Two-Stage Dynamic Average Consensus in Asymmetric Networks

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Abstract

Two-Stage Dynamic Average Consensus in Asymmetric Networks

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The main focus of this thesis is directed towards distributed control strategies for multi-agent systems. Given an asymmetric network of homogeneous agents with single-integrator dynamics and weighted links, it is desired to design a control rule for each agent using its local information as well as the information it receives from its neighbors to solve the average consensus problem. In other words, the global objective is to drive every agent's state to the average of the initial states of all agents (static average consensus) or the average of the reference inputs (dynamic average consensus). The main challenge, however, is to achieve these objectives in a general weighted network, i.e., when the graph representing the network is directed and each edge is weighted. To this end, a novel two-stage strategy is proposed, where in the first stage a mirror model is defined for every agent to compute its final state based on a standard consensus protocol. Then in the second stage, the standard update rule is adjusted for each agent accordingly to account for the discrepancy between the final state of its mirror model and the desired average consensus state. Simulations demonstrate the effectiveness of the proposed control strategies in different scenarios.

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Contribution of Authors

The results of this thesis will be submitted to a conference. As the supervisor, Dr. Aghdam provided guidance and direction throughout this work. All the research presented in this dissertation is performed by the author.

Table of Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation

Multi-agent systems have attracted a great deal of attention in the literature due to their applications in diverse fields such as the analysis of social media, coordination of a platoon of autonomous vehicles, formation flying of UAVs, and target tracking in wireless sensor networks, to name only a few [1,5,22,41,51,54]. In this type of networked control systems, it is often desired to use an update rule for each agent based on its local information and also the information it receives from its neighboring agents to achieve a global objective such as consensus, formation, and flocking [35,37,45,48]. Two classes of networks are studied in the control literature: symmetric and asymmetric [18,44], [12,39]. In a symmetric network, communication between neighboring agents is bidirectional, and hence, the graph representing the network is undirected [33], [47], [53]. In an asymmetric network, on the other hand, flow of information between agents is unidirectional, and therefore, the network is modeled by a directed graph (digraph) [3], [13]. A graph is called balanced if for each node the summation of the input weights be equal to the summation of the output links [39]. It is to be noted that in a graph model of a network, each agent is represented by a vertex (node) and the communication link between any pair of agents is represented by an edge (arc).

A class of multi-agent networks in which the communication links (edges) between the agents (nodes) are weighted is of special interest in emerging applications such as smart grids and acoustic underwater sensor networks [11, 15, 23]. For example, due to surface ambient noise, multi-path propagation and temperature fluctuation in an underwater acoustic sensor network, the environment is very unpredictable, and hence, communication among nodes in this type of network is not uniform [11], [53]. More precisely, the probability of node A receiving information from node B in the network is not necessarily the same as the probability of node B receiving information from node A. Thus, a random graph is often used to model such a network, which means that distinct edges in the graph may have different weights [14], [6].

Different control algorithms are proposed in the literature to achieve the global objectives mentioned earlier [17,39,56]. Since this thesis is mainly focused on the average consensus control, some background on the problem is provided in the next section.

1.2 Average Consensus Problem

Consensus is one of the fundamental objectives in the control of multi-agent systems. The objective of the consensus problem is to reach an agreement on the states of a group of agents using a distributed control strategy [26, 36, 40]. Two types of consensus problem are in particular important from the control perspective: static average control and dynamic average control. In the static average consensus problem, no reference signal is applied to the network, and the desired state that all agents' states are to converge to is the average of the initial states of all agents [13, 28]. In the dynamic consensus control problem, on the other hand, every agent is subject to a reference input (which can be time-varying, in general), and every agent's state is to approach the average of all reference inputs at any time instant [44, 56]. Both of these problems have been investigated extensively by several researchers in the past two decades, and various algorithms have been proposed for consensus control of multi-agent systems with undirected, directed and balanced directed graphs [3, 17, 56].

1.3 Thesis Contributions

Two main problems concerning the control of multi-agent networks are investigated in this dissertation. The first one is focused on average consensus in an unbalanced directed network with no external inputs. To solve the problem, a set of mirror models (one for each agent) is introduced, where each model uses the information available to the corresponding agents to compute the final state of the agent under the standard consensus algorithm. The result obtained is subsequently used to adjust the algorithm in such a way that the state of the agent asymptotically converges to the average of the initial states of all agents. The theoretical findings are validated by extensive simulations. Numerical examples also demonstrate that the proposed protocol can also achieve by dynamic average consensus for certain reference inputs.

The second problem studied in this dissertation is concerned with dynamic average consensus in an unbalanced directed network. It is assumed that every agent is subject to an external reference signal, and given the information flow constraint, each agent is not aware of the other agents' reference signal. The objective is to track the average of the reference signals at all times. To solve the problem, an approach similar to the static average consensus case is used but for a faster communication, the two-stage algorithm here consists of an inner stage and an outer stage. In the inner stage, the difference between the reference signals in two consecutive time instants is communicated between the agents, and in the outer stage the standard consensus algorithm is performed, using the results of the inner stage. The convergence analysis is provided, where it is shown that the state of every agent converges to an arbitrary small of the desired final state. Numerical simulations confirm the effectiveness of the proposed protocol.

1.4 Thesis Layout

The structure of the thesis is as follows:

- **Chapter 1** provides the motivation and some background information. It also outlines the contributions of the work.
- **Chapter 2** presents a novel two-stage distributed protocol to reach average consensus in an asymmetric multi-agent network.
- **Chapter 3** presents a two-stage solution to the dynamic average consensus problem in a weighted asymmetric network.
- **Chapter 4** provides concluding remarks as well as suggestions for future research directions.

Chapter 2

A Calibration-Based Average Consensus Protocol for Directed Multi-Agent Systems

In this chapter, a novel distributed protocol for consensus control of multi-agent networks is proposed. The objective is that every agent's state converges to the average of the initial states in a strongly connected unbalanced directed network of agents. The problem of average consensus has been investigated in the literature in the case of a connected undirected network as well as a strongly connected balanced directed network. However, the existing algorithms are not as effective in unbalanced directed networks. To address this shortcoming, a set of "mirror" models is defined to compute how much each agent contributes to the discrepancy between the final state and the average of the initial states. These values are subsequently used to adjust the control input of every agent in such a way that the network reaches average consensus asymptotically. The results are then extended to the case where there are constant reference signals. It is shown that using the proposed update rule and under some mild conditions, the agents' states converge to the average of the reference inputs. Numerical examples demonstrate the effectiveness of the proposed protocol in different scenarios.

2.1 Introduction

The consensus problem in multi-agent networks has been thoroughly studied in the control literature for both cases of undirected and directed graphs [12, 18, 27, 39, 42, 44, 47, 49, 54, 56]. In this type of problem, it is desired to reach an agreement in the states of the agents, with limited information communication between them. The consensus problem has applications in emerging fields such as sensor networks, control of a platoon of autonomous vehicles, and social networks, to name only a few [1, 5, 41]. As a special case of the general problem, static and dynamic average consensus in undirected and balanced directed networks have been investigated extensively in recent years for agents with different dynamics (e.g., single integrator or double integrator agents).

It is shown in [49] that an undirected network of agents with second-order dynamics can reach average velocity consensus using a suitable algorithm if the network graph is connected. For the case of single integrator agents, on the other hand, position average consensus can be achieved in an undirected network if and only if the network graph is connected [32]. In a directed network, however, the same objective is achieved if and only if the network graph is balanced [39]. In [40], a Kalman filter is employed for distributed estimation of reference inputs to achieve dynamic average consensus in an undirected multi-agent network. The use of Kalman filters in the consensus problem has also been investigated in [38], [36]. The authors in [16] propose an algorithm to track the average of reference signals in an undirected network by means of a multi-stage cascade consensus filter. It is shown in [42] that asymptotic consensus in a network of single integrators can be achieved if the union of the collection of directed interaction graphs across some time intervals has a spanning tree frequently enough. A dynamic average consensus algorithm is proposed in [56] under some conditions on the relative deviation of the reference signals. The authors in [12] investigate the distributed discrete average consensus problem in an undirected network, and show that for a time-invariant directed graph containing a directed spanning tree, the network reaches consensus, but not necessarily average consensus, in finite time. In [30] two algorithms are proposed to solve a distributed optimization problem with nonconvex velocity constraints, nonuniform position constraints and nonuniform step sizes in a multi-agent network for the case when the communication topology is jointly strongly connected and balanced. The dynamic average consensus problem for an undirected network of agents with double integrator dynamics is studied in [18], where it is shown that with reduced requirements on the velocity measurements using a distributed algorithm, one can achieve dynamic average consensus provided the input signals of the agents, their velocities and accelerations are all bounded. The authors in [54] propose an average consensus algorithm for a symmetric sensor network based on the maximum-likelihood estimate of the network parameters in the presence of measurement noise. [33], the dynamic average consensus problem in an undirected network is formulated as a static consensus problem in sufficiently small time intervals. A dynamic average consensus protocol is introduced in [27] for input signals with first- and second-order dynamics. The algorithm drives the agents to a sufficiently small neighborhood of the average of the reference signals at a pre-specified rate for a strongly connected network with weight-balanced topology. The authors in [44] investigate the consensus problem in an undirected network with switching topology.

While most of the existing results on consensus control of multi-agent systems are concerned with undirected or balanced directed networks, some algorithms have also been developed for unbalanced directed networks. Authors in [8] and [9] use a *surplus*-based algorithm for solving the average consensus problem in a strongly connected directed network for static and time-varying topologies, respectively, where each agent, in addition to updating its state, updates its corresponding surplus value. In [3], a proportional-integral algorithm is presented which yields dynamic average consensus for any initial state, provided node i has access to the ith column and ith row of the Laplacian matrix. However, such a requirement may not be realistic in a practical setting. In [13] and [3], distributed strategies are introduced to adapt the weights of a digraph in order to asymptotically reach a balanced graph. In the former paper, each node can adapt the weights of its outgoing links based on the weights of its incoming links. However, this poses some practical difficulties as far as the limited resources of the agents are concerned. In the present work, a novel approach is introduced to solve the static average consensus problem in a directed network without any requirements of having balanced weights. The proposed scheme is performed in two stages, and unlike some of the existing methods, it is completely distributed. In the first stage, namely calibration, every agent uses its locally available information to compute the discrepancy between the final states under the standard consensus protocol and the desired final state, which is the average of the initial states. Then in the second stage, an error compensation term is added to the standard consensus protocol in order to asymptotically reach the desired state and hence achieve the average consensus objective.

The remainder of this chapter is organized as follows. In Section 2.2, some preliminaries on graph theory and background information on average consensus are provided. The main contributions of this chapter are presented in Section 2.3, where the mirror models for the calibration phase are introduced, and it is shown how these models can be used to generate an error compensation term in order to achieve average consensus.

Then, in Section 2.4, simulation results are presented which confirm the effectiveness of the proposed average consensus protocol in different scenarios. Finally, conclusions are drawn in Section 2.5.

2.2 Preliminaries

Notation: Throughout this work, the set of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. Furthermore, $\mathbb{N}_n$ represents the finite set of integers $\{1, 2, ..., n\}$. All scalar variables are denoted by Italic fonts while all vectors and matrices are represented by bold fonts. The transpose of a vector or a matrix is denoted by a superscript T, and the (i, j) element of a matrix $\mathbf{M}$ is represented by m_{ij}. The $n \times n$ identity matrix is denoted by $\mathbf{I}_n$, the $n \times 1$ vector of all ones is denoted by $\mathbf{1}_n$, and a column vector of all zeros except a 1 at the ith position is represented by $\mathbf{e}_i$.

Consider a directed network with n nodes, whose information exchange topology is represented by the weighted digraph $\mathcal{G} = (V, E, \mathbf{W})$, where $V = \mathbb{N}_n$ is the set of vertices, $E \subseteq V \times V$ is the set of edges, and $\mathbf{W} \in \mathbb{R}^{n\times n}$ is a matrix whose elements represent the weight of the corresponding elements in the edge set E. The in-neighbor and out-neighbor sets associated with node i, for any $i \in V$, are defined as

$$N_i^{in} = \{j \in V \setminus \{i\} \,|(j, i) \in E\}\,, \tag{2.1}$$

$$N_i^{out} = \{j \in V \setminus \{i\} \,|(i,j) \in E\} \,. \tag{2.2}$$

For a digraph, the in-neighbor set of a node and in-degree Laplacian matrix of a network are hereafter referred to simply as the *neighbor set* and the *Laplacian matrix*, respectively. Note that the in-degree Laplacian of a weighted digraph $\mathcal{G}$ is a real matrix $\mathbf{L}$ defined as [32]

$$\mathbf{L}(\mathcal{G}) = \mathbf{\Delta}(\mathcal{G}) - \mathbf{W}(\mathcal{G}), \tag{2.3}$$

where Δ is a diagonal matrix whose diagonal element, $\delta_{ii},\ i \in V$, is equal to the sum of the in-degree of vertex v_i, i.e., the total number of incoming edges of v_i, and $\mathbf{W}(\mathcal{G})$ is the weight matrix of $\mathcal{G}$ whose (i,j) entry represents the weight of the link from node j to node i. In other words

$$\begin{aligned} &\mathbf{\Delta}(\mathcal{G}) = \text{diag}\{d_{in}(v_i)\}, \\ &d_{in}(v_i) = \sum_{j \in \mathbb{N}_i^{in}} w_{ij}, \end{aligned} \tag{2.4}$$

$$\mathbf{W}(\mathcal{G}) = \begin{cases} w_{ij}, & \text{if } (j,i) \in E, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

From equations (2.3) - (2.5), one can find the Laplacian matrix as follows

$$l_{ij} = \begin{cases} -w_{ij}, & \text{if } (j,i) \in E, \\ \sum_{k \neq i} w_{ik}, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Note that each node knows the weights of its incoming links (the corresponding row of the Laplacian matrix), which are by definition, its neighbors, but does not know the weights of its outgoing links.

A graph is said to be strongly connected if every vertex is reachable from every other vertex [32]. Strong connectivity is a fundamental requirement in cooperative control of multi-agent systems. It is assumed in this work that the digraph representing the network is strongly connected.

From equations (2.6)), it is obvious that sum of each row in the Laplacian matrix is equal to zero for a directed network (this is a well-known property of undirected network as well). This means that zero is an eigenavlue of the Laplacian mtrix and $\mathbf{p} = \mathbf{1}$ is its corresponding right eigenvector, i.e. of Laplacian matrix

$$\mathbf{L1} = 0. \tag{2.7}$$

Unlike undirected networks, however, in a directed network the sum of the elements of every columns is not zero unless it is weight balanced. In other words, $\mathbf{q} = \mathbf{1}^T$ is a left eigenvector of the Laplacian matrix associated with the zero eigenvalue in a directed network only if the network is balanced.

Let $\mathbf{x}[k] \in \mathbb{R}^n$ be a vector in discrete time domain, and $\bar{\mathbf{x}}[k]$ be a vector with equal elements. For any real positive constant ϵ, $\mathbf{x}[k]$ is said to be in the ϵ-neighborhood of $\bar{\mathbf{x}}[k]$ if there exists an integer k' such that

$$\| \mathbf{x}[k] - \bar{\mathbf{x}}[k] \|_\infty < \epsilon, \ \forall k > k'. \tag{2.8}$$

Consider a multi-agent system, and let the dynamics of each node in the discrete-time domain is described by

$$x_i[k+1] = x_i[k] + hu_i[k], \tag{2.9}$$

where $h > 0$ is the sampling time, and $x_i[k], u_i[k] \in \mathbb{R}$ are the state and control input of node i in the time interval $[kh, (k+1)h)$, $k \in \mathbb{N}$, respectively. Denote the average of the initial states of all nodes as

$$\bar{x}_0 = \frac{1}{n} \sum_{i=1}^{n} x_i[0]. \tag{2.10}$$

In the average consensus problem, it is desired that the state of every node converges to $\bar{x}_0$ as k increases. For the case of a balanced network with fixed topology (i.e., when the edge set E is fixed) and zero communication delay, it is well-known that the following control rule achieves average consensus [28, 31, 52]

$$u_i[k] = \sum_{j \in N_i^i n} w_{ij}(x_j[k] - x_i[k]). \tag{2.11}$$

Let $r_i[k]$ represent the reference signal applied to node i, for any $i \in V$. Then, $\mathbf{r}[k] \in \mathbb{R}^n$ is a vector consisting of all reference signals, and $\Delta r_i[k] = r_i[k] - r_i[k-1]$. Consider the dynamic average consensus problem with the following agent dynamics [44]

$$x_i[k+1] = x_i[k] + hu_i[k] + \Delta r_i, \tag{2.12}$$

where the initial state of node i is assumed to be $x_i[0] = r_i[-1]$. In this case, the objective is that the state of each node approaches $\bar{r}[k]$, as k increases.

Remark 2.1. Let $d_{max} = max_i\,(l_{ii})$ denote the maximum in-degree of the nodes of the network digraph. Then, for all sampling times $h \in (0, 1/d_{max})$, $(I - hL)$ is a nonnegative matrix.

2.3 Control Protocol: Average Consensus

Consider an asymmetric network consisting of n nodes, represented by the weighted digraph $\mathcal{G}$, and let the node dynamics be described by (2.9). The objective is to develop a distributed algorithm for reaching average consensus. For a fixed topology, it is known that the consensus protocol (2.11) reaches average consensus asymptotically if and only if the graph is strongly connected and balanced. However, using this update rule for a strongly connected unbalanced digraph results in a steady-state error, in general. The amount of this steady-state error is, in fact, a function of the initial states of the system and the link weights.

It is desired now to modify the update rule (2.11) for balanced directed networks in such a way that average consensus is achieved for a network with an unbalanced digraph. It is to be noted that since the control structure is distributed, each node has only access to its own information as well as that of its neighbors. Define n mirror models with identical dynamics, and denote the state of the ith model, $i \in \mathbb{N}_n$, by $\hat{\mathbf{q}}_i \in \mathbb{R}^n$. Let the dynamics of each model be described by

$$\begin{aligned} \hat{\mathbf{q}}_i[k] &= (\mathbf{I}_n - h\mathbf{L})\hat{\mathbf{q}}_i[k-1], \\ \hat{\mathbf{q}}_i[0] &= \mathbf{e}_i, \quad i \in \mathbb{N}_n. \end{aligned} \tag{2.13}$$

One can then write

$$\hat{\mathbf{q}}_i[k] = (\mathbf{I}_n - h\mathbf{L})^k \hat{\mathbf{q}}_i[0]. \tag{2.14}$$

It is shown in [39] that in an asymmetric network with the node dynamics (2.9), consensus is achieved asymptotically, i.e., the states of all nodes $x_1[k]$, ..., $x_n[k]$ become equal as k goes to infinity. It can be verified that for any positive real value ϵ, the states of all nodes reach the ϵ-neighborhood of the same steady-state value for a sufficiently large k. This steady-state value depends on the parameters of the network as well as the initial states. Now, let the steady-state value of $\hat{\mathbf{q}}_i[k]$ be denoted by $\hat{q}_{i,ss}\mathbf{1}_n$, and note that the dynamics of the mirror model (2.13) resembles that of the network from the viewpoint of node i. Note also that the average value of the elements of the initial state in (2.13) is equal to $1/n$. Thus, the difference between the final state and the average value of the initial states is $1/n - \hat{q}_{i,ss}$. This value will be referred to as the *average disagreement* for node i. It is important to note that the value of the average disagreement depends only on the Laplacian matrix $\mathbf{L}$ and sampling time h.

Remark 2.2. It is to be noted that in the calibration phase, each node only requires information from its in-neighbor nodes. More precisely, node i requires the information of the ith row of the Laplacian matrix along with the current states of its in-neighbor nodes at each time instant to generate the vector $\hat{\mathbf{q}}_i$ at the next time instant.

It is desired now to add a term to the dynamics of the nodes given by (2.9) in order to calibrate them in such a way that the average consensus is achieved.

Lemma 2.1. *Let the vector of steady state values obtained from the n mirror models be given by (2.13), i.e. $\hat{\boldsymbol{q}}_{ss} := [\hat{q}_{1,ss}\ \hat{q}_{2,ss}, \dots\ \hat{q}_{n,ss}]^T$. Then, $\hat{\boldsymbol{q}}_{ss}$ is, in fact, the left eigenvector corresponding to eigenvalue 1 of matrix $(\boldsymbol{I}_n - h\boldsymbol{L})$.*

Proof. Matrix $(\mathbf{I}_n - h\mathbf{L})$ is irreducible if and only if the digraph corresponding to the Laplacian L is strongly connected [4, 21, 55]. Since $(\mathbf{I}_n - h\mathbf{L})$ is a non-negative, square, and irreducible matrix whose all diagonal elements are positive, it is a *primitive matrix*

[19,55]. According to Perron-Frobenius theorem [25,55], for any primitive matrix, there exists a real positive number λ_p, that is an eigenvalue of $(\mathbf{I}_n - h\mathbf{L})$, such that the magnitude of any other eigenvalue (possibly complex) λ is strictly smaller than r, i.e. $|\lambda| < \lambda_p$. Given that $(\mathbf{I}_n - h\mathbf{L})$ has an eigenvalue at 1 (and hence so does $(\mathbf{I}_n - h\mathbf{L})^k$ for any integer k), which is, in fact, the eigenvalue with the largest magnitude, one can show that

$$\lim_{k\to\infty} \frac{(\mathbf{I}_n - h\mathbf{L})^k}{r^k} = \mathbf{p}_1\mathbf{q}_1^T, \tag{2.15}$$

where $r = 1$ and vectors $\mathbf{p}_1$ and $\mathbf{q}_1$ denote, respectively, the right and left eigenvectors corresponding to the eigenvalue 1 of the matrix $(\mathbf{I}_n - h\mathbf{L})$, normalized such that $\mathbf{q}_1^T\mathbf{p} = 1$. It is shown in [2] that every node in a strongly connected directed graph can obtain the information of the graph topology in finite time. Therefore, every node in the network considered here can obtain the steady state information (2.15) in finite time. It then follows from (2.14) and (2.15) that $\hat{\mathbf{q}}_i[k] = \mathbf{p}_1\mathbf{q}_1^T\hat{\mathbf{q}}_i[0]$. Note that for a digraph $\mathbf{p}_1 = \mathbf{1}$; therefore, $\hat{\mathbf{q}}_i[k] = \mathbf{1}\ [q_{11}\ q_{12}\ \cdots\ q_{1n}]\ \hat{\mathbf{q}}_i[0]$. Now it is straightforward to verify that $\hat{\mathbf{q}}_i[k] = [q_{1i}\ q_{1i}\ \cdots\ q_{1i}]^T$. ■

Theorem 2.1. *Let $\boldsymbol{L}$ represent the Laplacian of the digraph of an directed network with n agents, described by (2.9). Using the following control input*

$$u_i[k] = \sum_{j\in N_i^{in}} w_{ij}(x_j[k] - x_i[k]) + \nu_i\delta[k], \tag{2.16}$$

where $\delta[.]$ *is the unit impulse function and*

$$\nu = \begin{bmatrix} \frac{1}{h}\zeta_1 \\ \vdots \\ \frac{1}{h}\zeta_n \end{bmatrix} = \begin{bmatrix} \frac{1}{h}(\frac{1}{n\hat{q}_{1,ss}} - 1)x_1[0] \\ \vdots \\ \frac{1}{h}(\frac{1}{n\hat{q}_{n,ss}} - 1)x_n[0] \end{bmatrix}, \tag{2.17}$$

results in the average consensus in steady state.

Proof. Using the average disagreement term introduced earlier, define the deviation adjustment value ζ_i as follows

$$\zeta_i = x_i[0](\frac{1}{n} - \hat{q}_{i,ss})\frac{1}{\hat{q}_{i,ss}} = \frac{x_i[0]}{n\hat{q}_{i,ss}} - x_i[0]. \tag{2.18}$$

It follows from (2.9), (2.16), and (2.17) that

$$\mathbf{x}[k+1] = (\mathbf{I}_n - h\mathbf{L})^k\mathbf{x}[0] + (\mathbf{I}_n - h\mathbf{L})^k\nu\delta[k]. \tag{2.19}$$

Now, from (2.15)

$$\mathbf{x}[k+1] = \mathbf{p}_1\mathbf{q}_1^T\mathbf{x}[0] + \mathbf{p}_1\mathbf{q}_1^T\nu\delta[k], \tag{2.20}$$

or

$$\mathbf{x}[k+1] = \begin{bmatrix} \frac{\sum_{i=1}^{n} x_i[0]}{n} - \sum_{i=1}^{n} q_{1i} x_i[0] \\ \vdots \\ \frac{\sum_{i=1}^{n} x_i[0]}{n} - \sum_{i=1}^{n} q_{1i} x_i[0] \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n} q_{1i} x_i[0] \\ \vdots \\ \sum_{i=1}^{n} q_{1i} x_i[0] \end{bmatrix} \delta[k]. \tag{2.21}$$

It then follows that

$$\mathbf{x}[k+1] = \mathbf{1}\bar{x}_0. \tag{2.22}$$

■

Remark 2.3. It can easily be seen that using control input of (2.16) is equivalent as

$$\mathbf{x}_{new}[0] = \mathbf{Z}\mathbf{x}[0], \tag{2.23}$$

where matrix $\mathbf{Z}$ is

$$\mathbf{Z} = \begin{bmatrix} \frac{1}{n\hat{q}_{1,ss}} & 0 & \cdot & 0 \\ 0 & \frac{1}{n\hat{q}_{2,ss}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{n\hat{q}_{n,ss}} \end{bmatrix}. \tag{2.24}$$

Using the new vector of states results in static average consensus of an asymmetric network.

Corollary 2.1. *The consensus protocol (2.16) with initial states* $x_i[0] = r_i[-1],\ i \in \mathbb{N}_n$*, achieves dynamic average consensus for any constant reference signal* $\boldsymbol{r}$*.*

Proof. The proof is similar to that of Theorem 2.1, and is omitted here. ■

Corollary 2.2. *The consensus protocol (2.16) with the initial states* $x_i[0] = r_i[-1],\ i \in \mathbb{N}_n$*, and the reference signals satisfying* $\Delta r_i[k] = \Delta r_j[k], \forall i, j \in \mathbb{N}_n$*, for all time instances, achieves dynamic average consensus.*

Proof. An eigenvector of the Laplacian matrix corresponding to the zero eigenvalue is also equal to $\mathbf{1}$; hence, an eigenvector of the matrix $(\mathbf{I}_n - h\mathbf{L})$ corresponding to the eigenvalue 1 is equal to $\mathbf{1}$, i.e. $(\mathbf{I}_n - h\mathbf{L})\mathbf{1} = \mathbf{1}$. It then follows from equations (2.12) and (2.23) that

$$\begin{aligned}\mathbf{x}[k] &= (\mathbf{I}_n - h\mathbf{L})^k \mathbf{Z}\mathbf{x}_0 + (\mathbf{I}_n - h\mathbf{L})^{k-1}\boldsymbol{\Delta}\mathbf{r}[1] + (\mathbf{I}_n - h\mathbf{L})^{k-2}\boldsymbol{\Delta}\mathbf{r}[2] + ... + \boldsymbol{\Delta}\mathbf{r}[k] \\ &= \mathbf{1}\bar{x}_0 + \boldsymbol{\Delta}\mathbf{r}[1] + \boldsymbol{\Delta}\mathbf{r}[2] + ... + \boldsymbol{\Delta}\mathbf{r}[k] = \mathbf{r}[0] + \mathbf{r}[k] - \mathbf{r}[0] = \mathbf{r}[k].\end{aligned} \tag{2.25}$$

■

For the static case, the vector of states at time instance k is

$$\mathbf{x}_{new}[k] = (\mathbf{I}_n - h\mathbf{L})^k \mathbf{x}_0. \tag{2.26}$$

In the case of having reference signals, the vector of states can be shown as

$$\mathbf{x}_{new}[k] =(\mathbf{I}_n - h\mathbf{L})^k\mathbf{Z}\mathbf{x}_0 + (\mathbf{I}_n - h\mathbf{L})^{k-1}\mathbf{\Delta r}[0] + (\mathbf{I}_n - h\mathbf{L})^{k-2}\mathbf{\Delta r}[1] + \cdots \\ + (\mathbf{I}_n - h\mathbf{L})\mathbf{\Delta r}[k-2] + \mathbf{\Delta r}[k-1]. \tag{2.27}$$

Since all the elements in $\mathbf{\Delta r}[j]$ are the same

$$\mathbf{\Delta r}[j] = \mathbf{1}\Delta\bar{r}_j. \tag{2.28}$$

Thus, equation (2.27) yields

$$\mathbf{x}_{new}[k] =(\mathbf{I}_n - h\mathbf{L})^k\mathbf{Z}\mathbf{x}_0 + (\mathbf{I}_n - h\mathbf{L})^{k-1}\mathbf{1}\Delta\bar{r}_0 + (\mathbf{I}_n - h\mathbf{L})^{k-2}\mathbf{1}\Delta\bar{r}_1 + \cdots \\ + (\mathbf{I}_n - h\mathbf{L})\mathbf{1}\Delta\bar{r}_{k-2} + \mathbf{1}\Delta\bar{r}_{k-1}. \tag{2.29}$$

Since $(\mathbf{I}_n - h\mathbf{L})\mathbf{1} = \mathbf{1}$, it follows from equation (2.29) that

$$\mathbf{x}_{new}[k] = (\mathbf{I}_n - h\mathbf{L})^k\mathbf{Z}\mathbf{x}_0 + \mathbf{1}\Delta\bar{r}_0 + \mathbf{1}\Delta\bar{r}_1 + \cdots + \mathbf{1}\Delta\bar{r}_{k-1}. \tag{2.30}$$

From Theorem 2.1, $(\mathbf{I}_n - h\mathbf{L})^k\mathbf{Z}\mathbf{x}_0 = \mathbf{1}\bar{r}_{-1}$, thus

$$\mathbf{x}_{new}[k] = \mathbf{1}\bar{r}_{-1} + \mathbf{1}\Delta\bar{r}_0 + \mathbf{1}\Delta\bar{r}_1 + \cdots + \mathbf{1}\Delta\bar{r}_{k-1} = \mathbf{1}\bar{r}_{k-1}. \tag{2.31}$$

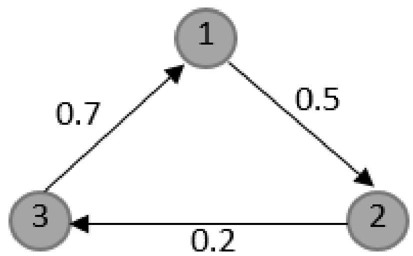

Figure 2.1: Digraph pf the multi-agent network in Examples 1, 2 and 5

2.4 Numerical Examples

In this section, some simulations are presented to demonstrate the efficiency of the proposed algorithm in achieving both static and dynamic average consensus in directed networks.

For the next two examples, a strongly connected network of three agents is considered, with a digraph depicted in Fig. 2.1. All three nodes perform the calibration procedure by computing the steady state value of $\hat{\mathbf{q}}_1$, $\hat{\mathbf{q}}_2$ and $\hat{\mathbf{q}}_3$ in the mirror models (2.13) for the initial states $[1\ 0\ 0]^T$, $[0\ 1\ 0]^T$ and $[0\ 0\ 1]^T$, as noted earlier. Using the resultant steady-state values $\hat{q}_{1,ss}$, $\hat{q}_{2,ss}$ and $\hat{q}_{3,ss}$, each node can find its corresponding deviation adjustment value introduced in (2.18), and subsequently obtain the update rule proposed in Theorem 2.1 to achieve average consensus.

Example 2.1. Let the initial states of node 1, 2 and 3 be, respectively 0, 0.6 and 0.3.

Let also the sampling time and weight matrix be $h = 0.01$ and

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0.7 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}. \tag{2.32}$$

Using the controller proposed in Theorem 2.1 the results given in Fig. 2.2 are obtained. This figure shows that the states of all three nodes converge to 0.3 as time increases, which means that the average consensus is achieved (as the average of the initial states is, in fact, equal to 0.3). It can be easily verified that matrix Z in this example is equal to

$$\mathbf{Z} = \begin{bmatrix} 1.9667 & 0 & 0 \\ 0 & 1.4048 & 0 \\ 0 & 0 & 0.5619 \end{bmatrix}, \tag{2.33}$$

and

$$\mathbf{x}_{new}[0] = \begin{bmatrix} 0 \\ 0.8429 \\ 0.1686 \end{bmatrix}. \tag{2.34}$$

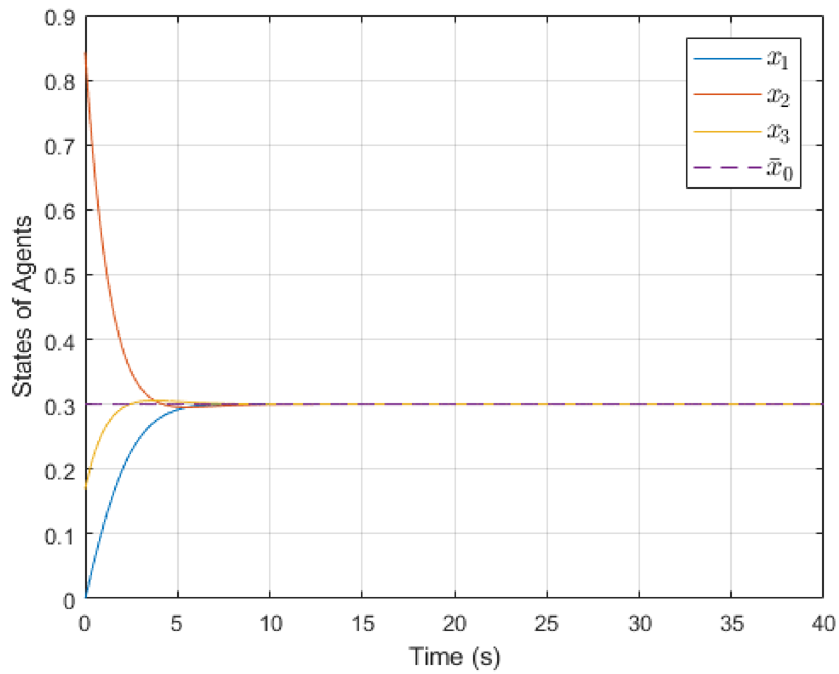

Figure 2.2: Convergence of the states of the multi-agent network of Example 1 to the average of the initial states using the proposed average consensus protocol

Example 2.2. Consider now the same network as the previous example, with a constant reference signal $\mathbf{r} = [3\ 2\ 4]^T$ and the initial state vector $\mathbf{x}[0] = \mathbf{r}$. Using equation (2.12) and under Assumption 3, the network reaches the average of the reference signals as illustrated in Fig. 2.3. For comparison, the trajectory of the dynamic average consensus protocol proposed in [17] is shown in Fig. 2.4. Note that [17] requires the ith node to have the information of both the ith row and ith column. However, this is not realistic in practice because in a distributed control structure, each agent has the information of its incoming edges (row i for agent i) not its outgoing edges (column i for agent i). In addition, the simulations demonstrate that the trajectory obtained by the method proposed in the present work is smoother than that resulted by using [17].

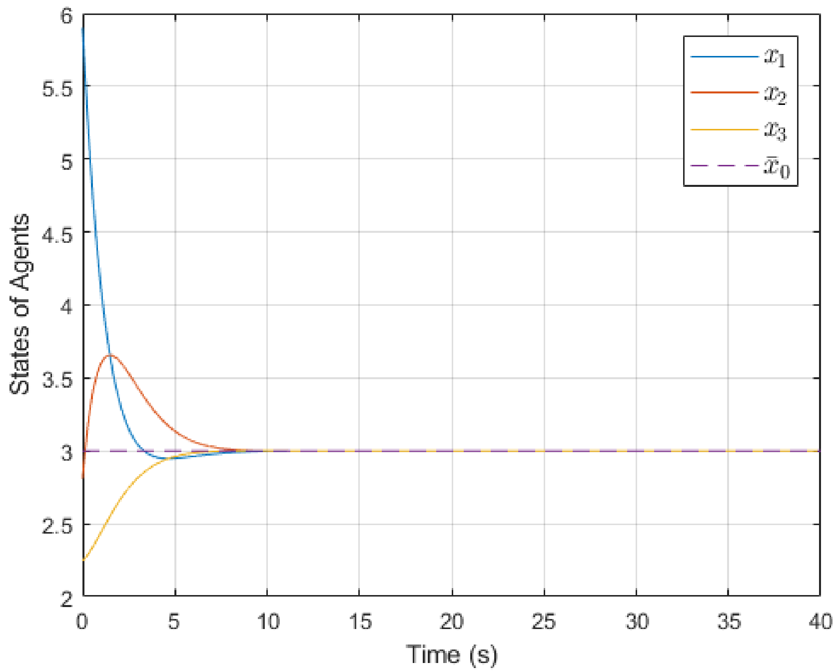

Figure 2.3: Convergence of the states of the multi-agent network of Example 2 to the average of the three constant reference inputs in $\mathbf{r}$ using the proposed average consensus protocol

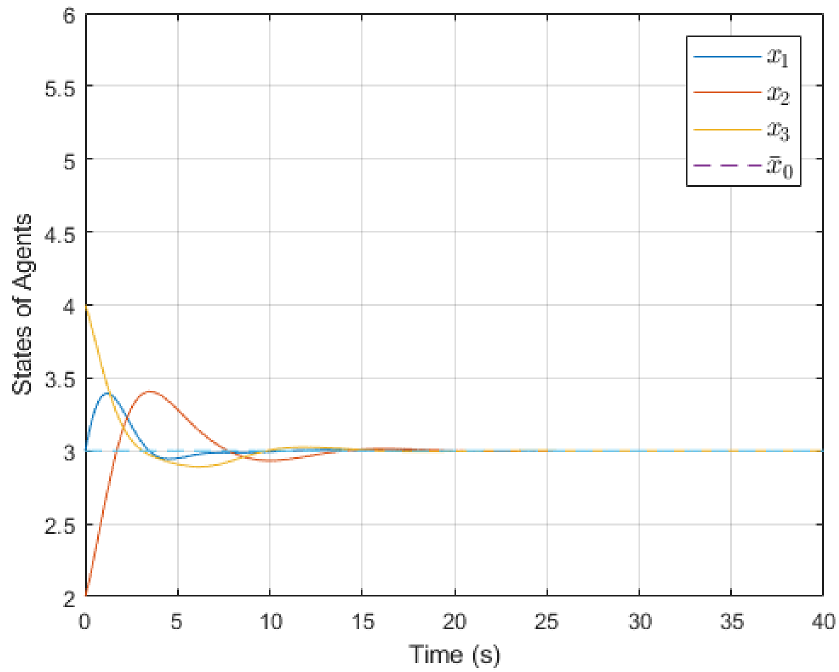

Figure 2.4: Convergence of the states of the multi-agent network in [17] to the average of the three constant reference inputs in $\mathbf{r}$

Example 2.3. Consider a network of four agents with the configuration depicted in Fig. 2.5, the initial values satisfying $\mathbf{x}[0] = \mathbf{x}[-1]$, sampling time $h = 0.01$, and the following reference signals

$$\begin{cases} r_1(t) = t + 1 + 5\sin t, \\ r_2(t) = t - 1 + 5\sin t, \\ r_3(t) = t + 5\sin t, \\ r_4(t) = t + 50 + 5\sin t. \end{cases} \tag{2.35}$$

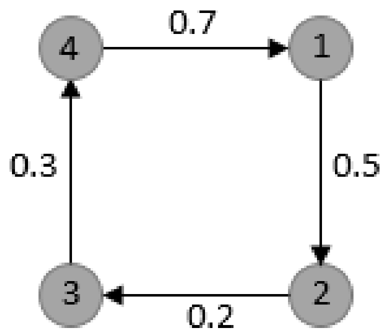

Figure 2.5: Configuration of the digraph of the multi-agent network in Example 3 and 4 communication graph with four nodes

One can easily verify that the above reference signals satisfy the conditions for the specific class of signals defined in [56]. Fig. 2.6 demonstrates the trajectory of agents' states with given reference signals under the proposed average consensus control protocol. This figure shows that in addition to the static case, control protocol of (2.12) is capable of handling the dynamic cases, where derivative of all reference signals in each time instance are the same $\Delta_{\forall i \in V} r_i[k] = \Delta_{\forall j \in V} r_j[k]$.

Example 2.4. Consider the same multi-agent network of Example 3 with the new

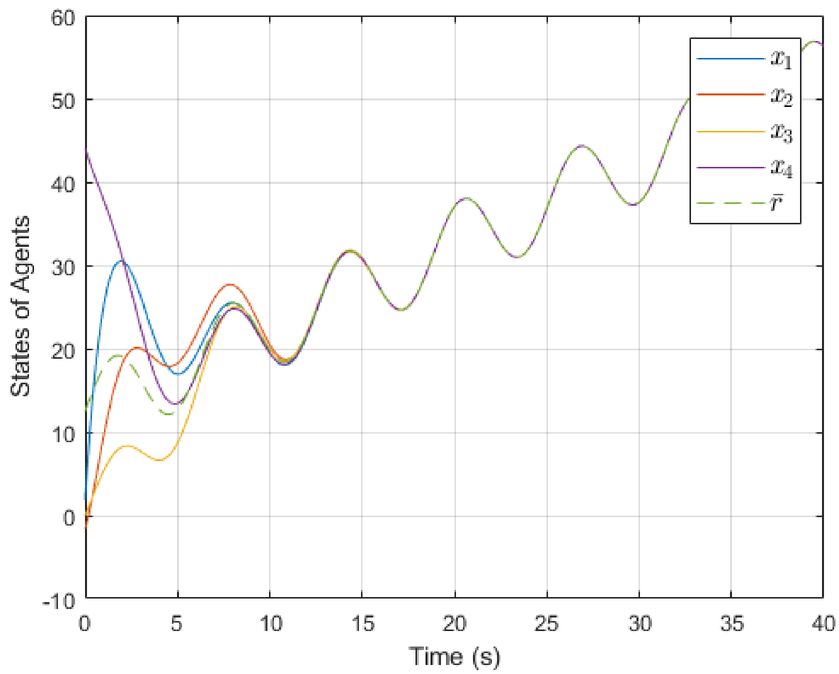

Figure 2.6: The convergence of the states of the multi-agent network of Example 3 to the average of the four time-varying reference inputs in **r** using the proposed average consensus protocol

reference signals given below

$$\begin{cases} r_1(t) = 5\sin t + \frac{10}{t+2} + 1, \\ r_2(t) = 5\sin t + \frac{10}{t+2} + 2, \\ r_3(t) = 5\sin t + \frac{10}{t+2} + 3, \\ r_2(t) = 5\sin t + 10\exp -t + 4. \end{cases} \tag{2.36}$$

Similar to the previous example, it can be easily verified that these reference signals also belong to the class of signals introduced in [56]. Fig. 2.7 shows the trajectory of agents' states with the above reference signals under the proposed average consensus protocol. It can be observed from this figure that the proposed control algorithm can reach dynamic average consensus when the derivative of all reference signals are the same in each time instance.

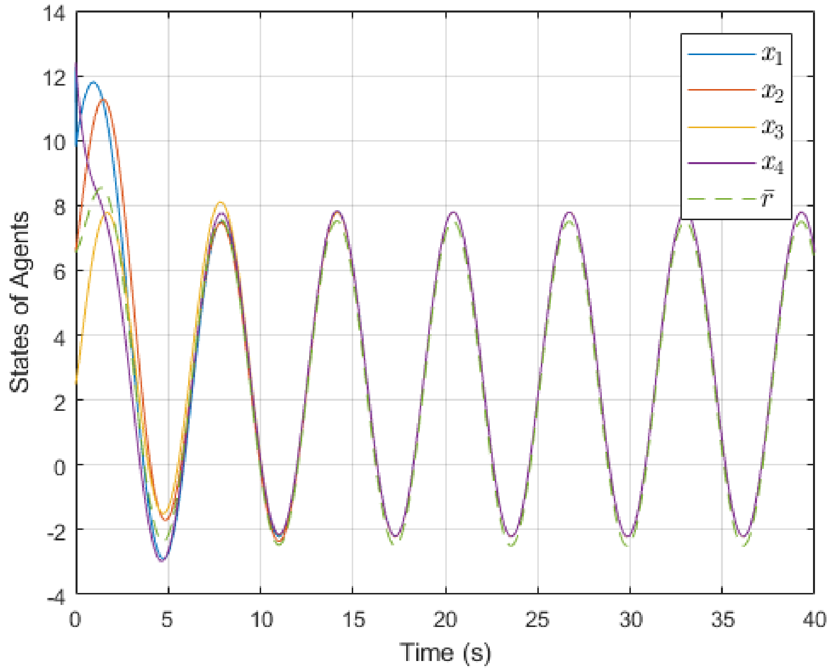

Figure 2.7: The convergence of the states of the multi-agent network of Example 4 to the average of the four time-varying reference inputs in $\mathbf{r}$ using the proposed average consensus protocol

Example 2.5. In this example, perturbation in the graph weight matrix and its effect on the performance of the proposed average consensus algorithm is investigated numerically. Consider the multi-agent system of Example 1, and let the weight matrix used in the calibration phase be given by

$$\mathbf{W} = \begin{bmatrix} 0 & 0 & 0.3 \\ 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \end{bmatrix}, \tag{2.37}$$

which is different from the actual weight matrix given by (2.32). Thus, matrix $\mathbf{Z}$ defined in (2.24) is computed based on an inexact weight matrix. The simulation results are depicted in Fig. 2.8, and show that while convergence is achieved, there is a discrepancy between the steady-state values and the desired final value.

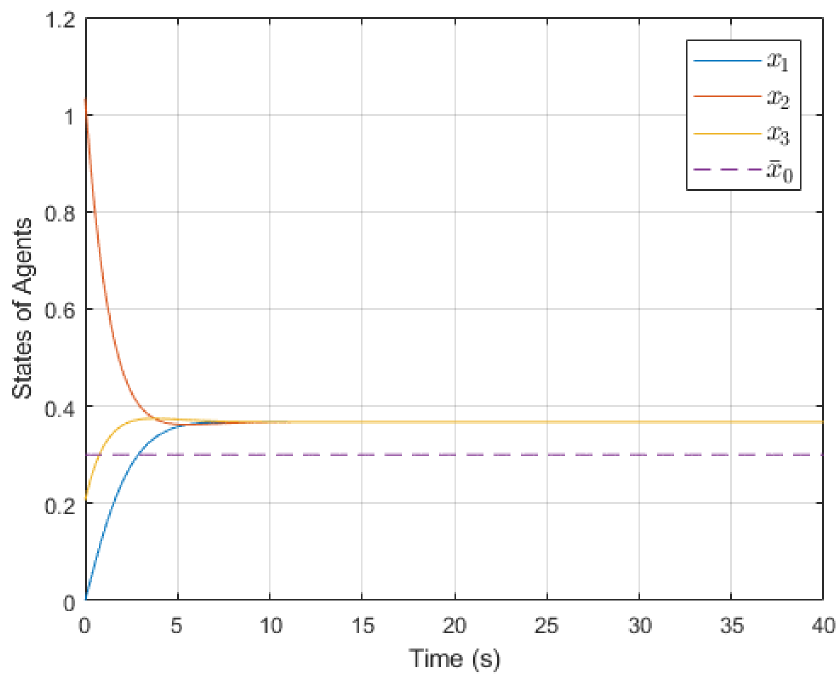

Figure 2.8: The steady-state error resulted from the weight matrix perturbation

It is to be noted that matrix $\mathbf{Z}$ is dependent on the left eigenvectors of the matrix $(\mathbf{I}_n - h\mathbf{L})$, Thus, due to the discrepancy in the actual weight matrix and the one used in the computations of the calibration phase, the agents' states in this example converge to $\sum_{i=1}^{n} \frac{q_i}{n\hat{q}_i} x_i[0]$ the average of the initial states. The sensitivity of the left eigenvector of a matrix to the perturbation is studied in [20].

2.5 Conclusions

In this chapter, the problem of consensus as well as some special cases of dynamic average consensus in asymmetric networks were investigated. The proposed scheme is distributed, and consists of a calibration phase followed by a novel average consensus rule. In the calibration phase, every agent uses its local information as well as the information it receives from its neighbors to compute an error term which is subsequently used in the update rule to compensate for the difference between the final state, obtained by using a standard consensus protocol, and the average of the initial states of all agents. The convergence analysis is carried out using the celebrated Perron-Frobenius theorem. The method is then extended to the case where each agent is subject to a class of reference signals under some mild conditions. The efficiency of the results is confirmed by simulations.

Chapter 3

A Dynamic Average Consensus Algorithm for Directed Multi-Agent Networks

In this chapter, a novel algorithm is presented to achieve dynamic average consensus in a directed network of homogeneous agents. Each agent receives an input signal which is not available to other agents. The objective is for every agent's state to track the average of all input signals at all times. The agents communicate information according to a directed communication graph. To achieve the objective, an algorithm is proposed where every agent initially shares information about the difference between the current value of its reference signal and the previous value. The procedure is repeated for certain number of times, and then the obtained value is used by the agents to update their state. Moreover, it is shown that using an error-compensating scheme similar to the one presented in Chapter 2, the steady-state error of the dynamic average consensus problem can be driven to a sufficiently small neighborhood of zero. The convergence analysis is provided using the Markov chain properties. Simulations are presented to verify the theoretical results.

3.1 Introduction

One of the important objectives in the control of the multi-agent systems with a reference signal for each agent is that the state of every agent tracks the average of all reference signals [27,44,56]. This problem is referred to as the *average dynamic consensus* problem, and has important applications, e.g., in the surveillance problem and target tracking in mobile sensor networks [10, 24].

Several techniques are proposed in the literature to solve the dynamic average consensus problem in multi-agent systems. In [12], the authors use a signum function to tackle the problem. A decomposed Kalman filter approach is used in [36] to achieve dynamic average consensus. In [7, 46, 47], the time required for reaching a sufficiently small neighborhood the steady-state is investigated in the context of Markov chain models using the so-called "mixing time" property. The authors in [47] use this property for the convergence analysis in undirected multi-agent systems.

In this chapter, a novel approach is proposed for dynamic average consensus control in multi-agent systems. Using the concept of Markov chain, an upper bound is derived for the disagreement vector in the dynamic average consensus control of a directed multi-agent network with unbalanced topology. The convergence analysis is provide to show that the state of every agent settles within a desired neighborhood of the average of the reference signals after a sufficiently long time. Numerical examples support the theoretical findings.

The remainder of the chapter is organized as follows. In Section 3.2, some preliminaries and background information concerning networked systems are provided. Then in Section 3.3, the main contribution of this chapter is presented and an upper bound is derived for the magnitude of the disagreement vector in the dynamic average consensus problem for a directed multi-agent network with unbalanced topology. In Section 3.4, simulation results are presented, and finally, concluding remarks are made in Section 3.5.

3.2 Preliminaries and Notation

In this section, some preliminaries on the dynamic average consensus problem and Markov chain are reviewed. Consider an asymmetric network of n agents (nodes), whose information exchange topology is represented by the weighted digraph $\mathcal{G} = (V, E, \mathbf{W})$, where $V = \mathbb{N}_n := \{1, ..., n\}$ is the set of vertices (nodes), $E \subseteq V \times V$ is the set of edges, and $\mathbf{W} \in \mathbb{R}^{n \times n}$ is the weight matrix associated with the edge set E. The in-neighbor and out-neighbor sets of node i for all $i \in \mathbb{N}_n$ in a digraph are defined as

$$N_i^{in} = \left\{ j \in V \setminus \{i\} \,|(j,i) \in E \right\}, \tag{3.1}$$

and

$$N_i^{out} = \left\{ j \in V \setminus \{i\} \,|(i,j) \in E \right\}. \tag{3.2}$$

As noted in the previous chapter, the Laplacian matrix for an undirected network is

$$\mathbf{L} = \Delta - \mathbf{W}, \tag{3.3}$$

or

$$l_{ij} = \begin{cases} -w_{ij}, & \text{if } (j,i) \in E, \\ \sum_{k \neq i} w_{ik}, & \text{if } j = i, \\ 0, & \text{otherwise}, \end{cases} \tag{3.4}$$

where w_{ij} is the (i,j) element of the weight matrix $\mathbf{W}$, which is, in fact, the weight of the edge from node j to i. The matrix given above is, in fact, the *in-degree* Laplacian matrix.

From equations (3.3) and (3.4), it is concluded that the row sum of the Laplacian matrix is equal to zero for a directed network (this is a well-known property of undirected networks as well). This means that zero is an eigenavlue of the Laplacian matrix and

$\mathbf{p} = \mathbf{1}$ is its corresponding right eigenvector, i.e.

$$\mathbf{L}\mathbf{1} = 0. \tag{3.5}$$

Unlike undirected networks, however, in a directed network the sum of the elements of every columns is not zero unless the digraph is weight-balanced. In other words, $\mathbf{q} = \mathbf{1}^T$ is a left eigenvector of the Laplacian matrix associated with the zero eigenvalue in a directed network only if the network is balanced.

3.2.1 Dynamic Average Consensus

In the dynamic average consensus problem, a reference input is applied to every agent, and the objective is that all agents track the average of the reference signals. Similar to the static average consensus problem, existing protocols for the dynamic average consensus of directed networks are mainly for those with a balanced graph. For the case of unbalanced networks, the standard consensus protocol leads to a steady-state tracking error. It is shown later in this work how the steady-state error can be reduced by using the properties of the Markov chain along with an error-compensating scheme similar to the one proposed in the previous chapter.

3.2.2 Disagreement Vector in Dynamic Average Consensus

To evaluate the performance of a consensus control protocol in the case of no reference signal, the concept of *disagreement vector* is introduced in [39] for balanced networks as follows

$$\mathbf{x}[k] = \alpha\mathbf{1} + \boldsymbol{\phi}[k], \tag{3.6}$$

where α represents the average state of the nodes at time k, i.e. $\alpha = \frac{\mathbf{x}_1[k]+\ldots+\mathbf{x}_n[k]}{n}$. The disagreement vector is used to determine how far the agents' states are from their steady-state value and what the rate of convergence is.

For the case of dynamic average consensus, the disagreement vector can be formulated as [17]

$$\boldsymbol{\phi}[k] = \mathbf{x}_{new}[k] - \frac{1}{n}\mathbf{1}\mathbf{1}^T\mathbf{r}[k], \tag{3.7}$$

where $\mathbf{r}[k]$ is the vector of all reference signals. A dynamic average consensus protocol is proposed later, which drives the above vector to an arbitrarily small neighborhood of zero. To this end, some important properties of the Markov chain are reviewed next.

3.2.3 Markov Chain

A Markov chain is a stochastic model for a sequence of events whose probability depends on the state attained in the previous event [29,34]. This can be described by the following dynamic equation

$$\boldsymbol{\mu}[k] = \boldsymbol{\mu}[k-1]\mathbf{P}, \tag{3.8}$$

where μ is a row vector of the probability distributions and $\mathbf{P}$ is the transition matrix. According to [43], for a Markov chain described by (3.8), given an initial distribution $\boldsymbol{\mu}[0]$, if the second largest eigenvalue of $\mathbf{P}$, denoted by λ_*, is strictly less than 1, then there exist a stationary distribution $\boldsymbol{\pi}$ for which $\lim_{k\to\infty} \| \boldsymbol{\mu}[k] - \boldsymbol{\pi} \| \to 0$. Moreover, for any given ϵ, one can determine how large k should be such that $\| \boldsymbol{\mu}[k] - \boldsymbol{\pi} \|_\infty \leq \epsilon$. In fact, there is a unique distribution $\boldsymbol{\pi}$ on $j \in \{0, 1, ..., n-1\}$ such that

$$| \mu_j[k] - \pi_j | \leq C_j k^{J-1} (\lambda_*)^{k-J+1}, \tag{3.9}$$

where C_j is a strictly positive constant and J is the size of the largest Jordan block of $\mathbf{P}$. This yields

$$\| \boldsymbol{\mu}[k] - \boldsymbol{\pi} \| \leq \mathbf{C} k^{J-1} \lambda_*^{k-J+1}. \tag{3.10}$$

According to [43], if matrix $\mathbf{P}$ is diagonalizable then $J = 1$, and equation (3.9) can be rewritten as

$$| \mu_j[k] - \pi_j | \leq (\sum_{m=1}^{n-1} | a^m w_j^m |) \lambda_*^k, \tag{3.11}$$

where $\mathbf{w}^0, ..., \mathbf{w}^{n-1}$ are the bases of left eigenvectors corresponding to $\lambda_0, ..., \lambda_{n-1}$, respectively, and where a^0, ..., a^{n-1} are the unique coefficients satisfying

$$\boldsymbol{\mu}[0] = a^0\mathbf{w}^0 + a^1\mathbf{w}^1 + ... + a^{n-1}\mathbf{w}^{n-1}. \tag{3.12}$$

3.3 Control Protocol: Dynamic Average Consensus

Consider an asymmetric network of n stationary nodes, whose topology is represented by a weighted digraph $\mathcal{G} = (V, E, \mathbf{W})$, where $V = \mathbb{N}_n$ is the set of vertices, $E \subseteq V \times V$ is the set of edges, and $\mathbf{W} \in \mathbb{R}^{n\times n}$ is a matrix whose entries represent the weight of the corresponding elements in the edge set E. It is desired now to develop a distributed algorithm to achieve dynamic average consensus for any type of reference signal under some mild conditions. To this end, a protocol is introduced which consists of an inner loop and an outer one. The inner loop aims at compensating for the error due to the reference signals under the unbalanced Laplacian matrix, and the outer loop compensates for the error due to the initial states under the unbalanced Laplacian matrix. The outer loop is described by

$$\begin{aligned} x_i[k+1] &= x_i[k] + hu_i[k] + \Delta\rho_i^{[i_{max}]}[k], \\ u_i[k] &= \sum_{j\in N_i^{in}} w_{ij}(x_j[k] - x_i[k]) + \nu_i x_i[0]\delta[k], \end{aligned} \tag{3.13}$$

while the inner loop is given by

$$\boldsymbol{\Delta\rho}^{[k'+1]}[k] = (\mathbf{I}_n - h'\mathbf{L})\boldsymbol{\Delta\rho}^{[k']}[k], \tag{3.14a}$$

$$\boldsymbol{\Delta\rho}^{[0]}[k] = \mathbf{Z}(\mathbf{r}[k] - \mathbf{r}[k-1]). \tag{3.14b}$$

In the above equation, $h' \leq h$ is the sampling time of the inner loop. Moreover, k is the outer loop iteration index, k' is the inner loop iteration index, and i_{max} denotes the maximum number of the inner loop iterations. The vector $\boldsymbol{\Delta\rho}^{[0]}[k]$ denotes the amount of increase in the reference signal from time $k-1$ to time k. Note that equation (3.14b) describes the evolution of $\boldsymbol{\Delta\rho}[k]$ in the inner loop.

Fig. 3.1 demonstrates the integration of the two loops in the proposed dynamic average consensus protocol. The structure given in this figure consists of multiple inner loops and one outer loop. Utilizing inner loops with a sufficiently small sampling time and suppressing the error term at each time step by applying the inner loop, the steady-state error of the original system will be significantly reduced.

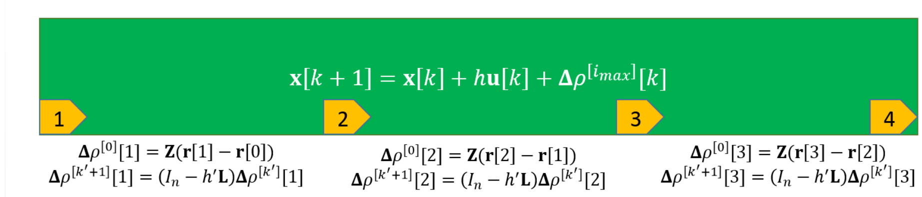

Figure 3.1: Schematic of the proposed two stage dynamic average consensus protocol

The disagreement vector in equation (3.7) can be written as

$$\boldsymbol{\phi}[k] = \mathbf{x}_{new}[k] - \frac{1}{n}\mathbf{1}\mathbf{1}^T\mathbf{x}[0] - \frac{1}{n}\sum_{i=0}^{k}\mathbf{1}\mathbf{1}^T\boldsymbol{\Delta}\mathbf{r}[k]. \tag{3.15}$$

Expanding the term x_{new} from equation (2.23) yields

$$\begin{aligned}\boldsymbol{\phi}[k] =& (\mathbf{I}_n - h\mathbf{L})^k\mathbf{Z}\mathbf{x}[0] + \sum_{i=0}^{k}(\mathbf{I}_n - h\mathbf{L})^{k-i}(\mathbf{I}_n - h'\mathbf{L})^{i_{max}}\mathbf{Z}\boldsymbol{\Delta}\mathbf{r}[k] \\ & - \frac{1}{n}\mathbf{1}\mathbf{1}^T\mathbf{Z}\boldsymbol{\Delta}\mathbf{r}[k] - \frac{1}{n}\sum_{i=0}^{k}\mathbf{1}\mathbf{1}^T\boldsymbol{\Delta}\mathbf{r}[k].\end{aligned} \tag{3.16}$$

From Chapter 2, the above equation can be simplified as

$$\boldsymbol{\phi}[k] = \sum_{i=0}^{k}(\mathbf{I}_n - h\mathbf{L})^{k-i}(\mathbf{I}_n - h'\mathbf{L})^{i_{max}}\mathbf{Z}\boldsymbol{\Delta}\mathbf{r}[k] - \frac{1}{n}\sum_{i=0}^{k}\mathbf{1}\mathbf{1}^T\boldsymbol{\Delta}\mathbf{r}[k]. \tag{3.17}$$

Similarly, if i_{max} is sufficiently large, each term of the two summation in the right side of the above equation approaches a small neighborhood of zero. However, there might be some limitation in terms of how large i_{max} could be chosen (depending on the rate of convergence of the inner loop). Regardless of the size of i_{max}, since the number of the iterations is finite in practice and the size of error decreases after each iteration, one can find an upper bound on the magnitude of the disagreement vector for each i_{max}.

It is desired to find an upper bound on the magnitude of $\boldsymbol{\phi}[k]$, and the number of summation terms in (3.17) in order to have sufficiently small error. To this end, it is noted that the entry P_{ij} of matrix $\mathbf{P}$ in equation (3.8) is the probability that state of node i becomes equal to that of node j in the next iteration. In contrast, in our problem, element T_{ij} of matrix $\mathbf{T} = (\mathbf{I}_n - h\mathbf{L})$, represents the probability that node j sends information to node i. In other words, although both $\mathbf{P}$ and $\mathbf{T}$ are row stochastic, in the former the sum of the outgoing links from each node is equal to one, while in the latter on the sum of the incoming links to each node is equal to one. For the case of an undirected graph, no transformation would be required. Taking the transpose of both sides of (3.8), one obtains

$$\boldsymbol{\mu}^T[k] = \mathbf{P}^T \boldsymbol{\mu}^T[k-1]. \tag{3.18}$$

On the other hand, the transpose of matrix $(\mathbf{I}_n - h\mathbf{L})^T$ in equation (3.18) is column-stochastic. According to [50], there exists a similarity transformation to convert the above matrix to it transposed form, i.e.

$$\mathbf{M}(\mathbf{I}_n - h\mathbf{L})\mathbf{M}^{-1} = (\mathbf{I}_n - h\mathbf{L})^T, \tag{3.19}$$

where $\mathbf{M}$ is an invertible matrix. The dynamics of the initial state can be expressed by

$$\mathbf{v}[k] = (\mathbf{I}_n - h\mathbf{L})\mathbf{v}[k-1], \quad \text{where } \mathbf{v}[0] = \mathbf{Z}\mathbf{x}[0]. \tag{3.20}$$

One can then rewrite the above equation as

$$\mathbf{y}_v[k] = \mathbf{y}_v[k-1](\mathbf{I}_n - h\mathbf{L}), \tag{3.21}$$

where $\mathbf{y}_v[k] := (\mathbf{M}\mathbf{v}[k])^T$. Now using the inequality (3.10), it can be concluded that

$$\| \mathbf{y}_v[k] - \boldsymbol{\pi}_{y_v} \| \leq \mathbf{C}_{y_v} k^{J-1} \lambda_*^{k-J+1}, \tag{3.22}$$

where $\mathbf{C}_{y_v}$ is a matrix of strictly positive entries and $\boldsymbol{\pi}_{y_v}$ is a stationary distribution as described in (3.10). Since $\mathbf{y}_v[k] = \mathbf{M}\mathbf{v}[k]$, thus

$$\begin{aligned} &\| \mathbf{M}\mathbf{v}[k] - \boldsymbol{\pi}_{y_v} \| \leq \mathbf{C}_{y_v} k^{J-1} \lambda_*^{k-J+1} \\ \rightarrow &\| \mathbf{v}[k] - \mathbf{M}^{-1}\boldsymbol{\pi}_{y_v} \| \leq \frac{\mathbf{C}_{y_v}}{\| \mathbf{M} \|} k^{J-1} \lambda_*^{k-J+1}, \end{aligned} \tag{3.23}$$

which results in

$$\| \mathbf{v}[k] - \boldsymbol{\pi}_v \| \leq \mathbf{C}_v k^{J-1} \lambda_*^{k-J+1}, \tag{3.24}$$

where $\mathbf{C}_v := \frac{\mathbf{C}_{y_v}}{\|\mathbf{M}\|}$. A procedure similar to (3.21)-(3.24) can now be used to obtain an upper bound on $\boldsymbol{\Delta\rho}$ in (3.14) after each iteration of the outer loop. Therefore, for the k-th outer loop

$$\| \mathbf{y}_{\Delta\rho^{[imax]}[k]} - \boldsymbol{\pi}_{y_{\boldsymbol{\Delta\rho}^{[imax]}[k]}} \| \leq \mathbf{C}_{y_{\Delta\rho^{[imax]}[k]}} k^{J-1} (\lambda_*)^{k+i_{max}}, \tag{3.25}$$

or

$$\| \boldsymbol{\Delta\rho}^{[i_{max}]}[k] - \boldsymbol{\pi}_{\Delta\rho^{[imax]}[k]} \| \leq \mathbf{C}_{\Delta\rho^{[imax]}[k]} k^{J-1} (\lambda_*)^{k+i_{max}}. \tag{3.26}$$

Note that to derive the upper bound in (3.24), it is required to find matrix $\mathbf{M}$ in the similarity transformation introduced in (3.19). Let $\mathbf{D}$ be a diagonal matrix with the eigenvalues of $\mathbf{T}$ on its main diagonal (not for simplicity and loss of generality, it was assumed that the Laplacian matrix does not have repeated eigenvalues, and hence the eigenvalues of matrix $\mathbf{T}$ are also distinct). It is straightforward to show that there are invertible matrices $\mathbf{S}$ and $\mathbf{R}$ such that

$$\begin{aligned} \mathbf{D} &= \mathbf{S}^{-1}\mathbf{T}\mathbf{S}, \\ \mathbf{D} &= \mathbf{R}^{-1}\mathbf{T}^{T}\mathbf{R}. \end{aligned} \tag{3.27}$$

Thus

$$\mathbf{T}\mathbf{S}\mathbf{R}^{-1} = \mathbf{S}\mathbf{R}^{-1}\mathbf{T}^{T} \rightarrow (\mathbf{S}\mathbf{R}^{-1})^{-1}\mathbf{T}(S\mathbf{R}^{-1}) = \mathbf{T}^{T}, \tag{3.28}$$

which yields

$$\mathbf{M} = \mathbf{R}\mathbf{S}^{-1}. \tag{3.29}$$

Note that matrices $\mathbf{S}$ and $\mathbf{R}$ can be obtained from right eigenvectors of $\mathbf{T}$ and $\mathbf{T}^{T}$, respectively. Moreover, from equation (3.27) it can be deduced that $\mathbf{R}^{-1} = \mathbf{S}^{T}$. As a result, equation (3.29) can be rewritten as

$$\mathbf{M} = (\mathbf{S}^{-1})^{T}\mathbf{S}^{-1}. \tag{3.30}$$

This implies that

$$\mathbf{T}^{T} = (\mathbf{S}\mathbf{S}^{T})^{-1}\mathbf{T}\mathbf{S}\mathbf{S}^{T}. \tag{3.31}$$

To derive the upper bound in (3.24) and (3.26), it is also required to find matrices $\mathbf{C}_{y_v}$

and $\mathbf{C}_{y_{\boldsymbol{\Delta\rho}}}$. Since matrix $\mathbf{T}$ is assumed to be diagonalizable, according to [43], equation (3.9) can be written as

$$| \mu_j[k] - \pi_j | \leq (\sum_{j=1}^{n-1} | a^m w_j^m |)\lambda_*^k, \tag{3.32}$$

where $\mathbf{w}^0, ..., \mathbf{w}^{n-1}$ are the bases of left eigenvectors corresponding to $\lambda_0, ..., \lambda_{n-1}$, respectively, and where $a^0, ..., a^{n-1}$ are the unique coefficients of the following linear combination

$$\boldsymbol{\mu}[0] = a^0\mathbf{w}^0 + a^1\mathbf{w}^1 + ... + a^{n-1}\mathbf{w}^{n-1}. \tag{3.33}$$

In order to find the said upper bound on the disagreement vector in equation (3.26), it is required to find a^i, w^i in the above equation for $i = 0, 1, ..., n-1$ in the k-th iteration. Replace $\boldsymbol{\mu}_0$ in (3.33) by $\mathbf{y}_{\boldsymbol{\Delta\rho}^{[0]}[k]} = \mathbf{M}\boldsymbol{\Delta\rho}^{[0]}[k]$ to obtain

$$\mathbf{M}\boldsymbol{\Delta\rho}^{[0]}[k] = a^0\mathbf{w}^0 + a^1\mathbf{w}^1 + ... + a^{n-1}\mathbf{w}^{n-1}. \tag{3.34}$$

Therefore, $a^0, a^1, ..., a^{n-1}$ can be found as

$$\mathbf{a}_{y_{\Delta\rho[k]}} = \mathbf{W}^{-1}\mathbf{M}\boldsymbol{\Delta\rho}^{[0]}[k], \tag{3.35}$$

where $\mathbf{W}$ is a matrix whose columns are the left eigenvectors of matrix $\mathbf{T}$. Define $\mathbf{W}^r =$

$[\mathbf{w}^1\ \mathbf{w}^2\ ...\ \mathbf{w}^{n-1}]$ and $\mathbf{a}^r_{y_{\Delta\rho[k]}} = [a^1\ a^2\ ...\ a^{n-1}]^T$. It then follows from (3.32) that

$$| \mathbf{y}_{\Delta\rho^{[i_{max}]}[k]} - \boldsymbol{\pi}_{y_{\boldsymbol{\Delta\rho}^{[i_{max}]}[k]}} \leq | \mathbf{W}^r \mathbf{a}^r_{y_{\Delta\rho[k]}} | \lambda_*^{k+i_{max}}. \tag{3.36}$$

The above inequality provides an upper bound on the disagreement error after each iteration of the outer loop. Similarly, one can find an upper bound on the initial state disagreement expressed in equation (3.24) as

$$| \mathbf{y}_v[k] - \boldsymbol{\pi}_{y_v[k]} \leq | \mathbf{W}^r \mathbf{a}^r_{y_v} | \lambda_*^k. \tag{3.37}$$

To find the overall bound on the disagreement, one needs to take the summation of the error bounds of all inner loops. To this end, let the state of the system at step k be written as

$$\begin{aligned} \mathbf{x}[k] =& (\mathbf{I}_n - h\mathbf{L})^k \mathbf{Z}\mathbf{x}_0 \\ & + (\mathbf{I}_n - h\mathbf{L})(\mathbf{I}_n - h'\mathbf{L})^{i_{max}} \mathbf{Z}\boldsymbol{\Delta}\mathbf{r}[0] \\ & + (\mathbf{I}_n - h\mathbf{L})^{k-1}(\mathbf{I}_n - h'\mathbf{L})^{i_{max}} \mathbf{Z}\boldsymbol{\Delta}\mathbf{r}[1] \\ & + \vdots \\ & + (\mathbf{I}_n - h\mathbf{L})(\mathbf{I}_n - h'\mathbf{L})^{i_{max}} \mathbf{Z}\boldsymbol{\Delta}\mathbf{r}[k-1] \\ & + (\mathbf{I}_n - h'\mathbf{L})^{i_{max}} \mathbf{Z}\boldsymbol{\Delta}\mathbf{r}[k], \end{aligned} \tag{3.38}$$

or equivalently

$$\mathbf{x}[k] = (\mathbf{I}_n - h\mathbf{L})^k Z x_0 + \sum_{i=0}^{k} (\mathbf{I}_n - h\mathbf{L})^{k-i} (\mathbf{I}_n - h'\mathbf{L})^{i_{max}} \mathbf{Z}\boldsymbol{\Delta}\mathbf{r}[k]. \tag{3.39}$$

For simplicity, let $h' = h$; then

$$\mathbf{x}[k] = (\mathbf{I}_n - h\mathbf{L})^k \mathbf{Z}\mathbf{x}_0 + \sum_{i=0}^{k} (\mathbf{I}_n - h\mathbf{L})^{k+i_{max}-i} \mathbf{Z}\boldsymbol{\Delta}\mathbf{r}[k]. \tag{3.40}$$

Multiplying the transformation matrix $\mathbf{M}$ to both sides of the above equation and using (3.20) yields

$$\mathbf{M}\mathbf{x}[k] = \mathbf{M}\mathbf{v}[k] + \sum_{i=0}^{k} \mathbf{M}\mathbf{Z}\boldsymbol{\Delta}\boldsymbol{\rho}^{[i_{max}]}[k-i], \tag{3.41}$$

or

$$\mathbf{M}\mathbf{x}[k] = \mathbf{y}_v[k] + \sum_{i=0}^{k} \mathbf{y}_{\Delta\rho^{[i_{max}]}[k-i]}. \tag{3.42}$$

From equations (3.36) and (3.42)

$$\begin{aligned} &\| \mathbf{y}_v[k] - \boldsymbol{\pi}_{y_v} \| + \sum_{i=0}^{k} \| \mathbf{y}_{\Delta\rho^{[i_{max}]}[k-i]} - \boldsymbol{\pi}_{y_{\boldsymbol{\Delta\rho}^{[i_{max}]}[k-i]}} \| \\ &\leq \| \mathbf{W}^r \mathbf{a}^r_{\mathbf{y}_v} \| (\lambda_*)^k + \sum_{i=0}^{k} (\| \mathbf{W}^r \mathbf{a}^r_{y_{\Delta\rho[k-i]}} \| (\lambda_*)^{k+i_{max}-i}), \end{aligned} \tag{3.43}$$

or equivalently

$$\begin{aligned}
&\| \mathbf{M}(\mathbf{I}_n - h\mathbf{L})^k \mathbf{Z} x[0] - \boldsymbol{\pi}_{y_v} \| \\
&+ \sum_{i=0}^{k} \| \mathbf{M}(\mathbf{I}_n - h\mathbf{L})^{k+i_{max}-i} \mathbf{Z} \boldsymbol{\Delta\rho}^{[i_{max}]}[k-i] - \boldsymbol{\pi}_{y_{\Delta\rho^{[i_{max}]}[k-i]}} \|, \\
&\leq \| \mathbf{W}^r \mathbf{a}^r_{y_v} \| (\lambda_*)^k + \sum_{i=0}^{k} (\| \mathbf{W}^r \mathbf{a}^r_{y_{\Delta\rho[k-i]}} \| (\lambda_*)^{k+i_{max}-i}).
\end{aligned} \tag{3.44}$$

Thus

$$\begin{aligned}
&\| \mathbf{v}[k] - \mathbf{M}^{-1} \boldsymbol{\pi}_{y_v} \| + \sum_{i=0}^{k} \| \boldsymbol{\Delta\rho}^{[i_{max}]}[k-i] - \mathbf{M}^{-1} \boldsymbol{\pi}_{y_{\Delta\rho^{[i_{max}]}[k-i]}} \| \\
&\leq \frac{\| \mathbf{W}^r \mathbf{a}^r_{y_v} \|}{\| \mathbf{M} \|} (\lambda_*)^k + \sum_{i=0}^{k} (\frac{\| \mathbf{W}^r \mathbf{a}^r_{y_{\Delta\rho[k-i]}} \|}{\| \mathbf{M} \|} (\lambda_*)^{k+i_{max}-i}).
\end{aligned} \tag{3.45}$$

The overall disagreement for any k can then be obtained as

$$\begin{aligned}
&\| \mathbf{x}[k] - \boldsymbol{\pi}_x \| \\
&\leq \| \mathbf{v}[k] - \boldsymbol{\pi}_v \| + \sum_{i=0}^{k} \| \boldsymbol{\Delta\rho}^{[i_{max}]}[k-i] - \boldsymbol{\pi}_{\boldsymbol{\Delta\rho}^{[i_{max}]}[k-i]} \| \\
&\leq \frac{\| \mathbf{W}^r \mathbf{a}^r_{y_v} \|}{\| \mathbf{M} \|} (\lambda_*)^k + \sum_{i=0}^{k} (\frac{\| \mathbf{W}^r \mathbf{a}^r_{y_{\Delta\rho[k-i]}} \|}{\| \mathbf{M} \|} (\lambda_*)^{k+i_{max}-i}).
\end{aligned} \tag{3.46}$$

Remark 3.1. It is to be noted that for the concept of Markov chain, some fundamental assumptions are required. These assumptions are given below.

i) $0 \leq (\mathbf{MZx}[0])_j = y_{v_j}[0] \leq 1$;

ii) $\sum_{j=0}^{n-1} y_{v_j}[0] = 1$;

iii) $0 \leq (\mathbf{MZ}\boldsymbol{\Delta\rho}[k])_j = (\mathbf{y}_{\Delta\rho^{[i_{max}]}[k-i]})_j \leq 1$, and

iv) $\sum_{j=0}^{n-1}(\mathbf{y}_{\Delta\rho^{[i_{max}]}[k-i]})_j = 1$.

Remark 3.2. The equations provided in this section can be properly scaled to make them effective for any initial state or reference signal.

3.4 Numerical Examples

In this section, some numerical simulations are provided to demonstrate the effectiveness of the proposed two-stage algorithm to achieve dynamic average consensus. Moreover, the verification of the proposed upper bound on the magnitude of the disagreement function is performed by simulations.

In Examples 3.1 and 3.2, it is assumed that conditions (i)-(iv) in Remark 3.1 are satisfied. However Example 3.3 and 3.4 consider the general case. The case of dynamic average consensus is first investigated in Example 3.1 to 3.3 in order to compare the disagreement value with the upper bound provided in equation (3.46). The effect of sampling time on the performance of the proposed dynamic average consensus strategy is then investigated in Example 3.4. It is to be noted that the state and reference signal of each agent in these examples are assumed to be scalars. Thus, Italic letters are used to represented them (instead of boldface Roman letters).

Example 3.1. Consider a network of four agents, each one subject to a first-order reference signal (as a function of time). Let the number of the inner loop iterations be $i_{max} = 400$, and the sampling time be $h \approx 0.0126$s (which means that $\frac{1}{h} \approx 79.58$). The graph representing this network is depicted in Fig. 3.2. In order to compare the actual disagreement values with the upper bound derived in Subsection 3.3, all four conditions of Remark 3.1 need to be satisfied. To this end, consider

$$\begin{cases} (\ y_{\Delta\rho})_1 = 0.1\frac{t}{h} + 0.2, \\ (\ y_{\Delta\rho})_2 = 0.4\frac{t}{h} + 0.8, \\ (\ y_{\Delta\rho})_3 = 0.2\frac{t}{h} + 0.4, \\ (\ y_{\Delta\rho})_4 = 0.3\frac{t}{h} + 0.6, \end{cases} \tag{3.47}$$

where $(\ y_{\Delta\rho})_j,\ j = 1, 2, ..., n$, is the transformed reference signal applied to the j-th agent. Respectively the corresponding reference signals $r_1, ..., r_4$ are as follows

$$\begin{cases} r_1 = 34.09t + 0.856, \\ r_2 = 0.001, \\ r_3 = 51.14t + 1.284, \\ r_4 = 36.53t + 0.9176, \end{cases} \tag{3.48}$$

where k represents the k-th iteration of the outer loop. Fig. 3.3 shows the trajectories of the states and reference signals of the agents, where $\bar{r}$ denotes the average of the reference signals. This figure demonstrates that agents track the average of the reference signals, and hence dynamic average consensus is achieved. Figs. 3.4 and 3.5 respectively show the disagreement upper bound for $(y_{\Delta\rho})_1, ..., (y_{\Delta\rho})_4$ and $x_1, ..., x_4$.

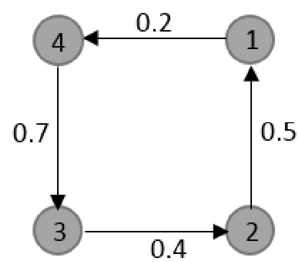

Figure 3.2: The graph representing the multi-agent network of Example 3.1

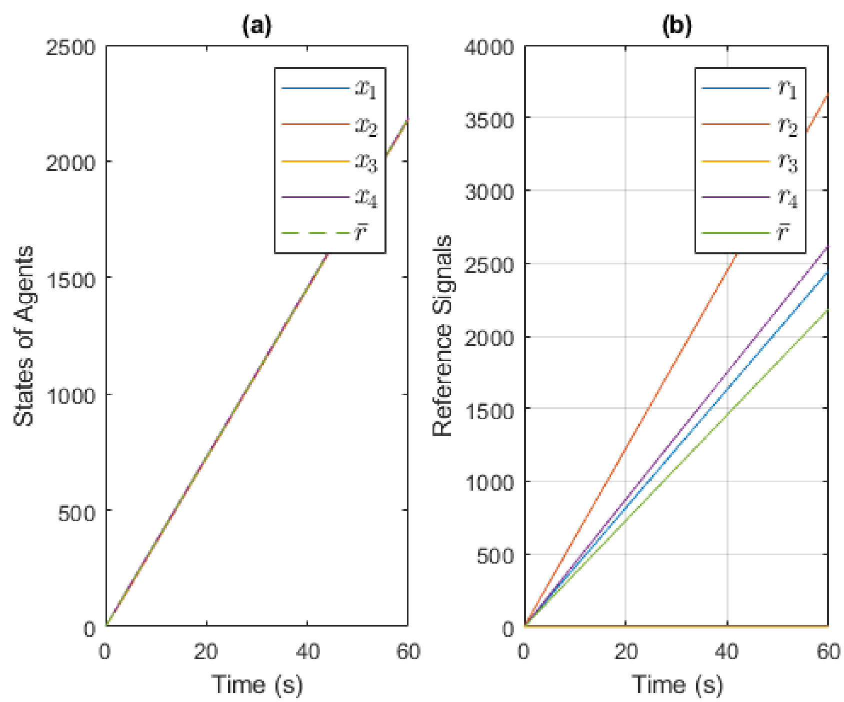

Figure 3.3: Trajectory of states and reference signals in Example 3.1

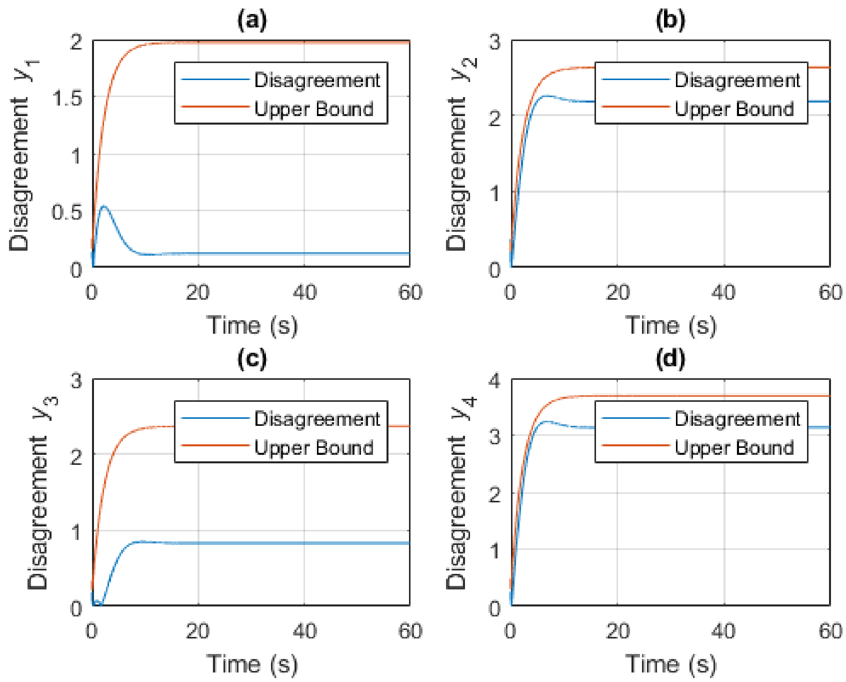

Figure 3.4: Disagreement values of the equivalent Markov chain system and their upper bounds in Example 3.1

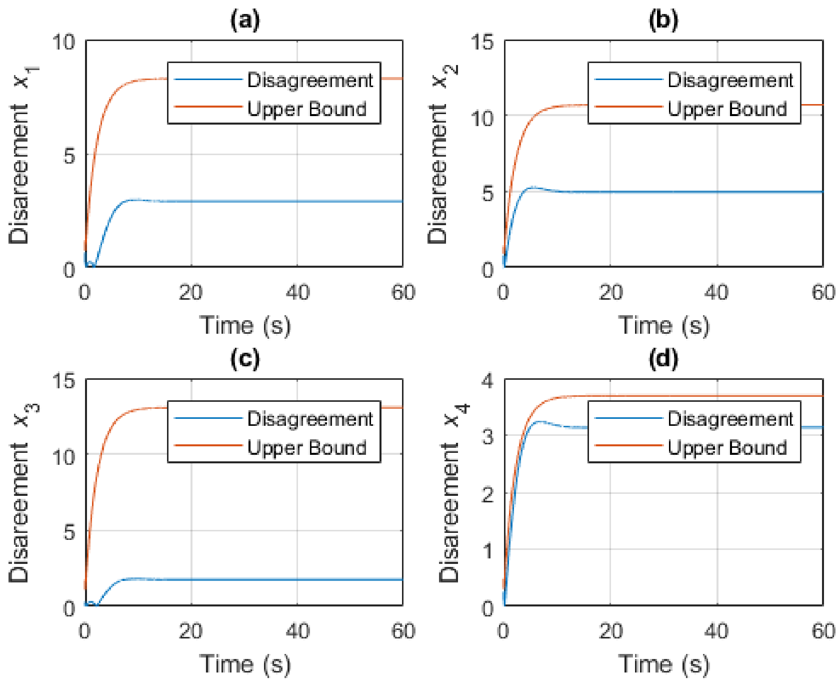

Figure 3.5: Disagreement values for the states of the agents and their upper bounds in Example 3.1

Example 3.2. In this example, the same multi-agent system as Example 3.1 is considered with the following $(y_{\Delta\rho})_1, ..., (y_{\Delta\rho})_4$, which satisfy conditions (i)-(iii) of Remark 3.1 but not condition (iv).

$$\begin{cases} (y_{\Delta\rho})_1 = 3.315t + 16.57\sin(t) + 0.2, \\ (y_{\Delta\rho})_2 = 3.315t + 16.57\sin(t) + 0.8, \\ (y_{\Delta\rho})_3 = 3.315t + 16.57\sin(t) + 0.4, \\ (y_{\Delta\rho})_4 = 3.315t + 16.57\sin(t) + 0.6, \end{cases} \tag{3.49}$$

where $(y_{\Delta\rho})_j,\ j=1,2,...,n$ is the transformed reference signal applied to the j-th agent. These functions satisfy conditions (i)-(iii) of Remark 3.1, but not condition (iv). The corresponding reference signals $r_1,...,r_4$ are, respectively, as follows

$$\begin{cases} r_1 = 7.103t + 35.51\sin(t), \\ r_2 = -13.19t - 65.95\sin(t) + 1.192, \\ r_3 = 21.31t + 106.5\sin(t) - 0.4279, \\ r_4 = 5.073t + 25.37\sin(t) + 0.7647. \end{cases} \tag{3.50}$$

Figs. 3.7 and 3.8 respectively show the disagreement upper bound for $(y_{\Delta\rho})_1,...,(y_{\Delta\rho})_4$ and $x_1,...,x_4$.

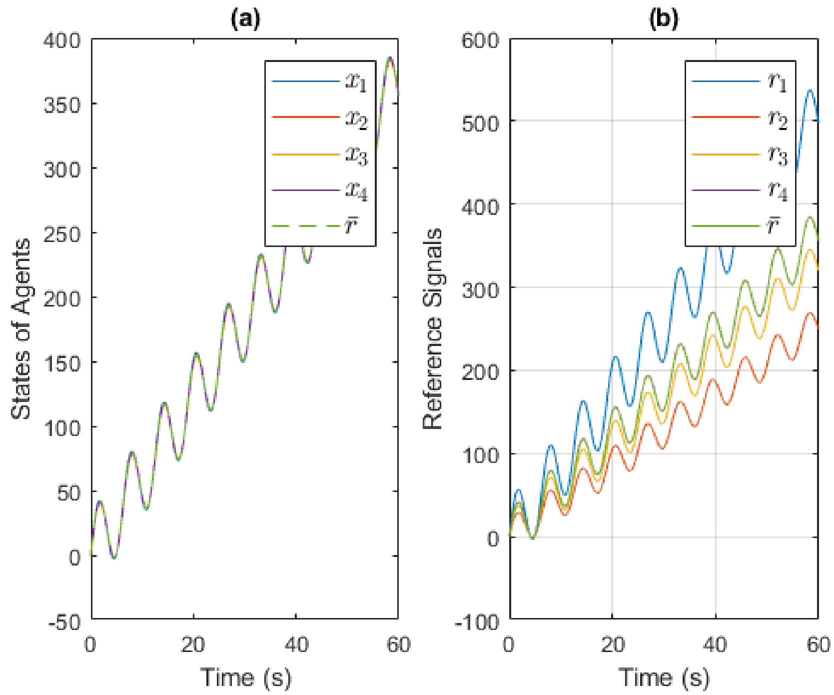

Figure 3.6: Trajectory of the states of agents and reference signals in Example 3.2

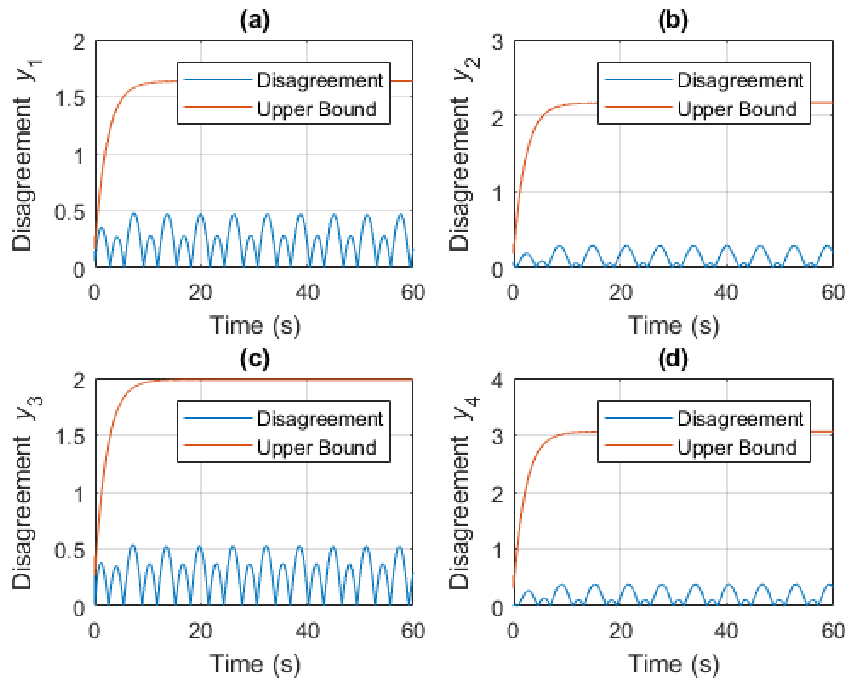

Figure 3.7: Disagreement values of the equivalent Markov chain system and their upper bounds in Example 3.2

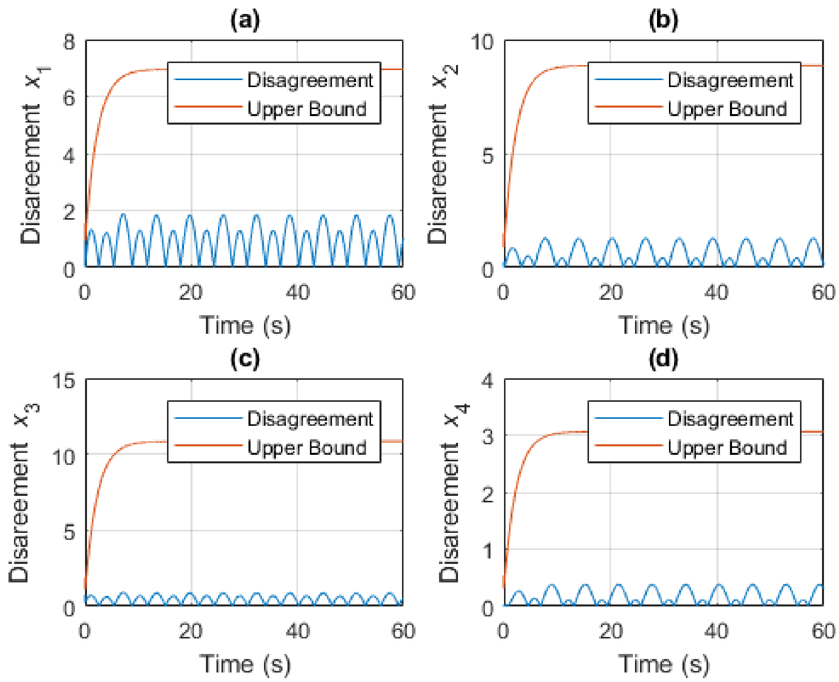

Figure 3.8: Disagreement values for the agents' states and their upper bounds in Example 3.2

Example 3.3. In this example, similar to the last two examples, a network of four agents subject to first-order reference signals (as a function of time) is considered but none of the conditions of Remark 3.1 applies in this case. The transformed signals in this example are given by

$$\begin{cases} (y_{\Delta\rho})_1 = 0.5t + 0.5, \\ (y_{\Delta\rho})_2 = 0.9t + 0.3, \\ (y_{\Delta\rho})_3 = 0.7t + 0.4, \\ (y_{\Delta\rho})_4 = 0.3t + 0.6, \end{cases} \tag{3.51}$$

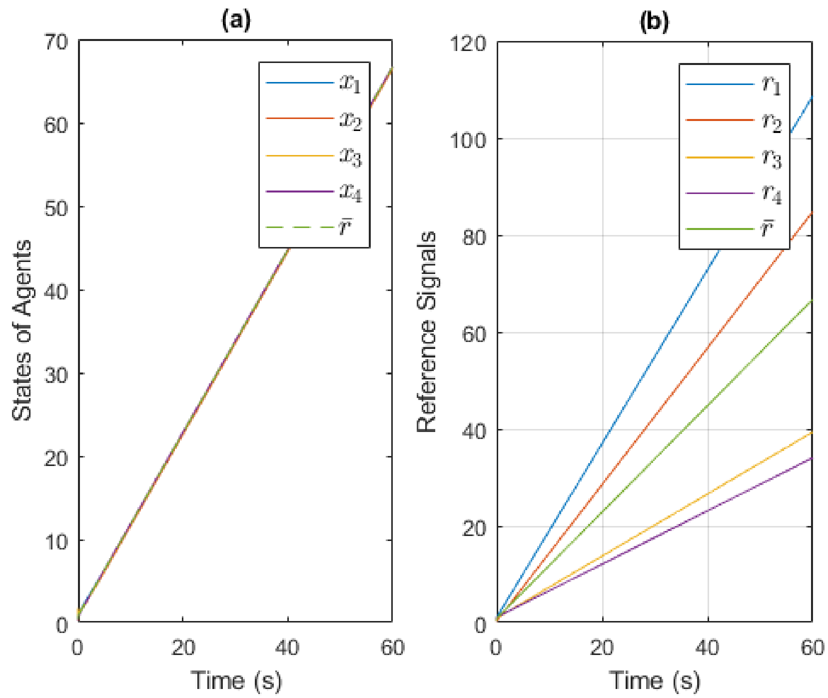

Figure 3.9: Trajectory of the states of agents and reference signals in Example 3.3

and the corresponding reference signals $r_1, ..., r_4$ are, respectively, as follows

$$\begin{cases} r_1 = 0.1793t + 1.002, \\ r_2 = 0.9t + 0.2887, \\ r_3 = 0.6405t + 0.9066, \\ r_4 = 0.549t + 1.091. \end{cases} \tag{3.52}$$

Fig. 3.9 demonstrates the convergence of the states of all agents to the average of the reference signals $\bar{r}$. Figs. 3.10 and 3.11 also provide the upper bounds for the states of the actual system and the transformed system, respectively.

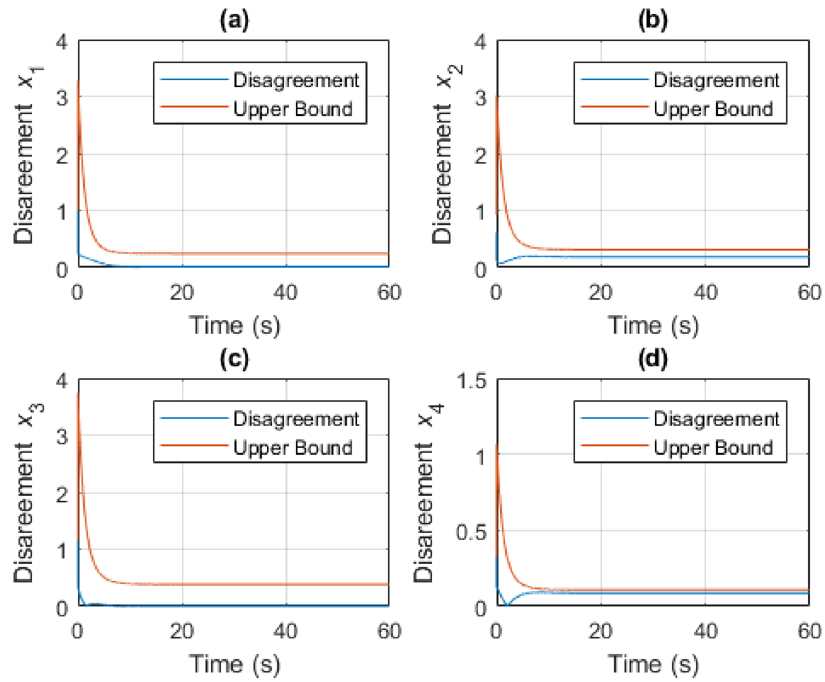

Figure 3.10: Disagreement values for the agents' states and their upper bounds in Example 3.3

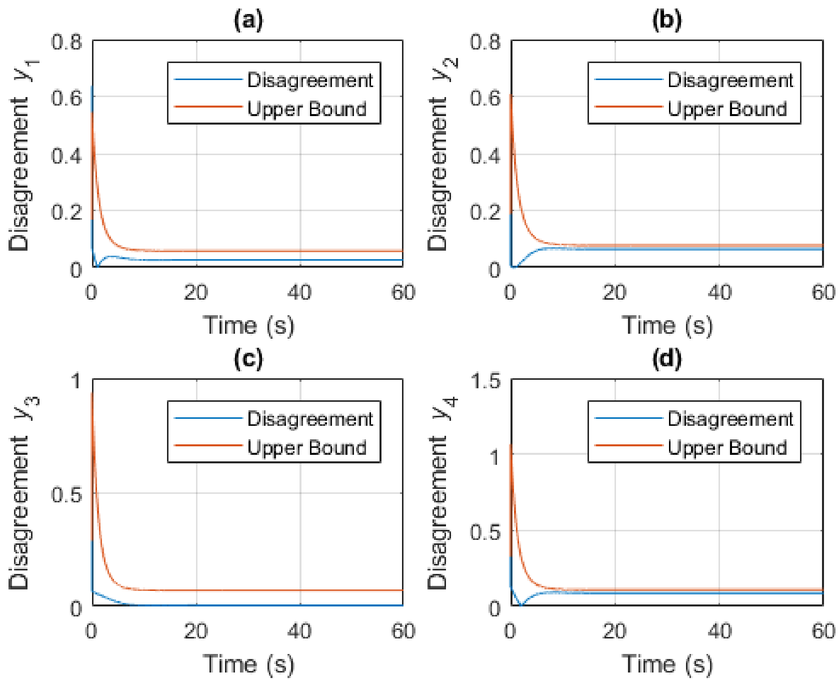

Figure 3.11: Disagreement values of the equivalent Markov chain system and their upper bounds in Example 3.3

Example 3.4. In this example, the effect of sampling time on the performance of the proposed dynamic average consensus control scheme with the same multi-agent system and reference inputs as the previous example is studied. Four different sampling times are considered, and the results are depicted in Fig. 3.12. It can be observed from these results that decreasing the sampling time improves the convergence speed of the dynamic average consensus control scheme. However, this improvement comes at the cost of more computations. This introduces a trade-off between convergence time and computational cost.

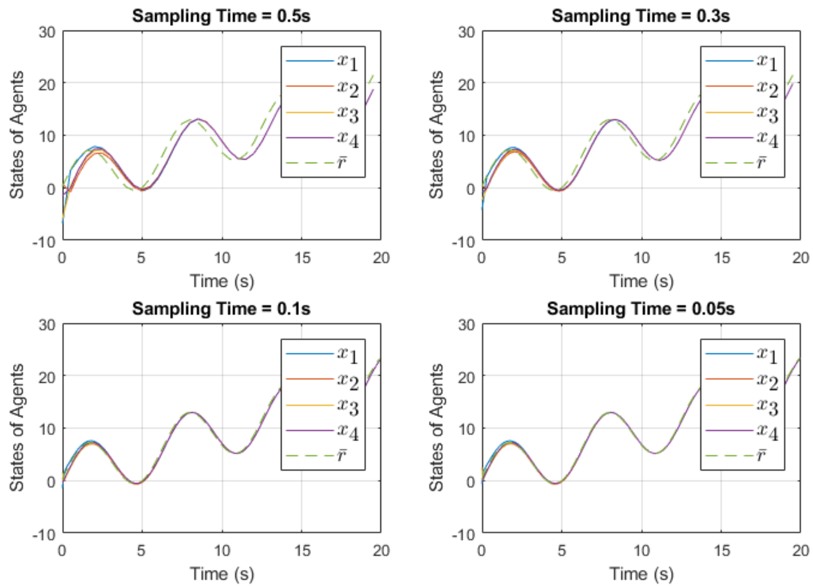

Figure 3.12: Disagreement values of the equivalent Markov chain system and their upper bounds in Example 3.3

3.5 Conclusions

A distributed control algorithm is proposed for dynamic average consensus in multi-agent systems. The network is directed, weighted, and unbalanced, in general. The objective is that the state of every agent tracks the average of the reference signals of all agents. An iterative algorithm is developed which operates in two phases. In the first phase, called inner stage, the difference between the reference signal of every agent in two consecutive time instants is communicated to their neighbors, and using an iterative strategy the disagreement vector with respect to the reference signals is suppressed. Then in the second phase, called outer stage, a standard update rule is used to reduce the error in the consensus problem due to the initial states. Simulations demonstrates the efficiency of the proposed algorithm using the Markov chain concept.

Chapter 4

Conclusions and Future Directions

In this thesis, the average consensus problem in a directed multi-agent network is studied, and a novel distributed algorithm is is introduced for the agents' states to reach the desired agreement. The proposed algorithm is proved to outperform existing methods in terms of the required information and/or steady-state performance for the class of unbalanced weighted directed networks. This is an important class of multi-agent systems which includes emerging applications such as underwater sensor networks.

Chapter 2 considers n homogeneous agents in an unbalanced weighted directed network, where the agents' states are desired to reach the average of the initial states. It is assumed that the directed network is strongly connected, and that each agent can receive information only from its in-neighbors. The salient feature of the proposed method is that it consists of two stages: (i) a calibration phase to find the discrepancy between the output of the standard consensus control protocol and the average of the initial states, and (ii) a properly adjusted consensus algorithm to account for the expected discrepancy in the final state of the agents. It is shown that under the proposed average consensus procedure, the disagreement vector (representing the difference between the final state of every agent and the average of the initial states) asymptotically converges to zero. It is also shown that this two-stage protocol can also achieve dynamic average consensus for the case of constant reference signals. Simulation results presented in this chapter are in line with the theoretical findings, and, in fact, demonstrate the effectiveness of the methods in a broader setting.

In Chapter 3, the same class of networks as the previous chapter is considered, but for the dynamic average consensus problem. In other words, it is assumed that a reference signal is applied to each agent, and the objective is that the state of every agent tracks the average of the reference signals under some mild conditions. Similar to the previous chapter, the main challenge here is to achieve the objective using only the locally available information, for a directed network with unbalanced weights. It is shown that using a Markov chain approach along with the error compensating method introduced in Chapter 2 for the state average consensus problem, the state of every agent tracks the average of the reference signals with an error bound which can arbitrarily small at the cost of higher computation complexity. The proposed dynamic average consensus protocol is scalable, and its effectiveness is verified by simulations in various scenarios.

4.1 Future Research Directions

Some suggestions for future research in this area are outlined below:

- applying the results obtained to the power optimization problem in underwater acoustic sensor networks, given the unbalanced structure of edge weights in such networks;
- extending the results to the case of a network with time-varying edge weights and/or switching topology;
- using the algorithms proposed in this work to the consensus-based training algorithms for the case of directed network of machines, where the privacy requirements prohibit the bi-directional information communication between agents, and
- performing robustness analysis to evaluate the impact of parameter uncertainty (e.g., in the elements of the weight matrix) on the steady-state results in both static and dynamic average consensus problems.

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