Worst-Case Valuation of Equity-Linked Products Using Risk-Minimizing Strategies

Emmanuel Sekyere Osei Mireku

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By: Emmanuel Sekyere Osei Mireku

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Signed by the final Examining Committee:

____________________________________ Thesis Supervisor

Dr. P. Gaillardetz

____________________________________ Examiner

Prof. J. Garrido

____________________________________ Examiner

Dr. F. Godin

Approved by ____

Chair of Department or Graduate Program Director

____

Dean of Faculty

____

Date

Abstract

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The global market for life insurance products has been stable over the years. However, equity-linked products which form about fifteen percent of the total life insurance market has experienced a decline in premiums written. The impact of model risk when hedging these investment guarantees has been found to be significant. We propose a framework to determine the worst case value of an equity-linked product through partial hedging using quantile and conditional value-at-risk measures. The model integrates both the mortality and the financial risk associated with these products to estimate the value as well as the hedging strategy. We rely on robust optimization techniques for the worst case hedging strategy. To demonstrate the versatility of the framework, we present numerical examples of point-to-point equity-indexed annuities in multinomial lattice dynamics.

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Chapter 1

Introduction

1.1 Background

Annuities are payments made at equal intervals of time. Traditionally, they serve as a type of investment for retirement purposes. Annuities whose returns are based on offering participation from the performance of some mutual fund or equity index are known as Equity-Linked Products. Here, the policyholder is an individual who makes an investment in the form of a single premium or periodic payments for a term and in return is guaranteed an accumulation on premium and some extra benefit at maturity based on the growth of the fund or index. Equity-linked products whose returns are tied to the performance of a mutual fund and guarantees a minimum rate of return are known as Variable Annuities (VAs) in the United States, Segregated Funds in Canada or Unit-Linked in some parts of Europe. Since its introduction in 1995 by Keyport Life Insurance Co., Equity-Index Annuities (EIAs) have become popular with stable sales over a short period of time. With an estimated sales of $69.6 billion by the fourth quarter of 2018, Equity-Indexed Annuities (EIAs) have been seen as the most innovative annuity product to ever hit the United States market. Variable Annuities on the other hand saw a $100.1 billion in sales in 2018, see LIMRA [2018].

Several models have been proposed for the pricing and valuation of equity-linked products.

In the Black-Scholes framework, Tiong [2000] considered the use of Esscher transforms to compute the prices of various types of EIAs in closed form by considering their embedded options. A more thorough discussion of these guaranteed investments is provided by Hardy [2003] where she considered the use of regime-switching log-normal index processes to compute the price of equity-linked products. Lee [2003] also develops a model to price EIAs using path-dependent options. The high volatility of the index creates an increase in the cost of the embedded options and triggers insurers to lower the participation rates. By considering path-dependent options, Lee [2003] derives a joint distribution function for the terminal time value and running maximum of a Brownian motion to use with Esscher transforms and propose direct pricing formulas.

Some variants of EIAs may be seen as American-type products. In this case, they are composed of embedded American options. Longstaff and Schwartz [2001] value American options by simulation. Since the optimal exercise strategy is mainly determined by the conditional expectation of the payoff from exercising the option immediately, by estimating the conditional expectation function for each exercise date, a complete specification of the optimal exercise strategy along each path is obtained. As such, they show that least square Monte-Carlo simulations can readily and accurately be applicable in path-dependent and multi-factor situations where traditional finite difference techniques cannot be used.

More recently, Quan et al. [2018] use classification and regression trees to develop the price of variable annuities using a large sample of variable annuity portfolios. They compare the predictive accuracy of tree-based models that includes traditional regression trees and observed that these models are generally efficient in producing more accurate predictions based on the synthetic data-set they generated.

Due to the various financial guarantees embedded in the structure of equity-linked products, traditional actuarial risk management practices cannot be extended to the valuation of EIAs. Gaillardetz and Lakhmiri [2011] consider a loaded premium that protects the issuers against the financial and mortality risks by obtaining a fair value contract using arbitrage-free theory and estimating a new participation rate using hedging errors on the investment loading. Boyle and Hardy [1997] also consider the reserving of maturity guarantees using the stochastic simulation of future investment returns and option pricing theory. In the former, they adopt the Wilkie model to calibrate and simulate future returns and compute both the expected cost and reserves for the guarantees at different risk levels and initial premiums. In the latter, they consider a modified Black-Scholes pricing model to dynamically hedge the guarantees for different maturities and volatility.

Equity-linked products are considered to be long term products with maturities spanning as long as 20 years. However, most hedging strategies in the literature work best for shorter maturities, considered to be self-financing and priced under market completeness assumptions. A market is said to be incomplete if some payoffs cannot be replicated by trading in marketed securities. As such, perfect transfer of risk is not possible. Staum [2007] iterates that the classic no-arbitrage theory of valuation in a complete market, based on the unique price of a self-financing replicating portfolio, is not adequate for non-replicable payoffs in incomplete markets.

Due to the presence of mortality and financial risk, and even in some cases surrender risk, EIAs are not suitable to be priced under market completeness. Bacinello [2003] models the policyholder surrender conditions of a unit-linked product where they extend an analysis of exogenous minimum guarantees and take into account the possibility that the surrender values can be determined endogenously. Møller [1998] also explains that due to market incompleteness the claims cannot be hedged completely by trading stocks and risk-free assets only. As such they propose risk-minimizing strategies and their associated intrinsic risk processes by extending the model to a situation where they can eliminate the risk completely to determine self-financing hedging strategies. The use of risk-minimizing strategies in hedging contingent claims has also been considered by Gaillardetz and Moghtadai [2017] where they made a comparison of the quadratic hedging strategy with two proposed strategies. The optimal iterated strategy, that seeks to minimize a risk measure and the allocation constraint minimizes the portfolio value based on some initial value or wealth.

Robust optimization involves several techniques that protect a decision-maker against parameter ambiguity and random uncertainty. In finance, the concept has been used in portfolio management and asset allocation. Scutellà and Recchia [2013] uses robust counterparts of the classical mean-variance and minimum-variance portfolio optimization problems to address uncertainty in portfolio asset allocation by focusing on robust optimization methods. After showing that errors in expected return estimates can lead to optimal portfolios with weights that are significantly different from the true optimal portfolio, Ceria and Stubbs [2006] introduce robust optimization to reduce some of the ill-effects of optimization caused by estimation error in expected return estimates.

1.2 Motivation

In analyzing the risk underlying investment guarantees, Augustyniak and Boudreault [2012] considered several econometric models and conducted out of sample analysis of these models compared to observed equity-linked returns during the financial crisis. They observe that tail risk measures vary significantly across the various models. More importantly, an analysis of the delta-hedging strategy, in addition, indicated large hedging errors. These observations stress the consideration of the impact of model risk when hedging investment guarantees.

Hardy et al. [2006] also validates some long-term models for equity-linked guarantees and observed that in comparing the results from two models, both had very similar likelihoods but their residuals gave very different capital requirements. Given that the Canadian Institute of Actuaries (CIA) and the American Academy of Actuaries (AAA) recommend the use of stochastic models for reserving losses on equity-linked insurance, including segregated funds and variable annuities, it is important to have a measure that could serve as a yardstick for both insurers and regulators to compare various model risks to, and hence the consideration for the worst-case value of equity-linked products.

The use of worst-case models in minimizing risk has been considered and proven to be efficient. Chen [2009] uses the worst-case model in an ϵ-arbitrage framework to compute the price of options and observed close matches when the results were compared to observed market prices. The concept has also been widely adapted in optimizing the returns of portfolios in finance, see Rockafellar et al. [2000]. Zhu and Fukushima [2009] and Cornuejols and Tütüncü [2006] also apply this concept to the conditional value-at-risk in portfolio management.

Our approach to estimate the worst-case value of a contract rests on the solution of a robust linear optimization problem defined by considering a multinomial lattice for the index structure coupled with product losses restricted by a risk measure. Our aim is to minimize the cost of establishing a portfolio that replicates our contingent claim by minimizing the risk under uncertainty in the index model. This approach allows us to develop the model using additional information on index return dynamics defined by adjusting the uncertainty sets used, and the flexibility of selecting different volatility which could also be allowed to follow the volatility index.

1.3 Thesis Overview

In Chapter 2, we describe the risks that are associated with equity-linked products. For index risk, we define an index model with discrete state space, specifically a lattice model and define how a trading strategy can be established from such a model. We further describe mortality risk and introduce the notations that will be used to define our losses. Since we use risk-minimizing strategies in our optimization, we discuss two known risk measures and describe how they can be adapted to suit our model. Lastly, we define a typical loss random variable for a portfolio that has both financial and mortality risks.

In Chapter 3, we discuss some hedging strategies and define our hedging model that needs to be optimized. In Chapter 4, we give a brief description of robust optimization, a technique we intend to use to solve our hedging optimization problem. We describe uncertainty in a model and define some uncertainty sets. We conclude the chapter by exploring some robust optimization strategies and formulate the robust counterpart of our worst-case hedging strategy.

In Chapter 5, we briefly explain how we can set up our capital requirement from the model. A concept that is very important to regulators. In Chapter 6, we describe the investment guarantees that come with equity-linked products and define the various indexing methods used in valuing equity-indexed annuities. We conclude the chapter by analyzing the numerical implications, and relevance of the model to different sampling techniques and uncertainty sets. We also discuss the sensitivity of the model to basic pricing parameters of equity-indexed annuities. We conclude with summary and recommendations to our work.

Chapter 2

Index, Mortality and Risk

In this chapter, we describe the structure of a typical index and mortality model and introduce the actuarial notations that will be used. We continue to talk about the risk measures that will be used in the next chapter and provide their characteristics and coherency. Finally, we define a typical portfolio loss function for an equity-linked product.

2.1 Index Model

Equity-linked products provide returns based on some stock index. There are several models used to imitate the stock index dynamics. Lattice pricing models over the years have been shown to be resilient in modeling stock indexes, interest rates, stock, and other securities. First introduced by Cox et al. [1979], the two-state (binomial) lattice approach proved to be a valuable model in the valuation of several financial securities. With various assumptions, their fundamental market process was shown to converge to the independent log-normal model. The model failed to work when market variations and extreme movement in the long-term are considered. To adjust the model by Cox et al. [1979], Boyle [1988] introduces a three-state lattice model whose risk-neutral probabilities were computed by matching the moments of the discrete model to some mean and variance of a continuous log-normal distribution. Since then several multi-period discrete multinomial lattice models have been developed with a number of continuous-time approximation models. For example, see Jabbour et al. [2004].

2.1.1 Lattice Model

In our financial model, we assume a finite market where all pertinent quantities take discrete values. As such, we consider the finite state space Ω such that for all $\psi \in \Omega$, $\mathbb{P}(\psi) > 0$. Let n, the time horizon in years within which all activities are expected to happen, be divided into N periods per year, each of length $\Delta = \frac{1}{N}$. Let the stock price process $\mathcal{S} = \{S_t : t = 0, \Delta, 2\Delta, \cdots, n\}$ be positive random variables such that the measured space S_t takes values in the finite set Ω and define r_t to be the annualized risk-free rate effective in the interval $[t, t+\Delta]$. We denote the time t price of an at-the-money option that matures at time n to be $P(t, n)$ and without loss of generality, let the index price at time $t = 0$ be one unit.
The assumption of a finite Ω helps develop a lattice model. For $k+1$ number of branches per lattice point in a tree and $i = 0, 1, 2, \cdots, tk/\Delta$, let $\psi_{i,t}$ be the event of realizations relative to node i at time t. In Figure 2.1, at $t = 0$, we denote $\psi_{0,0}$ to be the unique root node from which the lattice arises. At $t = \Delta$, we have $k+1$ possible states values for $\psi_{i,\Delta}$ with each connected to the root node. The lattice can be constructed to grow in such a way that, as an indication of history, each node at some time t is connected to the root node by a path. Next we let $\mathbb{P}[\psi_{i+j,t+\Delta}|\psi_{i,t}] = p_{i+j,t+\Delta|i,t}$ be the conditional probability of moving from state $\psi_{i,t}$ to $\psi_{i+j,t+\Delta}$, for $i = 0, 1, 2, \cdots, tk/\Delta$, $j = 0, 1, 2, \cdots, k$ and $t = 0, \Delta, \cdots, n - \Delta$.

2.1.2 Recombining Trees

We consider a multi-period multinomial recombining lattice model for our index dynamics. Here we can set the number of nodes or branches that can be created from the initial stock index and compute the various index price movements for each node through time. We keep in mind that we allow the tree nodes to recombine. For some stock index S_t at time t, the time $t+\Delta$ price is assumed to go up by a factor u, (uS_t) or down by a factor d, (dS_t) in a binomial framework. In a trinomial setting, the index either moves up (uS_t), remains the same (S_t)

or moves down (dS_t). Details of the trinomial lattice framework and the construction of probabilities can be found in Boyle [1988]. Furthermore, a typical multinomial model will have the time $t + \Delta$ indexes as

$$S_{t+\Delta}|S_t = S_t u^{k-j} d^j, \qquad \{j = 0, 1, \cdots, k\}, \tag{2.1}$$

S_t can be seen as a component of the state variable $\psi_{i,t}$. Also to ensure recombining of the trees for numerical tractability, we set $ud = 1$ and define $u = e^{\sigma\sqrt{\Delta/k}}$ and $d = 1/u$ for some σ. When $k = 1$, σ represents the volatility of the index returns for a discrete time increment Δ, see Cox et al. [1979]. The construction of the up and down factor using volatility allows us to capture the market dynamics which relies strongly on volatility. Also, because the tree is recombining by construction, the level of computation implied is manageable.

We construct the tree by starting with an initial index value at time $t = 0$. Setting the initial value to one makes it easy to re-scale the tree by simply multiplying by whichever initial value is desired. We then compute the first $k + 1$ branches of the initial value at time $t = \Delta$ using (2.1). After this, for each one of the $k + 1$ branches, we compute their respect successors for the second period at time $t = 2\Delta$ and so on until maturity at time n, resulting in a multinomial index tree with each node at $t + \Delta$ connected to at least one node at t.

2.1.3 Binomial Structure

One difficulty faced when constructing a multinomial lattice is the description of the probabilities of moving from one state to another. The choice of structure of the index in (2.1) is consistent with the convergence to the geometric Brownian motion. As stated, if $p_{i+j,t+\Delta|i,t}$ is the conditional probability to reach state $\psi_{i+j,t+\Delta}$ from $\psi_{i,t}$, then using a typical binomial pricing model, we can set $q = \frac{e^{\mu\Delta/k} - d}{u - d}$ for some drift μ such that the related binomial distribution simulates the geometric Brownian motion of the underlying index and hence the

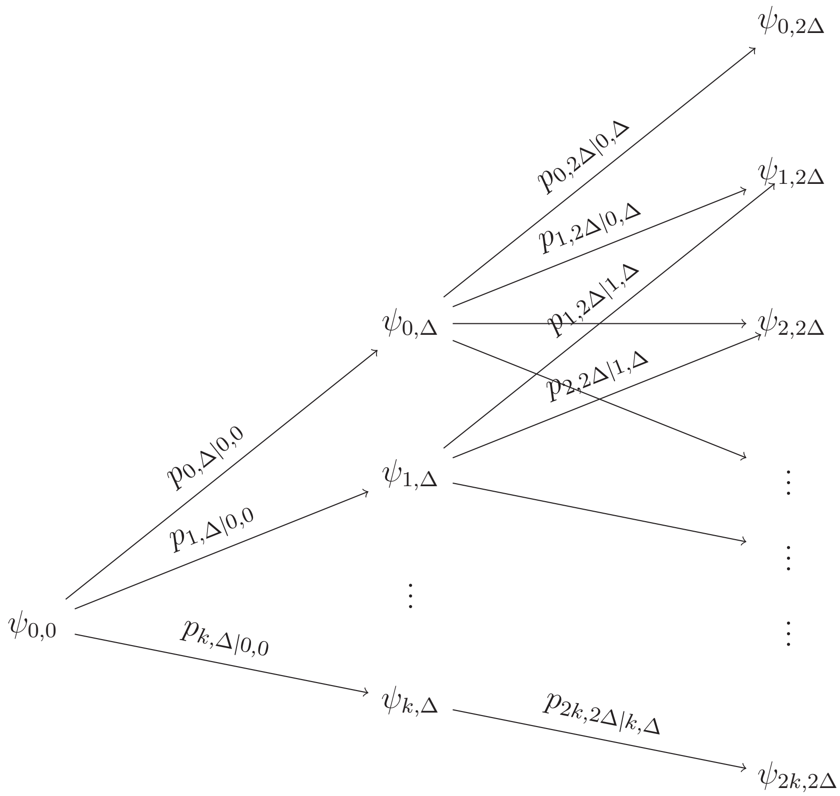

Figure 2.1: Multinomial lattice tree

probability $p_{i+j,t+\Delta|i,t}$ using a binomial structure is

$$p_{i+j,t+\Delta|i,t} = \binom{k}{j} q^{k-j}(1-q)^j \tag{2.2}$$

for $i = 0, 1, 2, \cdots, tk/\Delta$, $j = 0, 1, 2, \cdots, k$ and $t = 0, \Delta, \cdots, n - \Delta$. A diagram of a typical multinomial tree for our states and probabilities can be seen in Figure 2.1.

2.1.4 Trading Strategy

Bingham and Kiesel [2013] defines a trading strategy, or more importantly a dynamic portfolio $\mathcal{A}$ to be a stochastic vector process $\mathcal{A} = \{\mathcal{A}_{i,t}\}_{t=0}^{n} = \{[a_{i,t}, b_{i,t}, c_{i,t}]'\}_{t=0}^{n}$ in $\mathbb{R}^3$ to be adapted. Without loss of generality, we assume the portfolio comprises of three assets, $a_{i,t}$ dollar amount in an index, $b_{i,t}$ in a risk-free asset and $c_{i,t}$ in an option. It is however, possible to have multiple assets. Thus a trading strategy is simply a collection of the number of shares of an asset, which is determined at time t but are held in the portfolio at time $t+\Delta$. The values of $\mathcal{A}_{i,t}$ can be both negative and positive so that short-sales can be allowed and more importantly we assume that assets can be divided perfectly.

With the definition of a dynamic portfolio and the stock process, we can proceed to define the portfolio value at any time.

Definition 2.1.1 *The portfolio value $V_{i,t}^{(\mathcal{A})}$ at any time t is the scalar product of the trading strategies and the asset prices such that*

$$V_{i,t}^{(\mathcal{A})} = a_{i,t} + b_{i,t} + c_{i,t} \tag{2.3}$$

for $t = 0, \Delta, 2\Delta, \cdots, n$.

Thus $V_{i,t}^{(\mathcal{A})}$ can be seen as the wealth of value process generated by the trading strategy $\mathcal{A}$ and $V_{0,0}^{(\mathcal{A})}$, the initial investment or wealth of an investor.

Definition 2.1.2 *A trading strategy $\mathcal{A}$ is said to be self-financing if*

$$a_{i,t}\frac{S_{t+\Delta}}{S_t} + b_{i,t}e^{r_t\Delta} + c_{i,t}\frac{P(t+\Delta,n)}{P(t,n)} = a_{i,t+\Delta} + b_{i,t+\Delta} + c_{i,t+\Delta} \tag{2.4}$$

for $t = 0, \Delta, 2\Delta, \cdots, n$.

The definition of a self-financing portfolio implies that although the prices of assets or stocks change through time, the investor makes no injections or withdrawals from the portfolio. Thus the strategy is defined by re-allocating the number of asset shares at any point in time. This result is trivial in a discrete time setting but requires a little argument in a continuous time setting. An interesting feature of a self-financing strategy is that normalizing the asset prices has little or no economic effects.

Another feature that is paramount to the description of an index model is the concept of arbitrage. As several combinations of assets can be created from different trading strategies, we say a self-financing strategy creates an arbitrage opportunity if the initial portfolio value is zero but produces a non-negative final value almost surely and has a positive probability of positive final value. Bingham and Kiesel [2013] define arbitrage mathematically.

Definition 2.1.3 *Let Φ be a set of self-financing strategies, then a strategy $\mathcal{A} \in \Phi$ is an arbitrage strategy with respect to Φ if* $\mathbb{P}\{V_{0,0}^{(\mathcal{A})} = 0\} = 1$ *and the terminal wealth of $\mathcal{A}$ satisfies:* $\mathbb{P}\{V_{i,n}^{(\mathcal{A})} \geq 0\} = 1$ *and* $\mathbb{P}\{V_{i,n}^{(\mathcal{A})} > 0\} > 0$.

We say a market is complete if every contingent claim is attainable. That is, if we let $\mathcal{X}$ be any contingent claim, there exists a replicating self-financing strategy $\mathcal{A} \in \Phi$ such that $V_{i,n}^{(\mathcal{A})} = \mathcal{X}_{i,n}$. An arbitrage-free market is then said to be complete if and only if there exists a unique probability measure equivalent to $\mathbb{P}$ under which discounted asset prices are martingales. A concise proof of completeness in relation to the fundamental theorem of asset pricing in discrete time can be found in Bingham and Kiesel [2013] and that of continuous time in Björk [2009].

Time series regime switching methods can be extended to index models. Introduced in economics series, Hamilton [1989] proposed an adjustable approach to modeling changes in regimes by considering some parameters of an auto-regressive process as outcomes of some discrete state Markov process. Thus the probability of changing regime depends only on the current regime, not on the past observations of the process. A regime switching index model assumes that the index process switches randomly between some regimes. The rationale behind this assumption is that index prices may switch through time from say, a bullish market to a bearish market, a stable or low volatility state to an unstable or high volatility regime. Due to the uncertainty and randomness surrounding regime switching, stochastic modeling is used in dealing with such financial complexities. Caccia and Rémillard [2017] also apply the concept of regime switching to price and hedge options under discrete time auto-regressive hidden Markov models where they proposed a regime process to follow a Markov chain with some fixed transition matrix. They do this by first predicting the regime by assuming some stationary distributions, estimating parameters using the EM algorithm and conducting a global hedging using the regime predictions.

2.2 Mortality Model

In this section, we introduce the mortality process and notations in relation to equity-linked products. We adapt the notations from Dickson et al. [2013] and Gaillardetz and Moghtadai [2017].

There are types of mortality risk associated with equity-linked products. The systematic risk and the non-systematic risk. While systematic mortality risk is associated with the uncertainty due to a population's development of mortality rates, non-systematic mortality risk occurs when the number of deaths fluctuates around the probability of survival. The sale of a large number of homogeneous policies can mitigate the non-systematic risk but not the systematic risk.

When an equity-linked policy is issued, the insurer does not know the exact day the policy holder will die, surrender or terminate the policy. The only available future parameters may be the day the contract or policy matures. As such, there is the need to consider the mortality process of the policyholder during the term of the policy. If we let $(x) : x \geq 0$ denote a policyholder aged x, then we are interested in any future event that occurs after age x. We denote T_x to be the future lifetime random variable of (x). Thus the probability that life aged (x) dies before age $x + t$,

$$\mathbb{P}[T_x \leq t] = {}_tq_x, \tag{2.5}$$

and the probability that life aged (x) survives past age $x + t$ is

$$\mathbb{P}[T_x > t] = {}_tp_x. \tag{2.6}$$

The individual curtate future lifetime random variable of (x) is defined as the integer part of the future lifetime random variable and is denoted $K_x = \lfloor T_x \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function.

The probability function of K_x for $t = 0, 1, 2, \cdots$ is also defined as

$$\mathbb{P}[K_x = t] = \mathbb{P}[t \leq T_x \leq t+1] = {}_tp_x q_{x+t} \tag{2.7}$$

We consider l_0 mutually independent policyholders, and let (x_l) be the age of the i-th policyholder at the time issue, $l = 1, 2, \cdots, l_0$, then T_{x_l} is the time until the death of the l-th policyholder. If we consider the index framework in section (2.1) where the time to maturity is divided into N sub-periods, each of length $\Delta = \frac{1}{N}$, then the curtate future lifetime of (x) is

$$K_x = \lfloor NT_x \rfloor \Delta. \tag{2.8}$$

Also, we define n_l to be the contract maturity for the l-th policyholder and assume that a policyholder can exit the contract either through death or as the contract matures. Therefore, we can define the curtate time until exit as

$$K^*_{x_l} = \min(K_{x_l} + \Delta, n_l) \tag{2.9}$$

We denote the probability of death for the i-th policyholder in the period $[t, t+\Delta)$ as ${}_\Delta q_{x_l+t}$ and the probability that (x_l) remains in the cohort of policyholders for at least t years, $t = \{0, \Delta, 2\Delta, \cdots\}$ to be $\mathbb{P}[K^*_{x_l} \geq t] = {}_tp_{x_l}$ such that the probability that (x_l) exits the contract within period Δ having remained a policyholder for t years is $\mathbb{P}[K^*_{x_l} = t] = {}_{t|\Delta}q_{x_l} = {}_tp_{x_l}{}_\Delta q_{x_l+t}$. It is noteworthy to point out that death is not the only decrement in equity-linked contract and other decrements such as surrender could be considered as well. However, modeling policy holder surrender behavior has over the years proven to be difficult. Bacinello [2003] considers pricing with surrender option. We also assume exits from the contract occur at the end of periods.

Next, if we let $\mathcal{L}_t$ denote the group of policyholders with a contract in force and alive at time t, then we can define

$$\mathcal{L}_t = \sum_{l=1}^{l_0} \mathbf{1}_{\{\min(T_{x_l}, n_l) \geq t\}} \tag{2.10}$$

Let our state process be enlarged to comprise of both the cohort process as well as the index process with finite states $\psi_{i,\cdot} \in \Omega$ with probabilities $p_{i+j,\cdot|i,\cdot}$, for $i = 0, 1, \cdots, tk/\Delta$, $j = 0, 1, \cdots, k$. We define the cohort process to comprise of the number of policyholders that die in the period $[t, t + \Delta)$,

$$\mathcal{N}_t^{(d)} = \sum_{l=1}^{l_0} \mathbf{1}_{\{T_{x_l} \in [t, t+\Delta), n_l > t\}} \tag{2.11}$$

and the number of policyholders that terminate the contract at time t

$$\mathcal{N}_t^{(e)} = \sum_{l=1}^{l_0} \mathbf{1}_{\{T_{x_l} > t, n_l = t\}} \tag{2.12}$$

For further simplification, we assume that the l_0 independent policyholders are of the same age x, with mortality from the same distribution and they purchase the same contract. This assumption can however, be relaxed for different ages and contract lengths.

2.3 Risk Measures

A risk measure is a method of summarizing the riskiness of a random variable into a single number or a real-valued function. In loss reserving, Dickson et al. [2013] define a risk measure to be the function that is applied to some random loss to give a reserve value that reflects the riskiness of that loss. Actuaries are familiar with the use of premium principles as risk measures when setting policy premiums. Other common risk measures like the Value-at-Risk (VaR) and the Conditional Value-at-Risk ($CVaR$) have proven to be meaningful in business settings as well.

2.3.1 Coherent Risk Measures

Introduced by Artzner et al. [1999], if we define $\mathcal{G}$ to be the set of all risk, then a risk measure $\rho : \mathcal{G} \rightarrow \mathbb{R}$ is coherent if it obeys the following axioms;

- Translation invariance: for all $X \in \mathcal{G}$, risk-free rate r, and all real numbers α,

$$\rho(X + \alpha \cdot r) = \rho(X) - \alpha \tag{2.13}$$

- Sub-additivity: for all $X_1, X_2 \in \mathcal{G}$,

$$\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2) \tag{2.14}$$

- Positive homogeneity: for all $\lambda \geq 0$ and for all $X \in \mathcal{G}$,

$$\rho(\lambda X) = \lambda \rho(X) \tag{2.15}$$

- Monotonicity: for all $X, Y \in \mathcal{G}, X \leq Y$,

$$\rho(Y) \leq \rho(X). \tag{2.16}$$

2.3.2 Value-at-Risk

We let L be the discounted loss random variable and define the Value-at-Risk (VaR) of L at $c \in [0, 1]$ as

$$\rho(L) = VaR_c(L) = \inf\{y \in \mathbb{R} : \mathbb{P}[L \leq y] \geq c\} = \pi_c \tag{2.17}$$

for $t \in \{0, \Delta, 2\Delta, \cdots\}$. VaR_c, also known as the quantile risk measure may be seen as the amount that the discounted loss L would not exceed with some probability c. π_c may also be seen as the $100c$ percentile of the loss random variable.

2.3.3 Conditional Value-at-Risk

The quantile risk measure has some theoretical and practical problems associated with it. One problem is, it does not take into account the risk if the worst case $(1-c)$ event actually occurs. Recent applications introduce the conditional value-at-risk ($CVaR$) also determined with respect to some parameter $c \in [0,1]$. Given some probability measure on a continuous discounted loss L, the conditional value-at-risk or the $CVaR$ at c is defined as

$$\rho(L) = CVaR_c(L) = \mathbb{E}[L|L \geq \pi_c]. \tag{2.18}$$

Thus the $CVaR_c$ may be seen as the expected loss given that the loss lies in the worst $1-c$ part of the distribution of L, where the worst $1-c$ part is simply that part greater than the VaR_c.

We need to take care when the distribution of L is discrete or mixed and hence defined by some probability mass function. If the π_c falls in the probability mass, (that is $\exists\ \epsilon > 0 : \pi_{c+\epsilon} = \pi_c$), then if we consider losses strictly greater than π_c, we would be using less than the worst $1-c$ of the distribution. On the other hand, if we consider losses greater than or equal to π_c, we would be using more than the worst $1-c$ part of the distribution. We therefore, define the $CVaR_c$ approximation for discrete random losses as

$$CVaR_c(L) = \pi_c + \frac{\mathbb{E}[(L-\pi_c)\mathbf{1}_{\{L>\pi_c\}}]}{1-c}. \tag{2.19}$$

When studied in the framework of coherent risk measures, the Value-at-Risk lacks the sub-additivity property and therefore convexity in the case of general loss distributions with an exception to some special classes like the normal distribution, see Quaranta and Zaffaroni [2008]. This drawback creates inconsistencies with well-accepted principles like "diversification reduces risk" and also has even more problems when we consider numerical tractability. For this reason, Rockafellar et al. [2000] focuses on optimizing the conditional value-at-risk or conditional tail expectation and approximates it to be

$$CVaR_c(L) = \pi_c + \frac{1}{1-c} \sum_{y \in \mathbb{R}} p(y)[L(y) - \pi_c]^+ \tag{2.20}$$

where $p(y)$ is a probability mass function. Rockafellar et al. [2000] proved that (2.20) is convex and piece-wise linear with respect to π_c and although it may not be differentiable with respect to π_c, it can readily be minimized by some line search techniques or linear programming.

In relation to the lattice model, the discounted loss random variable is function of the state $\psi_{i,t}$, thus it depends on the period t and the node i. As such both the VaR_c and $CVaR_c$ depend on $\psi_{i,t}$.

2.4 Portfolio Loss Random Variable

Equity-linked products are long term investments. Due to this feature, they tend to be exposed to mortality and financial risks. As stated in the previous chapter, if we ignore the possibility of surrender and withdrawal options and assume that an individual exits a contract either by death or survives the term of the contract, then we can define $\mathcal{X}_{i,t}$ to be the payoff upon survival or death for policyholder (x) at some time K_x^* and assume that the insurer pays either the death benefit at the end of the period of death $(K_x + \Delta)$ or the survival benefit at the end of the contract (n).

Given the possible gains and expenses that can happen to a specific insurance contract, we need to determine the discounted loss random variable that can represent the aggregate losses for some insurance portfolio. Next, in order to hedge this insurance portfolio, we assume the insurer can invest in several financial instruments. Let $a_{i,t}$ be the dollar amount invested in some index shares, $b_{i,t}$ be the dollar amount invested in some risk-free asset and $c_{i,t}$ be the dollar amount invested in some other financial option all at node $\psi_{i,t}$, then we can define $\mathcal{A}_{i,t} = \{a_{i,t}, b_{i,t}, c_{i,t}\}$ to be the hedging strategy for the insurance portfolio,

$t \in \{0, \Delta, 2\Delta, \cdots, n - \Delta\}$ and $i = 0, 1, 2, \cdots, tk/\Delta$.

Thus, we denote $V_{i,t}$ to be the value of the aggregate hedge portfolio at node $\psi_{i,t}$ so that, the initial value of the hedge portfolio is $V_{0,0} = a_{0,0} + b_{0,0} + c_{0,0}$. The time t node i value of the hedge portfolio is $V_{i,t} = a_{i,t} + b_{i,t} + c_{i,t}$, for $t < n$. Furthermore, if we let $V_{i,t}^{-}$ denote the value of the hedge portfolio before any benefit is payed at state $\psi_{i,t+\Delta}$, then

$$V_{i,t}^{-} = a_{i,t}\frac{S_{t+\Delta}}{S_t} + b_{i,t}e^{r_t\Delta} + c_{i,t}\frac{P(t+\Delta, n)}{P(t, n)}. \tag{2.21}$$

for $t < n$ and $i = 0, 1, 2, \cdots, tk/\Delta$.

As such, if we consider the benefits paid out from the current period and risk from the remaining cohort of policyholders, we can now define the discounted loss random variable. Combining (2.11), (2.12) and (2.21) and conditioning on the process path, we define the conditional discounted loss random variable at state $\psi_{i,t}$ as,

$$L_{i,t} = e^{-r_t\Delta}\left(\mathcal{N}_t^{(d)}\mathcal{X}_{i+j,t+\Delta} + V_{i+j,t+\Delta} - V_{i,t}^{-}\right) \tag{2.22}$$

such that $V_{i+j,n} = \mathcal{N}_n^{(e)}\mathcal{X}_{i+j,n}$ for $t < n$, $i = 0, 1, 2, \cdots, tk/\Delta$, and $j = 0, 1, 2, \cdots, k$.

Chapter 3

Replicating Portfolio

In this chapter, we propose how the equity-linked products can be hedged under worst-case situations by minimizing an investment strategy coupled with some constraints on the loss functions. We discuss as well the various dynamic hedging techniques that will be used.

Hedging is an investment strategy taken to reduce the risk of losses in a portfolio. It usually involves taking the offsetting position in some future obligation or losses. A hedge portfolio often referred to as a replicating portfolio in finance is simply a portfolio of some assets or strategies whose value is able to offset some future losses. A replicating portfolio is said to be self-financing if no injections or withdrawals are made from the portfolio once it is established. That is, re-balancing of the portfolio is simply achieved by re-allocating the assets within the portfolio. In a complete market, it is possible to have a self-financing portfolio that perfectly replicates any payoff $\mathcal{X}$ at time n. However, in an incomplete market, there could exist some claims $\mathcal{X}$ which cannot be perfectly replicated with a self-financing portfolio. We can either replicate some attainable payoff that is close to $\mathcal{X}$ by global hedging or we can establish a non-self-financing portfolio that replicates $\mathcal{X}$ almost surely at time n by local hedging.

3.1 Super-replication

Another strategy for hedging the contingent claim $\mathcal{X}$ is to find a portfolio that eventually dominates the claim $\mathcal{X}$ at time n. For instance, if we assume $V_t = a_t + b_t$ and define the claim $\mathcal{X}$ to be a random variable at maturity n, then a super-replicating portfolio for the claim $\mathcal{X}$ is any self-financing portfolio such that $V_n \geq \mathcal{X}_n$. This implies that the price of the claim cannot exceed the price of the portfolio. Any self-financing strategy that satisfies $V_n \geq \mathcal{X}_n$ is sufficient to ensure super-replication. We note that there could exist infinitely many portfolios whose value could be greater than the contingent claim at maturity. As such an upper arbitrage bound for the price of the claim at time 0 is the super-replicating portfolio that results in the least cost for any long position in the claim $\mathcal{X}$. Thus

$$V_0^{super} = \inf\{V_0 | (a, b) \text{ is self-financing, } V_n \geq \mathcal{X}_n\}. \tag{3.1}$$

Also, under some conditions on the contingent claim, the index model and the presence of transactions costs, Chen et al. [2008] theorize that it may be possible to find a super-replicating portfolio that costs less than the replicating portfolio. However, when there are no transactions costs, a super-replicating portfolio must cost at least as much as a replicating portfolio. Soner also explores the use of dynamic programming to establish super-replicating portfolios in continuous time.

Davis and Clark [1994] in their paper describe why super-replicating strategies do not form the basis for a viable theory of option pricing in continuous time. They conjecture that the trivial buying and holding one share of the risky asset is the cheapest super-replicating strategy when transaction costs are strictly positive. Finally, they show that the only possible candidates for super-replicating strategies are those that track the Black-Scholes portfolio closely by introducing suitable reflecting barriers. However, if these sell and buy barriers are close together then the transaction cost will become large and super-replication will fail.

The construction of super-replicating portfolios has the condition that the value of such a portfolio should be greater than or equal to the contingent claim at maturity or at time n. Föllmer and Schied [2011] extends this condition to the value of the claim at any time t within the horizon of n. As such they define any self-financing strategy whose value process $V_t \geq \mathcal{X}_t$, almost surely for all t, as a super-hedging or super-replicating strategy for $\mathcal{X}$. The payoff $\mathcal{X}$ is attainable if and only if there exist τ and a super-replicating portfolio whose value process satisfies $V_\tau = \mathcal{X}_\tau$ almost surely. On the other hand, if $\mathcal{X}$ is not attainable, then the value process of the super-replicating portfolio satisfies $\mathbb{P}\{V_t \geq \mathcal{X}_t, \ \ \forall \ \ t\} > 0$. This implies that it is possible to construct a super-replicating portfolio even when the time to maturity is not fixed by constructing a portfolio with a minimal investment which super-replicates the claim at any time. A proof of this can be seen in Föllmer and Schied [2011]. It is important to note however that such a portfolio is typically not the arbitrage-free price for claim $\mathcal{X}$.

Before we formulate our hedging model for the value of an equity-linked product under the worst-case scenario, we consider the creation of a replicating portfolio under an American claim where the claim can be paid any time before maturity or at maturity. In a general framework, an American-type product is a type of security or contingency where the issuer of the product is likely to make claim payment at some period $t \leq n$. The holder of an American derivative security can exercise in any period before or at maturity and receive some payoff which is a function of the stock price at that period. In EIAs, a situation where the claim can be paid at any time t could arise when benefits are paid upon death or surrender. As such this can be seen as a variant of an American-type product.

In order to hedge such a claim, we construct a self-financing trading strategy $\mathcal{A}_t$ as in (2.4) such that for $V_t^{(\mathcal{A})}$, which is the corresponding value process, $V_0^{(\mathcal{A})} = x$, where x is the initial capital and $V_t^{(\mathcal{A})} \geq \mathcal{X}_t$ for all t. Next, in a framework where we assume investors behave rationally, that is, an investor's decision at any time t will be to make choices that result in the optimal level of benefit or utility. Thus if we consider constructing a strategy $\mathcal{A}$ for an American-type product, then at time n, the hedging strategy would need to cover the claim. Hence $V_n^{(\mathcal{A})} = \mathcal{X}_n$ would be a sufficient condition. At time $n - \Delta$, since there is a possibility to make a claim, the insurer may either pay $\mathcal{X}_{n-\Delta}$ or wait until maturity, in which case the expected claim payable at time n needs to be covered. Bingham and Kiesel [2013] use backward induction argument to show that the value, $V_t^{(\mathcal{A})}$ of a self-financing strategy under an American claim is the maximum of the current payoff at time t and the holding value, where the holding value is the expected claim value at the next time step, $t + \Delta$.

3.2 ϵ-arbitrage

The cost of a super-replicating portfolio is at least the cost of the replicating portfolio. The idea of ϵ-arbitrage is to find a portfolio, not necessarily self-financing, whose value at maturity matches the payoff $\mathcal{X}$ of the contingent claim within an error of ϵ. As such, if we define $\mathcal{A} = \{a, b\}$ to be a hedging strategy that invests a in a risky asset and b in a risk-less asset, then an ϵ-arbitrage hedging strategy can be defined as the strategy that seeks to minimize the error ϵ created by the mismatch between portfolio value and payoff of the contingent claim at maturity. That is the optimal ϵ-arbitrage hedging strategy for a claim with payoff $\mathcal{X}$ at maturity n can be

$$\mathcal{A}^* = \underset{a,b}{\operatorname{argmin}} \, ||\mathcal{X}_n - V_n||. \tag{3.2}$$

Chen [2009] try to find the portfolio that minimizes the worst-case ϵ-arbitrage between the portfolio value and the claim payoff over some uncertainty set of security returns using robust optimization. They formulate the optimization problem by considering a call option as the claim and establish a min-max objective under some uncertainty constraints. We discuss more the concept of robust optimization and its use in Finance in Chapter 4.

Bertsimas et al. [2001] provide a comprehensive framework for ϵ-arbitrage in both discrete and continuous time. Unlike Chen [2009], they consider the square root of the mean-squared replication error as a measure of success for creating an optimal replicating strategy due to its tractability and compute the least cost optimal replication strategy and a corresponding measure of the minimum replication error ϵ. They rely on stochastic dynamic programming algorithms to solve the optimization problem. It is worth noting that $\epsilon = 0$ creates perfect replication, although this might be an over-achievement (unrealistic in practice). They specify that ϵ may be seen as the degree of market incompleteness that provides a measure of the difficulty in replicating a portfolio. That said, dynamical market incompleteness may arise as a result of, but not limited to, the presence of taxes, transactions costs, short-sales, and borrowing restrictions.

3.3 Quadratic Hedging

Generally, investments in financial products involves risk. Schweizer [1988] considers a stochastic model where the price process is a semi-martingale and assumes market incompleteness to propose a sequential risk reduction hedging strategy in both discrete and continuous time. On the other hand, Föllmer and Schweizer [1988] propose a local risk-minimizing strategy to sequentially minimize the square error of the process and indicate that one should not expect an intrinsic risk to completely vanish. As such any adjustment of the fair price risk premium should be based on the *a priori* risk. Coleman et al. [2006] also consider discrete hedging of guarantees in variable annuities using local risk minimization and suggest that a joint model with both equity and interest rate leads to an effective reduction in risk. In this section, we discuss another hedging approach known as a quadratic hedging strategy. First introduced in the financial context by Bouleau and Lamberton [1989], this approach seeks a strategy that minimizes the sum of expected squared hedging errors. As such, using the notation of Gaillardetz and Moghtadai [2017], we can define the optimal quadratic hedging strategy at $\psi_{i,t}$ as

$$\mathcal{A}^*_{i,t} = \{a^*_{i,t}, b^*_{i,t}, c^*_{i,t}\} = \underset{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3}{\operatorname{argmin}} \mathbb{E}[L^2_{i,t}], \tag{3.3}$$

assuming $V_{i,t}$ produces the optimal portfolio value under the same minimization criterion, where $L_{i,t}$ is defined in (2.22). Thus for $t = n-\Delta, \cdots, 2\Delta, \Delta, 0$ and for $i = 0, 1, 2, \cdots, tk/\Delta$, we can iteratively solve for the initial hedging portfolio $\mathcal{A}^*_{0,0}$ by defining $V_{i,t} = a_{i,t} + b_{i,t} + c_{i,t}$ for $t < n$.

It is important to note that constructing the hedging strategy with the quadratic approach has a disadvantage over the use of a risk measure in that, the square of hedging errors does not distinguish the impact of positive losses from negative losses.

3.4 Risk Control

Due to the possibility that the insurer of an equity-linked product could make benefit payments within the contract horizon and the fact that these products require dynamic hedging strategies, Gaillardetz and Hachem [2019] in their paper explore some risk control strategies. They propose minimization with risk measures as either objectives or constraints in a dynamic programming setting. The risk measure is used in the constraints to limit the local risk while they minimize the portfolio value. On the other hand, the risk measure is used as an objective and requires additional local control modeling like limiting the value of the portfolio or setting initial values for the portfolio.

In order to derive an optimal hedging strategy, Rockafellar et al. [2000] use auxiliary variables to transform $CVaR$ into a linear expression and then minimize it using linear programming. Gaillardetz and Moghtadai [2017] also propose the minimization of hedge cost using local risk-minimizing strategies iteratively. They compare the iterative quadratic hedging, which minimizes the expected squared hedging errors, to the allocation constraint method, the optimal iterated method and, a no-hedge strategy.

Similar to the quadratic hedging strategy, the allocation constraints rather minimizes the VaR and $CVaR$ of the discounted losses instead of the squared errors. Since VaR and $CVaR$ are unbounded, it is possible to simply lower VaR or $CVaR$ at each iteration by increasing the value of the hedging portfolio. As such they introduce an initial capital value as a constraint and recursively minimize the risk measure at each period, by starting from the maturity until time 0. They obtain an optimal hedging strategy $\mathcal{A}_t^* = \{a_t^*, V_t - a_t^*\}$ for a typical two-asset strategy, where the available capital V_t is defined at the beginning of each period. Lastly, they propose the optimal iterated method, which also aims to minimize a risk measure iteratively instead of the squared errors. This method sets the strategy such that the risk measure is equal to a certain threshold γ. For instance, if the insurer sets the VaR level to be c and the threshold to be γ, then it implies that the hedging strategy requires the worst $(1-c)\%$ of losses to be γ, which provides more flexibility than the quadratic hedging strategy. This is because an increase in the risk level c makes the strategy more expensive, while an increase in the threshold γ makes the strategy cheaper.

The selection of the optimal portfolio at any time t is first found by obtaining all possible combinations of hedging strategies that set the risk measure to γ. A subset of this portfolio is selected numerically and the portion invested in the risk-free asset is estimated as the value that minimizes the square of the difference between the threshold γ and the loss at a given risk level. Details of the sequential approach to use the optimal iterated hedging strategy can be seen in Gaillardetz and Moghtadai [2017]. It is also worth noting that setting $\gamma = 0$ and $c \to 1$ implies that we want losses to be greater than 0. This leads to a super-replicating strategy. That is we ensure that the replicating portfolio is greater than the claim, $\mathcal{X}$ almost surely.

In an equity-linked product where we guarantee some payment upon the death of the policy holder, the claim at any time is a combination of the claim payable at death of the insured life and the value of expected discounted claims from future periods represented by the conditional discounted loss at time t in (2.22). However, unlike American-type products where the value of the strategy at time t is the maximum of the claim if payed at time t and the expected value of the claim at maturity, given that investors act rationally, the value of the strategy of and EIA is a combination of two contingencies. As such we need to relax the assumption of self-financing in order to construct a hedging strategy to cover the claims at all times.

We now formulate the model that needs to be minimized under the worst-case scenario to obtain the optimal hedging strategy or hedging value. For some portfolio hedging strategy $\mathcal{A}$, we apply the concept of the optimal iterated dynamic hedging strategy by Gaillardetz and Moghtadai [2017] to find the optimum portfolio recursively. We denote the optimal hedging replicating portfolio at time t by $\mathcal{A}^*_{i,t} = \{a^*_{i,t}, b^*_{i,t}, c^*_{i,t}\}$. Thus starting from time $t = n - \Delta$ and working backwards, we dynamically optimize the sum of the risky and risk-free assets of the replicating portfolio at each time step and node. We recall that the conditional probability of the index reaching state $\psi_{i+j,t+\Delta}$, given $\psi_{i,t}$ for $j = 0, 1, \cdots, k$, is $p_{i+j,t+\Delta|i,t}$.

Next, we restrict the $CVaR$ at some level c to be less than the threshold γ as in the optimal iterated method. While each state $\psi_{i,t}$ comprises of the index value as well as the loss function of the contract at that state, we define auxiliary/slack variables $u_{i+j,t+\Delta}$ to denote the positive difference between the loss at state i and the value-at-risk at some level c, all at time t. We then use Rockafellar et al. [2000] formulation of $CVaR$ in (2.20) and constraints defined by Gaillardetz et al. [2019].

$$\pi_c^+ - \pi_c^- + u_{i+j,t+\Delta} - L_{i,t} \geq 0, \qquad \forall\, j = 0, 1, 2, \cdots, k, \tag{3.4}$$

$$u_{i+j,t+\Delta} \geq 0, \qquad \forall\, j = 0, 1, 2, \cdots, k, \tag{3.5}$$

$$\pi_c^+, \pi_c^- \geq 0, \tag{3.6}$$

$$\pi_c^+ - \pi_c^- + \frac{1}{1-c} \sum_{j=0}^{k} u_{i+j,t+\Delta} p_{i+j,t+\Delta|i,t} \leq \gamma, \tag{3.7}$$

for $t \in \{0, \Delta, 2\Delta, \cdots, n-\Delta\}$ and $i = 0, 1, 2, \cdots, tk/\Delta$. Equation (3.4) as constraint ensures the positive difference between losses and π_c. (3.7) as constraint also controls the tail risk of losses with the threshold γ.

The hedge portfolio that we seek at any time t is the minimum replicating portfolio value that is able to cover any contingent claim at time t. As we explore ways to approach a worst-case solution to this hedging strategy, the introduction of the auxiliary variables in the expression of the conditional value-at-risk creates a non-linear relation in the constraints. Attempts to optimize with non-linear min-max algorithms were not reliable for a worst-case orientation. The use of Lagrange multipliers and Karush-Kuhn-Tucker (K-K-T) conditions requires our objective function to be continuous and at least twice differentiable, a condition our objective set function does not satisfy. Also, most non-linear optimization techniques require initial values as a starting point which usually results in local solutions. As such we rely on linear programming under robust optimization to obtain a solution for our worst-case dynamics.

Chapter 4

Review of Robust Optimization

We intend to find our worst case hedging strategy using linear programming, which is consistent with the approach used by Rockafellar et al. [2000]. However, we notice that the constraint in (3.7) is non-linear, since both the auxiliaries ($u_{i+j,t+\Delta}$) and probabilities ($p_{i+j,t+\Delta|i,t}$) are variables, when we consider a worst-case orientation. Hence a drawback to the use of the linear programming approach. We, therefore, resort to robust optimization techniques.

4.1 Description of Robustness

Robust optimization techniques are useful in Finance due to the uncertainty that is usually associated with the inputs to the problem at the time the problem needs to be solved. Other than just uncertainty, in some cases, these inputs may not be known or even inaccurate.

Cornuejols and Tütüncü [2006] describe robust optimization as the modeling of optimization problems with data uncertainty so as to achieve a solution that is guaranteed to be satisfactory for most or all possible realizations of the uncertain variables or parameters. This type of optimization can be seen as a proxy to either stochastic programming or sensitivity analysis. According to Cornuejols and Tütüncü [2006], robust optimization methods are useful when some of the problem parameters are estimated and carry estimation risk, there are constraints with uncertain parameters that must be satisfied regardless of the values of these parameters, the objective function or the optimal solutions are sensitive to deviations, and lastly when we cannot afford a low probability and high magnitude of risk.

In comparison to our model, we can observe that both the loss random variable and the index model will require some estimates and hence will carry estimation risks. Also, our objective is to minimize the worst-case risk, as such we expect a high probability with a low magnitude of risk, and this makes robust optimization a suitable tool for our optimization problem.

An important consideration in the robust optimization framework is the way in which we define or interpret robustness. Each different representation of uncertainty and description of robustness leads to different formulations of robust optimization. When the uncertainty is associated with the objective function, it affects the closeness of the generated solutions to the optimal one. This type of framework is known as objective robustness. On the other hand, when the uncertainty is associated with the model constraints, there is some risk in the feasibility of possible solutions. This type of framework is also known as constraints robustness. It is noteworthy that both objective and constraints robustness have a worst-case orientation, where we optimize the model solutions under some adverse conditions.

Cornuejols and Tütüncü [2006] categorize model robustness, where we seek just solutions that optimize the worst-case behaviors, as absolute robust solutions. Conversely, solutions that evaluate the worst-case behavior relative to some best possible solution under each scenario are called relative robust solutions. The notion of relativity may be measured by some regret function linked to a decision after the uncertainty is resolved. Another variant of relative robustness may measure the regret in terms of closeness of the model solution to the optimal solution and is very useful for multi-period problems, where revising a decision can be costly, typically portfolio re-balancing with transaction costs. Other robust optimization formulations like adjustable robust optimization can be found in literature, see Cornuejols and Tütüncü [2006].

It is interesting to note that many of these descriptions of robustness can be reformulated to a tractable form that reduces the complexity of finding an optimal solution. In light of the various description of robustness, our model can be seen as a constraint robust model since the uncertainty only appears in the constraint for $CVaR$. Also, since we are interested in the worst-case scenario of the value of the proposed hedging strategy, the model solution may be seen as an absolute robust solution.

4.2 Uncertainty Sets

In order to accurately optimize a model that can be classified as robust, it is important to have a thorough understanding of the behavior and influence of the uncertainty parameters. This will, in fact, affect the choice of strategy to adopt to find the optimal solution.

The formation of uncertainty sets depends heavily on the perception of future values of certain parameters. In some instances, statistical or Bayesian techniques are applied to historical data to generate alternative estimates of parameters. Cornuejols and Tütüncü [2006] lists four common types of uncertainty sets that tend to come up in a robust optimization framework. We denote $\mathcal{U}$ to be an arbitrary uncertainty set and p_j to be the possible values of the model uncertainty parameters. Then,

1. $\mathcal{U} = \{p_0, p_1, \cdots, p_k\}$, a set of finite number of scenarios generated for the possible values of the uncertain parameters.

2. $\mathcal{U} = \{p_j : l \leq p_j \leq u, \forall\ j = 0, 1, \cdots, k\}$, a set of intervals that defines each uncertain parameter.

3. $\mathcal{U} = \{p_j : p_j = p_0 + Mu, ||u|| \leq 1, \forall\ j = 0, 1, \cdots, k\}$, a set of ellipsoids that defines the (confidence) regions of each uncertain parameter and are particular useful for smoothing the optimal value function.

4. $\mathcal{U} = conv(p_0, p_1, \cdots, p_k)$, a set of polytopes with a convex number of scenarios generated for the possible values of the uncertain parameters.

The desired level of robustness often influences the choice and the size of $\mathcal{U}$. The type of $\mathcal{U}$ will, however, depend on the source of uncertainty and the sensitivity of the optimal solutions to this $\mathcal{U}$. Usually, we generate estimates of the true parameters by making assumptions about the stationarity of the random processes. Defining $\mathcal{U}$ in a reasonable way is the key issue to successful practical application. That is, we need to define the likelihood distribution of the uncertainty as well as their structure. For instance, knowing the bounds for interval and finite uncertainty or the scaling matrix for ellipsoidal uncertainty. Zhu and Fukushima [2009] propose estimating the distribution p_j's over $\mathcal{U}$ using historical data and then simulating different samples of the likelihood distribution to find the confidence intervals of the uncertainty parameters. Another is simply to use expert predictions to find the uncertainty parameters.

Tütüncü and Koenig [2004] describe how uncertainty sets can be generated by using bootstrapping and moving averages of historical data. Goldfarb and Iyengar [2003] also illustrate how ellipsoidal uncertainties can be generated by analyzing the confidence regions of some linear regression model and Rustem and Howe [2009] illustrate how finite uncertainty sets can be generated using some algorithms. It is also worth nothing that it is possible to determine uncertainty sets by equating model estimates to real-time financial market data and solving for the implicit uncertainty directly or by calibrating their mismatches.

4.3 Uncertainty in Linear Optimization

There are various types of optimization problems and the solutions to these problems rest mainly on the structure and description of the problem. An optimization problem is characterized by the structure and the data available to solve the problem. A linear optimization problem is of the form

$$\min_x \{c^T x : Ax \leq b\}, \tag{4.1}$$

such that for n number of variables and m number of constraints, $x \in \mathbb{R}^n$ is a vector of decision variables with a vector of coefficients $c \in \mathbb{R}^n$ forming the objective function. A is an $m \times n$ constraint matrix with vector $b \in \mathbb{R}^m$ as the bounds. The structure of the linear optimization is specified by the number of constraints and the number of variables. Ben-Tal et al. [2009] defines an uncertain linear optimization problem as the collection

$$\left\{ \min_x \{c^T x : Ax \leq b\} \right\}_{(c,A,b) \in \mathcal{U}} \tag{4.2}$$

of linear optimization instances of common structure with its data varying in some given uncertainty set $\mathcal{U} \subset \mathbb{R}^{m \times n}$. The uncertainty set is usually assumed to be parameterized in an affine structure.

The solution to a single linear optimization problem is governed by the concepts of the feasible or optimal solution and optimal value. However, uncertain linear optimization problems like (4.2) are not entirely defined by these concepts but rather by the underlying decision environment. We rely on the assumptions of a typical decision environment by Ben-Tal et al. [2009], which states that all decision variables should be assigned specific numerical values as a result of solving the problem before the actual data reveals itself. This first assumption is suitable for an index process environment since our hedging portfolio needs to be constructed before we observe market values. Secondly, the decision maker is fully responsible for consequences of the decisions to be made when, and only when, the actual data is within the specified uncertainty set $\mathcal{U}$ and lastly, the decision maker should not tolerate even small violations of the constraints when the data is in the uncertainty set. The second and third assumptions imply that the feasible solutions to (4.2) should satisfy all the constraints, whatever the realization of the data from the uncertainty set may be. Based on these assumptions for a decision environment, they define a feasible solution as

Definition 4.3.1 *A vector $x \in \mathbb{R}^n$ is a robust feasible solution to an uncertain linear optimization problem, if it satisfies all realizations of the constraints from the uncertainty set,* *that is,*

$$Ax \leq b, \quad \forall \ (c, A, b) \in \mathcal{U}. \tag{4.3}$$

With the constraints defined on our uncertainty set, the choice of a feasible objective value can be seen as the value immunized against uncertainty or the worst-case value and leads to the following definition.

Definition 4.3.2 *Given a candidate solution x, the robust value $\hat{a}(x)$ of the objective in an uncertain linear optimization problem at x is the largest value of the true objective function over all realizations of the data from the uncertainty set:*

$$\hat{a}(x) = \max_{(c,A,b)\in\mathcal{U}} c^T x. \tag{4.4}$$

Having defined the structure of a feasible solution for our objective function and the constraint with data uncertainty in our linear optimization model, we then seek the best robust value of the objective among all robust feasible solutions to the problem. As such Ben-Tal et al. [2009] defines the robust counterpart of an uncertain linear optimization problem as

Definition 4.3.3 *The Robust Counterpart of the uncertain linear optimization problem is the optimization problem*

$$\min_{x} \left\{ \hat{a}(x) = \max_{(c,A,b)\in\mathcal{U}} c^T x : Ax \leq b, \quad \forall \ (c, A, b) \in \mathcal{U}\} \right\}, \tag{4.5}$$

minimizing the robust value of the objective over all robust feasible solutions to the uncertain problem.

Hence, a robust optimal solution to an uncertain linear optimization problem is the optimal solution of its robust counterpart, and an optimal value of an uncertain linear optimization problem is the optimal value of its robust counterpart. Thus we can view the robust optimal solution as the best uncertainty immunized solution that can be associated with the uncertain linear problem. It is worth noting that we do not lose anything when we restrict an uncertain linear optimization program with certain objectives. For instance, if we have a certain objective function, then the robust counterpart of an uncertain linear optimization will be of the form

$$\min_x \left\{ c^T x : Ax \leq b, \quad \forall \ (A, b) \in \mathcal{U} \right\}, \tag{4.6}$$

with our uncertainty set now defined in the constraints only. We can further define our uncertainty for only a part of the constraints as well (for instance only A or only b) and still preserve the properties of the optimization problem provided the constraints in the uncertainty set are not violated. Ben-Tal et al. [2009] gives a step by step approach to constructing a constraint robust linear optimization with certain objective function as follows.

- preserve the original certain objective as it is and,
- replace every one of the original constraints $(Ax)_i \leq b \Leftrightarrow a_i^T x \leq b_i$ with its robust counterpart $a_i^T x \leq b_i \quad \forall \ [a_i, b_i] \in \mathcal{U}_i$, where a_i^T is the i-th row in A and $\mathcal{U}_i$ is the projection of $\mathcal{U}$ on the space of data of the i-th constraint.

In particular, the robust counterpart of an uncertain linear optimization problem with a certain objective remains appropriate when we extend the sets $\mathcal{U}_i$ of uncertain data of respective constraints to their closed convex hulls and extend $\mathcal{U}$ to the direct product of the resulting sets. Ben-Tal et al. [2009] proves that we lose nothing by assuming from the sets $\mathcal{U}_i$ of uncertain data of the constraints are closed and convex, and $\mathcal{U}$ is the direct product of these sets.

4.4 Tools for Robust Optimization

After describing the nature of uncertainty in a robust optimization model, the next step is to devise a strategy to solve the problem. Tools for solving robust optimization problems are essentially reformulation strategies that transform the problem into a deterministic one with no uncertainty. However, there usually arises a trade-off between tractability and economy.

In this case, while we look for a formulation that can be solved efficiently with standard optimization techniques, we need to ensure that the new formulation is not much bigger and complex than the original uncertainty problem.

4.4.1 Sampling

One strategy in dealing with uncertainty in a robust optimization setting is by sampling. This is done by sampling several scenarios of the uncertain parameters from the set of all possible values that the parameters can take. This results in a finite uncertainty set and sampling can be done by using distributional assumptions or random simulations. This technique can also be used for both objective and constraint robust models. Sampling is a very useful strategy because it involves little or no formulation thereby preserving the structural properties like convexity of the model although a large finite uncertainty set results in a large robust optimization problem, it is often tractable and easy to optimize.

4.4.2 Conic

Other strategies involve conic optimization and saddle-point characterizations. Conic optimization techniques are particularly useful when uncertainty sets are intervals and ellipsoids. This is due to the continuous nature of the uncertain parameters and such formulations are termed as semi-finite optimization since there will be a finite number of parameters but infinitely many constraints indexed by the uncertainty set. Semi-finite optimizations can be reformulated using finite sets of conic constraints.

4.4.3 Saddle-Point

Saddle-point techniques are also useful in solving objective robust problems. This is because, when the uncertainty lies in the objective function, saddle-point conditions can be used to characterize the robust problem provided it satisfies convexity assumptions. Interior-point algorithms can readily be used to find optimal solutions. Characterization and use of saddle-point for robust optimization, especially for special min-max problems can be found in Halldórsson and Tütüncü [2003], Tütüncü and Koenig [2004] and Rustem and Howe [2009].

Chapter 5

Robust Optimization

In our search for a measure that can serve as a yardstick for both insurers and regulators to compare various model risks with, we estimate the worst-case value of the hedge portfolio. The optimal portfolio that we seek at any time t is the minimum replicating portfolio value that is able to cover any contingent claim at state $\psi_{i,t}$. As such we seek $V_{i,t}^* = \min V_{i,t}$. We can, therefore, define the optimal replicating portfolio at state $\psi_{i,t}$ to be

$$\mathcal{A}_{i,t}^* = \{a_{i,t}^*, b_{i,t}^*, c_{i,t}^*\} = \underset{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3}{\operatorname{argmin}} \{V_{i,t}\} = \underset{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3}{\operatorname{argmin}} \{a_{i,t} + b_{i,t} + c_{i,t}\}, \tag{5.1}$$

subject to (3.4)-(3.7) as constraints. Instead of assuming a specific knowledge of the distribution of the random vector of probabilities, we assume that their values are only known to belong to some set $\mathcal{U}_{i,t}$. Thus $p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}$ for all $j = 0, 1, \cdots, k$, where $\sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} = 1$ and $p_{i+j,t+\Delta|i,t} \geq 0$.

We, therefore, define the worst-case value of a hedge replicating portfolio for some losses at $\psi_{i,t}$ with respect to $\mathcal{U}_{i,t}$ to be

$$WV_{i,t} = \max_{p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}} V_{i,t}^*. \tag{5.2}$$

As such for a worst-case scenario, while we seek to minimize the cost of the portfolio at each node and time, we as well want to maximize the probability of observing the loss.

5.1 Quadratic Approach

In this section, we introduce uncertainty in probabilities and apply robust optimization techniques to formulate a worse case quadratic hedging approach. Unlike (3.3) which seeks a hedging strategy that minimizes expected square hedging errors or losses, a worse case quadratic approach seeks a strategy that minimizes the sum of expected squared errors while maximizing over the probabilities associated with each node. Consequently, the worst-case quadratic hedging strategy at $\psi_{i,t}$ can be defined as

$$\mathcal{A}^*_{i,t} = \{a^*_{i,t}, b^*_{i,t}, c^*_{i,t}\} = \max_{p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}} \min_{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3} \mathbb{E}[(L_{i,t})^2]. \tag{5.3}$$

Also, the construction of the index model affords us to express the expected squared errors using the state probabilities. Thus given some uncertainty set $\mathcal{U}_{i,t}$,

$$\mathcal{A}^*_{i,t} = \{a^*_{i,t}, b^*_{i,t}, c^*_{i,t}\} = \max_{p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}} \min_{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3} \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t}(L_{i,t})^2, \tag{5.4}$$

for all $p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}$, $t \in 0, \Delta, \cdots, n-\Delta$, $i = 0, 1, 2, \cdots, tk/\Delta$ and $j = 0, 1, 2, \cdots, k$. To serve as a control for our hedging model, we explore the use of robust sampling to the worst-case quadratic hedging and compare the approach to worst-case robust linear optimization approach.

5.2 Risk Control Approach

In our worst-case risk control approach, the objective is to find the hedging strategy that maximizes across index movements, the minimum portfolio value. Setting (5.2) as our objective, we want to optimize

$$\max_{p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}} \min_{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3} V_{i,t}$$

$$\begin{aligned}
\text{s.t } & \pi_c^+ - \pi_c^- + u_{i+j,t+\Delta} - L_{i,t} \geq 0, && \forall j = 0, 1, 2, \cdots, k, \\
& u_{i+j,t+\Delta} \geq 0, && \forall j = 0, 1, 2, \cdots, k, \\
& \pi_c^+, \pi_c^- \geq 0, && \\
& \pi_c^+ - \pi_c^- + \frac{1}{1-c} \sum_{j=0}^{k} u_{i+j,t+\Delta} p_{i+j,t+\Delta|i,t} \leq \gamma, &&
\end{aligned}$$

for $t = 0, \Delta, 2\Delta, \cdots, n - \Delta$, $i = 0, 1, 2, \cdots, tk/\Delta$ and $j = 0, 1, 2, \cdots, k$, where $(\pi_c^+ - \pi_c^-)$ is the VaR_c at optimality and (3.7) is the risk measure $CVaR_c$ controlled by γ. A dynamic programming approach is used to optimize (5.2) recursively. Starting with the last period n, we optimize (5.2) subject to (3.4)-(3.7) and $\mathcal{U}_{i,n-\Delta}$ to obtain $\mathcal{A}_{i,n-\Delta}$. We then use the same set of equations to obtain $\mathcal{A}_{i,n-2\Delta}$ but this time we set $V_{i,n-\Delta} = a_{i,n-\Delta} + b_{i,n-\Delta} + c_{i,n-\Delta}$ and so on until we obtain the initial value of the hedge portfolio $V_{0,0}$ and strategy $\mathcal{A}^*_{0,0}$. However, one constraint, (3.7) of the proposed is non-linear. As such with the robust optimization techniques we can formulate our model to a more tractable form that can be solved easily with linear programming. Also, robust optimization affords us to solve the problem even with the uncertainty of the probability variables. To solve our worst-case optimization problem, we re-formulate (3.4)-(3.7) and (5.2) with the uncertainty set $\mathcal{U}_{i,t}$ in our robust framework.

Theorem 5.1 *Suppose* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *are nonempty compact convex sets in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively, and the function* $f(\boldsymbol{x},\boldsymbol{y})$ *is convex in* $\boldsymbol{x}$ *for any given* $\boldsymbol{y}$*, and concave in* $\boldsymbol{y}$ *for any given* $\boldsymbol{x}$*, that is for* $\lambda \in [0,1]$*; and for* $\boldsymbol{x}_1$*,* $\boldsymbol{x}_2$ *in* $\boldsymbol{X}$ *there exists* $\boldsymbol{x}_3$ *in* $\boldsymbol{X}$ *with*

$$f(\boldsymbol{x}_3, \boldsymbol{y}) \leq \lambda f(\boldsymbol{x}_1, \boldsymbol{y}) + (1-\lambda) f(\boldsymbol{x}_2, \boldsymbol{y}), \tag{5.5}$$

for all $\boldsymbol{y}$ *in* $\boldsymbol{Y}$*; and for* $\boldsymbol{y}_1$*,* $\boldsymbol{y}_2$ *in* $\boldsymbol{Y}$ *there exists* $\boldsymbol{y}_3$ *in* $\boldsymbol{Y}$ *with*

$$f(\boldsymbol{x}, \boldsymbol{y}_3) \geq \lambda f(\boldsymbol{x}, \boldsymbol{y}_1) + (1 - \lambda) f(\boldsymbol{x}, \boldsymbol{y}_2), \tag{5.6}$$

for all $\boldsymbol{x}$ *in* $\boldsymbol{X}$ *Then the*

$$\max_{\boldsymbol{y} \in \mathbf{Y}} \min_{\boldsymbol{x} \in \mathbf{X}} f(\boldsymbol{x}, \boldsymbol{y}) = \min_{\boldsymbol{x} \in \mathbf{X}} \max_{\boldsymbol{y} \in \mathbf{Y}} f(\boldsymbol{x}, \boldsymbol{y}). \tag{5.7}$$

Proof: see Borwein and Zhuang [1986] and Bazaraa et al. [2013].

Rockafellar et al. [2000] after making some adjustment to the $CVaR$ prove that (3.7) is convex in π and affine (concave) in $p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}$. As such based on (5.7), it suffices to say that our objective for the worst-case value

$$\max_{p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}} \min_{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3} V_{i,t} = \min_{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3} \max_{p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}} V_{i,t}.$$

5.3 Uncertainty Sets

The choice of the uncertainty set is vital to the type of solution obtained in the optimization. In fact, some types of uncertainty can render the optimization infeasible. It is important to note that description of uncertainty in robust optimization in finance usually relies on the underlying pricing dynamics, in our case the index model. In their model, Chen [2009] rely on the Central Limit Theorem (CLT) to construct a bound for the returns on the underlying asset using a parameter which depends on the risk aversion of the investor to construct a polyhedral uncertainty set. This technique generates solutions that rely on the uncertainty of market returns or the state space.

However, we intend to maximize over some probability space that characterizes the index movement dynamics. As such, we adapt sampling techniques to generate samples of our uncertainty set $\mathcal{U}_{i,t}$. For instance, for a index drift, μ of the state variables, we can construct an uncertainty set such that

$$\mathcal{U}_{i,t}^{(1)} = \left\{ p_{i+j,t+\Delta|i,t} \ : \ \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} = 1, \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} \psi_{i+j,t+\Delta} = \mu, p_{i+j,t+\Delta|i,t} \geq 0 \right\}, \quad (5.8)$$

for $i = 0, 1, 2, \cdots, tk/\Delta$ and $j = 0, 1, 2, \cdots, k$. More critically, we can also include variations such as volatility, σ in the state variables to construct a more restrictive uncertainty set. Thus we can define

$$\mathcal{U}_{i,t}^{(2)} = \left\{ p_{i+j,t+\Delta|i,t} \ : \ \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} = 1, \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} \psi_{i+j,t+\Delta} = \mu, \right.$$
$$\left. \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} (\psi_{i+j,t+\Delta} - \mu)^2 = \sigma^2, p_{i+j,t+\Delta|i,t} \geq 0 \right\}, \quad (5.9)$$

for $i = 0, 1, 2, \cdots, tk/\Delta$ and $j = 0, 1, 2, \cdots, k$.

Another way to define our uncertainty set is to allow the values of μ and σ to vary as well within some specified region. For instance, one can keep track of market returns and volatility in order to determine an appropriate value of mean and volatility to choose. However, since we cannot be certain that these values will be exact, we can further introduce some uncertainty by creating an ϵ region around the chosen values. As such, we could also consider our uncertainty with only μ as

$$\mathcal{U}_{i,t}^{(3)} = \left\{ p_{i+j,t+\Delta|i,t} \ : \ \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} = 1, \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} \psi_{i+j,t+\Delta} \in \mu \pm \epsilon_1, p_{i+j,t+\Delta|i,t} \geq 0, \epsilon_1 \geq 0 \right\}, \quad (5.10)$$

for $i = 0, 1, 2, \cdots, tk/\Delta$ and $j = 0, 1, 2, \cdots, k$. And with a second constraint involving the volatility,

$$\mathcal{U}_{i,t}^{(4)} = \Bigg\{ p_{i+j,t+\Delta|i,t} \;:\; \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} = 1, \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t}\psi_{i+j,t+\Delta} \in \mu \pm \epsilon_1,$$
$$\sum_{j=0}^{k} p_{i+j,t+\Delta|i,t}(\psi_{i+j,t+\Delta} - \mu)^2 \in (\sigma \pm \epsilon_2)^2, p_{i+j,t+\Delta|i,t} \geq 0, \epsilon_1, \epsilon_2 \geq 0 \Bigg\}, \tag{5.11}$$

for $i = 0, 1, 2, \cdots, tk/\Delta$ and $j = 0, 1, 2, \cdots, k$.

Lastly, we can also define uncertainty by considering a region around some specific probabilities determined uniquely. For this choice, we adopt the binomial structure in (2.2) under our physical measure. That is, we replace r with μ and define $q = \frac{e^{\mu\Delta/k} - d}{u-d}$. We can then create an uncertainty set by defining

$$\mathcal{U}_{i,t}^{(5)} = \Bigg\{ p_{i+j,t+\Delta|i,t} = q_j + \epsilon_j \;:\; q_j = \binom{k}{j} q^{k-j}(1-q)^j, \sum_{j=0}^{k} p_{i+j,t+\Delta|i,t} = 1, \sum_{j=0}^{k} \epsilon_j = 0 \Bigg\}, \tag{5.12}$$

for $i = 0, 1, 2, \cdots, tk/\Delta$ and $j = 0, 1, 2, \cdots, k$.

With the types of our uncertainty sets now defined, we can solve our optimization by

$$\max_{p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}} V_{i,t}^* \quad \text{for} \;\; i = 0, 1, 2, \cdots, tk/\Delta$$
$$\text{where} \;\; V_{i,t}^* = \min_{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3} V_{i,t} | \mathcal{U}_{i,t}^{(i)}$$
$$\text{such that (3.4)-(3.7) holds} \; \forall \;\; j = 0, 1, 2, \cdots, k$$

However, we notice that our optimization is a constraint robust optimization since the generated values of uncertain probabilities $p_{i+j,t+\Delta|i,t}$'s lie in the constraint (3.7). As such, if

$$\pi_c^+ - \pi_c^- + \frac{1}{1-c} \sum_{j=0}^{k} u_{i+j,t+\Delta} p_{i+j,t+\Delta|i,t} \leq \gamma,$$

holds, it follows that

$$\max_{p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}} \pi_c^+ - \pi_c^- + \frac{1}{1-c} \sum_{j=0}^{k} u_{i+j,t+\Delta} p_{i+j,t+\Delta|i,t} \leq \gamma,$$

also holds for all $i = 0, 1, 2, \cdots, tk/\Delta$. Hence we can finally re-formulate our worst-case optimization into a more tractable form with the help of the transformations in (5.7) and (4.5). Thus the robust counterpart becomes

$$\begin{aligned} \min_{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3} & V_{i,t} && && (5.13) \\ \text{s.t. } & \pi_c^+ - \pi_c^- + u_{i+j,t+\Delta} - L_{i,t} \geq 0, && \forall\, j = 0, 1, 2, \cdots, k, \\ & u_{i+j,t+\Delta} \geq 0, && \forall\, j = 0, 1, 2, \cdots, k, \\ & \pi_c^+, \pi_c^- \geq 0, \\ & \pi_c^+ - \pi_c^- + \frac{1}{1-c} \sum_{j=0}^{k} u_{i+j,t+\Delta} p_{i+j,t+\Delta|i,t} \leq \gamma, && \forall\; p_{i+j,t+\Delta|i,t} \in \mathcal{U}_{i,t}, \end{aligned}$$

for $i = 0, 1, 2, \cdots, tk/\Delta$, and $j = 0, 1, 2, \cdots, k$. Hence our optimal hedging strategy $\mathcal{A}_{i,t}^* = \{a_{i,t}^*, b_{i,t}^*, c_{i,t}^*\} = \operatorname{argmin}_{a_{i,t}, b_{i,t}, c_{i,t} \in \mathbb{R}^3} V_{i,t}$. Since we now have a method to eliminate the non-linearity in (3.7), the remainder of the problem is simplified and we adopt linear programming to solve our optimization problem. Although we simplify our optimization problem, using linear programming makes the problem no less tractable, since the accuracy of our results rests on the number of uncertainty sets we can generate. In Chapter 6, we apply the model to an equity-index annuity and investigate the numerical accuracy and consistency of the model.

5.4 Sampling Techniques

With our index tree defined, we proceed to describe two sampling techniques that will be considered for the worst-case analysis.

5.4.1 Homogeneous Sampling

The first sampling technique is the homogeneous sampling of the probabilities or our uncertainty set. We say homogeneous because we generate a single sample of the uncertainty set and assume the probabilities of the index dynamics remain the same throughout the period of the contract. Thus assuming we choose our uncertainty set based on $\mathcal{U}_{i,t}^{(1)}$ for instance, then we can generate several vectors, each containing the probabilities $p_{i+j,t+\Delta|i,t}$ for $i = 0, 1, 2, \cdots, tk/\Delta$, and $j = 0, 1, 2, \cdots, k$. Hence, if the probability of the index moving from state $\psi_{i,t}$ to the next period $\psi_{i+j,t+\Delta}$ is $p_{i+j,t+\Delta|i,t}$, then this probability remains the same for a movement from state $\psi_{i,t+\Delta}$ to state $\psi_{i+j,t+2\Delta}$ and so on until maturity. Thus for homogeneous sampling technique,

$$p_{i+j,t+\Delta|i,t} = p_{i+j,t+2\Delta|i,t+\Delta} = p_{i+j,t+3\Delta|i,t+2\Delta} = \cdots = p_{i+j,n|i,n-\Delta},$$

for $i = 0, 1, 2, \cdots, tk/\Delta$, $j = 0, 1, 2, \cdots, k$, and $t \in 0, \Delta, \cdots, n - \Delta$.

5.4.2 Non-homogeneous Sampling

Since we are interested in studying the behavior of the model to the uncertainty or these probabilities, we also explore a non-homogeneous sampling of the probabilities. Unlike homogeneous sampling, we explore changes in the probability vectors through time. These changes could be defined by a function of the initial sample generated and time or simply by sampling randomly the probability vectors of the uncertainty set at each node or branch in the index tree. It is noteworthy that the samples of the uncertainty should be generated from values of the state variables. Thus for a non-homogeneous sampling technique,

$$p_{i+j,t+\Delta|i,t} \neq p_{i+j,t+2\Delta|i,t+\Delta} \neq p_{i+j,t+3\Delta|i,t+2\Delta} \neq \cdots \neq p_{i+j,n|i,n-\Delta},$$

for $i = 0, 1, 2, \cdots, tk/\Delta$, $j = 0, 1, 2, \cdots, k$, and $t \in 0, \Delta, \cdots, n - \Delta$.

Lastly, we also explore a situation where we consider no uncertainty within the model and compute the state probabilities as in the binomial structure, 2.2. This is simply to serve as a control to compare our results from using the two sampling techniques on the model. We refer to this as the "no sampling" technique.

5.5 Capital Requirement

Capital requirement is the standardized requirement or amount that financial institutions need to hold. It is simply how much liquidity financial regulators require issuers of financial securities or investment and insurance products to have to be stable and cover future obligations. For instance, Pillar I of Solvency II defines the Solvency Capital Requirement (SCR) as the amount of capital required by an insurance company to cover losses that occur with a probability of 99.5% over the next 12 months (i.e. $\text{VaR}_{99.5\%}$). Insurers are faced with a wide range of risks like disability, surrender, mortality and interest rate uncertainties. Similar to Gaillardetz and Moghtadai [2017], in order to estimate the capital requirement, we need to evaluate the losses that pertain to the specific contract. These losses are as a result of the differences between the hedge portfolio $V_{i,t}$ and the insurer's obligations $\mathcal{X}_i(x,t)$ at some time t and happen whenever the insurer's portfolio is re-balanced. Thus the hedging errors either appear as income or losses to the insurer. Let $\mathcal{H}$ be the discounted hedging errors random variable and using (2.22) we define

$$\mathcal{H} = \sum_{k=0}^{nN-1} e^{-k\Delta r_{k\Delta}} L_{k\Delta}. \tag{5.14}$$

Equity-linked contracts require the policyholder to make an initial investment and are promised some returns or guarantee at a future time. As such, for a unit initial investment from the holder and initial contract value of $V_{0,0}$, it suffices to say that the insurer will be required to set up $V_{0,0} - 1$ to establish the replicating portfolio. However, with mismatches

$\mathcal{H}$, we define the capital requirement (CR) to be

$$CR = V_{0,0} + VaR_c(\mathcal{H}) - 1, \tag{5.15}$$

where c is the desired retention level for computing the capital requirement and can be set differently from that used in the linear optimization when estimating the initial value of the contract.

This implies that an insurer who holds the amount CR at the beginning of the contract will have approximately $(1-c)\%$ probability of positive losses over the term of the contract. This form of capital requirement implies that the insurer will have to invest the excess of fund from the difference between the initial premium and $V_{0,0}$ in the risk-free asset.

It is noteworthy that CR can be a negative value, which could imply no capital requirement needs to be set for the contract. One importance of estimating the capital requirement this way is that, unlike most financial products, equity-linked products are priced by tweaking some internal parameters like the participation rates or percentage of initial premium that goes into the guarantees, see Tiong [2000] and Boyle and Hardy [1997]. This way, an insurer could numerically set these critical parameters such that the capital requirement is acceptable for the insurer.

Chapter 6

Numerical Applications

In this chapter, we apply our model in the previous chapter to some type of equity-linked contract and discuss the results for various parameters in the model. We begin this chapter by introducing a brief introduction of equity-linked products currently in the market and describe one of the contracts known as the Equity-Index Annuity.

6.1 Investment Guarantees

The principal component for an equity-linked product is that it allows an investor to participate in some underlying fund, stock index or even a mixture of some funds. However, these products tend to be different from other financial products due to the insurance component it provides through guarantees that are agreed at the inception of the contract. It is the presence of these guarantees that creates risk for the insurer since the payout structure of the guarantees is tied to the performance of the underlying fund or index.

6.1.1 Guaranteed Minimum Living Benefits (GMLB)

One category that describes the payout structure of the equity-linked product is the Guaranteed Minimum Living Benefits (GMLB). This type of contract has its payout effected while the insured is alive. It can further be separated into Guaranteed Minimum Maturity Benefit (GMMB) which promises the insured some guaranteed amount when the contract matures. While the insured has an upside benefit when the underlying stock index performs well, the insured is protected from downside risk by the guaranteed amount which could be subject to regular increase or be fixed over the term of the contract. Guaranteed Minimum Accumulation Benefit (GMAB) provides a reentry option or contract renewal at the end of the initial contract, see Hardy [2003]. Guaranteed Minimum Withdrawal Benefit (GMWB) allows the policyholder to make periodic withdrawals from the fund until some premium that was paid at the commencement of the contract is depleted. However, the policyholder is allowed to make guaranteed withdrawals, at the cost of the insurer, even if the account depletion occurred before the end of the term of the contract. Milevsky and Salisbury [2006] presents the valuation of GMWB using both static and dynamic models. Unlike GMMB, Guaranteed Minimum Surrender Benefit (GMSB) guarantees the policyholder the cash value of the contract beyond some fixed date when the policy is surrendered. This type of contract may be seen as some return of premium contract. Finally, Guaranteed Minimum Income Benefit (GMIB) simply ensures that the amount accumulated by the fund can be converted into some annuity at a guaranteed rate.

6.1.2 Guaranteed Minimum Death Benefit (GMDB)

Contrary to GMLBs, the Guaranteed Minimum Death Benefit is a type of guarantee that promises the insured some guaranteed amount contingent on death during the term of the contract or some deferred period. This guaranteed amount could simply be the original premium or may be accumulated at some specified interest rate and is payable to the surviving beneficiaries of the insured life, see Hardy [2003].

The expenses charged by the insurer on these investment guarantees may be financed as some fixed fee or by the rate of participation in the index. A segregated fund contract is a special type of equity-linked product sold in Canada which requires a single premium to be partitioned and invested in some mutual funds for the policyholder which is independent of the insurer's fund and mostly has separate management. Variable Annuities and Unit-Linked Insurance are very similar to Segregated funds but are widely sold in the United States and the United Kingdom respectively.

6.2 Equity-Indexed Annuities (EIAs)

Next, we describe a special type of equity-linked product known as Equity-Indexed Annuity (EIA). While EIA guarantees some rate (between 1% to 3%) that accumulates annually until maturity on a portion (usually 87.5%) of the initial premium invested, at maturity, the policyholder benefits from additional returns which is linked to some increase in the stock index (typically S&P 500) over the period of the contract.

In contrast to variable annuities (specifically GMMB) or segregated funds (Canadian version), EIAs have shorter terms but can generally be considered as long-term financial derivatives. Hardy [2003] regards EIA as a call option on the underlying equity index, while variable annuities are rather put options. Also, unlike variable annuities, the indexes for EIAs are price indexes and not total return indexes which allow for dividend reinvestments. As such, EIAs give more risk and potential gains than fixed annuities but less risk and potential gains than variable annuities. Also, EIAs do not require policyholders to pay taxes on the earnings until a withdrawal is made and hence are tax-deferred.

6.2.1 EIA Indexing Methods

The design of EIA contracts is based on the indexing method used. The Point-to-Point (PTP) indexing method simply ignores any dynamics of the index during the term of the contract and compares the change in the index from the beginning to the end of the policy. Let G be the value of the accumulated guarantee at contract maturity ($t = n$) and α the participation rate in the index returns, then for a unit initial investment, the payoff at maturity for a typical PTP is the

$$\max\left[1+\alpha\left(\frac{S_n}{S_0}-1\right), G\right] = G + \left[\left(1+\alpha\left(\frac{S_n}{S_0}-1\right)\right) - G\right]^+. \tag{6.1}$$

Another indexing method is the Lookback or High Water Mark which differs slightly from the PTP by comparing the highest value of the index throughout the term of the contract with the initial stock index value at the beginning of the contract. Let $S_{max} = \max\{S_\Delta, S_{2\Delta}, \cdots, S_n\}$ such that the payoff of a high water mark EIA contract at maturity is

$$\max\left[1+\alpha\left(\frac{S_{max}}{S_0}-1\right), G\right] = G + \left[\left(1+\alpha\left(\frac{S_{max}}{S_0}-1\right)\right) - G\right]^+. \tag{6.2}$$

The last type of indexing method is the Annual Ratchet (Cliquet). Unlike the PTP method, the Clique method evaluates the index participation annually. Thus it disregards declines and compares the changes in the stock index every year. A typical the Annual Ratchet method could have a simple annual payoff

$$\max\left[\sum_{t=\Delta}^{n}\left[1+\max\left(\alpha\left(\frac{S_t}{S_{t-\Delta}}-1\right), 0\right)\right], G\right], \tag{6.3}$$

or compound annual payoff

$$\max\left[\prod_{t=\Delta}^{n}\left[1+\max\left(\alpha\left(\frac{S_t}{S_{t-\Delta}}-1\right), 0\right)\right], G\right]. \tag{6.4}$$

It is worthy to note that a fixed or varying cap could be placed on each of these methods to limit the maximum amount the policyholder can earn should the index rise throughout the contract.

In the valuation of EIAs, we face the mathematical challenge of pricing and risk-management due to the embedded options in these products. Tiong [2000] presents closed-form pricing formulas under the assumption that the market is complete - there exists a unique risk neutral measure and a constant risk-free force of interest. Index price process are also assumed to follow geometric Brownian motions. Lee [2003] further proposed explicit pricing formulas and proposes a floating-strike lookback option to tackle the increase in the cost of the embedded options due to the high volatility of the equity market.

Also Hardy [2003] explains that if one is interested in calculating the purchase price of the options cover from a third party or planning to use a dynamic hedging approach, the basic Black-Scholes-Merton results can be modified to value EIAs. On the other hand, if mortality and lapses are not taken into consideration, then there is no difference between valuing the option outside or within the EIA contract.

6.3 Numerical Examples

The description of various indexing methods used in the valuation of equity-index annuities provides an expression for the product payoff or contingent claim $\mathcal{X}$ in our model. We proceed to conduct an analysis of the results from the model obtained by considering different variations of our parameters.

For our base-case example, we adopt a simple point-to-point EIA with GMAB and GMDB payoffs as stated in (6.1). For illustration purposes, we assume all policyholders are age (50) at the time of purchasing the contract and mortality is assumed to follow the illustrative life table, see Bowers et al. [1986]. We assume there is no cap on the potential returns of the investment and a premium of $1 is paid at inception. The guaranteed amount is given by $G = \beta(1+g)^t$. This implies, we only guarantee $\beta\%$ of the initial premium at some growth rate of g at time t. Also, we set the participation rate in the index returns to be $\alpha\%$ and assume constant r, μ, and σ to be the annualized risk-free interest rate, the index average return, and the index volatility respectively. We also assume that the market is frictionless. That is, no transaction cost, no taxes and also our set of calculations does not include any allowance for expense charges and premium loading.

6.3.1 Worst-Case Analysis

Next, we consider variations in the worst-case value due to sampling with restrictions on the mean and both the mean and volatility. Also, we compare situations when sampling of the uncertainty set is performed once for the whole index model (homogeneous sampling) or performed at each node or branch of the index model (non-homogeneous sampling). We rely on Markov Chain Monte-Carlo (MCMC) simulation methods, specifically the *Hit-and-Run* sampler to generate samples of $p_{i+j,t+\Delta|i,t}$'s for $\mathcal{U}_{i,t}$ since it guarantees that all imputations satisfy the constraints of the uncertainty set. The *Hit-and-Run* sampler generates one continuous-state Markov chain sample path over some parameter space. Chen and Schmeiser [1996] provide a technical proof of the convergence of the sampler whiles Florescu and Viens [2008] reveal that this Monte Carlo method converges as the number of probability samples and the number of Monte Carlo samples increases. Berger and Chen [1993] also explain that this sampler is particularly useful when the parameter space is constrained. After several numerical computations, we found that we attain convergence in the initial worst-case value of the hedge portfolio with at least 4000 samples of $\mathcal{U}_{i,t}$.

For our worst-case analysis, we initially choose (5.8) and (5.9) as our uncertainty set and generate the samples for the probabilities. Starting at maturity or time n, we first compute the claims at n using S_0, S_n and the other parameters in (6.1). Then using the samples from (5.8) and (5.9), we solve the linear optimization (5.13) for the worst-case values at $V_{i,n-\Delta}$ as well as the optimal hedging solutions $\mathcal{A}^*_{i,n-\Delta} = \{a^*_{i,n-\Delta}, b^*_{i,n-\Delta}, c^*_{i,n-\Delta}\}$ for each node. Next, we combine these values to obtain (2.22) at $i, n-\Delta$ which is then used in the linear optimization alongside the samples generated to obtain the worst-case values at $V_{i,n-2\Delta}$ as well as the optimal hedging solutions $\mathcal{A}^*_{i,n-2\Delta} = \{a^*_{i,n-2\Delta}, b^*_{i,n-2\Delta}, c^*_{i,n-2\Delta}\}$. This dynamic programming approach is done recursively until we get to time 0 and obtain the worst-case values at $V_{0,0}$ and $\mathcal{A}^*_{0,0} = \{a^*_{0,0}, b^*_{0,0}, c^*_{0,0}\}$. Thus for some initial value of the contract $V_{0,0}$, $a^*_{0,0}$ is to be invested in the index, $b^*_{0,0}$ in a risk-free asset that earns returns at r_0 and the final part $c^*_{0,0}$ in buying a call option that matures at n. The price of the at-the-money call option is

l_0=1, β= 100%, g=0%, α= 50%, c=95%,r=4%, μ=8%, σ=20%, k=7, N=1 , γ=0									
	Sampling with μ				Sampling with μ and σ				No sampling
n	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	$V_{0,0}$
3	1.0084	0.1869	0.7965	0.0250	1.0034	0.1815	0.7961	0.0258	1.0021
5	0.9976	0.2460	0.7333	0.0184	0.9873	0.2297	0.7368	0.0209	0.9829
7	0.9855	0.2665	0.7023	0.0167	0.9738	0.2487	0.7058	0.0193	0.9678
10	0.9761	0.2837	0.6775	0.0149	0.9618	0.2617	0.6818	0.0183	0.9554

Table 6.1: Hedging strategy for different terms to maturities with homogeneous sampling.

computed using the Black-Scholes pricing model.

In comparing our sampling strategies, Table 6.1 shows the values computed from homogeneous sampling. In this case, we generated one set of samples for the uncertainty set and assumed uncertainty to be homogeneous with respect to time, that is, the distribution of the uncertainty set remains the same through time. Also, Table 6.1 further indicates a decrease when we sample with both mean and volatility to when we sample with only mean. For a 3-year maturity EIA, there is a decrease of 0.5% from sampling with μ to sampling with both μ and σ. The difference increases to 1.43% when we consider a 10-year the term to maturity contract. Contract values obtained through sampling with μ tends to be higher than sampling with both μ and σ because, while volatility is allowed to vary in the former, the latter has fixed volatility in the distributed values of the uncertainty set. As such while there may not be drastic changes in volatility in the short term, having fixed volatility for a long duration contract may understate the worst-case value of the contract. Although we do not model volatility, it is worth noting that this observation encourages the inclusion of index volatility models to further estimate the worst-case scenario.

On the other hand, Table 6.2 shows the values computed from sampling our uncertainty set

l_0=1, β= 100%, g=0%, α= 50%, c=95%, r=4%, μ=8%, σ=20%, k=7, N=1 , γ=0									
	Sampling with μ				Sampling with μ and σ				No sampling
n	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	$V_{0,0}$
3	1.0088	0.1865	0.7972	0.0251	1.0036	0.1820	0.7959	0.0258	1.0021
5	0.9986	0.2467	0.7335	0.0184	0.9875	0.2297	0.7370	0.0208	0.9829
7	0.9865	0.2671	0.7027	0.0167	0.9740	0.2487	0.7059	0.0194	0.9678
10	0.9767	0.2800	0.6802	0.0155	0.9620	0.2622	0.6815	0.0184	0.9554

Table 6.2: Hedging strategy for different terms to maturities with non-homogeneous sampling.

at each node of the index model. That is, for each node at every time point, we generate samples of the uncertainty set used to estimate the value of the hedge portfolio and insurers obligation. The differences in the values computed are similar to that of Table 6.1. However, one thing worth noting is the increase in the values of the initial hedge portfolio from homogeneous sampling to sampling at each node. The values increase by 0.04% for the 3 years contract through to 0.1% for the 10 years contract under the sampling with μ and also increase by 0.02% for the 3 years contract through to the 10 years contract under the sampling with both μ and σ. While the decrease from sampling with μ to sampling with both μ and σ could be attributed to the fixed volatility, the overall increase from single sampling to node-wise sampling is as a result of allowing for more variability at each node, and hence the greater chance of observing the worst-case situation in the sampled uncertainty set.

The last column in Tables 6.1 and 6.2 show the value of the initial hedge portfolio using the risk control method without the worst-case framework and uncertainty set. As expected, both sampling schemes used to compute the price under the worst-case environment show higher contract values as compared to having no sampling, with as large as 2.1% decrease

l_0=1, β= 100%, g=0%, α= 50%, c=95%, r=4%, μ=8%, σ=20%, k=7, N=1 , γ=0					
Sampling with binomial uncertainty					No sampling
n	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	$V_{0,0}$
3	1.0031	0.1817	0.7956	0.0258	1.0021
5	0.9861	0.2267	0.7381	0.0213	0.9829
7	0.9712	0.242	0.7088	0.0205	0.9678
10	0.9585	0.2547	0.6843	0.0195	0.9554

Table 6.3: Hedging strategy for different terms to maturities with binomial uncertainty sampling.

in the value, when valuation is not done under the worst-case for a 10 years contract term.

Table 6.3 shows the values computed from sampling our uncertainty by constructing a 1% region around the probabilities obtained from the binomial structure by using (5.12). In this case, we explore an $\epsilon = 1\%$ region around the probabilities used to obtain the values from no sampling and observe the usefulness of allowing uncertainty when considering worst-case valuation. It can be observed that although the initial values of the contract are not as high as when we include only μ or both μ and σ, they are slightly greater than when we do not include uncertainty with a 0.3% decrease in the value when valuation is done with the no sampling strategy for a 10 years contract term.

Table 6.4 shows the values obtained from using quadratic hedging to estimate the strategy for different terms to maturity of the contract. In this analysis, we use the uncertainty obtained by generating homogeneous samples of (5.8). With our losses defined in (2.22), we use dynamic programming to optimize the expected squared errors starting from contract maturity using (5.4) until we obtain the initial hedging strategy at $t = 0$.

l_0=1, β= 100%, g=0%, α= 50%,r=4%, μ=8%, σ=20%, k=7, N=1					
	Quadratic Hedging				No sampling
n	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	$V_{0,0}$
3	0.9913	0.1948	0.7711	0.0253	1.0021
5	0.9628	0.2662	0.6789	0.0178	0.9829
7	0.9357	0.2988	0.6220	0.0150	0.9678
10	0.8883	0.3495	0.5282	0.0106	0.9554

Table 6.4: Hedging strategy for different terms to maturities with the quadratic approach.

We observe that, in comparison to the rest of the worst-case analysis made, applying a worst-case dynamic to the quadratic approach yields the minimum initial value of the contract. There is a steady decrease from 1% to 7% for increasing maturities when we compare the quadratic hedging to the no sampling strategy. This decrease is attributed to the absence of the tail risk measures in the quadratic approach. While the squaring of the hedging errors imply that we cannot differentiate between positive and negative errors, the use of a tail risk measure like the $CVaR$ allows the model to consider those losses greater than some threshold. This accounts for the higher values of our worst-case hedging strategy as opposed to that of the quadratic strategy. Estimating the hedging strategy using the quadratic approach may be seen as a regression.

Tables 6.1 to 6.4 show the cost of establishing the hedge portfolio at the inception of the contract as well as the allocations to each investment strategy in the hedge portfolio, when increasing the term to maturity of the contracts. We use $3, 5, 7$ and 10 years for our analysis. In all tables, we observe that the value of the contract decreases as the term to maturity increases. This can be attributed to the use of the optimal iterated hedging strategy. As the

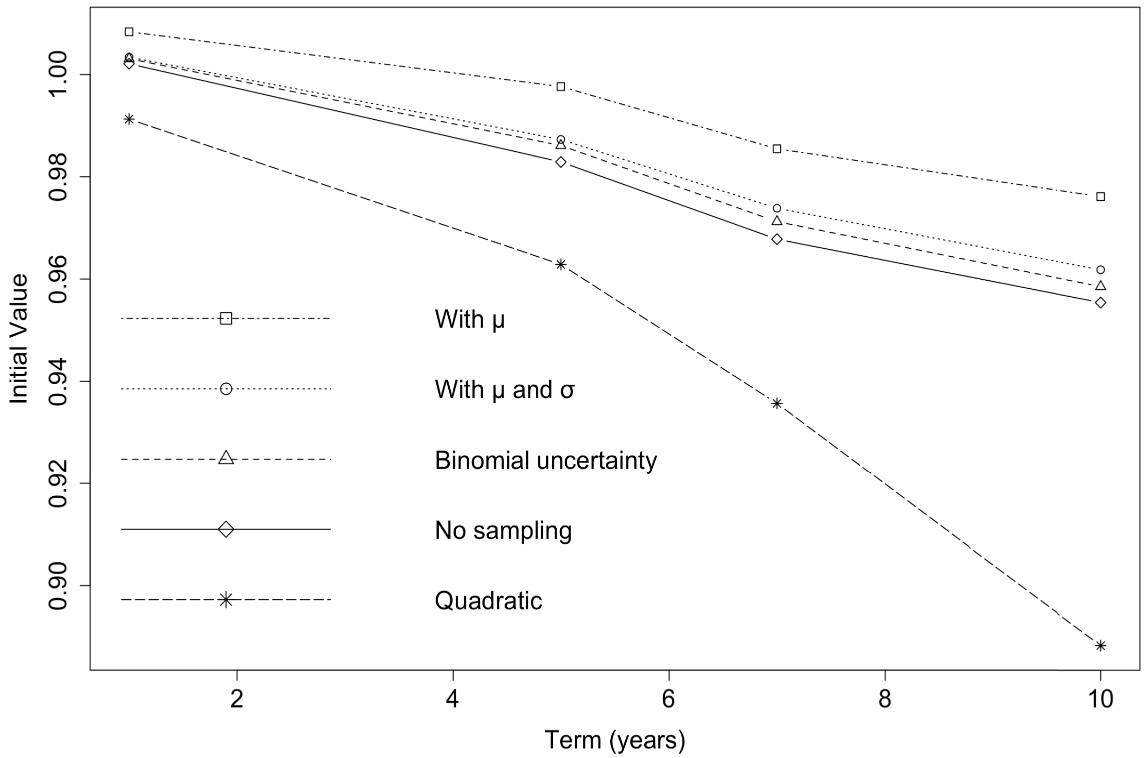

Figure 6.1: Initial values for different worst-case strategies and contract terms

term to maturity increases, there is a possibility for the index to grow exponentially if the probabilities are well set. As such the replicating portfolio takes advantage of this behavior in the index. Thus, the mismatches between the contact payoff and hedge portfolios tend to cancel out when hedging strategies are applied. This is in contrast to vanilla options, whose prices are increasing function of the time to maturity.

A 2 years increase in the term to maturity decreases the value of the contract by 1.2%, while a 5 years increase causes a decrease of 2.2% in the value of the contract. One reason that explains this decrease is that for longer maturities, the initial premium invested would have generally accumulated enough to absorb quite substantial market value fluctuations without falling below the guarantee level, which could trigger an increase in the value. This observation is consistent with why EIA's are considered long-term investments. It is clear that, based on these results, the insurer could target and reward long-term investors.

Another observation is the shift in investment strategy from the risk-free asset to the index asset under both sampling strategies as the term to maturity increases. This is due to the fact that equities have historically outperformed cash accounts and risk-free assets, over the long-term, and hence a diversified hedging strategy for a long term contract allocates a substantial amount to be invested in the index. The same can be said for sampling with both index average returns and volatility. Figure 6.1 gives a summary of the initial values for different worst-case strategies and term to contract maturities.

6.3.2 Sensitivity Analysis

We finally test the sensitivity of the model to some changes in the internal parameters. We discuss the results of estimates when we consider the sensitivity of the worst-case value of the contract to changes in index mean and volatility. Next, we also examine the behavior of the worst-case value to changes in parameters that are usually set by the insurer. For instance, while the mean and volatility are strictly determined by the index market, the choice of the desired level of retention, number of branches on the tree, number of trading period per year, number of policyholders, participation rate and percentage of premium guaranteed are determined by the insurer. As such we also discuss the impact of these parameters on the capital requirement as well as the initial value of the replicating portfolio.

Here, we consider the case where sampling the uncertainty set is done once (homogeneous) for the index model and compare sampling with fixed mean only and sampling with both mean and volatility fixed. Since we are interested in the sensitivity of the model to changes in the market, such as mean and volatility, we choose (5.10) and (5.11) as our uncertainty sets and $\epsilon_1 = 1\%$ for a fixed mean, and $\epsilon_2 = 5\%$ when we fix volatility. We then choose the running maximum for $10,000$ simulations of the hedging strategy.

Table 6.5 gives the initial values of the replicating portfolio as well as the asset allocations for different values of average returns, μ. As expected, a change in the average returns has very little or no effect on the initial value of the hedge portfolio. This indicates that the hedge portfolio is indeed able to cover the worst-case index dynamics since changes in the

l_0=1, β= 100%, g=0%, α= 50%, c=95%, r=4%, n=3, σ=20%, k=7, N=6 , γ=0								
	Sampling with μ				Sampling with μ and σ			
μ	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$
6%	1.0057	0.2754	0.7261	0.0042	1.0056	0.2773	0.7241	0.0041
8%	1.0055	0.2763	0.7251	0.0042	1.0055	0.2788	0.7226	0.0040
10%	1.0055	0.2783	0.7231	0.0041	1.0054	0.2803	0.7210	0.0040

Table 6.5: Hedging strategy for different means.

market movement have no effect on the value of the portfolio.

When the uncertainty set is conditioned on μ, we notice that a 2% change in the average returns has no effect on the value of the initial portfolio. The same can be said when we sample with both μ and σ. In both instances, we notice that while the initial worst-case value remains fairly constant, the hedging strategies suggest an increase in the investment of the index, $a_{0,0}$ by approximately 0.1% for a 2% increase in the average returns. This is due to the increasing nature of the index if the probabilities are well set, the strategy takes advantage of this increase to allocate more investment in the index returns while keeping the value of the contact immune to fluctuations in the index returns.

To test the effect of volatility on the worst-case value, Table 6.6 illustrates the initial values of the replicating portfolio and asset allocations when volatility increases from 15% to 30%. Although a change in the average return has no effect on the worst-case value of the contract, an increase in their variances causes an increase in the value of the hedge portfolio.

A 5% increase in σ increases the value of the contract by 2% and a 10% increase in σ increases the value of the contract by 4%. However, there is only an approximate reduction by 0.04% when we move from sampling with only mean to sampling with volatility. It is worth mentioning that the values in Table 6.6 are the running maximum of the simulations and for

l_0=1, β= 100%, g=0%, α= 50%, c=95%, r=4%, n=3, μ=8%, k=7, N=6 , γ=0								
	Sampling with μ				Sampling with μ and σ			
σ	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$
15%	1.0052	0.2791	0.7221	0.0040	1.0049	0.2760	0.7247	0.0042
20%	1.0238	0.2782	0.7406	0.0050	1.0234	0.2761	0.7422	0.0051
30%	1.0610	0.2860	0.7683	0.0068	1.0609	0.2854	0.7687	0.0068

Table 6.6: Hedging strategy for different volatility.

each fixed value of σ, we allow a 5% region around it. As such like any financial product, increasing volatility increases the worst-case value of the contract.

Once the optimal hedging portfolios are estimated, we also compute the capital requirements for each dynamic with (5.15) based on a $VaR_{99\%}$ by running $100,000$ simulations to compute the hedging errors (5.14). This ensures that the insurer will have a 1% probability of positive losses over the term of the contract with this capital requirement at the beginning of the contract.

Table 6.7 also shows values for the initial hedge portfolio for different retention levels, c, and their respective capital requirements. Notice that an increase in retention level from 20% to 50% causes a 99% approximate increase in the initial value when we sample with just the mean, but 1.7% increase when we fix volatility. Also, there is a relatively small change in the initial value of 0.1% for a 30% increase in the level of retention but an increase of 1.8% for a 20% change in the level of retention.

Since we constrain the $c-$conditional value-at-risk to be less than 0 in our optimization, it implies that we want the expected loss, given that the loss lies in the worst $1-c$ part of our loss distribution, to be less than 0. As such lower retention levels put less weight on the worst-case value of the contract whiles higher levels of retention put more weight on the

l_0=1, β= 100%, g=0%, α= 50%, r=4%, n=3, μ=8%, σ=20%, k=7, N=6 , γ=0										
	Sampling with μ					Sampling with μ and σ				
c	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR
20%	0.0145	0.0000	0.0000	0.0145	1.4047	0.0006	0.0000	0.0000	0.0006	0.8414
50%	1.0002	0.2864	0.7100	0.0038	0.0746	0.0176	0.0000	0.0000	0.0176	0.4040
80%	1.0012	0.2847	0.7126	0.0039	0.0743	0.8979	0.0000	0.8959	0.0020	0.3330
99%	1.0190	0.2428	0.7704	0.0058	0.0741	1.0181	0.2530	0.7609	0.0052	0.0735

Table 6.7: Hedging strategy and capital requirements for different retention levels.

worst-case values of the contract, when we maximize over a homogeneous uncertainty set. This explains why the initial value for $c = 20\%$ is less than that of $c = 99\%$. However, the capital requirement tends to decrease when the retention level increases. This is because, although fewer weights are assigned to the worst-case value when c is small, an expected loss in the worst 50% is riskier than an expected loss in the worst 1% part of the distribution. Hence, higher levels of retention yield higher contract values but a low capital requirement. Thus the choice of the level of retention is entirely based on the insurer's trade-off between contract value and capital requirement.

The use of a multinomial lattice for the index model makes it necessary to study the sensitivity of the initial value of the replicating portfolio and their capital requirements when we vary the number of branches or nodes. Table 6.8 illustrates this dynamic for an increasing number of branches, $k = 4, 9, 14, 19$. An increase in k by 5 increases the value of $V_{0,0}$ by approximately 0.7% and 0.05% for the capital requirement.

Also, an increase in the number of branches by 10 increases the initial value on the average by approximately 2.3%, but reduces as the branches get larger for both sampling techniques. The capital requirement also increases for larger values of k, but the difference reduces as

l_0=1, β= 100%, g=0%, α= 50%, c=95%, r=4%, n=3, μ=8%, σ=20%, N=6 , γ=0										
	Sampling with μ					Sampling with μ and σ				
k	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR
4	1.0047	0.2990	0.7029	0.0027	0.0756	0.8961	0.0000	0.8961	0.0000	0.0625
9	1.0114	0.2755	0.7318	0.0042	0.0761	1.0112	0.2764	0.7306	0.0042	0.0763
14	1.0303	0.2823	0.7442	0.0038	0.0864	1.0299	0.2819	0.7443	0.0038	0.0865
19	1.0315	0.2703	0.7566	0.0046	0.0869	1.0306	0.2683	0.7575	0.0047	0.0868

Table 6.8: Hedging strategy and capital requirements for different number of branches.

the branches get larger. It can be observed that when volatility is fixed, the variability of the model is reduced and as such, smaller number of branches, $k = 4$, allocates the entire portfolio to the risk-free asset. Due to our lack of computational power, we could not investigate the number of branches beyond which convergence in the value is achieved. The increase in values can be attributed to the wider range of market of movements which increases the value of the portfolio under the worst-case scenario.

Table 6.9 shows the sensitivity of the initial hedge portfolio to different frequencies of trading. Here we consider instances where we re-balance the hedge portfolio monthly ($N = 12$), quarterly ($N = 4$), semi-annually ($N = 2$) and annually ($N = 1$). When the hedge portfolio is re-balanced monthly, we observe that the value of the initial portfolio is slightly less than quarterly and the value increase as we trade less frequently. An increase in the value of Δ from monthly through to annual gives an increase in initial value from 0.3% to 0.1%.

Notice from Table 6.9 that when we fix volatility the overall value in $V_{0,0}$ and capital requirement decreases. Another interesting observation is that, when volatility is fixed, the variability of the model is reduced and hence, investing all the initial value in the risk-free asset is shown to be optimal when you consider re-balancing monthly or frequently. Also,

l_0=1, β= 100%, g=0%, α= 50%, c=95%, r=4%, n=3, μ=8%, σ=20%, k=7, γ=0										
	Sampling with μ					Sampling with μ and σ				
N	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR
12	1.0038	0.2964	0.7053	0.0021	0.0708	0.8964	0.0000	0.8964	0.0000	0.0602
4	1.0067	0.2642	0.7364	0.0061	0.0747	1.0061	0.2647	0.7353	0.0061	0.0749
2	1.0085	0.2329	0.7636	0.0121	0.0795	1.0073	0.2294	0.7653	0.0126	0.0783
1	1.0097	0.1881	0.7965	0.0250	0.0829	1.0084	0.1815	0.7960	0.0258	0.0831

Table 6.9: Hedging strategy and capital requirements for different number of re-balancing per year.

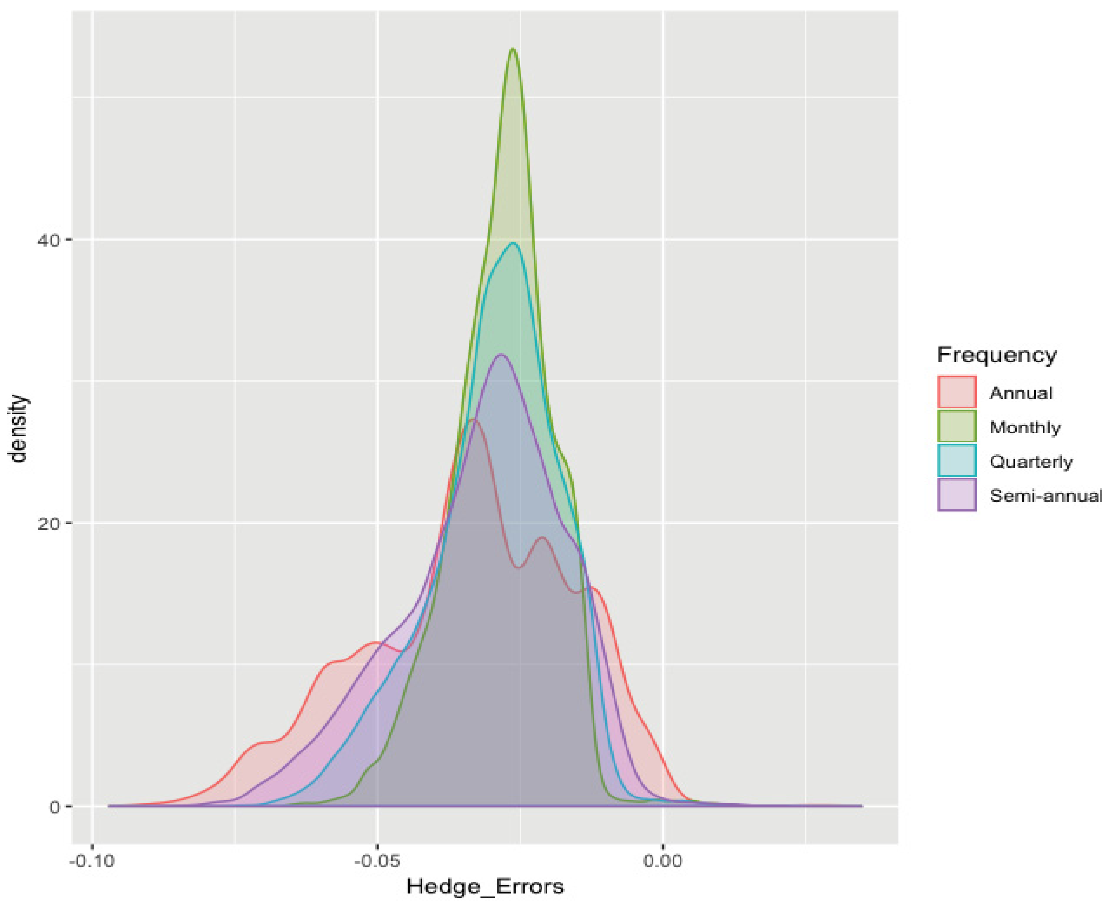

Figure 6.2: Hedging errors for different frequency of re-balancing

β= 100%, g=0%, α= 50%, c=95%, r=4%, n=3, μ=8%, σ=20%, k=7, N=6 , γ=0										
	Sampling with μ					Sampling with μ and σ				
l_0	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR
1	1.0054	0.2768	0.7245	0.0042	0.0729	1.0053	0.2794	0.7219	0.0040	0.0732
3	1.0053	0.2769	0.7241	0.0042	0.0694	1.0052	0.2793	0.7218	0.0040	0.0693
5	1.0052	0.2764	0.7245	0.0043	0.0694	1.0051	0.2788	0.7221	0.0042	0.0691
10	1.0046	0.2746	0.7255	0.0045	0.0694	1.0041	0.2780	0.7218	0.0043	0.0690

Table 6.10: Hedging strategy and capital requirements for different number of policyholders.

there is a significant reduction in the magnitude of hedging errors and this explains the lower value of capital requirement for the frequent re-balancing.

It is worth noting however that incurring transaction costs, bid-ask spreads and the like during re-balancing could make frequent re-balancing quite expensive, since increasing the frequency of trading in a year will also imply increasing the cost associated in the trading although it reduces the risk associated of the contract. We, however, require more capital if we intend to trade less frequently. This can be seen in the increasing trend of capital requirement as we balance less frequently. Figure 6.2 illustrates the density of the hedging errors and their respective re-balancing frequencies generated from $100,000$ simulations. The diagram shows that there is a significant reduction in the variance of the hedging errors as we increase the frequency of re-balancing.

Table 6.10 gives the normalized values of the initial hedge portfolio and capital requirement for different cohort sizes of $l_0 = 1, 3, 5$ and 10 homogeneous policyholders. As expected, the initial values of the contract tend to decrease when we increase the number of policyholders, although not by much. This is due to the reduction in non-systematic mortality risk as a result of diversification. We observe a 0.01% decrease in the initial value for every 2

l_0=1, g=0%, c=95%, r=4%, n=3, μ=8%, σ=20%, k=7, N=6 , γ=0											
		Sampling with μ					Sampling with μ and σ				
α	β	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR	$V_{0,0}$	$a_{0,0}$	$b_{0,0}$	$c_{0,0}$	CR
0.25	0.7	0.9282	0.2500	0.6782	0.0000	0.0996	0.9280	0.2500	0.6780	0.0000	0.0996
	0.9	0.9307	0.2321	0.6982	0.0004	0.0826	0.9305	0.2323	0.6979	0.0004	0.0823
0.50	0.7	0.9526	0.4953	0.4571	0.0001	0.2290	0.9522	0.4953	0.4567	0.0001	0.2287
	0.9	0.9726	0.3857	0.5843	0.0025	0.1273	0.9722	0.3842	0.5853	0.0026	0.1271
0.75	0.7	0.9836	0.6947	0.2876	0.0014	0.3092	0.9834	0.6940	0.2879	0.0014	0.3092
	0.9	1.0218	0.5256	0.4916	0.0046	0.1760	1.0216	0.5255	0.4916	0.0046	0.1755
1.0	0.7	1.0251	0.8529	0.1689	0.0033	0.3504	1.0250	0.8528	0.1689	0.0033	0.3503
	0.9	1.0757	0.6680	0.4013	0.0064	0.2266	1.0756	0.6673	0.4019	0.0064	0.2266

Table 6.11: Hedging strategy and capital requirements for different guarantees and participation rates.

extra policyholders we add to the contract, but a 0.06% decrease for accommodating 5 extra policyholders. Although there is a decrease in the initial value for an increasing number of policyholders, the capital requirement stays fairly constant. This can be attributed to the simulation of hedging errors when we computed the capital requirements. Since taking on more policyholders implies bearing more risk, there could be an overall decrease in the shared contract value per policy due to diversification but increasing risk requires higher capital and hence no reduction in capital requirement per policy. It is important to note that this observation is highly influenced by our capital requirement formula.

Generally, EIA's are priced by tuning some internal parameters. There is not much flexibility around the parameters that affect the index and mortality risk directly. As such, the participation rate and the percentage of initial premium guaranteed tend to be the best parameters to be tuned for desirable risk tolerance of capital requirement.

Table 6.11 shows the initial values, strategies and capital requirements for participation rates $\alpha = 25\%, 50\%, 75\%, 100\%$ and percentage of initial premium $\beta = 70\%, 90\%$. When the policy holder is allowed a 25% participation in the returns from the index, the value of the initial hedge portfolio increases by 0.25% with a 20% increase in the percentage of initial premium invested. There is also a 5.3% increase in $V_{0,0}$ for a 50% increase in the participation rate. Although an increase in participation rate increases the capital requirement, an increase β decreases the capital requirement. This is because, when the insurer allows a high participation in the index, the overall value of the contract increases since there is an increase in the insurer's obligation. Albeit, a high percentage of initial premium guaranteed reduces the insurer's obligation since there is little or no risk associated with the payment of the guarantee. As such it is favorable to set α low and β high to ensure a minimum capital requirement.

Other parameters like the growth of the guaranteed amount, the presence of a cap rate, surrender rate or a withdrawal option can also be considered in setting the fair price of an equity-linked product. These internal parameters can also be chosen based on some considerations for management expense ratios, loading, and other transaction costs.

Conclusion

In this thesis, we sought to present a partial worse case hedging strategy for an equity-linked product. We achieved this by defining a loss random variable which consisted of two main risks, the financial and mortality risks. While non-systematic mortality risk can be reduced by selling more policies to independent homogeneous lives, systematic mortality risk affects the whole population. The distribution of returns as well as volatility are major risks that affect the index or financial model. We assume the market is incomplete and frictionless and rely on the coherency of the conditional value-at-risk and the optimal iterated hedging strategy to locally minimize the cost of establishing a replicating portfolio that immunizes the worst-case value of the contract.

Our proposed strategy requires us to solve a linear optimization problem with data uncertainty. Thus we resort to robust optimization techniques. With our uncertainty defined with the probability of states in the index model in mind, we formulate the robust counterpart of our linear optimization and solve it using linear and dynamic programming approaches. The capital requirement is determined using the hedging errors of our risk minimization strategy, which also includes investments in the index, a risk-free asset, and a European call option, assuming investors behave rationally.

A detailed numerical analysis is performed for a point-to-point equity-indexed annuity with GMAB and GMDB investment guarantees based on a multinomial lattice tree for the index. Based on our analysis, we observe that sampling with average returns and allowing volatility to vary gives higher worst-case contract value than when we fix volatility. Also, the non-homogeneous sampling strategy showed slightly higher values than the homogeneous and binomial strategies. The quadratic strategy had the least worst-case contract values due to the absence of a tail risk measure. In analyzing the sensitivity of the model to changes in parameters, we observe that while the worst-case value of the contract increases with increasing volatility, they remain unchanged for changes in the average return, a feature very useful for setting reserves and capital requirements and also suitable to serve as a benchmark for comparing models.

To illustrate the reduction in non-systematic risk, we observe a reduction in the worst-case value per policyholder when we increase the number of policyholders. Also, our model is consistent with the concept that frequent re-balancing reduces the variance of hedging errors. Lastly, since equity-linked products are priced with internal parameters, we explore the changes in the worst-case value and capital requirements to different participation rates and percentage of premiums guaranteed and observed that it is best to set participation rates low and percentage of premium guaranteed high to ensure a minimum capital requirement. Our methodology may be extended to other equity-linked products such as variable annuities (segregated fund contracts in Canada) due to the similarities in their payoff structure. In the future, we intend to further explore this local risk-minimizing strategy with the conditional value-at-risk as the objective function instead of as a constraint. We also look to explore the use of the box and ellipsoidal uncertainty to converge at the worst-case value since sampling can be computationally extensive for longer maturities and higher trading frequencies.

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