* looks at robust facility location problems (for instance, Averbakh [15], Snyder [10]).

This second category falls under the scope of the current study. Therein, authors have worked along two directions for the design of resilient facility location models: *(ii-a)* fortification of a subset of facilities subject to some fortification number or budget constraints (e.g., Scaparra and Church [16]) and *(ii-b)* establishing some backup facilities (e.g., Lim *et al.* [4]).

We now review the papers dealing with category *(ii)*, starting with those that used the concept of fortification (category *(ii-a)*). In this category, authors consider single and overall multiple disruptions for the facility location problem (e.g., Losada *et al.* [17]). Different levels of disruption as partial and complete are modeled (e.g., Liberatore *et al.* [2]). In case of the capacitated facility location problem, disruption can reduce the capacity of some suppliers (Scaparra and Church [16]) or a facility can loose part of all of its capacity (Atoei *et al.* [18]). This has resulted in the so-called facility interdiction models (see, e.g., Church *et al.* [19] and Losada *et al.* [20] for a thorough review of them), in which a system, e.g., a facility location, a road or a link, is interdicted (due, e.g., to a strike) and cannot cover the demand of any customer. Fortifications then prevent facilities from being interdicted.

There are also several studies dealing with backup resources. None of them consider shared backup capacity. For instance, Lim *et al.* [4] consider hardening selected facilities and require each demand to have a backup assignment to a reliable (hardened) facility. Lin and Savachkin [4] introduce one layer of supplier backup, and facility fortification with variable reliability (up to total reliability) within a finite budget, using a nonlinear model, hence with a limited scalability for its solution. Losada *et al.* [17] investigate a r-interdiction uncapacitated median problem with facility recovery time and frequent disruptions using a bilevel model, solving using three different decomposition methods relying on Benders decomposition and super valid inequalities.

The studies dealing with backup facilities and facility fortifications typically used data instance sizes of 50 to 150 demand nodes (Church and Scaparra[21], Liberatore *et al.*

[22]) and 20 to 40 facility nodes (Losada *et al.* [17], Scaparra and Church [16]) to support demand nodes.

The objectives of the facility location problem under disruption are to minimize or overcome the impact of disruption, with the minimum amount of additional resources. It can expressed by identifying the disruption scenario entailing a maximum overall traveling distance in serving all customers with the use of a stochastic model (Losada *et al.* [20]). It can also be translated by maximizing the facility protection or fortification based on the first investment (Church and Scaparra [21]) or maximizing the recovery of disrupted facilities (e.g., Liberatore *et al.* [3]).

Different exact methods have been proposed in the literature to solve the facility location problem under disruption considering fortification and backup facilities (not shared capacity), such as Benders decomposition (Azad *et al.* [23], Losada *et al.* [17]), Lagrangian relaxation (Snyder and Duskin [24]), pre-processing techniques with the computation of lower and upper bounds. Heuristic methods have been also used in some studies (e.g., Liberatore *et al.* [22]). Different sizes of data sets (up to 316 users and 316 facilities) for facility location problem including or not fortification and backup resources are reported. Observe that those sizes vary with the objective and assumptions (e.g., uncapacitated vs capacitated), and above all, with heuristic vs. exact solutions.

2.3 Problem Statement

Consider a set of m facilities and let J be the set of their potential locations. Facility located in j has a Q_j capacity and a failure probability π_j. Let p be the maximum number of facilities to be opened. We assume that the selection of the p facilities will be constrained by a failure probability π_j of the facilities, so that the overall failure probability of these facilities does not exceed a given threshold ($\overline{\pi}$). A fortification budget B is available in order to harden some facility locations (or facility themselves), so that a user assigned to a fortified facility does not need a backup facility.

We denote by I the set of customers, assuming each customer $i \in I$ has demand D_i. Each customer is assigned a primary supplier. Primary customers of a given facility are assigned to potentially different backup suppliers if their primary facility is not fortified.

Let COST_{ij} be the transport cost of demand between facility location $j \in J$ and customer $i \in I$. Facility fortification cost with two components: s_j a fixed fortification cost that represents the cost to implement facility fortification (e.g., contract negotiation, overhead, personnel training) and r_j a cost associated with the unit reduction in the failure probability π_j of facility j as in Lim *et al.* [4]. Indeed, the r_j fortification cost varies with the amount of reliability improvement of the facility (e.g., acquisition and installation of units of protective measures, procurement, storage of backup inventory, hiring extra workforce).

For a given facility location j with a set I_j of assigned customers, the associated transportation cost can be written as follows:

$$\text{COST}_j = \sum_{i \in I_j} d_{ij} = \text{COST}_j^W + \text{COST}_j^B, \tag{2.1}$$

where COST_j^W is the transportation cost of the users associated with j as a primary facility, and COST_j^B is the transportation cost of the users associated with j as a backup facility if any.

We assume a single facility location disruption at a time, meaning we assume we have time to recover from a first facility failure before a second one occurs. Consequently, the probability of a simultaneous failure of both primary and backup suppliers is assumed negligible. In the sequel, we will propose two models: one model without backup resource sharing, and a second model with backup resource sharing, a concept that has been widely used in communication networks for already a long time ago, see, e.g., Ramaswami and Sivarajan [25] or Develder *et al.* [26]. Consider two users i_1 and i_2, each assigned to a different facility location for their primary supplier, say j_1 for i_1, and j_2 for i_2. Assume that both i_1 and i_2 have the same backup supplier, say j_3. When checking the capacity constraint for j_3, we only need $\max\{D_1, D_2\}$ since i_1 and i_2 will not need to recourse to facility in j_3 at the same time, hence the concept of shared backup capacity.

Additional assumptions:

- The events of facility failures are independent.
- For any customer, if the primary supplier fails, the backup supplier is available following the single facility failure assumption.
- If a facility is fortified, it becomes totally reliable. This is a very common assumption among the studies on facility location with fortification, see, e.g., Qin *et al.* [8], Liberatore *et al.* [2].
- If a facility fails, it becomes unavailable/interdicted (i.e., no partial failure).

2.4 Resilient Capacitated p-Median Problem

We propose a first decomposition model for the reliable p-median facility location problem. Decomposition involves two problems solved alternatively. The first one, the master problem, is defined in Sections 2.4.1-2.4.3 (overall concept, variables, parameters, optimization model) and the second one, called the pricing problem, in Section 2.4.4.1, while the overall solution scheme is depicted in Section 2.4.4.2.

2.4.1 Decomposition Scheme

The proposed model, called RpMP, relies on a decomposition scheme with the concept of configurations defined as follows.

Each configuration is associated with *(i)* one potential facility location (j) and *(ii)* a subset of users connected to this facility either as primary ($a_i^{\mathrm{W},c}$) or backup ($a_i^{\mathrm{B},c}$) supplier. For a given facility location j, let C_j be the set of all possible user facility configurations, where $c \in C_j$ is characterized by the subset of users assigned to a facility in location j, as defined by the values of the $a_i^{\mathrm{W},c}$ and $a_i^{\mathrm{B},c}$ parameters. Each configuration c is consequently characterized by two sets of parameters:

$a_i^{\mathrm{W},c} \in \{0,1\}$. $a_i^{\mathrm{W},c} = 1$ if customer i uses the facility of configuration c as a primary facility location, 0 otherwise.

$a_i^{\mathrm{B},c} \in \{0,1\}$. $a_i^{\mathrm{B},c} = 1$ if customer i uses the facility of configuration c as a backup facility location, 0 otherwise.

Other generic parameters that will be used in the decomposition model have already been defined in Section 2.3.

2.4.2 Variables

We use three sets of decision variables. The first set corresponds to the classical opening facility variables: $y_j = 1$ if facility location j is open, 0 otherwise. The second set is related to the fortified locations of facilities: $x_j = 1$ if facility location j or the facility in j is selected for fortification, 0 otherwise. The third set corresponds to the decomposition variables: $z_c = 1$ if user facility configuration c is selected in the optimal solution, 0 otherwise.

2.4.3 A Decomposition Model

We now describe the new decomposition model we propose for the resilient capacitated p-median problem.

$$\min \quad \sum_{c \in C} \mathrm{COST}_c z_c \tag{2.2}$$

where:

$$\mathrm{COST}_c = \sum_{i \in I} (a_i^{\mathrm{W},c} + a_i^{\mathrm{B},c}) d_{ij} \qquad c \in C_j, j \in J. \tag{2.3}$$

subject to:

$$\sum_{c \in C_j} z_c = y_j \qquad j \in J \tag{2.4}$$

$$x_j \leq y_j \qquad j \in J \tag{2.5}$$

$$\sum_{j \in J} y_j \leq p \tag{2.6}$$

$$\sum_{c \in C} a_i^{\mathrm{W},c} z_c = 1 \qquad i \in I \tag{2.7}$$

$$\sum_{c \in C_j} a_i^{\mathrm{W},c} z_c + \sum_{c \in C} a_i^{\mathrm{B},c} z_c \leq 2 - x_j \qquad i \in I, j \in J \tag{2.8}$$

$$\sum_{c \in C} a_i^{\mathrm{B},c} z_c \geq \sum_{c \in C_j} a_i^{\mathrm{W},c} z_c - x_j \qquad i \in I, j \in J \tag{2.9}$$

$$\sum_{j \in J} \pi_j (\sum_{c \in C_j} a_i^{\mathrm{W},c} z_c) \leq \sum_{j \in J} \pi_j (\sum_{c \in C_j} a_i^{\mathrm{B},c} z_c) - \sum_{c \in C} a_i^{\mathrm{B},c} z_c + 1 \qquad i \in I \tag{2.10}$$

$$\sum_{j \in J} \pi_j y_j \leq \overline{\pi} \tag{2.11}$$

$$\sum_{j \in J} (s_j + r_j \pi_j) x_j \leq B \tag{2.12}$$

$$y_j \in \{0, 1\} \qquad j \in J \tag{2.13}$$

$$x_j \in \{0, 1\} \qquad j \in J \tag{2.14}$$

$$z_c \in \{0, 1\} \qquad c \in C. \tag{2.15}$$

Constraints (2.4) check whether location j is opened for a facility, to be used either or both as primary or a backup supplier. If $y_j = 0$, no facility is opened in location j. If $y_j = 1$, one facility is opened in location j, and we make sure to select exactly one facility configuration in location j. Constraints (2.5) ensure that a given facility location can be considered for fortification (using variable x_j) only if facility j is opened. Constraint (2.6) sets the limit on the number of open facilities. Constraints (2.7) guarantee that each user i is assigned to a primary supplier.

Constraints (2.8) and (2.9) guarantee that each user i is assigned to a backup supplier, if its primary supplier is not a fortified facility. Indeed, assume without loss of generality that $j^w(i)$ is the primary supplier of user i. Then, $\sum_{c \in C_j} a_i^{\mathrm{W},c} z_c = 1$. Consequently, if $x_{j^w(i)} = 1$, then $\sum_{c \in C} a_i^{\mathrm{B},c} z_c = 0$. On the other hand, if $x_{j^w(i)} = 0$, then, according to constraint (2.9), $\sum\limits_{c \in C} a_i^{\mathrm{B},c} z_c \geq \sum\limits_{c \in C_{jw(i)}} a_i^{\mathrm{W},c} z_c = 1$, meaning that user i needs a backup facility, i.e., what we want to be required. Note that, due to the constraints in the pricing problem (see Section 2.4.4.1), a given facility location cannot be used both as a primary and a backup supplier.

Constraints (2.10) guarantee, for any user i, a selection of the primary supplier with a failure probability that is smaller than the failure of the backup supplier if the primary supplier is not fortified. Let us consider a particular user i. If its primary supplier is fortified, due to constraints (2.8) and (2.9), $\sum\limits_{c \in C} a_i^{\mathrm{B},c} z_c = 0$, and consequently $\sum\limits_{j \in J} \pi_j (\sum\limits_{c \in C_j} a_i^{\mathrm{B},c} z_c) = 0$ as well. This is turn implies that $\sum\limits_{j \in J} \pi_j (\sum\limits_{c \in C_j} a_i^{\mathrm{W},c} z_c) \leq 1$ which is always true due to constraints (2.7). On the other hand, if the primary supplier is not fortified, $\sum\limits_{j \in J} \pi_j (\sum\limits_{c \in C_j} a_i^{\mathrm{B},c} z_c) = 1$, and constraints (2.10) becomes:

$$\sum_{j \in J} \pi_j (\sum_{c \in C_j} a_i^{\mathrm{W},c} z_c) \geq \sum_{j \in J} \pi_j (\sum_{c \in C_j} a_i^{\mathrm{B},c} z_c). \tag{2.10'}$$

Constraint (2.11) takes into account the failure probability. This constraint checks that the total failure probability of opened facilities does not exceed $\overline{\pi}$. The value of $\overline{\pi}$ depends on the number of fortifications and its expression is discussed in Section 2.6.1. Constraint (2.12) enforces the fortification budget limit on the selection of fortified facilities. Remaining constraints define the domains of the variables.

2.4.4 Solution Process

As in most column generation ILP models, the first step is to be able to solve the linear relaxation using a column generation technique. It consists in solving alternatively the restricted master problem, i.e., the continuous relaxation of model (2.2)- (2.15) with a very limited number of user facility configurations, and the pricing problems, one for each potential facility location, i.e., the generation of a set of users associated with a given facility location, until an optimality condition (negative reduced cost) is satisfied, as to guarantee we have reached $z^{\star}_{\mathrm{LP}}$, the optimal value of the linear relaxation (reader is referred to, e.g., Chvatal [27] if not familiar with decomposition techniques). Before discussing the ILP solution (see Section 2.4.4.2), we next establish the expression of the pricing problem.

2.4.4.1 Pricing Problem of Model rpmp

The pricing problem ($\text{PP}_j^{\text{RPMP}}$) for a potential facility location j can be written as follows. Let $u_j^{(2.4)} \lesseqgtr 0$, $u_i^{(2.7)} \lesseqgtr 0$, $u_{ij}^{(2.8)} \leq 0$, $u_{ij}^{(2.9)} \geq 0$, and $u_{ij}^{(2.10)} \leq 0$, be the values of the dual variables associated with constraints (2.4), (2.7), (2.8), (2.9), and (2.10) respectively.

The objective function of the pricing problem, i.e., the reduced cost of variable z_j, is written as follows.

$$\min \overline{\text{COST}}_j^{\text{RPMP}} = \text{COST}_j^{\text{RPMP}} - u_j^{(2.4)} - \sum_{i \in I} u_i^{(2.7)} a_i^{\text{W}} - \sum_{i \in I} u_{ij}^{(2.8)} \left(a_i^{\text{B}} + a_i^{\text{W}} \right)$$
$$- \sum_{j' \in J : j' \neq j} \sum_{i \in I} u_{ij'}^{(2.8)} a_i^{\text{B}} - \sum_{i \in I} u_{ij}^{(2.9)} (a_i^{\text{B}} - a_i^{\text{W}})$$
$$- \sum_{j' \in J : j' \neq j} \sum_{i \in I} u_{ij'}^{(2.9)} a_i^{\text{B}} - \sum_{i \in I} u_i^{(2.10)} \left(\pi_j a_i^{\text{W}} + (1 - \pi_j) a_i^{\text{B}} \right),$$

where $\text{COST}_j = \text{COST}_c$ for $c \in C_j$, see (2.3).

Constraints, which identify the primary and secondary users of a new configuration associated with facility j, are as follows:

$$\sum_{i \in I} D_i \left(a_i^{\text{W}} + a_i^{\text{B}} \right) \leq Q_j \tag{2.16}$$
$$a_i^{\text{W}} + a_i^{\text{B}} \leq 1 \qquad i \in I \tag{2.17}$$
$$a_i^{\text{W}}, a_i^{\text{B}} \in \{0, 1\} \qquad i \in I. \tag{2.18}$$

Constraint (2.16) enforce the facility location capacity constraints, while constraints (2.17) prevent facility j from being both the primary and the backup supplier of a user.

2.4.4.2 Column Generation and Quality of the ILP Solutions

The flowchart of the column generation solution is depicted in Figure 2.1. Note that there are different pricing problems, one for each potential facility location. We solve them in a round robin order (while they could be solved in parallel if using multiple processors/threads). Rather than developing a costly branch-and-price (see, e.g., Barnhart

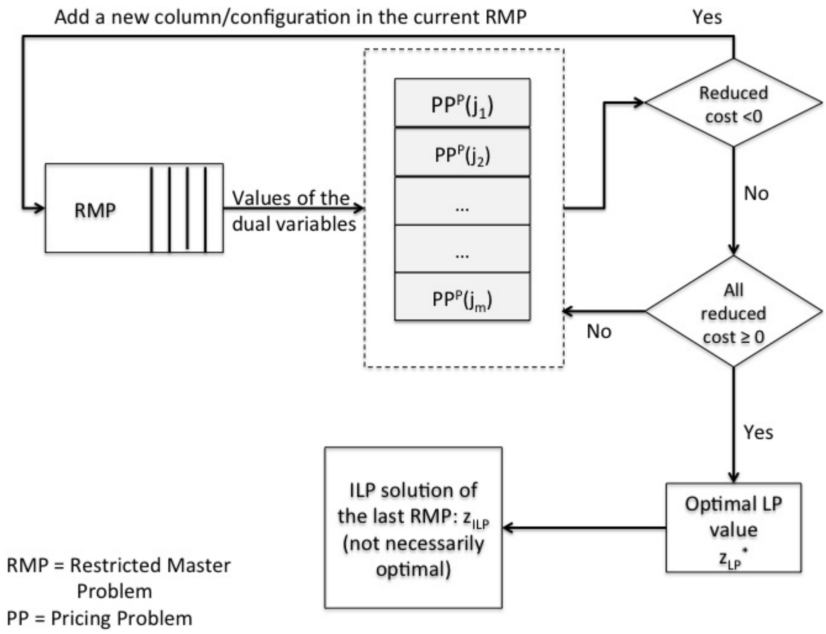


FIGURE 2.1: Flowchart of the Column Generation and ILP Solution

et al. [28] or Vanderbeck [29] for more details), we choose to solve exactly the last restricted master problem with the integrality constraints in order to get an ILP solution, denoted by $\tilde{z}_{\text{ILP}}$. As the result, the solution that is output is an ε-optimal solution, with

$$\varepsilon = \frac{\tilde{z}_{\text{ILP}} - z^{\star}_{\text{LP}}}{z^{\star}_{\text{LP}}}. \tag{2.19}$$

While Model RpMP already offers an improvement over the previous models of the literature in terms of scalability, we can go one step further with the investigation of the savings incurred by sharing the backup bandwidth. That is the purpose of Section 2.5.

2.4.5 Generation of Initial Solutions (Greedy$^{\text{rpmp}}$)

In order to provide a "warm" start to the column generation models, we use a greedy heuristic to set an initial set of columns to speed up the solution of model RpMP, called Greedy$^{\text{RpMP}}$.

In Model RpMP, each column is associated with a potential facility location, together with its set of assigned users, for which it is either a primary or a backup facility. In Greedy$^{\text{RpMP}}$, we follow these steps:

Step 1- Order the facility locations in the increasing order of their π_j values.

Step 2- Following the order in Step 1, fortify the facilities until the fortification budget is exhausted. We assume that fortification budget is such that we cannot fortify more than p facilities.

Step 3- For each user, identify the closest fortified facility. Order the users with respect to their increasing distance to a fortified facility.

Step 4- Assign users to their nearest fortified facility as primary facility, starting with the facility order of Step 1, and then with the users using the order defined in Step 3, subject to the facility capacity constraints.

Step 5- If less than p facilities have been opened, we may open additional facilities starting with those with the smallest π_j.

Step 6- If some users are still without a primary supplier, assign them to their closest open facility as primary supplier, considering the facility capacity constraints.

Step 7- For each user assigned to an unfortified facility, we assign them to the nearest open facility, other than their primary facility for their backup facility, taking into account capacity facility constraints.

2.5 Shared Backup Facility p-Median Problem

Let us revisit the example depicted in Figure 2.2. Therein, users i_1 and i_2 have different primary suppliers, while they share the same backup supplier (j_1). Under the assumption that at most one supplier will fail at a time, and that we have the time to fix the failure of a failure before another one occurs, backup resources of i_1 and i_2 can be shared, i.e., instead of requiring $D_1 + D_2$ with respect to facility location j_1, $\max\{D_1, D_2\}$ suffices as i_1 and i_2 will never require backup resources at the same time, see Figure 2.2.

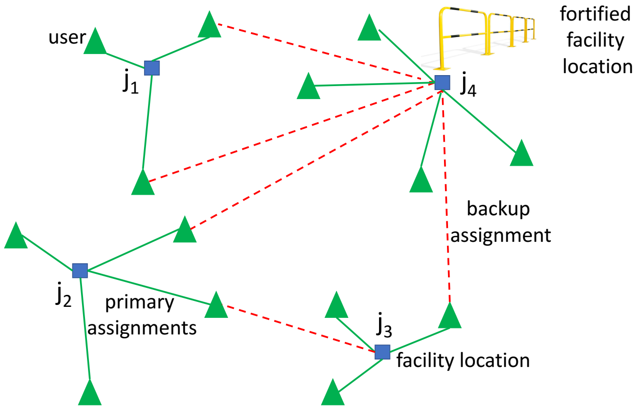


FIGURE 2.2: Sharing Backup Resources

We now revisit RPMP model as described in Section 2.4 in order to integrate resource backup sharing.

2.5.1 A One Step Nonlinear Model

We now describe Model SB1_RPMP, derived from Model RPMP, with resource backup sharing.

Due to the sharing, capacity constraints cannot any more be taken care in the pricing problem, as each pricing problem deals with a potential set of users for a single facility. We therefore divide the capacity values into the primary (Q_j^{W}) and the backup ones (Q_j^{B}) for any given facility j. Primary capacities (Q_j^{W}) are easy to compute, as they correspond to the sum of users' demand. We next need to compute the backup resource requirements (Q_j^{B}). In order to do so, we calculate the backup capacity for each non-fortified facility, say j'. As shown in Figure 2.3, $Q_{j'}^{\mathrm{B}}$ is the largest backup capacity which is required for a given facility failure.

In other words:

$$Q_{j'}^{\mathrm{B}} = \max_{j \in J} \{Q_{jj'}\} \qquad\qquad j, j' \in J, \tag{2.20}$$

where $Q_{jj'}$ is the backup facility that is required in location j' when facility located in j fails. Note that $Q_{jj'}$ is defined by the users assigned to j as their primary facility and to j' as their backup facility.

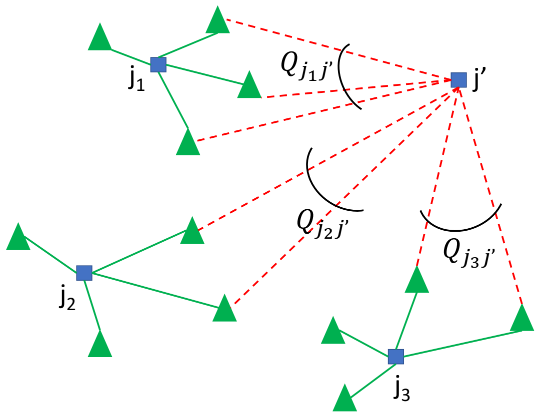


FIGURE 2.3: Computing the Shared Backup Resource Requirements

Capacity constraints can then be written as follows:

$$\sum_{c \in C_j} \sum_{i \in I} D_i \, a_i^{\mathrm{W}} z_c \leq Q_j^{\mathrm{W}} \qquad j \in J \tag{2.21}$$

$$Q_j^{\mathrm{W}} + Q_j^{\mathrm{B}} \leq Q_j \qquad j \in J \tag{2.22}$$

$$\sum_{i \in I} \sum_{c \in C_j} D_i a_i^{\mathrm{W},c} \sum_{c \in C_{j'}} a_i^{\mathrm{B},c} z_c^{\mathrm{B}} \leq Q_j^{\mathrm{B}} \qquad j, j' \in J : j \neq j' \tag{2.23}$$

$$Q_j^{\mathrm{W}}, Q_j^{\mathrm{B}}, Q_j \geq 0 \qquad j \in J. \tag{2.24}$$

Model SB2_RPMP is then defined by the set of constraints of Model RPMP with the addition of three sets of variables $(Q_j^{\mathrm{W}}, Q_j^{\mathrm{B}}, Q_j)$ and of constraints (2.21) to (2.24).

The pricing problem needs to be modified as well. We first update the expression of its objective, i.e., the reduced cost associated with variables z_c following the addition of constraints (2.21) to (2.23):

$$\min \ \overline{\mathrm{COST}_j}^{\mathrm{SB1\_RPMP}} = \overline{\mathrm{COST}_j}^{\mathrm{RPMP}} - u_j^{(2.21)} \sum_{i \in I} a_i^{\mathrm{W}} - \sum_{i \in I} D_i a_i^{\mathrm{W}} \sum_{j' \in J : j' \neq j} u_{jj'}^{(2.23)} a_i^{\mathrm{B}},$$

where $u_j^{(2.21)} \leq 0$ and $u_{jj'}^{(2.23)} \leq 0$ are the values of the new dual variables associated with constraints (2.21),and (2.23), respectively.

Constraints of the pricing problem are now restricted to constraints (2.17) and (2.18), i.e.,

$$a_i^{\mathrm{W}} + a_i^{\mathrm{B}} \leq 1 \qquad i \in I \tag{2.17}$$

$$a_i^{\mathrm{W}}, a_i^{\mathrm{B}} \in \{0, 1\} \qquad i \in I. \tag{2.18}$$

Observe the the reduced cost is now nonlinear, due to the variable product $a_i^{\mathrm{W}} a_i^{\mathrm{B}}$. While it is always possible to linearize, it results in additional variables and constraints, and therefore would greatly affect the scalability of the exact solution of Model SB1_RpMP.

In the next section we design a two step model, called SB2_RpMP, in order to go around the nonlinear objective function of the pricing problem of Model SB1_RpMP.

2.5.2 A Two Step Solution

Since Model SB1_RpMP in Section 2.5.1 is nonlinear, we now look at a two step model: first, we assign users to their primary facility, i.e., we compute a_{ij}^{W} with Model RpMP considering non-shared backup. Secondly, using the a_{ij}^{W} values, we identify a backup facility for users associated with a non fortified facility for their primary facility, considering shared backup capacity. We next describe the decomposition model, called SB2_RpMP, associated with the second step.

2.5.3 Decomposition Scheme

The proposed model SB2_RpMP relies on a decomposition scheme with a similar concept of configurations as for Model RpMP. Consequently, we will also use a column generation algorithm to solve it.

2.5.3.1 Backup Configuration

For Model SB2_RpMP, each configuration is associated with one potential facility location (j) and contains the set of users connected to this facility as backup ($a_i^{\mathrm{B},c}$) supplier. For a given facility location j, let C_j be the set of all possible facility location configurations, where $c \in C_j$ is characterized by the set of users assigned to a facility in location j, as defined by the values of the $a_i^{\text{B},c}$ parameters. Each configuration c is consequently characterized by the following set of parameters:

$a_i^{\text{B},c} \in \{0, 1\}$. $a_i^{\text{B},c} = 1$ if customer i uses the facility of configuration c as a backup facility location, 0 otherwise.

2.5.3.2 Variables

In order to write Model SB2_RPMP, we need three sets of variables. Firstly, we reuse the variables y_j, see Section 2.4.2 for their definition. Note that we do not need to introduce again the variables x_j (decision variables for the facility fortifications) as we assume that the fortification budget is always a limited one, which do not allow the fortification of all the facilities assigned as primary facilities to users. We then define two new sets of variables. The first one corresponds to a set of decision variables: $z_c^{\text{B}} \in \{0, 1\}$, where $z_c^{\text{B}} = 1$ if backup configuration c is selected in the optimal solution, 0 otherwise. The second set of variables define the required backup capacity for each open facility: $Q_j^{\text{B}} \geq 0$ for facility j.

Based on the solution of the first step, the assignment of the users to their primary facility is defined by a_{ij}^{W} with $a_{ij}^{\text{W}} = 1$ if user i is assigned to facility j as its primary facility, and 0 otherwise. It follows that the primary capacity (Q_j^{W}), i.e., the fraction of each facility capacity that is used for primary assignments of users, is as follows:

$$Q_j^{\text{W}} = \sum_{i \in I} a_{ij}^{\text{W}} \, D_i.$$

Information related to primary assignment of users entails the knowledge of the values of some y_j variables. Indeed, $y_j = 1$ for all facilities such that $a_{ij}^{\text{W}} = 1$ for a given user i.

In Model SB2_RPMP, we therefore divide the set J of facilities into two subsets: (i) J^{W}, the facilities which have been already selected to be opened and assigned to users as primary suppliers and (ii) J^U, the facilities which have not been selected to be opened yet and which are available for backup selection if needed, assuming $|J^{\text{W}}| < p$. Similarly, we divide the set i of users into two subsets: (i) $I^{\P}$, the users which are assigned a primary facility that is fortified, and therefore do not need any access to backup resources, (ii) $I^{\mathrm{U}} = I \setminus I^{\P}$, the users that need to be assigned a backup facility.

$$\min \sum_{c \in C} \textsc{cost}^{\mathrm{B}}_c \, z^{\mathrm{B}}_c \tag{2.25}$$

where

$$\textsc{cost}^{B}_c = \sum_{i \in I^{\mathrm{U}}} a^{\mathrm{B},c}_i d_{ij} \qquad c \in C_j.$$

subject to:

$$\sum_{c \in C_j} z^{\mathrm{B}}_c = y_j \qquad j \in J \setminus J^{\mathrm{W}} \tag{2.26}$$

$$\sum_{c \in C_j} z^{\mathrm{B}}_c \leq 1 \qquad j \in J^{\mathrm{W}} \tag{2.27}$$

$$\sum_{j \in J \setminus J^{\mathrm{W}}} y_j + |J^{\mathrm{W}}| \leq p \tag{2.28}$$

$$\sum_{c \in C} a^{\mathrm{B},c}_i z^{\mathrm{B}}_c \leq 2 - a^{\mathrm{W}}_{ij} \qquad i \in I^{\mathrm{U}}, j \in J^{\mathrm{W}} \tag{2.29}$$

$$\sum_{c \in C} a^{\mathrm{B},c}_i z^{\mathrm{B}}_c \geq a^{\mathrm{W}}_{ij} \qquad i \in I^{\mathrm{U}}, j \in J^{\mathrm{W}} \tag{2.30}$$

$$\sum_{j \in J \setminus J^{\mathrm{W}}} \pi_j y_j \leq \overline{\pi} - \sum_{j \in J^{\mathrm{W}}} \pi_j \tag{2.31}$$

$$Q^{\mathrm{W}}_j + Q^{\mathrm{B}}_j \leq Q_j \qquad j \in J \tag{2.32}$$

$$\sum_{i \in I^{\mathrm{U}}} D_i a^{\mathrm{W}}_{ij} \sum_{c \in C_{j'}} a^{\mathrm{B},c}_i z^{\mathrm{B}}_c \leq Q^{\mathrm{B}}_j \qquad j, j' \in J : j \neq j' \tag{2.33}$$

$$Q^{\mathrm{B}}_j \geq 0 \qquad j \in J \tag{2.34}$$

$$y_j \in \{0,1\} \qquad j \in J \setminus J^{\mathrm{W}}$$

$$z^{\mathrm{B}}_c \in \{0,1\} \qquad c \in C. \tag{2.35}$$

Constraints (2.26) identify the new facility locations which are open, for the sole purpose of backup facilities. For each already open facility j, constraints (2.27) make sure we do not select more than one facility configuration, which offers backup resources to users assigned to an unfortified primary facility. Constraints (2.28) make sure that we do not open more than p facilities. Constraints (2.29) and (2.30) are simplified versions of constraints (2.8) and (2.9) as we only consider users which are not assigned to a primary fortified facility, and therefore guarantee that each user $i \in I^{\mathrm{U}}$ is assigned to a backup supplier. Constraints (2.31) is similar to constraints (2.11), and make sure we do not exceed the failure probability bound. Constraints (2.32) enforce the facility capacities. Constraints (2.33) compute an estimate of the backup resource for each facility. Remaining constraints (2.34) - (2.35) define the domains of the variables.

2.5.4 Pricing Problem of Model with Known a_{ij}^{W}

The pricing problem is modified as follows. We now write the pricing problem (PP$_j$) for potential facility location j. Let $u_j^{(2.26)} \leqslant 0$, $u_j^{(2.27)} \leq 0$, $u_{ij}^{(2.29)} \leq 0$, $u_{ij}^{(2.30)} \geq 0$, and $u_{ij}^{(2.33)} \leq 0$ be the values of dual variables associated with constraints (2.26), (2.27), (2.29), (2.30) and (2.33) respectively.

$$\min \overline{\text{COST}_j}^{\text{SHARED}} = \text{COST}_j - u_j^{(2.26)} - u_j^{(2.27)} - \sum_{i \in I} u_{ij}^{(2.29)} a_i^{\mathrm{B}} - \sum_{i \in I} \Big(\sum_{j' \in J : j' \neq j} u_{ij'}^{(2.29)} a_i^{\mathrm{B}} \Big)$$
$$- \sum_{i \in I} u_{ij}^{(2.30)} a_i^{\mathrm{B}} - \sum_{i \in I} D_i a_{ij}^{\mathrm{W}} \sum_{j' \in J : j' \neq j} u_{jj'}^{(2.33)} a_i^{\mathrm{B}}$$

subject to:

$$a_{ij}^{\mathrm{W}} + a_i^{\mathrm{B}} \leq 1 \qquad i \in I \tag{2.36}$$
$$a_i^{\mathrm{B}} \in \{0, 1\} \qquad i \in I.$$

2.5.5 Generation of Initial Solutions (Greedy$^{\text{sb2_rpmp}}$)

In order to provide a "warm" start to the column generation models, we reuse a greedy heuristic in section 2.4.5 to set an initial set of columns to model SB2_RPMP, called GREEDY$^{\text{SB2_RPMP}}$.

In Model $\textsc{sb2\_RpMP}$, we re-use the primary allocation (i.e, $a_i^{\mathrm{W},c}$ obtained by Model $\textsc{RpMP}$(i.e., a_{ij}^{W}). Since the primary assignment is done using $\textsc{RpMP}$, we are only searching for the backup assignment after deciding for the fortifications. In heuristic $\textsc{Greedy}^{\textsc{sb2\_RpMP}}$ we rank the selected facilities as primary suppliers according to their failure probability. Next, we fortify opened facilities with highest failure probabilities until we exhaust the fortification budget. Then we go to the Step 7 in Heuristic $\textsc{Greedy}^{\textsc{RpMP}}$ to assign backup facilities.

2.6 Numerical Results

Models and algorithms proposed in the previous sections were tested on three data sets. We first described the data sets (Section 2.6.1). We next discuss the numerical results, i.e., the accuracy of the solutions and the computational times (Section 2.6.2), some characteristics of the solutions (Section 2.6.3) and then the impact of sharing the backup resources (Section 2.6.4).

2.6.1 Data Sets and Parameters

We considered the three data sets of Snyder and Daskin [24] with 49, 88 and 150 users and $m = n$, which can be found (demand and distance values) in the online appendix of Daskin [30]. We assumed that the transportation cost is proportional to the Euclidean distances.

In order to generate values for the fortification budget B, we re-used the formula of Li *et al.* [5] and Snyder and Daskin [24]. Let $\overline{B}$ be the overall fortification cost for all facilities, it leads to:

$$\overline{B} = \sum_{j \in J} (s_j + r_j \pi_j),$$

where s_j (fixed cost for fortification) is drawn using a uniform distribution $s_j \sim [500, 1500]$ and rounded to nearest integer as in Snyder and Daskin [24]. Following Lim *et al.* [4], r_j (hardening cost) is set as follows: $r_j = 0.2 \times s_j$. Failure probability π_j was randomly generated using a uniform distribution $\pi_j \sim [0, 0.05]$. Let PERCENT be the percentage of the number of fortifications and p be maximum number of potential facilities to open. We next defined the fortification budget as follows:

$$B = (\textsc{percent}/m)\,\overline{B}.$$

The value of the failure probability $\overline{\pi}$ depends on the number of fortifications and the failure probabilities (π_j):

$$\overline{\pi} = \sum_{j \in J} \pi_j \times \text{PERCENT}.$$

We generated the location capacities as follows. For each potential facility location j, we compute Q_j i.e., the capacity value as a randomly generated value in the interval $[2\overline{D}, 2.2\overline{D}]$. Let $\overline{D}$ be the average demand per facility location (under the assumption there are p facilities and the load is balanced among the facilities). Following Lorena and Senne [11], we defined:

$$\overline{D} = \lceil \theta \sum_{i \in I} D_i / p \rceil,$$

with θ equal to 0.9 as in Lorena and Senne [11].

For the remaining parameters, we used $p \in \{5, 10, 20, 30, 40\}$. Demand values are taken from Snyder and Daskin [24] for different test cases.

2.6.2 Performances of the Models

We exmine here the accuracy and the computational times of the solutions for both Models RpMP and SB2_RpMP, and their corresponding algorithms.

2.6.2.1 Model rpmp: Accuracy and Computational Times

We first report on the accuracy and the computational times of the solutions of Model RpMP. The cplex studio12.6 64G used to solve the model for test cases on the shared server named "Wolsey" with 700 GB memory and 20 CPU Intel(R) Xenon(R) CPU E7-4890 v2@ 2.80 GHz. Results are summarized in Table 2.1.

The first column contains the number of nodes (n), Which is equal to the number of users (m). Next column provides p, the maximum number of facilities which can be opened. Column 3 is presenting the number of fortified facilities.

Next two columns report the LP and ILP values, denoted by $z^\star_{\text{LP}}$ and $\tilde{z}_{\text{ILP}}$, respectively. While $z^\star_{\text{LP}}$ is the LP optimal solution, $\tilde{z}_{\text{ILP}}$ is only an upper bound on the optimal ILP value ($z^\star_{\text{ILP}}$) as we did not develop a branch-and-price algorithm, see Section 2.4.4.2 for the details on the solution scheme. Accuracy ϵ is then calculated using formula of equation (2.19) and it is reported in column entitled gap. In columns 7 to 9, we provide the number of initial, generated, and selected columns. Computational times (CPU) are reported in seconds in last column.

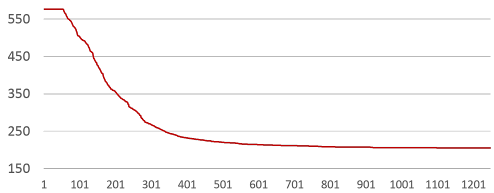
TABLE 2.1: Computational Times and Solution Accuracy (Model RPMP)

		Model RPMP							
# of nodes	p	# fortif.	$z^\star_{\text{LP}}$	$\tilde{z}_{\text{ILP}}$	gap (%)	# columns init.	gen.	select.	CPU (sec.)
49	5	3	298.5	336.1	12.5	5	897	5	821.3
		4	279.4	312.7	11.8		780		785.3
	10	4	258.8	286.5	10.7	10	1,458	10	1,058.6
		6	205.3	223.3	8.7		1,248		996.5
	20	6	172.4	183.3	6.3	20	1,976	20	1,435.4
		11	125.5	132.7	5.7		1,686		1,235.6
88	5	3	684.5	798.6	16.7	5	1,243	5	1,043.6
		4	576.4	657.4	14.1		1,054		983.3
	10	4	549.3	625.4	13.8	10	2,154	10	1,986.7
		6	466.8	527.3	12.9		1,983		1,483.7
	20	6	361.5	402.1	11.2	20	2,897	20	2,369.2
		11	287.6	313.7	9.1		2,346		2,115.3
150	20	6	845.6	988.3	16.9	20	2,045	20	1,015.6
		11	627.8	717.4	14.3		1,865		989.1
	30	11	486.1	546.6	12.4	30	2,495	30	2,447.7
		16	378.4	422.9	11.8		2,217		2,185.4
	40	16	372.6	414.6	11.3	40	2,922	40	3,236.7
		21	308.1	335.4	8.9		2,796		2,742.9

We observe that, as expected, the ILP values (transportation cost) as given by $\tilde{z}_{\text{ILP}}$, decrease as p increases. In addition, accuracy of the solutions improves as p increases.

However, the gap remains rather high for small values of p, as was already observed in Lorena and Senne [11], and Liang *et al.* [31] when using a similar decomposition scheme for capacitated p-median problems and production planning and facility location problem respectively. CPU time observed as similar to those reported in Liang *et al.* [31] for the production planning and facility location problem problem.

In order to understand better from where the rather large gaps are coming, we look into the convergence of the LP solution. Results are plotted in Figure 2.4 for two different data instances, the first one with 49 potential facility locations, 6 fortified facilities and p=10, second one with 88 potential facility location, 11 fortified facilities and p=20. We observe that in Figure 2.4(a) the LP solutions converge in about 1,200 iterations, with a very convergence starting after 800 iterations. In figure 2.4(b), the LP solution converges in about 2,800 iterations and slow convergence starts after about 1600 iterations.



(a) 49 potential facility locations, $p = 10$

(b) 88 potential facility locations, $p = 20$

FIGURE 2.4: Model RPMP: Convergence of the LP solutions

2.6.3 ILP Solutions for Model rpmp and Model sb2_rpmp

Figures 2.5 and 2.6 show the schematic ILP solution on the test case with 49 potential facilities, 6 fortified ones with p=10. As expected users connected to fortified facilities (i.e., facilities 28, 29, 32, 33, 34, and 43) do not have any backup supplier facility. But the users connected to non-fortified facilities (facilities 1,8,10, and 21) have a backup facility for spare resources in case of a disruption event.
In Figure 2.5, the users connected to facility number 8 as primary, have to connected to facility number 1 as backup facility. Because other closer facilities (such as facility number 33) doesn't have enough capacity to supply. But in Figure 2.6 since we are sharing the backup capacity there are more backup capacity in facility 33 to supply to users connected to facility number 8 as primary. Since facility number 33 is closer than the facility number 1 to the users connected to facility number 8, the users favoring to connect to facility number 33 as backup, up to facility number 33 has enough capacity.

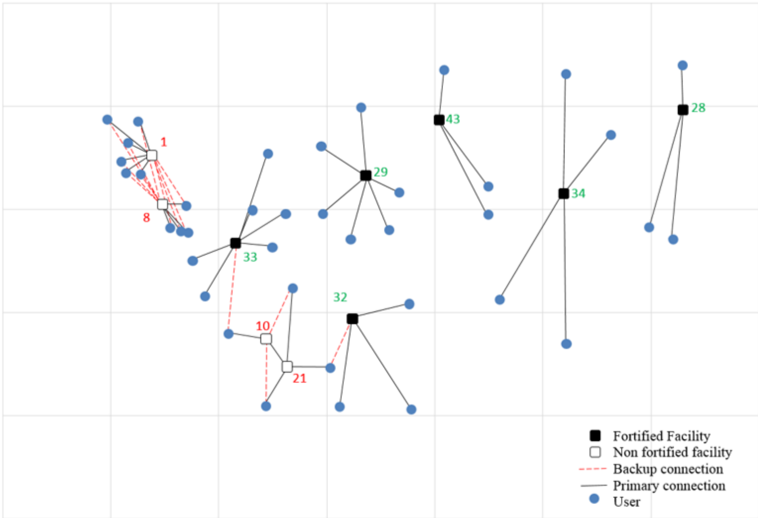


FIGURE 2.5: Model RpMP Location Solution for Test Case 49 Facilities with 6 Fortified Facilities and $p = 10$

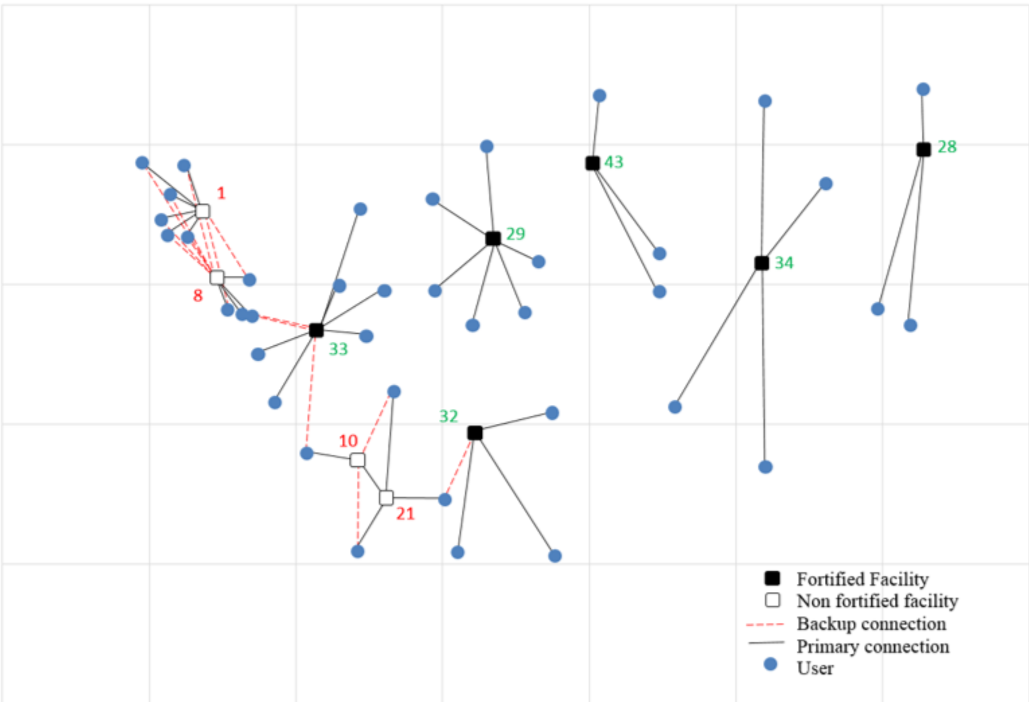


FIGURE 2.6: Model SB2_RpMP Location Solution for Test Case 49 Facilities with 6 Fortified Facilities and $p = 10$

2.6.3.1 Model Accuracy for Model sb2_rpmp

In this section we look into the results and accuracy of Model SB2_RPMP. They are summarized in Table 2.2. Recall that Model SB2_RPMP looks into backup allocation based on the known primary assignment from Model RPMP. In table 2.2 the number of nodes (n), the maximum number of open facilities (p) are presented in two first column. The number of fortifications are reported in column 3. The fourth and fifth columns report the optimal LP and the best ILP values, denoted by ($z^{B\star}_{LP}$) and ($\tilde{z}^{B}_{ILP}$). The primary transportation cost is known from the Model RPMP and it is shown in the table 2.3 column three. The column 6 represent the gap value of ($\tilde{z}^{B}_{ILP}$) from ($z^{B\star}_{LP}$) i.e., ε based on the equation 2.19. The next three columns show the number of initial, generated, selected columns. The last column shows the cpu time in seconds.

Both Lp and ILP solution values (backup transportation cost) is decreasing when p increasing in all cases. According to column 5 the gap is decreasing when p is increasing. In comparison between gaps in Table 2.2 and corresponding gaps in Table 2.1 shows that gaps in Model SB2_RPMP are less than gaps in Model RPMP in all test cases.

TABLE 2.2: Computational Times and Solution Accuracy (Model SB2_RPMP)

# of nodes	p	# of Fortifi.	backup component		gap	# columns			CPU
			$z^{B\star}_{LP}$	$\tilde{z}^{B}_{ILP}$	(%)	i	g	s	(sec.)
49	5	3	98.6	104.8	6.3	5	256	5	296
	10	6	52.4	55.2	5.4	10	345	10	390
	20	11	9.9	10.4	4.7	20	415	20	456
88	5	3	279.6	298.5	6.8	5	356	5	401
	10	6	179.4	190.4	5.9	10	442	10	486
	20	11	108.2	113.6	4.9	20	561	20	625
150	20	11	221.4	233.8	5.6	20	465	20	512
	30	16	111.6	115.8	3.8	30	572	30	625
	40	21	90.5	62.9	2.7	40	648	40	694

2.6.4 Resource Sharing

We now investigate the resource savings when backup resources are shared. Results are summarized in Table 2.3.

TABLE 2.3: Backup Capacity Comparison for Models RpMP and SB2_RpMP

# nodes	p	$z^{W\star}_{ILP}$	$z^{B\star}_{ILP}$		Q^{W}	Q^{B}		Save (%)	Fort.
			SB2_RpMP	RpMP		SB2_RpMP	RpMP		
49	5	231.3	104.8	104.8	24,705	5,684	5,684	0	3
	10	160.3	55.2	62.9	24,705	2,976	8,092	63	6
	20	116.8	10.4	15.8	24,705	1,085	6,751	84	11
88	5	500.1	298.5	298.5	8,213	2,965	2,965	0	3
	10	275.4	190.4	251.9	8,213	896	2,865	69	6
	20	280.8	113.6	227.2	8,213	728	2,805	74	11
150	20	483.6	233.8	233.8	5,820	2,646	2,646	0	11
	30	197.9	115.8	225.0	5,820	824	2,514	67	16
	40	173.3	92.9	162.1	5,820	315	1,781	82	21

Note that SB2_RpMP uses the assignment of users to primary facilities as determined by RpMP, does not necessarily correspond to the optimal assignment of users to their primary facilities under the scenario in which users shared their backup resources, if not primarily assigned to a facility with fortification. In other words, by using the same assignment of users to primary facilities, we do not favor SB2_RpMP, on the contrary. However, the selection of the facilities to be fortified may differ, but not their number. In practice, on the set of data we used, we observe no difference in the selection of the facilities to be fortified.

Comparing the ILP values of the two models, RpMP and SB2_RpMP, we observe that they are identical, meaning that the assignment of users to their backup facilities does not change. However, even if the backup facility remains the same for each user, the resulting overall amount of backup resources is dropping very significantly.

We compute the percentage of resource saving as follows:

$$\text{savings} \; = 1 - (Q^{B}_{\text{SB2\_RpMP}} / Q^{B}_{\text{RpMP}}) \times 100\%.$$

Where the used backup capacity for RpMP is calculated by:

$$Q^{\mathrm{B}} = \sum_{c \in C} \sum_{i \in I} D_i a_i^{\mathrm{B},c} z_c$$

and the used backup capacity for SB2_RpMP is calculated by:

$$Q^{\mathrm{B}} = \sum_{c \in C} \sum_{i \in I} D_i a_i^{\mathrm{B},c} z_c^{\mathrm{B}}$$

Note that those two values are over estimates of the backup resources. Results show that in small size test cases the Model RpMP, and Model SB2_RpMP, are performing almost same while in case of 49 nodes and $p = 10$ the Model SB2_RpMP, perform better than Model RpMP, and there are more resource saving.

2.7 Conclusion and Future Work

We investigated the facility location problem in the event of disruptions. We presented a first decomposition model (Model RpMP) for capacitated facility location problem subject to disruption. We next introduced the concept of shared protection with Model SB2_RpMPunder the assumption of single failure, i.e., no more than one failure at a time following an analogy with communication networks. We then observed that significant savings could be achieved by using shared protection.

In future work, we plan to explore a one-step optimization model and check whether additional savings are possible for the backup resources.

Chapter 3

Shared Capacitated Reliable p-Median Facility Location in Presence of Disruption

abstract

Recently one of the main concerns in logistics network is reliable facility location. Disruption, e.g., resource failures, natural disaster, can affect the reliability of facilities. The facilities can be protected and have backup in case of disruption event. To have a backup facility we need more resources or capacities to use. In this paper, we proposed a backup capacity sharing for facilities since there are capacity constraints in capacitated facility location. We revisit the facility location problem under disruptions with a column generation formulation in order to have facility fortification and backup sharing. We proposed a sharing backup model taking into account that not all failures occur at the same time, and therefore some backup resources can be shared. Intensive numerical experiments complete the paper, with some comparisons to previously proposed model. Conclusions are drawn in the last section.

keyword:

Reliable Facility Location (FLP), disruptions, column Generation, Backup Facility, Capacity sharing

3.1 Introduction

One of the main elements of logistics networks is the facilities that they supply the demand of users in logistics networks. The reliable facility location can deliver demands on-time to users. In real world there are disruption events that affects on reliability of the facilities so that the facility can not supply the users' demands on-time. Disruption is an unexpected, temporal event which leads to negative deviation from the planned outcome of a supply chain or logistics network (Brenner [32]). In facility location problem, disruption affects the facilities' reliability so that some users need to be directed to other facilities. To have a reliable facility location we can protect the facility in presence of disruption. The main application of facility protection or interdiction is in defense military logistics networks. Protected facilities should be either fortified or we have to consider a backup for the facility with the risk of disruption.

Recently, In context of facility location there are more attention on designing a resilient logistics network in presence of disruptions. Design of reliable logistics network can be accomplished by improvement of existing facilities in order to avoid disruption. Also a backup facility for users connected to the facility under disruption can be designed. We will focus on the disruptions which affecting facility locations.

The classical location problem named p-median problem (PMP) is looking for to open p facilities (medians) to minimize the sum of the distances from each user to its nearest facility on a network. The capacitated p-median problem (CPMP) considers capacities for the product or services to be given by each facility. The total demand by users cannot exceed total facilities capacities.

In this paper we examine a column generation approach to the capacitated facility location problem in presence of disruption which shared the backup capacity. First we propose the column generation which the identified restricted master problem (RMP) optimize the covering of 1-median clusters satisfying a set of capacity constraints. In this case the pricing problem outcomes are based on facilities. New columns are generated based on the sub-problems solution, which consider the restricted master dual variables and the clusters' capacities. The first model is nonlinear and we decided to change the configuration of columns based on the users instead of facilities. Then we propose the second model which the identified restricted master problem (RMP) optimizes the covering of 1 user clusters satisfying a set of capacity constraints considering shared backup capacity. In continue, in Section 3.2 we review the literature related to the problem and then in Section 3.3 We define the problem. In Section 3.4 we propose the models of shared backup capacitated facility problem (CFP) including shared backup CFP based on facility configuration and shared backup CFP based on user configuration Then we formulate model of fortification and shared backup CFP based on user configuration. In Section 3.5 we proposed the compact model of shared backup CFP. In Section 3.8 generation of the initial solutions, model accuracy, and solutions' comparisons are presented. In Last section the conclusion and future works are discussed.

3.2 Related Work

There are various study on facility location problem. Different problems proposed in literature, such as, Max-covering problem (Liberatore *et al.* [22]), p-center problem (Mladenović *et al.* [33]), and p-median problem (Lim *et al.* [4], Snyder and Daskin [24]). In this paper we have looked in to the p-median problem. The p-median problem is a facility location problem and one of the basic models in discrete location theory. Most of the location problems are classified as NP-hard (Mladenović *et al.* [34]). In this research we are looking in to the reliable facility locations in the presence of disruption. The variety of solutions' methods (heuristics and exacts) are proposed to solve this problem. In continue first we look into reliable facility location problem literature, then in detail we look into the literature related to facility locations problem under disruption and the models of literature. Finally the location problems and column generation stabilizing literature is reviewed.

3.2.1 Reliable Facility Location Problems

Reliable facility location problem has been studied in logistics networks. Different problem structures are considered in the literature. In most of the papers the capacity constraints are not considered and the problem is solved in un-capacitated p-median problem format (e.g., [4]). In this case there is no limitation of the resources in facilities to supply users' demands and they can support as much as users connected to them. The capacitated p-median problem (CPMP) [35] adds capacity constraints on the facilities, but has not yet been studied in the context of disruptions. In the literature there are two main directions to make facilities reliable in the presence of disruption: (i) In the events of disruption facilities can be hardened or fortified and protected to be disrupted (e.g., [21], [2], and [8]). Facility fortification is defined as if a facility is fortified, it becomes reliable and if there is a disruption event, the facility can support the users demand (e.g., [8] and [2]). (ii) In some cases authors considered backup facility for disrupted facility (e.g., [16], [20], and [4]). In this case if there is a disruption on primary supplier facility, there is a backup facility open to users demand. In continue we classify facility locations problem under disruption.

3.2.2 Classification of Reliable FLP Subject to Disruption

There are several parameters that should be considered in decision making about facility location in a case to make sure the facility is reliable. The incorrect or poor amusement of this parameters (e.g., demands, transportation times, available capacity and availability of facility) may makes the decision inaccurate. Disruption affects the logistics networks parameters as discussed before. Disruption can effects on the parameters such as recognizing demand point, distances or availability of facilities. Several cases for recognizing demand point or distances parameters which have been studied by authors and several models were developed [10]. Recently authors look into the facility location problem under disruption such as the disruption makes facility unavailable to process the users' demands. In continue we classify the related works made on reliable facility location under disruption that disruption make facility unavailable. This problem can be categorized based on the problem feature, objective, considering capacity constraints, Solution methods, data set size, and and facility setup cost.

3.2.2.1 Classification of Reliable FLP under Disruption Based on Features

In this section we provide a classification of reliable facility location under disruption. We are looking into the different features for this problem. In the literature two main features (strategies) are proposed in order to attain a reliable facility location in logistics networks under disruption events: facility fortification and backup supplier facility.
Recent decades several optimization models considering fortification of facilities under potential intentional interdiction scenarios come being studied[36]. Fortification or hardening the facility protects the facility in order to hedge against the most disruptive interdiction [21]. In this case based on the budget we fortify the facilities to protect them from disruption. (for instance, [21], [2], [8], [22], and [24]). In this case there is high cost to fortify facilities.
Backup supplier facility is proposed in the literature to supply the users connected to disrupted facilities. In this case instead of fortification of the facility we consider a back up supplier for the users so that if there is any disruption on primary supplier, the back up facility support the users' demand. (for instance, [16], [9], [20], [4], and [5]).
One of the main features in facility location problem is facility capacity constraint. In the literature, authors don't consider capacity for the facility to simplify the problem (for instance [21], [9], [37], [38], [4], and [5]). In some cases authors consider the capacity for the facilities (such as, [9], [8], [2], and [39]). In this case authors consider only back up facility in their problem feature not fortification (e.g., [9]). Liberatore *et al.* [2] and Qin *et al.* [8] considered capacity constraint for facilities in case there is only fortification.
In the literature different feature of facility location problem such as facility fortification, backup facility, and facility capacity, but because of complexity of problem the combination of three is not proposed in the literature.

3.2.2.2 Classification of Reliable FLP under Disruption Based on Formulations Models and Problem Objective

The facility location under disruption is modeled in different formulation. Most of the cases considering regular facility location problem (p-median) and some parts of the literature indicates solving the problem as max-covering problem. In the max-covering problem, the objective seeks the location of a number of facilities on a network in such a way that the covered population is maximized ([9], [20], and [22]).In the p-median problem, the objective is to minimize total transportation cost (time) between users and opened facilities. In the literature the transportation cost is euclidean distance between user and assigned supplier facility ([21], [38], [4], [5], [24]). The authors considered multi-objective function in their p-median modeling. In this case they consider the total euclidean distances between users and assigned facilities and in addition they maximize the interdiction cost ([16] and [2]). Also adding fortification cost minimization is added to the total euclidean distances between users and assigned facilities [8]. In some cases the fixed cost to open the facilities is also added to the objective function (e.g., [37], [4], and [39]).

Different solution models are proposed in the literature to solve this problem. There are two main categories of solutions in literature. The first categories are exact methods such as, Integer linear programming ([21] and [9]); tri-level ILP for multi-objective formulation ([16] and [2]); two stages stochastic and robust optimization ([8],[37], and [20]) and Branch and price ([38]). The second categories are Heuristics methods such as Lagrangian relaxation ([4],[5], and [10]) and greedy methods ([22] and [39]).

Different case sizes of problem are solved in the literature. In the most cases the largest size is 150 nodes or potential facilities to open which they are the large cities of United States ([21], [9], [38], [20], and [10]). Other size of 263 [22], 305 [2], and 316 [9] potential facilities to open are considered by authors. In most cases the number of users are equal the number of potential facilities to open([21], [16], [9], [8], and [38]). Different number of p (number of opened facilities) is considered by authors between 8 to 60 based on the number of potential facilities to open.

In continue we look in to the location problem and column generation.

3.2.3 Location Problems and Column Generation Stabilizing

In this section we summarized different methods to stabilizing the column generation in facility location problems. Authors worked on upper bound (e.g., Liang *et al.*[31]), lower bound (e.g., Lorena and Senne [11], Klose and Görtz [40]) or quality of initial dual estimations(e.g., Senne and Lorena [41], Klose and Drexl [11]) to improve the reported gap. Heursitics and exact methods are used for gaps improvement in the literature. The summary of the literature working on the gaps improvement for location problems and column generation are discussed briefly in the Table 3.1.

TABLE 3.1: Methods to Improve CG

Author	UB[1]	LB[2]	IDE[3]	method
Senne and Lorena [41]			×	Lagrangean/surrogate relaxation
Lorena and Senne [11]		×		Lagrangean/surrogate relaxation
Klose and Drexl [42]		×	×	heuristic based LR[4] & Subgradient
Klose and Görtz [40]		×	×	heuristic B&P[5]/capacity relaxation
Liang *et al.* [31]	×			relax-and-fix technique/heuristics

The literature reported different gap for facility location problem using column generation. The summary of different gaps are reported in the Table 3.2.

TABLE 3.2: Reported Gap in the Literature for FLP Using CG

Author	Gap reported
Senne and Lorena [41]	Dual gap
Lorena and Senne [11]	Best fiseable solution
Klose and Drexl [42]	UB found and LB found
Klose and Görtz [40]	LP bound and compact UB
Liang *et al.* [31]	Optimality gap (ILP)

3.3 Problem Statement

We denote by J the set of potential locations which are available to open, and p is the maximum number of open facilities. Let Q_j be the capacity of location j. Associated with each facility location j, there is a failure probability π_j such that $0 \leq \pi_j \leq 1$. A fortification budget B is available in order to harden some facility locations, so that a user assigned to a fortified facility does not need a backup facility. We denote by I the set of customers, and each customer $i \in I$ has demand D_i. Each customer is

[1] Uper bound
[2] Lower bound
[3] Quality of initial dual estimates
[4] Lagrangean Relaxation
[5] Branch and Price

assigned to a primary supplier and a different backup supplier. Let COST_{ij} be the transport cost of demands between facility location $j \in J$ and customer $i \in I$ (with the convention $\text{COST}_{ij} = 0$ for all $i \in I$). For a given facility location j with a set I_j (assigned customers to j), the transportation cost is sum of COST_{ij}, $i \in I_j$ and j. s_j is a fixed fortification cost and represents the cost for facility fortification implementation (e.g., contract negotiation, overhead, personnel training) and r_j is a cost associated with the unit reduction for failure probability (π_j) of facility (j) as discussed in [4]. Indeed, the r_j (fortification cost) varies with the amount of reliability improvement of the facility (e.g., acquisition and installation of the units of protective measures, procurement, storage of backup inventory, hiring extra workforce). Under the assumption of a single facility failure (we have time to recover from a facility failure before a new one occurs), we need to optimize the facility capacities. The probability of a simultaneous failure of its primary and backup supplier is negligible.

Considering two users i_1 and i_2 with demand of d_1 and d_2, each assigned to a different facility location for their primary supplier, say j_1 for i_1, and j_2 for i_2. Assume that both i_1 and i_2 have the same backup supplier, say j_3. When we are checking the capacity constraint for j_3, we take into account $\max\{d_1, d_2\}$ since i_1 and i_2 will not need to recourse to facility in j_3 at the same time. Let Q_1 as a required backup capacity for j_1 and Q_2 as a required backup capacity for j_2. In this case the backup capacity for j_3 is $\max\{Q_1, Q_2\}$ (share the backup capacity).

The additional assumptions can be listed in continue. Each user's demand is supplied by one primary facility. If a facility fails, it becomes unavailable. The occurrence of facility failures are independent. If the primary supplier fails for any user, the backup supplier will be available. If a facility is fortified, it will be fail protected.

In continue we formulate the problem by using column generation decomposition.

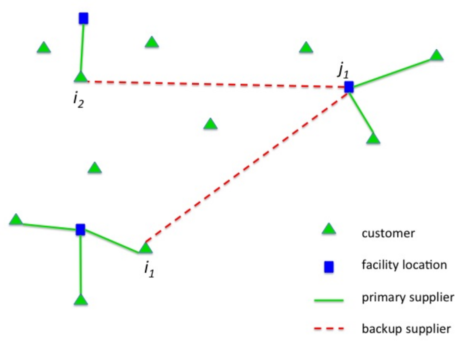
3.4 Shared Backup Capacitated Facility Problem (CFP)

There are two directions for modeling shared backup facility based on the column generation decomposition. (i) We select one facility for each configuration (Column). Then we assign some users to that particular facility either as primary or backup in each iteration of pricing problem. (ii) In the second direction we select a user for each configuration and will assign a facility as primary and the other facility as backup (if necessary). We will discuss both model as follows.

3.4.1 Shared Backup CFP Based on Facility Configuration (Model Sh_Rmp_j)

Considering the example depicted in Figure 3.1 users i_1 and i_2 have different primary suppliers, while they share the same backup supplier (j_1). Under the assumption that at most one supplier will fail at a time. In this case, for fixing the failure before occurrence of the another one we have enough time. Theretofore, the backup resources of i_1 and i_2 can be shared. Let D_1 and D_2 be i_1 and i_2 demands' correspondence. During this procedure, instead of inquiring the capacity of $D_1 + D_2$ with respect to facility location j_1, the backup capacity (Q_j^{B}) of j_1 can be $Q_j^{\text{B}} = \max\{D_1, D_2\}$.

FIGURE 3.1: Sharing Backup Resources



We should consider the capacity constraints in the master problem for sharing of the backup resources. The explanation of modeling the problem in Section 3.3 is discussed in continue.

3.4.1.1 Decomposition Scheme

The proposed model, called SH_RMP_J, relies on a decomposition scheme with the concept of configurations defined as follows. Each configuration is associated with one potential facility location (j) and contains the set of users connected to this facility either as primary ($a_i^{\mathrm{W},c}$) or backup ($a_i^{\mathrm{B},c}$) supplier. For a given facility location j, let C_j be the set of facility location configurations, where $c \in C_j$ is characterized by the set of users assigned to a facility in location j with $a_i^{\mathrm{W},c}$ and $a_i^{\mathrm{B},c}$. Each configuration c is characterized by the two sets of parameters:

$a_i^{\mathrm{W},c} \in \{0, 1\}$. $a_i^{\mathrm{W},c} = 1$ if customer i uses the facility of configuration c as a primary facility location, 0 otherwise.

$a_i^{\mathrm{B},c} \in \{0, 1\}$. $a_i^{\mathrm{B},c} = 1$ if customer i uses the facility of configuration c as a backup facility location, 0 otherwise.

Other parameters that will be used in the decomposition model have been defined in Section 3.3.

3.4.1.2 Variables

We use five sets of decision variables. The first set is related to opening location: $y_j = 1$ if facility location j is open, 0 otherwise. The second set is related to the fortified facility locations: $x_j = 1$ if facility location j or the facility in j is selected for fortification, 0 otherwise. The third set corresponds to the decomposition variables: $z_c = 1$ if configuration c is selected in the optimal solution, 0 otherwise. The fourth set corresponds to the primary capacity of facilities (j): $Q_j^{\mathrm{W}} \geq 0$ the primary capacities' required amount for j to supply demands of users connected to j as the primary supplier. The fifth set corresponds to the backup capacity of facilities (j): $Q_j^{\mathrm{B}} \geq 0$ the primary capacities' required amount for j to supply demands of users connected to j as the backup supplier.

3.4.1.3 A Decomposition Model

Now we formulate the problem as below:

$$\min \sum_{c \in C} \text{COST}_c z_c \tag{3.1}$$

where

$$\text{COST}_c = \sum_{i \in I} (a_i^{\mathrm{W},c} + a_i^{\mathrm{B},c}) d_{ij} \qquad c \in C_j.$$

Subject to:

$$\sum_{c \in C_j} z_c = y_j \qquad j \in J \tag{3.2}$$

$$x_j \leq y_j \qquad j \in J \tag{3.3}$$

$$\sum_{j \in J} y_j \leq p \tag{3.4}$$

$$\sum_{c \in C} a_i^{\mathrm{W},c} z_c = 1 \qquad i \in I \tag{3.5}$$

$$\sum_{c \in C_j} a_i^{\mathrm{W},c} z_c + \sum_{c \in C} a_i^{\mathrm{B},c} z_c \leq 2 - x_j \qquad i \in I, j \in J \tag{3.6}$$

$$\sum_{c \in C} a_i^{\mathrm{B},c} z_c \geq \sum_{c \in C_j} a_i^{\mathrm{W},c} z_c - x_j \qquad i \in I, j \in J \tag{3.7}$$

$$\sum_{j \in J} \pi_j (\sum_{c \in C_j} a_i^{\mathrm{W},c} z_c) \leq \sum_{j \in J} \pi_j (\sum_{c \in C_j} a_i^{\mathrm{B},c} z_c) - \sum_{c \in C} a_i^{\mathrm{B},c} z_c + 1 \qquad i \in I \tag{3.8}$$

$$\sum_{j \in J} \pi_j y_j \leq \overline{\pi} \tag{3.9}$$

$$\sum_{j \in J} (s_j + r_j \pi_j) x_j \leq B \tag{3.10}$$

$$\sum_{c \in C_j} \sum_{i \in I} D_i \, a_i^{\mathrm{W}} z_c \leq Q_j^{\mathrm{W}} \qquad j \in J \tag{3.11}$$

$$\sum_{i \in I} \sum_{c \in C_j} D_i a_i^{\mathrm{W}} \sum_{c \in C_{j'}} a_i^{\mathrm{B}} z_c \leq Q_j^{\mathrm{B}} \qquad j, j' \in J : j \neq j' \tag{3.12}$$

$$Q_j^{\mathrm{W}} + Q_j^{\mathrm{B}} \leq Q_j \qquad j \in J \tag{3.13}$$

$$Q_j^{\mathrm{W}}, Q_j^{\mathrm{B}} \geq 0 \qquad j \in J.$$

$$y_j \in \{0,1\} \qquad j \in J$$

$$x_j \in \{0,1\} \qquad j \in J$$

$$z_c \in \{0,1\} \qquad c \in C.$$

Constraints (3.2) check whether location j is opened for a facility, to be used either as a primary or a backup supplier, or both of them. Constraints (3.3) ensure that x_j can be considered for the fortification, just if it is open. Constraint (3.4) sets the limit on a number of facilities to open. Constraints (3.5) guarantee that each user i is assigned to a primary supplier. Constraints (3.6) and (3.7) guarantee that each user i is assigned to a backup supplier, if its primary supplier is not a fortified facility. Constraints (3.8) guarantee, if the primary supplier is not fortified, the failure probability of primary supplier selection for any user i is smaller than the failure of the backup supplier. Arguing a particular user i, if its primary supplier is fortified, due to constraints (3.6) and (3.7), $\sum_{c\in C} a_i^{\mathrm{B},c} z_c = 0$, and consequently $\sum_{j\in J} \pi_j(\sum_{c\in C_j} a_i^{\mathrm{B},c} z_c) = 0$ as well. This is turn implies that $\sum_{j\in J} \pi_j(\sum_{c\in C_j} a_i^{\mathrm{W},c} z_c) \leq 1$ which is always true due to constraints (3.5). On the other hand, if the primary supplier is not fortified, $\sum_{j\in J} \pi_j(\sum_{c\in C_j} a_i^{\mathrm{B},c} z_c) = 1$, and constraints (3.8) becomes:

$$\sum_{j\in J} \pi_j(\sum_{c\in C_j} a_i^{\mathrm{W},c} z_c) \geq \sum_{j\in J} \pi_j(\sum_{c\in C_j} a_i^{\mathrm{B},c} z_c). \tag{3.8'}$$

Constraint (3.9) takes into account the failure probability. This constraint controls the total failure probability of opened facilities based on the value of $\overline{\pi}$. The value of $\overline{\pi}$ is depended on the number of fortification and the calculation formula is mentioned in Section 3.8.1.

Constraint (3.10) enforces the fortification budget limit on the selection of fortified facilities. Constraints (3.11) take into account the amount of primary capacity based on the primary demand for each facility. Constraints (3.12) take into account the amount of backup capacity based on the maximum of backup demand for each facility. As it is shown in Figure 3.1, let the amount of backup demand for the users connected to facility j_1 as primary be equal $Q^{\mathrm{B}}_{j_1}$, for j_2 be $Q^{\mathrm{B}}_{j_2}$ and for j_3 be $Q^{\mathrm{B}}_{j_3}$. We are looking into the $\max\left\{Q^{\mathrm{B}}_{j_1}, Q^{\mathrm{B}}_{j_2}, Q^{\mathrm{B}}_{j_3}\right\}$ to calculate the $Q^{\mathrm{B}}_{j'}$.

Constraints (3.13) consider total available capacity for each facility as a summit limitation of the primary and backup capacity. The remaining constraints define the domains of the variables.

3.4.1.4 Pricing Problem

The pricing problem is modified as follows. We now write the pricing problem (PP_j) for potential facility location j. Let $u_j^{(3.2)} \lesseqgtr 0$, $u_i^{(3.5)} \lesseqgtr 0$, $u_{ij}^{(3.6)} \leq 0$, $u_{ij}^{(3.7)} \geq 0$, $u_i^{(3.8)} \leq 0$, $u_j^{(3.11)} \leq 0$, and $u_j^{(3.12)} \leq 0$, be the values of dual variables associated with constraints

(3.2), (3.5), (3.6), (3.7), (3.8), (3.11),and (3.12) respectively.

$$\begin{aligned}\min\ \overline{\text{COST}_j^{\text{SHARED}}} = \overline{\text{COST}_j} - u_j^{(3.2)} - \sum_{i\in I} u_i^{(3.5)} a_i^{\text{W}} - \sum_{i\in I} u_{ij}^{(3.6)}\left(a_i^{\text{B}} + a_i^{\text{W}}\right) \\ - \sum_{j'\in J: j'\neq j}\ \sum_{i\in I} u_{ij'}^{(3.6)} a_i^{\text{B}} - \sum_{i\in I} u_{ij}^{(3.7)} (a_i^{\text{B}} - a_i^{\text{W}}) \\ - \sum_{j'\in J: j'\neq j}\ \sum_{i\in I} u_{ij'}^{(3.7)} a_i^{\text{B}} - \sum_{i\in I} u_i^{(3.8)}\ (\pi_j a_i^{\text{W}} + (1-\pi_j) a_i^{\text{B}}) \\ - u_j^{(3.11)} \sum_{i\in I} a_i^{\text{W}} - \sum_{j\in J}\sum_{i\in I} u_j^{(3.12)} a_i^{\text{W}} - \sum_{i\in I} D_i a_i^{\text{W}} \sum_{j'\in J: j'\neq j} a_i^{\text{B}} u_{jj'}^{(3.12)}\end{aligned}$$

subject to:

$$\begin{aligned} a_i^{\text{W}} + a_i^{\text{B}} &\leq 1 \qquad i \in I \qquad (3.14)\\ a_i^{\text{W}}, a_i^{\text{B}} &\in \{0,1\} \qquad i \in I. \end{aligned}$$

The term "$\sum_{i\in I} D_i a_i^{\text{W}} \sum_{j'\in J: j'\neq j} a_i^{\text{B}} u_{jj'}^{(3.12)}$" is not linear in objective function of pricing problem. We propose another model for this problem based on the user configuration which is explained in next section.

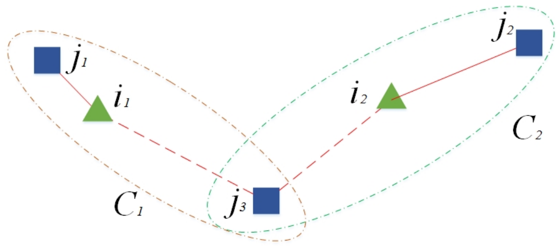
3.4.2 Shared Backup CFP Based on User Configuration (Model Sh_Rmp_i)

In this section we propose another decomposition for share backup facility location. To construct the shared backup and fortification decomposition we follow two phases. First we explain the model with backup for each user without any fortification. Then in Section stage (Section 3.4.3) we add fortification constraints to the model SH_RMP_I to ultimate the modeling.

We assume a users based configuration instead of facility based configuration (Section 3.4.1). In this case, in each configuration for each user there is a primary supplier and a backup supplier. In figure 3.2 the configuration C_1 contains user i_1 connected to j_1 as a primary supplier and j_3 as a backup supplier. Also, it represents that the configuration

c_2 contains i_2 connected to j_2 as a primary supplier and j_3 as a backup supplier. In this case i_1 and i_2 shared the backup capacity in j_3.

FIGURE 3.2: Different Configuration Based on Users



3.4.2.1 Decomposition Scheme

The proposed model, called SH_RMP_I, relies on a decomposition scheme with the concept of configurations. In the following the shared backup facility location is defined. Each configuration is associated with one user (i) and it contains a primary supplier facility ($a_j^{\mathrm{W},c}$) and a backup supplier facility ($a_j^{\mathrm{B},c}$). For a given user i, let C_i be the set of user configurations, where $c \in C_i$ is characterized by the set of facilities assigned to a user i with $a_j^{\mathrm{W},c}$ and $a_j^{\mathrm{B},c}$. Each configuration c is characterized by the two sets of parameters:

$a_j^{\mathrm{W},c} \in \{0,1\}$. $a_j^{\mathrm{W},c} = 1$ if the customer i uses the facility j in configuration c as a primary facility location, 0 otherwise.

$a_j^{\mathrm{B},c} \in \{0,1\}$. $a_j^{\mathrm{B},c} = 1$ if the customer i uses the facility j in configuration c as a backup facility location, 0 otherwise.

Other parameters that will be used in the decomposition model have been defined in Section 3.3.

3.4.2.2 Variables

We used five sets of decision variables which they were clarified in Section 3.4.1.2.

3.4.2.3 A Decomposition Model

Here we can formulate the problem as below:

$$\min \sum_{c\in C} \textsc{cost}_c z_c \tag{3.15}$$

where

$$\textsc{cost}_c = \sum_{j\in J}(a_j^{\mathrm{W},c} + a_j^{\mathrm{B},c})d_{ij} \qquad c \in C_i.$$

Subject to:

$$\sum_{c\in C_i}\sum_{j\in J} a_j^{\mathrm{W},c} z_c = 1 \qquad i \in I \tag{3.16}$$

$$\sum_{c\in C_i}\sum_{j\in J} a_j^{\mathrm{B},c} z_c = 1 \qquad i \in I \tag{3.17}$$

$$\sum_{c\in C}(a_j^{\mathrm{W},c} + a_j^{\mathrm{B},c})z_c \leq My_j \qquad j \in J \tag{3.18}$$

$$\sum_{j\in J} y_j \leq p \tag{3.19}$$

$$\sum_{i\in I} D_i \sum_{c\in C_i} a_j^{\mathrm{W},c} z_c \leq Q_j^{\mathrm{W}} \qquad j \in J \tag{3.20}$$

$$\sum_{i\in I} D_i \sum_{c\in C_i} a_{j'}^{\mathrm{W},c} a_j^{\mathrm{B},c} z_c \leq Q_j^{\mathrm{B}} \qquad j, j' \in J : j \neq j' \tag{3.21}$$

$$Q_j^{\mathrm{W}} + Q_j^{\mathrm{B}} \leq Q_j \qquad j \in J \tag{3.22}$$

$$Q_j^{\mathrm{W}}, Q_j^{\mathrm{B}} \geq 0 \qquad j \in J$$

$$z_c \in \{0, 1\} \qquad c \in C$$

$$y_j \in \{0, 1\} \qquad j \in J.$$

Constraints (3.16) ensure that each user only and only assigned to a facility as a primary supplier. Constraints (3.17) ensure that each user only and only assigned to a facility as a backup supplier. Constraints (3.18) and (3.19) are taking into account the number of facilities to open as primary and backup. The total demands assigned to a particular facility is controlled by the Constraints (3.20) for primary assignment and by the Constraints (3.21) for backup assignment. Constraints (3.22) ensure that the total amount of demands assign to a facility either primary or backup are less than the facilities' capacity.

3.4.2.4 Pricing Problem

The pricing problem is modified as follows. We now write the pricing problem (PP_i) for user i. Let $u_i^{(3.16)} \lesseqgtr 0$, $u_i^{(3.17)} \lesseqgtr 0$, $u_{ij}^{(3.18)} \leq 0$, $u_{ij}^{(3.20)} \leq 0$, $u_j^{(3.21)} \leq 0$, and be the values of dual variables associated with constraints (3.16), (3.17), (3.18), (3.20),and (3.21) respectively.

$$\begin{aligned}\min \ \overline{\text{COST}_i^{\text{SHARED}}} = \text{COST}_i - \sum_{j\in J} u_i^{(3.16)} a_j^{\text{W}} - \sum_{j\in J} u_i^{(3.17)} a_j^{\text{B}} \\ - \sum_{j\in J} u_{ij}^{(3.18)}(a_j^{\text{W}} + a_j^{\text{B}}) - \sum_{j\in J} u_{ij}^{(3.20)} a_j^{\text{W}} - \sum_{j\in J} \sum_{j'\in J: j\neq j'} D_i a_j^{\text{W}} a_{j'}^{\text{B}} u_j^{(3.21)}\end{aligned}$$

Where

$$\text{COST}_i = \sum_{j\in J} (a_j^{\text{W}} + a_j^{\text{B}}) d_{ij}$$

subject to:

$$\sum_{j\in J} a_j^{\text{W}} = 1 \tag{3.23}$$

$$\sum_{j\in J} a_j^{\text{B}} = 1 \tag{3.24}$$

$$a_j^{\text{W}} + a_j^{\text{B}} \leq 1 \qquad j \in J \tag{3.25}$$

$$a_j^{\text{W}}, a_j^{\text{B}} \in \{0,1\} \qquad i \in I$$

In the objective function of pricing problem we have nonlinear term that we will take care of that in next completed model. Constraint (3.23) ensure that for specific user i there is only one primary facility. Constraint (3.24) ensure that for specific user i there is only one backup facility. Constraint (3.25) ensure that for specific user i the primary facility and backup facility are different.

3.4.3 Fortification and Shared Backup CFP Based on User Configuration (Forti_Sh_Rmp_i)

In this section the fortification and shared backup CFP based on user configuration is finalized. The fortification scheme added to the model which is explained in Section 3.4.2.1. We used five sets of decision variables which were clarified in Section 3.4.1.2.

3.4.3.1 A Decomposition Model

At this stage we formulate the problem as hereunder:

$$\min \sum_{c \in C} \text{COST}_c z_c \tag{3.26}$$

where

$$\text{COST}_c = \sum_{j \in J} (a_j^{\text{W},c} + a_j^{\text{B},c}) d_{ij} \qquad c \in C_i.$$

Subject to:

$$\sum_{c \in C_i} \sum_{j \in J} a_j^{\text{W},c} z_c = 1 \qquad i \in I \tag{3.27}$$

$$\sum_{c \in C_i} \sum_{j' \in J : j \neq j'} a_{j'}^{\text{B},c} z_c + \sum_{c \in C_i} a_j^{\text{W},c} z_c \leq 2 - x_j \qquad i \in I, j \in J \tag{3.28}$$

$$\sum_{c \in C_i} \sum_{j' \in J : j \neq j'} a_{j'}^{\text{B},c} z_c \geq \sum_{c \in C_i} a_j^{\text{W},c} z_c - x_j \qquad i \in I, j \in J \tag{3.29}$$

$$x_j \leq y_j \qquad j \in J \tag{3.30}$$

$$\sum_{j \in J} (s_j + r_j \pi_j) x_j \leq B \tag{3.31}$$

$$\sum_{c \in C} (a_j^{\text{W},c} + a_j^{\text{B},c}) z_c \leq M y_j \qquad j \in J \tag{3.32}$$

$$\sum_{j \in J} y_j \leq p \tag{3.33}$$

$$\sum_{i \in I} D_i \sum_{c \in C_i} a_j^{\mathrm{W},c} z_c \leq Q_j^{\mathrm{W}} \qquad j \in J \tag{3.34}$$

$$\sum_{i \in I} D_i \sum_{c \in C_i} a_{j'}^{\mathrm{W},c} a_j^{\mathrm{B},c} z_c \leq Q_j^{\mathrm{B}} \qquad j, j' \in J : j \neq j' \tag{3.35}$$

$$Q_j^{\mathrm{W}} + Q_j^{\mathrm{B}} \leq Q_j \qquad j \in J \tag{3.36}$$

$$Q_j^{\mathrm{W}}, Q_j^{\mathrm{B}} \geq 0 \qquad j \in J$$

$$z_c \in \{0, 1\} \qquad c \in C$$

$$y_j \in \{0, 1\} \qquad j \in J$$

$$x_j \in \{0, 1\} \qquad j \in J.$$

Constraints (3.27) ensure that each user only assigned to a facility as a primary supplier. Constraints (3.28) guaranteed that each user only assigned to a facility as a backup supplier if there is no fortification. Constraints (3.29) ensure that each user only is not assigned to a facility as a backup supplier if there is fortification. In the Table 3.3 the accuracy of the constraints (3.28) and (3.29) are justified. In the Constraints (3.30) the facility may lead to be fortified if it is open. Constraint (3.31) limits budget for fortification. Constraints (3.32) and (3.33) control the number of facility to open as primary and backup.

The total demands assigned to a particular facility is controlled by the Constraints (3.34) for primary assignment and by the Constraints (3.35) for backup assignment. Constraint (3.36) ensure that the total amount of demand assign to a facility either primary or backup are less than the capacity of that facility.

3.4.3.2 Pricing Problem

The pricing problem is modified as follows. We now write the pricing problem (PP_i) for each user i. Let $u_i^{(3.27)} \lesseqgtr 0$, $u_{ij}^{(3.28)} \leq 0$, $u_{ij}^{(3.29)} \geq 0$, $u_j^{(3.32)} \leq 0$, $u_j^{(3.34)} \leq 0$, $u_j^{(3.35)} \leq 0$, and be the values of dual variables associated with constraints (3.27), (3.28), (3.29),

$\sum_{c\in C_i} a_j^{\text{W},c} z_c$	x_j	Constraint 3.28	Constraint 3.29
1	1	$\sum_{c\in C_i} \sum_{j'\in J: j\neq j'} a_j^{\text{B},c} z_c = 0$	redundant
1	0	redundant	$\sum_{c\in C_i} \sum_{j'\in J: j\neq j'} a_j^{\text{B},c} z_c = 1$
0	1	redundant	$\sum_{c\in C_i} \sum_{j'\in J: j\neq j'} a_j^{\text{B},c} z_c = 0$
0	0	redundant	$\sum_{c\in C_i} \sum_{j'\in J: j\neq j'} a_j^{\text{B},c} z_c = 1$

TABLE 3.3: Fortification Considering Two Constraints 3.28 & 3.29 for Specific User i

(3.32), (3.34),and (3.35) respectively.

$$
\begin{aligned}
\min\ \overline{\text{COST}_i^{\text{SHARED}}} = \text{COST}_i &- \sum_{j\in J} u_i^{(3.27)} a_j^{\text{W}} - \sum_{j\in J} u_{ij}^{(3.28)} (a_j^{\text{B}} + a_j^{\text{W}}) - \sum_{i'\in I: i'\neq i} \sum_{j\in J} u_{i'j}^{(3.28)} a_j^{\text{B}} \\
&- \sum_{j\in J} u_{ij}^{(3.29)} (a_j^{\text{W}} - a_j^{\text{B}}) - \sum_{i'\in I: i'\neq i} \sum_{j\in J} u_{i'j}^{(3.29)} a_j^{\text{B}} \\
&- \sum_{j\in J} u_j^{(3.32)} (a_j^{\text{W}} + a_j^{\text{B}}) - \sum_{j\in J} u_j^{(3.34)} D_i a_j^{\text{W}} - \sum_{j\in J} \sum_{j'\in J: j\neq j'} u_j^{(3.35)} D_i a_j^{\text{W}} a_{j'}^{\text{B}}
\end{aligned}
$$

Where

$$
\text{COST}_i = \sum_{j\in J} (a_j^{\text{W}} + a_j^{\text{B}}) d_{ij}
$$

In the objective function the last term ($\sum_{j\in J} \sum_{j'\in J} u_{jj'}^{(3.35)} a_j^{\text{W}} a_{j'}^{\text{B}}$) is non-linear and to linearize, we defined new binary variable $\alpha_{jj'}$ and we will replace it with term $a_j^{\text{W}} a_{j'}^{\text{B}}$ Then we add three constraint in the set of constraints (constraints 3.40, 3.41, 3.42). So the problem will be as continue.

$$
\begin{aligned}
\min\ \overline{\text{COST}_i^{\text{SHARED}}} = \text{COST}_i - \sum_{j\in J} u_i^{(3.27)} a_j^{\text{W}} \\
- \sum_{j\in J}(u_{ij}^{(3.28)} a_j^{\text{W}} + \sum_{j'\in J: j\neq j'} u_{ij'}^{(3.28)} a_{j'}^{\text{B}}) - \sum_{j\in J}(u_{ij}^{(3.29)} a_j^{\text{W}} - \sum_{j'\in J: j\neq j'} u_{ij'}^{(3.29)} a_{j'}^{\text{B}}) \\
- \sum_{j\in J} u_j^{(3.32)}(a_j^{\text{W}} + a_j^{\text{B}}) - \sum_{j\in J} u_j^{(3.34)} D_i a_j^{\text{W}} - \sum_{j\in J}\sum_{j'\in J: j\neq j'} u_{jj'}^{(3.35)} D_i \alpha_{jj'}
\end{aligned}
$$

Where

$$
\text{COST}_i = \sum_{j\in J}(a_j^{\text{W}} + a_j^{\text{B}}) d_{ij}
$$

.

subject to:

$$
\begin{aligned}
& a_j^{\text{W}} + a_j^{\text{B}} \leq 1 && j \in J && (3.37)\\
& \sum_{j\in J} a_j^{\text{W}} = 1 && && (3.38)\\
& \sum_{j\in J} a_j^{\text{B}} \leq 1 && && (3.39)\\
& a_j^{\text{W}} + a_{j'}^{\text{B}} - 1 \leq \alpha_{jj'} && j, j' \in J : j \neq j' && (3.40)\\
& \alpha_{jj'} \leq a_j^{\text{W}} && j, j' \in J : j \neq j' && (3.41)\\
& \alpha_{jj'} \leq a_{j'}^{\text{B}} && j, j' \in J : j \neq j' && (3.42)\\
& \sum_{j\in J} \pi_j a_j^{\text{W}} \leq \sum_{j\in J} \pi_j a_j^{\text{B}} && && (3.43)\\
& a_j^{\text{W}}, a_j^{\text{B}} \in \{0,1\} && j \in J &&\\
& \alpha_{jj'} \in \{0,1\} && j, j' \in J : j \neq j'. &&
\end{aligned}
$$

3.5 Compact Model

In this section we formulate the problem as compact model to analyze the column generation algorithm.

$$\min \sum_{i \in I} \sum_{j \in J} (a_{ij}^{\mathrm{W},c} + a_{ij}^{\mathrm{B},c}) d_{ij} \qquad i \in I, j \in J \tag{3.44}$$

Subject to:

$$\sum_{j \in J} a_{ij}^{\mathrm{W}} = 1 \qquad i \in I \tag{3.45}$$

$$\sum_{j' \in J : j \neq j'} a_{ij'}^{\mathrm{B}} + a_{ij}^{\mathrm{W}} \leq 2 - x_j \qquad i \in I, j \in J \tag{3.46}$$

$$\sum_{j' \in J : j \neq j'} a_{ij'}^{\mathrm{B}} \geq a_{ij}^{\mathrm{W}} - x_j \qquad i \in I, j \in J \tag{3.47}$$

$$x_j \leq y_j \qquad j \in J \tag{3.48}$$

$$\sum_{j \in J} (s_j + r_j \pi_j) x_j \leq B \tag{3.49}$$

$$\sum_{i \in I} (a_{ij}^{\mathrm{W}} + a_{ij}^{\mathrm{B}}) \leq M y_j \qquad j \in J \tag{3.50}$$

$$\sum_{j \in J} y_j \leq p \tag{3.51}$$

$$\sum_{i \in I} D_i a_{ij}^{\mathrm{W}} \leq Q_j^{\mathrm{W}} \qquad j \in J \tag{3.52}$$

$$\sum_{i \in I} D_i \alpha_{ijj'} \leq Q_j^{\mathrm{B}} \qquad j, j' \in J : j \neq j' \tag{3.53}$$

$$Q_j^{\mathrm{W}} + Q_j^{\mathrm{B}} \leq Q_j \qquad j \in J \tag{3.54}$$

$$a_{ij}^{\mathrm{W}} + a_{ij}^{\mathrm{B}} \leq 1 \qquad i \in I, j \in J \tag{3.55}$$

$$\sum_{j \in J} a_{ij}^{\mathrm{B}} \leq 1 \qquad i \in I \tag{3.56}$$

$$a_{ij}^{\mathrm{W}} + a_{ij'}^{\mathrm{B}} - 1 \leq \alpha_{ijj'} \qquad i \in I, j, j' \in J : j \neq j' \tag{3.57}$$

$$\alpha_{ijj'} \leq a_{ij}^{\mathrm{W}} \qquad i \in I, j, j' \in J : j \neq j' \tag{3.58}$$

$$\alpha_{ijj'} \leq a_{ij'}^{\mathrm{B}} \qquad i \in I, j, j' \in J : j \neq j' \tag{3.59}$$

$$\sum_{j \in J} \pi_j a_{ij}^{\mathrm{W}} \geq \sum_{j \in J} \pi_j a_{ij}^{\mathrm{B}} \qquad i \in I \tag{3.60}$$

$$y_j, x_j \in \{0,1\} \qquad j \in J$$

$$a_{ij}^{\mathrm{W}}, a_{ij}^{\mathrm{B}} \in \{0,1\} \qquad i \in I, j \in J$$

$$\alpha_{ijj'} \in \{0,1\} \qquad i \in I, j, j' \in J : j \neq j'$$

$$Q_j^{\mathrm{W}}, Q_j^{\mathrm{B}} \geq 0 \qquad j \in J.$$

Al the constraints (3.45) to (3.60) are the same concept of constraints in section 3.4.3, Model FORTI_SH_RMP_I. In continue we look into the numerical results based on the test cases.

3.6 Heuristics ILP Solution Based on Random Selection

To control lower bound of column generation, in each iteration of column generation, we decide about one of the facility to be opened. Then we add this facility to restricted master problem, as an opened and fortified facility. In this paper we call this method $Heuristic_{pp_1}$.In this method, the selected facility to be opened and fortified is called Heu_j. Heu_j is selected randomly among all potential facility and then in master problem we defined it as an opened and fortified facility.

3.7 Heuristics ILP Solution Based on Set of Columns from Pricing Problem

In this method, we follow the steps in the previous method in Section 3.6, but the facility selection is different from the $Heuristic_{pp_1}$ method. In this method, the selected facility to be opened and fortified is called Max_j. In continue we explain how we decide about one of facilities to open based on previous iteration.

In each iteration of column generation after finishing solving all pricing problems, we find the facility with most users connected to it. After finding the Max_j we add it to master problem.

In each iteration in heuristic method the founded Max_j can be an option to be open and fortified for next iteration in restricted master problem (RMP). In this case we add constraints to RMP in order to ensure the facility Max_j will be open and fortified. In this case we add these constraints to RMP:

$$y_{Max_j} = 1 \tag{3.61}$$

$$x_{Max_j} = 1 \tag{3.62}$$

Constraint (3.61) ensure that facility Max_j is opened. Constraint (3.62) ensure facility Max_j is fortified.

In this method, in each iteration we remove selected Max_j from candidate facilities for Max_j in next iteration, to eliminate the repetition of same solution but we always make sure that there is enough candidate to be Max_j.

3.8 Numerical Results

3.8.1 Data Sets and Parameters

Models and algorithms proposed in the previous sections were tested on three data sets used by [24], (can be found in the online appendix of [43]). These test cases contain 49, 88 and 150 users, with $m = n$, together with their demands and distance values. We assume that the transportation cost is proportional to the Euclidean distances.

We generated the location capacity values as follows. Let $\overline{D} = \sum_{i \in I} D_i/p$ be the average demand per facility location (under the assumption there are p facilities and load is balanced among the facilities), where the demand values are taken from [24] for $p = 5$, 10, 20,30 and 40. Then, for each potential facility location j, we randomly generated the Q_j capacity values in the interval $[2\overline{D}, 2.2\overline{D}]$.

The fixed cost s_j, following [24], are randomly drown from $U \sim [500, 1500]$ and rounded to the nearest integer. Following [4], the hardening cost is set as follows: $r_j = 0.2 \times s_j$. π_j the probability of failure is randomly generated by uniform distribution $U \sim [0, 0.05]$.

The budget for fortification is calculated as below:

$$\overline{B} = \sum_{j \in J} (s_j + r_j \pi_j)$$

and the budget depends on percentage of opened facilities we decide to fortify (f is the percentage of facility we decide to fortify). In this case p is number of facilities to open and the budget is:

$$B = \frac{\overline{B}}{f \times p}$$

3.8.2 Generation of Initial Solutions

We provide an initial set of columns to both implemented models. In Model SH_RMP_J, each column is associated to a facility location, together with its set of assigned users, for which it is either a primary or a backup facility. In Model FORTI_SH_RMP_I , each column is associated to a user, together with its facilities, for which it is either a primary or a backup facility. In continue we explain the heuristic initial solution algorithms.

3.8.2.1 Heuristic Initial Solution Algorithms

For initial solution, first we follow Algorithm 1 to assign the primary facilities to users. In this algorithm we sort the users based on the higher demand to lower demand. Then for each user we sort the facilities based on their distance to user i. Then we assign the user to closest facility, considering the capacity constraint for facilities and p.

Algorithm 1 *Primary User-Facility Assignment Algorithm*

Require: Set of facilities (J), Set of users (I), π_j, d_ij, B, Facility fortification cost ($Forti_j$), Q_j, $Demand_i$, p.

Ensure: Primary User-Facility Assignment

Sorting users based on the higher demand to lower demand

$Q_j^{\mathrm{w}} = 0$;

for $(i = 1\ ,\ i \leqslant I.length\ ,\ i++)$ **do**

Sorting the facilities based on their distance to user i;

for $(j = 1\ ,\ j \leqslant J.length\ ,\ j++)$ **do**

if $Demand_i \leqslant Q_j - Q_j^{\mathrm{w}}$ **then**

if $j \in OpenFacility$ **then**

assign $i \rightarrow j$ & $a_{ij}^{\mathrm{w}} = 1$;

$Q_j^{\mathrm{w}} = Q_j^{\mathrm{w}} + Demand_i$; Break to next i;

else if $OpenFacility.length \leqslant p$ **then**

add j to $OpenFacility$; assign $i \rightarrow j$ & $a_{ij}^{\mathrm{w}} = 1$;

$Q_j^{\mathrm{w}} = Q_j^{\mathrm{w}} + Demand_i$; Break to next i;

end if

end if

end for

end for

After primary user-facility assignment we fortify the opened facilities based on Algorithm 2. In this algorithm, first we sort the facility location in the increasing order of their π_j values. Then with the budget constraint consideration, we fortify as many locations as possible. For the fortified facility locations, assign as many primary users as possible taking into account the facility capacity constraints. If some users are still without a primary supplier, assign them in priority to the remaining unfortified facility location with the smallest π_j.

Algorithm 2 *Facility Fortification Algorithm*

Require: Set of facilities (J), Set of users (I), π_j, d_ij, B, Facility fortification cost ($Forti_j$), Q_j, $Demand_i$, p.

Ensure: Set of configuration for initial feasible solution

Sorting facilities based on their failure probability;

Replace the sorted facilities in the set of facilities (J); $j = 0$, $cost = 0$;

while $cost \leqslant B$ && $j \leqslant J.length$ **do**

if $j \in OpenFacility$ **then**

Fortifying facility j;

$cost = cost + Forti_j$;

end if

$j++$;

end while

Then for those users connected to facilities which they are not fortified we consider a backup facility. In this case we follow the Algorithm 3. In this algorithm we go through all users. For each user, if the user is connected to a non fortified facility, then we find the closest open facility with enough capacity. Then we assign user to that facility as a backup.

Algorithm 3 *Backup User-Facility Assignment Algorithm*

Require: Set of facilities (J), Set of users (I), π_j, d_ij, B, Q_j, $Demand_i$, p.

Ensure: Backup User-Facility Assignment

```
Q_j^B = Q_j − Q_j^W && BackupDemand_jj' = 0;          for all j, j' ∈ J;
for (i = 1 , i ⩽ I.length , i++) do
    for (k = 1 , k ⩽ OpenFacility.length , k++) do
        j = openFacility[k];
        if a_ij^W = 1 && j isn't fortified then
            for (l = 0 , l ⩽ OpenFacility.length , l++) do
                j' = openFacility[k];
                if j ≠ j' then
                    BackupDemand_jj' = BackupDemand_jj' + Demand_i;
                    if BackupDemand_jj' ⩽ Q_j'^B then Assign user i to j' (a_ij'^B = 1);
                    end if
                end if
            end for
        end if
    end for
end for
```

3.8.3 Model Accuracy

Our first report is on the computational times and the accuracy of the solutions. The results are summarized in Table 3.4 for SH_RMP_J. The three test cases' results are shown in this table. The first column is included the number of potential facilities which are available to open. p is the number of facilities which can be opened. The third column is included the number of fortified facilities. The $z^\star_{\text{LP}}$ is linear optimal solution for this problem. The $\tilde{z}^{\text{W}}_{\text{ILP}}$ is the integer optimal solution for primary user allocation and $\tilde{z}^{\text{B}}_{\text{ILP}}$ the integer optimal solution for backup user allocation. The gap calculated based on the formula: $\frac{(\tilde{z}^{\text{W}}_{\text{ILP}}+\tilde{z}^{\text{B}}_{\text{ILP}})-z^\star_{\text{LP}}}{(\tilde{z}^{\text{W}}_{\text{ILP}}+\tilde{z}^{\text{B}}_{\text{ILP}})}\%$. through the columns 7 to 9 in the table 3.4, the number of initial,generated, and selected columns are shown. the CPU usage (time to solve) is shown in last column.

In the table 3.4 the optimal solution (transportation cost) either LP or ILP is decreasing when the p increasing in both cases 49, 88 and 150 nodes. Number of generated columns are increasing when the size of network is increasing but they decreasing when the number of fortified facilities are increasing. The gap is increasing when the size of network getting big. With increasing the p and number of fortified facilities, the gap decreased. The CPU time is increasing when the size of network is enlarging.

TABLE 3.4: Computational Times and Solution Accuracies

Nodes	p	Forti fication	$z^{\star}_{\text{LP}}$	$z^{\star}_{\text{ILP}}$	gap (%)	# columns			CPU
						i	g	s	(sec.)
49	5	3	281.1	319.8	13.8%	5	795	5	965.1
	5	4	275.2	312.7	13.6%	5	686	5	856.4
	5	5	208.3	224.6	7.8%	5	432	5	532.8
	10	4	227.1	256.7	13.0%	10	1,239	10	1,442.3
	10	6	177.9	198.9	11.8%	10	1,105	10	1,351.2
	10	10	145.6	154.9	6.9%	10	936	10	1,168.1
	20	6	165.7	178.4	7.7%	20	1,532	20	1,861.0
	20	11	115.3	123.2	6.9%	20	1,453	20	1,678.1
	20	20	61.9	65.7	6.1%	20	1,201	20	1,436.3
88	5	3	636.6	753.6	18.4%	5	1,015	5	1,248.9
	5	4	559.2	657.4	17.6%	5	896	5	1,125.7
	10	4	541.0	624.9	15.5%	10	1,801	10	2,354.2
	10	6	337.4	387.3	14.8%	10	1,698	10	1,880.6
	20	6	327.6	371.4	13.4%	20	2,753	20	3,016.5
	20	11	230.4	256.6	11.4%	20	2,234	20	2,852.4
150	20	6	596.4	707.7	18.7%	20	1,856	20	2,278.9
	20	11	394.8	465.6	17.9%	20	1,765	20	2,072.1
	30	11	345.1	399.2	15.7%	30	2,356	30	3,144.3
	30	16	245.1	278.4	13.6%	30	2,158	30	2,884.6
	40	16	220.2	248.8	13.0%	40	2,874	40	3,862.5
	40	21	178.9	201.5	12.6%	40	2,685	40	3,258.4

3.8.4 Comparison of Optimal Solution Between Three Models

in Table 3.5 we compare results of three different models of non sharing backup model, shared backup model such the primary is known and shared back up model. z^{*}_{ILP} stands for total cost for primary and backup assignment, $z^{\text{W}}_{\text{ILP}}$ stands for costs for primary assignment and $z^{\text{B}}_{\text{ILP}}$ stands for costs for back up assignment. This table shows that cost increase when there is sharing. In case of sharing backup when the primary known and fixed we had improvement but more improvement is when we relax selecting primary while sharing the backup. In third model (shared backup model) for test case 88 nodes and $p = 5$ there is improvement in primary assignment since there is no improvement in back up sharing (compare to model sharing backup with known primary). In third model (shared backup model) for test case 49 nodes and $p = 20$ the backup cost is better than first model (non-shared backup model) but there is no improvement in backup assignment while there is improvement in primary and then there is improvement in total cost ($127.2 < 123.2$) which is benefit of primary assignment and sharing back at the same time to improve solution.

TABLE 3.5: Comparison of Optimal Solution Between Three Models

Nodes	p	Non-Shared backup model			Shared backup with known primary model			Shared backup model			Forti
		$z^{\star}_{\text{ILP}}$	$z^{\text{W}}_{\text{ILP}}$	$z^{\text{B}}_{\text{ILP}}$	$z^{\star}_{\text{ILP}}$	$z^{\text{W}}_{\text{ILP}}$	$z^{\text{B}}_{\text{ILP}}$	$z^{\star}_{\text{ILP}}$	$z^{\text{W}}_{\text{ILP}}$	$z^{\text{B}}_{\text{ILP}}$	
49	5	**336.1**	231.3	104.8	**336.1**	231.3	104.8	**319.8**	224.6	95.2	3
	10	**223.2**	160.3	62.9	**215.5**	160.3	55.2	**198.9**	159.3	39.6	6
	20	**132.6**	116.8	15.8	**127.2**	116.8	10.4	**123.2**	110.6	12.6	11
88	5	**798.6**	500.1	298.5	**798.6**	500.1	298.5	**753.6**	455.1	298.5	3
	10	**527.3**	275.4	251.9	**465.8**	275.4	190.4	**387.3**	276.2	111.1	6
	20	**508.0**	280.8	227.2	**394.4**	280.8	113.6	**256.6**	193.3	63.3	11
150	20	**717.4**	483.6	233.8	**717.4**	483.6	233.8	**465.6**	280.6	185.0	11
	30	**422.9**	197.9	225.0	**313.7**	197.9	115.8	**278.4**	197.7	80.7	16
	40	**335.4**	173.3	162.1	**266.2**	173.3	92.9	**201.5**	141.7	59.8	21

3.8.5 Comparison of Compact Model and Column Generation for Model Forti_Sh_Rmp_i

The Table 3.6 shows the results of different test cases when they are solved by compact model and column generation.

Table 3.6: Comparison of Compact Model and Column Generation

Nodes	p	Forti fication	Column Generation $\tilde{z}^{\star}_{\text{LP}}$	Column Generation $\tilde{z}^{\star}_{\text{ILP}}$	Compact Integer $z^{\star}_{\text{LP}}$	Compact Integer $z^{\star}_{\text{ILP}}$	
49	5	3	281.1	319.8	104.1	296.2	
	5	4	275.2	312.7	100.9	281.7	
	10	4	227.1	256.7	97.4	246.5	
	10	6	177.9	198.9	89.4	191.2	
	20	6	165.7	178.4	85.9	169.9	
	20	11	115.3	123.2	72.5	118.2	
88	5	3	636.6	753.6	121.3	655.4	
	5	4	559.2	657.4	119.3	581.1	
	10	4	541.0	624.9	118.3	554.9	
	10	6	337.4	387.3	109.2	368.4	
	20	6	327.6	371.4	105.8	341.6	
	20	11	230.4	256.6	95.3	241.6	
150	20	6	596.4	707.7	115.1	806.3	upper-bound after 70 min
	20	11	394.8	465.6	107.4	508.7	
	30	11	345.1	399.2	104.5	450.9	
	30	16	245.1	278.4	99.1	302.3	
	40	16	220.2	248.8	96.7	289.8	
	40	21	178.9	201.5	83.1	229.4	

As it is shown the compact model has better solution if we could access to optimal solution but thee is no optimal solution for large size test cases and we reached to upper bound after 70 minutes. In test case with 150 nodes the integer solution is an upper bound of integer optimal solution and the integer solution of column generation is better than compact model.

Also in Figure 3.9 the deviation of results from optimal solution for test cases 49 nodes (Figures 3.3 - 3.5) and 88 nodes (Figures 3.6 - 3.8) are shown. In these figures the the LP and ILP solution of column generation and compact model are shown. In all cases the ILP solution of column generation is bigger than the optimal solution (compact model ILP solution).

3.8.6 ILP Solution Comparison of Three Heuristics Models (CG_{cplex}, $Heuristic_{pp_1}$ and $Heuristic_{pp_2}$)

The Table 3.7 shows different ILP solutions from three heuristics methods. In the columns one to three the number of nodes, p, and the number of fortified facilities are indicated. The column three shows the optimal LP solution from column generation.

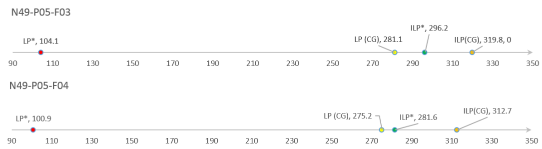


FIGURE 3.3: CG and Compact 4905

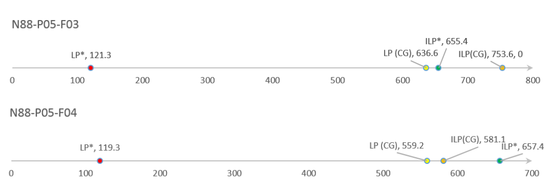


FIGURE 3.6: CG and Compact 8805

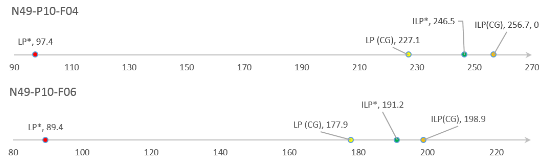


FIGURE 3.4: CG and Compact 4910

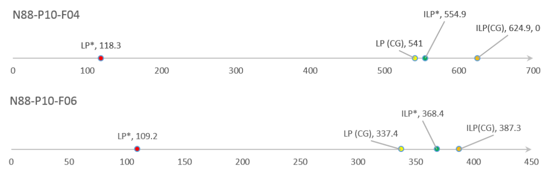


FIGURE 3.7: CG and Compact 8810

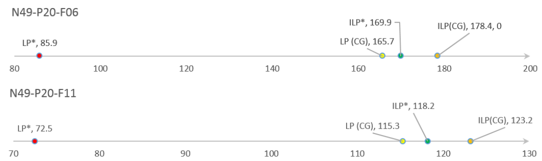


FIGURE 3.5: CG and Compact 4920

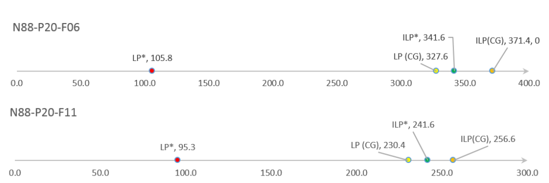


FIGURE 3.8: CG and Compact 8820

FIGURE 3.9: CG and Compact Comparison

In the columns four to seven, the ILP solutions using column generation, $Heuristic_{pp_1}$, $Heuristic_{pp_2}$ respectively.

As it is shown, the $Heuristic_{pp_1}$ has a small improvement of the ILP solution compare to the solution of $CG_{cplex_{ilp}}$. The $Heuristic_{pp_2}$ has a significant improvement of the $\tilde{z}^{\star}_{\text{ILP}}$ (ILP solution) for all test cases and less deviation from LP solution from column generation compare to other two columns. Improvements are more observable in larger case sizes such as test cases with 150 nodes.

Table 3.7: ILP Solution Comparison of Heuristics Models(CG_{cplex}, $Heuristic_{pp_1}$ and $Heuristic_{pp_2}$)

Nodes	p	Forti fication	$CG_{cplex_{lp}}$	$CG_{cplex_{ilp}}$	$Heuristic_{pp_1}$	$Heuristic_{pp_2}$
			$\tilde{z}^\star_{\text{LP}}$	$\tilde{z}^\star_{\text{ILP}}$	$\tilde{z}^\star_{\text{ILP}}$	$\tilde{z}^\star_{\text{ILP}}$
49	5	3	281.1	319.8	315.9	308.8
	5	4	275.2	312.7	304.7	296.8
	10	4	227.1	256.7	256.0	250.1
	10	6	177.9	198.9	196.8	193.0
	20	6	165.7	178.4	176.4	173.1
	20	11	115.3	123.2	122.7	119.9
88	5	3	636.6	753.6	744.9	705.6
	5	4	559.2	657.4	656.9	618.4
	10	4	541.0	624.9	620.2	598.9
	10	6	337.4	387.3	385.7	372.6
	20	6	327.6	371.4	369.1	351.2
	20	11	230.4	256.6	255.2	246.5
150	20	6	596.4	707.7	701.6	668.1
	20	11	394.8	465.6	460.0	428.7
	30	11	345.1	399.2	398.8	378.2
	30	16	245.1	278.4	277.5	263.4
	40	16	220.2	248.8	246.7	238.5
	40	21	178.9	201.5	198.3	191.3

3.8.7 Gap Comparison of Heuristics Models

In this section we analyze the gap of solution for different models (CG_{cplex}, $Heuristic_{pp_1}$, $Heuristic_{pp_2}$, and Cplex Solution of Compact Model) In the Table 3.8 There are two main category of gaps: (i) LP gaps (Columns four to six) and (ii) ILP gaps (columns seven to ten).

(i) In LP Gaps part we present three gaps:

$(i-1)$ $\%LP^*_{MIP}$ (in column four), which is the deviation of compact model LP optimal solution from ILP optimal solution: $\%LP^*_{MIP} = (|z^\star_{\text{LP}} - z^\star_{\text{ILP}}| / z^\star_{\text{ILP}}) \times 100\%$

$(i-2)$ $\%LP^*_{CG}$ (in column five), which is deviation of CG LP solution from best solution among H_{Cplex}, H_{pp_1} and H_{pp_2}: $\%LP^*_{CG} = (|\tilde{z}^\star_{\text{LP}} - \tilde{z}^\star_{\text{ILP}}| / \tilde{z}^\star_{\text{ILP}}) \times 100\%$,

$(i-3)$ $\%LB^*$ (in column six), which is deviation of best lower bound from best available solution: $\%LB^* = (\left|\tilde{z}^\star_{\text{LP}} - \tilde{z}^{Best}_{\text{ILP}}\right| / \tilde{z}^{Best}_{\text{ILP}}) \times 100\%$.

(ii)In ILP gaps part we present four gaps:

$(ii-1)$ $\%H_{Cplex}$ (in column seven), which is deviation of H_{Cplex} optimal solution from best available solution: $\%H_{Cplex} = (\left|\tilde{z}^{\star}_{CG_{Cplex}} - \tilde{z}^{Best}_{\text{ILP}}\right| / \tilde{z}^{Best}_{\text{ILP}}) \times 100\%$

$(ii-2)$ $\%H_{pp_1}$ (in column eight), which is deviation of $Heuristic_{pp_1}$ optimal solution from best available solution: $\%H_{pp_1} = (\left|\tilde{z}^{\star}_{H_{pp_1}} - \tilde{z}^{Best}_{\text{ILP}}\right| / \tilde{z}^{Best}_{\text{ILP}}) \times 100\%$

$(ii-3)$ $\%H_{pp_2}$ (in column nine), which is deviation of $Heuristic_{pp_2}$ optimal solution from best available solution: $\%H_{pp_2} = (\left|\tilde{z}^{\star}_{H_{pp_2}} - \tilde{z}^{Best}_{\text{ILP}}\right| / \tilde{z}^{Best}_{\text{ILP}}) \times 100\%$

$(ii-4)$ $\%MIP$ (in column ten), which is deviation of compact model optimal solution from best available solution: $\%MIP = (\left|\tilde{z}^{\star}_{MIP} - \tilde{z}^{Best}_{\text{ILP}}\right| / \tilde{z}^{Best}_{\text{ILP}}) \times 100\%$

TABLE 3.8: Comparison of Heuristics Model and Column Generation Model 1

Nodes	p	Fortification	LP gaps			Column Generation gaps			compact	
			$\%LP^*_{MIP}$	$\%LP^*_{CG}$	$\%LB^*$	$\%H_{Cplex}$	$\%H_{pp_1}$	$\%H_{pp_2}$	$\%MIP$	
49	5	3	64.9%	9.0%	5.1%	8.0%	6.7%	4.3%	0.0%	
	5	4	64.8%	7.3%	4.0%	7.9%	6.3%	3.5%	0.0%	
	10	4	60.5%	9.2%	7.9%	4.1%	3.9%	1.5%	0.0%	
	10	6	53.2%	7.8%	7.0%	4.0%	2.9%	0.9%	0.0%	
	20	6	49.4%	4.3%	2.5%	5.0%	3.8%	1.9%	0.0%	
	20	11	38.7%	3.8%	2.5%	4.2%	3.8%	1.4%	0.0%	
88	5	3	81.5%	9.8%	2.9%	15.0%	13.7%	7.7%	0.0%	
	5	4	79.5%	9.6%	3.8%	13.1%	13.0%	6.4%	0.0%	
	10	4	78.7%	9.7%	2.5%	12.6%	11.8%	7.9%	0.0%	
	10	6	70.4%	9.4%	8.4%	5.1%	4.7%	1.1%	0.0%	
	20	6	69.0%	6.7%	4.1%	8.7%	8.1%	2.8%	0.0%	
	20	11	60.6%	6.5%	4.6%	6.2%	5.6%	2.0%	0.0%	
150	20	6	85.7%	10.7%	10.7%	5.9 %	5.0%	0.0%	20.7%	upper-bound after 70 min
	20	11	78.9%	7.9%	7.9%	8.6 %	7.3%	0.0%	18.7%	
	30	11	76.8%	8.8%	8.8%	5.6 %	5.4%	0.0%	19.2%	
	30	16	67.2%	6.9%	6.9%	5.7 %	5.4%	0.0%	14.8%	
	40	16	66.6%	7.7%	7.7%	4.3 %	3.4%	0.0%	21.5%	
	40	21	63.8%	6.5%	6.5%	5.3 %	3.7%	0.0%	19.9%	

As it is shown in the table 3.8, the best solution for this problem is optimal solution from compact model in small size test cases. But when the size of problem become bigger the best upper bound (best ILP solution) is from model $Heuristic_{pp_2}$.

3.9 Conclusion and Future Work

We have presented a new integer-linear programming model for identifying optimal solution under assumption of facility sharing and facility fortification in logistics networks under disruption. We have tested this model on three different geographical data sets. In this paper the fortification is binary. If we fortified the facility there is no disruption which in real word is not a case. For future work we suggest to consider partial fortification and also partial disruption with different probability.

Chapter 4

Designing Reliable Facility Location and Routing in Logistics Network in Presence of Disruption Considering Backup Sharing : Column Generation Approach

abstract

With globalization has come the growing threat of disruption, which impacts the reliability of logistics networks. To ensure reliability, we must examine the elements of the logistics network. Two main network elements are the supplier facility location and routing between facilities and users. Disruption can affect either the locations (e.g., suppliers and warehouses) or the routes between suppliers and users. In this paper, we revisit the capacitated location-routing problem (LRP) under disruptions with a column generation decomposition. We include capacity constraints as well as a more accurate estimate of the required resources, taking into account that not all failures occur at the

same time, and therefore some backup resources can be shared. We consider one disruption at a time per facility and per route. To improve solutions for column generation in terms of gap and time, we implement new heuristics in pricing as well as a rounding off method. Intensive numerical experiments complete the paper. Conclusions are drawn in the last section.

keyword:

Reliable Locating-routing, LRP, Disruptions, Column Generation, Mixed Integer problem, Rounding off

4.1 Introduction

According to Prodhon and Prins [44], location and routing decisions are interdependent and studies have shown that the overall system cost may be excessive if they are tackled separately. The location-routing problem (LRP) integrates the two kinds of decisions. Given a set of potential depots with opening costs, a fleet of identical vehicles and a set of customers with known demands, the classical LRP consists in opening a subset of depots, assigning customers to them and determining vehicle routes, to minimize a total cost including the cost of open depots, the fixed costs of vehicles used, and the total cost of the routes [44].

To make facility reliable we can fortify facilities from Natural disaster such as earthquake and storms; or fire, electricity down, material shortage which can be considered as disruption in facilities.

In reality, various protection measures are available, such as installing structural reinforcements, adding built-in redundancies, improving monitoring and security guarding, buying insurance and using outsourcing. Most of the papers considering facility protection assume a context of deliberate attacks, i.e., where an intelligent adversary intentionally tries to interdict the facility network to maximize the losses and where, in contrast, a defender protects some of the most critical components to mitigate the effect of the attacks [45].

In this paper we look into the related work in section 4.2, then the problem statement

4.2.4 is defined. The column generation decomposition model for this problem is proposed in section 4.4. The column generation improvement methods, named Heuristics Max_j and Rounding off methods are explained in section 4.5 and section 4.6. To improve the column generation calculation time we proposed a heuristic method for pricing problem named pricing heuristics is explained in section 4.7. After we proposed are numerical results based on three main test cases in section 4.8. conclusion and further directions are discussed in section 4.9.

4.2 Related Work

4.2.1 Reliable Facility Location Models subject to Disruption

Decisions about facility location are either costly or difficult to reverse. The impact of decisions will remain for a long time horizon. The parameter estimation (e.g., costs, demands, transportation times) may be inaccurate due to poor measurements. Disruption effects on the logistics networks parameters as discussed before (e.g., time, cost, and availability of facility). In case of uncertainty of disruption, the parameters such as recognizing demand point or distances have been studied by authors and several models have been developed for facility location under uncertainty [10]. In the sequel, we classify the work made on the facility location under disruption and the solution methods related to the design of reliable facility location under disruption.

4.2.2 Classification of Facility Location Problems

There are various studies on the reliable facility location problems in logistics networks. Most of them considered the un-capacitated p-median problem (UFLP) [4]. Its objective is to find the optimal location based on the average distances between locations and users. The capacitated p-median problem (CPMP) [11] adds capacity constraints on the facilities, but has not yet been studied in the context of disruptions. To our knowledge, facility location under uncertainty can be divided by to two main categories: *(i)* uncertainty on customer demand, and travel time(cost)between facility locations and customers, *(ii)* uncertainty on availability of facility locations.

The facility location under uncertainty for category (i) are well reviewed by Snyder [10]. In this case, problems are categorized into three main categories. *(i-a)*, when there are certainties on demand or travel time values.

(i-b), when there are uncertainties on demand or travel time whose values are governed by probability distributions that are known by the decision maker. *(i-c)*, when demand or travel time is uncertain, and furthermore, no information about probabilities is known. In the context of *(ii)*, authors have worked along two directions for the design of resilient facility location models: *(ii-a)* fortification of a subset of facilities subject to some number or budget constraints (e.g., [16]) and *(ii-b)* establishing some backup facilities (e.g., [4]).

Facility location problem considering path between the potential facilities and users can be categorized in two orientation in the literature. First direction, those problems in which the facility supports several users. The vehicles leave the facilities and deliver material to users with a sequence and then return to facility. In second direction the models are considering direct direction between each facility to each users. The distance between the user and facility is the shortest path instead of euclidean distance.

4.2.3 Locating Problem

We first look into the papers related to the first category (category *(i)*). The methods to solve this problem are reviewed by Snyder [10]. The category *(i-a)* is the classical facility location problem. the category *(i-b)* is modeled as stochastic facility location problems (for instances, [13], [14]), and *(i-c)* is modeled as robust facility location problems (for instance, [15], [10]).

Now we review the papers dealing with category *(ii)*, starting with category *(ii-a)*, those which used the concept of fortification, or equivalently interdiction. In this case, single and multiple disruptions are defined in facility location problem (e.g., [17]). Different levels of disruption as partial and complete are modeled in this problem (e.g., [2]). In case of the capacitated problem the disruption can reduce the capacity of supplier [16]or facility can loose its capacity [18]. Different sizes of facility location fortification problem are discussed in the literature. For instances, 50 to 150 demand nodes [21], [22] and 20 to 40 facility nodes [17], [16] to support demand nodes are solved by authors.

The objectives of facility location under disruption are to minimize the impact of disruption. It can be minimizing the worst-case impact of disruption or worse-case disruption scenario (e.g., [20]). It can be facility protection or fortification based on the first investment [21]. Other objectives proposed by authors are minimizing of recovery of disrupted facilities (e.g., [3]).

The most method to solve the proposed models are Bender decomposition [23][17], Lagrangian relaxation [24], pre processing techniques based on the valid lower and upper bound, and heuristics methods [22].

In category *(ii-b)* we now review the papers dealing with the backup facility location. In this case, the authors considered for each user there is a primary facility and a backup facility or a layer of backup facilities [4]. Also there is no capacity constraint for potential facilities. [4] considered hardening selected facilities. They dividing facilities as unreliable and another that is reliable. Different size of problem is considered by authors. [5] solved the problem with size of 150 demand nodes and 30 to 50 supplier. [4] employed a data set of 263 nodes representing the largest cities in the contiguous 48 states in the United States. They solved the problem where different combination of reliable and unreliable facilities are considered (for instance, 11 unreliable facilities and 4 reliable facilities). The method, which is used in case of solving backup facility location models, is Lagrangian relaxation [5], [4].

4.2.4 Location and Routing Problem

Drexl and Schneider [46] defined the term location-routing problem(LRP)as a mathematical optimization problem where at least the following two types of decisions must be made interdependently:

(i) Which facilities out of a finite or infinite set of potential ones should be used(for a certain purpose)?

(ii) Which vehicle routes should bebuilt, i.e., which customer clusters should be formed and in which sequence should the customers in each cluster be visited by a vehicle from a given fleet(to perform a certain service)?

The classical location-routing problem (LRP) well studied by Nagy and Salhi [47] and then by Prodhon and Pins [44]. Recently the last survey of variant and extension of LRP is studied by Drexl and Schneider [46].

In the LRP problem can be classified based on the capacity consideration for the depot and vehicle. In this case there is classical LRP (in case of uncapacitated depot or vehicle) and CLRP or capacitated LRP.

Prodhon and Pins [44] Classified the LRP into four main categories:

(*i*) Classical location routing problems: (*i*-*a*)LRP with uncapacitated vehicle (rout), (*i*-*b*)LRP with uncapacitated depot (*i*-*c*)Capacitated LRP (CLRP).

(*ii*) Multi-echon location-routing problems: (*ii*-*a*) Two-echelon LRP, (*ii*-*b*) mobile depots, (*ii*-*c*)Truck and trailer routing problem (TTRP).

(*iii*) LRP with special or multiple objective functions.

(*iv*) Miscellaneous location-routing problems: (*iv*-*a*) Additional attributes on nodes and vehicles, (*iv*-*b*) Multi-period LRP, (*iv*-*c*) LRP with Inventory management, (*iv*-*d*) LRP with uncertain data (demand, customer presence, travel time).

We look into the summary of related problem including data sets, objectives and solution approaches in table 4.1.

In facility location problem usually the links between the nodes are Euclidean distance for simplification but in Location and Routing problem considering the links are effecting on the solution of problem. Different assumptions are considered in the literature:

In Melkote and Daskin [48], Daskin et al. [49] the desired number of candidate links are randomly selected and added to the network with a bias towards shorter links to emulate transportation networks. Euclidean distances are computed for each link and rounded to the nearest integer The specific distances are considered and all the links with the value less than that is available as links between nodes to route. For instance Lin and Kwok [50] considered Hamiltonian circuit whose over all cost is equal 74. Guerra et al. [51] considered available links which have specific criteria like the links with shortest trip but heavy loaded. Shaikh et al. [52] considered different network degree in Their problem and investigated the performance of solution in three different topology. Xie et al. [53] considering euclidean distances between all user and supplier but they considered capacity constraint on vehicles. In Table 4.1 we summarize the literature and proposed methods including comparison with our work.

Reference	Disruption		problem feature					Largest data size
			Facility		Path			
	Facility	Transport	Fortification	Backup	Fortification	Backup	Capacity Sharing	
This paper	Y	Y	Y	Y		Y	Y	150
Azad *et al.* [54]	Y	Y	Y		Y			150
Snyder and Daskin [24]	Y		Y					150
Church and Scaparra [21]	Y		Y					150
Scaparra *et al.* [9]	Y			Y				316
Lim *et al.* [4]	Y			Y				263
Liberatore *et al.* [22]	Y		Y					263
Scaparra and Church [16]	Y			Y				150
Losada *et al.* [20]	Y							150
Liberatore *et al.* [2]	Y		Y					305
Li *et al.* [5]	Y		Y	Y				150
Qin *et al.* [8]	Y		Y					rand.
Hernandez *et al.* [37]	Y							100

TABLE 4.1: Logistics Networks Disruption: Facility Location and Routing Problem

4.3 Problem Statement

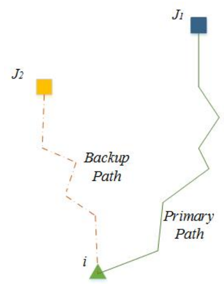
We denote by I the set of customers, J the set of potential locations for the facilities, and p the maximum number of facilities to open. Each customer $i \in I$ has demand D_i, and let Q_j be the capacity of location j. Let $\text{COST}_{Path_{ij}}$ be the transport cost of demands between facility location $j \in J$ and customer $i \in I$. For a given facility location j with a set I_j of assigned customers. Different degrees considered for the graphs (for instance, for graph with 49 nodes the degrees of 8,10, and 12 are considered). The available edge between two nodes is shortest rout (considering routes among links between two nodes). The routes between the customer i and facility j_1 ans j_2 is shown in figure 4.1. In this research we are looking for primary and backup facility for each user and also we are looking for primary and backup route for each user to primary and backup facility. We assume that there is one disruption on facility or edges at the time.

The setup cost s_j is a fixed cost required to implement facility fortification (the costs of contract negotiation, overhead, personnel training, etc.). The variable fortification cost varies with the amount of reliability improvement of the facility. (the cost of acquiring and installing the units of protective measures, the cost of procurement and storage of backup inventory, and the cost of hiring extra workforce, etc.).

We define r_j as the cost associated with the unit reduction in the failure probability of facility j. The total available fortification budget is equal to B.

LENGTH_ℓ is the value of edges (such as travel time, or travel cost) for each edge in set of edge which is calculated based on the euclidean distance between two nods of that edge.

FIGURE 4.1: Primary and Backup Route in Case of Route Disruption



4.4 Capacitated Reliable Facility and transportation Problem

We propose a new decomposition model in which the sets of configurations assigned primary and backup facility for each user is considered as one decision variable (z_c). The capacity constraint is considered in master problem to control both demand for primary and backup based on the selected facility capacity. Also in master problem we add constraint for fortification decision based on the available budgets. In pricing problem we consider assigning primary or backup supplier or facility for specific user in that column to minimize the route cost between user and facility.

4.4.1 Decomposition Scheme

he proposed model, called LRP_ShRouting_f, relies on a decomposition scheme with the concept of configurations defined as follows. Each configuration is associated with one user (i) and contains a primary supplier facility ($a_j^{\text{W},c}$) and may a backup supplier facility($a_j^{\text{B},c}$). For a given user i, let C_i be the set of user i configurations, where $c \in C_i$ is characterized by the user i assigned to the facility j as primary and j' as backup so that $j \neq j'$.

$a_j^{\mathrm{W},c} \in \{0,1\}$. $a_j^{\mathrm{W},c} = 1$ if customer uses the facility j in the configuration c as a primary facility location, 0 otherwise. $a_j^{\mathrm{B},c} \in \{0,1\}$. $a_j^{\mathrm{B},c} = 1$ if customer uses the facility j in the configuration c as a backup facility location, 0 otherwise. $\alpha_{jj'} \in \{0,1\}$. We defined new binary variable $\alpha_{jj'}$ and we will replace it with term $a_j^{\mathrm{W}} a_{j'}^{\mathrm{B}}$ in Constraint4.13 ($\sum\limits_{j \in J} \sum\limits_{j' \in J} D_i a_j^{\mathrm{W}} a_{j'}^{\mathrm{B}}$) which is not linear and to linearize, then we add three constraint in the set of constraints(constraints 4.17, 4.18, 4.20).

$p_\ell^{\mathrm{W}} \in \{0,1\}$. $p_\ell^{\mathrm{W}} = 1$ if customer uses the edge ℓ for routing to facility as primary, 0 otherwise.

$p_\ell^{\mathrm{B}} \in \{0,1\}$. $p_\ell^{\mathrm{B}} = 1$ if customer uses the edge ℓ for routing to facility as backup, 0 otherwise. Other parameters that will be used in the decomposition model have been defined in Section 4.3.

4.4.1.1 Variables

We used five set of decision variable: $z_c \in \{0,1\}$. $z_c = 1$ if configuration c is selected in the optimal solution, 0 otherwise. $x_j \in \{0,1\}$. $x_j = 1$ if facility location is selected and fortified, 0 otherwise. $y_j \in \{0,1\}$. $y_j = 1$ if facility location j is open and is used as a primary or a backup supplier, 0 otherwise. $Q_j^{\mathrm{W}} \geq 0$. the amount of capacity needed for primary demand. $Q_j^{\mathrm{B}} \geq 0$. the amount of capacity needed for backup demand.

4.4.2 Compact Model

The problem can be formulated as continue. We explain the role of each constraint in detail in column generation decomposition section.

In summary the objective function minimize the total traveling cost for the primary and backup paths. Constraints (4.2) to (4.4) take care of disjoint paths for primary and backup paths. Constraints (4.5) to (4.11) and (4.15) to (4.20) take car of primary and backup user-facility assignment, opening facilities, and fortification. Constraints (4.12) to (4.14) take care of primary capacity and shared backup capacity. The detail of constraints are explained in decomposition model in Sections 4.4.3 and 4.4.3.1.

$$\min \sum_{\ell \in L} ((p_\ell^{\mathrm{W}} + p_\ell^{\mathrm{B}}) \times \textsc{length}_\ell). \tag{4.1}$$

Subject to:

$$\sum_{\ell \in \omega^+(j)} p_\ell^{\mathrm{W}} - \sum_{\ell \in \omega^-(j)} p_\ell^{\mathrm{W}} = \begin{cases} 1 - a_{ij}^{\mathrm{W}} & \text{if } j = i \\ -a_{ij}^{\mathrm{W}} & \text{otherwise} \end{cases} \quad j \in J \tag{4.2}$$

$$\sum_{\ell \in \omega^+(j)} p_\ell^{\mathrm{B}} - \sum_{\ell \in \omega^-(j)} p_\ell^{\mathrm{B}} = \begin{cases} 1 - a_{ij}^{\mathrm{B}} & \text{if } j = i \\ -a_{ij}^{\mathrm{B}} & \text{otherwise} \end{cases} \quad j \in J \tag{4.3}$$

$$p_\ell^{\mathrm{W}} + p_\ell^{\mathrm{B}} \leq 1 \qquad \ell \in L \tag{4.4}$$

$$\sum_{j \in J} a_{ij}^{\mathrm{W}} = 1 \qquad i \in I \tag{4.5}$$

$$\sum_{j' \in J : j \neq j'} a_{ij'}^{\mathrm{B}} + a_{ij}^{\mathrm{W}} \leq 2 - x_j \qquad i \in I, j \in J \tag{4.6}$$

$$\sum_{j' \in J : j \neq j'} a_{ij'}^{\mathrm{B}} \geq a_{ij}^{\mathrm{W}} - x_j \qquad i \in I, j \in J \tag{4.7}$$

$$x_j \leq y_j \qquad j \in J \tag{4.8}$$

$$\sum_{j \in J} (s_j + r_j q_j) x_j \leq B \tag{4.9}$$

$$\sum_{i \in I} (a_{ij}^{\mathrm{W}} + a_{ij}^{\mathrm{B}}) \leq M y_j \qquad j \in J \tag{4.10}$$

$$\sum_{j \in J} y_j \leq p \tag{4.11}$$

$$\sum_{i \in I} D_i a_{ij}^{\mathrm{W}} \leq Q_j^{\mathrm{W}} \qquad j \in J \tag{4.12}$$

$$\sum_{i \in I} D_i \alpha_{ijj'} \leq Q_j^{\mathrm{B}} \qquad j, j' \in J : j \neq j' \tag{4.13}$$

$$Q_j^{\mathrm{W}} + Q_j^{\mathrm{B}} \leq Q_j \qquad j \in J \tag{4.14}$$

$$a^{\mathrm{W}}_{ij} + a^{\mathrm{B}}_{ij} \leq 1 \qquad i \in I, j \in J \tag{4.15}$$

$$\sum_{j \in J} a^{\mathrm{B}}_{ij} \leq 1 \qquad i \in I \tag{4.16}$$

$$a^{\mathrm{W}}_{ij} + a^{\mathrm{B}}_{ij'} - 1 \leq \alpha_{ijj'} \qquad i \in I, j, j' \in J : j \neq j' \tag{4.17}$$

$$\alpha_{ijj'} \leq a^{\mathrm{W}}_{ij} \qquad i \in I, j, j' \in J : j \neq j' \tag{4.18}$$

$$\alpha_{ijj'} \leq a^{\mathrm{B}}_{ij'} \qquad i \in I, j, j' \in J : j \neq j' \tag{4.19}$$

$$\sum_{j \in J} q_j a^{\mathrm{W}}_{ij} \geq \sum_{j \in J} q_j a^{\mathrm{B}}_{ij} \qquad i \in I \tag{4.20}$$

$$y_j, x_j \in \{0, 1\} \qquad j \in J \tag{4.21}$$

$$a^{\mathrm{W}}_{ij}, a^{\mathrm{B}}_{ij} \in \{0, 1\} \qquad i \in I, j \in J \tag{4.22}$$

$$\alpha_{ijj'} \in \{0, 1\} \qquad i \in I, j, j' \in J : j \neq j' \tag{4.23}$$

$$Q^{\mathrm{W}}_j, Q^{\mathrm{B}}_j \geq 0 \qquad j \in J \tag{4.24}$$

4.4.3 A Decomposition Model

We can formulate the problem as below:

$$\min \sum_{c \in C} \textsc{cost}_c z_c \tag{4.25}$$

where

$$\textsc{cost}_c = \sum_{\ell \in L} \textsc{length}_\ell (p^{\mathrm{W}}_{c,\ell} + p^{\mathrm{B}}_{c,\ell}). \qquad c \in C_i.$$

Subject to:

$$\sum_{c \in C_i} \sum_{j \in J} a_j^{\mathrm{W},c} z_c = 1 \qquad i \in I \tag{4.26}$$

$$\sum_{c \in C_i} \sum_{j' \in J: j \neq j'} a_{j'}^{\mathrm{B},c} z_c + \sum_{c \in C_i} a_j^{\mathrm{W},c} z_c \leq 2 - x_j \qquad i \in I, j \in J \tag{4.27}$$

$$\sum_{c \in C_i} \sum_{j' \in J: j \neq j'} a_{j'}^{\mathrm{B},c} z_c \geq \sum_{c \in C_i} a_j^{\mathrm{W},c} z_c - x_j \qquad i \in I, j \in J \tag{4.28}$$

$$x_j \leq y_j \qquad j \in J \tag{4.29}$$

$$\sum_{j \in J} (s_j + r_j q_j) x_j \leq B \tag{4.30}$$

$$\sum_{c \in C} (a_j^{\mathrm{W},c} + a_j^{\mathrm{B},c}) z_c \leq M y_j \qquad j \in J \tag{4.31}$$

$$\sum_{j \in J} y_j \leq p \tag{4.32}$$

$$\sum_{i \in I} D_i \sum_{c \in C_i} a_j^{\mathrm{W},c} z_c \leq Q_j^{\mathrm{W}} \qquad j \in J \tag{4.33}$$

$$\sum_{i \in I} D_i \sum_{c \in C_i} a_{j'}^{\mathrm{W},c} a_j^{\mathrm{B},c} z_c \leq Q_j^{\mathrm{B}} \qquad j, j' \in J : j \neq j' \tag{4.34}$$

$$Q_j^{\mathrm{W}} + Q_j^{\mathrm{B}} \leq Q_j \qquad j \in J \tag{4.35}$$

$$Q_j^{\mathrm{W}}, Q_j^{\mathrm{B}} \geq 0 \qquad j \in J \tag{4.36}$$

$$z_c \in \{0, 1\} \qquad c \in C \tag{4.37}$$

$$y_j \in \{0, 1\} \qquad j \in J \tag{4.38}$$

$$x_j \in \{0, 1\} \qquad j \in J \tag{4.39}$$

$$\tag{4.40}$$

Constraints (4.26) ensure that each user only assigned to a facility as a primary supplier. Constraints (4.27) ensure that each user only assigned to a facility as a backup supplier if there is no fortification. Constraints (4.28) ensure that each user only is not assigned to a facility as a backup supplier if there is fortification. In Constraints (4.29), the facility is fortified if it is selected to open. In Constraints (4.30), the facilities are fortified if the budget (B) limitation is satisfied. Constraints (4.31) and (4.32) ensure that the total number of facilities to open (primary and backup) are less than p. Constraints (4.33) control the total demand assigned to a particular facility as primary. Constraints (4.34) control the total demand assigned to a particular facility as backup. Constraints (4.35) ensure that the total amount of demand assign to a facility (either primary or backup) are less than the capacity of that facility.

4.4.3.1 Pricing Problem

We now write the pricing problem (PP_i) for each user i. Let $u_i^{(4.26)} \lesseqgtr 0$, $u_{ij}^{(4.27)} \leq 0$, $u_{ij}^{(4.28)} \geq 0$, $u_j^{(4.31)} \leq 0$, $u_j^{(4.33)} \leq 0$, $u_j^{(4.34)} \leq 0$, and be the values of dual variables associated with constraints (4.26), (4.27), (4.28), (4.31), (4.33),and (4.34) respectively.

$$
\begin{aligned}
\min \overline{\text{COST}_i^{\text{SHARED}}} = \text{COST}_i - \sum_{j \in J} u_i^{(4.26)} a_j^{\text{W}} - \sum_{j \in J} (u_{ij}^{(4.27)} a_j^{\text{W}} \\
+ \sum_{j' \in J : j \neq j'} u_{ij'}^{(4.27)} a_{j'}^{\text{B}}) - \sum_{j \in J} u_{ij}^{(4.27)} (a_j^{\text{B}} + a_j^{\text{W}}) - \sum_{i' \in I : i' \neq i} \sum_{j \in J} u_{ij'}^{(4.27)} a_j^{\text{B}} \\
- \sum_{j \in J} (u_{ij}^{(4.28)} a_j^{\text{W}} - \sum_{j' \in J : j \neq j'} u_{ij'}^{(4.28)} a_{j'}^{\text{B}}) - \sum_{j \in J} u_{ij}^{(4.28)} (a_j^{\text{W}} - a_j^{\text{B}}) \\
- \sum_{j' \in J : j' \neq j} \sum_{j' \in J} u_{ij'}^{(4.28)} a_j^{\text{B}} - \sum_{j \in J} u_j^{(4.31)} (a_j^{\text{W}} + a_j^{\text{B}}) - \sum_{j \in J} u_j^{(4.33)} D_i a_j^{\text{W}} \\
- \sum_{j \in J} \sum_{j' \in J : j \neq j'} u_{jj'}^{(4.34)} D_i \alpha_{jj'}
\end{aligned}
$$

where

$$
\text{COST}_i = \sum_{\ell \in L} \text{LENGTH}_\ell (p_\ell^{\text{W}} + p_\ell^{\text{B}}).
$$

subject to:

Flow conservation constraints for Primary and backup routes for primary assignment:

$$
\sum_{\ell \in \omega^+(j)} p_\ell^{\text{W}} - \sum_{\ell \in \omega^-(j)} p_\ell^{\text{W}} = \begin{cases} 1 - a_j^{\text{W}} \text{ if } j = i \\ -a_j^{\text{W}} \text{ otherwise} \end{cases} \quad j \in J \tag{4.41}
$$

$$
\sum_{\ell \in \omega^+(j)} p_\ell^{\text{B}} - \sum_{\ell \in \omega^-(j)} p_\ell^{\text{B}} = \begin{cases} 1 - a_j^{\text{B}} \text{ if } j = i \\ -a_j^{\text{B}} \text{ otherwise} \end{cases} \quad j \in J \tag{4.42}
$$

$$
p_\ell^{\text{W}} + p_\ell^{\text{B}} \leq 1 \qquad \ell \in L \tag{4.43}
$$

Primary and backup assignment

$$a_j^{\mathrm{W}} + a_j^{\mathrm{B}} \leq 1 \qquad j \neq i, j \in J \tag{4.44}$$

$$\sum_{j \in J} a_j^{\mathrm{W}} = 1 \tag{4.45}$$

$$\sum_{j \in J} a_j^{\mathrm{B}} \leq 1 \tag{4.46}$$

$$a_j^{\mathrm{W}} + a_{j'}^{\mathrm{B}} - 1 \leq \alpha_{jj'} \qquad j, j' \in J : j \neq j' \tag{4.47}$$

$$\alpha_{jj'} \leq a_j^{\mathrm{W}} \qquad j, j' \in J : j \neq j' \tag{4.48}$$

$$\alpha_{jj'} \leq a_{j'}^{\mathrm{B}} \qquad j, j' \in J : j \neq j' \tag{4.49}$$

Decision variables

$$p_l^{\mathrm{W}}, p_l^{\mathrm{B}} \in \{0, 1\} \qquad l \in L \tag{4.50}$$

$$a_j^{\mathrm{W}}, a_j^{\mathrm{B}} \in \{0, 1\} \qquad j \in J \tag{4.51}$$

$$\alpha_{jj'} \in \{0, 1\} \qquad j, j' \in J : j \neq j' \tag{4.52}$$

Constraints (4.41) ensure that each user is connected to a facility as a primary supplier with a route

Constraint (4.42) ensure that there is a route between user and facility as a backup supplier if there is a backup facility

Constraint (4.43) control the disjoint links in order to have disjoint user-facility route for primary and backup facilities

Constraint (4.44) ensure that the primary and backup facility is not the same

Constraint (4.45) ensure that there is one and only one primary facility for the user

Constraint (4.46) control to have not more than one backup facility if needed

Constraints (4.47-4.49)added to problem to linirize the quadratic term in last part of objective function and replace $\alpha_{jj'}$ istead of $a_j^{\mathrm{W}} a_j^{\mathrm{B}}$

4.5 Heuristics Max_j

In this section we purpose a heuristic method based on outputs in each iteration. In continue we explain the method in detail. The algorithm for this heuristic method is described in Algorithm 4. There are five main steps for this algorithm. The steps are explained in sections 4.5.1, 4.5.2, 4.5.3, 4.5.4, and 4.5.5.

Algorithm 4 *Heuristics Max_j Column Generation*

Require: $Sub\_solution$ set, I (set of Users), J (set of Facilities)
Ensure: Finding facility Max_j & the best solution of heuristics $Z^*_{H\_Maxj} = \infty$

```
1: procedure (())
2:     iter = 0
3:     while column generation termination! do
4:         Z^{iter}_{H_Maxj} = 0
5:         Empty Sub_solution
6:         Solve Restricted Master Problem
7:         for i ⩽ I do
8:             Solve PP_i (Pricing Problem)
9:             Sub_solution(i) = Output(PP_i) (add the column result from PP_i
               toSub_solution(i)).
10:        end for
11:        Max_j finder() function
12:        if Sub_solution is a feasible solution then
13:            Fine the objective value and update Z^{iter}_{H_Maxj}
14:        else
15:            construction function()
16:            Fine the objective value and update Z^{iter}_{H_Maxj}
17:        end if
18:        if Z^{iter}_{H_Maxj} < Z^*_{H_Maxj} then Z^*_{H_Maxj} = Z^{iter}_{H_Maxj}
19:        end if
20:    end whileiter + +
21: end procedure
```

4.5.1 Collection of Subset of Solution

in Algorithm 4 line 7 to 10 we describe how to collect a sub set of solution as an input for heuristic model.

In each iteration of column generation after solving restricted master problem (RMP), based on duals from RMP we solve the pp_i for each $i \in I$ and the output of that pricing is a column in order to add to RMP. We keep also this column to a subset of solution for this heuristic method. We do this for each $i \in I$ and keep the output in that subset of solution (named $Sub\_solution$). Afte we solve pricing for all $i \in I$ in this iteration we have a subset of solution which is same as created columns in order to add to RMP in next iteration.

4.5.2 Finding Max_j Facility

in Algorithm 5 we explain how to find the Max_j. and this function is called in Algorithm 4 line 11.

Max_j facility is a facility that has most assigned users, among all facilities. In order to find the Max_j we go through all the column in $Sub\_solution$. The supplier j with most user i assigned to it, is the Max_j.

Algorithm 5 Max_j *finder*

Require: $Sub\_solution$ set, I (set of Users), J (set of Facilities), $J_{Counter}$ set
Ensure: Finding facility Max_j

procedure (())
    for $j \leqslant J$ **do**
        $j_{Counter}[j] = 0$
        Counting the number of i assigned to j in $Sub\_solution$ set.
        Update the $j_{Counter}[j]$ based on number of i assigned to j.
    end for
    for $j \leqslant J$ **do**
        if $j_{Counter}[j]$ ¿ $j_{Counter}[Max_j]$ **then**
            $Max_j = j$.
        end if
    end for
    Return Max_j
end procedure

4.5.3 Feasibility Check of Solutions

in Algorithm 4 line 12 to 16 we describe this step. In this step we check this subset of solution feasibility with check function (mixed integer linear programming MILP). If this subset of solution is feasible we have a solution and we calculate the $Z_{H\_Maxj}^{iter}$ of that iteration. If it is not a feasible solution we go to next step to construct a feasible solution (Section 4.5.4)

4.5.4 Constructing a Feasible Solution

In this section we construct a feasible solution by using columns in $Sub\_solution$. The construction function is called in Algorithm 4 line 16. The detail of this function is explained in Algorithm 6. Also the schematic steps of this construction function is shown in figure 4.3 . To explain construction steps we explain in detail for an small instance with 5 nodes (number of facility and users), $p = 3$, and number of facility to fortify is 1. the steps for this instance is shown in figure 4.2. In following we explain about the steps of construction function based on 5 nodes instance.

Step1- In first row of figure 4.2 we show a random out put of set of pp_i solution. We name this out put, Sub_solution set and it shown in first row of figure 4.2. in $Sub\_solution$ set we show for each $i \in I$ there is primary supplier and backup supplier (if there is no fortification. For instance user 0 is assigned to facility 1 as primary and user 0 is assigned to supplier 2 as backup. User 3 is assigned to facility 1 as primary facility and there is no backup for user 3. This $Sub\_solution$ is not feasible because $p = 5$

We find the Max_j by using the Algorithm 5. The $Max_j = 1$ It means $j = 1$ supply more user in this subset of solution.

Step2- After finding Max_j we separate $Sub\_solution$ to two sets $Sub\_configuration$ and $Temp\_configuration$. $Sub\_configuration$ include those columns which i assigned to Max_j, and $Temp\_configuration$ include those columns which i didn't assign to Max_j.

As it is shown in second row of figure 4.2 facility 1 is fortified already because there is no backup for user 3 then we fortify facility 1 and we remove backup for other users that connected to this facility.

Step3- In this row of figure 4.2 in $Sub\_configuration$ the users 0,1, and 3 are connected to facility 1 and facility 1 is fortified since there is no back up for all users connected to this facility. Now we start adding columns from $Temp\_configuration$ to $Sub\_configuration$.

Step4- by adding column related to user 2 from $Temp\_configuration$ to $Sub\_configuration$, $p \leq 3$, capacities are supporting and number of fortification is ≤ 1. We add another column from $Temp\_configuration$ to $Sub\_configuration$.(column related to user 4)

Step5- by adding column related to user 4 from $Temp\_configuration$ to $Sub\_configuration$, will effect on $p(p = 5)$. To solve this problem we have to construct this column based on other existing column in $Sub\_configuration$. There is no enough capacity for facility 1 to assign more user to it. There are two option facility 0 and 3. Facility 3 is closest to user 4 and there is enough capacity to add more user. We assign user 4 to supplier 3.

Step6- Facility 3 is not fortified and we need to have a backup for user 4. The closes facility with enough capacity is facility 0. Then we assign user 4 to supplier 0 as backup supplier.

Step7- There is no more column to add and $Sub\_configuration$ is a feasible solution

Step8- based on feasible $Sub\_configuration$ we calculate $Z^{iter}_{H\_Maxj}$.

4.5.5 Finding $Z^{best}_{H\_Maxj}$

in Algorithm 4 line 17 we describe this step. In this step we find best or minimum $Z^{iter}_{H\_Maxj}$ among all outputs of heuristics. compare $Z^{iter}_{H\_Maxj}$ with the $Z^{best}_{H\_Maxj}$, for $(iter \in Ierations)$ if $(Z^{iter}_{H\_Maxj}$ ¡ $Z^{best}_{H\_Maxj})$ then $Z^{best}_{H\_Maxj} = Z_{H\_Maxj}$.

4.6 Rounding off

To control lower bound of column generation, in each iteration of column generation, we decide about one of the facility to be opened. In continue we explain how we decide about one of facilities to open based on previous iteration. First we decide about Max_j in section 4.6.1 and then in section 4.6.2 we add the Max_j to master problem

Algorithm 6 *Construction function*

Require: $Sub\_solution$ set,$Sub\_configuration$ set,$Temp\_configuration$ set, I (set of Users), J (set of Facilities), $J_{Counter}$, Capacity, Fortification budget, p (number of facility to open)

Ensure: Constructing a set of feasible solution

procedure (())
  for $c \leqslant Sub\_solution.length$ **do**
    if Sub_solution[c][Max_j]=1 (i assigned to Max_j) **then**
      $Sub\_configuration[c] = Sub\_solution[c]$ (keep this column for a feasible solution)
    else
      $Temp\_configuration[c] = Sub\_solution[c]$ (keep this column for constructing a feasible solution)
    end if
  end for
  for $f \leqslant Temp\_configuration.length$ **do**
    Add $Temp\_configuration[f]$ to $Sub\_configuration[f]$
    Check feasibility constraints (Capacity, Fortification budget, p)
    if $Sub\_configuration$ is not feasible **then**
      Assign the user in $Sub\_configuration[f]$ to closest facility which make $Sub\_configuration$ feasible
    end if
  end for
end procedure

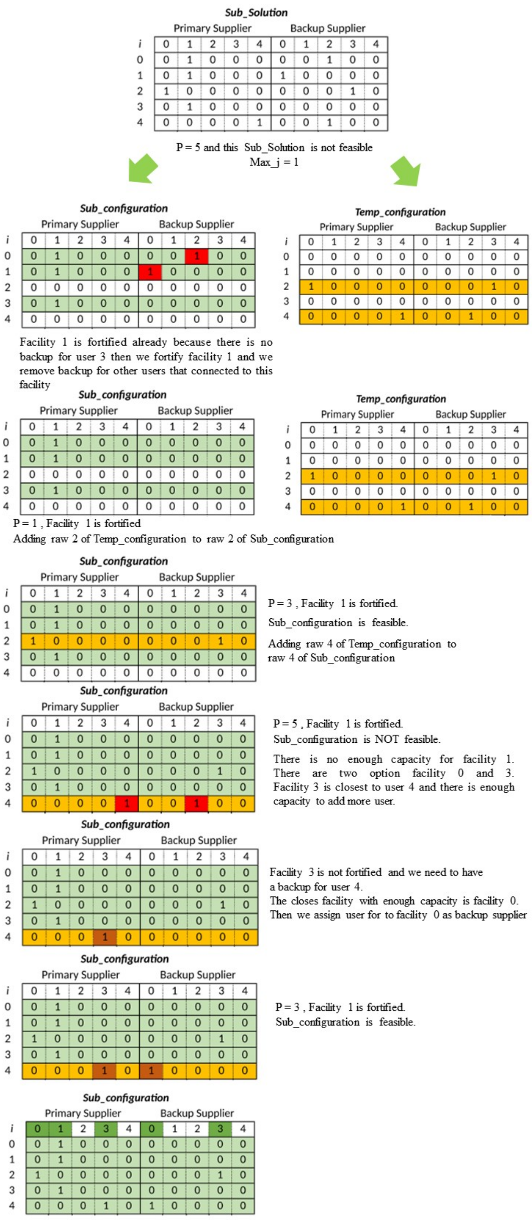
4.6.1 Finding the Max_j to Master Problem

In each iteration of column generation we find the facility with most users connected to it. We call this facility Max_j. The steps to find the Max_j is explained in the Algorithm 5. After finding the Max_j we add it to master problem as explained in next section (section 4.6.2).

4.6.2 Adding Max_j to Master Problem

In each iteration in heuristic method the founded Max_j can be an option to be open and fortified for next iteration in restricted master problem(RMP). In this case we add constraints to RMP in order to ensure the facility Max_j will be open and fortified. In

FIGURE 4.2: Heuristics Primary and Backup Assignment Construction Function Instance (Node:5, P:3, Fortify: 1)



this case we add these constraints to RMP:

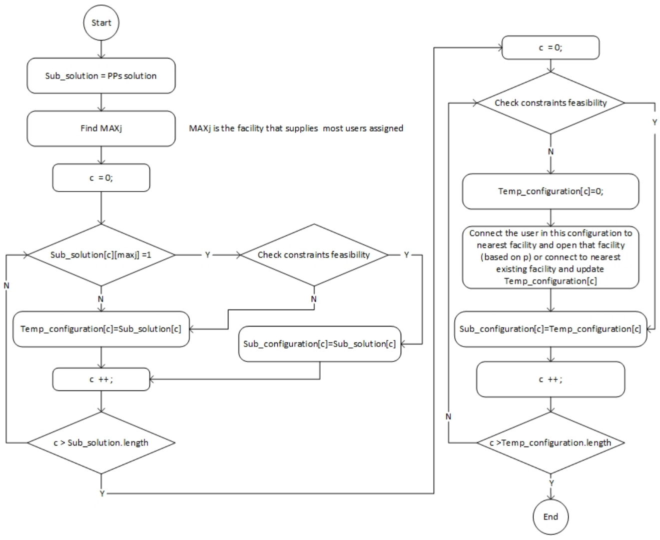
$$y_{Max_j} = 1 \tag{4.53}$$

$$x_{Max_j} = 1 \tag{4.54}$$

Constraint (4.53) ensure that facility Max_j is opened.

Constraint (4.54) ensure facility Max_j is fortified.

FIGURE 4.3: Heuristics Primary and Backup Assignment Construction Function



since there is constraint (4.29) to ensure fortified facility should be opened, we can eliminate constraint (4.53) and only add constraint (4.54). We can have only the constraint (4.54).

In this method, in each iteration we remove selected Max_j from candidate facilities for Max_j in next iteration, to eliminate the repetition the the same solution but we always make sure that there is enough candidate to be Max_j.

4.7 Pricing Heuristics

According to solutions and results for regular column generation, the calculation time is very high in large scale test-cases (88, 149 nodes). The most time consuming process in column generation is pricing solution process which it is very high. To improve the processing time we have to decrease the pricing calculation time. To do so we proposed a heuristic method for pricing problem.

4.7.1 Objective Function

In this method, for each user we decide about facility to open based on dual values. In this case we calculate the dual values related to each facility in each iteration.

We can write the pricing problem objective function of section 4.4.3.1 in this way:

$$\min \overline{\text{COST}_i^{\text{SHARED}}} = \text{COST}_i + \sum_{j\in J}(DUALCOST_j^W \times a_j^{\text{W}}) \\ + \sum_{j\in J}(DUALCOST_j^B \times a_j^{\text{B}}) + \sum_{j\in J}(DUALCOST_j^C \times \sum_{j'\in J:j\neq j'} \alpha_{jj'})$$

where

$$DUALCOST_j^{\text{W}} = -u_i^{(4.26)} - 2u_{ij}^{(4.27)} - 2u_{ij}^{(4.28)} - u_j^{(4.31)} - D_i u_j^{(4.33)}$$
$$DUALCOST_j^{\text{B}} = -u_{ij}^{(4.27)} - 2\sum_{j'\in J:j\neq j'} u_{ij'}^{(4.27)} - u_{ij}^{(4.28)} - 2\sum_{j'\in J:j\neq j'} u_{ij'}^{(4.28)} \\ - 2\sum_{j'\in J:j'\neq j} u_{ij'} + u_j^{(4.31)} - u_j^{(4.33)}$$
$$DUALCOST_j^{c} = \sum_{j'\in J:j\neq j'} u_{jj'}^{(4.34)} D_i$$

Since the value of $DUALCOST_j^C$ is very small and close to 0 we can eliminate this part.

4.7.2 $a_j^{\mathbf{w}}$ and $a_j^{\mathbf{b}}$ Decision

In each iteration for each user i we decide about primary and backup facility to connect. To decide about a_j^{W} to open, we check first to see is there remaining capacity to assign user. Then we find the smallest value of $DUALCOST_j^{\text{W}}$ among those facility which they sttil have capacity. For the backup facility also we follow the same way for primary but we check that the primary and back up is not the same. To decide about a_j^{B} to open we find the smallest value of $DUALCOST_j^{\text{B}}$.

4.7.3 Pricing Heuristics Algorithm

The algorithm of column generation with this pricing is shown in figure 4.4. In this algorithm we follow steps below:

step1 Solving Restricted Master Problem (RMP).

step2 Solving heuristic Pricing Problem (PP_i)for all users (round-robin).

step3 For each PP_i check if the reduced cost is negative go to step 4.

step4 Adding column to RMP

step5 If reduced cost for all PP_is from heuristic Pricing Problem (PP_i) is positive go to step6 otherwise go to step 1.

step6 Solving MIP Pricing Problem (PP_i)for each users and check if the reduced cost is positive go to next pricing. But if the reduced cost is negative for that PP_i go to step4

step7 If reduced cost for all PP_is from MIP Pricing Problem (PP_i) is positive go to step8.

step8 Solve the MIP restrict master problem to find the ILP solution.

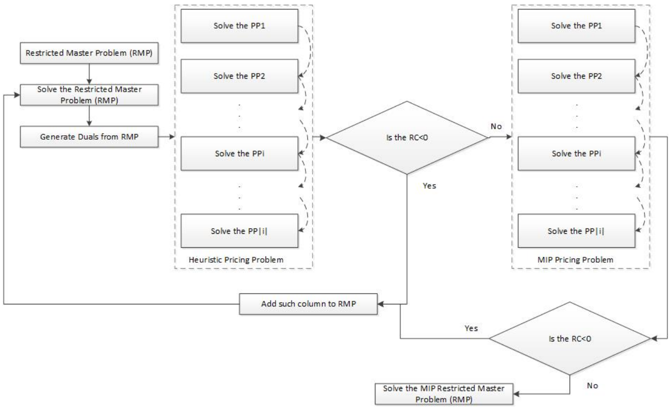
In step 6 we solve the ILP of pricing (section 4.4.3.1) to check if there is any more column or we stop the CG algorithm. In this case if even there is a pricing with a negative reduced cost we need to add more column to CG. that is the reason not to solve more ILP pricing problem and we get back to algorithm.

4.8 Numerical Results

4.8.1 Data Sets and Parameters

Models and algorithms proposed in the previous sections were tested on two data sets taken from [24], which can be found in the online appendix of [43], with 49, 88 and 150

FIGURE 4.4: Column Generation Algorithm Using a PP Heuristic Method



users, with $m = n$, together with their demand and distance values. We assume that the transportation cost is proportional to the Euclidean distances. We generated the location capacity values as follows. Let $\overline{D} = \sum_{i \in I} D_i/p$ be the average demand per facility location (under the assumption there are p facilities and load is balanced among the facilities), where the demand values are taken from [24] p=5, 10, 20,30 and 40. Then, for each potential facility location j, we computed the Q_j capacity value as a randomly generated value in the interval $[2\overline{D}, 2.2\overline{D}]$. The fixed cost s_j based on [24] are drown from $U \sim [500, 1500]$ and rounded to nearest integer. Following [4], the hardening cost is set as follows: $r_j = 0.2 \times s_j$. q_j the probability of failure is generated randomly by uniform distribution $U \sim [0, 0.05]$.

The budget for fortification is calculated as below: $\overline{B} = \sum_{j \in J}(s_j + r_j q_j)$ and the budget depends on percentage of opened facilities we decide to fortify (f is the percentage of facility we decide to fortify). In this case p is number of facilities to open and the budget is: $B = \dfrac{\overline{B}}{f \times p}$.

4.8.1.1 Decision on Edges in the Existing Graph

We considered different degrees of graph and we solved the problem for three different degrees. since we have the primary and back routes for those users connected to non-fortified facility, to consider the least value of degree we should make sure that the graph is connected and there are enough edges to connect users and facility node where the routes are disjoint and there is at least one route between selected user and facility.

Then we increase the degree value. For instance for test case 49 nodes the degrees are 8, 10, 12. In this case if the degree is less than 8 there are some missing routes between nodes.

4.8.1.2 Initial Solution

To process the solution for Column Generation we need to create an initial solution (Initial Columns) with feasible solution. to have better initial solution in terms of facility- user assignment we used the results from our previous chapter. In this case we have the columns for user-facility assignment in order to have shared backup facility and facility fortification but the results are based on the Euclidean distance and also we took care of disruption on facility only not routes. In this paper we are looking into the user-facility with shared backup facility and also routes between user and primary facility and backup facility to protect the network from disruption on facilities and routes. In order to take care of LRP under disruption, we find the shortest path between (based on integer linear programming) and add to column generation as initial column. To start column generation we have to set of columns to add. First one is user-facility assignment solutions which contained a_j^{W} and a_j^{B} values as input to restricted master problem. Second set of columns are routes between the users and primary/backup facilities which contain p_ℓ^{W} and p_ℓ^{B} values as input to restricted master problem.

4.8.2 Model Accuracy

We first report on the computational times and the accuracies of the solutions. There are three test cases with 49, 88, 150 nodes. In each test case, we analyse the solution with different values for p (for 49 nodes instance the $p = 5, 10, 20$), fortification degree (in 49 nodes instance; 3,4,6,11), different graph degree (in 49 nodes instance; 8,10,12). Results are summarized in Table 4.2, Table 4.3 and Table 4.4.Table 4.2 shows the results for 49 nodes, Table 4.3 shows the results for 88 nodes and Table 4.4 shows the results for 150 nodes.

In Tables 4.2, 4.3 and 4.4, in column two the name of test case is shown. The first two digits is CG and it stands for column generation. The second two digits is for number of nodes. The third two digits is p value. The fourth two digits is fortification degree and the fifth two digits is graph degree value. As it is shown below:

$$(CG\text{-}\ \#\ of\ nodes\ \text{-}\ p\ \text{-}\ fortification\ \text{-}\ degree)$$

In third column $z^{\star}_{\text{LP-CG}}$ is linear optimal solution for this problem in column generation. The $z^{\star}_{\text{ILP-CPLEX}}$ is the integer optimal solution for this problem. The fifth column shows the number of column generated through the column generation solution process for each test case. The gap calculated based on this formula: $\frac{z^{\star}_{\text{ILP-CPLEX}} - z^{\star}_{\text{LP-CG}}}{z^{\star}_{\text{ILP-CPLEX}}}\%$ and shown in sixth column. The CPU usage (time to solve) is shown in last column.

As it is shown the optimal value (transportation cost) either LP or ILP is decreasing when the p increasing in all cases. According to column five the gap is decreasing when the p is increasing. Based on the results in column five with increasing the p and the degree the number of generated column in column generation is increasing and it effects on CPU time. With increasing the fortification in The gap is increasing and effects to improvement of gap.

Table 4.2: Model Accuracy of Test Cases for 49 Nodes with Different Fortifications and Different Graph Degree

	Test case name	Z^*_{LP-CG}	$Z^*_{ILP-cplex}$	# column	Gap%	CPU (sec)
1	CG-49-05-03-08	321.2	393.0	429	18.27%	1,354.2
2	CG-49-05-03-10	310.5	378.5	418	17.97%	1,315.1
3	CG-49-05-03-12	289.2	351.1	412	17.63%	1,302.3
4	CG-49-05-04-08	309.5	385.1	401	19.63%	1,279.9
5	CG-49-05-04-10	305.6	371.2	393	17.67%	1,261.2
6	CG-49-05-04-12	293.2	352.8	390	16.89%	1,254.3
7	CG-49-10-04-08	274.8	335.3	541	18.04%	1,715.5
8	CG-49-10-04-10	263.2	315.7	534	16.63%	1,685.9
9	CG-49-10-04-12	253.5	300.1	532	15.53%	1,676.2
10	CG-49-10-06-08	215.1	262.3	502	17.99%	1,536.8
11	CG-49-10-06-10	208.4	249.7	496	16.54%	1,501.3
12	CG-49-10-06-12	207.5	245.5	496	15.48%	1,498.2
13	CG-49-20-06-08	189.5	227.7	625	16.78%	1,951.6
14	CG-49-20-06-10	182.2	217.5	605	16.23%	1,820.8
15	CG-49-20-06-12	174.4	207.1	601	15.79%	1,815.2
16	CG-49-20-11-08	138.5	165.4	586	16.26%	1,769.2
17	CG-49-20-11-10	134.6	159.4	575	15.56%	1,745.7
18	CG-49-20-11-12	130.1	151.8	570	14.30%	1,739.2

Table 4.3: Model Accuracy of Test Cases for 88 Nodes with Different Fortifications and Different Graph Degree

	Test case name	Z^*_{LP-CG}	$Z^*_{ILP-cplex}$	# column	Gap%	CPU (sec)
1	CG-88-05-03-08	626.5	804.7	688	22.14%	1,981.5
2	CG-88-05-03-10	607.5	776.4	607	21.75%	1,834.6
3	CG-88-05-03-12	608.8	772.4	607	21.18%	1,829.9
4	CG-88-05-04-08	574.5	729.3	610	21.23%	1,876.4
5	CG-88-05-04-10	571.2	711.6	526	19.73%	1,728.3
6	CG-88-05-04-12	570.8	701.8	525	18.67%	1,715.9
7	CG-88-10-04-08	627.6	806.4	761	22.17%	2,078.8
8	CG-88-10-04-10	614.8	771.7	721	20.33%	1,853.2
9	CG-88-10-04-12	599.5	722.6	703	17.04%	1,806.9
10	CG-88-10-06-08	461.4	578.7	689	20.27%	1,789.6
11	CG-88-10-06-10	451.8	556.3	601	18.78%	1,705.2
12	CG-88-10-06-12	443.4	525.6	598	15.64%	1,698.4
13	CG-88-20-06-08	424.2	528.8	918	19.78%	2,205.8
14	CG-88-20-06-10	373.2	456.8	850	18.30%	2,153.6
15	CG-88-20-06-12	368.9	435.8	850	15.35%	2,148.1
16	CG-88-20-11-08	315.1	389.7	852	19.14%	2,178.4
17	CG-88-20-11-10	280.6	338.4	792	17.08%	2,098.5
18	CG-88-20-11-12	276.5	328.3	783	15.78%	2,084.5

Table 4.4: Model Accuracy of Test Cases for 150 Nodes with Different Fortifications and Different Graph Degree

	Test case name	Z^*_{LP-CG}	$Z^*_{ILP-cplex}$	# column	Gap%	CPU (sec)
1	CG-150-20-06-08	789.5	1064.2	1,395	25.81%	3337.1
2	CG-150-20-06-10	768.5	1036.1	1,263	25.83%	3225.6
3	CG-150-20-06-12	759.9	967.5	1,138	21.46%	3105.8
4	CG-150-20-11-08	545.7	710.6	1,272	23.21%	3258.5
5	CG-150-20-11-10	532.8	684.1	1,123	22.12%	3085.3
6	CG-150-20-11-12	495.6	634.6	1,013	21.90%	2569.1
7	CG-150-30-11-08	438.5	572.9	1,512	23.46%	3585.6
8	CG-150-30-11-10	405.5	522.6	1,389	22.41%	3339.8
9	CG-150-30-11-12	387.1	479.2	1,389	19.22%	3312.9
10	CG-150-30-16-08	459.6	576.2	1,380	20.24%	3288.7
11	CG-150-30-16-10	407.3	504.6	1,243	19.28%	3212.8
12	CG-150-30-16-12	379.5	457.0	1,120	16.96%	3085.6
13	CG-150-40-16-08	385.5	486.8	1,675	20.81%	3677.6
14	CG-150-40-16-10	359.8	445.6	1,444	19.25%	3421.3
15	CG-150-40-16-12	348.5	428.1	1,329	18.59%	3298.2
16	CG-150-40-21-08	328.6	405.8	1,433	19.02%	3412.8
17	CG-150-40-21-10	312.5	384.1	1,319	18.64%	3275.1
18	CG-150-40-21-12	295.6	360.8	1,205	18.07%	3188.9

4.8.3 The Effect of Two Method on Gap Improvement

In this section we analyze the different solution for test case with 49 nodes with different criteria. Comparison of optimal solution between regular column generation and H_{Max_j} heuristics and Rounding off Max_j. In table 4.5 We summarize the results for test case 49. Second column shows the name of test cases, The third column shows the gaps for the $ILP-cplex$ solutions, The fourth column represents the gaps for the H_{Max_j} method (Method has been explained in section 4.5) and the last column shows the gaps when the Rounding off and H_{Max_j} methods applied in column generation.(Section 4.5 and 4.6). The results show that by using the H_{Max_j} alone there is no high improvement but when we add Rounding off method there are high impact on gaps.

Table 4.5: Gap Analysis for Three Methods: Cplex, H_{Max_j} and Rounding off

	Test case name	Gap %		
		Cplex	H_{Max_j}	H_{R+Max_j}
1	CG-49-05-03-08	18.27%	15.85%	12.19%
2	CG-49-05-03-10	17.97%	14.26%	11.94%
3	CG-49-05-03-12	17.63%	14.07%	11.67%
4	CG-49-05-04-08	19.63%	13.90%	10.83%
5	CG-49-05-04-10	17.67%	13.88%	10.43%
6	CG-49-05-04-12	16.89%	13.57%	10.36%
7	CG-49-10-04-08	18.04%	13.19%	9.93%
8	CG-49-10-04-10	16.63%	13.16%	9.74%
9	CG-49-10-04-12	15.53%	12.77%	9.56%
10	CG-49-10-06-08	17.99%	12.64%	8.19%
11	CG-49-10-06-10	16.54%	12.59%	8.11%
12	CG-49-10-06-12	15.48%	11.87%	8.10%
13	CG-49-20-06-08	16.78%	12.63%	7.92%
14	CG-49-20-06-10	16.23%	11.42%	7.84%
15	CG-49-20-06-12	15.79%	11.08%	7.53%
16	CG-49-20-11-08	16.26%	11.61%	7.42%
17	CG-49-20-11-10	15.56%	11.35%	7.36%
18	CG-49-20-11-12	14.30%	10.54%	7.34%

4.8.4 Heuristics Column Generation with Rounding off and H_{Max_j} Heuristics

In this section we calculate the heuristic method that combines rounding off and H_{Max_j} heuristics for all three test cases. Tables 4.6, 4.7, and 4.8 represent the results for combination of these two method.

In tables 4.6, 4.7, and 4.8 the second column represents the test case name, third column shows the LP solution and the fourth column shows the $ILP - cplex$ solution, and The third column shows $ILP - cplex$ gap. The fifth column presents the $Z^*_{H_{R+Max_j}}$ which is the ILP solution with using two method rounding off and H_{Max_j} heuristics. The last column shows the gap for $Z^*_{H_{R+Max_j}}$ solution.

Table 4.6: Solution Comparison of $ILP-cplex$ and $Z^*_{H_{R+Max_j}}$ for 49 Nodes

	Test case name	Z^*_{LP-CG}	$Z^*_{ILP-cplex}$	$Gap_{ILP-cplex}\%$	$Z^*_{H_{R+Max_j}}$	$Gap_{H_{R+Max_j}}\%$
1	CG-49-05-03-08	321.2	393.0	18.27%	365.8	12.19%
2	CG-49-05-03-10	310.5	378.5	17.97%	352.6	11.94%
3	CG-49-05-03-12	289.2	351.1	17.63%	327.4	11.67%
4	CG-49-05-04-08	309.5	385.1	19.63%	347.1	10.83%
5	CG-49-05-04-10	305.6	371.2	17.67%	341.2	10.43%
6	CG-49-05-04-12	293.2	352.8	16.89%	327.1	10.36%
7	CG-49-10-04-08	274.8	335.3	18.04%	305.1	9.93%
8	CG-49-10-04-10	263.2	315.7	16.63%	291.6	9.74%
9	CG-49-10-04-12	253.5	300.1	15.53%	280.3	9.56%
10	CG-49-10-06-08	215.1	262.3	17.99%	234.3	8.19%
11	CG-49-10-06-10	208.4	249.7	16.54%	226.8	8.11%
12	CG-49-10-06-12	207.5	245.5	15.48%	225.8	8.10%
13	CG-49-20-06-08	189.5	227.7	16.78%	205.8	7.92%
14	CG-49-20-06-10	182.2	217.5	16.23%	197.7	7.84%
15	CG-49-20-06-12	174.4	207.1	15.79%	188.6	7.53%
16	CG-49-20-11-08	138.5	165.4	16.26%	149.6	7.42%
17	CG-49-20-11-10	134.6	159.4	15.56%	145.3	7.36%
18	CG-49-20-11-12	130.1	151.8	14.30%	140.4	7.34%

4.8.5 Pricing Heuristic Solutions

In this section we show the results based on heuristics method for pricing in section 4.7. We solved this method for all three test cases 49, 88, 150 nodes. We solve test cases once by using only pricing heuristics in section 4.7that we call it PP_h. Then we combine this method (PP_h) with rounding off and H_{Max_j} That we call it $H_{PP_h+R+Max_j}$.

The tables 4.9, 4.10, and 4.11 summarize the results and compare the solution of column generation with PP_h and column generation with $H_{PP_h+R+Max_j}$.

The table 4.9 summarize the results for test case 49. The table 4.10 summarize the results for test case 88, and the table 4.11summarize the results for test case 150.

In the tables 4.9, 4.10, and 4.11, Second column states for test case name, third column shows the LP solution. The fourth and fifth columns show the ILP value and gap corresponding for the method PP_h (pricing heuristics). The sixth and seventh columns show

TABLE 4.7: Solution Comparison of $ILP - cplex$ and $Z^*_{H_{R+Max_j}}$ for 88 Nodes

	Test case name	Z^*_{LP-CG}	$Z^*_{ILP-cplex}$	$Gap_{ILP-cplex}\%$	$Z^*_{H_{R+Max_j}}$	$Gap_{H_{R+Max_j}}\%$
1	CG-88-05-03-08	626.5	804.7	22.14%	749.8	16.44%
2	CG-88-05-03-10	607.5	776.4	21.75%	718.8	15.48%
3	CG-88-05-03-12	608.8	772.4	21.18%	715.8	14.95%
4	CG-88-05-04-08	574.5	729.3	21.23%	663.2	13.37%
5	CG-88-05-04-10	571.2	711.6	19.73%	650.1	12.14%
6	CG-88-05-04-12	570.8	701.8	18.67%	648.8	12.02%
7	CG-88-10-04-08	627.6	806.4	22.17%	722.6	13.15%
8	CG-88-10-04-10	614.8	771.7	20.33%	706.6	12.99%
9	CG-88-10-04-12	599.5	722.6	17.04%	682.3	12.14%
10	CG-88-10-06-08	461.4	578.7	20.27%	529.6	12.88%
11	CG-88-10-06-10	451.8	556.3	18.78%	516.7	12.56%
12	CG-88-10-06-12	443.4	525.6	15.64%	502.7	11.80%
13	CG-88-20-06-08	424.2	528.8	19.78%	487.9	13.06%
14	CG-88-20-06-10	373.2	456.8	18.30%	424.2	12.02%
15	CG-88-20-06-12	368.9	435.8	15.35%	412.5	10.57%
16	CG-88-20-11-08	315.1	389.7	19.14%	358.4	12.08%
17	CG-88-20-11-10	280.6	338.4	17.08%	318.6	11.93%
18	CG-88-20-11-12	276.5	328.3	15.78%	309.7	10.72%

the ILP value and gap corresponding for the method $H_{PP_h+R+Max_j}$ (pricing heuristics + rounding off and H_{Max_j}).The last column shows the processing time.

4.9 Conclusion and Future Work

The different methods are implemented for column generation in reliable capacitated LRP in presence of Disruption. The results shows the regular column generation has high gap and long processing time. to improve these issues we implemented three heuristics methods.

For future work two pricing decomposition can improve the solution and processing time. In this case one pricing for the decision on which facility to open and second on for rout decision between users and facilities. Also different level of disruption can be considered. For instance we can consider traffics as a partial disruption on routes. Also

TABLE 4.8: Solution Comparison of $ILP - cplex$ and $Z^*_{H_{R+Max_j}}$ for 150 Nodes

	Test case name	Z^*_{LP-CG}	$Z^*_{ILP-cplex}$	$Gap_{ILP-cplex}\%$	$Z^*_{H_{R+Max_j}}$	$Gap_{H_{R+Max_j}}\%$
1	CG-150-20-06-08	789.5	1,064.2	25.81%	942.7	16.25%
2	CG-150-20-06-10	768.5	1,036.1	25.83%	914.2	15.94%
3	CG-150-20-06-12	759.9	967.5	21.46%	903.3	15.88%
4	CG-150-20-11-08	545.7	710.6	23.21%	644.6	15.34%
5	CG-150-20-11-10	532.8	684.1	22.12%	620.3	14.11%
6	CG-150-20-11-12	495.6	634.6	21.90%	575.4	13.87%
7	CG-150-30-11-08	438.5	572.9	23.46%	532.2	16.09%
8	CG-150-30-11-10	405.5	522.6	22.41%	475.1	14.65%
9	CG-150-30-11-12	387.1	479.2	19.22%	448.3	13.65%
10	CG-150-30-16-08	459.6	576.2	20.24%	522.6	13.64%
11	CG-150-30-16-10	407.3	504.6	19.28%	460.8	11.61%
12	CG-150-30-16-12	379.5	457.0	16.96%	426.7	11.06%
13	CG-150-40-16-08	385.5	486.8	20.81%	454.5	15.18%
14	CG-150-40-16-10	359.8	445.6	19.25%	419.5	14.23%
15	CG-150-40-16-12	348.5	428.1	18.59%	399.2	12.70%
16	CG-150-40-21-08	328.6	405.8	19.02%	378.5	13.18%
17	CG-150-40-21-10	312.5	384.1	18.64%	358.8	12.90%
18	CG-150-40-21-12	295.6	360.8	18.07%	335.4	11.87%

considering fortification on routes can be considered that it protect the route on events of disruption.

TABLE 4.9: Solution Comparison of CG with PP_h and CG with $H_{PP_h+R+Max_j}$ for 49 Nodes

	Test case name	Z^*_{LP-CG}	$Z^*_{ILP-PP_h}$	Gap%	$Z^*_{ILP-H_{PP_h+R+Max_j}}$	Gap%	CPU (sec)
1	CG-49-05-03-08	321.2	399.8	19.66%	367.5	12.60%	41.1
2	CG-49-05-03-10	310.5	380.3	18.35%	354.6	12.44%	45.6
3	CG-49-05-03-12	289.2	356.9	18.97%	329.6	12.26%	49.4
4	CG-49-05-04-08	309.5	386.2	19.86%	351.5	11.95%	53.8
5	CG-49-05-04-10	305.6	371.2	17.67%	345.6	11.57%	55.6
6	CG-49-05-04-12	293.2	361.7	18.94%	330.3	11.23%	60.8
7	CG-49-10-04-08	274.8	339.8	19.13%	308.1	10.81%	69.7
8	CG-49-10-04-10	263.2	324.6	18.92%	293.6	10.35%	72.4
9	CG-49-10-04-12	253.5	308.9	17.93%	281.4	9.91%	76.7
10	CG-49-10-06-08	215.1	263.9	18.49%	238.8	9.92%	85.4
11	CG-49-10-06-10	208.4	251.6	17.17%	231.2	9.86%	88.6
12	CG-49-10-06-12	207.5	248.8	16.60%	229.3	9.51%	90.8
13	CG-49-20-06-08	189.5	231.2	19.24%	208.3	9.03%	105.3
14	CG-49-20-06-10	182.2	221.3	17.67%	199.6	8.72%	114.9
15	CG-49-20-06-12	174.4	211.5	17.54%	190.9	8.64%	117.1
16	CG-49-20-11-08	138.5	170.5	18.77%	151.6	8.64%	121.8
17	CG-49-20-11-10	134.6	165.3	18.57%	147.3	8.62%	125.7
18	CG-49-20-11-12	130.1	159.4	18.38%	142.1	8.44%	128.9

Table 4.10: Solution Comparison of CG with PP_h and CG with $H_{PP_h+R+Max_j}$ for 88 Nodes

	Test case name	Z^*_{LP-CG}	$Z^*_{ILP-PP_h}$	Gap%	$Z^*_{ILP-H_{PP_h+R+Max_j}}$	Gap%	CPU (sec)
1	CG-88-05-03-08	626.5	814.5	23.08%	753.3	16.83%	185.6
2	CG-88-05-03-10	607.5	786.5	22.76%	724.6	16.16%	194.6
3	CG-88-05-03-12	608.8	782.4	22.19%	717.1	15.10%	215.6
4	CG-88-05-04-08	574.5	738.2	22.18%	672.8	14.61%	232.3
5	CG-88-05-04-10	571.2	716.6	20.29%	658.9	13.31%	256.4
6	CG-88-05-04-12	570.8	711.9	19.82%	651.9	12.44%	272.9
7	CG-88-10-04-08	627.6	819.6	23.43%	726.9	13.66%	285.6
8	CG-88-10-04-10	614.8	782.6	21.44%	711.6	13.60%	292.6
9	CG-88-10-04-12	599.5	731.6	18.06%	686.7	12.70%	307.9
10	CG-88-10-06-08	461.4	583.9	20.98%	532.4	13.34%	344.1
11	CG-88-10-06-10	451.8	561.5	19.54%	519.7	13.07%	425.1
12	CG-88-10-06-12	443.4	532.6	16.75%	506.8	12.51%	465.2
13	CG-88-20-06-08	424.2	536.8	20.98%	490.6	13.53%	523.7
14	CG-88-20-06-10	373.2	462.2	19.26%	428.5	12.91%	566.5
15	CG-88-20-06-12	368.9	441.8	16.50%	418.9	11.94%	582.9
16	CG-88-20-11-08	315.1	397.5	20.73%	364.1	13.46%	592.8
17	CG-88-20-11-10	280.6	346.8	19.09%	325.6	13.82%	617.5
18	CG-88-20-11-12	276.5	338.6	18.34%	312.6	11.55%	623.7

Table 4.11: Solution Comparison of CG with PP_h and CG with $H_{PP_h+R+Max_j}$ for 150 Nodes

	Test case name	Z^*_{LP-CG}	$Z^*_{ILP-PP_h}$	Gap%	$Z^*_{ILP-H_{PP_h+R+Max_j}}$	Gap%	CPU (sec)
1	CG-150-20-06-08	789.5	1117.9	29.38%	967.5	18.40%	761.4
2	CG-150-20-06-10	768.5	1082.5	29.01%	932.1	17.55%	780.5
3	CG-150-20-06-12	759.9	1054.3	27.92%	916.4	17.08%	824.6
4	CG-150-20-11-08	545.7	754.9	27.71%	655.4	16.74%	838.9
5	CG-150-20-11-10	532.8	725.8	26.59%	635.9	16.21%	859.6
6	CG-150-20-11-12	495.6	645.8	23.26%	583.4	15.05%	878.5
7	CG-150-30-11-08	438.5	585.9	25.16%	533.1	17.75%	901.2
8	CG-150-30-11-10	405.5	518.6	21.81%	482.9	16.03%	912.7
9	CG-150-30-11-12	387.1	488.7	20.79%	453.1	14.57%	925.8
10	CG-150-30-16-08	459.6	587.4	21.76%	527.7	12.91%	954.8
11	CG-150-30-16-10	407.3	518.2	21.40%	463.8	12.18%	967.8
12	CG-150-30-16-12	379.5	468.9	19.07%	431.2	11.99%	977.9
13	CG-150-40-16-08	385.5	504.6	23.60%	463.5	16.83%	988.6
14	CG-150-40-16-10	359.8	451.9	20.38%	427.1	15.76%	1,005.5
15	CG-150-40-16-12	348.5	436.2	20.11%	409.3	14.85%	1,014.7
16	CG-150-40-21-08	328.6	428.7	23.35%	382.6	14.11%	1,087.6
17	CG-150-40-21-10	312.5	390.8	20.04%	363.3	13.98%	1,115.5
18	CG-150-40-21-12	295.6	369.1	19.91%	339.7	12.98%	1,165.2

Chapter 5

Conclusions and Future Work

5.1 Conclusions

There are different components that should be considered in the reliable logistics network design stage. The main components of a logistics network are the supplier facility location to serve users and the route between the supplier facility and the customer users. To design a reliable logistics network, we must design a reliable suppler facility location and reliable routes between the users and the facilities.

A disruption event affects the reliability of the facility location and the routing. In presence of a disruption event, the supplier facility is unavailable to support the customer user's demand. A different method has been proposed in this thesis in order to design a reliable facility location, such as facility fortification or backup facility. Fortification is a high-cost facility protection against the disruption, and it requires a high budget to protect all facilities. In this case, by considering a backup facility for those non-fortified facilities, the logistics network is reliable. To consider the backup facility, the facility's capacity should be considered as a constraint.

In this thesis, we investigated the facility location under disruption and location and routing under disruption. We considered how disruption affects facility reliability and route availability. In this problem, a disruption event at the time has been considered.

In the case of disruption affecting the facility reliability, we considered the facility fortification for some facilities based on a limited budget. Then, we considered backup facility for the rest of the facilities. In this case, since we have one disruption occurring at the time and the capacity of the facilities are limited, we proposed a shared backup capacity model.

In the case of disruption affecting the facility and the route, we considered facility fortification and backup facility. In addition, we considered the backup disjoint route for backup facility in case of disruption on route.

In Chapter Two, we designed a decision model for the facility user assignment, and then we looked into the sharing backup capacity based on the primary assignment. After, we made a decision about the backup assignment based on the primary assignment. In this chapter, we proposed a new column generation decomposition. The columns are generated based on the facilities (j). Each column includes the users connected to a specific facility (j). Then, in the master problem, we decide which column should be selected to open facility correspond. In the sharing model, the primary assignments are known and each column includes the backup users assigned to a specific facility (j).

In Chapter Three, we proposed a new decomposition of facility location in order to decide on primary and backup assignment while considering backup capacity sharing. In this model, we modelled the column generation as the columns including the facility assignment to specific user(i). In this case, each column includes the primary and backup facility or supplier (if there are any.)

In Chapter Four, we looked into the location and routing problem (LRP). In this problem, we consider the disruptions on the facility and the route. There is only one disruption occurring at the time, either on the facility or the route. In this case, we consider the path between the facility and the user, instead of the Euclidean distance between the facility and the user. In this chapter, we consider the primary and the backup facility, alongside the primary and the backup route for each user-facility, to have a resilience logistics network.

5.2 Future Work

In this thesis, we focused on the facility location and the routing in event of disruption. We considered disruption to mean that the facility or the route will be unavailable. When a disaster occurs in the real world, the facility may become completely unavailable. But in most instances, like an electrical problem or a machine failure, the disruption is only a delay. Therefore, we consider it to be a partial disruption, rather than a complete disruption, and we can consider a percentage of availability for the facility. For instance, after a partial disruption, the facility is available with 30% availability. In future work for this problem, we can consider partial disruption and then model the problem by fortifying the facility or sharing backup.

In this thesis, we considered having one disruption event on facilities or routes, at the time, but it is possible to have several disruption (even complete or partial) on facilities and routes.

One of the main concerns about column generation with a high volume of decision variables is the processing time. We implemented the heuristics method to manage the processing time. Another way to manage the processing time is to model the problem as two pricing in column generation. In this way, we have a pricing and the decision variables for primary and backup facility assignment, and the second pricing is for deciding the routes between the user and the facility as primary and backup routes.

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