STATISTICAL AND DETERMINISTIC DYNAMICS OF MAPS WITH MEMORY

PAWEŁ GÓRA, ABRAHAM BOYARSKY, ZHENYANG LI, AND HARALD PROPPE

Abstract. We consider a dynamical system to have memory if it remembers the current state as well as the state before that. The dynamics is defined as follows: $x_{n+1} = T_\alpha(x_{n-1}, x_n) = \tau(\alpha \cdot x_n + (1-\alpha) \cdot x_{n-1})$, where τ is a one-dimensional map on $I = [0,1]$ and $0 < \alpha < 1$ determines how much memory is being used. T_α does not define a dynamical system since it maps $U = I \times I$ into I. In this note we let τ to be the symmetric tent map. We shall prove that for $0 < \alpha < 0.46$, the orbits of $\{x_n\}$ are described statistically by an absolutely continuous invariant measure (acim) in two dimensions. As α approaches 0.5 from below, that is, as we approach a balance between the memory state and the present state, the support of the acims become thinner until at $\alpha = 0.5$, all points have period 3 or eventually possess period 3. For $0.5 < \alpha < 0.75$, we have a global attractor: for all starting points in U except $(0,0)$, the orbits are attracted to the fixed point $(2/3, 2/3)$. At $\alpha = 0.75$, we have slightly more complicated periodic behavior.

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1. Introduction

In nonlinear discrete chaotic dynamical systems theory we study the statistical long term dynamics of iterated maps which depend only on the present state of the system. In this paper we consider dynamical systems which depend both on the present state as well as on one previous state. Such memory systems find applications in cellular automata and in modeling natural phenomena [1, 2].

Let τ be a piecewise, expanding map on I. We refer to it as the base map. At each step, the system remembers the current state x_n as well as one previous state x_{n-1}, which we refer to as the memory. Our dynamical system is defined by $x_{n+1} = T_\alpha(x_n) = \tau(\alpha \cdot x_n + (1-\alpha) \cdot x_{n-1})$, where $0 < \alpha < 1$ is a fixed number that specifies the ratio between the present state and the memory state. T_α does not define a dynamical system, since it is not a map of a space into itself. Rather, it denotes a process. To start a trajectory we need an initial point x_0 and its memory,

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which we consider to be bundled into the previous state x_{n-1}. When α is close to 0, the present state x_n is weighted down, acting as a perturbation on the memory state x_{n-1} which is dominant. However, when α is close to 1, the memory state is diminished and the resulting system behaves almost like a regular dynamical system, depending mostly on the present state.

In order to define an invariant measure for T_α, we consider the 2-dimensional map:

$$G_\alpha : [x_{n-1}, x_n] \mapsto [x_n, T_\alpha(x_n)] = [x_n, \tau(\alpha \cdot x_n + (1-\alpha) \cdot x_{n-1})] ,$$

i.e.,

$$G_\alpha(x, y) = [y, \tau(\alpha \cdot y + (1-\alpha) \cdot x)] .$$

The trajectory of G_α is:

$$[x_{-1}, x_0], [x_0, x_1], [x_1, x_2], [x_2, x_3], [x_3, x_4], \dots$$

If Π_1 is the projection on the first coordinate, we have

$$T^n_{\alpha, x_{-1}}(x_0) = \Pi_1(G^{n+1}_\alpha(x_{-1}, x_0)) \ , \ n = 1, 2, \dots ,$$

where $T_{\alpha, x_{-1}}$ means that the process T_α uses the initial history, x_{-1}.

Let us assume that G_α has an ergodic invariant measure ν_α on $\mathfrak{B}([0,1]^2)$. The measure ν_α defines a marginal measure μ_α on the first coordinate: $\mu_\alpha(A) = \nu_\alpha(A \times [0,1])$. In particular, if $\nu_\alpha = g_\alpha(x,y)dxdy$, i.e., an absolutely continuous measure with density $g_\alpha(x,y)$, then

$$\mu_\alpha = \left(\int_{[0,1]} g_\alpha(x,y) dy \right) dx$$

is also absolutely continuous with density $g_{\alpha,1}(x) = \int_{[0,1]} g_\alpha(x,y)dy$.

Since we assume that G_α is ν_α-ergodic, the Birkhoff Ergodic Theorem holds. Thus, for any integrable function f and almost every pair (x,y), we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(G^i_\alpha(x,y)) = \int f(x,y) d\nu_\alpha(x,y) .$$

If the function f depends only on the first coordinate, we can rewrite this as

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\Pi_1(G^i_\alpha(x,y))) = \int f(x) d\mu_\alpha(x) ,$$

that is,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i_\alpha(x_0, y_0)) = \int f(x) d\mu_\alpha(x) .$$

Since the limit is independent of the initial condition, the initial history x_{-1} used by T_α is irrelevant.

This shows that the marginal measure of the G_α-invariant measure determines the behavior of ergodic averages of trajectories of the process T_α. Thus, μ_α is a good candidate for an "invariant" measure of T_α.

In Section 2, we show that for certain α, G_α is expanding in both directions and establish the existence of an *acim* for the memory system defined by any piecewise expanding map τ. In Sections 3 – 6 we study the behavior of the memory system defined when the base map is the tent map τ. For $0 < \alpha < 0.46$, we prove the orbits of $\{x_n\}$ are described statistically by an acim. As α approaches 0.5 from below, that is, as we approach a balance between the memory state and the present state, the support of the acims become thinner until at $\alpha = 0.5$, all points have period 3 or eventually possess period 3. In Section 7, we consider $1/2 \leq \alpha \leq 3/4$. We prove that for $\alpha = 1/2$ all points (except two fixed points) are eventually periodic with period 3. For $\alpha = 3/4$ we prove that all points of the line $x + y = 4/3$ (except the fixed point) are of period 2 and all other points (except $(0, 0)$) are attracted to this line. For $1/2 < \alpha < 3/4$, we prove the existence of a global attractor: for all starting points in the square $[0, 1 \times [0, 1]$ except $(0, 0)$, the orbits are attracted to the fixed point $(2/3, 2/3)$.

Additional pictures illustrating the behaviour of the family G_α and Maple programs used in this study can be found at

http://www.mathstat.concordia.ca/faculty/pgora/G-map/.

2. Preliminary Results

In this section we show that for certain α, G_α is expanding in both directions.(We will ussually suppres the subscript α in the sequel.)

Let $\tau : I \to I$ be a piecewise expanding map is defined on the partition $\mathcal{P}$ with endpoints $\{a_0 = 0, a_1, a_2, \ldots, a_{q-1}, a_q = 1\}$. Let $I_i = [a_{i-1}, a_i]$, $i = 1, 2, \ldots, q$. Then, the map G_α is defined on the partition whose boundaries are the boundaries of the square $U = I^2$ and the lines

$$L_i^\alpha : y = \frac{a_i}{\alpha} - \frac{1-\alpha}{\alpha} x \ , \ i = 0, 2, \ldots, q \, .$$

Each of the lines L_0^α and L_q^α intersects $[0, 1]^2$ at only one point. Let R_i denote the region in $[0, 1]$ between the lines L_{i-1}^α and L_i^α, $i = 1, 2, \ldots, q$. The example for $\mathcal{P} = \{0, 0.25, 0.5, 0.8, 1\}$ is shown in Figure 1 a and the example for $\mathcal{P} = \{0, 0.5, 1\}$ is shown in Figure 1 b.

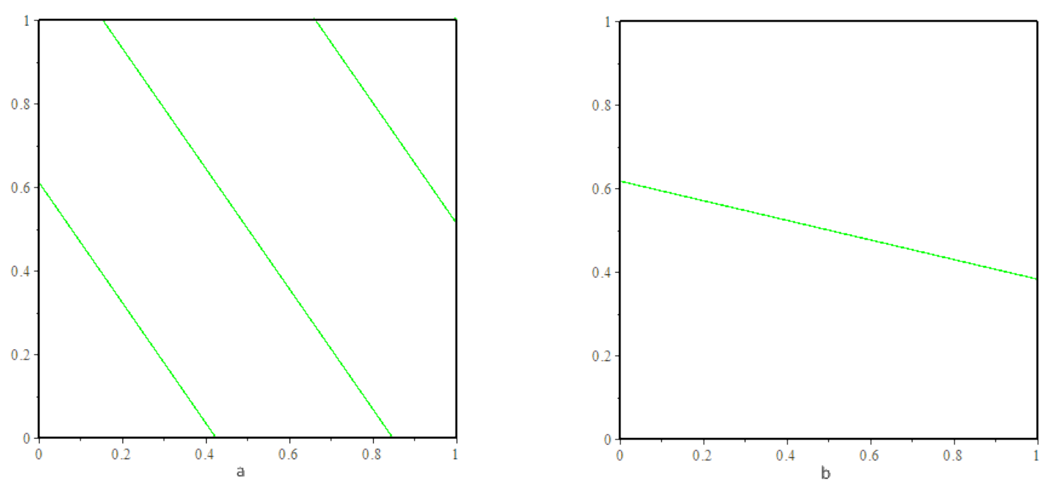

Figure 1. Examples of partitions for map G

Note that G_α is not piecewise expanding. However, we will show that G_α^2 is a piecewise expanding map for small values of α. The inverse branches of G_α^2

are of the form $(G_{\alpha,j} \circ G_{\alpha,k})^{-1} = G_{\alpha,k}^{-1} \circ G_{\alpha,j}^{-1}$. We have $D(G_{\alpha,k} \circ G_{\alpha,j})^{-1} = DG_{\alpha,j}^{-1} \circ G_{\alpha,k}^{-1} \cdot DG_{\alpha,k}^{-1}$. That is

$$DG_{\alpha,j}^{-1} \circ G_{\alpha,k}^{-1} \cdot DG_{\alpha,k}^{-1} = \begin{pmatrix} \frac{-\alpha}{1-\alpha} & \frac{1}{(1-\alpha)\tau_j'(\tau_j^{-1}(u))} \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{-\alpha}{1-\alpha} & \frac{1}{(1-\alpha)\tau_k'(\tau_k^{-1}(v))} \\ 1 & 0 \end{pmatrix}, \tag{2.1}$$

which is equal to

$$\begin{pmatrix} \frac{\alpha^2}{(1-\alpha)^2} + \frac{1}{(1-\alpha)\tau_j'(\tau_j^{-1}(u))} & \frac{-\alpha}{(1-\alpha)^2\tau_k'(\tau_k^{-1}(v))} \\ \frac{-\alpha}{1-\alpha} & \frac{1}{(1-\alpha)\tau_k'(\tau_k^{-1}(v))} \end{pmatrix}. \tag{2.2}$$

If α is chosen small enough, since τ is expanding, all the entries of the matrix can be made smaller than one (in absolute value), so the norm is smaller than one. This implies that G_α^2 is a piecewise expanding map. By [4] we have the existence of an acim.

One can immediately make the following observation.

Remark 1. *If $\alpha \approx 0$ (strong memory), then $G_\alpha(x,y) \approx (y, \tau(x))$ hence $G_\alpha^2(x,y) \approx (\tau(x), \tau(y))$, so G_α is likely to have an acim because τ has an acim and G_α is close to the product $\tau \times \tau$. On the other hand, if $\alpha \approx 1$ (weak memory), then $G_\alpha(x,y) \approx (y, \tau(y))$, which is independent of x and the orbit of any point $(x,y) \in U$ is approximately a subset of the graph of τ. In this case it is likely that there is an SRB measure, but that it is singular with respect to the 2D Lebesgue measure.*

We now show that in general G_α is not piecewise expanding. Suppose $\tau_j : I_j = (a_j, b_j) \to I$ is a monotonic branch of τ. Then G_α is piecewise monotonic on the strips $\{(x,y) : a_j < \alpha y + (1-\alpha)x < b_j\}$. If G_j is the branch of G_α corresponding to I_j, then the inverse of $G_{\alpha,j}$ is given by

$$G_{\alpha,j}^{-1}(u,v) = \left(\frac{\tau_j^{-1}(v) - \alpha u}{1-\alpha}, u \right) \tag{2.3}$$

Note that

$$D_{(u,v)} G_{\alpha,j}^{-1} = \begin{pmatrix} \frac{-\alpha}{1-\alpha} & \frac{1}{(1-\alpha)\tau_j'(\tau_j^{-1}(v))} \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

Such a matrix has Euclidean norm $\left\| DG_{\alpha,j}^{-1} \right\|_2 \geq 1$. Indeed, for a square matrix A, this norm is equal to $\sqrt{\lambda_{\max}(A^TA)}$, where $\lambda_{\max}(A^TA)$ denotes the maximum eigenvalue of the symmetric matrix A^TA. For us, A is given by (2.4) and is of the form

$$\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}. \tag{2.5}$$

Therefore, A^TA is of the form

$$\begin{pmatrix} 1+a^2 & ab \\ ab & b^2 \end{pmatrix}. \tag{2.6}$$

Note that the sum of the eigenvalues of a matrix is equal to the trace of the matrix, which for A^TA is $1 + a^2 + b^2$. This means that both eigenvalues cannot be smaller than 1. Therefore, $\lambda_{\max}(A^TA) \geq 1$, and G is not a piecewise expanding map (see [3], Remark 2.1 item 2) in the sense that all directions are contracted under the branches of the inverse of G.

3. τ IS THE SYMMETRIC TENT MAP.

In the sequel we study the dynamical system where the base map is

$$\tau(x) = \begin{cases} 2x & , \text{ for } \ 0 \le x < 1/2; \\ 2 - 2x & , \text{ for } \ 1/2 \le x \le 1. \end{cases}$$

and

$$G_\alpha(x, y) = (y, \tau(\alpha y + (1 - \alpha)x)).$$

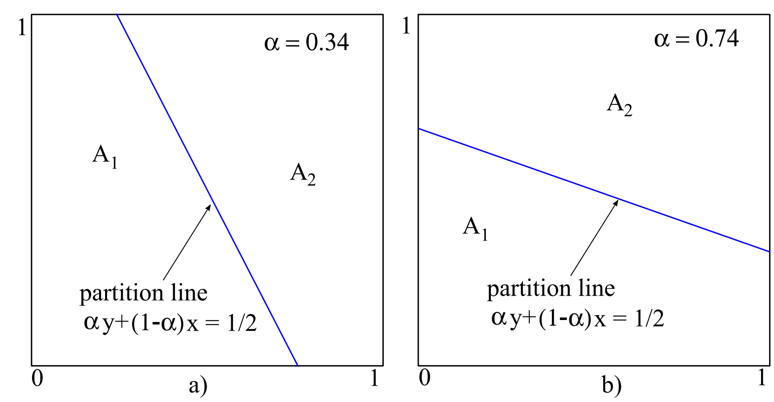

FIGURE 2. Partition into A_1 and A_2 for a) $\alpha = 0.34$ and b) $\alpha = 0.74$

4. CASE I: $0 \le \alpha < 1/2$.

Remark 2. *For $\alpha = 0$ we have*

$$(x, y) \xrightarrow{G} (y, \tau(x)) \xrightarrow{G} (\tau(x), \tau(y)),$$

so $G^2 = \tau \times \tau$ and preserves two-dimensional Lebesgue measure on the square $[0, 1] \times [0, 1]$.

In the sequel we consider only $\alpha > 0$.

Let A_1 denote the part of the square $[0, 1]\times[0, 1]$ below the line $\alpha y+(1-\alpha)x = 1/2$ and A_2 the part above this line. We now collect some simple facts.

Proposition 1. *If $(x, y) \in A_1$ and $\alpha y + (1 - \alpha)x > a$, $a < 1/2$, then the point $(w, z) = G(x, y)$ satisfies $\alpha z + (1 - \alpha)w > 2\alpha a$.*

Proof. We have

$$\alpha z + (1 - \alpha)w = \alpha(2\alpha y + 2(1 - \alpha)x) + (1 - \alpha)y = [2\alpha^2 - \alpha + 1]y + 2\alpha(1 - \alpha)x\,.$$

It is enough to see that $(2\alpha^2-\alpha+1)/\alpha = 2\alpha-1+1/\alpha > 2\alpha$ and $2\alpha(1-\alpha)/(1-\alpha) = 2\alpha$. □

Proposition 2. *If $(x, y) \in A_1$ and $G(x, y) \in A_1$ as well, and $\alpha y + (1 - \alpha)x > a$, $a < 1/2$, then the point $(w, z) = G^2(x, y)$ satisfies $\alpha z+(1-\alpha)w > (4\alpha^2-2\alpha+2)a$.*

Proof. We have

$$(w,z) = \big(2(1-\alpha)x + 2\alpha y, 4\alpha(1-\alpha)x + (4\alpha^2 - 2\alpha + 2)y\big) ,$$

and

$$\begin{aligned}\alpha z + (1-\alpha)w &= (-4\alpha^3 + 6\alpha^2 - 4\alpha + 2)x + (4\alpha^3 - 4\alpha^2 + 4\alpha)y \\ &= (4\alpha^2 - 2\alpha + 2)(1-\alpha)x + (4\alpha^2 - 4\alpha + 4)\alpha y\,.\end{aligned}$$

It is enough to see that $(4\alpha^2 - 4\alpha + 4) > (4\alpha^2 - 2\alpha + 2) > 1$. □

Proposition 3. *If $(x,y) \in A_2$, then the point $(w,z) = G(x,y)$ satisfies $\alpha z + (1-\alpha)w \geq 2\alpha^2$.*

Proof. We have

$$\alpha z + (1-\alpha)w = \alpha(2 - 2\alpha y - 2(1-\alpha)x) + (1-\alpha)y = 2\alpha + [1 - 2\alpha^2 - \alpha]y - 2\alpha(1-\alpha)x\,.$$

For $\alpha \in (0,1/2)$ the coefficient next to y is positive and that next to x is negative so the minimum is reached at $(1,0)$ and is equal to $2\alpha^2$. This completes the proof. □

Proposition 4. *If $(x,y) \in A_2$ and $G(x,y) \in A_1$ then the point $(w,z) = G^2(x,y)$ satisfies $\alpha z + (1-\alpha)w \geq 2\alpha(1-\alpha) \geq 2\alpha^2$.*

Proof. We have

$$\alpha z + (1-\alpha)w = -(4\alpha^2 - 2\alpha + 2)(1-\alpha)x - 4\alpha^3 y + 4\alpha^2 - 2\alpha + 2\,.$$

For $\alpha \in (0,1/2)$ both coefficients next to x and y are negative so the minimum is reached at $(1,1)$ and is equal to $2\alpha(1-\alpha) \geq 2\alpha^2$. This completes the proof. □

Proposition 5. *Let A_I denote the part of the square $[0,1] \times [0,1]$ above (to the right of) the line $\alpha y + (1-\alpha)x = 2\alpha^2$. Propositions 1-3 prove that the support of G-invariant measures (except the point measure at $(0,0)$) must lie in region A_I.*

Proof. Proposition 2 implies that every point of A_1, except $(0,0)$, enters A_2 after a finite number of steps. Let us consider a point $X_0 \in A_2$. By Proposition 3 its image $X_1 = G(X_0)$ stays above the line $\alpha y + (1-\alpha)x = 2\alpha^2$. Assuming that $X_1 \in A_1$, by Proposition 4 the point $X_2 = G(X_1)$ is also above this line. If $X_2 \in A_1$ the next image $X_3 = G(X_2) = G^2(X_1)$ is above the line $\alpha y + (1-\alpha)x = 2\alpha^2(4\alpha^2 - 2\alpha + 2)$ (by Proposition 2). Now, if $X_3 \in A_1$, the next image $X_4 = G(X_3) = G^2(X_2)$ is also above this line. We see that further points of the trajectory move up towards A_2 and none of them can go below the line $\alpha y + (1-\alpha)x = 2\alpha^2$. □

Remark 3. *For $0.24 < \alpha < 1/2$, if $(x,y) \in A_I$, then it reaches A_2 in at most 6 steps.*

We define some functions which we will use below. Let $G_i = G_{|A_i}$, $i = 1,2$, be the restrictions of G to regions A_1 and A_2, respectively. Let $S(x,y) = \alpha y + (1-\alpha)x$. Then, $A_1 = \{(x,y) : 0 \leq x, y \leq 1, S(x,y) < 1/2\}$ and $A_2 = \{(x,y) : 0 \leq x, y \leq 1, S(x,y) \geq 1/2\}$.

Let

$$D_1 = DG_1 = \begin{bmatrix} 0 & 1 \\ 2(1-\alpha) & 2\alpha \end{bmatrix} \quad , \quad D_2 = DG_1 = \begin{bmatrix} 0 & 1 \\ -2(1-\alpha) & -2\alpha \end{bmatrix} .$$

Theorem 1. *The map G admits an acim for $0 < \alpha \leq \alpha_1 \sim 0.24760367$*

We define α_1 as a root of the equation $16\alpha^4 + 16\alpha^3 - 52\alpha^2 + 48\alpha - 9 = 0$ in the interval $[0,1]$. It is explained below.

Proof. We will prove that G^2 satisfies the assumptions of Tsujii ([4]), i.e., it is piecewise analytic and expanding in the sense that for any vector v we have $|DG^2 v| > |v|$. We will do this by showing that the smaller singular value $s_2(\alpha)$ of the matrix $D_i D_j$, $i, j \in \{1, 2\}$ is above 1 for $0 < \alpha \leq 0.24760367$.

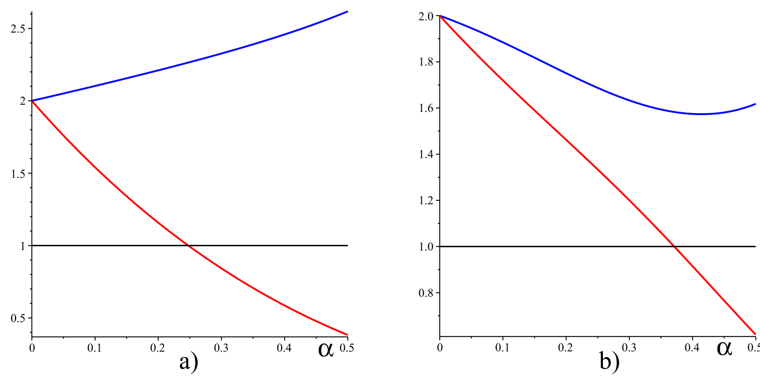

FIGURE 3. a) Singular values for matrices $D_2 D_1$ and $D_1 D_1$. The lower curve intersects level 1 at $\alpha_1 \sim 0.24760367$. b) Singular values for matrices $D_2 D_2$ and $D_1 D_2$. The lower curve intersects level 1 at ~ 0.3709557543.

The singular values of the matrices $D_2 D_1$ and $D_1 D_1$ are

$$\sigma_1(\alpha) = \sqrt{16\alpha^4 - 24\alpha^3 + 22\alpha^2 - 8\alpha + 4 + 2\sqrt{w_1(\alpha)}},$$

and

$$\sigma_2(\alpha) = \sqrt{16\alpha^4 - 24\alpha^3 + 22\alpha^2 - 8\alpha + 4 - 2\sqrt{w_1(\alpha)}},$$

where

$$w_1(\alpha) = 64\alpha^8 - 192\alpha^7 + 320\alpha^6 - 328\alpha^5 + 245\alpha^4 - 120\alpha^3 + 36\alpha^2.$$

They are shown in Figure 3 a). The lower curve intersects level 1 at at the root α_1 of $16\alpha^4 + 16\alpha^3 - 52\alpha^2 + 48\alpha - 9 = 0$, i.e., at $\alpha_1 \sim 0.24760367$.

The singular values of the matrices $D_2 D_2$ and $D_1 D_2$ are:

$$\sigma_1(\alpha) = \sqrt{16\alpha^4 - 8\alpha^3 + 6\alpha^2 - 8\alpha + 4 + 2\sqrt{w_2(\alpha)}},$$

and

$$\sigma_2(\alpha) = \sqrt{16\alpha^4 - 8\alpha^3 + 6\alpha^2 - 8\alpha + 4 - 2\sqrt{w_2(\alpha)}},$$

where

$$w_2(\alpha) = 64\alpha^8 - 64\alpha^7 + 64\alpha^6 - 88\alpha^5 + 69\alpha^4 - 24\alpha^3 + 4\alpha^2.$$

They are shown in Figure 3 b). The lower curve intersects level 1 at ~ 0.3709557543. This shows that at least for $0 < \alpha \leq \alpha_1 \sim 0.24760367$ the assumptions of [4] are satisfied and thus, G^2 and consequently also G admit an acim. □

5. Proof of the existence of acim for $\alpha > \alpha_1$

We will prove that high iterates of the map G expand all vectors. We will make estimates of the smaller singular value σ_2 of derivative matrix DG^n for large n. The general strategy is as follows: we will consider the admissible products of the derivative matrices $\prod_{j=1}^{n_s} D_{i_j}$, where $i_j \in \{1, 2\}$ and the length n_s depends on the sequence, for $\alpha \in (\alpha_s, \alpha_t)$, where (α_s, α_t) denotes contiguous intervals. The order of the matrices is natural, e.g., the sequence $D_1D_2D_2$ corresponds to the iteration $G_1G_2G_2$. We will consider sequences of the form $D_1^nD_2^m$, $n \le 3$, $m \ge 1$, since by Proposition 2 every point (except $(0, 0)$) visits region A_2. We will break the long sequence into short "good" sequences for which we can bound σ_2 from below by numbers larger that 1. Since

$$\sigma_2(AB) \ge \sigma_2(A)\sigma_2(B), \tag{5.1}$$

this will allow us to show that the σ_2 of a long product grows to infinity with the length n_s. Once we have a good estimate, we proceed as follows: we choose a large number M and find a sequence length n_s such that any admissible sequence of length n_s starting with D_2 has $\sigma_2 > M$. Then, adding at most three matrices D_1 at the beginning of the sequences and a corresponding number of matrices at the end (to keep the length of all sequences equal to $n_s + 3$) we will have derivative matrices of G^{n_s+3} for all non-transient points (we will prove that 3 is enough) and their σ_2's greater than 1. This proves that G^{n_s+3} on the set of non-transient points expands all vectors and in turn that G admits an acim.

Our proofs are based on symbolic calculations using Maple 17, but they are all finite calculations and "in principle" could be done using pen and paper.

Recall $G_i = G_{|A_i}$, $i = 1, 2$ are the restrictions of G to regions A_1 and A_2, respectively.

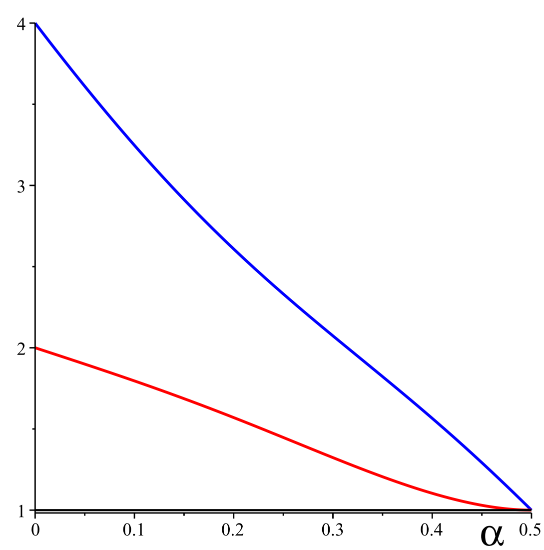

Figure 4. Singular values of $D_1D_2D_2$ or $D_2D_2D_2$.

The following result holds for all $0 < \alpha < 1/2$.

Proposition 6. *For any matrix M we have $\sigma_2(D_1M) = \sigma_2(D_2M)$. Also,*

$$\sigma_2(D_1D_2D_2) = \sigma_2(D_2D_2D_2) > 1, \tag{5.2}$$

for $0 < \alpha < 1/2$. More generally,

$$\sigma_2(D_1D_2^m) \geq \sigma_2(D_2D_2D_2) > 1, \text{ for } m = 2 + 3k, k \geq 1, 0 < \alpha < 1/2.$$

Proof. The singular values of the matrix B are square roots of the eigenvalues of the matrix B^TB, where B^T is the transpose of B. Since $D_1^TD_1 = D_2^TD_2$, the first claim follows. The graphs of the singular values of the matrices $D_1D_2D_2$ and $D_2D_2D_2$ are shown in Figure 4. Both singular values are above 1 for all $0 \leq \alpha < 1/2$. The last inequality follows from (5.1). □

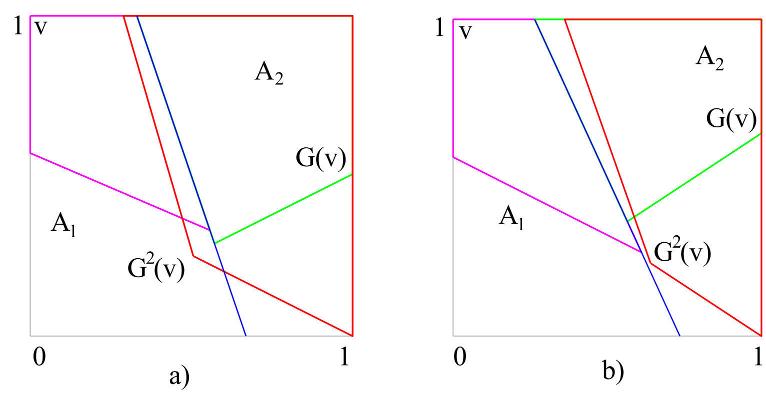

FIGURE 5. First two images of A_1 for a) $\alpha = 0.25290169942$ and b) $\alpha = 0.320169942$

Proposition 7. *For $\alpha > (\sqrt{5}-1)/4 \sim 0.3090169942$ a point in A_2 originating in A_1 must stay in A_2 for at least 2 steps.*

Proof. Figure 5 shows the first (green) and second (red) image of A_1. $G^{-1}(A_2) \cap A_1$ is bounded by magenta lines, the blue line is the partition line $S(x, y) = 1/2$. The important point is $v_2 = G(G(v)) = (2\alpha, 2\alpha(1 - 2\alpha))$ for $v = (0, 1)$. When $v_2 \in A_1$, then points can return to A_1 after one visit in A_2. When $v_2 \in A_2$, then a point coming from A_1 must stay in A_2 for at least 2 steps. $S(v_2) = 1/2$ for $\alpha = (\sqrt{5}-1)/4 \sim 0.3090169942$. □

Proposition 8. *The following estimates of $\sigma_2(D_1^nD_2^m)$ for various n and m were obtained using Maple 17:*

1) $\sigma_2(D_1D_2) > 1$ at least for $\alpha \leq 0.3709557543$;
2) $\sigma_2(D_1D_2D_2D_2) > 1$ at least for $\alpha \leq 0.3938896523$;
3) $\sigma_2(D_1D_1D_2) > 1$ at least for $\alpha \leq 0.3149466135$;
4) $\sigma_2(D_1D_1D_2D_2) > 1$ at least for $\alpha \leq 0.3758203590$;
5) $\sigma_2(D_1D_1D_2D_2D_2) > 1$ at least for $\alpha \leq 0.3506831157$;
6) $\sigma_2(D_1D_1D_1D_2) > 1$ at least for $\alpha \leq 0.3058009335$;
7) $\sigma_2(D_1D_1D_1D_2D_2) > 1$ at least for $\alpha \leq 0.3355882883$;
8) $\sigma_2(D_1D_1D_1D_2D_2D_2) > 1$ at least for $\alpha \leq 0.3312697596$;

Theorem 2. *The map G admits an acim for $\alpha_1 \leq \alpha \leq \alpha_2 \sim 0.2797707433$.*

We define α_2 as a root of the equation $8\alpha^4 - 8\alpha^3 + 8\alpha^2 = 1/2$ in $[0, 1]$. Again, it is explained below in Proposition 10.

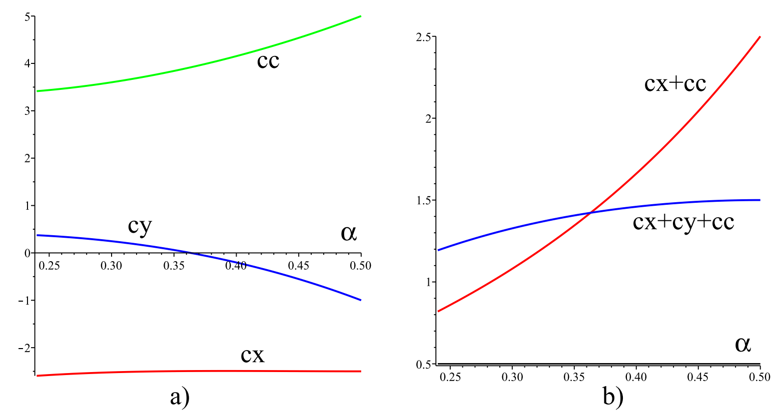

FIGURE 6. a) Functions cx, cy, cc in Proposition 9. b)Functions $cx + cc$ and $cx + cy + cc$ in Proposition 9.

First, we prove the following:

Proposition 9. *For $\alpha_1 \le \alpha \le \alpha_2$, a point originating in A_2 remains in A_1 for at most 3 steps.*

Proof. It is enough to show that $f(x, y) = S(G_1(G_1(G_1(G_2(x, y))))) \ge 1/2$. We have

$$f(x, y) = cx(\alpha)x + cy(\alpha)y + cc(\alpha),$$

where

$$cx(\alpha) = 16\alpha^5 - 40\alpha^4 + 52\alpha^3 - 36\alpha^2 + 12\alpha - 4;$$
$$cy(\alpha) = -16\alpha^5 + 16\alpha^4 - 12\alpha^3 - 8\alpha^2 + 4\alpha;$$
$$cc(\alpha) = 16\alpha^4 - 24\alpha^3 + 28\alpha^2 - 8\alpha + 4.$$

The functions cx, cy and cc are shown in Figure 6 a). We consider the worst case scenario, i.e., $y = 1$ and $x = 0$ where $cx > 0$ and $x = 1$ where $cx < 0$. Graphs of $cx + cc$ and $cx + cy + cc$ are shown in Figure 6 b). They both above 1/2 for $\alpha > 0.24$, and in particular for $\alpha_1 \le \alpha \le \alpha_2$. □

Proof of Theorem 2: By Proposition 9, Proposition 6 and estimates of Proposition 8 we see that, for α's in the interval $[\alpha_1\, \alpha_2]$, all admissible "basic" sequences of derivative matrices have σ_2 larger than 1. Note that we have

$$\sigma_2(D_1^n D_2^m \ge \sigma_2(D_1^n D_2^{m-3})\sigma_2(D_2^3) > \sigma_2(D_1^n D_2^{m-3}), \tag{5.3}$$

for $m > 3$ (Proposition 6). This shows that the general strategy described at the beginning of this section will work and proves the theorem.

Theorem 3. *The map G admits an acim for $\alpha_2 \le \alpha \le \alpha_3 = 1/3$.*

First, we prove the following:

Proposition 10. *For $\alpha_2 \le \alpha \le \alpha_3$ a point coming from A_2 can stay in A_1 for at most 2 steps.*

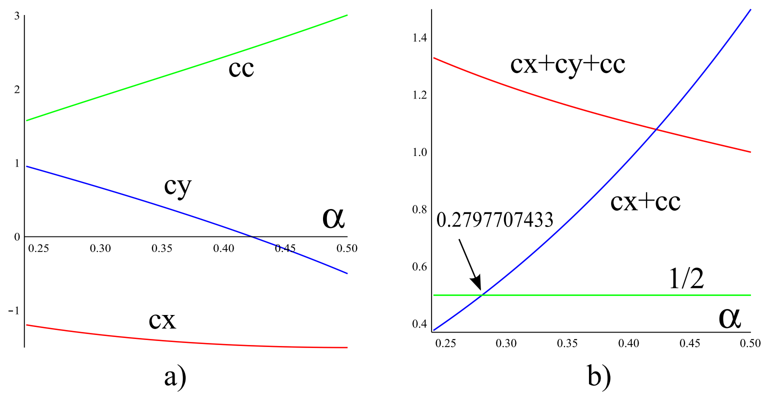

FIGURE 7. Functions cx, cy, cc and their sums in Proposition 10

Proof. The proof is similar to that of Proposition 9. It is enough to show that $f(x, y) = S(G_1(G_1(G_2(x, y)))) \geq 1/2$. We have

$$f(x, y) = cx(\alpha)x + cy(\alpha)y + cc(\alpha),$$

where

$$\begin{aligned} cx(\alpha) &= 8\alpha^4 - 16\alpha^3 + 16\alpha^2 - 8\alpha; \\ cy(\alpha) &= -8\alpha^4 + 4\alpha^3 - 2\alpha^2 - 4\alpha + 2; \\ cc(\alpha) &= 8\alpha^3 - 8\alpha^2 + 8\alpha. \end{aligned}$$

The functions cx, cy and cc are shown in Figure 7 a). Again, we consider the worst case scenario, i.e., $y = 1$ and $x = 0$ where $cx > 0$ and $x = 1$ where $cx < 0$. Graphs of $cx + cc$ and $cx + cy + cc$ are shown in Figure 7 b). They are both above $1/2$ for $\alpha > \alpha_2$. □

Proof of Theorem 3: Let us first consider the sequence $D_1D_1D_2$. By part 3) of Proposition 8 its σ_2 is larger than 1 until $\alpha \sim 0.3149466135$. By Proposition 7 the sequence is not admissible after $\alpha \sim 0.3090169942$. All other admissible "basic" sequences of derivative matrices have σ_2 larger than 1 for α's in the interval $[\alpha_2, \alpha_3]$. We used Proposition 10, Proposition 6 and estimates of Proposition 8 as well as inequality (5.3). This shows that the general strategy described at the beginning of this section will work and proves the theorem.

6. PROOF OF THE EXISTENCE OF ACIM FOR $\alpha > \alpha_3 = 1/3$

We will continue the estimates of σ_2 for "basic" admissible sequences.

Proposition 11. *For $\alpha > \alpha_3 = 1/3$ a point coming from A_2 can stay in A_1 for at most 1 step.*

Proof. Figure 8 shows the region $G(A_2) \cap A_1$ (outlined in green) and its image (outlined in red). The blue line is the partition line $S(x, y) = 1/2$. The important point is $G(w) = (\alpha/(\alpha + 1), 1)$ for $w = \left(\frac{\alpha+1/2}{\alpha+1}, \frac{\alpha}{\alpha+1}\right)$. When $G(w) \in A_1$, points coming from A_2 can stay in A_1 for two steps. When $G(w) \in A_2$, a point coming from A_2 can be in A_1 for only one step. $S(G(w)) = 2\alpha/(\alpha + 1)$ so $S(G(w)) = 1/2$ for $\alpha_3 = 1/3$. □

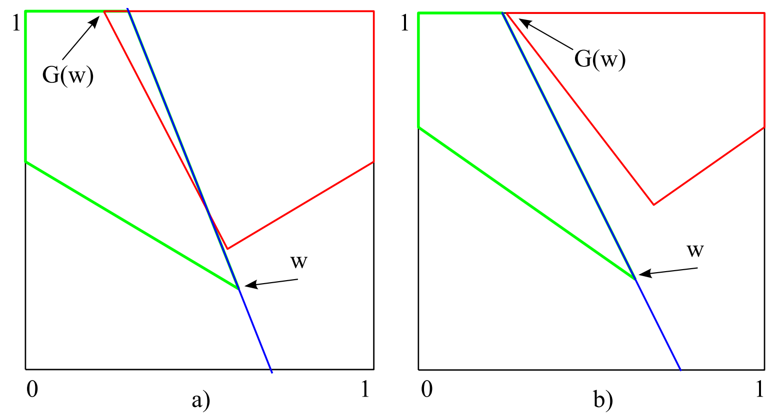

FIGURE 8. Region $G(A_2) \cap A_1$ and its image for a) $\alpha = 0.29$ and b) $\alpha = 0.34$

Proposition 7 and Proposition 11 imply that for $\alpha > 1/3$ basic admissible sequences are of the form $D_1 D_2^m$ with $m \geq 2$.

Proposition 12. *For $\alpha > \alpha_3 = 1/3$ we give estimates of σ_2 for basic admissible sequences. Again, the estimates are obtained using Maple 17.*

1) $\sigma_2(D_1 D_2^3) > 1$ at least for $\alpha \leq 0.3938896523$;
2) $\sigma_2(D_1 D_2^4) > 1$ at least for $\alpha \leq 0.4444154417$;
3) $\sigma_2(D_1 D_2^6) > 1$ at least for $\alpha \leq 0.4345268819$;
4) $\sigma_2(D_1 D_2^7) > 1$ at least for $\alpha \leq 0.4645618403$;

Corollary 1. *Propositions 7, 11 and 12 imply that our general strategy works for $\alpha \leq 0.3938896523$. For longer sequences we use inequality (5.3).*

Proposition 13. *For $\alpha > 0.3931078326$, the sequence $D_1 D_2 D_2 D_2$ is followed by $D_1 D_2 D_2$. We have*

$$\sigma_2(D_1 D_2^2 D_1 D_2^3) > 1 \quad \textit{for at least} \quad \alpha \leq 0.4160029431.$$

With the previous results this extends the interval of the existence of acim up to $\alpha \sim 0.4160029431$.

Proof. Figure 9 shows the first four images of $B = G(A_2) \cap A_1$ (green thick boundary). The blue line is the partition line $S(x, y) = 1/2$. The images are consecutively $G(B)$ (red), $G^2(B)$ (blue), $G^3(B)$ (brown). The set $G^3(B) \cap A_2$ is bounded by thick brown lines and represents points which stay in A_2 for 3 steps. Its image is bounded by green lines. The set we are interested in is the triangle $C = G(G^3(B) \cap A_2) \cap A_1$, namely the points which after three steps in A_2 go to A_1.

Further images of the triangle C are shown in Figure 10 for a) $\alpha = 0.391$ and b) $\alpha = 0.394$. The important point is $G^7(w)$ (the same point w as in the proof of Proposition 11). When $G^7(w) \in A_2$, then some points of C stay in A_2 longer than twice. When $G^7(w) \in A_1$, all points of C stay in A_2 exactly for two steps. Equation $S(G^7(w)) = 1/2$ is equivalent to $192\alpha^7 + 192\alpha^6 - 336\alpha^5 - 144\alpha^4 + 256\alpha^3 - 128\alpha^2 + 53\alpha - 11 = 0$ with a root $\alpha \sim 0.3931078326$. Since $0.3931078326 < 0.3938896523$ for $\alpha > 0.3931078326$ we replace estimate 1) of Proposition 12 with estimate of Proposition 13 which holds up to $\alpha \sim 0.4160029431$. □

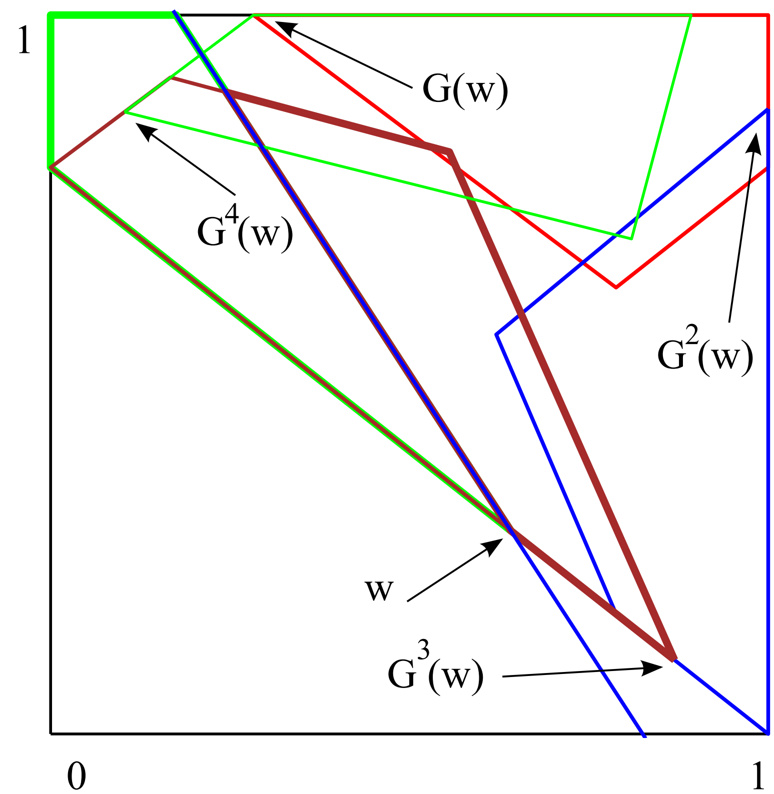

FIGURE 9. Four first images of $G(A_2) \cap A_1$, $\alpha > 0.39$

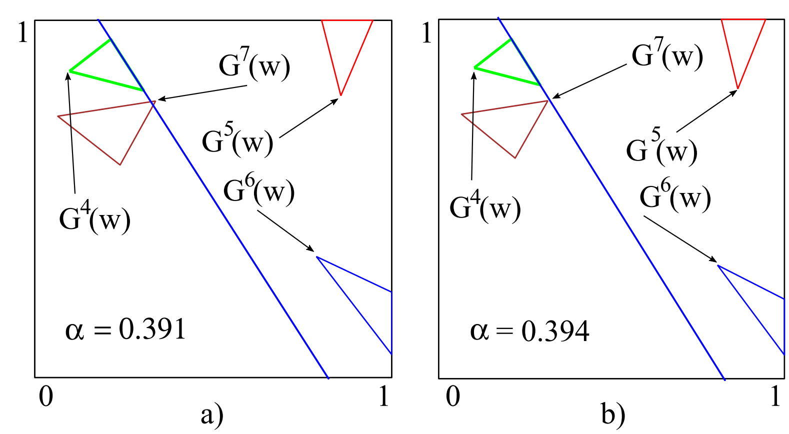

FIGURE 10. Further images of $G(G^3(B) \cap A_2) \cap A_1$ for a) $\alpha = 0.391$ and b) $\alpha = 0.394$

Proposition 14. *For* $\alpha > 0.3510763028$ *group* $D_1 D_2^4$ *is not admissible. For* $\alpha > 0.4284630893$ *group* $D_1 D_2^3$ *is not admissible. The following estimates hold:*

1) $\sigma_2(D_1 D_2^2 D_1 D_2^2 D_1 D_2^3) > 1$ *at least for* $\alpha \leq 0.4315221884$*;*

2) $\sigma_2(D_1D_2^2D_1D_2^3D_1D_2^2D_1D_2^3) > 1$ *at least for* $\alpha \leq 0.4584009011$*;*

3) $\sigma_2(D_1D_2^4D_1D_2^2D_1D_2^3) > 1$ *for all* $\alpha < 0.5$*. Although the group* $D_1D_2^4$ *may not be admissible, this inequality can be used for estimates.*

4) $\sigma_2(D_1D_2^5D_1D_2^2D_1D_2^3) > 1$ *at least for* $\alpha \leq 0.4456891654$*;*

5) $\sigma_2(D_1D_2^6D_1D_2^2D_1D_2^3) > 1$ *at least for* $\alpha \leq 0.4624281766$*.*

For $n = 3k + i$*,* $i = 4, 5, 6$*, we have*

(6.1)
$$\sigma_2(D_1D_2^nD_1D_2^2D_1D_2^3) = \sigma_2(D_2D_2^nD_1D_2^2D_1D_2^3) \geq \sigma_2^k(D_2^3)\sigma_2(D_1D_2^iD_1D_2^2D_1D_2^3).$$

With the previous results this extends the interval of the existence of acim up to $\alpha \sim 0.4345268819$ *by estimate 3) of Proposition 12).*

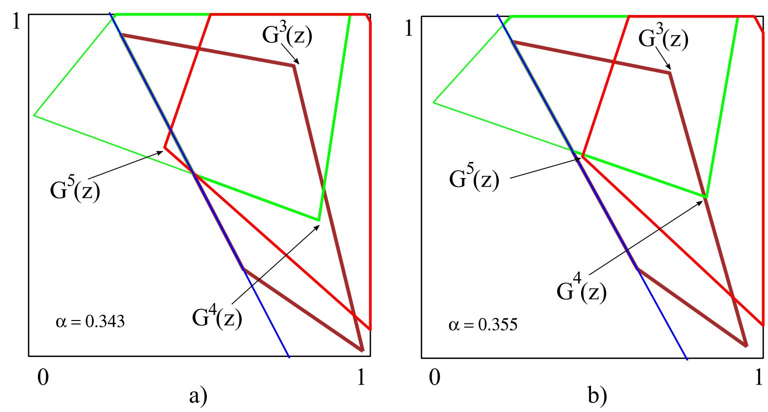

FIGURE 11. Further images of $C_1 = G(G^3(B) \cap A_2) \cap A_2$ (thick brown), for a) $\alpha = 0.343$ and b) $\alpha = 0.355$.

Proof. First, the estimates 1)–5) show that the basic admissible sequences starting with $D_1D_2^3$ (followed by $D_1D_2^2$ in view of Proposition 13) have $\sigma_2 > 1$ up to $\alpha \sim 0.4315221884$.

Now, we will show that groups $D_1D_2^4$ and $D_1D_2^3$ are not admissible above some α's. Figure 11 shows further images of $C_1 = G(G^3(B) \cap A_2) \cap A_2$ (thick brown), where $B = G(A_2) \cap A_1$ (thick green) shown in Figure 9. The first image $G(C_1)$ is bounded in green. These are points which were 3 steps in A_2, some of them are in A_1, some stay for the fourth step in A_2. The region bounded in red is the image $G(G(C_1) \cap A_2)$ (thick green), the points which were in A_2 for 4 steps. For $\alpha = 0.343$ (a)) some of them land in A_1, for $\alpha = 0.355$ (b)) the whole image is in A_2. The important point is $G^5(z)$, where $z = (0, 2\alpha)$ is a vertex of B. Equation $S(G^5(z)) = 1/2$ is equivalent to $\alpha^6 + 8\alpha^5 - 8\alpha^4 - 40\alpha^3 - 48\alpha^2 - 96\alpha + 320 = 0$ with a root $\alpha \sim 0.3510763028$.

Figure 12 shows the set $G^3(B) \cap A_2$ (thick brown), the set of point which stayed in A_2 for three steps. $B = G(A_2) \cap A_1$ as in the proof of Proposition 13 and point w is also the same as there. The image $G(G^3(B) \cap A_2)$ is bounded in green. The important point is $G^4(w)$. When $G^4(w) \in A_1$, then some points can go to A_1 after three steps in A_2. When $G^4(w) \in A_2$, then all points which stayed 3 times in A_2 stay there for at least two more steps (4 times in A_2 were excluded

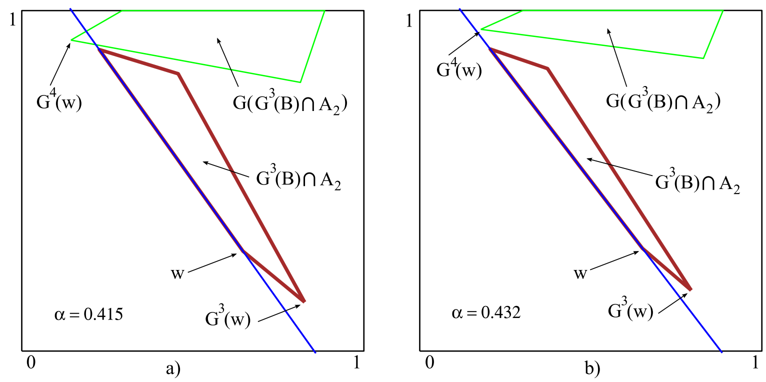

FIGURE 12. The image of $G^3(B) \cap A_2$ for a) $\alpha = 0.415$ and b) $\alpha = 0.432$.

in the previous part of the proof). The equation $S(G^4(w)) = 1/2$ is equivalent to $24\alpha^4 + 12\alpha^3 - 36\alpha^2 + 9\alpha + 1 = 0$ with a root $\alpha \sim 0.4284630893$.

Once the the sequence $D_1D_2^3$ is rendered inadmissible, the worst estimate is $\alpha \sim 0.4345268819$, estimate 3) of Proposition 12. □

To further improve the range of α's for which G has an acim we have to consider sequences starting with sequence $D_1D_2^6$.

Proposition 15. *Above $\alpha \sim 0.4345268819$ the sequence $D_1D_2^6$ is followed by the sequence $D_1D_2^2$ or $D_1D_2^5$. After $\alpha \sim 0.4397492527$ the only possibility is $D_1D_2^2$.*

Proof. The blue quadrangle in Figure 13 is $G^3(G^3(B) \cap A_2)$, i.e., it is the third image of brown quadrangle of Figure 12. These are images of points which (for our range of α's) were for 5 steps in A_2. The green triangle $O_6 = G(G^3(G^3(B) \cap A_2) \cap A_2) \cap A_1$ are the points which went to A_1 after 6 steps in A_2. Figure 13 shows the images $G(O_6)$ (bigger red), $G^2(O_6)$ (blue) and $G^3(O_6)$ (partially brown, partially red). The points in $G^3(O_6) \cap A_1$ (brown part of $G^3(O_6)$) correspond to group $D_1D_2^2D_1D_2^6$. Figure 13 shows also three consecutive images of $T = G^3(O_6) \cap A_2$ (small red triangles). In particular $G^3(T)$ is completely inside A_1. These points correspond to the group $D_1D_2^5D_1D_2^6$. This proves the first claim of the proposition.

Figure 14 shows O_6 and its images $G(O_6)$, $G^2(O_6)$ and $G^3(O_6)$ for parameters $\alpha = 0.434$ (part a)) and $\alpha = 0.441$ (part b)). For larger α's the image $G^3(O_6)$ is completely in A_1, which means that after group $D_1D_2^6$ there must be group $D_1D_2^2$. The group $D_1D_2^5D_1D_2^6$ is no longer admissible. The important point is $G^{10}(w)$, where w is the point used already in Propositions 14 and 13. The equation $S(G^{10}(w) = 1/2$ is equivalent to $1536\alpha^{10} + 3840\alpha^9 - 2688\alpha^8 - 7296\alpha^7 + 4128\alpha^6 + 3840\alpha^5 - 3504\alpha^4 + 992\alpha^3 - 160\alpha^2 - 5\alpha + 11 = 0$ with a root $\alpha \sim 0.4397492527$. □

Proposition 16. *Above $\alpha \sim 0.4546258153$ the sequence $D_1D_2^6$ becomes inadmissible. For this range of α the sequence $D_1D_2^7$ is also inadmissible.*

Proof. Figure 15 shows the quadrangle $B_1 = G^3(G^3(B) \cap A_2) \cap A_2$ (thick blue), the set of points which stay in A_2 for 6 steps. The images $G(B_1)$ (brown) and $G^2(B_1))$ (green) are also shown. Part a) is for $\alpha = 0.451$ and part b) for $\alpha = 0.456$. For larger α both images are completely inside A_2. This means that the sequences $D_1D_2^6$ and $D_1D_2^7$ are inadmissible. The important point is $G^7(w)$ for the same

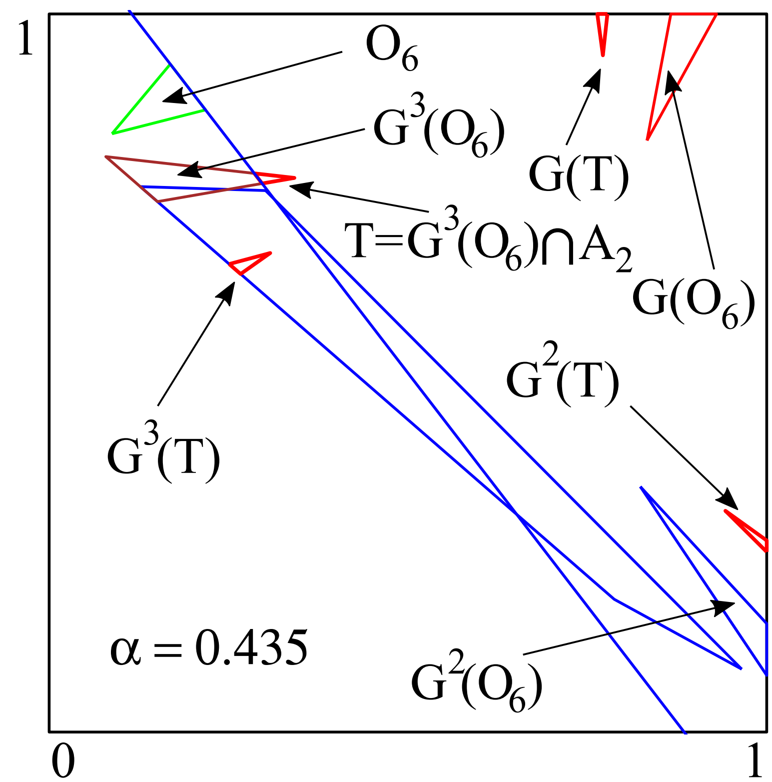

FIGURE 13. Images of points which stayed for 6 steps in A_2.

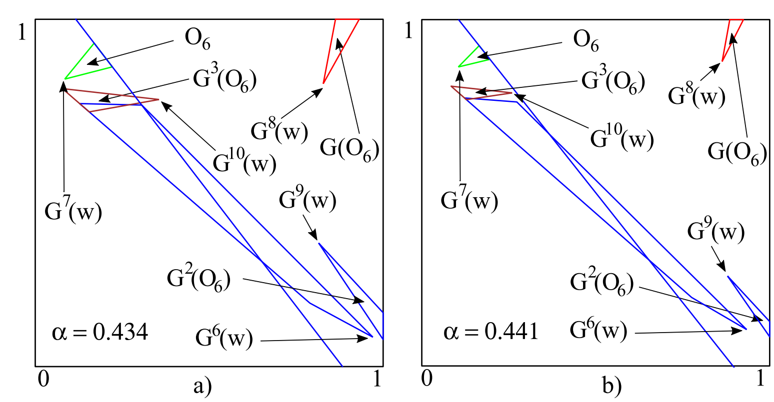

FIGURE 14. When the sequence $D_1 D_2^5 D_1 D_2^6$ becomes inadmissible.

point w as before. The equation $S(G^7(w)) = 1/2$ is equivalent to $192\alpha^7 + 384\alpha^6 - 432\alpha^5 - 480\alpha^4 + 480\alpha^3 - 69\alpha + 11 = 0$ with a root $\alpha \sim 0.4546258153$. □

Proposition 17. *We have proved the existence of acim for alpha's up to $\alpha \sim 0.4345268819$ (Proposition 14). We have the following estimates on the σ_2's of sequences starting with $D_1 D_2^6$:*

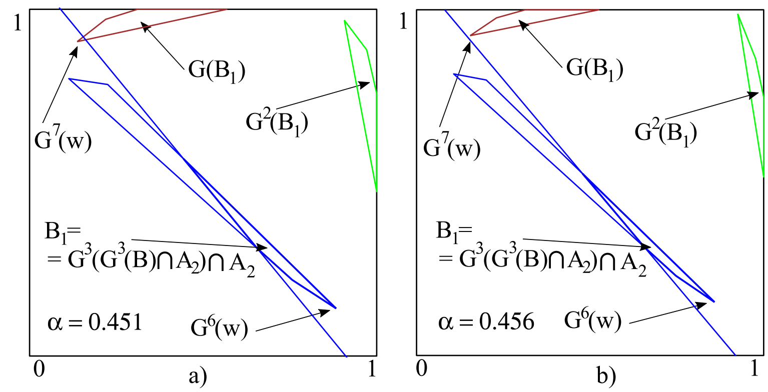

FIGURE 15. Sequence $D_1D_2^6$ becomes inadmissible.

1) $\sigma_2(D_1D_2^5D_1D_2^6) > 1$ *at least for* $\alpha \leq 0.4487890698$*;*
2) $\sigma_2(D_1D_2^2D_1D_2^6) > 1$ *at least for* $\alpha \leq 0.4451846371$*;*
3) $\sigma_2(D_1D_2^2D_1D_2^2D_1D_2^6) > 1$ *at least for* $\alpha \leq 0.4527916100$*;*
4) $\sigma_2(D_1D_2^4D_1D_2^2D_1D_2^6) > 1$ *for all* $\alpha < 0.5$*. Although the group* $D_1D_2^4$ *maybe not admissible, this inequality can be used for useful estimates.*
5) $\sigma_2(D_1D_2^5D_1D_2^2D_1D_2^6) > 1$ *at least for* $\alpha \leq 0.4600595036$*;*
6) $\sigma_2(D_1D_2^6D_1D_2^2D_1D_2^6) > 1$ *at least for* $\alpha \leq 0.4718920017$*.*
These estimates and previous results extend the range of the existence of acim up to $\alpha \sim 0.4527916100$*.*

Proof. Estimate 1) together with Proposition 15 tell us that all sequences starting with $D_1D_2^5D_1D_2^6$ have $\sigma_2 > 1$ as long as they are admissible. All other sequences starting with $D_1D_2^6$ must start with $D_1D_2^2D_1D_2^6$. Using inequality (6.1) and estimates 2)–6) we see that they all have $\sigma_2 > 1$ at least up to $\alpha \sim 0.4527916100$. □

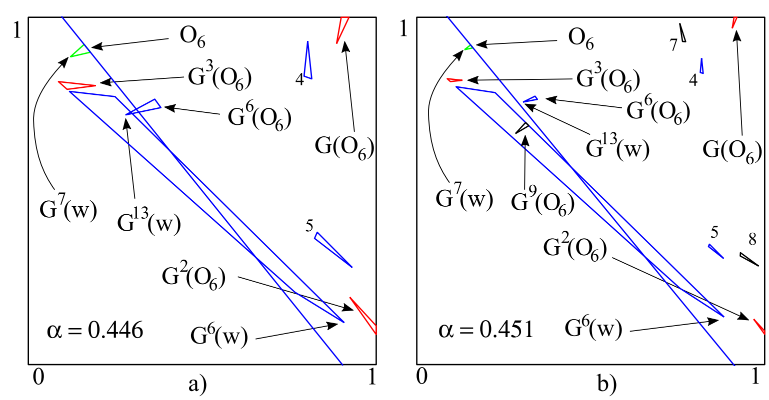

FIGURE 16. Images of O_6: a) 6 images for $\alpha = 0.446$, b) 9 images for $\alpha = 0.451$.

We want to push α higher to make the sequences starting with $D_1D_2^6$ inadmissible. First, we will find out what comes after the sequence $D_1D_2^2D_1D_2^6$ for $\alpha > 0.4527916100$.

Proposition 18. *After $\alpha \sim 0.4496432201$ after the sequence $D_1D_2^6$ comes the sequence $D_1D_2^5D_1D_2^2$.*

Proof. Figure 16 a) shows 6 consecutive images of triangle O_6 (introduced in Proposition 15), the set of points which leave A_2 after staying in it for six steps, for $\alpha = 0.446$. The triangle $G^3(O_6)$ is completely in A_1. This corresponds to the sequence $D_1D_2^2D_1D_2^6$, whose necessity was proved in Proposition 15. The triangle $G^6(O_6)$ intersects the partition line so some points leave A_2 at this moment, some continue staying in A_2.

Part b) of the same figure show the same 6 images of O_6 and 3 next images, for $\alpha = 0.451$. Some images have full descriptions, some only numbers. For this α triangle $G^6(O_6)$ is completely inside A_2 so all of its points continue staying in A_2. The triangle $G^9(O_6)$ is completely in A_1. This shows that for this range of α's after group $D_1D_2^6$ there must be group $D_1D_2^5D_1D_2^2$.

The important point is $G^{13}(w)$ (the same w as before), the left most vertex of $G^6(O_6)$. The equation $S(G^{13}(w)) = 1/2$ implies $12288\alpha^{13} + 36864\alpha^{12} - 12288\alpha^{11} - 86016\alpha^{10} + 16128\alpha^9 + 84480\alpha^8 - 43392\alpha^7 - 23360\alpha^6 + 36288\alpha^5 - 19456\alpha^4 + 2816\alpha^3 + 1984\alpha^2 - 869\alpha + 91 = 0$ with a root $\alpha \sim 0.4496432201$. □

Theorem 4. *The map G admits an acim for α up to at least $\alpha \sim 0.4600595036$.*

Proof. In Proposition 17 we proved existence of an acim up to $\alpha \sim 0.4527916100$. After this value, by Proposition 18 the offending sequence $D_1D_2^2D_1D_2^2D_1D_2^6$ is no longer admissible. The lowest estimate we need now is estimate 5) of Proposition 17. Thus, the existence of an acim is proved for α's up to $\alpha \sim 0.4600595036$. □

Remark 4. *For α's above $\alpha \sim 0.4600595036$ the sequence $D_1D_2^6$ is no longer admissible.*

The exact estimates for $\alpha > 0.4600595036$ become more and more complicated. We hope to find some more abstract way to prove that G satisfies the expanding conditions of [4]. We performed numerical experiments estimating $\sigma_2(x_0, N) = \sigma_2\left(\prod_{k=0}^N DG(G^k(x_0))\right)$ for millions of initial points x_0. Instead of calculating σ_2 directly, we used estimate (see, e.g., [5])

$$\sigma_2\left(\prod_{k=0}^N M_k\right) \geq \frac{\det\left(\prod_{k=0}^N M_k\right)}{\left|\left|\prod_{k=0}^N M_k\right|\right|_F} = \frac{\prod_{k=0}^N \det M_k}{\left|\left|\prod_{k=0}^N M_k\right|\right|_F}, \tag{6.2}$$

where $\|M\|_F = \sqrt{m_{1,1}^2 + m_{1,2}^2 + m_{2,1}^2 + m_{2,2}^2}$ is the Frobenius norm of the matrix M. Since all M_k's are either D_1 or D_2 and $\det D_1 = \det D_2$, the calculations of right hand side of (6.2) are very stable. All trials showed that for $\alpha < 1/2$ the quantity $\sigma_2(x_0, N)$ grows to infinity as N increases. This provides numerical evidence for expanding properties of G and the existence of acim.

The Figures 17–18 show the support of acim (or conjectured acim) for $\alpha = 0.3, 0.4, 0.43, 0.46, 0.49, 0.495$. The pictures were obtained by computer plotting 10^6 iterates long trajectory of G_α after skipping the first $1.5 \cdot 10^6$ iterations. The experiments show that the obtained support is independent of the typical initial point.

For α's in a very narrow window around $\alpha = 0.493$ (of radius approximately 10^{-6}), the support of conjectured acim looks very different from typical, see Figure

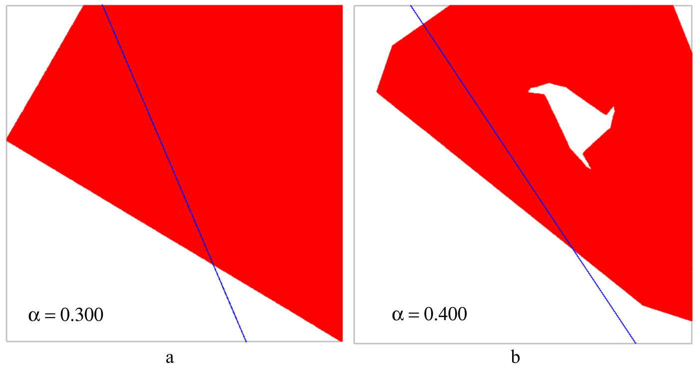

FIGURE 17. Support of acim for $\alpha = 0.3$ and $\alpha = 0.4$.

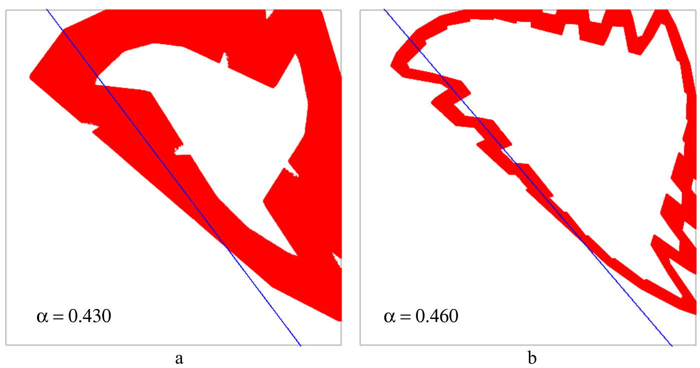

FIGURE 18. Support of acim for $\alpha = 0.43$ and $\alpha = 0.46$.

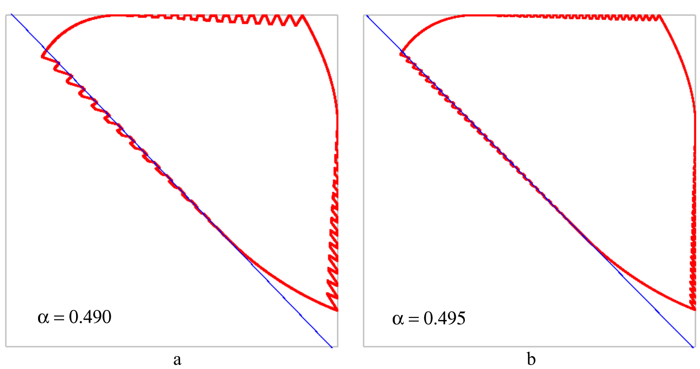

FIGURE 19. Support of conjectured acim for $\alpha = 0.49$ and $\alpha = 0.495$.

20. It consists of 175 clusters which under action of G move by 58 positions in the clockwise direction. Since $3 \cdot 58 = 174$, G^{175} preserves every cluster. Figure 20 b shows one of the clusters (pointed out by an arrow in part a). It shows $500 \cdot 10^6$ iterations of G^{175}, after skipping $35 \cdot 10^6$ initial iterations. Parts of the image were

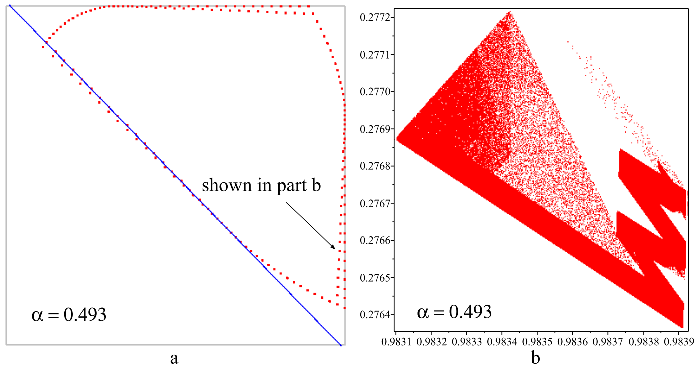

FIGURE 20. a: Support of conjectured acim for $\alpha = 0.493$. b: Close-up of one of the clusters in part a.

showing up extremely slowly. We observed similar behaviour for $\alpha = 0.4883$ (106 clusters moving by 35 positions), $\alpha = 0.4943$ (214 clusters moving by 71 positions) and $\alpha = 0.4973$ (448 clusters moving by 149 positions). Probably there are many other windows of α with similar behaviour.

7. Deterministic Behaviour of Memory Map for $1/2 \le \alpha \le 3/4$

7.1. $\alpha = 1/2$. Let $\alpha = 1/2$. In particular, we have

$$\tau(1 - x/2) = 2 - 2(1 - x/2) = x\,.$$

Assume $y \ge 1 - x$ or $x + y \ge 1$ or $(x + y)/2 \ge 1/2$. Then,

$$\begin{aligned} G(x, y) &= (y, \tau((x + y)/2)) = (y, 2 - x - y)\,, \\ G^2(x, y) &= G(y, 2 - x - y) = (2 - x - y, \tau(1 - x/2)) = (2 - x - y, x)\,, \\ G^3(x, y) &= G(2 - x - y, x) = (x, \tau(1 - y/2)) = (x, y)\,. \end{aligned} \tag{7.1}$$

This shows that any such point is periodic with period 3. The only fixed point in this region is $(2/3, 2/3)$. (Another one is $(0, 0)$ and there is no more fixed points)

If $y < 1 - x$, then we have to show that any such point except $(0, 0)$ eventually goes to the upper triangle $y \ge 1 - x$. Note that if $G(x, y) = (0, 0)$, then $(x, y) = (0, 0)$. Also, $G(x, 0) = (0, \tau(x/2))$, so we can consider only points with $y > 0$. Then, as long as the second coordinate is less than 1 minus the first, we have

$$(x, y) \mapsto (y, x + y) \mapsto (x + y, x + 2y) \mapsto (x + 2y, 2x + 3y) \mapsto \dots$$

It is clear that the sum of the coordinates grows on each step at least by the value y so eventually it goes above 1, which means that the point goes to the upper triangle.

7.2. $\alpha = 3/4$. Let $\alpha = 3/4$ and let us assume that $x + y = 4/3$ or $3x + 3y = 4$. Then,

$$G(x, y) = \left(y, \tau\left(\frac{3}{4}y + \frac{1}{4}x\right)\right)\,.$$

We have $\frac{3}{4}y + \frac{1}{4}x = \frac{1}{4}(3y + 3x - 2x) = 1 - x/2 \ge 1/2$ so

$$\tau\left(\frac{3}{4}y + \frac{1}{4}x\right) = 2 - \frac{1}{4}(6y + 6x - 4x) = 2 - 2 + x = x\,.$$

Thus, for such points

$$G(x, y) = (y, x),$$

so each of them is periodic with period 2, except for the fixed point $(2/3, 2/3)$.

We will prove the following:

Theorem 5. *For $\alpha = 3/4$ any point, except $(0,0)$ is either periodic (period 2 or 1) or eventually periodic or attracted to the line $x + y = 4/3$.*

The line $y = 2/3 - x/3$ (or equivalently $y + x/3 = 2/3$) partitions square $[0,1] \times [0,1]$ into two parts on which G is defined differently: A_1 below the line and A_2 above it. We partition region A_2 further into three parts: B_1 between the lines $y = -x/2 + 5/6$ and $y = -x/2 + 7/6$, B_2 between $y = -x/2 + 5/6$ and the partition line and B_3 above the line $y = -x/2 + 7/6$, see Figure 21.

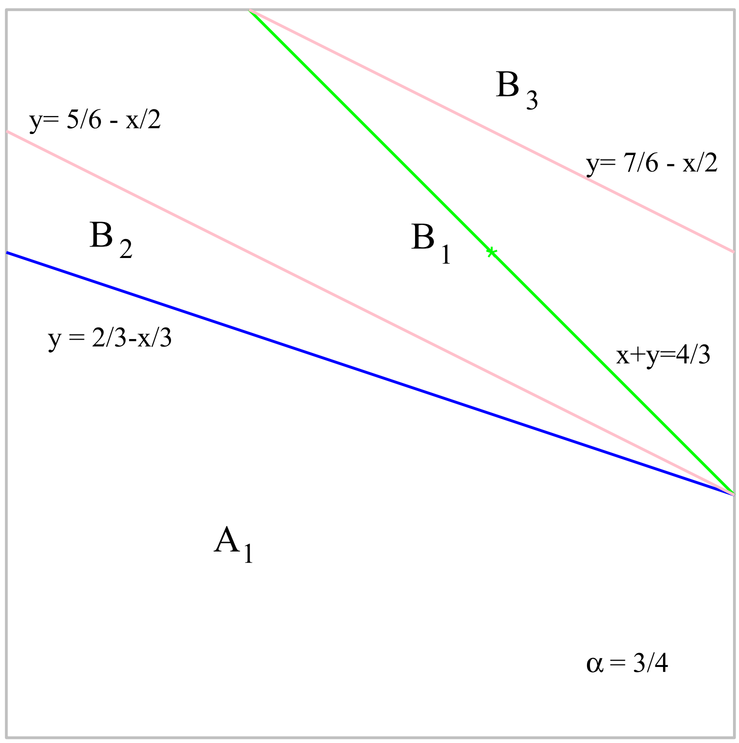

FIGURE 21. Regions for $\alpha = 3/4$.

Let $(x, y) \in A_1 \setminus \{(0,0)\}$. If $(x, y) = (x, 0)$, then $G(x, 0) = (0, x/2)$, so we can assume that $y > 0$. It is easy to calculate that

$$G(x, y) = \left(y, \frac{3}{2}y + \frac{1}{2}x\right),$$

with the sum of second coordinate plus one third of the first coordinate equal to $\frac{5}{2}y + \frac{1}{6}x$ so on each step this sum grows by at least y and eventually every such point will move to the upper half of the square $y + x/3 > 2/3$.

Consider now the region B_1 inside $[0,1]\times[0,1]$ between the lines $y=-x/2+5/6$ and $y=-x/2+7/6$. It contains the line $L: x+y=4/3$ of periodic points. The derivative matrix in this region is constant and has eigenvalues $-1,-1/2$ and corresponding eigenvectors $v_1=[-1,1]$ and $v_2=[-2,1]$. Every point in B_1 can be written uniquely as $[2/3,2/3]+tv_1+sv_2=[2/3-t-2s,2/3+t+s]$ for $[t,s]\in E$, some compact neighbourhood of $[0,0]$. We have

$$\begin{aligned} G\left([2/3-t-2s,2/3+t+s]\right) &= [2/3+t+s,2/3-t-s/2] \\ &= [2/3,2/3]-tv_1-s/2v_2 \end{aligned}$$

and since v_1 is parallel to L, this means the distance to L is divided by 2.Thus, every point in B_1 is attracted to the periodic line.

Let us consider B_2 now. We will show that $G(B_2)\subset B_3$. Let $(x,y)\in B_2$. Then, $y<5/6-x/2$ and $G(x,y)=(w,z)=(y,2-(3/2)y-(1/2)x)$. We will show that $z>7/6-w/2$, or

$$2-\frac{3}{2}y-\frac{1}{2}x>\frac{7}{6}-\frac{1}{2}y\,,$$

which is exactly our assumption. Thus, $G(B_2)\subset B_3$.

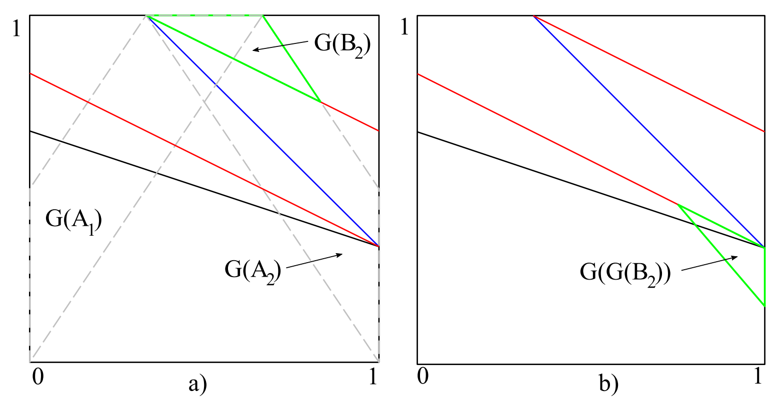

FIGURE 22. Images $G(B_2)$ and $G(G(B_2))$, $\alpha=3/4$.

In Figure 22 a) we see the image $G(B_2)$ (green) and both images $G(A_1)$ and $G(A_2)$ (grey dashed). The points outside $G(A_1)\cup G(A_2)$ are transient and unimportant for dynamics because they are eventually mapped into $G(A_1)\cup G(A_2)$. Thus, the only part of B_3 we will study is the image $G(B_2)$. In Figure 22 b) we see the image $G(G(B_2))$ (green). It consists of two parts, upper $G^2(B_2)\cap A_2$ and lower $G^2(B_2)\cap A_1$.

In Figure 23 a) we see the image $G(G^2(B_2)\cap A_2)$ (magenta) of the upper part of $G^2(B_2)$. We have $G(G^2(B_2)\cap A_2)\subset G(B_2)\subset A_2$ so further iterations of these points will be similar to that of whole $G(B_2)$. In Figure 23 b) we see the image $G(G^2(B_2)\cap A_1)$ (magenta) of the lower part of $G^2(B_2)$. We see that the points of $G(G^2(B_2)\cap A_1)$ are either in B_1 (and then their future iterates are attracted to the line $x+y=4/3$) or they are inside $G(B_2)$ above the line $y=-x/2+7/6$ (upper red). The lowest point of $G(G^2(B_2)\cap A_1)$ is $(1/6,3/4)$ and belongs to the line $y=-x/2+5/6$ (lower red).

Under the action of G every point in A_2 gets closer to the line $x+y=4/3$ (blue). To show that every point of $G(B_2)$ is attracted to this line, it is enough to show

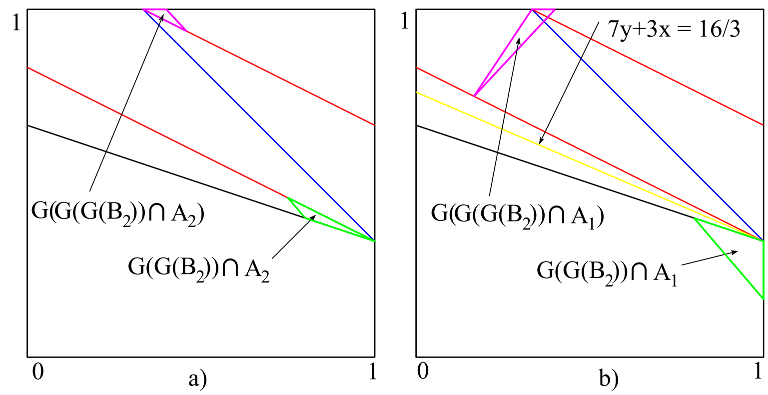

FIGURE 23. Images of a) the upper part and b) the lower part of $G(G(B_2))$

that for any point $(x, y) \in G^2(B_2) \cap A_1$ its image $(z, w) = (y, \frac{3}{2}y + \frac{1}{2}x)$ is either in B_1 or is closer to the line $x + y = 4/3$ than (x, y). Using the formula for the distance of a point from a line we have to check that

$$|x + y - 4/3| > |z + w - 4/3|.$$

Since the point (x, y) is below the partition line we have $|x+y-4/3| = 4/3-x-y$. Since the point (z, w) is above line $y = -x/2 + 7/6$ (upper red) we have $|z + w - 4/3| = z+w-4/3$. Thus, our condition is equivalent to $4/3-x-y > z+w-4/3$, or

$$4/3 - x - y > y + \frac{3}{2}y + \frac{1}{2}x - 4/3, \text{ or } y < -\frac{3}{7}x + \frac{16}{21}. \tag{7.2}$$

The line $y = -\frac{3}{7}x + \frac{16}{21}$ (yellow) intersects the partition line $y = -\frac{1}{3}x + \frac{2}{3}$ at the point $(1, 1/3)$ and for $x \in (0, 1)$ is above it. Thus, all points in $G^2(B_2) \cap A_1$ satisfy the condition (7.2). This proves Theorem 5.

7.3. $1/2 < \alpha < 3/4$. Let $1/2 < \alpha < 3/4$. We will prove that the fixed point $x_0 = (2/3, 2/3)$ is the global attractor attracting all points except $(0, 0)$. The derivative matrix at x_0 is

$$D = \begin{bmatrix} 0 & 1 \\ -2(1-\alpha) & -2\alpha \end{bmatrix},$$

with eigenvalues $e_1 = -\alpha + \sqrt{\alpha^2 + 2\alpha - 2}$, $e_2 = -\alpha - \sqrt{\alpha^2 + 2\alpha - 2}$ which are complex for $1/2 < \alpha < \sqrt{3} - 1$ and real for $\sqrt{3} - 1 \le \alpha < 3/4$. In the interval $(1/2, \sqrt{3}-1)$ their moduluses are equal to $|e_1| = |e_2| = \sqrt{2(1-\alpha)}$ and less than 1. In the interval $[\sqrt{3} - 1, 3/4)$ eigenvalue e_2 has larger modulus equal $|e_2| = -e_2 = \alpha + \sqrt{\alpha^2 + 2\alpha - 2}$ also less than 1. Thus, x_0 is an attracting fixed point.

We will now prove a few facts. Recall that A_1 denote the part of the square $[0, 1] \times [0, 1]$ below the line $\alpha y + (1-\alpha)x = 1/2$ and A_2 the part above this line.

We extend Proposition 1 to :

Proposition 19. *If $(x, y) \in A_1$ and $\alpha y + (1-\alpha)x > a$, $a < 1/2$, then the point $(w, z) = G(x, y)$ satisfies $\alpha z + (1-\alpha)w > 2\alpha a$, holds also for the $1/2 < \alpha < 3/4$.*

Proposition 20. *If $(x, y) \in A_2$, then the point $(w, z) = G(x, y)$ satisfies $\alpha z + (1-\alpha)w \ge 1-\alpha$.*

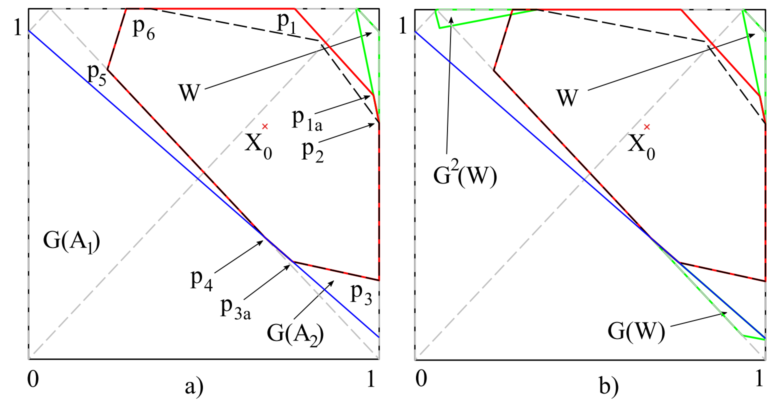

FIGURE 24. Trapping region T for $1/2 < \alpha \leq\sim 0.593$. Case $\alpha = 0.533$ is shown.

Proof. We have

$$\alpha z+(1-\alpha)w = \alpha(2-2\alpha y-2(1-\alpha)x)+(1-\alpha)y = 2\alpha-[2\alpha^2+\alpha-1]y-2\alpha(1-\alpha)x\,.$$

The inequality

$$2\alpha - [2\alpha^2 + \alpha - 1]y - 2\alpha(1-\alpha)x \geq 1-\alpha\,,$$

is equivalent to

$$[2\alpha^2 + \alpha - 1]y + 2\alpha(1-\alpha)x \leq 3\alpha - 1\,.$$

For $\alpha > 1/2$ the left hand side of the inequality is an increasing function of x and y with maximum at $(1,1)$ equal to $3\alpha - 1$. This completes the proof. □

Let A_I' denote the part of the square $[0,1]\times[0,1]$ above the line $\alpha y+(1-\alpha)x = 1-\alpha$. Propositions 1 and 20 prove that $G(A_I') \subset A_I'$, i.e., the region A_I' is G-invariant. It follows from Proposition 1 that every point of A_1, except $(0,0)$, enters A_I' after a finite number of steps.

Proposition 21. *For every* $(x,y) \in A_1 \cap A_I'$ *we have* $G(x,y) \in A_2$ *or* $G^2(x,y) \in A_2$.

Proof. Applying Proposition 19 twice and Proposition 20 for $a = 1-\alpha$, it is enough to show that

$$(2\alpha)^2(1-\alpha) > 1/2\,.$$

Let $f(\alpha) = \alpha^2(1-\alpha)$. It is easy to check that on interval $[1/2, 3/4]$ function f is concave with maximum at $2/3$. We have $f(1/2) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$. Also, $f(3/4) = \frac{9}{16}\cdot\frac{1}{4} = \frac{9}{64} > \frac{1}{8}$. This completes the proof. □

Proposition 22. *For* $1/2 < \alpha \leq (\sqrt{33}-1)/8 \sim 0.5930703309$*, the fixed point* $X_0 = (2/3, 2/3)$ *attracts all points except* $(0,0)$.

Proof. We will construct a trapping region $T \subset A_2$, containing X_0, such that $G(T) \subset T$. Every point whose trajectory stays in A_2 is attracted to X_0, since $G_{|A_2}$ is an affine map with an attracting point X_0. We will prove that every point of A_2 eventually enters T. From Proposition 1 we know that every point except $(0,0)$ eventually enters A_2.

Construction of T: The trapping region T is shown in Figure 24 a). It is a polygon with vertices $p_1, p_{1a}, p_2, p_3, p_{3a}, p_4, p_5$ and p_6 (red). Its image $G(T)$ is bounded by

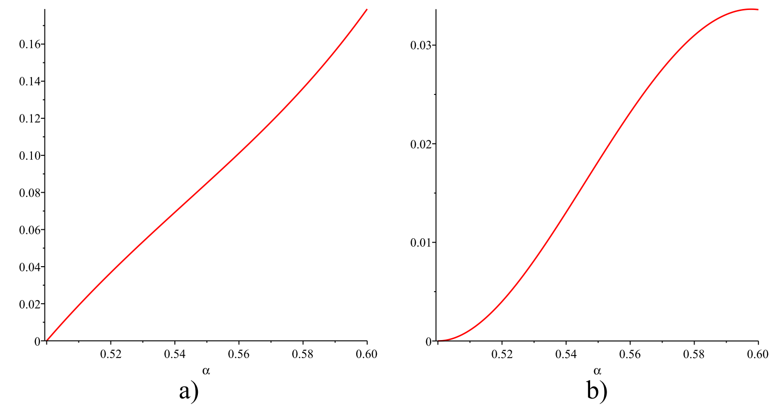

FIGURE 25. a)The graph of $z-t$ and b) of $y(z_i)-y_w$ for the proof of Proposition 23

black dashed line. We will describe the choice of the vertices. Let $G_i = G_{|A_i}$, $i = 1, 2$. The large quadrangles bounded by dashed grey lines are the sets $G(A_1)$ and $G(A_2)$. We do not need to consider the points outside $G(A_1) \cup G(A_2)$ as they are transient and their images eventually go into trapping region or the region bounded by green lines. The green quadrangle (it looks like a triangle) is the set $W = G_2^{-1}(A_1) \cap G(A_2)$, the non-transient points of A_2 which go in one step to A_1. Point p_2 is the lowest vertex of W. Then, consecutively $p_6 = G_2^{-1}(p_2)$, $p_{3a} = G_2^{-1}(p_6)$ and $p_{1a} = G_2^{-1}(p_{3a})$. For the point p_5 we have $p_3 = G_2^{-1}(p_5)$ and $p_1 = G_2^{-1}(p_3)$. The point p_5 is chosen on the boundary of $G(A_2)$ in such a way that its image $G(p_5)$ lies to the left of the line connecting p_1 and p_{1a}. Finally, p_4 is the intersection of the lower boundary of $G(A_2)$ and the partition line (blue). We also have $p_4 = G(p_2)$. By construction, every vertex of T goes into T. Since T is convex, we have $G(T) \subset T$.

The only thing we have to prove is that any point of W (non-transient points going out of A_2) eventually enters the trapping region T. In Figure 24 b) we see that the second image $G^2(W)$ is a thin quadrangle (looking like a triangle) adjacent to the upper boundary of the square $[0,1] \times [0,1]$. The lowest point of $G^2(W)$ is the point $(2\alpha(2\alpha-1), 8\alpha^3 - 8\alpha + 4)$. Its most to the right point is $(\alpha/(\alpha+1), 1)$. We will prove in Proposition 23 that for any point (x, y) with $x \le x_w = \alpha/(\alpha+1)$ and $y \ge y_w = 8\alpha^3 - 8\alpha + 4$ and its third image $(z, w) = G^3(x, y)$ the difference $z - x$ is larger than some positive constant depending on α and $w \ge y_w$ unless $(z, w) \in T$. This shows that any point of $G^2(W)$ eventually enters T, and completes the proof of Proposition 22. □

Proposition 23. *Let* $1/2 < \alpha \le (\sqrt{33}-1)/8 \sim 0.5930703309$. *Let point* (x, y) *satisfies* $x \le x_w = \alpha/(\alpha+1)$ *and* $y \ge y_w = 8\alpha^3 - 8\alpha + 4$. *Then, for its third image* $(z, w) = G^3(x, y)$ *the difference* $z - x$ *is larger then some positive constant depending on* α. *If* $(z, w) \notin T$, *then* $w \ge y_w$.

Proof. Let $(x, y) = (t, 1-s)$ satisfy the assumptions. The third iterate G^3 on such point is equal either $G_1 \circ G_2 \circ G_2$ or $G_2 \circ G_2 \circ G_2$. The first coordinate of $(z, w) = G^3(x, y)$ does not depend on the whether the last map applied is G_1 or

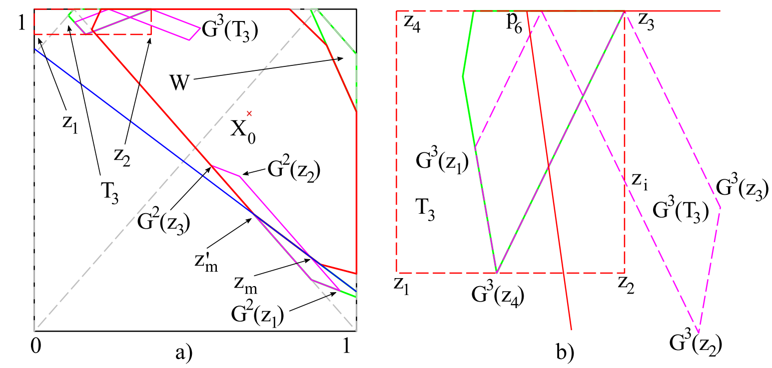

FIGURE 26. a) T_3 and its images, b) enlargement of T_3 and $G^3(T_3)$.

G_2. We have $z - t = ct(\alpha)t + cs(\alpha)s + cc(\alpha)$, where

$$ct = -4\alpha^2 + 4\alpha - 1 < 0 \ , \ cs = -4\alpha^2 - 2\alpha + 2 < 0 \ , \ cc = 2\alpha(2\alpha - 1) > 0.$$

Since both $ct(\alpha)$ and $cs(\alpha)$ are negative $z - t$ has the least value when both t and s are maximal, i.e., $t = x_w$ and $s = 1 - y_w$. Then,

$$z - t = 2(2\alpha - 1)(8\alpha^4 + 4\alpha^3 - 4\alpha^2 - 5\alpha + 3) > 0.$$

The graph of $z - t$ is shown in Figure 25.

To prove the second claim we will consider the images of the rectangle T_3 (see Figures 25 b) and 26) with vertices $z_1 = (0, y_w)$, $z_2 = (x_w, y_w)$, $z_3 = (x_w, 1)$ and $z_4 = (0, 1)$. The second image $G_2^2(T_3)$ has the vertices $G^2(z_1), G^2(z_4) \in A_1$ and $G^2(z_2), G^2(z_3) \in A_2$. Its sides intersect partition line at points z_m between $G^2(z_1)$ and $G^2(z_2)$ and z'_m between $G^2(z_3)$ and $G^2(z_4)$. The image $G^3(z_4)$ lies on the lower side of the rectangle T_3 and the image $G^3(z_1)$ is higher. The images

$$G(z_m) = ((8\alpha^4 - 12\alpha^3 - 6\alpha^2 + 17\alpha - 6)/(\alpha + 1), 1)$$

and $G(z'_m)$ are on the top side of the square. The image

$$G^3(z_2) = \Big(\frac{2}{\alpha+1}\left(16\alpha^6 + 24\alpha^5 - 16\alpha^4 - 26\alpha^3 + 12\alpha^2 + 7\alpha - 3\right), \\ -64\alpha^6 - 64\alpha^5 + 128\alpha^4 + 40\alpha^3 - 100\alpha^2 + 36\alpha - 2\Big).$$

The line $L(G^3(z_2), G(z_m))$ intersects right hand side of T_3 at the point $z_i = (x_w, 16\alpha^4 - 32\alpha^3 + 38\alpha - 27 + 6/\alpha)$ with the second coordinate larger than y_w. This shows, that the points of $G^3(T_3)$ lie either in T_3 or in T. Together with the first claim this shows that every point of T_3 eventually enters T. □

We continue to prove that the fixed point $(2/3, 2/3)$ is a global attractor for other intervals of parameter $\alpha \in (1/2, 3/4)$.

Proposition 24. *For* $(\sqrt{33} - 1)/8 < \alpha \le \sqrt{33}/12 + 1/4 \sim 0.7287135539$*, the fixed point* $X_0 = (2/3, 2/3)$ *attracts all points except* $(0, 0)$.

Proof. The general plan of the proof is the same as for Proposition 22. We construct a trapping region T and show that some (fourth or fifth) image of $W = G_2^{-1}(A_1) \cap G(A_2)$ falls into T.

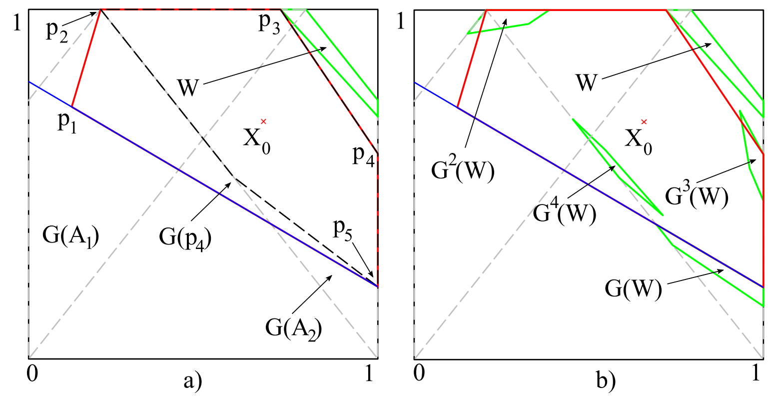

FIGURE 27. $\alpha = 0.63$ (case ii)) a) Trapping region T (red) and its image $G(T)$ (dashed black). b) Region W and its images, $G^4(W) \subset T$.

Construction of the trapping region T: T is a pentagon with the vertices: p_3 which is the upper left vertex of W, $p_5 = G(p_3)$, $p_2 = G(p_5)$, $p_4 = G(p_2)$, and $p_1 = G_2^{-1}(p_3)$. Since, for α in the considered interval, $G(p_4) \in T$, we $G(T) \subset T$, i.e., T is a trapping region. Figure 27 a) shows the trapping region T (red) and its image $G(T)$ (dashed black). The green quadrangle is $W = G_2^{-1}(A_1) \cap G(A_2)$.

Below, we will show that fifth or fourth image of W is a subset of T. We consider subintervals of α.

i) $(\sim 0.5930703309, \sim 0.5970091680)$

$\alpha = (\sqrt{33} - 1)/8 \sim 0.5930703309$ is the largest α for which the sides of W and $G^3(W)$ which are on the line $x = 1$ intersect. For $\alpha \in (\sim 0.5930703309, \sim 0.5970091680)$, W and $G^3(W)$ still intersect (the highest vertex of $G^3(W)$ is in W). (~ 0.5970091680 is a root of $16\alpha^5 - 16\alpha^3 + 10\alpha^2 - 9\alpha + 4 = 0$.) This causes a minimal "spill off" of $G^4(W)$ outside T. See Figure 28. We also see there that $G^5(W) \subset T$.

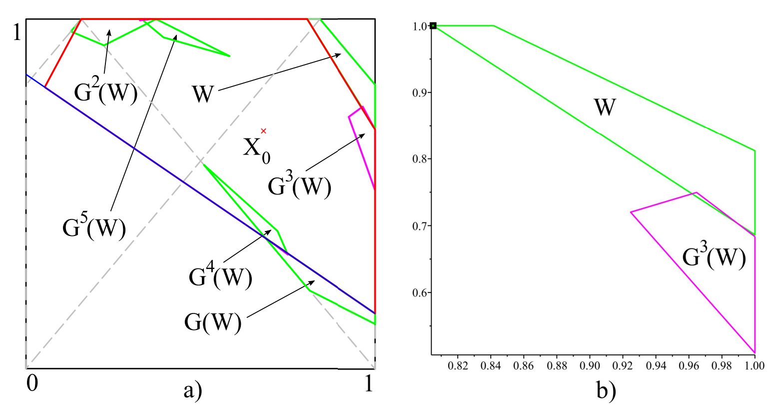

FIGURE 28. $\alpha = 0.594$ (case i)) a) Region W and its images in green except for $G^3(W)$ in magenta, $G^5(W) \subset T$. b) Enlargement of the intersection of W and $G^3(W)$ which causes $G^4(W) \not\subset T$.

ii) $(\sim 0.5970091680, \sim 0.6513878188)$

For $\alpha \in (\sim 0.5970091680, (\sqrt{13}-1)/4 =\sim 0.6513878188)$ the set W and $G^3(W)$ no longer intersect and $G^4(W) \subset T$. See Figure 27 b). Value $\alpha = (\sqrt{13}-1)/4$ is the point where W stops to be a quadrangle and starts to be just a triangle.

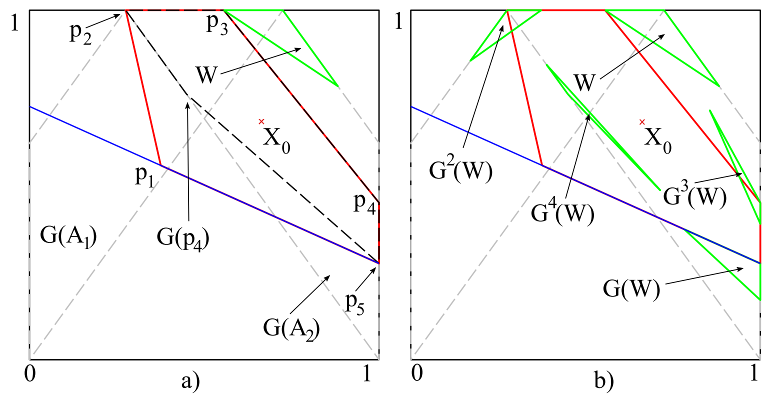

FIGURE 29. $\alpha = 0.69$ (case iii)) a) Trapping region T (red) and its image $G(T)$ (dashed black). b) Region W and its images, $G^4(W) \subset T$.

iii)$(\sim 0.6513878188, \sim 0.7287135539)$

For α between ~ 0.6513878188 and $1/4+\sqrt{(33)}/12 =\sim 0.7287135539$, the region W is a triangle and $G^4(W) \subset T$. See Figure 29. Part a) shows the trapping region T (red) and its image $G(T)$ (dashed black). Part b) shows region W and its images, $G^4(W) \subset T$. For α approaching 0.7287135539 the top vertex of $G^4(W)$ approaches boundary of T but stays in T as it is the image of the lowest vertex of $G^3(W)$ which is already in T. For α above ~ 0.7287135539 the image $G(p_4)$ goes outside the line $L(p_1, p_2)$ and T is no longer a trapping region. □

For the next interval of parameter α we have to make a "micro" adjustment of T adding to its construction two more vertices $G(p_4)$ and $G^2(p_4)$.

Proposition 25. *For* $\sim 0.7287135539 < \alpha \leq\sim 0.7360241475$*, the fixed point* $X_0 = (2/3, 2/3)$ *attracts all points except* $(0, 0)$*.* ~ 0.7360241475 *is the root of* $4\alpha^4 - 8\alpha^3 + 14\alpha^2 - 13\alpha + 4 = 0$*. Above this value of* α *sets* W *and* $G^2(W)$ *intersect.*

Proof. Again, we construct a trapping region T and show that fourth image of $W = G_2^{-1}(A_1) \cap G(A_2)$ falls into T. The construction of T is a micro adjustment of the construction from Proposition 24, it is almost not visible on pictures. We add two more vertices , $p_{1a} = G(p_4)$ and $p_{3a} = G^2(p_4)$, to the the construction and T becomes a heptagon (seven angles figure). Since $G(p_{3a})$ is inside such constructed T, and T is convex, we have $G_2(T) \subset (T)$. See Figure 30. Part a) shows the trapping region T (red) and its image $G(T)$ (dashed black). The green triangle is the region W. $G(p_{3a})$ stays inside T up to $\alpha =\sim 0.7464180853$ but earlier another problem arises. At $\alpha =\sim 0.7360241475$ the image $G^2(W)$ starts intersecting with W and this needs another approach.

Figure 30 b) shows W and its images with $G^4(W) \subset T$. Two upper vertices of $G^4(W)$ are on the boundary of T since the corresponding vertices of $G^3(W)$ are

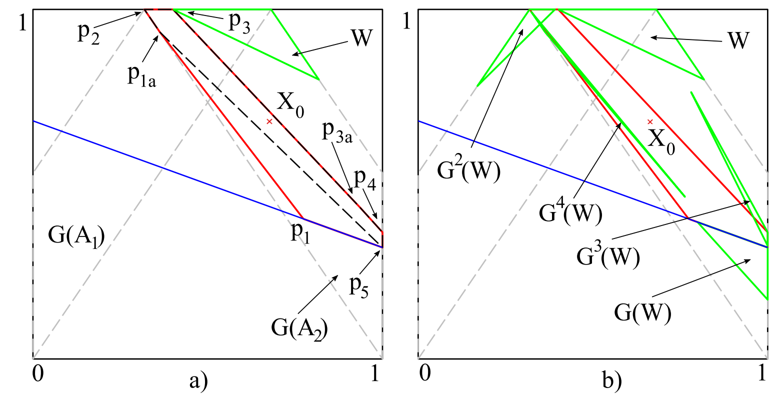

FIGURE 30. $\alpha = 0.734$ a) the trapping region T (red) and its image $G(T)$ (dashed black). b) shows W and its images with $G^4(W) \subset T$.

already on the boundary of T. This is better visible on the Figure 31 b) presenting T, $G^3(W)$ and $G^4(W)$. Figure 31 a) shows the old trapping region of Proposition 24 and the points $G(p_4)$, $G^2(p_4)$ both outside this region as well as the point $G^3(p_4) = G(p_{3a})$ well inside T.

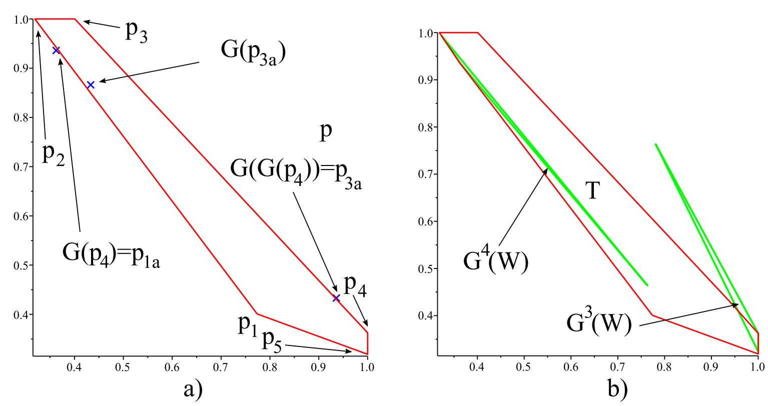

FIGURE 31. $\alpha = 0.734$ a)the old trapping region of Proposition 24 and the points $G(p_4)$, $G^2(p_4)$, $G^3(p_4)$. b) enlarged T, $G^3(W)$ and $G^4(W)$.

Now, we will consider the last subinterval of α's for which X_0 is an almost global attractor.

Proposition 26. *For* $\sim 0.7360241475 < \alpha < 3/4$*, the fixed point* $X_0 = (2/3, 2/3)$ *attracts all points except* $(0, 0)$*.*

Proof. For $\sqrt{3} - 1 < \alpha < 3/4$, $\sqrt{3} - 1 =\sim 0.732050808$, the eigenvalues of DG_2 are real and both between -1 and $-1/2$. They are $\lambda_{1,2} = -\alpha \pm \sqrt{\alpha^2 + 2\alpha - 2}$. The corresponding eigenvectors are $v_{1,2} = [(-\alpha \pm \sqrt{\alpha^2 + 2\alpha - 2})^{-1}, 1]$.

Since α's up to ~ 0.7360241475 were already considered, we will study only the interval $(\sim 0.7360241475, 3/4)$. The trapping region will be constructed using

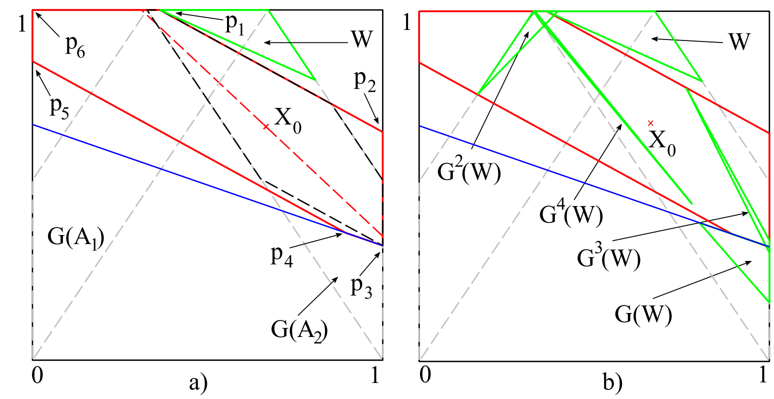

FIGURE 32. $\alpha = 0743$ a) Trapping region T (red) and its image $G(T)$ (dashed black). The dashed red line is an eigenline going through X_0. b) Region W and its images (green), $G^4(W) \subset T$.

the the vector v_1, see Figure 32 a). Let p_1 be the left upper vertex of $W = G_2^{-1}(A_1) \cap G_2(A_2)$ and $p_4 = G_2^{-1}(p_1)$ its preimage on the partition line. T is the part of A_2 between the lines L_1, L_4 going through points p_1 and p_4, respectively, and parallel to the vector v_1. Thus, T is a hexagon with vertices p_1, $p_2 = L_1 \cap \{x = 1\}$, $p_3 = \text{partitionline} \cap \{x = 1\}$, p_4, $p_5 = L_4 \cap \{x = 0\}$ and $p_6 = (0, 1)$. T is a trapping region, $G(T) \subset T$, by construction since its sides are the eigenlines and eigenvalues have absolute values less than one. Figure 32 a) shows the trapping region T (red) and its image $G(T)$ (dashed black). The dashed red line is an eigenline (parallel to v_2) going through X_0.

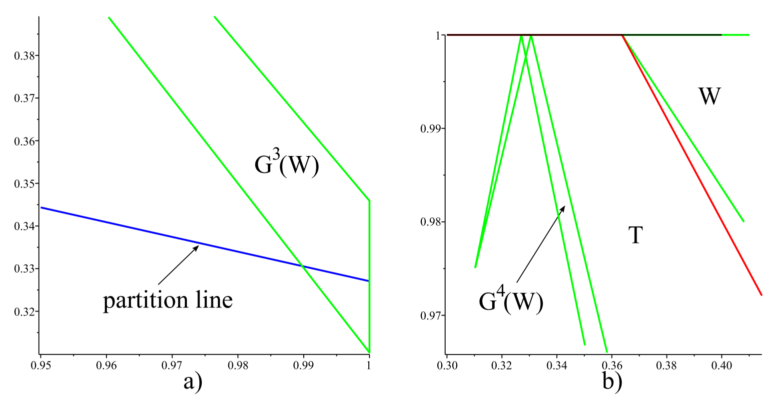

FIGURE 33. $\alpha = 0743$ a) Lower part of $G^3(W)$ and b) upper part of $G^4(W)$.

In Figure 32 b) we see region W and its images (green). We see that $G^4(W) \subset T$. It can be proven that the lowest vertex of $G^4(W)$ touches the line $L(p_4, p_5)$ first time for $\alpha = 3/4$. Lower part of $G^3(W)$ and upper part of $G^4(W)$ are shown more precisely in Figure 33 a) and b), respectively. Since $G^3(W)$ crosses the partition line, its image $G^4(W)$ is "broken". □

Propositions 22, 24, 25 and 26 together prove the following:

Theorem 6. *For $1/2 < \alpha < 3/4$, the fixed point $X_0 = (2/3, 2/3)$ attracts all points except $(0,0)$, so it is an almost global attractor.*

□

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