Self-nolar Planar Polytopes: When Finding the Polar is Rotating by Pi

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ABSTRACT

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John-Mark Fortier

The impetus for our work was a preprint by Alathea Jensen, titled self-polar polytopes, [2]. In the preprint, Jensen describes an intriguing method to add vertices to a self-polar polytope while maintaining self-polarity. This method, applied exclusively to self-nolar polytopes in $\mathbb{R}^2$, is our main focus for our work here. We expound upon the method, as well as clarify the underlining theoretical framework it was derived from. In doing so, we have built up our own set-up and framework and proved the theoretical steps independently, often differently than the original paper. In addition, we prove some noteworthy properties of self-nolar sets such as: all self-nolar sets are convex, the family of all self-nolar sets is uncountable, and the set of all self-nolar planar polytopes is dense in the set of all self-nolar planar sets. We also give proofs concerning the length of the boundary of a self-nolar set with smooth boundary, the center of mass of self-nolar polytopes and the Mahler volume product. Moreover, we prove an original theorem that can be used as a practical method to construct self-nolar polytopes.

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Notations

Let $A \subseteq \mathbb{R}^2$ be a set in the Euclidean plane. The following is a quick guide to the notations that will be used throughout the thesis.

- $\mathbb{R}^+$ denotes all positive real numbers.
- $conv(A)$ denotes the convex hull A.
- $closure(A)$ denotes the closure of A.
- $x \cdot y$ denotes the standard inner product for $x, y \in \mathbb{R}^2$.
- For any $x \in \mathbb{R}^2$, $||x||$ denotes the standard, Euclidean norm of x in $\mathbb{R}^2$.
- $[A] = closure(conv(A \cup \{0\}))$.
- $A^o = \{y \in \mathbb{R}^2 \mid x \cdot y \leq 1, \quad \forall x \in A\}$ denotes the polar set of A.
- Let $\hat{u}$ be a unit vector and let d be a strictly positive scalar. We denote a line in $\mathbb{R}^2$, with orthogonal directed distance $d(\hat{u})$ from the origin, as

$$\begin{aligned} H(\hat{u}, d) &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} = d\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v = 1, v = \frac{\hat{u}}{d}\} \\ &= H(v). \end{aligned}$$

- For every $H(\hat{u}, d)$ in $\mathbb{R}^2$, we associate two mutually exclusive halfplanes:

$$\begin{aligned} H^-(\hat{u}, d) &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} < d\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v < 1, v = \frac{\hat{u}}{d}\} \\ &= H^-(v), \end{aligned}$$

  and

$$\begin{aligned} H^+(\hat{u}, d) &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} > d\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v > 1, v = \frac{\hat{u}}{d}\} \\ &= H^+(v). \end{aligned}$$

Note that $H^+(\hat{u}, d)$ and $H^-(\hat{u}, d)$ do not contain the boundary line $H(\hat{u}, d)$. We will denote the respective union of each halfplane with there boundary line, rendering them closed sets, as:

$$\begin{aligned} \overline{H^-(\hat{u}, d)} &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} \le d\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v \le 1, v = \frac{\hat{u}}{d}\} \\ &= \overline{H^-(v)}, \end{aligned}$$

and

$$\begin{aligned} \overline{H^+(\hat{u}, d)} &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} \ge d\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v \ge 1, v = \frac{\hat{u}}{d}\} \\ &= \overline{H^+(v)}. \end{aligned}$$

Chapter 1

Preliminaries

1.1 Introduction

The basic object of study in this work are convex sets in $\mathbb{R}^n$ that are equal to their polar sets under an orthogonal transformation. These sets have been used in the past as a tool to investigate diverse mathematical inquiries. For example, they have been used to establish the chromatic number of distance graphs on spheres [3]. In addition, Radon curves which are boundaries of planar sets that are equal to their polar under a $\frac{\pi}{2}$ rotation, proved useful in providing a scalar product in some metric spaces that did not have a natural one [7]. More recently Jensen has investigated the existence, construction, facial structure, and practical applications of polytopes that are orthogonal transformations of their respective polar sets [2]. Such polytopes are referred to as self-polar by Jensen.

The primary motivation for our work here was a preprint by Jensen, titled self-polar polytopes, [2]. In the preprint, special attention was given to polytopes that are equal to their polars under the negative identity transformation $-I$, which is equivalent to a π-rotation. Jensen refers to these ploytopes as negatively self-polar, but, for brevity, we call them self-nolar. In Section 7 of the preprint, Jensen describes an intriguing method to add vertices to a self-polar polytope while maintaining self-polarity. This method, applied exclusively to self-nolar polytopes in $\mathbb{R}^2$, is the cornerstone of our work here. Although the method is sound, there was a need to expound upon it, as well as clarify the underlining theoretical framework it was derived from. In doing so, we have built our own set-up and framework and proved the theoretical steps independently, often differently than the original paper. Along the way, we discovered that, in fact, Figure 8 of the preprint, which is suppose to illustrate the method, did so incorrectly.

We have organized our work into three chapters. The first chapter, titled *Preliminaries*, gives the necessary background information for understanding and characterizing convex polytopes that is needed to comprehend our work in subsequent chapters. The second chapter, titled *Self-nolar Planar Sets*, is where we expounded on Jensen's method to add vertices to a self-nolar polytope while maintaining self-nolarity. Here we clarify the underlying theoretical framework by stringing together a sequence of original theorems, lemmas and corollaries that fit together neatly to ultimately prove a theorem from which the method is derived. In addition, we prove how the method can be used to reduce or preserve the number of vertices of a self-nolar polytope while maintaining self-nolarity. Lastly, we prove that self-nolar planar polytopes must have an odd number of vertices, a result also known to Jensen, but obtained via a different proof. The third chapter, titled *Finer Properties of Self-nolar Planar Sets*, is the last chapter. Here, we prove some noteworthy properties of self-nolar sets such as: all self-nolar sets are convex, the family of all self-nolar sets is uncountable, and the set of all self-nolar planar polytopes is dense in the set of all self-nolar planar sets. We also give proofs concerning the length of the boundary of a self-nolar set with smooth boundary, the center of mass of self-nolar polytopes and the Mahler volume product. Moreover, we prove an original theorem that can be used as a practical method to construct self-nolar polytopes. This entire chapter is, to the best of our knowledge, new and original.

1.2 Convex Sets

Definition 1.2.1. *Let $A \subseteq \mathbb{R}^2$ be a set in the plane. We say that A is a convex set if $\lambda x + \mu y \in A$ whenever $x, y \in A$ and for any real numbers $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$.*

Definition 1.2.2. *Let $m > 0$ be a positive integer and let $A \subseteq \mathbb{R}^2$ be a convex set containing points $a_1, ..., a_m$. If $\lambda_1, ..., \lambda_m \geq 0$ with $\lambda_1 + ... + \lambda_m = 1$, we call $\lambda_1 a_1 + ... + \lambda_m a_m$ a convex combination of the points $a_1, ..., a_m$ in A. More generally, a convex combination of points in A is a combination of points (not necessarily distinct) of A as above for some fixed $m > 0$.*

1.2.1 Properties of Convex sets

a. The intersection of an arbitrary family of convex sets is convex.

b. Every convex combination of points of a convex set belongs to that set.

c. Some examples of convex sets are points, lines, rays, line segments and half-planes.

For proof of these properties, we refer the reader to [1] and [9]

Convex Hulls

Definition 1.2.3. *Let $A \subseteq \mathbb{R}^2$, then $conv(A)$ is the intersection of all convex sets in $\mathbb{R}^2$ containing A. We call $conv(A)$ the convex hull of A.*

1.2.2 Properties of Convex Hulls

a. $conv(A)$ is the smallest convex set containing A.

b. $conv(A)$ is the set of all convex combinations of points in A.

c. If A is a finite set of points, then $conv(A)$ is compact

For proof of these properties, we refer the reader to [9].

Definition 1.2.4. *Let* $A \subseteq \mathbb{R}^2$ *be a set in the plane and denote by* $[A] = closure(conv(A \cup \{0\}))$.

Here closure refers to the inclusion of all limit points under the Euclidean metric. Also, since the closure of any set contains all its limit points, and a set that contains all its limit points is said to be closed, $[A]$ is a closed set.

1.3 Polar of a Set

Definition 1.3.1. *Let* $A \subseteq \mathbb{R}^2$, *then* $A^o = \{y \in \mathbb{R}^2 \mid x \cdot y \leq 1, \quad \forall x \in A\}$. *The set* A^o *is the called the polar set of* A.

1.3.1 Properties of the Polar of a Set

For any $A, B \subseteq \mathbb{R}^2$,

a. $A^o = [A^o]$

b. $A^o = [A]^o$

c. $A^{oo} = [A]$

d. $A^{ooo} = A^o$

e. $A \subset B \implies B^o \subset A^o$

f. $(A \cup B)^o = A^o \cap B^o$

g. $(A \cap B)^o = [A^o \cup B^o]$

h. If A is bounded, then A^o contains the origin in its interior.

i. If A contains the origin in its interior, then A^o is bounded.

j. If A is closed, convex and contains the origin, then $A^{oo} = [A] = A$.

For proof of these elementary properties, we refer the reader to [1] and [9].

1.4 Convex Polytopes in $\mathbb{R}^2$

Definition 1.4.1. *Let $\hat{u}$ be a unit vector and let d be a strictly positive scalar. We define a line in $\mathbb{R}^2$, with orthogonal directed distance $d(\hat{u})$ from the origin, as*

$$\begin{aligned} H(\hat{u}, d) &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} = d\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v = 1, v = \frac{\hat{u}}{d}\} \\ &= H(v). \end{aligned}$$

For every $H(\hat{u}, d)$ in $\mathbb{R}^2$, we can associate two mutually exclusive halfplanes:

$$\begin{aligned} H^-(\hat{u}, d) &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} < d\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v < 1, v = \frac{\hat{u}}{d}\} \\ &= H^-(v) \end{aligned}$$

and

$$\begin{aligned} H^+(\hat{u}, d) &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} > d\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v > 1, v = \frac{\hat{u}}{d}\} \\ &= H^+(v). \end{aligned}$$

Note that $H^+(\hat{u}, d)$ and $H^-(\hat{u}, d)$ do not contain the boundary line $H(\hat{u}, d)$. We will denote the respective union of each halfplane with there boundary line, rendering them closed sets, as:

$$\begin{aligned}
\overline{H^-(\hat{u}, d)} &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} \leq d\} \\
&= \{x \in \mathbb{R}^2 \mid x \cdot v \leq 1, v = \frac{\hat{u}}{d}\} \\
&= \overline{H^-(v)}
\end{aligned}$$

and

$$\begin{aligned}
\overline{H^+(\hat{u}, d)} &= \{x \in \mathbb{R}^2 \mid x \cdot \hat{u} \geq d\} \\
&= \{x \in \mathbb{R}^2 \mid x \cdot v \geq 1, v = \frac{\hat{u}}{d}\} \\
&= \overline{H^+(v)}.
\end{aligned}$$

1.4.1 Characterizing Convex Polytopes

We define a polytope in $\mathbb{R}^2$ as any closed planar figure that is bounded by line segments. We will be restricting our focus to a subset of polytopes that are also convex sets, hence called convex polytopes.

There are two well established ways to characterize convex polytopes in $\mathbb{R}^2$:

a. **Characterization by Vertices**

   Any convex polytope can be described as the convex hull of a finite set of points [9]. For a given convex polytope P, there are many finite sets of points whose convex hull is P. However, we can always find a unique finite set whose convex hull is P, [9]. We refer to this set as the set of minimal points in the sense that if we were to remove any point(s) from the set and then take the convex hull of the remaining points, it would not yield P. We denote the set of minimal points as $V(P)$, and the refer to its elements as the vertices of P. By indexing the vertices of P counter-clockwise, with respect to its representation in the plane, we may then define an edge E of P to be the closed line segment with two consecutive vertices of P as endpoints.

b. **Characterization by Halfplanes**

Any convex polytope can also be described as the intersection of all the elements of a finite set whose elements are closed halfplanes, [9]. It is also the case that any nonempty finite intersection of halfplanes which is bounded, meaning it does not contain any sequences of points whose positions vectors have norms tending to infinity, is a convex polytope, [9]. We refer to a finite intersection of halfplanes that is not necessarily bounded as a polytopal set. If a given polytopal set is not bounded, we do not consider it a polytope. For a given convex polytope P, we can always find a unique finite set whose elements are halfplanes such that the intersection of all its elements is P, [9]. We refer to this set as the set of minimal intersections in the sense that if we were to remove any closed halfspace from the set and then take the intersection of the remaining halfspaces, it would not yield P. The closed halfplanes contained in the set of minimal intersection for a given polygon P are called essential halfplanes and their respective boundary lines are called essential support lines.

From here on, when we refer to a polytope, the reader may assume that it is a subset of $\mathbb{R}^2$, that it is closed, convex, that it contains the origin in its interior and that we are using either the set of minimal points or the set of minimal intersection to describe it.

1.4.2 Relationship Between Vertices and Essential Support Lines

Let P be a polytope described as

$$\begin{aligned} P &= \bigcap_{i\in\{1,2\ldots,k\}} \overline{H^-(\hat{u}_i, d_i)} \\ &= \bigcap_{i\in\{1,2\ldots,k\}} \overline{H^-(v_i)} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v_i \le 1, 1 \le i \le k\}, \end{aligned}$$

where the essential support lines are indexed counter-clockwise with respect to their representation on a Cartesian plane. Some immediate consequences of characterizing P in this way are the following:

a. The intersection of two consecutive essential support lines is a vertex of P.

b. Every essential support line contains exactly two consecutive vertices

c. Let E_i be an edge of P, then $E_i \subset H(v_j)$ if and only if $i = j$.

d. The number of vertices of P is equal to the number of essential supporting half-planes and/or essential supporting lines.

Chapter 2

Self-nolar Planar Sets

2.1 Introduction

A stronger, older and more widely used definition of self-polar, in view of Jensen's definition [2], is: a set A in $\mathbb{R}^n$ for which $A = A^o$. It is natural to wonder if there exist sets in $\mathbb{R}^n$ that are self-polar, with respect to the aforementioned definition. It turns out that the only set with this property is the Euclidean unit ball centered at the origin. We present below its proof as in [2]:

Theorem 2.1.1. *The only set A in $\mathbb{R}^n$ for which $A = A^o$ is the unit ball, which is defined as $A = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$.*

Proof. Let B denote the unit ball, that is $B = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$. Clearly, from the definition of the polar operation $B = B^o$. Now suppose there exist some other set A in $\mathbb{R}^n$ such that $A = A^o$. For all $x \in A$, we have $x \in A^o$. It follows that $x \cdot x \leq 1$, which implies that $||x|| \leq 1$. Hence $A \subseteq B = B^o$. We then have that $B \subseteq A^o = A$. By double inclusion, we may conclude that $A = B$. $\square$

So, the only self-polar set in $\mathbb{R}^2$ is the unit disk centered at the origin. In view of this, in order to try to obtain a larger class of sets, we relax this condition and consider sets in $\mathbb{R}^2$ that are equal to their polar set under the negative identity transformation

$-I$, which is equivalent to a π-rotation. In other words, we are going to focus on the sets with the following property: A a set in $\mathbb{R}^2$ with $A = -A^o$. These sets have been referred to in the literature as negatively self-polar, but, mainly for brevity, we will call them self-nolar sets. Moreover, since convex polytopes are well understood, simple to characterize, and form a dense set in the class of all planar convex bodies, we will be exploiting the theory of convex polytopes by focusing our investigation on self-nolar sets that are convex polytopes.

2.2 Self-nolar Polytopes

Definition 2.2.1. *Let $C \subseteq \mathbb{R}^2$. If $C = -C^o$, then C is said to be self-nolar.*

2.2.1 Erecting the Theoretical Framework

Lemma 2.2.1. *Let $\overline{\mathcal{N}}$ be the set of all self-nolar sets. If $C \in \overline{\mathcal{N}}$, then C is convex.*

Proof. Let $C \in \overline{\mathcal{N}}$, then

$$\begin{aligned} C = -C^o & \\ &= -\{y \in \mathbb{R}^2 \mid x \cdot y \leq 1, \quad \forall x \in C\} \\ &= \{y \in \mathbb{R}^2 \mid -x \cdot y \leq 1, \quad \forall x \in C\} \end{aligned}$$

So, $-C^o$ can be characterized as the intersection of halfplanes. It is well-known that halfplanes are convex sets and that any arbitrary intersection of convex sets is convex. This implies that $-C^o$ is convex and since $C = -C^o$, we may conclude that C is convex.

□

A basic question one may ask about self-nolar polytopes is: do any such set exists? To answer this question, we will use a theorem that can be used to verify whether or not a given polytope is self-nolar. To prove this theorem, we commence with the following lemma.

Lemma 2.2.2. *Let* $P = \{x \in \mathbb{R}^2 \mid x \cdot v_i \leq 1,\ 1 \leq i \leq k\}$*, for some fixed positive integer* k*, be a polytope, then* $P^o = conv(\{v_i \mid 1 \leq i \leq k\})$*.*

Proof. Let $Q = \{v_i \in \mathbb{R}^2 \mid 1 \leq i \leq k\}$, then by definition

$$\begin{aligned} Q^o &= \{x \mid x \cdot v_i \leq 1,\ 1 \leq i \leq k\} \\ &= P, \end{aligned}$$

which implies that

$$\begin{aligned} Q^{oo} &= [Q] \\ &= closure(conv(Q \cup \{0\})) \\ &= P^o. \end{aligned}$$

Since P is bounded and contains the origin in its interior, P^o contains the origin in its interior. It follows that $conv(Q \cup \{0\}) = conv(Q)$, otherwise we would have a contradiction with the fact that the origin is in interior of P^o. In addition, $Q \cup \{0\}$ is a finite set of points, which implies that $conv(Q \cup \{0\})$ is compact and therefore closed. It follows that $closure(conv(Q \cup \{0\})) = conv(Q \cup \{0\})$.

We may now conclude that

$$\begin{aligned} P^o &= closure(conv(Q \cup \{0\})) \\ &= conv(Q \cup \{0\}) \\ &= conv(Q) \\ &= conv(\{v_i \mid 1 \leq i \leq k\}). \end{aligned}$$

□

2.2.2 Characterization Theorem

We will now state and prove the aforementioned theorem that we will use to determine if a given polytope is self-nolar.

Theorem 2.2.1. *Let P be a planar polytope defined, equivalently, by*

$$\begin{aligned} P &= \bigcap_{i \in \{1,2\ldots,k\}} \overline{H^-(\hat{u}_i, d_i)} \\ &= \bigcap_{i \in \{1,2\ldots,k\}} \left\{ \overline{H^-(v_i)} \mid v_i = \frac{\hat{u}_i}{d_i} \right\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v_i \leq 1, 1 \leq i \leq k\}. \end{aligned}$$

Then, P is self-nolar if and only if $-v_i$ is a vertex of P and $\overline{H^-(v_i)}$ is an essential halfplane with associated essential support line $H(v_i)$.

Proof. " $\Rightarrow$ "

Assume $P = -P^o$. By Lemma 2.2.2, we have that $P^o = conv(\{v_i \mid 1 \leq i \leq k\})$.

Thus

$$\begin{aligned} P &= -P^o \\ &= -conv(\{v_i \mid 1 \leq i \leq k\}) \\ &= conv(\{-v_i \mid 1 \leq i \leq k\}). \end{aligned}$$

Here $\{-v_i \mid 1 \leq i \leq k\}$ must be the set of vertices of P, since the number of essential supporting halfplanes must equal the number of vertices of P. It follows that $-v_i$ is a vertex of P for all $i \in \{1, 2, ..., k\}$.

" $\Leftarrow$ "

Assume $-v_i$ is a vertex of P for all $i \in \{1, 2, ..., k\}$. It follows that

$$\begin{aligned} P &= conv(\{-v_i \mid 1 \leq i \leq k\}) \\ &= -conv(\{v_i \mid 1 \leq i \leq k\}). \end{aligned}$$

By Lemma 2.2.2, we have that $P^o = conv(\{v_i \mid 1 \leq i \leq k\})$. From this we conclude that $P = -P^o$. $\square$

2.2.3 An Example of a Self-nolar Pentagon

Now, we will provide an example of a self-nolar polytope. Consider the polytope P characterized by the intersection of the following essential supporting halfplanes:

$$\begin{aligned} P &= \bigcap_{i\in\{1,2\dots,5\}} \overline{H^{-}(\hat{u}_i, d_i)} \\ &= \bigcap_{i\in\{1,2\dots,5\}} \left\{ \overline{H^{-}(v_i)} \mid v_i = \frac{\hat{u}_i}{d_i} \right\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v_i \leq 1, 1 \leq i \leq 5\}, \end{aligned}$$

where

$$\begin{aligned} v_1 &= (-1,1) = \frac{\hat{u}_1}{d_1} = \frac{(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}{\frac{1}{\sqrt{2}}} \\ v_2 &= (1,1) = \frac{\hat{u}_2}{d_2} = \frac{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}{\frac{1}{\sqrt{2}}} \\ v_3 &= (1,0) = \frac{\hat{u}_3}{d_3} = \frac{(1,0)}{1} \\ v_4 &= (0,-1) = \frac{\hat{u}_4}{d_4} = \frac{(0,-1)}{1} \\ v_5 &= (-1,0) = \frac{\hat{u}_5}{d_5} = \frac{(-1,0)}{1}. \end{aligned}$$

To determine the vertices of P, we find the intersection point of each pair of consecutive support lines:

$$\begin{aligned} H(v_1) \cap H(v_2) &= (0,1) = -v_4 \\ H(v_2) \cap H(v_3) &= (1,0) = -v_5 \\ H(v_3) \cap H(v_4) &= (1,-1) = -v_1 \\ H(v_4) \cap H(v_5) &= (-1,-1) = -v_2 \\ H(v_5) \cap H(v_1) &= (-1,0) = -v_3. \end{aligned}$$

It follows that $-v_i$ is a vertex of P, for all $i \in \{1, 2, ..., 5\}$. Based on this result we may now conclude, by employing Theorem 2.2.1, that P is self-nolar, that is $P = -P^o$.

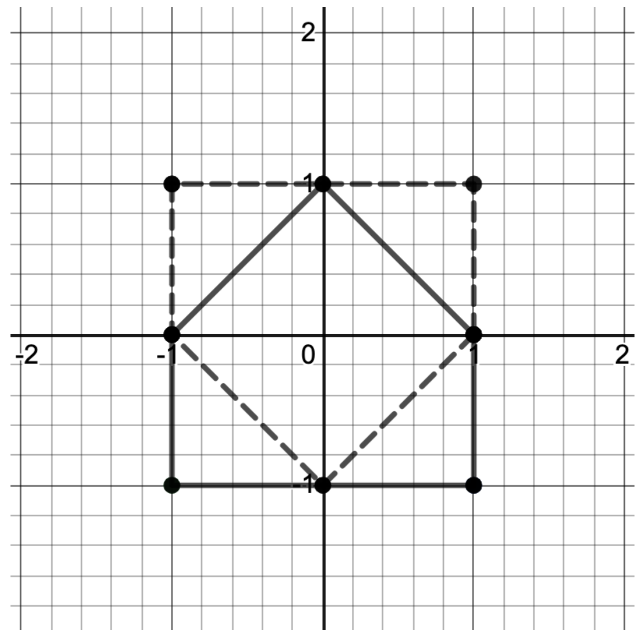

Figure 2.1: A self-nolar pentagon (solid line boundary) and its polar set (dashed line boundary).

2.3 Constructing a Self-nolar Polytope Based on a Pre-existing One

With the previous example, have shown the existence of, at least, one self-nolar polytope. Therefore, the cardinality of the set containing all self-nolar polytopes is at least 1. In pursuit of determining the precise cardinality of the set of all self-nolar polytopes we will construct, by way of a sequence of novel proofs, a theorem that one can use as a method to build a new self-nolar polytope based on one already known to exist. It is through a recursive use of this method that we will explore the cardinality of the set of all self-nolar polytopes, denoted for simplicity $\mathcal{N}$.

In order to provide insight, we will now give a naive description regarding how the method operates on an existing self-nolar polytope P to construct from it another self-nolar polytope T.

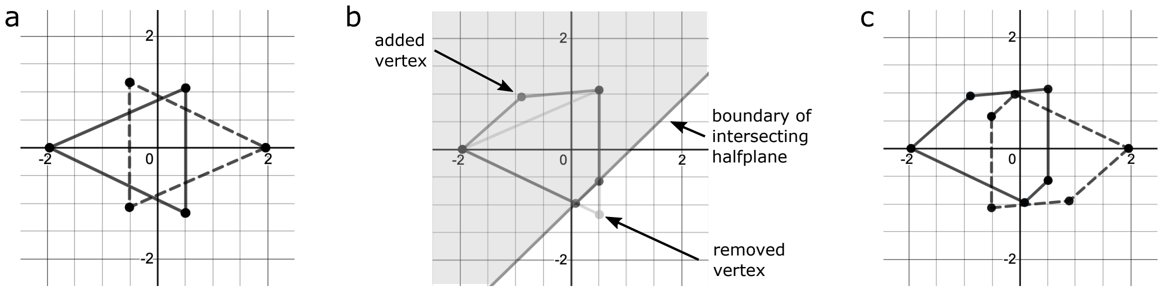

Figure 2.2: Constructing a self-nolar ploytope from a pre-existing one.

a. A self-nolar triangle P (solid line boundary) and its polar set P^o(dashed line boundary)

b. Altering P(light solid line boundary) to obtain a self-nolar pentagon T(dark solid line boundary)

c. T (solid line boundary) and its polar set T^o(dashed line boundary)

Intuitively, the method alters P by first "cutting" off one of its vertices with a new edge while simultaneously "adding" a new vertex. This is accomplished by intersecting P with a halfplane $H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$ (of direction different than any direction of hyperplane in the minimal set) such that $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ intersects two consecutive edges of P. Secondly, in order to ensure self-nolarity, the method "adds" a carefully chosen new vertex: $v_* = d_*\left(-\hat{u}_*\right)$, which is actually the negative polar set of $H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$. This is done by taking the convex hull of the set containing point $v_* = d_*\left(-\hat{u}_*\right)$ and $P \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$. Ultimately, the method constructs the set $T = conv(P \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right) \cup \{v_*\})$, which will be proved to be self-nolar.

We will now prove the theorem that provides the theoretical framework of the method described above. We construct our argument by first proving a sequence of lemmas and corollaries, that at first glance may seem unrelated or inconsequential, but ultimately fit neatly together to form our proof. Let us begin.

Lemma 2.3.1. *Let points x_{int} and y_{int} be the x and y intercepts of line $H(\hat{u}, d) = H(v)$ (if they exist), where $v = \dfrac{\hat{u}}{d}$. Let m be the slope of $H(v)$ and let Q_i be the i^{th} quadrant belonging to the Cartesian plane, where $i \in \{1, 2, 3, 4\}$.*

a. *If $y_{int} < 0$ and $m < 0$, then the point of position vector $-v$ is included in the interior of Q_1.*

b. *If $y_{int} < 0$ and $m > 0$, then the point of position vector $-v$ is included in the interior of Q_2.*

c. *If $y_{int} > 0$ and $m < 0$, then the point of position vector $-v$ is included in the interior of Q_3.*

d. *If $y_{int} > 0$ and $m > 0$, then the point of position vector $-v$ is included in the interior of Q_4.*

e. *If $y_{int} < 0$ and $m = 0$, then the point of position vector $-v$ is included in the interior of the positive y-axis.*

f. *If $x_{int} < 0$ and m is undefined, then the point of position vector $-v$ is included in the interior of the positive x-axis.*

g. *If $y_{int} > 0$ and $m = 0$, then the point of position vector $-v$ is included in the interior of the negative y-axis.*

h. *If $x_{int} > 0$ and m is undefined, then the point of position vector $-v$ is included in the interior of the negative x-axis.*

Proof. Assume $H(v)$ has $y_{int} < 0$ and $m < 0$. This implies that $x_{int} < 0$. Now, consider the triangle $\triangle\left((0,0),(0,y_{int}),(x_{int},0)\right)$. By construction, this is a right triangle in Q_3 having line segment $\overline{y_{int}, x_{int}}$ as its hypotenuse.

Let $\overline{(0,0),s}$ be a line segment such that $s \in H(v)$ with $\overline{(0,0),s}$ and $H(v)$ orthogonal. From elementary geometry, $\overline{(0,0),s}$ can be consider the altitude of $\triangle\,(0, y_{int}, x_{int})$ whose foot intersects the hypotenuse $\overline{y_{int}, x_{int}}$ at point s with $s \neq (0, y_{int})$ and $s \neq (x_{int}, 0)$. This implies that s (considered as a vector) is the orthogonal directed distance from the origin to $H(v)$. It follows that $s = d(\hat{u})$. Clearly, s is strictly in Q_3, which implies that s rotated by π, which is equivalent to multiplying s by the negative identity matrix $-I$ to obtain $-s$, is strictly in Q_1.

From $s = d(\hat{u})$, we obtain $-v = \frac{(-\hat{u})}{d} = \frac{-s}{d^2}$. Since $-s$ is strictly in Q_1, it follows that $\frac{-s}{d^2} = -v$ is strictly in Q_1. This proves a.

By rotating $H(v)$ by $\frac{\pi}{2}$ and repeating a similar argument used to prove a, we prove b.

By rotating $H(v)$ by π and repeating a similar argument used to prove a, we prove c.

By rotating $H(v)$ by $\frac{2\pi}{3}$ and repeating a similar argument used to prove a, we prove d.

Assume now that $H(v)$ has $y_{int} < 0$ and $m = 0$. Clearly, $d(\hat{u}) = y_{int}(0, -1)$, implying that $v = \frac{-(0,-1)}{y_{int}} = (0, \frac{1}{y_{int}})$. It follows that v is on the positive y-axis. This proves e.

By rotating $H(v)$ by $\frac{\pi}{2}$ and repeating a similar argument used to prove e, we prove f.

By rotating $H(v)$ by π and repeating a similar argument used to prove e, we prove g.

By rotating $H(v)$ by $\frac{2\pi}{3}$ and repeating a similar argument used to prove e, we prove h. $\square$

Corollary 2.3.1. *Let $H(v)$ be a line in $\mathbb{R}^2$ which does not pass through the origin, then $-v \notin H(v)$.*

Proof. Without any loss of generality, let $H(v)$ be below and parallel to the x-axis. It follows that $H(v)$ has $y_{int} < 0$ and $m = 0$. By Lemma 2.3.1, $-v$ is on the positive y-axis. This implies that $H(v)$ and $-v$ are separated by the x-axis. From this, we conclude that $-v \notin H(v)$. □

Lemma 2.3.2. *If $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ is a line containing the point $p = \dfrac{-\hat{u}}{d}$ with $0 < \hat{u}_* \cdot (-\hat{u}) \leq 1$, then $v_* = d_*\left(-\hat{u}_*\right) \in H\left(\hat{u}, d\right)$.*

Proof. Without any loss of generality, let point $p = \frac{-\hat{u}}{d}$ be located on the positive y-axis with $p \in H\left(\hat{u}_*, \frac{1}{d_*}\right)$.

Case one: If $\hat{u}_* \cdot (-\hat{u}) = 1$, then we have $\frac{1}{d_*} = \frac{1}{d}$, which implies $d_* = d$. Thus, $v_* = d_*\left(-\hat{u}_*\right) = d\left(\hat{u}\right) \in H\left(\hat{u}, d\right)$. Case two: If $0 < \hat{u}_* \cdot (-\hat{u}) < 1$, then we can construct the two similar triangles:

$$\triangle\left(0, \frac{\hat{u}_*}{d_*}, \frac{-\hat{u}}{d}\right) \sim \triangle\Bigg(0, d\left(\hat{u}\right), l(-\hat{u}_*)\Bigg). \tag{2.1}$$

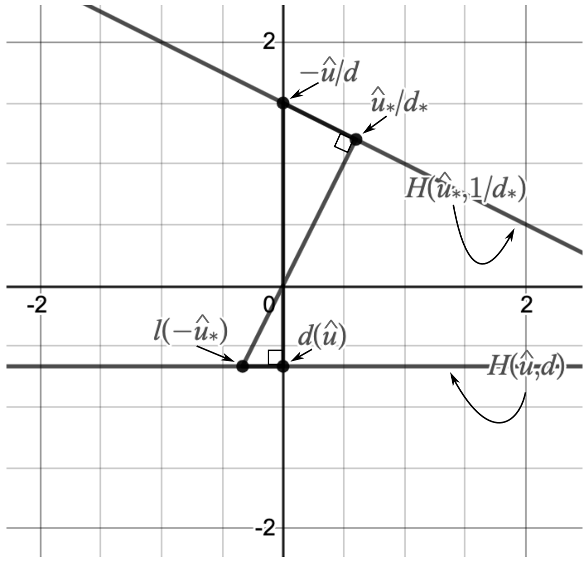

Figure 2.3: Illustration of similar triangles

These triangles are similar because, by construction, they have two sets of congruent angles:

$$\angle\left(0, \frac{\hat{u}_*}{d_*}, \frac{-\hat{u}}{d}\right) \equiv \angle\left(0, d\left(\hat{u}\right), l\left(-\hat{u}_*\right)\right) \tag{2.2}$$

and

$$\angle\left(\frac{-\hat{u}}{d}, 0, \frac{\hat{u}_*}{d_*}\right) \equiv \angle\left(l\left(-\hat{u}_*\right), 0, d\left(\hat{u}\right)\right). \tag{2.3}$$

Therefore, we have that

$$\frac{\frac{1}{d}}{l} = \frac{\frac{1}{d_*}}{d} \implies l = d_*. \tag{2.4}$$

Let $v_l = l\left(-\hat{u}_*\right)$. By construction, $v_l \in H\left(\hat{u}, d\right)$. Then,

$$v_l = l\left(-\hat{u}_*\right) = d_*\left(-\hat{u}_*\right) = v_* \tag{2.5}$$

Hence, $v_* = v_l \in H\left(\hat{u}, d\right)$. $\square$

Corollary 2.3.2. *If $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ is a line containing a fixed vertex $-v_j = \dfrac{-\hat{u}_j}{d_j}$ of a self-nolar polytope $P = \{x \in \mathbb{R}^2 \mid x \cdot v_i \leq 1, 1 \leq i \leq k\}$ with $0 < \hat{u}_* \cdot (-\hat{u}_j) \leq 1$, then $v_* = d_*\left(-\hat{u}_*\right) \in H\left(\hat{u}_j, d_j\right)$, where $H\left(\hat{u}_j, d_j\right)$ is an essential support line of P.*

Proof. Suppose $-v_j = \frac{-\hat{u}_j}{d_j}$ is a vertex of P. By Theorem 2.2.1, $H\left(\hat{u}_j, d_j\right)$ is an essential support line of P, and by Lemma 2.3.2, $v_* \in H(\hat{u}_j, d_j)$. $\square$

Lemma 2.3.3. *Let* $p = \dfrac{-\hat{u}_j}{d_j}$ *be the intersecting point of arbitrary yet distinct lines* $H(\hat{u}_h, d_h)$ *and* $H(\hat{u}_{h+1}, d_{h+1})$. *If* $p \in H\left(\hat{u}_*, \frac{1}{d_*}\right)$, *where* $\hat{u}_*$ *belongs to the convex cone with two boundary rays that originate at the origin and have direction unit vectors* $\hat{u}_h$ *and* $\hat{u}_{h+1}$ *respectively, then* $v_* = d_*(-\hat{u}_*)$ *belongs to the line segment, contained in* $H(\hat{u}_j, d_j)$, *with end points* $p_h = \dfrac{-\hat{u}_h}{d_h}$ *and* $p_{h+1} = \dfrac{-\hat{u}_{h+1}}{d_{h+1}}$.

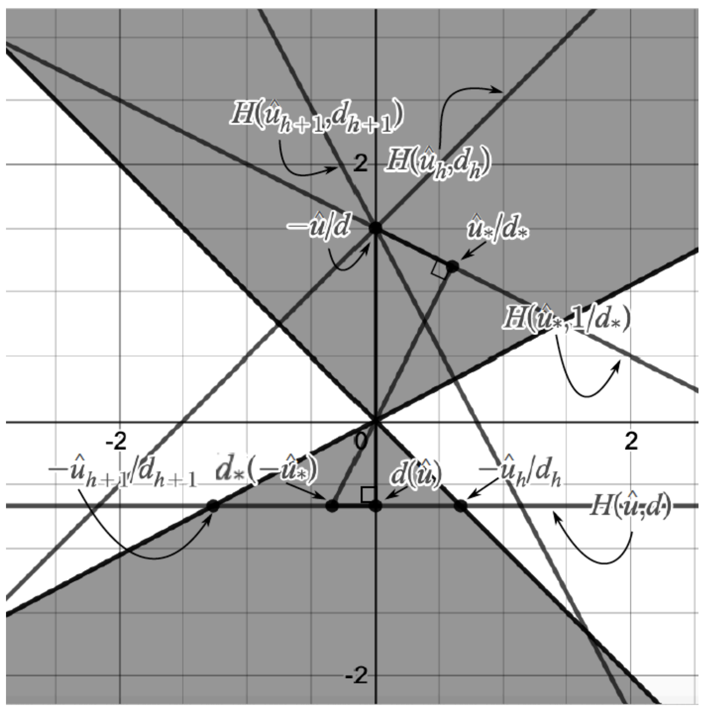

Figure 2.4: Illustration of Lemma 2.3.3

Proof. Let $H(\hat{u}_{h+1}, d_{h+1})$ and $H(\hat{u}_h, d_h)$ intersect at $p = \frac{-\hat{u}_j}{d_j}$. Suppose $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ contains point p, and that its unit normal $\hat{u}_*$ belongs to the cone with two boundary rays that originate at the origin and have direction unit vectors that are the unit normals of $H(\hat{u}_h, d_h)$ and $H(\hat{u}_{h+1}, d_{h+1})$, thus $\hat{u}_h$ and $\hat{u}_{h+1}$ respectively. This implies, by symmetry with respect to the origin, that $v_* = d_*(-\hat{u}_*)$ belongs to the convex cone with two boundary rays that originate at the origin and have unit direction $-\hat{u}_h$ and $-\hat{u}_{h+1}$ respectively. Now, considering the extreme cases when v_* is contained in the boundary of the convex cone it belongs to; we have that if $-\hat{u}_* = -\hat{u}_h$, then $v_* = p_h$ and if $-\hat{u}_* = -\hat{u}_{h+1}$, then $v_* = p_{h+1}$. By Lemma 2.3.2, $v_* \in H\left(\hat{u}_j, d_j\right)$. From this, it follows that v_* belongs to the line segment, contained in $H\left(\hat{u}_j, d_j\right)$, with endpoints p_h and p_{h+1}. $\square$

Corollary 2.3.3. *In addition to the premises of Lemma 2.3.3, let $H\left(\hat{u}_h, d_h\right)$ and $H\left(\hat{u}_{h+1}, d_{h+1}\right)$ be two consecutive essential support lines of self-nolar polytope $P = \{x \in \mathbb{R}^2 \mid x \cdot v_i \leq 1, 1 \leq i \leq k\}$, such that their point of intersection $p = \frac{-\hat{u}_j}{d_j} = -v_j$ is a vertex of P. If E_j is the edge of P contained in essential support line $H\left(\hat{u}_j, d_j\right)$, then $v_* \in E_j$.*

Proof. By Lemma 2.3.3, v_* is contained on the line segment with endpoints p_h and p_{h+1}. By Theorem 2.3.2, $p_h = -v_h$ and $p_{h+1} = -v_{h+1}$ are vertices of P. In addition, Corollary 2.3.2 implies that $-v_h$ and $-v_{h+1}$ are contained in the essential hyperplane $H\left(\hat{u}_j, d_j\right)$. Therefore, the line segment with endpoints $-v_h$ and $-v_{h+1}$ is the edge E_j of P. It follows that $v_* \in E_j$.

$\square$

Lemma 2.3.4. *Let P be the polytope defined by*

$$P = \bigcap_{i \in \{1,2\ldots,k\}} \overline{H^-\left(\hat{u}_i, d_i\right)},$$

and consider the associated polyhedron Q, obtained by removing the jth constraint from P, defined thus as:

$$Q = \bigcap_{i \in \{1,2\ldots,j-1,j+1,\ldots,k\}} \overline{H^-\left(\hat{u}_i, d_i\right)}.$$

If

$$v \in Q \cap H^+ \left(\hat{u}_j, d_j\right),$$

then

$$\left(conv(P \cup \{v\}) \setminus P\right) \subset H^+ \left(\hat{u}_j, d_j\right).$$

Proof. Clearly, all points in P as well as v belong to Q. Since Q is a polytope, any convex combination of v with any point in P must also belong to Q. It follows that

$$(conv(P \cup \{v\})) \subset Q.$$

This implies that

$$(conv(P \cup \{v\}) \setminus P) \subset (Q \setminus P).$$

We also know that

$$(Q \setminus P) \subset \left(Q \cap H^+ \left(\hat{u}_j, d_j\right)\right) \subset H^+ \left(\hat{u}_j, d_j\right).$$

From this, we may conclude that

$$\left(conv(P \cup \{v\}) \setminus P\right) \subset H^+ \left(\hat{u}_j, d_j\right).$$

$\square$

Lemma 2.3.5. *Let P and Q be as defined in the previous lemma. Then*

$$Q \cap H^+ \left(\hat{u}_j, d_j\right) = \overline{H^- \left(\hat{u}_{j-1}, d_{j-1}\right)} \cap \overline{H^- \left(\hat{u}_{j+1}, d_{j+1}\right)} \cap H^+ \left(\hat{u}_j, d_j\right).$$

Proof. In the halfplane $H^- \left(\hat{u}_j, d_j\right)$, it is clear that Q contains exactly the same vertices as P. Recall that the vertices of polytope P, and therefore Q, are located only where two consecutive essential supporting lines intersect. It follows that each of the essential supporting lines $H\left(\hat{u}_{j-1}, d_{j-1}\right)$ and $H\left(\hat{u}_{j+1}, d_{j+1}\right)$ contains only one vertex of Q in the halfspace $\overline{H^{-}\left(\hat{u}_j, d_j\right)}$. Now, considering each of the supporting hyperplanes $H\left(\hat{u}_i, d_i\right)$ of Q in $\overline{H^{-}\left(\hat{u}_j, d_j\right)}$, apart from $H\left(\hat{u}_{j+1}, d_{j+1}\right)$ and $H\left(\hat{u}_{j-1}, d_{j-1}\right)$, each contains two vertices of Q. Since P was defined by the intersection of essential supporting halfspaces, Q is a also defined by the intersection of essential supporting halfspaces. As such, each $H\left(\hat{u}_i, d_i\right)$ of Q contains at most two vertices of Q; one vertex if it contains a boundary ray of Q or two vertices if it contains an edge of Q. In consequence, only $H\left(\hat{u}_{j+1}, d_{j+1}\right)$ and $H\left(\hat{u}_{j-1}, d_{j-1}\right)$ contribute to boundary of Q in $H^{+}\left(\hat{u}_j, d_j\right)$. This, in conjunction with the definition of Q, implies that $Q \cap H^{+}\left(\hat{u}_j, d_j\right) = \overline{H^{-}\left(\hat{u}_{j-1}, d_{j-1}\right)} \cap \overline{H^{-}\left(\hat{u}_{j+1}, d_{j+1}\right)} \cap H^{+}\left(\hat{u}_j, d_j\right)$.

□

Lemma 2.3.6. *Let Q be defined as in the previous lemma. If $H\left(\hat{u}_{j-1}, d_{j-1}\right)$ and $H\left(\hat{u}_{j+1}, d_{j+1}\right)$ do not intersect in $H^{+}\left(\hat{u}_j, d_j\right)$, then $H\left(\hat{u}_{j-1}, d_{j-1}\right) \cap H^{+}\left(\hat{u}_j, d_j\right)$ and $H\left(\hat{u}_{j+1}, d_{j+1}\right) \cap H^{+}\left(\hat{u}_j, d_j\right)$ each belong to boundary rays of Q.*

Proof. By Lemma 2.3.5, $H\left(\hat{u}_{j-1}, d_{j-1}\right)$ and $H\left(\hat{u}_{j+1}, d_{j+1}\right)$ are the only two essential support lines that contribute to the boundary of Q in $\overline{H^{-}\left(\hat{u}_j, d_j\right)}$. They each contain exactly one vertex of Q in $H^{-}\left(\hat{u}_j, d_j\right)$. If $H\left(\hat{u}_{j-1}, d_{j-1}\right)$ and $H\left(\hat{u}_{j+1}, d_{j+1}\right)$ do not intersect in $H^{+}\left(\hat{u}_j, d_j\right)$, it follows that they each respectively contain only one vertex of Q and there both contained in $H^{-}\left(\hat{u}_j, d_j\right)$. This implies that, in $H^{+}\left(\hat{u}_j, d_j\right)$, every point belonging to $H\left(\hat{u}_{j-1}, d_{j-1}\right)$ or $H\left(\hat{u}_{j+1}, d_{j+1}\right)$ is a boundary point of Q. From this, we may conclude that $H\left(\hat{u}_{j-1}, d_{j-1}\right) \cap H^{+}\left(\hat{u}_j, d_j\right)$ and $H\left(\hat{u}_{j+1}, d_{j+1}\right) \cap H^{+}\left(\hat{u}_j, d_j\right)$ each belong to boundary rays of Q.

□

Lemma 2.3.7. *Let P and Q be as previously defined with the addition that P is negatively self polar, which implies that $-v_{j-1} = \frac{-\hat{u}_{j-1}}{d_{j-1}}$, $-v_j = \frac{-\hat{u}_j}{d_j}$ and $-v_{j+1} = \frac{-\hat{u}_{j+1}}{d_{j+1}}$*

are three consecutive vertices of P*. Furthermore, the line segments* $\overline{(-v_{j-1}),(-v_j)} = E_h$ *and* $\overline{(-v_j),(-v_{j+1})} = E_{h+1}$ *are two consecutive edges of* P*.*

Assuming that $H(\hat{u}_{j-1}, d_{j-1})$ *and* $H(\hat{u}_{j+1}, d_{j+1})$ *do not intersect in* $H^+(\hat{u}_j, d_j)$*, if the hyperplane* $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ *intersects edges* E_h *and* E_{h+1} *at points* p_1 *and* p_2 *respectively, such that* $conv(\{-v_j, p_1, p_2\})$ *does not contain the origin, then*

$$(conv(P \cup \{v_*\}) \setminus P) \subset H^+(\hat{u}_j, d_j).$$

Proof. Without any loss of generality let $-v_j$ be located on the positive y-axis. Since the $conv(\{-v_j, p_1, p_2\})$ does not contain the origin, $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ must contain the following three points; p_1, p_2 and $p_y = \frac{-\hat{u}_y}{d_y}$ which is a point on the y-axis that is between $-v_j$ and the origin.

For a fixed p_y, in the extreme cases, $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ will contain either $-v_{j-1}$ or $-v_{j+1}$, but never both, because that would imply, by Lemma 2.3.2, that $H(\hat{u}_{j-1}, d_{j-1})$ and $H(\hat{u}_{j+1}, d_{j+1})$ do intersect in $H^+(\hat{u}_j, d_j)$, which contradicts our assumption.

Now, let us define $H(\hat{u}_\alpha, d_\alpha)$ to be the line that contains point p_y and $-v_{j-1}$, and let $H(\hat{u}_\beta, d_\beta)$ be the line that contains points p_y and $-v_{j+1}$.

By construction, we have that $p_y \in H\left(\hat{u}_*, \frac{1}{d_*}\right)$, where $\hat{u}_*$ belongs to the cone with two boundary rays that originate at the origin and have direction unit vectors $\hat{u}_\alpha$ and $\hat{u}_\beta$ respectively. It follows, by Lemma 2.3.3, that v_* is contained in the line segment with endpoints $\frac{-\hat{u}_\alpha}{d_\alpha}$ and $\frac{-\hat{u}_\beta}{d_\beta}$. In addition, by Lemma 2.3.2, $\frac{-\hat{u}_\alpha}{d_\alpha} \in H(\hat{u}_{j-1}, d_{j-1})$, $\frac{-\hat{u}_\beta}{d_\beta} \in H(\hat{u}_{j+1}, d_{j+1})$ and we have that $v_* \in H(\hat{u}_y, d_y) \subset H^+(\hat{u}_j, d_j)$. This implies, by Lemma 2.3.6, that the endpoints $\frac{-\hat{u}_\alpha}{d_\alpha}$ and $\frac{-\hat{u}_\beta}{d_\beta}$ of the line segment containing v_* are boundary points of Q. This result, combined with Q being convex, implies that $v_* \in Q \cap H^+(\hat{u}_j, d_j)$. We may now conclude, by Lemma 2.3.4, that

$(conv(P \cup \{v_*\}) \setminus P) \subset H^+(\hat{u}_j, d_j)$.

$\square$

Lemma 2.3.8. *Let P and Q be as previously defined and, in addition, define q_1 to be the point of intersection of $H(\hat{u}_{j-1}, d_{j-1})$ and $H(\hat{u}_j, d_j)$, and q_2 to be the intersection of $H(\hat{u}_{j+1}, d_{j+1})$ and $H(\hat{u}_j, d_j)$. If $H(\hat{u}_{j-1}, d_{j-1})$ and $H(\hat{u}_{j+1}, d_{j+1})$ intersect in $H^+(\hat{u}_{j-1}, d_{j-1})$ at point v, then line segments $\overline{v, q_1}$ and $\overline{v, q_2}$ belong to Q.*

Proof. Since $H(\hat{u}_{j-1}, d_{j-1})$ and $H(\hat{u}_{j+1}, d_{j+1})$ are consecutive essential hyperplanes of Q, their intersection point v is a vertex of Q. By a similar argument, q_1 and q_2 are vertices of P. On the other hand, P is subset of Q, which implies that q_1 and q_2 are contained in Q. By the convexity of Q, line segments $\overline{v, q_1}$ and $\overline{v, q_2}$ belong to Q.

$\square$

Lemma 2.3.9. *Let P and Q be as previously defined with the addition that P is negatively self polar, which implies that $-v_{j-1} = \frac{-\hat{u}_{j-1}}{d_{j-1}}$, $-v_j = \frac{-\hat{u}_j}{d_j}$ and $-v_{j+1} = \frac{-\hat{u}_{j+1}}{d_{j+1}}$ are three consecutive vertices of P. Furthermore, the line segments $\overline{(-v_{j-1}), (-v_j)} = E_h$ and $\overline{(-v_j), (-v_{j+1})} = E_{h+1}$ are two consecutive edges of P.*

Assuming that $H(\hat{u}_{j-1}, d_{j-1})$ and $H(\hat{u}_{j+1}, d_{j+1})$ intersect in $H^+(\hat{u}_j, d_j)$, if the hyperplane $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ intersects edges E_h and E_{h+1} at points p_1 and p_2 respectively, such that $conv(\{-v_j, p_1, p_2\})$ does not contain the origin, then

$$(conv(P \cup \{v_*\}) \setminus P) \subset H^+(\hat{u}_j, d_j).$$

Proof. Without any loss of generality, let $-v_j$ be located on the positive y-axis. Since the $conv\{-v_j, p_1, p_2\}$ does not contain the origin, $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ must contain the following three points; p_1, p_2 and $p_y = \frac{-\hat{u}_y}{d_y}$ which is a point on the y-axis that is strictly between v_2 and the origin.

For a fixed p_y, in the extreme cases $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ will contain either $-v_{j-1}$ or $-v_{j+1}$. If it contains both, then, by Theorem 2.3.2, v_* is the point of intersection of $H\left(\hat{u}_{j-1}, d_{j-1}\right)$ and $H\left(\hat{u}_{j+1}, d_{j+1}\right)$. Let us call the p_y in this situation $p_{y'} = \frac{-\hat{u}_{y'}}{d_{y'}}$. It is clear that for all $H\left(\hat{u}_*, \frac{1}{d_*}\right)$, p_y must be between v_j and $p_{y'}$, which implies that all $H\left(\hat{u}_y, d_y\right)$ are contained in $\overline{H^+\left(\hat{u}_j, d_j\right)} \cap \overline{H^-\left(\hat{u}_{y'}, d_{y'}\right)}$.

Now, let us define $H\left(\hat{u}_\alpha, d_\alpha\right)$ to be the line that contains point p_y and $-v_{j-1}$, and let $H\left(\hat{u}_\beta, d_\beta\right)$ be the line that contains point p_y and $-v_{j+1}$.

By construction, we have that $p_y \in H\left(\hat{u}_*, \frac{1}{d_*}\right)$ where $\hat{u}_*$ belongs to the cone with two boundary rays that originate at the origin and have direction unit vectors $\hat{u}_\alpha$ and $\hat{u}_\beta$ respectively. It follows, by Lemma 2.3.3, that v_* is contained in the line segment with endpoints $\frac{-\hat{u}_\alpha}{d_\alpha}$ and $\frac{-\hat{u}_\beta}{d_\beta}$. In addition, by Lemma 2.3.2, $\frac{-\hat{u}_\alpha}{d_\alpha} \in H\left(\hat{u}_{j-1}, d_{j-1}\right)$, $\frac{-\hat{u}_\beta}{d_\beta} \in H\left(\hat{u}_{j+1}, d_{j+1}\right)$ and that $v_* \in H\left(\hat{u}_y, d_y\right) \subset \overline{H^+\left(\hat{u}_j, d_j\right)} \cap \overline{H^-\left(\hat{u}_{y'}, d_{y'}\right)}$. This implies, by Lemma 2.3.8, that the endpoints $\frac{-\hat{u}_\alpha}{d_\alpha}$ and $\frac{-\hat{u}_\beta}{d_\beta}$ of the line segment containing v_* are boundary points of Q. This result, combined with Q being convex, implies that $v_* \in Q \cap H^+\left(\hat{u}_j, d_j\right)$. We may now conclude by Lemma 2.3.4 that $(conv(P \cup \{v_*\}) \setminus P) \subset H^+\left(\hat{u}_j, d_j\right)$.

□

Corollary 2.3.4. *The net effect the set operation $conv(P \cup v_*)$ has on P is adding points that are strictly in $H^+\left(\hat{u}_j, d_j\right)$.*

Proof. By combining Lemmas 2.3.7 and 2.3.9, we see that $(conv(P \cup v_*) \setminus P) \subset H^+\left(\hat{u}_j, d_j\right)$ holds true whether or not $H\left(\hat{u}_{j-1}, d_{j-1}\right)$ and $H\left(\hat{u}_{j+1}, d_{j+1}\right)$ intersect in $H^+\left(\hat{u}_j, d_j\right)$. This implies that the sole effect the set operation $conv(P \cup v_*)$ has on P is adding points that are strictly in $H^+\left(\hat{u}_j, d_j\right)$.

□

Lemma 2.3.10. *Let P, q_1, q_2 and v_* be as previously defined, then $conv(P \cup v_*) = P \cup conv(\{q_1, q_2, v_*\})$.*

Proof. By Corollary 2.3.4, $conv(P \cup v_*)$ alters P by adding points strictly in $H^+(\hat{u}_j, d_j)$. The points that are added to P are convex combinations of v_* and points in P. Let $E_j = \overline{q_1, q_2}$ be the edge of P that is contained in $H(\hat{u}_j, d_j)$. Suppose that $x \in conv(P \cup v_*) \setminus P$ and $x \notin conv(\{q_1, q_2, v_*\})$, then x must be a convex combination of some point $s \in P \subset \overline{H^-(\hat{u}_j, d_j)}$ and $v_* \in H^+(\hat{u}_j, d_j)$. It follows that the line segment $\overline{s, v_*} \subset conv(P \cup v_*)$ and that there exist $w \in \overline{s, v_*}$ such that $w \in H(\hat{u}_j, d_j)$. If $w \in E_j$, then $w \in conv(\{q_1, q_2, v_*\})$. Since $E_j = \overline{q_1, q_2}$, we have that $x \in conv(\{q_1, q_2, v_*\})$, which is a contradiction.

If $w \notin E_j$, then this would imply that $conv(P \cup v_*)$ added the point $w \notin H^+(\hat{u}_j, d_j)$ to P, which is a contradiction as well.

In consequence, if $x \in conv(P \cup v_*) \setminus P$, then $x \in conv(\{q_1, q_2, v_*\})$. It follows that $conv(P \cup v_*) \subseteq P \cup conv\{q_1, q_2, v_*\}$. In addition, it is clear that $P \cup conv\{q_1, q_2, v_*\} \subseteq conv(P \cup v_*)$. We may conclude, by double inclusion, that $conv(P \cup v_*) = P \cup conv(\{q_1, q_2, v_*\})$.

□

Lemma 2.3.11. *Let $\{v_*\}^o$ be the polar set of the set $\{v_*\}$, then $-\{v_*\}^o = \overline{H^-\left(\hat{u}_*, \frac{1}{d_*}\right)}$. Here the notation $-A$ stands for $-A = \{x \in \mathbb{R}^2 \mid -x \in A\}$ and uses the vector space structure of the Euclidean plane.*

Proof. By definition, $\{v_*\}^o = \{x \in \mathbb{R}^2 \mid x \cdot v_* \leq 1\} = \{x \in \mathbb{R}^2 \mid x \cdot d_*(-\hat{u}_*) \leq 1\}$. It follows that:

$$\begin{aligned} -\{v_*\}^o &= -\{x \in \mathbb{R}^2 \mid x \cdot v_* \leq 1\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot d_*(\hat{u}_*) \leq 1\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot (\hat{u}_*) \leq \frac{1}{d_*}\} \\ &= \overline{H^-\left(\hat{u}_*, \frac{1}{d_*}\right)}. \end{aligned}$$

$\square$

Theorem 2.3.1. *Let f_x and g_x be mappings whose domain and codomain are the powerset of $\mathbb{R}^2$. Explicitly, for some fixed $x \in \mathbb{R}^2$, define $f_x(S) = conv(S \cup \{x\})$ and $g_x(S) = (S \cap (-\{x\}^o))$.*

If P and v_* are as defined in the previous theorem, then the following equalities hold

$$\begin{aligned} f_{v_*}(g_{v_*}(P)) &= conv((P \cap (-\{v_*\}^o)) \cup \{v_*\}), \\ g_{v_*}(f_{v_*}(P)) &= conv(P \cup \{v_*\}) \cap (-\{v_*\}^o), \\ f_{v_*}(g_{v_*}(P)) &= g_{v_*}(f_{v_*}(P)). \end{aligned}$$

Proof. By Lemma 2.3.10, $conv(P \cup v_*) = P \cup conv(\{q_1, q_2, v_*\})$. Moreover,

$$\begin{aligned} g_{v_*}(f_{v_*}(P)) &= (conv(P \cup \{v_*\}) \cap (-\{v_*\}^o)) \\ &= (P \cup conv(\{q_1, q_2, v_*\})) \cap (-\{v_*\}^o) \\ &= (P \cap (-\{v_*\}^o)) \cup (conv\{q_1, q_2, v_*\} \cap (-\{v_*\}^o) \\ &= (P \cap (-\{v_*\}^o)) \cup conv\{q_1, q_2, v_*\} \\ &= g_{v_*}(P) \cup conv(\{q_1, q_2, v_*\}). \end{aligned}$$

We claim that $g_{v_*}(P) \cup conv(\{q_1, q_2, v_*\}) = conv(g_{v_*}(P) \cup \{v_*\})$. We will prove that this claim is true by showing the double inclusion.

Recall that q_1 and q_2 are vertices of P that are contained in $g_{v_*}(P) = (P \cap (-\{v_*\}^o))$. Also, $g_{v_*}(f_{v_*}(P))$ is the intersection of two convex sets, which implies that $g_{v_*}(P) \cup conv(\{q_1, q_2, v_*\})$ is convex as well.

Suppose $p \in (g_{v_*}(P) \cup conv(\{q_1, q_2, v_*\}))$. If $p \in g_{v_*}(P)$, then it is clear that $p \in conv(g_{v_*}(P) \cup \{v_*\})$. If $p \in conv(\{q_1, q_2, v_*\})$, then p is a convex combination of q_1, q_2 and v$_*$. Since q_1 and q_2 are points in $g_{v_*}(P)$, we may conclude that $p \in$

$conv(g_{v_*}(P) \cup \{v_*\})$. It follows that $(g_{v_*}(P) \cup conv(\{q_1, q_2, v_*\})) \subset (conv(g_{v_*}(P) \cup \{v_*\})$. Suppose $p \in conv(g_{v_*}(P) \cup \{v_*\})$, then p is a convex combination of points in $g_{v_*}(P)$ and v_*. Since $g_{v_*}(P) \cup conv(\{q_1, q_2, v_*\})$ is convex and contains all points in $g_{v_*}(P)$ and v_*, it must contain p. It follows that $(conv(g_{v_*}(P) \cup \{v_*\}) \subset (g_{v_*}(P) \cup conv(\{q_1, q_2, v_*\}))$. By double inclusion, we have shown that $g_{v_*}(P) \cup conv(\{q_1, q_2, v_*\}) = conv(g_{v_*}(P) \cup \{v_*\})$.

Now, we have that:

$$\begin{aligned} g_{v_*}(f_{v_*}(P)) &= g_{v_*}(P) \cup conv(\{q_1, q_2, v_*\}) \\ &= conv(g_{v_*}(P) \cup \{v_*\}) \\ &= conv((P \cap (-\{v_*\}^o)) \cup \{v_*\}) \\ &= f_{v_*}(g_{v_*}(P)). \end{aligned}$$

□

Lemma 2.3.12. *If P and v_* are as previously defined, then*

$$\begin{aligned} conv(P \cup \{v_*\}) &= closure(conv((P \cup \{v_*\}) \cup \{0\})) \\ &= [(P \cup \{v_*\})] \end{aligned}$$

and

$$\begin{aligned} conv(P \cap (-\{v_*\}^o)) \cup \{v_*\}) &= closure(conv((P \cap (-\{v_*\}^o) \cup \{v_*\}) \cup \{0\})) \\ &= [(P \cap (-\{v_*\}^o)) \cup \{v_*\}]. \end{aligned}$$

Proof. Recall that P contains the origin by definition. From this, it follows that $conv(P \cup \{v_*\}) = conv((P \cup \{v_*\}) \cup \{0\})$.

Now, we will show by double inclusion that $conv(P \cup \{v_*\}) = conv(V(P) \cup \{v_*\})$, where $V(P)$ is the set of vertices of P. It is clear that $conv(V(P) \cup \{v_*\}) \subset conv(P \cup \{v_*\})$.

Suppose $x \in conv(P \cup \{v_*\})$, then x is a convex combination of some point $p \in P$ and v_*. Since P is a convex polytope, p is a convex combination of points in $V(P)$. This implies that $p \in conv(V(P) \cup \{v_*\})$. Thus p and v_* are in $conv(V(P) \cup \{v_*\})$, implying that $x \in conv(V(P) \cup \{v_*\})$. We may conclude that $conv(P \cup \{v_*\}) \subset conv(V(P) \cup \{v_*\})$. By double inclusion, we have that $conv(P \cup \{v_*\}) = conv(V(P) \cup \{v_*\})$.

From the above result, we see that $conv(P \cup \{v_*\})$ can be characterized as the convex hull of a finite set of points. It follows that $conv(P \cup \{v_*\})$ is a convex polytope. It is well known that polytopes are closed sets from which we may assert that $conv(P \cup \{v_*\})$ is a closed set. Now, recalling that a closed set is equal to its closure, we my conclude that

$$\begin{aligned} conv(P \cup \{v_*\}) &= closure(conv((P \cup \{v_*\}) \cup \{0\})) \\ &= [(P \cup \{v_*\})]. \end{aligned}$$

To prove the second string of set equality, we first remark that $P \cap (-\{v_*\}^o)$ can be characterized as an intersection of halfplanes and is therefore a polytope. Since P is bounded, $(P \cap (-\{v_*\}^o))$ must be bounded too. We may now conclude that $P \cap (-\{v_*\}^o)$ is a polytope. By repeating the same argument as above with the replacement of P with $(P \cap (-\{v_*\}^o))$, we prove that

$$\begin{aligned} conv(P \cap (-\{v_*\}^o)) \cup \{v_*\}) &= closure(conv((P \cap (-\{v_*\}^o) \cup \{v_*\}) \cup \{0\})) \\ &= [(P \cap (-\{v_*\}^o)) \cup \{v_*\}]. \end{aligned}$$

$\square$

Theorem 2.3.2. *Let P and v_* be as previously defined. Define $T = [P \cup \{v_*\}] \cap (-\{v_*\}^o)$, then T is a self-nolar polytope, that is $T = -T^o$.*

Proof.

$$\begin{aligned}
T &= [P \cup \{v_*\}] \cap (-\{v_*\}^o) \\
&= conv((P \cup \{v_*\}) \cap (-\{v_*\}^o) \\
&= conv(P \cap (-\{v_*\}^o) \cup \{v_*\}) \\
&= [(P \cap (-\{v_*\}^o)) \cup \{v_*\}] \\
&= ((P \cap (-\{v_*\}^o)) \cup \{v_*\})^{oo} \\
&= ((-P^o \cap (-\{v_*\}^o)) \cup \{v_*\})^{oo} \\
&= ((-P \cup (-\{v_*\}))^o \cup \{v_*\})^{oo} \\
&= ((-P \cup (-\{v_*\}))^{oo} \cap \{v_*\}^o)^o \\
&= -((P \cup (\{v_*\}))^{oo} \cap (-\{v_*\}^o))^o \\
&= -([P \cup \{v_*\}] \cap (-\{v_*\}^o))^o \\
&= -T^o
\end{aligned}$$

□

It is Theorem 2.3.2, or more evidently the upcoming Corollary 2.3.7, which can be used as a method to construct a new self-nolar polytope T based on one already known to exist P.

Under the assumptions of Theorem 2.3.2, recall that $-v_{j-1}$, $-v_j$ and $-v_{j+1}$ are three consecutive vertices of self-nolar polytope $P = \{x \in \mathbb{R}^2 \mid x \cdot v_i \leq 1, 1 \leq i \leq k\}$ such that $\overline{(-v_{j-1}), (-v_j)} = E_h \subset H(\hat{u}_h, d_h)$ and $\overline{(-v_j), (-v_{j+1})} = E_{h+1} \subset H(\hat{u}_{h+1}, d_{h+1})$. The segments E_h and E_{h+1} are edges of P, while $H(\hat{u}_h, d_h)$ and $H(\hat{u}_{h+1}, d_{h+1})$ are essential support lines, where $-v_j = H(\hat{u}_h, d_h) \cap H(\hat{u}_{h+1}, d_{h+1})$.

In addition, the boundary line $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ of halfplane $\overline{H^-\left(\hat{u}_*, \frac{1}{d_*}\right)}$ intersects E_h and E_{h+1} at points p_1 and p_2 respectively, such that $conv(\{-v_j, p_1, p_2\})$ does not contain the origin.

Corollary 2.3.5. *Let T be as previously defined, then*

$$T =$$

$$(conv\{(-v_1), ..., (-v_k)\} \setminus conv\{p_1, p_2, (-v_j)\}) \cup conv\{p_1, p_2\} \cup conv\{q_1, q_2, v_*\}.$$

Proof. By Theorem 2.3.1 and Lemma 2.3.12, we have that

$$\begin{aligned} T &= [P \cup \{v_*\}] \cap (-\{v_*\}^o) \\ &= conv((P \cup \{v_*\}) \cap (-\{v_*\}^o) \\ &= (P \cap (-\{v_*\}^o)) \cup conv\{q_1, q_2, v_*\}. \end{aligned}$$

We also know, by Lemma 2.3.11, that $-\{v_*\}^o = \overline{H^-\left(\hat{u}_*, \frac{1}{d_*}\right)}$ and, by construction, we have $P \cap \overline{H^-\left(\hat{u}_*, \frac{1}{d_*}\right)} = (P \setminus conv(\{p_1, p_2, (-v_j)\}) \cup conv(\{p_1, p_2\})$. In addition, $P = conv(\{(-v_1), ..., (-v_k)\})$, since P is a polytope.
It follows that

$$T =$$

$$(conv\{(-v_1), ..., (-v_k)\} \setminus conv\{p_1, p_2, (-v_j)\}) \cup conv\{p_1, p_2\} \cup conv\{q_1, q_2, v_*\}.$$

$\square$

Corollary 2.3.6. *Consistent with previous notation, we have*

$$(conv\{(-v_1), ..., (-v_k)\} \setminus conv\{p_1, p_2, (-v_j)\}) \cup conv\{p_1, p_2\}$$
$$= conv\{(-v_1), ..., (-v_{j-1}), (-v_{j+1}), ..., (-v_k), p_1, p_2, \}.$$

Proof. By Corollary 2.3.5,

$$(conv(\{(-v_1), ..., (-v_k)\} \setminus conv\{p_1, p_2, (-v_j)\}) \cup conv\{p_1, p_2\}$$
$$= (P \cap (-\{v_*\}^o)).$$

Since $(P \cap (-\{v_*\}^o))$ is the intersection of two convex sets, we have that

$$(conv\{(-v_1), ..., (-v_k)\} \setminus conv\{p_1, p_2, (-v_j)\}) \cup conv\{p_1, p_2\}$$

is convex.

Suppose that:

$$x \in (conv\{(-v_1), ..., (-v_k)\} \setminus conv\{p_1, p_2, (-v_j)\}) \cup conv\{p_1, p_2\}.$$

In general, x must be a convex combination of the form

$$x = a_1(-v_1) + ... + a_j(-v_j) + ... + a_k(-v_k) + b_1 p_1 + b_2 p_2.$$

Clearly, $x \neq -v_j$. It follows, by convexity of

$$(conv\{(-v_1), ..., (-v_k)\} \setminus conv\{p_1, p_2, (-v_j)\}) \cup conv\{p_1, p_2\},$$

that $a_j = 0$ and that x is restricted to a convex combination of the form

$$x = a_1(-v_1) + ... + a_{j-1}(-v_{j-1}) + a_{j+1}(-v_{j+1})... + a_k(-v_k) + b_1p_1 + b_2p_2,$$

which are precisely all the elements of

$$conv\{(-v_1), ..., (-v_{j-1}), (-v_{j+1}), ..., (-v_k), p_1, p_2\}.$$

We may now conclude that

$$(conv\{(-v_1), ..., (-v_k)\} \setminus conv\{p_1, p_2, (-v_j)\}) \cup conv\{p_1, p_2\}$$
$$= conv\{(-v_1), ..., (-v_{j-1}), (-v_{j+1}), ..., (-v_k), p_1, p_2\}.$$

□

Corollary 2.3.7. *Assuming previous notation still valid, we have*

$$T = conv(\{(-v_1), ..., (-v_{j-1}), (-v_{j+1}), ..., (-v_k), p_1, p_2, v_*\}).$$

Proof. By Corollaries 2.3.5 and 2.3.6,

$$T =$$
$$conv(\{(-v_1), ..., (-v_{j-1}), (-v_{j+1}), ..., (-v_k), p_1, p_2\}) \cup conv\{q_1, q_2, v_*\}.$$

T is convex. From this fact, it follows that if $x \in T$, then x must be a convex combination of the form

$$x = a_1(-v_1)+...+a_{j-1}(-v_{j-1})+a_{j+1}(-v_{j+1})...+a_k(-v_k)+b_1p_1+b_2p_2+b_3v_*+b_4q_1+b_5q_2.$$

We know from Lemma 2.3.9 that q_1 and q_2 are vertices of self-nolar polytope P, which are contained in its essential support line $H(\hat{u}_j, d_j)$. By definition of $(-v_j)$, $(-v_j) \notin H(\hat{u}_j, d_j)$, implying that $(-v_j) \neq q_1$ and $(-v_j) \neq q_2$. Thus, $q_1, q_2 \in \{(-v_1), ..., (-v_{j-1}), (-v_{j+1}), ..., (-v_k)\}$. Now, by combining like terms in the convex combination expression of x, we can simplify the expression to

$$x = a_1(-v_1) + ... + a_{j-1}(-v_{j-1}) + a_{j+1}(-v_{j+1}) ... + a_k(-v_k) + b_1 p_1 + b_2 p_2 + b_3 v_*.$$

These convex combinations forming x are precisely the elements of

$$conv\{(-v_1), ..., (-v_{j-1}), (-v_{j+1}), ..., (-v_k), p_1, p_2, v_*\}.$$

We may now conclude that

$$T = conv\{(-v_1), ..., (-v_{j-1}), (-v_{j+1}), ..., (-v_k), p_1, p_2, v_*\}.$$

□

2.3.1 Altering the Number of Vertices While Maintaining Self-nolarity

Interestingly, the number of vertices T will have is dependent on the number of vertices P has and on where $H(\hat{u}_*, \frac{1}{d_*})$ intersects the boundary of P. We will now state and prove a few propositions regarding the number of vertices of T .

Lemma 2.3.13. *Suppose P has $k \geq 5$ vertices. If $p_1 = (-v_{j-1})$ and $p_2 = (-v_{j+1})$, then T will have $k - 2$ vertices.*

Proof. Assume that $p_1 = (-v_{j-1})$ and $p_2 = (-v_{j+1})$.

Clearly, the set vertices of P contained in $H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$ is $\{-v_i \mid i = 1, 2, ..., k\} \setminus \{(-v_{j-1}), (-v_j), (-v_{j+1}), q_1, q_2\}$. By Theorem 2.3.1, it follows that

$$T \cap H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right) = P \cap H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right).$$

Thus the set of vertices of T contained in $H^-(\hat{u}_j, d_j) \cap H^-(\hat{u}_*, \frac{1}{d_*})$ is also $\{-v_i \mid i = 1, 2, ..., k\} \setminus \{(-v_{j-1}), (-v_j), (-v_{j+1}), q_1, q_2\}$. We may conclude that there are exactly $k-5$ vertices of T belonging to $H^-(\hat{u}_j, d_j) \cap H^-(\hat{u}_*, \frac{1}{d_*})$.

By construction, p_1 and p_2 are vertices of T belonging to $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ and, in addition, T does not contain any points in $H^+\left(\hat{u}_*, \frac{1}{d_*}\right)$. This implies that the only vertices of T in $\overline{H^+\left(\hat{u}_*, \frac{1}{d_*}\right)}$ are $p_1 = (-v_{j-1})$ and $p_2 = (-v_{j+1})$. We may conclude that there are exactly 2 vertices of T belonging to $\overline{H^+\left(\hat{u}_*, \frac{1}{d_*}\right)}$. Now, the only points belonging to T in $\overline{H^+(\hat{u}_j, d_j)}$ are the ones belonging to $conv(v_*, q_1, q_2)$, which implies that v_*, q_1 and q_2 are the only potential vertices of T in $\overline{H^+(\hat{u}_j, d_j)}$. By construction, we know that v_* is a vertex of T and that q_1 and q_2 are vertices of P belonging to $H(\hat{u}_j, d_j)$. By Theorem 2.2.1 and Corollary 2.3.2, $H(\hat{u}_h, d_h)$ and $H(\hat{u}_{h+1}, d_{h+1})$ are the essential support lines of P associated with q_1 and q_2, respectively. But since $p_1 = (-v_{j-1})$ and $p_2 = (-v_{j+1})$, $H(\hat{u}_h, d_h)$ and $H(\hat{u}_{h+1}, d_{h+1})$ do not contain edges of T, so they cannot be essential hyperplanes of T. It follows from Theorem 2.2.1 that q_1 and q_2 are not vertices of T. We may conclude that there is exactly 1 vertex of T belonging to $\overline{H^+(\hat{u}_j, d_j)}$. Note that, since the number of vertices of T is $k \geq 5$, the vertices in these three sets are unique and that the union of the three sets is $\mathbb{R}^2$. Thus, by summing the number vertices of T in these three set, we may conclude that T has exactly $(k-5) + 2 + 1 = k - 2$ vertices.

□

Lemma 2.3.14. *Suppose P has $k \geq 5$ vertices. If $p_1 = (-v_{j-1})$ and $p_2 \neq (-v_{j+1})$ or if $p_1 \neq (-v_{j-1})$ and $p_2 = (-v_{j+1})$, then T will have k vertices as well.*

Proof. Assume that $p_1 = (-v_{j-1})$ and $p_2 \neq (-v_{j+1})$. Clearly, the set vertices of P contained $H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$ is $\{-v_i \mid i = 1, 2, ..., k\} \setminus \{(-v_j), (-v_{j+1}), q_1, q_2\}$. By Theorem 2.3.1, it follows that

$$T \cap H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right) = P \cap H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right).$$

Thus, the set of vertices of T contained in $H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$ is also $\{-v_i \mid i = 1, 2, ..., k\} \setminus \{(-v_j), (-v_{j+1}), q_1, q_2\}$. We may conclude that there are exactly $k-4$ vertices of T belonging to $H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$.

By construction, $p_1 = (-v_{j-1})$ and p_2 are vertices of T belonging to $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ and, in addition, T does not contain any points in $H^+\left(\hat{u}_*, \frac{1}{d_*}\right)$. This implies that the only vertices of T in $\overline{H^+\left(\hat{u}_*, \frac{1}{d_*}\right)}$ are $p_1 = (-v_{j-1})$ and p_2. We may conclude that there are exactly 2 vertices of T belonging to $\overline{H^+\left(\hat{u}_*, \frac{1}{d_*}\right)}$. Now, the only points belonging to T in $\overline{H^+(\hat{u}_j, d_j)}$ are the ones belonging to $conv(v_*, q_1, q_2)$, which implies that v_*, q_1 and q_2 are the only potential vertices of T in $\overline{H^-(\hat{u}_j, d_j)}$. By construction, we know that v_* is a vertex of T and that q_1 and q_2 are vertices of P belonging to $H(\hat{u}_j, d_j)$. By Theorem 2.2.1 and Corollary 2.3.2, $H(\hat{u}_h, d_h)$ and $H(\hat{u}_{h+1}, d_{h+1})$ are the essential support lines of P associated with q_1 and q_2 respectively. But, since $p_1 = (-v_{j-1})$, $H(\hat{u}_h, d_h)$ does not contain an edge of T and therefore cannot be an essential support line of T. However, $\overline{p_2, -(v_{j+1})}$ is an edge of T contained in $H(\hat{u}_{h+1}, d_{h+1})$, implying that $H(\hat{u}_{h+1}, d_{h+1})$ is an essential support line of T. It follows from Theorem 2.2.1 that q_1 is not a vertex of T but q_2 is a vertex of T. We may conclude that there are exactly 2 vertices of T belonging to $\overline{H^+(\hat{u}_j, d_j)}$.

Note that, since the number of vertices of T is $k \geq 5$, the vertices in these three sets are unique and that the union of the three sets is $\mathbb{R}^2$. Thus, by summing the number vertices of T in these three set, we may conclude that T has exactly $(k-4)+2+2 = k$ vertices.

Assume that $p_1 \neq (-v_{j-1})$ and $p_2 = (-v_{j+1})$. By interchanging the role between p_1 and p_2 in the previous argument, we may once again conclude the T has exactly k vertices.

$\square$

Lemma 2.3.15. *Suppose P has $k \geq 5$ vertices. If $p_1 \neq (-v_{j-1})$ and $p_2 \neq (-v_{j+1})$, then T will have $k+2$ vertices.*

Proof. Assume that $p_1 \neq (-v_{j-1})$ and $p_2 \neq (-v_{j+1})$. Clearly, the set vertices of P contained $H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$ is $\{-v_i \mid i = 1, 2, ..., k\} \setminus \{(-v_j), q_1, q_2\}$ By theorem 2.3.1, it follows that

$$T \cap H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right) = P \cap H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right).$$

Thus the set of vertices of T contained in $H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$ is also $\{-v_i \mid i = 1, 2, ..., k\} \setminus \{(-v_j), q_1, q_2\}$. We may conclude that there are exactly $k-3$ vertices of T belonging to $H^-(\hat{u}_j, d_j) \cap H^-\left(\hat{u}_*, \frac{1}{d_*}\right)$.

By construction, p_1 and p_2 are vertices of T belonging to $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ and, in addition, T does not contain any points in $H^+\left(\hat{u}_*, \frac{1}{d_*}\right)$. This implies that the only vertices of T in $\overline{H^+\left(\hat{u}_*, \frac{1}{d_*}\right)}$ are p_1 and p_2. We may conclude that there are exactly 2 vertices of T belonging to $\overline{H^+(\hat{u}_*, \frac{1}{d_*})}$. Now, the only points belonging to T in $\overline{H^+(\hat{u}_j, d_j)}$ are the ones belonging to $conv(v_*, q_1, q_2)$, which implies that v_*, q_1 and q_2 are the only potential vertices of T in $\overline{H^-(\hat{u}_j, d_j)}$. By construction, we know that v_* is a vertex of T and that q_1 and q_2 are vertices of P belonging to $H(\hat{u}_j, d_j)$. By Theorem 2.2.1 and corollary 2.3.2, $H(\hat{u}_h, d_h)$ and $H(\hat{u}_{h+1}, d_{h+1})$ are the essential support lines of P associated with q_1 and q_2 respectively.The line segment $\overline{p_1, -(v_{i-1})}$ is an edge of T contained in $H(\hat{u}_h, d_h)$ and $\overline{p_2, -(v_{i+1})}$ is an edge of T contained in $H(\hat{u}_{h+1}, d_{h+1})$, implying that $H(\hat{u}_h, d_h)$ and $H(\hat{u}_{h+1}, d_{h+1})$ are both an essential support lines of T.

It follows by Theorem 2.2.1 that q_1 and q_2 are vertices of T. We may conclude that there are exactly 3 vertices of T belonging to $\overline{H^+(\hat{u}_j, d_j)}$. Note that, since the number of vertices of T is $k \geq 5$, the vertices in these three sets are unique and that the union of the three sets is $\mathbb{R}^2$. Thus, by summing the number vertices of T in these three set, we may conclude that T has $(k-3)+2+3 = k+2$ vertices.

□

2.3.2 Parity Restriction on the Vertices

Another question that can be asked about self-nolar polytopes is if there is a parity restriction on the vertices. We begin to answer this question by first proving that a self-nolar polytope cannot have exactly 4 vertices.

Lemma 2.3.16. *There does not exist a self-nolar polytope with exactly* 4 *vertices.*

Proof. Suppose

$$\begin{aligned} P &= \bigcap_{i \in \{1,2,3,4\}} \overline{H^-(\hat{u}_i, d_i)} \\ &= \bigcap_{i \in \{1,2,3,4\}} \left\{ \overline{H^-(v_i)} \mid v_i = \frac{\hat{u}_i}{d_i} \right\} \\ &= \{x \in \mathbb{R}^2 \mid x \cdot v_i \leq 1, 1 \leq i \leq 4\} \end{aligned}$$

is a self-nolar polytope. By Theorem 2.2.1, $P = conv(\{-v_i \mid 1 \leq i \leq 4\})$, where each v_i is a vertex of P. In other words, P has exactly 4 vertices.

Without any loss of generality, let $-v_1$ be located on the positive y-axis. This implies, by Theorem 2.2.1 and the fact that each essential support line contains exactly one edge, that there exists an edge E_1 of P that is below the x-axis and parallel to it.

Case one: Assume P has all 4 vertices on or above the x-axis.The edge E_1 of P is located below the x-axis and contains 2 additional vertices. This implies that P

must have in fact 6 vertices, which is a contradiction.

Case two: Assume P has exactly 3 vertices on or above the x-axis. The edge E_1 of P is located below the x-axis and contains 2 additional vertices. This implies that P must have 5 vertices, which is again a contradiction with our assumption.

Case three: Assume P has exactly 2 vertices on or above the x-axis. The edge E_1 of P is located below the x-axis and contains 2 additional vertices. This implies that there exists another vertex of P, besides $-v_1$, located in the first or second quadrant. Without any loss of generality, let it be in the second quadrant. let us now locate and label the vertices of P using our predefined counterclockwise method. We first have $-v_1$ on the positive y-axis, $-v_2$ in the second quadrant, $-v_3$ in the third quadrant and lastly $-v_4$ in the fourth quadrant. Here, we label and locate the associated 4 edges of P, again using our predefined counterclockwise method:

$$\begin{aligned}
E_1 &= \overline{(-v_3),(-v_4)} \\
E_2 &= \overline{(-v_4),(-v_1)} \\
E_3 &= \overline{(-v_1),(-v_2)} \\
E_4 &= \overline{(-v_2),(-v_3).}
\end{aligned}$$

Now, the slope m of $E_2 = \overline{(-v_4),(-v_1)}$ must be negative, otherwise it would imply the P contains the origin in its boundary, which is a contradiction. This implies that the essential support line $H(v_2)$, containing E_2, has $y_{int} > 0$ and $m < 0$. By Lemma 2.3.1, $-v_2$ is strictly in Q_3. This contradicts $-v_2$ being in Q_2.

Case four: Assume P has exactly 1 vertex on or above the x-axis. The edge E_1

of P is located below the x-axis and contains 2 additional vertices. This implies that there exists another vertex of P in the third or fourth quadrant. Without loss of generality, let it be in the fourth quadrant. Let us now locate and label the vertices of P using our predefined counterclockwise method. We first have $-v_1$ on the positive y-axis, $-v_2$ in the third quadrant, $-v_3$ in the fourth quadrant and, lastly, $-v_4$ also in the fourth quadrant. Here, we label and locate the associated 4 edges of P, again using our predefined counterclockwise method:

$$\begin{aligned}
E_1 &= \overline{(-v_2),(-v_3)} \\
E_2 &= \overline{(-v_3),(-v_4)} \\
E_3 &= \overline{(-v_4),(-v_1)} \\
E_4 &= \overline{(-v_1),(-v_2)}.
\end{aligned}$$

Now, the slope m of $E_3 = \overline{(-v_4),(-v_1)}$ must be negative, otherwise it would imply the P contains the origin in its boundary, which is a contradiction. So, this implies the essential support line $H(v_3)$,containing E_3, has $y_{int} > 0$ and $m < 0$. By Lemma 2.3.1, $-v_3$ is strictly in Q_3. This contradicts $-v_3$ being in Q_4.
This concludes the proof of the lemma.

□

Lemma 2.3.17. *Every self-nolar polytope* $P = \{x \in \mathbb{R}^2 \mid x \cdot v_i \leq 1, 1 \leq i \leq k\}$ *with* $k \geq 5$ *has three consecutive vertices* $-v_{i-1}$,$-v_i$ *and* $-v_{i+1}$ *such that* $(0,0) \notin conv(\{-v_{i-1}, -v_i, -v_{i+1}\})$.

Proof. If $(0,0) \notin conv(\{-v_{i-1}, -v_i, -v_{i+1}\})$ for all $i \in \{1, 2, ..., k\}$, then the lemma is trivially true.

Assume that there exists, at least, one set of three consecutive vertices $\{-v_{i+1}, -v_{i+2}, -v_{i+3}\}$ such that $(0,0) \in conv\{-v_{i+1}, -v_{i+2}, -v_{i+3}\}$. Since $k \geq 5$, we may assume that $\{-v_{i-1}, -v_i, -v_{i+1}, -v_{i+2}, -v_{i+3}\}$ is the set of unique vertices of P where no two are the same. It follows, by convexity of P, that

$$conv(\{-v_{i-1}, -v_i, -v_{i+1}\}) \cap conv(\{-v_{i+1}, -v_{i+2}, -v_{i+3}\}) = \{-v_{i+1}\}.$$

Since $-v_{i+1}$ is a vertex of P, we have that $-v_{i+1} \neq (0,0)$. Recalling that $(0,0) \in conv\{-v_{i+1}, -v_{i+2}, -v_{i+3}\}$, we may conclude that $(0,0) \notin conv\{-v_{i-1}, -v_i, -v_{i+1}\}$.

□

The following result has been also proved in [2], Theorem 4.4, by different methods.

Theorem 2.3.3. *If $P = conv\{v_1, v_2, ..., v_k\}$ is a self-nolar polytope, then k is an odd integer or, in other words, P has an odd number of vertices.*

Proof. We have previously confirmed the existence of self-nolar polytopes by constructing a self-nolar pentagon. By Lemma 2.3.16, we know the $k \neq 4$. So, let $k = 2n$ with $3 \leq n$. Now, combining the results of Theorem 2.3.2, Lemma 2.3.17 and Lemma 2.3.13, they imply that we can construct a new self-nolar polytope T_1 that has two less vertices than P. If we iterate this process $n-2$ times, where the i^{th} iteration alters T_{i-1} into T_i, we will obtain a new self-nolar polytope T_{n-2} that has $2(n-2) = 2n-4$ less vertices than P. This implies that T_{n-2} is a self-nolar polytope that has $2n-(2n-4) = 4$ vertices, which contradicts Lemma 2.3.16. From these facts, we conclude that each self-nolar polytope must have an odd number of vertices.

□

Chapter 3

Finer Properties of Self-Nolar Planar Sets

3.1 Introduction

In this chapter we prove some noteworthy properties of self-nolar sets such as the fact that all self-nolar sets are convex, that the family of all self-nolar sets is uncountable, and that the set of all self-nolar planar polytopes is dense in the set of all self-nolar planar sets. We also give proofs concerning the length of the boundary of a self-nolar set with smooth boundary, the center of mass of self-nolar polytopes and the Mahler product. Moreover, we prove an original theorem, theorem 3.2.2, that can be used as a practical method to construct self-nolar polytopes.

3.2 Topological Properties

3.2.1 Cardinality

Theorem 3.2.1. *The set of all self-nolar sets is uncountable.*

Proof. Consider once again the pentagon P (figure 2.1) defined by

$$
\begin{aligned}
P &= \bigcap_{i\in\{1,2\ldots,5\}} \overline{H^-\left(\hat{u}_i, d_i\right)} \\
&= \bigcap_{i\in\{1,2\ldots,5\}} \left\{ \overline{H^-(v_i)} \mid v_i = \frac{\hat{u}_i}{d_i} \right\} \\
&= \{x \in \mathbb{R}^2 \mid x \cdot v_i \leq 1, 1 \leq i \leq 5\},
\end{aligned}
$$

where

$$
\begin{aligned}
v_1 &= (-1,1) = \frac{\hat{u}_1}{d_1} = \frac{(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}{\frac{1}{\sqrt{2}}} \\
v_2 &= (1,1) = \frac{\hat{u}_2}{d_2} = \frac{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}{\frac{1}{\sqrt{2}}} \\
v_3 &= (1,0) = \frac{\hat{u}_3}{d_3} = \frac{(1,0)}{1} \\
v_4 &= (0,-1) = \frac{\hat{u}_4}{d_4} = \frac{(0,-1)}{1} \\
v_5 &= (-1,0) = \frac{\hat{u}_5}{d_5} = \frac{(-1,0)}{1}.
\end{aligned}
$$

We previously determined that P is self-nolar and that $V(P) = \{-v_i \mid 1 \leq i \leq 5\}$. To determine the vertices of P, we found the intersection point of each pair of consecutive support lines:

$$
\begin{aligned}
H(v_1) \cap H(v_2) &= (0,1) = -v_4 \\
H(v_2) \cap H(v_3) &= (1,0) = -v_5 \\
H(v_3) \cap H(v_4) &= (1,-1) = -v_1 \\
H(v_4) \cap H(v_5) &= (-1,-1) = -v_2 \\
H(v_5) \cap H(v_1) &= (-1,0) = -v_3.
\end{aligned}
$$

Let us now intersect P with $\overline{H^-\left(\hat{u}_*, \frac{1}{d_*}\right)}$ such that $\hat{u}_* = (0,1)$ and $\frac{1}{4} \leq \frac{1}{d_*} \leq \frac{1}{2}$.

It follows that $v_* = d_*(-\hat{u}_*) = (0, -d_*)$, where $2 \leq d_* \leq 4$. Since $-v_3, -v_4$ and $-v_5$ are three consecutive vertices of P, $\overline{-v_3, -v_4}$ and $\overline{-v_4, -v_5}$ are edges of P. By construction, $H\left(\hat{u}_*, \frac{1}{d_*}\right)$ intersects edge $\overline{-v_3, -v_4}$ at point $p_* = (1 - \frac{1}{d_*}, \frac{1}{d_*})$ and edge $\overline{-v_4, -v_5}$ at point $q_* = (\frac{1}{d_*} - 1, \frac{1}{d_*})$. For a fixed d_*, by Corollary 2.3.7 and Lemma 2.3.15, we may assert that $T_* = conv\{-v_1, -v_2, -v_3, -v_5, p_*, q_*, v_*\}$ is a self-nolar polygon with seven vertices. Recall that the cardinality of a nonempty interval of the real line is uncountable. From this fact, it follows that the set of choices for d_* is uncountable. Now, let us choose two distinct real numbers d_* and d_{**}, such that $2 \leq d_*, d_{**} \leq 4$, and consider the two associated self-nolar polygons, $T_* = conv\{-v_1, -v_2, -v_3, -v_5, p_*, q_*, v_*\}$ and $T_{**} = conv\{-v_1, -v_2, -v_3, -v_5, p_{**}, q_{**}, v_{**}\}$ respectively. Note that $\{-v_3, -v_5, p_*, q_*, p_{**}, q_{**}\}$ is a subset of the unit disc and that $||-v_1|| = ||-v_2|| = \sqrt{2}$. In addition, we have that $||v_*|| = d_*$ and $||v_{**}|| = d_{**}$. This implies that if $x \in \{-v_1, -v_2, -v_3, -v_5, p_*, q_*, p_{**}, q_{**}\}$, then $||x|| \leq \sqrt{2}$ and if $x \in \{v_*, v_{**}\}$, then $2 \leq ||x|| \leq 4$. Suppose that T_* and T_{**} are congruent up to some rigid rotation. It follows that for some $x \in \{-v_1, -v_2, -v_3, -v_5, p_*, q_*, v_*\}$, we have that $2 \leq ||x|| = ||v_{**}|| \leq 4$. Suppose $x \in \{-v_1, -v_2, -v_3, -v_5, p_*, q_*\}$, this leads to the contradiction that $||x|| \leq \sqrt{2}$ and $2 \leq ||x|| \leq 4$. Suppose $||x|| = ||v_*|| = d_*$, since $||v_{**}|| = d_{**}$, this implies that $d_* = d_{**}$ which contradicts d_* and d_{**} being two distinct real numbers. Thus, we may conclude that T_* and T_{**} are not congruent up to some rigid rotation. Finally, given that the set of all choices of d_* is uncountable and that no two distinct choices are associated (through the application of Corollary 2.3.7) with congruent (up to some rigid rotation) self-nolar polytopes, we may conclude that the set of self-nolar polytopes constructed by all possible choices of d_* is uncountable. This implies that the set of all self-nolar sets is uncountable.

$\square$

3.2.2 Density

In this section we ultimately prove that that set of all self-nolar polytopes is dense in the set of all self-nolar sets.

Lemma 3.2.1. *Let C be a self-nolar set and let B be the closed unit disc.*

a. If C is a subset of the unit disc B, then C is the unit disc B.

b. If the unit disc B is a subset of C, then C is the unit disc B.

Proof. Suppose $C \subseteq B$. Since C is self-nolar, we have that

$$B = B^o \subseteq C^o = -C.$$

This implies that

$$B = -B \subseteq -(-C) = C.$$

By double inclusion, we have proved claim a.

Suppose $C \subseteq B$. Since C is self-nolar, we have that

$$B = B^o \subseteq C^o = -C.$$

This implies that

$$B = -B \subseteq -(-C) = C.$$

By double inclusion, we have proved claim b.

□

Lemma 3.2.2. *If C is as self-nolar set with boundary $Bd(C)$, then there exist two boundary points $n, m \in Bd(C)$ such that $m = -\frac{n}{||n||^2}$, with $n \cdot m = -1$.*

Proof. Choose $n \in Bd(C)$ such that $||n||$ is the maximal Euclidean distance from the origin to $Bd(C)$. Let $||n||B$ be the disc centered at the origin with radius $||n||$. Clearly we have the $C \subseteq ||n||B$. This implies that

$$\begin{aligned}
\frac{1}{||n||}B &= (||n||B)^o \subseteq C^o \\
-\frac{1}{||n||}B &= -(||n||B)^o \subseteq -C^o = C.
\end{aligned}$$

Since

$$\frac{1}{||n||}B = -\frac{1}{||n||}B,$$

we then have

$$\frac{1}{||n||}B \subseteq C.$$

Now, define $m = -\frac{n}{||n||^2}$, so that $||m|| = \frac{1}{||n||}$. Since $\frac{1}{||n||}B \subseteq C$, we have $m \in C$. Suppose $m \notin Bd(C)$, then there exist $x = l(-\frac{n}{||n||}) \in Bd(C)$ such that $||x|| = l > \frac{1}{||n||} = ||m||$. It follows, by self-nolarity of C, that $-x = l(\frac{n}{||n||}) \in Bd(C^o)$ with $||-x|| = l > \frac{1}{||n||} = ||m||$. This implies that

$$-x \cdot n = ||-x||\,||n||\cos(0) = l||n|| > ||m||\,||n|| = \frac{1}{||n||}\,||n|| = 1,$$

which contradicts $-x \in Bd(C^o)$.

As such, we may now conclude that $m, n \in Bd(C)$ with $m = -\frac{n}{||n||^2}$ and

$$n \cdot m = ||n||\,||m||\cos(\pi) = ||n||\,\frac{1}{||n||}\,(-1) = 1(-1) = -1.$$

$\square$

Corollary 3.2.1. *In addition to the above notation, let* $m = d\hat{u}$ *and* $n = -\frac{\hat{u}}{d}$ *for* *some $d \in \mathbb{R}^+$ and some unit vector $\hat{u}$. Then*

$$C = -C^o \subset \overline{H^-(-\hat{u}, \frac{1}{d})} \cap \overline{H^-(\hat{u}, d)}.$$

Proof. By definition, we have $C^o = \{x \mid x \cdot c \leq 1, \forall c \in C\}$ implying that

$$\begin{aligned} -C^o \subset \{x \mid x \cdot (-m) \leq 1\} &= \{x \mid x \cdot d(-\hat{u}) \leq 1\} \\ &= \{x \mid x \cdot (-\hat{u}) \leq \frac{1}{d}\} \\ &= \overline{H^-(-\hat{u}, \frac{1}{d})}. \end{aligned}$$

Similarly,

$$\begin{aligned} -C^o \subset \{x \mid x \cdot -n \leq 1\} &= \{x \mid x \cdot (\frac{\hat{u}}{d}) \leq 1\} \\ &= \{x \mid x \cdot (\hat{u}) \leq d\} \\ &= \overline{H^-(\hat{u}, d)}. \end{aligned}$$

It follows that

$$-C^o \subset \overline{H^-(-\hat{u}, \frac{1}{d})} \cap \overline{H^-(\hat{u}, d)}.$$

By assumption, $C = -C^o$, which allows us to conclude that

$$C = -C^o \subset \overline{H^-(-\hat{u}, \frac{1}{d})} \cap \overline{H^-(\hat{u}, d)}.$$

. □

Lemma 3.2.3. *Let C is a self-nolar set, $\theta \in [0, 2\pi]$ such that $\hat{u} = \hat{u}(\theta) = (\cos(\theta), \sin(\theta))$ is a unit vector in $\mathbb{R}^2$ and $d_1, d_2 \in \mathbb{R}^+$. If $d_1(\hat{u}), d_2(-\hat{u}) \in Bd(C)$, where $d_1(\hat{u})$ and*

$d_2(-\hat{u})$ *are scalar multiplications, then* $d_1 d_2 \leq 1$.

Proof. Let $d_1(\hat{u}), d_2(-\hat{u}) \in Bd(C)$. Since C is a self-nolar set, we have that $d_1(-\hat{u}), d_2(\hat{u}) \in Bd(C^o)$. By definition, $C^o = \{x \mid x \cdot c \leq 1, \forall c \in C\}$ implying that

$$d_1(\hat{u}) \cdot d_2(\hat{u}) = |d_1||d_2|cos(0) = d_1 d_2 \leq 1.$$

$\square$

Definition 3.2.1. *Let* $\overline{U} = \{(x,y) \mid y \geq 0\}$, $\overline{L} = \{(x,y) \mid y \leq 0\}$, $\hat{j} = (0,1)$, $N \in \mathbb{R}^+$, $k \in \mathbb{N}$ *such that* $k \geq 3$ *and define*

$$Q = \lim_{N\to\infty} \bigcap_{i\in\{1,2\ldots,k\}} \left(\overline{H^-(\hat{u}_i, d_i)} \cap \overline{H^-(-\hat{j}, N)}\right)$$
$$= \lim_{N\to\infty} \left(conv\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\} \cup \right.$$
$$\left. \left(\overline{H^-(\hat{j}, \frac{1}{N})} \cap \overline{H^-\left(\hat{u}_{1*}, \frac{1}{d_{1*}}\right)} \cap \overline{H^-\left(\hat{u}_{(k-1)*}, \frac{1}{d_{(k-1)*}}\right)} \cap \overline{H^-(-\hat{j}, N)}\right)\right)$$

to be a convex polyhedron with the assumptions that:

a. $d_1\hat{u}_1 = \frac{\hat{u}_{1*}}{d_{1*}} = (-\frac{1}{d}, 0)$ *and* $d_k\hat{u}_k = \frac{\hat{u}_{(k-1)*}}{d_{(k-1)*}} = (d, 0)$,

b. $\frac{\hat{u}_{i*}}{d_{i*}} \in \overline{H^-\left(\hat{u}_{1*}, \frac{1}{d_{1*}}\right)} \cap \overline{H^-\left(\hat{u}_{(k-1)*}, \frac{1}{d_{(k-1)*}}\right)} \cap \overline{U}$, *for* $1 \leq i \leq (k-1)$,

c. $\frac{\hat{u}_{i*}}{d_{i*}} = H(\hat{u}_i, d_i) \cap H(\hat{u}_{i+1}, d_{i+1})$, *for* $1 \leq i \leq (k-1)$,

d. $H(\hat{u}_i, d_i)$ *has a positive y-intercept, for* $2 \leq i \leq (k-1)$.

Equivalently, as $N \to \infty$,

$$Q = \bigcap_{i\in\{1,2\ldots,k\}} \overline{H^-(\hat{u}_i, d_i)}$$
$$= conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\}) \cup \left(\overline{L} \cap \overline{H^-\left(\hat{u}_{1*}, \frac{1}{d_{1*}}\right)} \cap \overline{H^-\left(\hat{u}_{(k-1)*}, \frac{1}{d_{(k-1)*}}\right)}\right).$$

Lemma 3.2.4. *If Q and $\overline{L}$ are defined as above, then*

$$\begin{aligned} -Q^o &= conv(\{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\}) \\ &= \bigcap_{i\in\{1,2\ldots,(k-1)\}} \overline{H^-\left(-\hat{u}_{i*}, d_{i*}\right)} \cap \overline{L}. \end{aligned}$$

Proof. It follows, by the properties of the polar set listed in Chapter 1, and Theorem 2.2.2, that:

$$\begin{aligned} -Q^o &= \lim_{N\to\infty} conv(\{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\} \cup \{\frac{\hat{j}}{N}\}) \\ &= \lim_{N\to\infty} \bigcap_{i\in\{1,2\ldots,(k-1)\}} \overline{H^-\left(-\hat{u}_{i*}, d_{i*}\right)} \cap conv\{(0,-N), d_{1*}(-\hat{u}_{1*}), d_{(k-1)*}(-\hat{u}_{(k-1)*}), (0,\frac{1}{N})\}. \end{aligned}$$

As $N \to \infty$,

$$\begin{aligned} -Q^o &= conv(\{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\}) \\ &= \bigcap_{i\in\{1,2\ldots,(k-1)\}} \overline{H^-\left(-\hat{u}_{i*}, d_{i*}\right)} \cap (\overline{L} \cap \overline{H^-\left(-\hat{u}_{1*}, d_{1*}\right)} \cap \overline{H^-\left(-\hat{u}_{(k-1)*}, d_{(k-1)*}\right)}) \\ &= \bigcap_{i\in\{1,2\ldots,(k-1)\}} \overline{H^-\left(-\hat{u}_{i*}, d_{i*}\right)} \cap \overline{L}. \end{aligned}$$

□

Definition 3.2.2. *With the same notations as in the Definition 3.2.1 of Q, define*

$$\begin{aligned} \widetilde{-Q^o} &= \bigcap_{i\in\{1,2\ldots,(k-1)\}} \overline{H^-\left(-\hat{u}_{i*}, d_{i*}\right)} \\ &= conv(\{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq (k-1)\}) \cup \left(\overline{U} \cap \overline{H^-\left(-\hat{u}_1, \frac{1}{d_1}\right)} \cap \overline{H^-\left(-\hat{u}_k, \frac{1}{d_k}\right)}\right). \end{aligned}$$

Theorem 3.2.2. *If Q and $\widetilde{-Q^o}$ are as defined above, then*

$$\begin{aligned} Q \cap \widetilde{-Q^o} &= \bigcap_{i \in \{1,2\ldots,(k-1)\}} \overline{H^-(-\hat{u}_{i*}, d_{i*})} \cap \bigcap_{i \in \{1,2\ldots,k\}} \overline{H^-(\hat{u}_i, d_i)} \\ &= conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \le i \le (k-1)\} \cup \{\frac{-\hat{u}_i}{d_i} \mid 1 \le i \le k\}) \end{aligned}$$

is a self-nolar polytope.

Proof. By assumptions a, b in Definition 3.2.1 and Lemma 2.3.1, for $1 \le i \le (k-1)$, we have

$$\frac{\hat{u}_{i*}}{d_{i*}} \in \overline{H^-\left(\hat{u}_{1*}, \frac{1}{d_{1*}}\right)} \cap \overline{H^-\left(\hat{u}_{(k-1)*}, \frac{1}{d_{(k-1)*}}\right)} \cap \overline{U}$$

and, for $1 \le i \le k$,

$$\frac{-\hat{u}_i}{d_i} \in \overline{H^-\left(-\hat{u}_1, \frac{1}{d_1}\right)} \cap \overline{H^-\left(-\hat{u}_k, \frac{1}{d_k}\right)} \cap \overline{L},$$

where

$$\overline{H^-\left(\hat{u}_{1*}, \frac{1}{d_{1*}}\right)} \cap \overline{H^-\left(\hat{u}_{(k-1)*}, \frac{1}{d_{(k-1)*}}\right)} = \overline{H^-\left(-\hat{u}_1, \frac{1}{d_1}\right)} \cap \overline{H^-\left(-\hat{u}_k, \frac{1}{d_k}\right)}.$$

The latter equality implies that

$$Q \cap \widetilde{-Q^o} = conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \le i \le (k-1)\}) \cup conv(\{\frac{-\hat{u}_i}{d_i} \mid 1 \le i \le k\}).$$

By the properties of polar sets, it follows that

$$\begin{aligned}
-(Q \cap \widetilde{-Q^o})^o &= \left(conv(\{\frac{-\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\})\right)^o \cap \left(conv(\{\frac{\hat{u}_i}{d_i} \mid 1 \leq i \leq k\})\right)^o \\
&= \left[\bigcap_{i \in \{1,2\ldots,(k-1)\}} \overline{H^-(-\hat{u}_{i*}, d_{i*})}\right] \cap \left[\bigcap_{i \in \{1,2\ldots,k\}} \overline{H^-(\hat{u}_i, d_i)}\right] \\
&= \widetilde{-Q^o} \cap Q \\
&= Q \cap \widetilde{-Q^o},
\end{aligned}$$

implying that $Q \cap \widetilde{-Q^o}$ is a self-nolar set.

In addition, since Q and $\widetilde{-Q^o}$ are convex, we have that

$$Q \cap \widetilde{-Q^o} = conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\}) \cup conv(\{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\})$$

is convex as well. It follows that

$$\begin{aligned}
Q \cap \widetilde{-Q^o} &= conv\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\} \cup conv\{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\} \\
&= conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\} \cup \{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\}).
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
Q \cap \widetilde{-Q^o} &= \left[\bigcap_{i \in \{1,2\ldots,(k-1)\}} \overline{H^-(-\hat{u}_{i*}, d_{i*})}\right] \cap \left[\bigcap_{i \in \{1,2\ldots,k\}} \overline{H^-(\hat{u}_i, d_i)}\right] \\
&= conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\} \cup \{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\})
\end{aligned}$$

is a self-nolar polytope.

□

Theorem 3.2.3. *Let $\overline{\mathcal{N}}$ be the set of all self-nolar convex sets, and $\mathcal{N}$ be the set of all self-nolar polytopes. If $C \in \overline{\mathcal{N}}$, then for all $\epsilon > 0$, there exist $P \in \mathcal{N}$, such that*

$d_H(P, C) < \epsilon$.

Here $d_H(A, B)$ denotes the Hausdorff distance between sets A and B.

Proof. Let $C \in \overline{\mathcal{N}}$. By Theorem 3.2.2, there exist two boundary points $n, m \in Bd(C)$ such that $m = -\frac{n}{||n||^2}$, with $n \cdot m = -1$. Without any loss of generality, we may assume that m and n lie on the x-axis such that $m = d\hat{i}$ and $n = -\frac{\hat{i}}{d}$ for some $d \in \mathbb{R}^+$ and $\hat{i} = (1, 0)$. By Corollary 3.2.1, we have that

$$C = -C^o \subset \overline{H^-(-\hat{i}, \frac{1}{d})} \cap \overline{H^-(\hat{i}, d)}.$$

Now, let $\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \le i \le r\} \subset Bd(C)$, where

$$Q = conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid i \in \{1, 2..., k, ..., r\}\}) = \bigcap_{i \in \{1,2...,k,...,r\}} \overline{H^-(\hat{u}_i, d_i)} \subseteq C$$

such that

a. $d_H(Q, C) < \epsilon_*$, for a given $\epsilon_* \in \mathbb{R}^+$,

b. $Q_{\overline{U}} = Q \cap \overline{U} = conv\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \le i \le (k-1)\}$,

c. $d_1\hat{u}_1 = \frac{\hat{u}_{1*}}{d_{1*}} = (-\frac{1}{d}, 0) = n$ and $d_k\hat{u}_k = \frac{\hat{u}_{(k-1)*}}{d_{(k-1)*}} = (d, 0) = m$,

d. $\frac{\hat{u}_{i*}}{d_{i*}} \in \overline{H^-\left(\hat{u}_{1*}, \frac{1}{d_{1*}}\right)} \cap \overline{H^-\left(\hat{u}_{(k-1)*}, \frac{1}{d_{(k-1)*}}\right)} \cap \overline{U}$, for $1 \le i \le (k-1)$,

e. $\frac{\hat{u}_{i*}}{d_{i*}} = H(\hat{u}_i, d_i) \cap H(\hat{u}_{i+1}, d_{i+1})$, for $1 \le i \le (k-1)$,

f. $H(\hat{u}_i, d_i)$ has a positive y-intercept, for $2 \le i \le (k-1)$.

It follows that

$$-Q^o = \bigcap_{i \in \{1,2...,k,...,r\}} \overline{H^-(-\hat{u}_{i*}, d_{i*})} = conv(\{\frac{-\hat{u}_i}{d_i} \mid i \in \{1, 2..., k, ..., r\}\}).$$

By Theorem 2.3.1,

$$-Q^o_{\overline{L}} = -Q^o \cap \overline{L} = conv(\{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\}).$$

Since $d_H(Q, C) < \epsilon_*$, we have that

$$\begin{aligned}
(1-\epsilon_*)Q \subseteq C \subseteq (1+\epsilon_*)Q \quad &\Longrightarrow \quad \left(\frac{1}{1+\epsilon_*}\right)Q^o \subseteq C^o \subseteq \left(\frac{1}{1-\epsilon_*}\right)Q^o \\
&\Longrightarrow \quad \left(\frac{1}{1+\epsilon_*}\right)(-Q^o) \subseteq -C^o \subseteq \left(\frac{1}{1-\epsilon_*}\right)(-Q^o) \\
&\Longrightarrow \quad \left(\frac{1}{1+\epsilon_*}\right)(-Q^o) \subseteq C \subseteq \left(\frac{1}{1-\epsilon_*}\right)(-Q^o), \\
&\Longrightarrow \quad \left(1-\left(\frac{\epsilon_*}{1+\epsilon_*}\right)\right)(-Q^o) \subseteq C \subseteq \left(1+\left(\frac{\epsilon_*}{1-\epsilon_*}\right)\right)(-Q^o).
\end{aligned}$$

This, in conjunction with the fact that, $\forall \epsilon \in (0, \frac{1}{2})$,

$$\left(1-\left(\frac{\epsilon_*}{1-\epsilon_*}\right)\right) \geq 1-2\epsilon_*$$

and

$$\left(1+\left(\frac{\epsilon_*}{1-\epsilon_*}\right)\right) \leq 1+2\epsilon_*,$$

implies that $d_H(-Q^o, C) < 2\epsilon_*$.

Let $\epsilon = 2\epsilon_*$, then

$$\begin{aligned}
d_H(Q, C) < \epsilon \quad &\Longrightarrow \quad d_H(Q_{\overline{U}}, C \cap \overline{U}) < \epsilon \\
d_H(-Q^o, C) < \epsilon \quad &\Longrightarrow \quad d_H(-{Q^o}_{\overline{L}}, C \cap \overline{L}) < \epsilon.
\end{aligned}$$

Define

$$
\begin{aligned}
P &= Q_{\overline{U}} \cup -Q^{o}{}_{\overline{L}} \\
&= conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\}) \cup conv(\{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\}).
\end{aligned}
$$

Since $d_H(Q_{\overline{U}}, C \cap \overline{U}) < \epsilon$, $d_H(-Q^{o}{}_{\overline{L}}, C \cap \overline{L}) < \epsilon$ and $P = Q_{\overline{U}} \cup -Q^{o}{}_{\overline{L}}$, it is the case that $d_H(P, C) < \epsilon$. In addition, by Theorem 3.2.2, we can conclude that

$$
\begin{aligned}
P &= conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\}) \cup conv(\{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\}). \\
&= conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\} \cup \{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\}) \\
&= \bigcap_{i \in \{1,2\ldots,(k-1)\}} \overline{H^{-}\left(-\hat{u}_{i*}, d_{i*}\right)} \cap \bigcap_{i \in \{1,2\ldots,k\}} \overline{H^{-}\left(\hat{u}_i, d_i\right)}
\end{aligned}
$$

is a self-nolar polytope.

$\square$

3.3 A Practical Method to Construct Self-nolar Polytopes

Theorem 3.2.2 enables us to construct self-nolar polytopes in a simple way. The first step is to choose two finite sets of points $M = \{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\}$ and $\{d_1\hat{u}_1, d_k\hat{u}_k\}$ that satisfy:

a. $d_1\hat{u}_1 = \frac{\hat{u}_{1*}}{d_{1*}} = (-\frac{1}{d}, 0)$ and $d_k\hat{u}_k = \frac{\hat{u}_{(k-1)*}}{d_{(k-1)*}} = (d, 0)$,

b. $\frac{\hat{u}_{i*}}{d_{i*}} \in \overline{H^{-}\left(\hat{u}_{1*}, \frac{1}{d_{1*}}\right)} \cap \overline{H^{-}\left(\hat{u}_{(k-1)*}, \frac{1}{d_{(k-1)*}}\right)} \cap \overline{U}$, for $1 \leq i \leq (k-1)$,

c. $\frac{\hat{u}_{i*}}{d_{i*}} = H\left(\hat{u}_i, d_i\right) \cap H\left(\hat{u}_{i+1}, d_{i+1}\right)$, for $1 \leq i \leq (k-1)$,

d. $H\left(\hat{u}_i, d_i\right)$ has a positive y-intercept, for $2 \leq i \leq (k-1)$.

e. $\bigcup_{i\in\{1,2\ldots,(k-2)\}} \overline{\frac{\hat{u}_{i*}}{d_{i*}}, \frac{\hat{u}_{(i+1)*}}{d_{(i+1)*}}}$ is part of the boundary of a convex polytope.

From a, c and d we can easily determine the set of points $N = \{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\}$. Then, by the previous theorem, we have that

$$M \cap N \quad = \quad conv(\{\frac{\hat{u}_{i*}}{d_{i*}} \mid 1 \leq i \leq (k-1)\} \cup \{\frac{-\hat{u}_i}{d_i} \mid 1 \leq i \leq k\})$$

is a self-nolar polytope.

The following figures of self-nolar polytopes were constructed with the aforementioned method.

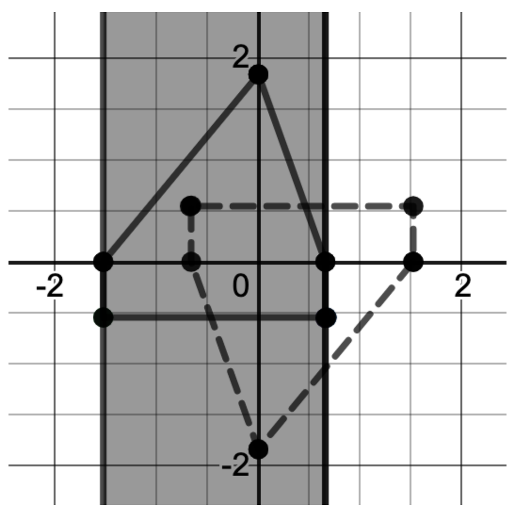

Figure 3.1: A self-nolar pentagon (solid boundary line) and its polar set (dashed boundary line).

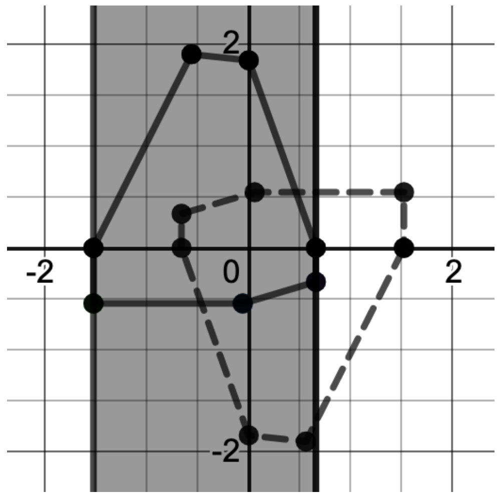

Figure 3.2: A self-nolar heptagon (solid boundary line) and its polar set (dashed boundary line).

3.4 Metric Property: Boundary Length

The following facts will be used in the next lemma. Recall that for a convex set C in $\mathbb{R}^2$, $h : [0, 2\pi] \to \mathbb{R}$ is the support function of C, and is 2π periodic, where:

$$\begin{aligned} h(\hat{u}) &= h(\hat{u}(\theta)) = \max\{x \cdot \hat{u} \mid x \in C\} \\ &= d, \end{aligned}$$

such that $H^-(\hat{u}, d)$ is an essential halfplane of C. Note that if C contains the origin of the plane, then $d > 0$ in all directions. In addition, $r : [0, 2\pi] \to \mathbb{R}$ is the radial function of C, and is 2π periodic, where:

$$\begin{aligned} r(\hat{u}) &= r(\hat{u}(\theta)) \\ &= d, \end{aligned}$$

such that $d(\hat{u})$ is a boundary point of C. It is a well known fact that the reciprocal of the radial function of a convex body C is the support function of its polar C^o and vice versa. So $h^o(\hat{u}(\theta)) = \frac{1}{r(\hat{u}(\theta))}$, where $h^o(\hat{u}(\theta))$ is the support function of C^o. Another well established fact is that for a smooth convex curve, its length is simply the integral of its support function over $[0, 2\pi]$. Proof of these facts can be found in [6].

Lemma 3.4.1. *Let C be a self-nolar set with smooth boundary $Bd(C)$ and boundary length $L(Bd(C))$, then $L(Bd(C)) \geq 2\pi$, with equality only when C is the unit disk B.*

Proof. Since C is self-nolar with smooth $Bd(C)$, then $Bd(C^o)$ is smooth and $L(Bd(C)) = L(Bd(C^o)) = \int_0^{2\pi} h^o(\hat{u}(\theta))d\theta = \int_0^{2\pi} h^o(-\hat{u}(\theta))d\theta$. From this, coupled with the AM-GM inequality, the fact that $h^o(\hat{u}(\theta)) = \frac{1}{r(\hat{u}(\theta))}$ and Lemma 3.2.3, it follows that

$$\begin{aligned}
2(L(Bd(C))) &= L(Bd(C^o)) + L(Bd(C^o)) \\
&= \int_0^{2\pi} (h^o(\hat{u}(\theta)) + h^o(-\hat{u}(\theta)))d\theta \\
&\geq \int_0^{2\pi} 2\sqrt{(h^o(\hat{u}(\theta))h^o(-\hat{u}(\theta))}d\theta \\
&= 2\int_0^{2\pi} \frac{1}{\sqrt{(r(\hat{u}(\theta))r(-\hat{u}(\theta))}}d\theta \\
&\geq 2(2\pi).
\end{aligned}$$

So, $(L(Bd(C))) \geq 2\pi$. For equality to hold, $(r(\hat{u}(\theta))r(-\hat{u}(\theta)) = 1$, for all directions $\hat{u}$, and we must have equality in AM-GM inequality, hence $h^o(\hat{u}(\theta)) = h^o(-\hat{u}(\theta))$. It follows that $r(\hat{u}(\theta)) = 1$ in all directions and, thus, C is a disk.

$\square$

3.5 A Connection Between Self-nolar Polytopes and the Mahler Product

In this section we estabish a connection between self-nolar polytopes and the Mahler product by explicitly finding a self-nolar polytope that minimizes the Mahler product in $\mathbb{R}^2$.

Lemma 3.5.1. *If T is a self-nolar triangle with the origin $(0,0)$ as its center of mass, height H and base B, then T is the unique equilateral triangle (up to a rigid rotation) with $H = \frac{3}{\sqrt{2}}$ and $B = \frac{2\sqrt{3}}{\sqrt{2}}$.*

Proof. Let $h, b \in \mathbb{R}^+$. Without any loss of generality, let point $(0, h)$ be a vertex of triangle T. For T to be self-nolar the two other vertices must belong to line $y = \frac{-1}{h}$. So, let $(b, \frac{-1}{h})$ be another vertex of T. By Theorem 2.2.1, T is self-nolar if and only if $(\frac{-(h^2+1)}{bh^2}, \frac{-1}{h})$ is also a vertex of T. So, $(0, h)$, $(b, \frac{-1}{h})$ and $(\frac{-(h^2+1)}{bh^2}, \frac{-1}{h})$ are the three vertices of T. It is well known that if (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of a triangle, then its center of mass is $(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3})$. Thus, for T to have the origin as its center of mass, we must have

$$\begin{cases} 0 + b - \frac{(h^2+1)}{bh^2} = 0 \\ h - \frac{2}{h} + 0 = 0 \end{cases}$$

implying that $h = \sqrt{2}$ and $b = \frac{\sqrt{3}}{\sqrt{2}}$. It follows that the vertices of T are $(0, \sqrt{2})$,$(\frac{\sqrt{3}}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(-\frac{\sqrt{3}}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. By inspection, we may conclude that T is and equilateral triangle with $H = \frac{3}{\sqrt{2}}$ and $B = \frac{2\sqrt{3}}{\sqrt{2}}$. $\square$

Note that based on this lemma we can say, loosely speaking, that generally a self-nolar triangle will not have the origin as its center of mass.

For the next lemma some background information is needed. Let C be a compact convex set in $\mathbb{R}^2$ with area $A(C)$. The Mahler product is defined as the minimum, for $x \in C$, of $A(C)A((C-x)^o)$. The unique point where this minimum is attained is called the Santaló point of C. It is well known that x is the Santaló point for C if and only if the origin is the center of mass for $(C-x)^o$, [8]. In 1939, Mahler proved in [4] that $A(C)A((K-x)^o) \geq \frac{27}{4}$ and, in 1991, Meyer showed that equality holds only for triangles, [5]. In the following lemma we explicitly construct a self-nolar triangle, with both its center of mass and Santaló point being the origin, that minimizes the Mahler product.

Lemma 3.5.2. *There exists a self-nolar polytope that minimizes the Mahler product.*

Proof. Let T be the self-nolar triangle from Lemma 3.5.1. From a direct calculation we find that $A(C) = \frac{(\frac{3}{\sqrt{2}})(\frac{2\sqrt{3}}{\sqrt{2}})}{2} = \frac{3\sqrt{3}}{2}$. Since T is self-nolar with the origin as its center of mass, this implies that T^o also has the origin as its center of mass and $A(T^o) = \frac{3\sqrt{3}}{2}$. It follows that the origin is the Santaló point for C and that $A(T)A((T)^o = \frac{27}{4}$. Therefore, T minimizes the Mahler product.

$\square$

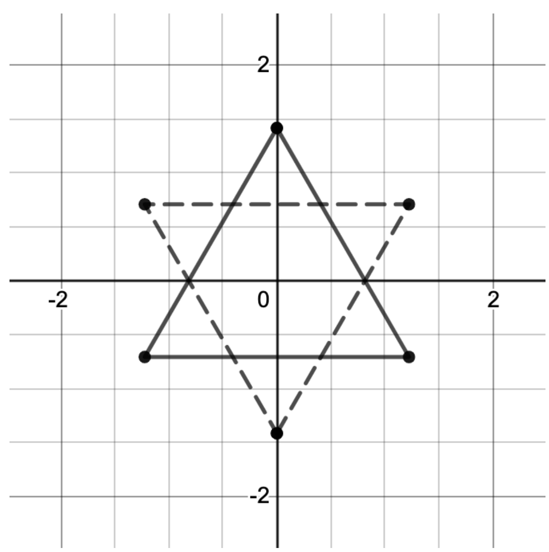

Figure 3.3: A self-nolar triangle (solid line boundary), that minimizes the Mahler product, and its polar set(dashed line boundary).

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